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FACULTY OF SCIENCES

DEPARTMENT OF PHYSICS

COURSE HANDOUT

Physics of materials point

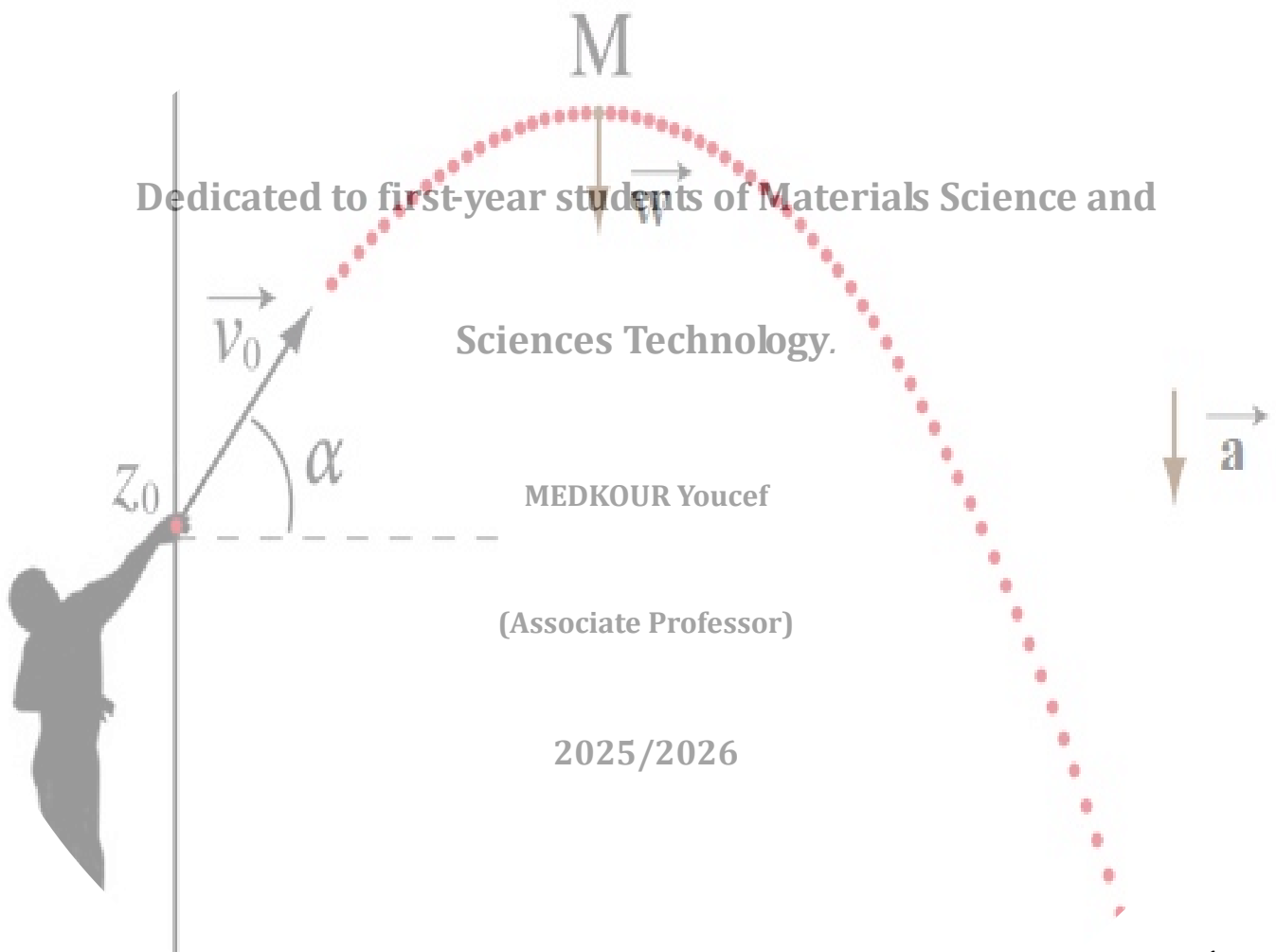


TABLE OF CONTENTS

MATHEMATICAL REMINDERS

I.1. Dimensional Analysis.....	01
I.1. The International System of Units (SI).....	01
I.2. Notion of Dimension.....	01
I.3. Homogeneity of a Result.....	02
I.4. The Dimension of a Physical Quantity.....	02
I.2. Uncertainty Calculations.....	02
I.2.1. Absolute Uncertainty and Relative Uncertainty.....	03
I.2.2. Uncertainty Theorem.....	04
I.2.3. Presentation of Results.....	04
I.3. Vector Analysis.....	04
I.3.1. Scalars and Vectors.....	04
I.3.2. Magnitude and Unit Vector of a Vector.....	05
I.3.3. Scalar Product (Dot Product).....	05
I.3.4. Vector Product (Cross Product).....	06
I.3.5. The Nabla Operator.....	06

CHAPTER I: Kinematics of a Material Point

I.1. Introduction.....	09
.....	
I.2. Frame of Reference.....	09
I.2.1. Main Galilean Frames of Reference.....	09
I.2.2. Position, Time Equation, and Trajectory.....	10
I.3. Main Coordinate Systems..... ;;;	10
I.3.1. Cartesian Coordinates.....	10
I.3.2. Polar Coordinates.....	10
I.3.3. Cylindrical Coordinates.....	11
I.3.4. Spherical Coordinates.....	12
I.3.5. Curvilinear Abscissa and Frenet Frame.....	13
I.4. Velocity Vector of a Point.....	15
I.5. Acceleration Vector of a Point.....	15
I.6. Expressions of Velocity and Acceleration in Coordinate Systems.....	17
I.6.1. Cartesian Coordinates.....	17
I.6.2. Polar Coordinates.....	18
I.6.3. Cylindrical Coordinates.....	19
I.6.4. Spherical Coordinates.....	20

I.6.5. Curvilinear Coordinates (Frenet Frame).....	23
I.7. Application.....	24
I.8. Relative Motion.....	24
I.8.1. Composition of Velocities.....	26
I.8.2. Composition of Accelerations.....	27

CHAPTER II: Dynamics of a Material Point

II.1. Introduction.....	30
II.1.1. Mass.....	30
II.1.2. Notion of Force.....	30
II.1.3. Momentum.....	30
.....	
II.2. Newton's Laws..	31
II.2.1. Principle of Inertia (Newton's First Law).....	31
II.2.1.1. Conservation of Momentum.....	31
II.2.2. Fundamental Principle of Dynamics (Newton's Second Law).....	32
II.2.2.1. Fundamental Principle of Dynamics in a Non-Galilean Frame of Reference.....	32
II.2.3. Principle of Action and Reaction (Newton's Third Law).....	33
II.3.1. Differential Equations of Motion.....	34
II.4. Example of Forces.....	35
II.4.1. Weight Force.....	35
II.4.2. Newtonian Gravitational Forces.....	35
II.4.3. Friction Forces.....	35
II.4.3.1. Solid Friction.....	36
II.4.3.2. Viscous Friction.....	36
II.4.4. Tension Forces.....	37
II.5. Angular Momentum.....	37
II.5.1. Properties of Angular Momentum.....	37
II.5.2. Moment of a Force (Torque).....	39
II.5.3. Angular Momentum Theorem.....	39
II.6. Applications.....	39
II.6.1. Uniform Motions.....	39
II.6.2. Uniformly Varied Motion.....	40
II.6.3. Arbitrary Motions.....	42

CHAPTER III: Work and Energy

III.1. Introduction.....	46
III.2. Work of a Force.....	46
III.2.1. Constant Force on a Rectilinear Displacement.....	46
III.2.2. Elementary Work.....	47
III.3. Energy.....	47

III.3.1. Kinetic Energy.....	47
III.3.2. Potential Energy.....	49
III.3.2.1. Conservative Forces and Potential Energies.....	49
III.3.2.1. Examples of Conservative Forces and Potential Energies.....	50
III.3.2.2. Non-Conservative Force.....	52
III.4. Power of a Force.....	52
III.5. Mechanical Energy.....	53
Reference	54

MATHEMATICAL REMINDERS

1. Dimensional Analysis:

Physical sciences use quantities that can be measured or located. These physical quantities can be associated with a numerical value that translates their intensity. But the essential characteristic of a physical quantity is its unit, which specifies its nature. Some quantities (in particular those corresponding to the base units) have obvious dimensions. For more complex cases, one must be able to find the dimension of a physical quantity from elementary physical laws.

1. The International System of Units (SI):

Not all units have the same status. Some have been chosen as fundamental. Consequently, all other units can be expressed as a combination of the fundamental units (in the form of a multiplication of units). The fundamental system contains seven base units:

Quantity	Symbol	Name
Length	m	Meter
Mass	<i>kg</i>	Kilogram
Time	s	Second
Electric Current	<i>A</i>	Ampere
Temperature	<i>K</i>	Kelvin
Luminous Intensity	<i>cd</i>	Candela
Amount of Substance	<i>mol</i>	Mole

2. Notion of Dimension:

Each unit is attached to a measurable physical quantity that defines its nature, called dimension. However, one must not confuse unit and dimension. For example, the meter is associated with the dimension of length, just like the yard or the marine mile.

- **Example:**

Velocity v is defined by the relation

$$x = \frac{v}{t} \dots\dots\dots (01)$$

The distance x is expressed in meters, we write:

$$[x] = L \dots\dots\dots(02)$$

The symbol $[d]$ designates the dimension of length. Equation (02) is **an equation of dimensions**; it expresses that the dimension of the quantity designated by the symbol d is a length.

Relation (01) can also be written as:

$$x = x/t \dots\dots\dots(03)$$

We deduce that a velocity is homogeneous to the product of a distance by a time. We write the dimensional equation (03) to translate this property:

$$[v] = [d] \times [t]^{(-1)} = L \times T^{(-1)} \dots\dots\dots(04)$$

T represents the dimension of time.

L represents the dimension of length.

3. Homogeneity of a Result

In physics, one only compares homogeneous quantities, meaning those whose dimensions are equal. The use of dimensions allows one to know if a literal expression is homogeneous or not. This allows for the search for potential errors. But dimensional analysis allows one to rediscover, to guess physical laws when the theoretical resolution is too complex.

❖ Rules of Homogeneity:

- ✚ One can only add terms that are homogeneous.
- ✚ The argument of a transcendental mathematical function (*exp, ln, cos, sin, tan...*) is necessarily dimensionless.
- ✚ One should avoid replacing the symbol of a quantity by its numerical value.
- ✚ A vector can only be added to a vector and not to a scalar.

4. The Dimension of a Quantity

The dimension of a quantity consists of finding its dimension as a function of the base dimensions (L: length, M: mass, T: Time, etc.) or how the unit of a physical quantity decomposes into the base units.

- **Example 1:**

What is the dimension of an electrical resistance R ?

We will use the following formula:

$$P = R \times I^2 \dots\dots\dots (05)$$

P , R , and I designate respectively power, resistance, and electric current. Since electric current is a base dimension, the problem is to finding the dimension of power as a function of the base dimensions.

In mechanics, we know that:

$$P = F \times v \dots\dots\dots (06)$$

F and v designate respectively force and velocity, and furthermore, force is homogeneous to the product of a mass by an acceleration ($F = m \times a$), we therefore write:

$$[F] = M \times L \times T^{(-2)} \dots\dots\dots (07)$$

$$[v] = L \times T^{(-1)} \dots\dots\dots (08)$$

We have:

$$[P] = [F] \times [v] = (M \times L \times T^{(-2)}) \times (L \times T^{(-1)}) \dots\dots\dots (09)$$

From which we derive

$$[R] = M \times L^2 \times T^{(-3)} \dots\dots\dots (10)$$

We can therefore write the following conversion:

$$1\Omega = 1Kg m^2 s^{-3} A^{-2}$$

2. Uncertainty Calculations:

The words error and uncertainty have distinctly different meanings. The measurement error of a certain quantity is the difference between the measured value and the true value. The uncertainty of a measurement translates the scientific attempts to estimate the importance of the error made.

2. 1. Absolute Uncertainty and Relative Uncertainty:

The absolute error Δx is the maximum error that one is likely to make in the evaluation of x . The absolute uncertainty is therefore expressed in the units of the measured quantity. The relative uncertainty $(\Delta x)/x$ represents the importance of the error relative to

the measured quantity. The relative uncertainty has no units and is generally expressed in % ($100 \times (\Delta x)/x$).

2. 2. Uncertainty Theorem:

- **Absolute Uncertainty of an Algebraic Sum:** The absolute uncertainty on an algebraic sum of measurements is the sum of the absolute uncertainties on each of the terms. For example: for the experimental relation $f = x + 2y - z$ we find:

$$\Delta S = \Delta A + 2 \times \Delta B + \Delta C = \Delta x + 2\Delta y + \Delta z \quad (11)$$

- **Relative Uncertainty of a Product or Quotient:** The relative uncertainty on the product and/or quotient of independent measurements is the sum of the relative uncertainties (affected by the necessary coefficients). For example: for the

experimental relation $f = (2x - 1)\sqrt{\frac{y}{z^{-3}}}$ we find:

$$\frac{\Delta f}{f} = \frac{2\Delta x}{(2x - 1)} + \frac{1}{2} \frac{\Delta y}{y} + \frac{3}{2} \frac{\Delta z}{z} \quad (12)$$

2. 3. Presentation of Results:

The measurement result is always written in the form:

$$f = f_0 \pm \Delta f \text{ (Unit)} \quad (13)$$

Where X : exact value, X_0 : approximate (measured one) value and ΔX : absolute uncertainty.

3. Vector Analysis

3. 1. Scalars and Vectors:

Many quantities can be uniquely determined by their magnitude. These quantities are called scalars and are represented by a single real number. These are, for example, lengths, temperatures, masses... On the other hand, when one wishes to define, for example, a displacement, we need to specify the distance traveled during this displacement but also the direction in which it occurs. We say that displacement is a vector quantity. Velocities and forces are other examples of vector quantities.

3. 2. Magnitude and Unit Vector of a Vector

Let a vector $\vec{A}$ in space with Cartesian coordinates, we can write it:

$$\vec{A} = A_x \vec{i} + A_y \vec{j} + A_z \vec{k} \dots\dots\dots (14)$$

Where A_x , A_y , and A_z are the (Cartesian) components. And $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the basis vectors.

