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SOLVING THE AFFINE VARIATIONAL INEQUALITIES PROBLEM USING OPTIMIZATION TECHNIQUES

Presented by

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Dedication

To my dear parents,

my brothers,

my sisters,

and their children.

To all those who love me and whom I love.

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List of publications

- A. Noui, Z. Kebaili and M. Achache. "An Efficient DCA Algorithm for Solving Non-Monotone Affine Variational Inequality Problem". Nonlinear Dynamics and Systems Theory (NDST), **24**(4) (2024), 410-418.

Glossary of notation

Problem classes

VIP	:	Variational Inequality Problem.
AVI	:	Affine Variational Inequality.
MAVI	:	Monotone Affine Variational Inequality.
NMAVI	:	Non Monotone Affine Variational Inequality.
CP	:	Complementarity Problem.
LCP	:	Linear Complementarity Problem.
MP	:	Mathematical Programming.
LO	:	Linear Optimization.
QO	:	Quadratic Optimization.

Notations

KKT	:	Karush-Kuhn-Tucker.
IPA	:	Interior-Point Algorithm.
IPM	:	Interior-Point Method.
IPC	:	Interior-Point Condition.
DC	:	Difference of Convex functions programming.
DCA	:	Difference of Convex functions Algorithm.
AET	:	Algebraic Equivalent Transformation.

Spaces and Orthants

$\mathbb{R}$	:	Set of real numbers.
$\mathbb{R}^n$	:	n-dimensional real space.
$\mathbb{R}^n_+$	:	Nonnegative orthant in $\mathbb{R}^n$.
$\mathbb{R}^n_{++}$	:	Positive orthant in $\mathbb{R}^n$.
$\mathbb{R}^{n\times n}$	:	Space of all $n \times n$ real squared matrices.
$\mathbb{S}^{n\times n}$	:	Space of all $n \times n$ real symmetric squared matrices.

Vectors

x_i	:	$i-th$ component of vector x.
x^T	:	Transpose of vector x.
$x \geq 0$	:	$x_i \geq 0, \forall i$.
$x > 0$	:	$x_i > 0, \forall i$.
$\|x\|$	:	Euclidean norm $(x^T x)^{\frac{1}{2}}$ of $x \in \mathbb{R}^n$.
$\|x\|_\infty$	:	Maximum norm $\max_{i=1:n} \|x_i\|$ of $x \in \mathbb{R}^n$.
$x^T y = \langle x, y \rangle$	:	Standard inner product $\sum_{i=1}^{n} x_i y_i$ of two vectors $x, y \in \mathbb{R}^n$.
xy	:	Hadamard product $(x_1 y_1, \ldots, x_n y_n)^T$.
$\frac{x}{y}$	:	Component wise division $\left(\frac{x_1}{y_1}, \ldots, \frac{x_n}{y_n}\right)^T$ $(y \neq 0)$
e	:	Vector of ones $(1, \ldots, 1)^T$.

Matrices

I	:	Identity matrix.
A	:	Matrix with entries a_{ij}.
A^T	:	Transposed matrix of A.
A^{-1}	:	Inverse of a regular matrix A.
$\lambda_{\max}(A)$	:	Largest eigenvalues of $A \in \mathbb{R}^{n \times n}$.
$X = diag(x)$	:	Diagonal matrix with diagonal elements equal to the components of the vector x, $(X_{ii} = x_i)$.

Functions

∇f	:	Gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$.
$\nabla^2 f$	:	Hessian matrix of f.
$\frac{\partial f_i}{\partial x_j}(x)$	:	Partial derivative of f_i with respect to x_j at the point x.

General introduction

The variational inequalities problem (VIP) first appeared in the mid-sixties, developed by specialists in partial differential equations. In particular, Stampacchia and Hartman who formalized and solved a large number of these equations using this methodology [69, 94].

In the last decades, VIP has a growing interest and in particular the class of affine variational inequalities problem (AVI) which will be our main goal in this thesis, where we are motivated by the fact that: On the one hand, the theory of AVI encompasses several important mathematical problems, such as: systems of linear equations, the Linear Complementarity Problem (LCP) and certain types of convex optimization problem (Linear Optimization (LO) and Quadratic Optimization (QO)). On the other hand, AVI is known for its practical importance through its miscellaneous applications to translate very important practical models, namely, equilibrium models in economics, transport networks in operation research, elasticity problems in mechanics and game theory. For more details we refer to the following references [22, 28, 72, 73, 77, 93] as well as the theses of H. Grar and Z. Kebaili [43, 56].

Nowadays, AVI still a good subject because it is the most coveted research topics in the field of numerical optimization. The goal was to find a richer and less restrictive theory and develop efficient algorithms for solving AVIs. For its solution, a number of iterative methods have been proposed and analyzed, such as projection methods [42, 61], methods based on optimization techniques [54, 56, 79] and interior-point methods [2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 26, 42, 47, 48]. However, it should be noted that these methods have major theoretical and numerical drawbacks, among others: the strong convergence assumptions and the excessive cost of iteration dominated by the calculation of the necessary projection. A promising avenue for addressing these challenges lies in exploiting the relationship between AVI and Quadratic Programming (QP). The latter allows AVI to be reformulated as optimization problems, opening the door to advanced techniques from convex interior-point methods (IPM) and non-convex optimization techniques (DC programming and DCA).

In IPMs, selecting appropriate search directions is crucial to ensuring feasibility, convergence, and maintaining polynomial complexity bounds. To do so, many techniques have been used for this purpose. Peng [83, 84] introduced the notion of

self-regular function. Also, in 2004, Bai et al [20] proposed Kernel function. The reader can consult the following references, Bouafia [23, 24], Touil [96, 97] and Zerari [101]. Meanwhile, in 2003, for solving LO, Darvay [30] proposed an innovative method to generate efficient search directions by applying the square root function to both sides of an Algebraically Equivalent Transformation (AET) of the centering equation that defines the central-path. He then applied the full-Newton method to the transformed system. Yields a new search direction to proof proved the polynomial complexity, namely $\mathcal{O}\left(\sqrt{n}\log\frac{n}{\varepsilon}\right)$. Later, Achache [1] extended this approach to QP, including Convex Quadratic Optimization (CQO) [1, 75] and the LCP [46].
Further developments include Kheirfam and Nasrollahi's [59] extension of the square root-based technique to integer powers, as well as Darvay and Takàcs's [31] introduction of a new function based on AET to design a primal-dual IPM for LO [98]. Similarly, Kheirfam and Haghighi [58] developed a primal-dual IPM for the $P_*(\kappa)$-LCP. Additional related works can be found in [3, 33, 46, 75]. Currently, AET technique has garnered significant research interest, with the development of new AET to describe novel primal-dual interior-point methods IPMs emerging as a key focus for researchers. In 2011, Zhang and Xu [102] introduced a specific search direction for LO.
In this thesis, we propose a full-Newton step path-following interior-point algorithm (IPA) for solving monotone affine variational inequalities problems (MAVI). The algorithm employs an AET technique, induced by a univariate function $\psi(x) = t^{\frac{q}{2}}$, to reformulate the centering equations that define the central-path. By applying Newton's method to this transformed system, we derive a new search direction for solving the MAVI. Under suitably defined neighborhood thresholds that govern the size of the central-path's neighborhood and the reduction of the barrier parameter, we establish that the proposed IPA is well-defined and exhibits local quadratic convergence. Numerical results are provided to demonstrate the efficiency and effectiveness of the proposed algorithm.

Next, we deal with the hard AVI, where we lose the monotonicity and convexity of the mapping. Therefore, the usual used methods are difficult to apply them to solve non-monotone affine variational inequalities problems (NMAVI). We focus on the smooth [15, 36, 39] and non-smooth [14, 35, 36, 82] optimization techniques. Among many methods, the DC programming and the DC algorithm, DCA play a great role to solve efficiently this type of this problems. Originally, this methods were proposed in a preliminary by Pham Dinh Tao in 1985 and significantly developed since 1994, as documented in references [63, 65, 66, 87, 88, 71], and in related works cited therein. DC programming and DCA have now become classics and are widely used tools in

non-convex optimization by many researchers where numerous studies have demonstrated their effectiveness, as evidenced by references [55, 62, 70, 90, 92].
Our motivation stems from the proven success of DCA in solving a broad class of smooth and non-smooth, large-scale, non-convex problems across various scientific and engineering domains. DCA has often been shown to outperform traditional optimization techniques in terms of robustness, convergence behavior, and even the ability to reach global solutions.
We reformulate the NMAVI as an optimization problem and show that it can be expressed as a DC program. A typical DC program involves minimizing a function expressed as the difference of two convex functions, $f(x) = g(x) - h(x)$, over a convex set. Notably, DCA is constructed using the individual convex components g and h, rather than the non-convex function f itself. We propose a tailored DC formulation of the NMAVI and explore the theoretical and practical advantages brought by this decomposition.
For studying the efficiency of the optimization model as well as the corresponding DC formulation and resulting DCA schemes, we test the algorithm on several benchmark data.
This thesis is composed of three chapters and is organized as follows :
The introductory chapter contains notions and reminders of some concepts of convex analysis and MP useful for this study. In this chapter, we also give an expanded presentation of the general variational inequalities problem, followed by its various reformulations and the main results of relative existence and uniqueness.
The second chapter is committed for solving a MAVI problem by a full-Newton step interior-point algorithm, based on AET .
The third chapter is devoted to the theoretical and numerical study of our DCA approach for solving a NMAVI problem. As well as a brief introduction on the foundations of DC programming and its DCA.
The thesis ends with a general conclusion on the results obtained and some prospects.

CHAPTER 1

Useful reminders and position of the variational inequalities problem

1.1 Introduction

In this chapter, we briefly recall some important concepts that will be used for the treatment of our work. More details be found in Keraghel's monograph [57] and other references [18, 37, 74, 80].

1.2 Elements of convex analysis

In this part, we recall the most important convex analysis results that are necessary for the development of this work.

Affine set

A subset C of $\mathbb{R}^n$ is said to be affine if

$$\forall \lambda \in \mathbb{R}, \forall x, y \in C : (1-\lambda)x + \lambda y \in C.$$

Affine hull

Let C be a convex set of $\mathbb{R}^n$, the linear variety generated by C (which mean the affine envelope of C), denoted by $aff(C)$ is the smallest affine subset of $\mathbb{R}^n$ containing C defined by :

$$aff(C) = \left\{ x \in \mathbb{R}^n : x = \sum_{i=1}^{n} \lambda_i x_i, \lambda_i \in \mathbb{R}, x_i \in C, \text{ with } \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

Affine function

A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be affine on $C \subseteq \mathbb{R}^n$ if

$$\forall \lambda \in \mathbb{R}, \forall x, y \in C : f\left((1-\lambda)x + \lambda y\right) \leq (1-\lambda)f(x) + \lambda f(y).$$

Convex set

We say that a subset C of $\mathbb{R}^n$ is convex if

$$\forall \lambda \in [0,1], \forall x, y \in C : (1-\lambda)x + \lambda y \in C.$$

- A convex set of the form

$$P = \{x \in \mathbb{R}^n : Ax \leq b\}, \text{ where } A \in \mathbb{R}^{m \times n} \text{ and } b \in \mathbb{R}^m$$

is called a convex polyheder (or simply a polyheder).
A convex set of the form

$$S_n = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1, x_i \geq 0,\ i = 1 : n \right\}$$

is called the n-simplex.

Convex function

Let C be a convex subset of $\mathbb{R}^n$ and $f : C \longrightarrow \mathbb{R}$ a function. We say that f is **convex** on C if:

$$\forall \lambda \in [0,1], \ \ \forall x, y \in C : \ f\left((1-\lambda)x + \lambda y\right) \leq (1-\lambda)f(x) + \lambda f(y).$$

f is said to be strictly convex if the inequalities above is strict

$$\forall x \neq y, \forall \lambda \in (0,1).$$

Convex hull

The convex hull of a set $C \subseteq \mathbb{R}^n$, denoted by $conv(C)$, is the set of finite convex combinations of elements of C, that is:

$$conv(C) = \left\{ x \in \mathbb{R}^n : x = \sum_{i=1}^{k \leq n+1} \lambda_i x_i, \lambda_i \in \mathbb{R}_+,\ x_i \in C, \text{ with } \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

Proper function

The function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be proper if $f(x) > -\infty,\ \forall x \in \mathbb{R}^n$ and $dom(f) = \{x \in \mathbb{R}^n : f(x) < +\infty\} \neq \varnothing$.

Lower semicontinuous function

A function $f : \mathbb{R}^n \to \overline{\mathbb{R}} = (-\infty, +\infty)$ is said to be lower semicontinuous (lsc) at a point $x^0 \in \mathbb{R}^n$ if and only if

$$\forall \varepsilon > 0, \exists \delta > 0 : \left\| x - x^0 \right\| \leq \delta \Rightarrow f(x) \geq f(x^0) - \varepsilon.$$

We denote by $\Gamma_0(\mathbb{R}^n)$ the set of all lower semicontinuous proper convex functions on $\mathbb{R}^n$.

Conjugate function

Let $f \in \Gamma_0(\mathbb{R}^n)$. The conjugate function of f, denoted by f^* is defined by:

$$f^*(y) = \sup\{\langle x, y\rangle - f(x) : x \in \mathbb{R}^n\}, \ \ \forall y \in \mathbb{R}^n.$$

Coercive function

A function $f : C \subseteq \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be coercive on C if there exists $x^0 \in C$, such that:

$$\lim_{\|x\| \to +\infty} \frac{\langle f(x), x - x^0\rangle}{\|x\|} = +\infty, \ \forall x \in C.$$

Relative interior

The relative interior of a convex set $C \subseteq \mathbb{R}^n$ denoted by $ri(C)$ is the interior of C relative to its affine hull $aff(C)$, define by:

$$ri(C) = \{x \in aff(C) \ : \exists r > 0 : \ B(x, r) \cap aff(C) \subseteq C\},$$

where

$$B(x, r) = \{y \in \mathbb{R}^n : \|x - y\| \leq r\},$$

is the closed ball centered at x with radius r.

