REPUBLIQUE ALGERIENNE DEMOCRATIQUE ET POPULAIRE

MINISTERE DE L'ENSEIGNEMENT SUPERIEUR
ET DE LA RECHERCHE SCIENTIFIQUE

Université Ferhat Abbas Sétif 1

Handout
Waves and Vibrations:
Course

Module : **Waves and vibrations**

Level : **The second year of the Bachelor's degree**

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Foreword

This course handout is intended for second-year undergraduate students (LMD system) specializing in chemistry and physics. It covers a foundational module on the oscillations of mechanical and electrical systems, a topic that has seen significant growth in recent years. It has led to the considerable development of techniques for solving physical problems in various fields. This document provides a detailed course with worked examples. It is divided into two main parts: "Vibrations and Mechanical Waves," which are further divided into five chapters. These chapters are listed below:

CHAPTER I : FREE UNDAMPED OSCILLATIONS AT ONE DEGREE OF FREEDOM

CHAPTER II : DAMPED FREE OSCILLATIONS WITH ONE DEGREE OF FREEDOM

CHAPTER III : DAMPED FORCED OSCILLATIONS WITH ONE DEGREE OF FREEDOM

CHAPTER IV : N DEGREES OF FREEDOM

CHAPTER V : WAVES

The second part contains one chapter recommended to introduce the basic phenomena related to the propagation of mechanical waves in different material media. To this end, we have taken the vibrating cord as a model.

The objective sought is to provide students with elements that will allow them to enrich their knowledge on the one hand, and on the other hand, help them to better master the problems they may encounter in this module.

Dr SAOUDI Amer

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CHAPTER I :FREE UNDAMPED OSCILLATIONS AT ONE DEGREE OF FREEDOM

CHAPTER I :FREE OSCILLATIONS NOT DAMPED WITH ONE DEGREE OF FREEDOM

I.1 General information on oscillations

* Vibrations or oscillations are movements or changes of state of physical systems, which are repeated more or less regularly over time.

* A mechanical oscillation or vibration is a movement that a system performs around its equilibrium position (rest position) when the latter is moved away from this position or given an initial speed.

* When the movement of the system is fast, we speak of vibration, when it is slower, we speak of oscillation.

* The main property of oscillations is the elasticity or inertia of the system induced by an internal force called the restoring force. This force has the role of recalling or returning the system to its initial equilibrium position (rest position).

* The oscillatory or vibratory phenomenon can exist in solids, fluids or vacuum and at the microscopic (atomic) or macroscopic and cosmic (planetary) scale.

*The oscillatory or vibratory phenomenon can be mechanical or electromagnetic.

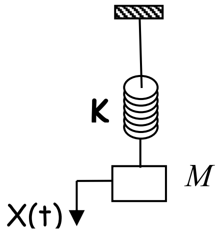
Examples of mechanical oscillators

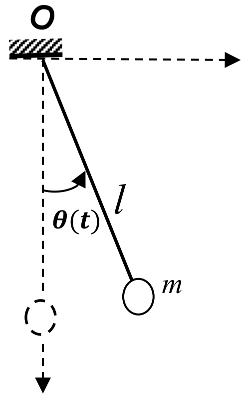
- Mass (M), spring (K): system (M,K)

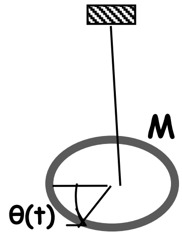
- Simple pendulum: (m,l) system

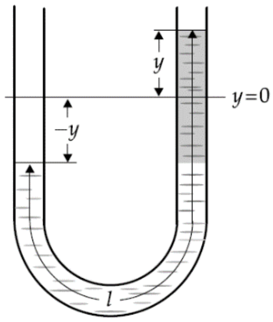
- Torsion pendulum: (disc M, wire)

- Hydraulic system: (U-tube, liquid)
- Bird wings movement, rotating machine,
- vibration of tree branches under the effect of the wind.



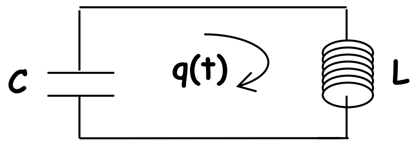




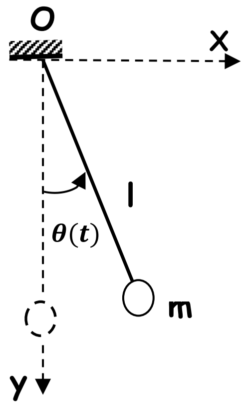


Examples of electromagnetic oscillators

- Electrical circuit Coil L, capacitance C: system (L,C)
- Luminous phenomena (radiation)



* A mechanical oscillation or vibration is said to be periodic if the motion of the system around its equilibrium position repeats itself identically at intervals of time réguliers $\mathbf{T_0}$ called period of the motion. This quantity is often used for oscillations while for vibrations, we use the quantity $\mathbf{f_0 = 1/T_0}$ called frequency which is more practical.



Physically:

$\mathbf{T_0}$ (in seconds) defines the time taken to complete a movement cycle.

$\mathbf{f_0}$ (in second^{-1} or Hertz) defines the number of motion cycles per second

* a periodic oscillation (or vibration) is characterized by a time function q(t) called the equation of motion such that $q(t) = q(t + T_0)$.

*** q(t) can be:**

- Angle of a pendulum (rotation).
- Position of a body on an axis (translation).
- Displacements of slices of a liquid or gas in a centenary.
- Electric field and magnetic field of a radio wave or light.
- Current or voltage in an electric circuit.

* The simplest periodic oscillations are called harmonic or sinusoidal oscillations having the equation of motion **$q(t)=A\cos(\omega_0 t+\varphi_0)$**

Where the constants:

- A is the amplitude of the movement (in m or rd …….) depending on the nature of the movement (translation, rotation, electric……..)
  - **$\omega_0=2\pi/T_0$** (in rd/s) is the proper pulsation of the movement.
  - **$\omega_0 t + \varphi_0$** (in rd) is the phase at time t.
  - **φ_0** (in rd) is the initial phase at t=0.

* A mechanical oscillator is called a **harmonic** oscillator when, once removed from its equilibrium position by a distance **q** (**$q=x$** for translation or **$q=\theta$** angle for rotation), it is subjected to a restoring force opposite and proportional to the distance x (or θ) such that

$F=-C\mathbf{q}$. : C is positive constant

***Undamped** free oscillations are oscillations performed only under the effect of the **internal forces** of the system (weight of the masses $\mathbf{P_m}$ or stiffness force of the springs $\mathbf{F_K}$) in the absence of any exciting **external forces $\mathbf{F_{ext}}$** or friction **$\mathbf{f\alpha}$**.

* **Undamped** free oscillations are periodic harmonic oscillations and called linear if the oscillation amplitude is low.

I.2 Degree of freedom

The degree of freedom is the number "S" of independent generalized coordinates, necessary and sufficient to determine precisely the position of the oscillatory system at time t. Such that

$$\mathbf{S = 3N - n}$$

Where **N** is the number of particles constituting the system (e.g. point masses).

n is the number of geometric links (mathematical relations between generalized coordinates).

* Generalized coordinates are the set of related and independent coordinates of the system.

Example: Simple pendulum (point mass m and inextensible wire l)

Here we have

* **N = 1** the mass m therefore generalized coordinates (x_m, y_m, z_m)

* **n =2** relations 1 : $x_m^2 + y_m^2 = l^2$ or $x_m = l\sin\theta$ and $y_m = l\cos\theta$

relation 2 : z_m = Constant

which gives that S = 3N – n = 3.1 – 2 = 1 ⟶ S = 1 so the system considered has one degree of freedom: dof =1

Conclusion: To follow the evolution of the system's motion over time, we need only one coordinate, i.e. **q(t) = x(t)** or **q(t) = y(t)** or **q(t) = θ(t)**

But since the motion of this system is rotational around the O axis, we prefer the coordinate **q(t) = θ(t)** as the variable of motion in time.

Now it is more convenient and practical to write **x** instead **of x(t), y** instead of **y(t), z** instead of **z(t)** and **q** instead of **q(t).**

* A system with **p** degrees of freedom **(S = p)** i.e. **p** independent generalized coordinates are required to follow the evolution of the movement of the system over time.

* A system with **p** degrees of freedom **(S = p)** is a system having p differential equations, **p** equations of motion and **p** proper frequencies.

1.3 Oscillation conditions

a) Existence of an equilibrium position (q_0) or state of equilibrium

For a system to be in static equilibrium, it is necessary and sufficient that :

* The resultant of the forces ($\vec{F}$) or the moment ($\overrightarrow{\mathcal{M}}$) resulting from the forces exerted on the system is zero : $\vec{F} = \sum_i \vec{F} = 0$ **(translational movement)**

Ou $\overrightarrow{\mathcal{M}} = \sum_i \overrightarrow{M_i} = 0$ **(rotational movement)**

This means that the system is subject only to internal forces derived from a potential (the weight of the masses or the restoring force of the system's springs) where the potential energy **U(q)** of the system is extremal at the position q_0 such that: $\left.\frac{\partial U}{\partial q}\right|_{q=q0} = 0$

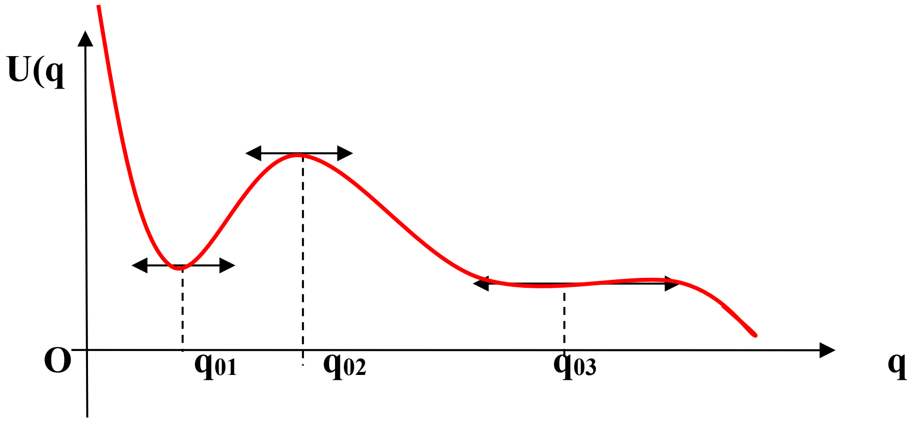
b) Stable equilibrium position (q_0)

* An equilibrium position **(q_0)** is said to be **stable** if the point slightly away from it tends to return to it, in this case:

$$U(q_0) \text{ is minimal } \Leftrightarrow \left.\frac{\partial U}{\partial q}\right|_{q=q_0} = 0 \text{ and } \left.\frac{\partial^2 U(q)}{\partial q^2}\right|_{q=q_0} > 0$$

* An equilibrium position is said to be **unstable** if the point slightly deviating from it tends to deviate further from it, in this case

$$\mathbf{U(q_0)} \text{ is maximal } \Leftrightarrow \left.\frac{\partial U}{\partial q}\right|_{q=q_0} = \mathbf{0} \quad \text{and} \quad \left.\frac{\partial^2 U(q)}{\partial q^2}\right|_{q=q_0} \leq \mathbf{0}$$

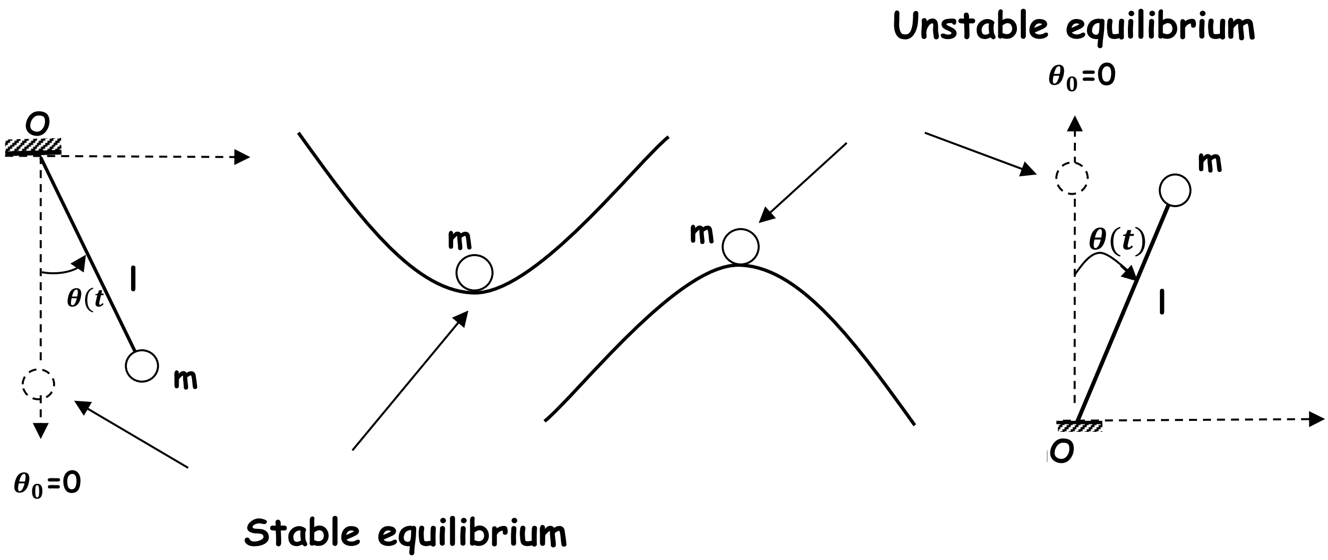


$$q_{03} : \left.\frac{\partial^2 U(q)}{\partial q^2}\right|_{q=q_{03}} = \mathbf{0}$$

$\mathbf{q_{01}}$ stable equilibrium position

$\mathbf{q_{02}}$ unstable equilibrium position

$\mathbf{q_{03}}$ inflection point the force ($\vec{\mathbf{F}}$) does not change sign when passing through position



I.4 Differential equation of motion

There are different methods for determining the differential equations of the oscillatory motion of a mechanical system, we cite:

• Method of the fundamental principle of dynamics (or equilibrium or Newton method)

• Energy method.

• Lagrange method.

We remind you that:

*During all the chapters of this “vibration” part, we will use two methods:

- *the Fundamental Principle of Dynamics* (FPD) method which is a vector method.

- the Lagrange method which is a scalar method.

* In this chapter: the oscillators studied are free and undamped with one degree of freedom (**1 dof** or **S=1**) i.e (i.e : that's to say). (**q(t)= x(t)** or **θ(t)** or **i(t)**) so we have only one differential equation of motion to determine, only one equation of motion and only one proper frequency.

1.4.1 Method of the fundamental principle of dynamics

This method is based on Newton's second law for determining the differential equation of motion of the system.

* If the (mechanical) system is in translational motion

Newton's second law ⟶ $\sum_i \vec{F}_{ext\,i} = M\vec{\ddot{x}}$

* If the (mechanical) system is in rotational motion

Newton's second law ⟶ $\sum_i \vec{\mathcal{M}}_{ext\,i} = \frac{d\vec{L}}{dt} = \vec{J\ddot{\theta}}$

or $\vec{\mathcal{M}}$ moment of external forces, $\vec{L}$ angular momentum of the system, **J** moment of inertia of the system, $\ddot{\theta}$ angular acceleration.

N.B **:**

In our course, we always consider that:

***The amplitudes of the oscillations of the systems studied are small.**

***The masses of the springs are always neglected.**

To determine the differential equation, we consider the mass-spring mechanical oscillator (**M, K**) represented by the figure opposite.

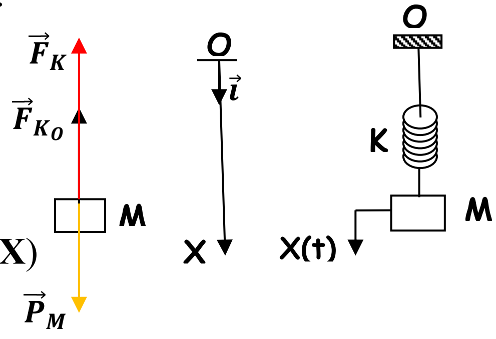
Here we have:

* Undamped free oscillator (harmonic oscillator)

* Degree of freedom **dof =1 (S=1)**

* Vertical translational movement along the axis (**OX**)

* **q(t) = x(t)= x, $\dot{q}$(t) = $\dot{x}$(t)= $\dot{x}$ and $\ddot{q}$(t) = $\ddot{x}$(t)= $\ddot{x}$**



$$\textbf{PFD} \longrightarrow \sum_i \vec{F}_{ext\,i} = M\ddot{\vec{x}}$$

a) **<u>Static state</u>** (Equilibrium state or Resting state)

FPD

$$\longrightarrow \sum_i \vec{F}_{ext\,i} = \vec{0} \longrightarrow \vec{P}_M + \vec{F}_{K_0} = \vec{0} \longrightarrow Mg - Kx_0 = 0 \qquad \textbf{(1)}$$

x_0 elongation of the spring **K** at equilibrium under the effect of the weight of **M.**

b) **<u>Dynamic state</u>** (State of motion)

We shift the mass **M** of x from its equilibrium and then leave the system to itself

FPD

$$\longrightarrow \sum_i \vec{F}_{ext\,i} = M\ddot{\vec{x}} \longrightarrow \vec{P}_M + \vec{F}_{K_0} + \vec{F}_K = M\ddot{\vec{x}} \longrightarrow \vec{F}_K = M\ddot{\vec{x}} \qquad \textbf{(2)}$$

$\longrightarrow$ **- Kx = M$\ddot{x}$ ou M$\ddot{x}$ + Kx = 0 ou $\ddot{x} + \frac{K}{M}x = 0$ ou $\ddot{x} + \omega_0^2 x = 0$** **(3)**

with $\boldsymbol{\omega_0 = \sqrt{\frac{K}{M}}}$ **(4)** called a **proper pulsation** of the movement.

The word '**proper**' is used because the system only **oscillates** under the effect of its own **internal** or **intrinsic forces**.

The equation **(3)** is called the **differential equation of the movement** of the system considered. This equation is linear, of the second order, homogeneous and with constant coefficients.

1.4.2 Lagrange method

The Lagrange method is a very practical method for determining the differential equations of motion of a mechanical oscillator, particularly in the case of oscillators with complicated parameterization. It is the expression of the **FPD** in an energy form. This method is based on the determination of the Lagrange function or the Lagrangian **L** such that the differential equation for a system with **dof = 1** is written:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] - \frac{\partial L}{\partial q} = \begin{cases} \sum_{j=1}^{n} \left|\vec{F}_{ext\,j}\right| & \longrightarrow \text{Translational Motion System} \\ \sum_{j=1}^{n} \left|\vec{\mathcal{M}}_{ext\,j}\right| & \longrightarrow \text{Rotational Motion System} \\ 0 & \longrightarrow \text{Conservative system} \end{cases} \quad (5)$$

q Independent generalized coordinate.

