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Speciality: Optimization and Control

Theme :

NEW TECHNIQUES FOR DETERMINING SEARCH DIRECTIONS OF INTERIOR POINT ALGORITHMS IN OPTIMIZATION

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Glossary of abbreviations and notations

MP	:	Mathematical Programming;
LP, LO	:	Linear Programming, Linear Optimization;
QP	:	Quadratic Programming;
CQP, CQO	:	Convex Quadratic Programming, Convex Quadratic Optimization;
SDO	:	Semidefinite Optimization;
CQSDO	:	Convex Quadratic Semidefinite Optimization;
PSD	:	Positive Semidefinite;
PD	:	Positive Definite;
IP	:	Interior Point;
IPMs	:	Interior Point Methods;
IPA	:	Interior Point Algorithm;
IPC	:	Interior Point condition;
AET	:	Algebraic Equivalent Transformation;
T(s)	:	The required time (in seconds) to obtain an optimal solution;
Iter	:	The number of iterations produced by the algorithm to obtain an optimal solution;
$\mathbb{R}$	:	The set of real numbers ;
$\mathbb{R}^n$	:	The real n-dimensional space;
$\mathbb{R}^n_+$	:	The nonnegative orthant in $\mathbb{R}^n$;
$\mathbb{R}^n_{++}$	:	The positive orthant in $\mathbb{R}^n$;
$\mathbb{R}^{n\times n}$	:	The space of $n \times n$ real squared matrices;

$\mathbb{S}^n$	:	The cone of symmetric matrices;
$\mathbb{S}^n_+$	:	The cone of symmetric positive semidefinite matrices;
$\mathbb{S}^n_{++}$	:	The cone of symmetric positive definite matrices;
e	:	$= (1, \ldots, 1)^T$; Vector of ones;
x^T	:	$= (x_1, \ldots, x_n)$; The transpose of a vector x with components x_i;
xy	:	$= (x_1y_1, \ldots, x_ny_n)^T$; Hadamard product;
x^Ty	:	$= \sum_{i=1}^{n} x_iy_i$; The standard inner product of two vectors $x, y \in \mathbb{R}^n$;
$\sqrt{x}$	:	$= (\sqrt{x_1}, \ldots, \sqrt{x_n})^T$, $(x \geq 0)$;
x^{-1}	:	$= (\frac{1}{x_1}, \ldots, \frac{1}{x_n})^T$ $(x_i \neq 0)$;
$\frac{x}{y}$	:	$= (\frac{x_1}{y_1}, \ldots, \frac{x_n}{y_n})^T$ $(y_i \neq 0)$;
$\lVert x \rVert$	:	The Euclidean norm of $x \in \mathbb{R}^n$;
$\lVert x \rVert_\infty$	:	$= \max_{i=1,\ldots,n} \lvert x_i \rvert$; The maximum norm of $x \in \mathbb{R}^n$;
$\min(x)$	:	The minimal component of the vector x;
I_n	:	Identity matrix of order n;
$Diag(x)$	:	The diagonal matrix with diagonal elements equal to the components of the vector x with $X_{ii} = x_i$;
$\det(M)$	:	The determinant of a square matrix M;
$\mathrm{Tr}(M)$	:	The trace of a square matrix M;
$\lambda(M)$	:	The vector of eigenvalues of a matrix M;
$\lambda_{\min}(M)$	:	The smallest eigenvalue of a matrix M;
$\lambda_{\max}(M)$	:	The largest eigenvalue of a matrix M;
$A \bullet B$	:	$= \langle A, B \rangle = \mathrm{Tr}(AB) = \sum_{i,j} a_{ij}b_{ij}$; The inner product on $\mathbb{S}^n$ of two matrices A and B;

$\|M\|_F$	:	$= \sqrt{M \bullet M} = \sqrt{\text{Tr}(M^2)} = \sqrt{\sum_{i=1}^{n} \lambda_i^2(M)}$; The Frobenius norm of a symmetric matrix M;
$A \succeq B$	:	indicates that the matrix $A - B$ belongs to $\mathbb{S}^n_+$;
$A \succ B$	:	indicates that the matrix $A - B$ belongs to $\mathbb{S}^n_{++}$;
$Q^{1/2}$	:	The symmetric square root of $Q \in \mathbb{S}^n_{++}$; denoted also as $\sqrt{Q}$;
$A \sim B$	:	$\Leftrightarrow A = ZBZ^{-1}$ for some invertible matrix Z; means the similarity between A and B;
∇f	:	The gradient of a function $f : \mathbb{R}^n \to \mathbb{R}^n$;
$\nabla^2 f$	:	The Hessian matrix of f;
$\dfrac{\partial f_i}{\partial x_j}(x)$	:	The partial derivative of f_i at x_j;

Introduction

Optimization is a fundamental concept that pervades various fields, playing a crucial role in decision-making, resource allocation, and problem-solving. At its core, optimization involves the process of finding the best possible solution from a set of feasible alternatives, with the objective of either maximizing or minimizing an objective function while satisfying certain constraints. Understanding the different types of optimization problems, their characteristics, applications and methodologies is essential for addressing complex challenges and driving innovation across industries.

Optimization problems are characterized by their ability to optimize objectives while considering constraints. These problems can vary widely in complexity, involving linear or nonlinear objective functions, as well as linear or nonlinear constraints. Despite their complexity, optimization problems find applications in numerous fields, including finance, engineering, logistics, and operations research. For instance, in finance, optimization techniques are used for portfolio management and asset allocation, while in engineering, they are employed for process optimization and system design. The importance of optimization lies in its ability to enhance efficiency, improve decision-making processes, and maximize resources, ultimately leading to cost savings and competitive advantages for organizations.

Optimization problems come in various forms and complexities, reflecting the diverse challenges encountered in real-world scenarios. These problems span across a spectrum, ranging from simple linear objectives with linear constraints to more complex quadratic, nonlinear or even stochastic objectives with nonlinear or stochastic constraints. Linear programming, convex quadratic programming, semidefinite optimization, and other types of optimization problems represent different facets of this spectrum.

Linear Optimization (LO) focuses on optimizing linear objective functions subject to linear equality and inequality constraints. LO has widespread applications in resource allocation, production planning, and transportation logistics. Its simplicity and efficiency make it a popular choice for addressing optimization problems in various industries.

Convex Quadratic Optimization (CQO) involves optimizing quadratic objective functions subject to linear constraints. Quadratic programming (QP) is utilized in portfolio optimization, control system design, and structural optimization. The convexity property of quadratic functions ensures the existence of globally optimal solutions, making QP particularly valuable in practical applications.

Semidefinite Optimization (SDO) extends linear programming to problems with linear matrix inequality constraints. It is applied in robust control, signal processing, and combinatorial optimization. Semidefinite optimization techniques offer powerful tools for solving complex optimization problems involving structured matrices and non-convex constraints.

Convex Quadratic Semidefinite Optimization (CQSDO) combines the properties of convex quadratic programming and semidefinite optimization. This hybrid approach is used in machine learning, quantum information theory, and computational biology. By leveraging both convexity and semidefinite relaxation, convex quadratic semidefinite optimization enables efficient solutions to challenging optimization problems.

Indeed, LO is often considered one of the simplest optimization problems due to its linear objective function and linear constraints. Despite its apparent simplicity, LO has extensive implications and applications across various fields. In economics, for instance, the importance of LO was recognized with the Nobel Prize in Economics awarded to Kantorovich [37] and Koopmans [51] in 1976.

In 1947, Dantzig proposed the well-known simplex method for solving LO problems [15]. Before 1984, every LO problem was solved using the simplex method [15] or a variant thereof. Research efforts were made to develop alternative methods, but none of the proposed methods improved upon the simplex method.

In the 1970s, complexity theory became integral to LO, raising questions about the theoretical limits of solving linear programs. Klee and Minty's discovery [47] of an example showcasing the exponential nature of the simplex method's worst-case complexity highlighted the need for more efficient algorithms.

Khachiyan's breakthrough in 1979 with the ellipsoid method demonstrated a polynomial-time algorithm for linear programming [40]. However, the ellipsoid method, while theoretically significant, was not as practical as the simplex method in real-world applications.

In 1984, Karmarkar proposed an innovative interior point method with polynomial convergence, competing with the simplex method for solving large-scale LO problems [38]. This method and its variants that were developed subsequently are now called interior-point methods (IPMs). For a survey, we refer to [5, 16, 62, 65, 85]. Megiddo [53] and Sonnevend [67] were the first to recognize the relevance of the central path for LO. The authors in [66] investigated the first primal-dual path-following IPM for LO problems with full-Newton step. This technique has extensively extended to other optimization problems (e.g., [44, 48, 81]).

In IPMs, the determination of search directions plays a key role in the feasibility, convergence and complexity bound of the algorithm. In fact, one can apply kernel functions technique to achieve this, this last requires two types of iterations, inner iterations and outer iterations (e.g. [13, 10, 12]). Moreover, Darvay [18] introduces a new method for finding efficient search directions. He applied the square root function on both sides of an algebraic equivalent transformation (AET) on the centering equation of the system which defines the central path. Then he used the full-Newton method to the resulting system. This method is extended to other optimization problems such as: CQO [1], SDO [77], second-order cone optimization (SOCO)[78] and symmetric optimization (SO) [79]. Moreover, Kheirfam and Nasrollahi in [46] extended this technique which is based on the square root function to the integer powers of this function. Furthermore, Based on the AET strategy, Darvay and Takàcs in [19] considered a new function to present a new primal-dual IPM for L0. In the same way, Kheirfam and Haghighi in [45] presented a new primal-dual IPM for $P_*(\kappa)$-linear complementarity. For more related papers, we refer to [2, 31, 56, 21].

Currently, the AET technique has become a wide research interest, and the search for a new AET to describe a new primal-dual IPM has become an important motivation for researchers. In 2011, Zhang and Xu [93] proposed a specific search direction for LO. They considered the equivalent form $v^2 = v$ of the centering equation, and they transformed it into the form $xs = \mu v$. After that, they assumed that the variance vector is fixed and they applied full-Newton's method. Based on this new AET, Darvay and Takàcs [20] proposed another technique to obtain a new descent direction for solving LO. They applied the function $\psi(t) = t^2$ on both sides of the nonlinear equation $v^2 = v$. Next, they used full-Newton's method to get the new search direction. The authors proved both the theoretical and numerical effectiveness of their method compared to other existing techniques. Furthermore, the same authors in [68] extended this approach to symmetric optimization problems. Later, Kheirfam [41] extended the method to $P_*(\kappa)$-horizontal linear complementarity problems, while Guerra [32] applied it to the SDO case. For more related papers about Darvay and Takàcs' technique, we refer to (see e.g., [23, 42, 43]).

Motivated by the above-mentioned works, many questions naturally arise. Chief among these inquiries is the reevaluation of the method proposed in [20] by incorporating new functions. Additionally, there arises a compelling need to explore the potential for extending the works of Darvay and Takàcs [20] and Zhang and Xu [93] to a broader spectrum of optimization problems such as: CQO, SDO and CQSDO. By doing so, we aim to unlock new dimensions of applicability and efficacy in optimization methodologies. These questions hold paramount significance as they catalyze the advancement of research and innovation within the realm of optimization.

This thesis aims to propose, develop and analyze new interior point algorithms (IPAs) based on new descent directions in optimization. We explore the possibility of extending some IPAs from LO to more general problems, such as CQO, SDO and CQSDO. Therefore, we include some new results related to efficient complexity of the corresponding algorithms.

Short Out line of the Thesis.

The thesis contains six chapters, followed by a bibliography which are organized as follows:

Chapter 1: A mathematical background on convex analysis, matrix theory and mathematical programming is stated which will be utile throughout this thesis.

Chapter 2: We recall the concept of the central path, outlining its properties. Then, we derive the classical Newton search direction for LO. Furthermore, we introduce a pioneering IPA for LO, which relies on the algebraic equivalent transformation technique for determining search directions. The chapter ends with a discussion of the most recent modification to this technique.

Chapter 3: Building upon the work of Darvay and Takàcs [20] which is based on the technique of algebraic transformation, we propose two new functions $\psi(t)$ to enhance the performance of path-following algorithms. The first function is $\psi(t) = t^{\frac{7}{4}}$, where we have shown that the corresponding algorithm converges after $O\left(\sqrt{n}\log\frac{n+\frac{3}{\sqrt[7]{2}}}{\varepsilon}\right)$ iterations. The second function is defined by $\psi(t) = t^{\frac{3}{2}}$, where we have proven that the corresponding algorithm converges after $O\left(\sqrt{n}\log\frac{n+\sqrt[3]{4}}{\varepsilon}\right)$ iterations. Numerical tests specifically on some problem from the Netlib test collections were conducted to consolidate the obtained theoretical results.

Chapter 4: We introduce new primal-dual IPAs for CQO. The first proposed algorithm is based on an extension of the techniques presented in the work of Darvay and Takàcs for LO [20], while the other on it's first idea presented in the work of Zhang and Xu [93]. We demonstrate that the presented methods solve efficiently the CQO problem within polynomial time. Notably, the short-step algorithms achieve the best-known iteration bound. Moreover, we present a comparative numerical study to prove the efficiency of our proposed algorithms.

Chapter 5: We present a novel primal-dual IPA tailored for SDO. Drawing inspiration from Zhang and Xu's approach to linear optimization, our method extends their technique. The symmetrization of the search direction is based on the Nesterov-Todd scaling scheme. We shown that our short-step algorithm achieves the best-known iteration bound. Namely, $O(\sqrt{n}\log\frac{n}{\varepsilon})$ iterations. Furthermore, we conduct a comprehensive numerical study, focusing on some SDO applications to underscore the effectiveness of our proposed algorithm.

Chapter 6: We introduce a primal-dual IPA for CQSDO. This algorithm is based on an extension of the technique presented in the work of Zhang and Xu for LO [93]. The symmetrization of the search direction is based on the Nesterov-Todd scaling scheme. Our analysis shows that this method solves efficiently the problem within polynomial time. Notably, the obtained short-step algorithm achieves the best-known iteration bound, namely $O(\sqrt{n}\log\frac{n}{\varepsilon})$ iterations. The numerical experiments conclude that the newly proposed algorithm is not only polynomial but requires a number of iterations clearly lower than that obtained theoretically.

Finally, we end this thesis by a general conclusion and suggestions for future work.

CHAPTER 1

Fundamental notions

In this chapter, we will introduce certain concepts and results of matrix calculus and some properties of symmetric matrices (especially, positive semidefinite matrices), as well as basic notions of convex analysis and mathematical programming that will be useful later. For more details we refer to [8, 39, 61, 63, 84].

1.1 Matrix theory

In this section, we provide some useful notions concerning matrix analysis. We start with $\mathbb{M}_n$, the set of square matrices of order n with real coefficients:

$$\mathbb{M}_n = \{A \in \mathbb{R}^{n \times n}\}.$$

and the set $\mathbb{S}^n$, which denotes the space of symmetric matrices of $\mathbb{M}_n$:

$$\mathbb{S}^n = \{A \in \mathbb{M}_n \mid A^T = A\}.$$

1.1.1 Eigenvalues and Spectrum

1. Let $A \in \mathbb{M}_n$ be a matrix. Then, A is regular (or invertible) if and only if $\det(A) \neq 0$, where $\det(A)$ is the determinant of the matrix A.

2. The polynomial $p_A(\lambda)$, defined by $p_A(\lambda) = \det(A - \lambda I)$, is called the characteristic polynomial of A. The roots of $p_A(\lambda)$ are called the eigenvalues of A.

3. Let $\lambda_1, \cdots, \lambda_m$ be the eigenvalues of the matrix A of order n. Then, the characteristic polynomial can be represented as:

$$p_A(\lambda) = p_0 + p_1\lambda + \cdots + p_{n-1}\lambda^{n-1} + \lambda^n = (\lambda - \lambda_1)^{n_1} \cdots (\lambda - \lambda_m)^{n_m},$$

where n_i are positive integers such that $\sum_{i=1}^{m} n_i = n$. The number n_i is the algebraic multiplicity of the eigenvalue λ_i, $i = 1, \ldots, m$.

4. An eigenvalue is called simple if its multiplicity is equal to 1.

5. It is important to note that the determinant of a matrix A is the product of its eigenvalues. i.e., $\det(A) = \prod_{i=1}^{n} \lambda_i$.

6. The spectrum of $A \in \mathbb{M}_n$, for $n \geq 1$, is the set of eigenvalues of A:

$$\text{sp}(A) = \{\lambda \in \mathbb{R} \mid \det(\lambda I - A) = 0\}.$$

7. The spectral radius of A is the largest eigenvalue of A in absolute value:

$$\rho(A) = \max_{\lambda \in \text{sp}(A)} |\lambda|.$$

8. If $A \in \mathbb{S}^n$, then

(a) All eigenvalues of A, denoted $\lambda_i(A)$, are real.

(b) There exists an orthonormal basis in which A is diagonalizable, i.e., there exists an orthogonal matrix P (called the transition matrix) such that $A = PDP^T$, where D is a diagonal matrix. The columns of P are the eigenvectors of A and the eigenvalues (λ_i) of A are the diagonal coefficients of D. In this case, the matrices A and D are said to be similar.

9. For our purposes, it is appropriate to write the eigenvalues in increasing order:

$$\lambda_{\text{min}}(A) = \lambda_1(A) \leq \lambda_2(A) \leq \ldots \leq \lambda_n(A) = \lambda_{\text{max}}(A).$$

10. Let A and B be two matrices in $\mathbb{S}^n$, then:

(a) $\lambda_{\text{min}}(A + B) \geq \lambda_{\text{min}}(A) + \lambda_{\text{min}}(B)$.

(b) $\lambda_{\text{max}}(A + B) \leq \lambda_{\text{max}}(A) + \lambda_{\text{max}}(B)$.

11. Let $A \in \mathbb{M}_n$, then:

    (a) If A is orthogonal (i.e., $A^T A = I$), then the eigenvalues of A are -1 or 1.

    (b) A is regular (or invertible) if and only if $0 \notin$ sp(A).

12. Let $A \in \mathbb{M}_n$ be an invertible matrix, then:

    (a) λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T.

    (b) λ is an eigenvalue of A if and only if $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

    (c) λ is an eigenvalue of A if and only if $\lambda + \mu$ is an eigenvalue of $A + \mu I_n$.

    (d) λ is an eigenvalue of A if and only if $\mu\lambda$ is an eigenvalue of μA.

1.1.2 Trace, inner product and norm

1. The trace of a matrix $A \in \mathbb{M}_n$ is defined as the sum of its diagonal elements

$$\text{Tr}(A) = \sum_{i=1}^{n} a_{ii}, \quad \forall A \in \mathbb{M}_n.$$

2. The trace of a matrix $A \in \mathbb{M}_n$, equals the sum of its eigenvalues. i.e., $\text{Tr}(A) = \sum_{i=1}^{n} \lambda_i$.

3. For all $A, B, C \in \mathbb{M}_n$ and $\alpha, \beta \in \mathbb{R}$, the following properties hold:

    (a) $\text{Tr}(\alpha A + \beta B) = \alpha \text{Tr}(A) + \beta \text{Tr}(B)$ (Linearity).

    (b) $\text{Tr}(A^T) = \text{Tr}(A)$.

    (c) $\text{Tr}(A^2) \leq \text{Tr}(A^T A)$.

    (d) $\text{Tr}(AB) = \text{Tr}(BA)$ (Invariance under permutation).

    (e) $\text{Tr}(AB) \leq \frac{1}{2}(\text{Tr}(A^2 + B^2))$ for symmetric matrices A and B.

    (f) $\text{Tr}(ABC) = \text{Tr}(CAB) = \text{Tr}(BCA) \neq \text{Tr}(ACB)$.

    (g) $\text{Tr}(BAB^{-1}) = \text{Tr}(A)$.

4. The usual inner product of two vectors x and y in $\mathbb{R}^n$ is defined by:

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i = x^T y.$$

5. Similarly, an inner product is defined on the set of real square matrices. The inner product of two matrices $A, B \in \mathbb{M}_n$ is defined by:

$$A \bullet B = \langle A, B \rangle = \text{Tr}(A^T B) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ij} = B \bullet A.$$

Now, let's state the properties of the inner product $A \bullet B$. For this, consider $A, B, C \in \mathbb{M}_n$ and $\alpha, \beta \in \mathbb{R}$:

(a) $\langle A + B, C \rangle = \langle A, C \rangle + \langle B, C \rangle$.

(b) $\langle \alpha A, \beta B \rangle = \alpha\beta \langle A, B \rangle$.

6. The vector norm is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^+$, denoted by $|| \cdot ||$, and satisfies the following conditions:

(a) $\forall x \in \mathbb{R}^n : ||x|| = 0 \iff x = 0$,

(b) $\forall \alpha \in \mathbb{R}, \forall x \in \mathbb{R}^n : ||\alpha x|| = |\alpha| ||x||$.

(c) $\forall x, y \in \mathbb{R}^n : ||x + y|| \leq ||x|| + ||y||$.

7. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. The following are the commonly used vector norms:

(a) The ℓ_1 norm (also known as the Manhattan norm or taxicab norm) is defined as:

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|.$$

(b) The ℓ_2 norm (also known as the Euclidean norm) is defined as:

$$\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

(c) The ℓ_∞ norm (also known as the maximum norm or Chebyshev norm) is defined as:

$$\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

8. For any vector $\mathbf{x} \in \mathbb{R}^n$, we have:

$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \sqrt{n} \|\mathbf{x}\|_\infty.$$

$$\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq \sqrt{n} \|\mathbf{x}\|_2.$$

9. Let $A, B \in \mathbb{M}_n$. The mapping $||\cdot|| : \mathbb{M}_n \to \mathbb{R}^+$ is called the matrix norm and satisfies the following conditions:

   (a) $||A|| = 0 \iff A = 0$.

   (b) $||\alpha A|| = |\alpha|||A||$, where $\alpha \in \mathbb{R}$.

   (c) $||A + B|| \leq ||A|| + ||B||$.

   (d) $||AB|| \leq ||A||||B||$.

10. For any matrix $A \in \mathbb{M}_n$, we have:

    (a) $||A||_1 = \max\limits_{j=1}^{n} \sum\limits_{i=1}^{n} |a_{ij}|$.

    (b) $||A||_\infty = \max\limits_{i=1}^{n} \sum\limits_{j=1}^{n} |a_{ij}|$.

    (c) $||A||_2 = \sqrt{\max\limits_{i=1}^{n} |\lambda_i|} = \sqrt{\rho(A^T A)}$. This is called the Euclidean norm, where $\lambda_i, i = 1, \ldots, n$, are the eigenvalues of the matrix $A^T A$.

11. $||A||_F = \sqrt{A \bullet A} = \sqrt{\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} a_{ij}^2}, \quad \forall A \in \mathbb{M}_n$. This is called the Frobenius norm, it satisfies the following properties:

    (a) $||A||_F = ||A^T||_F$.

    (b) $||A||_2 \leq ||A||_F \leq \sqrt{n}||A||_2$.

    (c) $||\alpha A||_F = |\alpha|||A||_F \ \forall \alpha \in \mathbb{R}$.

    (d) $||A + B||_F \leq ||A||_F + ||B||_F$ (Triangle inequality).

    (e) $||AB||_F \leq ||A||_F||B||_F$.

    (f) $||A + B||_F^2 + ||A - B||_F^2 = 2(||A||_F^2 + ||B||_F^2)$ (Parallelogram identity).

    (g) $\langle A, B\rangle = \frac{1}{4}(||A + B||_F^2 - ||A - B||_F^2)$.

    (h) $\langle A, B\rangle = \frac{1}{2}(||A + B||_F^2 - ||A||_F^2 - ||B||_F^2)$.

    (i) If $\langle A, B\rangle = 0$, then $||A + B||_F^2 = ||A - B||_F^2 = ||A||_F^2 + ||B||_F^2$ (Pythagorean theorem).

12. We note that if $A \in \mathbb{S}^n$, then the following results can be easily obtained:

$$||A||_F = \sqrt{\text{Tr}(A^2)} = \sqrt{\sum_{i=1}^{n} \lambda_i^2(A)}.$$

$$||A||_2 = \sqrt{\lambda_{\max}(A^2)} = \max_{i=1} |\lambda_i(A)| = \rho(A).$$

For any matrix norm, we have:

$$\rho(A) \leq ||A||.$$

1.1.3 Positive semidefinite matrices and their properties

In this subsection, we are interested on symmetric positive semidefinite matrices.

Definition 1.1. A matrix $A \in \mathbb{M}_n$ is said to be:

- **Positive semidefinite** (written as $A \succeq 0$) if $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n$. We denote by $\mathbb{S}_+^n$ the set of symmetric positive semidefinite matrices.
- **Positive definite** (written as $A \succ 0$) if $x^T Ax > 0$ for all $x \in \mathbb{R}^n \backslash \{0\}$. We denote by $\mathbb{S}_{++}^n$ the set of symmetric positive definite matrices.

Theorem 1.2. *For a matrix $A \in \mathbb{S}^n$, the following propositions are equivalent:*

1. $A \in \mathbb{S}_+^n$ *(resp. $A \in \mathbb{S}_{++}^n$).*
2. $\lambda_i(A) \geq 0$ *(resp. $\lambda_i(A) > 0$) for all $i = 1, \ldots, n$.*
3. *There exists a matrix $C \in \mathbb{M}_n$ such that $A = C^T C$ (resp. there exists a regular matrix $C \in \mathbb{M}_n$ such that $A = C^T C$).*
4. *All leading principal minors of A are positive (resp. strictly positive).*

Properties

1. For $A, B \in \mathbb{S}_+^n$:
   (a) $A \succeq B$ implies $A - B \succeq 0$.
   (b) $A + B \succeq B$.
   (c) $A^{1/2} B A^{1/2} \succeq 0$.

(d) $\text{Tr}(AB) \leq \text{Tr}(A)\text{Tr}(B)$.

(e) $\text{Tr}(AB) \geq 0$.

2. It is easy to observe that a matrix $A \in \mathbb{S}^n_{++}$ if and only if $A^{-1} \in \mathbb{S}^n_{++}$, since the eigenvalues of A^{-1} are $\frac{1}{\lambda_i(A)}$ for all $i = 1, \ldots, n$.

3. Any principal sub-matrix of a semidefinite positive (resp. definite) matrix is also positive semidefinite (resp. definite).

4. For any $A \in \mathbb{S}^n_+$, there exists $i \in \{1, \ldots, n\}$ such that $a_{ii} = \max_{j \in \{1,\ldots,n\}} |a_{ij}|$.

5. If $A \in \mathbb{S}^n_+$ and $a_{ii} = 0$ for some $i \in \{1, \ldots, n\}$, then $a_{ij} = 0$ for all $j \in \{1, \ldots, n\}$.

6. Let $B \in \mathbb{M}_n$ be an invertible matrix. If $A \in \mathbb{S}^n_+$ (resp. $\mathbb{S}^n_{++}$)$\Longleftrightarrow B^T AB \in \mathbb{S}^n_+$ (resp. $\mathbb{S}^n_{++}$).

7. The following equivalence holds:

$$A \succeq 0 \Longleftrightarrow A + B \succeq 0, \forall B \succeq 0.$$

8. Also,

$$A \succ 0 \Longleftrightarrow A + B \succ 0, \forall B \succ 0.$$

9. For $A, B \in \mathbb{S}^n_+$,

$$A \bullet B = 0 \Longleftrightarrow AB = 0 \Longleftrightarrow \frac{1}{2}(AB + BA) = 0.$$

10. For $A, B \in \mathbb{S}^n$:

(a) If $A \succeq 0$, then $\|A\|_F \leq \text{Tr}(A)$ and $n(\det(A))^{\frac{1}{n}} \leq \text{Tr}(A)$.

(b) If $C, D \in \mathbb{S}^n$ such that $C - A \succeq 0$ and $D - B \succeq 0$, then $\text{Tr}(AB) \leq \text{Tr}(CD)$.

(c) $A \succeq B \Longleftrightarrow C^T AC \succeq C^T BC$ for all $C \in \mathbb{M}_n$.

(d) If $A \succeq I_n$, then A is invertible, and $I_n \succeq A^{-1}$.

(e) If $B \succeq A \succ 0$, then B is invertible ($B \succ 0$) and $A^{-1} \succeq B^{-1}$.

11. $\lambda_{\text{min}}(A)\lambda_{\text{max}}(B) \leq \lambda_{\text{min}}(A)\,\text{Tr}(B) \leq A \bullet B \leq n\lambda_{\text{max}}(A)\,\text{Tr}(B) \leq n^2\lambda_{\text{max}}(A)\lambda_{\text{max}}(B)$,

Theorem 1.3. *For $A \in \mathbb{S}^p_{++}$, $C \in \mathbb{S}^n$, and $B \in \mathbb{R}^{p\times n}$, then*

$$\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \succeq 0 \iff C - B^T A^{-1} B \succeq 0.$$
$$\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \succ 0 \iff C - B^T A^{-1} B \succ 0.$$

1. The matrix $S = C - B^T A^{-1} B$ is called the Schur complement of A. In particular, for any $x \in \mathbb{R}^n$ and $X \in \mathbb{S}^n$:

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0 \iff X - xx^T \succeq 0.$$

Proposition 1.4. *For $A \in \mathbb{S}^n_+$, there exists a unique matrix $B \in \mathbb{S}^n_+$ such that $A = BB = B^2$. Often, B is called the square root of A, and we often write $B = A^{1/2}$. Moreover, rank$(A) =$ rank(B).*

Cholesky Factorization

Definition 1.5. For $A \in \mathbb{S}^n_{++}$, there exists a unique lower triangular invertible matrix L such that $A = LL^T$.

Diagonally Dominant Matrix

Definition 1.6. A matrix $A \in \mathbb{M}_n$ is:

1. Diagonally dominant if $|a_{ii}| \geq \sum_{j\neq i=1}^{n} |a_{ij}|$ for all $i = 1, \ldots, n$.

2. Strictly diagonally dominant if $|a_{ii}| > \sum_{j\neq i=1}^{n} |a_{ij}|$ for all $i = 1, \ldots, n$.

Theorem 1.7. *If $A \in \mathbb{S}^n$ is strictly diagonally dominant and all diagonal elements are strictly positive, then A is positive definite.*

1.2 Convex analysis

The convexity plays an important role in mathematical optimization theory. The notion of convexity takes two forms: a convex set and convex function.

1.2.1 Convex sets

Definition 1.8. • A set C in $\mathbb{R}^n$ $(C \neq \emptyset)$ is called:

1. Affine set if

$$\lambda x + (1-\lambda)y \in C, \forall x, y \in C, \forall \lambda \in \mathbb{R}.$$

2. Convex set if

$$\lambda x + (1-\lambda)y \in C, \forall x, y \in C, \forall \lambda \in [0,1].$$

• A polyhedron convex is defined as

$$C = \{x \in \mathbb{R}^n_+ \ / \ Ax \leq b, \ Cx = d\}.$$

Where $A \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{p\times n}$, $b \in \mathbb{R}^m$, $d \in \mathbb{R}^p$.

• A point $x \in C$ is called **extremal point** of C if

$$\forall x_1, x_2 \in C, \forall \lambda \in]0,1[: x = (1-\lambda)x_1 + \lambda x_2 \Rightarrow x = x_1 = x_2.$$

Definition 1.9. Let C be a non-empty convex set, d is called **an admissible direction** of C if

$$\forall x \in C, \forall \lambda \geq 0 : x + \lambda d \in C \text{ with } d \in \mathbb{R}^n \ (d \neq 0).$$

Definition 1.10. Convex combination of m points $x_1, \ldots, x_m$ of $\mathbb{R}^n$ is the form

$$x = \sum_{j=1}^{m} \lambda_j x_j,$$

with $\lambda_j \geq 0, \forall j = 1, \ldots, m$ and $\sum_{j=1}^{m} \lambda_j = 1.$

Definition 1.11. Let $X \subseteq \mathbb{R}^n$ be a non-empty convex set. The **convex hull** of X, denoted $Conv(X)$, is the smallest convex set containing X, i.e.,

$$Conv(X) = \left\{ \sum_{i=1}^{k} \lambda_i x_i : \ k \ \in \mathbb{N}^*, \ \sum_{i=1}^{k} \lambda_i = 1, \ \lambda_i \in \mathbb{R}_+, \ x_i \in X \right\}.$$

1.2.2 Convex function

Definition 1.12. Let $f : C \to \mathbb{R}$ be a function with $C \subset \mathbb{R}^n$ a non-empty convex set . We say that f is:

- Affine if

$$f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y),\ \forall x, y \in C,\ \forall \lambda \in \mathbb{R}.$$

- Convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y),\ \forall x, y \in C,\ \forall \lambda \in [0,1].$$

  or

$$f\left(\sum_{i=1}^{m} \lambda_i x_i\right) \leq \sum_{i=1}^{m} \lambda_i f(x_i),\ \forall m \in \mathbb{N},\ \forall \lambda_i \geq 0 \ /\ \sum_{i=1}^{m} \lambda_i = 1,\ \forall x_i \in \mathbb{R}^n.$$

- Strictly convex if

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y),\ \forall x, y \in C,\ x \neq y,\ \forall \lambda \in]0,1[.$$

- Strongly convex if $\exists \alpha > 0$, with $\lambda \in]0,1[$, $\forall x, y \in C$ and $x \neq y$ we have:

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y) - \frac{\alpha}{2}\lambda(1-\lambda)\|x-y\|^2.$$

- Quasiconvex if

$$\forall x, y \in C, \forall \lambda \in [0,1] : f(\lambda x + (1-\lambda)y) \leq \max(f(x), f(y)).$$

- Concave if $(-f)$ is convex, i.e.,

$$f(\lambda x + (1-\lambda)y \geq \lambda f(x) + (1-\lambda)f(y), \forall x, y \in C, \forall \lambda \in [0,1].$$

- Coercive if

$$\lim_{\|x\| \to +\infty} f(x) = +\infty.$$

Remark 1.13. f is strongly convex $\Rightarrow$ f is strictly convex $\Rightarrow$ f is convex $\Rightarrow$ f is quasiconvex.

1.2.3 Characterization of differentiable convex function

For once or twice differentiable functions, there are some additional criteria for verifying convexity. A useful alternative characterization of convexity for differentiable functions is given in the following properties.

Definition 1.14. Let $f : C \subseteq \mathbb{R}^n \to \mathbb{R}$ be a continuous differentiable function at a point $x \in C$.

- The **gradient** for differentiable function f is the vector of its partial derivatives

$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \cdots, \frac{\partial f(x)}{\partial x_n}\right)^T.$$

- The **Hessian** for twice continuous differentiable function f is the symmetric matrix of M_n, noted $H(x) = \nabla^2 f(x)$ with $H_{ij}(x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)(x)$; $i, j = 1, 2, \ldots, n$, or

$$H(x) = \begin{pmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{pmatrix}.$$

Theorem 1.15. *Let $f : C \subseteq \mathbb{R}^n \to \mathbb{R}$ be a differentiable function, the following propositions are equivalents:*

1. *f is convex.*

2. $\forall x, y \in C$, $f(y) \geq f(x) + < \nabla f(x), y - x >$.

3. *The gradient of f is a monotone operator:*

$$\forall x, y \in C, < \nabla f(y) - \nabla f(x), y - x > \geq 0.$$

Remark 1.16. f is strictly convex over C if and only if the above inequalities are strict whenever $x \neq y$.

Proposition 1.17. *Let $f : C \subseteq \mathbb{R}^n \to \mathbb{R}$ of class $\mathcal{C}^2$.*

- *f is convex if and only if $\nabla^2 f(x)$ is positive semidefinite on C.*

- *f is strictly convex if and only if $\nabla^2 f(x)$ is positive definite on C.*

1.3 Newton's method for solving nonlinear systems

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable nonlinear function. Newton's method is an iterative approach designed to find a point $x \in \mathbb{R}^n$ such that $F(x) = 0$.

At each iteration x_k, the method approximates F linearly around x_k and determines the next iterate x_{k+1} by solving for the zero of this linear approximation. Given that J represents the Jacobian matrix of F, the approximation is expressed as:

$$F(x_k + \Delta x_k) \approx F(x_k) + J(x_k)\Delta x_k,$$

where the Newton direction Δx_k is chosen so that this linear approximation is set to zero. This yields the following update rule:

$$x_{k+1} = x_k + \Delta x_k,$$

with the Newton step Δx_k given by:

$$\Delta x_k = -J(x_k)^{-1} F(x_k).$$

The method converges to a solution provided that the initial guess x_0 is sufficiently close to a zero of F. Newton's method is renowned for its quadratic convergence near the solution, making it a powerful tool in solving nonlinear systems.

