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Analysis 1

intended for first year Mathematics and Computer Science students

- Theory of sets
- Real sequences
- Real functions
- Continuity
- Derivability
- Elementary functions

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Contents

Introduction

The aim of this course is to bridge the gap between the knowledge of analysis accumulated in high school and the fundamentals that will form one of the pillars of mathematical analysis training in the bachelor's degree. Given that recruitment to the first year of analysis is fairly heterogeneous, it seems a good idea to start by recalling the basic notions that will be used throughout this course, so as not to lose anyone along the way. Where necessary, at the start of each chapter, we'll remind you of what you're supposed to know by the end of the course. As far as possible, we'll also try to provide the essential results of each chapter on a single page, so as to summarize the knowledge you need to master in order to move on to the next chapter. We will provide as many examples and figures as necessary afin order to achieve a better understanding of the course. We'll also try to point out the pitfalls that everyone can fall into, either through inattention or poor mastery of the course.

This course brings together the notes from the analysis module dedicated to first-year students in the mathematics and computer science stream. It includes set theory, real sequences, real functions of one real variable, real functions of one real variable: continuity, real functions of one real variable: derivability and elementary functions their relations.

The general idea behind the first chapter is to handle real numbers properly, you first need to know a few basic rules. These rules all have a name afin order to identifier them right away, so you know what you're talking about. A bit like a map, this lets us know what we're seeing, where we're going and how we're getting there. Let's start with the simple, familiar rules.

In the second chapter we give the notion of a sequence is central in Analysis. Among sequences, convergent sequences form a special important class. They are introduced and studied in this chapter. Further, based on this, the notion of a convergent series is introduced.

For the third chapter: In this chapter the notion of The basics of functions, and Some properties of functions are presented

The forth chapter: the key notion of a continuous function is introduced, followed by several important theorems about continuous functions.

The fifth chapter:The notion of a differentiable function is developed, and several important theorems about differentiable functions are presented.

For the last chapter: We use power series to strictly define the Exponential, Logarithmic, and Trigonometric functions and describe their properties.

Each chapter is followed by a list of exercises, usually illustrating a point made earlier. It goes without saying that these exercises complement the course.

Notations

If E is a set, note by:

$P(E)$	Part of E.
$A^c = C_E^A$	$A's$complement in E
$\varnothing$	Empty set
Id_x	Identity application
inf	the lower bound,
sup	the upper bound
$\lim_{n\to+\infty}$	the limit when n tends towards $+\infty$
$\mathbb{R}$	the set of real numbers
$\mathbb{N}$	the set of natural numbers
$x \to x_0$	x tends towardsx_0
$f'(x)$	The first derivative of $f(x)$
$f^{(n)}(x)$	n^{th}The n^{th} derivative of $f(x)$

Let I be an interval of $\mathbb{R}$, we have the following notations:

C^n	"n" times derivable functions
$C^n(I)$	The set of functions of class C^n on I
$C^\infty(I)$	The set of functions of class C^∞ on I
e^x	exponential
ln	Neperian logarithm

L.D	limited development
$\int_a^b$	Bounded integral
$\int$	Infinite integral
O.D.E	Ordinary differential equations
iff	if only if

Chapter 1

Theory of Sets

1.1 Sets

1.1.1 Inclusion

Definition 1

Let E and F be two sets : We say that E is included in F(or F contains E)and we note $E \subset F$(or $F \supset E$)

If: any element x of E belongs to F: i.e.,

$$\forall x \in E, x \in F$$

We say that E is a part of F.

Definition 2

Let E be a set, all parts of E constitute a new set of parts of E, denoted $P(E)$ [Part of E] .

Example 1

Let

$$E = \{0, 1\}, P(E) = \{\varnothing, \{0\}, \{1\}, E\}.$$

1/ $A \in P(E) \Leftrightarrow A \subset E$
$\{x\} \in P(E) \Leftrightarrow x \subset E$
2/ We note $E \not\subset F$. Inclusion negation i.e., $\exists x \in E, x \notin F$
3/ $E = F \Leftrightarrow \{E \subset F, F \subset E\}$
4/ $\varnothing \subset E, E \subset E, E \subset G, G \subset F \Leftrightarrow E \subset F$

1.1.2 Operations in P(E)

Definition 3

Let E be a set, A and B be two parts of E, we define the following parts:

1/ $A^c = C_E^A = \{x \in E, x \notin A\}$ (Complementary of A in E)
2/ $A \cup B = \{x \in E; x \in A \text{ or } x \in B\}$ (Meeting of A et B)
3/ $A \cap B = \{x \in E; x \in A \text{ et } x \in B\}$ (Intersection of A et B)
4/ $A - B = \{x \in E; x \in A \text{ et } x \notin B\}$ (Difference A minus B)($A \backslash B$, $A \cap B^c$)
5/ $A \triangle B = \{(A \backslash B) \cup (B \backslash A)\}$ (Symmetrical difference)

Definition 4

Let E be a set, A and B two parts of E.
We say that A and B are disjoint if $A \cap B = \varnothing$.

Example 2

1/ A and A^c are disjoint i.e., $A \cap A^c = \varnothing$
2/ Let A, B and C be three parts of E

$C_E \varnothing = E,$ $\qquad C_E E = \varnothing$
$C_E(C_E A) = A,$ $\qquad (A^c)^c = A$
$A \cup A = A,$ $\qquad A \cap A = A$
$A \cup E = E,$ $\qquad A \cap E = A$
$A \cup B = A \Leftrightarrow B \subset A$
$A \cap B = A \Leftrightarrow A \subset B$
$(A \cup B) \cup C = A \cup (B \cup C)$
$(A \cap B) \cap C = A \cap (B \cap C)$
$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
$C \cup (A \cap B) = (C \cup A) \cap (C \cap B)$

Morgan's Law

$$1/\ (A \cup B)^c = A^c \cap B^c \Leftrightarrow \begin{cases} (A \cup B)^c \subset A^c \cap B^c \\ A^c \cap B^c \subset (A \cup B)^c \end{cases}$$

$$(A \cup B)^c = A^c \cap B^c \Leftrightarrow \begin{cases} \forall x \in (A \cup B)^c \\ x \notin A \cup B \\ x \notin A \text{ and } x \notin B \\ x \in A^c \text{ and } x \in B^c \\ x \in A^c \cap B^c \end{cases}$$

2/ $(A \cap B)^c = A^c \cup B^c$
3/ $C_E A = E \backslash A$
4/ $A \backslash \varnothing = A$
5/ $A \backslash B = \varnothing$
6/ $A \backslash B = A \cap B^c = A \backslash (A \cap B)$

Definition 5

Let E be a set, $P(E)$:a part of E.

We say that $P(E)$ is a partition of E if:

1/ $\forall A \in P(E) : A \neq \varnothing$
2/ $\forall A, B \in P(E) : A \neq B \Rightarrow A \cap B = \varnothing$
3/ $\forall x \in E, \exists A \in P(E) : x \in A$

1.2 The sets: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$

* The simplest are the natural integers, noted by $\mathbb{N}$:

$$\mathbb{N} = \{0, 1, 2, ...\}, \quad \mathbb{N}^* = \mathbb{N} - \{0\}.$$

* Then the relative integers, denoted $\mathbb{Z}$:

$$\mathbb{Z} = \{....., -2, -1, 0, 1, 2, ...\}, \quad \mathbb{Z}^* = \mathbb{Z} - \{0\}.$$

* The third category of numbers is that of rational numbers $\mathbb{Q}$:

$$\mathbb{Q} = \left\{ \frac{p}{q} \quad \text{or} \quad p \quad \text{and} \quad q \quad \text{are relative numbers} \right\}.$$

Example 3

$x^2 = 2$ does not have solutions in $\mathbb{Q}$. Then $\sqrt{2} \notin \mathbb{Q}$. Non-rational numbers know as irrationals.

* The set of real numbers denoted by $\mathbb{R}$ is the set of rational and irrational numbers. We have the following conclusion:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

Aximatic of $\mathbb{R}$:

$$\mathbb{R}_+ = \{x \in \mathbb{R}/x \geq 0\}.$$
$$\mathbb{R}_+^* = \{x \in \mathbb{R}/x > 0\}.$$

The same thing for $\mathbb{R}_-, \mathbb{R}_-^*$:

$$\mathbb{R}_- = \{x \in \mathbb{R}/x \leq 0\}.$$
$$\mathbb{R}_-^* = \{x \in \mathbb{R}/x < 0\}.$$

1.3 Relations

Generalities

Let E and F be two sets. We call the Cartisian product of E and F the set noted: $E \times F$ of pairs (x, y) such that $x \in E$ and $y \in F$:

$$E \times F = \{(x, y), x \in E \text{ and } y \in F\}.$$

Remark 1

$$\text{if:}(x, y) = (x', y') \Leftrightarrow \begin{cases} x = x' \\ y = y' \end{cases}$$

Definition 6

Let E be a set, a relation from E to E is called a binary relation on E.

Definition 7

A relation R on E is said:
1/ Reflexive:

$$\forall x \in E : xRx.$$

2/ Symmetric:

$$\forall (x,y) \in E^2 : xRy \Longrightarrow yRx.$$

3/ Anti-symmetric:

$$\forall (x,y) \in E^2 : xRy \text{ and } yRx \Longrightarrow x = y.$$

4/ Transitive:

$$\forall (x,y,z) \in E^3 : xRy \text{ and } yRz \Longrightarrow xRz.$$

Example 4

1/ Equality in a set is reflexive, symmetric, anti-symmetric and transitive.
2/ The inclusion $(\subset)$ in the set $P(E)$ is: reflexive, anti-symmetric and transitive.

1.3.1 Equivalence relation

Definition 8

Let R be a binary relation in E. we say that R is an equivalence relation, if: R is reflexive, symmetric and transitive.

Example 5

Equality in a set is an equivalence relation.

Definition 9

Let R be an equivalence relation on E, for all $x \in E$. The subset of E denoted by $\dot{x}$ is called the equivalence class of modulo R, defined by:

$$\dot{x} = \{y \in E, xRy\}.$$

1.3.2 Order relation

Definition 10

Let R be a binary relation in E. We say that R is an order relation, if R is reflexive, antisymmetric and transitive.

Remark 2
The order relation is often denoted $(\leq)$. The pair$(E, \leq)$ (where E is a set, $(\leq)$ is an order relation): is called an ordered set.

Definition 11

Let$(E, \leq)$ an order set, $(\leq)$ is a total order relation, if any two elements of E are comparable i.e.,

$$\forall (x, y) \in E^2 : x \leq y \text{ or } y \leq x$$

If not, the order is partial.

Example 6

The set $P(E)$ containing at least two elements $\begin{cases} E = \{0, 1\}, \\ P(E) = \{\varnothing, \{0\}, \{1\}, E\} \end{cases}$

Inclusion is a partial-order relation in $P(E)$:

if $(a, b) \in E^2, a \neq b$, So $\begin{matrix} \{a\} \not\subseteq \{b\} \\ \{b\} \not\subseteq \{a\}. \end{matrix}$

1.4 The set of reals $\mathbb{R}$

1.4.1 Axiomatic characterization of $\mathbb{R}$

There are two applications of $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ denoted $+(sum)$ and $\bullet(product)$

$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$
$(x, y) \to f(x, y) = x + y$
$(x, y) \to g(x, y) = x \bullet y$

$\begin{matrix} + : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ (x, y) \to x + y \end{matrix}$ is called the addition in $\mathbb{R}$.

$\begin{matrix} \bullet : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ (x, y) \to x \bullet y \end{matrix}$ is called the product in $\mathbb{R}$.

These two applications verify the following axions for $(x, y, z) \in \mathbb{R}^3$:

1/ $(x + y) + z = x + (y + z)$ associability.

2/ $x + y = y + x$.

3/ $\exists 0 \in \mathbb{R} / \forall x \in \mathbb{R} : \ x + 0 = 0 + x = x$(0 is the neutral element for $+$ in $\mathbb{R}$).

4/ $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ / $x + y = y + x = 0 \Leftrightarrow y = -x$ is the symmetric of x in $\mathbb{R}$ by the $+$ law.

5/ $x \bullet y = y \bullet x$.

6/ $(x \bullet y) \bullet z = x \bullet (y \bullet z)$.

7/ $x \bullet (y + z) = x \bullet y + x \bullet z$

8/ $\exists 1 \in \mathbb{R} / \forall x \in \mathbb{R} : \ x \bullet 1 = 1 \bullet x = x$(1 is the neutral element for $\bullet$ in $\mathbb{R}$).

9/ $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ / $x \bullet y = y \bullet x = 1 \Leftrightarrow y = \frac{1}{x}$ is the inverse of x in $\mathbb{R}$ by $\bullet$ law

$\mathbb{R}$ with these two laws $(+, \bullet)$.

$(\mathbb{R}, +, \bullet)$ is a commutative field, $\mathbb{R}$ is ordered.

1.4.2 Axion of upper bound and lower bound in $\mathbb{R}$

Let $\mathbb{R}$ be the field ordered by $(\leq)$, Let A be a part of $\mathbb{R}$ and M, m two elements of $\mathbb{R}$

Definition 12

1/ M is a majorant of A in $\mathbb{R}$, if

$$\forall x \in A, x \leq M.$$

Example 7

$$A =]0,1], \quad M \in [1, +\infty[$$

2/ m is a minorant of A in $\mathbb{R}$, if

$$\forall x \in A, x \geq m.$$

Example 8

$$A =]0,1], \quad m \in]-\infty, 0]$$

3/ M is a majorant of A in $\mathbb{R}$ and if $M \in A$. So M is the maximum of A in $\mathbb{R}$ and note by:

$$M = maxA.$$

Example 9

$$A =]0,1], \quad M = maxA = 1 \in A.$$

4/ m is a minorant of A in $\mathbb{R}$ and if $m \in A$. So m is the minimum of A in $\mathbb{R}$ and note by:

$$m = \min A.$$

Example 10

$$A =]0,1], \quad m = \min A = \nexists (0 \notin A).$$

5/ An upper bound of A in $\mathbb{R}$ is the smallest majorant of A in $\mathbb{R}$, if it exists we note :

$$M = \sup A.$$

Example 11

$$A =]0,1], \quad M = supA = 1.$$

6/ A lower bound of A in $\mathbb{R}$ is the greatest minorant of A in $\mathbb{R}$, if it exists we note :

$$m = \inf A.$$

Example 12

$$A =]0,1]\,, \quad m = \inf A = 0.$$

Example 13

Let $A =]0,1] \cup]2,3]$, $maxA = \sup A = 3$.
$minA\nexists, \inf A = 0$.

7 $/A$ is bounded in $\mathbb{R}$, if it is major and minor at the same time:

$$\forall x \in A, m \leq x \leq M.$$

Example 14

$$A =]0,1]\,, \quad \forall x \in A, 0 < x \leq 1.$$

1.4.3 Characterization of the sup inf

Proposition 1

Let A be a non-empty subset of $\mathbb{R}$.

$$M = \sup(A) \Leftrightarrow \begin{cases} \text{i)} & \forall x \in A, \quad x \leq M \\ \text{ii)} & \forall \epsilon > 0,\ \exists x_0 \in A : M - \epsilon < x_0 \end{cases}$$

$$m = \inf(A) \Leftrightarrow \begin{cases} \text{i)} & \forall x \in A, \quad x \geq m \\ \text{ii)} & \forall \epsilon > 0,\ \exists x_0 \in A : x_0 < m + \epsilon \end{cases}$$

Example 15

$$F = \left\{2 + \tfrac{1}{n}, n \in \mathbb{N}^*\right\} \Leftrightarrow \begin{cases} \forall y_n \in F : y_n = 2 + \frac{1}{n} \\ \text{we have:} 0 < \frac{1}{n} \leq 1 \\ 2 < 2 + \frac{1}{n} \leq 3 \\ \sup F = \max F = 3 \\ \inf F = 2?, \min F\nexists \end{cases}$$

$$\inf F = 2 \Leftrightarrow \begin{cases} \forall x \in F : x \geq 2 \\ \forall \epsilon > 0, \exists x_0 \in F : 2 + \epsilon > x_0 \end{cases}$$

We have: $2 + \epsilon > x_0 \Leftrightarrow 2 + \frac{1}{n} < 2 + \epsilon \Leftrightarrow n > \frac{1}{\epsilon}$
So just take $n = \left[\frac{1}{\epsilon}\right] + 1$, so:$\inf F = 2$ ($[.]$: the integer part)

1.4.4 Archimedean properties and their consequences

We show that the set $\mathbb{R}$ verifies the following principle

$$\forall x \in \mathbb{R}, \exists n \in \mathbb{N} : n > x.$$

This property is also written as follows

$$\forall x > 0, \forall y \in \mathbb{R}, \exists n \in \mathbb{N}^* : n > x.$$

As a first consequence of Archimedes-principle, we show that:

$$\forall x \in \mathbb{R}, \exists n \in \mathbb{Z} : n \leq x < n + 1.$$

Theorem 1.4.1

$\mathbb{R}$ is Archimedean, in other words

$$\forall x \in \mathbb{R}, \exists n \in \mathbb{N}^* : n \geq x.$$

Or in similar way

$$\forall x \in \mathbb{R}, \forall a \in \mathbb{R}_+^*, \exists n \in \mathbb{N}^* : na \geq x.$$

Or:

$$\forall y \in \mathbb{R}, \forall x \in \mathbb{R}_+^*, \exists n \in \mathbb{N}^* : nx \geq y.$$

1.4.5 Intervals in $\mathbb{R}$

Definition 13

For $(a, b) \in \mathbb{R}^2$ and $a \leq b$. We define in $\mathbb{R}$ the following intervals:

Bounded intervals

$I_1 = [a, b] = \{x \in \mathbb{R}/a \leq x \leq b\}$.
$minI_1 = a$ $\quad$ $\max I_1 = b$.
$I_2 = [a, b[= \{x \in \mathbb{R}/a \leq x < b\}$.
$minI_2 = a$ $\quad$ $\max I_2 = \nexists$.
$I_3 =]a, b] = \{x \in \mathbb{R}/a < x \leq b\}$.
$minI_3 = \nexists$ $\quad$ $\max I_3 = b$.
$I_4 =]a, b[= \{x \in \mathbb{R}/a < x < b\}$.
$minI_4 = \nexists$ $\quad$ $\max I_4 = \nexists$.

Unbounded intervals

$I_5 = [a, +\infty[= \{x \in \mathbb{R}/a \leq x\}$.
$minI_5 = a$ $\quad$ $\max I_5 = \nexists$.
$I_6 =]a, +\infty[= \{x \in \mathbb{R}/a < x\}$.
$minI_6 = \nexists$ $\quad$ $\max I_6 = \nexists$.
$I_7 =]-\infty, a] = \{x \in \mathbb{R}/x \leq a\}$.
$minI_7 = \nexists$ $\quad$ $\max I_7 = a$.
$I_8 =]-\infty, a[= \{x \in \mathbb{R}/x < a\}$.
$minI_8 = \nexists$ $\quad$ $\max I_8 = \nexists$.
$I_9 =]-\infty, +\infty[= \mathbb{R}$.
$minI_9 = \nexists$ $\quad$ $\max I_9 = \nexists$.

Closed intervals

$I_1 = [a, b] = \{x \in \mathbb{R}/a \leq x \leq b\}$.
$minI_1 = a$ $\quad$ $\max I_1 = b$.

$I_5 = [a, +\infty[= \{x \in \mathbb{R}/a \leq x\}$.
$minI_5 = a$ $\max I_5 = \nexists$.
$I_7 =]-\infty, a] = \{x \in \mathbb{R}/x \leq a\}$.
$minI_7 = \nexists$ $\max I_7 = a$.
$I_9 =]-\infty, +\infty[= \mathbb{R}$.
$minI_9 = \nexists$ $\max I_9 = \nexists$.

Open intervals
$I_4 =]a, b[= \{x \in \mathbb{R}/a < x < b\}$.
$minI_4 = \nexists$ $\max I_4 = \nexists$.
$I_6 =]a, +\infty[= \{x \in \mathbb{R}/a < x\}$.
$minI_6 = \nexists$ $\max I_6 = \nexists$.
$I_8 =]-\infty, a[= \{x \in \mathbb{R}/x < a\}$.
$minI_8 = \nexists$ $\max I_8 = \nexists$.
$I_9 =]-\infty, +\infty[= \mathbb{R}$.
$minI_9 = \nexists$ $\max I_9 = \nexists$.

Semi-open(semi-closed) intervals
$I_2 = [a, b[= \{x \in \mathbb{R}/a \leq x < b\}$.
$minI_2 = a$ $\max I_2 = \nexists$.
$I_3 =]a, b] = \{x \in \mathbb{R}/a < x \leq b\}$.
$minI_3 = \nexists$ $\max I_3 = b$.
$\mathbb{R}\backslash \{0\}$ $\mathbb{R}_+ = [0, +\infty[$ $\mathbb{R}_- =]-\infty, 0]$.

Definition 14

every interval of the following type is called an open interval respectively(closed) of center a

$$]a-h, a+h[= \begin{array}{l} \{x \in \mathbb{R}/a-h < x < a+h\} \\ \{x \in \mathbb{R}./-h < x-a < +h\} \\ \{x \in \mathbb{R}/\left|x-a\right| < +h\}. \end{array}$$

1.5 The absolute value

Definition 15

The absolute value of the real x is the positive real noted $|x|$ defined by

$$|x| = \begin{cases} x & \text{if x} \geq 0 \\ -x & \text{if x } \langle\ 0 \end{cases}$$

Example 16

$\forall a > 0, \forall x \in \mathbb{R}$ 1/ $|x| \leq a \Leftrightarrow -a \leq x \leq a \Leftrightarrow x \in [-a, a]$.
2/ $|x| < a \Leftrightarrow -a < x < a \Leftrightarrow x \in]-a, a[$.
3/ $|x| \geq a \Leftrightarrow a \leq x$ and $x \leq -a \Leftrightarrow x \in]-\infty, -a] \cup [a, +\infty[$.
4/ $|x| > a \Leftrightarrow a < x$ and $x < -a \Leftrightarrow x \in]-\infty, -a[\cup]a, +\infty[$.
For all $(x, y) \in \mathbb{R}^2$:
1/ $|x \cdot y| = |x| \cdot |y|$.
2/ $|x + y| \leq |x| + |y|$ (Triangular inequality).
3/ $||x| - |y|| \leq |x + y|$.
4/ $||x| - |y|| \leq |x - y|$.

1.6 Integer part

Definition 16

The integer part of a real x is the greatest relative number less than or equal to x. It is denoted $E(x)$ or $[x]$.
By definition, there is a unique relative integer $n \in \mathbb{Z}$ such that:

$$n = E(x) \leq x \leq E(x + 1) = E(x) + 1 = n + 1.$$

Example 17

$E(5, 6) = 5, \quad E(-5, 6) = -6.$

1.6.1 The properties of the integer part

$1 - \forall x \in \mathbb{R}, \forall p \in \mathbb{Z} : E(x + p) = E(x) + p.$

$2 - \forall x, y \in \mathbb{R} : x \leq y \Rightarrow E(x) \leq E(y).$

$3 - \forall x, y \in \mathbb{R} : E(x) + E(y) \leq E(x + y) \leq E(x) + E(y) + 1.$

$4 - \forall x \in \mathbb{R}, \forall n \in \mathbb{N}^* : E(\frac{E(nx)}{n}) = E(x).$

$5 - \forall x \in \mathbb{R} : E(x) + E(-x) = \begin{cases} 0 & if x \in \mathbb{Z} \\ -1 & if not \end{cases}$

1.7 Applications

Definition 17

application Let E and F be two sets. We call an application from E to F, every relation f from E to F that has $x \in E$ associated with an unique at most one element $y \in F$ i.e., every an antecedent has an unique image.