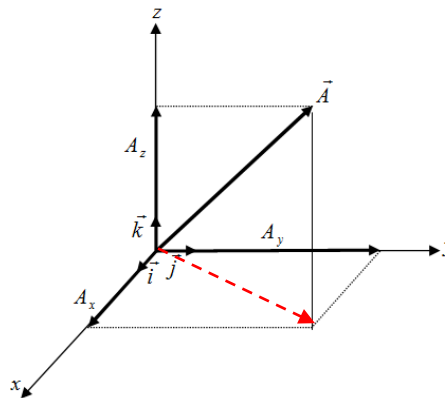


Figure.1: Components of a vector in Cartesian coordinate space.

We define the magnitude of the vector by the scalar quantity $A = \|\vec{A}\|$:

$$A = \|\vec{A}\| = \sqrt{A_x^2 + A_y^2 + A_z^2} \dots\dots\dots (15)$$

We express the unit vector u along V in the form:

$$\vec{u}_A = \frac{\vec{A}}{\|\vec{A}\|} \dots\dots\dots (16)$$

$\vec{u}_A$ is a unit vector, with the same direction and the same sense as the vector $\vec{A}$.

3. 3. Scalar Product (Dot Product)

The scalar product of a vector $\vec{A}$ by a vector $\vec{B}$, which we will denote $\vec{A} \cdot \vec{B}$ is defined as follows:

$$\vec{A} \cdot \vec{B} = \|\vec{A}\| \cdot \|\vec{B}\| \cdot \cos \alpha \dots\dots\dots (17)$$

θ : the angle formed by the two vectors A and B .

- ✓ **The scalar product $A \cdot B$ is the projection of vector A onto B .**

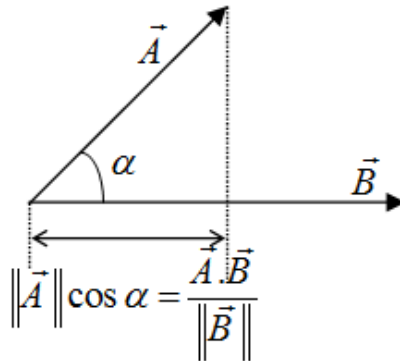


Figure.2: The scalar product of two vectors A and B

- ✓ **Analytical Expression of the Scalar Product:**

We define the vectors $\vec{A}$ and $\vec{B}$ in an orthonormal Cartesian frame as follows:

$\vec{A} = A_x \vec{i} + A_y \vec{j} + A_z \vec{k}$ and $\vec{B} = B_x \vec{i} + B_y \vec{j} + B_z \vec{k}$ the scalar product is written:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \dots\dots\dots (18)$$

The scalar product satisfies the following laws:

- $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
- $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$
- $\lambda (\vec{A} \cdot \vec{B}) = (\lambda \vec{A}) \cdot \vec{B} = \vec{A} \cdot (\lambda \vec{B})$, λ : constant.
- $\vec{A} \cdot \vec{B} = 0 \Rightarrow \vec{A} \perp \vec{B}$

3.4. Vector Product (Cross Product)

The vector product of a vector $\vec{A}$ by a vector $\vec{B}$, which we will denote $\vec{A} \wedge \vec{B}$ is defined as follows:

$$\vec{A} \wedge \vec{B} = \|\vec{A}\| \cdot \|\vec{B}\| \cdot \sin \alpha \vec{u} \dots\dots\dots (19)$$

α : the angle formed by the two vectors $\vec{A}$ and $\vec{B}$ and $\vec{u}$ is a unit vector indicating the direction of the vector $\vec{w} = \vec{A} \wedge \vec{B}$. The resulting vector $\vec{w}$ is perpendicular to the plane formed by the vectors $\vec{A}$ and $\vec{B}$.

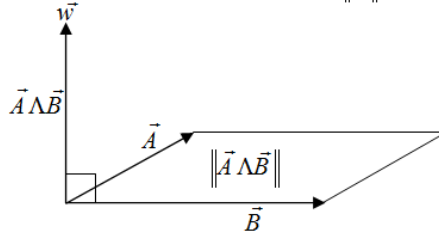


Figure.3: The vector product of two vectors $\vec{A}$ and $\vec{B}$.

❖ **Analytical Expression of the Vector Product** We define the vectors $\vec{A}$ and $\vec{B}$ in an orthonormal Cartesian frame as follows:

$$\vec{A} = A_x \vec{i} + A_y \vec{j} + A_z \vec{k} \text{ and } \vec{B} = B_x \vec{i} + B_y \vec{j} + B_z \vec{k}$$

the vector product is written:

$$\vec{A} \wedge \vec{B} = \begin{vmatrix} +\vec{i} & -\vec{j} & +\vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = + (A_y B_z - A_z B_y) \vec{i} - (A_x B_z - A_z B_x) \vec{j} + (A_x B_y - A_y B_x) \vec{k}$$

..... (20)

This is often represented by a determinant.

The vector product satisfies the following laws:

- $\vec{A} \wedge \vec{B} = \text{Vecteur}$
- $\|\vec{A} \wedge \vec{B}\| = S_p$: S_p area of the parallelogram formed by the two vectors
- $\vec{A} \wedge \vec{B} = -\vec{B} \wedge \vec{A}$
- $\vec{A} \wedge (\vec{B} + \vec{C}) = \vec{A} \wedge \vec{B} + \vec{A} \wedge \vec{C}$
- $\lambda (\vec{A} \wedge \vec{B}) = (\lambda \vec{A}) \wedge \vec{B} = \vec{A} \wedge (\lambda \vec{B})$, λ : constant.

$$\blacktriangleright \vec{A} \wedge \vec{B} = 0 \Rightarrow \vec{A} // \vec{B}$$

1.1. 3.5. The Nabla Operator $\vec{\nabla}$

The nabla operator is a differential operator. Its expression in Cartesian coordinates is given by:

$$\vec{\nabla} = \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \dots\dots\dots (21)$$

The operator $\vec{\nabla}$ can be applied either to a scalar or to a vector. Application to a scalar or scalar function defines the gradient of that function.

$$\vec{\nabla} f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k} \dots\dots\dots (22)$$

The application of the operator $\vec{\nabla}$ to a vector $\vec{A}$ by the scalar product defines the divergence of this vector.

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \text{div } \vec{A} \dots\dots\dots (23)$$

The application of the nabla operator to a vector by the vector product defines the rotational of this vector.

$$\vec{\nabla} \wedge \vec{A} = \begin{vmatrix} +\vec{i} & -\vec{j} & +\vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = + \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \vec{i} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) \vec{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \vec{k} \dots\dots\dots (24)$$

CHAPTER I:

KINEMATICS OF A

MATERIAL POINT

I.1. Introduction:

The subject of point kinematics is to study the motion of a point over time, independently of the causes that produce the motion. The objectives are the determination of kinematic quantities such as acceleration vectors, velocity, position, and the time equation of the trajectory of this point relative to a frame of reference chosen by the observer.

I.2. Frame of Reference:

The study of the motion of a point necessarily implies the simultaneous presence of the point and an observer who analyzes the motion of this point. The observer is the basis of the study of motion, because depending on their position relative to the moving object, their conclusions regarding the nature of the motion will vary greatly. The motion of an object can only be defined relative to a reference. It is therefore necessary to define what is called a frame of reference or reference body in which the observer is fixed. A reference body is understood to be a set of points all fixed relative to each other. There are potentially an infinite number of possible frames of reference. However, we distinguish Galilean frames of reference.

I.2.1. Main Galilean Frames of Reference

- **Copernican Frame** (Copernicus 1473-1543): This frame is defined by three axes originating from the center of the solar system and directed towards three suitably chosen fixed stars. This system is used to study the motion of planets and interplanetary spacecraft.
- **The Geocentric Frame:** This frame is defined by three axes originating from the center of inertia of the Earth and directed towards three fixed stars of the Copernican frame. This frame is used to study the motion of the moon and satellites rotating around the Earth.
- **The Terrestrial Frame:** This frame is defined by three perpendicular axes originating from any point on Earth. This frame is used to study the motion of bodies linked to the Earth. In this frame, the Earth is fixed, thus constituting a Galilean frame.

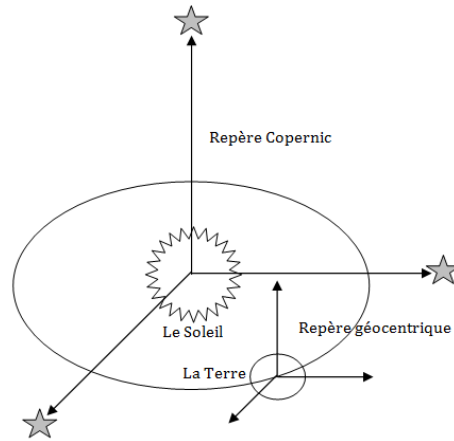


Figure I.1: The different Galilean frames.

I.2.2. Position, Time Equation, and Trajectory:

The position of a material point $\mathbf{M}$ in a frame of reference is defined using the position vector $\overrightarrow{OM}$, where O is a fixed point in the frame of reference. O can be the observer but it is not obligatory; the only condition is that O is fixed relative to the latter. We must therefore give the components of $\overrightarrow{OM}$ in the chosen projection basis, namely in the chosen coordinate system. The motion of the material point $\mathbf{M}$ leads to studying its evolution over time and to giving the time equation of the motion by $\overrightarrow{OM}(t)$ or its components in the chosen projection basis. The curve described by $\mathbf{M}$ during the motion is called the trajectory. It is important to note that all these definitions are relative to the chosen frame of reference.

I.3. Main Coordinate Systems:

For a more precise study of the motion of a moving point in a frame of reference, we are led to define its position and its vector quantities, velocity and acceleration. It will therefore be necessary to choose a coordinate system and use the corresponding basis.

I.3.1. Cartesian Coordinates:

A point $\mathbf{M}$ is located by its spatial coordinates (x, y, z) corresponding to the algebraic measure of the projection of $\mathbf{M}$ successively onto the three axes of the frame. These coordinates are of the same nature and homogeneous to a length. The orthonormal basis

associated with this system of axes is denoted $(\vec{i}, \vec{j}, \vec{k})$. This is a basis that does not change over time (these vectors keep the same direction, the same sense, and the same magnitude over time).

Knowing the position vector $\vec{OM}$ also allows locating the point M . The components of this vector, in the Cartesian basis $(\vec{i}, \vec{j}, \vec{k})$, correspond to the coordinates of point M :

$$\vec{OM} = x \vec{i} + y \vec{j} + z \vec{k} \dots\dots\dots (I.01)$$

x: abscissa, y: ordinate, and z: altitude, are the Cartesian coordinates of point M .

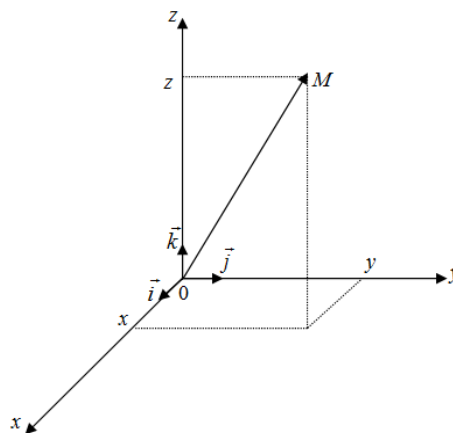


Figure I.2: Cartesian coordinates.

I.3.2. Polar Coordinates:

The point M is perfectly located if we know the distance $OM = \rho$ and the angle θ that the segment OM makes with the axis Ox .

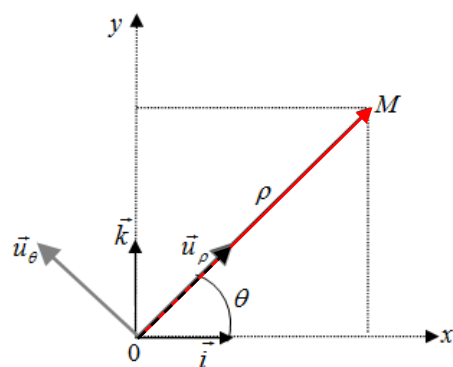


Figure I.3: Polar coordinates.

The length of the segment OM corresponds to the radial coordinate (denoted r or ρ). The angle θ is the angular coordinate. This angle is measured relative to the abscissa axis Ox. The position vector $\overrightarrow{OM}$ can be written as:

$$\overrightarrow{OM} = \rho \vec{u}_\rho \dots\dots\dots (I.02)$$

The basis of the polar coordinate system ($(\vec{u}_\rho, \vec{u}_\theta)$ $\vec{u}_\rho$ radial vector, $\vec{u}_\theta$ orthoradial vector) is a basis defined from the position of point **M**. If the point is moving, the vector $\vec{u}_\rho$ (and consequently $\vec{u}_\theta$) changes direction: the basis is moving in the frame. These vectors can be represented anywhere in space but are often represented either at the origin point **O** or at the point **M** itself.

- **Relation between Polar and Cartesian Coordinates:**

Using the notations from Figure I.3, the relations between the Cartesian and polar coordinate systems are:

Cartesian $(x, y) \rightarrow$ Polar (ρ, θ)

$$x = \rho \cos \theta \qquad y = \rho \sin \theta$$

$$\vec{i} = \cos \theta \vec{u}_\rho - \sin \theta \vec{u}_\theta \qquad \vec{j} = \sin \theta \vec{u}_\rho + \cos \theta \vec{u}_\theta$$

Polar $(r, \theta) \rightarrow$ Cartesian (x, y)

$$OM = \rho = \sqrt{x^2 + y^2} \qquad \tan \theta = \frac{y}{x}$$

$$\cos \theta = \frac{x}{\rho} = \frac{x}{\sqrt{x^2 + y^2}} \qquad \sin \theta = \frac{y}{\rho} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\vec{u}_\rho = \cos \theta \vec{i} + \sin \theta \vec{j} \qquad \vec{u}_\theta = -\sin \theta \vec{i} + \cos \theta \vec{j}$$

I.3.3. Cylindrical Coordinates:

To obtain the cylindrical coordinate system, it is sufficient to complement the polar coordinate system (in the plane xOy) with the axis Oz with its Cartesian coordinate z.

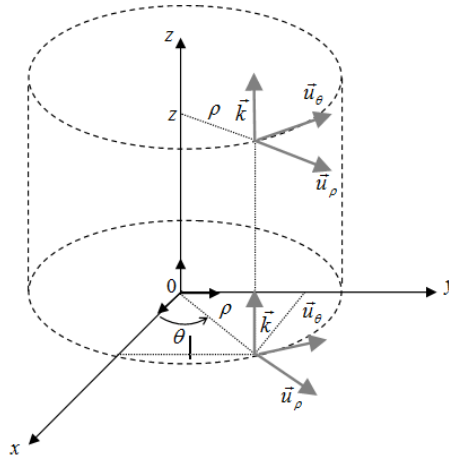


Figure I.4: The cylindrical coordinate system.