$\dot{q}$ Independent generalized velocity.

$\vec{F}_{ext\,j}$ External forces acting on the system.

$\overrightarrow{\mathcal{M}}_{ext\,j}$ Moments of external forces acting on the system.

L Lagrange function or Lagrangian.

$$L = T - U \qquad (6)$$

T Kinetic energy of the system due to the velocity of the oscillating point (mass).

U Potential energy of the system due to the deformation of a solid (spring) or to the variation of position of a point (mass) in a gravitational field.

As we are dealing here with free undamped oscillations under the effect of **internal forces** only, therefore our oscillator is a **conservative system** and the only independent generalized coordinate is **q(t) = x(t)= x** and ⟶ $\dot{q}(t) = \dot{x}(t) = \dot{x}$ $\ddot{q}(t) = \ddot{x}(t) = \ddot{x}$ so:

$$(5) \longrightarrow \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right] - \frac{\partial L}{\partial x} = 0 \qquad (7)$$

$$* \; T = T_M = \frac{1}{2}MV_M^2 = \frac{1}{2}M\dot{x}^2 \qquad (8)$$

$$* \; U = U_K = \frac{1}{2}Kx_K^2 = \frac{1}{2}Kx^2$$

$$\longrightarrow L = \frac{1}{2}M\dot{x}^2 - \frac{1}{2}Kx^2 \qquad (9)$$

N.B: The potential energy U_M of M's weight is not included in the expression of **U** because the effect of the weight on the movement is compensated by the restoring force of the spring at equilibrium (in other words, M's weight deforms the spring at equilibrium).

So from **(7)** and **(9)** we will have

$\frac{\partial L}{\partial \dot{x}} = \mathbf{M\dot{x}} \longrightarrow \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) = \mathbf{M\ddot{x}}$ and $\frac{\partial L}{\partial x} = \mathbf{- Kx}$ **(10)**

from **(7)** and **(10)** $\longrightarrow$ $\mathbf{M\ddot{x} + Kx = 0}$ or $\ddot{x} + \omega_0^2 x = 0$ **(11)**

The differential equation **(11)** is the same equation obtained by the **FPD** method

*This differential equation is of the type "homogeneous differential equation of the second order, linear and with constant coefficients.

1.5. Equation of motion (solution of the differential equation)

Mathematically the differential equation **(4)** or **(11)** has as solution in the form

$\mathbf{x(t) = Aexp(\alpha t)} \longrightarrow \dot{x}(t) = \alpha A\exp(\alpha t) = \alpha x(t) \longrightarrow \ddot{x}(t) = \alpha^2 x(t)$ **(12)**

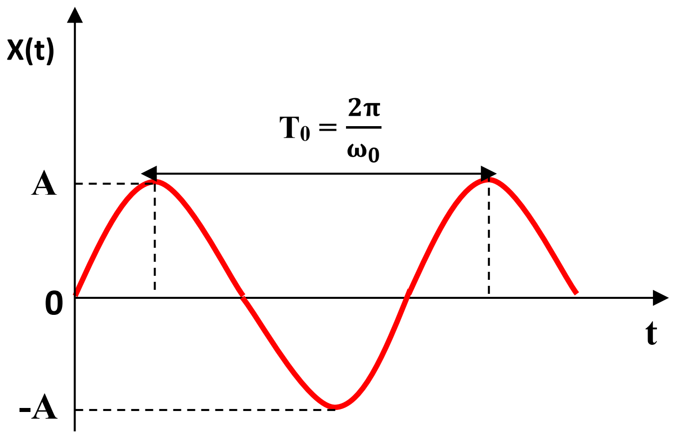
(12) in **(11)** $\longrightarrow$ $\alpha^2 + \omega_0^2 = 0 \longrightarrow \alpha = \pm j\ \omega_0$ with $j^2 = -1$

Which gives

$\mathbf{x(t) = A\cos(\omega_0 t + \varphi_0)}$ **(a)** or $\mathbf{x(t) = B\sin(\omega_0 t + \varphi_0)}$ **(b)**

or $\mathbf{x(t) = A_1\exp(j\omega_0 t) + A_2\exp(-j\ \omega_0 t)}$ **(13)** **(c)**

* Equation **(a)** or **(b)** or **(c)** is called the equation of motion of the system under study.



***A, B** or **(A_1 , A_2)** Constants and Amplitudes of movement.

* ω_0 Proper pulsation of the movement.

* $\boldsymbol{\omega_0 t + \varphi_0}$ Phase at time t.

* $\boldsymbol{\varphi_0}$ Initial phase at **t = 0.**

* **T_0** Proper period of the movement.

* The constants **A, B**, **(A_1 , A_2**) and $\boldsymbol{\varphi_0}$ are determined by the initial conditions **x(0)** and **$\dot{x}(0)$.**

1.6. Total energy of motion

The total energy **E_T** of an undamped free oscillator (harmonic oscillator) is conserved since the total force (internal forces $\vec{P}_M$ and $\vec{F}_K$**) derives from a potential.**

So we can write **E_T = T + U =Cte** **(14)**

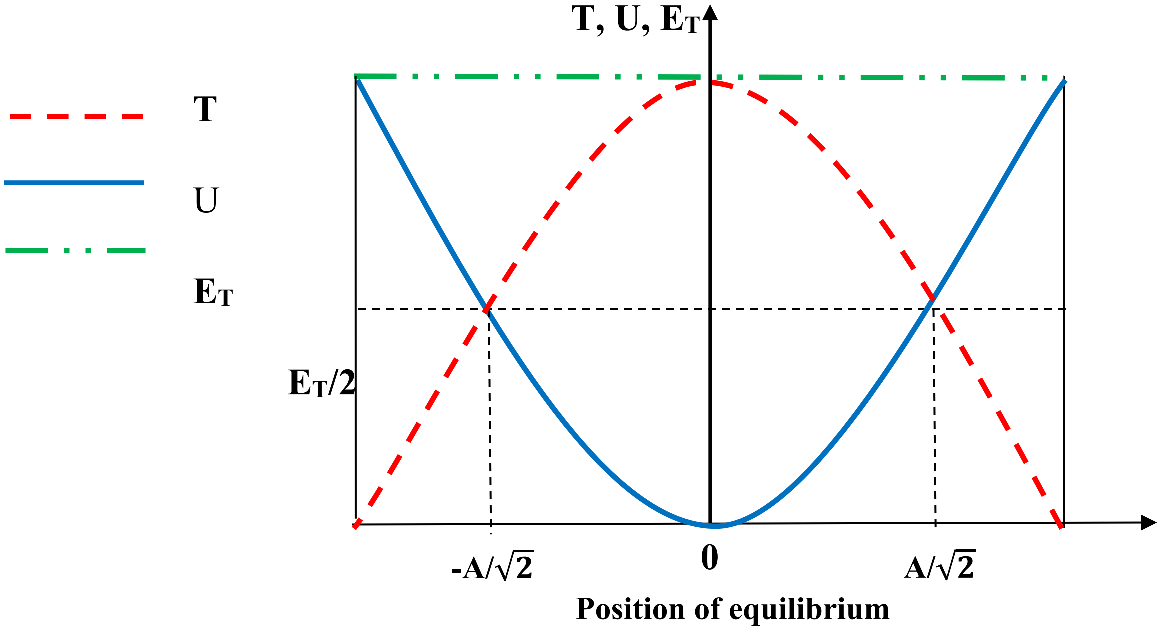
for **$x(t) = A\cos(\omega_0 t + \varphi_0)$**

We will have: $T = \frac{1}{2}M\dot{x}^2 = \frac{1}{2}MA^2\omega_0{}^2 sin^2(\omega_0 t + \varphi_0)$ **(15)**

$U = \frac{1}{2}Kx^2 = \frac{1}{2}KA^2 cos^2(\omega_0 t + \varphi_0)$ **(16)**

and as $\omega_0 = \sqrt{\frac{K}{M}} \longrightarrow \omega_0{}^2 = \frac{K}{M}$

Which gives that $E_T = \frac{1}{2}KA^2 = \frac{1}{2}MA^2\omega_0{}^2 = Cte \quad \forall\, t$



Conclusion

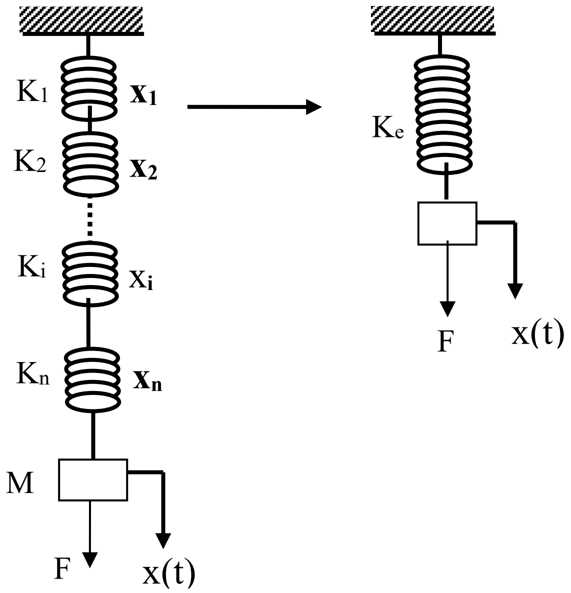
During movement there is no loss of total energy $\mathbf{E_T}$ ($\mathbf{E_T}$ conserved with constant amplitude **A**), but there is an exchange of energy between kinetic energy **T** and potential energy **U**, i.e. everything that is lost in kinetic energy is transformed into potential energy and vice versa.

1.7. Equivalent systems

Equivalent springs

In vibratory oscillatory systems, we can encounter different spring mounting configurations (we recall here that the masses of the springs are always considered negligible).

*1st configuration: **Springs in series**



$x = \sum_i^n x_i = x_1 + x_2 + ... + x_i + ... + x_n$

$F = F_1 = F_2 = ... = F_i ...... = F_n$

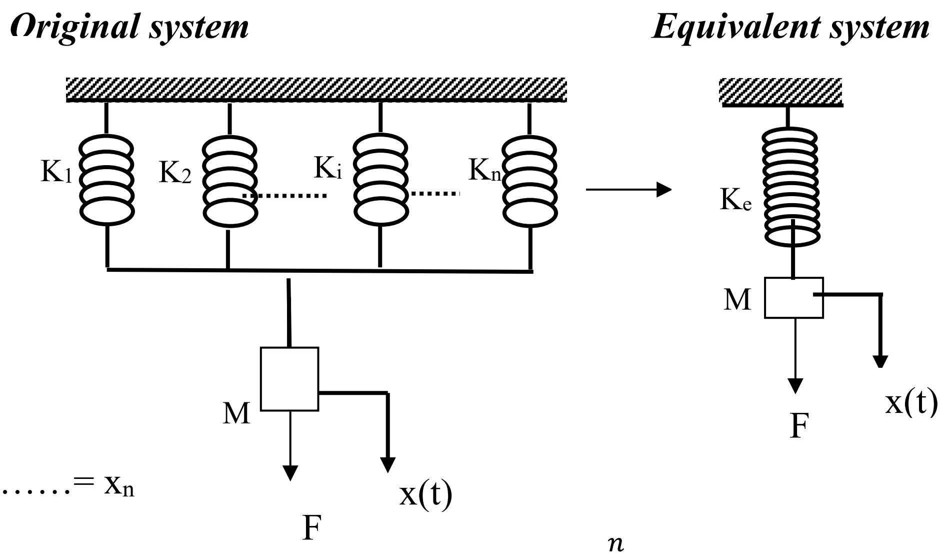
$F = K_1x_1 = K_2x_2 = ... = K_ix_i = ... = k_nx_n$

$F = k_ex$

From this we will have

$$\frac{1}{K_e} = \sum_{i=1}^{n} \frac{1}{K_i}$$

*2nd configuration: Parallel sprin



$x = x_1 = x_2 = ... = x_i ...... = x_n$

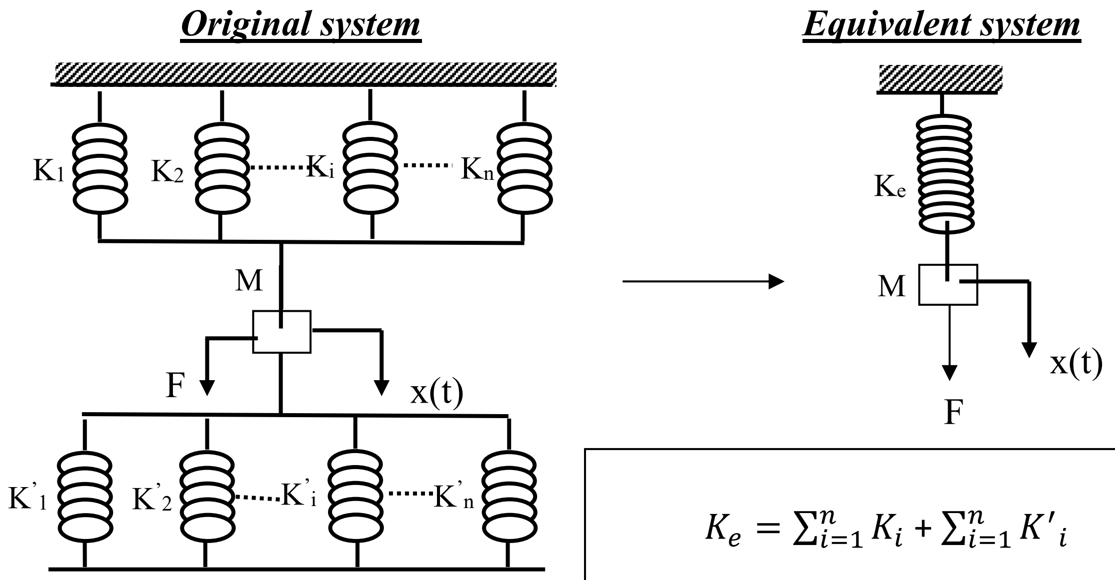
$F = \sum_i^n F_i = F_1 + F_2 + ... + F_i + ... + F_n$

$F = K_1x + K_2x + ... + K_ix + ... + k_nx = K_ex$

$$K_e = \sum^{n} K_i$$

From this we will have

Other examples of parallel springs

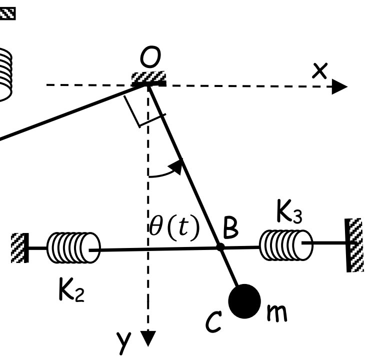


$$K_e = \sum_{i=1}^{n} K_i + \sum_{i=1}^{n} K'_i$$

1.8. Analogy between mass-spring mechanical system (M,K) and coil-capacitor electrical system (L, C)

MECHANICAL SYSTEM (M, K)	ELECTRICAL SYSTEM (L, C)
Linear displacement **x**	Electric charge **q**
Linear velocity **ẋ**	Electric current **i**
Linear acceleration $\ddot{x}$	Current variation **q̈**
Mass **M**	Inductance **L**
Stiffness constant **K**	Inverse of capacitance **1/C**
Inertial force **M** $\ddot{x}$	potential difference between the terminals of coil **L** $\ddot{q}$
Restoring force **Kx**	potential difference between the limits of the capacity **q/C**
Kinetic energy T $=\frac{1}{2}M\dot{x}^2$	Magnetic energy $E_{mag} = \frac{1}{2}L\dot{q}^2$
Potential energy **U** $= \frac{1}{2}Kx^2$	Electric energy $E_{élect} = \frac{1}{2C}q^2$
Proper pulse $\omega_0 = \sqrt{\frac{K}{M}}$	Proper pulse $\omega_0 = \sqrt{\frac{1}{LC}}$

Exercise



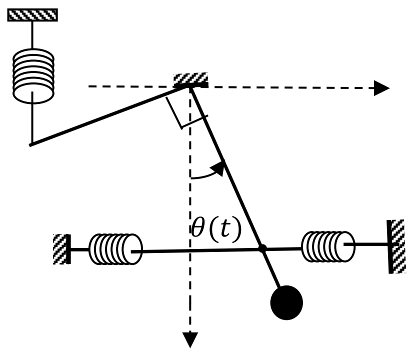
Consider the following system composed of two perpendicular rods ***OA*** and ***OC*** of negligible masses welded to the axis of rotation ***O,*** three springs of stiffnesses ***K_1, K_2*** and ***K_3*** and a point mass m (see diagram opposite). The rod ***OC*** is shifted from its equilibrium position ***(θ=0)*** (vertical rod ***OC***) by a small angle ***θ*** and then the system is left to itself. The oscillations of the system around the axis ***O*** are of small amplitude and take place in a vertical plane. We give ***OA= a*** and ***OB=3b/4***, ***OC= b***, ***K_1 = K*** and **$K_2 = K_3 = K/2$.**

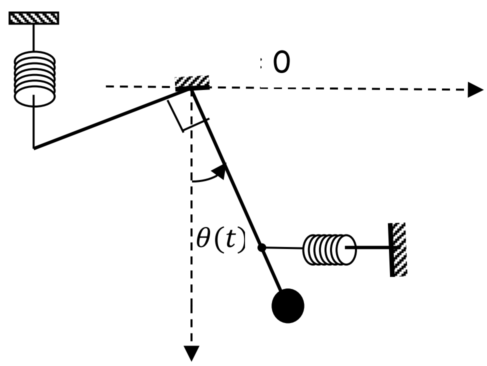
1°) Calculate the kinetic ***(T)*** and potential ***(U)*** energies of the system.

2°) Using Lagrange's principle, establish the differential equation of the system's motion and deduce the proper pulsation **ω_0.**

3°) Give the equation ***θ(t)*** of the motion under the following initial conditions:
$\theta(0) = \theta_0$ and **$\dot{\theta}(0) = 0$**.

<u>Solution</u>





Original System **Equivalent System**

The two springs **K_2** and **K_3** are parallel. ⟶ **$K_e = K_2 + K_3 = K$**

But **K_1** and **K_e** are neither in series nor in parallel.

1°) Calculation of the kinetic energy T and potential energy U of the system

* Our system is **an undamped free oscillator**

* **dof = 1** $\longrightarrow$ **q(t) =** $\boldsymbol{\theta(t)}$

* rotational movement.