Definition 18

function Let E and F be two sets. We call a function from E to F, every relation f from E to F that has $x \in E$ associated with at most one element $y \in F$ i.e., every an antecedent has at most image.

· E is the set of definition.

· F is the set of images.

· We say that $y = f(x)$ is the image of x by f and x is the antecedent of y by f.

· The set $\Gamma = \{(x, f(x)), x \in E\}$ called the graph of the function f.

f is often denoted by:

$$f : E \to F$$
$$x \mapsto y = f(x).$$

Example 18

The application:

$$f : \mathbb{R}^* \to \mathbb{R}^*$$
$$x \mapsto \frac{1}{x}.$$

Every an antecedent has an unique image
The function:

$$f : \mathbb{R} \to \mathbb{R}^*$$
$$x \mapsto \frac{1}{x}.$$

The antecedent $x = 0$ does not have an image.

Example 19

The application:

$$f : E \to E$$
$$x \mapsto x = Id_x(x).$$

Surjection, Injection, Bijection

Definition 19

An application $f : E \to F$ is called a:

1/ Surjective: if:

$$\forall y \in F, \exists x \in E, y = f(x).$$

i.e., every element y of F has an antecedent x in E by f.

Example 20

$$f : [1, +\infty[\to [0, +\infty[$$
$$x \mapsto x^2 - 1.$$

Let $y \in [0, +\infty[$, we find $x \in [1, +\infty[$ such that $y = x^2 - 1$
So: $x = \sqrt{y+1} \in [1, +\infty[$ exist and unique.

2/ Injective:

$$\forall (x, x') \in E^2, f(x) = f(x') \Rightarrow x = x'.$$

i.e., two different elements have two different images.

Example 21

$$f : [1, +\infty[\to [0, +\infty[$$
$$x \mapsto x^2 - 1.$$

Let $(x, x') \in ([1, +\infty[)^2 : f(x) = f(x') \Rightarrow x^2 - 1 = x'^2 - 1 \Rightarrow x = x'$.

Example 22

$$f : \mathbb{R}^* \to \mathbb{R}^*$$
$$x \mapsto \frac{1}{x^2}.$$

We have: $f(1) = f(-1) = 1 \Rightarrow x \neq x'$.
So this application is not injective.

3/ Bijective: If it is surjective and injective at the same time.
i.e., every element y of F is the image of a unique element x of E.

$$\forall y \in F, \exists! x \in E, y = f(x).$$

Example 23

$$f : [1, +\infty[\to [0, +\infty[$$
$$x \mapsto x^2 - 1.$$

Let $y \in [0, +\infty[$, we find $x \in [1, +\infty[$ such that $y = x^2 - 1$.
So: $x = \sqrt{y+1} \in [1, +\infty[$ exist and unique.

1.7.1 Composite applications

Let E, F and G be three sets $f : E \to F$ and $g : F \to G$.
The application from $g \circ f$ the application from E to G is called the composite application:

Example 24

$$(g \circ f)(x) = g(f(x)), \forall x \in E.$$

$$f(x) = logx,\ g(y) = \sqrt{y^2+1}$$
$$(g \circ f)(x) = g(f(x)) = \sqrt{(logx)^2+1}.$$

Remark 3
If f and g are both injectives respectively (surjectives, bijectives). So $g \circ f$ is also injective respectively(surjective, bijective).

1.7.2 Reciprocal application

If $f : E \to F$ is a bijective application from E to F, we define an application from F to E.By associating with every element y of F its antecedent x in E.
This application is called the reciprocal application of f and is denoted by f^{-1}.

$$\forall (x, y) \in E \cdot F : \quad y = f(x) \Leftrightarrow x = f^{-1}(y).$$
$$f^{-1}(f(x)) = x \quad f(f^{-1}(y)) = y.$$
$$f^{-1} \text{ is also bijective } \quad \Leftrightarrow (f^{-1})^{-1} = f.$$

Example 25

$$f : [1, +\infty[\to [0, +\infty[$$
$$x \mapsto x^2 - 1.$$

Let $y \in [0, +\infty[$, we find $x \in [1, +\infty[$ such that $y = x^2 - 1$
So: $x = \sqrt{y+1} \in [1, +\infty[$ exist and unique.
So this an application is bijective, then it admit a reciprocal application f^{-1}.

$$f^{-1} : [0, +\infty[\to [1, +\infty[$$
$$x \mapsto \sqrt{x+1}.$$

Direct or reciprocal images of parts by an application

Definition 20
Let E and F be two sets $f : E \to F$ an application : $A \subset E, B \subset F$.

· The direct image of A by f denoted $f(A)$ is the subset of F containing the images of the elements of A by:

$$f(A) = \{f(x) \in F / x \in A\}.$$

Example 26

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto x^2.$$

Let $A = [-2, 3]$,
$$f(A) = \{f(x) \in \mathbb{R}/x \in [-2, 3]\},$$
we have $x \in [-2, 3]$ so $x^2 \in [0, 9]$
So: $f(A) = [0, 9]$.

Remark 4
If $A = \emptyset$ can $f(A) = \emptyset$ No, because f is an application, every antecedent has unique image

· The reciprocal image of B by f denoted $f^{-1}(B)$ is the subset of E containing the antecedents of the elements of B by f.

$$f^{-1}(B) = \{x \in E/f(x) \in B\}.$$

Example 27

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto x^2.$$

Let $B = \{-1, 4\}$,
$$f^{-1}(B) = \{x \in \mathbb{R}/f(x) \in \{-1, 4\}\},$$
we have: $x^2 = -1, x^2 = 4$ so $x = \pm 2$
So: $f^{-1}(B) = \{-2, 2\}$.

Example 28

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto x^2.$$

Let $B = [-1, 4]$,
$$f^{-1}(B) = \{x \in \mathbb{R}/f(x) \in [-1, 4]\},$$
we have $x^2 \geq -1, x^2 \leq 4$ so $x^2 + 1 \geq 0, x^2 - 4 \leq 0$
So: $f^{-1}(B) = [-2, 2]$.

Some properties

Let $f : E \to F$ and $A, B \subset E$.

$1/A \subset B \Rightarrow f(A) \subset f(B)$.

$2/f(A \cup B) = f(A) \cup f(B)$.

$3/f(A \cap B) = f(A) \cap f(B)$.

$4/A \subset B \Rightarrow f^{-1}(A) \subset f^{-1}(B)$

$5/f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

$6/f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

$7/f^{-1}(A^c) = \left(f^{-1}(A)\right)^c$.

1.8 Solved exercises

Exercise 1:

I. Let A and B be two subsets of a set E. Show that :

$$(A \cup B)^c = (A)^c \cap (B)^c,$$

and

$$(A \cap B)^c = (A)^c \cup (B)^c.$$

II. Establish the following equalities:

$$A - B = A \cap B^c = (AUB) - B$$

$$(AUB) = (A - B) \cup B = (B - A) \cup A.$$

Solution:

I. First, if $X \subset E$, we note $X^c = E - X$: the complementary of X in E.

1)We have:

$$\forall x \in E : x \in (A \cup B)^c \Leftrightarrow x \notin A \cup B$$

$$\Leftrightarrow x \notin A \text{ and } x \notin B$$

$$\Leftrightarrow x \in A^c \text{ and } x \in B^c$$

$$\Leftrightarrow x \in A^c \ \cap \ B^c$$

$$i.e., \forall x \in E : x \in (A \cup B)^c \Leftrightarrow x \in A^c \ \cap \ B^c$$

$$\text{So: } (A \cup B)^c = A^c \ \cap \ B^c.$$

2) We have:

$$\forall x \in E : x \in (A \cap B)^c \Leftrightarrow x \notin A \cap B.$$

$$\Leftrightarrow x \notin A \text{ or } x \notin B$$

$$\Leftrightarrow x \in A^c \text{ or } x \in B^c$$

$$\Leftrightarrow x \in A^c \ \cup \ B^c$$

$$i.e., \forall x \in E : x \in (A \cap B)^c \Leftrightarrow x \in A^c \ \cup \ B^c$$

$$\text{hence: } (A \cap B)^c = A^c \ \cup \ B^c$$

Reminder:

$$x \in A \ \cup \ B \Leftrightarrow x \in A \text{ or } x \in B$$

$$x \notin A \ \cup \ B \Leftrightarrow x \notin A \text{ and } x \notin B$$

$$x \in A \ \cap \ B \Leftrightarrow x \in A \text{ and } x \in B$$

$$x \notin A \ \cap \ B \Leftrightarrow x \notin A \text{ or } x \notin B.$$

$II.3)$ We have:

$$\forall x \in E : x \in A - B \Leftrightarrow x \in A \text{ and } x \notin B$$

$$\Leftrightarrow x \in A \text{ and } x \in B^c$$

$$\Leftrightarrow x \in A \ \cap \ B^c$$

$$i.e., \forall x \in E : x \in A - B \Leftrightarrow x \in A \ \cap \ B^c.$$

From this: $A - B = A \ \cap \ B^c$.

4) We have:

Let $x \in E : x \in A - B$ So $x \in A$ and $x \notin B$

as: $A \subset A \cup B$, So, we have: $x \in A \cup B$ and $x \notin B$

From this: $x \in (A \cup B) - B$ So: $A - B \subset (A \cup B) - B$.

Let $x \in E : x \in (A \cup B) - B$ so $x \in A \cup B$ and $x \notin B$

as: $x \in A \cup B$, so, $x \in A$ or $x \in B$ or: $x \notin B$, so: $x \in A$

As a result: $x \in A$ and $x \notin B$, from this: $x \in A - B$

So: $(A \cup B) - B \subset A - B$ then : $A - B = (A \cup B) - B$.

5)

$$* \, (A - B) \cup B = (A \ \cap \ B^c) \cup B \, (\text{From } (3))$$

$$= (A \ \cup \ B) \cap (B^c \cup B)$$

$$= (A \ \cup \ B) \cap E = A \ \cup \ B$$

$$* \, (B - A) \cup A = (B \ \cap \ A^c) \cup A \, (\text{from } (3))$$

$$= (B \ \cup \ A) \cap (A^c \cup A)$$

$$= (B \ \cup \ A) \cap E = B \ \cup \ A = A \ \cup \ B$$

$$\text{So: } (A - B) \cup B = (B - A) \cup A = A \ \cup \ B$$

Exercise 2:

Let A and B be two bounded non-empty parts of $\mathbb{R}$. Show that

if $A \cap B \neq \emptyset$, then $A \cap B$ is bounded and moreover :

$$sup(infA, infB) \leq inf(A \cap B)$$

and

$$inf(A \cap B) \leq sup(A \cap B) \leq inf(supA, supB)$$

Solution:

$\emptyset \neq A, B \subset \mathbb{R}$ bounded, $A \cap B \neq \emptyset$

*$A \cap B$ is bounded, as A, B are bounded, then there exist m, m', M, M' such that::

$$\begin{cases} \forall a \in A, m \leq a \leq M \\ \\ \forall b \in B, m' \leq b \leq M' \end{cases}$$

Posing: $\overline{m} = \min\{m, m'\}, \overline{M} = \max\{M, M'\}$

We obtain: $\forall x \in A \cap B : \overline{m} \leq x \leq \overline{M}$, from which: $A \cap B$ is bounded.

*Show that:

$$sup(infA, infB) \leq inf(A \cap B)$$

and

$$inf(A \cap B) \leq sup(A \cap B) \leq inf(supA, supB)$$

First of all, here are the important notes.

-To show that: $\sup X \leq \alpha$, it suffices to show that α

is a majorant of X, since $\sup X$ is the smallest majorant of X.

-To show that: $\inf X \geq \beta$, it suffices to show that β is a minorant of X, since $\inf X$

is the largest minorant of X.

*show that:

$$sup(infA, infB) \leq inf(A \cap B),$$

and

$$inf(A \cap B) \leq sup(A \cap B) \leq inf(supA, supB).$$

Therefore, it suffices to show that $inf(A \cap B)$ is a majorant of $\{infA, infB\}$ i.e., it suffices to show that: $infA \leq inf(A \cap B)$ and $infB \leq inf(A \cap B)$ this again means that

it suffices to show that $infA$ is a minorant of $A \cap B$ and $infB$ is a minorant of $A \cap B$.

Indeed, we know that:

$$\begin{cases} \forall a \in A, infA \leq a \,(i-e : infA \text{ is a minorant of } A) \\ \\ \forall b \in B, infB \leq b \,(i-e : infB \text{ is a minorant of } B) \end{cases}$$

Or: $A \cap B \subset A$ et $A \cap B \subset B$, so, in particular.

We also have: $\begin{cases} \forall a \in A \cap B, infA \leq a \\ \\ \forall b \in A \cap B, infB \leq b \end{cases}$

Hence: $\begin{cases} infA \text{ is a minorant of } A \cap B \Rightarrow infA \leq inf(A \cap B) \\ \\ infB \text{ is a minorant of } A \cap B \Rightarrow infB \leq inf(A \cap B). \end{cases}$

So: $inf(A \cap B)$ is a majorant of $\{infA, infB\}$, *i.e.*, $sup(infA, infB) \leq inf(A \cap B)$.

*Show that:

$$inf(A \cap B) \leq sup(A \cap B) \leq inf(supA, supB)$$

First, here's the reminder:

$$\sup X = \alpha \Leftrightarrow \begin{cases} \forall x \in X : x \leq \alpha \\ \\ \forall \epsilon > 0, \exists x_0 \in X : \alpha - \epsilon < x_0 \end{cases}$$

$$\inf X = \beta \Leftrightarrow \begin{cases} \forall x \in X : x \geq \beta \\ \\ \forall \epsilon > 0, \exists x_0 \in X : \beta + \epsilon > x_0 \end{cases}$$

*As: $A \cap B \neq \emptyset$,then, exists $x \in A \cap B$.

Therefore, we have: $x \leq sup(A \cap B)$ and $x \geq inf(A \cap B)$.

i.e.,$inf(A \cap B)x \leq sup(A \cap B)$. or: $inf(A \cap B) \leq sup(A \cap B)$

*It suffices to show that $sup(A \cap B)$ is a minorant of $\{supA, supB\}$, because $inf(supA, supB)$

is the largest of the minirants of $\{supA, supB\}$.

That is, it is sufficient to show that $supA \geq sup(A \cap B), supB \geq sup(A \cap B)$.

This again means that it suffices to show that $\begin{cases} supA \text{ is a majorant of } A \cap B \\ \\ supB \text{ is a majorant of} A \cap B \end{cases}$

In fact, we know that:

$$\begin{cases} \forall a \in A, \sup A \geq a \, (i.e., \sup A \text{is a majorant of } A) \\ \\ \forall b \in B, \sup B \geq b \, (i.e., \sup B \text{ is a majorant of } B) \end{cases}$$

Or: $A \cap B \subset A$ and $A \cap B \subset B$,

so, in particular

We also have: $\begin{cases} \forall a \in A \cap B, \sup A \geq a \\ \\ \forall b \in A \cap B, \sup B \geq b \end{cases}$

Hence: $\begin{cases} \sup A \text{ is a majorant of } A \cap B \Rightarrow \sup A \geq \sup(A \cap B) \\ \\ \sup B \text{ is a majorant of } A \cap B \Rightarrow \sup B \leq \sup(A \cap B) \end{cases}$

So: $\sup(A \cap B)$ is a majorant of $\{\sup A, \sup B\}$, $i.e., \sup(A \cap B) \leq \inf(\sup A, \sup B)$

Then: $inf(A \cap B) \leq \sup(A \cap B) \leq \inf(\sup A, \sup B)$.

Exercise 3:

For each of the following sets give, if they exist, the upper and lower bounds, the maximum and the minimum.

$A =]0, 3], B = [0, 1] \cup]2, 3]$

$C = \left\{ \frac{1}{x}, 1 \leq x \leq 2 \right\}, D = \left\{ \frac{1}{x}, 1 < x < 2 \right\}$

$E = \left\{ -\frac{1}{x}, 1 \leq x \leq 2 \right\}, F = \left\{ 2 + \frac{1}{n}, n \in \mathbb{N}^* \right)$

Solution:

$$* \; A =]0, 3] \Leftrightarrow \begin{cases} \forall x \in A : 0 < x \leq 3 \\ \\ \sup A = \max A = 3 \\ \\ \inf A = 0, \min A \nexists \end{cases},$$

$$* \; B = [0, 1] \cup]2, 3] \Leftrightarrow \begin{cases} \forall x \in B : 0 \leq x \leq 3 \\ \\ \sup B = \max B = 3 \\ \\ \inf B = \min B = 0. \end{cases}$$

$$* \ C=\left\{\frac{1}{x},1\leq x\leq 2\right\}\Leftrightarrow\begin{cases}\forall x\in C:1\leq x\leq 2\Rightarrow \frac{1}{2}\leq\dfrac{1}{x}\leq 1\\ \\ \sup C=\max C=1\\ \\ \inf C=\min C=\dfrac{1}{2}\end{cases}$$

$$* \ D=\left\{\frac{1}{x},1< x< 2\right\}\Leftrightarrow\begin{cases}\forall x\in D:1< x< 2\Rightarrow \dfrac{1}{2}<\dfrac{1}{x}< 1\\ \\ \sup D=1,\max D\nexists\\ \\ \inf D=\dfrac{1}{2},\min D\nexists\end{cases}$$

$$* \ E=\left\{-\tfrac{1}{x},1\leq x\leq 2\right\}\Leftrightarrow\begin{cases}\forall x\in E:1\leq x\leq 2\Rightarrow -1\leq-\dfrac{1}{x}\leq -\dfrac{1}{2}\\ \\ \sup E=\max E=-\dfrac{1}{2}\\ \\ \inf E=\min E=-1\end{cases}$$

$$*F=\left\{2+\tfrac{1}{n},n\in\mathbb{N}^*\right\}\Leftrightarrow\begin{cases}\forall y_n\in F:y_n=2+\dfrac{1}{n}\\ \\ \text{we have:}0<\dfrac{1}{n}\leq 1\\ \\ 2<2+\dfrac{1}{n}\leq 3\\ \\ \sup F=\max F=3\\ \\ \inf F=2?,\min F\nexists\end{cases}$$

$$\inf F=2\Leftrightarrow\begin{cases}\forall x\in F:x\geq 2\\ \\ \forall\epsilon>0,\exists x_0\in F:2+\epsilon>x_0\end{cases}$$

We have: $2+\epsilon>x_0\Leftrightarrow 2+\dfrac{1}{n}<2+\epsilon\Leftrightarrow n>\dfrac{1}{\epsilon}$

So just take $n=\left[\dfrac{1}{\epsilon}\right]+1$, so:$\inf F=2$.

Exercise 4:

Let $f:\mathbb{R}\to\mathbb{R}$ a function defined by:

$$f(x)=\frac{x^2+2x+2}{x^2+2x+3}$$

Without studying the variation of f find the minimum of f on $\mathbb{R}$.

Solution:

We have: $f(x) = 1 - \dfrac{1}{x^2+2x+3} = 1 - \dfrac{1}{(x+1)^2+2}$

we have: $(x+1)^2 + 2 \geq 2 \Rightarrow 0 < \dfrac{1}{(x+1)^2+2} \leq \dfrac{1}{2} \Rightarrow \dfrac{1}{2} \leq f(x) < 1.$

for $x = -1$, we have: $f(x) = \dfrac{1}{2}$the minimum of f on $\mathbb{R}$. **Exercise 5:**

Using the triangular inequality $|a+b| \leq |a| + |b|$, show that

$$\forall (x,y) \in \mathbb{R}^2, ||x| - |y|| \leq |x-y|.$$

Solution:

We have: $\begin{cases} |x| = |x-y+y| \leq |x-y| + |y| \\ \\ |y| = |y-x+x| \leq |x-y| + |x| \end{cases}$

$\Leftrightarrow \begin{cases} |x| - |y| \leq |x-y| \\ \\ |y| - |x| \leq |x-y| \end{cases}$

$\Leftrightarrow \begin{cases} |x| - |y| \leq |x-y| \\ \\ |x| - |y| \geq -|x-y| \end{cases}$

$\Leftrightarrow -|x-y| \leq |x| - |y| \leq |x-y| \Leftrightarrow \forall (x,y) \in \mathbb{R}^2, ||x| - |y|| \leq |x-y|$

Exercise 6:

Let f be an application of $E \to F$:

Let A and B be two parts of E and Frespectively: We define the direct image of A by :

$$f(A) = \{y \in F / \exists x \in A, f(x) = y\},$$

and the reciprocal image of B by:

$$f^{-1}(B) = \{x \in E, f(x) \in B\}.$$

Let A, A_1, A_2 be parts of E, and B, B_1, B_2 parts of F: Show the following relations.

$1/ A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$

$2/ f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$

$3/ f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$

$4/ B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$

$5/ f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

Solution:

$1/\forall y \in f(A_1), \exists x \in A_1 / y = f(x)$

or: $x \in A_1 \subset A_2 \Rightarrow x \in A_2 \Rightarrow f(x) \in f(A_2)$

$\Rightarrow y \in f(A_2)$ hence: $f(A_1) \subset f(A_2)$

$A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$.

$2/\forall y \in f(A_1 \cap A_2), \exists x \in A_1 \cap A_2 / y = f(x)$.

$$\begin{cases} \Rightarrow \exists x \in A_1 / y = f(x) \\ \\ \text{et} \Rightarrow \exists x \in A_2 / y = f(x) \end{cases}$$

$$\begin{cases} \Rightarrow y \in f\left(A_1\right) \\ \\ \text{and} \Rightarrow y \in f\left(A_2\right) \end{cases} \Rightarrow y \in f(A_1) \cap f(A_2)$$

So: $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$.

$4/\forall x \in f^{-1}(B_1), f\left(x\right) \in B_1$.

or: $B_1 \subset B_2 \Rightarrow f\left(x\right) \in B_2$

$\Rightarrow x \in f^{-1}(B_2)$.

So: $f^{-1}(B_1) \subset f^{-1}(B_2)$.

Chapter 2

Real sequences

Definition 21

A sequence is an application $U : \mathbb{N} \to \mathbb{R}$, for $n \in \mathbb{N}$. We denote $U(n)$ by U_n and call it $n^{ième}$ term where general term

$$U : \mathbb{N} \to \mathbb{R}$$
$$n \mapsto U(n) = U_n.$$

Example 29

$U_n = \sqrt{n}, n \geq 0$ is the sequence of terms $0, \sqrt{1}, \sqrt{2}$,........,

Sequences can be defined in two different ways.
1. Directly by a formula, usually a function f, and we have for all,

$$n \in \mathbb{N}, U_n = f(n),$$

which is called an explicit formulation of the sequence.

Example 30

$$U : \mathbb{N} \to \mathbb{R}$$
$$n \mapsto U(n) = U_n = 2n + 1 = f(n).$$

2.By expressing U_{n+1} in terms of the preceding term U_n and definishing an initial value, as for example:

$$U_{n+1} = f(U_n),$$
$$u_0 = a.$$

This is known as a recurrence formulation.