1.4 Mathematical programming

Mathematical programming (MP) constitutes a vast and rich domain in numerical analysis, addressing several important practical problems.

In general, a mathematical program is an optimization problem with constraints of the form:

$$\begin{cases} \min f(x) \\ x \in C, \end{cases} \tag{MP}$$

where $C = \{x \in \mathbb{R}^n \;/\; h_j(x) \leq 0,\ j = 1, \ldots, m,\ g_i(x) = 0,\ i = 1, \ldots, p\}$, with f, g_i, h_j are functions from $\mathbb{R}^n$ to $\mathbb{R}$.

We call f the objective function and C the set of feasible or admissible solutions.

1.4.1 Classification of a mathematical program

The classification of (MP) depends on various factors, and key aspects include the convexity of the objective function and constraints. Here are some common classifications:

1. Unconstrained convex optimization: f convex, $C = \mathbb{R}^n$ (no constraint).
2. Linear programming: f linear, C affine set.
3. Quadratic programming: f quadratic, C affine set.
4. Linearly constrained convex problems: f convex, C affine set.
5. Convex programming: f convex, C convex set.
6. Semidefinite programming: programs where the feasible set is defined by linear matrix inequalities.

1.4.2 Existence and uniqueness of optimal solutions

We are interested in identifying points in convex set C at which the function f attains a (local or global) minimum.

- **Local minimum**: Let $f : C \to \mathbb{R}$. A point $x^* \in C$ is a local minimum of (MP) if there exists a neighborhood $V(x^*)$ of x^* such that

$$f(x^*) \leq f(x), \forall x \in V(x^*).$$

- **Global minimum:** A point $x^* \in C$ is a global minimum of (MP) if

$$f(x^*) \leq f(x), \forall x \in C.$$

Remark 1.18. Local and global maximum can be defined similarly by just reverting the inequalities.

Proposition 1.19. *Let C be a convex set and $f : C \longrightarrow \mathbb{R}$ be a convex function. If $x^*$is a local minimum of (MP), then x^* is a global minimum of f over C.*

Existence of a solution

Theorem 1.20 (Weierstrass [9]). *Let C be a compact subset of $\mathbb{R}^n$ and $f : C \longrightarrow \mathbb{R}$ be continuous on C. Then, there exists at least $x^* \in C$ such that $f(x^*) \leq f(x)$ for all $x \in C$.*

Corollary 1.21. *If C is non-empty and closed and f is continuous and coercive on C, then (MP) has a global optimal solution.*

Uniqueness of a solution

Theorem 1.22. *[9] Let C be a non-empty convex subset of $\mathbb{R}^n$ and f a strictly convex function on C. Then, (MP) admits at most an optimal solution.*

1.4.3 Constraints qualification

Definition 1.23. We say that the constraint $h_i(x) \leq 0$ is active or saturated at $\overline{x} \in C$ if $h_i(\overline{x}) = 0$. We introduce then the set

$$I(\overline{x}) = \{i : h_i(\overline{x}) = 0\}.$$

By definition, an equality constraint is saturated.

Here are three classical qualification conditions:
- **Slater (1950):** If C is convex (i.e., h_i are convex and g_j affine) and $\text{int}(C) \neq \emptyset$, then the constraints are qualified at every feasible point.
- **Karlin (1959):** If C is a convex polyhedron (i.e., h_i, g_j are affine), then the constraints are qualified at every feasible point.
- **Mangasarian-Fromovitz (1967):** If the gradients of all constraints saturated at $\overline{x} \in C$ are linearly independent, then the constraints are qualified at $\overline{x}$.

1.4.4 Optimality conditions

Theorem 1.24. ***(Karush-Kuhn-Tucker (K.K.T))*** *Suppose that one of the preceding constraint qualification conditions is satisfied at the point $\overline{x} \in C$. A necessary condition for f to have a* *local minimum at $\overline{x}$ is that there exist $\lambda \in \mathbb{R}_+^m$ and $\mu \in \mathbb{R}^p$ such that:*

$$\begin{cases} \nabla h(\overline{x}) + \sum_{i=1}^{m} \lambda_i \nabla h_i(\overline{x}) + \sum_{j=1}^{p} \mu_j \nabla g_j(\overline{x}) = 0, \\ \lambda_i h_i(\overline{x}) = 0, \quad i = 1, \ldots, m, \\ g_j(\overline{x}) = 0, \quad j = 1, \ldots, p. \end{cases}$$

The λ_i and μ_j are called the Karush-Kuhn-Tucker multipliers.

Remark 1.25. - If the constraints are not qualified at $\overline{x}$, the KKT conditions do not apply.
- If (MP) is convex, the KKT conditions are both necessary and sufficient for $\overline{x}$ to be a global minimum.

1.5 Linear programming

Without loss of generality, a linear program (LP) can be presented in the following standard form

$$\begin{cases} \min c^T x \\ Ax = b, \\ x \geq 0, \end{cases} \tag{LP}$$

where A is a real matrix of dimensions (m, n) assumed to be full rank (rank$(A) = m < n$), $b \in \mathbb{R}^m$ and $c, x \in \mathbb{R}^n$.

The set of feasible solutions $S = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$ is a closed convex polyhedron.

The dual of (LP) is given by:

$$\begin{cases} \max b^T y \\ A^T y + s = c, \\ s \geq 0, \end{cases} \tag{LD}$$

with $s \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$.

Proposition 1.26. *[61] A feasible and bounded linear program (with a bounded objective) has at least one optimal solution, located on the boundary of the feasible set.*

Proposition 1.27. *[61] If x^* is an optimal solution of (LP) , then x^* is a vertex of the feasible set of (LP).*

1.5.1 Methods of resolution

1) Simplex method

Developed by G. Dantzig in the late 1940s [15], the simplex method systematically moves along the boundary of the feasible set from one adjacent vertex to another, reducing the objective value until reaching the optimum. A simple optimality criterion allows identifying the optimal vertex. The algorithm converges in a finite number of iterations, not exceeding C_n^m under the assumption that all visited vertices's are non-degenerate.

In degenerate cases, the algorithm may cycle, but suitable techniques exist to prevent this phenomenon. Despite its potential cycling issue, the simplex method exhibits excellent numerical behavior in practice, as evidenced by its widespread applications in solving a broad class of practical problems. The theoretical complexity of the simplex method is exponential, on the order of $O(2^n)$ operations.

2) Interior point methods

These methods are introduced in the late 1950s for solving nonlinear mathematical programs, IPMs did not gain much enthusiasm for LO at first due to the quasi-total dominance of the simplex method. However, after the appearance of Karmarkar's algorithm in 1984 for LO [38], IPMs underwent a significant revolution, resulting in over 3000 (in the 2000's year) publications within a few years. There are three fundamental classes of IPMs, namely: affine methods, potential reduction methods and central path methods.

a) Affine methods (Dikin 1967): In 1967, Dikin [27] was the first to introduce this type of IPMs for LO. After the appearance of Karmarkar's article [38] in 1984 for LO, several researchers dedicated their work to affine methods, including Barnes [6], Cavalier and Soyster [14], Vanderbei and Freedman [74]. The convergence of these methods was initially studied by Dikin, who proved that the algorithm converges under the assumption of non-degeneracy. Subsequently, Tseng and Luo [73] demonstrated that the method converges even in the degenerate case. There are three types of affine methods: primal, dual, and primal-dual.

For more details on affine methods we refer to [27, 36].

b) Potential reduction methods: The potential function plays a crucial role in the development of IPMs. Reducing this function directly leads to the reduction of the objective function. Karmarkar's algorithm applied to the LO problem in standard form uses a potential function of the form: $n\ln(c^Tx-Z)-\sum_{i=1}^{n}\ln(x_i)$, where Z is a lower bound on the optimal objective value. Karmarkar proves the convergence and polynomiality of his algorithm by showing that this function is reduced by at least a constant at each iteration. Since 1987, researchers introduced primal-dual type potential functions, among which the one by Todd and Ye [69] defined as: $\Phi_\rho(x,s)=\rho\ln(x^Ts)-\sum_{i=1}^{n}\ln(x_is_i)$, where $\rho>n$. This function played a significant role in the development of potential reduction algorithms after 1988. The algorithms corresponding to these methods have polynomial complexity.

For more details on the method mentioned in this paragraph, we refer to [36, 85].

c) Central path methods: As the name suggests, this method is based on tracking a central trajectory. The central trajectory method involves staying within a certain neighborhood of the central trajectory using Newton iterations.

The central trajectory method was first studied for linear programming by Bayer and Lagarias [7], and later by Meggido [54], Gonzaga [30], Monteiro and Adler [55], Kojima et al. [49], Roos and Vial [64]...etc.

Due to their attractive properties, including polynomial complexity and superlinear convergence, many researchers have proposed interesting extensions to solve other important optimization problems (see e.g., [3, 4, 28]).

1.6 Quadratic Programming

Quadratic programming is known for its diverse applications in various fields. Often, it serves as an intermediate procedure for nonlinear programs, as seen in methods like Sequential Quadratic Programming (SQP).

Without loss of generality, a quadratic program can be presented in the following form

$$\begin{cases} \min f(x)=\frac{1}{2}x^TQx+c^Tx \\ Ax=b, \\ x\geq 0, \end{cases} \tag{QP}$$

where Q is a symmetric matrix of order n, $b \in \mathbb{R}^m$, $c, x \in \mathbb{R}^n$, and $A \in \mathbb{R}^{m \times n}$ is full rank ($\text{rank}(A) = m < n$).

Recall that the set of constraints $S = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$ forms a closed convex polyhedron, and the objective function f is infinitely differentiable. (QP) is convex if and only if f is convex, in which case the matrix Q is positive semidefinite.

Methods of resolution

We can classify the resolution methods of (QP) into two categories in accordance with their principles: simplicial methods (Wolfe method [83]) and IPMs [3, 82] which are extensions of the algorithms proposed for the linear case.

Due to the attractive properties of central path methods including polynomial complexity and superlinear convergence. In this thesis, we are interested on primal-dual IPMs of the central path type to solve some optimization problems such as: LO, CQO, SDO and CQSDO. In the next chapter, we will provide an overview of these methods for LO.

CHAPTER 2

Interior point algorithms using the algebraic equivalent transformation method for linear optimization

In this chapter, we recall the concept of the central path, outlining its properties. We then derive the classical Newton search direction for linear optimization (LO). Furthermore, we introduce a pioneering interior point algorithm (IPA) for LO, which relies on the algebraic equivalent transformation technique for determining search directions. In this context, a comparative numerical study was reported which is published in Euro-Tbilisi Mathematical Journal [86]. The chapter ends with a discussion of the most recent modification to this technique.

2.1 Linear optimization problem

We reconsider the linear programming (LP) problem in the following standard form

$$\begin{cases} \min \ c^T x \\ Ax = b, \\ x \geq 0, \end{cases} \tag{LP}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $c, x \in \mathbb{R}^n$.

The dual problem of (LP) can be written in the following form

$$\begin{cases} \max\ b^T y \\ A^T y + s = c, \\ s \geq 0,\ s \in \mathbb{R}^n, \\ y \in \mathbb{R}^m. \end{cases} \tag{LD}$$

We assume that the pair (LP) and (LD) satisfy the conditions belows:

- The matrix A is a full rank row, i.e., $Rank(A) = m < n$.
- There exist (x^0, y^0, s^0) such that:

$$Ax^0 = b, A^T y^0 + s^0 = c, x^0 > 0, s^0 > 0. \tag{IPC}$$

This last condition (IPC) is named the interior point condition for (LP) and (LD).

We denote by:

- $F_{(LP)} = \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}$, the set of feasible primal solutions of (LP).

- A vector $x \in F_{(LP)}$ is called a feasible solution of (LP).

- A vector $x^* \in F_{(LP)}$ that minimizes the objective function of (LP) is called an optimal solution of (LP).

- The set $F_{(LP)}$ is bounded if the objective function is bounded on $F_{(LP)}$.

- $F^0_{(LP)} = \{x \in \mathbb{R}^n : Ax = b,\ x > 0\}$, the set of strictly feasible primal solutions of (LP).

- $F_{(LD)} = \left\{y \in \mathbb{R}^m : A^T y + s = c,\ s \geq 0\right\}$, the set of feasible dual solutions of (LD).

- A vector $y^* \in F_{(LD)}$ that maximizes the objective function of (LD) is called an optimal solution of (LD).

- $F^0_{(LD)} = \left\{y \in \mathbb{R}^m : A^T y + s = c,\ s > 0\right\}$, the set of strictly feasible dual solutions of (LD).

- $F^0 = F^0_{(LP)} \times F^0_{(LD)}$, the set of strictly feasible primal-dual solutions of (LP) and (LD).

Here are some fundamental results of duality in linear programming

1. If one of the problems (LP) and (LD) has an optimal solution, the other also has one, and their corresponding optimal values are equal.

2. If one of the problems has an unbounded optimal value, the other does not have an optimal solution.

Theorem 2.1 (Weak Duality [61]). *If x and (y, s) are feasible solutions for (LP) and (LD) respectively, then*

$$c^T x \geq b^T y.$$

Theorem 2.2 (Strong Duality [61]). *If $\overline{x}$ and $(\overline{y}, \overline{s})$ are feasible solutions corresponding to a finite optimal value for (LP) and (LD) such that*

$$c^T \overline{x} = b^T \overline{y},$$

then $\overline{x}$ is an optimal primal solution for (LP), and $\overline{y}$ is an optimal dual solution for (LD).

Remark 2.3. It is easy to notice that if $\overline{x}$ and $(\overline{y}, \overline{s})$ are feasible solutions for (LP) and (LD) respectively, then the following property holds:

$$c^T \overline{x} = b^T \overline{y} \Leftrightarrow \overline{x}^T \overline{s} = 0 \Leftrightarrow \overline{xs} = 0.$$

2.2 The classical central path method

For (LP) problem, we associate the following perturbed problem:

$$\begin{cases} \min \ f_\mu(x) \\ Ax = b, \\ x > 0, \end{cases} \tag{LP_μ}$$

where f_μ is the perturbed function defined by:

$$f_\mu(x) = c^T x - \mu \sum_{i=1}^{n} \log(x_i),$$

and μ is a strictly positive barrier parameter.

Properties of f$_\mu$(x)[36]:

1. The function $f_\mu(x)$ is strictly convex (Indeed, $\nabla^2 f_\mu(x) = \mu X^{-2}$ is a positive definite matrix, because $X = diag(x_1, ..., x_n)$ is a positive definite matrix and $\mu > 0$).

2. If $F^0_{(LP)}$ and $F^0_{(LD)}$ are non-empty, then for any $\mu > 0$, the perturbed problem (LP_μ) has a unique solution, denoted $x(\mu)$, and called the "central point".

3. When $\mu \to 0$, $x(\mu) \to x^*$ the optimal solution of (LP).

4. The function $\mu \to (x(\mu), y(\mu), s(\mu))$ defines the central path of (LP_μ) which we denote by:
$$T_C = \{(x(\mu), y(\mu), s(\mu)) : \mu > 0\}.$$

5. $x(\mu)$ is uniquely defined by the following Karush-Khun-Tucker optimality conditions:
$$\begin{cases} c - \mu X^{-1}e - A^T y = 0, \\ Ax = b, \; x > 0, \end{cases}$$

where $y \in \mathbb{R}^m$ is the Lagrange multiplier associated with the constraint $Ax = b$ of the problem (LP_μ) and e is the all-one vector of length n, the previous system becomes:
$$\begin{cases} Ax = b, \; x > 0, \\ A^T y + s = c, \; s > 0, \\ xs = \mu e, \; \mu > 0, \end{cases} \tag{S_μ}$$
with
$$xs = Xs = (x_1 s_1, x_2 s_2, ..., x_n s_n)^T,$$
denotes the Hadamard product of the vectors x and s.
Note that (S_μ) corresponds to the complementarity conditions for a linear primal-dual problem.

The system (S_μ) also denotes the optimality conditions for the following dual perturbed problem
$$\begin{cases} \max \; b^T y + \mu \sum_{i=1}^{n} \log(s_i) \\ A^T y + s = c, \\ s > 0. \end{cases} \tag{LD_μ}$$

Indeed, the Karush-Kuhn-Tucker optimality conditions for the problem (LD_μ) are given by
$$\begin{cases} b - Ax = 0, \\ \mu S^{-1}e - x = 0, \\ A^T y + s = c, \end{cases} \tag{$S\prime_\mu$}$$
where x is the Lagrange multiplier associated with the constraint $A^T y + s = c$ and $S = diag(s_1, ..., s_n)$. Hence, ($S\prime_\mu$) is equivalent to (S_μ).
(S_μ) is a system of nonlinear equations, Newton's method is one of the most techniques used for its resolution.

At each μ, we find a solution $(x(\mu), y(\mu), s(\mu))$ close to the central path (proximity condition).

Definition 2.4. The solution $(x(\mu), y(\mu), s(\mu))$ is said to be close to the central trajectory if it belongs to the set:

$$T_C(\theta) = \{(x, y, s) \in F^0_{(LP)} \times F^0_{(LD)} / \|xs - \mu e\| \leq \theta\mu, 0 < \theta < 1\},$$

where $\|.\|$ is the Euclidean norm.

The system (S_μ), can be written as $F(x, y, s) = 0$, where

$$F(x, y, s) = \begin{pmatrix} Ax - b \\ A^T y + s - c \\ xs - \mu e \end{pmatrix}.$$

Newton's iteration is defined by $(x_+, y_+, s_+) = (x, y, s) + (\Delta x, \Delta y, \Delta s)$, where $(\Delta x, \Delta y, \Delta s)$ is the solution of the linear system:

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta s \end{pmatrix} = - \begin{pmatrix} 0 \\ 0 \\ xs - \mu e \end{pmatrix}. \quad \text{(LS)}$$

Theoretically, we assume that we have an initial strictly feasible primal-dual solution close to the central path. For any $\mu > 0$, (S_μ) admits a unique solution denoted as $(x(\mu), y(\mu), s(\mu))$ called the μ-center of (LP) and (LD) [67]. The set of all μ-centers constructs the central path. The limit of the central path as μ tends to zero exists and gives the optimal solution for both problems (LP) and (LD).

2.3 Recent descent directions based on the algebraic equivalent transformation

In their works, Darvay [17, 18] defined a new method to find search directions for IPMs based on an algebraic equivalent transformation (AET) technique applied to the centrality equation $\frac{xs}{\mu} = e$,. The principle of Darvay's method is to replace this last equation in (S_μ) by the following equation $\varphi\left(\frac{xs}{\mu}\right) = \varphi(e)$, where φ is an invertible function, i.e., φ^{-1} exists.

The system (S_μ) becomes as follow:

$$\begin{cases} Ax = b,\ x > 0, \\ A^T y + s = c,\ s > 0, \\ \varphi\left(\frac{xs}{\mu}\right) = \varphi(e),\ \mu > 0. \end{cases} \tag{2.1}$$

The application of Newton's method to the nonlinear system (2.1), gives

$$\begin{cases} A\Delta x = 0, \\ A^T\Delta y + \Delta s = 0, \\ \frac{s}{\mu}\varphi'(\frac{xs}{\mu})\Delta x + \frac{x}{\mu}\varphi'(\frac{xs}{\mu})\Delta s = \varphi(e) - \varphi(\frac{xs}{\mu}). \end{cases} \tag{2.2}$$

where φ' denotes the derivative of φ.

By introducing the following notations:

$$v = \sqrt{\frac{xs}{\mu}}, d_x = \frac{v\Delta x}{x} \text{ and } d_s = \frac{v\Delta s}{s},$$

we can obtain easily

$$\mu v(d_x + d_s) = s\Delta x + x\Delta s \text{ and } d_x d_s = \frac{\Delta x \Delta s}{\mu}.$$

Therefore, the linear system (2.2) can be written in the following form:

$$\begin{cases} \bar{A}d_x = 0, \\ \bar{A}^T\Delta_y + d_s = 0, \\ d_x + d_s = p_v, \end{cases} \tag{2.3}$$

where

$$p_v = \frac{\varphi(e) - \varphi(v^2)}{v\varphi'(v^2)}, \text{ and } \bar{A} = \frac{1}{\mu}Adiag\left(\frac{x}{v}\right).$$

Next, some values of the vector p_v related to different choices of the function φ are stated.

Functions $\varphi(t)$	The vector p_v
$\varphi(t) = t$	$(v^{-1} - v)$ (Roos et al. [65], the classical method),
$\varphi(t) = \sqrt{t}$	$2(e - v)$ (Darvay [18]),
$\varphi(t) = t - \sqrt{t}$	$\dfrac{2(v - v^2)}{2v - e}$, $v > \frac{e}{2}$ (Darvay and Takács [19]),
$\varphi(t) = \dfrac{\sqrt{t}}{2(1+\sqrt{t})}$	$e - v^2$ (Kheirfam and Haghani [45]),
$\varphi_q(t) = t^{\frac{q}{2}}, q \in \mathbb{N}$	$\dfrac{2}{q}(v^{1-q} - v)$ (Kheirfam and Nasrollahi [46]).

The value of p_v for different functions φ.

In [86], a comparative numerical results are presented, where we use a different functions in the AET technique. In the next section we present other approaches for defining search directions.

2.4 Other ways to determine search directions

In [93], Zhang and Xu proposed a specific search direction for LO. They considered the equivalent form $v^2 = v$ of the centering equation and they transformed it into the form $xs = \mu v$. After that, they assumed that the variance vector v is fixed and they applied Newton's method.

Based on this new AET, Darvay and Takàcs [20] proposed another technique to obtain a new descent direction for LO. The proposed method was as follows:

for $x, s > 0$ and $\mu > 0$, from the third equation of system (S_μ) we deduce that

$$xs = \mu e \Leftrightarrow \frac{xs}{\mu} = e \Leftrightarrow \sqrt{\frac{xs}{\mu}} = e \Leftrightarrow \frac{xs}{\mu} = \sqrt{\frac{xs}{\mu}}.$$

In other words,

$$xs = \mu e \Leftrightarrow \frac{xs}{\mu} = e \Leftrightarrow v^2 = e \Leftrightarrow v = e \Leftrightarrow v^2 = v,$$

where, $\frac{xs}{\mu}$ denotes the Hadamard product of the vectors x and s divided by $\mu > 0$, hence $\frac{xs}{\mu} = \left(\frac{x_1 s_1}{\mu}, \frac{x_2 s_2}{\mu}, ..., \frac{x_n s_n}{\mu}\right)^T > 0$ and $\sqrt{\frac{xs}{\mu}}$ is the vector obtained by taking square roots of the components of $\frac{xs}{\mu}$.

The perturbed central path system (S_μ) can be equivalently stated as follows:

$$\begin{cases} Ax = b, \\ A^T y + s = c, \\ \sqrt{\frac{xs}{\mu}} = \frac{xs}{\mu}. \end{cases} \tag{2.4}$$

Applying the AET method to (2.4), we obtain

$$\begin{cases} Ax = b, \\ A^T y + s = c, \\ \psi\left(\sqrt{\frac{xs}{\mu}}\right) = \psi(\frac{xs}{\mu}). \end{cases} \tag{2.5}$$

where, ψ is defined , invertible and continuously differentiable on the interval (k^2, ∞), with $0 \le k < 1$, such that $2t\psi'(t^2) - \psi'(t) > 0, \forall t > k^2$.

This last system (2.5) can be written in the form $f(x, y, s) = 0$, where

$$f(x,y,s) = \begin{pmatrix} Ax - b \\ A^T y + s - c \\ \psi\left(\sqrt{\frac{xs}{\mu}}\right) - \psi(\frac{xs}{\mu}) \end{pmatrix}. \tag{2.6}$$

Applying Newton's method to this system we get: $x_+ = x + \Delta x$, $y_+ = y + \Delta y$, $s_+ = s + \Delta s$, where $(\Delta x, \Delta y, \Delta s)$ is the solution of the linear system:

$$\begin{cases} A\Delta x = 0, \\ A^T \Delta y + \Delta s = 0, \\ \frac{1}{\mu}(s\Delta x + x\Delta s) = \dfrac{-\psi\left(\frac{xs}{\mu}\right) + \psi\left(\sqrt{\frac{xs}{\mu}}\right)}{\psi'(\frac{xs}{\mu}) - \frac{1}{2\sqrt{\frac{xs}{\mu}}}\psi'(\sqrt{\frac{xs}{\mu}})}. \end{cases} \tag{2.7}$$

Defining the scaled vector v and the scaled search directions d_x and d_s according to

$$v = \sqrt{\frac{xs}{\mu}}, \; d_x = \frac{v\Delta x}{x} \text{ and } d_s = \frac{v\Delta s}{s}. \tag{2.8}$$

Hence, we obtain

$$\frac{1}{\mu}(s\Delta x + x\Delta s) = v(d_x + d_s), \tag{2.9}$$

and

$$d_x d_s = \frac{\Delta x \Delta s}{\mu}. \tag{2.10}$$

Obviously, with these notations, the scaled feasible Newton system of (2.7) can be expressed as:

$$\begin{cases} \overline{A}d_x = 0, \\ \overline{A}^T \Delta y + d_s = 0, \\ d_x + d_s = p_v. \end{cases} \tag{2.11}$$

Where

$$\overline{A} = \frac{1}{\mu} A diag\left(\frac{x}{v}\right),$$

and

$$p_v = \frac{2\psi(v) - 2\psi(v^2)}{2v\psi'(v^2) - \psi'(v)}. \tag{2.12}$$

Here, $diag(\frac{x}{v})$ is a diagonal matrix, which contains on its main diagonal the elements of the vector $\frac{x}{v}$ respectively in the original order.

Darvay and Takàcs [20] consider the function:

$$\psi : \left(\frac{1}{\sqrt{2}}, +\infty\right) \to \mathbb{R}, \quad \psi(t) = t^2,$$

thus

$$p_v = \frac{v - v^3}{2v^2 - e}.$$

The condition $2t\psi'(t^2) - \psi'(t) > 0, \forall t > k^2$ is satisfied in the case where $k^2 = \frac{1}{\sqrt{2}}$.

Throughout this chapter, we use a proximity measure of the central path defined by:

$$\delta(v) = \delta(xs, \mu) = \frac{\|p_v\|}{2} = \frac{1}{2}\left\|\frac{v - v^3}{2v^2 - e}\right\|.$$

The generic representation of this algorithm is given as follows:

Primal-dual algorithm for LO

Input

a proximity parameter $0 < \tau < 1$;

an accuracy parameter $\varepsilon > 0$;

a fixed barrier update parameter θ, $0 < \theta < 1$;

a strictly feasible (x^0, y^0, s^0) such that $\delta(x^0 s^0, \mu^0) < \tau$, where $\mu^0 = \frac{(x^0)^T s^0}{n}$;

begin

$x := x^0; y := y^0; s := s^0; \mu := \mu^0$;

while $x^T s \geq \varepsilon$ do

$\mu := (1-\theta)\mu$;

solve the system (2.11) via (2.8) to obtain $(\Delta x, \Delta y, \Delta s)$;

take $x := x + \Delta x; y := y + \Delta y; s := s + \Delta s$;

end

Figure 2.1: Generic algorithm

In order to facilitate the convergence analysis of the algorithm, let us define the vector

$$q_v = d_x - d_s.$$

Then, using the above equation and the third equation of (2.11) we have

$$d_x = \frac{1}{2}(p_v + q_v) \text{ and } d_s = \frac{1}{2}(p_v - q_v).$$

This implies

$$d_x d_s = \frac{p_v^2 - q_v^2}{4}. \tag{2.13}$$

Since $d_x^T d_s = d_x^T \left(-\overline{A}^T \Delta y\right) = -\left(\overline{A} d_x\right)^T \Delta y = 0$, then

$$\|q_v\| = \|p_v\|. \tag{2.14}$$

In the next, we present some results related to algorithm complexity analysis.

Lemma 2.5 (Analysis of strict feasibility [20]). *If $\delta = \delta(xs, \mu) < 1$ and $v > \frac{1}{\sqrt{2}}e$, then*

$$x_+ > 0 \quad \textit{and} \quad s_+ > 0,$$

i.e., the new iterations are strictly feasible.

Lemma 2.6 (Quadratic convergence of proximity measure [20]). *If* $\delta = \delta(xs, \mu) < \frac{1}{\sqrt{2}}$ *and* $v > \frac{1}{\sqrt{2}}e$, *then*

$$v_+ = \sqrt{\frac{x_+ s_+}{\mu}} > \frac{1}{\sqrt{2}}e \quad \textit{and} \quad \delta(x_+ s_+; \mu) \leq \frac{5\delta^2}{1-2\delta^2}\sqrt{1-\delta^2},$$

which proves that the full Newton step ensures local quadratic convergence of the proximity measure.

Lemma 2.7 (The effect of a full Newton step on the duality gap [20]). *Let* $\delta = \delta(xs, \mu)$. *Then,*

$$(x_+)^\top s_+ \leq \mu(n + 8\delta^2),$$

for all $n \in \mathbb{N}^*$.

Lemma 2.8 (The well-defined algorithm [20]). *Let* $\delta = \delta(xs, \mu) < \frac{1}{\sqrt{2}}$, $v > \frac{1}{\sqrt{2}}e$, *and* $\mu_+ = (1-\theta)\mu$, *where* $0 < \theta < 1$. *Furthermore, let* $v_{++} = \sqrt{\frac{x_+ s_+}{\mu_+}}$. *Then,* $v_{++} > \frac{1}{\sqrt{2}}e$ *and*

$$\delta(v_{++}) = \delta(x_+ s_+, \mu_+) < \frac{\sqrt{1-\delta^2}}{2\sqrt{1-\theta}(1-2\delta^2+\theta)}(\theta\sqrt{n} + 10\delta^2).$$

Moreover, if $\delta < \frac{1}{10}$ *and* $\theta = \frac{1}{12\sqrt{n}}$, *then* $\delta(x_+ s_+, \mu_+) < \frac{1}{10}$.

Lemma 2.9 (The complexity analysis [20]). *Suppose* (x^0, s^0) *are strictly feasible,* $\mu^0 = \frac{(x^0)^T s^0}{n}$, *and* $\delta(x^0 s^0, \mu^0) < \frac{1}{\sqrt{2}}$. *Let* x^k *and* s^k *be the vectors obtained after* k *iterations. Then, for every*

$$k \geq \left[\frac{1}{\theta}\log\frac{\mu^0(n+4)}{\varepsilon}\right],$$

we have $(x^k)^T s^k \leq \varepsilon$.

Theorem 2.10. *[20] Suppose that* $x^0 = s^0 = e$. *For the default values* $\theta = \frac{1}{12\sqrt{n}}$ *and* $\tau = \frac{1}{10}$, *the algorithm given in Figure 2.1 requires no more than*

$$12\sqrt{n}\log\frac{\mu^0(n+4)}{\varepsilon}$$

iterations. The resulting vectors satisfy $\left(x^k\right)^T s^k < \varepsilon$.

In the next chapter, primal-dual IPAs for LO based on new search directions are presented using two new functions: $\psi(t) = t^{\frac{7}{4}}$ and $\psi(t) = t^{\frac{3}{2}}$.

CHAPTER 3

New full-Newton step interior point algorithms for linear optimization

Building upon the work of Darvay and Takàcs [20] which is based on the technique of algebraic transformation, we propose two new functions $\psi(t)$ to enhance the performance of path-following algorithms. The first function is $\psi(t) = t^{\frac{7}{4}}$, where we have shown that the corresponding algorithm converges after $O\left(\sqrt{n}\log\frac{n+\frac{3}{\sqrt[7]{2}}}{\varepsilon}\right)$ iterations. The second function is defined by $\psi(t) = t^{\frac{3}{2}}$, where we have proven that the corresponding algorithm converges after $O\left(\sqrt{n}\log\frac{n+\sqrt[3]{4}}{\varepsilon}\right)$ iterations. Numerical tests specifically on some problem from the Netlib test collections were conducted to consolidate the obtained theoretical results. The set of those results were published in RAIRO Oper. Res. [87] and Stat. Optim. Inf. Comput. [89].

3.1 First new full-Newton step interior point algorithm based on the function $\psi(t) = t^{7/4}$

In this section, we reconsider the technique introduced by Darvay and Takàcs [20] with our new function $\psi(t) = t^{\frac{7}{4}}$. The aim of this technique is to obtain better theoretical and numerical results.

Let consider $\psi : \left(\left(\frac{1}{2}\right)^{\frac{4}{7}}, \infty\right) \to \mathbb{R}$, such that $\psi(t) = t^{\frac{7}{4}}$. From (2.12), we get

$$p_v = \frac{8v - 8v^{\frac{11}{4}}}{14v^{\frac{7}{4}} - 7e}. \tag{3.1}$$

The condition $2t\psi'(t^2) - \psi'(t) > 0, \forall t > k^2$ is satisfied in this case, when $k^2 = \left(\frac{1}{2}\right)^{\frac{4}{7}}$. We give a proximity measure to the central path as follows:

$$\delta(v) = \delta(xs, \mu) = \frac{\|p_v\|}{2} = \frac{4}{7}\left\|\frac{v - v^{\frac{11}{4}}}{2v^{\frac{7}{4}} - e}\right\|. \tag{3.2}$$

3.1.1 Analysis of the algorithm

In the following lemma, we state a condition which ensures the feasibility of the generated point after a full-Newton step x_+ and s_+, where $x_+ = x + \Delta x$ and $s_+ = s + \Delta s$.

Lemma 3.1. *Let $\delta = \delta(xs, \mu) < 1$ and $v > \frac{1}{2^{\frac{4}{7}}}e$. Then*

$$x_+ > 0 \text{ and } s_+ > 0.$$

Proof. For each $0 \leq \alpha \leq 1$ denote $x_+(\alpha) = x + \alpha\Delta x$ and $s_+(\alpha) = s + \alpha\Delta s$. Hence,

$$x_+(\alpha)s_+(\alpha) = xs + \alpha(x\Delta s + s\Delta x) + \alpha^2\Delta x\Delta s. \tag{3.3}$$

Now, in view of (2.9) and (2.10) we have

$$\frac{1}{\mu}x_+(\alpha)s_+(\alpha) = v^2 + \alpha v(d_x + d_s) + \alpha^2 d_x d_s, \tag{3.4}$$

also from (2.11) and (2.13), we can write

$$\frac{1}{\mu}x_+(\alpha)s_+(\alpha) = (1-\alpha)v^2 + \alpha(v^2 + vp_v) + \alpha^2\left(\frac{p_v^2}{4} - \frac{q_v^2}{4}\right). \tag{3.5}$$

In addition, from (3.1) we obtain

$$v^2 + vp_v = v^2 + \frac{8v^2 - 8v^{\frac{15}{4}}}{14v^{\frac{7}{4}} - 7e} = \frac{6v^{\frac{15}{4}} + v^2}{14v^{\frac{7}{4}} - 7e}. \tag{3.6}$$

Let's consider the function: $f(x) = \dfrac{6x^{\frac{15}{4}} + x^2}{14x^{\frac{7}{4}} - 7}$, for $x > \frac{1}{2^{\frac{4}{7}}}$. We have $f(x) \geq f(1)$, so $f(x) \geq 1$. Using this result, we get

$$v^2 + vp_v \geq e, \tag{3.7}$$

this implies that

$$\begin{aligned}
\frac{1}{\mu}x_+(\alpha)s_+(\alpha) &\geq (1-\alpha)v^2 + \alpha e + \alpha^2\left(\frac{p_v^2}{4} - \frac{q_v^2}{4}\right) \\
&\geq (1-\alpha)v^2 + \alpha e + \alpha^2\left(\frac{p_v^2}{4} - \frac{q_v^2}{4}\right) - \alpha\frac{p_v^2}{4} \\
&\geq (1-\alpha)v^2 + \alpha e + \alpha(\alpha-1)\frac{p_v^2}{4} - \alpha^2\frac{q_v^2}{4} \\
&\geq (1-\alpha)v^2 + \alpha\left[e - \left((1-\alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right)\right].
\end{aligned}$$

In addition, we have

$$\begin{aligned}
\left\|(1-\alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right\|_\infty &\leq (1-\alpha)\frac{\|p_v^2\|_\infty}{4} + \alpha\frac{\|q_v^2\|_\infty}{4} \\
&\leq (1-\alpha)\frac{\|p_v\|^2}{4} + \alpha\frac{\|q_v\|^2}{4} = \delta^2,
\end{aligned}$$

where $\|.\|_\infty$ marks the Chebychev norm (or l_∞ norm).
Also, as $\delta < 1$ we get

$$\left\|(1-\alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right\|_\infty < 1,$$

then

$$e - \left[(1-\alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right] > 0.$$

Hence, $x_+(\alpha)s_+(\alpha) > 0$ for each $0 \leq \alpha \leq 1$, which means that the linear functions of α, $x_+(\alpha)$ and $s_+(\alpha)$ do not change sign on the interval $[0, 1]$. Therefor $x_+(0) = x > 0$ and $s_+(0) = s > 0$ give $x_+(1) = x_+ > 0$ and $s_+(1) = s_+ > 0$. This means that the full-Newton step is strictly feasible. □

The following lemma will be useful in the next part of the analysis.