Subgradient and subdifferential

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function on $\mathbb{R}^n$ and $x^0 \in dom(f)$. We call the sub-gradient of f at the point x^0 any vector $g \in \mathbb{R}^n$ satisfying:

$$f(x) \geq f(x^0) + \left\langle x - x^0, g \right\rangle, \ \forall x \in \mathbb{R}^n.$$

The set of all the sub-gradients of f at x^0 is called the sub-differential of f at the point x^0, wjch is denoted by:

$$\partial f(x^0) = \left\{ g \in \mathbb{R}^n : f(x) \geq f(x^0) + \left\langle x - x^0, g \right\rangle, \forall x \in \mathbb{R}^n \right\}.$$

If f is differentiable at x^0, then the subdifferential reduces to a single element, the gradient of f at x^0,

$$\partial f(x^0) = \left\{\nabla f(x^0)\right\}.$$

The domain of the subdifferential of f, denoted by $dom(\partial f)$, is defined by

$$dom(\partial f) = \{x \in \mathbb{R}^n : \partial f(x) \neq \varnothing\}.$$

For a convex function f on $\mathbb{R}^n \cup \{+\infty\}$ and any $x \in dom(\partial f)$, the set $\partial f(x)$ is a nonempty, closed and convex subset of $\mathbb{R}^n$.

Proposition 1.1. *If $f \in \Gamma_0(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ then:*

$$y \in \partial f(x) \Leftrightarrow x \in \partial f^*(x) \Leftrightarrow \langle x, y\rangle = f(x) + f^*(y),$$

ε-subgradient and ε-subdifferential

Let $\varepsilon > 0$ be a strictly positive real number, $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function and $x^0 \in dom(f)$. We call the ε-subgradient of f at the point x^0 any vector $g \in \mathbb{R}^n$ satisfying:

$$f(x) \geq f(x^0) + \left\langle x - x^0, g\right\rangle - \varepsilon,\ \forall x \in \mathbb{R}^n$$

The set of all the ε-subgradients of f at x^0 is called the ε-subdifferential of f at the point x^0, which is denoted by:

$$\partial_\varepsilon f(x^0) = \left\{g \in \mathbb{R}^n : f(x) \geq f(x^0) + \left\langle x - x^0, g\right\rangle - \varepsilon,\ \forall x \in \mathbb{R}^n\right\}.$$

1.3 Mathematical programming

A mathematical program (MP) is an optimization problem of the following general form:

$$\begin{cases} \min f(x) \\ g_i(x) \leq 0,\ i = 1 : m, \\ h_j(x) = 0,\ j = 1 : p, \\ x \in C \subseteq \mathbb{R}^n, \end{cases} \tag{MP}$$

where $f,\ g_i$ and h_j are defined functions of $\mathbb{R}^n$ on $\mathbb{R}$ and $C \neq \varnothing$.
The set

$$S = \{x \in C : g_i(x) \leq 0,\ h_j(x) = 0, i = 1 : m,\ j = 1 : p\}$$

is called the feasible solutions set.
A feasible solution that minimizes the objective function on S is said to be a global optimal solution of the MP. We denote by $\arg\min_S f(x)$ the set of global optimal solutions.

A point $x^* \in S$ is a local optimal solution of the MP if there exists a neighborhood $\mathcal{V}$ of x^* such that $f(x^*) \leq f(x), \forall x \in \mathcal{V} \cap S$ and we denote $loc\min\limits_{S} f(x)$ the set of all local optimal solutions of the MP.
We still have $\arg\min\limits_{S} f(x) \subseteq loc\min\limits_{S} f(x)$ and we have equality between the two sets if the MP is convex.

Remark 1.1. *The optimization problem MP consists of one of the following cases:*
- Either to look for an optimal point (local or global).
- Or if such a point does not exist, we so look for a lower bound to f.
- Or to establish that f is not bounded below on S in which case we adopt the convention $\inf\limits_{S} f(x) = -\infty$.
- When S is empty we put by convention $\inf\limits_{S} f(x) = +\infty$.

1.3.1 Optimization problems classification

Optimization problems are classified according to the characteristics of the functions f, g_i, h_j namely convexity and differentiability. In this regard, the MP is convex if f and g_i are convex and h_j are affine. If the latter are all differentiable, we say that the MP is a differentiable program. The class of differentiable convex mathematical programs is the best developed model. In the absence of convexity or differentiability the problem becomes difficult to handle.

1.3.2 Constraints qualification

We say that the constraint $g_i(x) \leq 0$ is an active or saturated at $x^* \in S$, if it satisfies $g_i(x^*) = 0$, in this regard, an equality constraint is saturated by definition.
A point $x^* \in C$ is regular if the gradients of the saturated constraints at x^* are linearly independent. In this case, we say that the constraints are qualified at x^*.
There are also two usual criteria for qualification in any point of view, namely:
- If all constraints are affine (the qualification automatically holds).
- If C is defined only by inequalities, we have the following Slater criterion: $g_i(x)$ is convex for all $i = 1 : m$, and there exists a point x^0 such that $g_i(x^0) < 0,\ \forall i = 1 : m$, then the constraints are qualified. ($ri(S) \neq \varnothing$)

Remark 1.2. *The complete resolution of the MP deals with the following points in order:*
The existence (and possibly the uniqueness) of an optimal solution.
The solution characterization (these are the conditions of optimality).
The development of algorithms to calculate this solution.

1.3.3 Existence and Uniqueness

Theorem 1.1. *[18]*
Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function and $S \subseteq \mathbb{R}^n$ a nonempty subset. The MP admits at least one global optimal solution $x^* \in S$, if one of the following two conditions is satisfied:
1. S is compact (closed and bounded) ***(Weierstrass theorem)***
2. The function f is coercive $\left(\lim_{\|x\|\to+\infty} f(x) = +\infty\right)$ on C closed.

Remark 1.3. *The uniqueness of an optimal solution is generally a consequence of S convex and f strictly convex.*

1.3.4 Optimal conditions

Necessary conditions

Theorem 1.2. *(First-order necessary optimality condition)*
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. If f achieves a minimizer (local or global) at $x^\star \in \mathbb{R}^n$ then:

$$\nabla f(x^\star) = 0.$$

Sufficient conditions

Theorem 1.3. *(First-order sufficient optimality condition)*
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex differentiable function. $x^\star$ is a global minimizer of f on $\mathbb{R}^n$, if

$$\nabla f(x^\star) = 0.$$

Theorem 1.4. *(Second-order sufficient condition optimality)*
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice differentiable function. If $\nabla f(x^\star) = 0$ and $\nabla^2 f(x^\star)$ is positive definite, then $x^\star$ is a strict local minimizer of f.

Theorem 1.5. *(Karush-Kuhn-Tucker (KKT) conditions)[78]*
If x^* is a local optimal solution of the MP satisfying one of the previous constraints qualification conditions, then there are multipliers $\lambda \in \mathbb{R}^m_+$ and $\mu \in \mathbb{R}^p$ such that:

$$\begin{cases} \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) = 0 & \text{(optimality)} \\ \\ \lambda_i g_i(x^*) = 0, \quad i = 1:m & \text{(complementarity)} \\ h_j(x^*) = 0, \quad j = 1:p \end{cases}$$

Remarks 1.1.
If the qualification condition is not satisfied, the KKT conditions do not apply (the existence of multipliers in this case is not guaranteed)
If the MP is convex, then the KKT conditions are both necessary and sufficient for x^* to be a global minimizer.

1.3.5 Classes of problems in mathematical programming

Linear programming

This kind of programming is one of the most important optimization techniques used in operations research. It is of course necessary to avoid forcing any model to be linear. In contrast, a very large number of models constitute extensions of linear programs. It can be defined as a mathematical technique for solving management problems and particularly those where the manager must determine, in the face of different possibilities, the optimal use of the company's resources to achieve specific objective such as maximizing profit or minimizing cost.
A linear programming (LP) expressed in its standard form as follows:

$$\begin{cases} \min c^T x \\ Ax = b, \\ x \geq 0. \end{cases} \tag{LP}$$

where $c \in \mathbb{R}^n,\ b \in \mathbb{R}^m,\ A \in \mathbb{R}^{n \times n}$.
We associate to LP its dual problem:

$$\begin{cases} \max b^T y \\ A^T y \leq c, \\ y \in \mathbb{R}^m. \end{cases} \tag{DL}$$

Proposition 1.2. *[18]*
A feasible and bounded linear program (bounded objective function) has at least one optimal solution located on the boundary of the feasible domain.

Quadratic programming

Quadratic Programming (QP) has numerous for its multiple applications in several fields. QP are often used as intermediate procedures for nonlinear programs. this is the case, among others, with successive quadratic programming methods (SQP).

We can present a QP as follows:

$$\begin{cases} \min\left(\frac{1}{2}x^TQx + c^Tx\right) \\ Ax = b, \\ x \geq 0, \end{cases} \tag{QP}$$

where Q is a symmetric square matrix of order n, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$ of full rank $(rank(A) = m < n)$. Recall that the set of constraints is a convex and closed polyheder and the objective function is continuously differentiable.

Remark 1.4. *If Q is positive definite matrix, then the QP is strictly convex, and we show, in this case that its optimal solution, if it exists, is unique.*

Nonlinear programming

Nonlinear programming is the search for the optimum of a nonlinear function over a given feasible set.
The nonlinear problems are presented systematically in the form:

$$\begin{cases} \min f(x) \\ g(x) = 0, \\ h(x) \leq 0, \end{cases} \tag{NLP}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^n \to \mathbb{R}^p$.
If at least one of the functions f, g or h is non-convex, then the problem within the field of global optimization context.

1.4 Variational inequalities problem

Given a continuous mapping F of $\mathbb{R}^n$ in $\mathbb{R}^n$ and nonempty closed convex subset C of $\mathbb{R}^n$. The variational inequalities problem denoted by (VIP) consists of :

$$\begin{cases} \text{Find } x \in C, \quad \text{such that} \\ \langle F(x), y - x\rangle \geq 0, \text{ for all } y \in C. \end{cases} \tag{VIP}$$

1.4.1 Problems related to the variational inequalities problem

Variational inequalities allow the mathematical representation of a multitude of scientific and practical models On the practical level, it can be found in several fields through its applications to translate very important practical models (equilibrium models in economics, transport networks in operations research. elasticity problems in mechanics, game theory). It is also viewed as a unifying framework for several significant mathematical problems, the following examples are given here as an indication :

Nonlinear equations systems

The simplest case is the one where $C = \mathbb{R}^n$, the VIP is reduced to solving a nonlinear equations system $F(x) = 0$. Moreover, if F is affine, the system becomes linear.

Unconstrained convex differentiable optimization problem

If $F = \nabla f$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex, differentiable function then, the VIP is nothing other than the necessary and sufficient optimality condition of a solution x^* for the unconstrained optimization problem:

$$\begin{cases} \min f(x) \\ x \in \mathbb{R}^n, \end{cases}$$

expressed by:

$$\nabla f(x^*) = 0.$$

Nonlinear complementarity problem

If $C = \mathbb{R}^n_+$, then the VIP takes the formulation called nonlinear complementarity problem denoted by (NCP):

$$\begin{cases} \text{Find } x \in \mathbb{R}^n_+, \quad \text{such that} \\ F(x) \geq 0 \text{ and } F(x)^T x = 0. \end{cases} \tag{NCP}$$

If $F = \nabla f$, with f convex and differentiable, this becomes the necessary and sufficient optimality condition for the simple problem:

$$\begin{cases} \min f(x) \\ x \geq 0. \end{cases}$$

Linear complementarity problem

When F is an affine function: $F(x) = Mx + q, \quad M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, the VIP coincides with the linear complementarity problem denoted by (LCP):

$$\begin{cases} \text{Find } x \in \mathbb{R}^n_+, \quad \text{such that} \\ Mx + q \geq 0 \text{ and } (Mx + q)^T x = 0, \end{cases} \tag{LCP}$$

which in turn generalizes LP and QP.

Constrained convex differentiable optimization problem

The class of differentiable convex programming (CP) constitutes an interesting special case of the VIP.
Indeed, let the problem:

$$\begin{cases} \min f(x) \\ x \in C, \end{cases} \tag{CP}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex differentiable function and C is a nonempty closed, convex subset of $\mathbb{R}^n$.
If $F = \nabla f$, the first order necessary and sufficient optimality condition for the (CP) is expressed by the variational inequalities:

$$\langle \nabla f(x), y - x \rangle \geq 0, \ \forall y \in C,$$

whose resolution is equivalent to that of the CP.

1.4.2 Classification of VIP

The VIPs are typically classified according to the properties of the mapping F.
In this regard, we give the following definitions [27, 36, 49].
Let $F : \mathbb{R}^n \to \mathbb{R}^n$.

Monotone mapping

F is monotone if

$$\langle F(x) - F(y), x - y \rangle \geq 0, \quad \forall x, y \in \mathbb{R}^n.$$

Strictly monotone mapping

F is strictly monotone if

$$\langle F(x) - F(y), x - y \rangle > 0, \quad \forall x, y \in \mathbb{R}^n \text{ and } x \neq y.$$

Strongly monotone mapping

F is strongly monotone if there exists $\gamma > 0$ such that:

$$\langle F(x) - F(y), x - y \rangle \geq \gamma \|x - y\|^2, \quad \forall x, y \in \mathbb{R}^n.$$

Coercive mapping

F is coercive if there exists $x^0 \in \mathbb{R}^n$ and $s > 0$ such that:

$$\|x\| > s \Rightarrow \left\langle F(x), x - x^0 \right\rangle > 0, \quad \forall x \in \mathbb{R}^n.$$

Strongly coercive mapping

F is strongly coercive if there exists $x^0 \in \mathbb{R}^n$ such that:

$$\lim_{\|x\|\to\infty} \frac{\langle F(x), x - x^0\rangle}{\|x - x^0\|} = +\infty.$$

Remarks 1.2.
F is strongly monotone $\Rightarrow$ F is strongly coercive $\Rightarrow$ F is coercive.
The following results are derived from the relationship between the differentiability of the mapping F and its Jacobian, so, if F is a differentiable mapping on an open convex set C of $\mathbb{R}^n$, then:
F is monotone on C if and only if $\nabla F(x)$ is positive semidefinite.
F is strictly monotone on C if and only if $\nabla F(x)$ is positive definite.