Example 31

$$\begin{cases} u_0 = 2 \\ U_{n+1} = f(U_n) = U_n + 3 \end{cases}$$

There are two classic sequences that we come across quite often, aithmetic sequences and geometric sequences.

2.1 Two classic sequences

Definition 22

Arithmetic sequences
An arithmetic sequence is every sequence $(U_n)_{n\in\mathbb{N}}$ for which there exists $a \in \mathbb{R}$, called the reason of the sequence, such that
for any $n \in \mathbb{N} : U_{n+1} = a + U_n$.

Example 32

$$\begin{cases} u_0 = 0 \\ U_{n+1} = U_n + 2. \end{cases}$$

We have: $u_0 = 0, u_1 = u_0 + 2 = 0 + 2 = 2, u_2 = u_1 + 2 = 2 + 2 = 4, ...$
An arithmetic sequence with the ratio $a = 2$.

Definition 23

Geometric sequences
A geometric sequence is all sequence $(U_n)_{n\in\mathbb{N}}$for which there exists $r \in \mathbb{R}$, called the reason of the sequence, such that,

$$\text{for any } n \in \mathbb{N} : U_{n+1} = rU_n.$$

Example 33

$$\begin{cases} u_0 \leq 2 \\ U_{n+1} = \frac{1}{2}U_n + 1 \end{cases}$$

We have: $v_n = u_n - 2$. So:

$$v_{n+1} = u_{n+1} - 2 = \frac{1}{2}U_n + 1 - 2 = \frac{1}{2}v_n.$$

A geometric sequence with the ration $r = \frac{1}{2}$.

Proposition 2

Two classic sequences Explicit formulation of arithmetic and geometric sequences
1. The general term of an arithmetic sequence of reason a and first term U_0 is

$$U_n = U_0 + na.$$

Example 34

$$\begin{cases} u_0 = 0 \\ U_{n+1} = U_n + 2. \end{cases}$$

We have: $u_0 = 0, u_1 = u_0 + 2 = 0 + 2 = 2, u_2 = u_1 + 2 = 2 + 2 = 4, ...$
An arithmetic sequence with the ration $a = 2$
So: $U_n = U_0 + na \Rightarrow U_n = 0 + n2 = 2n.$

2. The general term of a geometric sequence with reason r and first term U_0 is $U_n = U_0 r^n$.

Example 35

$$\begin{cases} u_0 \leq 2 \\ U_{n+1} = \frac{1}{2}U_n + 1. \end{cases}$$

We have: $v_n = u_n - 2$. So:

$$v_{n+1} = u_{n+1} - 2 = \frac{1}{2}U_n + 1 - 2 = \frac{1}{2}v_n.$$

A geometric sequence with the ration $r = \frac{1}{2}$

So:$v_n = v_0 r^n = (u_0 - 2)\left(\frac{1}{2}\right)^2$.

A null sequence is one whose terms approach zero. This a particular case of a sequence tending to a limit.

2.2 Convergence of a sequence

The sequence (U_n) is said to be convergent if there exists a real l such that:

$$\forall \varepsilon > 0, \exists N_0 \in \mathbb{N}, \forall n \geq N_0 \Rightarrow |U_n - l| < \varepsilon.$$

We say that (U_n) converges to l.

Example 36

$U_n = \frac{2n-1}{n+1}$,show that (U_n) converges to 2 we have:

$$|U_n - 2| = \left|\frac{2n-1}{n+1} - 2\right| = \left|\frac{-3}{n+1}\right| < \varepsilon \Rightarrow \frac{3}{n+1} < \varepsilon$$
$$n > \frac{3-\varepsilon}{\varepsilon}$$
$$\exists N_0 = \left[\frac{3-\varepsilon}{\varepsilon}\right] + 1/\forall n \geq N_0 : |U_n - 2| < \varepsilon$$
(U_n) converges to 2.

Theorem 2.2.1

All sequences converge to a limit l which is unique.

Proof Assume that $U_n \to l_1$ et $U_n \to l_2 (l_1 \neq l_2)$:

$$\lim_{n\to+\infty} U_n = l_1 \Leftrightarrow \forall \varepsilon_1 > 0, \exists N_1 \in \mathbb{N}, \forall n \geq N_1 \Rightarrow |U_n - l_1| < \varepsilon_1$$

$$\lim_{n\to+\infty} U_n = l_2 \Leftrightarrow \forall \varepsilon_2 > 0, \exists N_2 \in \mathbb{N}, \forall n \geq N_2 \Rightarrow |U_n - l_2| < \varepsilon_2$$

$$|l_1 - l_2| = |l_1 - U_n + U_n - l_2| \leq |U_n - l_1| + |U_n - l_2| < \varepsilon_1 + \varepsilon_2$$

$$\Rightarrow |l_1 - l_2| < \varepsilon \Rightarrow \lim_{n\to+\infty} (l_1 - l_2) = 0 \Rightarrow l_1 = l_2.$$

2.3 Finite and infinite limits

Definition 24

The sequence (U_n) has a limit $l \in \mathbb{R}$, if for any $\varepsilon > 0$, there exists an integer N_0 such that: $\forall n \geq N_0$ then: $|U_n - l| < \varepsilon$ i.e.,

$$\lim_{n\to+\infty} U_n = l \Leftrightarrow \forall \varepsilon > 0, \exists N_0 \in \mathbb{N}, \forall n \geq N_0 \Rightarrow |U_n - l| < \varepsilon.$$

We say that U_n tends towards l from a certain rank.

Example 37

$$\underset{n\to+\infty}{lim} \frac{1}{2n-1} = 0 \Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n : \left(n > N \Rightarrow \left|\frac{1}{2n-1} - 0\right| < \epsilon\right)$$

Let $\epsilon > 0$: We have (we can assume from the beginning that$N \geq 1$), then $n \geq 1$ and $2n - 1 > 0$.

$$\left|\frac{1}{2n-1} - 0\right| < \epsilon \Leftrightarrow \frac{1}{2n-1} < \epsilon \Leftrightarrow \frac{1}{\epsilon} < 2n - 1 \Leftrightarrow n > \frac{1}{2} + \frac{1}{2\epsilon}.$$

Just choose

$$N = max\left(\left[\frac{1}{2} + \frac{1}{2\epsilon}\right] + 1, 1\right) = \left[\frac{1}{2} + \frac{1}{2\epsilon}\right] + 1.$$

Definition 25

U_n tends towards $(+\infty)$ iff:

$$\lim_{n\to+\infty} U_n = +\infty \Leftrightarrow \forall A > 0, \exists N_0 \in \mathbb{N}, \forall n \geq N_0 \Rightarrow U_n \geq A.$$

Example 38

$$\underset{n\to+\infty}{lim}\ 2^{n^{\frac{1}{2}}} = +\infty \Leftrightarrow \forall A > 0, \exists N \in \mathbb{N}, \forall n : \Big(n > N \Rightarrow 2^{n^{\frac{1}{2}}} > A\Big)$$

Let $A > 0$: we find $N \in \mathbb{N}$ such that $\forall n : n > N \Rightarrow 2^{n^{\frac{1}{2}}} > A$

Let $n \in \mathbb{N}$, we have:

$2^{n^{\frac{1}{2}}} > A \Rightarrow \ln 2^{n^{\frac{1}{2}}} > \ln A \Leftrightarrow n^{\frac{1}{2}} > \frac{\ln A}{\ln 2} \Leftrightarrow n > \left(\frac{\ln A}{\ln 2}\right)^2.$

Just take $N = \left[\left(\frac{\ln A}{\ln 2}\right)^2\right] + 1.$

Definition 26

U_n tends towards $(-\infty)$ iff:

$$\lim_{n\to+\infty} U_n = -\infty \Leftrightarrow \forall A > 0, \exists N_0 \in \mathbb{N}, \forall n \geq N_0 \Rightarrow U_n \leq -A.$$

Definition 27

*(U_n) is a convergent sequence if it has a finite limit.

* It is divergent if it does not have a limit, or if the limit is infinity.

Example 39

1/ $U_n = (-1)^n = \begin{cases} 1 & \text{if n even} \\ -1 & \text{if n odd} \end{cases}$

2/ $U_n = n \lim_{n\to+\infty} U_n = +\infty$.

2.4 Major sequences, minor sequences and bounded sequences

1/(U_n) isupper bound, if:

$$\exists M \in \mathbb{R}, \forall n \in \mathbb{N}, U_n \leq M.$$

Example 40

$$\begin{cases} u_0 = 4 \\ U_{n+1} = \sqrt{5U_n} \end{cases}$$

We have: $0 < u_n < 5$. So:

$$5u_n < 25 \Rightarrow \sqrt{5U_n} < 5$$

(U_n) is upper bounded by 5.

2/(U_n) is lower bound, if:

$$\exists m \in \mathbb{R}, \forall n \in \mathbb{N}, U_n \geq m.$$

Example 41

$$\begin{cases} u_0 = 4 \\ U_{n+1} = \sqrt{5U_n}. \end{cases}$$

We have: $4 \leq u_n$. So:

$$20 \leq 5u_n \Rightarrow 4 \leq \sqrt{20} \leq \sqrt{5U_n}.$$

(U_n) is lower bounded by 4

3/(U_n) is bounded, if:

$$\exists M \in \mathbb{R}, \forall n \in \mathbb{N}, |U_n| \leq M.$$

Example 42

$$\begin{cases} u_0 = 4 \\ U_{n+1} = \sqrt{5U_n}. \end{cases}$$

According by (1) and (2). We have:

$$-5 < 4 \leq u_n < 5 \Rightarrow |u_n| < 5.$$

2.5 Increasing, decreasing and monotone sequences

1/(U_n) is an increasing(strictly increasing) sequence, if:

$$\forall n \in \mathbb{N}, U_{n+1} \geq U_n \Rightarrow U_{n+1} - U_n \geq 0 (U_{n+1} > U_n \Rightarrow U_{n+1} - U_n > 0).$$

Example 43

$$\begin{cases} u_0 = 4 \\ U_{n+1} = \sqrt{5U_n}. \end{cases}$$

We have: $u_n < 5$. So:

$$u_{n+1} - u_n = \sqrt{5U_n} - u_n = \frac{u_n(5 - u_n)}{\sqrt{5U_n} - u_n} > 0.$$

So: (u_n) is strictly increasing

2/(U_n) is a decreasing sequence(strictly decreasing) if:

$$\forall n \in \mathbb{N}, U_{n+1} \leq U_n \Rightarrow U_{n+1} - U_n \leq 0 (U_{n+1} < U_n \Rightarrow U_{n+1} - U_n < 0).$$

Example 44

$U_n = \frac{1}{n^2}, n \in \mathbb{N}^*$

$U_{n+1} - U_n = \frac{1}{(n+1)^2} - \frac{1}{n^2} = \frac{-(2n+1)}{n^2(n+1)^2} < 0 \Rightarrow (U_n)$ is decreasing

or: $\frac{U_{n+1}}{U_n} = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} = \left(\frac{n}{n+1}\right)^2 \leq 1 \Rightarrow (U_n)$ is decreasing

3/ (U_n) is a monotone(strictly monotone) sequence if the sequence is increasing or decreasing

4/ (U_n) is a constant sequence, if:

$$\forall n \in \mathbb{N}, U_{n+1} = U_n.$$

Remark 5

(U_n) is an increasing sequence if $\forall n \in \mathbb{N} : U_{n+1} \geq U_n$

If (U_n) is a sequence with positive terms, it is increasing, if and only if

$\forall n \in \mathbb{N} : \frac{U_{n+1}}{U_n} \geq 1.$

Proposition 3

All convergent sequence is bounded i.e.,

$$\exists M \in \mathbb{N} / \forall n \in \mathbb{N} : |U_n| \leq M.$$

Proof

$\lim_{n \to +\infty} U_n = l \Leftrightarrow \forall \varepsilon > 0, \exists N_0 \in \mathbb{N}, \forall n \geq N_0 \Rightarrow |U_n - l| < \varepsilon.$

$|U_n| = |U_n - l + l| \leq |U_n - l| + |l| < \varepsilon + |l|\,.$

$\exists M = \max(U_0, U_1, U_2, ..., \varepsilon + |l|), \forall n \in \mathbb{N} : |U_n| \leq M.$

The reciprocal is not always true

Example 45

Let $U_n = 1 + (-1)^n, \forall n \geq 0$

we have : $|U_n| = |1 + (-1)^n| \leq 1 + |(-1)^n| \leq 2$

$|U_n| \leq 2 \Rightarrow (U_n)$ is bounded

$$\lim_{n \to +\infty} U_n = \lim_{n \to +\infty} (1 + (-1)^n) = \begin{cases} 2 & \text{if n is even} \\ 0 & \text{if n is uneven} \end{cases}$$

Theorem 2.5.1

1/ if (U_n) converges to l and if $U_n \geq 0, \forall n \in \mathbb{N}$. Then $l \geq 0$

2/ if(U_n) converges to l and if $U_n \leq 0, \forall n \in \mathbb{N}$. So $l \leq 0$.

Example 46

$$U_n = \frac{n+2}{3n+1}, \forall n \in \mathbb{N}, U_n \geq 0$$

$$\lim_{n \to +\infty} U_n = \lim_{n \to +\infty} \frac{n+2}{3n+1} = \frac{1}{3} \geq 0$$

Proposition 4

1/if (U_n) is increasing and upper bounded it is convergent

and $\lim_{n \to +\infty} U_n = \sup(U_n)$.

2/ if (U_n) is decreasing and lower bounded it is convergent

and $\lim_{n \to +\infty} U_n = \inf(U_n)$

Example 47

$$\begin{cases} |U_0| \leq 1 \\ U_{n+1} = \frac{1+U_n^2}{2}, \forall n \geq 0 \end{cases}$$

Let's show that $(U_n)$is increasing and upper bounded

1/$\forall n \in \mathbb{N} : U_{n+1} - U_n = \frac{1+U_n^2}{2} - U_n = \frac{(1-U_n)^2}{2} \geq 0 \Rightarrow (U_n)$ is increasing

2/$\forall n \in \mathbb{N} : |U_n| \leq 1$(by reccurence)

$*$for $n = 0 : |U_0| \leq 1$

$*$We assume that $|U_n| \leq 1$ and we show that $|U_{n+1}| \leq 1$.

We have: $|U_n| \leq 1 \Rightarrow |U_{n+1}| = \left|\frac{1+U_n^2}{2}\right| = \frac{1}{2}(|1 + U_n^2|) \leq \frac{1}{2}(|1| + |U_n^2|) \leq 1$

(U_n) is increasing and upper bounded by 1 so it is convergent$(U_n \to l)$

$U_{n+1} = \frac{1+U_n^2}{2} \Rightarrow \lim_{n \to +\infty} U_{n+1} = \lim_{n \to +\infty} \frac{1+U_n^2}{2} \Rightarrow l = \frac{1+l^2}{2} \Rightarrow (l-1)^2 = 0 \Rightarrow l = 1.$

Proposition 5

Comparison of sequences and limits

Let $(U_n)_{n\in\mathbb{N}}$, and $(V_n)_{n\in\mathbb{N}}$ be two sequences.

1. If $\lim\limits_{n\to+\infty} U_n = +\infty$ and if there exists $N \in \mathbb{N}$ such that:
$$\forall n \geq N : U_n \leq V_n, \text{ then } \lim_{n\to+\infty} V_n = +\infty,$$
2. If $\lim\limits_{n\to+\infty} U_n = -\infty$ and if there exists $N \in \mathbb{N}$ such that:
$$\forall n \geq N : V_n \leq U_n, \text{ then } \lim_{n\to+\infty} V_n = -\infty.$$

Example 48

1) $U_n = \sum_{k=1}^{n} \dfrac{1}{3+|\sin k|\sqrt{k}}$. We have:

$$|\sin k| \leq 1 \Leftrightarrow 3+|\sin k|\sqrt{k} \leq 3+\sqrt{k} \leq 3+\sqrt{n}$$

$$\frac{1}{3+\sqrt{n}} \leq \frac{1}{3+|\sin k|\sqrt{k}} \Rightarrow \sum_{k=1}^{n}\frac{1}{3+\sqrt{n}} \leq \sum_{k=1}^{n}\frac{1}{3+|\sin k|\sqrt{k}}$$
$$\frac{n}{3+\sqrt{n}} \leq U_n.$$

We have:
$$\lim_{n\to+\infty}\frac{n}{3+\sqrt{n}} = +\infty . SO : \lim_{n\to+\infty} U_n = +\infty.$$

Proposition 6

Let $(U_n), (V_n)$ and (W_n) are three sequences such that, from a certain rank

$U_n \leq V_n \leq W_n$ and (U_n) and (W_n) have the same limit l. So (V_n) are also

converging to the same limit l.

Example 49

$U_n = \frac{5n-3}{2n}, V_n = \frac{5n+3(-1)^n}{2n}, W_n = \frac{5n+3}{2n}$

(U_n) and (W_n) converges to $l = \frac{5}{2}$.

we have: $\frac{5n-3}{2n} \leq \frac{5n+3(-1)^n}{2n} \leq \frac{5n+3}{2n} \Rightarrow \lim\limits_{n\to+\infty} V_n = \lim\limits_{n\to+\infty} \frac{5n+3(-1)^n}{2n} = \frac{5}{2}$.

Proposition 7

Let (U_n) be a sequence converging to "0" and if (V_n) is a bounded sequence then the sequence $(U_n \cdot V_n)$ converges to "0".

Example 50

Let $W_n = \frac{\sin(n+1)}{n^2} \to l = 0$

we have: $\begin{cases} U_n = \sin(n+1) \Rightarrow |\sin(n+1)| \leq 1 \\ V_n = \frac{1}{n^2} \to 0. \end{cases}$

2.6 Extracted sequences(sub-sequences)

Definition 28

Let (U_n) be a real sequence and $(n_k)_{k\in\mathbb{N}}$ a strictly increasing extracted sequence. Then (U_{n_k}) is a sub-sequence of (U_n).

Example 51

$U_n = (-1)^n . \frac{n}{n+1}$

*an even-numbered subsequence: $U_{2k} = (-1)^{2k} . \frac{2k}{2k+1} \to 1.$

*an uneven-numbered subset: $U_{2k+1} = (-1)^{2k+1} . \frac{2k+1}{2k+2} \to -1.$

Theorem 2.6.1

If the sequence (U_n) converges to "l". All extracted sequences converge to "l", the converse is false.

Example 52

Let $U_n = (-1)^n = \begin{cases} U_{2k} = (-1)^{2k} \to 1 \\ U_{2k+1} = (-1)^{2k+1} \to -1. \end{cases}$

Results

1/ if (U_n) admits two extracted sequences converging to two different limits. Then the sequence (U_n) is divergent.

2/ if (U_n) admits a divergent extracted sequence. Then the sequence(U_n) is divergent.

3/ if (U_n) is a Cauchy sequence and admits an extracted sequence converges to "l". Then the sequence (U_n) is converges to "l".

Bolzano's Theorem: Every bounded sequence in $\mathbb{R}$ admits a convergent sub-sequence.

Example 53

Let $U_n = (-1)^n \Rightarrow |U_n| \leq 1 \Rightarrow \begin{cases} (U_{2k}) \text{ is a sub-sequence converging to "1"} \\ (U_{2k+1}) \text{ is a sub-sequence converging to " - 1".} \end{cases}$

2.7 Adjacent sequences

Let (U_n) and (V_n) be two sequences, they are adjacent if and only if

1/(U_n) is increasing.

2/(V_n) is decreasing.

3/ $\lim\limits_{n \to +\infty} (U_n - V_n) = 0$.

Proposition 8

if (U_n) and (V_n) two adjacent sequences. Then they converge to the same limit l.

Example 54

$$\begin{cases} U_0 = 1 \\ U_n = U_{n-1} + \frac{1}{n!} \\ V_n = U_n + \frac{1}{n!n} \end{cases}$$

We have:* $U_n - U_{n-1} = \dfrac{1}{n!} > 0 \Rightarrow (U_n)$ is increasing

* $V_n - V_{n-1} = (U_n - U_{n-1}) + \left(\dfrac{1}{n!n} - \dfrac{1}{(n-1)!(n-1)}\right)$

$= \dfrac{1}{n!} + \dfrac{1}{n!n} - \dfrac{1}{(n-1)!(n-1)} = \dfrac{n(n-1) + (n-1) - n^2}{n!n(n-1)} = \dfrac{-1}{n!n(n-1)} \Rightarrow (V_n)$ is decreasing

* $\lim\limits_{n \to +\infty} (U_n - V_n) = \lim\limits_{n \to +\infty} \dfrac{1}{n!n} = 0$ So: (U_n) and (V_n) are adjacent.

2.8 Cauchy sequences

Definition 29

*A sequence (U_n) is called of Cauchy if and only if:

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall n, m > N_\varepsilon \Rightarrow |U_n - U_m| < \varepsilon$$

*A sequence(U_n) is not of Cauchy if and only if:

$$\exists \varepsilon > 0, \forall N_\varepsilon \in \mathbb{N}, \exists n, m > N_\varepsilon \wedge |U_n - U_m| \geq \varepsilon$$

Theorem 2.8.1

Every convergent sequence is of Cauchy, and the converse is true in $\mathbb{R}$

Example 55

1) $U_n = (-1)^n + \dfrac{1}{n}$ is not Cauchy

We take: $n = 2N_\varepsilon + 1, m = 2N_\varepsilon$.