The projection of $\mathbf{M}$ onto the axis Oz gives the height z . The projection of point $\mathbf{M}$ in the plane xOy is located in polar coordinates (ρ, θ) . The cylindrical coordinates of point $\mathbf{M}$ are: (ρ, θ, z) and the position vector is given by (I.03):

$$OM = \rho \vec{u}_\rho + z \vec{k} \dots\dots\dots (I.03)$$

$$\|OM\| = OM = \sqrt{\rho^2 + z^2} = \sqrt{x^2 + y^2 + z^2}$$

I.3.4. Spherical Coordinates:

Spherical coordinates allow locating a point on a sphere of radius ρ . This is typically the location of a point on Earth for which it is then sufficient to specify two angles: latitude and longitude.

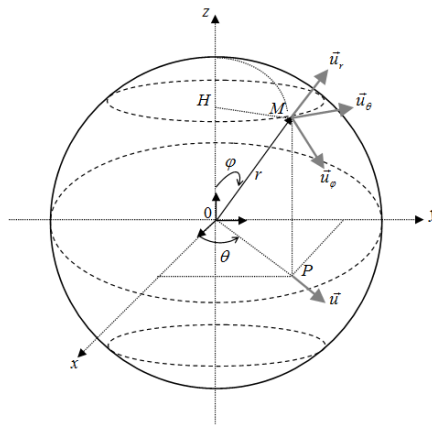


Figure I.5: The spherical coordinate system.

The radial coordinate ρ corresponds to the distance from the origin O of the frame to the point M . The angular coordinate θ corresponds to the angle that OM makes with the axis Oz ; this angle, between 0 and π , is called the colatitude (complementary angle to the latitude) or zenith. The angular coordinate φ corresponds to the angle that the plane defined by the axis Oz and OM makes with the axis Ox ; this angle, between 0 and 2π , is called the longitude or azimuth. The position vector is expressed by equation (I.04):

$$\overrightarrow{OM} = r \vec{u}_r \dots\dots\dots (I.04)$$

The vectors $(\vec{u}_r, \vec{u}_\theta, \vec{u}_\varphi)$ form a direct orthonormal basis. This basis is moving in the frame.

- **Relation between Spherical and Cartesian Coordinates:**

The projection H of point M onto the axis Oz gives the height:

$$z = OH = r \cos \varphi$$

If P is the projection of M onto the plane xOy we have:

$$OP = r \sin \varphi$$

The coordinates x and y of point M are those of point P , that is:

$$x = OP \cos(\varphi) = \rho \sin(\theta) \cos(\varphi)$$

$$y = OP \sin(\varphi) = \rho \sin(\theta) \sin(\varphi)$$

The unit vector u along OP has the expression:

$$\vec{u} = \cos \theta \vec{i} + \sin \theta \vec{j}$$

The unit vector u_θ is directly perpendicular to u_φ . It makes an angle $\left(\theta + \frac{\pi}{2}\right)$ with the axis Oz and is written:

$$\vec{u}_\theta = -\sin \theta \vec{i} + \cos \theta \vec{j}$$

The unit vector $\vec{u}_r$ has the expression:

$$\vec{u}_r = \sin \varphi \vec{u} + \cos \varphi \vec{k} = \sin \varphi [\cos \theta \vec{i} + \sin \theta \vec{j}] + \cos \varphi \vec{k} = \sin \varphi \cos \theta \vec{i} + \sin \varphi \sin \theta \vec{j} + \cos \varphi \vec{k}$$

Finally, the unit vector $\vec{u}_\varphi$ is directly perpendicular to $\vec{u}_r$ and is written:

$$\begin{aligned} \vec{u}_\varphi &= \cos \varphi \vec{u} - \sin \varphi \vec{k} = \cos \varphi [\cos \theta \vec{i} + \sin \theta \vec{j}] - \sin \varphi \vec{k} \\ &= \cos \varphi \cos \theta \vec{i} + \cos \varphi \sin \theta \vec{j} - \sin \varphi \vec{k} \end{aligned}$$

I.3.5. Curvilinear Abscissa and Frenet Frame:

When the trajectory followed by point M is known, it is possible to locate the point on the curve representing this trajectory. We choose an origin point w on the oriented curve and define the curvilinear abscissa s as the algebraic measure on the curve of the distance wM : $S = \overline{wM}$ (measure on the curve).

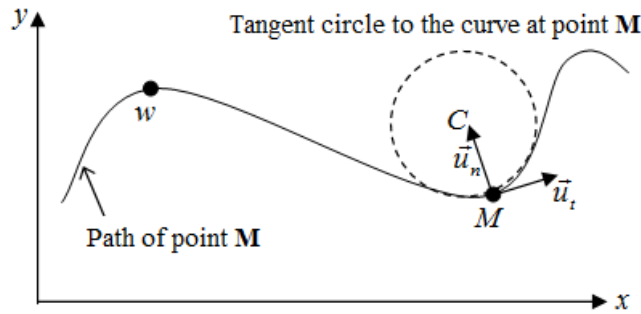


Figure I.6: Curvilinear Abscissa.

The circle with center C and radius R that locally touches the trajectory of the point at M is called the osculating circle. The radius ρ of this circle then corresponds to the radius of curvature of the trajectory at the point considered. At each point M on the curve, we define the Frenet frame $(\vec{u}_t, \vec{u}_n)$ with:

$\vec{u}_t$: Unit vector tangent to the curve at M and oriented in the chosen positive direction.

$\vec{u}_n$: is a vector perpendicular to the vector t , and oriented towards the center of curvature (from M to C). This basis is moving in the frame.

I.4. Velocity Vector of a Point:

The instantaneous velocity can be defined as an average velocity between the position $M(t)$ of the moving point at time t , and the position $M(t+\Delta t)$ of this same point at time $t+\Delta t$ where Δt represents a very short duration. This average velocity tends more towards the instantaneous velocity at time t as the duration Δt tends towards zero. The position vector $\overrightarrow{OM} = \overrightarrow{OM}(t)$ is a function of time and the instantaneous velocity then corresponds to the derivative with respect to time of the position vector:

$$v(t) = \lim_{\Delta t \rightarrow 0} \frac{M(t + \Delta t) - M(t)}{\Delta t} \dots\dots\dots (I.05)$$

When we consider an elementary duration dt (infinitely small), the moving point goes from a position $M(t)$ to a position $M(t+dt)$ infinitely close to $M(t)$. The corresponding elementary displacement can be written as:

$$\overrightarrow{MM'} = \overrightarrow{OM'} - \overrightarrow{OM} = d\overrightarrow{OM} \dots\dots\dots (I.06)$$

The elementary duration is chosen sufficiently small so that the average velocity over the elementary displacement coincides with the instantaneous velocity. With these notations, the velocity vector v , derivative of the position vector $\overrightarrow{OM}$, is written as:

$$v(t) = \frac{d\overrightarrow{OM}}{dt} \dots\dots\dots (I.07)$$

When the point $M(t+dt)$ tends towards the point $M(t)$, the chord $M(t)M(t+dt)$ tends towards the tangent to the trajectory of the point $M(t)$. The velocity vector is a vector tangent to the trajectory at the point considered. Using the definition of the instantaneous velocity vector, the elementary displacement is written as:

$$d\overrightarrow{OM} = v(t)dt \dots\dots\dots (I.08)$$

The velocity vector can be written as the derivative of the position vector or as the ratio of an elementary displacement to the corresponding elementary duration. This velocity vector can have different expressions depending on the coordinate system chosen to study the motion of the point in a given frame.

I.5. Acceleration Vector of a Point:

Just as the velocity vector accounts for the variation of the position vector with respect to time, the acceleration vector will account for the variations of the velocity vector with respect to time. The acceleration vector therefore corresponds to the derivative with respect to time of the velocity vector, that is, also to the second derivative of the position vector:

$$\vec{a} = \frac{d\vec{v}(t)}{dt} = \frac{d^2\vec{OM}}{dt^2} \dots\dots\dots (1.09)$$

The acceleration is the derivative of the velocity vector or the second derivative of the position vector with respect to time.

I.6. Expressions of Velocity and Acceleration in Coordinate Systems:

I.6.1. Cartesian Coordinates:

- **Expression of the velocity vector in Cartesian coordinates:** The basis $(\vec{i}, \vec{j}, \vec{k})$ consists of vectors fixed in the frame: their direction, sense, and magnitude do not change over time. Using the expression of the position vector in Cartesian coordinates and the rules for differentiating a sum of functions, we have:

$$\begin{aligned} \vec{v}(t) = \frac{d\vec{OM}}{dt} &= \frac{d(x\vec{i} + y\vec{j} + z\vec{k})}{dt} \\ &= \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} \dots\dots\dots (1.10) \\ &= \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k} \end{aligned}$$

From the relation for $\vec{v}(t)$, we deduce the elementary displacement vector:

$$d\vec{OM} = \vec{v}(t)dt = dx\vec{i} + dy\vec{j} + dz\vec{k} \dots\dots\dots (1.11)$$

The elementary displacement in Cartesian coordinates therefore corresponds to an elementary displacement dx along the direction $\vec{i}$, then an elementary displacement dy along the direction $\vec{j}$, and finally an elementary displacement dz along the direction $\vec{k}$.

- **Expression of the acceleration vector in Cartesian coordinates:** From the definition (I.09) of the acceleration vector and the expression (I.10) of the velocity vector, we have:

$$\vec{a} = \frac{d\vec{v}(t)}{dt} = \frac{d^2 O\vec{M}}{dt^2} \dots\dots\dots (I.12)$$

$$\begin{aligned} \vec{a} &= \frac{d(\dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k})}{dt} \dots\dots\dots (I.13) \\ &= \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k} \end{aligned}$$

$$\|\vec{a}(t)\| = \vec{a} = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}$$

I.6.2. Polar Coordinates:

- **Expression of the velocity vector in polar coordinates:** The basis $(\vec{u}_\rho, \vec{u}_\theta)$ consists of moving vectors in the frame: these vectors change direction over time. Using the expression of the position vector in polar coordinates and the rules for differentiating a product of functions, we have:

$$\vec{v}(t) = \frac{dO\vec{M}}{dt} = \frac{d(\rho\vec{u}_\rho)}{dt} = \dot{\rho}\vec{u}_\rho + \rho\dot{\vec{u}}_\rho \dots\dots\dots (I.14)$$

According to the expression, the vector $\vec{u}_\rho$ appears as a function of the angular coordinate θ itself a function of time during the motion of the point M. Applying the rules of differentiation to the expression of the vector $\vec{u}_\rho$ gives:

$$\begin{aligned} \dot{\vec{u}}_\rho &= \frac{d(\cos\theta\vec{i} + \sin\theta\vec{j})}{dt} \dots\dots\dots (I.15) \\ &= -\dot{\theta}\sin\theta\vec{i} + \dot{\theta}\cos\theta\vec{j} = \dot{\theta}(-\sin\theta\vec{i} + \cos\theta\vec{j}) \end{aligned}$$

From the relation $du_\rho / d\theta$, we finally obtain:

$$\dot{\vec{u}}_\rho = \dot{\theta}\vec{u}_\theta \dots\dots\dots (I.16)$$

Similarly for the vector $\vec{u}_\theta$:

$$\begin{aligned}\dot{\vec{u}}_\theta &= \frac{d(-\sin\theta \vec{i} + \cos\theta \vec{j})}{dt} \dots\dots\dots (I.17) \\ &= -\dot{\theta} \cos\theta \vec{i} - \dot{\theta} \sin\theta \vec{j} = -\dot{\theta}(\cos\theta \vec{i} + \sin\theta \vec{j})\end{aligned}$$

From the relation of $\vec{u}_\rho$, we finally obtain:

$$\dot{\vec{u}}_\theta = -\dot{\theta} \vec{u}_\rho \dots\dots\dots (I.18)$$

Returning to the expression $\vec{v} = (dr/dt) \vec{u}_\rho + r (d\vec{u}_\rho/dt)$ and using the previous result, we have:

$$\vec{v}(t) = \dot{\rho} \vec{u}_\rho + \rho \dot{\theta} \vec{u}_\theta \dots\dots\dots (I.19)$$

$$\|\vec{v}(t)\| = v = \sqrt{v_\rho^2 + v_\theta^2} = \sqrt{\dot{\rho}^2 + (\rho \dot{\theta})^2}$$

The quantities v_ρ and v_θ are respectively the radial and orthoradial components of the velocity vector in the polar basis.

From the relations for $d\vec{v}(t)$ and $d\vec{OM}$, the elementary displacement vector in polar coordinates is expressed by:

$$\begin{aligned}d\vec{OM} = d\vec{v}(t)dt &= \left(\frac{d\rho}{dt} \vec{u}_\rho + \rho \frac{d\theta}{dt} \vec{u}_\theta \right) dt \dots\dots\dots (I.20) \\ &= d\rho \vec{u}_\rho + \rho d\theta \vec{u}_\theta\end{aligned}$$

Any elementary displacement $d\vec{OM}$ from a point M is thus decomposed into a radial displacement dr along $\vec{u}_\rho$ and an orthoradial displacement $\rho d\theta$ along $\vec{u}_\theta$.

- **Expression of the acceleration vector in polar coordinates:** From the definition of the acceleration vector and the expression (I.19) of the velocity vector, we have:

$$\begin{aligned}\vec{a}(t) &= \frac{d\vec{v}(t)}{dt} = \frac{d(\dot{\rho} \vec{u}_\rho + \rho \dot{\theta} \vec{u}_\theta)}{dt} \dots\dots\dots (I.21) \\ &= \ddot{\rho} \vec{u}_\rho + \dot{\rho} \dot{\vec{u}}_\rho + \dot{\rho} \dot{\theta} \vec{u}_\theta + \rho \ddot{\theta} \vec{u}_\theta + \rho \dot{\theta} \dot{\vec{u}}_\theta\end{aligned}$$

Using the usual rules for differentiating a product and the expressions for $(\dot{\vec{u}}_\rho, \dot{\vec{u}}_\theta)$, we have:

$$\vec{a}(t) = \ddot{\rho} \vec{u}_\rho + \dot{\rho} \dot{\theta} \vec{u}_\theta + \dot{\rho} \ddot{\theta} \vec{u}_\theta + \rho \ddot{\theta} \vec{u}_\theta + \rho \dot{\theta} (-\dot{\theta} \vec{u}_\rho)$$

Grouping and ordering the different results, we obtain the expression:

$$\vec{a}(t) = (\ddot{\rho} - \rho \dot{\theta}^2) \vec{u}_\rho + (\rho \ddot{\theta} + 2\dot{\rho} \dot{\theta}) \vec{u}_\theta \dots\dots\dots (I.22)$$

The first term $a_\rho = (\ddot{\rho} - \rho \dot{\theta}^2)$ corresponds to the radial component of the acceleration and the second term $a_\theta = (\rho \ddot{\theta} + 2\dot{\rho} \dot{\theta})$ is its orthoradial component.