* **coordinates in the given reference frame** (**XOY**)

$$\vec{r}_m \begin{cases} b\sin\theta \\ b\cos\theta \end{cases} \longrightarrow \vec{v}_m \begin{cases} b\dot{\theta}\cos\theta \\ -b\dot{\theta}\sin\theta \end{cases} \longrightarrow \vec{r}_B \begin{cases} \frac{3}{4}b\sin\theta \\ \frac{3}{4}b\cos\theta \end{cases} \longrightarrow x_B = \frac{3}{4}b\sin\theta \ ,$$

$$\vec{r}_A \begin{cases} -a\cos\theta \\ a\sin\theta \end{cases} \quad , \quad y_A = a\sin\theta$$

Kinetic Energy T: $T_m = \frac{1}{2}mV_m^2 = \frac{1}{2}mb^2\dot{\theta}^2$

***Potentiel energy U:**

$$U_{Ke} + U_{K1} + U_m = \frac{1}{2}Kx_B^2 + \frac{1}{2}Kx_B^2 + mgh_m$$

$$\frac{1}{2}K\left(\frac{9}{16}b^2\sin^2\theta\right) + \frac{1}{2}Ka^2\sin^2\theta - mgb\cos\theta + cste$$

*U_m is included in the expression for **U** because the weight of **m** does not deform any of the springs in the system considered at equilibrium, and this weight changes height during the movement.

* spring K_1 deforms vertically and spring K_e deforms horizontally. **($\theta \ll$)**

2°) Determining the differential equation of motion

The Lagrange principle $\longrightarrow$ $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) = 0 \qquad (1)$$

$$L = T - U = \frac{1}{2}mb^2\dot{\theta}^2 - \frac{1}{2}K\left(\frac{9}{16}b^2\sin^2\theta\right) - \frac{1}{2}Ka^2\sin^2\theta + mgb\cos\theta + cste \qquad (2)$$

(2) and (1) ⟶

$$\frac{\partial L}{\partial\dot{\theta}} = mb^2\dot{\theta} \longrightarrow \frac{d}{dt}\left[\frac{\partial L}{\partial\dot{\theta}}\right] = mb^2\ddot{\theta}, \qquad \frac{\partial L}{\partial\theta} = -\frac{9b^2}{16}K\sin\theta\cos\theta - Ka^2\sin\theta\cos\theta - mgb\sin\theta \qquad (3)$$

de (1) and (3) ⟶ $\ddot{\theta} + \frac{9}{16}Kb^2\sin\theta\cos\theta + Ka^2\sin\theta\cos\theta + mgb\sin\theta = 0$

or $\ddot{\theta} + \sin\theta\left(\frac{9}{16}Kb^2\cos\theta + Ka^2\cos\theta + mgb\right) = 0$ **original differential of motion**

as $\theta \ll$ **(low amplitude oscillations)** ⟶ $\cos\theta \approx 1$ **and** $\sin\theta \approx \theta$ **(5)**

(4) and **(5)** $\ddot{\theta} + \theta\left(\frac{9}{16}Kb^2 + Ka^2 + mgb\right) = 0$ **(6a)**

approximate differential equation of motion

or

$\ddot{\theta} + \omega_0^2\theta = 0$ **(6b)** with $\omega_0 = \sqrt{\dfrac{\frac{9}{16}Kb^2 + Ka^2 + mgb}{mb^2}}$ **(7)**

proper pulsation of the movement

<u>3°) Determining the equation θ(t) of motion</u>

The equation of the form **(6a)** or **(6b)** is a homogeneous, second-order, linear differential equation with constant coefficients. has as its solution :

$$\theta(t) = A\cos(\omega_0 t + \varphi_0)$$

(8)

With the constants **(A, φ_0)** are determined by the initial conditions

$\theta(0) = \theta_0$ et $\dot{\theta}(0) = 0$. **(9)**

(8) ⟶ $\dot{\theta}(t) = -A\omega_0 \sin(\omega_0 t + \varphi_0)$ **(10)**

from **(8), (9) and (10)**

⟶ $$\begin{cases} \theta(0) = A\cos(\varphi_0) = \theta_0 & (11a) \\ \dot{\theta}(0) = -A\,\omega_0 \sin(\varphi_0) = 0 & (11b) \end{cases}$$

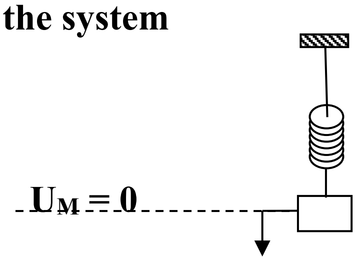
(11b) ⟶ $\sin(\varphi_0) = 0$ ⟶ $\varphi_0 = n\pi$ **with** $n \in \mathbb{Z}$ **(12)**

(12) and (11a) ⟶ $A = \theta_0$ **(13)**

and from **(8), (12) and (13)** ⟶ $\theta(t) = \theta_0 \cos(\omega_0 t + n\pi) = \theta_0 \cos(\omega_0 t)$ **(14)**

equation of motion of the system

Lagrange differential equation



***dof = 1** ⟶ $q(t) = y(t)$

*** undamped free oscillator**

*** At equilibrium we have $Mg - Ky_0 = 0$ (1) y_0 spring elongation K at equilibrium**

Lagrange ⟶ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \left(\frac{\partial L}{\partial y}\right) = 0$ **(2)**

With $L = T - U$ (3) with $T = T_M = \frac{1}{2}M\dot{y}^2$ and $U = U_K + U_M$

1st case **the weight of M changes height during movement** $\longrightarrow$ **U_M is involved in U**

$$\longrightarrow \quad U = U_K + U_M = \frac{1}{2}K(y + y_0)^2 - Mgy \quad (4)$$

$$\longrightarrow \quad L = T - U = \frac{1}{2}M\dot{y}^2 - \frac{1}{2}K(y + y_0)^2 + Mgy \quad (5)$$

2nd case **the weight of M deforms the K spring at equilibrium** $\longrightarrow$ **U_M is not involved in U** $\longrightarrow$

$$U = U_K = \frac{1}{2}Ky^2 \quad (6)$$

$$\longrightarrow \quad L = T - U = \frac{1}{2}M\dot{y}^2 - \frac{1}{2}Ky^2 \quad (7)$$

1st case

(2) and (5) $\longrightarrow \frac{\partial L}{\partial \dot{y}} = M\dot{y} \longrightarrow \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{y}}\right] = M\ddot{y}$

$$\frac{\partial L}{\partial y} = -K(y + y_0) + Mg = -Ky - Ky_0 + Mg \quad (8)$$

(2) and (8) $\longrightarrow M\ddot{y} + Ky + Ky_0 - Mg = 0 \longrightarrow M\ddot{y} + Ky = 0$ **(9)**

2nd case

(2) and (7) $\longrightarrow \frac{\partial L}{\partial \dot{y}} = M\dot{y} \longrightarrow \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{y}}\right] = M\ddot{y}$ and $\frac{\partial L}{\partial y} = -Ky$ **(10)**

(2) and (10) $\longrightarrow M\ddot{y} + Ky = 0$ **(11)**

We find the same result, but the 2nd case is shorter and simpler.

CHAPTER II : DAMPED FREE OSCILLATIONS WITH ONE DEGREE OF FREEDOM

CHAPTER II :DAMPED FREE OSCILLATIONS WITH ONE DEGREE OF FREEDOM

II.1 Introduction

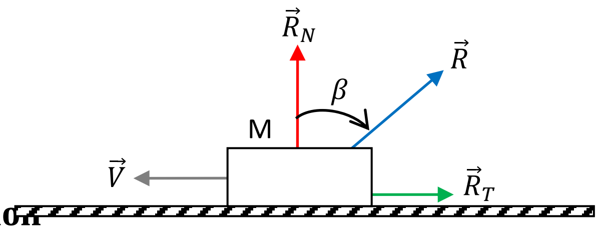
In reality, all vibratory (or oscillatory) systems, whether natural or physical, vibrate under the effects of frictional forces of different origins. These vibratory movements are called free-damped vibratory (or oscillatory) movements. They are characterised by a reduction in their oscillation amplitudes or by dissipation in the form of heat of their total energies over time due to these frictional forces. These forces do not derive from the potential. These oscillations of the system are said to be non-periodic.

*** Damped due to the presence of frictional forces and free due to the absence of external exciting forces.**

II.2 Types of frictional forces

a) Dry friction forces (or friction between two surfaces)

These friction forces appear during sliding between two contact surfaces of two solids.



$\vec{\boldsymbol{R}}$ Reaction due to contact of the two surfaces

$\overrightarrow{\boldsymbol{R_T}}$ Tangential Reaction Vector defines **dry friction**

$\overrightarrow{\boldsymbol{R_N}}$ Vector Normal reaction

$\vec{\boldsymbol{V}}$ Velocity vector of **M**

***Dynamic** friction (V ≠ 0) : $\left\|\overrightarrow{\boldsymbol{R_T}}\right\| = \boldsymbol{\mu_d}\left\|\overrightarrow{\boldsymbol{R_N}}\right\| = \left\|\overrightarrow{\boldsymbol{R_N}}\right\|\mathbf{tg}(\boldsymbol{\beta})$

* **Static** friction (V=0) : $\left\|\overrightarrow{\boldsymbol{R_T}}\right\| = \boldsymbol{\mu_S}\left\|\overrightarrow{\boldsymbol{R_N}}\right\|$

With $\boldsymbol{\mu_d < \mu_S}$

$\boldsymbol{\mu_d}$, $\boldsymbol{\mu_S}$ are the dynamic and static coefficients of friction respectively.

b) Solid friction forces (or internal friction)

These forces originate in the friction that occurs inside solid matter and in the crystalline structure (between molecules).

c) Viscous friction forces (or fluid friction)

This type of friction generates forces that appear when the system moves in a fluid or viscous medium (gas or liquid: example: air, water, oil, milk......). In this case, the viscous friction force $\vec{f}_\alpha$, depending on the amplitude of the velocity, is as follows:

$\vec{f}_\alpha = -\alpha\vec{V}$ if the velocity **V** of the system's motion is low (viscous motion).

$\vec{f}_{\alpha'} = -\alpha'\overrightarrow{V^2}$ if the velocity **V** of the system's motion is high (turbulent motion).

The sign (-) means that the frictional force opposes the movement of the system.
$\boldsymbol{\alpha}$ and $\boldsymbol{\alpha'}$ **viscous friction coefficients** depending on the density and viscosity of the fluid and the shape of the moving solid. The equation with dimensions of $\boldsymbol{\alpha}$ is:

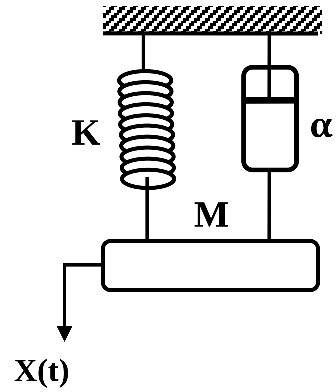
$[\alpha] = [\frac{f_\alpha}{V}] = \frac{M.L.T^{-2}}{L.T^{-1}} = M.T^{-1}$ which gives its international unit is **Kg.S**$^{-1}$

In view of the complexity of the dependence of these various friction forces on the parameters of the system's motion, in this chapter we are concerned only with the simplest form of viscous friction, where the friction forces are proportional to the velocity of the system's ı $\vec{f}_\alpha = -\alpha\vec{V}$ ıat

with V low.

II.3 Determining the differential equation of motion

Let's always consider the system or mechanical oscillator mass (M), spring (K) and damper (**α**), i.e. oscillator (**M, K, α**). Viscous friction is symbolised by the damper in the diagram below



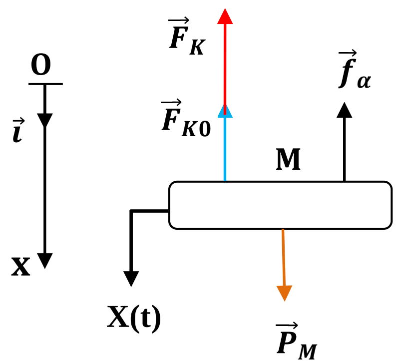
The oscillator considered is represented by the diagram opposite

II.3.1 Method of the Fundamental Principle of Dynamics

* Damped free oscillator

* Degree of freedom **dof =1** ⟶ **q(t) = x(t)** ⟶ $\dot{q} = \dot{x}$ and $\ddot{q} = \ddot{x}$

* Vertical translational movement along the axis (**OX**)



FPD ⟶ $\sum_i \vec{F}_{ext\,i} = M\ddot{\vec{x}}$ **(1)**

a) <u>Static state</u> (Equilibrium state or Resting state)

FPD ⟶ $\sum_i \vec{F}_{ext\,i} = \vec{0}$ ⟶ $\vec{P}_M + \vec{F}_{K_0} = \vec{0}$

⟶ $Mg\vec{\imath} - Kx_0\vec{\imath} = \vec{0}$ ⟶ $Mg - Kx_0 = 0$

x_0 elongation of the spring **K** at equilibrium under the effect of the weight of **M.**

b) <u>Dynamic state</u> (State of motion)

We shift the mass M of x from its equilibrium and then leave the system to itself

FPD ⟶ $M\ddot{\vec{x}} = \sum_i \vec{F}_{ext\,i}$ ⟶ $M\ddot{\vec{x}} = \vec{P}_M + \vec{F}_{K_0} + \vec{F}_K + \vec{f}_\alpha$

(2)

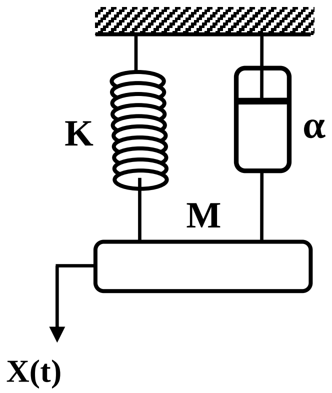
$\longrightarrow$ $M\ddot{\vec{x}} = \vec{F}_K + \vec{f}_\alpha \longrightarrow$ $M\ddot{x}\vec{\imath} = -Kx\vec{\imath} - \alpha\dot{x}\vec{\imath}$

(3)

$\longrightarrow$ $M\ddot{x} + \alpha\dot{x} + Kx = 0$ **(4)**

differential equation of motion of the system

II.3.2 Lagrange method



* Damped free oscillator

* Degree of freedom **dof =1** $\longrightarrow$ **q(t) = x(t)**

According to Lagrange's principle the differential equation sought is in the general form

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] - \frac{\partial L}{\partial q} = -\frac{\partial D_\alpha}{\partial \dot{q}} \qquad (1)$$

With

$$D_\alpha = \frac{1}{2}\alpha V^2 \qquad (2)$$

energy dissipation function

For our considered oscillator we have **q(t) = x(t)= x,** $\dot{q}$**(t) =** $\dot{x}$**(t)=** $\dot{x}$ **and** $\ddot{q}$**(t) =** $\ddot{x}$**(t)=** $\ddot{x}$

(1) will be in the form

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right] - \frac{\partial L}{\partial x} = -\frac{\partial D_\alpha}{\partial \dot{x}} \qquad (3)$$

and $D_\alpha = \frac{1}{2}\alpha\dot{x}^2$ **(4)**

Calculation

*** L = T – U** ⟶ **T = T$_M$ = $\frac{1}{2}M\dot{x}^2$ and U = $\frac{1}{2}Kx^2$** ⟶

L = $\frac{1}{2}M\dot{x}^2 - \frac{1}{2}Kx^2$ **(5)**

*** U$_M$** does not intervene in the expression of **U** because the weight of **M** deforms the spring **K** at equilibrium.

of **(3) , (4)** and **(5)** ⟶ $\frac{\partial L}{\partial \dot{x}} = M\dot{x}$ ⟶ $\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right] = M\ddot{x}$, $\frac{\partial L}{\partial x} = -Kx$ and $\frac{\partial D_\alpha}{\partial \dot{x}} = \alpha\dot{x}$

$M\ddot{x} + \alpha\dot{x} + Kx = 0$ **(6)**

of **(3)** and **(6)** ⟶ $M\ddot{x} + Kx = -\alpha\dot{x}$ **(7)**

or

(7) same differential equation found by applying the **FPD**.

The differential equation **(4)** or **(7)** of the motion of the system can be written in the form

(7) ⟶ $\ddot{x} + \frac{\alpha}{M}\dot{x} + \frac{K}{M}x = 0$ **or**

$\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = 0$ **(8)**

with **differential equation of motion of the system**

$\omega_0 = \sqrt{\frac{K}{M}}$ **proper pulsation** of the movement in (**rd/s**).

$\delta = \frac{\alpha}{2M}$ **:friction or damping factor** of the movement in **(s^{-1})**

II.4 Determination of the equation of motion

This equation of motion is only the solution of the differential equation **(8)** which is a second-order linear differential equation with constant coefficients without second member. The solution of this equation is of the form:

$x(t) = x = Ae^{rt}$ ⟶ $\dot{x} = Are^{rt}$ ⟶ $\ddot{x} = Ar^2e^{rt}$ **(9)**

(9) in **(8)** ⟶ $r^2 + 2\delta r + \omega_0^2 = 0$ **(10)**

(10) has two solutions or two roots $\begin{cases} r_1 = -\delta + \sqrt{\delta^2 - \omega_0^2} \\ \text{and} \\ r_2 = -\delta - \sqrt{\delta^2 - \omega_0^2} \end{cases}$ **(11)**

so **x(t) = x** equation of motion and solution of the differential equation **(8)** its expression depends on the relationship between $\boldsymbol{\delta}$ and $\boldsymbol{\omega_0}$**.**

<u>* First case:</u> $\delta < \omega_0$ ⟶ **low friction or damping**

(11) ⟶ $\begin{cases} r_1 = -\delta + j\sqrt{\omega_0{}^2 - \delta^2} \\ \text{and} \\ r_2 = -\delta - j\sqrt{\omega_0{}^2 - \delta^2} \end{cases}$ **with** $j^2 = -1$

(12)

(9) and **(12)** ⟶ $x(t) = x = A_1e^{r_1t} + A_2e^{r_2t} = A_1e^{(-\delta + j\sqrt{\omega_0{}^2 - \delta^2})t} + A_2e^{(-\delta - j\sqrt{\omega_0{}^2 - \delta^2})t}$

Or

$x(t) = x = Ae^{-\delta t}\sin(\omega_a t + \varphi)$ **(13)**

Equation of motion case ($\boldsymbol{\delta < \omega_0}$)

* The quantity $\mathbf{A}e^{-\delta t}$ is called the amplitude or envelope of the motion which decays exponentially over time due to viscous friction (α). From this, the motion is not purely periodic but is said to be pseudoperiodic by the fact that the friction is low (**$\delta < \omega_0$**) where the system will oscillate around its position for some time before stopping.

with $\mathbf{T_a} = \frac{2\pi}{\omega_a}$ (in **s**) is the pseudo-period of the movement

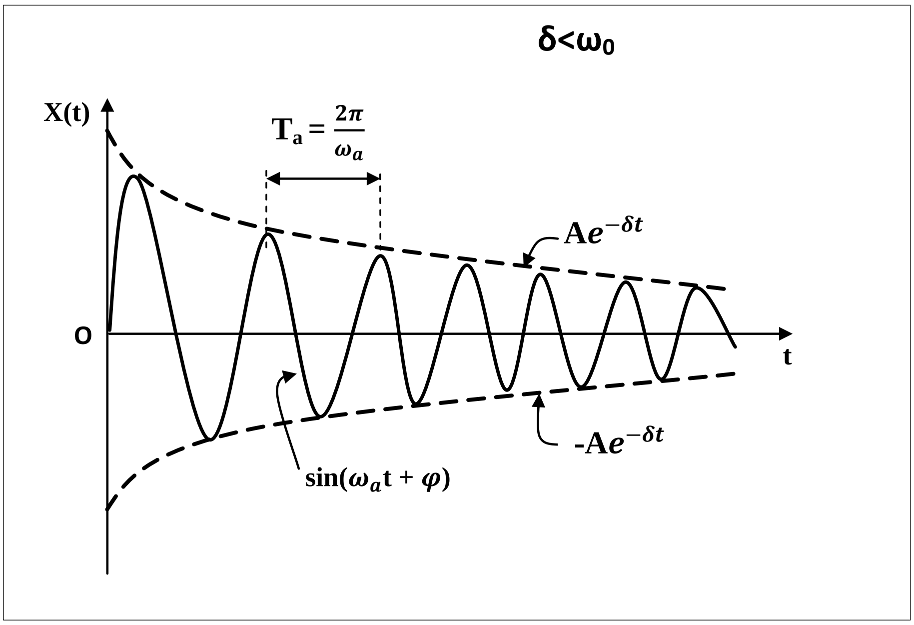
$$\omega_a = \sqrt{\omega_0^2 - \delta^2}$$

and $\omega_0 = \frac{2\pi}{T_0}$ (in **rd/s**) is called pseudo pulsation of movement.