We have: $n > m > N_\varepsilon$:

$$|U_n - U_m| = |U_{2N_\varepsilon+1} - U_{2N_\varepsilon}| = \left|(-1)^{2N_\varepsilon+1} + \frac{1}{2N_\varepsilon + 1} - (-1)^{2N_\varepsilon} - \frac{1}{2N_\varepsilon}\right|$$

$$|U_n - U_m| = \left|-1 + \frac{1}{2N_\varepsilon + 1} - 1 - \frac{1}{2N_\varepsilon}\right| = \left|-2 - \frac{1}{2N_\varepsilon\,(2N_\varepsilon + 1)}\right| = \left|2 + \frac{1}{2N_\varepsilon(2N_\varepsilon+1)}\right| > 2$$

$\exists \varepsilon = 2 \Rightarrow (U_n)$ is not of Cauchy

Example 56

2) $U_n = \cos\frac{1}{n}$ is of Cauchy

$$|U_n - U_m| = \left|\cos\frac{1}{n} - \cos\frac{1}{n}\right|$$

$$\cos A - \cos B = -2\sin\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right)$$

$$|U_n - U_m| = \left|2\sin\left(\frac{m-n}{2nm}\right)\sin\left(\frac{m-n}{2nm}\right)\right|$$

$$|U_n - U_m| \leq 2\left|\sin\left(\frac{m-n}{2nm}\right)\right| \left|\sin\left(\frac{m-n}{2nm}\right)\right|$$

$$|U_n - U_m| \leq 2\left|\sin\left(\frac{m+n}{2nm}\right)\right| \left(\text{ car } \left|\sin\left(\frac{m-n}{2nm}\right) \leq 1\right|\right)$$

$$|U_n - U_m| \leq 2\left|\frac{m+n}{2nm}\right| \left(\text{because } \sin\left(\frac{m+n}{2nm}\right) \leq \frac{m+n}{2nm}\right)$$

$$|U_n - U_m| \leq 2\left|\frac{1}{2}\left(\frac{1}{m} + \frac{1}{n}\right)\right|$$

$$|U_n - U_m| \leq \left(\frac{1}{m} + \frac{1}{n}\right) \leq \frac{1}{N_\varepsilon} + \frac{1}{N_\varepsilon} = \frac{2}{N_\varepsilon} < \varepsilon$$

$$\Rightarrow N_\varepsilon > \frac{2}{\varepsilon} \Rightarrow N_\varepsilon = \left[\frac{2}{\varepsilon}\right] + 1.$$

Properties

1) Every convergent sequence is a Cauchy sequence

2) (U_n) is not a Cauchy sequence$\Rightarrow(U_n)$ is divergent

3) Every Cauchy sequence is bounded

Cauchy criterion of convergence

A real sequence is convergent in $\mathbb{R}$: If and only if it is of Cauchy

2.9 Recurrent sequences

Definition 30

A recurrent sequence is a sequence defined by: $\begin{cases} U_0 \text{ given} \\ U_{n+1} = f(U_n). \end{cases}$

$f : \mathbb{N} \to \mathbb{R}$ is an application. Studying the nature of the sequence is like

studying convergence if (U_n) converges to l , it is verified: $l = f(l) \Leftrightarrow l - f(l) = 0.$

Example 57

Study the nature of the sequence (U_n)defined by: $\begin{cases} U_0 &= \frac{1}{2} \\ U_{n+1} &= U_n^2 + \frac{3}{16}. \end{cases}$

We have: $U_1 = U_0^2 + \frac{3}{16} = \left(\frac{1}{2}\right)^2 + \frac{3}{16} = \frac{7}{16} > 0 / U_2 = U_1^2 + \frac{3}{16} = \left(\frac{7}{16}\right)^2 + \frac{3}{16} = \frac{97}{256} > 0.$

*By recurrence, we show: $U_n > 0, \forall n \in \mathbb{N}$ We have:$U_0 = \frac{1}{2} > 0.$

We assume that $U_n > 0$ and we show that $U_{n+1} > 0.$

We have: $U_n > 0 \Rightarrow U_n^2 + \frac{3}{16} > 0 \Rightarrow U_{n+1} > 0$

* The monotonicity of (U_n) $U_{n+1} - U_n = U_n^2 - U_n + \frac{3}{16} = \left(U_n - \frac{1}{4}\right)\left(U_n - \frac{3}{4}\right) > 0$ So $\left[0, \frac{1}{4}\right]$.

2.10 Solved exercises

Exercise 1.

Using the definition of limit show that:

1) $\lim\limits_{n\to+\infty} \frac{1}{2n-1} = 0,$

2) $\lim\limits_{n\to+\infty} 2^{n^{\frac{1}{2}}} = +\infty$

Solution.

1) We have:
$\lim\limits_{n\to+\infty} \frac{1}{2n-1} = 0 \Leftrightarrow \forall\epsilon > 0, \exists N \in \mathbb{N}, \forall n : \left(n > N \Rightarrow \left|\frac{1}{2n-1} - 0\right| < \epsilon\right)$

Let $\epsilon > 0$:We have (we can assume from the beginning that$N \geq 1$),

then $n \geq 1$ and $2n - 1 > 0$.

$$\left|\frac{1}{2n-1} - 0\right| < \epsilon \Leftrightarrow \frac{1}{2n-1} < \epsilon \Leftrightarrow \frac{1}{\epsilon} < 2n - 1 \Leftrightarrow n > \frac{1}{2} + \frac{1}{2\epsilon}.$$

Just choose

$N = max\left(\left[\frac{1}{2} + \frac{1}{2\epsilon}\right] + 1, 1\right) = \left[\frac{1}{2} + \frac{1}{2\epsilon}\right] + 1.$

2)We have:

$\lim\limits_{n\to+\infty} 2^{n^{\frac{1}{2}}} = +\infty \Leftrightarrow \forall A > 0, \exists N \in \mathbb{N}, \forall n : \left(n > N \Rightarrow 2^{n^{\frac{1}{2}}} > A\right).$

Let $A > 0$: we find $N \in \mathbb{N}$ such that $\forall n : n > N \Rightarrow 2^{n^{\frac{1}{2}}} > A$.

Let $n \in \mathbb{N}$. We have:

$$2^{n^{\frac{1}{2}}} > A \Rightarrow \ln 2^{n^{\frac{1}{2}}} > \ln A \Leftrightarrow n^{\frac{1}{2}} > \frac{\ln A}{\ln 2} \Leftrightarrow n > \left(\frac{\ln A}{\ln 2}\right)^2.$$

Just take $N = \left[\left(\frac{\ln A}{\ln 2}\right)^2\right] + 1$.(Where $[\alpha]$ denotes the integer part of α).

Exercice 2:

Let $(u_n)_{n\geq 0}$ be a sequence defined by the recurrence relation

$$u_{n+1} = \frac{1}{2}u_n + 1,$$

and u_0 is given.

1-Show that if $u_0 \leq 2$ then for all $n \geq 0$, $u_n \leq 2$ and that the sequence is monotonic.

2-Deduce that the sequence is convergent and determine its limit.

3-Show that if$u_0 \geq 2$ then for all $n \geq 0, u_n \geq 2$ and that the sequence is monotonic.

4-Deduce that the sequence is convergent and determine its limit.

5-We put $v_n = u_n - 2$: Show that the sequence (v_n) is a geometric sequence of reason $\frac{1}{2}$.

6-Deduce an expression for u_n as a function of n and u_0.

Find the result of the first two questions.

7-Deduce $\lim\limits_{n \to +\infty} \frac{\sum\limits_{k=0}^{n} u_k}{n}$.

Solution.

1-By recurrence $u_0 \leq 2$ and let's show that $u_n \leq 2$ leads to $u_{n+1} \leq 2$,

$$u_{n+1} = \frac{1}{2}u_n + 1 \leq \frac{1}{2} \cdot 2 + 1 = 2$$

So for all $n \geq 0$,$u_n \leq 2$,

$$u_{n+1} - u_n = \frac{1}{2}u_n + 1 - u_n = -\frac{1}{2}u_n + 1 = \frac{2 - u_n}{2}. \geq 0$$

Then the sequence (u_n) is increasing.

2-The sequence is increasing and upper bounded by 2 so it converges to a limit l which verifies

$$u_{n+1} = \frac{1}{2}u_n + 1 \Leftrightarrow l = \frac{1}{2}l + 1 \Leftrightarrow l = 2.$$

3- By recurrence $u_0 \geq 2$ and let's show that $u_n \geq 2$ leads to $u_{n+1} \geq 2$.

$$u_{n+1} = \frac{1}{2}u_n + 1 \geq \frac{1}{2} \cdot 2 + 1 = 2$$

Then for all $n \geq 0, u_n \geq 2$.

$$u_{n+1} - u_n = \frac{1}{2}u_n + 1 - u_n = -\frac{1}{2}u_n + 1 = \frac{2 - u_n}{2} \leq 0.$$

So the sequence (u_n) is decreasing.

4-The sequence is decreasing and lower bounded by 2, so it converges to a limit l which verifies

$$u_{n+1} = \frac{1}{2}u_n + 1 \Leftrightarrow l = \frac{1}{2}l + 1 \Leftrightarrow l = 2$$

5-

$$v_{n+1} = u_{n+1} - 2 = \frac{1}{2}u_n - 1 = \frac{1}{2}(u_n - 2) = \frac{1}{2}v_n.$$

So (v_n) is a geometric sequence of reason $\frac{1}{2}$.

6-From 5. we deduce that for all $n \geq 0$ That:

$$v_n = \frac{1}{2^n}v_0 = \frac{1}{2^n}(u_0 - 2)$$

So for all $n \geq 0$:

$$u_n = v_n + 2 = \frac{u_0}{2^n} - \frac{1}{2^{n-1}} + 2$$

$$\lim_{n \to +\infty} \left(\frac{u_0}{2^n} - \frac{1}{2^{n-1}} \right) = 0 \Leftrightarrow \lim_{n \to +\infty} u_n = 2.$$

7-

$$\sum_{k=0}^{n} u_k = \sum_{k=0}^{n} (v_n + 2) = (v_0 + 2) + (v_1 + 2) + ... + (v_n + 2)$$
$$= v_0 + v_1 + ... + v_n + 2(n+1)$$
$$\sum_{k=0}^{n} u_k = v_0 + \tfrac{1}{2}v_0 + ... + \tfrac{1}{2^n}v_0 + 2(n+1)$$
$$\sum_{k=0}^{n} u_k = v_0 \left(1 + \frac{1}{2} + ... + \frac{1}{2^n}\right) + 2(n+1)$$
$$\sum_{k=0}^{n} u_k = v_0 \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} + 2(n+1) = 2v_0 \left(1 - \frac{1}{2^{n+1}}\right) + 2(n+1)$$
$$\sum_{k=0}^{n} u_k = 2v_0 - \frac{v_0}{2^{n-1}} + 2(n+1)$$
$$\lim_{n \to +\infty} \frac{\sum_{k=0}^{n} u_k}{n} = \lim_{n \to +\infty} \frac{2v_0 - \frac{v_0}{2^{n-1}} + 2(n+1)}{n} = 2.$$

Exercise 3:

Show that the sequence $(u_n)_{n \geq 0}$ with general term u_n defined by :

$$u_n = \frac{1 \times 3 \times ....(2n+1)}{3 \times 6...(3n+3)}$$

Is convergent and determine its limit.

Solution.

We can express u_{n+1} as a function of u_n :

$$u_{n+1} = \frac{1 \times 3 \times ...(2n+1) \times (2n+3)}{3 \times 6...(3n+3) \times (3n+6)} = \frac{1 \times 3 \times ...(2n+1)}{3 \times 6...(3n+3)} \times \frac{2n+3}{3n+6}$$

$$u_{n+1} = u_n \times \frac{2n+3}{3n+6}.$$

If there is a limit l it verifies:

$u_{n+1} = u_n \times \frac{2n+3}{3n+6} \Leftrightarrow l = l \times \frac{2}{3} \Leftrightarrow l\left(1 - \frac{2}{3}\right) = 0 \Leftrightarrow l = 0.$

It remains to show that the sequence of general term u_n converges.

It is more than clear that $u_n > 0$, the sequence islower bounded.

just look at the quotient$\dfrac{u_{n+1}}{u_n}$:

$\dfrac{u_{n+1}}{u_n} = \dfrac{2n+3}{3n+6} < \dfrac{2n+4}{3n+6} = \dfrac{2(n+2)}{3(n+2)} = \dfrac{2}{3} < 1.$

So the sequence with general term u_n is decreasing and lower bounded, so it converges.

The only possible limit is 0.

Exercise 4:

Let $(u_n)_{n\geq1}$be the sequence defined by $\begin{cases} u_1 = \sqrt{2} \\ u_{n+1} = \sqrt{u_n + 2}, \forall n \geq 1 \end{cases}$

1-Show that : $\forall n \in \mathbb{N}^*$, $u_n \leq 2$.

2-Study the monotony of the sequence (u_n).

3-Study the nature of (u_n), and determine its limit if it is convergent.

Solution.

1-We have: $\begin{cases} u_1 = \sqrt{2} \\ u_{n+1} = \sqrt{u_n + 2}, \forall n \geq 1. \end{cases}$

Let's show that $u_n \leq 2$ (by recurrence).

For $n = 1$ (the relation is true).

Assume that $u_n \leq 2$ and show that $u_{n+1} \leq 2$.

We have: $u_n \leq 2 \Leftrightarrow u_n + 2 \leq 4 \Leftrightarrow \sqrt{u_n + 2} \leq 2 \Leftrightarrow u_{n+1} \leq 2$.

Finally: $u_n \leq 2$,$\forall n \geq 1$.

2-The monotony:

On the one hand, according to(1) we have $\forall n \geq 1 : \ u_n \leq 2$, therefore the sequence (u_n) is increased.

On the other hand: (u_n) is a recurring sequence defined by:

$u_{n+1} = \sqrt{u_n + 2} = f\left(u_n\right)$ where $f(x) = \sqrt{x + 2}$.

We have $\forall x > -2 : \ f'(x) = \frac{1}{2\sqrt{x+2}} > 0$. So f is increased

As $u_1 = \sqrt{2}$, and $u_2 = \sqrt{u_1 + 2} = \sqrt{\sqrt{2} + 2}$.

So $u_2 > u_1$, from which (u_n) is increased.

Finally, since the sequence $(u_n)$is increasing and upper bounded, then it is convergent.

3- The limit:

First: since (u_n) is convergent, then $\exists l \in \mathbb{R}$, $\underset{n \to +\infty}{lim} u_n = l$.

As $u_{n+1} = \sqrt{u_n + 2} = f\left(u_n\right)$ where $f(x) = \sqrt{x + 2}$, so $f(l) = l$.

We have $f(l) = l \Leftrightarrow \sqrt{l + 2} = l \Leftrightarrow l^2 - l - 2 = 0 \Leftrightarrow (l + 1)\,(l - 2) = 0$.

But $\forall n \geq 1 : \ u_n \geq 0$ (in fact : by recurrence $u_1 = \sqrt{2} > 0$, and($u_n \geq 0 \Rightarrow u_{n+1} = \sqrt{u_n + 2} \geq 0$)

Hence $l = 2$, i.e., $\underset{n \to +\infty}{lim} u_n = 2$.

<u>Exercise 5:</u>

Use Cauchy's criterion to study the nature of general term sequences.

1)$u_n = cos\dfrac{1}{n}$.

$2)v_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \dfrac{1}{n}$

<u>**Solution.**</u>

1)We have:

$$|u_p - u_q| = \left|\cos\frac{1}{p} - \cos\frac{1}{q}\right| = \left|-2\sin\frac{1}{2}\left(\frac{1}{p} - \frac{1}{q}\right)\sin\frac{1}{2}\left(\frac{1}{p} + \frac{1}{q}\right)\right|.$$

But we have:

$$\sin x \le |x|, \forall x \in \mathbb{R}.$$

So:

$$\left|\cos\frac{1}{p} - \cos\frac{1}{q}\right| \le \left|2\frac{1}{2}\left(\frac{1}{p} - \frac{1}{q}\right)\frac{1}{2}\left(\frac{1}{p} + \frac{1}{q}\right)\right| = \left|\frac{1}{2p^2} - \frac{1}{2q^2}\right|$$

$$\left|\cos\tfrac{1}{p} - \cos\frac{1}{q}\right| \le \left|\frac{1}{2p^2}\right| + \left|\frac{1}{2q^2}\right| \text{(Triangular inequality).}$$

We have:

$$q \ge p \ge n_0 \Rightarrow \tfrac{1}{q} \le \frac{1}{p} \le \frac{1}{n_0}.$$

So:

$$\left|\cos\tfrac{1}{p} - \cos\tfrac{1}{q}\right| \le \tfrac{1}{2n_0^2} + \tfrac{1}{2n_0^2} = \tfrac{1}{n_0^2} < \epsilon \Rightarrow n_0 > \tfrac{1}{\sqrt{\epsilon}}.$$

We take:

$$n_0 = \left[\tfrac{1}{\sqrt{\epsilon}}\right] + 1$$

The sequence $(u_n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence, so it is convergent.

$*) \lim\limits_{n\to+\infty}\left(\cos\frac{1}{n}\right) = \cos\left(\lim\limits_{n\to+\infty}\frac{1}{n}\right) = 1.$

2) We have:

$$|v_{2n} - v_n| = \left|\left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} + \frac{1}{n+1} + ... + \frac{1}{2n}\right) - \left(1 + \frac{1}{2} + \tfrac{1}{3} + ... + \frac{1}{n}\right)\right|.$$

$|v_{2n} - v_n| = \frac{1}{n+1} + ... + \frac{1}{2n}.$

We have:

$n + 1 < 2n, n + 2 < 2n, ..., 2n < 2n.$

So:

$$\tfrac{1}{n+1} + ... + \tfrac{1}{2n} > \tfrac{1}{2n} + ... + \tfrac{1}{2n} > n\tfrac{1}{2n}.$$

$|v_{2n} - v_n| > n\frac{1}{2n} = \frac{1}{2}$, wherever N;

there exists $\epsilon = \frac{1}{2}, p = n > N, q = 2n > N$, such that $|v_q - v_p| > \epsilon$.

The sequence $(v_n)_{n\in N^*}$ is not a Cauchy sequence, so it is divergent.

Exercise 6:

Let (u_n) et (v_n) be two sequences defined by:

$u_n = \sum\limits_{k=1}^{n} \frac{1}{k!}, \quad v_n = u_n + \frac{1}{n \cdot n!}$.

Show that (u_n) and (v_n) are two adjacent sequences.

Solution.

*)The monotony of (u_n):

$$u_n = \sum_{k=1}^{n} \frac{1}{k!}, \quad v_n = u_n + \frac{1}{n \cdot n!}$$

$$u_{n+1} - u_n = \sum_{k=1}^{n+1} \frac{1}{k!} - \sum_{k=1}^{n} \frac{1}{k!}$$

$$u_{n+1} - u_n = \left(1 + \frac{1}{2!} + ... + \frac{1}{n!} + \frac{1}{(n+1)!}\right) - \left(1 + \frac{1}{2!} + ... + \frac{1}{n!}\right)$$

$$u_{n+1} - u_n = \frac{1}{(n+1)!} > 0.$$

So (u_n) is strictly increasing.

*)The monotony of (v_n) :

$$v_{n+1} - v_n = (u_{n+1} - u_n) + \left(\frac{1}{(n+1) \cdot (n+1)!} - \frac{1}{n \cdot n!}\right)$$

$$v_{n+1} - v_n = \frac{1}{(n+1)!} + \frac{n - (n+1)^2}{n\,(n+1) \cdot (n+1)!} = \frac{-1}{n\,(n+1) \cdot (n+1)!} < 0.$$

So (v_n) is strictly decreasing.

$$* \lim_{n\to+\infty} (v_n - u_n) = \lim_{n\to+\infty} \frac{1}{n \cdot n!} = 0.$$

So (u_n) and $(v_n)$are two adjacent sequences.

Exercise 7:

Let $(u_n)_n$ be defined by:

$u_n = (-1)^n + \frac{1}{n}, \forall n \in \mathbb{N}^*$.

1-Show that (u_n) does not converge, that this sequence is bounded.

2-Can we extract a convergent sub-sequence?

Solution.

1) Let's show that (u_n) does not converge and is bounded.

As $u_n = (-1)^n + \frac{1}{n}, \forall n \in \mathbb{N}^*$, then consider the two subsequences (v_n) and (w_n)

defined by:

$$v_n = u_{2n} = (-1)^{2n} + \frac{1}{2n} = 1 + \frac{1}{2n}$$
$$w_n = u_{2n+1} = (-1)^{2n+1} + \frac{1}{2n+1} = -1 + \frac{1}{2n+1}.$$

We have:

* $\underset{n\to+\infty}{lim} v_n = \underset{n\to+\infty}{lim} \left(1 + \frac{1}{2n}\right) = 1.$

* $\underset{n\to+\infty}{lim} w_n = \underset{n\to+\infty}{lim} \left(-1 + \frac{1}{2n+1}\right) = -1.$

As

$\underset{n\to+\infty}{lim} v_n \neq \underset{n\to+\infty}{lim} w_n$, then the sequence (u_n) is not convergent(i.e., it is divergent)

*) Let $n \in \mathbb{N}^*$: As

$-1 \leq (-1)^n \leq 1.$

$0 < \dfrac{1}{n} \leq 1.$

So

$$-1 < (-1)^n + \tfrac{1}{n} \leq 2.$$

i.e.,

$-1 < u_n \leq 2.$

We have $\forall n \in \mathbb{N}^*$:$-1 < u_n \leq 2$; hence the sequence (u_n) is bounded.

2) Since the sequence (u_n) is bounded , then by Bolzano-Weiestrass's Theorem, we can always extract from (u_n) a convergent sub-sequence.

Chapter 3

Real functions of a real variable

Definition 31

A numerical function of a real variable is an application
$f : I \to \mathbb{R}(I \subset \mathbb{R})$. We denote by $D(f)$: Definition domain of f

$$\left\{ \begin{array}{l} f : I \to \mathbb{R} \\ \quad x \mapsto 3x \end{array} \right. \text{ is a numerical function}$$

Graphic representation:

Si f s a numerical function of a real variable, the graph of f notes G is the subset of $\mathbb{R}$ defined by:

$$G(f) = \{(x, f(x)), x \in D(f)\}.$$

3.1 Some properties of functions

Limits of functions respect algebraic operations.

3.1.1 Algebraic operations

Definition 32

Operations on functions
If $f$and g are two functions defined on the same interval $I \subset \mathbb{R}$, then we have the following results:
1. **Sum**: the sum function$f + g$is defined for all real x on the interval I by :

$$(f + g)(x) = f(x) + g(x).$$

2. **Product:** the product function fg is defined for all real x in the interval I by :

$$(fg)(x) = f(x)g(x).$$

3. **Quotient:** when the function g does not annul on the interval I, the quotient function $\frac{f}{g}$ is defined for any real x of I by:

$$(\frac{f}{g})(x) = \frac{f(x)}{g(x)}. \quad g(x) \neq 0.$$

3.2 The restriction

Definition 33

Restriction
Let f be an application defined on an interval I of $\mathbb{R}$. Let I_0 be an interval of $\mathbb{R}$ included in I. The restriction of f to I_0, denoted $f|I_0$, is the function defined on I_0 by:

$$\forall x \in I_0, f_{I_0} = f(x) \ .$$

The successive application of mappings leads to the notion of the composition of functions.

3.3 The composition

Definition 34

Composition of functions
Let f be a function defined from an interval I of $\mathbb{R}$ to values in an interval J of $\mathbb{R}$. Let g be a function defined from interval J of $\mathbb{R}$ to an interval K of $\mathbb{R}$. The composite function of functions f and g is the new function that we write $g \circ f$ (and read g round f) defined for all x in interval I by:

$$(g \circ f)(x) = g(f(x)),$$

and can be written as follows:

$$g \circ f : I \to J \to K \quad x \longmapsto f(x) \longmapsto g(f(x)).$$

Example 58

$$(g \circ f)(x) = g(f(x)), \forall x \in E$$
$$f(x) = lnx,\ g(y) = \sqrt{y^2 + 1}$$
$$(g \circ f)(x) = g(f(x)) = \sqrt{(lnx)^2 + 1}.$$

3.4 Parity of a Function

3.4.1 Even and odd functions

Let $f : I \to \mathbb{R}$ be a numerical function

a) It is said to be even if:

$$\forall x \in I \text{ et } \forall (-x) \in I, f(-x) = f(x)$$

Example 59

$f(x) = \cos x,\ f(x) = x^2$ are two even functions.