I.6.3. Cylindrical Coordinates:

- **Expression of the velocity vector in cylindrical coordinates:** It is sufficient to add the component along the axis Oz to the polar coordinate system to obtain the expression of the velocity vector. The basis vector $\vec{k}$ not depending on time, we have:

$$\begin{aligned} \vec{v}(t) &= \frac{d\vec{OM}}{dt} = \frac{d(\rho \vec{u}_\rho + z \vec{k})}{dt} \dots\dots\dots (I.23) \\ &= \dot{\rho} \vec{u}_\rho + \rho \dot{\theta} \vec{u}_\theta + \dot{z} \vec{k} \end{aligned}$$

$$\|\vec{v}(t)\| = v = \sqrt{v_\rho^2 + v_\theta^2 + v_z^2} = \sqrt{\dot{\rho}^2 + (\rho \dot{\theta})^2 + \dot{z}^2}$$

Combining an elementary displacement in polar coordinates with an elementary displacement dz along the axis k, we find the elementary displacement vector in cylindrical coordinates:

$$d\vec{OM} = d\rho \vec{u}_\rho + \rho d\theta \vec{u}_\theta + dz \vec{k} \dots\dots\dots (I.24)$$

- **Expression of the acceleration vector in cylindrical coordinates:** It is sufficient to add the term corresponding to the derivation of the height z:

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = (\ddot{\rho} - \rho\dot{\theta}^2)\vec{u}_\rho + (\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta})\vec{u}_\theta + \dot{z}\vec{k} \dots\dots\dots (I.25)$$

$$\vec{a}(t) = (a_\rho = (\ddot{\rho} - \rho\dot{\theta}^2), a_\theta = (\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta}), a_z = \dot{z})$$

I.6.4. Spherical Coordinates:

- **Expression of the velocity vector in spherical coordinates:** The basis $(\vec{u}_r, \vec{u}_\theta, \vec{u}_\varphi)$ consists of moving vectors in the frame: these vectors change direction over time. Using the expression of the position vector in spherical coordinates and the rules for differentiating a product of functions, we have:

$$\vec{v}(t) = \frac{dOM}{dt} = \frac{d(r\vec{u}_r)}{dt} = \dot{r}\vec{u}_r + r\dot{\vec{u}}_r \dots\dots\dots (I.26)$$

The expressions of the unit vectors of the basis $(\vec{u}_r, \vec{u}_\theta, \vec{u}_\varphi)$ in Cartesian coordinates are:

$$\vec{u}_r = \sin\varphi\cos\theta\vec{i} + \sin\varphi\sin\theta\vec{j} + \cos\varphi\vec{k} \dots\dots\dots (I.27)$$

$$\vec{u}_\theta = -\sin\theta\vec{i} + \cos\theta\vec{j} \dots\dots\dots (I.28)$$

$$\vec{u}_\varphi = \cos\varphi\cos\theta\vec{i} + \cos\varphi\sin\theta\vec{j} - \sin\varphi\vec{k} \dots\dots\dots (I.29)$$

Expression of the derivative of $\vec{u}_r$:

$$\dot{\vec{u}}_r = \dot{\varphi}\cos\varphi\cos\theta\vec{i} - \dot{\theta}\sin\varphi\sin\theta\vec{i} + \dot{\varphi}\cos\varphi\sin\theta\vec{j} + \dot{\theta}\sin\varphi\cos\theta\vec{j} - \dot{\varphi}\sin\varphi\vec{k} \dots\dots\dots (I.30)$$

Expression of the derivative of $\vec{u}_\varphi$:

$$\begin{aligned} \dot{\vec{u}}_\varphi &= -\dot{\varphi}\sin\varphi\cos\theta\vec{i} - \dot{\theta}\cos\varphi\sin\theta\vec{i} - \dot{\varphi}\sin\varphi\sin\theta\vec{j} + \dot{\theta}\cos\varphi\cos\theta\vec{j} - \dot{\varphi}\cos\varphi\vec{k} \\ \dot{\vec{u}}_\varphi &= -\dot{\varphi}\vec{u}_r + \dot{\theta}\cos\varphi\vec{u}_\theta \dots\dots\dots (I.31) \end{aligned}$$

Expression of the derivative of $\vec{u}_\theta$:

$$\begin{aligned} \dot{\vec{u}}_\theta &= -\dot{\theta}\cos\theta\vec{i} - \dot{\theta}\sin\theta\vec{j} \\ &= -\dot{\theta}(\cos\theta\vec{i} + \sin\theta\vec{j}) \dots\dots\dots (I.32) \end{aligned}$$

To complete the demonstration, we must find the term $(\cos\theta\vec{i} + \sin\theta\vec{j})$. Let's return to the expressions for $\vec{u}_r$ and $\vec{u}_\varphi$. Multiply the first by $\sin\varphi$ and the second by $\cos\varphi$, we obtain:

$$\vec{u}_r \sin \varphi = \sin^2 \varphi \cos \theta \vec{i} + \sin^2 \varphi \sin \theta \vec{j} + \sin \varphi \cos \varphi \vec{k}$$

$$\vec{u}_\varphi \cos \varphi = \cos^2 \varphi \cos \theta \vec{i} + \cos^2 \varphi \sin \theta \vec{j} - \cos \varphi \sin \varphi \vec{k}$$

Add the two expressions to obtain:

$$\vec{u}_r \sin \varphi + \vec{u}_\varphi \cos \varphi = (\cos \theta \vec{i} + \sin \theta \vec{j})$$

Replace the term $(\cos \theta \vec{i} + \sin \theta \vec{j})$ in the expression for $\dot{\vec{u}}_\theta$, we obtain:

$$\begin{aligned}\dot{\vec{u}}_\theta &= -\dot{\theta} \cos \theta \vec{i} - \dot{\theta} \sin \theta \vec{j} \\ &= -\dot{\theta} (\sin \varphi \vec{u}_r + \cos \varphi \vec{u}_\varphi)\end{aligned}$$

$$\dot{\vec{u}}_\theta = -\dot{\theta} (\sin \varphi \vec{u}_r + \cos \varphi \vec{u}_\varphi) \dots\dots\dots (I.33)$$

Returning to the expression $\vec{v} = (dr/dt) \vec{u}_r + r (d\vec{u}_r/dt)$ and using the previous results, we obtain:

$$\vec{v}(t) = \frac{d\vec{OM}}{dt} = \frac{d(r\vec{u}_r)}{dt} = \dot{r}\vec{u}_r + r(\dot{\varphi}\vec{u}_\varphi + \dot{\theta}\sin\varphi\vec{u}_\theta)$$

$$\vec{v}(t) = \dot{r}\vec{u}_r + r\dot{\vec{u}}_r = \dot{r}\vec{u}_r + r\dot{\theta}\sin\varphi\vec{u}_\theta + r\dot{\varphi}\vec{u}_\varphi \dots\dots\dots (I.34)$$

$$\|\vec{v}(t)\| = v = \sqrt{v_r^2 + v_\theta^2 + v_\varphi^2} = \sqrt{\dot{r}^2 + (r\dot{\theta}\sin\varphi)^2 + (r\dot{\varphi})^2}$$

The elementary radial displacement dr along $\vec{u}_r$ (the point moves away from the origin).
The elementary displacement along $\vec{u}_\varphi$ (the point moves along the meridian) The colatitude changes from φ to $\varphi + d\varphi$. The elementary displacement $r \sin \varphi d\theta$ along $\vec{u}_\theta$ (the point moves along the parallel). We thus obtain the expression:

$$d\vec{OM} = \vec{v}(t)dt = dr\vec{u}_r + r \sin \varphi d\theta \vec{u}_\theta + r d\varphi \vec{u}_\varphi \dots\dots\dots (I.35)$$

- **Expression of the acceleration vector in spherical coordinates:** From the definition of the acceleration vector and the expression (I.34) of the velocity vector, we have:

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \frac{d(\dot{r}\vec{u}_r + r\dot{\theta}\sin\varphi\vec{u}_\theta + r\dot{\varphi}\vec{u}_\varphi)}{dt}$$

$$\begin{aligned}\vec{a}(t) = & \ddot{r} \vec{u}_r + \dot{r} \dot{\vec{u}}_r + \dot{r} \dot{\theta} \sin \varphi \vec{u}_\theta + r \ddot{\theta} \sin \varphi \vec{u}_\theta + r \dot{\theta} \dot{\varphi} \cos \varphi \vec{u}_\theta \\ & + r \dot{\theta} \sin \varphi \dot{\vec{u}}_\theta + \dot{r} \dot{\varphi} \vec{u}_\varphi + r \ddot{\varphi} \vec{u}_\varphi + r \dot{\varphi} \dot{\vec{u}}_\varphi\end{aligned}$$

Using the usual rules for differentiating a product and the expressions for $d\vec{u}(\vec{u}_r, \vec{u}_\theta)$ we have:

$$\begin{aligned}\vec{a}(t) = & \ddot{r} \vec{u}_r + \dot{r} \dot{\vec{u}}_r + \dot{r} \dot{\theta} \sin \varphi \vec{u}_\theta + r \ddot{\theta} \sin \varphi \vec{u}_\theta + r \dot{\theta} \dot{\varphi} \cos \varphi \vec{u}_\theta \\ & + r \dot{\theta} \sin \varphi \dot{\vec{u}}_\theta + \dot{r} \dot{\varphi} \vec{u}_\varphi + r \ddot{\varphi} \vec{u}_\varphi + r \dot{\varphi} \dot{\vec{u}}_\varphi\end{aligned}$$

Grouping and ordering the different results, we obtain the expression:

$$\begin{aligned}\vec{a}(t) = & (\ddot{r} - r\dot{\varphi}^2 - r\dot{\theta}^2 \sin^2 \varphi) \vec{u}_r \\ & + (r\ddot{\theta} \sin \varphi + 2r\dot{\theta} \sin \varphi + 2r\dot{\theta} \cos \varphi) \vec{u}_\theta \dots\dots\dots (I.36) \\ & + (r\ddot{\varphi} + \dot{r}\dot{\varphi} - r\dot{\theta}^2 \sin \varphi \cos \varphi) \vec{u}_\varphi\end{aligned}$$

(Where $\omega_\theta = d\theta/dt$, $\alpha_\theta = d^2\theta/dt^2$, $\omega_\varphi = d\varphi/dt$, $\alpha_\varphi = d^2\varphi/dt^2$)

I.6.5. Curvilinear Coordinates (Frenet Frame):

- **Expression of the velocity vector in curvilinear coordinates** (Frenet frame):
When the position of point **M** is varied in an elementary way while describing the trajectory. The curvilinear abscissa of point **M** goes from s to $s + ds$ between time t and time $t + dt$. The elementary displacement of point **M** is tangent to the trajectory and is then written as:

$$d\vec{OM} = ds \vec{u}_t \dots\dots\dots (I.37)$$

The velocity vector has the expression:

$$\vec{v}(t) = \frac{d\vec{OM}}{dt} = \frac{ds}{dt} \vec{u}_t = \dot{s} \vec{u}_t \dots\dots\dots (I.38)$$

The quantity $v = \dot{s}$ corresponds to the algebraic value of the velocity (positive if the point moves in the chosen positive direction).

- **Expression of the acceleration vector in curvilinear coordinates** (Frenet frame): In the Frenet frame, the velocity is written as: $\vec{v}(t) = \frac{ds}{dt} \vec{u}_t$. The acceleration vector is obtained by differentiating the velocity vector with respect

to time:

$$\vec{a}(t) = \frac{ds}{dt} \vec{u}_t + \dot{s} \vec{u}_t$$

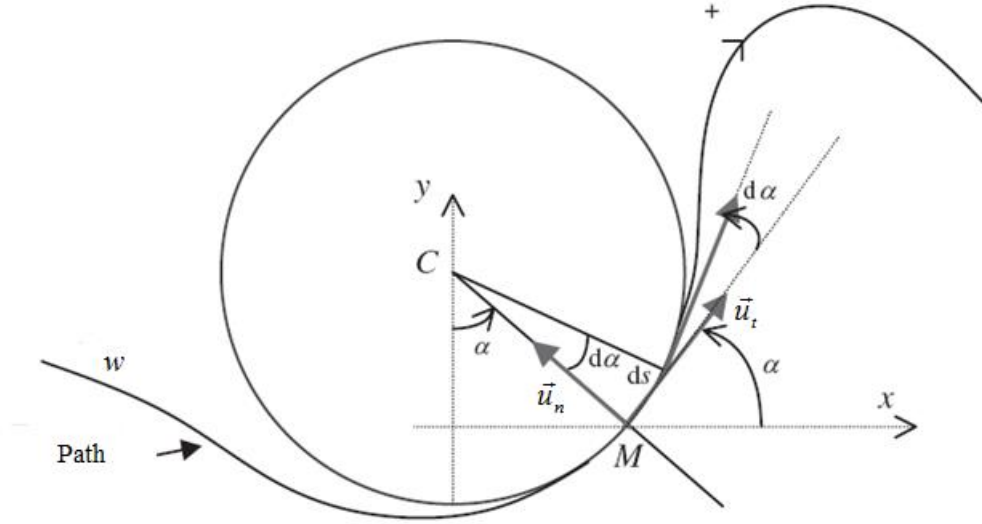


Figure 1.8: Frenet frame and elementary displacement $d(OM)$.

At a time t , at point M of the trajectory, the basis vector $\vec{u}_t$ makes an angle α with the direction of the x axis. At time $t+dt$, this vector rotates by an angle $d\alpha$. The derivative, with respect to time, of this unit vector is therefore given by:

$$\frac{d\vec{u}_t}{dt} = \dot{\alpha} \vec{u}_n$$

With $CM = \rho$ is the radius of the osculating circle tangent to the curve at point M , we have:

$$ds = CM d\alpha = \rho d\alpha \Rightarrow \frac{d\alpha}{dt} = \dot{\alpha} = \frac{1}{\rho} \frac{ds}{dt} = \frac{\dot{s}}{\rho} = \frac{v}{\rho}$$

Finally:

$$\dot{s} \frac{d\vec{u}_t}{dt} = \dot{s} \dot{\alpha} = \dot{s} \frac{\dot{s}}{\rho} = \frac{v^2}{\rho} \vec{u}_n$$

The expression of the acceleration vector in the Frenet frame is:

$$\vec{a}(t) = \ddot{s}\vec{u}_t + \frac{v^2}{\rho}\vec{u}_n = \frac{dv}{dt}\vec{u}_t + \frac{v^2}{\rho}\vec{u}_n \dots\dots\dots (I.39)$$

It can be verified that this result is always true regardless of the concavity of the trajectory.