1.4.3 Interesting reformulation of the VIP

In the following paragraph, we will give an equivalent reformulation (optimization problem) to the variational inequalities problem.
It can be seen without difficulty that $x \in C$ is a solution of the VIP if and only if x is a solution of the following constrained optimization problem:

$$\begin{cases} \min F(x)^T y \\ \quad y \in C. \end{cases}$$

Indeed:

$$\begin{aligned} x \text{ is a VIP solution} &\Leftrightarrow \langle F(x), y - x\rangle \geq 0,\ \forall y \in C \\ &\Leftrightarrow F(x)^T y \geq F(x)^T x,\ \forall y \in C \\ &\Leftrightarrow x = \arg\min_{y\in C} F(x)^T y. \end{aligned}$$

Remark 1.5. *Note that this last optimization problem is a LP in the case where C is a polyhederon. Therefore, several algorithms known by their effectiveness can be used for its resolution. Unfortunately, this is not the case because the value $F(x)$ in the objective function is unknown.*

1.4.4 Existence, uniqueness and characterization of solutions for the VIP

The majority of the basic existence and uniqueness results that have been established rely considerably on the theory of optimization, and depend largely on the properties of the mapping F and the set C.
So, before giving the main results, it is worth recalling some useful notions namely the lower semi-continuity and coercivity of mapping.

Theorem 1.6. *[17, 36]*
Let C be convex, compact and nonempty subset of $\mathbb{R}^n$ and let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a monotone and continuous mapping on C, then the solutions set of VIP is nonempty and convex.

Corollary 1.1. *[17, 36]*
If C is convex, closed and nonempty subset of $\mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ is monotone, semi-continuous $\left(\forall x, y \in C,\ \text{the application: } t \mapsto (y-x)^T F\left(x + t\left(y - x\right)\right)\ \text{is continuous on } [0,1]\right)$ and coercive on C , then the solutions set of VIP is nonempty and compact.

Theorem 1.7. *[17, 36]*
If C is convex, closed and nonempty subset of $\mathbb{R}^n$ and F is strongly monotone on C, then there exists a unique solution of VIP.

1.4.5 KKT system of variational inequalities [95]

If $C = \{x \in \mathbb{R}^n, h(x) = 0,\ g(x) \leq 0\} \subseteq \mathbb{R}^n$ is a nonempty set, where $h : C \longrightarrow \mathbb{R}^p$ and $g : C \longrightarrow \mathbb{R}^m$ are continuously differentiable functions. The following system is called the KKT system of VIP:

$$\begin{cases} F(x) + \nabla h(x)y + \nabla g(x)z = 0, \\ h(x) = 0, \\ g(x) \leq 0, z \geq 0, g(x)^T z = 0. \end{cases}$$

A triple $(x, y, z) \in \mathbb{R}^{n+p+m}$ satisfying the previous system, is called a KKT point of the VIP.
There is a strong relationship between the KKT system and the solution of the VIP [35, 49, 82]. Namely, if $x^* \in C$ is a solution of the VIP and a constraint qualification holds, then multipliers vectors $y^* \in \mathbb{R}^p$ and $z^* \in \mathbb{R}^m$ exist such that $(x^*, y^*, z^*) \in \mathbb{R}^{n+p+m}$ is a KKT point of the VIP. Conversely, if h_i for $i = 1 : p$ are affine and g_i are convex for $i = 1 : m$ and if (x^*, y^*, z^*) solves the KKT system, then x^* solves the VIP.

CHAPTER 2

A feasible algorithm for solving monotone affine variational inequalities based on algebraic equivalent transformation

2.1 Introduction

We are mainly interested to solve the monotone affine variational inequalities problem (MAVI), which is to:

$$\begin{cases} \text{Find } x \in C, \quad \text{such that} \\ \langle Mx + q, y - x \rangle \geq 0, \ \text{for all } y \in C, \end{cases} \tag{MAVI}$$

where C is a nonempty, convex polyheder represents the constraint set defined as:

$$C = \{x \in \mathbb{R}^n : Ax \geq b\},$$

with $M \in \mathbb{R}^{n \times n}$ positive semidefinite matrix, $A \in \mathbb{R}^{m \times n}$ of full row rank $(rank(A) = m < n)$, $q \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ representing the data of the problem.
In order to solve the problems of the VIP in general, there are several popular approaches. The initial methods were known as projection methods which were based on the notion of the fixed point [25, 44, 45, 60, 66].
Another traditional method to solving VIP is nonlinear optimization methods [13, 15, 52, 54, 66, 99], that is, reformulating the VIP into an equivalent optimization problem with or without constraints. In addition, VIP can be solved with other methods, some of which are briefly mentioned below as an indication :
Interior point methods [26, 42]
Penalty methods [17, 56].
Alternating direction methods [38].

The transformation of the MAVI into a system of equations via its KKT conditions provides the essential framework for the effective application of interior-point methods IPM. This KKT reformulation casts the MAVI as a structured system of nonlinear equations and complementarity conditions, specifically capturing the optimality requirements through primal feasibility, dual feasibility, and complementary condition. It is precisely, form a system demanding both equality and inequalities constraints, that interior-point methods are uniquely designed to solve. IPA address the polynomial complexity inherent in the complementarity conditions not by enumerating active constraints, but by employing algebraic equivalent transformation to enforce strict feasibility and using Newton like iterations to solve a sequence of perturbed, smooth nonlinear equation systems descendants from the KKT conditions. Therefore, the KKT system generated from the MAVI serves as the direct input and foundation upon which interior-point algorithms operate, leveraging the inherent structure, especially the affine nature of the problem, to efficiently find solutions by navigating a central-path within the feasible region towards the KKT point satisfying the original variational inequalities.

2.2 Over view of interior-point methods

Interior point methods form a class of algorithms that allow solving mathematical optimization problems. They have the advantage of being polynomial when applied to linear, convex quadratic, semi-defined optimization problems and more generally to convex optimization problems. These algorithms are extensions of the methods developed for linear programming (affine, projective and central-path). The interior-point methods are divided into several families:

2.2.1 Affine methods (optimization on ellipsoids)

The ellipsoid method is an iterative method used to minimize convex functions. This method is known as being the first polynomial complexity algorithm discovered to solve linear optimization problems. The algorithm constructs a sequence of smaller and smaller ellipsoids, which contain the minimizer.
Arkadi Nemirovski, David B. Yudin, Shor develop in 1972, the ellipsoid method for (nonlinear) convex optimization problems. It is practically the Karmarkar algorithm with no potential function and no projective transformation, an affine transformation is used and the non-negativity constraint is replaced by an ellipsoid which contains the new iterate one. The algorithm is of a simple structure, unfortunately, it is not easy to demonstrate polynomiality [98].

2.2.2 Potential reducing methods

The potential function plays a big role in the development of interior-point methods. Karmarkar's algorithm [53] applied to the linear program in standard form uses a potential function of the form :

$$\rho \log \left(c^T - Z\right) - \sum_{i=1}^{n} \log(x_i),$$

where $\rho = n + 1$ and Z is a lower bound of the optimal value of the objective. Karmarkar proves the convergence and the polynomiality of his algorithm by showing that this function is reduced at each iteration by at least one constant. Since 1987, researchers have been introducing primal-dual potential functions, among which, that of Tanabe, Todd and Ye defined by :

$$\Phi_\rho(x, z) = \rho \log \left(x^T z\right) - \sum_{i=1}^{n} \log(x_i z_i),$$

for $\rho > n$. This function played a very important role in the development of potential reduction algorithms after 1988. The algorithms corresponding to these methods have a polynomial complexity, they require $\mathcal{O}(\sqrt{n}|\log \varepsilon|)$ iterations to reduce the duality jump ($x^T z \leq \varepsilon$, ε is a given precision). (The books of Roos et al [91], Ye [100], and Wright [98] are important for the development of modern methods of interior-points).

2.2.3 Central-path methods

These methods are the direct result of a large part of the strenuous studies carried out by several researchers towards the end of the 80s, and fully developed at the beginning of the 90s. They have the most aesthetic theoretical properties, polynomial complexity and a super linear and sometimes quadratic convergence. These qualities of comfort place this class of methods at the center of the primary interest of researchers, to effectively solve mathematical programs with constraints. The strategy of the central-path methods consists in looking for approximate solutions for a nonlinear system, following a neighborhood of the central-path (continuous arc of strictly feasible points) by obtaining a decreasing sequence of the duality jump (or in an equivalent way of the parameter μ which tends towards zero at the limit). Let us now describe these methods of interior-point of central-path type for a MAVI.

2.3 A feasible full-Newton step IPA for MAVI

Throughout this chapter, we assume that the interior-point condition (IPC) holds for the MAVI, which mean: there exists $(x^0, z^0, \lambda^0) \in int(C)$ such that:

$$\begin{cases} Mx^0 - A^T z^0 = -q, \ z^0 > 0, \\ Ax^0 - \lambda^0 = b, \ \lambda^0 > 0. \end{cases}$$

Finding a solution for (MAVI) is equivalent to solving the following system, which represents the KKT optimality conditions of MAVI which be expressed as follows:

$$\begin{cases} Mx - A^T z = -q, \\ Ax - \lambda = b, \\ z \geq 0, \lambda \geq 0, z^T \lambda = 0. \end{cases} \tag{2.1}$$

Lemma 2.1. *Let $x, y \in \mathbb{R}^n_+$, so we have:*

$$x^T y = 0 \Leftrightarrow xy = 0.$$

Then, the problem (MAVI) is equivalent to the following equations:

$$\begin{cases} Mx - A^T z = -q, \\ Ax - \lambda = b, \\ z \geq 0, \lambda \geq 0, z\lambda = 0. \end{cases} \tag{2.2}$$

2.3.1 Central-path of MAVI

The main idea is to replace the complementarity condition $z\lambda = 0$ in (2.2) by the parameterized equation $z\lambda = \mu e$, where $\mu > 0$. Hence, we obtain the system of equations:

$$\begin{cases} Mx - A^T z = -q, \\ Ax - \lambda = b, \\ z \geq 0, \lambda \geq 0, z\lambda = \mu e. \end{cases} \tag{2.3}$$

If the IPC holds, then system (2.3) has a unique solution for each fixed $\mu > 0$ denoted by $(x(\mu), y(\mu), z(\mu))$. The set $\{(x(\mu), y(\mu), z(\mu)) : \mu > 0\}$ is called the μ-center (central-path) of the MAVI. The set of μ-centers is called the central-path of the MAVI, for more details see [41]. If μ tends to zero, then the limit of the central-path exists and since the limit point satisfies the complementarity condition, the limit yields an optimal solution for MAVI.

2.3.2 New modified search direction of the MAVI by using AET

Similar to [1, 30], the AET technique for computing the new search direction for MAVI is simply based on replacing the non linear equation $z\lambda = \mu e$ by the new one:

$$\psi\left(\frac{z\lambda}{\mu}\right) = \psi\left(e\right),$$

where $\psi : (0, +\infty) \longrightarrow \mathbb{R}$ is a differentiable and invertible function.
Using the function ψ, then (2.3) can be rewritten as follows:

$$\begin{cases} Mx - A^T z = -q, \\ Ax - \lambda = b, \\ \psi\left(\dfrac{z\lambda}{\mu}\right) = \psi(e), \end{cases} \tag{2.4}$$

where ψ is applied coordinate-wisely. Applying Newton's method to the modified system (2.4), we obtain the following system:

$$\begin{cases} M\Delta x - A^T \Delta z = 0, \\ A\Delta x - \Delta\lambda = 0, \\ \psi'_z \Delta z + \psi'_\lambda \Delta\lambda = \psi(e) - \psi\left(\dfrac{z\lambda}{\mu}\right), \end{cases} \tag{2.5}$$

where

$$\psi'_z = \frac{\lambda}{\mu}\psi'\left(\frac{z\lambda}{\mu}\right) \quad \text{and} \quad \psi'_\lambda = \frac{z}{\mu}\psi'\left(\frac{z\lambda}{\mu}\right),$$

and ψ'_a means the derivative of ψ with respect to a.
So, the system (2.5) becomes:

$$\begin{cases} M\Delta x - A^T \Delta z = 0, \\ A\Delta x - \Delta\lambda = 0, \\ \dfrac{\lambda}{\mu}\psi'\left(\dfrac{z\lambda}{\mu}\right)\Delta z + \dfrac{z}{\mu}\psi'\left(\dfrac{z\lambda}{\mu}\right)\Delta\lambda = \psi(e) - \psi\left(\dfrac{z\lambda}{\mu}\right), \end{cases} \tag{2.6}$$

but

$$\frac{1}{\mu}\psi'\left(\frac{z\lambda}{\mu}\right)(\Lambda\Delta z + Z\Delta\lambda) = \psi(e) - \psi\left(\frac{z\lambda}{\mu}\right), \tag{2.7}$$

then

$$\Lambda\Delta z + Z\Delta\lambda = \frac{\mu\left(\psi(e) - \psi\left(\dfrac{z\lambda}{\mu}\right)\right)}{\psi'\left(\dfrac{z\lambda}{\mu}\right)}, \tag{2.8}$$

so we have the system:

$$\begin{cases} M\Delta x - A^T\Delta z = 0, \\ A\Delta x - \Delta\lambda = 0, \\ \Lambda\Delta z + Z\Delta\lambda = \dfrac{\mu\left(\psi(e) - \psi\left(\dfrac{z\lambda}{\mu}\right)\right)}{\psi'\left(\dfrac{z\lambda}{\mu}\right)}, \end{cases} \tag{2.9}$$

to simplify matters, we introduce the following notations:

$$v := \sqrt{\frac{z\lambda}{\mu}} \text{ and } d := \sqrt{\frac{z}{\lambda}}, \tag{2.10}$$

so, the scaling directions are defined as:

$$d_z = \frac{v\Delta z}{z} \text{ and } d_\lambda = \frac{v\Delta\lambda}{\lambda}, \tag{2.11}$$

in addition, we have:

$$\Lambda\Delta z + Z\Delta\lambda = \mu v\left(d_z + d_\lambda\right), \tag{2.12}$$

now, due to (2.4) and (2.10), the scaled form of system (2.9) is given by:

$$\begin{cases} M\Delta x - \overline{A}^T d_z = 0, \\ \dfrac{1}{\mu}\overline{A}\Delta x - d_\lambda = 0, \\ d_z + d_\lambda = p_v, \end{cases} \tag{2.13}$$

where

$$p_v = \frac{\psi(e) - \psi\left(v^2\right)}{v\psi'\left(v^2\right)}, \tag{2.14}$$

with $\overline{A} = \sqrt{\mu}DA$ and $D = diag(d)$.
We consider the AET introduced by the power function:

$$\psi(t) = t^{\frac{1}{2}} = \sqrt{t},$$

which is introduced by Darvay in [29, 30, 32] for LO and $P_*(\kappa)-$LCP and extended by Achache in [1] for convex quadratic programming (CQP). this yields:

$$p_v = 2\left(e - v\right), \tag{2.15}$$

moreover, system (2.9) becomes:

$$\begin{cases} M\Delta x - A^T\Delta z = 0, \\ A\Delta x - \Delta\lambda = 0, \\ \Lambda\Delta z + Z\Delta\lambda = 2\mu\left(\sqrt{\dfrac{z\lambda}{\mu}} - \dfrac{z\lambda}{\mu}\right). \end{cases} \tag{2.16}$$