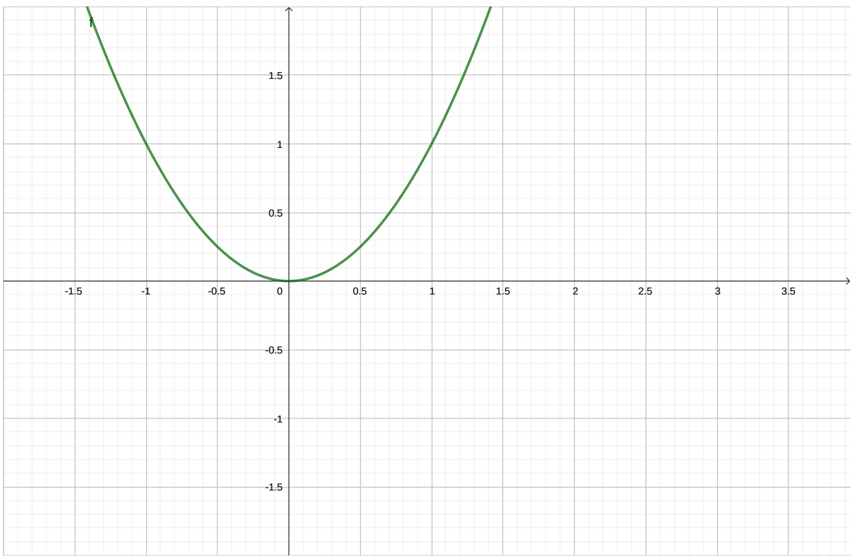


Figure 3.1: x^2

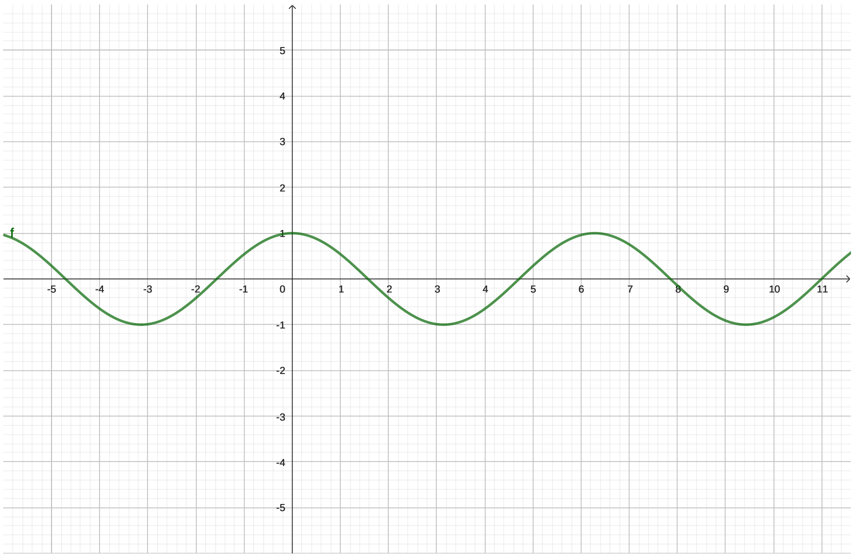


Figure 3.2: cos x

b) It is said to be odd if:

$$\forall x \in I \quad \text{and } \forall(-x) \in I, f(-x) = -f(x).$$

Example 60

$f(x) = \sin x,\ f(x) = x^3$ are two odd functions.

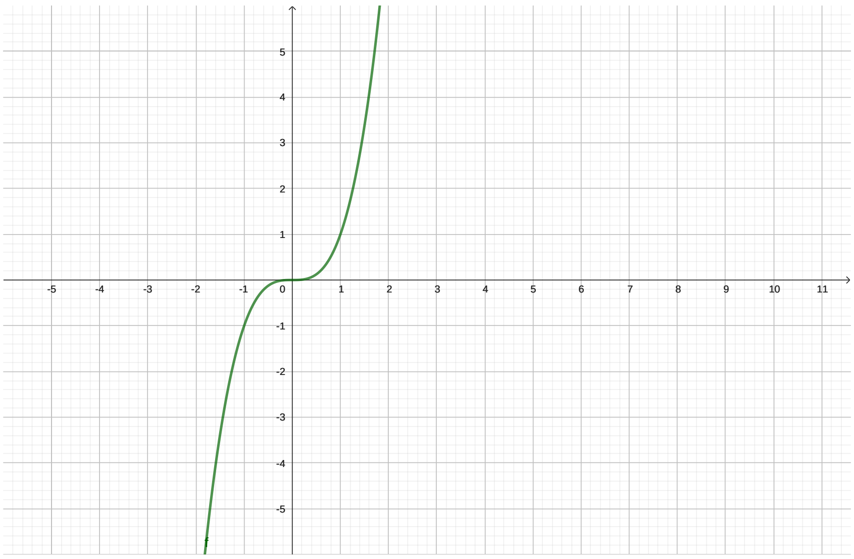


Figure 3.3: x^3

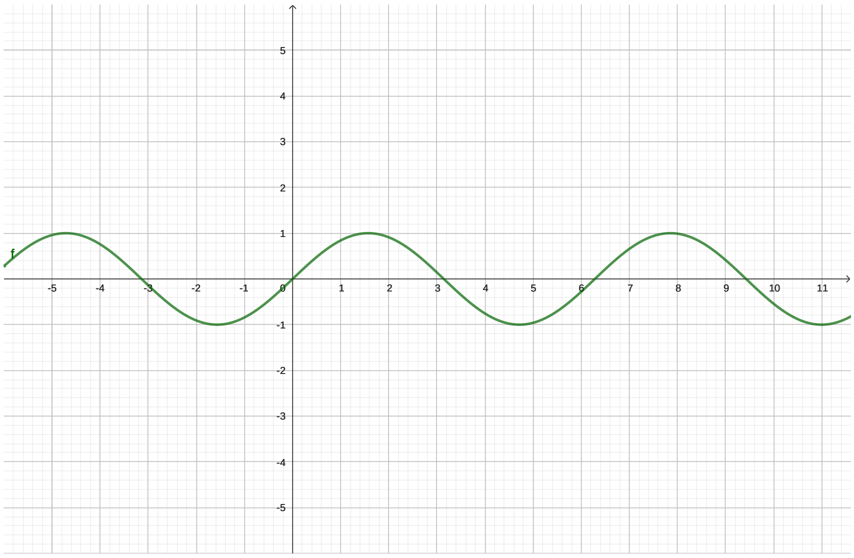


Figure 3.4: sin x

3.5 Periodicity of a Function

It is said to be periodic of period $p (p \in \mathbb{R}^*)$ if:

$$\forall x \in I \text{ and } \forall\, (x+p) \in I, f(x+p) = f(x).$$

Example 61

$f : x \mapsto \sin x, g : x \mapsto \cos x$. Where $\quad T = 2\pi \quad$ the period.
$1. f(x+2\pi) = \sin(x+2\pi) = \sin x$.
$g(x+2\pi) = \cos(x+2\pi) = \cos x$.
$2. h : x \mapsto \tan x, T = \pi, 2\pi \quad$ the period because : $\tan(x+\pi) = \dfrac{\sin(x+\pi)}{\cos(x+\pi)} = \dfrac{-\sin x}{-\cos(x)} = \tan x$.

3.6 Monotonic functions

Let $\ f : I \to \mathbb{R}$ be a numerical function
a) It is said to be increasing (respective, strictly increasing) if,

$$\forall x, y \in I : x \leq y \Rightarrow f(x) \leq f(y)$$
$$\forall x, y \in I : x < y \Rightarrow f(x) < f(y)$$

b) It is said to be decreasing (respective, strictly decreasing) if,

$$\forall x, y \in I : x \leq y \Rightarrow f(x) \geq f(y)$$
$$\forall x, y \in I : x < y \Rightarrow f(x) > f(y)$$

c) f is monotone if it is increasing or decreasing.

Example 62

*) $f(x) = x^3$ is monotonic
*) $f(x) = x^2$ is not monotonic

3.7 Bounded functions

1/ We call majorant of $f : I \to \mathbb{R}$ every real M such that

$$\forall x \in I : \ f(x) \leq M.$$

2/ We call minorant of $f : I \to \mathbb{R}$is every real m such that

$$\forall x \in I : \ f(x) \geq m.$$

3/ f is said to be majored (minored) if it has a majorant (minorant).
4/ f is bounded if it is major and minor at the same time.

Example 63

$$1.\forall x \in \mathbb{R} : |\sin x| \leq 1 \quad \text{and} \quad |\cos x| \leq 1.$$

$$So : x \mapsto \sin x \quad \text{and} \quad x \mapsto \cos x \text{are bounded functions.}$$

$$2. f(x) = \frac{1}{x^2+1}, \forall x \in \mathbb{R} : x^2 + 1 \geq 1 \Rightarrow 0 < \frac{1}{x^2+1} \leq 1. \quad \text{so f is bounded.}$$

Remark 6
if f is bounded then it reaches its upper and lower bounds which are:

$$\sup_{x \in I} f(x) = M, \ \inf_{x \in I} f(x) = m$$
$$\sup_{x \in I} f(x) = M \Leftrightarrow \forall x \in I, \forall \varepsilon > 0, \exists x_0 \in I : M - \varepsilon \leq f(x_0) \leq M$$
$$\inf_{x \in I} f(x) = m \Leftrightarrow \forall x \in I, \forall \varepsilon > 0, \exists x_0 \in I : m \leq f(x_0) \leq m + \varepsilon$$

Example 64

$$f(x) = \sin x \Leftrightarrow \begin{cases} \sup_{x \in \mathbb{R}} f(x) = \sin \frac{\pi}{2} = 1 \\ \inf_{x \in \mathbb{R}} f(x) = \sin\left(-\frac{\pi}{2}\right) = -1 \end{cases}$$

3.8 Injective, Surjective and Bijective functions

Definition 35

Let $f : I \to \mathbb{R}$ be a real function.

1/ We say that f is injective if

$$\forall (x, y) \in I \times I, f(x) = f(y) \Rightarrow x = y$$
$$\Leftrightarrow \forall (x, y) \in I \times I, x \neq y \Rightarrow f(x) \neq f(y).$$

Remark 7
Every monotone function is injective.

Example 65

$$f : [1, +\infty[\rightarrow [0, +\infty[$$
$$x \mapsto x^2 - 1$$

Let $(x, x') \in ([1, +\infty[)^2 : f(x) = f(x') \Rightarrow x^2 - 1 = x'^2 - 1 \Rightarrow x = x'$.

Example 66

$$f : \mathbb{R}^* \rightarrow \mathbb{R}^*$$
$$x \mapsto \frac{1}{x^2}$$

We have: $f(1) = f(-1) = 1 \Rightarrow x \neq x'$.
So this application is not injective.

2/ We say that f is surjective if

$$\forall y \in \mathbb{R}, \exists x \in I, y = f(x).$$

Example 67

$$f : [1, +\infty[\rightarrow [0, +\infty[$$
$$x \mapsto x^2 - 1$$

Let $y \in [0, +\infty[$, we find $x \in [1, +\infty[$ such that $y = x^2 - 1$
So: $x = \sqrt{y + 1} \in [1, +\infty[$ exist and unique.

3/ We say that f is bijective if it is injective and surjective at the same time, i.e.,

$$\forall y \in I, \exists! x \times I, y = f(x)$$

In this case we can define a new function:

$$f^{-1} : \mathbb{R} \rightarrow I$$
$$\forall y \in \mathbb{R} : f^{-1}(y) = x$$

x is the only element such that $y = f(x)$

Example 68

$$f : [1, +\infty[\to [0, +\infty[$$
$$x \mapsto x^2 - 1$$

Let $y \in [0, +\infty[$, we find $x \in [1, +\infty[$ such that $y = x^2 - 1$
So: $x = \sqrt{y+1} \in [1, +\infty[$ exist and unique.

3.9 The limit of a function

3.9.1 Limit in $x_0 \in \mathbb{R}$

Definition 36

$f : I \to \mathbb{R}$ is a function, let $x_0 \in I$. We say that f has a limit $l \in \mathbb{R}$ when x tends to x_0, if

$$\lim_{x \to x_0} f(x) = l \Leftrightarrow \forall \varepsilon > 0, \exists \eta > 0, \forall x \in I : |x - x_0| < \eta \Rightarrow |f(x) - l| < \varepsilon.$$

Example 69

Using the definition, show that $\lim_{x \to 1} (3x - 2) = 1$.

$\lim_{x \to 1} (3x - 2) = 1 \Leftrightarrow \forall \varepsilon > 0, \exists \eta > 0, \forall x \in I, |x - 1| < \eta \Rightarrow |(3x - 2) - 1| < \varepsilon$
We have: $|(3x - 2) - 1| = |3x - 3| = 3|x - 1| < \varepsilon \Rightarrow |x - 1| < \frac{\varepsilon}{3}$,
just take$\eta = \dfrac{\varepsilon}{3}$.

Example 70

Using the definition, show that $\lim_{x \to 2} (x^2 - 1) = 3$.

$\lim_{x \to 2} (x^2 - 1) = 3 \Leftrightarrow \forall \varepsilon > 0, \exists \eta > 0, \forall x \in I, |x - 2| < \eta \Rightarrow |(x^2 - 1) - 3| < \varepsilon$
we have: $|(x^2 - 1) - 3| = |x^2 - 4| = |(x - 2)(x + 2)| < \varepsilon$
we have: $|x - 2| < 1 \Rightarrow -1 < x - 2 < 1 \Rightarrow 1 < x < 3 \Rightarrow -5 < 3 < x + 2 < 5 \Rightarrow |x + 2| < 5$
$|(x - 2)(x + 2)| < \varepsilon \Rightarrow |x - 2| < \dfrac{\varepsilon}{5}$,
just take$\eta = \min(1, \dfrac{\varepsilon}{5})$.

Example 71

Proving that $\lim_{x \to 0} x \sin \frac{1}{x} = 0$.

$\lim_{x \to 0} x \sin \frac{1}{x} = 0 \Leftrightarrow \forall \varepsilon > 0, \exists \eta > 0, \forall x \in I, |x - 0| < \eta \Rightarrow \left|x \sin \frac{1}{x} - 0\right| < \varepsilon$
we have: $\left|x \sin \frac{1}{x} - 0\right| = \left|x \sin \frac{1}{x}\right| = |x| \left|\sin \frac{1}{x}\right| \leq |x| < \varepsilon$ $\left(\text{car: } \left|\sin \frac{1}{x}\right| \leq 1\right)$
just take $\eta = \varepsilon$.

3.9.2 Left and Right Limits

Definition 37

We say that f has a limit on the right (resp, a left) at the point x_0 if :

$$\lim_{x \to x_{0+}} f(x) = l \Leftrightarrow \forall \varepsilon > 0, \exists \eta > 0, \forall x \in I : x_0 < x < x_0 + \eta \Rightarrow |f(x) - l| < \varepsilon \ (resp, x_0 - \eta < x < x_0)$$

Example 72

$$f(x) = \frac{|x|}{x} \quad x_0 = 0$$
$$\lim_{x \to x_{0+}} f(x) = \lim_{x \to 0^+} \frac{x}{x} = 1$$
$$\lim_{x \to x_{0-}} f(x) = \lim_{x \to 0^-} \frac{-x}{x} = -1$$

So f(0) does not exist.

3.9.3 Infinite limits

1/ Let $f : I \to \mathbb{R}$be a real function. We say that f has a limit $l \in \mathbb{R}$ when $x \to +\infty$ if:

$$\lim_{x \to +\infty} f(x) = l \Leftrightarrow \forall \varepsilon > 0, \exists M \in \mathbb{R}, \forall x \in I, x \geq M \Rightarrow |f(x) - l| < \varepsilon$$

2/ We say that f has a limit $l \in \mathbb{R}$ when $x \to -\infty$ if:

$$\lim_{x \to -\infty} f(x) = l \Leftrightarrow \forall \varepsilon > 0, \exists M \in \mathbb{R}, \forall x \in I, x \leq M \Rightarrow |f(x) - l| < \varepsilon$$

3/ We say that f tends to ∞ when $x \to x_0$ if:

$$\lim_{x \to x_0} f(x) = +\infty \Leftrightarrow \forall M \in \mathbb{R}, \exists \eta > 0, \forall x \in I, |x - x_0| < \eta \Rightarrow f(x) \geq M$$

$$\lim_{x \to x_0} f(x) = -\infty \Leftrightarrow \forall M \in \mathbb{R}, \exists \eta > 0, \forall x \in I, |x - x_0| < \eta \Rightarrow f(x) \leq -M$$

4/ Let $f : I \to \mathbb{R}$ be a real function. We say that f does not have a limit ∞ when $x \to \infty$ if:

$$\lim_{x \to +\infty} f(x) = +\infty \Leftrightarrow \forall M > 0, \exists \eta > 0, \forall x \in I, x > \eta \Rightarrow f > M$$

$$\lim_{x \to +\infty} f(x) = -\infty \Leftrightarrow \forall M > 0, \exists \eta > 0, \forall x \in I, x > \eta \Rightarrow f < -M$$

$$\lim_{x \to -\infty} f(x) = +\infty \Leftrightarrow \forall M > 0, \exists \eta > 0, \forall x \in I, x < -\eta \Rightarrow f > M$$

$$\lim_{x \to -\infty} f(x) = -\infty \Leftrightarrow \forall M > 0, \exists \eta > 0, \forall x \in I, x < -\eta \Rightarrow f < -M$$

Example 73

$\underset{x\to+\infty}{lim}\dfrac{1}{x+1}=0.$

$\underset{x\to+\infty}{lim}\dfrac{1}{x+1}=0\Leftrightarrow\forall\varepsilon>0,\exists M\in\mathbb{R},\forall x\in I,x\geq M\Rightarrow\left|\dfrac{1}{x+1}-0\right|<\varepsilon$

We have: $\left|\dfrac{1}{x+1}-0\right|=\dfrac{1}{x+1}$

And: $x\geq M\Rightarrow\dfrac{1}{x+1}\leq\dfrac{1}{M+1}<\varepsilon$

just take: $M>\dfrac{1}{\varepsilon}-1$

3.10 Operations on limits

Let $f:I\to\mathbb{R}$ and $g:I\to\mathbb{R}$ be two real functions and $x_0,l_1,l_2\in\mathbb{R}$
Assume that $\underset{x\to x_0}{lim}f(x)=l_1$ and $\underset{x\to x_0}{lim}g(x)=l_2$. Then

$$\underset{x\to x_0}{lim}(f+g)(x)=\underset{x\to x_0}{lim}f(x)+\underset{x\to x_0}{lim}g(x)=l_1+l_2$$
$$\underset{x\to x_0}{lim}(f\times g)(x)=\underset{x\to x_0}{lim}f(x)\times\underset{x\to x_0}{lim}g(x)=l_1\cdot l_2$$
$$\underset{x\to x_0}{lim}(\lambda\times f)(x)=\lambda\times\underset{x\to x_0}{lim}f(x)=\lambda l_1\quad\forall\lambda\in\mathbb{R}$$
$$\forall x\in I,g(x)\neq0:\ \underset{x\to x_0}{lim}\frac{f(x)}{g(x)}=\frac{\underset{x\to x_0}{lim}f(x)}{\underset{x\to x_0}{lim}g(x)}=\frac{l_1}{l_2}.$$

3.11 Limit of a compound function

Let $f:I\to J$ and $g:J\to\mathbb{R}$ be two real functions and$x_0,l_1,l_2\in\mathbb{R}$, if:

$$\underset{x\to x_0}{lim}f(x)=l_1\quad\text{and}\quad\underset{x\to l_1}{lim}g(x)=l_2,\quad\text{Then}:\quad\underset{x\to x_0}{lim}(g\circ f)(x)=\underset{x\to x_0}{lim}g(f(x))=l_2.$$

Example 74

$f(x)=\ln x\Rightarrow\underset{x\to1}{lim}f(x)=0$

$g(y)=\sqrt{y^2+1}\Rightarrow\underset{x\to0}{lim}g(x)=1$

$(g\circ f)(x)=g(f(x))=\sqrt{(\ln x)^2+1}\Rightarrow\underset{x\to x_0}{lim}(g\circ f)(x)=\underset{x\to x_0}{lim}\sqrt{(\ln x)^2+1}=1.$

3.12 Comparison Relations(Landau's Notations)

Let f,g be two functions defined on the same interval I in a neighborhood of the point x_0 (left and right of$x_0\in\bar{\mathbb{R}}$) with $\bar{\mathbb{R}}=[-\infty,+\infty]$. We say that f is negligible compared to g in the neighborhood of x_0 if a function exists ε defined on I with:

$$f=\varepsilon\times g\quad\text{and}\ \underset{x\to x_0}{lim}\varepsilon(x)=0.$$

It is symbolized by :$f(x) = O_{x_0} g(x)$.
We call $f = O(g)$ Landau notation.

Remark 8 (I)
the function g does not annul in the neighborhood of x_0. So

$$f = O(g) \Leftrightarrow \underset{x \to x_0}{lim} \frac{f(x)}{g(x)} = 0.$$

Example 75

$$x = O_{+\infty} x^2, \quad \exists \varepsilon \quad \text{with} \quad \varepsilon = \frac{1}{x}.$$

$$x = \frac{1}{x} x^2, \quad \text{such that } \underset{x \to +\infty}{lim} \varepsilon(x) = 0.$$

$$x = O_{+\infty} x^2 \Leftrightarrow \underset{x \to x_0}{lim} \frac{x}{x^2} = 0.$$

3.13 Lipschitz functions

Let $k \in \mathbb{R}_+$. We say that $f : I \to \mathbb{R}$ is k Lipschitz(or it is lipshitz with respect to k) if:

$$\forall (x, y) \in I^2, |f(x) - f(y)| \leq k |x - y.|$$

Example 76

$f(x) = x^2, I = [0, 1]$.
$\forall x, y \in [0, 1], |f(x) - f(y)| = |x^2 - y^2| = |(x + y)(x - y)| = |x + y| |x - y| = (x + y) |x - y| \leq 2 |x - y|$
$\begin{cases} x \leq 1 \\ y \leq 1 \Rightarrow (x + y) \leq 2 \end{cases}$
So: f is Lipschitz .

3.14 Indeterminate Forms

Let f and g be two real functions.

$1/ f + g \quad$ with $\quad f \to +\infty \quad$ and $\quad g \to -\infty$.

$2/ f \cdot g \quad$ with $\quad f \to 0 \quad$ and $\quad g \to \infty$.

$3/ \dfrac{f}{g} \quad$ with $\quad f \to 0 \quad$ and $\quad g \to 0$.

$4/ \dfrac{f}{g} \quad$ with $\quad f \to \infty \quad$ and $\quad g \to \infty$.

$5/ f^g \quad$ with $\quad f \to 1 \quad$ and $\quad g \to \infty$.

$6/f^g$ with $f \to 0$ and $g \to 0.$

$7/f^g$ with $f \to \infty$ and $g \to 0.$

3.15 Algebraic methods for eliminating indeterminate Forms

3.15.1 Simplification of algebraic expression

Example 77

$$\lim_{x\to 1} \frac{x^2-1}{x^2+3x+1} = \frac{0}{0}$$

$$\lim_{x\to 1} \frac{x^2-1}{x^2+3x+1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-2)(x-1)} = \lim_{x\to 1} \frac{x+1}{x-2} = -1.$$

3.15.2 Putting in factor

Example 78

$$\lim_{x\to +\infty} \frac{x^2+1}{x+2} = \frac{\infty}{\infty}$$

$$\lim_{x\to +\infty} \frac{x^2+1}{x+2} = \lim_{x\to +\infty} \frac{x^2\left(1+\frac{1}{x^2}\right)}{x\left(1+\frac{2}{x}\right)} = \lim_{x\to +\infty} x = +\infty.$$

3.15.3 Using the conjugate quantity

Example 79

$$\lim_{x\to +\infty} \left(\sqrt{x+1} - \sqrt{x}\right) = +\infty - \infty$$

$$\lim_{x\to +\infty} \frac{\left(\sqrt{x+1}-\sqrt{x}\right)\left(\sqrt{x+1}+\sqrt{x}\right)}{\left(\sqrt{x+1}+\sqrt{x}\right)} = \lim_{x\to +\infty} \frac{x+1-x}{\left(\sqrt{x+1}+\sqrt{x}\right)} = \lim_{x\to +\infty} \frac{1}{\left(\sqrt{x+1}+\sqrt{x}\right)} = 0.$$

3.15.4 Using the ln and exponential functions

$$exp(ln) = Id \quad / \quad ln(exp) = Id$$

Example 80

$\underset{x\to 0}{lim}(1+x)^{\frac{1}{x}} = \underset{x\to 0}{lim}\exp\left(\ln(1+x)^{\frac{1}{x}}\right) = \underset{x\to 0}{lim}\exp\left(\frac{1}{x}\ln(1+x)\right)$

We have: $\underset{x\to 0}{lim}\frac{\ln(1+x)}{x} = 1$,

So: $\underset{x\to 0}{lim}\exp\left(\frac{1}{x}\ln(1+x)\right) = \underset{x\to 0}{lim}\exp(1) = \exp(1)$

because: $\underset{x\to 0}{lim}\frac{\ln(1+x)}{x} = \underset{x\to 0}{lim}\frac{f(x)-f(0)}{x-0} = f'(0)$,

$f(x) = \ln(1+x) \Rightarrow f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = 1$.