(in **rd/ s**) **proper pulsation** of the movement and **T_0** proper period in **(s)**

* The quantity **$\sin(\omega_a t + \varphi)$** is called **the vibrational part of the motion**.

* The constants **(A, φ)** are determined by the initial conditions of the motion **x(0)** and $\dot{x}(0)$.



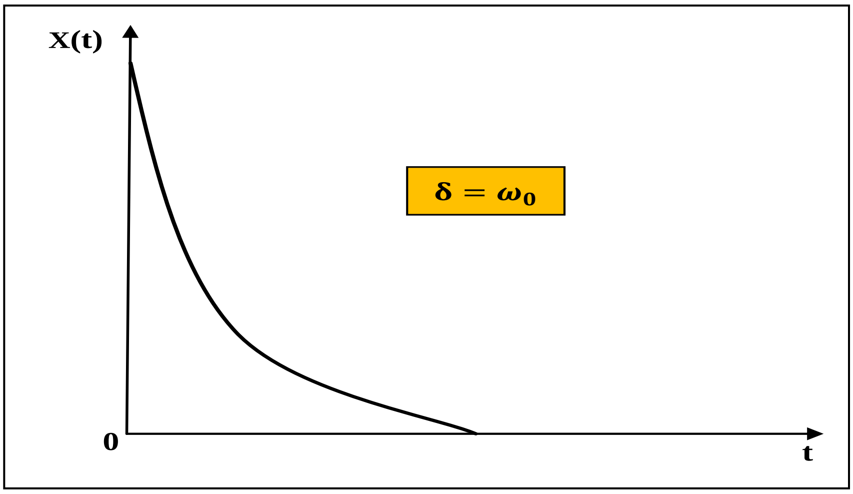
***Second case :** $\delta = \omega_0$ ⟶ **friction or critical damping**

(11) ⟶ $r_1 = r_2 = -\delta$ **(14)**

(9) and (14) ⟶ $$x(t) = x = (B_1 + B_2 t)\, e^{-\delta t} \qquad (15)$$

↑ **equation of motion case** $(\delta = \omega_0)$

* The motion is **non-periodic**. The motion is said to be critical where the system will **not oscillate at all** and it returns **very quickly** to its equilibrium position once shifted from this position.



* The constants **(B_1, B_2)** are determined by the initial conditions of the motion $x(0)$ **et** $\dot{x}(0)$.

*** Third case:** $\delta > \omega_0$ ⟶ **strong friction or damping**

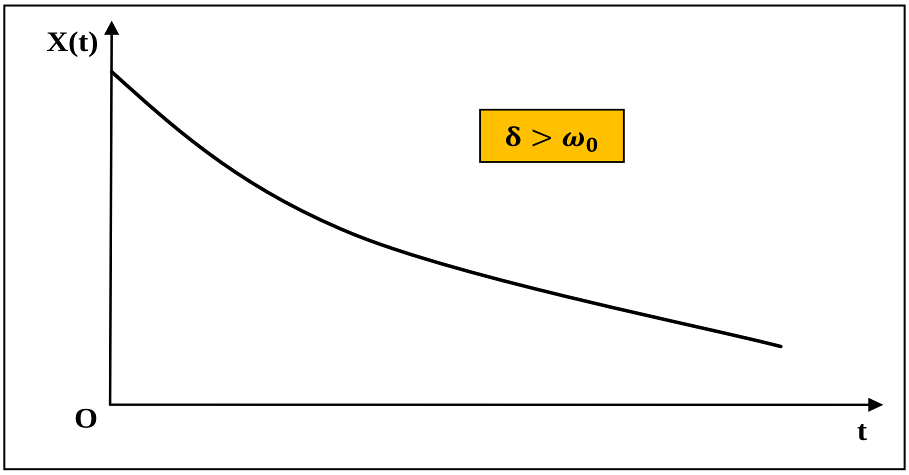
(11) ⟶ $$\begin{cases} r_1 = -\delta + \sqrt{\delta^2 - \omega_0^2} \\ \text{et} \\ r_2 = -\delta - \sqrt{\delta^2 - \omega_0^2} \end{cases} \qquad (16)$$

equation of motion case $(\delta > \omega_0)$

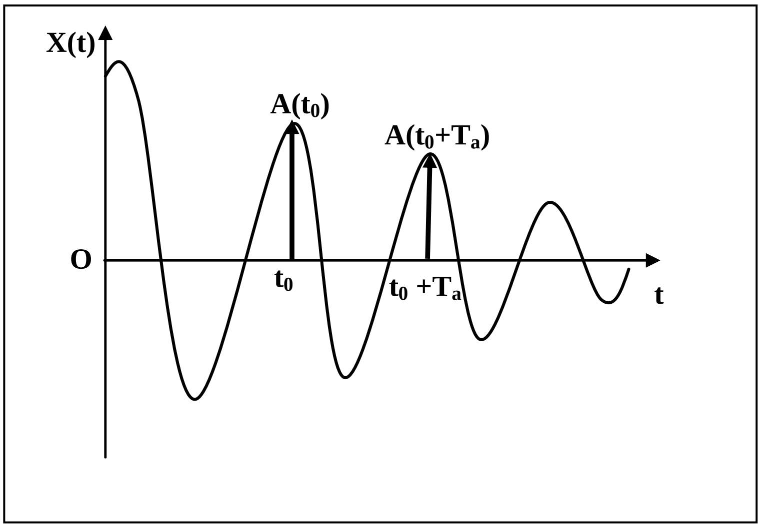
(9) and **(16)** ⟶ $$x(t) = x = C_1 e^{r_1 t} + C_2 e^{r_2 t} = e^{-\delta t}\left(C_1 e^{\left(\sqrt{\delta^2 - \omega_0^2}\right)t} + C_2 e^{-\left(\sqrt{\delta^2 - \omega_0^2}\right)t}\right) \qquad (17)$$

***** The motion is **aperiodic**. The system will not oscillate at all and it returns **very slowly** to its equilibrium position once shifted from this position due to **friction** or **strong braking**.

***** the constants **(C_1, C_2)** are determined by the initial conditions of the motion $\boldsymbol{x(0)}$ **and** $\boldsymbol{\dot{x}(0)}$.



<u>Remarks or Noticed</u> :



From the three cases of viscous friction types discussed in this chapter: low, critical and strong friction, we see that the most useful and practical is the case of weak friction where the mechanical system has a chance to oscillate even in a pseudoperiodic manner. This situation allows us to follow as a function of time the evolution of certain physical properties of the system, in particular the variation or dissipation of its total energy under the effect of friction via two important parameters, namely the **logarithmic decrement (D)** and **the quality coefficient (Q).**

II.5 Logarithmic decrement and quality coefficient

II.5.1 Logarithmic decrement

It is defined as the logarithm of two successive amplitudes of weakly damped oscillations (**$\delta < \omega_0$**) of the same sign.

$$D = \ln\left(\frac{A(t_0)}{A(t_0+T_a)}\right) = \ln\left(\frac{Ae^{-\delta t_0}}{Ae^{-\delta(t_0+T_a)}}\right)$$

$$\longrightarrow \quad D = \delta T_a \approx \delta T_0$$

or $T_0 = \frac{2\pi}{\omega_0}$

II.5.2 Quality coefficient

It is defined as the ratio between the maximum energy (**E_{max}**) stored by the system and the energy lost **ΔE** during an oscillation cycle (**T_a**).

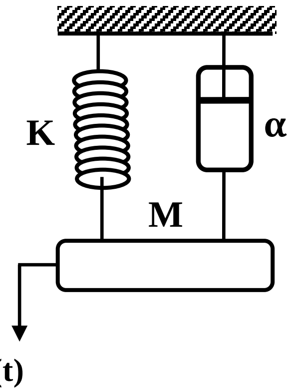
$$Q = 2\pi \frac{E_{max}}{|\Delta E|} \approx 2\pi \frac{\frac{1}{2}KA^2}{\left|-\frac{1}{2}KA^2 2\delta T_0\right|} = \frac{\omega_0}{2\delta}$$

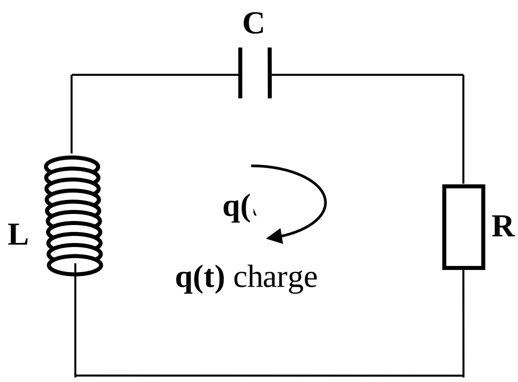
The energy lost by viscous friction is transformed into heat and propagates in the system and its surroundings.

* In conclusion we can say: For a damped free oscillator to be able to oscillate several times before stopping, that is to say that the loss of its total energy during the pseudooscillations is **very slow**, it is necessary to have a large quality coefficient **Q** or very low friction (**$\delta \ll$**).

II.6 Analog electrical system

Mechanical system (M, K, α) **Analog electrical system (L, C, R)**





* $\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = 0$ ⟶ $\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = 0$

* $\omega_0 = \sqrt{\frac{K}{M}}$ ⟶ $\omega_0 = \sqrt{\frac{1}{LC}}$

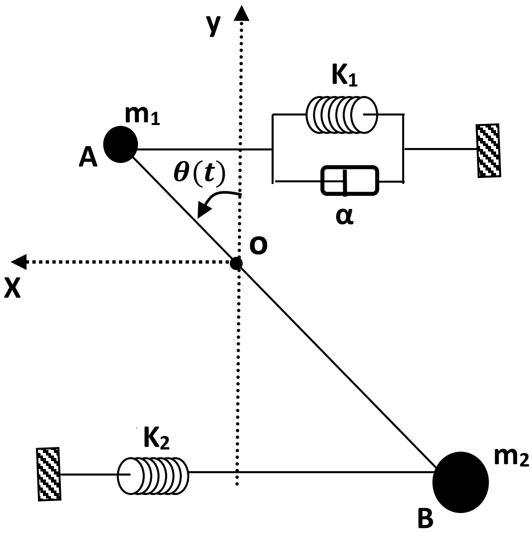
* $\delta = \frac{\alpha}{2M}$ ⟶ $\delta = \frac{R}{2L}$

* **Friction α :** ⟶ **Resistance R**

Low-critical-strong **small-critical (medium)-big**

*** friction force $f_\alpha = \alpha V$** ⟶ **voltage across terminals of R : $U_R = R\frac{dq}{dt}$**

Exercise



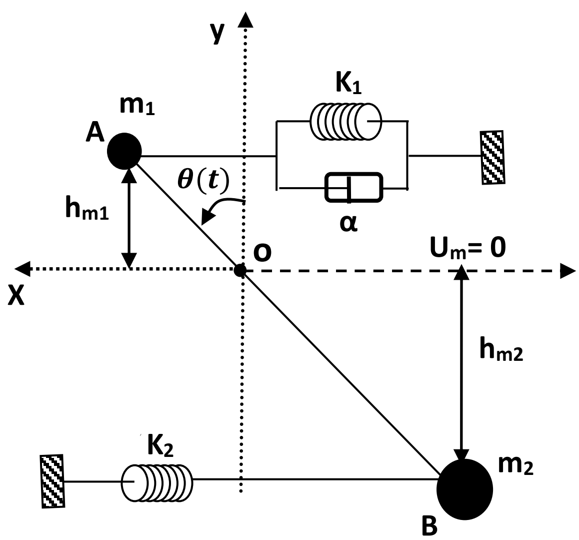
Consider the mechanical system shown opposite, consisting of a homogeneous rod **AB** of negligible mass and length ***l*** oscillating without friction, in a vertical plane, about a fixed axis perpendicular to the plane of motion at ***O***, springs of stiffness K_1, K_2 and a damper of viscous friction coefficient ***α*** all attached to the rod at points **A** and **B** (see figure). The rod is moved from its equilibrium point ***θ=0*** (vertical rod **AB**) by a small angle ***θ*** and then the system is left to its own devices. **OA = a**, **OB = 2a**,$m_2 = 2m_1 = 2m$ and $K_1 = K_2 = K$.

1°) Calculate the kinetic energy **T**, potential energy **U** and dissipation energy **D** of the system.

2°) Using Lagrange's principle establish the differential equation of motion of the system and deduce the proper pulsation ω_0 and the coefficient of friction ***δ*** of the motion.

3°) Give the expression of the equation ***θ (t)*** of motion in the case of low friction and under the following initial conditions: $\theta(0)=\theta_0$ and $\dot{\theta}(0 = 0)$

Solution



* the two springs K_1 and K_2 are neither in series nor in parallel.

1°) **calculation of the kinetic energy T, potential U and dissipation Dα of the system.**

* Our system is a **damped free** oscillator

* **dof = 1** ⟶ **q(t) =** $\theta(t)$

* rotational movement.

a) <u>Kinetic Energiy T :</u> $T = T_{m1} + T_{m2} = \frac{1}{2}mV_{m1}^2 + \frac{1}{2}mV_{m2}^2$,

$V_{m1} = ?$ et $V_{m2} = ?$

b) **<u>Potential Energy U :</u>** $U = U_{K1} + U_{K2} + U_{m1} + U_{m2} + cste$

$$U = \frac{1}{2}K_1x_{m1}^2 + \frac{1}{2}K_2x_{m2}^2 + m_1gh_{m1} - m_2gh_{m2} + cste$$

$x_{m1} = ?$, $x_{m2} = ?$, $h_{m1} = ?$ et $h_{m2} = ?$

*U_{m1} and U_{m2} intervene in the expression of **U** because their weights do not deform the two springs at equilibrium and change height during the movement.

c) <u>Dissipation energy Dα:</u> $D_\alpha = \frac{1}{2}\alpha V^2 = \frac{1}{2}\alpha V_{m1}^2$, $V_{m1} = ?$

*** coordinates in the given reference frame** (XOY)

$\overrightarrow{r_{m1}} \begin{cases} a\sin\theta \\ a\cos\theta \end{cases}$ → $\overrightarrow{v_{m1}} \begin{cases} a\dot{\theta}\cos\theta \\ -a\dot{\theta}\sin\theta \end{cases}$, $\overrightarrow{r_{m2}} \begin{cases} -b\sin\theta \\ -b\cos\theta \end{cases}$ $\overrightarrow{v_{m2}} \begin{cases} -b\dot{\theta}\cos\theta \\ b\dot{\theta}\sin\theta \end{cases}$

T ⟶

$$T = T_{m1} + T_{m2} = \frac{1}{2}mV_{m1}^2 + \frac{1}{2}mV_{m2}^2 = \frac{1}{2}m_1a^2\dot{\theta}^2 + \frac{1}{2}m_2b^2\dot{\theta}^2 = \frac{9}{2}ma^2\dot{\theta}^2 \quad (2)$$

$$U = \frac{1}{2}K_1x_{m1}^2 + \frac{1}{2}K_2x_{m2}^2 + m_1gy_{m1} - m_2g|y_{m2}| + cste$$

$$= \frac{1}{2}K(a\sin\theta)^2 + \frac{1}{2}K(-b\sin\theta)^2 + m_1ga\cos\theta - m_2gb\cos\theta + cste$$

U ⟶

$$U = \frac{5}{2}K(a\sin\theta)^2 - 3\,mga\cos\theta + cste \quad (3)$$

$$D_\alpha = \frac{1}{2}\alpha V_{m1}^2 = \frac{1}{2}\alpha a^2\dot{\theta}^2$$

$D_\alpha \longrightarrow$ **(4)**

2°) Determining the differential equation of motion

* Our system is a **damped free** oscillator
* **dof = 1** $\longrightarrow$ **q(t) =** $\boldsymbol{\theta(t)}$
* rotational movement.