Therefore, the new unique modified search directions $(\Delta x, \Delta z, \Delta\lambda)$ are obtained by solving system (2.16). Moreover, the new iterate is computed by taking a full-Newton step as follows:

$$x_+ := x + \Delta x, \quad z_+ := z + \Delta z, \quad \lambda_+ := \lambda + \Delta\lambda. \tag{2.17}$$

Remark 2.1. *By choosing function $\psi(t)$ appropriately, the system (2.16) can be used to define a class of new search directions. For example, some values of the vector p_v related to different choices of the function ψ are stated.*

Function $\psi(t)$	*Vector* p_v
$\psi(t) = t$	$(v^{-1} - v)$, *(Roos and al. [91]),*
$\psi(t) = \sqrt{t}$	$2(e - v)$, *(Darvay [30]),*
$\psi(t) = t - \sqrt{t}$	$\frac{2(v - v^2)}{2v - e}$, $v > \frac{e}{2}$ *(Darvay and Takács [33]),*
$\psi(t) = \log t$	$-2v \log v$, *(Pan [81]),*
$\psi(t) = \frac{\sqrt{t}}{2(1+\sqrt{t})}$	$e - v^2$, *(Kheirfam and Haghani [58]),*
$\psi(t) = t^{\frac{q}{2}}, q \geq 1$	$\frac{2}{q}(v^{1-q} - v)$ *(Kheirfam and Nasrollahi [59]),*
$\psi(t) = t^{\frac{3}{2}}$	$\frac{2}{3}(v^{-2} - v)$ *(Moussaoui and Achache [29, 75]).*
$\psi(t) = t^{\frac{5}{2}}, q \geq 1$	$\frac{2}{5}(v^{-4} - v)$ *(Grimes and Achache [46]),*

The value of p_v for different functions ψ.

These functions will be taken in account in our future works for solving VIP.

2.3.3 The algorithm's proximity measure

To measure the quality of the solution found, according to (2.15), we define the notion of centralization or norm-based proximity measure as follows:

$$\Gamma(v) := \Gamma(z\lambda; \mu) := \frac{\|p_v\|}{2} = \|e - v\|. \tag{2.18}$$

It is clear that:

$$\Gamma(v) = 0 \Leftrightarrow v = e \Leftrightarrow z\lambda = \mu e.$$

Consequently, the value of $\Gamma(v)$ represents the distance between the achievable point (z, λ) and the central-path for a certain μ.
In addition, we define the vector q_v by

$$q_v = d_z - d_\lambda, \tag{2.19}$$

then

$$d_z d_\lambda = \frac{p_v^2 - q_v^2}{4}, \tag{2.20}$$

and

$$\|q_v\| \leq \|p_v\|, \tag{2.21}$$

Furthermore, we define the τ-neighborhood of the central-path as follows:

$$\mathcal{N}(\tau, \mu) = \{(z, \lambda), z \geq 0, \lambda \geq 0, z\lambda = 0 \text{ and } \Gamma(z\lambda; \mu) \leq \tau\},$$

where $\tau > 0$ is a threshold (default value) and $\mu > 0$ is fixed.

2.3.4 The central-path full-Newton step IPA for MAVI

Here is how the central-path following IPA functions for MAVI problem: we suppose that a strictly feasible initial point $(x^0,\ z^0,\ \lambda^0) \in \mathcal{N}(\tau, \mu^0)$ exists for certain known $\mu^0 > 0,\ \left(\text{with } \tau = \frac{1}{2}\right)$.
A full-Newton step between consecutive iterations is described as: $(x_+, z_+, \lambda_+) = (x + \Delta x, z + \Delta z, \lambda + \Delta\lambda)$ where the Newton directions $\Delta x,\ \Delta z$ and $\Delta\lambda$ are solutions of the linear system (2.16). Then it updates the parameter μ by the factor $(1-\beta)$ with $0 < \beta = \frac{1}{2\sqrt{n}} < 1$, and target a new μ-center and so on. This procedure is repeated until the stopping criterion $n\mu \geq \varepsilon$ is satisfied for a given accuracy parameter ε.
Therefore, the full-Newton step IPA for MAVI is stated as follows.

Algorithm 2.1 Primal-dual central-path full-Newton step IPA for MAVI

Begin
Initialize:
$k = 0,\ \mu^0 > 0$ and a strictly feasible initial point (x^0, z^0, λ^0);
an accuracy parameter $\varepsilon > 0$;
a barrier update parameter $0 < \beta < 1,\ \ \beta = \frac{1}{2\sqrt{n}}$;
a threshold parameter $0 < \tau < 1$, default $\tau = \frac{1}{2}$;
Ensure: $(x^0, z^0, \lambda^0) \in \mathcal{N}(\tau, \mu^0)$
while $n\mu \geq \varepsilon$ **do**
    $\mu := (1 - \beta)\mu$;
    Compute $(\Delta x, \Delta z, \Delta\lambda)$ from the system (2.16)
    Set $x^{k+1} := x^k + \Delta x^k$; $z^{k+1} := z^k + \Delta z^k$; $\lambda^{k+1} := \lambda^k + \Delta\lambda^k$;
    Set $k := k + 1$;
end while
End

2.3.5 Convergence analysis

This section focuses on proving the new default conditions under which the previous algorithm is well-defined, converges quadratically to an optimal solution locally, and solves the MAVI with polynomial complexity. We start by presenting some preliminary results that will be useful for the later analysis of the algorithm.

Lemma 2.2. *Let* $(\Delta x, d_z, d_\lambda)$ *be the unique solution of (2.13), and let* $\mu > 0$ *with* $\Gamma := d(z\lambda; \mu)$. *Then*

$$0 \leq d_z^T d_\lambda \leq 2\Gamma^2, \tag{2.22}$$

and

$$\|d_z d_\lambda\|_\infty \leq \Gamma^2, \quad \|d_z d_\lambda\| \leq \sqrt{2}\Gamma^2. \tag{2.23}$$

Proof. For the first part of (2.22) we have:

$$d_z^T d_\lambda = d_\lambda^T d_z = \frac{1}{\mu}(\Delta\lambda)^T \Delta z = \frac{1}{\mu}\left(A\Delta x\right)^T \Delta z = \frac{1}{\mu}(\Delta x)^T A^T \Delta z$$

$$= \frac{1}{\mu}(\Delta x)^T A^T \left(A^T\right)^{-1} M\Delta x = \frac{1}{\mu}(\Delta x)^T M \Delta x \geq 0,$$

since M is positive semidefinite.
The second part, follows trivially from the following equality:

$$4\Gamma^2 = \|p_v\|^2 = \|d_z + d_\lambda\|^2 = \|d_z\|^2 + \|d_\lambda\|^2 + 2d_z^T d_\lambda \geq 2d_z^T d_\lambda.$$

For the first claim in (2.23), since $d_z d_\lambda = \frac{1}{4}\left(\left(d_z + d_\lambda\right)^2 - \left(d_z - d_\lambda\right)^2\right)$, then we have

$$\|d_z d_\lambda\|_\infty \leq \frac{1}{4}\max\left(\|d_z + d_\lambda\|_\infty^2, \|d_z - d_\lambda\|_\infty^2\right) \leq \frac{1}{4}\max\left(\|d_z + d_\lambda\|^2, \|d_z - d_\lambda\|^2\right),$$

since $d_z^T d_\lambda \geq 0$, then $\|d_z + d_\lambda\|^2 \geq \|d_z - d_\lambda\|^2$, thus:

$$\|d_z d_\lambda\|_\infty \leq \frac{1}{4}\|d_z + d_\lambda\|^2 = \frac{1}{4}\|p_v\|^2 = \Gamma^2$$

In the other hand, we have:

$$\begin{aligned}
\|d_z d_\lambda\|^2 &= \frac{1}{16}\left(\left\|\left(d_z + d_\lambda\right)^2 - \left(d_z - d_\lambda\right)^2\right\|^2\right) \\
&\leq \frac{1}{16}\left(\left\|\left(d_z + d_\lambda\right)^2\right\|^2 - \left\|\left(d_z - d_\lambda\right)^2\right\|^2\right) \\
&\leq \frac{1}{16}\left(\|d_z + d_\lambda\|^4 - \|d_z - d_\lambda\|^4\right) \\
&\leq \frac{1}{8}\|d_z + d_\lambda\|^4 = \frac{1}{8}\|p_v\|^4 = 2\Gamma^4,
\end{aligned}$$

hence, $\|d_z d_\lambda\| \leq \sqrt{2}\Gamma^2$. □

In the next lemma, we show the feasibility of a full-Newton step under the condition $\Gamma < 1$ throughout the algorithm.

Lemma 2.3. *Let (x, z, λ) be a strictly feasible point for the problem (MAVI) and let $\Gamma(z\lambda; \mu) < 1$. Then $z_+ := z + \Delta z > 0$ and $\lambda_+ := \lambda + \Delta\lambda > 0$ are strictly feasible.*

Proof. For each $\alpha \in [0, 1]$ and (z, λ) a strictly feasible point for MAVI, we have

$$z(\alpha)\lambda(\alpha) = (z + \alpha\Delta z)(\lambda + \alpha\Delta\lambda) = z\lambda + \alpha\left(z\Delta\lambda + \lambda\Delta z\right) + \alpha^2\Delta z\Delta\lambda. \tag{2.24}$$

In addition with the notation in (2.11), we have

$$z\Delta\lambda + \lambda\Delta z = \mu v\left(d_z + d_\lambda\right), \tag{2.25}$$

and

$$\mu d_z d_\lambda = \Delta z\Delta\lambda, \tag{2.26}$$

so

$$z_+(\alpha)\lambda_+(\alpha) = \mu\left(v^2 + \alpha v\left(d_z + d_\lambda\right) + \alpha^2 d_z d_\lambda\right) = \mu\left((1-\alpha)v^2 + \alpha\left(v^2 + v\left(d_z + d_\lambda\right) + \alpha d_z d_\lambda\right)\right),$$

and from (2.15), we have

$$v_i + \frac{(p_v)_i}{2} = 1, \text{ for all } \ i = 1:n,$$

and thus

$$v_i^2 + v_i(p_v)_i = 1 - \frac{(p_v)_i^2}{4},$$

thereby

$$(z_+)_i(\alpha)(\lambda_+)_i(\alpha) = \mu\left(\left(1-\alpha\right)v_i^2 + \alpha\left(v_i^2 + v_i(p_v)_i\right) + \frac{\alpha^2}{4}\left((p_v)_i^2 - (q_v)_i^2\right)\right),$$

and

$$(v_+)_i^2 = \left(1-\alpha\right)v_i^2 + \alpha\left(1 - (1-\alpha)\frac{(p_v)_i^2}{4} - \alpha\frac{(q_v)_i^2}{4}\right), \tag{2.27}$$

thus the inequalities $(z_+)_i(\alpha)(\lambda_+)_i(\alpha) > 0$ holds if

$$\max_i \left|(1-\alpha)\frac{(p_v)_i^2}{4} - \alpha\frac{(q_v)_i^2}{4}\right| < 1,$$

using (2.18) and (2.21) we get:

$$\begin{aligned}
\max_i \left|(1-\alpha)\frac{(p_v)_i^2}{4} - \alpha\frac{(q_v)_i^2}{4}\right| &\leq (1-\alpha)\left\|\frac{p_v}{4}\right\|_\infty^2 + \alpha\left\|\frac{q_v}{4}\right\|_\infty^2 \\
&\leq (1-\alpha)\frac{\|p_v\|^2}{4} + \alpha\frac{\|q_v\|^2}{4} \\
&\leq \frac{\|p_v\|^2}{4} = \left(\frac{\|p_v\|}{2}\right)^2 = \Gamma^2 < 1.
\end{aligned}$$

Hence, $(z_+)_i(\alpha)(\lambda_+)_i(\alpha) > 0$ for each $0 \leq \alpha \leq 1$. Since $(z_+)_i(\alpha)$ and $(\lambda_+)_i(\alpha)$ are linear function of α, then they do not change sign on the interval $[0, 1]$. For $\alpha = 0$, we have $(z_+)_i(0) > 0$ and $(\lambda_+)_i(0) > 0$. This leads to $(z_+)_i(1) > 0$ and $\lambda_+(1) > 0$. □

In the next lemma we state that under the condition $\Gamma(z\lambda; \mu) < 1$, the full-Newton step is quadratically convergent.

Lemma 2.4. *Let $z_+ := z + \Delta z$ and $\lambda_+ := \lambda + \Delta\lambda$ be the iterate obtained after a full Newton step. Suppose $\Gamma(z\lambda; \mu) < 1$, then*

$$\Gamma(z_+\lambda_+; \mu) \leq \frac{\Gamma^2}{1 + \sqrt{1 - \Gamma^2}}.$$

Thereby $\Gamma(z_+\lambda_+; \mu) < \Gamma^2$, which means quadratic convergence of the full-Newton step.