3.15.5 Equivalence relation

1/ Let $x_0 \in \mathbb{R}$ and let $Q : I \to \mathbb{R}$ be a function that does not cancel on I except perhaps at x_0. Let f a function of $I \to \mathbb{R}$

a)We say that f is dominated by Q in the vicinity of x_0 if we can write

$$f(x) = Q(x) \cdot U(x),$$

where: $U : I \to \mathbb{R}$is a function bounded in the vicinity of x_0, this is equivalent to: $\frac{f(x)}{Q(x)}$ is bounded, and we write:

$$f = O(Q)$$

$f(x)$ is dominated by $Q(x)$.

Example 81

1 $f(x) = x \cdot \sin\frac{1}{x} = O(x)$ $\underset{x\to 0}{lim} x \cdot \sin\frac{1}{x} \sim \underset{x\to 0}{lim} x = 0$

$2/f(x) = \ln(x^2+1) = O(\ln x)$ in the vicinity of $(+\infty)$ $\underset{x\to +\infty}{lim}\ln(x^2+1) \sim \underset{x\to +\infty}{lim}\ln x = +\infty$.

b) We say that$f$ is negligible in front of Q in the vicinity of x_0, if we can write:

$$f(x) = Q(x) \cdot \varepsilon(x),$$

where: $\varepsilon : I \to \mathbb{R}$ an infinitely small function such that $\underset{x\to x_0}{lim}\varepsilon(x) = 0$.

This is equivalent to $\underset{x\to x_0}{lim}\frac{f(x)}{Q(x)} = 0$, and we write:

$$f = O(Q)$$

$f(x)$is negligible in front of$Q(x)$.

Example 82

$1/\ln(x) = O(x)$ in the vicinity of $+\infty$ $\underset{x\to+\infty}{lim} \frac{\ln(x)}{x} = 0 \sim \underset{x\to+\infty}{lim} \frac{1}{x} = 0.$

$2/x^{n+1} = O(x^n)$ in the vicinity of 0 $\underset{x\to 0}{lim} \frac{x^{n+1}}{x^n} = 0 \sim \underset{x\to+\infty}{lim} \frac{1}{x^n} = 0.$

c) Let f and g be two functions of $I \to \mathbb{R}$ which do not cancel on I except perhaps at x_0. We say that f is equivalent to g in the vicinity of x_0, if we can write:

$$f(x) = Q(x) \cdot (1 + \varepsilon(x))$$

Where: $\varepsilon : I \to \mathbb{R}$ an infinitely small function such that: $\underset{x\to x_0}{lim}\, \varepsilon(x) = 0,$

this is equivalent to $\underset{x\to x_0}{lim} \frac{f(x)}{Q(x)} = 1$, and we write:

$$f \sim Q$$

$$f \underset{x_0}{\sim} Q \quad \text{in the vicinity of} x_0.$$

Example 83

$\sin x \underset{0}{\sim} x,\ \ e^x - 1 \underset{0}{\sim} x,\ \ 1 - \cos x \underset{0}{\sim} \frac{1}{2}x^2,\ \ \tan(x) \underset{0}{\sim} x,\ \ln(1+x) \underset{0}{\sim} x$

Exercice:

$$*\underset{x\to 0}{lim} \frac{(1-\cos x)\ln(1+x)}{x^2 \tan(x)} = \frac{0}{0}.$$

$$\underset{x\to 0}{lim} \frac{(1-\cos x)\ln(1+x)}{x^2 \tan(x)} \sim \underset{x\to 0}{lim} \frac{\frac{1}{2}x^2 \cdot x}{x^2 \cdot x} = \frac{1}{2}.$$

Exercice:

$$*\underset{x\to 0}{lim} e^{\frac{(1-\cos x)\sin x}{x^2 \tan(x)}} = e^{\frac{0}{0}}$$

$$\underset{x\to 0}{lim} e^{\frac{(1-\cos x)\sin x}{x3}} \sim \underset{x\to 0}{lim} e^{\frac{\frac{1}{2}x^2 \cdot x}{x3}} = e^{\frac{1}{2}}.$$

Exercice:

$$*\underset{x\to 0}{lim} \frac{\ln(\cos x)}{(1-\cos 2x)} = \frac{0}{0}.$$

We have: $\ln(\cos x) = \ln\left(1 + \overbrace{(\cos x - 1)}\right) = \ln(1+X) \underset{0}{\sim} X = \cos x - 1$

$1 - \cos 2x \underset{0}{\sim} \frac{1}{2}(2x)^2.$ Then: $\underset{x\to 0}{lim} \frac{\ln(\cos x)}{(1-\cos 2x)} \sim \underset{x\to 0}{lim} \frac{\cos x - 1}{2x^2} = \underset{x\to 0}{lim} \frac{-\frac{1}{2}x^2}{2x^2} = \frac{-1}{4}$

Exercice:

* Give the equivalence at infinity for the following function:

$$f(x) = (1+x)^{\frac{1}{x+1}} - x^{\frac{1}{x}} = \exp\left(\frac{1}{x+1}\ln(1+x)\right) - \exp\left(\frac{1}{x}\ln(x)\right)$$

$$f(x) = \exp\left(\frac{1}{x}\ln(x)\right)\left[\exp\left(\frac{1}{x+1}\ln(1+x) - \frac{\ln(x)}{x}\right) - 1\right]$$

$$\exp\left[\frac{x\ln\left(1+\frac{1}{x}\right) - \ln(x)}{x(x+1)}\right] \underset{\infty}{\sim} \exp\left[\frac{-\ln(x)}{x^2}\right]$$

$$f(x) = \exp\left(\frac{1}{x}\ln(x)\right)\left[\exp\left(\frac{1}{x+1}\ln(1+x) - \frac{\ln(x)}{x}\right) - 1\right] \underset{\infty}{\sim} \exp\left[\frac{-\ln(x)}{x}\right] - 1 \to 0.$$

3.16 Solved exercises

Exercise 1:

Determine the domain of definition of the following functions.

$1)f(x)=\frac{x+1}{1-e^{\frac{1}{x}}}, 2)f(x)=\frac{1}{\sqrt{\sin x}}, 3)f(x)=\sqrt{x^2-1}e^{\frac{1}{1-x}}, 4)f(x)=(1+\ln x)^{\frac{1}{x}}$

Solution:

Determine the domain of definition.

$1)f(x)=\frac{x+1}{1-e^{\frac{1}{x}}} \to D_f=\,]-\infty,0[\cup]0,+\infty[$

$2)f(x)=\frac{1}{\sqrt{\sin x}} \to D_f=\bigcup_{k\in\mathbb{Z}}]2k\pi,\pi+2k\pi[\,.$

$3)f(x)=\sqrt{x^2-1}e^{\frac{1}{1-x}} \to D_f=\,]-\infty,-1]\cup]1,+\infty[\,.$

$4)f(x)=(1+\ln x)^{\frac{1}{x}}=e^{\frac{1}{x}\ln(1+\ln x)} \to D_f=\,]e^{-1},+\infty[\,.$

Exercise 2:

Using the definition of the limit of a function, show that:

$1)\lim\limits_{x\to4}(2x-1)=7,\quad 2)\lim\limits_{x\to+\infty}\dfrac{3x-1}{2x+1}=\dfrac{3}{2},\quad 3)\lim\limits_{x\to+\infty}\ln x=+\infty,\quad 4)\lim\limits_{x\overset{>}{\to}-3}\dfrac{4}{x+3}=+\infty.$

Solution:

Using the definition of the limit of a function, show that:

$1)\lim\limits_{x\to4}(2x-1)=7 \Leftrightarrow \forall\epsilon>0,\exists\alpha>0,\forall x\in\mathbb{R}:|x-4|<\alpha\Rightarrow|2x-8|<\epsilon.$

We have: $|2x-8|<\epsilon \Leftrightarrow 2|x-4|<\epsilon \Leftrightarrow |x-4|<\frac{\epsilon}{2}$.

So just take $\alpha=\frac{\epsilon}{2}$.

$2)\lim\limits_{x\to+\infty}\frac{3x-1}{2x+1}=\dfrac{3}{2} \Leftrightarrow \forall\epsilon>0,\exists\alpha>0,\forall x\in\mathbb{R}:x>\alpha\Rightarrow\left|\frac{3x-1}{2x+1}-\frac{3}{2}\right|<\epsilon.$

We have: $\left|\dfrac{3x-1}{2x+1}-\frac{3}{2}\right|<\epsilon \Leftrightarrow \dfrac{5}{4x+2}<\epsilon \Leftrightarrow x>\dfrac{5-2\epsilon}{4\epsilon}.$

So just take $\alpha=\left|\frac{5-2\epsilon}{4\epsilon}.\right|$

$3)\lim\limits_{x\to+\infty}\ln x=+\infty \Leftrightarrow \forall A>0,\exists\alpha>0,\forall x\in\mathbb{R}:x>\alpha\Rightarrow\ln x>A.$

We have: $\ln x>A\Rightarrow x>e^A$.

So just take $\alpha=e^A$.

4) $\lim\limits_{x \overset{>}{\to} -3} \frac{4}{x+3} = +\infty \Leftrightarrow \begin{cases} \forall A > 0, \exists \alpha > 0, \forall x \in \mathbb{R} : |x+3| < \alpha \Rightarrow \frac{4}{x+3} > A \\ \Leftrightarrow \forall A > 0, \exists \alpha > 0, \forall x \in \mathbb{R} : -3 < x < -3 + \alpha \Rightarrow \dfrac{4}{x+3} > A. \end{cases}$

We have: $\dfrac{4}{x+3} > A \Leftrightarrow x < \frac{4}{A} - 3.$

So just take $\alpha = \dfrac{4}{A}.$

Exercise 3:

We consider the two functions f and g defined on $\mathbb{R}$. by

$$f(x) = \begin{cases} \dfrac{x}{1+e^{\frac{1}{x}}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} xe^{\frac{1}{x}}, & \text{if } x < 0 \\ 0, & \text{if } x = 0. \\ x^2 \ln\left(1 + \dfrac{1}{x}\right), & \text{if} x > 0. \end{cases}$$

Study the continuity of functions f and g.

Solution:

$$*f(x) = \begin{cases} \dfrac{x}{1+e^{\frac{1}{x}}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \rightarrow D_f =]-\infty, +\infty.[$$

f is continuous on $]-\infty, 0[\cup]0, +\infty.[$

$$\lim_{x \overset{<}{\to} 0} f(x) = \lim_{x \overset{<}{\to} 0} \frac{x}{1+e^{\frac{1}{x}}} = 0.$$

$$\lim_{x \overset{>}{\to} 0} f(x) = \lim_{x \overset{>}{\to} 0} \frac{x}{1+e^{\frac{1}{x}}} = 0.$$

And $f(0) = 0$, then f is continuous at "0", so f is continuous on $\mathbb{R}$.

$$*g(x) = \begin{cases} xe^{\frac{1}{x}}, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ x^2 \ln\left(1 + \dfrac{1}{x}\right), & \text{if } x > 0 \end{cases} \quad D_g =]-\infty, +\infty[$$

g is continuous on $]-\infty, 0[\cup]0, +\infty.[$

$$\lim_{x \overset{<}{\to} 0} g(x) = \lim_{x \overset{<}{\to} 0} xe^{\frac{1}{x}} = 0.$$

$$\lim_{x \overset{>}{\to} 0} g(x) = \lim_{x \overset{>}{\to} 0} x^2 \ln\left(1 + \frac{1}{x}\right) = \lim_{x \overset{>}{\to} 0} x^2 \left[\ln(1+x) - \ln x.\right]$$

$$=\lim_{x \overset{>}{\to} 0} x^2 \ln(1+x) - \lim_{x \overset{>}{\to} 0} x^2 \ln x = 0 \left(\text{because } \ln(1+x) \underset{0}{\sim} x\right).$$

and $g(0) = 0$, then g is continuous at "0", so g is continuous on $\mathbb{R}$.

Exercise 4:

Study the prolongation by continuity of the following functions on $\mathbb{R}$.

$$a) f(x) = \cos\frac{1}{x}, \quad b) f(x) = x \cdot e^{arctg\left(\frac{1}{x^2}\right)}, \quad c) f(x) = \frac{\sin \pi x}{1-x}.$$

Solution:

$a) f(x) = \cos\frac{1}{x} \to D_f = \mathbb{R}^*.$

$\lim_{x\to 0} f(x) = \lim_{x\to 0} \cos\frac{1}{x} = \nexists.$

Therefore, it cannot be prolonged by continuity into $x_0 = 0$.

$b) f(x) = x \cdot e^{arctg\left(\frac{1}{x^2}\right)}, \to D_f = \mathbb{R}^*.$

$\lim_{x\to 0} f(x) = \lim_{x\to 0} x \cdot e^{arctg\left(\frac{1}{x^2}\right)} = 0, \left(\text{because} \lim_{x\to 0} arctg\left(\frac{1}{x^2}\right) = \frac{\pi}{2}\right).$

Therefore f admits a prolongation by continuity at $x_0 = 0$ as follows:

$$g(x) = \begin{cases} x \cdot e^{arctg\left(\frac{1}{x^2}\right)}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$c) f(x) = \frac{\sin \pi x}{1-x} \to D_f = \mathbb{R} - \{1\}.$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{\sin \pi x}{1-x} = \lim_{y=x-1\to 0} \frac{\sin \pi(y+1)}{-y} = \lim_{y\to 0} \frac{\sin \pi y}{y}$$
$$= \lim_{y\to 0} \frac{\pi y}{y} = \pi, \left(\text{because } \sin \pi y \underset{0}{\sim} \pi y\right).$$

Therefore f admits a prolongation by continuity at $x_0 = 1$ as follows:

$$g(x) = \begin{cases} \frac{\sin \pi x}{1-x}, & \text{if } x \neq 1 \\ \pi, & \text{if } x = 1. \end{cases}$$

Exercise 5:

1*Show that the function f is uniformly continuous on $[0, +\infty[$ and that the function g is not continuous on $]0, 1]$, where $f(x) = \frac{x+1}{x+2}$ and $g(x) = \frac{1}{x}$.

2*Show that every continuous application f of a segment $[a, b]$ in it self has a fixed point.

Solution:

1)* We have: $f(x) = \frac{x+1}{x+2}$.

Let x et y be two positive reals.

$$|f(x)-f(y)|=\left|\frac{x+1}{x+2}-\frac{y+1}{y+2}\right|=\left|\frac{x-y}{(x+2)(y+2)}\right|=\frac{|x-y|}{(x+2)(y+2)}.$$

$$\left\{\begin{array}{l} x\geq 0\Leftrightarrow x+2\geq 2. \\ \\ y\geq 0\Leftrightarrow y+2\geq 2. \end{array}\right. \Rightarrow (x+2)(y+2)\geq 4.$$

$\Rightarrow \frac{1}{(x+2)(y+2)}\leq \frac{1}{4}.$

Hence:$|f(x)-f(y)|\leq \frac{1}{4}|x-y|.$

With:$0<\frac{1}{4}<1$, then f is contracting so it is uniformly continuous on $[0,+\infty[.$

*$g(x)=\frac{1}{x}\Leftrightarrow g'$ is not uniformly continuous on $]0,1].$

$g(x)=\frac{1}{x}\Leftrightarrow \exists\epsilon>0,\forall\alpha>0,\exists x_1,x_2\in\,]0,1]:|x_1-x_2|<\alpha\wedge|g(x_1)-g(x_2)|\geq\epsilon.$

We have:$\forall\alpha>0,\exists n\in\mathbb{N}^*$, such that:

$$x_1=\frac{1}{2n}\in\,]0,1]\,,x_2=\frac{1}{n}\in\,]0,1]\,.$$

Hence:$|x_1-x_2|=\frac{1}{2n}$, so that $|x_1-x_2|<\alpha$, it is sufficient to take $n=\left[\frac{1}{2\alpha}\right]+1.$

And since $|g(x_1)-g(x_2)|=n$, then just take $\epsilon=\frac{1}{2}.$

2*

We have: $\left\{\begin{array}{l} f:[a,b]\to[a,b] \\ \\ x\mapsto f(x)=x \end{array}\right.$

We put $g(x)=f(x)-x.$

*g is continuous on $[a,b].$

*We have: $\forall x\in[a,b]:a\leq f(x)\leq b$, so

$$\left\{\begin{array}{l} a\leq f(a)\leq b. \\ \\ a\leq f(b)\leq b. \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} 0\leq f(a)-a\leq b-a. \\ \\ a-b\leq f(b)-b\leq 0. \end{array}\right.$$

$$\Leftrightarrow\left\{\begin{array}{l} 0\leq g(a) \\ g(b)\leq 0 \end{array}\right. \Rightarrow g(a)\cdot g(b)\leq 0.$$

and therefore, according to the intermediate value theorem:

$\exists c\in[a,b]:g(c)=0\Rightarrow\exists c\in[a,b]:f(c)=c.$

and therefore f has a fixed point in $[a,b].$

Chapter 4

Real functions of a real variable: Continuity

4.1 Continuous functions at a point

Definition 38

Let $f : I \to \mathbb{R}$ be a function, x_0 a point of I We say that f is continuous at x_0,if and only if $\begin{cases} f(x_0) \text{ is defined " its value is finite"} \\ \lim_{x\to x_0} f(x) = f(x_0) \end{cases}$

Using the definition:

$$\lim_{x\to x_0} f(x) = f(x_0) \Leftrightarrow \forall\varepsilon > 0, \exists\eta > 0, \forall x \in I : |x - x_0| < \eta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

Example 84

$$f(x) = \sin x$$

$\lim_{x\to x_0} \sin x = \sin(x_0) \Leftrightarrow \forall\varepsilon > 0, \exists\eta > 0, \forall x \in I : |x - x_0| < \eta \Rightarrow |\sin x - \sin(x_0)| < \varepsilon$

We have:

$$\sin x - \sin(x_0) = 2\sin\left(\frac{x - x_0}{2}\right)$$

So:

$$|\sin x - \sin(x_0)| = 2\left|\sin\left(\frac{x - x_0}{2}\right)\cos\left(\frac{x + x_0}{2}\right)\right| \le 2\left|\sin\left(\frac{x - x_0}{2}\right)\right| (because: \left|\cos\left(\frac{x + x_0}{2}\right)\right| \le 1$$

We find:

$$|\sin x - \sin(x_0)| \le 2\left|\sin\left(\frac{x - x_0}{2}\right)\right| \le 2\left|\frac{x - x_0}{2}\right| = |x - x_0| < \varepsilon$$

(because: $\left|\sin\left(\frac{x - x_0}{2}\right)\right| \le \left|\frac{x - x_0}{2}\right|$)

Just take: $\eta = \varepsilon$.

4.2 Continuous functions on an interval

Definition 39

A function defined on I is said to be continuous on I, if it is continuous at every point on this interval. The set of continuous functions on I is denoted by $C(I)$

4.3 Some examples of discontinuous functions

Definition 40

A function $f : I \to \mathbb{R}$ is said to be discontinuous at x_0, if it is not continuous at this point.

Example 85

Let $f(x) = \begin{cases} \frac{1}{x}, & \text{if x} \neq 0 \\ a & \text{if x} = 0\,(a \in \mathbb{R}^*) \end{cases}$

We have: $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \frac{1}{x} = \infty \neq f(0) \Rightarrow f(x)$ is not continuous at $x_0 = 0$

Example 86

Let $f(x) = \begin{cases} \sin\frac{1}{x}, & \text{if x} \neq 0 \\ 0 & \text{if x} = 0 \end{cases}$

Definition 41

We have: $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \sin\frac{1}{x} = \nexists \Rightarrow f(x)$ is not continuous at $x_0 = 0$

4.4 Operations on continuous functions

Let f and g be two functions defined on I and continuous at $x_0 \in I$, then the functions: $\alpha \cdot f\,(\alpha \in \mathbb{R})\,,\ f + g,\ f \cdot g,\ \frac{f}{g}($ if $g(x) \neq 0)$ are continuous at x_0.

4.5 Continuity of a compound function

Theorem 4.5.1

Let $f : I \to J$ and $g : J \to \mathbb{R}$ be two functions, if f is continuous at x_0 of I and g is continuous in $y = f(x_0)$. Then the compound function $g \circ f$is continuous at x_0 of I

$$(g \circ f)(x) = g\,[f(x)]\,.$$

Example 87

$$f(x)=\ln x \Rightarrow \underset{x\to 1}{lim} f(x)=0=f(1)$$
$$g(y)=\sqrt{y^2+1} \Rightarrow \underset{x\to 1}{lim} g(x)=1=g(f(1))$$
$$(g\circ f)(x)=g(f(x))=\sqrt{(\ln x)^2+1} \Rightarrow \underset{x\to x_0}{lim}(g\circ f)(x)=\underset{x\to x_0}{lim}\sqrt{(\ln x)^2+1}=1=g(f(1)).$$

4.6 Uniform Continuity

A function f defined on interval I is uniformly continuous on I if:

$$\forall\varepsilon>0,\exists\eta>0,\forall x,y\in I:|x-y|<\eta\Rightarrow|f(x)-f(y)|<\varepsilon.$$

Example 88

$f(x)=\sqrt{x}, I=[1,+\infty[$

Let $x,y\in I: |f(x)-f(y)|=\left|\sqrt{x}-\sqrt{y}\right|=\left|\dfrac{x-y}{\sqrt{x}+\sqrt{y}}\right|$

we have: $\begin{cases} x\geq 1 & \Rightarrow \sqrt{x}\geq 1 \\ y\geq 1 & \Rightarrow \sqrt{y}\geq 1 \end{cases} \quad \Rightarrow \sqrt{x}+\sqrt{y}\geq 2$

So: $|f(x)-f(y)|=\dfrac{|x-y|}{\sqrt{x}+\sqrt{y}}\leq\dfrac{|x-y|}{2}<\varepsilon$

just take $\eta=2\varepsilon$

So f is uniformly continuous (U.C) on $[1,+\infty[$.

Remark 9

(1) If f is uniformly continuous on I so f is a continuous on I, the reciprocal is false

(2) Every function f is a continuous in interval closed and bounded I so f is uniformly continuous on I.