The normal component of the acceleration $a_n = \frac{v^2}{\rho}$ is always positive: it is always turned towards the center of curvature of the trajectory at the point considered. It indicates that the direction of the velocity vector changes and is more significant the smaller the radius of curvature (tight turn). If the motion is rectilinear (infinite radius of curvature) this term is zero. The tangential component of the acceleration $a_t = \frac{dv}{dt}$, if the motion is uniform this term is zero.

I.8. Application:

- **Example 1:** A mobile in rectilinear motion has an acceleration $a = 1/t^2$. Its initial velocity at time $t=0$ at point O is zero.

a) What is its instantaneous velocity $v(t)$?

b) What is its instantaneous position $x(t)$?

- **Example 2:** A mobile **M** describes a trajectory with parametric equations:

$$\begin{cases} x = \cos(2t) \\ y = \sin(2t) \\ z = 2t \end{cases}$$

a) Graphically represent the trajectory of the motion of point **M**.

b) Express the velocity and acceleration of **M** in Cartesian and then cylindrical coordinates.

c) Verify that the magnitudes of the velocity and acceleration do not depend on the coordinate system used.

- **Example 3:** A point moves on the surface of a sphere of radius R . $\varphi = \frac{\pi}{6}$ and $\theta = t^2$, are respectively the spherical coordinates colatitude and longitude of this point.

- Find the velocity and acceleration of this mobile in the spherical basis, and their magnitudes.
- Find the velocity and acceleration in the Cartesian basis and then calculate their magnitudes again and verify that they coincide with the results of question 'a'.
- What is the trajectory of point M? Represent it qualitatively.

I.9. Relative Motion:

A motion is absolute if it is described relative to an absolute frame of reference (at absolute rest). The Earth can be assimilated with a very good approximation to an absolute frame of reference. A motion is relative if it is described relative to a moving frame of reference (relative frame).

Let a frame of reference (R) considered as fixed and (R') a frame of reference moving relative to (R). Let us associate with the frame of reference (R) the fixed basis ($\vec{i}, \vec{j}, \vec{k}$) and with the frame (R') the basis $(\vec{i}', \vec{j}', \vec{k}')$ moving relative to (R). If M is a moving point in (R), we will call:

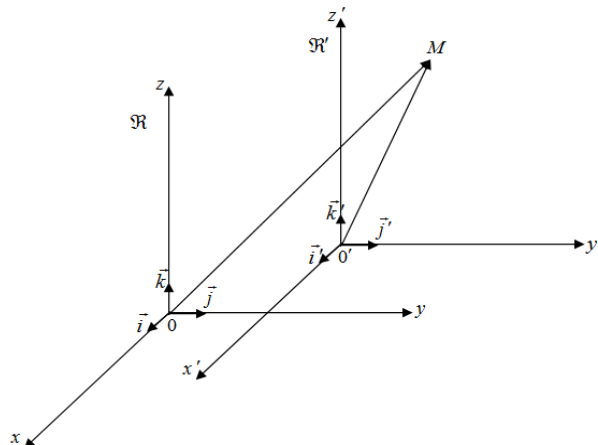


Figure I.9: Absolute and relative frames of reference.

- ✓ *Relative motion*, the motion of M relative to the frame (R') .
- ✓ *Absolute motion*, the motion of M relative to the frame (R) .
- ✓ *Motion of transport*, the motion of (R') relative to the frame (R) .

I.9.1. Composition of Velocities:

In the fixed frame of reference (R) , the position vector is written:

$$\overrightarrow{OM} = x \vec{i} + y \vec{j} + z \vec{k} \dots\dots\dots (I.40)$$

We deduce the velocity vector $\vec{v}_a$ of point M with respect to the frame of reference (R) :

$$\vec{v}_a = \frac{d(\overrightarrow{OM})}{dt} = \dot{x} \vec{i} + \dot{y} \vec{j} + \dot{z} \vec{k} \dots\dots\dots (I.41)$$

v_a : The absolute velocity of $M(x, y, z)$ relative to the frame (R) .

We proceed similarly in the moving frame of reference (R') to obtain the expression of the velocity vector of point M with respect to the frame of reference (R') :

$$\vec{v}_r = \frac{d(\overrightarrow{OM'})}{dt} = \dot{x}' \vec{i}' + \dot{y}' \vec{j}' + \dot{z}' \vec{k}' \dots\dots\dots (I.42)$$

$\vec{v}_r$: The relative velocity of $M(x', y', z')$ relative to the frame (R') .

However:

$$\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M} = \overrightarrow{OO'} + x' \vec{i}' + y' \vec{j}' + z' \vec{k}' \dots\dots\dots (I.43)$$

As:

$$\vec{v} = \frac{d(\overrightarrow{OM})}{dt} = \frac{d(\overrightarrow{OO'})}{dt} + \frac{d(\overrightarrow{O'M})}{dt}$$

$$\underbrace{\frac{d(\overrightarrow{OM})}{dt}}_{\vec{v}_a} = \frac{d(\overrightarrow{OO'})}{dt} + \overbrace{x' \dot{\vec{i}}' + y' \dot{\vec{j}}' + z' \dot{\vec{k}}'}^{\vec{v}_t} + \underbrace{\dot{x}' \vec{i}' + \dot{y}' \vec{j}' + \dot{z}' \vec{k}'}_{\vec{v}_r} \dots\dots\dots (I.44)$$

And

$$\vec{v}_e = \frac{d(\overrightarrow{OO'})}{dt} + x' \dot{i}' + y' \dot{j}' + z' \dot{k}' = \dot{x}_o \vec{i} + \dot{y}_o \vec{j} + \dot{z}_o \vec{k} + x' \dot{i}' + y' \dot{j}' + z' \dot{k}' \dots\dots\dots (I.45)$$

$\vec{v}_e$ the transport velocity of $R'(x_o', y_o', z_o')$ relative to the frame (R).

We deduce the relation of the law of composition of velocities:

$$\vec{v}_a = \vec{v}_e + \vec{v}_r \dots\dots\dots (I.46)$$

When the transport motion is in translational motion, the vectors $\vec{i}'$, $\vec{j}'$ and $\vec{k}'$ keep fixed directions so that: $\dot{i}' = \dot{j}' = \dot{k}' = \vec{0}$.

I.9.2. Composition of Accelerations:

We take the expressions obtained previously for the velocity vector of M with respect to the frame of reference (R), and we differentiate with respect to time to obtain the acceleration of M with respect to the frame of reference (R)

$$\begin{aligned} \vec{a}_a &= \frac{d\vec{v}_a}{dt} = \frac{d}{dt} \frac{d(\overrightarrow{OM})}{dt} \\ &= \frac{d^2(\overrightarrow{OO'})}{dt^2} + x' \ddot{i}' + y' \ddot{j}' + z' \ddot{k}' + \dot{x}' \dot{i}' + \dot{y}' \dot{j}' + \dot{z}' \dot{k}' + 2(x' \dot{i}' + y' \dot{j}' + z' \dot{k}') \end{aligned} \dots\dots\dots (I.47)$$

We obtain:

$$\underbrace{\vec{a}_a}_{\text{accélération absolue}} = \underbrace{\vec{a}_r}_{\text{accélération relative}} + \underbrace{\vec{a}_e}_{\text{accélération d'entraînement}} + \underbrace{\vec{a}_c}_{\text{accélération de Coriolis}}$$

With:

$$\left\{ \begin{array}{l} \vec{a}_a = \frac{d^2(\vec{OM})}{dt^2} \\ \vec{a}_e = \frac{d^2(\vec{OO'})}{dt^2} + x' \ddot{\vec{i}}' + y' \ddot{\vec{j}}' + z' \ddot{\vec{k}}' \dots\dots\dots (I.48) \\ \vec{a}_r = \ddot{x}' \vec{i}' + \ddot{y}' \vec{j}' + \ddot{z}' \vec{k}' \\ \vec{a}_c = 2 \left(\dot{x}' \dot{\vec{i}}' + \dot{y}' \dot{\vec{j}}' + \dot{z}' \dot{\vec{k}}' \right) \end{array} \right.$$

Where $\vec{a}_c$ represents the Coriolis acceleration or complementary acceleration. If the transport motion is a translation without rotation of (R'), $\vec{a}_c = \vec{0}$. If (R') is rotating relative to the frame (R), we define the rotation vector as:

$$\vec{\omega} = \omega \vec{k} \dots\dots\dots(49)$$

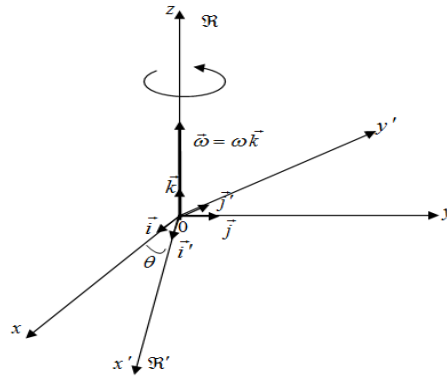


Figure I.10: Relative frame in rotation around a fixed axis of an absolute frame

The differentiation of the rotating unit vectors ($\vec{i}'$, $\vec{j}'$ and $\vec{k}'$) leads to:

$$\dot{\vec{i}}' = \frac{d\vec{i}'}{dt} = \vec{\omega} \wedge \vec{i}'$$

$$\dot{\vec{j}}' = \frac{d\vec{j}'}{dt} = \vec{\omega} \wedge \vec{j}'$$

$$\dot{\vec{k}}' = \frac{d\vec{k}'}{dt} = \vec{\omega} \wedge \vec{k}'$$

Replacing in the previous equations in equation (I.48). The two laws of composition of velocities and accelerations take the following forms:

$$\vec{v}_a = \vec{v}_r + \vec{v}_e \Leftrightarrow \underbrace{\frac{d(\vec{OM})}{dt}}_{\vec{v}_a} = \underbrace{\frac{d(\vec{OO'})}{dt}}_{\vec{v}_e} + \vec{\omega} \wedge \vec{OM} + \underbrace{\frac{d(\vec{O'M})}{dt}}_{\vec{v}_r} \dots\dots\dots(50)$$

Where

$$\underbrace{\frac{d^2(\vec{OM})}{dt^2}}_{\vec{a}_a} = \left[\underbrace{\frac{d^2(\vec{OO'})}{dt^2}}_{\vec{a}_e} + \frac{d\vec{\omega}}{dt} \wedge \vec{OM} + \vec{\omega} \wedge (\vec{\omega} \wedge \vec{OM}) \right] + \underbrace{\frac{d^2(\vec{O'M})}{dt^2}}_{\vec{a}_r} + 2\vec{\omega} \wedge \vec{v}_r \dots\dots\dots(51)$$

 **The Coriolis Effect:** In the case of a body moving in a horizontal plane, the Coriolis acceleration $2\vec{\omega} \wedge \vec{v}_r$ deviates the trajectory from the straight line: to the right in the northern hemisphere, and to the left in the southern hemisphere. The effect increases towards the poles. As an example of the Coriolis effect, let us mention the swirling of the wind in a hurricane. If a center of low pressure develops in the atmosphere, the wind will blow radially towards the center; however, the Coriolis acceleration deviates the air molecules to the right of their trajectory, in the northern hemisphere, and this results in a swirling motion counterclockwise, as shown in the figure.

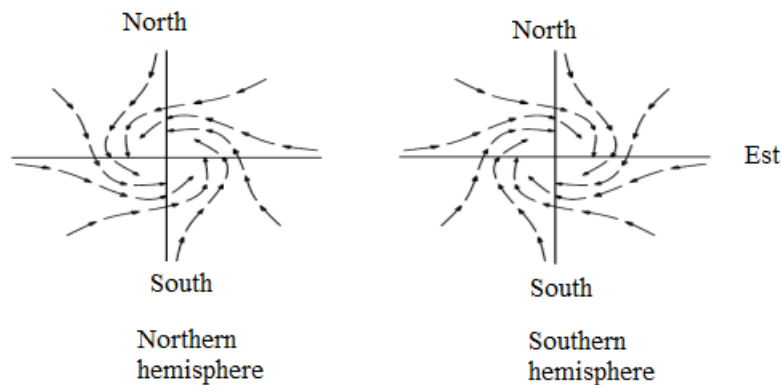


Figure I.11 : Coriolis Effect

CHAPTER II

DYNAMICS OF THE MATERIAL POINT

1. Introduction:

Dynamics is the study of the relationships between forces and the motions they produce. It describes how bodies move under the action of external forces. In contrast to kinematics, which describes *how* a body moves, dynamics seeks to explain *why* it moves and introduces the concepts of mass and force.

1.1. Mass:

The mass of a system characterizes the quantity of matter it contains. It is invariable in Newtonian mechanics. In relativistic mechanics, it depends on velocity through the expression:

$$m = m_0 \left(\sqrt{1 - \frac{v^2}{c^2}} \right)^{-1} \dots\dots\dots (II.01)$$

m: mass at velocity v

m_0 : rest mass

$c \approx 3.10^8 \text{ ms}^{-1}$: speed of light.

II.1.2. Notion of Force:

Any action that modifies the velocity vector of a material point, both in magnitude and direction, characterizes a force. A force is therefore a vector quantity. The velocity vector varies in a direction that is that of the force vector. The magnitude of the force vector indicates the importance of the effect. This definition highlights the vector nature of forces, and therefore their additivity, in addition to providing a simple physical interpretation. We distinguish two main categories of forces:

- Forces at a distance: electromagnetic interaction force, gravitational interaction force, etc.
- Contact forces: force of the reaction of a solid support, force of the tension of a spring, etc.