The Lagrange principle $\longrightarrow$
differential equation sought

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \left(\frac{\partial L}{\partial\theta}\right) = -\frac{\partial D_\alpha}{\partial\dot{\theta}} \quad (5)$$

with $L = T - U = ?$ **et** $D_\alpha = ?$

$T = \frac{9}{2}ma^2\dot{\theta}^2$ **et** $U = \frac{5}{2}K(a\sin\theta)^2 - 3\,mga\cos\theta$,

$$L = \frac{9}{2}ma^2\dot{\theta}^2 - \frac{5}{2}K(a\sin\theta)^2 + 3\,mga\cos\theta \quad \text{and} \quad D_\alpha = \frac{1}{2}\alpha a^2\dot{\theta}^2 \qquad (6)$$

from **(5) and (6)** $\longrightarrow$ $\frac{\partial L}{\partial\dot{\theta}} = 9ma^2\dot{\theta} \longrightarrow \frac{d}{dt}\left[\frac{\partial L}{\partial\dot{\theta}}\right] = 9ma^2\ddot{\theta}$

$$\frac{\partial L}{\partial\theta} = -5Ka^2\sin\theta\cos\theta - 3mga\sin\theta \quad \text{and} \quad \frac{\partial D_\alpha}{\partial\dot{\theta}} = \alpha a^2\dot{\theta} \qquad (7)$$

from **(5) and (7)** $\longrightarrow$ $9ma^2\ddot{\theta} + 5Ka^2\sin\theta\cos\theta + 3mga\sin\theta = -\alpha a^2\dot{\theta}$

$$9ma^2\ddot{\theta} + \alpha a^2\dot{\theta} + 5Ka^2\sin\theta\cos\theta + 3mga\sin\theta = 0$$

or

$$9ma^2\ddot{\theta} + \alpha a^2\dot{\theta} + (5Ka^2\cos\theta + 3mga)\sin\theta = 0 \qquad (8)$$

original differential equation of motion

as **θ ≪ (low amplitude oscillations)** → **$\cos\theta \approx 1$ and $\sin\theta \approx \theta$ (9)**

from (8) and (9) →

$$9ma^2\ddot{\theta} + \alpha a^2\dot{\theta} + (5Ka^2 + 3mga)\theta = 0$$

(10a)

or in the form $\ddot{\theta} + 2\delta\dot{\theta} + \omega_0^2\theta = 0$ **(10b)** **approximate differential equation of motion**

$$\delta = \frac{\alpha a^2}{18ma^2} = \frac{\alpha}{18m}$$

movement friction factor

with

$$\omega_0 = \sqrt{\frac{5Ka^2 + 3mga}{9ma^2}}$$

proper pulsation of the movement

3°) <u>Equation θ(t) of motion</u>

The differential equations of motion **(10a)** or **(10b)** are second-order linear differential equations

with constant coefficients and no second member.

Solution of **(10b)** and for the case of **low friction ($\delta < \omega_0$)** → **pseudoperiodic movement**

→ $$\theta(t) = \theta = Ae^{-\delta t}\sin(\omega_a t + \varphi)$$ **(11)**

$$\omega_a = \sqrt{\omega_0^2 - \delta^2}$$

pseudopulsation of motion.

The Constants **(A , φ)** are determined by initial conditions $\begin{cases} \theta(0) = \theta_0 \\ \dot{\theta}(0) = 0 \end{cases}$ **(12)**

(4) ⟶ $\dot{\theta}(t) = -\delta A e^{-\delta t}\sin(\omega_a t + \varphi) + \omega_a A e^{-\delta t}\cos(\omega_a t + \varphi)$ **(13)**

(4) , (5) et (6) ⟶ $\begin{cases} \theta(0) = A\sin(\varphi) = \theta_0 \\ \dot{\theta}(0) = -\delta A\sin(\varphi) + \omega_a A\cos(\varphi) = 0 \end{cases}$ **(14)**

from **(7)** ⟶ $tg(\varphi) = \frac{\omega_a}{\delta}$ ⟶ $\varphi = Arctg(\frac{\omega_a}{\delta})$ **(15a)**

$$A = \frac{\theta_0}{sin(\varphi)} = \frac{\theta_0}{sin(Arctg(\frac{\omega_a}{\delta}))}$$ **(15b)**

⟶ $\theta(t) = \theta = Ae^{-\delta t}\sin(\omega_a t + \varphi)$

CHAPTER III :DAMPED FORCED OSCILLATIONS WITH ONE DEGREE OF FREEDOM

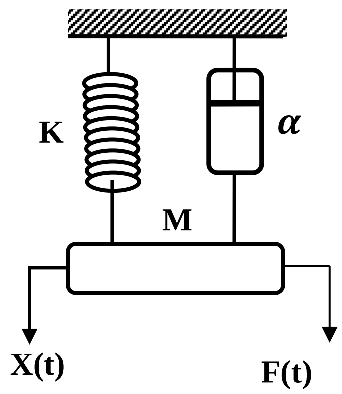
CHAPTER III :DAMPED FORCED OSCILLATIONS WITH ONE DEGREE OF FREEDOM

III.1 Introduction

We saw in the previous chapter that the presence of friction (viscous friction) causes the system's mechanical energy to be dissipated in the form of heat during movement, forcing the system to stop after a certain time. To compensate for this loss of energy, and therefore to give the system a permanent oscillatory motion, we need to supply the system with an external source of energy by subjecting it to an **external force F(t),** called the **exciting force**, which is collinear with the motion of the system. In this case, the oscillations of the system are called **damped forced oscillations.** This phenomenon of forced oscillations is of great importance in practice and applies to all types of oscillator: mechanical, acoustic, electrical, optical, thermodynamic, etc.....

III.2 Differential equation of motion

As before, consider the mechanical system composed of a mass (**M**), a spring (**K**), a viscous friction damper and the external exciting force **F(t).**



The whole is represented by the mechanical system shown in the figure opposite.

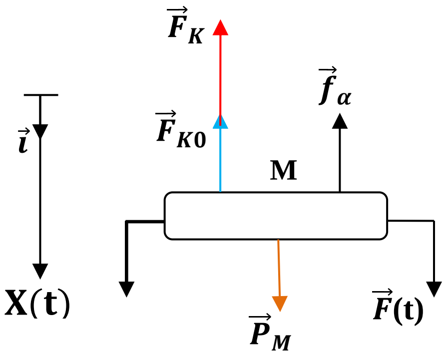
III.2.1 Method of the fundamental principle of dynamics (FPD)

* Damped forced oscillator

* Degree of freedom **dof =1** ⟶ **q(t) = x(t)** ⟶ $\dot{q} = \dot{x}$

and $\ddot{q} = \ddot{x}$

* Vertical translation movement along axis **(OX)**



PFD ⟶ $\sum_i \vec{F}_{ext\,i} = M\ddot{\vec{x}}$ (1)

a) <u>Static state</u> <u>(equilibrium or resting state)</u>

PFD ⟶ $\sum_i \vec{F}_{ext\,i} = \vec{0}$ ⟶ $\vec{P}_M + \vec{F}_{K_0} = \vec{0}$

⟶ $Mg\vec{i} - Kx_0\vec{i} = \vec{0} \rightarrow Mg\vec{i} - Kx_0\vec{i} = \vec{0}$

x_0 elongation of the spring **K** at equilibrium under the effect of the weight of **M**.

b) <u>Dynamic state</u> <u>(state of movement)</u>

The mass **M** of **x** is shifted from its equilibrium position and the system is left to its own devices.

PFD ⟶ $M\ddot{\vec{x}} = \sum_i \vec{F}_{ext\,i}$ ⟶ $M\ddot{\vec{x}} = \vec{P}_M + \vec{F}_{K_0} + \vec{F}_K + \vec{f}_\alpha + \vec{F}(t)$ **(2)**

⟶ $M\ddot{\vec{x}} = \vec{F}_K + \vec{f}_\alpha + \vec{F}(t) \rightarrow M\ddot{x}\vec{i} = -Kx\vec{i} - \alpha\dot{x}\vec{i} + F(t)\vec{i}$ **(3)**

$$M\ddot{x} + \alpha\dot{x} + Kx = F(t) \qquad (4)$$

differential equation of motion

III.2.2 Lagrange method

According to Lagrange's principle, the differential equation of motion of a damped forced mechanical oscillator with degree of freedom **dof =1** ⟶ **q(t)** is of the following general form :

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] - \frac{\partial L}{\partial q} = -\frac{\partial D_\alpha}{\partial \dot{q}} + F_q(t) \qquad (5)$$

where

L : Lagrangian or Lagrange function with **L = T – U**

$D_\alpha = \frac{1}{2}\alpha v^2$ **: energy dissipation function** and **velocity** of the point of application of friction α

$F_q(t)$ **: excitatory external generalised force linked to the independent generalised coordinate q(t) = q**

The quantity $F_q(t) = \begin{cases} \text{the exciting force } F(t) \text{ if the movement is translational} \\ \text{the moment of } F(t) \text{ if the movement is rotational} \end{cases}$

such as $F_q(t) = \vec{F}(t).\frac{\partial \vec{r}}{\partial q}$, $\vec{r}$ position vector of the point of application of $\vec{F}(t)$

For our oscillator considered previous system (**M, K, α and F(t)**) we have**:**

* **q(t) = x(t)= x, $\dot{q}$(t) = $\dot{x}$(t)= $\dot{x}$ and $\ddot{q}$(t) = $\ddot{x}$(t)= $\ddot{x}$**

from where **(5)** ⟶ $$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right] - \frac{\partial L}{\partial x} = -\frac{\partial D_\alpha}{\partial \dot{x}} + F_x(t) \qquad (6)$$

with

$$L = T_M - U_K = \frac{1}{2}M\dot{x}^2 - \frac{1}{2}Kx^2$$

*** U_M** does not intervene in the expression of **U** because the weight of **M** deforms the spring **K** at equilibrium.

$$D_\alpha = \frac{1}{2}\alpha v_M^2 = \frac{1}{2}\alpha \dot{x}^2 \qquad \textbf{(7)}$$

* $F_x(t) = \vec{F}(t)\frac{\partial \vec{r}_M}{\partial x} = F(t)\vec{\imath}\frac{\partial x\vec{\imath}}{\partial x} = F(t)\vec{\imath}\cdot\vec{\imath} = F(t)$ (translational movement)

from **(6)** and **(7)** ⟶ $\frac{\partial L}{\partial \dot{x}} = M\dot{x}$ ⟶ $\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right] = M\ddot{x}$ **(8)**

$\frac{\partial L}{\partial x}$ = -Kx and $\frac{\partial D_\alpha}{\partial \dot{x}} = \alpha\dot{x}$ and $F_x(t) = F(t)$

from **(6)** and **(8)** ⟶ $M\ddot{x} + \alpha\dot{x} + Kx = F(t)$ **(9)**

differential equation of motion of the system

found before by the FPD (4)

III.3 Equation of motion

Equation **(4)** or **(9)** is a second-order linear differential equation with constant coefficients and a second member. This equation can be written as

$$\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = B(t) \qquad \textbf{(10)}$$

(

(10)

Where $\omega_0 = \sqrt{\frac{K}{M}}$ **proper pulsation of** movement

$\delta = \frac{\alpha}{2M}$ **friction or motion damping factor**

$B(t) = \frac{F(t)}{M}$ **term dependent on the exciting force F (t)**

The solution x(t) (equation of motion) of this equation is the sum of two terms such that**:**

$$\mathbf{x(t) = x_h(t) + x_P(t)} \qquad \mathbf{(11)}$$

$\mathbf{x_h(t)}$ **: homogeneous solution** with no second member such that

$$\ddot{x}_h + 2\delta\dot{x}_h + \omega_0^2 x_h = 0 \qquad \mathbf{(12)}$$

The solutions $\mathbf{x_h(t)}$ (see Chapter II) depend on the relationship between ω_0 and δ ($\delta < \omega_0$, $\delta = \omega_0$, **and** $\delta > \omega_0$) where the limit of $\mathbf{x_h(t)}$ in all cases is always zero when **t** tends to infinity.

$\mathbf{x_P(t)}$ **: particular solution** with second member such that

$$\ddot{x}_P + 2\delta\dot{x}_P + \omega_0^2 x_P = F(t) \qquad \mathbf{(13)}$$

$\mathbf{x_P(t)}$ its mathematical form depends on that of **F(t).**

In reality, the motion **x(t)** of the system is composed of two regimes:

* The first regime is called the transient regime and occurs at the start of the movement for such that $\mathbf{x(t) = x_h(t) + x_P(t)}$

This regime disappears when **t** increases since $\mathbf{x_h(t)} \propto e^{-\delta t}$ tends to zero for **t**≫

* The second regime is called **steady state** and occurs when $\mathbf{t \gg t_0}$ such that $\mathbf{x(t) \approx x_P(t)}$

This regime is permanent since it is due to the exciting force **F(t)** being permanently applied.

III.4 Amplitude and initial phase of steady state

Case of sinusoidal F(t) excitation

$$F(t) = F_0 cos(\Omega\, t) \qquad \mathbf{(14)}$$

where Ω is the excitation pulsation **F(t)**

When the component $x_h(t)$ becomes truly negligible, only the particular solution remains, which is the solution imposed by the excitation function. We say that we are in a forced regime or steady state.

The exciting forces the mechanical system to follow a time evolution equivalent to its own. So if $\mathbf{F_{ext}}$ is a sinusoidal function of angular frequency **Ω**; then the particular solution $x_P(t)$ will be a sinusoidal solution of the same angular frequency **Ω**.

The oscillations of the mass m are not necessarily in phase with the exciting force and present a phase shift noted **φ**. The particular solution corresponding to the permanent regime is therefore written:

$$\mathbf{x_p(t) = A\cos(\Omega t + \varphi)} \qquad (15)$$

For practical reasons, it is convenient to use complex notation. The complex quantity associated with $x_p(t)$ is written as:

$$\mathbf{X_p(t) = A\, e^{j(\Omega t + \varphi)} \quad and \quad F_{ext} = F_0 e^{j\Omega t}} \qquad (16)$$

Determining the quantities **A** and **φ** amounts to finding the amplitude and phase of the complex number $\mathbf{X_p(t)}$.

III.4.1 Calculation of amplitude A

$\mathbf{X_p(t)}$ Check the differential equation with second member:

$$\mathbf{\ddot{X}_p + 2\delta\dot{X}_p + \omega_0^2 X_p = \frac{F_0}{m} X_p = Be^{j\Omega t}} \qquad (17)$$

Let us calculate the first derivative and then the second derivative of $\mathbf{X_p(t)}$.

$$X_p(t) = A\, e^{j(\Omega t+\varphi)} \Rightarrow \begin{cases} \dot{X}_p(t) = Aj\Omega\, e^{j(\Omega t+\varphi)} = j\Omega\, X_p(t) \\ \ddot{X}_p(t) = Aj^2\,\Omega^2\, e^{j(\Omega t+\varphi)} = -\Omega^2\, X_p(t) \end{cases} \qquad (18)$$

We replace in the differential equation we find:

$$-\Omega^2\, X_p(t) + 2\delta j\Omega\, X_p(t) + \omega_0^2\, X_p(t) = B\, e^{j\Omega t}$$

$$\Rightarrow \left[\left(\omega_0^2 - \Omega^2\right) + 2\delta\Omega j\right] X_p(t) = \left[\left(\omega_0^2 - \Omega^2\right) + 2\delta\Omega j\right] A\, e^{j(\Omega t+\varphi)}$$

$$= B\, e^{j\Omega t}\left[\left(\omega_0^2 - \Omega^2\right) + 2\delta\Omega j\right] Ae^{j\varphi} = B$$

We divide by ‘’ $e^{j\varphi}$ ‘’ and we find:

$$\left[\left(\omega_0^2 - \Omega^2\right) + 2\delta\Omega j\right] A = Be^{-j\varphi} \qquad (19)$$

The complex conjugate of this equation is as follows:

$$\left[\left(\omega_0^2 - \Omega^2\right) - 2\delta\Omega j\right] A = Be^{j\varphi} \qquad (20)$$

$$(19)\text{et}\,(20) \Rightarrow A^2\left[\left(\omega_0^2 - \Omega^2\right)^2 + (2\delta\Omega)^2\right] = B^2$$

$$\Rightarrow \boxed{A = \frac{B}{\sqrt{\left[\left(\omega_0^2 - \Omega^2\right)^2 + (2\delta\Omega)^2\right]}}}$$

III.4.2 Calculating the phase shift φ :

$$\left[\left(\omega_0^2 - \Omega^2\right) + 2\delta\Omega j\right] A = \begin{cases} Be^{-j\varphi} \\ B(\cos\varphi - j\sin\varphi) \end{cases} \Leftrightarrow \begin{cases} A\left(\omega_0^2 - \Omega^2\right) = B\cos\varphi \\ 2\delta\Omega A = -B\sin\varphi \end{cases}$$

$$\Rightarrow \text{tg}\varphi = \frac{-2\delta\Omega}{\left(\omega_0^2 - \Omega^2\right)} \Rightarrow \boxed{\varphi = \text{Arctg}\left[\frac{-2\delta\Omega}{\left(\omega_0^2 - \Omega^2\right)}\right]}$$

So :

$$x_p(t) = \frac{B}{\sqrt{\left[\left(\omega_0^2 - \Omega^2\right)^2 + (2\delta\Omega)^2\right]}} \cos\left[\Omega t + \text{Arctg}\left[\frac{-2\delta\Omega}{\left(\omega_0^2 - \Omega^2\right)}\right]\right]$$

Remarks :

• The general solution to the differential equation is written as: $\mathbf{x(t) = x_h(t) + x_P(t)}$.

• $\mathbf{x_h(t)}$ is called a homogeneous solution characterising a transient regime that disappears exponentially with time. When the transitory regime disappears: $x(t) \simeq x_P(t)$.

• $\mathbf{x_P(t)}$ is called a **particular solution** of amplitude $\mathbf{A = \frac{B}{\sqrt{\left[\left(\omega_0^2 - \Omega^2\right)^2 + (2\delta\Omega)^2\right]}}}$ which characterises a steady state (stationary) because it exists as long as the external force ($\mathbf{F_{ext}}$) is applied. We note the dependence of the amplitude **A** of pulsation $\mathbf{\Omega}$.

• The solution x(t) will therefore often have a characteristic appearance like that shown in the figure below.

III.5. Study of the steady state: amplitude resonance phenomenon.