Example 89

$$f(x)=\frac{1}{x}, I=]0,1]$$
$$\exists\varepsilon>0,\forall\eta>0,\exists x,y\in I:|x-y|<\eta\wedge|f(x)-f(y)|\geq\varepsilon$$
$$\text{We have:}\eta>0,\exists n\in\mathbb{N}:\frac{1}{n}<\eta \quad \text{(Archimed's therorem)}$$
$$\text{we take}: x=\frac{1}{n} \quad y=\frac{1}{2n}\Rightarrow|x-y|=\left|\frac{1}{n}-\frac{1}{2n}\right|=\frac{1}{n}<\eta.$$
$$\text{And:} |f(x)-f(y)|=|n-2n|=|n|\geq 1=\varepsilon$$
$$\text{So:}\quad \exists\varepsilon=1,\forall\eta>0,\exists x=\frac{1}{n},y=\frac{1}{2n}\in I:|x-y|<\eta\wedge|f(x)-f(y)|\geq 1.$$

So f is not uniformly continuous (U.C) on $]0,1]$ but is continuous on $]0,1]$.

4.7 Extension by Continuity

Let I be an interval of $\mathbb{R}, x_0 \in \mathbb{R}$ and let $f : I - \{x_0\} \to \mathbb{R}$ be a function. If f has a finite limit at x_0 : $l = \lim_{x\to x_0} f(x)$. Then the function $g : I \to \mathbb{R}$ defined by: $g(x) = \begin{cases} f(x), & \text{if} x \neq x_0 \\ l, & \text{if} x = x_0 \end{cases}$

g is continuous at x_0, and is called the extension by continuity of f at x_0.

Example 90

Let $f(x) = x.\sin\frac{1}{x}$ and be defined on $\mathbb{R} - \{x_0\}$
$\lim_{x\to 0} f(x) = \lim_{x\to 0} x.\sin\frac{1}{x} = \lim_{x\to 0} x = 0$
because: We have: $-1 \leq \sin\frac{1}{x} \leq 1 \Rightarrow -x \leq x\cdot\sin\frac{1}{x} \leq x \Rightarrow -\lim_{x\to 0} x \leq \lim_{x\to 0} x.\sin\frac{1}{x} \leq \lim_{x\to 0} x$

f is extend-able at $x_0 = 0$ by g: $g(x) = \begin{cases} x.\sin\frac{1}{x}, \text{if } x \neq 0 \\ 0, \quad \text{if } x = 0 \end{cases}$

g is the extension by continuity of f at $x_0 = 0$

Example 91

Let $f(x) = \frac{\sin x}{x}$ on $\mathbb{R}^*$
$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1$

f is extend-able at $x_0 = 0$ by g: $g(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$

g is the extension by continuity of f at $x_0 = 0$

4.8 Theorems on continuous functions

Theorem 4.8.1

Every continuous function $f : [a, b] \to \mathbb{R}$ is bounded, i.e.,

$$\sup_{x\in[a,b]} |f(x)| < \infty$$

Theorem 4.8.2

All continuous functions $f : [a, b] \to \mathbb{R}$ reaches its bounds at least once i.e., there exist $x_1, x_2 \in [a, b]$ such that:

$$\sup_{x\in[a,b]} |f(x)| = f(x_1) \qquad \inf_{x\in[a,b]} |f(x)| = f(x_2)$$

Theorem 4.8.3

If $f : [a, b] \to \mathbb{R}$ is continuous and if $f(a)$ and $f(b)$ are of different signs (if $f(a) \cdot f(b) < 0$).
Then there exists at least $c \in \,]a, b[\,$ $(a < c < b)$ such that $f(c) = 0$

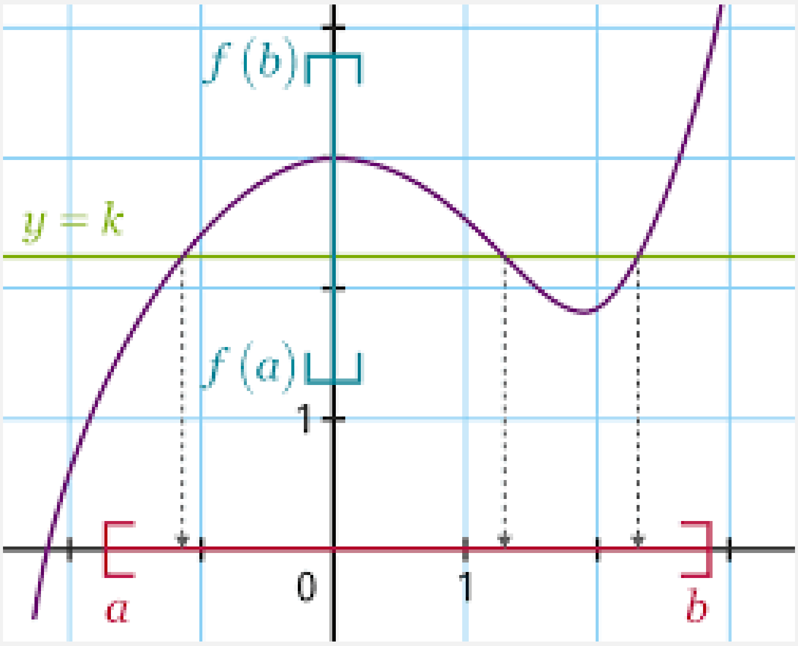


Figure 4.1: Intermediate value theorem

Example 92

Let $f(x) = x^5 - 3x + 1$ be defined on $[0, 1]$

We have: $\left.\begin{array}{r} f(0) = 1 \\ f(1) = -1 \end{array}\right\} f(0) \cdot f(1) < 0$

So: $\exists c \in \,]0, 1[\, : f(c) = 0$ root of equation $x^5 - 3x + 1 = 0$

4.9 Intermediate value theorem

Let $f : I \to \mathbb{R}$ be a continuous function, I an interval. Let $f(x_1)$ and $f(x_2)$ be two arbitrary values such that $x_1 < x_2$.
Let y be a real point between $f(x_1)$ and $f(x_2)$ $(f(x_1) < y < f(x_2))$,
There exists $x_0 \in \,]x_1, x_2[\,$, such that : $y = f(x_0)$.

4.10 Fixed point theorem

Definition 42

Let:$\varphi : I \to I$ be a function. We say that ξ is a fixed point of a function φ if:

$$\varphi(\xi) = \xi.$$

Theorem 4.10.1

Let:$\varphi : [a, b] \to [a, b]$ is continuous, we say that φ has at least a fixed point on $[a, b]$ i.e.,

$$\exists \xi \in [a, b] \qquad \varphi(\xi) = \xi.$$

Example 93

Consider the function $f(x) = x$ on a closed interval $[0, 1]$. This function has the fixed point $x = 0$ and $x = 1$, since both satisfy $f(x) = x$.

4.11 Reciprocal function theorem

Let I be an interval of $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be a continuous, strictly monotonic function. Then: the reciprocal function $f^{-1} : f(I) \to I$ exists and is continuous.

Example 94

$$f : [1, +\infty[\to [0, +\infty[$$
$$x \mapsto x^2 - 1$$

We have: f is continous on $[1, +\infty[$
And: $f'(x) = 2x > 0$on $[1, +\infty[$ is strictly positive
So: f^{-1} exists and is continous

$$f^{-1} : [0, +\infty[\to [1, +\infty[$$
$$x \mapsto \sqrt{x + 1}.$$

4.12 Solved exercises

Exercise 1:

Calculate the following limits.

$\lim\limits_{x\to-\infty}\frac{\sqrt{\frac{1+x}{x^3}}}{\sin\frac{1}{x}}$, $\lim\limits_{x\to+\infty}x^2\sin\frac{1}{x}$, $\lim\limits_{x\to-\infty}\frac{x^4+1}{\cot g\frac{1}{x}}$, $\lim\limits_{x\to0}\frac{2-\cos x-\cos 2x}{tg^2x}$, $\lim\limits_{x\to0}\frac{\sqrt{1+x}-1}{tgx}$.

Solution:
We Calculate the following limits.

$* \lim\limits_{x\to-\infty}\frac{\sqrt{\frac{1+x}{x^3}}}{\sin\frac{1}{x}}=\lim\limits_{x\to-\infty}\frac{\sqrt{\frac{1}{x^2}\left(\frac{1}{x}+1\right)}}{\frac{1}{x}\cdot\frac{\sin\frac{1}{x}}{\frac{1}{x}}}=\lim\limits_{x\to-\infty}\frac{\left|\frac{1}{x}\right|\sqrt{\frac{1}{x}+1}}{\frac{1}{x}\cdot\frac{\sin\frac{1}{x}}{\frac{1}{x}}}=\lim\limits_{x\to-\infty}\frac{-\frac{1}{x}\sqrt{\frac{1}{x}+1}}{\frac{1}{x}\cdot\frac{\sin\frac{1}{x}}{\frac{1}{x}}}=\lim\limits_{x\to-\infty}\frac{-\sqrt{\frac{1}{x}+1}}{\frac{\sin\frac{1}{x}}{\frac{1}{x}}}=-1.$

$* \lim\limits_{x\to+\infty}x^2\sin\frac{1}{x}=\lim\limits_{x\to+\infty}x^2\frac{1}{x}\cdot\frac{\sin\frac{1}{x}}{\frac{1}{x}}=\lim\limits_{x\to+\infty}x=+\infty.$

$* \lim\limits_{x\to-\infty}\frac{x^4+1}{\cot g\frac{1}{x}}=\lim\limits_{x\to-\infty}\frac{x^4+1}{\frac{1}{tg\frac{1}{x}}}=\lim\limits_{x\to-\infty}(x^4+1)\cdot tg\frac{1}{x}=\lim\limits_{x\to-\infty}(x^4+1)\frac{1}{x}\cdot\frac{tg\frac{1}{x}}{\frac{1}{x}}=\lim\limits_{x\to-\infty}\frac{x^4+1}{x}=$
$-\infty.$

$*\lim\limits_{x\to0}\frac{2-\cos x-\cos 2x}{tg^2x}=\lim\limits_{x\to0}\frac{1-\cos x+1-\cos 2x}{tg^2x}=\lim\limits_{x\to0}\frac{x^2\cdot\left(\frac{1-\cos x}{x^2}+\frac{1-\cos 2x}{4x^2}\cdot4\right)}{tgx\cdot tgx}=\lim\limits_{x\to0}\frac{x^2\cdot\left(\frac{1-\cos x}{x^2}+\frac{1-\cos 2x}{4x^2}\cdot4\right)}{x\cdot\frac{tgx}{x}\cdot x\cdot\frac{tgx}{x}}.$
$=\lim\limits_{x\to0}\frac{\frac{1-\cos x}{x^2}+\frac{1-\cos 2x}{4x^2}\cdot4}{\frac{tgx}{x}\cdot\frac{tgx}{x}}=\lim\limits_{x\to0}\frac{\frac{1}{2}+\frac{1}{2}\cdot4}{1}=\frac{5}{2}.$

$*\lim\limits_{x\to0}\frac{\sqrt{1+x}-1}{tgx}=\lim\limits_{x\to0}\frac{x}{x\cdot\frac{tgx}{x}}\frac{1}{\sqrt{1+x}+1}=\frac{1}{2}.$

Exercise 2:

Calculate the following limits using the equivalent functions and say if they can be extended by continuity.

$\lim\limits_{x\to3}\frac{\sin 2\pi x}{\sin 5\pi x}$, $\lim\limits_{x\to\frac{\pi}{2}}\frac{\log\sin^2x}{\left(\frac{\pi}{2}-x\right)}$, $\lim\limits_{x\to0}\frac{1}{1-2\cos 2x}\log(\cos x)$, $\lim\limits_{x\to-\infty}\sin\frac{1}{x}tg\left(\frac{2x}{4x+3}\right)$
$\lim\limits_{x\to0^+}\log x\cdot\log\left[1+\log(1+x)\right]$, $\lim\limits_{x\to+\infty}\sqrt{4x+1}\log\left[1-\frac{\sqrt{x+1}}{x+2}\right]$

Solution:
$*\lim\limits_{x\to3}\frac{\sin 2\pi x}{\sin 5\pi x}=\frac{0}{0}.$

We take: $y=x-3$.

$\lim\limits_{x\to3}\frac{\sin 2\pi x}{\sin 5\pi x}=\lim\limits_{y\to0}\frac{\sin 2\pi(y+3)}{\sin 5\pi(y+3)}=\lim\limits_{y\to0}\frac{\sin 2\pi y}{-\sin 5\pi y}=\lim\limits_{y\to0}\frac{2\pi y}{-5\pi y}=\frac{-2}{5}(because:\sin x\underset{0}{\sim}x)$(therefore admits a continuity extension).

The extendable function is $:g(x)=\begin{cases}\frac{\sin 2\pi x}{\sin 5\pi x}, \text{if } x\neq0\\ \dfrac{-2}{5}, \text{if } x=0\end{cases}$

$*\lim_{x\to\frac{\pi}{2}} \frac{\log\sin^2 x}{\left(\frac{\pi}{2}-x\right)} = \frac{0}{0}$.

We take: $y = \frac{\pi}{2} - x$.

$\lim_{x\to\frac{\pi}{2}} \frac{\log\sin^2 x}{\left(\frac{\pi}{2}-x\right)} = \lim_{y\to 0} \frac{\log\cos^2 y}{y} = \lim_{y\to 0} \frac{\log\left(1-\frac{1}{2}y^2\right)^2}{y} (because : 1-\cos x \underset{0}{\sim} \frac{1}{2}x^2)$.

$= \lim_{y\to 0} \frac{\log\left(1-y^2+\frac{1}{4}y^4\right)}{y} = \lim_{y\to 0} \frac{-y^2+\frac{1}{4}y^4}{y} = \lim_{y\to 0} \left(-y+\frac{1}{4}y^3\right) = 0 (because : \ln(1-y) \underset{0}{\sim} y)$ (therefore admits a extension by continuity).

The extendable function is :$g(x) = \begin{cases} \frac{\log\sin^2 x}{\left(\frac{\pi}{2}-x\right)}, \text{if } x \neq \frac{\pi}{2} \\ 0, \text{if } x = \frac{\pi}{2}. \end{cases}$

$*\lim_{x\to 0} \frac{1}{1-2\cos 2x} \log(\cos x) = \lim_{x\to 0} \frac{1}{1-\cos 2x - \cos 2x} \log(\cos x) = \lim_{x\to 0} \frac{1}{\frac{1}{2}4x^2+\frac{1}{2}4x^2-1} \log\left(1-\frac{1}{2}x^2\right)$.

$= \lim_{x\to 0} \frac{1}{4x^2-1}\left(-\frac{1}{2}x^2\right) = 0$ (therefore admits an extension by continuity).

The extendable function is :$g(x) = \begin{cases} \frac{1}{1-2\cos 2x}\log(\cos x), \text{if } x \neq 0 \\ 0, \text{if } x = 0. \end{cases}$

$*\lim_{x\to -\infty} \sin\frac{1}{x} tg\left(\frac{2x}{4x+3}\right) = \lim_{x\to -\infty} \frac{1}{x} tg\left(\frac{2x}{4x+3}\right) = \lim_{x\to -\infty} \frac{1}{x} tg\left(\frac{1}{2}\right) = 0.$(therefore does not admit an extension by continuity).

$*\lim_{x\to 0^+} \log x \cdot \log[1+\log(1+x)] = \lim_{x\to 0^+} \log x \cdot \log[1+x] = \lim_{x\to 0^+} \log x \cdot x = 0$(therefore admits an extension by continuity).

The extendable function is :$g(x) = \begin{cases} \log x \cdot \log[1+\log(1+x)], \text{if } x \neq 0 \\ 0, \text{if } x = 0. \end{cases}$

$*\lim_{x\to +\infty} \sqrt{4x+1}\log\left[1-\frac{\sqrt{x+1}}{x+2}\right] = \lim_{y=\frac{1}{x}\to 0} \sqrt{\frac{4}{y}+1}\log\left[1-\frac{\sqrt{\frac{1}{y}+1}}{\frac{1}{y}+2.}\right]$

$= \lim_{y=\frac{1}{x}\to 0} \sqrt{\frac{4}{y}+1}\log\left[1-\frac{\sqrt{\frac{1}{y^2}(y+y^2)}}{\frac{1}{y}\left(1+\frac{2}{y}\right)}\right] = \lim_{y=\frac{1}{x}\to 0} \sqrt{\frac{4}{y}+1}\log\left[1-\frac{\frac{1}{y}\sqrt{y+y^2}}{\frac{1}{y}\left(1+\frac{2}{y}\right)}\right]$.

$= \lim_{y=\frac{1}{x}\to 0} -\sqrt{\frac{4}{y}+1}\frac{\sqrt{y+y^2}}{\left(1+\frac{2}{y}\right)} = \lim_{y=\frac{1}{x}\to 0} -\frac{\sqrt{y+4y+4+y^2}}{1+\frac{2}{y}} = -2.$(therefore does not admit an extension by continuity)

Exercise 3:

Let the function f be defined on $\mathbb{R}$ by:

$\begin{cases} |x|+x+1, \text{if } x \leq 1 \\ \\ \sqrt{x}\left(x^2+2\right), \text{if } x > 1. \end{cases}$

1) Show that f is continuous on $\mathbb{R}-\{1\}$

2) Study the continuity of f in 1.

3) Deduce the continuity of f on its definition set.

Solution:

1) Let's show that f is continuous on $\mathbb{R}-\{1\}$.

* $x \to |x|+x+1$ is continuous on$]-\infty, 1]$.

* $x \to \sqrt{x}\left(x^2+2\right)$ is continuous on $]1,+\infty[$.

So: f is continuous on $\mathbb{R}-\{1\}$.

2) Study the continuity of f in 1.

$$\begin{cases} \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (|x|+x+1) = 3 \\ \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \left(\sqrt{x}\left(x^2+2\right)\right) = 3 \end{cases}$$

We have: $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x)$, so f is continuous at :$x_0 = 1$

3) Deduce the continuity of f on its set of definition

We have: f is continuous on $\mathbb{R}-\{1\}$ and f is continuous at: $x_0 = 1$ so it is continuous on $\mathbb{R}$.

Exercise 4:

Let: $f: \mathbb{R} \to \mathbb{R}$, continues at "0" such that $\forall x \in \mathbb{R}, f(x) = f(2x)$.

Show that f is constant!

Hint: set $x \in \mathbb{R}$ and take $y = \frac{x}{2}$.

Solution:

We have: $f(x) = f(2x) \Leftrightarrow f(x) = f\left(\frac{x}{2}\right)$ .... (1).

So: $f(x) = f\left(\frac{x}{2}\right) = f\left(\frac{x}{4}\right) = f\left(\frac{x}{8}\right) = ...f\left(\frac{x}{2^n}\right)$.

assume that: $f(x) = f\left(\frac{x}{2^n}\right)$ and show that: $f(x) = f\left(\frac{x}{2^{n+1}}\right)$.

From $(1): f\left(\frac{x}{2^n}\right) = f\left(\frac{x}{2^{n+1}}\right)$, therefore:$f(x) = f\left(\frac{x}{2^{n+1}}\right)$.

Chapter 5

Real functions of a real variable: Differentiability

5.1 Derivative of a function at a point

Definition 43

Let $f : I \to \mathbb{R}$ be a function, $x_0 \in I$.
We say that $f$is derivable at x_0 if the finite limit and exists: $\lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$, this limit is unique and is called the derivative number f at x_0, and we denote $f'(x_0), D_f(x_0), \frac{df}{dx}(x_0)$
Using the previous definition, we can write

$$\frac{f(x)-f(x_0)}{x-x_0} = f'(x_0) \cdot \varepsilon(x), \text{ with : } \lim_{x \to x_0} \varepsilon(x) = 0$$

Thus:

$$\begin{aligned} f(x) - f(x_0) &= f'(x_0)(x - x_0) + \varepsilon(x)(x - x_0) \\ f(x) &= f(x_0) + f'(x_0)(x - x_0) + \varepsilon(x)(x - x_0) \\ &\text{with : } \lim_{x \to x_0} \varepsilon(x) = 0 \end{aligned}$$

If :$x_0 = 0, f(x_0) = a, f'(x_0) = b$.

So: $f(x) = a + bx$ polynomial of degree (1).

Example 95

Let $f(x) = \sqrt{x}, x_0 = 1$
$\lim\limits_{x \to 1} \frac{f(x)-f(1)}{x-1} = \lim\limits_{x \to 1} \frac{\sqrt{x}-1}{x-1} = \lim\limits_{x \to 1} \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(x-1)(\sqrt{x}+1)} = \lim\limits_{x \to 1} \frac{(x-1)}{(x-1)(\sqrt{x}+1)}$
$\lim\limits_{x \to 1} \frac{f(x)-f(1)}{x-1} = \lim\limits_{x \to 1} \frac{1}{\sqrt{x}+1} = \frac{1}{2} = f'(1)$
$f'(1) = \frac{1}{2} \Rightarrow f$ is derivable at $x_0 = 1$.

Example 96

Let $f(x) = \begin{cases} x^2\left(2 + x \cdot \sin\frac{1}{x}\right), \text{if } x \neq 0 \\ 0 \qquad\qquad\qquad \text{if } x = 0 \end{cases}$

We have: $\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 1}\frac{x^2\left(2+x\cdot\sin\frac{1}{x}\right)-0}{x-0} = \lim\limits_{x\to 1}\frac{x^2\left(2+x\cdot\sin\frac{1}{x}\right)}{x} =$
$\lim\limits_{x\to 1} x\left(2 + x \cdot \sin\frac{1}{x}\right) = 0($ because $\sin\frac{1}{x}$ is bounded)
$f'(0) = 0 \Rightarrow f$ is derivable at $x_0 = 0$.

5.2 Geometric interpretation

f is derivable at x_0 the line with equation $y = f(x_0) - f\prime(x_0)(x - x_0)$ is called the tangent to the graph of f at the point $M_0\left(x_0, f(x_0)\right)$, geometric interpretation with respect to the slope of M_0M.

* Show that the derivative of f is equal to the slope of the tension at the point $M_0\left(x_0, f(x_0)\right)$.

if α is the angle formed between the $\overrightarrow{OX}$ axis and the tangent at M_0. Then $tg\alpha = f\prime(x_0)$.

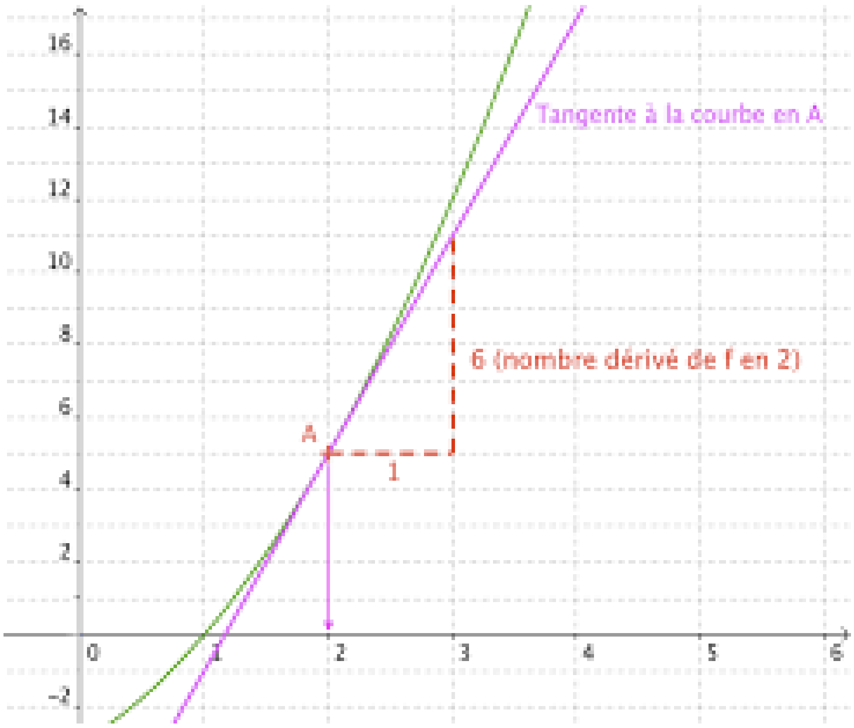


Figure 5.1: Derivative of a function at a point

$$tg\alpha = \frac{f(x) - f(x_0)}{x - x_0}$$

5.3 Left and right derivatives

Also interpreted by considering two tangents to the right and left of the point M_0 , if they are not equal, the graph of f represents an angular point at M_0.