Lemma 3.2. *[19, Lemma 5.2] Let $f : [d, \infty) \to (0, \infty)$ be a decreasing function with $d > 0$. Furthermore, let us consider the positive vector v of length n such that $min(v) > d$. Then*

$$\left\|f(v)\left(e - v^2\right)\right\| \leq f(\min(v))\left\|e - v^2\right\| \leq f(d)\left\|e - v^2\right\|.$$

Now, in Lemma 3.3 we show the quadratic convergence of the full-Newton step.

Lemma 3.3. *Suppose that $\delta = \delta(xs, \mu) < \frac{1}{2^{\frac{4}{7}}}$ and $v > \frac{1}{2^{\frac{4}{7}}}e$, then*

$$v_+ > \frac{1}{2^{\frac{4}{7}}}e \quad \textit{and} \quad \delta(x_+s_+, \mu) \leq 18\delta^2,$$

which proves that the full-Newton step ensures a local quadratic convergence of the proximity measure.

Proof. We know from Lemma 3.1 that $x_+ > 0$ and $s_+ > 0$, then $v_+ = \sqrt{\frac{x_+ s_+}{\mu}}$ is well defined. From $\delta = \delta(xs, \mu) < \frac{1}{2^{\frac{4}{7}}}$, we get

$$\sqrt{1-\delta^2} > \sqrt{1 - \frac{1}{2^{\frac{8}{7}}}} > \frac{1}{2^{\frac{4}{7}}}. \tag{3.8}$$

Let $\alpha = 1$. Then from (3.5) it follows that

$$v_+^2 = \frac{x_+ s_+}{\mu} = v^2 + v p_v + \frac{p_v^2}{4} - \frac{q_v^2}{4}. \tag{3.9}$$

From (3.7), we obtain

$$v_+^2 = v^2 + v p_v + \frac{p_v^2}{4} - \frac{q_v^2}{4} > e - \frac{q_v^2}{4},$$

hence

$$\min(v_+^2) \geq 1 - \frac{\|q_v\|_\infty^2}{4} \geq 1 - \frac{\|q_v\|^2}{4} \geq 1 - \delta^2,$$

and this relation yields

$$\min(v_+) \geq \sqrt{1-\delta^2}. \tag{3.10}$$

From (3.8) and (3.10), we obtain : $v_+ > \frac{1}{2^{\frac{4}{7}}} e$.

This completes the first part of the proof.

Now, we introduce the notation

$$\delta(v_+) = \delta(x_+ s_+, \mu) = \frac{\|p_{v_+}\|}{2} = 4 \left\| \frac{\left(v_+ - v_+^{\frac{11}{4}}\right)}{\left(14 v_+^{\frac{7}{4}} - 7e\right)\left(e - v_+^2\right)} (e - v_+^2) \right\|.$$

From (3.9), we have

$$\begin{aligned} e - v_+^2 = e - \frac{x_+ s_+}{\mu} &= e - \left[(v^2 + v p_v) + \frac{p_v^2}{4} - \frac{q_v^2}{4} \right] \\ &= \frac{q_v^2}{4} - \left[(v^2 + v p_v) + \frac{p_v^2}{4} - e \right] \\ &= \frac{q_v^2}{4} - \frac{p_v^2}{4} \left[\frac{4(v^2 + v p_v)}{p_v^2} + e - \frac{4}{p_v^2} \right]. \end{aligned}$$

Applying (3.1) and (3.6), we obtain

$$\begin{aligned} e - v_+^2 &= \frac{q_v^2}{4} - \frac{p_v^2}{4} \left[\frac{100 v^{\frac{11}{2}} - 60 v^{\frac{15}{4}} + 9v^2 - (14 v^{\frac{7}{4}} - 7e)^2}{16\left(v - v^{\frac{11}{4}}\right)^2} \right] \\ &= \frac{q_v^2}{4} - \frac{p_v^2}{4} \left[\frac{112\left(v - v^{\frac{11}{4}}\right)^2 - 12 v^{\frac{11}{2}} + 164 v^{\frac{15}{4}} - 103 v^2 - \left(14 v^{\frac{7}{4}} - 7e\right)^2}{16\left(v - v^{\frac{11}{4}}\right)^2} \right] \\ &= \frac{q_v^2}{4} - \frac{p_v^2}{4} \left[7e + \frac{164 v^{\frac{15}{4}} - 12 v^{\frac{11}{2}} - 103 v^2 - \left(14 v^{\frac{7}{4}} - 7e\right)^2}{16\left(v - v^{\frac{11}{4}}\right)^2} \right]. \end{aligned}$$

We can prove that $0 \leq \left[7e + \dfrac{164v^{\frac{15}{4}} - 12v^{\frac{11}{2}} - 103v^2 - \left(14v^{\frac{7}{4}} - 7e\right)^2}{16\left(v - v^{\frac{11}{4}}\right)^2}\right] \leq 7e, \forall v > \frac{1}{2^{\frac{4}{7}}}e$, so these imply that

$$\begin{aligned} \|e - v_+^2\| &\leq \left\|\frac{q_v^2}{4}\right\| + \left\|\frac{p_v^2}{4}\left[7e + \frac{164v^{\frac{15}{4}} - 12v^{\frac{11}{2}} - 103v^2 - \left(14v^{\frac{7}{4}} - 7e\right)^2}{16\left(v - v^{\frac{11}{4}}\right)^2}\right]\right\| \\ &\leq \frac{\|q_v\|^2}{4} + 7\frac{\|p_v\|^2}{4} = 8\delta^2. \end{aligned} \tag{3.11}$$

Let's consider the function: $f(t) = \dfrac{\left(t - t^{\frac{11}{4}}\right)}{\left(14t^{\frac{7}{4}} - 7\right)\left(1 - t^2\right)}$, for all $t > \frac{1}{2^{\frac{4}{7}}}$, $t \neq 1$. Because $f'(t) < 0$, so f is decreasing. Hence, in view of Lemma 3.2 and (3.10), we obtain

$$\delta(x_+ s_+, \mu) < 4\frac{\sqrt{1 - \delta^2} - \left(1 - \delta^2\right)^{\frac{11}{8}}}{\left(14(1 - \delta^2)^{\frac{7}{8}} - 7\right)\delta^2}\|e - v_+^2\|. \tag{3.12}$$

From (3.11) and (3.12) we deduce

$$\delta(x_+ s_+, \mu) < 32\frac{\sqrt{1 - \delta^2} - \left(1 - \delta^2\right)^{\frac{11}{8}}}{\left(14(1 - \delta^2)^{\frac{7}{8}} - 7\right)\delta^2}\delta^2.$$

Now, if we take $f(\delta) = 32\dfrac{\sqrt{1 - \delta^2} - \left(1 - \delta^2\right)^{\frac{11}{8}}}{\left(14(1 - \delta^2)^{\frac{7}{8}} - 7\right)\delta^2}$ for $\delta < \frac{1}{2^{\frac{4}{7}}}$, then we obtain that $f(\delta) < f(\frac{1}{2^{\frac{4}{7}}}) < 18$, and we conclude that

$$\delta(x_+ s_+, \mu) \leq 18\delta^2,$$

This completes the proof. □

In the following lemma, we analyze the effect of the full-Newton step on the new duality gap:

Lemma 3.4. *Let* $\delta = \delta(xs, \mu)$. *Then, the duality gap satisfies:*

$$(x_+)^T s_+ \leq \mu(n + 6\delta^2), \textit{ for all } n \in \mathbb{N}^*.$$

Proof. From (3.6), we have

$$\begin{aligned}
v^2+vp_v &= \frac{6v^{\frac{15}{4}}+v^2}{14v^{\frac{7}{4}}-7e}\\
&= \frac{6v^{\frac{15}{4}}+v^2+(14v^{\frac{7}{4}}-7e)-(14v^{\frac{7}{4}}-7e)}{14v^{\frac{7}{4}}-7e}\\
&= e+\frac{6v^{\frac{15}{4}}+v^2-14v^{\frac{7}{4}}+7e}{14v^{\frac{7}{4}}-7e}\\
&= e+\frac{p_v^2}{4}\left[\frac{4\left(6v^{\frac{15}{4}}+v^2-14v^{\frac{7}{4}}+7e\right)\left(14v^{\frac{7}{4}}-7e\right)}{64\left(v-v^{\frac{11}{4}}\right)^2}\right]\\
&= e+\frac{p_v^2}{4}\left[\frac{84v^{\frac{11}{2}}-42v^{\frac{15}{4}}+14v^{\frac{15}{4}}-7v^2-\left(14v^{\frac{7}{4}}-7e\right)^2}{16\left(v-v^{\frac{11}{4}}\right)^2}\right]\\
&= e+\frac{p_v^2}{4}\left[\frac{96\left(v-v^{\frac{11}{4}}\right)^2+164v^{\frac{15}{4}}-103v^2-12v^{\frac{11}{2}}-\left(14v^{\frac{7}{4}}-7e\right)^2}{16\left(v-v^{\frac{11}{4}}\right)^2}\right]\\
&= e+\frac{p_v^2}{4}\left[6e+\frac{164v^{\frac{15}{4}}-103v^2-12v^{\frac{11}{2}}-\left(14v^{\frac{7}{4}}-7e\right)^2}{16\left(v-v^{\frac{11}{4}}\right)^2}\right].
\end{aligned}$$

Since $\left[6e+\frac{164v^{\frac{15}{4}}-103v^2-12v^{\frac{11}{2}}-\left(14v^{\frac{7}{4}}-7e\right)^2}{16\left(v-v^{\frac{11}{4}}\right)^2}\right]\leq 6e$, because $\frac{164v^{\frac{15}{4}}-103v^2-12v^{\frac{11}{2}}-\left(14v^{\frac{7}{4}}-7e\right)^2}{16\left(v-v^{\frac{11}{4}}\right)^2}\leq 0$, for all $v>\frac{1}{2^{\frac{4}{7}}}e$.
Then

$$v^2+vp_v\leq e+6\frac{p_v^2}{4}. \tag{3.13}$$

Using (2.14), (3.5) and (3.13), we obtain

$$\begin{aligned}
(x_+)^T s_+ &= \mu\sum_{i=1}^{n}(v_{+_i})^2\\
&= \mu\sum_{i=1}^{n}\left(v_i^2+v_i(p_v)_i+\frac{(p_v)_i^2}{4}-\frac{(q_v)_i^2}{4}\right)\\
&\leq \mu\sum_{i=1}^{n}\left(1+6\frac{(p_v)_i^2}{4}\right)+\mu\left(\frac{\|p_v\|^2-\|q_v\|^2}{4}\right)\\
&\leq \mu n+6\mu\frac{\|p_v\|^2}{4}\\
&\leq \mu\left(n+6\delta^2\right).
\end{aligned}$$

This completes the proof. □

The next lemma shows that the algorithm is well-defined.

Lemma 3.5. *Let $\delta = \delta(x, s; \mu) < \frac{1}{2^{\frac{4}{7}}}$, $v > \frac{1}{2^{\frac{4}{7}}}e$ and $\mu_+ = (1-\theta)\mu$, where $0 < \theta < 1$, then, $v_{++} = \sqrt{\frac{x_+ s_+}{\mu_+}} > \frac{1}{2^{\frac{4}{7}}}e$, and*

$$\delta(x_+ s_+, \mu_+) < \frac{4}{7}\left[\frac{\sqrt{1-\delta^2}\left((1-\theta)^{\frac{7}{8}} - (1-\delta^2)^{\frac{7}{8}}\right)(8\delta^2 + \theta\sqrt{n})}{2(1-\theta)^{\frac{3}{2}}(1-\delta^2)^{\frac{7}{8}} - 2\sqrt{1-\theta}(1-\delta^2)^{\frac{15}{8}} - (1-\theta)^{\frac{19}{8}} + (1-\theta)^{\frac{11}{8}}(1-\delta^2)}\right].$$

Moreover, if $\delta < \frac{1}{8}$ and $\theta = \frac{1}{10\sqrt{n}}$, then $\delta(x_+ s_+, \mu_+) < \frac{1}{8}$.

Proof. From Lemma 3.3 we have $v_+ > \frac{1}{2^{\frac{4}{7}}}e$. Then

$$v_{++} = \sqrt{\frac{x_+ s_+}{\mu_+}} = \sqrt{\frac{x_+ s_+}{\mu(1-\theta)}} = \frac{1}{\sqrt{1-\theta}}v_+ > \frac{1}{2^{\frac{4}{7}}}e. \tag{3.14}$$

This last inequality follows from $0 < \theta < 1 \Rightarrow \frac{1}{\sqrt{1-\theta}} > 1$.

Now, from (3.2), we have

$$\delta(v_{++}) = \frac{\|p_{v_{++}}\|}{2} = \frac{1}{2}\left\|\frac{8v_{++} - 8v_{++}^{\frac{11}{4}}}{14v_{++}^{\frac{7}{4}} - 7e}\right\|$$
$$= 4\left\|\frac{\left(v_{++} - v_{++}^{\frac{11}{4}}\right)}{\left(14v_{++}^{\frac{7}{4}} - 7e\right)\left(e - v_{++}^2\right)}\left(e - v_{++}^2\right)\right\|. \tag{3.15}$$

Let us compute the three expressions of the previous norm, from (3.14) we obtain

$$v_{++} - v_{++}^{\frac{11}{4}} = v_{++}\left(e - v_{++}^{\frac{7}{4}}\right)$$
$$= \frac{v_+}{\sqrt{1-\theta}}\left(e - \frac{v_+^{\frac{7}{4}}}{(1-\theta)^{\frac{7}{8}}}\right)$$
$$= \frac{v_+}{(1-\theta)^{\frac{11}{8}}}\left((1-\theta)^{\frac{7}{8}}e - v_+^{\frac{7}{4}}\right). \tag{3.16}$$

$$14v_{++}^{\frac{7}{4}} - 7e = \frac{14v_+^{\frac{7}{4}}}{(1-\theta)^{\frac{7}{8}}} - 7e$$
$$= \frac{14v_+^{\frac{7}{4}} - 7(1-\theta)^{\frac{7}{8}}e}{(1-\theta)^{\frac{7}{8}}}. \tag{3.17}$$

$$e - v_{++}^2 = \frac{(1-\theta)e - v_+^2}{(1-\theta)}. \tag{3.18}$$

From (3.17) and (3.18), we have

$$\left(14v_{++}^{\frac{7}{4}} - 7e\right)\left(e - v_{++}^2\right) = \left[\frac{14v_+^{\frac{7}{4}} - 7(1-\theta)^{\frac{7}{8}}e}{(1-\theta)^{\frac{7}{8}}}\right]\left[\frac{(1-\theta)e - v_+^2}{(1-\theta)}\right]$$
$$= \frac{14(1-\theta)v_+^{\frac{7}{4}} - 14v_+^{\frac{15}{4}} - 7(1-\theta)^{\frac{15}{8}}e + 7(1-\theta)^{\frac{7}{8}}v_+^2}{(1-\theta)^{\frac{15}{8}}}. \tag{3.19}$$

Then, from (3.16), (3.18) and (3.19), we obtain

$$\frac{\left(v_{++}-v_{++}^{\frac{11}{4}}\right)\left(e-v_{++}^{2}\right)}{\left(14v_{++}^{\frac{7}{4}}-7e\right)\left(e-v_{++}^{2}\right)}=\frac{\left[v_{+}(1-\theta)^{\frac{15}{8}}\left((1-\theta)^{\frac{7}{8}}e-v_{+}^{\frac{7}{4}}\right)\right]\left[(1-\theta)e-v_{+}^{2}\right]}{(1-\theta)^{\frac{11}{8}}(1-\theta)\left[14(1-\theta)v_{+}^{\frac{7}{4}}-14v_{+}^{\frac{15}{4}}-7(1-\theta)^{\frac{15}{8}}e+7(1-\theta)^{\frac{7}{8}}v_{+}^{2}\right]}$$
$$=\frac{v_{+}\left((1-\theta)^{\frac{7}{8}}e-v_{+}^{\frac{7}{4}}\right)\left((1-\theta)e-v_{+}^{2}\right)}{\sqrt{1-\theta}\left(14(1-\theta)v_{+}^{\frac{7}{4}}-14v_{+}^{\frac{15}{4}}-7(1-\theta)^{\frac{15}{8}}e+7(1-\theta)^{\frac{7}{8}}v_{+}^{2}\right)}. \tag{3.20}$$

Let us consider the function

$$f(x)=\frac{x\left((1-\theta)^{\frac{7}{8}}-x^{\frac{7}{4}}\right)}{\sqrt{1-\theta}\left(14(1-\theta)x^{\frac{7}{4}}-14x^{\frac{15}{4}}-7(1-\theta)^{\frac{15}{8}}+7(1-\theta)^{\frac{7}{8}}x^{2}\right)},\ x>\frac{1}{2^{\frac{4}{7}}}.$$

We have $f'(x)<0$, then the function f is decreasing for all $x>\frac{1}{2^{\frac{4}{7}}}$. From Lemma 3.2, (3.15) and (3.20), we deduce that

$$\delta(x_{+}s_{+},\mu_{+})<4\frac{\sqrt{1-\delta^{2}}\left((1-\theta)^{\frac{7}{8}}-\sqrt{1-\delta^{2}}^{\frac{7}{4}}\right)\left\|(1-\theta)e-v_{+}^{2}\right\|}{\sqrt{1-\theta}\left(14(1-\theta)\sqrt{1-\delta^{2}}^{\frac{7}{4}}-14\sqrt{1-\delta^{2}}^{\frac{15}{4}}-7(1-\theta)^{\frac{15}{8}}+7(1-\theta)^{\frac{7}{8}}(1-\delta^{2})\right)}. \tag{3.21}$$

According to (3.11), we get

$$\|(1-\theta)e-v_{+}^{2}\|\leq\|e-v_{+}^{2}\|+\|\theta e\|\leq 8\delta^{2}+\theta\sqrt{n}. \tag{3.22}$$

Hence, by using (3.21) and (3.22) we get

$$\delta(x_{+}s_{+},\mu_{+})<\frac{4}{7}\left[\frac{\sqrt{1-\delta^{2}}\left((1-\theta)^{\frac{7}{8}}-(1-\delta^{2})^{\frac{7}{8}}\right)\left(8\delta^{2}+\theta\sqrt{n}\right)}{2(1-\theta)^{\frac{3}{2}}(1-\delta^{2})^{\frac{7}{8}}-2\sqrt{1-\theta}(1-\delta^{2})^{\frac{15}{8}}-(1-\theta)^{\frac{19}{8}}+(1-\theta)^{\frac{11}{8}}(1-\delta^{2})}\right],$$

which proves the first part of the lemma.

Now, suppose that $\delta<\frac{1}{8}$ and $\theta=\frac{1}{10\sqrt{n}}$. Let's consider the function

$$h(\delta)=\frac{\sqrt{1-\delta^{2}}\left((1-\theta)^{\frac{7}{8}}-(1-\delta^{2})^{\frac{7}{8}}\right)}{2(1-\theta)^{\frac{3}{2}}(1-\delta^{2})^{\frac{7}{8}}-2\sqrt{1-\theta}(1-\delta^{2})^{\frac{15}{8}}-(1-\theta)^{\frac{19}{8}}+(1-\theta)^{\frac{11}{8}}(1-\delta^{2})},$$

we obtain that $h'(\delta) > 0$, so h is increasing for each $\delta < \frac{1}{8}$, then

$$h(\delta) \leq h\Big(\frac{1}{8}\Big), \tag{3.23}$$

where

$$h\Big(\frac{1}{8}\Big) = \frac{a\sqrt{1-\theta}^{\frac{7}{4}} - a^{\frac{11}{4}}}{2a^{\frac{7}{4}}\sqrt{1-\theta}^{3} - 2a^{\frac{15}{4}}\sqrt{1-\theta} - \sqrt{1-\theta}^{\frac{19}{4}} + a^2\sqrt{1-\theta}^{\frac{11}{4}}},$$

such as $a = \sqrt{1 - \frac{1}{8^2}}$, $\theta = \frac{1}{10\sqrt{n}}$. Using $n \geq 1$ we get $\theta \leq \frac{1}{10}$, which is equivalent to

$$h\Big(\frac{1}{8}\Big) \leq 0.89. \tag{3.24}$$

Moreover, we have

$$8\delta^2 + \theta\sqrt{n} = 8\delta^2 + \frac{1}{10} < \frac{1}{8} + \frac{1}{10} = \frac{9}{40}. \tag{3.25}$$

Finally, using (3.23), (3.24) and (3.25), we get

$$\delta(x_+s_+, \mu_+) < \frac{4}{7}h\left(\frac{1}{8}\right)(8\delta^2 + \theta\sqrt{n}) < \frac{4}{7} \times 0.89 \times \frac{9}{40} = 0.1144 < \frac{1}{8}.$$

Which completes the second part of the proof. □

The next lemma gives an upper bound on the number of iterations.

Lemma 3.6. *Assume that the pair (x^0, s^0) is strictly feasible, $\mu^0 = \frac{(x^0)^T s^0}{n}$ and $\delta(x^0s^0, \mu^0) < \frac{1}{2^{\frac{4}{7}}}$. Moreover, let x^k and s^k be the vectors obtained after k iterations. Then the inequality $\left(x^k\right)^T s^k < \varepsilon$ is satisfied when*

$$k \geq \frac{1}{\theta}\log\left[\frac{\mu^0(n + \frac{3}{\sqrt[7]{2}})}{\varepsilon}\right].$$

Proof. After k iterations, we have $\mu^k = (1-\theta)^k\mu^0$. From Lemma 3.4 and $\delta(xs, \mu) < \frac{1}{2^{\frac{4}{7}}}$, we get

$$(x^k)^T s^k \leq \mu^k\left[n + \frac{3}{\sqrt[7]{2}}\right] = \mu^0(1-\theta)^k\left[n + \frac{3}{\sqrt[7]{2}}\right].$$

Hence, the inequality $\left(x^k\right)^T s^k < \varepsilon$ holds if

$$\mu^0(1-\theta)^k\left[n + \frac{3}{\sqrt[7]{2}}\right] \leq \varepsilon$$

$$\Longleftrightarrow \log(1-\theta)^k + \log\mu^0\left[n + \frac{3}{\sqrt[7]{2}}\right] \leq \log\varepsilon$$

$$\Longleftrightarrow -k\log(1-\theta) \geq \log\frac{\mu^0\left[n + \frac{3}{\sqrt[7]{2}}\right]}{\varepsilon}.$$

As $\theta \leq -\log(1-\theta)$, we see that the last inequality is valid only if

$$k\theta \geq \log \frac{\mu^0 \left[n + \frac{3}{\sqrt[7]{2}} \right]}{\varepsilon}$$
$$\Longleftrightarrow k \geq \frac{1}{\theta} \log \frac{\mu^0 \left[n + \frac{3}{\sqrt[7]{2}} \right]}{\varepsilon}.$$

This completes the proof. □

Theorem 3.7. *Suppose that $x^0 = s^0 = e$. If we consider the default values for θ and τ, we obtain that the algorithm given in Figure 2.1 requires no more than*

$$10\sqrt{n} \log \left(\frac{n + \frac{3}{\sqrt[7]{2}}}{\varepsilon} \right)$$

iterations. The resulting vectors satisfy $\left(x^k\right)^T s^k < \varepsilon$.

Proof. Since $x^0 = s^0 = e$, then $\mu^0 = \frac{\left(x^0\right)^T s^0}{n} = 1$ and if we take $\theta = \frac{1}{10\sqrt{n}}$ in Lemma 3.6, the result holds. □

3.2 Second new full-Newton step interior point algorithm based on the function $\psi(t) = t^{3/2}$

In this section, we restrict our analysis to the case $\psi : \left(\frac{1}{\sqrt[3]{4}}, \infty \right) \to \mathbb{R}$, such that $\psi(t) = t^{\frac{3}{2}}$. This yields:

$$p_v = \frac{4v - 4v^{\frac{5}{2}}}{6v^{\frac{3}{2}} - 3e}, \tag{3.26}$$

The condition $2t\psi'(t^2) - \psi'(t) > 0, \forall t > k^2$ is satisfied in this case, when $k^2 = \frac{1}{\sqrt[3]{4}}$.

For the analysis of the algorithm, we define a norm-based proximity measure $\delta(xs, \mu)$ as follows:

$$\delta(v) = \delta(xs, \mu) = \frac{\|p_v\|}{2} = \frac{2}{3} \left\| \frac{v - v^{\frac{5}{2}}}{2v^{\frac{3}{2}} - e} \right\|. \tag{3.27}$$

In the next subsection, we present some results related to algorithm complexity analysis.

3.2.1 Complexity Analysis

The following lemma shows the feasibility of the full-Newton step under the conditions $\delta(xs, \mu) < 1$ and $v > \frac{1}{\sqrt[3]{4}} e$.

Lemma 3.8. *Suppose that $\delta(xs, \mu) < 1$ and $v > \frac{1}{\sqrt[3]{4}}e$. Then the full-Newton step is strictly feasible, hence:*

$$x_+ > 0 \text{ and } s_+ > 0.$$

Proof. For each $0 \leq \alpha \leq 1$ denote $x_+(\alpha) = x + \alpha\Delta x$ and $s_+(\alpha) = s + \alpha\Delta s$. Hence,

$$x_+(\alpha)s_+(\alpha) = xs + \alpha(s\Delta x + x\Delta s) + \alpha^2\Delta x\Delta s.$$

Now, in view of (2.9) and (2.10) we have

$$\frac{1}{\mu}x_+(\alpha)s_+(\alpha) = \frac{xs}{\mu} + \alpha v(d_x + d_s) + \alpha^2 d_x d_s. \tag{3.28}$$

Also from (2.11) and (2.13), we can write

$$\frac{1}{\mu}x_+(\alpha)s_+(\alpha) = v^2 + \alpha v p_v + \alpha^2\left(\frac{p_v^2 - q_v^2}{4}\right),$$

so

$$\frac{1}{\mu}x_+(\alpha)s_+(\alpha) = (1-\alpha)v^2 + \alpha(v^2 + vp_v) + \alpha^2\left(\frac{p_v^2 - q_v^2}{4}\right). \tag{3.29}$$

In addition, from (3.26) we obtain

$$v^2 + vp_v = \frac{2v^{\frac{7}{2}} + v^2}{6v^{\frac{3}{2}} - 3e}. \tag{3.30}$$

Now, let's consider the function: $f(x) = \frac{2x^{\frac{7}{2}}+x^2}{6x^{\frac{3}{2}}-3}$, with $x > \frac{1}{\sqrt[3]{4}}$. We have $f(x) \geq f(1)$, so $f(x) \geq 1$. Using this result, we get

$$v^2 + vp_v \geq e. \tag{3.31}$$

Then

$$\begin{aligned}
\frac{1}{\mu}x_+(\alpha)s_+(\alpha) &\geq (1-\alpha)v^2 + \alpha e + \alpha^2\left(\frac{p_v^2}{4} - \frac{q_v^2}{4}\right) \\
&\geq (1-\alpha)v^2 + \alpha e + \alpha^2\left(\frac{p_v^2}{4} - \frac{q_v^2}{4}\right) - \alpha\frac{p_v^2}{4} \\
&\geq (1-\alpha)v^2 + \alpha e + \alpha(\alpha-1)\frac{p_v^2}{4} - \alpha^2\frac{q_v^2}{4},
\end{aligned}$$

so

$$\frac{1}{\mu}x_+(\alpha)s_+(\alpha) \geq (1-\alpha)v^2 + \alpha\left(e - \left((1-\alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right)\right). \tag{3.32}$$

To get the inequality $x_+(\alpha)s_+(\alpha) > 0$, it suffices to prove that $\left\|(1-\alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right\|_\infty < 1$. In this way, we have

$$\begin{aligned}\left\|(1-\alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right\|_\infty &\leq (1-\alpha)\frac{\|p_v^2\|_\infty}{4} + \alpha\frac{\|q_v^2\|_\infty}{4} \\ &\leq (1-\alpha)\frac{\|p_v\|^2}{4} + \alpha\frac{\|q_v\|^2}{4},\end{aligned}$$

from (2.14), we get

$$\begin{aligned}\left\|(1-\alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right\|_\infty &\leq (1-\alpha)\frac{\|p_v\|^2}{4} + \alpha\frac{\|p_v\|^2}{4} \\ &\leq \frac{\|p_v\|^2}{4} = \delta^2 < 1.\end{aligned}$$

Hence, $x_+(\alpha)s_+(\alpha) > 0$ for each $0 \leq \alpha \leq 1$, which means that the linear functions of α, $x_+(\alpha)$ and $s_+(\alpha)$ do not change sign on the interval $[0, 1]$ and for $\alpha = 0$, we have $x_+(0) = x > 0$ and $s_+(0) = s > 0$. This leads to $x_+(1) = x_+ > 0$ and $s_+(1) = s_+ > 0$. This means that the full-Newton step is strictly feasible. □

The local quadratic convergence of the full-Newton step is proved in the following lemma.

Lemma 3.9. *Let $\delta = \delta(xs, \mu) < \frac{1}{\sqrt[3]{4}}$ and $v > \frac{1}{\sqrt[3]{4}}e$. Then $v_+ = \sqrt{\frac{x_+s_+}{\mu}} > \frac{1}{\sqrt[3]{4}}e$ and*

$$\delta(x_+s_+, \mu) < 8\delta^2,$$

which means local quadratic convergence of the full- Newton step.

Proof. We know from Lemma 3.8 that $x_+ > 0$ and $s_+ > 0$, then $v_+ = \sqrt{\frac{x_+s_+}{\mu}}$ is well defined.

Let $\alpha = 1$. Then from (3.29), it follows that

$$v_+^2 = v^2 + vp_v + \frac{p_v^2}{4} - \frac{q_v^2}{4}. \tag{3.33}$$

Using (3.33) and the inequality (3.31), we obtain

$$v_+^2 \geq e + \frac{p_v^2}{4} - \frac{q_v^2}{4} \geq e - \frac{q_v^2}{4},$$

hence

$$\min(v_+^2) \geq 1 - \frac{\|q_v^2\|_\infty}{4} \geq 1 - \frac{\|q_v\|^2}{4} \geq 1 - \delta^2,$$

and this relation yields

$$\min(v_+) \geq \sqrt{1-\delta^2}. \tag{3.34}$$

Since $\delta < \frac{1}{\sqrt[3]{4}}$, then

$$\sqrt{1-\delta^2} > \sqrt{1-\frac{1}{\sqrt[3]{16}}} > \frac{1}{\sqrt[3]{4}},$$

using this last inequality and (3.34), we get

$$v_+ > \frac{1}{\sqrt[3]{4}}e.$$

This completes the first part of the proof.

Now, from (3.33) and (3.30) we have

$$\begin{aligned}
\|e - v_+^2\| &= \left\| e - \left(\left(v^2 + vp_v\right) + \frac{p_v^2}{4} - \frac{q_v^2}{4} \right) \right\| \\
&\leq \left\| \frac{q_v^2}{4} \right\| + \left\| e - \left(v^2 + vp_v\right) - \frac{p_v^2}{4} \right\| \\
&\leq \left\| \frac{q_v^2}{4} \right\| + \left\| e - \frac{2v^{\frac{7}{2}} + v^2}{6v^{\frac{3}{2}} - 3e} - \frac{p_v^2}{4} \right\| \\
&\leq \left\| \frac{q_v^2}{4} \right\| + \left\| \frac{6v^{\frac{3}{2}} - 3e - 2v^{\frac{7}{2}} - v^2}{6v^{\frac{3}{2}} - 3e} - \frac{p_v^2}{4} \right\| \\
&\leq \left\| \frac{q_v^2}{4} \right\| + \left\| \left[\frac{4 \times \left(6v^{\frac{3}{2}} - 3e - 2v^{\frac{7}{2}} - v^2\right)}{\left(6v^{\frac{3}{2}} - 3e\right) \times p_v^2} - e \right] \frac{p_v^2}{4} \right\| \\
&\leq \left\| \frac{q_v^2}{4} \right\| + \left\| \left[-\frac{16v^5 + v^2 - 8v^{\frac{7}{2}} - 36v^3 + 36v^{\frac{3}{2}} - 9e}{4\left(v - v^{\frac{5}{2}}\right)^2} \right] \frac{p_v^2}{4} \right\| \\
&\leq \left\| \frac{q_v^2}{4} \right\| + \left\| \left[\frac{16v^5 + v^2 - 8v^{\frac{7}{2}} - 36v^3 + 36v^{\frac{3}{2}} - 9e}{4\left(v - v^{\frac{5}{2}}\right)^2} \right] \frac{p_v^2}{4} \right\|.
\end{aligned}$$

On one hand, we have

$$\frac{16v^5 + v^2 - 8v^{\frac{7}{2}} - 36v^3 + 36v^{\frac{3}{2}} - 9e}{4\left(v - v^{\frac{5}{2}}\right)^2} \geq 0, \forall v > \frac{1}{\sqrt[3]{4}}e.$$

On the other hand, we have

$$\frac{16v^5 + v^2 - 8v^{\frac{7}{2}} - 36v^3 + 36v^{\frac{3}{2}} - 9e}{4\left(v - v^{\frac{5}{2}}\right)^2} = 5e + K(v),$$

where

$$K(v) = \frac{36v^{\frac{3}{2}} - 36v^3 - 9e + 32v^{\frac{7}{2}} - 4v^5 - 19v^2}{4\left(v - v^{\frac{5}{2}}\right)^2}.$$

Since

$$K(v) \leq 0, \forall v > \frac{1}{\sqrt[3]{4}}e,$$

then, we conclude that

$$0 \leq \frac{16v^5 + v^2 - 8v^{\frac{7}{2}} - 36v^3 + 36v^{\frac{3}{2}} - 9e}{4\left(v - v^{\frac{5}{2}}\right)^2} \leq 5e,$$

which implies that

$$\|e - v_+^2\| \leq \left\|\frac{q_v^2}{4}\right\| + 5\left\|\frac{p_v^2}{4}\right\| = 6\delta^2. \tag{3.35}$$

Now, by the definition of δ we have

$$\delta(v_+) = \delta(x_+s_+, \mu) = \frac{\|p_{v_+}\|}{2} = \frac{2}{3}\left\|\frac{\left(v_+ - v_+^{\frac{5}{2}}\right)}{\left(2v_+^{\frac{3}{2}} - e\right)(e - v_+^2)}(e - v_+^2)\right\|.$$

Let's consider the function: $f(t) = \frac{\left(t - t^{\frac{5}{2}}\right)}{\left(2t^{\frac{3}{2}} - 1\right)(1 - t^2)}$, for all $t > \frac{1}{\sqrt[3]{4}}$, $t \neq 1$. Since $f'(t) < 0$, so f is decreasing.

Hence, in view of Lemma 3.2, we obtain

$$\delta(x_+s_+, \mu) \leq \frac{2}{3}\frac{\left((1-\delta^2)^{\frac{1}{2}} - (1-\delta^2)^{\frac{5}{4}}\right)}{\delta^2\left(2(1-\delta^2)^{\frac{3}{4}} - 1\right)}\|e - v_+^2\|.$$

From this last inequality and (3.35), we deduce

$$\delta(x_+s_+, \mu) \leq \frac{4\left((1-\delta^2)^{\frac{1}{2}} - (1-\delta^2)^{\frac{5}{4}}\right)}{\delta^2\left(2(1-\delta^2)^{\frac{3}{4}} - 1\right)}\delta^2. \tag{3.36}$$

Now, if we take $g(\delta) = \frac{4\left((1-\delta^2)^{\frac{1}{2}} - (1-\delta^2)^{\frac{5}{4}}\right)}{\delta^2\left(2(1-\delta^2)^{\frac{3}{4}} - 1\right)}$ for $\delta < \frac{1}{\sqrt[3]{4}}$, then we obtain that $g(\delta) \leq g(\frac{1}{\sqrt[3]{4}}) < 8$, and we conclude that

$$\delta(x_+s_+, \mu) < 8\delta^2.$$

This completes the proof. □

The next lemma examines the effect of the full-Newton step on the duality gap.