III.5.1. The variation of the amplitude as a function of the force pulsation for different values of δ :

Let $\mathbf{A(\Omega)}$ be the amplitude of the particular solution characterising the (forced) steady state:

$$\mathbf{A(\Omega) = \frac{B}{\sqrt{\left[\left(\omega_0^2 - \Omega^2\right)^2 + (2\delta\Omega)^2\right]}}}$$

$$\mathbf{A(\Omega)} = \frac{\mathbf{B}}{\omega_0^2\sqrt{\left[\left(1-\frac{\Omega^2}{\omega_0^2}\right)^2 + (2\delta)^2\left(\frac{\Omega}{\omega_0^2}\right)^2\right]}} = \frac{\mathbf{B/\omega_0^2}}{\sqrt{\left[\left(1-\left(\frac{\Omega}{\omega_0}\right)^2\right)^2 + \left(\frac{2\delta}{\omega_0}\right)^2\left(\frac{\Omega}{\omega_0}\right)^2\right]}}$$

$$= \frac{\mathbf{A_0}}{\sqrt{\left[\left(1-\left(\frac{\Omega}{\omega_0}\right)^2\right)^2 + (2\xi)^2\left(\frac{\Omega}{\omega_0}\right)^2\right]}}$$

Such as : $A_0 = \frac{B}{\omega_0^2}$ ***and*** $\xi = \frac{\delta}{\omega_0}$ ***with*** $B = \frac{F_0}{m} \Rightarrow A_0 = \frac{F_0}{m\omega_0^2} = \frac{F_0}{K}$

$$\frac{\mathbf{A(\Omega)}}{\mathbf{A_0}} = \frac{1}{\sqrt{\left[\left(1-\left(\frac{\Omega}{\omega_0}\right)^2\right)^2 + (2\xi)^2\left(\frac{\Omega}{\omega_0}\right)^2\right]}}$$

Let's put: $\mathbf{r} = \frac{\Omega}{\omega_0} \Rightarrow \mathbf{A(r)} = \frac{\mathbf{A_0}}{\sqrt{[(1-\mathbf{r}^2)^2 + (2\xi\mathbf{r})^2]}}$

and let's look for the maximum value of $A(r)$.

$$\mathbf{A(r)_{max}} \Leftrightarrow \frac{\mathbf{dA(r)}}{\mathbf{dr}} = 0 \Leftrightarrow \mathbf{A_0}\frac{\mathbf{d}}{\mathbf{dr}}[(1-\mathbf{r}^2)^2 + (2\xi\mathbf{r})^2]^{-\frac{1}{2}} = 0\ ;\ \mathbf{A_0} \neq 0$$

$$\Rightarrow \frac{\mathbf{d}}{\mathbf{dr}}[(1-\mathbf{r}^2)^2 + (2\xi\mathbf{r})^2]^{-\frac{1}{2}} = 0$$

$$\Rightarrow -\frac{1}{2}[(1-r^2)^2 + (2\xi r)^2]^{-\frac{3}{2}}[2(1-r^2)(-2r) + 8\xi^2 r] = 0 \Rightarrow -4r\,[1-r^2-2\xi^2]$$

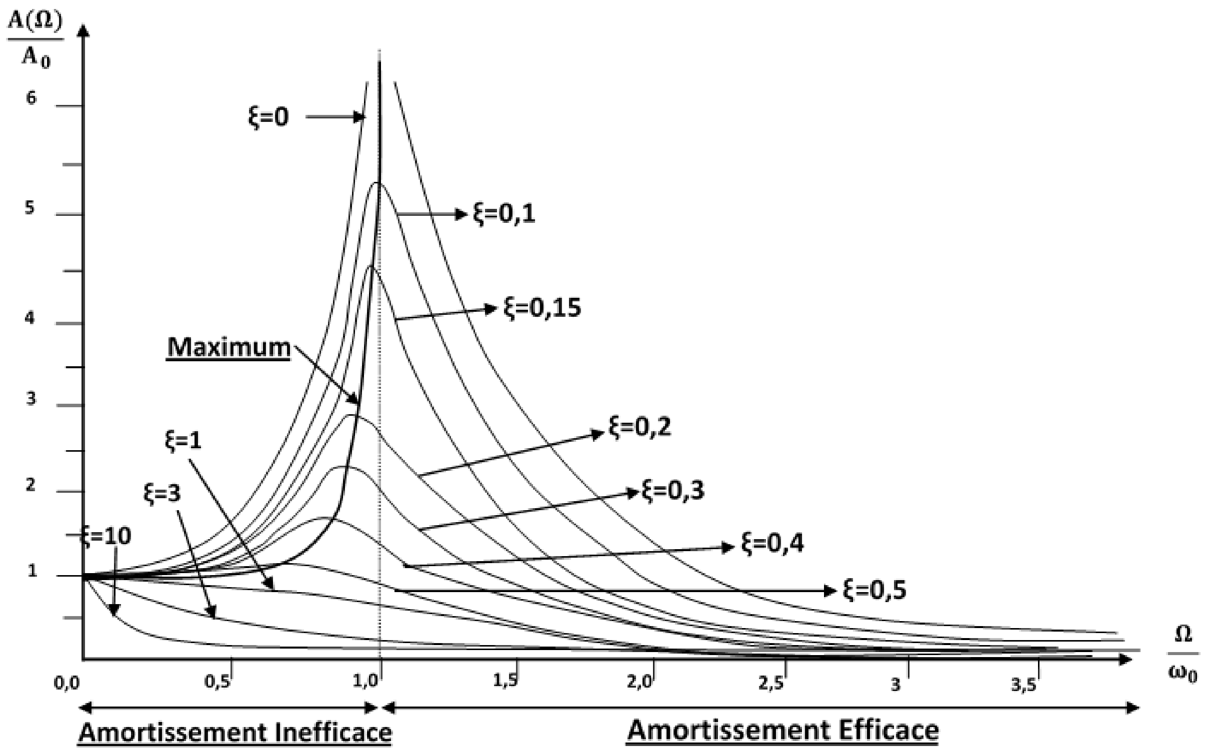
$$= 0 \Rightarrow \begin{cases} r \neq 0; \left(\frac{\Omega}{\omega_0} \neq 0\right) \\ r_{max} = \left(\frac{\Omega_R}{\omega_0}\right) = \sqrt{1-2\xi^2} \end{cases} \quad \text{So } \frac{A(\Omega)}{A_0} \text{ is maximal for } \left(\frac{\Omega_R}{\omega_0}\right)$$

$$= \sqrt{1-2\xi^2}$$

There is a maximum at pulsation $\Omega_R = \omega_0\sqrt{1-2\xi^2} = \sqrt{\omega_0^2 - 2\delta^2}$ only if the damping is sufficiently low that $\delta < \omega_0/\sqrt{2}$. At this pulsation the system enters into resonance and the amplitude $A(\Omega)$ is at its maximum; it is equal to:

$$A(\Omega)_{max} = \frac{A_0}{2\xi\sqrt{1-\xi^2}} = \frac{A_0\omega_0^2}{2\delta\sqrt{\omega_0^2-\delta^2}} = \frac{B}{2\delta\sqrt{\omega_0^2-\delta^2}}$$

The amplitude variation as a function of the force pulsation for different values of ξ is shown in the following figure:



III.5.2. Study of resonance

§- For $\boldsymbol{\Omega_R = \omega_0\sqrt{1-2\xi^2} = \sqrt{\omega_0^2 - 2\delta^2}}$: the amplitude of vibration reaches a maximum, we say that there is **resonance**.

§- At the start of a resonant movement, when the force is applied to the system. Most of the energy supplied during each cycle is stored in the system, which gradually increases the amplitude of its oscillations to a maximum value. This value remains as long as the energy supplied by the external force remains.

§- The lower the damping, the sharper the curve and the greater the maximum; if there is insufficient damping, there is nothing to limit the amplitude of the oscillations, with the risk of destroying the system: the system goes into resonance. The consequences can be serious. Two well-known cases can be cited: • On 18 April 1850 in Angers, a regiment crossing a suspension bridge in cadence (harmonious) caused its destruction.

- On 7 November 1940, six months after its inauguration, the suspension bridge in Tacoma (United States) was destroyed by the effects of gusts of wind which, although not particularly violent (60 km/h), were regular.

§- In the case of low damping (), the resonance frequency is very little different from the natural pulsation, . In this case, the amplitude of vibration at resonance :

$$A(\Omega)_{max} = \frac{A_0}{2\xi} = \frac{A_0\omega_0}{2\delta} = \frac{B}{2\delta\omega_0}$$

For low damping, $\mathbf{A(\Omega)_{max}}$ is therefore inversely proportional to δ.

III.5.3. Conclusions :

Depending on the value of ξ we have 3 possible cases:

<u>1st case</u>: Low frequencies*:* $\boldsymbol{\xi \ll 1\ (\Omega \ll \omega_0) \Rightarrow \begin{cases} A \approx A_0 = \frac{F_0}{K} \\ \varphi = 0 \end{cases}}$

<u>2nd case</u>: High frequencies*:* $\boldsymbol{\xi \gg 1\ (\Omega \gg \omega_0) \Rightarrow \begin{cases} A \approx 0 \\ \varphi = -\pi \end{cases}}$

<u>3rd case</u>: Resonance*:*

$$\xi = 1\ (\Omega \approx \omega_0 \approx \Omega_R) \Rightarrow \begin{cases} \Omega_R = \omega_0\sqrt{1-2\xi^2};\ A(\Omega)_{max} = \dfrac{A_0}{2\xi\sqrt{1-\xi^2}} \\ \varphi = -\dfrac{\pi}{2} \end{cases}$$

Remarks: • If $\xi=0$ (system is undamped): the amplitude tends towards infinity but in reality, the systems are all damped so the amplitude is never infinite.

• If ξ very low (weakly damped system): $A(\Omega)_{max} \approx \frac{A_0}{2\xi}$; $(\Omega \approx \omega_0 \approx \Omega_R)$.

III. 5.4. Resonance phenomenon and quality factor

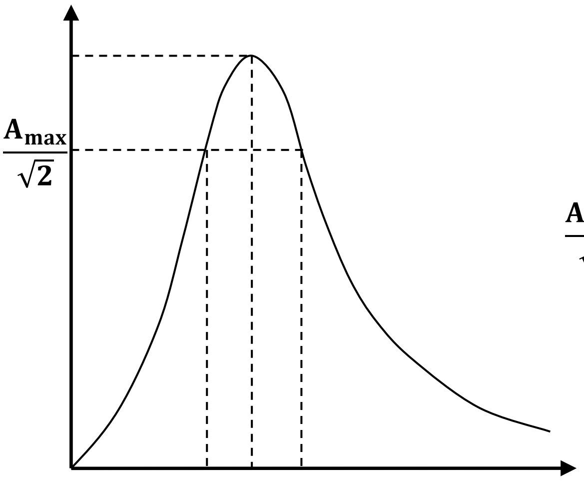
§- The resonance phenomenon appears when the pulsation of the external excitation approaches the natural pulsation of the system.

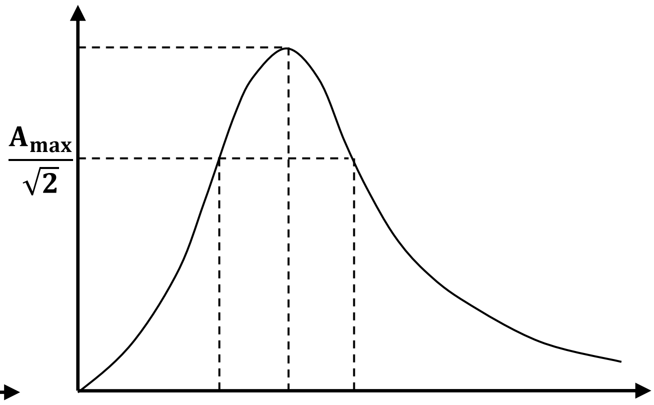
§- At resonance, the vibration amplitude increases until it exceeds the elastic limits of mechanical systems, producing their destruction.

§- In electrical systems, this phenomenon makes it possible to calculate the quality factor **Q**, which increases when the maximum amplitude increases $Q = \frac{A_{max}}{A_0} \approx \frac{1}{2\xi}$

§- Another practical method to determine the quality factor: $Q = \frac{\omega_0}{\Omega_2 - \Omega_1}$

§- To characterize the sharpness (intensity) of the response of an oscillator as a function of the pulsation, we define a bandwidth: $\Omega_2 - \Omega_1$

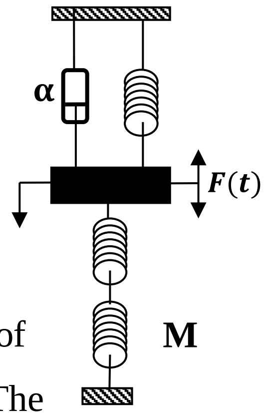




Conclusions: • When ξ increases ⟹ **Q** decreases ⟹ $\Omega_2 - \Omega_1$ increases ⟹ the resonance curve is wider ⟹ decrease of the resonance amplitude therefore of the quality factor too.

• The ends of the bandwidth correspond to an amplitude of value $\sqrt{2}$ times smaller than at resonance.

Exercise



Given a mechanical system consisting of a mass **M**, springs of stiffnesses K_1, K_2, K_3 and a damper of viscous friction coefficient α all attached to the mass **M** (see diagram opposite). The mass **M** is subjected to an external force $F(t) = F_0 \sin(\Omega t)$. The oscillatory motion of about its equilibrium position is described by the function **y(t)**. The amplitude of the oscillations and the viscous friction are small. We give $K_1 = K/2$ and $K_2 = K_3 = K$.

1°) Give the equivalent mechanical system (simplified system) of the given system.

2°) Cite the forces involved in the movement of **M**, specifying the effect of each on the movement. **Justify your answer**.

3°) Using the fundamental principle of dynamics **(FPD)**, establish the differential equation of the motion of **M** as a function of **y(t)** and deduce the proper pulsation ω_0 and the friction coefficient **δ** of the motion.

4°) Give the **general equation y(t)** of the motion of **M**, specifying the different terms of the equation.

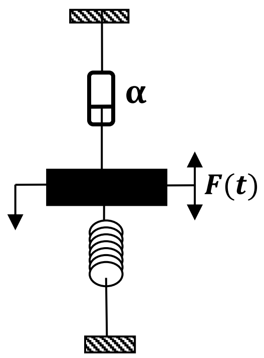
5°) What are the conditions that must be imposed on **F(t)** so that the system oscillates in an optimal (maximal) way? In this case, give the response (in terms of magnitudes) of the system to this force.

Solution

1°) Equivalent System

K_2 and **K_3** in series ⟶ $Ke_1 = K_2K_3/(K_2+K_3) = K/2$

K_1 and **Ke_1** in parallel ⟶ $Ke = K_1 + Ke_1 = K$



Equivalent system

2°) Definitions and effects of the forces involved in the movement of M.

* Weight of M: $\overrightarrow{P_M}$ force of gravity PM = Mg

* Spring force K : $\overrightarrow{F_K}$ module restoring force $F_K = Ky$, force deriving from a potential.

*Damper force α : $\overrightarrow{f\alpha}$ modulus viscous friction force $f_\alpha = \alpha\dot{y}$ force not derived from a potential but from dissipation energy D_α.

*External force $\overrightarrow{F(t)}$: exciting force (force that supplies energy to the system).

3°) Differential equation of movement

* dof = 1 :q(t) = y(t) , Damped forced oscillations and translational movement along the (OY) axis

FPD (Newton's2nd law) : $\sum_i \vec{F}i = M\vec{\ddot{y}}$ (1)

* Static state: FPD $\sum_i \vec{F}i = \vec{O}$ $\overrightarrow{P_M} + \overrightarrow{F_{K_0}} = \vec{O}$

* Dynamic state

FPD: $\sum_i \vec{F}i = M\vec{\ddot{y}}$ $M\vec{\ddot{y}} = \overrightarrow{P_M} + \overrightarrow{F_{K_0}} + \overrightarrow{F_K} + \overrightarrow{f_\propto} + \overrightarrow{F(t)}$

$M\ddot{y} + \propto \dot{y} + Ky = F_0 \sin(\Omega t)$ (2)

Ou $\ddot{y} + 2\delta\dot{y} + \omega_0^2 y = B_0 \sin(\Omega t)$ (3) with $\omega_0 = [\ K/M\ ]1/2$, $\delta = \propto/2M$ and $B_0 = F_0/M$

4°- Equation y(t) of motion

$y(t) = y_h(t) + y_P(t) = Ce^{-\delta t} \sin(\omega_a t + \varphi_0) + A(\Omega)\sin(\Omega t + \Phi(\Omega))$

$y_h(t) = Ce^{-\delta t}\sin(\omega_a t + \varphi_0)$ with $\omega_a = (\omega_0^2 - \delta^2)^{\frac{1}{2}}$: pseudo-pulsation

$y_P(t) = A(\Omega)\sin(\Omega t+\Phi(\Omega))$: permanent regime

$A(\Omega) = \frac{B_0}{\left[(\omega_0^2 - \Omega^2)^2 + 4\delta^2\Omega^2\right]^{1/2}}$, $\Phi(\Omega = -\text{Arctg}\left[\frac{2\delta\Omega}{\omega_0^2 - \Omega^2}\right]$ and (C, φ_0) determined by the initial conditions: (y(0) and $\dot{y}(0)$)

5°) For the system to oscillate optimally (maximally), it must:

* Force frequency $\Omega = \Omega r = [\omega_0^2 - 2\delta^2]^{1/2}$ resonance pulsation with $0 < \delta < \frac{\omega_0}{\sqrt{2}}$

* The intensity F0 of F(t) is maximum and finite

The system response $\longrightarrow$ In this case, steady state will be in terms of maximum values where

* $A(\Omega)$ $\longrightarrow$ $A_{max} = A(\Omega = \Omega r) = \frac{B_0}{2\delta[\omega_0^2 - \delta^2]^{1/2}}$ and finished

* $\Phi(\Omega)$ $\longrightarrow$ $\Phi(\Omega r) = -\text{Arctg}\left[\frac{[\omega_0^2 - \delta^2]^{1/2}}{\delta}\right] \sim -\pi/2$

CHAPTER IV :N DEGREES OF FREEDOM

CHAPTER IV :N DEGREES OF FREEDOM

IV-1.Introduction

This chapter examines conservative mechanical systems with an arbitrary number N of degrees of freedom, where the various components oscillate around their equilibrium positions and interact with one another. At any given moment, the system's configuration is defined by generalized coordinates, the number of which equals the number of degrees of freedom. Additionally, N relations between the different amplitude factors are obtained.

IV-2. Study of a Mechanical System with N Degrees of Freedom and Fixed Ends

We examine the behavior of a chain of N oscillators arranged in series between two fixed supports (see figure below). All oscillators are assumed to be identical, each consisting of a point mass m and a spring of stiffness k.

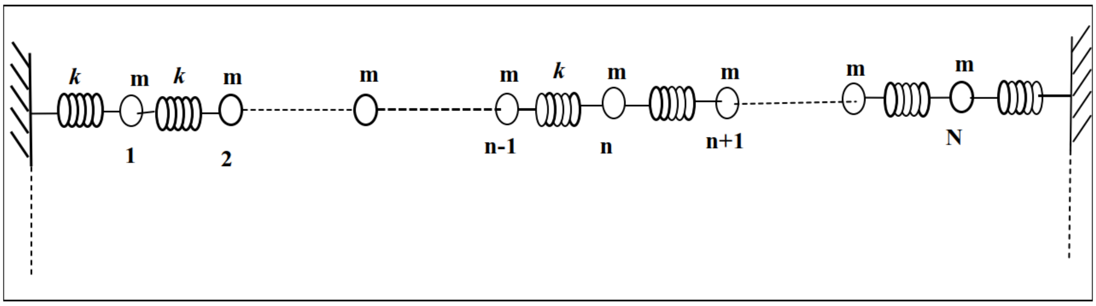


Figure 1. Analysis of a mechanical system with N degrees of freedom and fixed ends

IV.2.1 Derivation of the Differential Equation of Motion for Mass n

The motion of mass n is studied by applying the fundamental principle of dynamics (Newton's second law). $\sum \vec{F} = m\vec{\gamma}_n$

The weight of mass n is negligible.