II.1.3. Momentum:

Momentum is defined as the product of the mass and the velocity of a body. It is a vector quantity; its direction and sense are those of the displacement; it characterizes the state of motion relative to a frame of reference.

$$\vec{p} = m\vec{v} \dots\dots\dots (\text{II.02})$$

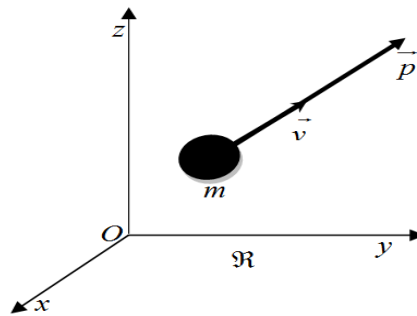


Figure II.1: Momentum $\vec{p}$ of a mass m , animated by a velocity $\vec{v}$ relative to a frame of reference $\mathfrak{R}$.

This vector depends on the frame of reference in which the velocity is expressed. It is collinear to the velocity of the point and is expressed in kg m/s.

II.2. Newton's Laws:

Newton's laws establish the relationships between the forces acting on a body and the variation of their momentum. Newton's first law is the principle of inertia, the second is the fundamental principle of dynamics, and the third is the principle of action and reaction.

II.2.1. Principle of Inertia (Newton's First Law):

Newton's first law (principle of inertia) accounts for the inertia of a body: "a body, on which no force acts ($\Sigma F = 0$), does not modify its momentum (p)". Since $p = m v$, then $v =$ constant. The principle of inertia states that:

In a Galilean frame of reference $\mathfrak{R}$, the center of inertia of any mechanically isolated material system is either at rest or in uniform rectilinear motion.

It is important to note that according to this principle, if a system is mechanically isolated, that is, if this system undergoes no action or undergoes compensated actions, then the motion of the particular point that is its center of inertia is rectilinear and uniform. It follows that a system can therefore be in motion even if it undergoes no action. It can just as well be at rest. The principle stipulates, however, that if the system mechanically

isolated in the Galilean frame of reference is in motion, then the motion of its center of inertia is necessarily rectilinear and uniform.

The principle of inertia therefore applies only in certain frames of reference called Galilean. *A Galilean frame of reference is defined as a frame of reference in which the principle of inertia applies.*

II.2.1.1. Conservation of Momentum:

The application of the principle of inertia leads to the law of conservation of the system's momentum:

$$\vec{v} = \overrightarrow{const} \Rightarrow \vec{p} = \overrightarrow{const} \Rightarrow \frac{d\vec{p}}{dt} = \vec{0} \dots\dots\dots (II.03)$$

The momentum vector is conserved if the principle of inertia is verified.

Note: the principle of inertia only predicts the motion of the center of inertia.

II.2.2. Fundamental Principle of Dynamics (Newton's Second Law):

If a system is not mechanically isolated, that is, if it undergoes an uncompensated action, the principle of inertia tells us that its momentum cannot be constant in time. The fundamental principle of dynamics allows us to link the cause (uncompensated actions) to the observed effect (variable momentum). It describes how the forces acting on a body modify its velocity.

$$\sum \vec{F}_{ext} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} \dots\dots\dots (II.04)$$

Since the mass of the system is assumed constant in time, it follows that the fundamental relation of dynamics or **RFD** can be written in the form:

$$\sum \vec{F}_{ext} = m \frac{d\vec{v}}{dt} = m\vec{a} \dots\dots\dots (II.05)$$

II.2.2.1. Fundamental Principle of Dynamics in a Non-Galilean Frame of Reference:

Let R be a Galilean frame of reference and R' another frame of reference in motion relative to R. Is R' a Galilean frame of reference?

If R' is in uniform rectilinear translation relative to R, then the absolute acceleration is equal to the relative acceleration. In this case, if M is a free material point in R, meaning its acceleration is zero in this frame, then its acceleration is also zero in R'. R' is therefore a Galilean frame of reference.

If R' is not in uniform rectilinear translation relative to R, but is animated by a rectilinear translation that is not uniform, or is animated by a rotation relative to R, then the absolute and relative accelerations are not equal. Therefore, if M is a free material point in R, then its acceleration in R' will not be zero. M cannot then be considered a free material point in R'. The frame of reference R' is not a Galilean frame of reference.

a. Fundamental Principle of Dynamics in a Galilean Frame of Reference:

Let R be a Galilean frame of reference and R' another frame of reference in motion relative to the frame R. The fundamental principle of dynamics applies in R:

$\vec{F} = m\vec{a}$ where $\vec{a}$ is the acceleration of the material point in the frame of reference R and $\vec{F} = \sum \vec{F}_{ext}$ is the resultant of the forces acting on the material point M.

We assume here that the frame of reference R' is also a Galilean frame of reference. We deduce that the frame of reference R' is animated by a uniform rectilinear translation relative to the frame of reference R. Consequently $\vec{a}_R = \vec{a}_{R'}$ meaning the absolute acceleration is equal to the relative acceleration. We can therefore write in the frame of reference R':

$$\vec{F} = m\vec{a}_{R'} \dots\dots\dots (II.06)$$

b. Fundamental Principle of Dynamics in a Non-Galilean Frame of Reference:

We assume here that the frame of reference R' is not a Galilean frame of reference. The fundamental principle of dynamics can therefore only be written in the frame of reference R where the law of composition of accelerations is written as: $\vec{a}_a = \vec{a}_r + \vec{a}_e + \vec{a}_c$ which gives the absolute acceleration as a function of the relative, transport, and Coriolis accelerations. The fundamental principle of dynamics is therefore written in R:

$$m\vec{a}_a = m\vec{a}_e + m\vec{a}_r + m\vec{a}_c = \vec{F} \dots\dots\dots (II.07)$$

Or alternatively

$$m\vec{a}_r = \vec{F}' = \vec{F} + \vec{F}_e + \vec{F}_c \dots\dots\dots (II.08)$$

Where $\vec{F}_e = -m\vec{a}_e$ is the transport force.

And $\vec{F}_c = -m\vec{a}_c$ is the Coriolis force (or complementary force).

These two forces are often called *Pseudoforces* because they are not real in the sense that the term force is reserved for the situation where there is interaction between the material point considered and another material point. The terms $\vec{F}_e$ and $\vec{F}_c$ absolutely do not fall into this category, hence the term *Pseudoforce*. However, the consequence of these *Pseudoforces* is very real.

II.2.3. Principle of Action and Reaction (Newton's Third Law):

Knowing that for every force $\vec{F}$ exerted on a body corresponds a second force $\vec{F}'$ of the same magnitude but opposite in direction applied to another body, we write: $F_{A \rightarrow B} = -\vec{F} = -\vec{F}'$. Where $\vec{F}_{1 \rightarrow 2} = -\vec{F}_{2 \rightarrow 1}$

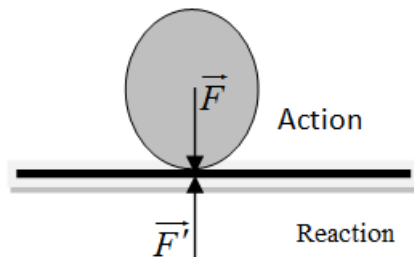


Figure II.2: Principle of Action and Reaction

Two bodies exert on each other forces equal in magnitude and opposite in direction.

II.3.1. The Differential Equations of Motion

The study of the motion of a material point aims to determine the time equations of the trajectory, that is, the law of evolution of the components of the position of the material point as a function of time. The time equations of the trajectory can only be obtained if the acceleration of this point is known beforehand. It is by taking stock of the actions that act on the material point that we determine, through the fundamental relation of dynamics, the acceleration of the material point. We then obtain the differential equation

of the motion of the material point, that is, an equation that relates the acceleration, velocity, and instantaneous position of the point to the time variable t .

We will distinguish several types of differential equations according to their forms. As a non-exhaustive example, we find the following differential equations:

- **First-order differential equation:**

$$x'(t) + \frac{x(t)}{\tau} = \eta$$

η : is a constant. If the initial condition is $x(0) = x_0$, the solution of the equation

(II) is of the form:

$$x(t) = \eta\tau + \left(x_0 - \eta\tau \exp\left(-\frac{t}{\tau}\right) \right)$$

- **Second-order differential equation:**

$$x''(t) + \omega_0^2 x(t) = 0$$

The solution of the equation (II) is of the form:

$$x(t) = A \cos(\omega t + \varphi)$$

The solution of differential equations depends strongly on the initial conditions of the motion through which the time equations of the motion can be determined.

II.4. Example of Forces:

In this section, we describe some common forces.

II.4.1. Weight Force:

Force of attraction exerted by the mass of the Earth on a body (gravitation). It is proportional to the mass of the body.

$P = m g$, P : weight force (N), m : mass of the body (kg), and g : acceleration due to gravity (m/s^2).

II.4.2. Newtonian Gravitational Forces:

The force of gravitation or gravitational interaction force is the force exerted by one mass m_1 on another mass m_2 . This interaction force follows a law stated by Newton in 1650.

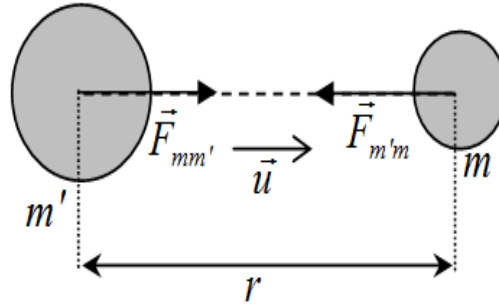


Figure II.3: Gravitational forces of an object of mass m' on an object of mass m .

The two bodies attract each other due to their masses and the inverse square of their distance according to a direction that passes through their centers of mass. The law of universal attraction is expressed analytically as follows:

$$\vec{F}_{mm'} = G \frac{mm'}{r^2} \vec{u} \dots\dots\dots (II.09)$$

$\vec{u}$: unit vector and $G = 6.67 \cdot 10^{11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$ gravitational interaction constant.

II.4.3. Friction Forces:

Friction forces are forces that appear either during the motion of an object or when this object is subjected to a force that tends to want to move it. Friction opposes the displacement of objects in motion. There are two types of friction:

- Solid friction (solid-solid contact).
- Viscous friction (solid-fluid contact).

II.4.3.1. Solid Friction (dry friction, sliding friction):

Force that appears at the contact surface of two solid bodies and which opposes their relative motion. This force acts parallel to the contact surface and is proportional to the normal reaction of the contact surface. The coefficient of proportionality is called the coefficient of friction.

- **Kinetic Friction (sliding):** acts on a body in motion that slides along the contact surface. This friction force $\vec{f}$ is equal to the product of the normal reaction $\vec{N}$ to the contact surface by the coefficient of kinetic friction μ_c .

$$F_f = \mu_c N$$

$$\frac{f}{N} = \tan \beta = \mu_c \dots\dots\dots (II.10)$$

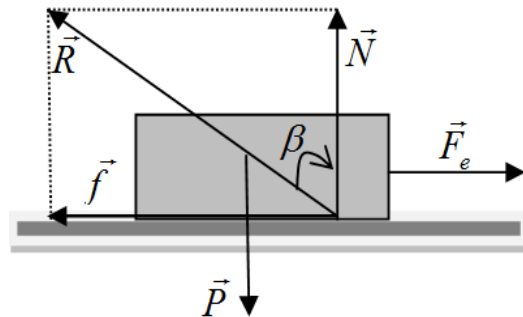


Figure II.4: Solid in motion on a support under the action of an external force.

- **Static Friction:** when a body rests on a contact surface, sliding only occurs if an external force is applied, tangential to the contact surface, with a magnitude greater than a certain value $\vec{F}_s$, called static friction. The maximum value of this friction force does not depend on the size of the contact surfaces, it is proportional to the normal component $\vec{R}_N$ of the force applying the body to its support and to the coefficient of static friction[μ_s]:

$$F_s \leq F_{s \max} = \mu_s R_N \dots\dots\dots (II.11)$$

II.4.3.2. Viscous Friction:

When a solid moves in a fluid (gas like air or liquid like water), it undergoes, from the fluid, friction forces. The resultant of these actions is a force vector proportional to the velocity vector of the object's displacement.

$$\vec{F} = -k . \vec{v} \dots\dots\dots (II.12)$$

α : positive constant. This force only exists if there is motion.

II.4.4. Tension Forces:

Tension force or restoring force. The simplest example is the restoring force of a spring.

$$\vec{F} = -k . (l - l_0) \vec{u} \dots\dots\dots (II.13)$$

k: elongation coefficient (spring stiffness coefficient).

When an operator pulls on one end of a string (the other end being fixed), it becomes taut. Simultaneously, the string exerts a resistance or an action on the operator. This action of the string on the operator is called the tension of the string. It only exists if the string is taut under the effect of an external action.

For a cord of negligible mass supporting an object of mass m at rest, the tension of the cord opposes the weight of the mass m according to the principle of reciprocal actions. It takes the same value at every point of the cord.

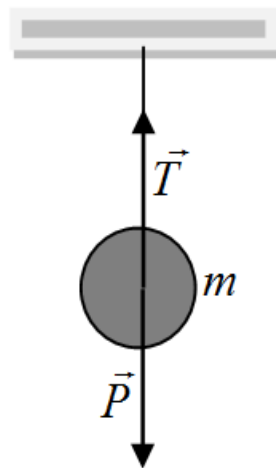


Figure II.5: Tension of a string.

When the cord is elastic, the tension of the cord can be expressed as a function of the state of stretching of the cord and increases linearly with its elongation. The elongation coefficient is called the stiffness k of the cord. A typical example of an elastic cord is the spring. The tension force of a spring of unstretched length l_0 and stretched to length l is written as:

$$\vec{T} = -k.(l - l_0) \vec{u} \dots\dots\dots (II.14)$$

With u a unit vector in the direction of the deformation. The sign $(-)$ in this relation means that the tension force of the spring is a restoring force and that it opposes the deformation.

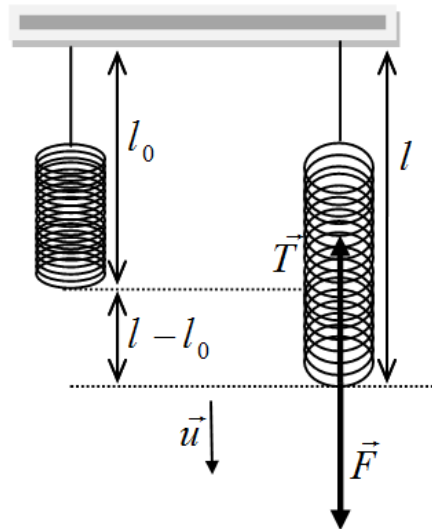


Figure II.6: Tension of a spring.