Lemma 3.10. *Let* $\delta = \delta(xs, \mu)$*. Then, the duality gap satisfies*

$$(x_+)^T s_+ \leq \mu(n + 4\delta^2).$$

Proof. From (3.30) we have

$$\begin{aligned}
v^2 + vp_v &= \frac{2v^{\frac{7}{2}} + v^2}{6v^{\frac{3}{2}} - 3e} \\
&= e + \frac{2v^{\frac{7}{2}} + v^2 - 6v^{\frac{3}{2}} + 3e}{6v^{\frac{3}{2}} - 3e} \\
&= e + \frac{4\left(2v^{\frac{7}{2}} + v^2 - 6v^{\frac{3}{2}} + 3e\right)}{\left(6v^{\frac{3}{2}} - 3e\right)p_v^2} \times \frac{p_v^2}{4} \\
&= e + \frac{12v^5 - 3v^2 - 36v^3 + 36v^{\frac{3}{2}} - 9e}{4\left(v - v^{\frac{5}{2}}\right)^2} \times \frac{p_v^2}{4} \\
&= e + \left[4e + \frac{36v^{\frac{3}{2}} - 36v^3 - 9e + 32v^{\frac{7}{2}} - 4v^5 - 19v^2}{4\left(v - v^{\frac{5}{2}}\right)^2}\right] \times \frac{p_v^2}{4} \\
&\leq e + 4\frac{p_v^2}{4},
\end{aligned}$$

because

$$\frac{36v^{\frac{3}{2}} - 36v^3 - 9e + 32v^{\frac{7}{2}} - 4v^5 - 19v^2}{4\left(v - v^{\frac{5}{2}}\right)^2} \leq 0, \forall v > \frac{1}{\sqrt[3]{4}}e.$$

Then

$$(x_+)^T(s_+) \leq \mu(n + 4\delta^2).$$

Which completes the proof. □

The following lemma investigates the effect on the proximity measure after a main iteration of the algorithm.

Lemma 3.11. *Let* $\delta = \delta(xs, \mu) < \frac{1}{\sqrt[3]{4}}$*,* $v > \frac{1}{\sqrt[3]{4}}e$ *and* $\mu_+ = (1-\theta)\mu$*, where* $0 < \theta < 1$*. In addition, let* $v_{++} = \sqrt{\frac{x_+ s_+}{\mu_+}}$*. Then* $v_{++} > \frac{1}{2^{\frac{4}{7}}}e$*, and*

$$\delta(x_+ s_+, \mu_+) < \frac{2}{3\sqrt{1-\theta}}\left[\frac{\left((1-\delta^2)^{\frac{1}{2}}(1-\theta)^{\frac{3}{4}} - (1-\delta^2)^{\frac{5}{4}}\right)(6\delta^2 + \theta\sqrt{n})}{2(1-\theta)(1-\delta^2)^{\frac{3}{4}} - 2(1-\delta^2)^{\frac{7}{4}} + (1-\theta)^{\frac{3}{4}}(1-\delta^2) - (1-\theta)^{\frac{7}{4}}}\right].$$

Moreover, if $\delta < \frac{1}{6}$ *and* $\theta = \frac{1}{7\sqrt{n}}$*, then* $\delta(x_+ s_+, \mu_+) < \frac{1}{6}$*.*

Proof. From Lemma 3.9 we have $v_+ > \frac{1}{\sqrt[3]{4}}e$. Then

$$v_{++} = \sqrt{\frac{x_+ s_+}{\mu_+}} = \sqrt{\frac{x_+ s_+}{(1-\theta)\mu}} = \frac{1}{\sqrt{1-\theta}}v_+ > \frac{1}{\sqrt[3]{4}}e. \tag{3.37}$$

This last inequality follows from $0 < \theta < 1 \Rightarrow \frac{1}{\sqrt{1-\theta}} > 1$.

Now, from the definition of δ, we write

$$\delta(v_+) = \delta(x_+s_+, \mu_+) = \frac{\|p_{v_{++}}\|}{2} = \frac{2}{3}\left\|\frac{\left(v_{++} - v_{++}^{\frac{5}{2}}\right)}{\left(2v_{++}^{\frac{3}{2}} - e\right)(e - v_{++}^2)}(e - v_{++}^2)\right\|.$$

Let us compute the three expressions of the previous norm. From (3.37) we obtain

$$\begin{aligned} v_{++} - v_{++}^{\frac{5}{2}} &= \frac{1}{\sqrt{1-\theta}}v_+ - \left(\frac{1}{\sqrt{1-\theta}}v_+\right)^{\frac{5}{2}} \\ &= \frac{1}{(1-\theta)^{\frac{5}{4}}}\left[(1-\theta)^{\frac{3}{4}}v_+ - v_+^{\frac{5}{2}}\right], \\ 2v_{++}^{\frac{3}{2}} - e &= \frac{2v_+^{\frac{3}{2}}}{(1-\theta)^{\frac{3}{4}}} - e, \\ e - v_{++}^2 &= \frac{1}{(1-\theta)}\left[(1-\theta)e - v_+^2\right]. \end{aligned}$$

Then

$$\left(2v_{++}^{\frac{3}{2}} - e\right)(e - v_{++}^2) = \frac{\left[2(1-\theta)v_+^{\frac{3}{2}} - 2v_+^{\frac{7}{2}} + (1-\theta)^{\frac{3}{4}}v_+^2 - (1-\theta)^{\frac{7}{4}}\right]}{(1-\theta)^{\frac{7}{4}}}. \tag{3.38}$$

And

$$\left(v_{++} - v_{++}^{\frac{5}{2}}\right)(e - v_{++}^2) = \frac{(1-\theta)^{\frac{3}{4}}v_+ - v_+^{\frac{5}{2}}}{(1-\theta)^{\frac{9}{4}}}\left[(1-\theta)e - v_+^2\right]. \tag{3.39}$$

These two last equalities give

$$\delta(v_+) = \frac{2}{3}\left\|\frac{\left((1-\theta)^{\frac{3}{4}}v_+ - v_+^{\frac{5}{2}}\right)\left[(1-\theta)e - v_+^2\right]}{2(1-\theta)^{\frac{3}{2}}v_+^{\frac{3}{2}} - 2(1-\theta)^{\frac{1}{2}}v_+^{\frac{7}{2}} + (1-\theta)^{\frac{5}{4}}v_+^2 - (1-\theta)^{\frac{9}{4}}}\right\|. \tag{3.40}$$

Let us consider the function

$$f(t) = \frac{(1-\theta)^{\frac{3}{4}}t - t^{\frac{5}{2}}}{2(1-\theta)^{\frac{3}{2}}t^{\frac{3}{2}} - 2(1-\theta)^{\frac{1}{2}}t^{\frac{7}{2}} + (1-\theta)^{\frac{5}{4}}t^2 - (1-\theta)^{\frac{9}{4}}}, \forall t > \frac{1}{\sqrt[3]{4}}.$$

After some calculation, we obtain $f'(t) < 0$ for all $n \in \mathbb{N}^*$and $t > \frac{1}{\sqrt[3]{4}}$, then the function f is decreasing. From lemma 3.2 and (3.40), we deduce that

$$\delta(x_+s_+, \mu_+) < \frac{2\left[(1-\theta)^{\frac{3}{4}}(1-\delta^2)^{\frac{1}{2}} - (1-\delta^2)^{\frac{5}{4}}\right]\left\|\left[(1-\theta)e - v_+^2\right]\right\|}{3\sqrt{1-\theta}\left[2(1-\theta)(1-\delta^2)^{\frac{3}{4}} - 2(1-\delta^2)^{\frac{7}{4}} + (1-\theta)^{\frac{3}{4}}(1-\delta^2) - (1-\theta)^{\frac{7}{4}}\right]}. \tag{3.41}$$

According to (3.35), we get

$$\|(1-\theta)e - v_+^2\| \leq \|e - v_+^2\| + \|\theta e\| \leq 6\delta^2 + \theta\sqrt{n}.$$

Hence, by using this last inequality in (3.41), we get

$$\delta(x_+s_+,\mu_+) < \frac{2}{3\sqrt{1-\theta}}\left[\frac{\left((1-\delta^2)^{\frac{1}{2}}(1-\theta)^{\frac{3}{4}} - (1-\delta^2)^{\frac{5}{4}}\right)(6\delta^2+\theta\sqrt{n})}{2(1-\theta)(1-\delta^2)^{\frac{3}{4}} - 2(1-\delta^2)^{\frac{7}{4}} + (1-\theta)^{\frac{3}{4}}(1-\delta^2) - (1-\theta)^{\frac{7}{4}}}\right].$$

which proves the first part of the lemma.

Now, suppose that $\delta < \frac{1}{6}$ and $\theta = \frac{1}{7\sqrt{n}}$. Let's consider the function

$$f(\delta) = \frac{(1-\delta^2)^{\frac{1}{2}}(1-\theta)^{\frac{3}{4}} - (1-\delta^2)^{\frac{5}{4}}}{2(1-\theta)(1-\delta^2)^{\frac{3}{4}} - 2(1-\delta^2)^{\frac{7}{4}} + (1-\theta)^{\frac{3}{4}}(1-\delta^2) - (1-\theta)^{\frac{7}{4}}},$$

we obtain $f'(\delta) > 0$, so f is increasing for each $\delta < \frac{1}{6}$, then

$$f(\delta) \leq f\left(\frac{1}{6}\right), \tag{3.42}$$

where

$$f\left(\frac{1}{6}\right) = \frac{\left(\frac{35}{36}\right)^{\frac{1}{2}}(1-\theta)^{\frac{3}{4}} - \left(\frac{35}{36}\right)^{\frac{5}{4}}}{2(1-\theta)(\frac{35}{36})^{\frac{3}{4}} - 2(\frac{35}{36})^{\frac{7}{4}} + (1-\theta)^{\frac{3}{4}}(\frac{35}{36}) - (1-\theta)^{\frac{7}{4}}}.$$

Also, we have for all $n \in \mathbb{N}^*$, $3\sqrt{1-\theta} > 0$ and $2\left(6\delta^2+\theta\sqrt{n}\right) = 2\left(6\delta^2+\frac{1}{7}\right) < 2\left(\frac{1}{6}+\frac{1}{7}\right)$, then

$$\frac{2\left(6\delta^2+\theta\sqrt{n}\right)}{3\sqrt{1-\theta}} < \frac{2\left(\frac{1}{6}+\frac{1}{7}\right)}{3\sqrt{1-\theta}} = \frac{13}{63\sqrt{1-\theta}}. \tag{3.43}$$

According to (3.42) and (3.43) we obtain

$$\delta(x_+s_+,\mu_+) < \frac{13}{63}g(\theta), \tag{3.44}$$

with

$$g(\theta) = \frac{\left(\frac{35}{36}\right)^{\frac{1}{2}}(1-\theta)^{\frac{3}{4}} - \left(\frac{35}{36}\right)^{\frac{5}{4}}}{2(1-\theta)^{\frac{3}{2}}(\frac{35}{36})^{\frac{3}{4}} - 2(1-\theta)^{\frac{1}{2}}(\frac{35}{36})^{\frac{7}{4}} + (1-\theta)^{\frac{5}{4}}(\frac{35}{36}) - (1-\theta)^{\frac{9}{4}}},$$

if $n \geq 1$ then $0 < \theta \leq \frac{1}{7}$. The function g is continuous and decreasing on $0 < \theta \leq \frac{1}{7}$. Consequently

$$g(\theta) < g(0) = \frac{\left(\frac{35}{36}\right)^{\frac{1}{2}} - \left(\frac{35}{36}\right)^{\frac{5}{4}}}{2\left(\frac{35}{36}\right)^{\frac{3}{4}} - 2(\frac{35}{36})^{\frac{7}{4}} + \left(\frac{35}{36}\right) - 1}. \tag{3.45}$$

Finally, using (3.44) and (3.45), we get

$$\delta(x_+s_+,\mu_+) < \frac{13}{63} \times \frac{\left(\frac{35}{36}\right)^{\frac{1}{2}} - \left(\frac{35}{36}\right)^{\frac{5}{4}}}{2\left(\frac{35}{36}\right)^{\frac{3}{4}} - 2(\frac{35}{36})^{\frac{7}{4}} + \left(\frac{35}{36}\right) - 1} = 0.1598 < \frac{1}{6}.$$

Which completes the second part of the proof. □

The next lemma gives a bound on the number of iterations.

Lemma 3.12. *Suppose that the pair* (x^0, s^0) *is strictly feasible,* $\mu^0 = \frac{(x^0)^T s^0}{n}$ *and* $\delta(x^0 s^0, \mu^0) < \frac{1}{\sqrt[3]{4}}$*. Moreover, let* x^k *and* s^k *be the vectors obtained after* k *iterations. Then the inequality* $\left(x^k\right)^T s^k < \varepsilon$ *is satisfied when*

$$k \geq \frac{1}{\theta} \log \left[\frac{\mu^0 (n + \sqrt[3]{4})}{\varepsilon} \right].$$

Proof. After k iterations, we have $\mu^k = (1-\theta)^k \mu^0$. From Lemma 3.10 and $\delta(xs, \mu) < \frac{1}{\sqrt[3]{4}}$, we get

$$(x^k)^T s^k < \mu^k \left(n + 4 \left(\frac{1}{\sqrt[3]{4}} \right)^2 \right) = \mu^0 (1-\theta)^k \left(n + \sqrt[3]{4} \right).$$

Hence, the inequality $\left(x^k\right)^T s^k < \varepsilon$ holds if

$$\begin{aligned} & \mu^0 (1-\theta)^k \left(n + \sqrt[3]{4} \right) \leq \varepsilon \\ \iff \quad & \log(1-\theta)^k + \log \mu^0 \left(n + \sqrt[3]{4} \right) \leq \log \varepsilon \\ \iff \quad & -k \log(1-\theta) \geq \log \frac{\mu^0 \left(n + \sqrt[3]{4} \right)}{\varepsilon}. \end{aligned}$$

As $\theta \leq -\log(1-\theta)$, then the last inequality is valid if

$$k\theta \geq \log \frac{\mu^0 \left(n + \sqrt[3]{4} \right)}{\varepsilon} \iff k \geq \frac{1}{\theta} \log \frac{\mu^0 \left(n + \sqrt[3]{4} \right)}{\varepsilon}.$$

This completes the proof. □

Theorem 3.13. *Suppose that* $x^0 = s^0 = e$*. If we consider the default values for* θ *and* τ*, we obtain that the algorithm represented in Figure 2.1 requires no more than*

$$7\sqrt{n} \log \frac{\left(n + \sqrt[3]{4} \right)}{\varepsilon}$$

iterations. The resulting vectors satisfy $\left(x^k\right)^T s^k < \varepsilon$.

Proof. Since $x^0 = s^0 = e$, we get $\mu^0 = \frac{(x^0)^T s^0}{n} = 1$. If we replace $\theta = \frac{1}{7\sqrt{n}}$ in Lemma 3.12, the result holds. □

3.3 Numerical experiments

In this section, we conduct comparative numerical tests on the algorithm depicted in Figure 2.1, utilizing three distinct choices for the function ψ. Specifically, we explore our

new functions $\psi(t) = t^{\frac{7}{4}}$ and $\psi(t) = t^{\frac{3}{2}}$ as proposed in papers [86] and [89] respectively, alongside the function introduced by Darvay and Takàcs [20], defined as $\psi(t) = t^2$.
Our objective is to assess the algorithm's efficiency presented in Figure 2.1, by utilizing these three functions independently. Additionally, we aim to evaluate the suitability of our proposed functions in comparison to that of Darvay and Takàcs.
For the numerical tests, we consider eight fixed-size examples and one variable-size example taken from the literature [11]. After that, we solve some problems from the Netlib test collection [29]. The implementation is carried out in Matlab R2009b, using the following parameters:

- The accuracy parameter $\varepsilon = 10^{-4}$.
- $\theta \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ for the first eight fixed size examples.
- $\theta = 0.7$ for the variable size example and the Netlib problems.

The obtained results will be presented in comparative tables where we note by:

- $Iter$: the number of iterations necessary for optimality.
- $T(s)$: the execution time in seconds.
- $M1$, $M2$ and $M3$: the primal-dual interior point algorithms based on the functions $\psi(t) = t^{\frac{3}{2}}$, $\psi(t) = t^{\frac{7}{4}}$ and $\psi(t) = t^2$, respectively.

3.3.1 Examples with fixed size

Example 3.14. $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & -3 \end{pmatrix}$, $c = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^T$, $b = \begin{pmatrix} 1 & 0.5 \end{pmatrix}^T$.
Where: $x^0 = \begin{pmatrix} 0.5 & 0.27 & 0.14 & 0.09 \end{pmatrix}^T$, $s^0 = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^T$, $y^0 = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$.

Example 3.15. $A = \begin{pmatrix} 2 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$, $c = \begin{pmatrix} 3 & -1 & 1 & 0 & 0 & 0 \end{pmatrix}^T$, $b = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$.
Where: $x^0 = \begin{pmatrix} 0.0658 & 0.1326 & 0.1330 & 0.2677 & 0.1330 & 0.2664 \end{pmatrix}^T$, $y^0 = \begin{pmatrix} -2 & -2 & -3 \end{pmatrix}^T$,
$s^0 = \begin{pmatrix} 10 & 4 & 6 & 1 & 5 & 1 \end{pmatrix}^T$.

Example 3.16. $A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$, $c = \begin{pmatrix} -5 & -5 & 0 & 0 & 0 \end{pmatrix}^T$, $b = \begin{pmatrix} 8 & 7 & 3 \end{pmatrix}^T$.

Where: $x^0 = \begin{pmatrix} 2.2534 & 1.5743 & 1.9185 & 1.5976 & 1.4256 \end{pmatrix}^T$, $s^0 = \begin{pmatrix} 1 & 3 & 2 & 2 & 2 \end{pmatrix}^T$, $y^0 = \begin{pmatrix} -2 & -2 & -2 \end{pmatrix}^T$.

Example 3.17. $A = \begin{pmatrix} -1 & 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & -3 & 2 & 0 & 1 & 0 \\ -3 & 2 & 1 & 0 & 0 & 0 & 1 \\ 3 & 5 & 4 & 0.5 & 0 & 0 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 2 & -9 & -2 & -0.5 & 0 & 0 & 0 \end{pmatrix}^T$,

$b = \begin{pmatrix} 1 & 2 & 1 & 12.5 \end{pmatrix}^T$.

Where: $x^0 = s^0 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T$, $y^0 = \begin{pmatrix} -1 & -1 & -1 & -1 \end{pmatrix}^T$.

Example 3.18. $A = \begin{pmatrix} 0 & 1 & 2 & -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & -2 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & -1 & -2 & 0 & 0 & 0 & 1 \\ 1 & 3 & 4 & 2 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 4 \\ 10 \\ 1 \\ 1 \\ 11 \end{pmatrix}$,

$c = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T$.

Where: $x^0 = s^0 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T$, $y^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$.

Example 3.19. $A = \begin{pmatrix} 1 & 0 & -4 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 5 & 3 & 1 & 0 & -1 & 3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 4 & 5 & -3 & 3 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & 1 & -5 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & -3 & 2 & -1 & 4 & 5 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$,

$c = \begin{pmatrix} -9 & -4 & 4 & -7 & -2 & -6 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$, $b = \begin{pmatrix} 3 & 12 & 7 & -2 & 6 & 10 \end{pmatrix}^T$.

Where : $x^0 = s^0 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T$,

$y_0 = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}^T$.

Example 3.20. $A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}, b = \begin{pmatrix} 5 \\ 5 \\ 4 \\ 4 \\ 4 \\ 3 \\ 3 \end{pmatrix},$

$c = \begin{pmatrix} 0.5 & 0.8 & -2 & -0.5 & 0.82 & -1.98 & 0.5 & 0.82 & -1.98 & 0.5 & 0.82 & -1.98 & 0.5 & 0.82 \end{pmatrix}^T.$

Where: $x^0 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T,$

$y^0 = \begin{pmatrix} -0.5 & -0.2 & -3 & -1 & 0.02 & 0.02 & 1 \end{pmatrix}^T,$

$s^0 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T.$

Example 3.21. $A = \begin{pmatrix} 3 & -5 & 2 & -8 & -5 & 2 & 6 & 10 & 5 & 2 \\ 2 & -8 & -6 & -7 & 6 & 5 & 5 & 6 & -9 & -8 \\ 6 & 1 & 9 & 5 & -2 & 3 & 10 & -5 & 6 & -7 \\ 6 & 3 & -2 & 1 & -4 & 7 & -8 & 10 & 8 & 4 \\ 1 & -3 & -1 & 5 & -8 & -4 & 4 & 1 & 4 & -2 \\ 4 & -1 & 3 & -2 & 5 & -8 & 4 & 1 & -6 & 0 \\ 4 & -2 & 6 & -8 & 8 & 8 & -2 & -1 & 6 & 7 \\ 8 & 0 & -9 & -5 & 5 & 4 & -8 & -6 & 5 & 5 \end{pmatrix}, b = \begin{pmatrix} 49 \\ 0 \\ 22 \\ 84 \\ -42 \\ -14 \\ 108 \\ 15 \end{pmatrix}, c =$

$\begin{pmatrix} 103 & -39 & -131 & -140 & 236 & 134 & -164 & 7 & -49 & 70 \end{pmatrix}^T.$

Where: $x^0 = \begin{pmatrix} 1 & 2 & 2 & 1 & 4 & 5 & 2 & 4 & 2 & 4 \end{pmatrix}^T, \quad s^0 = \begin{pmatrix} 8 & 10 & 5 & 4 & 9 & 1 & 4 & 6 & 5 & 1 \end{pmatrix}^T,$

$y^0 = \begin{pmatrix} -9 & 10 & -6 & 5 & 0 & 3 & 10 & 7 \end{pmatrix}^T$

The obtained results of these eight fixed size examples are summarized in Tables 3.1 and 3.2.

		$M1$		$M2$		$M3$	
Example	θ	$Iter$	$T(s)$	$Iter$	$T(s)$	$Iter$	$T(s)$
	0.1	94	0.0132	94	0.0152	94	0.1412
	0.3	28	0.0071	28	0.0077	29	0.0084
Example 3.14	0.5	15	0.0067	16	0.0070	17	0.0075
	0.7	11	0.0058	13	0.0062	15	0.0067
	0.9	10	0.0023	12	0.0032	15	0.0040
	0.1	99	0.0155	99	0.0168	99	0.0606
	0.3	30	0.0083	30	0.0085	30	0.0086
Example 3.15	0.5	16	0.0072	17	0.0073	18	0.0075
	0.7	11	0.0067	14	0.0070	16	0.0073
	0.9	10	0.0057	13	0.0061	16	0.0071
	0.1	115	0.0126	115	0.0137	115	0.0183
	0.3	35	0.0079	35	0.0085	35	0.0103
Example 3.16	0.5	19	0.0072	19	0.0075	21	0.0075
	0.7	13	0.0070	16	0.0073	18	0.0074
	0.9	12	0.0066	15	0.0071	18	0.0074
	0.1	106	0.0164	106	0.0170	107	0.0174
	0.3	32	0.0089	32	0.0093	32	0.0097
Example 3.17	0.5	17	0.0082	18	0.0084	19	0.0087
	0.7	12	0.0064	14	0.0072	17	0.0074
	0.9	11	0.0059	14	0.0070	17	0.0073
	0.1	109	0.0182	109	0.0194	109	0.0208
	0.3	33	0.0092	33	0.0106	33	0.0124
Example 3.18	0.5	18	0.0071	18	0.0077	20	0.0089
	0.7	12	0.0069	15	0.0075	17	0.0086
	0.9	11	0.0067	14	0.0072	17	0.0084

Table 3.1: Numerical results of the fixed size LO examples

		$M1$		$M2$		$M3$	
Example	θ	$Iter$	$T(s)$	$Iter$	$T(s)$	$Iter$	$T(s)$
Example 3.19	0.1	112	0.0219	112	0.0221	112	0.0224
	0.3	34	0.0099	34	0.0104	34	0.0106
	0.5	18	0.0082	19	0.0083	20	0.0085
	0.7	13	0.0078	15	0.0081	18	0.0083
	0.9	11	0.0074	14	0.0078	17	0.0081
Example 3.20	0.1	113	0.0247	113	0.0256	113	0.1867
	0.3	34	0.0129	34	0.0117	34	0.0120
	0.5	18	0.0083	19	0.0093	20	0.0091
	0.7	13	0.0080	15	0.0087	18	0.0088
	0.9	12	0.0085	15	0.0089	18	0.0086
Example 3.21	0.1	131	0.0469	131	0.0482	131	0.0501
	0.3	40	0.0150	40	0.0173	40	0.0216
	0.5	22	0.0092	22	0.0094	24	0.0096
	0.7	15	0.0085	18	0.0087	21	0.0090
	0.9	14	0.0073	17	0.0076	21	0.0081

Table 3.2: Numerical results of the fixed size LO examples

3.3.2 Example with variable size

Example 3.22. (Cube example): $n = 2m$,

$$A[i,j] = \begin{cases} 1 & \text{if } j = i \text{ or } j = i + m \\ 0 & \text{otherwise} \end{cases}, \quad c[i] = \begin{cases} -1 & \text{if } i = 1, ..., m \\ 0 & \text{if } i = m + 1, ..., n \end{cases},$$

$b[j] = 2$ for $j = 1, ..., m$

Where: $x^0[i] = 1$, for $i = 1, ..., n$, $y^0[j] = -2$ for $j = 1, ..., m$,

$$s^0[i] = \begin{cases} 1 & \text{if } i = 1, ..., m \\ 2 & \text{if } i = m + 1, ..., n \end{cases},$$

The obtained results are summarized in the Table 3.3.

	$M1$		$M2$		$M3$	
(m, n)	$Iter$	$T(s)$	$Iter$	$T(s)$	$Iter$	$T(s)$
$(25, 50)$	14	0.0701	17	0.0756	21	0.0876
$(50, 100)$	15	0.1845	18	0.1962	22	0.2788
$(100, 200)$	16	0.8154	19	0.8451	23	0.9747
$(150, 300)$	16	2.1723	19	2.4143	23	3.2392
$(300, 600)$	17	14.0970	20	17.5707	24	24.0428
$(400, 800)$	17	32.9461	21	42.0424	25	50.8862
$(500, 1000)$	17	64.9954	21	80.9903	25	97.2056
$(1000, 2000)$	18	524.3113	22	689.5115	26	743.5775

Table 3.3: Numerical results of LO Example 3.22

3.3.3 Netlib problems

The obtained numerical results for some Netlib problems are summarized in Table 3.4.

		$M1$		$M2$		$M3$	
Problem	(m, n)	$Iter$	$T(s)$	$Iter$	$T(s)$	$Iter$	$T(s)$
Afiro	$(27, 51)$	15	0.0552	19	0.0828	22	0.0941
Sc50a	$(50, 78)$	09	0.1734	11	0.1956	16	0.2909
Sc50b	$(50, 78)$	09	0.1245	11	0.1325	16	0.3071
Blend	$(74, 114)$	16	0.5323	20	0.6779	24	0.9493
Share2b	$(96, 162)$	18	1.1874	23	1.9154	27	2.3279
Scsd1	$(77, 760)$	15	40.9947	19	48.7907	23	60.3563
Bandm	$(305, 472)$	22	43.2515	26	51.2125	29	71.4231
Scsd6	$(141, 1350)$	16	284.3564	19	302.2165	23	377.7166

Table 3.4: Numerical results for some Netlib LO problems

3.4 Comments

Based on the numerical tests conducted on examples of various dimensions as well as on some problems from the Ntelib tests collection, we have arrived at the following conclusions:

For the fixed size examples:

- The numerical results show that the number of iterations and the execution time necessary for optimality of the algorithm given in Figure 2.1 depends on the values of the parameter θ. It is quite surprising that $\theta = 0.9$ gives the smallest number of iteration and the minimal time in the three choices of $\psi(t)$.

- The number of iterations required for optimality using our new functions $\psi(t) = t^{7/4}$ and $\psi(t) = t^{3/2}$ is significantly lower than that of the function given by Darvay and Takàcs $\psi(t) = t^2$. Similarly, the computation time is slightly improved with our new functions compared to $\psi(t) = t^2$.

- The function $\psi(t) = t^{\frac{3}{2}}$ offers better results compared to the two other functions $\psi(t) = t^{\frac{7}{4}}$ and $\psi(t) = t^2$.

For the variable size and the Netlib problems:

- The numerical results show the efficiency of our new functions for problems with large size as well as for the Netlib collection. Indeed, the number of iterations and the computation time are naturally reduced in the case of $\psi(t) = t^{\frac{3}{2}}$ and $\psi(t) = t^{\frac{7}{4}}$ compared to the case of $\psi(t) = t^2$.

The results obtained in all examples demonstrate the effectiveness of our new functions $\psi(t) = t^{7/4}$ and $\psi(t) = t^{3/2}$. This confirms our theoretical results and strengthens our objective as expressed by the proposed algorithmic complexities.

3.5 Conclusion

In this chapter, we have described new primal-dual path-following methods to solve linear programs. Our approach is a reconsideration of Darvay and Takàcs' technique [20] with using new functions $\psi(t) = t^{\frac{7}{4}}$ and $\psi(t) = t^{\frac{3}{2}}$. We showed that the obtained algorithms solve the linear problem in polynomial time. Moreover, we provided some numerical experiments that prove the efficiency of our proposed algorithms. The obtained results motivating us to extend Darvay and Takàcs' algorithm for other optimization problems such as quadratic programming and semidefinite programming. This is the concept of the next chapters.

CHAPTER 4

Efficient primal-dual interior point algorithms for convex quadratic optimization

This chapter introduces new primal-dual interior point algorithms for convex quadratic optimization (CQO). The first proposed algorithm is based on an extension of the techniques presented in the work of Darvay and Takàcs [20] for LO while the other on it's first idea presented in the work of Zhang and Xu [93]. We demonstrate that the presented methods solve efficiently the CQO within polynomial time. Notably, the short-step algorithms achieve the best-known iteration bound. Moreover, we present comparative numerical study to prove the efficiency of our proposed algorithms. A part of those results were accepted for publication in J. Inf. Optim. Sci. [88].

4.1 Convex quadratic optimization problem

We reconsider the convex quadratic programming (CQP) problem in its standard form

$$\begin{cases} \min \frac{1}{2}x^TQx + c^Tx \\ Ax = b, \\ x \in \mathbb{R}^n_+. \end{cases} \tag{CQP}$$

Where Q represents a symmetric positive semidefinite matrix with dimensions $n \times n$, $A \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$.

The dual problem of (CQP) can be expressed as

$$\begin{cases} \max b^T y - \frac{1}{2} x^T Q x \\ A^T y - Qx + s = c, \\ y \in \mathbb{R}^m, \ s \in \mathbb{R}^n_+. \end{cases} \tag{CQD}$$

Denoted by:

- $F_{(CQP)} = \{x \in \mathbb{R}^n : Ax = b, \ x \geq 0\}$, the set of feasible primal solutions of (CQP).

- A vector $x \in F_{(CQP)}$ is called a feasible solution of (CQP).

- A vector $x^* \in F_{(CQP)}$ that minimizes the objective function of (CQP) is called an optimal solution of (CQP).

- $F^0_{(CQP)} = \{x \in \mathbb{R}^n : Ax = b, \ x > 0\}$, the set of strictly feasible primal solutions of (CQP).

- $F_{(CQD)} = \left\{(y, s) \in \mathbb{R}^m \times \mathbb{R}^n : A^T y - Qx + s = c, \ s \geq 0\right\}$, the set of feasible dual solutions of (LD).

- A vector $y^* \in F_{(CQD)}$ that maximizes the objective function of (CQD) is called an optimal solution of (CQD).

- $F^0_{(CQD)} = \left\{(y, s) \in \mathbb{R}^m \times \mathbb{R}^n : A^T y - Qx + s = c, \ s > 0\right\}$, the set of strictly feasible dual solutions of (LD).

- $\overline{F^0} = F^0_{(CQP)} \times F^0_{(CQD)}$, the set of strictly feasible primal-dual solutions of (CQP) and (CQD).

Here are some fundamental results of duality in convex quadratic programming.

Theorem 4.1 (Weak duality [82])**.** *If $x \in F_{(CQP)}$ and $(y, s) \in F_{(CQD)}$, then*

$$\frac{1}{2} x^T Q x + c^T x \geq -\frac{1}{2} x^T Q x + b^T y.$$

Theorem 4.2 (Strong duality [82])**.** *Let $x^* \in F_{(CQP)}$ and $(y^*, s^*) \in F_{(CQD)}$ such that*

$$\frac{1}{2} (x^*)^T Q x^* + c^T x^* = -\frac{1}{2} (x^*)^T Q x^* + b^T y^*,$$

then x^* and (y^*, s^*) are optimal solutions for (CQP) and (CQD) respectively.

Definition 4.3 (Duality gap [82])**.** For $x \in F_{(CQP)}$ and $(y, s) \in F_{(CQD)}$, the quantity

$$\frac{1}{2} x^T Q x + c^T x - \left(-\frac{1}{2} x^T Q x + b^T y \right) = x^T s,$$

is called the duality gap of the two problems (CQP) and (CQD) at (x, y, s).

In the following theorem, we express the condition for x^* and (y^*, s^*) to be optimal solutions of (CQP) and (CQD) respectively in terms of complementarity condition.

Theorem 4.4. *[82] If $x^* \in F_{(cqp)}$ and $(y^*, s^*) \in F_{(CQD)}$, then x^* and (y^*, s^*) are optimal solutions of (CQP) and (CQD) respectively if and only if*

$$(x^*)^T s^* = 0 \Leftrightarrow x^* s^* = 0 \Leftrightarrow x_i^* s_i^* = 0, \quad \forall i = 1, \ldots, n.$$

Remark 4.5. The existence of an optimal solution for one of the problems (CQP) and (CQD) implies the optimality of the other, and their optimal values are equal. If one of the two problems has an unbounded optimal value, the other has no solution.

4.2 The classical central path method

Suppose that:

- The matrix A is a full rank row, i.e., $Rank(A) = m < n$.
- $\overline{F^0} \neq \emptyset$, i.e., a strictly feasible primal-dual point exists.

This last condition is called the interior point condition for (CQP) and (CQD).
Under these assumptions, finding an optimal solution for both (CQP) and (CQD) is the same as solving a system of nonlinear equations:

$$\begin{cases} Ax = b, \\ A^T y - Qx + s = c, \\ xs = 0, \quad x, s \geq 0. \end{cases} \tag{4.1}$$

The main concept behind primal-dual path-following IPMs is to replace the last equation in system (4.1), which is called the complementarity equation with the parameterized equation $xs = \mu e$, where e represents a vector of all ones with a length of n and $\mu > 0$. This leads us to consider the system below

$$\begin{cases} Ax = b, \\ A^T y - Qx + s = c, \\ xs = \mu e, \quad x, s > 0. \end{cases} \tag{4.2}$$

If the previous assumptions holds, then for a fixed $\mu > 0$, the μ-center given by Sonnevend [67] is the unique solution of system (4.2). For different μ, the central path generate a sequence which converges to a primal-dual optimal solution for both problems (CQP) and (CQD), when $\mu \to 0$.

4.3 New search direction based on Zhang and Xu's technique

This section presents a new class of search direction for CQP based on Zhang and Xu's method for LO [93].