Therefore: $\vec{T}_1 + \vec{T}_2 = m\vec{\gamma}_n$

By projecting along the OZ axis, we obtain:

$T_{1Z} - T_{2Z} = 0$

that is : $T_2 \cos\alpha_2 - T_1 \cos\alpha_1 = 0$

Since the angles α_1 and α_2 are less than 10 degrees (small amplitudes), we may assume:

$\cos\alpha_1 = 1$ and $\cos\alpha_2 = 1$

Thus : $T_1 = T_2 = T_0$ **(1)**

Projecting along the Ox axis yields: $T_{2x} - T_{1x} = m\ddot{x}_n$

That $T_2 \cos\alpha_2 - T_1 \cos\alpha_1 = m\ddot{x}_n$ **(2)**

$\tan\alpha_2 = \frac{x_{n+1} - x_n}{d}$ **(3)**

and $\tan\alpha_1 = \frac{x_n - x_{n-1}}{d}$: **(4)**

Substituting relations (IV-1), (IV-3), and (IV-4) into (IV-2) gives:

$$T_0 \frac{x_{n+1} - x_n}{d} - T_0 \frac{x_n - x_{n-1}}{d} = m\ddot{x}_n$$

Thus, the differential equation governing the motion of mass n is:

$\frac{T_0}{d}(x_{n+1} + x_{n-1} - 2x_n) = m\ddot{x}_n$ **(5)**

IV.2.1.1 Calculation of the dispersion relation

The ends of the mass are fixed. The solution x_n of the differential equation **(5)** is given by:

$x_n(t) = A\sin(nkd)\sin(\omega t + \varphi)$

Where $\boldsymbol{k}$ represents the magnitude of the wave vector.

The same will apply for the displacements $\boldsymbol{x_{n+1}}$ and $\boldsymbol{x_{n-1}}$**:**

$$x_{n+1}(t) = A\sin((n+1)kd)\sin(\omega t + \varphi)$$

and: $$x_{n-1}(t) = A\sin((n-1)kd)\sin(\omega t + \varphi)$$

We sum the two displacements $\boldsymbol{x_{n+1}}$ and $\boldsymbol{x_{n-1}}$:

$$x_{n+1} + x_{n-1} = A\sin((n+1)kd)\sin(\omega t + \varphi) + A\sin((n-1)kd)\sin(\omega t + \varphi)$$

$$x_{n+1} + x_{n-1} = A\sin(\omega t + \varphi)\,[\sin((n+1)kd) + \sin((n-1)kd)]$$

$$x_{n+1} + x_{n-1} = 2A\sin(kd)\sin(nkd)\sin(\omega t + \varphi)$$

Thus: $$x_{n+1} + x_{n-1} = 2\cos(kd)\,x_n \qquad \textbf{(6)}$$

and: $$\ddot{x_n} = \omega^2 x_n \qquad \textbf{(7)}$$

We substitute **(6)** and **(7)** into expression **(5)**, and obtain:

$$\omega^2 = \frac{2T_0}{md}(1 - \cos(kd))$$

We know that: $1 - \cos(kd) = 2sin^2\frac{kd}{2}$

We thus obtain : $\omega^2 = \frac{4T_0}{md}sin^2\frac{kd}{2}$ **(8)**

The displacement amplitude at these two points is zero:

$A\sin(kL) = 0$

Thus: $\sin(kL) = \sin(p\pi)$

Hence: $k_p = p\frac{\pi}{L}$ with $p \in N^*$

For each mode, there is a natural frequency, and formula **(8)** takes the generalized form:

$$\omega_p^2 = \frac{4T_0}{md} sin^2 \frac{k_p d}{2}$$

$$\omega_p = 2\sqrt{\frac{T_0}{md}} \left|\sin\frac{k_p d}{2}\right|$$

With $\omega_p = \sqrt{\frac{T_0}{md}}$

VI.3. Study of a mechanical system with N degrees of freedom and free ends

Such a medium is represented schematically by an infinite succession of masses m, separated at rest by a distance d, connected by springs with spring constants k (see **Figure 2**).

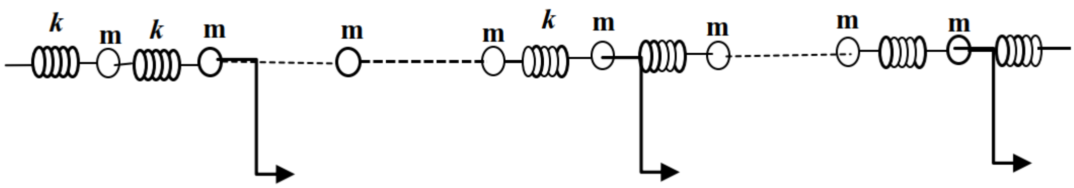


Figure 2. Study of a mechanical system with N degrees of freedom and free ends.

IV.3.1. Derivation of the differential equation of motion for mass n

Consider the masses with indices n-1, n, and n+1. At time *t*, their displacements from their respective equilibrium positions are x_{n-1}, x_n, and x_{n+1}, using the notation from Figure **2**.

At equilibrium, the masses are separated by a distance d. Mass n is subjected to two opposing forces of magnitude $k(d-I_0)$, where I_0 is the natural length of a spring.

Out of equilibrium, the spring between masses n and n+1 has a length $d+x_{n+1}-x_n$. It exerts on mass n the force: $\vec{F}_1 = k(d + x_{n+1} - x_n - I_0)\vec{\imath}$

Similarly, the spring between masses n and n-1 has a length $d+x_n-x_{n-1}$. It exerts on mass n the force: $\vec{F}_2 = -k(d + x_n - x_{n-1} - I_0)\vec{\imath}$

The equation of motion for mass n is given by:

$$m\ddot{x}_n\vec{\imath} = \vec{F}_1 + \vec{F}_2$$

$$m\ddot{x}_n\vec{\imath} = k\vec{\imath}(d + x_{n+1} - x_n - I_0 - d - x_n + x_{n-1} + I_0)$$

Thus: $m\ddot{x}_n = k(x_{n+1} + x_{n-1} - 2x_n)$

Or equivalently:

$$\ddot{x}_n + \frac{2k}{m}x_n = \frac{k}{m}(x_{n+1} + x_{n-1})$$

IV.3.2. Derivation of the dispersion relation

We are looking for solutions of the form :

$$x_n(t) = A\sin(\omega t - nkd)$$

Similarly, for the displacements x_{n+1} and x_{n-1}:

$$x_{n+1}(t) = A\sin(\omega t - (n+1)kd)$$

$$x_{n-1}(t) = A\sin(\omega t - (n-1)kd)$$

And the acceleration is:

$$\ddot{x}_{n+1}(t) = -A\omega^2\sin(\omega t - nkd)$$

We calculate $x_{n+1}+x_{n-1}$:

$$x_{n+1}(t) + x_{n-1}(t) = A\sin(\omega t - (n+1)kd) + A\sin(\omega t - (n-1)kd)$$

By taking the product of the sum of the sines, we obtain

$$x_{n+1}(t) + x_{n-1}(t) = 2A\sin(\omega t - nkd)\cos(kd)$$

Thus: $x_{n+1}+x_{n-1}=2\cos(kd)x_n$

Substituting into the equation of motion:

$$-A\omega^2\sin(\omega t - nkd) = \frac{k}{m}[2\cos(kd)A\sin(\omega t - nkd) - 2A\sin(\omega t - nkd)$$

$$-\omega^2 = \frac{2k}{m}[\cos(kd) - 1]$$

$$\omega^2 = \frac{2k}{m}[1 - \cos(kd)]$$

Where :

$$\omega^2 = \frac{2k}{m}\left[1 - \cos(\frac{\omega d}{v})\right] \qquad \textbf{(9)}$$

With : $k = \frac{\omega}{v}$

We replace the angular frequency with its value as a function of the frequency, we have :

$$4\pi^2 v^2 m = 4k sin^2\left(\frac{\pi d}{\lambda}\right)$$

$$k = \frac{2\pi}{\lambda}$$

We ask $v_0^2 = \frac{k}{\pi^2 m}$, v_0 is called resonance frequency

$$v = v_0 \sin\frac{\pi d}{\lambda}$$

Vibrations propagate only if $< v_0$. This part of the course will be covered in detail in a forthcoming book on waves.

In formula **(9)**, the term ***cos(kd)*** depends on ***k*** , the cosine varying between the limits **-1** and **1**; the permissible values for ω are : $\omega_{min} = 0$

and $\omega_{max} = 2\sqrt{\frac{k}{m}}$

In the case where kd is small, we can make an approximate calculation of the dispersion relation.

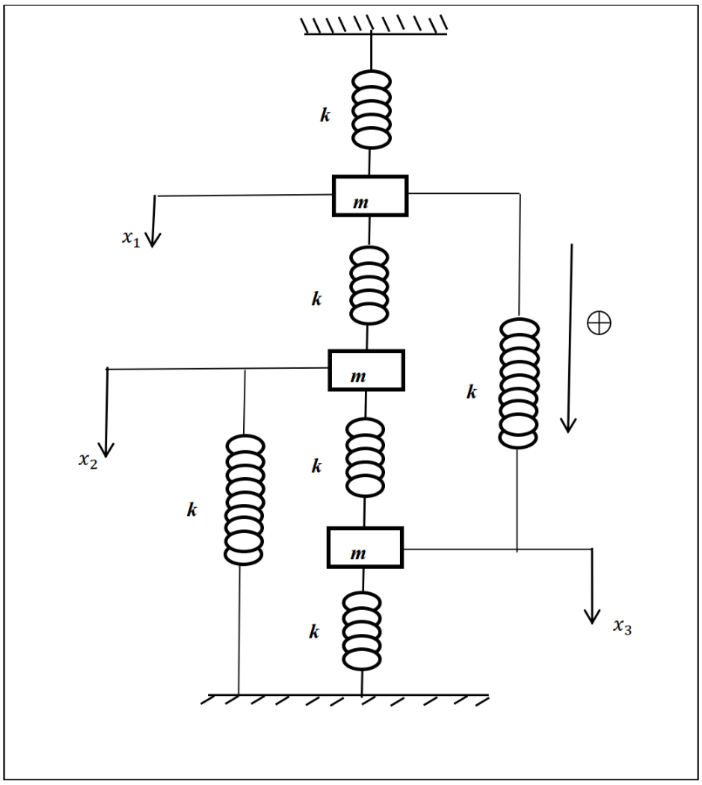
$$\omega^2 = \frac{2k}{m}(1 - \cos(kd)) = \frac{2k}{m}\frac{k^2 d^2}{2}$$

either $\omega = kd\sqrt{\frac{k}{m}}$

The propagation velocity takes the form: $v = d\sqrt{\frac{k}{m}}$

Exercise :

Consider the system of three identical masses **m** connected by springs with the spring constants indicated in the following figure:



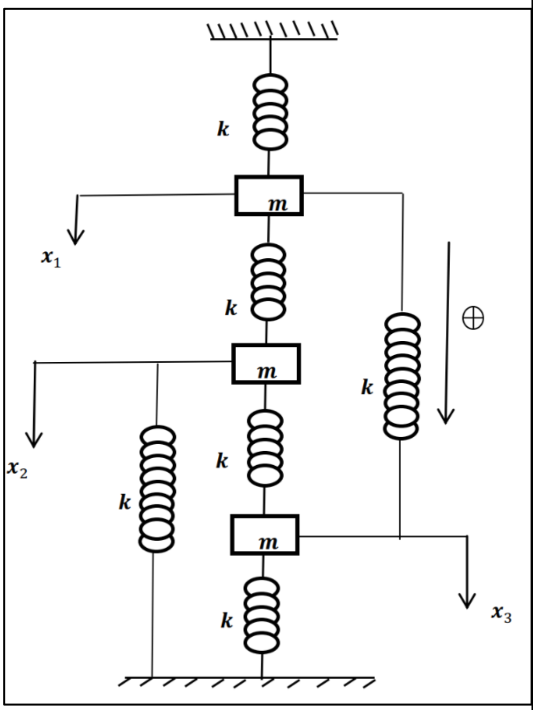
1- Establish the differential equations governing the motion of the system using Newton's method.

2- Calculate the three propres pulsations using the matrix method.

Solution

1. Differential equations of motion of the system using Newton's method:

Applying the fundamental principle of dynamics (Newton's second law) for translational motion:



Projecting in the positive direction yields:

$$m\ddot{x}_1 = -kx_1 - k(x_1 - x_3) - k(x_1 - x_2) \qquad (1)$$

$$m\ddot{x}_2 = -kx_2 - k(x_2 - x_1) - k(x_2 - x_3) \qquad (2)$$

$$m\ddot{x}_3 = -kx_3 - k(x_3 - x_1) - k(x_3 - x_2) \qquad (3)$$

2- Calculate the three propres pulsations

using the matrix method:

$$\begin{cases} m\ddot{x}_1 + 3kx_1 - kx_2 - kx_1 = 0 \\ m\ddot{x}_2 + 3kx_2 - kx_3 - kx_1 = 0 \\ m\ddot{x}_3 + 3kx_3 - kx_1 - kx_2 = 0 \end{cases}$$

We are looking for solutions of the form:

$$x_1(t) = A\cos(\omega t + \varphi_1) \quad with \;\; A > 0$$
$$x_2(t) = B\cos(\omega t + \varphi_2) \quad with \;\; B > 0$$
$$x_3(t) = A\cos(\omega t + \varphi_3) \quad with \;\; C > 0$$

In complex notation, we have:

$$\begin{cases} x_1(t) = A\, e^{j(\omega t + \varphi_1)} \\ x_2(t) = B\, e^{j(\omega t + \varphi_2)} \\ x_3(t) = C\, e^{j(\omega t + \varphi_3)} \end{cases}$$

The system of equations below is written as:

$$\begin{cases} (3k - m\omega^2)A - kB - kC = 0 \\ -kA + (3k - m\omega^2)B - kC = 0 \\ -kA - kB + (3k - m\omega^2)C = 0 \end{cases}$$

This homogeneous system admits non-zero solutions if and only if the determinant is zero:

$$\begin{vmatrix} (3k - m\omega^2) & -k & -k \\ -k & (3k - m\omega^2) & -k \\ -k & -k & (3k - m\omega^2) \end{vmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence:

$$(3k - m\omega^2)[(3k - m\omega^2)(3k - m\omega^2) - k^2] + k[-k(3k - m\omega^2) - k^2] + k[k^2 + k(3k - m\omega^2)] = 0$$

that is to say:

$$m^3\omega^6 - 9km^2\omega^4 + 24k^2m\omega^2 - 16k^3 = 0 \Rightarrow \begin{cases} \omega_1 = \sqrt{\dfrac{k}{m}} \\ \omega_2 = \sqrt{4\dfrac{k}{m}} \\ \omega_3 = \sqrt{4\dfrac{k}{m}} \end{cases}$$

If we set: $\omega_0 = \sqrt{\frac{k}{m}}$, we get:

$$\omega_1 = \omega_0, \quad \omega_2 = 2\omega_0, \qquad \omega_3 = 2\omega_0,$$

CHAPTER V : WAVES

CHAPTER V : WAVES

V-1 Introduction :

In the vibratory phenomena discussed in the previous chapters, we were interested in phenomena or physical quantities that depended on a single variable, time. We are now going to look at a whole series of phenomena that are described by a function that depends on both time t and a space variable, x for example.

These phenomena are governed by a partial differential equation, called a wave equation or a one-dimensional propagation equation of the form :

$$\frac{\partial^2 s}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 s}{\partial t^2} = 0 \qquad (\mathbf{1})$$

With $\boldsymbol{v}$ is a physical quantity which has the dimensions of a velocity and will subsequently be called propagation velocity.

V .2 Definition and types:

A wave corresponds to a displacement of energy that manifests itself by correlated oscillations between them, in the space crossed, producing on its passage a reversible variation of the local physical properties of the medium. It moves with a determined speed that depends on the characteristics of the propagation medium.

There are three main types of waves:

• **Mechanical waves** propagate through material media whose substance deforms. Restoring forces then reverse the deformation. For example, sound waves propagate through air molecules that collide with their neighbors.

• **Electromagnetic waves** do not require a physical medium. Instead, they consist of periodic oscillations of electric and magnetic fields originally generated by charged particles, and can therefore travel through a vacuum;

• **Gravitational waves** also do not require a medium. They are deformations of the geometry of space-time that propagate.

Our interest will be focused on mechanical waves that propagate in a single direction, which can be classified as transverse and longitudinal.

• **Longitudinal waves:** the points of the propagation medium move locally according to the direction of the disturbance (typical example: compression or decompression of a spring, sound in a medium without shear: water, air, etc.)

• **Transverse waves**: the points of the propagation medium move locally perpendicular to the direction of the disturbance, so that an additional quantity must be used to describe them (typical example: deformation of a live wire, earthquake waves, electromagnetic waves). This is described as polarization.

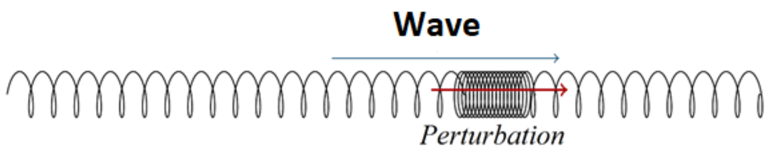


Figure 1: Longitudinal waves

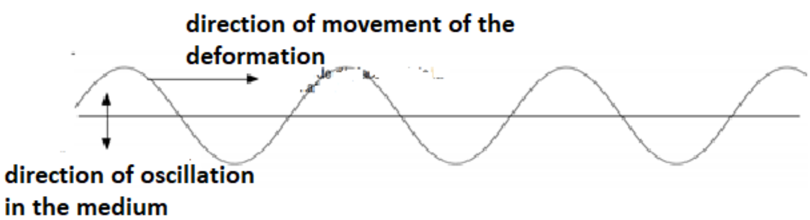


Figure 2: Transverse wave

Solution of the propagation equation

The general solution of the one-dimensional wave equation **(1)** is given by :

$$s(x,t) = F\left(t - \frac{x}{v}\right) + G\left(t + \frac{x}{v}\right) \qquad (\mathbf{2})$$

The functions *F(t - x/v)* and *G(t + x/v)* are functions whose nature is fixed by the boundary conditions imposed on the solution *s(x,t)*, such that *F(t - x/v)* represents a travelling part or wave while *G(t + x/v)* is a reflected part or wave which can be zero.

Note:

Any function that can be put into the form ***F(t - x/v)*** or ***F(t - x/v) + G(t + x/v)*** represents a wave function where ***v*** is the speed of propagation.