Proof. Setting $\alpha = 1$ in (2.27), we get

$$(v_+)_i^2 = 1 - \frac{(q_v)_i^2}{4}, \text{ for all } i = 1 : n, \tag{2.28}$$

so

$$\min_i (v_+)_i^2 \geq 1 - \frac{\|q_v\|_\infty^2}{4} \geq 1 - \frac{\|q_v\|^2}{4} \geq 1 - \frac{\|p_v\|^2}{4} = 1 - \Gamma^2,$$

thus

$$\min_i (v_+)_i \geq \sqrt{1 - \Gamma^2}. \tag{2.29}$$

On the other hand, we have

$$\begin{aligned}
\Gamma^2(z_+\lambda_+; \mu) &= \sum_{i=1}^{n} (1 - (v_+)_i)^2 \\
&= \sum_{i=1}^{n} (1 - (v_+)_i)^2 \frac{(1 + (v_+)_i)^2}{(1 + (v_+)_i)^2} \\
&= \sum_{i=1}^{n} \frac{\left(1 - (v_+)_i^2\right)^2}{(1 + (v_+)_i)^2} \\
&\leq \frac{1}{\left(1 + \min_i (v_+)_i\right)^2} \sum_{i=1}^{n} \left(1 - (v_+)_i^2\right)^2,
\end{aligned}$$

using (2.29), we get:

$$\Gamma^2(z_+\lambda_+; \mu) \leq \frac{1}{\left(1 + \sqrt{1 - \Gamma^2}\right)^2} \sum_{i=1}^{n} \left(1 - (v_+)_i^2\right)^2,$$

using (2.18),(2.21) and (2.28) we get:

$$\begin{aligned}\Gamma^2(z_+\lambda_+;\mu) &\leq \frac{\sum_{i=1}^{n}(q_v)_i^4}{4^2\left(1+\sqrt{1-\Gamma^2}\right)^2}\\ &\leq \frac{\left(\sum_{i=1}^{n}(q_v)_i^2\right)^2}{4^2\left(1+\sqrt{1-\Gamma^2}\right)^2} = \frac{1}{\left(1+\sqrt{1-\Gamma^2}\right)^2}\left(\frac{\|q_v\|^2}{4}\right)^2\\ &\leq \frac{1}{\left(1+\sqrt{1-\Gamma^2}\right)^2}\left(\frac{\|p_v\|^2}{4}\right)^2,\end{aligned}$$

hence

$$\Gamma^2(z_+\lambda_+;\mu) \leq \frac{\Gamma^2}{\left(1+\sqrt{1-\Gamma^2}\right)^2}.$$

□

The next Lemma discusses the influence on the proximity measure of the Newton process followed by a step along the central path.

Lemma 2.5. *Let* $\Gamma(z_+\lambda_+;\mu) < 1$ *and* $\mu_+ = (1-\beta)\,\mu$*, where* $0 < \beta < 1$*. Then*

$$\Gamma(z_+\lambda_+;\mu_+) \leq \frac{\beta\sqrt{n}+\Gamma}{1-\beta+\sqrt{(1-\beta)(1-\Gamma)}}.$$

Furthermore, if $\Gamma > \frac{1}{2}$, $\beta = \frac{1}{2\sqrt{n}}$ *and* $n \geq 4$*, then we get*

$$\Gamma(z_+\lambda_+;\mu_+) \leq \frac{1}{2}.$$

Proof. We have

$$\begin{aligned}\Gamma^2(z_+\lambda_+;\mu_+) &= \left\|e-\sqrt{\frac{z_+\lambda_+}{\mu_+}}\right\|^2\\ &= \frac{1}{1-\beta}\left\|\left(\sqrt{1-\beta}\right)e-v_+\right\|^2\\ &= \frac{1}{1-\beta}\sum_{i=1}^{n}\left(\sqrt{1-\beta}-(v_+)_i\right)^2\\ &= \frac{1}{1-\beta}\sum_{i=1}^{n}\left(\sqrt{1-\beta}-(v_+)_i\right)^2\frac{\left(\sqrt{1-\beta}+(v_+)_i\right)^2}{\left(\sqrt{1-\beta}+(v_+)_i\right)^2}\\ &= \frac{1}{1-\beta}\sum_{i=1}^{n}\frac{\left((1-\beta)-(v_+)_i^2\right)^2}{\left(\sqrt{1-\beta}+(v_+)_i\right)^2}\\ &\leq \frac{1}{1-\beta}\frac{1}{\left(\sqrt{1-\beta}+\min_i(v_+)_i\right)^2}\sum_{i=1}^{n}\left((1-\beta)-(v_+)_i^2\right)^2,\end{aligned}$$

using (2.28) and (2.29) we obtain:

$$\Gamma(z_+\lambda_+;\mu_+) \leq \frac{1}{\sqrt{1-\beta}}\frac{1}{\left(\sqrt{1-\beta}+\sqrt{1-\Gamma^2}\right)}\sqrt{\sum_{i=1}^{n}\left(-\beta+\frac{(q_v)_i^2}{4}\right)^2},$$

now, in view of (2.18) and (2.26) we deduce that:

$$\begin{aligned}\Gamma(z_+\lambda_+;\mu_+) &\leq \frac{1}{1-\beta+\sqrt{(1-\beta)(1-\Gamma^2)}}\left[\beta\sqrt{n}+\left(\frac{\|q_v\|}{2}\right)\right]\\ &\leq \frac{1}{1-\beta+\sqrt{(1-\beta)(1-\Gamma^2)}}\left[\beta\sqrt{n}+\left(\frac{\|p_v\|}{2}\right)\right],\end{aligned}$$

finally we get

$$\Gamma(z_+\lambda_+;\mu_+) \leq \frac{1}{1-\beta+\sqrt{(1-\beta)(1-\Gamma^2)}}\left[\beta\sqrt{n}+\Gamma\right].$$

□

The next lemma provides an upper bound on the duality gap after performing a full-Newton step.

Lemma 2.6. *Let $z_+ := z + \Delta z$ and $\lambda_+ := \lambda + \Delta\lambda$ be strictly feasible point. Then*

$$z_+^T\lambda_+ = \mu\left(n - \frac{\|q_v\|^2}{4}\right),$$

hence

$$z_+^T\lambda_+ \leq \mu n.$$

Proof. We have seen from the previous lemmas that

$$(v_+)_i^2 = 1 - \frac{(q_v)_i^2}{4}, \text{ for all } i = 1:n,$$

thereby

$$\sum_{i=1}^{n}(v_+)_i^2 = \sum_{i=1}^{n}\left(1-\frac{(q_v)_i^2}{4}\right) = n - \frac{\|q_v\|^2}{4},$$

since

$$z_+^T\lambda_+ = \mu\sum_{i=1}^{n}(v_+)_i^2,$$

it follows that

$$z_+^T\lambda_+ = \mu\left(n - \frac{\|q_v\|^2}{4}\right).$$

□

Lemma 2.7. *Assume that z^0 and λ^0 are strictly feasible, $\mu^0 = \dfrac{(z^0)^T \lambda^0}{n}$ and $\Gamma(z^0\lambda^0; \mu) < \dfrac{1}{2}$. Moreover let z^k and λ^k be the vectors obtained after k iterations. Then the inequalities $\left(z^k\right)^T \lambda^k \leq \varepsilon$ is satisfied for*

$$k \geq \left[\frac{1}{\beta} \log \frac{(z^0)^T \lambda^0}{\varepsilon}\right].$$

Proof. We have after k iterations $\mu^k = (1-\beta)^k \mu^0$.
Using lemma 2.6, it follows that

$$\left(z^k\right)^T \lambda^k \leq n\mu^k = (1-\beta)^k \left(z^0\right)^T \lambda^0.$$

Hence the inequalities $\left(z^k\right)^T \lambda^k \leq \varepsilon$ holds if $(1-\beta)^k \left(z^0\right)^T \lambda^0 \leq \varepsilon$. By taking logarithms of both sides, we obtain

$$k \log(1-\beta) + \log \left(z^0\right)^T \lambda^0 \leq \log \varepsilon.$$

The estimate for the logarithm function

$$-\log(1-\beta) \geq \beta \quad \text{for all } 0 \leq \beta < 1.$$

Therefore, the inequalities $\left(z^k\right)^T \lambda^k \leq \varepsilon$ is fulfilled if $k \geq \dfrac{1}{\beta} \log \dfrac{(z^0)^T \lambda^0}{\varepsilon}$. □

Theorem 2.1. *Suppose that the pair (z^0, λ^0) is strictly feasible and let $\mu^0 = \dfrac{(z^0)^T \lambda^0}{n}$. If $\beta = \dfrac{1}{2\sqrt{n}}$, then the algorithm (2.1) requires at most*

$$\left[2\sqrt{n} \log \frac{(z^0)^T \lambda^0}{\varepsilon}\right]$$

iterations. For the resulting vectors we have $\left(z^k\right)^T \lambda^k \leq \varepsilon$.

Proof. The proof follows directly from lemma (2.7) by putting $\beta = \dfrac{1}{2\sqrt{n}}$ and $\mu^0 = \dfrac{(z^0)^T \lambda^0}{n}$. □

2.4 Numerical implementation

In this section, we present some numerical results for the algorithm 2.1 applied to a set of problems that have been previously studied in the literature. All codes were written in **Matlab R2016a** and run on a personal computer **RTM i5-8265U** with **CPU**

1.8 GHz and **8.00 GB RAM** memory. Here, we display the following notations: "Iter" and "CPU" to denote the number of iterations and the elapsed time produced by our algorithm 2.1 . To improve the numerical results, we have relaxed the parameter by choosing a suitable value of μ^0. We also use $n\mu \geq 10^{-6}$ as a stopping criterion for all examples.

2.4.1 Examples

Example 2.1. *[76]*
Consider the two-person game, where the problems of player 1 and player 2 are defined by:

$$P_1(x_2): \quad \underset{x_1}{\text{minimize}} \; x_1^2 - x_1x_2 - x_1$$
$$\text{subject to} \quad x_1 \geq 0,$$
$$x_1 + x_2 \leq 1,$$

and

$$P_2(x_1): \quad \underset{x_2}{\text{minimize}} \; x_2^2 - \frac{1}{2}x_1x_2 - 2x_2$$
$$\text{subject to} \quad x_2 \geq 0,$$
$$x_1 + x_2 \leq 1,$$

respectively.
The set of generalized Nash equilibrium consists of infinitely many vectors

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t \\ 1-t \end{pmatrix}, \quad 0 \leq t \leq \frac{2}{3}.$$

On the other hand, the corresponding mapping $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is given by:

$$F(x) = Mx + q = \begin{pmatrix} 2x_1 - x_2 - 1 \\ \frac{1}{2}x_1 + 2x_2 - 2 \end{pmatrix}, \quad C = \left\{x \in \mathbb{R}^2_+ : x_1 + x_2 \leq 1\right\},$$

and the solution of MAVI is uniquely given by:

$$x^* = \left(\frac{4}{11}, \frac{7}{11} \right)^T \approx \left(0.3636, 0.6363 \right)^T.$$

n	$Iter$	CPU	$\|x^* - x\|$	$\Gamma(z\lambda;\mu)$
2	36	0.045416	$3.1248e-04$	0.0233

Table 2.1: Example 2.1

Example 2.2. ***(Harker's example)***

This problem is taken from [51]. There are two players and they solve the following problems

$$P_1(x_2): \quad \underset{x_1}{\text{minimize}} \;\; x_1^2 + \frac{8}{3}x_1x_2 - 34x_1$$
$$\text{subject to} \quad 0 \leq x_1 \leq 10,$$
$$x_1 + x_2 \leq 15,$$

and

$$P_2(x_1): \quad \underset{x_2}{\text{minimize}} \;\; x_2^2 - \frac{5}{4}x_1x_2 - 24.25x_2$$
$$\text{subject to} \quad 0 \leq x_2 \leq 10,$$
$$x_1 + x_2 \leq 15,$$

respectively.

The set of generalized Nash equilibrium consists of infinitely many vectors

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t \\ 15 - t \end{pmatrix}, \quad 9 \leq t \leq 10$$

The corresponding mapping $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is represented as:

$$F(x) = Mx+q = \begin{pmatrix} 2x_1 + \frac{8}{3}x_2 - 34 \\ -\frac{5}{4}x_1 + 2x_2 - 24.25 \end{pmatrix}, \quad C = \{x \in \mathbb{R}^2_+ : x_1+x_2 \leq 15, 0 \leq x_i \leq 10, i = 1, 2\}.$$

Since F is strongly monotone on $\mathbb{R}^2$, MAVI has a unique solution, which is given by:

$$x^* = \begin{pmatrix} 5 \,, & 9 \end{pmatrix}^T.$$

n	$Iter$	CPU	$\|x^* - x\|$	$\Gamma(z\lambda;\mu)$
2	50	0.091028	0.0021	0.0330

Table 2.2: Example 2.2

Example 2.3. ***(River basin pollution game)***

Consider the three person river basin pollution game, where the problem of player $i \in \{1, 2, 3\}$ is defined by

$$P_i(x_{-1}): \quad \underset{x_i}{\text{minimize}} \;\; (\alpha_i x_i + 0.01(x_1 + x_2 + x_3) - \chi_i)x_i$$
$$\text{subject to} \quad x_i \geq 0,$$

$$3.25x_1 + 1.25x_2 + 4.125x_3 \leq 100,$$
$$2.29x_1 + 1.562x_2 + 2.8125x_3 \leq 100,$$

with $\alpha_1 = 0.01,\ \alpha_2 = 0.05,\ \alpha_3 = 0.01,\ \chi_1 = 2.9,\ \chi_2 = 2.88,\ \chi_3 = 2.85.$
The corresponding mapping $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *and* $C \subseteq \mathbb{R}^3$ *are given by:*

$$F(x) = Mx + q = \begin{pmatrix} 0.04x_1 + 0.01x_2 + 0.01x_3 - 2.9 \\ 0.01x_1 + 0.12x_2 + 0.01x_3 - 2.88 \\ 0.01x_1 + 0.01x_2 + 0.04x_3 - 2.8 \end{pmatrix},$$

$$C = \left\{ x \in \mathbb{R}^3_+ : \begin{pmatrix} 3.25 & 1.25 & 4.125 \\ 2.2915 & 1.5625 & 2.8125 \end{pmatrix} x \leq \begin{pmatrix} 100 \\ 100 \end{pmatrix} \right\}.$$

Since F *is strongly monotone on* $\mathbb{R}^3$*, MAVI has a unique solution, which is given by*

$$x^* = \left(\frac{4673}{221},\ \frac{5754}{359},\ \frac{567}{208} \right)^T \approx \left(21.14,\ 16.03,\ 2.73 \right)^T.$$

n	$Iter$	CPU	$\|x^* - x\|$	$\Gamma(z\lambda;\mu)$
3	85	0.020751	$2.1558e-06$	0.0203

Table 2.3: Example 2.3

Example 2.4.
Consider the following MAVI problem where its data are given by:

$$M = \begin{pmatrix} 6 & 6 & 4 & 3 & 2 \\ 8 & 21 & 14 & 10 & 12 \\ 4 & 14 & 13 & 5 & 9 \\ 4 & 10 & 5 & 6 & 5 \\ 3 & 12 & 8 & 4 & 10 \end{pmatrix},\ q = \begin{pmatrix} -20.5 \\ -64.5 \\ -44.5 \\ -29.5 \\ -36.5 \end{pmatrix},\ A = \begin{pmatrix} -2 & -1 & -1 & 0 & 0 \\ -1 & -2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 \end{pmatrix},\ b = \begin{pmatrix} 8 \\ 7 \\ 3 \end{pmatrix}.$$