Note that for $x, s > 0$ and $\mu > 0$, the vector $v = \sqrt{\frac{xs}{\mu}} = \left(\sqrt{\frac{x_1 s_1}{\mu}}, \sqrt{\frac{x_2 s_2}{\mu}}, \ldots, \sqrt{\frac{x_n s_n}{\mu}}\right)^T > 0$ is well defined. From the third equation of system (4.2), we deduce that

$$xs = \mu e \Leftrightarrow \frac{xs}{\mu} = e \Leftrightarrow v^2 = e \Leftrightarrow v = e \Leftrightarrow v^2 = v. \tag{4.3}$$

By transforming the left-hand side of the equation $v^2 = v$ into the xs space, we can derive the following equation

$$xs = \mu v. \tag{4.4}$$

Now, the perturbed central path can be equivalently stated as follows

$$\begin{cases} Ax = b, \\ A^T y + s - Qx = c, \\ xs = \mu v. \end{cases} \tag{4.5}$$

This last system (4.5) can be written in the form $F(x, y, s) = 0$, where

$$F(x, y, s) = \begin{pmatrix} Ax - b \\ A^T y + s - Qx - c \\ xs - \mu v \end{pmatrix} \tag{4.6}$$

According to Zhang and Xu's idea [93], assuming that the variance vector v is fixed and applying Newton's method to this system, we obtain: $x_+ = x + \Delta x,\ y_+ = y + \Delta y,$ $s_+ = s + \Delta s$, where $(\Delta x, \Delta y, \Delta s)$ is the solution of the linear system

$$\begin{cases} A\Delta x = 0, \\ A^T \Delta y + \Delta s - Q\Delta x = 0, \\ \frac{1}{\mu}(s\Delta x + x\Delta s) = \mu v - xs. \end{cases} \tag{4.7}$$

Defining the scaled search directions

$$d_x = X^{-1} V \Delta x \text{ and } d_s = S^{-1} V \Delta s, \tag{4.8}$$

with

$$X = diag(x) \ \text{ and } S = diag(s),$$

where, $diag(x)$ and $diag(s)$ are diagonals matrices, which contains on their main diagonal the elements of x and s respectively in the original order.

Hence, we obtain

$$s\Delta x + x\Delta s = \mu v(d_x + d_s), \tag{4.9}$$

and

$$d_x d_s = \frac{\Delta x \Delta s}{\mu}. \tag{4.10}$$

Obviously, with these notations, the scaled feasible Newton system of (4.7) can be expressed as:

$$\begin{cases} \overline{A} d_x = 0, \\ \overline{A}^T \Delta y + d_s - \overline{Q} d_x = 0, \\ d_x + d_s = p_v, \end{cases} \tag{4.11}$$

where

$$p_v = e - v, \tag{4.12}$$

and

$$\overline{A} = \frac{1}{\mu} A V^{-1} X, \ \overline{Q} = \mu V^{-1} X Q V^{-1} X. \tag{4.13}$$

We define a proximity measure as follows:

$$\delta(v) = \delta(xs, \mu) = \|p_v\| = \|e - v\|. \tag{4.14}$$

Thus we have

$$\delta(xs, \mu) = 0 \Longleftrightarrow v = e \Longleftrightarrow xs = \mu e.$$

Hence, the value of $\delta(v)$ can be considered as a measure for the distance between the given pair (x, y, s) and the μ-center $(x(\mu), y(\mu), s(\mu))$.

Now, the generic representation of the obtained algorithm is given as follows:

Generic Primal-dual IPM for CQP

Input:

a proximity parameter $0 < \tau < 1$ (default $\tau = \frac{1}{2}$);

an accuracy parameter $\varepsilon > 0$;

an update parameter θ, $0 < \theta < 1$ (default $\theta = \frac{1}{7\sqrt{n}}$);

a strictly feasible point (x^0, y^0, s^0) such that $\delta(x^0 s^0, \mu^0) < \tau$, where $\mu^0 = \frac{(x^0)^T s^0}{n}$;

begin

$x := x^0$; $y := y^0$; $s := s^0$; $\mu := \mu^0$;

while $x^T s \geq \varepsilon$ **do**

Set $\mu := (1 - \theta)\,\mu$;

Solve the system (4.11) via (4.8) to obtain $(\Delta x, \Delta y, \Delta s)$;

Compute $x := x + \Delta x$; $y := y + \Delta y$; $s := s + \Delta s$;

end while

end.

Figure 4.1: First Generic algorithm for CQP

Note that, from (4.11), we have

$$d_x^T d_s = d_x^T \left(\overline{Q} d_x - \overline{A}^T \Delta y\right) = d_x^T \overline{Q} d_x - \left(\overline{A} d_x\right)^T \Delta y \geq 0, \tag{4.15}$$

this last inequality holds because $\overline{A} d_x = 0$ and $\overline{Q}$ is a positive semidefinite matrix. Thus, the directions are not orthogonal in CQP, in contrast with LP case where the directions are orthogonal. So, this makes a different analysis.

In the next subsection, we present some results related to algorithm complexity analysis.

4.3.1 Analysis of the algorithm

In this subsection, we describe the effects of a full-Newton step of a μ-update and prove the local convergence of the algorithm. Finally, we conclude with the complexity result of our algorithm.

We first recall some useful lemmas, which will be used later in the analysis of the algorithm.

Lemma 4.6. *Let $\delta = \delta(xs, \mu)$ and $(d_x, \Delta y, d_s)$ be a solution of (4.11) . Then one has*

$$0 \leq d_x^T d_s \leq \frac{1}{2}\delta^2, \tag{4.16}$$

and

$$\|d_x d_s\|_\infty \leq \frac{1}{4}\delta^2 \text{ and } \|d_x d_s\| \leq \frac{1}{2\sqrt{2}}\delta^2. \tag{4.17}$$

Proof. From (4.14), the last equation of (4.11), (4.15) and the following equality

$$\delta^2 = \|p_v\|^2 = \|d_x + d_s\|^2 = \|d_x\|^2 + \|d_s\|^2 + 2d_x^T d_s,$$

the first part of the lemma was fallows. For the second statement, we have

$$d_x d_s = \frac{1}{4}\left((d_x + d_s)^2 - (d_x - d_s)^2\right),$$

and

$$\|d_x + d_s\|^2 = \|d_x - d_s\|^2 + 4d_x^T d_s,$$

this means that

$$\|d_x - d_s\| \leq \|d_x + d_s\|. \tag{4.18}$$

Moreover, we have

$$\begin{aligned}
\|d_x d_s\|_\infty &= \frac{1}{4}\left\|(d_x + d_s)^2 - (d_x - d_s)^2\right\|_\infty \\
&\leq \frac{1}{4}\max\left(\|d_x + d_s\|_\infty^2, \|d_x - d_s\|_\infty^2\right) \\
&\leq \frac{1}{4}\max\left(\|d_x + d_s\|^2, \|d_x - d_s\|^2\right) \\
&\leq \frac{1}{4}\|d_x + d_s\|^2.
\end{aligned}$$

Hence

$$\|d_x d_s\|_\infty \leq \frac{1}{4}\delta^2.$$

This proves the first part of the second statement of the lemma.

For the second part, using (4.18) we have

$$\begin{aligned}
\|d_x d_s\|^2 &= e^T (d_x d_s)^2 \\
&= \frac{1}{16} e^T \left((d_x + d_s)^2 - (d_x - d_s)^2\right)^2 \\
&= \frac{1}{16}\left\|(d_x + d_s)^2 - (d_x - d_s)^2\right\|^2 \\
&\leq \frac{1}{16}\left(\left\|(d_x + d_s)^2\right\|^2 + \left\|(d_x - d_s)^2\right\|^2\right) \\
&\leq \frac{1}{16}\left(\|d_x + d_s\|^4 + \|d_x - d_s\|^4\right) \\
&\leq \frac{1}{8}\|d_x + d_s\|^4 = \frac{1}{8}\delta^4.
\end{aligned}$$

This implies that

$$\|d_x d_s\| \leq \frac{1}{2\sqrt{2}}\delta^2,$$

which completes the second part of the proof of the lemma. □

Lemma 4.7. *[93, Lemma 4.2] One has*

$$1 - \delta \leq v_i \leq 1 + \delta, \ \forall i = 1, ..., n. \tag{4.19}$$

In the next lemmas, we show under the condition $\delta(xs, \mu) < 2\sqrt{2} - 2$ that the full-Newton step is strictly feasible.

Lemma 4.8. *Let $x_+ = x + \Delta x$ and $s_+ = s + \Delta s$ be the generated point after a full-Newton step. Hence $x_+ > 0$ and $s_+ > 0$ if and only if $v + d_x d_s > 0$.*

Proof. For each $0 \leq \alpha \leq 1$ denote $x_+(\alpha) = x + \alpha\Delta x$ and $s_+(\alpha) = s + \alpha\Delta s$. Hence,

$$x_+(\alpha)s_+(\alpha) = xs + \alpha(s\Delta x + x\Delta s) + \alpha^2 \Delta x \Delta s.$$

Now, in view of (4.9), (4.10) and (4.11) we have:

$$\begin{aligned} x_+(\alpha)s_+(\alpha) &= \mu\left(v^2 + \alpha v\,(e - v) + \alpha^2 d_x d_s\right) \\ &= \mu\left((1-\alpha)v^2 + \alpha v + \alpha^2 d_x d_s\right). \end{aligned} \tag{4.20}$$

Suppose that $v + d_x d_s > 0$, which is equivalent to $d_x d_s > -v$. Substitution gives

$$x_+(\alpha)s_+(\alpha) > \mu(1-\alpha)\left(v^2 + \alpha v\right).$$

Hence, $x_+(\alpha)s_+(\alpha) > 0$ for each $0 \leq \alpha \leq 1$, which means that the linear functions of α, $x_+(\alpha)$ and $s_+(\alpha)$ do not change sign on the interval $[0, 1]$ and for $\alpha = 0$ we have $x_+(0) = x > 0$ and $s_+(0) = s > 0$. This leads to $x_+(1) = x_+ > 0$ and $s_+(1) = s_+ > 0$. This means that the full-Newton step is strictly feasible. □

Lemma 4.9. *If $\delta = \delta(xs, \mu) < 2\sqrt{2} - 2$. Then the primal-dual full-Newton step is strictly feasible.*

Proof. On one hand, by Lemma 4.8, x_+ and s_+ are strictly feasible if $v + d_x d_s > 0$. This last inequality holds if $\|d_x d_s\|_\infty < \min(v)$.
On the other hand, It follows from Lemma 4.6 and Lemma 4.7, that one has

$$\|d_x d_s\|_\infty \leq \frac{1}{4}\delta^2 \text{ and } 1 - \delta \leq \min(v).$$

It is clear that the inequality $\|d_x d_s\|_\infty < \min(v)$ holds for $\frac{1}{4}\delta^2 < 1 - \delta$, which is equivalent to $\delta < 2\sqrt{2} - 2$. Thus, $v + d_x d_s > 0$ holds if $\delta < 2\sqrt{2} - 2$. This completes the proof. □

In the next lemma, we show the local convergence of the full-Newton step.

Lemma 4.10. *Let* $\delta = \delta(xs, \mu) < 2\sqrt{2} - 2$, *then*

$$\delta(x_+ s_+, \mu) \leq \delta + \frac{1}{2\sqrt{2}}\delta^2,$$

which means local convergence of the full-Newton step.

Proof. For convenience, we may write $v_+ = \sqrt{\frac{x_+ s_+}{\mu}}$.
Let $\alpha = 1$. From (4.20) it follows that

$$v_+^2 = v + d_x d_s. \tag{4.21}$$

By Lemma 4.6 and Lemma 4.7, we get

$$\begin{aligned} \min\left((v_i)_+^2\right) &\geq \min(v) - \|d_x d_s\|_\infty \\ &\geq 1 - \delta - \frac{1}{4}\delta^2. \end{aligned}$$

Therefore,

$$\min\left((v_i)_+\right) \geq \sqrt{1 - \delta - \frac{1}{4}\delta^2}. \tag{4.22}$$

From (4.14), we have

$$\begin{aligned} \delta(v_+) &= \delta(x_+ s_+, \mu) = \|e - v_+\| \\ &= \left\| \frac{1}{e + v_+}\left(e - v_+^2\right) \right\| \\ &= \sqrt{\sum_{i=1}^n \left(\frac{1}{1 + (v_i)_+}\left(1 - (v_i)_+^2\right)\right)^2} \\ &\leq \sqrt{\left(\frac{1}{1 + \min\left((v_i)_+\right)}\right)^2 \sum_{i=1}^n \left(1 - (v_i)_+^2\right)^2} \\ &\leq \frac{1}{1 + \min\left((v_i)_+\right)} \|e - v_+^2\| \\ &\leq \frac{1}{1 + \min\left((v_i)_+\right)} \|e - v - d_x d_s\| \\ &\leq \frac{1}{1 + \min\left((v_i)_+\right)} \left[\|e - v\| + \|d_x d_s\|\right] \\ &\leq \frac{1}{1 + \min\left((v_i)_+\right)} \left[\delta + \|d_x d_s\|\right]. \end{aligned}$$

From Lemma 4.6 and (4.22) it follows that

$$\delta(v_+) \leq \frac{1}{1 + \sqrt{1 - \delta - \frac{1}{4}\delta^2}} \left[\delta + \frac{1}{2\sqrt{2}}\delta^2\right] \leq \delta + \frac{1}{2\sqrt{2}}\delta^2.$$

This completes the proof. □

In the next lemma, we give the change in the duality gap after taking a full-Newton step.

Lemma 4.11. *Let* $\delta = \delta(xs, \mu)$. *Then*

$$(x_+)^T s_+ \leq \mu \left(n(1+\delta) + \frac{1}{2}\delta^2 \right).$$

Proof. We know from (4.20) (For $\alpha = 1$), Lemma 4.6 and Lemma 4.7, that

$$\begin{aligned} (x_+)^T s_+ &= e^T (x_+ s_+) \\ &= \mu e^T (v + d_x d_s) \\ &= \mu \left(e^T v + d_x^T d_s \right) \\ &\leq \mu \left(n(1+\delta) + \frac{1}{2}\delta^2 \right), \end{aligned}$$

this completes the proof. □

The next lemma shows that the algorithm is well defined.

Lemma 4.12. *Let* $\delta = \delta(xs, \mu) < 2\sqrt{2} - 2$ *and* $\mu_+ = (1-\theta)\mu$, *where* $0 < \theta < 1$. *Then*

$$\delta(\tilde{v}) = \delta(x_+ s_+, \mu_+) < \frac{\theta\sqrt{n} + \delta + \frac{1}{2\sqrt{2}}\delta^2}{\sqrt{1-\theta}\left(\sqrt{1-\theta} + \sqrt{1 - \delta - \frac{1}{2\sqrt{2}}\delta^2} \right)}.$$

Moreover, if $\delta < \frac{1}{2}$ *and* $\theta = \frac{1}{7\sqrt{n}}$, *then* $\delta(\tilde{v}) < \frac{1}{2}$.

Proof. According to the definition of δ in (4.14), (4.21), (4.22) and Lemma 4.6, we get

$$\begin{aligned}
\delta(x_+s_+,\mu_+) &= \left\| e - \sqrt{\frac{x_+s_+}{\mu_+}} \right\| \\
&= \frac{1}{\sqrt{(1-\theta)}} \left\| \sqrt{(1-\theta)}e - v_+ \right\| \\
&= \frac{1}{\sqrt{(1-\theta)}} \left\| \frac{(1-\theta)e - v_+^2}{\sqrt{(1-\theta)}e + v_+} \right\| \\
&= \frac{1}{\sqrt{(1-\theta)}} \sqrt{\sum_{i=1}^{n} \left(\frac{1}{\sqrt{(1-\theta)} + (v_i)_+} \left(1-\theta-(v_i)_+^2\right) \right)^2} \\
&\leq \frac{1}{\sqrt{(1-\theta)}} \sqrt{\left(\frac{1}{\sqrt{(1-\theta)} + \min\left((v_i)_+\right)} \right)^2 \sum_{i=1}^{n} \left(1-\theta-(v_i)_+^2\right)^2} \\
&\leq \frac{1}{\sqrt{(1-\theta)}\left(\sqrt{(1-\theta)} + \min\left((v_i)_+\right)\right)} \left\|(1-\theta)e - v_+^2\right\| \\
&\leq \frac{1}{\sqrt{(1-\theta)}\left(\sqrt{(1-\theta)} + \sqrt{1-\delta-\frac{1}{2\sqrt{2}}\delta^2}\right)} \|(1-\theta)e - v - d_xd_s\| \\
&\leq \frac{1}{\sqrt{(1-\theta)}\left(\sqrt{(1-\theta)} + \sqrt{1-\delta-\frac{1}{2\sqrt{2}}\delta^2}\right)} \left[\|(1-\theta)e - v\| + \|d_xd_s\|\right] \\
&\leq \frac{1}{\sqrt{(1-\theta)}\left(\sqrt{(1-\theta)} + \sqrt{1-\delta-\frac{1}{2\sqrt{2}}\delta^2}\right)} \left[\|-\theta e\| + \|e - v\| + \|d_xd_s\|\right] \\
&\leq \frac{\theta\sqrt{n} + \delta + \frac{1}{2\sqrt{2}}\delta^2}{\sqrt{1-\theta}\left(\sqrt{1-\theta} + \sqrt{1-\delta-\frac{1}{2\sqrt{2}}\delta^2}\right)},
\end{aligned}$$

which proves the first part of the lemma.

Now, suppose that $\delta < \frac{1}{2}$ and $\theta = \frac{1}{7\sqrt{n}}$. Then, we obtain

$$\begin{aligned}
\delta(x_+s_+,\mu_+) &\leq \frac{\theta\sqrt{n} + \delta + \frac{1}{2\sqrt{2}}\delta^2}{\sqrt{1-\theta}\left(\sqrt{1-\theta} + \sqrt{1-\delta-\frac{1}{2\sqrt{2}}\delta^2}\right)} \\
&< \frac{\frac{1}{7} + \frac{1}{2} + \frac{1}{8\sqrt{2}}}{\sqrt{1-\theta}\left(\sqrt{1-\theta} + \sqrt{1-\delta-\frac{1}{2\sqrt{2}}\delta^2}\right)}.
\end{aligned}$$

Using $n \geq 2$, we get $\sqrt{1-\theta} = \sqrt{1-\frac{1}{7\sqrt{n}}} \geq \sqrt{1-\frac{1}{7\sqrt{2}}}$. Also, $\delta < \frac{1}{2}$ gives $\sqrt{1-\delta-\frac{1}{2\sqrt{2}}\delta^2} > \sqrt{\frac{1}{2}-\frac{1}{8\sqrt{2}}}$. Taking all these inequalities into consideration, we conclude

$$\begin{aligned}
\delta(x_+,s_+;\mu_+) &< \frac{\frac{1}{7} + \frac{1}{2} + \frac{1}{8\sqrt{2}}}{\sqrt{1-\frac{1}{7\sqrt{2}}}\left(\sqrt{1-\frac{1}{7\sqrt{2}}} + \sqrt{\frac{1}{2}-\frac{1}{8\sqrt{2}}}\right)} \\
&= 0.485\,14 < \frac{1}{2}.
\end{aligned}$$

This completes the proof. □

Lemma 4.13. *Assume that the pair (x^0, s^0) is strictly feasible, $\mu^0 = \frac{(x^0)^T s^0}{n}$ and $\delta(x^0 s^0, \mu^0) < \frac{1}{2}$. Moreover, let x^k and s^k be the vectors obtained after k iterations. Then, the inequality $\left(x^k\right)^T s^k < \varepsilon$ is satisfied when*

$$k \geq \left\lceil \frac{1}{\theta} \log \left(\frac{\mu^0 \left(\frac{3}{2}n + \frac{1}{8}\right)}{\varepsilon} \right) \right\rceil .$$

Proof. After k iterations we have $\mu^k = (1-\theta)^k \mu^0$. From Lemma 4.11 and $\delta(xs, \mu) < \frac{1}{2}$, we get

$$\left(x^k\right)^T s^k < \mu^k \left(n\left(1+\delta\right) + \frac{1}{2}\delta^2 \right) = (1-\theta)^k \mu^0 \left(\frac{3}{2}n + \frac{1}{8} \right).$$

Hence, the inequality $\left(x^k\right)^T s^k < \varepsilon$ holds if

$$(1-\theta)^k \mu^0 \left(\frac{3}{2}n + \frac{1}{8} \right) \leq \varepsilon.$$

By taking logarithms of both sides, we obtain

$$k \log(1-\theta) + \log\left(\mu^0 \left(\frac{3}{2}n + \frac{1}{8} \right) \right) \leq \log\left(\varepsilon\right).$$

As $\theta \leq -\log(1-\theta)$, we see that the inequality is valid if

$$k\theta \geq \log\left(\mu^0 \left(\frac{3}{2}n + \frac{1}{8} \right) \right) - \log\left(\varepsilon\right) = \log\left(\frac{\mu^0 \left(\frac{3}{2}n + \frac{1}{8}\right)}{\varepsilon} \right).$$

Hence,

$$k \geq \left\lceil \frac{1}{\theta} \log \left(\frac{\mu^0 \left(\frac{3}{2}n + \frac{1}{8}\right)}{\varepsilon} \right) \right\rceil .$$

This completes the proof. □

Theorem 4.14. *Suppose that $x^0 = s^0 = e$. If we consider the default values for θ and τ, we obtain that the algorithm given in Figure 4.1 requires no more than*

$$O\left(\sqrt{n} \log \frac{\left(\frac{3}{2}n + \frac{1}{8}\right)}{\varepsilon} \right)$$

interior-point iterations. The resulting vectors satisfy $\left(x^k\right)^T s^k < \varepsilon$.

Proof. Since $x^0 = s^0 = e$, we get $\mu^0 = \frac{(x^0)^T s^0}{n} = 1$. So, from Lemma 4.13, the result holds. □

4.4 New search direction based on Darvay and Takàcs' technique

In this section, we generalize the technique introduced by Darvay and Takàcs [20] to get new search direction for CQP. Since $(x, s, \mu) > 0$, we deduce from (4.3) that

$$\frac{xs}{\mu} = e \Leftrightarrow \sqrt{\frac{xs}{\mu}} = e \Leftrightarrow \frac{xs}{\mu} = \sqrt{\frac{xs}{\mu}},$$

with, $\frac{xs}{\mu} = \left(\frac{x_1s_1}{\mu}, \frac{x_2s_2}{\mu}, ..., \frac{x_ns_n}{\mu}\right)^T > 0$ and $\sqrt{\frac{xs}{\mu}} = \left(\sqrt{\frac{x_1s_1}{\mu}}, \sqrt{\frac{x_2s_2}{\mu}}, ..., \sqrt{\frac{x_ns_n}{\mu}}\right)^T$. Now, we can express the perturbed system (4.2) as follows

$$\begin{cases} Ax = b, \\ A^Ty - Qx + s = c, \\ \frac{xs}{\mu} = \sqrt{\frac{xs}{\mu}}. \end{cases} \tag{4.23}$$

In according with Darvay and Takàcs' idea, let's define a continuously differentiable function ψ on (k^2, ∞), such that $0 \leq k < 1$, and $2t\psi'(t^2) - \psi'(t) > 0, \forall t > k^2$. Applying the AET technique on system (4.23), we obtain

$$\begin{cases} Ax = b, \\ A^Ty - Qx + s = c, \\ \psi(\frac{xs}{\mu}) = \psi\left(\sqrt{\frac{xs}{\mu}}\right) \end{cases} \tag{4.24}$$

Newton iteration applied to system (4.24) define: $x_+ = x + \Delta x, y_+ = y + \Delta y, s_+ = s + \Delta s$, where $(\Delta x, \Delta y, \Delta s)$ is the search direction solution of the following linear system

$$\begin{cases} A\Delta x = 0, \\ A^T\Delta y - Q\Delta x + \Delta s = 0, \\ \frac{1}{\mu}(s\Delta x + x\Delta s) = \dfrac{-\psi\left(\frac{xs}{\mu}\right) + \psi\left(\sqrt{\frac{xs}{\mu}}\right)}{\psi'(\frac{xs}{\mu}) - \frac{1}{2\sqrt{\frac{xs}{\mu}}}\psi'(\sqrt{\frac{xs}{\mu}})}. \end{cases} \tag{4.25}$$

From the scaled search directions d_x and d_x defined as (4.8), system (4.25) can be represented as

$$\begin{cases} \overline{A}d_x = 0, \\ \overline{A}^T\Delta y - \overline{Q}d_x + d_s = 0, \\ d_x + d_s = p_v, \end{cases} \tag{4.26}$$

where

$$p_v = \frac{2\psi(v) - 2\psi(v^2)}{2v\psi'(v^2) - \psi'(v)},$$

the matrices $\overline{A}$ and $\overline{Q}$ are defined as in (4.13).
Let $\psi : \left(\frac{1}{\sqrt{2}}, \infty\right) \to \mathbb{R}$, $\psi(t) = t^2$. So

$$p_v = \frac{v - v^3}{2v^2 - e}. \tag{4.27}$$

The condition $2t\psi'(t^2) - \psi'(t) > 0, \forall t > k^2$ is holds for $k^2 = \frac{1}{\sqrt{2}}$. Introducing a proximity measure

$$\delta(v) = \delta(xs, \mu) = \frac{\|p_v\|}{2} = \frac{1}{2}\left\|\frac{v - v^3}{2v^2 - e}\right\|, \tag{4.28}$$

Now, we describe the generic algorithm in Figure 4.2.

Generic Primal-dual IPM for CQP

Input:

a proximity parameter $0 < \tau < 1$ (default $\tau = \frac{1}{10}$);

an accuracy parameter $\varepsilon > 0$;

an update parameter θ, $0 < \theta < 1$ (default $\theta = \frac{1}{12\sqrt{n}}$);

a strictly feasible point (x^0, y^0, s^0) such that $\delta(x^0 s^0, \mu^0) < \tau$, where $\mu^0 = \frac{(x^0)^T s^0}{n}$; $v^0 = \sqrt{\frac{x^0 s^0}{\mu^0}} > \frac{e}{\sqrt{2}}$;

begin

$(x, y, s) := (x^0, y^0, s^0)$; $\mu := \mu^0$

while $x^T s \geq \varepsilon$ **do**

Take $\mu := (1 - \theta)\mu$;

Solve the system (4.26) via (4.8) to obtain $(\Delta x, \Delta y, \Delta s)$;

Compute $x := x + \Delta x$; $y := y + \Delta y$; $s := s + \Delta s$;

end while

end.

Figure 4.2: Second Generic algorithm for CQP

The next section contains some results of the complexity analysis of our algorithm.

4.4.1 Convergence and complexity analysis

Lemma 4.15. *Let* $q_v = d_x - d_s$. *Then,*

$$\|q_v\| \leq \|p_v\|. \tag{4.29}$$

Proof. From the last equation of system (4.26), we obtain

$$d_x = \frac{p_v + q_v}{2} \text{ and } d_s = \frac{p_v - q_v}{2}.$$

Those give

$$d_x d_s = \frac{p_v^2 - q_v^2}{4}, \tag{4.30}$$

on the other hand, we have

$$\|p_v\|^2 = \|d_x + d_s\|^2 = \|d_x - d_s\|^2 + 4d_x^T d_s = \|q_v\|^2 + 4d_x^T d_s.$$

Since $\overline{A}d_x = 0$ and $\overline{Q}$ is a positive semidefinite matrix, then

$$d_x^T d_s = d_x^T \left(\overline{Q}d_x - \overline{A}^T \Delta y\right) = d_x^T \overline{Q} d_x - \left(\overline{A}d_x\right)^T \Delta y \geq 0,$$

which gives $\|q_v\| \leq \|p_v\|$. □

Lemma 4.16. *If* $v > \frac{1}{\sqrt{2}}e$ *and* $\delta = \delta(xs, \mu) < 1$. *The full-Newton step is strictly feasible, i.e.,*

$$(x_+, s_+) > 0.$$

Proof. Let $\alpha \in [0, 1]$. Denote by $(x_+(\alpha), s_+(\alpha)) = (x + \alpha\Delta x, s + \alpha\Delta s)$. So,

$$x_+(\alpha)s_+(\alpha) = xs + \alpha(s\Delta x + x\Delta s) + \alpha^2 \Delta x \Delta s.$$

In view of (4.9) and (4.10), we can observe that

$$\frac{x_+(\alpha)s_+(\alpha)}{\mu} = \frac{xs}{\mu} + \alpha v(d_x + d_s) + \alpha^2 d_x d_s. \tag{4.31}$$

Based on (4.26) and (4.30), we express

$$\frac{x_+(\alpha)s_+(\alpha)}{\mu} = (1-\alpha)v^2 + \alpha(v^2 + vp_v) + \alpha^2 \left(\frac{p_v^2 - q_v^2}{4}\right). \tag{4.32}$$

In addition, from (4.27) we obtain

$$v^2 + vp_v = \frac{v^4}{2v^2 - e} \geq e. \tag{4.33}$$

This last inequality follows directly from

$$\left(v^2 - e\right)^2 \geq 0 \Leftrightarrow v^4 \geq 2v^2 - e.$$

Then

$$\begin{aligned} \frac{x_+(\alpha)s_+(\alpha)}{\mu} &\geq (1-\alpha)v^2 + \alpha e + \alpha^2\left(\frac{p_v^2}{4} - \frac{q_v^2}{4}\right) - \alpha\frac{p_v^2}{4} \\ &\geq (1-\alpha)v^2 + \alpha\left(e - \left((1-\alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right)\right). \end{aligned} \tag{4.34}$$

$x_+(\alpha)s_+(\alpha) > 0$ when $\left\|(1-\alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right\|_\infty < 1$. Consequently,

$$\begin{aligned} \left\|(1-\alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right\|_\infty &\leq (1-\alpha)\frac{\|p_v^2\|_\infty}{4} + \alpha\frac{\|q_v^2\|_\infty}{4} \\ &\leq (1-\alpha)\frac{\|p_v\|_2^2}{4} + \alpha\frac{\|q_v\|_2^2}{4}, \end{aligned}$$

from (4.29), we get

$$\begin{aligned} \left\|(1-\alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right\|_\infty &\leq (1-\alpha)\frac{\|p_v\|_2^2}{4} + \alpha\frac{\|p_v\|_2^2}{4} \\ &\leq \frac{\|p_v\|_2^2}{4} = \delta^2 < 1. \end{aligned}$$

So, $x_+(\alpha)s_+(\alpha) > 0 \; \forall \alpha \in [0,1]$. This indicates that the linear functions of α, $x_+(\alpha)$ and $s_+(\alpha)$ do not switch signs within the interval $[0,1]$ and for $\alpha = 0$, we have $(x_+(0), s_+(0)) = (x, s) > 0$. As a result, it follows that $(x_+(1), s_+(1)) = (x_+, s_+) > 0$. This completes the proof. □

In the next result, we give the conditions of quadratically convergence of the proximity δ.

Lemma 4.17. *Suppose $v > \frac{1}{\sqrt{2}}e$ and $\delta = \delta(xs, \mu) < \frac{1}{\sqrt{2}}$. So $v_+ = \sqrt{\frac{x_+s_+}{\mu}} > \frac{1}{\sqrt{2}}e$ also*

$$\delta(x_+s_+, \mu) \leq \frac{5\delta^2}{1 - 2\delta^2}\sqrt{1 - \delta^2}.$$

Proof. According to Lemma 4.16, we have $x_+, s_+ > 0$, this means that $v_+ = \sqrt{\frac{x_+s_+}{\mu}}$ is well defined.

Let's set $\alpha = 1$. Consequently, from equation (4.32), it can be deduced that

$$v_+^2 = \frac{x_+s_+}{\mu} = v^2 + vp_v + \frac{p_v^2}{4} - \frac{q_v^2}{4}. \tag{4.35}$$

Now, in view of (4.33) we have

$$v^2 + vp_v = e + \frac{(v^2-e)^2}{2v^2-e}. \tag{4.36}$$

From (4.27), (4.35) and (4.36) we may write

$$\begin{aligned} v_+^2 &= e + \frac{(v^2-e)^2}{2v^2-e} + \frac{1}{4}\left(\frac{v-v^3}{2v^2-e}\right)^2 - \frac{q_v^2}{4} \\ &= e + \frac{(v^2-e)^2}{2v^2-e} + \frac{(v(e-v^2))^2}{4(2v^2-e)^2} - \frac{q_v^2}{4} \\ &= e + \frac{(v^2-e)^2}{2v^2-e}\left(e + \frac{v^2}{4(2v^2-e)}\right) - \frac{q_v^2}{4} \\ &= e + \frac{(v^2-e)^2(9v^2-4e)}{4(2v^2-e)^2} - \frac{q_v^2}{4}. \end{aligned} \tag{4.37}$$

Using (4.27) and (4.37) we get

$$\begin{aligned} e - v_+^2 &= \frac{q_v^2}{4} - \frac{(v^2-e)^2(9v^2-4e)}{4(2v^2-e)^2} \\ &= \frac{q_v^2}{4} - \frac{(9v^2-4e)}{v^2}\cdot\frac{v^2(e-v^2)^2}{4(2v^2-e)^2} \\ &= \frac{q_v^2}{4} - \frac{(9v^2-4e)}{v^2}\cdot\frac{v^2+v^6-2v^4}{4(2v^2-e)^2} \\ &= \frac{q_v^2}{4} - \frac{(9v^2-4e)}{v^2}\cdot\frac{(v-v^3)^2}{4(2v^2-e)^2} \\ &= \frac{q_v^2}{4} - \frac{(9v^2-4e)}{v^2}\cdot\frac{p_v^2}{4}. \end{aligned} \tag{4.38}$$

Since $\frac{9v^2-4e}{v^2} < 9e$, for all $v > \frac{1}{\sqrt{2}}e$, and from (4.29), we get

$$\left\|e - v_+^2\right\| \le \left\|\frac{q_v^2}{4}\right\| + \left\|\frac{(9v^2-4e)}{v^2}\cdot\frac{p_v^2}{4}\right\| < \frac{\|p_v\|^2}{4} + 9\frac{\|p_v\|^2}{4} = 10\delta^2. \tag{4.39}$$

Besides these, we know that $\frac{p_v^2}{4} > 0$ and we also have from (4.33) that $v^2+vp_v = \frac{v^4}{2v^2-e} \ge e$. As a result, these inferences indicate that, based on (4.35)

$$v_+^2 = v^2 + vp_v + \frac{p_v^2}{4} - \frac{q_v^2}{4} \ge e - \frac{q_v^2}{4}.$$

Hence

$$\min(v_+^2) \ge 1 - \frac{\|q_v\|_\infty^2}{4} \ge 1 - \frac{\|q_v\|^2}{4} \ge 1 - \frac{\|p_v\|^2}{4} \ge 1 - \delta^2,$$

and this relation yields

$$\min(v_+) \ge \sqrt{1-\delta^2}. \tag{4.40}$$

On the other hand, from $\delta < \frac{1}{\sqrt{2}}$, it can be deduced that $\sqrt{1-\delta^2} > \frac{1}{\sqrt{2}}$. In view of (4.40), we get $v_+ > \frac{e}{\sqrt{2}}$. Moreover, using the definition of δ, we get

$$\delta\left(v_+\right) = \delta(x_+s_+, \mu) = \frac{1}{2}\left\| \frac{v_+}{2v_+^2 - e}\left(e - v_+^2\right)\right\|.$$

Let $f(t) = \frac{t}{2t^2-1}$ for all $t > \frac{1}{\sqrt{2}}$. We have $f'(t) < 0$, this implies that f is decreasing for all $t > \frac{1}{\sqrt{2}}$. Therefore, by using (4.39), (4.40) and Lemma 3.2, we obtain

$$\begin{aligned}
\delta\left(v_+\right) &\leq \frac{1}{2} f(\sqrt{1-\delta^2})\left\|\left(e - v_+^2\right)\right\| \\
&\leq \frac{1}{2}\frac{\sqrt{1-\delta^2}}{2\left(1-\delta^2\right)-1}\left\|\left(e - v_+^2\right)\right\| \\
&\leq \frac{5\delta^2}{1-2\delta^2}\sqrt{1-\delta^2}.
\end{aligned}$$

This completes the proof. □

The lemma bellow examines the strong duality gap.

Lemma 4.18. *Let* $\delta = \delta(xs, \mu)$. *So*

$$\left(x_+\right)^T s_+ \leq \mu\left(n + 9\delta^2\right).$$

Moreover, if $\delta < \frac{1}{\sqrt{2}}$, *then* $\left(x_+\right)^T s_+ < \mu\left(n + \frac{9}{2}\right)$.