V.3 Progressive sinusoidal wave

Consider a progressive wave propagating in the direction of the ***x*** axis, such that the point of abscissa ***x = 0*** is subjected to a sinusoidal vibration of the form

$$s(x=0,t) = S_0\, cos(\omega t) \qquad (\mathbf{3})$$

The point located at the abscissa **x >0** will have the same vibration as that of the point **x = 0** but with a delay equal to ***x/v***:

$$s(x,t) = S_0\, cos\left[\omega\left(t - \frac{x}{v}\right)\right] \qquad (\mathbf{4})$$

This expression constitutes the definition of a progressive sinusoidal (or harmonic) wave; it can be written in the form:

$$s(x,t) = S_0\, cos[\omega t - \varphi(x)] \qquad (\mathbf{5})$$

Where $\boldsymbol{\varphi(x) = \frac{\omega}{v} x}$ represents the phase shift related to the propagation time $\frac{x}{v}$. We say that ***φ(x)*** represents the phase shift due to propagation. The sinusoidal progressive wave is written in the following form which highlights the double periodicity (in time and in space):

$$s(x,t) = S_0 \cos\left[2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)\right] \qquad (\mathbf{6})$$

The quantity $\boldsymbol{T = \frac{2\pi}{\omega}}$ is the time period while the quantity $\boldsymbol{\lambda = vT}$ is the wavelength which constitutes the spatial period. We can easily verify that:

$$s(x, t + nT) = s(x,t) \qquad (\mathbf{7})$$

$$s(x + n\lambda, t) = s(x,t) \qquad (\mathbf{8})$$

Where n is an integer.

The progressive wave is often written:

$$s(x,t) = S_0 \cos[\omega t - kx] \qquad (\mathbf{9})$$

Where $\boldsymbol{k = \frac{\omega}{v} = \frac{2\pi}{\lambda}}$ is called the modulus of the wave vector which is expressed in m^{-1}.

We very often use the complex notation of a sinusoidal progressive wave:

$$s(x,t) = S_0 e^{i(\omega t - kx)} \qquad (\mathbf{10})$$

V.4 Superposition of two progressive sinusoidal waves

V.4.1 Case of two waves of the same frequency propagating in the same direction

Let us consider two waves of the same frequency and direction of propagation, with respective amplitudes S_1 and S_2, and respective phases φ_1 and φ_2. The resulting wave will then be:

$$s(x,t) = S_1 e^{j(\omega t - kx + \varphi_1)} + S_2 e^{j(\omega t - kx + \varphi_2)} = S e^{j(\omega t - kx + \varphi)} \qquad \textbf{(11)}$$

Or in real notation:

$$s(x,t) = S cos(\omega t - kx + \varphi) \qquad \textbf{(12)}$$

with

$$S = \sqrt{S_1^2 + S_2^2 + 2 S_1 S_2 \cos(\varphi_1 - \varphi_2)} \quad ,$$

$$\varphi = Arctg\left(\frac{S_1 \sin(\varphi_1) + S_2 \sin(\varphi_2)}{S_1 \cos(\varphi_1) + S_2 \cos(\varphi_2)}\right) \qquad (\mathbf{13})$$

The superposition of two harmonic waves of the same frequency, and which propagate in the same direction, gives another progressive harmonic wave of the same frequency, amplitude S and phase φ.

V.4.2 Case of two waves of the same frequency propagating in opposite directions

If, on the other hand, we superimpose two harmonic waves of the same frequency but propagating in opposite directions, the result is completely different. Indeed, in this case:

$$s(x,t) = S_1 e^{j(\omega t - kx + \varphi_1)} + S_2 e^{j(\omega t + kx + \varphi_2)}$$

$$= \left[S_1 e^{j\varphi_1} e^{-jkx} + S_2 e^{j\varphi_2} e^{+jkx}\right] e^{j\omega t} \qquad (\mathbf{14})$$

and the resulting wave cannot be written as a simple progressive wave. An important special case occurs when the two amplitudes are identical. If we note:

$$S_1 = S_2 = S_0 \qquad (\mathbf{15})$$

We have:

$$s(x,t) = 2S_0 \cos\left(kx + \frac{\varphi_1 - \varphi_2}{2}\right) e^{j\left(\omega t + \frac{\varphi_1 + \varphi_2}{2}\right)} \qquad (\mathbf{16})$$

and therefore in real notation:

$$s(x,t) = 2S_0 \cos\left(kx + \frac{\varphi_1 - \varphi_2}{2}\right) \cos\left(\omega t + \frac{\varphi_1 + \varphi_2}{2}\right) \qquad (\mathbf{17})$$

This mode of vibration is very different from a progressive wave, since all the points ***x*** on the cord vibrate in phase with different amplitudes depending on the position ***x*** of the point in question.

$$S_x = 2S_0 \cos\left(kx + \frac{\varphi_1 - \varphi_2}{2}\right) \qquad (\mathbf{18})$$

In particular, there is a series of very specific points for which: $\boldsymbol{S_x = 0}$ or $\boldsymbol{S_x = 2S_0}$ called nodes and antinodes in the case of standing waves in a cord.

$$S_x = 0 \Rightarrow kx_N + \frac{\varphi_1 - \varphi_2}{2} = (2n+1)\frac{\pi}{2} \Rightarrow$$

$$x_N = \left[\left(n + \frac{1}{2}\right) - \frac{\varphi_1 - \varphi_2}{2\pi}\right]\frac{\lambda}{2} \qquad (\mathbf{19})$$

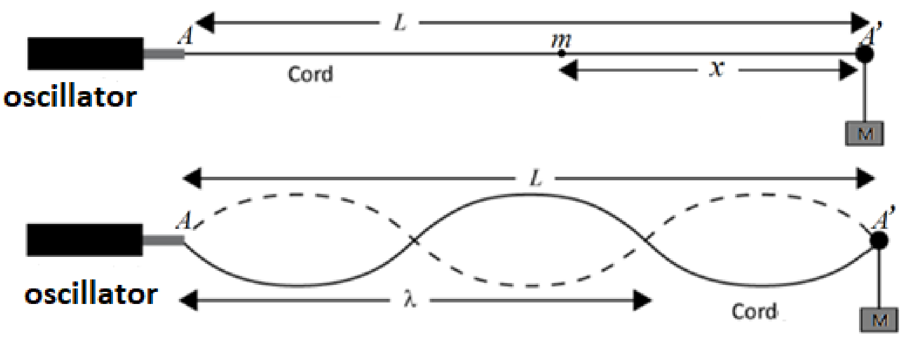
$$S_x = 2S_0 \Longrightarrow kx_{AN} + \frac{\varphi_1 - \varphi_2}{2} = n\pi \Longrightarrow$$

$$x_{AN} = \left[n - \frac{\varphi_1 - \varphi_2}{2\pi}\right]\frac{\lambda}{2} \qquad \textbf{(20)}$$

with : $n = 0, \pm1, \pm2, ...$

Between each pair of nodes there is a belly where the vibration amplitude is maximum and equal to $\boldsymbol{2S_0}$. We also note that the interval between two nodes is equal to half a wavelength $\boldsymbol{\lambda/2}$.

V.4.3 Standing waves in a cord:



The progressive wave is the superposition of two waves, one progressive and one regressive in a stretched cord (the following diagram).
Let the progressive wave be of the form at point **A`** :

$$S_1(x,t) = S_0\cos(\omega t)$$

The regressive wave in the same point **A`** : $S_2(x,t)$

Point A` is immobile under the action of the two waves, so:

$$S(x,t) = S_1(x,t) + S_2(x,t) = 0 \rightarrow S_2(x,t) = -S_1(x,t) = S_0 cos\omega(t+\pi)$$

Point **m** oscillates under the action of the two waves.

The motion of **m** along the progressive wave :

The point **m** is in advance of motion with respect to **A´**

$$S_1(x,t) = S_0 cos[\omega(t-\Delta t)] = S_0\ cos\left[2\pi\left(\frac{t}{T}+\frac{x}{\lambda}\right)\right]$$

The movement of m along the regressive wave :

$$S_2(x,t) = S_0 cos[\omega(t-\Delta t)+\pi] = S_0\ cos\left[2\pi\left(\frac{t}{T}-\frac{x}{\lambda}\right)+\pi\right]$$

The movement of **m** :

$$S(x,t) = S_1(x,t) + S_2(x,t) = S_0\ cos\left[2\pi\left(\frac{t}{T}+\frac{x}{\lambda}\right)\right] + S_0\ cos\left[2\pi\left(\frac{t}{T}-\frac{x}{\lambda}\right)+\pi\right]$$

$$S(x,t) = 2S_0\ cos\left(2\pi\frac{x}{\lambda}-\frac{\pi}{2}\right)cos\left(2\pi\frac{t}{T}+\frac{\pi}{2}\right)$$

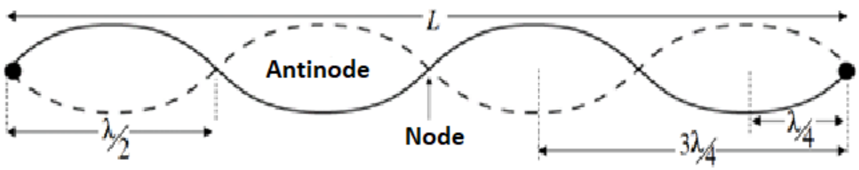
Note that the amplitude of point m is a function of $\boldsymbol{S_0}$, so we can find values of $\boldsymbol{x}$ for which **m** is stationary or of maximum amplitude :

$$Point\ m\ is\ a\ node: \qquad 2S_0\ cos\left(2\pi\frac{x}{\lambda}-\frac{\pi}{2}\right) = 0 \rightarrow 2\pi\frac{x}{\lambda}-\frac{\pi}{2} = (2k+1)\frac{\pi}{2}$$

$$\rightarrow 2\pi\frac{x}{\lambda} = (k+1)\pi \rightarrow x = (k+1)\frac{\lambda}{2}$$

$Point\ m\ is\ a\ antinodes$:

$$2S_0\ cos\left(2\pi\frac{x}{\lambda} - \frac{\pi}{2}\right) = \mp 1 \rightarrow 2\pi\frac{x}{\lambda} - \frac{\pi}{2} = k\pi \rightarrow 2\pi\frac{x}{\lambda} = k\pi + \frac{\pi}{2} \rightarrow x = (2k+1)\frac{\lambda}{4}$$



Standing waves are established in a taut cord by using an electric oscillator that produces vibrations in the cord. The setup is illustrated in **Figure 3**. The tension in the cord is equal to the weight of the masses suspended at the end of the cord. The tension can be changed by changing the suspended mass. The amplitude and frequency of the wave can be adjusted by changing the signal output from the function generator that controls the vibrations of the oscillator.

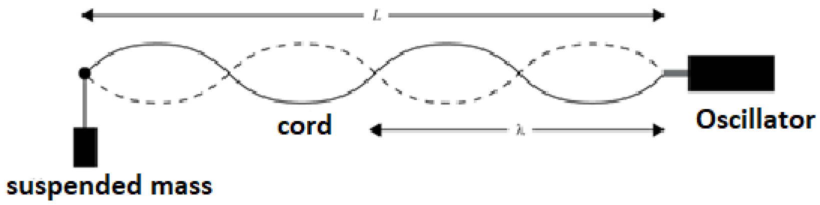


Figure 3: Cord and oscillator assembly.

Wave velocity and cord density

For any wave having wavelength $\boldsymbol{\lambda}$ and frequency $\boldsymbol{f}$, the velocity of the wave $\boldsymbol{v}$, is given by:

$$v = \lambda f \qquad (\mathbf{21})$$

The velocity of a wave is given by its wavelength and frequency. For a wave in a cord, the velocity is also related to the tension **($\boldsymbol{T}$)** in the cord and its linear density **($\boldsymbol{\mu}$)** as follows:

$$v = \sqrt{\frac{T}{\mu}} \qquad (\mathbf{22})$$

The linear density ($\boldsymbol{\mu}$) corresponds to the mass of the cord per unit length. The applied tension ($\boldsymbol{T}$) is given by the suspended mass ($\boldsymbol{m}$) multiplied by the gravitational acceleration ($\boldsymbol{T} = \boldsymbol{mg}$).

To produce standing waves, the length (L) of the cord must be an integer multiple of half the wavelength ($\boldsymbol{\lambda}$):

$$L = n\frac{\lambda}{2} \qquad (\mathbf{23})$$

The fundamental modes are identified by the number $\boldsymbol{n}$. **Figure 4** illustrates the first four modes of vibration.

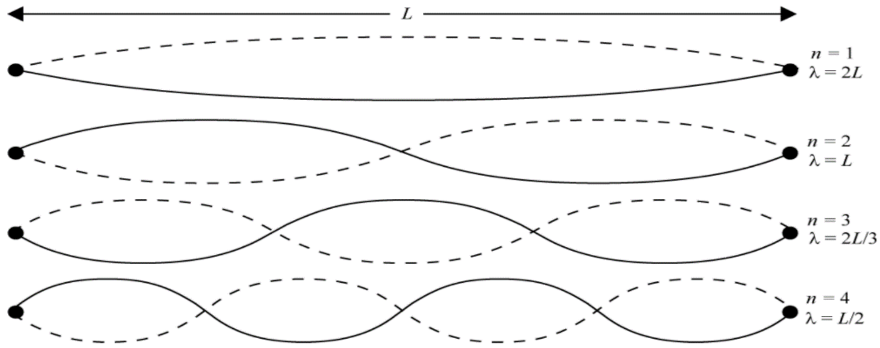


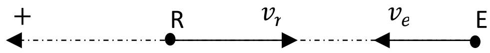
Figure 4 – The first four fundamental modes of vibration of a cord.

V.5 Doppler effect:

In the case of sound waves, the Doppler effect can be seen, for example, in the perception of the pitch of the sound of a car engine or the siren of an emergency vehicle. The sound is different depending on whether you are inside the vehicle (the transmitter being stationary in relation to the receiver), or whether the vehicle is moving towards the receiver (the sound being higher pitched) or away from it (the sound being lower pitched). It should be noted, however, that the variation in the pitch of the sound in this example is due to the position of the observer in relation to the trajectory of the mobile.

To simplify the study, we assume that the velocity of the transmitter v_eand the velocity of the receiver v_will follow the same trajectory. The positive direction is that of wave propagation.

By convention, speeds are counted as positive in the direction of propagation of the signal (from the transmitter (source) to the receiver). Thus, a positive $\boldsymbol{v_e}$ and negative $\boldsymbol{v_r}$ speed means that the source and receiver are moving towards each other, while a negative $\boldsymbol{v_e}$ and positive $\boldsymbol{v_r}$ speed means that they are moving away from each other.

Approximation between source and receiver 

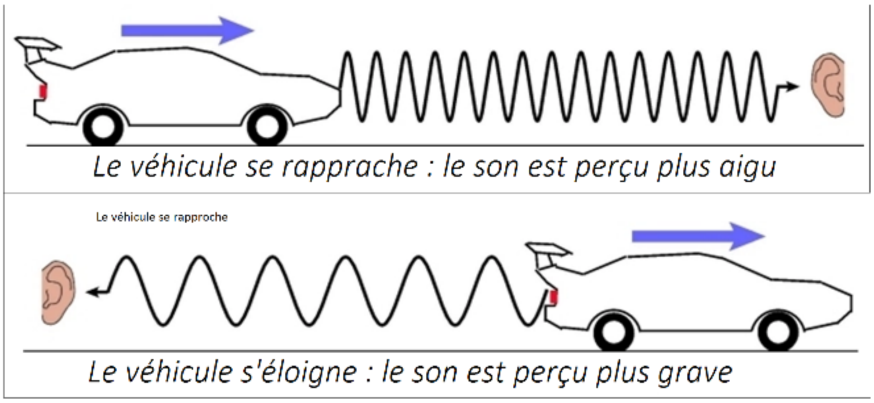
Distance between source and receiver 

Figure 5 : case of variation of the measured frequency $\boldsymbol{f_r}$

If $\boldsymbol{f_e}$ is the frequency of the wave in the transmitter's frame of reference and **c** is the propagation velocity of the wave, then the receiver will receive a wave of frequency $\boldsymbol{f_r}$.

For a fixed (immobile) receiver, the formula is written:

Case 1: source approaching the receiver:



$$f_r = \frac{f_s}{1 - v/c} \qquad (\mathbf{24})$$

Case 2: source moving away from the receiver:

$$f_r = \frac{f_s}{1 + v/c} \qquad (\mathbf{25})$$

With: $\boldsymbol{v}$: velocity of the source and **c**: velocity of sound

Note:

For electromagnetic waves one must take into account the theory of relativity.

Exercise :

A transverse sinusoidal wave propagating on a cord of length ***L***, of linear mass ***μ***, stretched with a tension T_0 has the following wave equation:

$$s(x,t) = 12\,sin\left[\frac{\pi}{6}\left(\frac{t}{3} - 3x\right)\right],\quad s\ and\ x\ in\ meters, t\ in\ seconds$$

1- What is a transverse wave?

2- Show that ***s(x,t)*** is a wave function and specify the direction and sense of propagation.

3- Determine the pulsation ***ω***, the propagation velocity ***v*** and the modulus of the wave vector ***k*** of this wave and deduce its two temporal periods ***T*** and spatial ***λ*** .

4- With what tension must the cord be stretched so that the propagation velocity of the wave is twice the previous propagation velocity? We give $v = \sqrt{T_0/\mu}$

Solution

1°) A transverse sine wave is a wave obtained from a sinusoidal disturbance which propagates in a medium (cord) where its particles (points) move under the effect of the disturbance, in a manner perpendicular to the direction of propagation of the wave.

2°) $s(x,t)$ is a wave function

$$s(x,t) = 12\,sin\left[\frac{\pi}{6}(t/3 - 3x)\right] = 12sin\left[\frac{\pi}{18}(t - 9x)\right] = 12\,sin\left[\frac{\pi}{18}(t - x/(1/9))\right]$$

$$\equiv F(t - x/v)$$

s(x ,t) is a wave function of propagation velocity $v = 1/\,9\ (m/s)$.

ou

* those who verified by the d'Alembert equation (Wave propagation equation)

$$\frac{\partial^2 S(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 S(x,t)}{\partial t^2}$$

$$\Rightarrow \begin{cases} \dfrac{\partial S^2(x,t)}{\partial x^2} = -(\dfrac{\pi}{2})^2 S(x,t) \\ \dfrac{1}{v^2}\dfrac{\partial S^2(x,t)}{\partial t^2} = -(\dfrac{\pi}{18})^2 \dfrac{1}{v^2} s(x,t) \end{cases} \textit{the velocity of propagation } v$$

$$= \frac{1}{9} m/s$$

* The wave propagates in the (x) direction and in the positive sense of the (x)

3°) <u>determination of wave parameters:</u>

pulsation $\omega = \pi/18\ (rd/s)$

velocity of propagation $v = 1/\ 9\ (m/s)$,

wave vector modulus $k = \omega/v = \pi/2\ (m^{-1})$,

time period $T = 2\,\pi/\omega = 36\ (s)$

space period $\lambda = v \cdot T = 4\ (m)$.

4°) The svelocity of propagation of the wave on the cord is given by $v = \sqrt{T_0/\mu}$

$$v_1 = 2v = 2\sqrt{T_0/\mu} = \sqrt{T_1/\mu} \Rightarrow T_1 = 4T_0$$

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