For this example, the initial point for the algorithm is taken as

$$\begin{aligned} x^0 &= \left(-10.7572,\ -172.9028,\ 45.1953,\ 172.8340,\ 106.1104 \right)^T, \\ z^0 &= \left(101.4658,\ 8.0245,\ 29.6031 \right)^T, \\ \lambda^0 &= \left(141.2219,\ 176.7288,\ 63.7924 \right)^T. \end{aligned}$$

And the optimal solution of MAVI is:

$$\begin{aligned} x^* &= \left(-0.4054,\ -11.2331,\ 4.0438,\ 13.6780,\ 8.2331 \right)^T, \\ z^* &= \left(8.3277,\ 0.0000,\ 3.1201 \right)^T, \\ \lambda^* &= \left(0.0000,\ 2.1935,\ 0.0000 \right)^T. \end{aligned}$$

n	$Iter$	CPU	$\Gamma(z\lambda;\mu)$
5	70	0.044684	0.0248

Table 2.4: Example 2.4

Example 2.5.
Consider the following MAVI problem where its data is given by:

$$M = \begin{pmatrix} 8 & 9 & 13 & 13 & 5 & 11 & 9 & 10 \\ 8 & 10 & 15 & 15 & 7 & 12 & 10 & 12 \\ 13 & 15 & 26 & 26 & 10 & 20 & 13 & 21 \\ 13 & 15 & 26 & 26 & 10 & 20 & 12 & 20 \\ 5 & 7 & 10 & 10 & 5 & 9 & 5 & 8 \\ 11 & 12 & 20 & 20 & 9 & 19 & 13 & 15 \\ 9 & 10 & 13 & 12 & 5 & 13 & 16 & 13 \\ 10 & 12 & 21 & 20 & 8 & 15 & 13 & 22 \end{pmatrix}, \; q = \begin{pmatrix} -8.265 \\ -9.3033 \\ -14.835 \\ -14.4633 \\ -5.995 \\ -12.4133 \\ -10.015 \\ -12.3033 \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & -2 & 1 & -2 & 0 & 2 & 1 & 0 \\ 3 & 2 & -5 & 1 & -1 & 5 & 4 & 1 \\ 1 & 1 & 2 & 1 & 2 & 2 & 1 & 1 \\ -2 & 5 & 1 & 2 & 3 & 1 & 2 & -8 \end{pmatrix}, \; b = \begin{pmatrix} 5 \\ 8 \\ 4 \\ 2 \end{pmatrix}.$$

For this example, the initial point for the algorithm is taken as

$$\begin{aligned} x^0 &= \left(\; 39.1549 \, , \;\; -4.5134 \, , \;\; -2.0536 \, , \;\; -52.7869 \, , \;\; 67.6146 \, , \;\; 38.0347 \, , \;\; 0.8625 \, , \;\; -20.0662 \; \right)^T, \\ z^0 &= \left(\; 33.8051 \, , \;\; 24.7638 \, , \;\; 48.2978 \, , \;\; 20.7058 \; \right)^T, \\ \lambda^0 &= \left(\; 223.6338 \, , \;\; 163.8616 \, , \;\; 165.8424 \, , \;\; 192.6292 \; \right)^T. \end{aligned}$$

The optimal solution of MAVI is:

$$\begin{aligned} x^* &= \left(\; 1.3432 \, , \;\; -0.4569 \, , \;\; -0.3419 \, , \;\; -0.5476 \, , \;\; 1.3574 \, , \;\; 0.8398 \, , \;\; 0.3101 \, , \;\; -0.3595 \; \right)^T, \\ z^* &= \left(\; 0.9780 \, , \;\; 0.5424 \, , \;\; 0.3779 \, , \;\; 0.4783 \; \right)^T, \\ \lambda^* &= 1.0e-06 * \left(\; 0.1214 \, , \;\; 0.2189 \, , \;\; 0.3143 \, , \;\; 0.2483 \; \right)^T. \end{aligned}$$

n	$Iter$	CPU	$\Gamma(z\lambda;\mu)$
8	123	0.050321	0.0094

Table 2.5: Example 2.5

Example 2.6. *(Large-scale example)*
Let us consider the data of MAVI, where $n = 2m$ *and*

$$M = \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 5 & 6 & \cdots & 6 \\ 2 & 6 & 9 & \cdots & 10 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 6 & 10 & \cdots & 4n-3 \end{pmatrix}, \quad q = -Me + e,$$

$$A = \begin{pmatrix} I & I \end{pmatrix}, \quad b = \begin{pmatrix} 2, & 2, & 2, & \cdots, & 2 \end{pmatrix}^T.$$

n	$Iter$	CPU	$\Gamma(z\lambda;\mu)$
10	136	0.111560	0.0083
50	344	0.134468	0.0034
100	424	0.415295	0.0023
500	2660	177.335594	0.0010
700	3214	527.254652	0.0008

Table 2.6: Example 2.6

2.5 Conclusion

This chapter presented a feasible full-Newton step interior-point algorithm designed to solve MAVI. The method leverages AET, specifically adopting Darvay's choice $\psi(t) = \sqrt{t}$, to reformulate the complementarity conditions and obtain a modified Newton search direction. This approach ensures strict feasibility and enables full-Newton steps without line searches, enhancing computational efficiency. Theoretically, the algorithm achieves quadratic local convergence under the proximity condition $\Gamma < 1$ and guarantees polynomial complexity of $\left[2\sqrt{n}\log\frac{(z^0)^T\lambda^0}{\varepsilon}\right]$ iterations, as rigorously proven through key lemmas and theorem 2.1.
Numerical experiments validate the algorithm's efficiency across diverse test problems, including Nash equilibrium games (Harker's example and river basin pollution models) and large-scale problems (up to $n = 700$).

CHAPTER 3

DCA algorithm for solving non-monotone affine variational inequalities problem

3.1 Introduction

This chapter develops a novel approach based on the difference of two convex functions algorithm DCA to solve a specific problem, known as the non-monotone affine variational inequalities problem (NMAVI) through its reformulation as a DC optimization problem.

3.2 Position of the non-monotone affine variational inequalities problem

Recall from subsection 1.3 that the VIP consists to

$$\begin{cases} \text{Find } x \in C, \quad \text{such that} \\ \langle F(x), y - x \rangle \geq 0, \ \text{for all } y \in C, \end{cases} \tag{3.1}$$

The non-monotone affine variational inequalities problem denoted by (NMAVI) is defined as:

$$\begin{cases} \text{Find } x \in C, \quad \text{such that} \\ \langle Mx + q, y - x \rangle \geq 0, \ \text{for all } y \in C, \end{cases} \tag{3.2}$$

where C is a nonempty convex polyheder representing the constraint set defined as:

$$C = \{x \in \mathbb{R}^n : Ax \geq b\},$$

with $M \in \mathbb{R}^{n \times n}$ is symmetric not necessarily positive semidefinite matrix, $A \in \mathbb{R}^{m \times n}$, $q \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ representing the data of the problem.

3.2.1 Fundamental existence results for solutions of NMAVI

Theorem 3.1. *[36]*
If C is a bounded polyheder, a solution of NMAVI exists for any matrix M and the vector q.

Theorem 3.2. *[89]*
Let C be an unbounded, nonempty polyheder, with recession cone:

$$C^{\infty} = \{d \in \mathbb{R}^n : Ad \leq 0\}.$$

The NMAVI admits a solution if there exists a vector $x^0 \in C$ such that:

$$\lim_{\|x\| \to \infty} \frac{\langle Mx + q, x - x^0 \rangle}{\|x\|} \geq 0, \ \forall x \in C$$

and for every nonzero $d \in C^{\infty}$ with $\langle Md, d \rangle = 0$, it holds that

$$\langle q, d \rangle \geq 0.$$

3.3 Formulation of NMAVI as an optimzation problem

In what follows, we present a formulation of NMAVI that has been discussed in [50]. It consists that when the matrix M is symmetric, the NMAVI problem is equivalent to the first-order conditions of the following quadratic program:

$$\begin{cases} \min f(x) = \frac{1}{2} x^T M x + q^T x \\ x \in C = \{x \in \mathbb{R}^n : Ax \geq b\} \end{cases} \tag{3.3}$$

Remark 3.1. *To solve the obtained quadratic program (generally non-convex), we use an algorithm called DCA (Difference of Two Convex Functions Algorithm), first proposed by T. Pham Dinh [85] in 1985. We present a reminder of this algorithm in the subsection below.*

3.3.1 General principle of DC programming and DCA

In this section, we briefly present the basic concepts of DC programming (difference of two convex functions) and DCA. For more details, can refer to [16, 64, 65, 80, 86].

DC function

Definition 3.1. *Let $C \neq \varnothing$ be a convex subset of $\mathbb{R}^n$ and $f : C \to \mathbb{R} \cup \{+\infty\}$ a function. The function f is said to be DC on C if it can be written as the difference of two convex functions g and h on C as follows:*

$$f(x) = g(x) - h(x), \forall x \in C.$$

We say that $g-h$ is a DC decomposition of f, and that g and h are DC components of f. If f is a DC function on a convex set C and admits a DC decomposition as $f = g-h$ then for any finite convex function ϕ on C, $(g+\phi)-(h+\phi)$ also provides a DC decomposition of f. Thus, any DC function admits an infinity of DC decompositions.
The set of DC functions on C is denoted by DC(C).
DC(C) is large enough to include most of practical the objective functions and is stable under standard optimization operations.

DC programming

Let $\Gamma_0(\mathbb{R}^n)$ be the set of all lower semicontinuous proper convex functions on $\mathbb{R}^n$.

Definition 3.2. *We call a DC program any problem of optimization of the form:*

$$\inf\{f(x) = g(x) - h(x) : x \in \mathbb{R}^n\}, \qquad (\mathcal{P}_{dc})$$

where g, $h \in \Gamma_0(\mathbb{R}^n)$. $(\mathcal{P}_{dc})$ is an unconstrained optimization problem.
An optimization problem with constraints (nonempty closed convex set C) of the form:

$$\inf\{f(x) = g(x) - h(x) : x \in C\},$$

is equivalent to $(\mathcal{P}_{dc})$ via the addition of the indicator function of C to g.

$$\inf\{F(x) = \varphi(x) - h(x) : x \in \mathbb{R}^n\}, \qquad (\mathcal{P}_{dc})$$

whith

$$\varphi(x) = g(x) + \mathcal{X}_C,$$

and

$$\mathcal{X}_C = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & otherwise. \end{cases}$$

DC Duality

Definition 3.3. *[65]*
Consider the DC program:

$$\alpha = \inf\{g(x) - h(x) : x \in \mathbb{R}^n\}, \qquad (\mathcal{P}_{dc})$$

where g, $h \in \Gamma_0(\mathbb{R}^n)$. According to the convention $(+\infty) - (+\infty) = (+\infty)$, the dual problem of $(\mathcal{P}_{dc})$ is therefore:

$$\alpha = \inf\{h^*(y) - g^*(y) : y \in \mathbb{R}^n\}, \qquad (\mathcal{D}_{dc})$$

where g^*, $h^* \in \Gamma_0(\mathbb{R}^n)$ are respectively the convex conjugate functions of g and h.

$(\mathcal{D}_{dc})$ is also a DC program. In addition, $(\mathcal{P}_{dc})$ and $(\mathcal{D}_{dc})$ have the same optimal value and we can observe the perfect symmetry between these two problems: the dual of $(\mathcal{D}_{dc})$ is exactly $(\mathcal{P}_{dc})$.
The following results give some properties concerning the solutions of $(\mathcal{P}_{dc})$ and $(\mathcal{D}_{dc})$.

Theorem 3.3. *Let* $g,\ h \in \Gamma_0(\mathbb{R}^n)$.
(i) x^* *is a global solution of* $(\mathcal{P}_{dc})$ *if and only if :*

$$\alpha = (g-h)(x^*) \leq (h^* - g^*)(y), \forall y \in \mathbb{R}^n.$$

(ii) y^* *is a global solution of* $(\mathcal{D}_{dc})$ *if and only if:*

$$\alpha = (h^* - g^*)(y^*) \leq (g-h)(x), \forall x \in \mathbb{R}^n.$$

Theorem 3.4. *Let* $g,\ h \in \Gamma_0(\mathbb{R}^n)$
(i) $\inf\{g(x) - h(x) : x \in dom(g)\} = \inf\{h^*(y) - g^*(y) : y \in dom(h^*)\}$.
(ii) If y^* *is a minimizer of* $h^* - g^*$*, then any* $x^* \in \partial g^*(y^*)$ *is a minimizer of* $g - h$*. Conversely, if* x^* *is a minimizer of* $g - h$*, then any* $y^* \in \partial h(x^*)$ *is a minimizer of* $h^* - g^*$.

This theorem shows that the solution of one of the two problems $(\mathcal{P}_{dc})$ and $(\mathcal{D}_{dc})$ implies that of the other.

Remarks 3.1.
It is clear that DCA applies to convex functions g *and* h*, and not to the function* f *itself. We can thus see how the DCA mechanism works for non-differentiable DC programs (*f *is a non-differentiable DC function). And since a DC function admits an infinity of DC decomposition, there will be as many DCAs. The choice of the appropriate DC is crucial because it determines the essential qualities (speed, robustness, globality of the calculated solutions) of the resulting DCA. Theoretically, the problem of optimal DC decompositions remains to be defined and studied.*
In practice, we are looking for DC decompositions that are well adapted to the specific structure of the problem being treated so that the two sequences $\left\{x^k\right\}$ *and* $\left\{y^k\right\}$*are obtained at a lower cost in computing time, if they are not explicit. Here, perhaps more than elsewhere, reformulation techniques are omnipresent. It is important to note that with appropriate DC decompositions, DCA makes it possible to find, as special cases, most of the standard methods in convex, non-convex programming.*
On the other hand, a convex program is a DC program for which DCA converges to a solution (local which is also global): in this way DCA makes it possible to build an infinity of algorithms for convex programming, which could be more efficient than existing methods.

It is important to note that with appropriate DC decompositions, DCA makes it possible to find, as special cases, most of the standard methods in convex, non-convex programming.
To our knowledge, DCA is currently among the few algorithms in non-convex programming algorithms capable of dealing with very large-scale problems and remains the one of the most efficient algorithm for non-differentiable non-convex programs.