Proof. Firstly, from (4.36), we get

$$\begin{aligned}
v^2 + vp_v &= e + \left(\frac{4\left(2v^2 - e\right)}{v^2}\right)\left(\frac{v^2\left(e - v^2\right)^2}{4\left(2v^2 - e\right)^2}\right) \\
&= e + \left(8e - \frac{4e}{v^2}\right)\frac{p_v^2}{4} \\
&\leq e + 8\frac{p_v^2}{4}, \qquad (4.41)
\end{aligned}$$

and we know that

$$\begin{aligned}
\left(x_+\right)^T s_+ &= \mu\sum_{i=1}^{n}\left(v_{+_i}\right)^2 \\
&= \mu\sum_{i=1}^{n}\left(v_i^2 + v_i\left(p_v\right)_i + \frac{\left(p_v\right)_i^2}{4} - \frac{\left(q_v\right)_i^2}{4}\right) \\
&= \mu\sum_{i=1}^{n}\left(v_i^2 + v_i\left(p_v\right)_i\right) + \mu\left(\frac{\left\|p_v\right\|^2 - \left\|q_v\right\|^2}{4}\right) \\
&\leq \mu\sum_{i=1}^{n}\left(1 + 8\frac{\left(p_v\right)_i^2}{4}\right) + \mu\left(\frac{\left\|p_v\right\|^2 - \left\|q_v\right\|^2}{4}\right) \\
&\leq \mu n + 8\mu\frac{\left\|p_v\right\|^2}{4} + \mu\frac{\left\|p_v\right\|^2}{4} \\
&\leq \mu\left(n + 9\delta^2\right).
\end{aligned}$$

Now, if $\delta < \frac{1}{\sqrt{2}}$ then

$$(x_+)^T s_+ < \mu \left(n + \frac{9}{2} \right).$$

This completes the proof. □

Now, in the next lemma, we show that the algorithm is well defined.

Lemma 4.19. *Suppose that, $v > \frac{1}{\sqrt{2}}e, \mu_+ = (1-\theta)\mu$, with $0 < \theta < 1$ and $\delta = \delta(xs, \mu) < \frac{1}{\sqrt{2}}$. In addition, let $\tilde{v} = \sqrt{\frac{x_+ s_+}{\mu_+}}$. So, $\tilde{v} > \frac{1}{\sqrt{2}}e$ and*

$$\delta(\tilde{v}) = \delta(x_+ s_+, \mu_+) < \frac{\sqrt{1-\delta^2}}{2\sqrt{1-\theta}\,(1-2\delta^2+\theta)} \left(\theta\sqrt{n} + 10\delta^2\right).$$

Furthermore, when $\delta < \frac{1}{10}$ and $\theta = \frac{1}{12\sqrt{n}}$, then $\delta(\tilde{v}) < \frac{1}{10}$.

Proof. By using Lemma 4.17, we have $v_+ > \frac{1}{\sqrt{2}}e$. From $\tilde{v} = \sqrt{\frac{x_+ s_+}{\mu_+}}$, we deduce

$$\tilde{v} = \sqrt{\frac{x_+ s_+}{\mu_+}} = \frac{1}{\sqrt{1-\theta}} v_+ > \frac{1}{\sqrt{2}} e.$$

This last inequality follows from $0 < \theta < 1$.

Now, we have

$$\delta(\tilde{v}) = \frac{1}{2} \left\| \frac{\tilde{v} - \tilde{v}^3}{2\tilde{v}^2 - e} \right\|, \tag{4.42}$$

where

$$2\tilde{v}^2 - e = \frac{2v_+^2 - (1-\theta)e}{1-\theta}, \tag{4.43}$$

and

$$\begin{aligned} \tilde{v} - \tilde{v}^3 &= \tilde{v}(e - \tilde{v}^2) \\ &= \frac{1}{\sqrt{1-\theta}} v_+ \left(e - \frac{1}{1-\theta} v_+^2 \right) \\ &= \frac{1}{(1-\theta)\sqrt{1-\theta}} v_+ \left((1-\theta)\, e - v_+^2 \right). \end{aligned} \tag{4.44}$$

Then

$$\begin{aligned} \frac{\tilde{v} - \tilde{v}^3}{2\tilde{v}^2 - e} &= \frac{1}{(1-\theta)\sqrt{1-\theta}} v_+ \left((1-\theta)\, e - v_+^2 \right) \frac{1-\theta}{2v_+^2 - (1-\theta)e} \\ &= \frac{1}{\sqrt{1-\theta}} \frac{v_+}{2v_+^2 - (1-\theta)e} \left((1-\theta)\, e - v_+^2 \right). \end{aligned} \tag{4.45}$$

Let $f(t) = \frac{t}{2t^2 - (1-\theta)}$ for all $t > \frac{1}{\sqrt{2}}$. Since $f'(t) < 0$, we conclude that f is decreasing for all $t > \frac{1}{\sqrt{2}}$. Moreover, considering Lemma 3.2 and (4.45), we get

$$\delta(\tilde{v}) = \frac{1}{2} \left\| \frac{\tilde{v} - \tilde{v}^3}{2\tilde{v}^2 - e} \right\| \leq \frac{\sqrt{1-\delta^2}}{2\sqrt{1-\theta}\,(1-2\delta^2+\theta)} \left\| (1-\theta)\, e - v_+^2 \right\|. \tag{4.46}$$

From (4.39) we have

$$\left\|(1-\theta)\,e-v_+^2\right\| \leq \|-\theta e\| + \left\|e-v_+^2\right\| < \theta\sqrt{n}+10\delta^2. \tag{4.47}$$

Hence, by using (4.46) and (4.47) we obtain

$$\delta\left(\tilde{v}\right) < \frac{\sqrt{1-\delta^2}}{2\sqrt{1-\theta}\,(1-2\delta^2+\theta)}\left(\theta\sqrt{n}+10\delta^2\right),$$

with that, we conclude the first part of the proof.

Now, assume that $\delta < \frac{1}{10}$ and $\theta = \frac{1}{12\sqrt{n}}$. In addition, we have $\sqrt{1-\delta^2} < 1$ and $\theta > 0$. Then, we get

$$\begin{aligned}\delta\left(\tilde{v}\right) &< \frac{\theta\sqrt{n}+10\delta^2}{2\sqrt{1-\theta}\,(1-2\delta^2)} = \frac{\frac{1}{12}+10\delta^2}{2\sqrt{1-\theta}\,(1-2\delta^2)} \\ &< \frac{1}{2\sqrt{1-\theta}}\frac{50}{49}\left(\frac{1}{12}+\frac{1}{10}\right) = \frac{1}{2\sqrt{1-\theta}}\left(\frac{55}{294}\right).\end{aligned}$$

For $n \in \mathbb{N}^*$, we get $2\sqrt{1-\theta} = 2\sqrt{1-\frac{1}{12\sqrt{n}}} = \sqrt{4-\frac{1}{3\sqrt{n}}} \geq \sqrt{\frac{11}{3}}$. Taking all these inequalities into consideration, we conclude

$$\delta\left(\tilde{v}\right) < \sqrt{\frac{3}{11}}\left(\frac{55}{294}\right) = \frac{5\sqrt{33}}{294} < \frac{1}{10}.$$

With that, we have completed the proof. □

Lemma 4.20. *Assume a strictly feasible point (x^0, s^0) with $\delta(x^0s^0, \mu^0) < \frac{1}{\sqrt{2}}$, and $\mu^0 = \frac{\left(x^0\right)^T s^0}{n}$. Additionally, (x^k, s^k) is the obtained pair vectors after k iterations. So $\left(x^k\right)^T s^k < \varepsilon$ holds when*

$$k \geq \left\lceil \frac{1}{\theta}\log\left(\frac{\mu^0\left(n+\frac{9}{2}\right)}{\varepsilon}\right)\right\rceil.$$

Proof. After k iterations, $\mu^k = (1-\theta)^k\mu^0$. From Lemma 4.18 and $\delta(xs, \mu) < \frac{1}{\sqrt{2}}$, it becomes

$$\left(x^k\right)^T s^k < \mu^k\left(n+\frac{9}{2}\right) = (1-\theta)^k\mu^0\left(n+\frac{9}{2}\right).$$

Hence, $\left(x^k\right)^T s^k < \varepsilon$ is satisfied if

$$\mu^0\left(n+\frac{9}{2}\right)(1-\theta)^k \leq \varepsilon.$$

By introducing logarithms of both sides, and take into consideration $\theta \leq -\log(1-\theta)$, the above inequality holds when

$$k\theta \geq \log\left(\mu^0\left(n+\frac{9}{2}\right)\right) - \log\left(\varepsilon\right) = \log\left(\frac{\mu^0\left(n+\frac{9}{2}\right)}{\varepsilon}\right).$$

Hence,

$$k \geq \left[\frac{1}{\theta} \log \left(\frac{\mu^0 \left(n + \frac{9}{2}\right)}{\varepsilon} \right) \right].$$

This completes the proof. □

Theorem 4.21. *Suppose that $x^0 = s^0 = e$ and consider the default values of θ and τ, the algorithm presented in Figure 4.2 requires no more than*

$$O\left(\sqrt{n} \log \frac{\left(n + \frac{9}{2}\right)}{\varepsilon} \right)$$

interior-point iterations. Furthermore $\left(x^k\right)^T s^k < \varepsilon$.

Proof. The case $x^0 = s^0 = e$ gives $\mu^0 = \frac{\left(x^0\right)^T s^0}{n} = 1$. Substitute this value into Lemma 4.20, the result remains valid. □

4.5 Numerical experiments

In this section, we present some numerical results of our algorithms given in Figures 4.1 and 4.2 as well as the algorithm given in [13] which is based in the technique of the kernel function. For the numerical tests, we consider four fixed-size examples and one variable-size example taken from the literature [12]. Moreover, we solve some problems from the quadprog test collection (`https://CRAN.R-project.org/package=quadprog`). The numerical results are obtained using MATLAB R2009b environment. In the algorithm given in [13], we take the best choice of its parameters which are considered as $\tau = 5n$,

$$\psi_{pq}(t) = \frac{t^2 - 1}{2} - \tanh^p(1) \int_1^t \coth^p(x) e^{qc(\coth(x) - \coth(1))} dx, \text{ with } p = 2, q = 1,$$

and

$$c = \frac{1}{\coth^2(1) - 1}.$$

Moreover, we choose a step size $\alpha = \min(\alpha_x, \alpha_s)$ with

$$\alpha_x = \min_{i=1..n} \begin{cases} -\frac{x_i}{\Delta x_i}, & \text{if } \Delta x_i < 0, \\ 1, & \text{elsewhere,} \end{cases} \quad \text{and } \alpha_s = \max_{i=1..n} \begin{cases} -\frac{s_i}{\Delta s_i}, & \text{if } \Delta s_i < 0, \\ 1, & \text{elsewhere.} \end{cases}$$

In the three algorithms, we take $\varepsilon = 10^{-4}$.

The obtained results will be presented in comparative tables where we note by:

- $Iter$: the number of iterations necessary for optimality.
- $T(s)$: the execution time in seconds.
- $M1$: the algorithm presented in Figure 4.1.
- $M2$: the algorithm presented in Figure 4.2.
- $M3$: the algorithm presented in [13].

4.5.1 Examples with fixed size

The examples are taken from the literature [12]. We take $\theta \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$.

Example 4.22. $A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad c = \begin{pmatrix} -2 \\ -4 \\ 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

The starting points are:
$x^0 = \begin{pmatrix} 0.3262 & 1.3261 & 0.3477 \end{pmatrix}^T, \quad y^0 = \begin{pmatrix} 0 & -2.0721 \end{pmatrix}^T,$
$s^0 = \begin{pmatrix} 0.7247 & 0.7247 & 2.0722 \end{pmatrix}^T.$

Example 4.23. $A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 5 & 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, c = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, Q = \begin{pmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$

The starting points are:
$x^0 = \begin{pmatrix} 0.9683 & 0.5775 & 0.4543 & 1.1444 \end{pmatrix}^T, y^0 = \begin{pmatrix} -0.9184 & -1.1244 \end{pmatrix}^T,$
$s^0 = \begin{pmatrix} 0.7612 & 0.9141 & 0.9185 & 1.1244 \end{pmatrix}^T.$

Example 4.24. $A = \begin{pmatrix} 1 & 1.2 & 1 & 1.8 & 0 \\ 3 & -1 & 1.5 & -2 & 1 \\ -1 & 2 & -3 & 4 & 2 \end{pmatrix}, b = \begin{pmatrix} 9.31 \\ 5.45 \\ 6.60 \end{pmatrix}, c = \begin{pmatrix} 1 \\ -1.5 \\ 2 \\ 1.5 \\ 3 \end{pmatrix},$

$$Q = \begin{pmatrix} 20 & 1.2 & 0.5 & 0.5 & -1 \\ 1.2 & 32 & 1 & 1 & 1 \\ 0.5 & 1 & 14 & 1 & 1 \\ 0.5 & 1 & 1 & 15 & 1 \\ -1 & 1 & 1 & 1 & 16 \end{pmatrix}.$$

The starting points are:

$$x^0 = \begin{pmatrix} 2.4539 & 0.7875 & 1.5838 & 2.4038 & 1.3074 \end{pmatrix}^T,$$
$$y^0 = \begin{pmatrix} 20.5435 & 9.4781 & 4.3927 \end{pmatrix}^T,$$
$$s^0 = \begin{pmatrix} 7.1215 & 7.9763 & 8.3150 & 6.8686 & 7.9750 \end{pmatrix}^T.$$

Example 4.25. $A = \begin{pmatrix} 1 & -1 & 1.9 & 1.25 & 1.2 & 0.4 & -0.7 & 1.06 & 1.5 & 1.05 \\ 1.3 & 1.2 & 0.15 & 2.15 & 1.25 & 1.5 & 0.4 & 1.52 & 1.3 & 1 \\ 1.5 & -1.1 & 3.5 & 1.25 & 1.8 & 2 & 1.95 & 1.2 & 1 & -1 \end{pmatrix},$

$$b = \begin{pmatrix} 11.651 \\ 16.672 \\ 21.295 \end{pmatrix}, c = \begin{pmatrix} -0.5 \\ -1 \\ 0 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ -1 \\ -0.5 \\ -1 \end{pmatrix}, Q = \begin{pmatrix} 30 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 21 & 0 & 1 & -1 & 1 & 0 & 1 & 0.5 & 1 \\ 1 & 0 & 15 & -0.5 & -2 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & -0.5 & 30 & 3 & -1 & 1 & -1 & 0.5 & 1 \\ 1 & -1 & -2 & 3 & 27 & 1 & 0.5 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 16 & -0.5 & 0.5 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0.5 & -0.5 & 8 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 0.5 & 1 & 24 & 1 & 1 \\ 1 & 0.5 & 1 & 0.5 & 1 & 0 & 1 & 1 & 39 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 11 \end{pmatrix}.$$

The starting points are:

$$x^0 = \begin{pmatrix} 0.9491 & 0.6121 & 1.8477 & 1.8115 & 1.2511 & 2.5211 & 1.5062 & 1.5658 & 0.8207 & 1.1289 \end{pmatrix}^T,$$
$$y^0 = \begin{pmatrix} 4.3800 & 19.9367 & 4.5679 \end{pmatrix}^T,$$
$$z^0 = \begin{pmatrix} 3.8902 & 4.4625 & 3.9788 & 3.6606 & 3.9018 & 3.5561 & 3.8760 & 3.7190 & 3.9138 & 4.3396 \end{pmatrix}^T.$$

The obtained results of these four fixed size examples are summarized in Table 4.1.

		$M1$		$M2$		$M3$	
Example	θ	$Iter$	$T(s)$	$Iter$	$T(s)$	$Iter$	$T(s)$
	0.1	91	0.0189	90	0.0116	826	0.2476
	0.3	29	0.0111	28	0.0103	252	0.1010
Example 4.22	0.5	16	0.0082	17	0.0087	133	0.0713
	0.7	10	0.0081	15	0.0083	60	0.0512
	0.9	06	0.0076	15	0.0080	30	0.0445
	0.1	106	0.2983	105	0.2810	791	0.3097
	0.3	32	0.0281	32	0.0229	238	0.1265
Example 4.23	0.5	17	0.0084	19	0.0091	126	0.0873
	0.7	11	0.0070	17	0.0089	77	0.0712
	0.9	07	0.0031	17	0.0033	42	0.0711
	0.1	128	0.2866	127	0.2751	1125	0.4870
	0.3	39	0.0182	39	0.0173	342	0.1755
Example 4.24	0.5	21	0.0089	23	0.0112	144	0.1247
	0.7	13	0.0082	20	0.0107	88	0.0933
	0.9	07	0.0071	20	0.0090	42	0.0860
	0.1	127	0.0367	126	0.0380	1206	0.9543
	0.3	39	0.0267	38	0.0254	288	0.3072
Example 4.25	0.5	21	0.0116	22	0.0142	152	0.2101
	0.7	12	0.0091	20	0.0117	96	0.1692
	0.9	07	0.0079	20	0.0227	63	0.1629

Table 4.1: Numerical results of the fixed size CQP Examples

Next, in Table 4.2 some obtained results are summarized for some problems from the quadprog test collection.

		$M1$		$M2$		$M3$	
Example	θ	$Iter$	$T(s)$	$Iter$	$T(s)$	$Iter$	$T(s)$
Tame	0.1	67	0.0105	66	0.0099	1534	0.6624
	0.3	21	0.0089	20	0.0086	420	0.5190
	0.5	11	0.0048	13	0.0073	198	0.2818
	0.7	07	0.0042	11	0.0058	90	0.1333
	0.9	04	0.0039	11	0.0051	24	0.0609
Genhs28	0.1	73	0.0168	72	0.0166	1936	0.9018
	0.3	23	0.0078	22	0.0086	576	0.4120
	0.5	12	0.0067	14	0.0077	304	0.2677
	0.7	08	0.0062	12	0.0075	121	0.1969
	0.9	05	0.0059	12	0.0073	30	0.1283
Hs51	0.1	87	0.0314	86	0.0295	750	1.7687
	0.3	27	0.0073	26	0.0071	228	0.9756
	0.5	14	0.0054	16	0.0055	120	0.7354
	0.7	09	0.0051	14	0.0054	50	0.4356
	0.9	05	0.0049	14	0.0052	30	0.3617
Zecevic2	0.1	74	0.02010	73	0.0140	1190	0.1681
	0.3	23	0.0083	22	0.0074	306	0.0704
	0.5	12	0.0065	14	0.0090	85	0.0538
	0.7	08	0.0061	12	0.0065	50	0.0452
	0.9	05	0.0048	12	0.0061	30	0.0411

Table 4.2: Numerical results of some quadprog problems

Now, let's provide an example with a variable size to demonstrate the efficiency of our obtained algorithms when dealing with problems of large sizes.

4.5.2 Example with variable size

In this example, we take $\theta = 0.7$.

Example 4.26. [12] Assume that $n = 2m$, $A[i,j] = \begin{cases} 1 & \text{if } i = j \text{ or } i+m = j \\ 0 & \text{otherwise} \end{cases}$,
$b[i] = 2\ \forall\, 1 \leq i \leq m$, $c[i] = \begin{cases} -1 & \forall\, 1 \leq i \leq m \\ 0 & \forall\, m+1 \leq i \leq n \end{cases}$, $Q = I_{n\times n}$.
The starting points are:
$x^0[i] = x^0[i+m] = 1$, $s^0[i] = 1$, $s^0[i+m] = 2$, $y^0[i] = -1\ \forall\, 1 \leq i \leq m$.
The obtained primal-dual optimal solutions are: $x^*[i] = \frac{3}{2}$, $x^*[i+m] = \frac{1}{2}$, $s^*[i] = s^*[i+m] = 0$, $y^*[i] = \frac{1}{2}\ \forall\, 1 \leq i \leq m$.

In Table 4.3, we provide a summary of the Example 4.26 results for various sizes of (m, n).

	$M1$		$M2$		$M3$	
(m,n)	$Iter$	$T(s)$	$Iter$	$T(s)$	$Iter$	$T(s)$
$(10, 20)$	12	0.1270	20	0.2897	108	0.3119
$(25, 50)$	13	0.0838	21	0.3162	108	0.7721
$(50, 100)$	13	0.1962	21	0.2250	108	1.6101
$(100, 200)$	13	0.2030	22	0.2571	140	4.0470
$(250, 500)$	15	13.8259	24	22.6768	140	17.3496
$(500, 1000)$	15	91.9980	25	152.7817	176	75.6636
$(600, 1200)$	15	158.2562	25	313.8644	176	123.1605
$(750, 1500)$	16	349.8760	26	651.8636	176	222.2253

Table 4.3: Numerical results of CQP Example 4.26

Comments:

Based on the numerical tests conducted on examples of various dimensions as well as on some problems from the quadprog tests collection, we observe that:

- The iterations number and the time required for achieving optimality with the three algorithms are influenced by the values of the parameters θ. Notably, $\theta = 0.9$ offers the smallest number of iterations with minimal time.

- The number of iterations required for optimality using our new approach, $M2$, is significantly lower than that of $M1$ and $M3$ if the value of θ is in the range $\{0.1, 0.3\}$. Otherwise, if $\theta \in \{0.5, 0.7, 0.9\}$, our other new approach $M1$ gives the smallest number of iterations. This becomes particularly apparent when the problem's dimensions are large, as illustrated in Table 4.3.
- The method $M1$ remains robust regardless of θ variations, making it a reliable choice.
- In Table 4.3, the method $M3$ demonstrates its effectiveness compared to our proposed approaches, M1 and M2, by significantly reducing the time required to solve problems with large dimensions. However, the number of iterations is higher than our approach $M1$.

The numerical experiments demonstrate the efficiency of our proposed algorithms. This efficiency is measured by the small iterations number and the reduced time to obtain the optimal primal-dual solution. Indeed, the number of iterations recorded in all examples tested is clearly lower than the theoretical number.

4.6 Conclusion

In this chapter, we have introduced new primal-dual interior point methods for solving convex quadratic programming problems. Our approaches are a generalization of the works of Darvay and Takàcs [20] and Zhang and Xu [93] for linear optimization. We demonstrated that the resulting algorithms require polynomial time to solve the considered problem. Additionally, we conducted numerical experiments, the results of which were acceptable and encouraging. Finally, we highlight that implementing the algorithm with the update parameter θ significantly reduces the number of iterations required by our proposed algorithms and improves their real numerical performance. The obtained results consolidate and confirm our theoretical purposes.

By exploiting the advantages of the previous technique in LO [93], in the next two chapters, we extended it to more general problems such as semidefinite optimization.

CHAPTER 5

Efficient primal-dual interior point algorithm for semidefinite optimization

In this chapter, we present a novel primal-dual interior point algorithm tailored for linear semidefinite optimization (SDO). Drawing inspiration from Zhang and Xu's approach to linear optimization, our method extends their technique. The symmetrization of the search direction is based on the Nesterov-Todd scaling scheme. We demonstrate the efficiency of our method, showcasing its ability to solve problems within polynomial time. Notably, our short-step algorithm achieves the best-known iteration bound of $O(\sqrt{n}\log\frac{n}{\varepsilon})$. Furthermore, we conduct a comprehensive numerical study, focusing on some applications of semidefinite programming to underscore the effectiveness of our proposed algorithm. The set of those results were in revision on Oper. Res. Lett review [90].

5.1 Semidefinite optimization problem

A semidefinite program (SDP) is a type of mathematical optimization problem that involves optimizing a linear objective function over the cone of positive semidefinite matrices. The standard form of SDP is expressed as follows

$$\begin{cases} \min C \bullet X \\ A_i \bullet X = b_i \quad \text{for} \quad i = 1, \ldots, m, \\ X \succeq 0. \end{cases} \tag{SDP}$$

Where $b \in \mathbb{R}^m$, the matrices C and $A_i,\ i = 1, \ldots, m$, are given and belong to the linear space of $n \times n$ symmetric matrices $\mathbb{S}^n$.
The dual problem of (SDP), can be expressed as follows

$$\begin{cases} \max b^t y \\ \sum_{i=1}^m y_i A_i + S = C, \\ S \succeq 0. \end{cases} \tag{DSDP}$$

Definition 5.1.
- A point X is said to be feasible for (SDP) if: $A_i \bullet X = b_i$ for $i = 1, \ldots, m$ and $X \succeq 0$.
- A point X is said to be strictly feasible for (SDP) if it is feasible for (SDP) and satisfies $X \succ 0$.
- A point (y, S) is said to be feasible for (DSDP) if: $\sum_{i=1}^m y_i A_i + S = C$ and $S \succeq 0$.
- A point (y, S) is said to be strictly feasible for (DSDP) if it is feasible for (DSDP) and satisfies $S \succ 0$.

Theorem 5.2 (Weak Duality). *If X and (y, S) are feasible solutions of (SDP) and (DSDP) respectively, then we always have*

$$C \bullet X - b^T y = X \bullet S \geq 0.$$

This difference is called the duality gap.

Contrary to linear programming and convex quadratic programming, it is not always true that the optimality of two problems (SDP) and (DSDP) implies that $X \bullet S = 0$. (see the example of Vandenberghe and Boyd [8, 70, 75, 92]).
The next theorem gives the conditions that ensure strong duality and the existence of primal-dual solutions.

Theorem 5.3 (Strong Duality). *Let*

$$p^* = \min\{C \bullet X : A_i \bullet X = b_i \quad for \quad i = 1, \ldots, m, X \succeq 0\},$$

and

$$q^* = \max\{b^T y : A^T y + S = C, S \succeq 0\}.$$

So,

1. *If (SDP) is strictly feasible with p^* finite, then $p^* = q^*$, and this value is attained for (DSDP).*

2. *If (DSDP) is strictly feasible with q^* finite, then $p^* = q^*$, and this value is attained for (SDP).*

3. *If (SDP) and (DSDP) are both strictly feasible, then $p^* = q^*$, and these two values are attained for both problems.*

5.2 The classical central path method

Assume that:

- The matrices A_i, $i = 1, \ldots, m$, are linearly independent.
- The problems (SDP) and $(DSDP)$ satisfy the interior-point condition (IPC), i.e., there exist $(X^0 \succ 0, y^0, S^0 \succ 0)$ such that

$$A_i \bullet X^0 = b_i, \forall i = 1, \ldots, m; \quad \sum_{i=1}^{m} y_i^0 A_i + S^0 = C. \tag{IPC}$$

It's well known that under these assumptions, finding an optimal solution of both problems (SDP) and $(DSDP)$ is equivalent to solving the following system

$$\begin{cases} A_i \bullet X = b_i, \quad i = 1, \ldots, m, \ X \succeq 0, \\ \sum_{i=1}^{m} y_i A_i + S = C, \ S \succeq 0, \\ XS = 0. \end{cases} \tag{5.1}$$

The two first equations of system (5.1) are called feasibility conditions of (SDP) and $(DSDP)$, respectively. The last one is named the complementarity condition.

The basic idea of primal-dual IPMs is to replace the complementarity condition in (5.1), by the parameterized equation $XS = \mu I$, with $X, S \succ 0$ and $\mu > 0$. So, we consider the following system

$$\begin{cases} A_i \bullet X = b_i, \quad i = 1, \ldots, m, \ X \succ 0, \\ \sum_{i=1}^{m} y_i A_i + S = C, \ S \succ 0, \\ XS = \mu I. \end{cases} \tag{5.2}$$

Under the previous assumptions, for each $\mu > 0$, the system (5.2) has a unique solution denoted by $(X(\mu), y(\mu), S(\mu))$ which is called the μ-center of both problems (SDP) and $(DSDP)$ [50, 57]. The set of all μ-centers defines a homotopy which is called the central path of (SDP) and $(DSDP)$. If μ goes to zero, then the limit of the central path exists and since the limit satisfies the complementarity condition, the limit yields a primal-dual optimal solution of both problems (SDP) and $(DSDP)$ [34].

5.3 New search direction

This section presents a new class of search direction for SDO based on Zhang and Xu's method for LO [93].

First, let's consider the Nesterov-Todd (NT)-symmetrization scheme [58, 59] defined as

$$P = X^{1/2}(X^{1/2}SX^{1/2})^{-1/2}X^{1/2} = S^{-1/2}(S^{1/2}XS^{1/2})^{1/2}S^{-1/2}. \tag{5.3}$$

Furthermore, we define $D = P^{1/2}$ where $P^{1/2}$ denotes the symmetric square root of P. The matrix D is used to scale both matrices X and S to the same matrix V defined by

$$V = \frac{1}{\sqrt{\mu}}D^{-1}XD^{-1} = \frac{1}{\sqrt{\mu}}DSD = \frac{1}{\sqrt{\mu}}(D^{-1}XSD)^{1/2}.$$

Note that the matrices D and V are symmetric and positive definite. Moreover, we have

$$V^2 = \left(\frac{1}{\sqrt{\mu}}D^{-1}XD^{-1}\right)\left(\frac{1}{\sqrt{\mu}}DSD\right) = D^{-1}\frac{XS}{\mu}D.$$

Also, for $X, S \succ 0$ and $\mu > 0$, from the third equation of the system (5.2) we deduce that

$$XS = \mu I \Leftrightarrow \frac{XS}{\mu} = I \Leftrightarrow V^2 = I. \tag{5.4}$$

Moreover, we have

$$XS = \mu I \Leftrightarrow V = \frac{1}{\sqrt{\mu}}(D^{-1}XSD)^{1/2} = \frac{1}{\sqrt{\mu}}(D^{-1}\mu ID)^{1/2} = I. \tag{5.5}$$

From (5.4) and (5.5), we deduce

$$V^2 = V.$$

Transforming the left-hand side of the equation $V^2 = V$ to the XS space, we obtain

$$XS = \mu DVD^{-1}.$$

Then, system (5.2) can be rewritten in the following form

$$\begin{cases} A_i \bullet X = b_i, \quad i = 1, \ldots, m, \ X \succ 0, \\ \sum\limits_{i=1}^{m} y_i A_i + S = C, \ S \succ 0, \\ XS = \mu D V D^{-1}. \end{cases} \tag{5.6}$$

According to Zhang and Xu's idea [93], we assume that the variance matrix V is fixed. The application of Newton's method to the system (5.6) produces the following system of equations for the search direction ΔX, Δy and ΔS

$$\begin{cases} A_i \bullet (X + \Delta X) = b_i, \quad i = 1, \ldots, m, \\ \sum\limits_{i=1}^{m} (y_i + \Delta y_i) A_i + (S + \Delta S) = C, \\ (X + \Delta X)(S + \Delta S) = \mu D V D^{-1}. \end{cases} \tag{5.7}$$

If we neglect $\Delta X \Delta S$, then since $A_i \bullet X = b_i, \forall i = 1, \ldots, m$, and $\sum\limits_{i=1}^{m} y_i A_i + S = C$, the system (5.7) can be rewritten as follows

$$\begin{cases} A_i \bullet \Delta X = 0, \quad i = 1, \ldots, m, \\ \sum\limits_{i=1}^{m} \Delta y_i A_i + \Delta S = 0, \\ \Delta X + X \Delta S S^{-1} = \mu D V D^{-1} S^{-1} - X. \end{cases} \tag{5.8}$$

It is clear that ΔS is symmetric due to the second equation of the system (5.8). However, a crucial observation is that ΔX is not necessarily symmetric, because $X \Delta S S^{-1}$ may not be symmetric. Several researchers have proposed methods for symmetrizing the third equation in (5.8) such that the resulting new system had a unique symmetric solution. Among them, we consider the symmetrization scheme yielding the NT-direction [58, 59] defined in (3.37).

In the NT-scheme, we replace the term $X \Delta S S^{-1}$ in the third equation of (5.8) by $P \Delta S P^T$. The system (5.8) becomes

$$\begin{cases} A_i \bullet \Delta X = 0, \quad i = 1, \ldots, m, \\ \sum\limits_{i=1}^{m} \Delta y_i A_i + \Delta S = 0, \\ \Delta X + P \Delta S P^T = \mu D V D^{-1} S^{-1} - X. \end{cases} \tag{5.9}$$

Let us further define

$$\overline{A}_i = \frac{1}{\sqrt{\mu}} D A_i D, \ \forall i = 1, \ldots, m, \ D_X = \frac{1}{\sqrt{\mu}} D^{-1} \Delta X D^{-1}, \ D_S = \frac{1}{\sqrt{\mu}} D \Delta S D. \tag{5.10}$$

Obviously, with these notations, the scaled NT search directions $(D_X, \Delta y, D_S)$ can be expressed as

$$\begin{cases} \overline{A}_i \bullet D_X = 0, \quad i = 1, \dots, m, \\ \sum_{i=1}^{m} \Delta y_i \overline{A}_i + D_S = 0, \\ D_X + D_S = P_V. \end{cases} \tag{5.11}$$

Where

$$P_V = I - V. \tag{5.12}$$

For the analysis of the algorithm, we define a proximity measure as follows:

$$\delta(V) = \delta(X, S; \mu) = \|P_V\|_F = \|I - V\|_F. \tag{5.13}$$

Due to the first two equations of the system (5.11), D_X and D_S are orthogonal. Thus

$$D_X \bullet D_S = D_S \bullet D_X = 0 \tag{5.14}$$

Then, we can easily verify that

$$\delta(V) = 0 \Leftrightarrow V = I \Leftrightarrow D_X = D_S = 0 \Leftrightarrow XS = \mu DVD^{-1}. \tag{5.15}$$

Hence, the value of $\delta(V)$ can be considered as a measure for the distance between the given pair (X, y, S) and the μ-center $(X(\mu), y(\mu), S(\mu))$.

5.3.1 The generic primal–dual IPM for SDO

The generic representation of this algorithm is given in Figure 5.1 as follows:

In the next section, we present some results related to algorithm complexity analysis.

5.4 Analysis of the algorithm

In this section, we describe the effects of a full-NT step of a μ-update and prove the local convergence of the algorithm. Finally, we conclude with the complexity result of our algorithm.

We first recall some useful lemmas, which will be used later.

Lemma 5.4. *([25, Lemma 6.1]) Let $X(\alpha) = X + \alpha\Delta X$ and $S(\alpha) = S + \alpha\Delta S$. Suppose that $X \succ 0$ and $S \succ 0$. If*

$$\det(X(\alpha)S(\alpha)) > 0, \ \forall 0 \le \alpha \le \tilde{\alpha},$$

then $X(\tilde{\alpha}) \succ 0$ and $S(\tilde{\alpha}) \succ 0$.

Generic Primal-dual IPM for SDO

Input:

a proximity parameter $0 < \tau < 1$ (default $\tau = \frac{1}{2}$);

an accuracy parameter $\varepsilon > 0$;

an update parameter θ, $0 < \theta < 1$ (default $\theta = \frac{1}{7\sqrt{n}}$);

a strictly feasible point (X^0, y^0, S^0) and $\mu^0 = \frac{X^0 \bullet S^0}{n}$ such that $\delta(X^0, y^0, S^0) < \tau$;

begin

$X := X^0$; $y := y^0$; $S := S^0$; $\mu := \mu^0$

while $X \bullet S \geq \varepsilon$ **do**

$\mu := (1 - \theta)\,\mu$;

solve the system (5.11) via (5.10) to obtain $(\Delta X, \Delta y, \Delta S)$;

$X := X + \Delta X; y := y + \Delta y; S := S + \Delta S$;

end while

end.

Figure 5.1: Generic algorithm for SDO

Lemma 5.5. *([25, Lemma 6.2]) Let $Q \in \mathbb{S}^n_{++}$, and $M \in \mathbb{R}^{n\times n}$ be skew-symmetric, i.e., $M = -M^T$. Then,* $\det(Q + M) > 0$. *Moreover, if the eigenvalues of $Q + M$ are real, then*

$$0 < \lambda_{\min}(Q) \leq \lambda_{\min}(Q + M) \leq \lambda_{\max}(Q + M) \leq \lambda_{\max}(Q).$$

Lemma 5.6. *([26, Lemma 6.3.2]) Let $D_{XS} = \frac{1}{2}(D_X D_S + D_S D_X)$ be the symmetric part of $D_X D_S$, then we have*

$$\|D_{XS}\|_F \leq \frac{1}{2\sqrt{2}} \|P_V\|_F^2 .$$

In the next lemma, we give some basic properties about the proximity measure $\delta(V)$.