II.5. Angular Momentum:

Analogous to momentum, an angular momentum is defined which accounts for a rotational motion. It is also a vector quantity that points in the direction of the axis of rotation. The angular momentum $\vec{L}$ (orbital moment) of a material point with radius vector $\vec{r}$, animated by a momentum $\vec{p} = m\vec{v}$, is:

$$\vec{L} = m\vec{r} \wedge \vec{v} = \vec{r} \wedge \vec{p} \dots\dots\dots (\text{II.15})$$

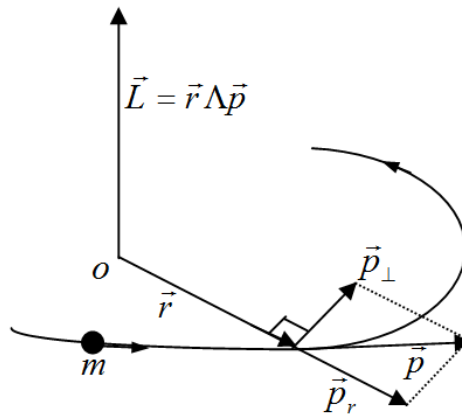


Figure II.7: Angular momentum L_O of a material point M.

II.5.1. Properties of Angular Momentum:

The angular momentum of a material point is a vector perpendicular to the velocity and the radius vector. Its magnitude is equal to: $L = r p \sin \alpha$, where α is the angle between the radius vector and the momentum vector.

The angular momentum can be expressed as a function of the orbital velocity $\vec{v}$ or the rotation vector $\vec{\omega}$.

$$\vec{L} = m\vec{r} \wedge \vec{v} = m\vec{r} \wedge (\vec{\omega} \wedge \vec{r}) \dots\dots\dots (II.16)$$

This double vector product can also be written as:

$$\vec{L} = mr^2 \vec{\omega} - (\vec{r} \cdot \vec{\omega}) \vec{r} \dots\dots\dots (II.17)$$

If the radius vector and the rotation vector are orthogonal (case of the rotation of a disk around its axis), we have $\vec{r} \cdot \vec{\omega} = 0$:

$$\vec{L} = mr^2 \vec{\omega} = I \vec{\omega} \dots\dots\dots (II.18)$$

The angular momentum is then parallel to the rotation vector, hence the axis of rotation, and perpendicular to the plane of the trajectory. The quantity $I = mr^2$ is the moment of inertia of the material point of mass m.

II.5.2. Moment of a Force:

The moment of a force is the product of the force and its lever arm relative to a fixed point around which the body can rotate. A force can set a body in translational motion, and the moment of a force can induce a rotational motion:

$$\vec{M}_o(\vec{F}) = \vec{r} \wedge \vec{F} \dots\dots\dots (II.19)$$

II.5.3. Theorem of Angular Momentum:

Analogous to Newton's second law, the variation of angular momentum is equal to the moment of the external forces applied to a point of mass m:

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \wedge m\vec{v} + \vec{r} \wedge \frac{d\vec{p}}{dt} \dots\dots\dots (II.20)$$

The first term on the right is zero because $\frac{d\vec{r}}{dt} = \vec{v}$ is parallel to $\vec{p} = m \vec{v}$, and the vector product of a vector with itself is zero. The term $\frac{d\vec{p}}{dt}$ is given by the applied force :

$$\frac{d\vec{L}}{dt} = \vec{r} \wedge \frac{d\vec{p}}{dt} = \vec{r} \wedge m\vec{a} = \vec{r} \wedge \vec{F} = \vec{M}_0(\vec{F}) \dots\dots\dots (II.21)$$

The theorem of angular momentum determines all rotational motions. If the material point is subjected to a central force of the same direction as the radius vector $\vec{r}$, then the moment of this force is zero and the angular momentum is conserved in magnitude and direction. Gravitational attraction is a central force; for the motion of planets around the Sun, it follows from the conservation of angular momentum.

II.6. APPLICATIONS

II.6.1. Uniform Motions:

The system under study is a material point **M** of mass **m**. The frame of reference in which the motion is studied is a Galilean frame.

- **Uniform Rectilinear Motion:** The system is subjected to no force or to a set of forces whose resultant is zero. The fundamental principle of dynamics applied to the material point **M** leads to:

$$\sum \vec{F}_{ext} = m\vec{a} = \vec{0}$$

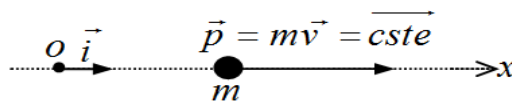


Figure II.8: Uniform rectilinear motion

By definition, a motion is uniform rectilinear if $\vec{v} = \overline{cste}$. The initial conditions are as follows: at $t = 0$, $\vec{v}(0) = v_0 \vec{i}$ et $OM(0) = x_0$.

It follows that the time equation of the motion is:

$$\dot{x} = v_0 \Rightarrow x(t) = v_0 t + x_0$$

- **Uniform Circular Motion:** The force balance is such that there is no force applied to M or a set of forces whose resultant moment about a fixed point O is zero. The theorem of angular momentum is written as:

$$\sum \vec{M}_{/O}(\vec{F}_{ext}) = \vec{0} \Rightarrow I_{\Delta} \frac{d^2\theta}{dt^2} = 0 \Rightarrow \frac{d\theta}{dt} = cste = \omega_0$$

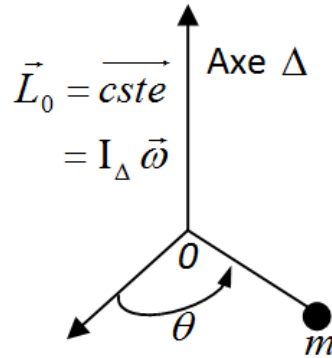


Figure II.9: Uniform circular motion

By definition, a motion is uniform circular if $\vec{\omega} = \frac{d\theta}{dt} \vec{k} = \omega_0 \vec{k}$. It follows that the time equation of the motion is:

$$d\theta = \omega_0 dt \Rightarrow \theta(t) = \omega_0 t + \theta_0$$

II.6.2. Uniformly Varied Motion

- **Uniformly Varied Rectilinear Motion:** The force balance reduces to a set of forces whose resultant ΣF is constant. The fundamental principle of dynamics leads to:

$$\sum \vec{F}_{ext} = m\vec{a} = \vec{F}$$

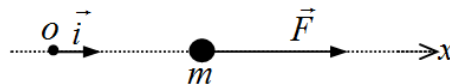


Figure II.10: Uniformly varied rectilinear motion

The projection onto the direction of motion gives us: $a = \frac{F}{m} = cste$. With the initial conditions:

at $t = 0$, $\vec{v}(0) = v_0 \vec{i}$ et $OM(0) = x_0$, we have, $OM(0) = x_0$,

$$\ddot{x} = \frac{F}{m} \Rightarrow \dot{x} = \frac{F}{m}t + v_0$$

This leads to the time equation:

$$x(t) = \frac{F}{2m}t^2 + v_0t + x_0$$

- **Uniformly Varied Circular Motion:** The force balance reduces to a set of forces whose resultant moment about a fixed point O is constant. The application of the theorem of angular momentum leads to:

$$I_{\Delta} \ddot{\theta} = \sum \vec{M}_{/O}(\vec{F}_{ext}) = \overrightarrow{cste} = F \cdot r$$

(where I_{Δ} is the moment of inertia and α is the angular acceleration)

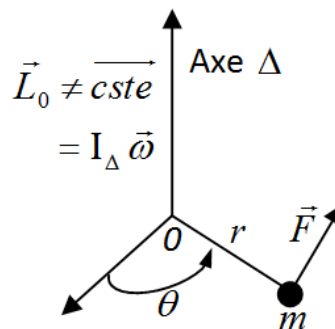


Figure II.11: Uniformly varied circular motion

With $r = OM$. We therefore have

$$\ddot{\theta} = \frac{F \cdot r}{mr^2} = \frac{F}{mr} = cste$$

Considering that at $t = 0$, $\dot{\theta}(0) = \dot{\theta}_0$ et $\theta(0) = \theta_0$, it follows:

$$\dot{\theta} = \frac{F}{mr}t + \dot{\theta}_0$$

This leads to the following time equation:

$$\theta(t) = \frac{F}{2mr} t^2 + \dot{\theta}_0 t + \theta_0$$

II.6.3. Arbitrary Motions

- **Braked Fall of a Body:** We assume that the body of mass $\mathbf{m}$ is decelerated during its fall by a viscous-type friction force. We study the problem in a terrestrial frame assumed to be Galilean. The fall occurring on a single axis, we limit ourselves to one basis vector (Figure II. 12). We will assume that the mass $\mathbf{m}$ falls with an initial velocity of zero from a position $x = 0$ to $t = 0$.

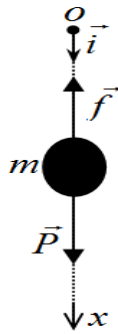


Figure II.12: Braked fall of a body

The applied external forces are: the weight $\vec{P}$ and the friction force $\vec{f} = -k \vec{v}$. The application of the fundamental relation of dynamics leads to:

$m \frac{d\vec{v}}{dt} = \vec{P} + \vec{f} = m\vec{g} - k \vec{v}$ Which, when projected onto the axis Oy of the motion, gives the differential equation of the motion of the mass $\mathbf{m}$:

$$\frac{dv_x}{dt} + \frac{k}{m} v_x = g \text{ or}$$

This equation is a first-order differential equation with constant coefficients and a constant right-hand side. The solution method consists of calculating a solution to the homogeneous equation and adding a particular solution independent of time. The

homogeneous equation is written as: $\frac{dv_x}{dt} + \frac{k}{m}v_x = 0$. The solution of this equation results in the velocity of the mobile varying exponentially according to a law of the type:

$$v_x = A \exp\left(-\frac{k}{m}t\right)$$

We look for a particular solution of the differential equation that does not depend on time. We therefore have:

$$\frac{dv_x}{dt} = 0 \Rightarrow v_x = \frac{mg}{k}$$

The general solution of this equation is therefore written as:

$$v_x = \frac{mg}{k} + A \exp\left(-\frac{k}{m}t\right)$$

It only remains to determine the constant A by returning to the boundary conditions of this motion which impose that $v(0) = 0$. It follows therefore:

$$\frac{mg}{k} + A = 0 \Rightarrow A = -\frac{mg}{k}$$

This leads to:

$$v_x = \frac{mg}{k} \left(1 - \exp\left(-\frac{k}{m}t\right)\right)$$

We can thus see that the velocity increases progressively to reach a limiting velocity when time tends to infinity (figure 3.19). The limiting velocity of fall, which is logically never reached, is given by:

$$v_{\lim} = \frac{mg}{k}$$

$$x(t) = \frac{mg}{k} \left(t - \frac{mg}{k} \left(\exp\left(-\frac{k}{m}t\right) - 1 \right) \right) \mathbf{S}$$

- **simple Pendulum:** Let us consider a mass $\mathbf{m}$ mobile around a fixed axis. The distance from the mass $\mathbf{m}$ to the axis of rotation is called l . We consider the motion

of the mass system $\mathbf{m}$ relative to a Galilean frame $R(O, x, y, z, t)$. The chosen basis is the moving basis $(\vec{u}_\rho, \vec{u}_\theta)$.

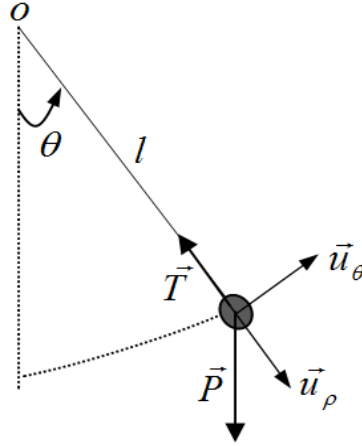


Figure II.13: Representation of a simple pendulum.

The applied external forces are the weight $\vec{P} = m\vec{g}$ and the tension $\vec{T}$ of the cord. The fundamental relation of dynamics leads to:

$$\begin{aligned}\vec{P} + \vec{T} &= m\vec{a} \\ \vec{P} + \vec{T} &= m(-l\dot{\theta}^2 \vec{u}_\rho + l\ddot{\theta} \vec{u}_\theta)\end{aligned}$$

In projection onto the basis vectors, the weight and tension are written as:

$$\begin{aligned}\vec{P} &= mg \cos \theta \vec{u}_\rho - mg \sin \theta \vec{u}_\theta \\ \vec{T} &= -T \vec{u}_\rho\end{aligned}$$

which leads to radial and orthoradial compounments:

$$\begin{cases} mg \cos \theta - T = -ml\dot{\theta}^2 \\ -mg \sin \theta = ml\ddot{\theta} \end{cases}$$

From the second equation, it is possible to write the differential equation of the motion of the mass $\mathbf{m}$:

$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0$$

This equation is a non-linear differential equation because of the presence of the sine term. The solution is therefore not easy to obtain except under certain conditions where the equation can be approximated by a linear equation. This condition is satisfied when the angle θ is small, meaning when the sine is approximable by the angle, i.e., $\sin \theta \approx \theta$. In this case, the differential equation becomes:

$$\ddot{\theta} + \frac{g}{l} \theta = 0$$

The solution of this equation is written as:

$$\theta(t) = \theta_{\max} \sin(\omega_0 t + \varphi)$$

provided we set: $\omega_0^2 = \frac{g}{l}$.

The previous differential equation could have been obtained directly by the theorem of moments calculated at **O**, namely:

$$ml^2 \ddot{\theta} \vec{k} = \vec{M}_{/O}(\vec{P}) + \vec{M}_{/O}(\vec{T}) = \vec{M}_{/O}(\vec{P}) \Rightarrow ml^2 \ddot{\theta} = -mgl \sin \theta$$

CHAPTER III:

WORK AND

ENERGY

III.1. Introduction:

Work and energy are essential quantities in the description of physical processes. Energy manifests itself in various forms (thermal, chemical, electromagnetic, electrical, nuclear, mechanical...). Total energy is conserved and through work, it can be exchanged between bodies. The various forms of energy can transform into one another. Energy exchanges are generally irreversible: for example, mechanical energy can be completely transformed into thermal energy, the reverse is not true, because it is accompanied by a loss of heat.

III.2. Work of a Force

III.2.1. Constant Force on a Rectilinear Displacement:

Consider an object assimilated to a material point **M** moving on a straight portion, from a point **A** to a point **B**, and subjected to a force **F** constant during the displacement.

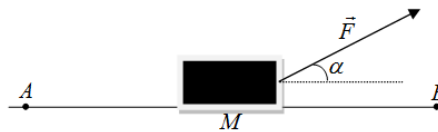


Figure III.1: Displacement of an object subjected to a constant force

By definition, the work of a constant force $\vec{F}$ on a rectilinear displacement **AB** is equal to the scalar product of the force vector by the displacement vector:

$$W = \vec{F} \cdot \overrightarrow{AB} = F AB \cos \alpha \dots\dots\dots (III.01)$$

With α the angle between the force and the displacement.