Optimality conditions in DC programming

We know that in convex optimization, $x^* \in \mathbb{R}^n$ minimizes a convex function $f \in \Gamma_0(\mathbb{R}^n)$ if and only if :

$$0 \in \partial f(x^*).$$

In DC programming, the necessary and sufficient condition for global optimality is formulated using the ε-subdifferentials of g and h.

Theorem 3.5. *[65]*
Let $g, h \in \Gamma_0(\mathbb{R}^n)$ and $x^* \in \mathbb{R}^n$.
x^* is a global minimizer of $g - h$ if and only if:

$$\partial_\varepsilon h(x^*) \subset \partial_\varepsilon g(x^*), \quad \forall \varepsilon > 0.$$

Definition 3.4. *(polyhedral function) [65]*
Let $f \in \Gamma_0(\mathbb{R}^n)$, then f is polyhedral convex if $dom(f)$ is nonempty plyedral convex and

$$f(x) = sup\left\{\left\langle a^i, x\right\rangle - b^i, i = 1 : m\right\} + \mathcal{X}_{dom(f)}, \ \forall x \in \mathbb{R}^n,$$

where $a^i \in \mathbb{R}^n, b^i \in \mathbb{R}, \forall i = 1 : m$.

Proposition 3.1. *[65]*
If $f \in \Gamma_0(\mathbb{R}^n)$ is polyhedral convex, then:
1- $\partial f(x)$ is nonempty convex polyheder for all $x \in dom(f)$.
2- f^* is also polyhedral convex and $dom(\partial f) = dom(f)$.
Moreover, if f is finite everywhere then:

$$dom(f^*) = conv\left\{a^i, i = 1 : m\right\},$$

and

$$f^*(y) = \min\left\{\sum_{i=1}^m \lambda_i b^i : y = \sum_{i=1}^m \lambda_i a^i, \ \sum_{i=1}^m \lambda_i = 1, \lambda_i > 0, \forall i = 1 : m\right\}.$$

Definition 3.5. *(polyhedral DC program) [65]*
Polyhedral DC programs are DC programs in which one of the DC components (g or h) is polyhedral convex.

Definition 3.6. *(Local minimizer) [65]*
Let $g,\ h \in \Gamma_0(\mathbb{R}^n)$. A point $x^* \in dom(g) \cap dom(h)$ is a local minimizer of $g - h$ if there exists a neighborhood $\mathcal{V}(x^*)$ of x^* such that:

$$g(x^*) - h(x^*) \leq g(x) - h(x),\ \forall x \in \mathcal{V}(x^*) \cap dom(g).$$

Definition 3.7. *(Critical point) [65]*
A point $x^* \in \mathbb{R}^n$ is said to be a critical point or a generalized KKT point of $g - h$ if

$$\partial g(x^*) \cap \partial h(x^*) \neq \varnothing.$$

We note:

$$\mathcal{P} = \{x \in \mathbb{R}^n : \partial h(x) \subset \partial g(x)\} \text{ and } \mathcal{D} = \{y \in \mathbb{R}^n : \partial g^*(y) \subset \partial h^*(y)\},$$

the set of global solutions of the problem $(\mathcal{P}_{dc})$ and $(\mathcal{D}_{dc})$ respectively.

Theorem 3.6. *(Necessary condition of local optimality)[66]*
If x^* is a local minimizer of $g - h$, then $x^* \in \mathcal{P}$. This condition is sufficient if h is polyhedral. Moreover, if f is locally convex at x^*, in particular if h is polyhedral and differentiable at x^*, then x^* is a local solution.

Theorem 3.7. *(Sufficient condition of local optimality) [66]*
If $x^* \in dom(g) \cap dom(h)$, admits a neighborhood $\mathcal{V}(x^*)$ such that :

$$\partial h(x^*) \cap \partial g(x^*) \neq \varnothing,\ \forall x \in \mathcal{V}(x^*) \cap dom(g),$$

then x^* is a local minimizer of $g - h$.

Theorem 3.8. *[66]*
(i) $\partial h(x^*) \subset \partial g(x^*),\ \forall x^* \in \mathcal{P}$ and $\partial g^*(y^*) \subset \partial h^*(y^*),\ \forall y^* \in \mathcal{D}$.
(ii) (Transport of global minimizers)

$$\bigcup_{x^* \in \mathcal{P}} \partial h(x^*) \subset \mathcal{D} \subset dom(h^*).$$

The first inclusion becomes equality if g^* is subdifferentiable on $\mathcal{D}$ (in particular if $\mathcal{D} \subset ri(dom(g^*))$ or if g^* is subdifferentiable on $dom(h^*)$).
In this case:

$$\mathcal{D} \subset \left(dom(\partial g^*) \cap\ dom(\partial h^*)\right).$$

By duality

$$\bigcup_{y^* \in \mathcal{D}} \partial g^*(y^*) \subset \mathcal{P} \subset dom(g).$$

The first inclusion becomes equality if h is subdifferentiable in $\mathcal{P}$ (in particular if $\mathcal{P} \subset ri(dom(h))$ or if h is subdifferentiable on $dom(g)$).
In this case,

$$\mathcal{P} \subset (dom(\partial g) \cap dom(\partial h)) .$$

(iii) (Transport of local minimizers) Let $x^* \in dom(\partial h)$ be a local minimizer of $g - h$ and $y^* \in \partial h(x^*)$. Suppose that $\mathcal{V}(x^*)$ is a neighborhood of x^* satisfying:

$$g(x) - h(x) \geq g(x^*) - h(x^*), \ \forall x \in \mathcal{V}(x^*) \cap dom(g).$$

If $x^* \in int(dom(h)), \ y^* \in int(dom(g^*))$ and $\partial g^*(y^*) \subset \mathcal{V}(x^*)$, so y^* is a local minimizer of $h^* - g^*$.

DC Algotithms (DCA)

Based on the local optimality condition and DC duality, the DCA consists of the construction of two sequences $\left\{x^k\right\}$ and $\left\{y^k\right\}$, candidates to be optimal solutions of the primal and dual DC programs, respectively, in such a way that the sequences $\left\{g(x^k) - h(x^k)\right\}$ and $\left\{h^*(y^k) - g^*(y^k)\right\}$ are decreasing and $\left\{x^k\right\}$ $\left(\text{resp. } \left\{y^k\right\}\right)$ converges to a feasible primal solution x^* (resp. a feasible dual solution y^*) satisfying the local optimality conditions and

$$x^* \in \partial g^*(y^*), \ \ y^* \in \partial h(x^*).$$

The two sequences $\left\{x^k\right\}$ and $\left\{y^k\right\}$ are determined in such a way that:
$x^{k+1}, \ (k \geq 0)$ is a solution of the convex problem :

$$\min \left\{g(x) - \left(h(x^k) + \left\langle x - x^k, y^k \right\rangle\right) : x \in \mathbb{R}^n\right\}. \qquad (\mathcal{P}_k)$$

$y^{k+1}, \ (k \geq 0)$ is a solution of the convex problem:

$$\min \left\{h^*(y) - \left(g^*(y^k) + \left\langle y - y^k, x^{k+1} \right\rangle\right) : y \in \mathbb{R}^n\right\}. \qquad (\mathcal{D}_k)$$

The interpretation of DCA is simple: at each iteration k, the second component of the primal problem $(\mathcal{P}_k)$ (resp. dual problem $(\mathcal{D}_k)$) is replaced by its affine minorant :

$$h_k(x) = h(x^k) + \left\langle x - x^k, y^k \right\rangle,$$

in the vicinity of x^k, defined by a sub-gradient y^k from h to x^k (resp. $g_k^*(y) = g^*(y^k) + \left\langle y - y^k, \ x^{k+1} \right\rangle$ in the vicinity of y^k, defined by a sub-gradient x^{k+1} from g^* to y^k). The set of solutions of the convex problem obtained $(\mathcal{P}_k)$ (resp. $(\mathcal{D}_k)$) is nothing other than $\partial g^*(y^k)$ (resp. $\partial h(x^{k+1})$),

$$x^{k+1} \in \partial g^*(y^k), \ \ y^{k+1} \in \partial h(x^{k+1}).$$

The process is then repeated until convergence. In practice, one of the two conditions is often used:

$$\|x^{k+1} - x^k\| \leq \varepsilon \quad \text{or} \quad \|(g-h)(x^{k+1}) - (g-h)(x^k)\| \leq \varepsilon$$

Algorithm 3.1 Generic DCA scheme

1: **Begin**
2: **Initialize**
3: Let $x^0 \in dom(g)$ be a best guess;
4: $k = 0$;
5: **Repeat**
6: Compute $y^k \in \partial h(x^k)$;
7: Compute $x^{k+1} \in \partial g^*(y^k)$;
8: $k := k + 1$
9: **Until the convergence**.
10: **End**

The convergence results of DCA are mentioned in the references [65, 66, 87, 88].

3.3.2 Choice of decomposition

For a general quadratic program that is not necessarily convex, several decompositions are proposed in [63, 66, 67], which is why it is necessary to choose the best decomposition that accelerates the convergence of the algorithm.
In our work, we propose the following DC decomposition :

$$f(x) = g(x) - h(x),$$

where:

$$\begin{cases} g(x) = \mathcal{X}_C(x) + \dfrac{\rho}{2}\|x\|^2 + \langle q, x\rangle, \\ h(x) = \dfrac{\rho}{2}\|x\|^2 - \dfrac{1}{2}\langle Mx, x\rangle, \end{cases} \quad \text{and } \rho \geq \lambda_{\max},$$

where, $\lambda_{\max}$ represents the largest eigenvalue of the matrix M, while $\mathcal{X}_C$ represents the indicator function associated with the set C. It is evident that both g and h are convex functions, rendering problem (3.1) a DC program in the standard form:

$$\begin{cases} \min\left(g(x) - h(x)\right), \\ x \in \mathbb{R}^n. \end{cases} \tag{3.4}$$

Following the generic DCA scheme, along with its properties and theoretical foundation detailed in [50, 63, 65, 68], at each step $k \geq 0$, the computation of y^k is performed as

$$y^k = \left(\nabla h\left(x^k\right)\right)^T = (\rho I - M)\, x^k, \tag{3.5}$$

and subsequently, the unique solution denoted as x^{k+1} is determined for the convex minimization problem:

$$\begin{cases} \min \left\{ g(x) - \left(h\left(x^k\right) + \left\langle x - x^k, y^k \right\rangle \right) \right\}, \\ x \in \mathbb{R}^n, \end{cases}$$

so, at each step $k \geq 0$, the point is computed as follows:

$$x^{k+1} = Proj_C \left(x^k - \frac{1}{\rho} \left(Mx^k + q \right) \right).$$

This is the unique solution corresponding to the problem (3.4). The latter can be equivalently expressed as:

$$\min \left\{ \left\| x - \frac{1}{\rho} \left(y^k - q \right) \right\|^2, \ Ax \geq b \right\}, \tag{3.6}$$

with $y^k = (\rho I - M) x^k$.
The primary operation at each iteration of the algorithm involves solving a quadratic program. The application of the DCA to (3.4) can be outlined as follows.

Description of the DCA algorithm with our decomposition

Algorithm 3.2 DCA for NMAVI

1: **Begin**
2: **Step 0**. Let $\varepsilon > 0$;
3: Let $x^0 \in C$ be a starting point;
4: Set $k = 0$;
5: **Step 1**. Compute $y^k = (\rho I - M) x^k$;
6: **Step 2**. Compute x^{k+1} as an optimal solution of the following optimization program: $\min \left\{ \left\| x - \frac{1}{\rho} \left(y^k - q \right) \right\|^2, \ Ax \geq b \right\}$;
7: **if** $\left\| x^{k+1} - x^k \right\| \leq \varepsilon$ **then** stop ;
8: **else** Set $k := k + 1$ and go to **Step 1**.
9: **end if**
10: **end**

Remark 3.2. *The main operation at each iteration of the algorithm consists of solving quadratic program.*

3.3.3 Convergence of the algorithm

Theorem 3.9. *[66]*
The DCA generates a sequence $\left\{x^k\right\}$ *in* C *such that the sequence* $\left\{f(x^k)\right\}$ *is decreasing.*

Theorem 3.10. *[66]*
If the optimal value of problem (3.4) is finite, then the sequence $\left\{x^k\right\}$ *converges to* x^* *satisfying the necessary local optimality condition* $\partial h\left(x^*\right) \subset \partial g\left(x^*\right)$.

3.4 Numerical implementation

To provide a better understanding of the algorithm 3.2 performance, we implemented it in **Matlab R2016a** and run on a personal computer **RTM i5-8265U** with **CPU 1.8 GHz** and **8.00 G RAM** memory, and we applied it to a set of examples that have been previously studied in the literature. For each example tested, we gave the matrices M and A, the vectors q and b, $\lambda_{\max}$, the set of exact solutions and two initial points x_1^0 feasible and x_2^0 not feasible. These examples were tested with various values of ρ to see its influence on the behavior of the DCA algorithm. In our implementation, we set the tolerance parameter $\varepsilon = 10^{-6}$.
Note that to find x^{k+1} the solution of the convex program in step 2 of the DCA algorithm, we used the solver (quadprog) which is a procedure from the Matlab optimization toolbox designed to solve convex quadratic programs.
Here, we display the following notations: "Iter" and "CPU" to denote the number of iterations and the elapsed time produced by the algorithm 3.2.

3.4.1 Examples

Example 3.1. *[68]*
Consider the following NMAVI problem, where its data is given by:

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix},\ q = \begin{pmatrix} -1 \\ 0 \end{pmatrix},\ A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix},\ b = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}.$$

The optimal solution set is $\left\{(0,2)^T, (1,0)^T\right\}$.
In the implementation, we take $\lambda_{\max} = 1$ *and the starting point* $x^0 \in \left\{(0.5,0.5)^T, (-10,-10)^T\right\}$.
The numerical results obtained by the algorithm are summarized in the Table 3.1.