Lemma 5.7. *For any $i = 1, ..., n$, we have*

$$1 - \delta(V) \leq \lambda_i(V) \leq 1 + \delta(V).$$

Proof. From (5.13), we have

$$\begin{aligned} \delta(V) &= \|I - V\|_F \\ &= \sqrt{\sum_{i=1}^{n} (1 - \lambda_i(V))^2} \\ &\geq |1 - \lambda_i(V)|, \ \forall i = 1, ..., n. \end{aligned}$$

Using this last inequality, the result easily follows. □

In the following lemma, we state a condition that ensures the feasibility of the full NT-step.

Lemma 5.8. *Let (X, S) be a strictly feasible primal-dual points. Hence $X_+ = X + \Delta X \succ 0$ and $S_+ = S + \Delta S \succ 0$, if and only if $V + D_{XS} \succ 0$.*

Proof. We introduce a step length $\alpha \in [0,1]$ and we define

$$X(\alpha) = X + \alpha\Delta X \quad \text{and} \quad S(\alpha) = S + \alpha\Delta S. \tag{5.16}$$

Thus $X(0) = X,\ X(1) = X_+$ and one can introduce similar notations for S, hence $V = \frac{1}{\sqrt{\mu}}(D^{-1}XSD)^{1/2} \succ 0$. Applying (5.10) and (5.16), we have

$$\begin{aligned} X(\alpha)S(\alpha) &= XS + \alpha(X\Delta S + \Delta XS) + \alpha^2\Delta X\Delta S \\ &= \mu D\left(V^2 + \alpha\left(D_X V + V D_S\right) + \alpha^2 D_X D_S\right) D^{-1} \\ &\sim \mu\left(V^2 + \alpha\left(D_X V + V D_S\right) + \alpha^2 D_X D_S\right), \end{aligned}$$

thus

$$X(\alpha)S(\alpha) = Q(\alpha) + M(\alpha). \tag{5.17}$$

Where

$$Q(\alpha) = \mu\left(V^2 + \frac{1}{2}\alpha\left(D_X V + V D_S + V D_X + D_S V\right) + \frac{1}{2}\alpha^2\left(D_X D_S + D_S D_X\right)\right),$$

and

$$M(\alpha) = \mu\left(\frac{1}{2}\alpha\left(D_X V + V D_S - V D_X - D_S V\right) + \frac{1}{2}\alpha^2\left(D_X D_S - D_S D_X\right)\right).$$

It's easy to show that the matrix $M(\alpha)$ is skew-symmetric, for each $0 \leq \alpha \leq 1$. Lemma 5.5 implies that the determinant of the matrix $X(\alpha)S(\alpha)$ is positive if the matrix $Q(\alpha) \succ 0$. To this end, from (5.11) and (5.12), we have

$$\begin{aligned} Q(\alpha) &= \mu\left(V^2 + \frac{1}{2}\alpha\left(D_X V + V D_S + V D_X + D_S V\right) + \frac{1}{2}\alpha^2\left(D_X D_S + D_S D_X\right)\right) \\ &= \mu\left(V^2 + \frac{1}{2}\alpha\left(\left(D_X + D_S\right)V + V\left(D_X + D_S\right)\right) + \alpha^2 D_{XS}\right) \\ &= \mu\left(\left(1-\alpha\right)V^2 + \alpha\left(V^2 + \frac{1}{2}\left(P_V V + V P_V\right)\right) + \alpha^2 D_{XS}\right) \\ &= \mu\left(\left(1-\alpha\right)V^2 + \alpha\left(V^2 + \frac{1}{2}\left((I-V)V + V(I-V)\right) + \alpha D_{XS}\right)\right), \end{aligned}$$

so

$$Q(\alpha) = \mu\left((1-\alpha)\, V^2 + \alpha\,(V + \alpha D_{XS})\right). \tag{5.18}$$

Suppose that $V + D_{XS} \succ 0$, then, $Q(1) = \mu\,(V + D_{XS}) \succ 0$ and $Q(0) = \mu V^2 \succ 0$. Furthermore, $V + D_{XS} \succ 0$ implies that $\alpha D_{XS} \succ -\alpha V$ for any $0 < \alpha < 1$, then $Q(\alpha) \succ \mu\,(1-\alpha)\,(V^2 + \alpha V)$. This means that $Q(\alpha) \succ 0$. Thus $\det\left(X(\alpha)S(\alpha)\right) > 0$. Moreover, since $X = X(0) \succ 0$ and $S = S(0) \succ 0$, Lemma 5.4 implies that $X_+ = X(1) \succ 0$ and $S_+ = S(1) \succ 0$ for $\tilde{\alpha} = 1$. This complete the proof of the lemma. □

Corollary 5.9. *The new iterate after a full NT-step is certainly strictly feasible if*

$$\|D_{XS}\|_F < \lambda_{\min}(V).$$

Proof. Lemma 5.8 implies that $X_+ \succ 0$ and $S_+ \succ 0$, if and only if $V + D_{XS} \succ 0$. We have,

$$V + D_{XS} \succ 0 \Leftrightarrow \lambda_{\min}(V + D_{XS}) > 0. \tag{5.19}$$

In the other hand, we have

$$\lambda_{\min}(V + D_{XS}) \geq \lambda_{\min}(V) - |\lambda_{\min}(D_{XS})| \geq \lambda_{\min}(V) - \|D_{XS}\|_F. \tag{5.20}$$

Thus, the inequality (5.19) holds if $\|D_{XS}\|_F < \lambda_{\min}(V)$. This completes the proof of the corollary. □

Lemma 5.10. *Let $\delta = \delta(X, S; \mu)$ be defined as (5.13), $\mu > 0$ and (X, S) be any pair of positive definite matrices. If $\delta < \sqrt{2}(\sqrt{\sqrt{2}+1} - 1)$, then the full NT-step for SDO is strictly feasible, hence $X^+ \succ 0$ and $S^+ \succ 0$.*

Proof. It follows from lemma 5.6 and lemma 5.7, that

$$\|D_{XS}\|_F \leq \frac{1}{2\sqrt{2}}\delta^2 \quad \text{and} \quad 1 - \delta \leq \lambda_{\min}(V).$$

It is clear that the inequality $\|D_{XS}\|_F < \lambda_{\min}(V)$ holds for $\frac{1}{2\sqrt{2}}\delta^2 < 1 - \delta$, which is equivalent to $\delta < \sqrt{2}(\sqrt{\sqrt{2}+1} - 1)$. By Corollary 5.9, the new iterates after a full-NT step are strictly feasible, which completes the proof. □

The next lemma shows the local convergence of the full NT-step.

Lemma 5.11. *Let* $\delta = \delta(X, S; \mu) < \sqrt{2}(\sqrt{\sqrt{2}+1} - 1)$*, then*

$$\delta(X^+, S^+; \mu) \le \delta + \frac{1}{2\sqrt{2}}\delta^2,$$

which means local convergence of the full NT-step.

Proof. Let $\alpha = 1$. Then from (5.17) it follows that

$$V_+^2 \sim \frac{X_+ S_+}{\mu} = V + D_{XS} + M, \tag{5.21}$$

with

$$M = \frac{1}{2}\left(D_X V + V D_S - V D_X - D_S V + D_X D_S - D_S D_X\right).$$

It should be noted that M is a skew-symmetric matrix. Lemma 5.5 implies that

$$\lambda_{\min}(V_+^2) \ge \lambda_{\min}(V + D_{XS}).$$

Using Lemma 5.6, Lemma 5.7 and (5.20), we get

$$\begin{aligned} \lambda_{\min}(V_+^2) &\ge \lambda_{\min}(V) - \|D_{XS}\|_F \\ &\ge 1 - \delta - \frac{1}{2\sqrt{2}}\delta^2. \end{aligned}$$

Thus

$$\lambda_{\min}(V_+) \ge \sqrt{1 - \delta - \frac{1}{2\sqrt{2}}\delta^2}. \tag{5.22}$$

On the other hand, using the definition of δ, we have

$$\begin{aligned} \delta(V_+) &= \delta(X_+, S_+; \mu) = \|I - V_+\|_F \\ &= \left\|(I + V_+)^{-1}(I - V_+^2)\right\|_F \\ &= \sqrt{\sum_{i=1}^{n}\left(\frac{1}{1+\lambda_i(V_+)}\left(1 - \lambda_i^2(V_+)\right)\right)^2} \\ &\le \sqrt{\sum_{i=1}^{n}\left(\frac{1}{1+\lambda_{\min}(V_+)}\left(1 - \lambda_i^2(V_+)\right)\right)^2} \\ &\le \frac{1}{1+\lambda_{\min}(V_+)}\sqrt{\sum_{i=1}^{n}\left(1 - \lambda_i^2(V_+)\right)^2} \\ &\le \frac{1}{1+\lambda_{\min}(V_+)}\left\|I - V_+^2\right\|_F \\ &\le \frac{1}{1+\lambda_{\min}(V_+)}\|I - V - D_{XS}\|_F \\ &\le \frac{1}{1+\lambda_{\min}(V_+)}\left(\|I - V\|_F + \|D_{XS}\|_F\right), \end{aligned}$$

then

$$\delta\left(V_{+}\right) \leq \frac{1}{1+\lambda_{\min}\left(V_{+}\right)}\left[\delta+\frac{1}{2\sqrt{2}}\delta^{2}\right]. \tag{5.23}$$

Now, in view of (5.22) and (5.23) we get

$$\begin{aligned}\delta\left(V_{+}\right) &\leq \frac{1}{1+\sqrt{1-\delta-\frac{1}{2\sqrt{2}}\delta^{2}}}\left[\delta+\frac{1}{2\sqrt{2}}\delta^{2}\right] \\ &\leq \delta+\frac{1}{2\sqrt{2}}\delta^{2}.\end{aligned}$$

This completes the proof. □

In the next lemma, we give a majorization of the duality gap after taking a full NT-step.

Lemma 5.12. *Let* $\delta=\delta(X,S;\mu)$*. Then the duality gap satisfies*

$$X_{+}\bullet S_{+} \leq \mu n\left(1+\delta\right).$$

Proof. Since M is a skew-symmetric matrix, using (5.14) then

$$\begin{aligned}X_{+}\bullet S_{+} &= \mu \mathrm{Tr}\left(V_{+}^{2}\right) \\ &= \mu \mathrm{Tr}\left(V+D_{XS}+M\right) \\ &= \mu \mathrm{Tr}\left(V\right) \\ &= \mu\sum_{i=1}^{n}\lambda_{i}\left(V\right) \\ &\leq \mu n\lambda_{\max}\left(V\right).\end{aligned}$$

This last inequality and Lemma 5.7 give

$$X_{+}\bullet S_{+} \leq \mu n\left(1+\delta\right).$$

This completes the proof. □

The next lemma shows that the algorithm is well-defined.

Lemma 5.13. *Let* $\delta=\delta(X,S;\mu)<\sqrt{2}(\sqrt{\sqrt{2}+1}-1)$ *and* $\mu_{+}=(1-\theta)\mu$*, where* $0<\theta<1$*. Then*

$$\delta(X_{+},S_{+};\mu_{+}) \leq \frac{\theta\sqrt{n}+\delta+\frac{1}{2\sqrt{2}}\delta^{2}}{\sqrt{1-\theta}\left(\sqrt{1-\theta}+\sqrt{1-\delta-\frac{1}{2\sqrt{2}}\delta^{2}}\right)}.$$

Moreover, if $\delta<\frac{1}{2}$*,* $\theta=\frac{1}{7\sqrt{n}}$ *and* $n\geq 2$*, then* $\delta(X_{+},S_{+};\mu_{+})<\frac{1}{2}$*.*

Proof. From the definition of δ, we obtain

$$\begin{aligned}
\delta(X_+, S_+; \mu_+) &= \left\| I - \sqrt{\frac{X_+ S_+}{\mu_+}} \right\|_F \\
&= \frac{1}{\sqrt{(1-\theta)}} \left\| \sqrt{(1-\theta)} I - V_+ \right\|_F \\
&= \frac{1}{\sqrt{(1-\theta)}} \left\| \left(\sqrt{(1-\theta)} I + V_+ \right)^{-1} \left((1-\theta) I - V_+^2 \right) \right\|_F \\
&\le \frac{1}{\sqrt{(1-\theta)} \left(\sqrt{(1-\theta)} + \lambda_{\min}(V_+) \right)} \left\| (1-\theta) I - V_+^2 \right\|_F \\
&\le \frac{1}{\sqrt{(1-\theta)} \left(\sqrt{(1-\theta)} + \sqrt{1 - \delta - \frac{1}{2\sqrt{2}}\delta^2} \right)} \left\| (1-\theta) I - V - D_{XS} \right\|_F \\
&\le \frac{1}{\sqrt{(1-\theta)} \left(\sqrt{(1-\theta)} + \sqrt{1 - \delta - \frac{1}{2\sqrt{2}}\delta^2} \right)} \left[\| (1-\theta) I - V \|_F + \| D_{XS} \|_F \right] \\
&\le \frac{1}{\sqrt{(1-\theta)} \left(\sqrt{(1-\theta)} + \sqrt{1 - \delta - \frac{1}{2\sqrt{2}}\delta^2} \right)} \left[\| -\theta I \|_F + \| I - V \|_F + \| D_{XS} \|_F \right] \\
&\le \frac{\theta\sqrt{n} + \delta + \frac{1}{2\sqrt{2}}\delta^2}{\sqrt{1-\theta} \left(\sqrt{1-\theta} + \sqrt{1 - \delta - \frac{1}{2\sqrt{2}}\delta^2} \right)},
\end{aligned}$$

which proves the first part of the lemma.

Now, suppose that $\delta < \frac{1}{2}$ and $\theta = \frac{1}{7\sqrt{n}}$. Then, we obtain

$$\begin{aligned}
\delta(X_+, S_+; \mu_+) &\le \frac{\theta\sqrt{n} + \delta + \frac{1}{2\sqrt{2}}\delta^2}{\sqrt{1-\theta} \left(\sqrt{1-\theta} + \sqrt{1 - \delta - \frac{1}{2\sqrt{2}}\delta^2} \right)} \\
&< \frac{\frac{1}{7} + \frac{1}{2} + \frac{1}{8\sqrt{2}}}{\sqrt{1-\theta} \left(\sqrt{1-\theta} + \sqrt{1 - \delta - \frac{1}{2\sqrt{2}}\delta^2} \right)}.
\end{aligned}$$

Using $n \ge 2$, we get $\sqrt{1-\theta} = \sqrt{1 - \frac{1}{7\sqrt{n}}} \ge \sqrt{1 - \frac{1}{7\sqrt{2}}}$. Also, $\delta < \frac{1}{2}$ gives $\sqrt{1 - \delta - \frac{1}{2\sqrt{2}}\delta^2} > \sqrt{\frac{1}{2} - \frac{1}{8\sqrt{2}}}$. Taking all these inequalities into consideration, we conclude

$$\begin{aligned}
\delta(X_+, S_+; \mu_+) &< \frac{\frac{1}{7} + \frac{1}{2} + \frac{1}{8\sqrt{2}}}{\sqrt{1 - \frac{1}{7\sqrt{2}}} \left(\sqrt{1 - \frac{1}{7\sqrt{2}}} + \sqrt{\frac{1}{2} - \frac{1}{8\sqrt{2}}} \right)} \\
&= 0.485\,14 < \frac{1}{2}.
\end{aligned}$$

This completes the proof. □

In the next lemma, we give an upper bound for the total number of iterations produced by the algorithm given in Figure 5.1.

Lemma 5.14. *Assume that the pair* X^0*and* S^0 *are strictly feasible,* $\mu^0 = \frac{X^0 \bullet S^0}{n}$ *and* $\delta(X^0, S^0; \mu^0) < \frac{1}{2}$*. Moreover, let* X^k *and* S^k *be the matrices obtained after* k *iterations. Then, the inequality* $X^k \bullet S^k < \varepsilon$ *is satisfied when*

$$k \geq \frac{1}{\theta} \log \left(\frac{\frac{3}{2}\mu^0 n}{\varepsilon} \right).$$

Proof. After k iterations we have $\mu^k = (1-\theta)^k \mu^0$. From Lemma 5.12 and $\delta(X, S; \mu) < \frac{1}{2}$, we get

$$X^k \bullet S^k \leq \mu^k n \left(1 + \delta\right) < (1-\theta)^k \frac{3}{2} \mu^0 n.$$

Hence, the inequality $X^k \bullet S^k < \varepsilon$ holds if

$$(1-\theta)^k \frac{3}{2} \mu^0 n \leq \varepsilon.$$

By taking logarithms of both sides, we obtain

$$k \log(1-\theta) \leq \log\left(\varepsilon\right) - \log\left(\frac{3}{2}\mu^0 n\right).$$

As $\theta \leq -\log(1-\theta)$, we see that the inequality is valid if

$$k\theta \geq \log\left(\frac{3}{2}\mu^0 n\right) - \log\left(\varepsilon\right).$$

Hence,

$$k \geq \frac{1}{\theta} \log \left(\frac{\frac{3}{2}\mu^0 n}{\varepsilon} \right).$$

This completes the proof. □

Theorem 5.15. *Suppose that* $\mu^0 = \frac{2}{3}$*. If we consider the default values for* θ *and* τ*, we obtain that the algorithm given in Figure 5.1 requires no more than*

$$O\left(\sqrt{n} \log \frac{n}{\varepsilon}\right)$$

interior-point iterations. Hence, the resulting vectors satisfy $X^k \bullet S^k < \varepsilon$*. This means that the currently best-known iteration bound for the algorithm with small-update method is archived.*

Proof. By replacing $\mu^0 = \frac{2}{3}$ in Lemma 5.14, the result holds. □

5.5 Numerical experiments

In order to compare the efficiency of our algorithm with the existing methods, we offer a comparative numerical study between our approach presented in this paper and the algorithm given by L. Guerra in 2022 [32]. Different values of the update barrier θ are presented to show their influence in reducing the number of iterations produced by the two algorithms as well as the time necessary for optimality. The implementation is manipulated in Matlab R2018a. The accuracy parameter taken is $\varepsilon = 10^{-4}$.

Note that L. Guerra [32] took in her approach the function $\psi(t) = t^p, p \geq 2$ and showed that the minimal number of iterations is achieved for $p = 2$. For this reason, in our computational study, we consider $p = 2$. In the following tables of results, we note by Itr G and Itr Z the iteration number produced by Guerra's approach [32] and our approach, respectively. Furthermore, TG(s) and TZ(s) represent the execution time necessary for optimality in seconds using Guerra's approach and our approach, respectively.

Example 5.16. [76]

Consider the following SDP problem

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & -1 \\ 0 & -1 & 1 & -1 & -2 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & -2 & 2 & 0 \\ 0 & 2 & 1 & 0 & 2 \\ -2 & 1 & -2 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 2 & 2 & -1 & -1 & 1 \\ 2 & 0 & 2 & 1 & 1 \\ -1 & 2 & 0 & 1 & 0 \\ -1 & 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & 0 & -2 \end{pmatrix}, C = \begin{pmatrix} 3 & 3 & -3 & 1 & 1 \\ 3 & 5 & 3 & 1 & 2 \\ -3 & 3 & -1 & 1 & 2 \\ 1 & 1 & 1 & -3 & -1 \\ 1 & 2 & 2 & -1 & -1 \end{pmatrix}, b = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix}.$$

We take: $X^0 = I$, $y^0 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ and $S^0 = I$ as a feasible starting points. The numerical results of this problem are summarized in Table 5.1.

Example 5.17. [32] Let us consider the SDP problem with variable size. We take $n = 2m$, and

$$A_k(i,j) = \begin{cases} 1 & \text{if } i = j = k \\ 1 & \text{if } i = j = m + k \\ 0 & \text{otherwise} \end{cases}, k = 1, ..., m.$$

θ	Itr G	TG(s)	Itr Z	TZ(s)
0.1	104	0.2614	105	0.2684
0.3	32	0.0332	33	0.0335
0.5	20	0.0245	18	0.0230
0.7	18	0.0232	11	0.0180
0.9	17	0.0212	06	0.0166

Table 5.1: Numerical results of SDP problem 5.16

$$C(i,j) = \begin{cases} -1 & \text{if } i = j = 1, ..., m \\ 0 & \text{otherwise} \end{cases}, i, j = 1, ..., n.$$

and

$$b(i) = 2, i = 1, ..., m.$$

We consider the following starting points

$$X^0 = \begin{cases} 2-\gamma & \text{if } i = j = 1, ..., m \\ \gamma & \text{if } i = j = m+1, ..., n \\ 0 & \text{otherwise} \end{cases},$$
$$y^0(i) = -\frac{1}{\gamma}, i = 1, ..., m,$$

and

$$S^0(i,j) = \begin{cases} -1+\frac{1}{\gamma} & \text{if } i = j = 1, ..., m \\ \frac{1}{\gamma} & \text{if } i = j = m+1, ..., n \\ 0 & \text{otherwise} \end{cases},$$

where, $\gamma = 2 - \sqrt{2}$. An exact optimal solution of Problem 5.17 is given by

$$X^*(i,j) = \begin{cases} 2 & \text{if } i = j = 1, ..., m \\ 0 & \text{otherwise} \end{cases},$$
$$y^*(i) = -1, i = 1, ..., m,$$

and

$$S^*(i,j) = \begin{cases} 1 & \text{if } i = j = m+1, ..., n \\ 0 & \text{otherwise} \end{cases}.$$

The numerical results of this problem are summarized in Table 5.2.

(m, n)	θ	Itr G	TG(s)	Itr Z	TZ(s)
(5,10)	0.1	111	0.21	112	0.18
	0.3	34	0.07	35	0.06
	0.5	21	0.06	19	0.04
	0.7	19	0.04	12	0.03
	0.9	18	0.04	07	0.02
$(15, 30)$	0.1	121	25.54	122	25.47
	0.3	37	7.71	38	8.23
	0.5	22	4.90	21	4.62
	0.7	20	4.34	13	3.07
	0.9	20	4.21	08	1.59
$(25, 50)$	0.1	126	423.36	127	593.69
	0.3	39	178.05	39	182.72
	0.5	23	108.98	21	102.40
	0.7	21	99.98	13	43.98
	0.9	21	98.14	08	38.86
$(40, 80)$	0.1	131	8409.86	132	8451.87
	0.3	40	2774.87	41	2885.30
	0.5	24	1649.02	22	1510.23
	0.7	22	1536.65	14	988.40
	0.9	21	1448.37	08	558.71

Table 5.2: Numerical results of SDP problem 5.17

Example 5.18 (Random SDP [52])**.** The test problem is generated as follows: After inputting two positive integers m, n, MATLAB language generates m matrices $R_i \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, m$ randomly. Then we take $A_i = \frac{R_i^T + R_i}{2}$, $b_i = \text{Tr}(A_i)$, and $C = \sum_{i=1}^{m} A_i + I$ to obtain an SDP and it's dual with an initial strictly feasible primal-dual point $(X^0, y^0, S^0) = (I, e, I)$.

The numerical results of this problem are summarized in Table 5.3.

(m,n)	θ	Itr G	TG(s)	Itr Z	TZ(s)
(05,15)	0.1	115	1.37	116	1.04
	0.3	35	0.33	36	0.34
	0.5	21	0.23	20	0.23
	0.7	14	0.24	08	0.16
	0.9	13	0.20	06	0.11
(10,35)	0.1	123	84.47	124	89.29
	0.3	38	22.82	38	22.89
	0.5	23	13.86	21	12.51
	0.7	11	12.56	07	7.90
	0.9	11	7.96	05	6.80
(05,50)	0.1	126	510.26	127	514.43
	0.3	39	152.41	39	153.39
	0.5	13	99.79	11	82.07
	0.7	10	69.98	07	46.79
	0.9	10	78.21	05	39.17
(40,40)	0.1	124	172.34	125	178.32
	0.3	38	61.03	39	62.96
	0.5	23	30.34	21	27.87
	0.7	13	43.89	08	11.43
	0.9	13	30.69	06	09.24
(30,50)	0.1	126	523.33	127	546.04
	0.3	39	160.68	39	163.83
	0.5	23	116.18	21	93.03
	0.7	11	106.98	07	78.14
	0.9	11	104.63	05	28.64

Table 5.3: Numerical results of Random SDP problem 5.18

Example 5.19 (Max-Cut problem [52]).

$$\begin{cases} \min L \bullet X \\ \text{diag}(X) = \frac{e}{4}, \\ X \succeq 0. \end{cases}$$

where $L = A - \text{Diag}(Ae)$, e is the vector of all components equal to 1, and A is the weighted adjacency matrix of a graph [35]. We choose the following feasible starting point:

$$X^0 = \frac{1}{4}I, \quad y^0 = \text{abs}(L)e, \quad S^0 = L - \text{Diag}(y_0).$$

The numerical results of this problem are summarized in Table 5.4.

(m, n)	θ	Itr G	TG(s)	Itr Z	TZ(s)
(10,10)	0.1	136	0.45	137	0.47
	0.3	42	0.12	42	0.15
	0.5	25	0.11	23	0.10
	0.7	11	0.08	07	0.07
	0.9	11	0.07	06	0.06
(15,15)	0.1	143	1.84	144	1.89
	0.3	44	0.65	44	0.55
	0.5	26	0.34	24	0.31
	0.7	13	0.29	08	0.18
	0.9	13	0.26	07	0.13
(30,30)	0.1	156	54.36	157	61.68
	0.3	48	16.27	48	16.94
	0.5	27	13.65	25	12.14
	0.7	14	11.57	10	8.47
	0.9	14	6.47	09	5.32

Table 5.4: Numerical results of Max-Cut problem 5.19

Example 5.20 (Educational testing problem (ETP) [52]).

$$\begin{cases} \max e^T y, \\ A - \text{Diag}(y) \succeq 0, \\ y \geq 0. \end{cases}$$

where $A \in \mathbb{S}_m^{++}$. This problem can readily be expressed as a dual form of SDP, involving symmetric matrices of dimension $n \times n$, where $n = 2m$. We choose the following feasible starting point:

$$X^0 = \begin{bmatrix} 3I_m & 0 \\ 0 & 2I_m \end{bmatrix}, \quad y^0 = 0.4\lambda_{\min}(A)e, \quad S^0 = \begin{bmatrix} A - \text{Diag}(y^0) & 0 \\ 0 & \text{Diag}(y^0) \end{bmatrix}.$$

The numerical results of this problem are summarized in Table 5.5.

(m, n)	θ	Itr G	TG(s)	Itr Z	TZ(s)
(08,16)	0.1	128	1.51	129	1.62
	0.3	39	0.43	40	0.55
	0.5	24	0.29	22	0.24
	0.7	22	0.28	14	0.17
	0.9	21	0.25	08	0.13
(15,30)	0.1	134	35.71	135	36.05
	0.3	41	11.18	42	11.19
	0.5	25	6.40	23	6.36
	0.7	22	5.99	14	3.98
	0.9	22	5.03	08	1.83
(30,60)	0.1	141	1576.90	142	1587.81
	0.3	43	471.47	44	478.23
	0.5	26	283.62	24	264.02
	0.7	23	250.15	15	162.33
	0.9	23	239.79	08	81.13

Table 5.5: Numerical results of ETP problem 5.20

Comments:

The numerical results show that the number of iterations and the execution time necessary for optimality in the two approaches depends on the values of parameter θ. It is quite surprising that when the value of θ increases, the number of iterations and the computational time decrease. On the other hand, our algorithm offers better numerical results in terms of the number of iterations and the computation time than that of Guerra algorithm [32]. This observation is confirmed more and more when the value of θ and the size of the problem increase.

5.6 Conclusion

In this chapter, we have described a new primal-dual path-following method to solve semidefinite programs. Our approach is a generalization of [93] for linear optimization. We have shown that the best result of iteration bounds for small-update methods was achieved, namely $O(\sqrt{n}\log\frac{n}{\varepsilon})$-iterations. Moreover, we presented some numerical results, which proved the efficiency of our algorithm. Finally, we point out that the implementation with the update parameter θ reduced significantly the number of iterations produced by this algorithm and leads it to reach its real numerical performances. These numerical results consolidate and confirm our theoretical purpose.

CHAPTER 6

Efficient primal-dual interior point algorithm for convex quadratic semidefinite optimization

In this chapter, we introduce a primal-dual interior point algorithm for convex quadratic semidefinite optimization. This algorithm is based on an extension of the technique presented in the work of Zhang and Xu for linear optimization. The symmetrization of the search direction is based on the Nesterov-Todd scaling scheme. Our analysis demonstrates that this method solves efficiently the problem within polynomial time. Notably, the short-step algorithm achieves the best-known iteration bound, namely $O(\sqrt{n}\log\frac{n}{\varepsilon})$-iterations. The numerical experiments conclude that the newly proposed algorithm is not only polynomial but requires a number of iterations clearly lower than that obtained theoretically. The set of those results were published in J. Appl. Math. Comput. [91].

6.1 The central path

In this section, we recall the concept of the central path with its accompanying properties, and then proceed to derive the classical search direction for the convex quadratic semidefinite optimization (CQSDO).

We consider the CQSDO problem defined on $\mathbb{S}^n$ by

$$\begin{cases} \min C \bullet X + \frac{1}{2} X \bullet Q(X) \\ A_i \bullet X = b_i \quad \text{for} \quad i = 1, \ldots, m, \\ X \succeq 0, \end{cases} \tag{CQSDP}$$

and its dual problem

$$\begin{cases} \max b^T y - \frac{1}{2} X \bullet Q(X) \\ \sum_{i=1}^{m} y_i A_i + S - Q(X) = C, \\ S \succeq 0. \end{cases} \tag{CQSDD}$$

Where $b \in \mathbb{R}^m$, the matrices C and $A_i,\ i = 1, \ldots, m$, are given and belong to the linear space of $n \times n$ symmetric matrices $\mathbb{S}^n$ and $Q : \mathbb{S}^n \to \mathbb{S}^n$ is a self-adjoint linear operator on $\mathbb{S}^n$, i.e., $A \bullet Q(B) = B \bullet Q(A)\ \forall A, B \in \mathbb{S}^n$.

Throughout the chapter, we assume that:

- The matrices A_i, $i = 1, \ldots, m$, are linearly independent.
- The transformation Q is monotone, i.e., $X \bullet Q(X) \geq 0$ for all $X \in \mathbb{S}^n$.
- The problems ($CQSDP$) and ($CQSDD$) satisfy the interior-point condition (IPC), i.e., there exist $(X^0 \succ 0, y^0, S^0 \succ 0)$ such that

$$A_i \bullet X^0 = b_i, \forall i = 1, \ldots, m; \quad \sum_{i=1}^{m} y_i^0 A_i + S^0 - Q(X^0) = C. \tag{IPC}$$

It is well-known that, under the previous assumptions, finding an optimal solution of both problems ($CQSDP$) and ($CQSDD$) is equivalent to solve the Karush–Kuhn–Tucker optimality conditions of ($CQSDP$) and ($CQSDD$)

$$\begin{cases} A_i \bullet X = b_i, \quad i = 1, \ldots, m,\ X \succeq 0, \\ \sum_{i=1}^{m} y_i A_i + S - Q(X) = C,\ S \succeq 0, \\ XS = 0. \end{cases} \tag{6.1}$$

The basic idea of primal-dual IPMs is to replace the third equation in system (6.1), the so-called complementarity condition for ($CQSDP$) and ($CQSDD$), by the parameterized equation $XS = \mu I$, with $\mu > 0$. This substitution results in the following system

$$\begin{cases} A_i \bullet X = b_i, \quad i = 1, \ldots, m, \; X \succ 0, \\ \sum_{i=1}^{m} y_i A_i + S - Q(X) = C, \; S \succ 0, \\ XS = \mu I. \end{cases} \tag{6.2}$$

Under the previously mentioned assumptions, for each $\mu > 0$, the system (6.2) has a unique solution denoted by $(X(\mu), y(\mu), S(\mu))$ called the μ-center of both problems $(CQSDP)$ and $(CQSDD)$ [60]. The set of all μ-centers defines a homotopy called the central path of $(CQSDP)$ and $(CQSDD)$. Essentially, the central path represents a continuous trajectory of solutions parameterized by μ, where each μ-center corresponds to a specific point along this path. As μ approaches zero, the central path converges to a limit solution, which satisfies the complementarity condition. This limit solution serves as a primal-dual optimal solution for both problems $(CQSDP)$ and $(CQSDD)$.

6.2 New search direction

This section introduces a novel class of search direction for CQSDO inspired by Zhang and Xu's method for LO as described in [93]. First, let's consider the Nesterov-Todd symmetrization scheme as defined in [58, 59] by

$$P = X^{1/2}(X^{1/2}SX^{1/2})^{-1/2}X^{1/2} = S^{-1/2}(S^{1/2}XS^{1/2})^{1/2}S^{-1/2}. \tag{6.3}$$

Furthermore, we define $D = P^{1/2}$ where $P^{1/2}$ denotes the symmetric square root of P. The matrix D is used to scale both matrices X and S to the same matrix V defined by

$$V = \frac{1}{\sqrt{\mu}} D^{-1}XD^{-1} = \frac{1}{\sqrt{\mu}} DSD = \frac{1}{\sqrt{\mu}} (D^{-1}XSD)^{1/2}.$$

It's important to note that the matrices D and V are symmetric and positive definite. Moreover, we have

$$V^2 = \left(\frac{1}{\sqrt{\mu}} D^{-1}XD^{-1}\right)\left(\frac{1}{\sqrt{\mu}} DSD\right) = D^{-1}\frac{XS}{\mu}D.$$

Also, from the third equation of the system (6.2), we deduce that

$$XS = \mu I \Leftrightarrow \frac{XS}{\mu} = I \Leftrightarrow V^2 = I. \tag{6.4}$$

Additionally, we have

$$XS = \mu I \Leftrightarrow V = \frac{1}{\sqrt{\mu}}(D^{-1}XSD)^{1/2} = \frac{1}{\sqrt{\mu}}(D^{-1}\mu ID)^{1/2} = I. \tag{6.5}$$

Based on equations (3.7) and (6.5), we deduce

$$V^2 = V.$$

Transforming the left-hand side of this last equation to the XS space, we obtain

$$XS = \mu DVD^{-1}.$$

In light of this, we can express system (6.2) in the following form

$$\begin{cases} A_i \bullet X = b_i, \quad i = 1, \dots, m, \ X \succ 0, \\ \sum_{i=1}^{m} y_i A_i + S - Q(X) = C, \ S \succ 0, \\ XS = \mu DVD^{-1}. \end{cases} \tag{6.6}$$

According to Zhang and Xu's idea [93], we assume that the variance matrix V is fixed. Applying Newton's method to the system (6.6) yields the following set of equations for the search direction ΔX, Δy and ΔS

$$\begin{cases} A_i \bullet \Delta X = 0, \quad i = 1, \dots, m, \\ \sum_{i=1}^{m} \Delta y_i A_i + \Delta S - Q(\Delta X) = 0, \\ \Delta X + X \Delta S S^{-1} = \mu DVD^{-1} S^{-1} - X. \end{cases} \tag{6.7}$$

It is clear that ΔS is symmetric, as implied by the second equation of the system (6.7). However, a crucial observation is that ΔX is not necessarily symmetric, because $X \Delta S S^{-1}$ may not be symmetric. Several researchers have proposed methods for symmetrizing the third equation in (6.7) such that the resulting new system had a unique symmetric solution. In this context, we will consider the symmetrization scheme yielding the NT-direction [58, 59] defined in (3.37).

In the Nesterov-Todd scheme, we replace the term $X \Delta S S^{-1}$ in the last equation of (6.7) by $P \Delta S P^T$. The system (6.7) becomes

$$\begin{cases} A_i \bullet \Delta X = 0, \quad i = 1, \dots, m, \\ \sum_{i=1}^{m} \Delta y_i A_i + \Delta S - Q(\Delta X) = 0, \\ \Delta X + P \Delta S P^T = \mu DVD^{-1} S^{-1} - X. \end{cases} \tag{6.8}$$

Let us further define

$$\overline{A}_i = \frac{1}{\sqrt{\mu}} D A_i D, \ \forall i = 1, ..., m, \ D_X = \frac{1}{\sqrt{\mu}} D^{-1} \Delta X D^{-1}, \ D_S = \frac{1}{\sqrt{\mu}} D \Delta S D. \tag{6.9}$$

Obviously, with the notations (6.9) and from (6.8), the scaled NT search directions $(D_X, \Delta y, D_S)$ can be represented as follows

$$\begin{cases} \overline{A}_i \bullet D_X = 0, \quad i = 1, \dots, m, \\ \sum\limits_{i=1}^{m} \Delta y_i \overline{A}_i + D_S - \bar{Q}(D_X) = 0, \\ D_X + D_S = P_V. \end{cases} \tag{6.10}$$

Where

$$P_V = I - V, \tag{6.11}$$

and

$$\bar{Q}(D_X) = DQ(DD_XD)D = \frac{1}{\sqrt{\mu}} DQ(\Delta X)D.$$

Remark 6.1. Note that, In the CQSDO case and due to the first two equations of the system (6.10), we have

$$D_X \bullet D_S = \frac{1}{\mu} \Delta X \bullet Q(\Delta X) \geq 0. \tag{6.12}$$

This means that D_X and D_S are not orthogonal.