1 *Joule* is the work provided when a body of 1 kg is displaced by 1 m by a force of 1 N. 1 J
 $[W] = \text{Joule} = N \cdot m = kg \cdot m^2 \cdot s^{-2}$.

The work is either positive, zero, or negative depending on the direction of the force $\vec{F}$ relative to the displacement. If $\vec{F}$ is perpendicular to $\overrightarrow{AB}$ the work is zero, the force $\vec{F}$ does not contribute to moving the object. When the force opposes the

displacement, it is resistive and the work is negative. When the force is driving, the work is positive.

$W > 0$: displacement and force have components in the same sense ($\cos \alpha > 0$).

$W < 0$: displacement and force have components in opposite senses ($\cos \alpha < 0$).

III.2.2. Elementary Work:

In the case where the force F varies during the displacement which can be arbitrary, it is no longer possible to use the previous expression. Indeed, the force can constantly change orientation and intensity over the considered displacement. To calculate the work, the path AB is then decomposed into a succession of elementary displacements $d\vec{l} = MM'$, infinitely small and rectilinear. On any one of these elementary paths, the force vector F can be considered constant and the previous definition can be applied. The expression of the elementary work on such an elementary displacement can therefore be written as:

$$dW_{M \rightarrow M'}(\vec{F}) = \vec{F} \cdot d\vec{l} \dots\dots\dots(III.02)$$

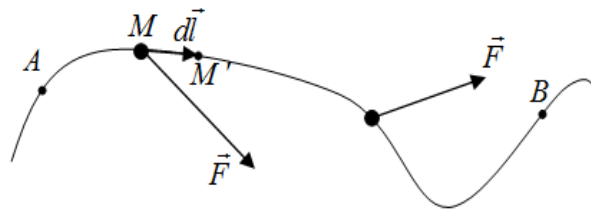


Figure III.2: Displacement of an object subjected to a variable force.

To obtain the total work of the force over the total displacement AB , it suffices to add the elementary works when going from point A to point B . The summation is continuous, which leads to:

$$W_{A \rightarrow B} = \int_A^B \vec{F} \cdot d\vec{l}$$

III.3. Energy:

A characteristic quantity of the state of a body (state of motion, temperature, deformation). The energy of a body can be modified by the work it performs or that is supplied to it. This work then modifies its state (position, velocity, acceleration, temperature, deformation).

III.3.1. Kinetic Energy:

The kinetic energy of a body measures the work that this body can perform by its motion. Consider a material point M moving, in a Galilean frame R , under the action of a set of external forces. The motion of this point is governed by the fundamental principle of dynamics, namely:

$$\sum \vec{F}_{ext} = m\vec{a} = m \frac{d\vec{v}}{dt} \dots\dots\dots(III.03)$$

During an elementary displacement $d\vec{l}$, the sum of the elementary works of the external forces is given by:

$$\sum \vec{F}_{ext} \cdot d\vec{l} = m \frac{d\vec{v}}{dt} \cdot d\vec{l} = m\vec{v}d\vec{v} \dots\dots\dots(III.04)$$

By integrating this relation over a path **AB** we obtain:

$$m \int_{v_A}^{v_B} \vec{v} \cdot d\vec{v} = \sum \int_A^B \vec{F}_{ext} \cdot d\vec{l} = W_{A \rightarrow B}(\vec{F}) \dots\dots\dots(III.05)$$

Namely:

$$\frac{1}{2} m (v_B^2 - v_A^2) = W_{A \rightarrow B}(\vec{F}) \dots\dots\dots(III.06)$$

The work provided $W_{A \rightarrow B}(\vec{F})$ depends only on the initial velocity v_A and final velocity v_B of the body. Therefore $W_{A \rightarrow B}(\vec{F})$ is the work to be done to increase the velocity of a body from its initial value v_A to its final velocity v_B .

The kinetic energy of a body whose velocity relative to a given frame of reference is:

$$E_{cin} = \frac{1}{2} m v^2 \dots\dots\dots(III.07)$$

It measures the work necessary to bring a body at rest ($v_0 = 0$) to the final velocity v .

- **Theorem of Kinetic Energy:** according to equation (III.06), we see that the variation of kinetic energy of the material point between two positions is equal to the work of all the forces applied to this point, which constitutes a theorem called the theorem of kinetic energy.

In a Galilean frame of reference, the variation of kinetic energy of a material point subjected to a set of external forces between a position A and a position B is equal to the sum of the works of these forces between these two points.

$$\Delta E_{cin} = E_{cin}(B) - E_{cin}(A) = W_{A \rightarrow B}(\vec{F}_{ext}) \dots\dots\dots(III.08)$$

III.3.2. Potential Energy:

In general, an energy that depends only on the position of a body in space and does not depend on its velocity is called potential energy.

III.3.2.1. Conservative Forces and Potential Energies:

A force is conservative when its work between two points **A** and **B** does not depend on the path followed but only on the position of these two points. It then derives from a potential energy, a quantity that characterizes energetically the point **M** in each position. We write:

$$W_{A \rightarrow B}(\vec{F}^{Cons}) = E_p(A) - E_p(B) = -\Delta E_p \dots\dots\dots(III.09)$$

This relation leads, by expressing the work, to the integral definition of potential energy:

$$E_p(B) = E_p(A) - \int_A^B \vec{F} \cdot d(\vec{OM}) \dots\dots\dots(III.10)$$

From the integral expression (III.10), it is possible to deduce the differential definition of potential energy by showing the elementary work of the conservative force namely:

$$dE_p = -dW_{A \rightarrow B}(\vec{F}^{Cons}) = -\vec{F}^{Cons} d\vec{l} \dots\dots\dots(III.11)$$

Finally, the differential of potential energy can be expressed as a function of the gradient of E_p :

$$dE_p = \overrightarrow{grad}(E_p) d\vec{l} \dots\dots\dots(III.12)$$

We arrive at the local definition of potential energy:

$$\overrightarrow{\text{grad}} (Ep) = -\vec{F}^{Cons} \dots\dots\dots(\text{III.13})$$

III.3.2.1. Examples of Conservative Forces and Potential Energies:

- **Weight:**

The most classic example is weight, which derives from the gravitational potential energy. Consider a mass **m** moving from a point **A** at altitude z_A to a point **B** at altitude z_B and let's calculate the work of the weight of this body during this displacement. The displacement from **A** to **B** is assumed to be arbitrary, meaning that the path leading from **A** to **B** can take different trajectories.

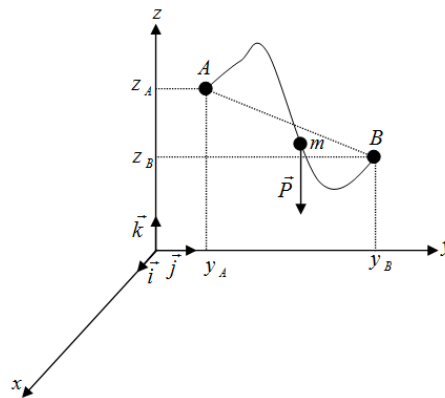


Figure III.3: Work of the weight of a body

Weight is a force constant in magnitude and direction. We thus obtain the following expression for the work of the weight:

$$W_{A \rightarrow B} = \int_A^B \vec{P}(M) d\vec{l} = \vec{P}(M) \overrightarrow{AB} = -mg(z_B - z_A) = -\Delta E_p$$

This result can be obtained using the coordinates of points **A** and **B** and the components of the vector **AB** in a Cartesian frame (O, x, y, z) . By orienting the Oz axis upwards, we can write:

$$\vec{P} = 0\vec{i} + 0\vec{j} - mg\vec{k}$$

and

$$\overrightarrow{AB} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} + (z_B - z_A)\vec{k}$$

Namely:

$$W_{A \rightarrow B} = \vec{P}(M) \overrightarrow{AB} = -mg(z_B - z_A)$$

- **Restoring Force of a Spring (Elastic Force):**

Consider a spring of stiffness k , rest length l_0 , at the end of which a mass m is attached. The spring and the mass are on a horizontal plane and we are only interested in the tension of the spring. The elastic force $\vec{T}$, i.e., the tension force of the spring, is a force that varies with the state of stretching of the spring x . It is therefore not a constant force during the displacement.

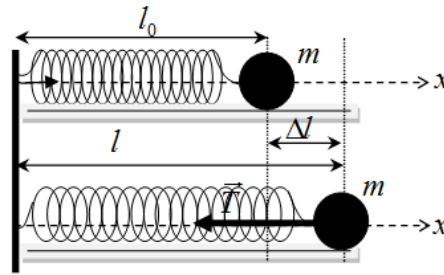


Figure III.4. Work of an elastic force.

To calculate the work of this force we need to calculate the elementary work of this force on an infinitely small displacement on which we will consider the force to be constant. With the orientation conventions of the vectors in figure III.4, the tension is expressed as follows:

$$\vec{T} = -k \Delta l \vec{i} = -k (l - l_0) \vec{i} = -k x \vec{i}$$

The elementary work of the elastic force $\vec{T}$, when the mass goes from a position x to a position $x + dx$, is therefore given by:

$$dW_{x \rightarrow x+dx} = \vec{T} d\vec{l} = -kx dx = -d\left(\frac{1}{2}kx^2\right)$$

When the point of application goes from a position x_1 to a position x_2 , the work of the elastic force is therefore:

$$W_{x_1 \rightarrow x_2} = \int_{x_1}^{x_2} \vec{T} d\vec{l} = -kx dx = -\frac{1}{2}k(x_2^2 - x_1^2) = -\Delta E_{p, \text{élastique}}$$

We notice that the work of this force does not depend on the path followed but only on the initial and final position of the spring.

III.3.2.2. Non-Conservative Force:

A force is conservative if its work on a displacement **AB** depends only on the position of points **A** and **B**, not on the path followed between **A** and **B**. Any constant force is conservative (e.g., weight, electric force, ...), but any conservative force is not necessarily constant (e.g., restoring force of a spring). Among others, friction forces are non-conservative.

- **Example of Non-Conservative Force:**

Friction forces are not conservative. Indeed, for example for a fluid friction force:

$$dW = \vec{f} d\vec{l} = -\lambda v_x dx = -\lambda v_x^2 dt$$

This relation cannot be written in the form of a differential, so one cannot define a potential energy. The work of conservative forces can easily be taken into account by introducing the concept of potential energy. Friction forces (dissipative forces) are not conservative and the work they perform depends on the path taken. Moreover, friction always opposing displacement, the work associated with it is negative. This work is generally dissipated as heat. In the case of solid friction:

$$W = \vec{f}_f d\vec{l} = -f_f x = -\mu_c N x$$

III.4. Power of a Force:

By introducing the definition of the elementary work of a force performed between instants t and $t+dt$, it is possible to define an instantaneous power $P(t)$ by:

$$P(t) = \frac{dW}{dt} = \vec{F} \cdot \vec{v} \dots\dots\dots (III.14)$$

This quantity is expressed in the International System of Units in **Watts** using the symbol **W**. It is therefore clear that the elementary work can also be expressed from the power of the force and written as:

$$dW = \vec{F} \cdot d\vec{l} = \vec{F} \cdot \vec{v} dt = P(t) dt \dots\dots\dots (III.15)$$

Which leads to the following expression for the work of a force:

$$W_{A \rightarrow B} = \int_A^B \vec{F} \cdot d\vec{l} = \int_A^B P(t) dt \dots\dots\dots (III.16)$$

III.4. Mechanical Energy:

We now introduce a new function particularly useful in all mechanics problems, the mechanical energy of a system. To define this function, we start from the theorem of kinetic energy in which we show the work of conservative forces and that of non-conservative forces, namely:

$$E_c(B) - E_c(A) = \sum W_{A \rightarrow B}(\vec{F}_{\text{conservative}}) + \sum W_{A \rightarrow B}(\vec{F}_{\text{Non conservative}}) \dots\dots\dots (III.17)$$

Calling E_p the total potential energy, sum of the potential energies from which each conservative force derives, we can write:

$$E_c(B) - E_c(A) = -[E_p(B) - E_p(A)] + \sum W_{A \rightarrow B}(\vec{F}_{\text{Non conservative}}) \dots\dots\dots (III.18)$$

Which, by moving the potential energy to the left-hand side, leads to:

$$[E_c(B) - E_c(A)] + [E_p(B) - E_p(A)] = \sum W_{A \rightarrow B}(\vec{F}_{\text{Non conservative}}) \dots\dots\dots (III.19)$$

It is possible to introduce a new state function called mechanical energy E_m of the system by setting:

$$E_m = E_c + E_p \dots\dots\dots (III.20)$$

The introduction of this function allows presenting the energy balance of a system in a very simple way by the following relation:

$$E_{B.méc} - E_{A.méc} = \sum W_{A \rightarrow B}(\vec{F}_{\text{Non conservative}}) \dots\dots\dots (III.21)$$

Which leads to the theorem of mechanical energy.

The variation of mechanical energy of a system between two points A and B is equal to the sum of the works of the non-conservative external forces applied to this system.

Non-conservative forces being resistive forces, the mechanical energy of a system can only decrease over time. However, when a system is mechanically isolated (that is, for a system that undergoes no non-conservative external force), the mechanical energy is conserved. If:

$$E_{\text{méc}}(B) - E_{\text{méc}}(A) = 0 \dots\dots\dots (\text{III.22})$$

References:

➤ R. DOUGLAS GREGORY, Classical Mechanics, Cambridge University Press, 2006

- T. COUSIN et H. PERODEAU, "Physique MPSI-PTSI", Dunod, Paris, 2011.
- A. GIBAUD et M. HENRY, "Cours de physique mécanique du point", 2^{ème} édition, Dunod, Paris, 2007.
- N. BILLY, J. DESBOIS, M.A. DUVAL, M. ELIAS, P. MONCEAU, A. PLASZCZYNSKI et M. TOULMONDE," CAPES de Sciences physiques, tome 1 – physique cours et exercices ", Éditions Belin, Paris, 2004.
- M.N. SANZ, F. CLAUSSET ET A. E. BADEL," PHYS-EN-UN ,1^{ère} année MPSI - PCSI – PTSI", 3e édition, Dunod, Paris,2002, 2003, 2008.
- H. Stöcker, F. Jundt et G. Guillaume,"Toute la physique", Dunod, Paris, 1999