Example 3.2. *[68]*
The data of the NMAVI is given by:

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ q = \begin{pmatrix} -1 \\ 0 \end{pmatrix},\ A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$$

	x_1^0			x_2^0		
ρ	$Iter$	CPU	$\|x^* - x\|$	$Iter$	CPU	$\|x^* - x\|$
0.5	3	0.0155	$4.7662e-12$	12	0.0630	$4.7662e-12$
1	3	0.0178	$4.0169e-08$	13	0.0701	$4.0169e-08$
1.5	4	0.0289	$8.0071e-22$	16	0.0773	$8.2056e-22$
2	4	0.0184	$1.6405e-08$	19	0.0724	$1.6405e-08$
4	6	0.0525	$7.6866e-10$	30	0.2047	$7.6866e-10$
10	12	0.0853	$2.8620e-08$	60	0.3669	$2.8620e-08$
100	103	0.5227	$4.1437e-07$	476	3.5156	$4.1445e-07$

Table 3.1: Example 3.1

The optimal solution set is $\left\{(2,-1)^T,(2,1)^T,(2,0)^T\right\}$.
In the implementation, we take $\lambda_{\max}=1$ *and the starting point* $x^0 \in \left\{(4,0)^T,(-0.5,-10)^T\right\}$.
The numerical results obtained by the algorithm are summarized in the Table 3.2.

	x_1^0			x_2^0		
ρ	$Iter$	CPU	$\|x^* - x\|$	$Iter$	CPU	$\|x^* - x\|$
0.01	2	0.0224	$3.2274e-10$	3	0.0278	0
0.1	2	0.0246	$8.5649e-09$	3	0.0583	$4.1880e-10$
0.5	2	0.0095	$8.0150e-20$	3	0.0199	$1.3846e-09$
1	2	0.0093	$1.1056e-10$	5	0.0390	0
2	3	0.1899	$4.0237e-08$	8	0.0457	$0.0000e-20$
10	12	0.1395	$1.1931e-07$	26	0.1926	$3.4204e-08$
100	111	0.6902	$7.5119e-07$	210	0.7789	$2.1271e-08$

Table 3.2: Example 3.2

Example 3.3. *[40]*
Consider the following NMAVI problem, with its data:

$$M=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ q=\begin{pmatrix} 6.5\\ -1\\ -2\\ -3\\ -2\\ -1 \end{pmatrix},$$

$$A = \begin{pmatrix} -1 & -2 & -8 & -1 & -3 & -5 \\ 8 & 4 & 2 & -2 & -4 & 1 \\ -2 & -0.5 & -0.2 & 3 & 1 & 4 \\ -0.2 & -2 & -0.1 & 4 & -2 & -2 \\ 0.1 & 0.5 & -2 & -5 & 5 & -3 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; b = \begin{pmatrix} -16 \\ 1 \\ -24 \\ -12 \\ -3 \\ -10 \\ -10 \\ -10 \\ -10 \\ -10 \\ -10 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The optimal solution set is $\left\{ \begin{pmatrix} 0\,, & 3.6403\,, & 0\,, & 2.9029\,, & 1.9388\,, & 0 \end{pmatrix}^T \right\}$.
In the implementation, we take $\lambda_{\max} = 1$,
and the starting point $x^0 \in \left\{ (1,1,0,0,0,0)^T, (0,0,0,0,0,-2)^T \right\}$.
The numerical results obtained by the algorithm are summarized in the Table 3.3.

	x_1^0			x_2^0		
ρ	$Iter$	CPU	$\|x^* - x\|$	$Iter$	CPU	$\|x^* - x\|$
0.5	2	0.0104	$5.5175e-05$	3	0.0227	$5.5175e-05$
1	3	0.02012	$5.5138e-05$	4	0.0225	$5.5138e-05$
1.5	4	0.0274	$5.5137e-05$	5	0.0426	$5.5137e-05$
2	5	0.0286	$5.5129e-05$	6	0.0482	$5.5129e-05$
4	9	0.0428	$5.5138e-05$	10	0.0689	$5.5138e-05$
10	20	0.0896	$5.5138e-05$	24	0.1231	$5.5138e-05$
100	188	0.8618	$5.5135e-05$	222	1.0282	$5.5135e-05$

Table 3.3: Example 3.3

Example 3.4. *[71]*

The data of the NMAVI is deduced from a non-convex quadratic program, where:

$$M = \begin{pmatrix} 263 & -97 & 62 & 217 & 52 & 621 & 935 & 258 & -61 & -10 \\ -97 & 299 & -17 & 9 & -4 & -123 & -17 & -40 & -3 & 37 \\ 62 & -17 & 178 & 71 & -118 & -83 & -110 & 9 & -56 & 42 \\ 217 & 9 & 71 & 143 & -5 & 842 & 228 & 42 & 58 & -41 \\ 52 & -4 & -118 & -5 & 177 & 102 & -15 & 120 & 13 & -52 \\ 621 & -123 & -83 & 842 & 102 & 219 & 574 & 22 & 73 & -53 \\ 935 & -17 & -110 & 228 & -15 & 574 & 457 & 154 & -25 & 84 \\ 258 & -40 & 9 & 42 & 120 & 22 & 154 & 473 & 18 & -29 \\ -61 & -3 & -56 & 58 & 13 & 73 & -25 & 18 & -4 & -79 \\ -10 & 37 & 42 & -41 & -52 & -53 & 84 & -29 & -79 & 224 \end{pmatrix}, \; q = \begin{pmatrix} -20 \\ -314 \\ 46 \\ -83.45 \\ -128.7 \\ 41.3 \\ 43.85 \\ 341.8 \\ 34.05 \\ -34.6 \end{pmatrix}.$$

$$A = \begin{pmatrix} I \\ -I \end{pmatrix}, \quad b = \begin{pmatrix} 0\,, & 0\,, & 0\,, & 0\,, & 0\,, & -1\,, & -1\,, & -1\,, & -1\,, & -1 \end{pmatrix}^T,$$

The optimal solution set is $\begin{pmatrix} 0\,, & 1\,, & 0.5\,, & 0\,, & 0.75\,, & 0\,, & 0\,, & 0.6\,, & 1\,, & 0.49 \end{pmatrix}^T$.

In the implementation, we take $\lambda_{\max} = 2081.7$ *and the starting point*

$$x^0 \in \left\{(-1, \ldots, -1)^T, (0.5, \ldots, 0.5)^T\right\}.$$

The numerical results obtained by the algorithm are summarized in the Table 3.4.

	x_1^0			x_2^0		
ρ	$Iter$	CPU	$\|x^* - x\|$	$Iter$	CPU	$\|x^* - x\|$
500	40	0.06440	$9.8042e-07$	97	0.02951	$9.3394e-07$
1000	81	0.0308	$9.7328e-07$	262	0.2974	$9.3586e-07$
2000	3	0.0545	$9.8586e-07$	4	0.0646	$9.7380e-07$
2081.7	3	0.0581	$9.9799e-07$	4	0.0652	$9.7380e-07$
2500	4	0.0711	$9.9741e-07$	7	0.1137	$9.6929e-07$
3000	7	0.1216	$9.9036e-07$	7	0.1178	$9.6089e-07$

Table 3.4: Example 3.4

Example 3.5. *[40] (Large-scale Example)*

Consider the following quadratic program:

$$\begin{cases} \min \left[-\sum_{i=1}^{n} x_i^2\right] \\ \textit{Subject to} \\ \sum_{i=1}^{j} x_i \leq j, \; j = 1, \; 2:n \\ x_i \geq 0, \; i = 1, \; 2:n \end{cases}$$

the data of the corresponding NMAVI problem are

$$M = \begin{pmatrix} -2 & 0 & 0 & \cdots & 0 \\ 0 & -2 & 0 & \cdots & 0 \\ 0 & 0 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -2 \end{pmatrix}, \; q = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$A = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ -1 & -2 & 0 & \cdots & 0 \\ -1 & -2 & -3 & \cdots & 0 \\ -1 & -2 & -3 & \cdots & -n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \; b = \begin{pmatrix} -1 \\ -2 \\ -3 \\ \vdots \\ -n \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The results of this example are quoted in the Table 3.5 for various values of the dimension n.

n	$Iter$	CPU
5	2	0.01026
10	2	0.0292
50	2	0.0751
100	2	0.1048
300	2	0.4333
700	2	2.2656
1000	2	6.1662
1500	2	19.3652
3000	2	207.7703

Table 3.5: Example 3.5

3.4.2 General comments

In this paragraph, we make the following observations:
It is noticed that the DCA algorithm converges for an initial point x^0 that is feasible or not feasible, but it works better for the initial points that are feasible.
It is also noted that it is possible to take values of ρ larger than $\lambda_{\max}$ and the results

obtained in this case in general are better than those obtained for the values of ρ which are smaller than $\lambda_{\max}$ in terms of number of iterations and calculation time. This is explained by the fact that the theoretical condition $\rho \geq \lambda_{\max}$ is not sufficient.

3.5 Conclusion

This chapter has presented a novel approach for solving non-monotone affine variational inequalities problem NMAVI by reformulating it as non-convex quadratic program and applying DC programming and DCA. The key contribution lies in the proposed DC decomposition that effectively transforms the problem into a sequence of convex quadratic subproblems, each solved via projection onto the feasible set. Theoretical analysis confirms the convergence of the algorithm to a solution that satisfying the necessary local optimality conditions. Extensive numerical experiments on various test problems, including large-scale problem, demonstrate the algorithm's efficiency. Notably, the method exhibits consistent performance across different problem sizes and achieves solutions with high accuracy, highlighting its practical applicability for solving challenging NMAVI problems.

General conclusion and future works

In this thesis, we presented two methods of optimization techniques for solving some classes of affine variational inequalities problems (both monotone and non-monotone cases).
Firstly, we are interested on the resolution of monotone affine variational inequalities problems MAVI, we adopted a type of interior-point method based on the use of algebraically equivalent transformation AET technique, induced by the univariante function $\psi(t) = \sqrt{t}$ applied to the central-path to offer new search directions given by Darvay and Takàcs [33] and Achache [1]. We conducted a comprehensive theoretical study on the analysis and complexity of the algorithm resulting from this function. The tests we carried out show that the numerical behaviour is quite appreciable, which places this approach at the forefront of modern developments related to MAVI up to the present day.
Secondly, for solving the non-monotone affine variational inequalities problems, we studied a non-convex programming approach based on DC programming and DCA algorithms to solve the NMAVI, an optimization model associated with the NMAVI was considered for which DCA algorithms have been used.
Based on an appropriate decomposition of this model, we developed a simple DCA algorithmic scheme. It consists of successively solving convex quadratic programs. Numerical experiments on several test problems prove the effectiveness of the proposed approach.

Future work

In this section, an interesting topics of research in the future are :
. Extension of the **algorithm** 2.1 to the class of monotone nonlinear variational inequalities problem.
. Using other functions of algebraically transformation.
. Extension of the **algorithm** 3.1 to the class of non-monotone non linear variational inequalities problem.
. Introduce other DC decompositions for the second approach.

Appendix

Newton-Raphson's method for solving nonlinear systems

Among the most popular methods applied for solving the nonlinear system of equations, is Newton's method, in the following we describe its principle.
Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a differentiable nonlinear function such that :

$$F(x) = (f_1(x), f_2(x), ..., f_n(x))^T .$$

This method is an iterative procedure whose objective is to find a point $x \in \mathbb{R}^n$ such that $F(x) = 0$. If J is the Jacobian matrix of F then, at the iteration k we write :

$$F(x^k + \Delta x^k) \approx F(x^k) + J(x^k)\Delta x^k,$$

where

$$J(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix},$$

and the Newton direction x^k, is chosen in such way that this linear approximation is equal to zero. So we put

$$x^{k+1} = x^k + \Delta x^k,$$

with

$$\Delta x^k = -J(x^k)^{-1} F(x^k).$$

Convergence towards a solution is guaranteed from the moment the initial point x^0 is in a sufficiently close neighborhood to one of the zeros of F.
The choice of Newton's method is very important from the point of view of the methods of the interior-points because of its numerical efficiency.

The Hessian matrix is a matrix of second order partial derivatives, defined as follow:

$$H(x) = \begin{pmatrix} \frac{\partial^2 f_1}{\partial x_1^2} & \frac{\partial^2 f_1}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f_1}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f_2}{\partial x_2 \partial x_1} & \frac{\partial^2 f_2}{\partial x_2^2} & \cdots & \frac{\partial^2 f_2}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f_n}{\partial x_n \partial x_1} & \frac{\partial^2 f_n}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f_n}{\partial x_n^2} \end{pmatrix},$$

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ملخص

تتعلق هذه الأطروحة بحل مسألة المتراجحات التغيرية التألفية. بالنسبة للحالة الرتيبة، استخدمنا طريقة النقطة الداخلية بناء على تقنية التحولات الجبرية. بالنسبة للحالة غير الرتيبة، اقترحنا طريقة تستند على البرمجة من خلال المشكلة التربيعية المرتبطة بها والتي تمر عبر شرط المثالية.

أجرينا دراسة نظرية وخوارزمية لهاتين الطريقتين وأظهرنا تقارب كل خوارزمية. كما قدمنا أيضا معدل حدودية تكلفة التعقيد الخوارزمي للطريقة الأولى، وقد عززنا النتائج التي حصلنا عليها بتجارب عددية مهمة للغاية.

كلمات مفتاحية: مشكلة المتراجحات التغيرية التألفية، طرق النقاط الداخلية البدائية والثنائية، التحويل الجبري المكافئ، البرمجة التربيعية، طريقة البرمجة ذات الفروق المحدبة.، خوارزمية البرمجة ذات الفروق المحدبة.

Résumé

Cette thèse concerne la résolution du problème des inégalités variationnelle affine. Pour le cas monotone, nous avons utilisé une méthode du point intérieur basée sur la technique des transformations algébriques. Pour le cas non monotone, nous avons proposé une méthode DC via son problème quadratique associé passant par la condition d'optimalité. Nous avons réalisé une étude théorique et algorithmique pour les deux méthodes et montré la convergence de chaque algorithme correspondant à son approche. Nous avons également donné le taux de complexité algorithmique de la première méthode. Les résultats obtenus ont été consolidés par des expériences numériques très significatives.

Mots-clés : Problème des inégalités variationnelle affine, Méthodes du point intérieur primal-dual, Transformation algébrique, Sens de descente, Programmation quadratique, Méthode DC, Algorithme DC.

Abstract

This thesis concerns the resolution of the affine variational inequalities problem. For the monotone case, we used an interior-point method based on the algebraic transformation technique. For the non-monotone case, we proposed a DC method via its associated quadratic problem passing through the optimality condition. We carried out a theoretical and algorithmic study for the two methods and shown the convergence of each algorithm corresponding to its approach. We also gave the algorithmic complexity rate of the first method. The results obtained have been consolidated by very significant digital experiments.

Keywords: Affine variational inequalities problem, Primal-dual interior-point methods, Algebraic transformation, Direction of descent, Quadratic programming, DC method, DC Algorithm.