Now, For the analysis of the algorithm and according to (6.10), we define a proximity measure as follows

$$\delta(V) = \delta(X, S; \mu) = \|P_V\|_F = \|I - V\|_F. \tag{6.13}$$

It's clear that

$$\delta(V) = 0 \Leftrightarrow V = I \Leftrightarrow XS = \mu DVD^{-1}. \tag{6.14}$$

Hence, the value of $\delta(V)$ can be considered as a measure for the distance between the given pair (X, y, S) and the μ-center $(X(\mu), y(\mu), S(\mu))$.

Now, we can describe the algorithm more formally. The algorithm starts with a strictly feasible initial point (X^0, y^0, S^0) such that $\delta(X^0, S^0; \mu^0) < \tau$ where $0 < \tau < 1$ for an arbitrary parameter $\mu^0 > 0$. The full Nesterov-Todd step between successive iterates for the system (6.6) is defined as $(X^+, y^+, S^+) = (X + \Delta X, y + \Delta y, S + \Delta S)$ where the directions ΔX, Δy and ΔS are solutions for the linear system (6.8). Then it updates the parameter μ by the factor $1 - \theta$ with $0 < \theta < 1$, and targets a new $\mu-$center and so on. This procedure is repeated until the stopping criterion $X \bullet S \leq \varepsilon$ is satisfied for a given accuracy parameter ε.

Therefore, the generic path-following Nesterov-Todd step interior point algorithm for CQSDO is described in Figure 6.1 as follows.

Generic Primal-dual IPM for CQSDO

Input:

a proximity parameter $0 < \tau < 1$ (default $\tau = \frac{1}{2}$);

an accuracy parameter $\varepsilon > 0$;

an update parameter θ, $0 < \theta < 1$ (default $\theta = \frac{1}{7\sqrt{n}}$);

a strictly feasible point (X^0, y^0, S^0) and $\mu^0 = \frac{X^0 \bullet S^0}{n}$ such that $\delta(X^0, y^0, S^0) < \tau$;

begin

$X := X^0;\ y := y^0;\ S := S^0;\ \mu := \mu^0$

while $X \bullet S \geq \varepsilon$ **do**

$\mu := (1 - \theta)\,\mu$;

solve the system (6.10) via (6.9) to obtain $(\Delta X, \Delta y, \Delta S)$;

$X := X + \Delta X;\ y := y + \Delta y;\ S := S + \Delta S$;

end while

end.

Figure 6.1: Generic algorithm for CQSDO

6.3 Convergence Analysis

In the following section, we will present some results of complexity analysis.

To facilitate the analysis of algorithm given in Figure 6.1, we initially present the following technical results, which will be used later.

Lemma 6.2. *([25, Lemma 6.1]) Suppose that $X \succ 0$ and $S \succ 0$. If*

$$\det\left(X(\alpha)S(\alpha)\right) > 0,\ \forall 0 \leq \alpha \leq \tilde{\alpha},$$

then $X(\tilde{\alpha}) \succ 0$ and $S(\tilde{\alpha}) \succ 0$.

Lemma 6.3. *([25, Lemma 6.2]) Let $Q \in \mathbb{S}^n_{++}$, and $M \in \mathbb{R}^{n \times n}$ be skew-symmetric, i.e., $M = -M^T$. Then, $\det(Q + M) > 0$. Moreover, if the eigenvalues of $Q + M$ are real, then*

$$0 < \lambda_{\min}(Q) \leq \lambda_{\min}(Q + M) \leq \lambda_{\max}(Q + M) \leq \lambda_{\max}(Q).$$

Lemma 6.4. *([25, Lemma 6.3.2]) Let $D_{XS} = \frac{1}{2}\left(D_X D_S + D_S D_X\right)$, the symmetric part of $D_X D_S$, then we have*

$$\left\|D_{XS}\right\|_F \leq \frac{1}{2\sqrt{2}} \left\|P_V\right\|_F^2.$$

Lemma 6.5. *Let* $(D_X, \Delta y, D_S)$ *be a solution of* (6.10) *and* $\mu > 0$. *If* $\delta = \delta(X, S; \mu)$, *then*

$$0 \leq D_X \bullet D_S \leq \frac{1}{2}\delta^2. \tag{6.15}$$

Proof. For the left hand side of (6.15), it follows directly from (6.12). For the right hand side of it, from (6.10), we have

$$\delta^2 = \|D_X + D_S\|_F^2 = \|D_X\|_F^2 + \|D_S\|_F^2 + 2D_X \bullet D_S,$$

this means that

$$\delta^2 \geq 2D_X \bullet D_S,$$

consequently,

$$0 \leq D_X \bullet D_S \leq \frac{1}{2}\delta^2.$$

This completes the proof. □

In the next lemma, we give some basic properties about the proximity measure $\delta(V)$.

Lemma 6.6. *For any* $i = 1, ..., n$, *we have*

$$1 - \delta(V) \leq \lambda_i(V) \leq 1 + \delta(V).$$

Proof. From (6.13), we have

$$\begin{aligned} \delta(V) &= \|I - V\|_F \\ &= \sqrt{\sum_{i=1}^{n} (1 - \lambda_i (V))^2} \\ &\geq |1 - \lambda_i (V)|, \ \forall i = 1, ..., n. \end{aligned}$$

Using this last inequality, the result easily follows. □

In the following lemma, we state a condition which ensures the feasibility of the full Nesterov-Todd step.

Lemma 6.7. *Let* (X, S) *be a strictly feasible primal-dual solution of* ($CQSDP$) *and* ($CQSDD$). *Hence,* $X^+ \succ 0$ *and* $S^+ \succ 0$ *if and only if* $V + D_{XS} \succ 0$.

Proof. Let

$$X^+ = X + \Delta X \text{ and } S^+ = S + \Delta S. \tag{6.16}$$

We introduce a step length $\alpha \in [0,1]$ and we define

$$X(\alpha) = X + \alpha\Delta X \text{ and } S(\alpha) = S + \alpha\Delta S. \tag{6.17}$$

Thus $X(0) = X,\ X(1) = X^{+}$ and one can introduce similar notations for S. Applying (6.9) and (6.17), we have

$$\begin{aligned} X(\alpha)S(\alpha) &= XS + \alpha(X\Delta S + \Delta XS) + \alpha^2 \Delta X \Delta S. \\ &= \mu D\left(V^2 + \alpha\left(D_X V + V D_S\right) + \alpha^2 D_X D_S\right) D^{-1} \\ &\sim \mu\left(V^2 + \alpha\left(D_X V + V D_S\right) + \alpha^2 D_X D_S\right) \\ &= Q(\alpha) + M(\alpha). \end{aligned} \tag{6.18}$$

Where

$$Q(\alpha) = \mu\left(V^2 + \frac{1}{2}\alpha\left(D_X V + V D_S + V D_X + D_S V\right) + \frac{1}{2}\alpha^2\left(D_X D_S + D_S D_X\right)\right),$$

and

$$M(\alpha) = \mu\left(\frac{1}{2}\alpha\left(D_X V + V D_S - V D_X - D_S V\right) + \frac{1}{2}\alpha^2\left(D_X D_S - D_S D_X\right)\right).$$

The matrix $M(\alpha)$ is skew-symmetric, for each $0 \leq \alpha \leq 1$. Lemma 6.3 implies that the determinant of $X(\alpha)S(\alpha)$ is positive if the matrix $Q(\alpha) \succ 0$. To this end, from (6.10) and (6.11), we have

$$\begin{aligned} Q(\alpha) &= \mu\left(V^2 + \frac{1}{2}\alpha\left(D_X V + V D_S + V D_X + D_S V\right) + \alpha^2 D_{XS}\right) \\ &= \mu\left(\left(1-\alpha\right)V^2 + \alpha\left(V^2 + \frac{1}{2}\left(P_V V + V P_V\right)\right) + \alpha^2 D_{XS}\right) \\ &= \mu\left(\left(1-\alpha\right)V^2 + \alpha\left(V^2 + \frac{1}{2}\left((I-V)V + V(I-V)\right) + \alpha D_{XS}\right)\right) \\ &= \mu\left(\left(1-\alpha\right)V^2 + \alpha\left(V + \alpha D_{XS}\right)\right). \end{aligned} \tag{6.19}$$

Suppose that $V + D_{XS} \succ 0$, then, $Q(1) = \mu\left(V + D_{XS}\right) \succ 0$, also $Q(0) = \mu V^2 \succ 0$. Furthermore, $V + D_{XS} \succ 0$ implies that $\alpha D_{XS} \succ -\alpha V$ for any $0 < \alpha < 1$, then $Q(\alpha) \succ \mu\left(1-\alpha\right)\left(V^2 + \alpha V\right) \succ 0$, which means that $Q(\alpha) \succ 0$ for all $0 \leq \alpha \leq 1$. Lemma 6.3 implies that $\det\left(X(\alpha)S(\alpha)\right) > 0$. Moreover, since $X = X(0) \succ 0$ and $S = S(0) \succ 0$. Lemma 6.2 implies that $X^{+} = X(1) \succ 0$ and $S^{+} = S(1) \succ 0$ for $\tilde{\alpha} = 1$. This completes the proof of the lemma. □

Lemma 6.8. *Let $\delta = \delta(X, S; \mu)$ as defined in (6.13), $\mu > 0$ and (X, S) be any pair of positive definite matrices. If $\delta < \sqrt{2}\left(\sqrt{\sqrt{2}+1}-1\right)$, then the full NT-step for CQSDO is strictly feasible, hence $X^+ \succ 0$ and $S^+ \succ 0$.*

Proof. On one hand, by Lemma 6.7 $X^+ \succ 0$ and $S^+ \succ 0$ if and only if $V + D_{XS} \succ 0$. This last holds for $\|D_{XS}\|_F < \lambda_{\min}(V)$. In fact

$$V + D_{XS} \succ 0 \Leftrightarrow \lambda_{\min}(V + D_{XS}) > 0. \tag{6.20}$$

Since,

$$\begin{aligned}\lambda_{\min}(V + D_{XS}) &\geq \lambda_{\min}(V) - |\lambda_{\min}(D_{XS})| \\ &\geq \lambda_{\min}(V) - \|D_{XS}\|_F. \end{aligned} \tag{6.21}$$

Then, (6.20) holds for $\|D_{XS}\|_F < \lambda_{\min}(V)$.

On the other hand, it follows from Lemma 6.4 and Lemma 5.7, that one has

$$\|D_{XS}\|_F \leq \frac{1}{2\sqrt{2}}\delta^2 \text{ and } 1 - \delta \leq \lambda_{\min}(V).$$

It is easily verified that $\|D_{XS}\|_F < \lambda_{\min}(V)$ certainly holds for $\frac{1}{2\sqrt{2}}\delta^2 < 1 - \delta$, which is equivalent to $\delta < \sqrt{2}\left(\sqrt{\sqrt{2}+1}-1\right)$. Thus, $V + D_{XS} \succ 0$ holds if $\delta < \sqrt{2}\left(\sqrt{\sqrt{2}+1}-1\right)$. This completes the proof. □

The next lemma, shows the influence of a full NT-step on the duality gap.

Lemma 6.9. *Let $\delta = \delta(X, S; \mu)$. Then the duality gap satisfies*

$$X^+ \bullet S^+ \leq \mu\left(n(1+\delta) + \frac{1}{2}\delta^2\right).$$

Proof. Since M is a skew-symmetric matrix, Then

$$\begin{aligned} X^+ \bullet S^+ &= \mu \text{Tr}\left(V_+^2\right) \\ &= \mu \text{Tr}(V + D_{XS} + M) \\ &= \mu[\text{Tr}(V) + \text{Tr}(D_{XS})] \\ &= \mu\left[\text{Tr}(V) + \frac{1}{2}\text{Tr}(D_X D_S) + \frac{1}{2}\text{Tr}(D_S D_X)\right] \\ &= \mu\sum_{i=1}^{n}\lambda_i(V) + \text{Tr}(D_X D_S), \text{ because } \text{Tr}(D_X D_S) = \text{Tr}(D_S D_X). \\ &\leq \mu n\lambda_{\max}(V) + D_X \bullet D_S. \end{aligned}$$

This last inequality, Lemma 6.5 and Lemma 6.6 give

$$X^+ \bullet S^+ \leq \mu \left(n\,(1+\delta) + \frac{1}{2}\delta^2 \right),$$

which completes the proof. □

The next lemma shows that the algorithm is well defined.

Lemma 6.10. *Let* $\delta = \delta(X, S; \mu) < \sqrt{2}\left(\sqrt{\sqrt{2}+1} - 1\right)$ *and* $\mu_+ = (1-\theta)\mu$, *where* $0 < \theta < 1$. *Then*

$$\delta(X^+, S^+; \mu_+) \leq \frac{\theta\sqrt{n} + \delta + \frac{1}{2\sqrt{2}}\delta^2}{\sqrt{1-\theta}\left(\sqrt{1-\theta} + \sqrt{1-\delta-\frac{1}{2\sqrt{2}}\delta^2}\right)}.$$

Furthermore, if $\delta < \frac{1}{2}$, $\theta = \frac{1}{7\sqrt{n}}$ *and* $n \geq 2$, *then* $\delta(X^+, S^+; \mu_+) < \frac{1}{2}$.

Proof. Let $\alpha = 1$. Then from (6.18), we obtain

$$V_+^2 \sim \frac{X^+S^+}{\mu} = V + D_{XS} + M, \tag{6.22}$$

with

$$M = \frac{1}{2}\left(D_X V + V D_S - V D_X - D_S V + D_X D_S - D_S D_X\right),$$

it should be noted that M is a skew-symmetric matrix. Lemma 6.3 implies that

$$\lambda_{\min}(V_+^2) \geq \lambda_{\min}(V + D_{XS}).$$

Using Lemma 6.4, Lemma 6.6 and (6.21), we get

$$\begin{aligned}\lambda_{\min}(V_+^2) &\geq \lambda_{\min}(V) - \|D_{XS}\|_F \\ &\geq 1 - \delta - \frac{1}{2\sqrt{2}}\delta^2.\end{aligned}$$

Thus

$$\lambda_{\min}(V_+) \geq \sqrt{1 - \delta - \frac{1}{2\sqrt{2}}\delta^2}. \tag{6.23}$$

Now, from the definition of δ, we get

$$
\begin{aligned}
\delta(X^+,S^+;\mu_+) &= \left\| I - \sqrt{\frac{X^+S^+}{\mu_+}} \right\|_F \\
&= \frac{1}{\sqrt{(1-\theta)}} \left\| \sqrt{(1-\theta)} I - V_+ \right\|_F \\
&= \frac{1}{\sqrt{(1-\theta)}} \left\| \left(\sqrt{(1-\theta)} I + V_+ \right)^{-1} \left((1-\theta) I - V_+^2 \right) \right\|_F \\
&\leq \frac{\left\| (1-\theta) I - V_+^2 \right\|_F}{\sqrt{(1-\theta)} \left(\sqrt{(1-\theta)} + \lambda_{\min}(V_+) \right)} \\
&\leq \frac{\| (1-\theta) I - V - D_{XS} - M \|_F}{\sqrt{(1-\theta)} \left(\sqrt{(1-\theta)} + \lambda_{\min}(V_+) \right)} \\
&\leq \frac{\| (1-\theta) I - V \|_F + \| D_{XS} + M \|_F}{\sqrt{(1-\theta)} \left(\sqrt{(1-\theta)} + \lambda_{\min}(V_+) \right)}.
\end{aligned} \tag{6.24}
$$

We have

$$
\| D_{XS} + M \|_F^2 = \text{Tr}\left((D_{XS} + M)^2 \right),
$$

but $M = -M^T$, then

$$
\| D_{XS} + M \|_F^2 = \text{Tr}(D_{XS}^2) - \text{Tr}(MM^T),
$$

$MM^T$is symmetric positive semidefinite, then Tr$(MM^T) \geq 0$ and this means that

$$
\| D_{XS} + M \|_F^2 \leq \text{Tr}(D_{XS}^2) = \| D_{XS} \|_F^2 .
$$

From this last inequality. (6.13), (6.23) and (6.24), we deduce

$$
\begin{aligned}
\delta(X^+,S^+;\mu_+) &\leq \frac{\| -\theta I \|_F + \| I - V \|_F + \| D_{XS} \|_F}{\sqrt{(1-\theta)} \left(\sqrt{(1-\theta)} + \sqrt{1 - \delta - \frac{1}{2\sqrt{2}} \delta^2} \right)} \\
&\leq \frac{\theta \sqrt{n} + \delta + \frac{1}{2\sqrt{2}} \delta^2}{\sqrt{1-\theta} \left(\sqrt{1-\theta} + \sqrt{1 - \delta - \frac{1}{2\sqrt{2}} \delta^2} \right)}.
\end{aligned}
$$

Which proves the first part of the lemma.

Now, suppose that $\delta < \frac{1}{2}$ and $\theta = \frac{1}{7\sqrt{n}}$. Then, we obtain

$$
\begin{aligned}
\delta(X^+,S^+;\mu_+) &\leq \frac{\theta \sqrt{n} + \delta + \frac{1}{2\sqrt{2}} \delta^2}{\sqrt{1-\theta} \left(\sqrt{1-\theta} + \sqrt{1 - \delta - \frac{1}{2\sqrt{2}} \delta^2} \right)} \\
&< \frac{\frac{1}{7} + \frac{1}{2} + \frac{1}{8\sqrt{2}}}{\sqrt{1-\theta} \left(\sqrt{1-\theta} + \sqrt{1 - \delta - \frac{1}{2\sqrt{2}} \delta^2} \right)}.
\end{aligned}
$$

Using $n \geq 2$, we get $\sqrt{1-\theta} = \sqrt{1-\frac{1}{7\sqrt{n}}} \geq \sqrt{1-\frac{1}{7\sqrt{2}}}$. Also, $\delta < \frac{1}{2}$ gives $\sqrt{1-\delta-\frac{1}{2\sqrt{2}}\delta^2} > \sqrt{\frac{1}{2}-\frac{1}{8\sqrt{2}}}$. Taking all these inequalities into consideration, we conclude

$$\begin{aligned}\delta(X^+, S^+; \mu_+) &< \frac{\frac{1}{7}+\frac{1}{2}+\frac{1}{8\sqrt{2}}}{\sqrt{1-\frac{1}{7\sqrt{2}}}\left(\sqrt{1-\frac{1}{7\sqrt{2}}}+\sqrt{\frac{1}{2}-\frac{1}{8\sqrt{2}}}\right)} \\ &= 0.485\,14 < \frac{1}{2}.\end{aligned}$$

This completes the proof. □

In the next lemma, we give an upper bound for the total number of iterations produced by the algorithm given in Figure 6.1.

Lemma 6.11. *Assume that the pair (X^0, S^0) is strictly feasible, $\mu^0 = \frac{X^0 \bullet S^0}{n}$ and $\delta(X^0, S^0; \mu^0) < \frac{1}{2}$. Moreover, let $X^k$and S^k be the matrices obtained after k iterations. Then, the inequality $X^k \bullet S^k < \varepsilon$ is satisfied when*

$$k \geq \frac{1}{\theta} \log\left(\frac{\frac{3}{2}\mu^0 n}{\varepsilon}\right).$$

Proof. After k iterations we have $\mu^k = (1-\theta)^k \mu^0$. From Lemma 6.9 and since $\delta < \frac{1}{2}$, we get

$$X^k \bullet S^k \leq \mu\left(n\left(1+\delta\right)+\frac{1}{2}\delta^2\right) < (1-\theta)^k \mu^0 \left(\frac{3}{2}n+\frac{1}{8}\right).$$

Hence, the inequality $X^k \bullet S^k < \varepsilon$ holds if

$$(1-\theta)^k \mu^0 \left(\frac{3}{2}n+\frac{1}{8}\right) \leq \varepsilon.$$

By taking logarithms of both sides, we obtain

$$\begin{aligned}k\log(1-\theta) &\leq \log\left(\varepsilon\right) - \log\left(\mu^0\left(\frac{3}{2}n+\frac{1}{8}\right)\right) \\ &\leq \log\left(\varepsilon\right) - \log\left(\frac{3}{2}\mu^0 n\right).\end{aligned}$$

As $\theta \leq -\log(1-\theta)$, we see that the inequality is valid if

$$k\theta \geq \log\left(\frac{3}{2}\mu^0 n\right) - \log\left(\varepsilon\right).$$

Hence,

$$k \geq \frac{1}{\theta} \log\left(\frac{\frac{3}{2}\mu^0 n}{\varepsilon}\right).$$

This completes the proof. □

Theorem 6.12. *Suppose that $\mu^0 = \frac{2}{3}$. If we consider the default values for θ and τ, we obtain that the algorithm given in Figure 6.1 requires no more than*

$$O\left(\sqrt{n}\log\frac{n}{\varepsilon}\right),$$

interior-point iterations. The resulting matrices satisfy $X^k \bullet S^k < \varepsilon$. This means that the currently best known iteration bound for the algorithm with small-update method is archived.

Proof. Replacing $\mu^0 = \frac{X^0 \bullet S^0}{n} = \frac{2}{3}$ in Lemma 6.11, the result holds. □

6.4 Numerical experiments

This section presents numerical results obtained by applying our algorithm to a set of CQSDO test problems taken from the reference [33]. These test problems include evaluations of the nearest correlation matrix (NCM). It's worth noting that NCM is an important problem within the realm of finance. Throughout our numerical experiments, we investigate various values for the update barrier parameter θ to demonstrate their impact on reducing the number of iterations required by our algorithm. The implementation was executed using Matlab. For all experiments, we set the barrier update parameter θ to values within the range $\{0.1, 0.3, 0.5, 0.7, 0.9\}$, while maintaining an accuracy parameter $\varepsilon = 10^{-4}$. We denote by "$Iter$" the number of iterations performed by the algorithm, "$T(s)$" the time in seconds necessary to get an approximate optimal solution and "Z^*" the optimal value.

Example 6.13. We consider the following CQSDO problem with $m = 2, n = 3, Q(X) = HXH$ such that

$$H = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, C = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

We consider the following starting points

$$X^0 = \begin{pmatrix} 0.4 & 0 & 0 \\ 0 & 1.4 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}, y^0 = \begin{pmatrix} 0 \\ -1.5 \end{pmatrix}, S^0 = \begin{pmatrix} 0.3 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 1.5 \end{pmatrix}.$$

The obtained primal-dual optimal solution is

$$X^* = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 0 \end{pmatrix}, y^* = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, S^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The obtained optimal value is: $Z^* = -4.5$.

The numerical results of this problem are summarized in Table 6.1

θ	0.1	0.3	0.5	0.7	0.9
$Iter$	88	28	16	10	06
$T(s)$	0.0692	0.0483	0.0254	0.0364	0.0247

Table 6.1: Results of CQSDO Example 6.13

Example 6.14. Let us consider the CQSDO problem with variable size. We take $n = 2m$,

$$Q(X) = X, C[i,j] = \begin{cases} -1 & \text{if } i = j, i = 1, ..., m \\ 0 & \text{otherwise} \end{cases},$$

$$A_i[j,k] = \begin{cases} 1 & \text{if } j = k = i \text{ or } j = k = i + m \\ 0 & \text{otherwise} \end{cases}, b[i] = 2 \text{ for } i = 1, ..., m.$$

We consider the following starting points

$$X^0[i,j] = \begin{cases} \frac{5}{4} & \text{if } i = j = 1, ..., m \\ \frac{3}{4} & \text{if } i = j = m+1, ..., n \\ 0 & \text{otherwise} \end{cases},$$

$y^0[i] = -\frac{7}{12}$ for $i = 1, ..., m$,

and

$$S^0[i,j] = \begin{cases} \frac{5}{6} & \text{if } i = j = 1, ..., m \\ \frac{4}{3} & \text{if } i = j = m+1, ..., n \\ 0 & \text{otherwise} \end{cases},$$

An exact optimal solution of Example 6.14 is given by

$$X^* = \begin{cases} 1.5 & \text{if } i = j = 1, ..., m \\ 0.5 & \text{if } i = j = m+1, ..., n \\ 0 & \text{otherwise} \end{cases},$$

$y^*(i) = 0.5$ for $i = 1, ..., m$, and $S^* = 0_{n \times n}$.

The obtained optimal value is: $Z^* = \frac{-m}{4} = \frac{-n}{8}$.

The numerical results of this problem for different sizes (m, n) are summarized in Table 6.2

θ	0.1		0.3		0.5		0.7		0.9	
m	$Iter$	$T(s)$	$Iter$	$T(s)$	$Iter$	$T(s)$	$Iter$	$T(s)$	$Iter$	$T(s)$
05	112	2.0231	34	1.2912	19	0.1996	13	0.1580	10	0.0891
10	119	17.8144	37	5.9705	20	3.4575	14	2.3935	11	1.6814
25	126	3896.84	39	1252.95	21	738.57	14	408.84	11	355.45

Table 6.2: Results of CQSDO Example 6.14

6.4.1 The nearest correlation matrix problem

The nearest correlation matrix problem (NCM) [71, 72, 80] is a fundamental optimization problem that arises in various fields, including finance, statistics, and data analysis. A correlation matrix is a square matrix that captures the relationships between variables, with diagonal elements representing perfect correlation with themselves and off-diagonal elements lying between -1 and 1.

The NCM problem involves optimizing a given matrix to find the correlation matrix that is closest to it while still adhering to the constraints of being a valid correlation matrix. This optimization is crucial in applications such as portfolio optimization and risk management, where ensuring the validity of correlation matrices is essential for accurate analysis and decision-making.

Now, we delve into the mathematical formulation of the NCM problem. Specifically, we aim to minimize a distance or discrepancy measure, such as the Frobenius norm, between the given matrix and the sought correlation matrix K while enforcing constraints that ensure positive semidefiniteness and diagonal elements equal to 1. Thus, the NCM problem can be formulated as

$$\min_{X} \left\{ \frac{1}{2} \|L(X-K)\|_F^2 : \mathrm{diag}(X) = e,\ X \in \mathbb{S}_+^n \right\}, \tag{NCM}$$

where, $K \in \mathbb{S}^n, L : \mathbb{S}^n \to \mathbb{S}^n$ is a self-adjoint linear operator on $\mathbb{S}^n$ and e is the all-one vector of length n.

The NCM problem can be reformulated as a CQSDO with

$$C = -L^2(K) = -L(L(K)),\ Q(X) = L^2(X),\ b = e,$$

and for $i = 1, ..., n$:

$$A_i[j,k] = \begin{cases} 1 & if\ i = j = k, \\ 0 & otherwise. \end{cases}$$

Let's consider the example bellow:

Example 6.15. Let the NCM problem for the data bellows: $m = n = 3$,

$$K = \begin{pmatrix} 0 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 0 \end{pmatrix} \text{ and } L(X) = X,$$

Then $Q(X) = L^2(X) = X$ and $C = -L^2(K) = -K$, this gives

$$C = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, b = e.$$

We take $X^0 = I$, $y^0 = \begin{pmatrix} -1.5 \\ -0.5 \\ -1.5 \end{pmatrix}, S^0 = \begin{pmatrix} 2.5 & 1 & 1 \\ 1 & 2.5 & 1 \\ 1 & 1 & 2.5 \end{pmatrix}.$

The obtained primal-dual optimal solution is

$$X^* = \begin{pmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 1 & -0.5 \\ -0.5 & -0.5 & 1 \end{pmatrix}, y^* = \begin{pmatrix} 0.5 \\ 1.5 \\ 0.5 \end{pmatrix}, S^* = \begin{pmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{pmatrix}.$$

The obtained optimal value is: $Z^* = 0.25$.

The numerical results of this NCM problem are summarized in Table 6.3

θ	0.1	0.3	0.5	0.7	0.9
$Iter$	108	34	19	13	10
$T(s)$	0.1107	0.0454	0.0450	0.0429	0.0370

Table 6.3: Results of CQSDO Example 6.15

Example 6.16. We take the same data for the previous example with

$L(X) = U^{\frac{1}{2}} X U^{\frac{1}{2}}$ such that $U = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix} \in \mathbb{S}^n_{++}.$

Then, $Q(X) = UXU$ and $C = -L^2(K) = -Q(K) = -UKU = \begin{pmatrix} -4 & 1 & 5 \\ 1 & 1 & 1 \\ 5 & 1 & -4 \end{pmatrix}$.

We take $X^0 = I$, $y^0 = \begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix}$, $S^0 = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$.

The obtained primal-dual optimal solution is

$$X^* = \begin{pmatrix} 1 & -0.7187 & 0.0441 \\ -0.7187 & 1 & -0.7187 \\ 0.0441 & -0.7187 & 1 \end{pmatrix}, \; y^* = \begin{pmatrix} 0.9365 \\ 1.1987 \\ 0.9365 \end{pmatrix},$$

$$S^* = \begin{pmatrix} 0.0016 & 0.0023 & 0.0016 \\ 0.0023 & 0.0033 & 0.0023 \\ 0.0016 & 0.0023 & 0.0016 \end{pmatrix}.$$

The obtained optimal value is: $Z^* = -3.25$.

The numerical results of this NCM problem are summarized in Table 6.4

θ	0.1	0.3	0.5	0.7	0.9
$Iter$	111	35	19	13	10
$T(s)$	0.1036	0.0459	0.0295	0.0114	0.0112

Table 6.4: Results of CQSDO Example 6.16

Comments:

The numerical results show that the iteration numbers of the algorithm and the execution time necessary for optimality depend on the values of the parameter θ. Specifically, as the value of θ increases, both the number of iterations and the computational time decrease. Indeed, the number of iterations recorded in all the tested examples is clearly lower than the obtained theoretical number.

6.5 Conclusion

We have introduced a novel primal-dual path-following method designed for solving convex quadratic semidefinite programs. Our approach represents an extension of the technique presented in the work of Zhang and Xu [93] for linear optimization. We have shown that our algorithm can efficiently solve the problem within polynomial time and that the correspondent short-step algorithm has the best-known iteration bound, namely $O\left(\sqrt{n}\log\frac{n}{\varepsilon}\right)$ iterations. Furthermore, we have presented a set of numerical tests that are not only acceptable but also quite encouraging in their performance. Notably, we have highlighted that by incorporating the update parameter θ into the implementation, we can significantly reduce the number of iterations required by the algorithm, thus aligning it more closely with its real numerical performance. These numerical results consolidate and confirm our theoretical results.

General conclusion and future works

In this thesis, we have presented a theoretical and numerical study of interior point algorithms for solving different optimization problems, such as linear optimization, convex quadratic optimization, semidefinite optimization and convex quadratic semidefinite optimization.

Firstly, we are interested in the resolution of linear programming, we adopted one of these types of interior point methods which is based on a new direction of Newton given by Darvay and Takàcs [20] and we proposed two functions $\psi(t) = t^{\frac{7}{4}}$ and $\psi(t) = t^{\frac{3}{2}}$. We conducted a comprehensive theoretical study on the analysis and complexity of the algorithms resulting from these functions. We thus established comparative numerical experiments of our results with those of Darvay and Takàcs specifically in some problems from the set of Netlib tests collection. The obtained results for our new functions are significant and encouraging.

Secondly, we adopt the fundamental analysis employed in Zhang and Xu's study [93] and Darvay and Takàcs' study [20] for LO to the CQP case to formulate a novel primal-dual path-following interior-point algorithms for CQP. The best iteration bound for the algorithms with small-update method was archived. Additionally, we established comparative numerical experiments of our results with another existing method [13], specifically in some problems from quadprog tests collection. The obtained results for our approach based on the extension of Zhang and Xu's work [93] are significant and encouraging.

Thirdly, leveraging the advantages of the previous technique [93], we extended it to more general problems such as semidefinite optimization. Specifically:

- For linear semidefinite optimization, we applied the fundamental analysis and demonstrated that our algorithm achieves the best iteration bounds for small-update methods, specifically $O(\sqrt{n}\log\frac{n}{\varepsilon})$ iterations. Our extensive numerical study, focusing on various SDO applications, validated the efficiency of our proposed algorithm.

- For convex quadratic semidefinite optimization (CQSDO), our extended technique efficiently solves CQSDO problems within polynomial time. The corresponding short-step algorithm also achieves the best-known iteration bound, $O(\sqrt{n}\log\frac{n}{\varepsilon})$ iterations. The numerical tests performed not only met expectations but also showed highly encouraging performance.

Finally, we have highlighted that by incorporating the update parameter θ into the implementation, we can significantly reduce the number of iterations required by all our obtained algorithms, thus aligning it more closely with its real numerical performance.

These different works are important contributions that allow for improving the complexity of algorithms and the numerical behavior of primal-dual interior point methods in different optimization problems mentioned previously. The set of some obtained results are published in international journals [86, 87, 89, 90] and others are accepted to published in [88, 91].

Several interesting topics remain for further research:

1. The extension of algorithms presented in chapter 3 for other optimization problems such as semidefinite problems.

2. The development of an infeasible full-Newton step interior point algorithm for linear programming and convex quadratic programming based on our algorithms presented in chapter 3 and chapter 4.

3. The search directions used in the SDO and CQSDO cases are all based on the Nesterov-Todd symmetrization scheme. It may be possible to design similar algorithms using other symmetrization schemes to obtain an improving polynomial-time complexity bound.

4. The extension of the study presented in chapters 5 and 6 to semidefinite linear complementarity problems.

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ملخص:

تتناول هذه الأطروحة حل مسائل الأمثلة باستخدام طرق النقاط الداخلية الأولية-الثنوية. من خلال استخدام تحويلات جبرية مكافئة للمعادلات الوسطية، أجرينا دراسة نظرية وخوارزمية لمسائل الأمثلة الأربع التالية: البرمجة الخطية، البرمجة التربيعية المحدبة، البرمجة نصف معرفة الخطية والبرمجة نصف معرفة التربيعية المحدبة. في كل مسألة و عبر تحويلات جبرية متنوعة، قمنا بإثبات تقارب الخوارزميات المقترحة وتوفير معدل حدودية تكلفة خوارزمياتها.

تم تعزيز النتائج المحصل عليها من خلال تجارب عددية مختلفة مميزة و ذات أهمية بالغة.

كلمات مفتاحية : البرمجة الخطية، البرمجة التربيعية المحدبة، البرمجة نصف معرفة، طرق النقاط الداخلية الأولية-الثنوية، التحويل الجبري، اتجاه الانحدار.

Abstract:

This thesis deals with solving optimization problems using primal-dual interior point methods. By employing algebraic transformations of centrality equations, we conducted a theoretical and algorithmic study on four optimization problems: linear programming, convex quadratic programming, linear semidefinite programming and convex quadratic semidefinite programming. For each problem, through various algebraic transformations, we demonstrated the convergence of the proposed algorithms and provided the rates of their polynomial algorithmic complexities.

The obtained results are reinforced by highly significant numerical experiments.

Keywords: Linear programming, Convex quadratic programming, Semidefinite programming, Primal-dual interior point method, Algebraic transformation, Descent direction.

Résumé :

Cette thèse concerne la résolution de quelques problèmes d’optimisation par des méthodes de point intérieure primale-duale. Moyennant la technique des transformations algébriques des équations de centralité, nous avons fait une étude théorique et algorithmique sur quatre problèmes d’optimisation à savoir : la programmation linéaire, la programmation quadratique convexe, la programmation semi-définie linéaire et la programmation semi-définie quadratique convexe. Dans chaque problème, à travers des différentes transformations algébriques, nous avons montré la convergence des algorithmes proposés et donné le taux de leurs complexités algorithmiques polynomiales.

Les résultats obtenus ont été consolide par des expérimentations numériques très significatives.

Mots clés : Programmation linéaire, Programmation quadratique convexe, Programmation semi-définie, Méthodes de point intérieure primale-duale, Transformation Algébrique, Direction de descente.