Example 97

$f(x)=\sqrt{x^3+x^2}, D_f=[-1,\infty[$

The derivability at x_0

if $x\in[-1,0[$:
$f'_g(0)=\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0}\frac{|x|\sqrt{x+1}-0}{x-0}=\lim\limits_{x\to 0}\frac{-x\sqrt{x+1}}{x}=\lim\limits_{x\to 0}-\sqrt{x+1}=-1.$

if $x\in[0,\infty[$:
$f'_d(0)=\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0}\frac{|x|\sqrt{x+1}-0}{x-0}=\lim\limits_{x\to 0}\frac{x\sqrt{x+1}-0}{x-0}=\lim\limits_{x\to 0}\sqrt{x+1}=1.$

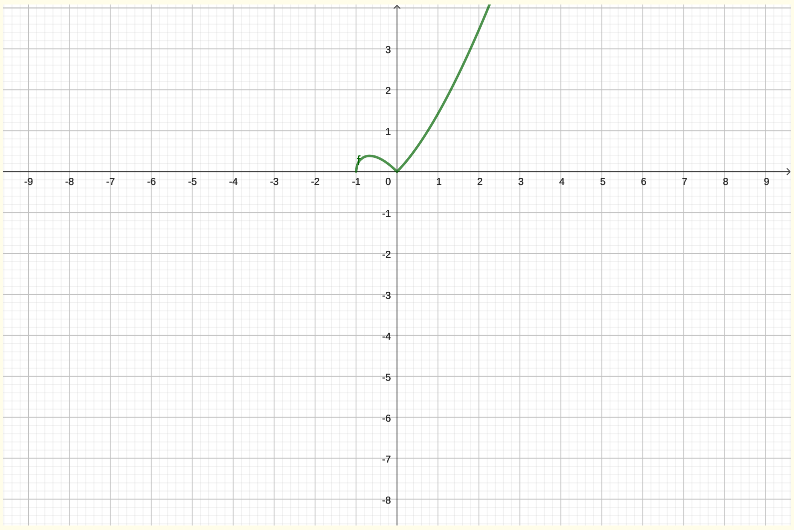


Figure 5.2: Left and right derivatives

5.4 Derivability on an interval

5.4.1 Derivative function

Let $f: I\to\mathbb{R}$ be a function, it is said to be derivable on I, if it is derivable at all point I

The application: $\begin{cases} I\to\mathbb{R} \\ x\mapsto f'(x)\end{cases}$ is called the derivative function

If f is well defined on I. Then it is derivable on I.

Example 98

Let $f(x)=\sqrt{x+1}$ on$[-1,+\infty[$

$f'(x)=\frac{1}{2\sqrt{x+1}}$ on $]-1,+\infty[$.

5.5 Differentiability

Let f be a function defined in a vicinity V of the point x_0, it is said to be differentiable at x_0 if there exists a number α such that: we have:

$$\forall h \neq 0, \quad f(x_0+h) - f(x_0) = \alpha h + h\varepsilon(h) \quad \text{with:} \lim_{h \to 0} \varepsilon(h), x - x_0 = h.$$

Theorem 5.5.1

A function f is differentiable at x_0 if it is derivable at this point.

5.6 Operations on derivative functions

Definition 44

Let $f, g : I \to \mathbb{R}$ be two functions derivable at $x_0 \in I$.

1/ $\alpha \cdot f$ is derivable at x_0 :

$$(\alpha \cdot f)'(x) = \alpha \cdot f'(x)$$

2/ $f + g$ is derivable at x_0 :

$$(f+g)'(x) = f'(x) + g'(x)$$

3/ $f \cdot g$ is derivable at x_0 :

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

4/ $\frac{f}{g}$ is derivable at x_0 :

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x).}$$

5.7 The derivative of a composite function

Let $f : I \to J$ be a function derivable at $x_0 \in I$, $g : J \to \mathbb{R}$ a function derivable at $f(x_0)$ So $g \circ f$ is derivable at x_0, and we write

$$(g \circ f)'(x_0) = (g(f(x_0)))' = f'(x_0) \cdot g'(f(x_0)).$$

Example 99

$f(x) = \sqrt{x^2+1}$

We take: $g(x) = \ln(x) \to f'(x) = \frac{1}{x}$.

Then:$(g \circ f)(x) = \ln\sqrt{x^2+1} \Rightarrow (g \circ f)'(x_0) \frac{2x}{2\sqrt{x^2+1}} \cdot \frac{1}{\sqrt{x^2+1}} = \frac{x}{x^2+1}$.

5.8 Derivative of a reciprocal function

Theorem 5.8.1

Let f be a continuous bijection of $I \to \mathbb{R}$ is derivable at $x_0 \in I$ and such that $f'(x_0) \neq 0$. Then the reciprocal function f^{-1} is derivable at the point $f(x_0)$ and has the derivative.

$$\left(f^{-1}\right)'(f(x_0)) = \frac{1}{f'(x_0)}$$
$$\left(f^{-1}\right)'(f(x_0)) = \lim_{x \to x_0} \frac{f^{-1}(f(x)) - f^{-1}(f(x_0))}{f(x) - f(x_0)} = \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{f'(x_0)}.$$

Example 100

$$f : [1, +\infty[\to [0, +\infty[$$
$$x \mapsto x^2 - 1$$

Let $y \in [0, +\infty[$, we find $x \in [1, +\infty[$ such that $y = x^2 - 1$

So: $x = \sqrt{y+1} \in [1, +\infty[$ exist and unique

$$\left(f^{-1}\right)'(f(x_0)) = \frac{1}{f'(x_0)} = \frac{1}{2x_0}.$$

5.9 Derivability and continuity

Proposition 9

If f is a differentiable function at x_0. Then it is continuous at this point.

Remark 10

The reciprocal of the proposition is not true.

Example 101

Let $f(x) = \sqrt{x}$
$f(x)$ is continuous at $x_0 = 0$ but is not differentiable.

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\sqrt{x}}{x} = \lim_{x \to 0} \frac{1}{\sqrt{x}} = \infty$$

5.10 Higher-order derivatives

If the function f' has a derivative. This is called the second derivative [second-order derivative of f] denoted f''. By recurrence, we define the successive derivatives of f.

5.10.1 n^{th}-order derivative of f

denoted $f^{(n)}$ is the derivative of the function $f^{(n-1)}$.

$$f^{(n)} = \left(f^{(n-1)}\right)'.$$

Example 102

Find the derivatives $n^{ième}$ $\left(f^{(n)}\right)$ of $f(x) = \sin x$

$$\begin{aligned}
&n = 1 : f'(x) = \cos x = \sin\left(x + \tfrac{\pi}{2}\right)\\
&n = 2 : f''(x) = -\sin x = \cos\left(x + \tfrac{\pi}{2}\right) = \sin\left(x + 2\cdot\tfrac{\pi}{2}\right)\\
&n = 3 : f^{(3)}(x) = -\sin\left(x + \tfrac{\pi}{2}\right) = \cos\left(x + 2\cdot\tfrac{\pi}{2}\right) = \sin\left(x + 3\cdot\tfrac{\pi}{2}\right)\\
&\vdots\\
&n = n : f^{(n)}(x) = -\sin\left(x + (n-2)\tfrac{\pi}{2}\right) = \cos\left(x + (n-1)\cdot\tfrac{\pi}{2}\right) = \sin\left(x + n\cdot\tfrac{\pi}{2}\right).
\end{aligned}$$

5.11 C^n class functions

* Let $n \in \mathbb{N}^*$. A function $f : I \to \mathbb{R}$ is of class C^n (or n time continuous derivable) If it is n times derivable and if $f^{(n)}$ is continuous on I.

* We say that f is of class C^0, if it is continuous on I.

* We say that f is of class C^∞ , if it is infinitely differentiable on I ,*i.e.*, $f^{(n)}$ exists for all n.

* Let $C^n(I)$: be the set of functions of class C^n on I.

* Let $C^\infty(I)$: The set of functions of class C^∞ on I.

5.12 Leibniz formula: Derivative n^{th} for the product of two functions

Theorem 5.12.1

If f and g admit derivatives at the point x_0. Then the function $f \cdot g$ admits a derivative $n^{ième}$ at the point x_0 and we write:

$$(f\cdot g)^{(n)} = \sum_{k=0}^{n} C_n^k f^{(k)} \cdot g^{(n-k)} \quad \text{with:} C_n^k = \frac{n!}{(n-k)!k!} \quad C_n^0 = C_n^n = 1$$
$$(f\cdot g)^{(n)} = C_n^0 f^{(0)}\cdot g^{(n)} + C_n^1 f^{(1)}\cdot g^{(n-1)} + C_n^2 f^{(2)}\cdot g^{(n-2)} + \cdots C_n^n f^{(n)}\cdot g^{(0)}$$

Example 103

Let $f(x) \cdot g(x) = e^x \cdot \sin x$

$$f^{(n)}(x) = e^x \quad g^{(n)}(x) = \sin\left(x + n\frac{\pi}{2}\right)$$
$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} C_n^k f^{(k)} \cdot g^{(n-k)} = \sum_{k=0}^{n} C_n^k e^x \sin\left(x + (n-k)\frac{\pi}{2}\right)$$
$$\text{with:} C_n^k = \frac{n!}{(n-k)!k!} \quad C_n^0 = C_n^n = 1$$
$$(f \cdot g)^{(n)} = C_n^0 f^{(0)} \cdot g^{(n)} + C_n^1 f^{(1)} \cdot g^{(n-1)} + C_n^2 f^{(2)} \cdot g^{(n-2)} + \cdots C_n^n f^{(n)} \cdot g^{(0)}$$

$$(f \cdot g)^{(n)} = C_n^0 e^x \sin\left(x + (n)\frac{\pi}{2}\right) + C_n^1 e^x \sin\left(x + (n-1)\frac{\pi}{2}\right) + \ldots + C_n^n e^x \sin\left(x + (n-n)\frac{\pi}{2}\right)$$

$$(f \cdot g)^{(n)} = e^x \left[\sin\left(x + (n)\frac{\pi}{2}\right) + n \sin\left(x + (n-1)\frac{\pi}{2}\right) + \ldots + \sin(x)\right]$$

5.13 Theorems for differentiable functions

Rolle's Theorem: Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$, derivable on $]a, b[$ and such that $f(a) = f(b)$. Then there exists $c \in\,]a, b[$ such that : $f'(c) = 0$.

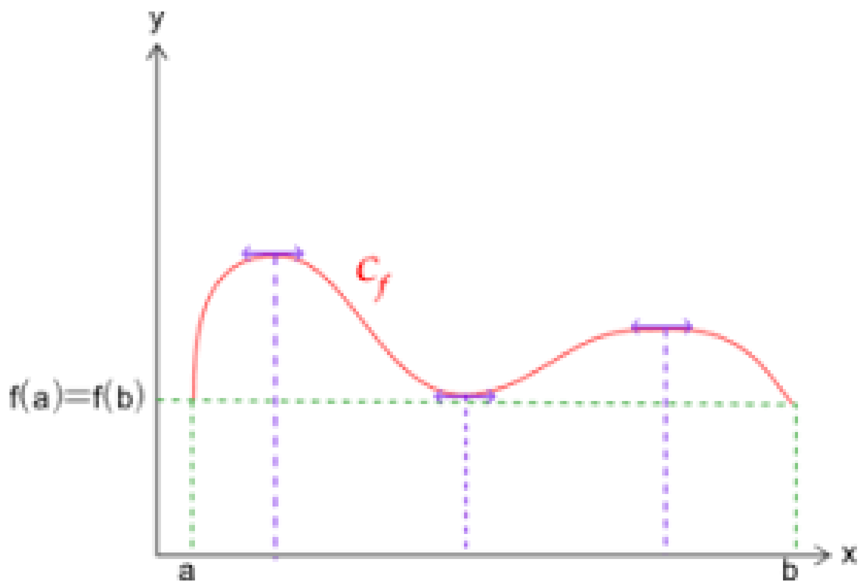


Figure 5.3: Rolle's Theorem

Remark 11
The assumptions of Rolle's theorem are essential for its validity.

Example 104

$f(x) = x^4 + 1, [a, b] = [-1, 1]$,

we have that f is continuous on $[-1, 1]$ and differentiable on $]-1, 1[$,

$f(-1) = f(1) = 2$.

So $\exists]-1, 1[, f\prime(c) = 0 \Rightarrow 4x^3 = 0 \Rightarrow c = 0$

Example 105

f defined on $[0,1]$ *by* : $\begin{cases} 1-x & \text{if } x \neq 0 \\ 0 & \text{if} x = 0 \end{cases}$

$\lim\limits_{x\to 0} f(x) = 1 \neq f(0)$, so f is not continuous at $x_0 = 0$. So, we can't apply Rolle's theorem.

Example 106

f defined on $[-1,1]$ *by* : $f(x) = |x| = \begin{cases} -x & \text{if } x \in [-1,0] \\ x & \text{if } x \in]0,1] \end{cases}$

f is not derivable at $x_0 = 0$. So, we can't apply Rolle's theorem.

Mean value theorem (Finite increments theorem)

Let $f : [a,b] \to \mathbb{R}$ be is a continuous function on $[a,b]$, derivable on $]a,b[$.

Then there exists $c \in]a,b[$ such that : $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Generalization

Let f and g be two functions: $[a,b] \to \mathbb{R}$ such that: $\forall x \in [a,b] : |f'(x)| \leq g'(x)$, f and g are derivable.

So: $|f(b) - f(a)| \leq g(b) - g(a)$.

We have:

$$f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow |f'(c)| = \left|\frac{f(b)-f(a)}{b-a}\right| \leq g'(c) = \frac{g(b)-g(a)}{b-a}$$
$$\Rightarrow |f(b) - f(a)| \leq g(b) - g(a).$$

Example 107

$$f(x) = 3 - x^2 \quad \text{is a continuous on} \quad [0,2], \text{derivable on}]0,2[.$$
$$\text{Then there exists} \quad c \in]0,2[: f'(c) = \frac{f(2)-f(0)}{2-0} = \frac{-1-3}{2-0} = \frac{-4}{2} = -2.$$

Application: Error calculation

If we take $f(a)$ to be the approximate value of $f(a+h)$ and if we know how to determine

a major M of $|f'(x)|$ for $x \in [a, a+h]$.

By the formula of finite increments, we obtain a majorant of error comise.

$$|f(b) - f(a)| = |f'(c)(b-a)| \leq M(b-a)$$
$$|f(a+h) - f(a)| \leq M \cdot h.$$

Example 108

$\sqrt{123} \simeq 11, \sqrt{121} = 11$
Using the formula of finite increments for the function $f(x) = \sqrt{x}$, we find:

$|f(a+h) - f(a)| = |f(123) - f(121)| = |f(121+2) - f(121)| = \frac{1}{\sqrt{123}} \leq \frac{1}{11}$.

because:$f(b) - f(a) = f'(c)(b-a) \Rightarrow f(123) - f(121) = \frac{1}{2\sqrt{c}} \cdot 2$.

5.14 Hospital rule

Let f and g be two functions continuous and derivable on I except at the point x_0 and are verifying:

1/ $\lim\limits_{x \to x_0} f(x) = 0 = \lim\limits_{x \to x_0} g(x)$.

2/ $\forall x \in I - \{x_0\} : g'(x) \neq 0$.

If:$\frac{f'(x)}{g'(x)}$ admits a limit l at the point x_0, then:$\frac{f(x)}{g(x)}$ admits a limit l at the point x_0.

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \lim_{x \to x_0} \frac{f(x)}{g(x)} = l.$$

We have: $\lim\limits_{x \to x_0} \frac{f'(x)}{g'(x)} = \lim\limits_{x \to x_0} \frac{\frac{f(x)-f(x_0)}{x-x_0}}{\frac{g(x)-g(x_0)}{x-x_0}} = \lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \lim\limits_{x \to x_0} \frac{f(x)}{g(x)}$.

Example 109

$$\lim_{x \to 0} \frac{\cos x - 1}{x^2} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{f''(x)}{g''(x)} = \lim_{x \to 0} \frac{-\cos x}{2} = \frac{-1}{2}.$$

5.15 Inflexion point

Let f be a function defined and 2 times derivable on an interval $]a, b[$ of $\mathbb{R}$, if there exists a point c of $]a, b[$ such that $f''(c) = 0$ and f'' changes sign in a neighborhood of c from positive to negative or vice versa. Then c is an inflection point for f.

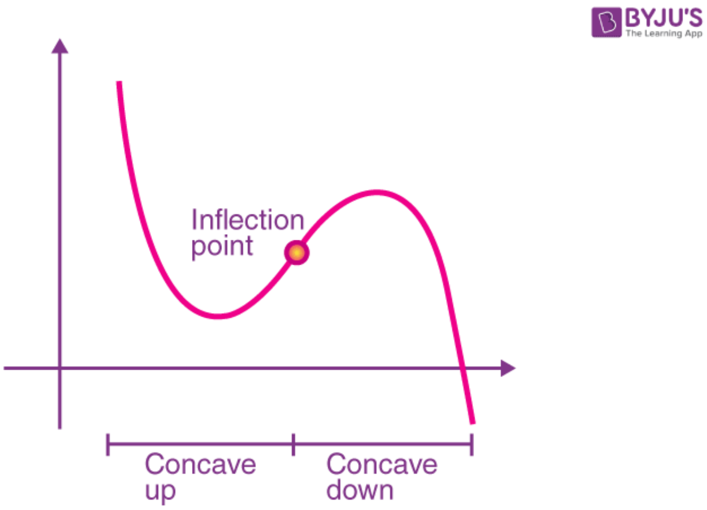


Figure 5.4: Inflexion point

Example 110

For the function $f(x) = x^3$, the second derivative is $f"(x) = 6x$. Setting this equal to zero gives $x = 0$, which is an inflection point, because the curve changes from concave down to concave up at $x = 0$.

5.16 Taylor's formula

Let $f : I \to \mathbb{R}$ be a function derivable at $x_0 \in I$, it can be written (in the neighborhood ofx_0) .

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + R(x)$$

$$\text{with: } R(x) = \varepsilon(x)(x - x_0) \text{ and } \lim_{x \to x_0} \varepsilon(x) = 0$$

$R(x)$: is called the rest.

This means that f can be approximated by a polynomial of degree 1.

$$f(x) = P_1(x) + R(x).$$

$P_1(x)$: polynomial of degree 1 in $(x - x_0)$.

$P_1(x) \mapsto f(x_0) + f'(x_0)(x - x_0)$.

The error comised by this approximation tends to "0" when $x \to x_0$.

5.17 Taylor's generalized formula

This result shows that "n" times derivable functions can be approximated by "n"(in the neighborhood of degree polynomials and we write.

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x)$$

$$f(x) = P_n(x) + R_n(x).$$

$P_n(x)$: polynomial of degree n at $(x - x_0)$.

$$P_n(x) \mapsto f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

$R_n(x)$: is called the rest

Example 111

$f(x) = \exp x, x_0 = 0.$

We have: $f(x) = f(0) + \frac{f'(0)}{1!}(x - 0) + \frac{f''(0)}{2!}(x - 0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x - 0)^n + R_n(x).$

So: $f(x) = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + R_n(x).$

5.18 Taylor's formula with Lagrange remainder

Theorem 5.18.1

Let $f : [a, b] \to \mathbb{R}$ be a function such that $f \in C^n([a, b]), f^{(n)}$ is derivable on $]a, b[$ and let $x_0 \in [a, b]$. Then $\forall x_0 \neq x \in [a, b]$:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

$$f(x) = P_n(x) + R_n(x).$$

"c" is a point between x_0 and x
The term: $\frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$ is called the Lagrange remainder.

Example 112

$f(x) = \exp x, x_0 = 0.$

We have: $f(x) = f(0) + \frac{f'(0)}{1!}(x - 0) + \frac{f''(0)}{2!}(x - 0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x - 0)^n + R_n(x).$

So: $f(x) = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + \exp c \frac{x^{n+1}}{(n+1)!}.$

5.19 Mac-Laurin's formula with Lagrange remainder

When $x_0 = 0$ in the Taylor lagrange formula. We obtain Mac-Laurin's formula with Lagrange remainder:

$$f(x) = f(0) + \frac{f'(0)}{1!}(x) + \frac{f"(0)}{2!}(x)^2 + ... + \frac{f^{(n)}(0)}{n!}(x)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x)^{n+1}$$
$$f(x) = P_n(x) + R_n(x)$$

$R_n(x)$: is called Lagrange remainder.

5.20 Mac-Laurin's formula with Young remainder

When $x_0 = 0$ in the Taylor lagrange formula. We obtain Mac-Laurin's formula with Young remainder.

$$f(x) = f(0) + \frac{f'(0)}{1!}(x) + \frac{f"(0)}{2!}(x)^2 + ... + \frac{f^{(n)}(0)}{n!}(x)^n + O(x^n)$$
$$f(x) = P_n(x) + R_n(x)$$

$O(x^n) = x^n \epsilon(x)$ and $\lim\limits_{x \to 0} \varepsilon(x) = 0 = \lim\limits_{x \to 0} \frac{O(x^n)}{x^n}$: is called Young remainder.

Example 113

$$f(x) = 1 + x + x^2 + x^3 + x^3 \cos x$$
$$f(x) = 1 + x + x^2 + 2x^3 + x^3(\cos x - 1)$$
$$\lim_{x \to 0} \varepsilon(x) = \lim_{x \to 0}(\cos x - 1) = 0.$$

Exercice: Using Mac-Laurin of order "n" show that:

1/ $e^x \geq 1 + \frac{x}{1!} + \frac{(x)^2}{2!} + ... + \frac{(x)^n}{n!}$

2/ Using Mac-Laurin of order "4", show that: $\frac{3}{8} < e^x < 3$

3/ Show that: $\frac{1}{(n+1)!} < e^x - \left(1 + \frac{x}{1!} + \frac{(x)^2}{2!} + ... + \frac{(x)^n}{n!}\right) < \frac{3}{(n+1)!}$

Example 114

$f(x) = \cos x$ order "5" with Mac-Laurin's

$\cos x = 1 - \frac{(x)^2}{2!} + \frac{(x)^4}{4!} - \frac{\sin(Ox)}{5!}(x)^5$

$R_5(x) = -\frac{\sin(Ox)}{5!}(x)^5.$

5.20.1 Local extremum

Definition 45

point x_0 such that $f'(x_0) = 0$ is called a critical point.

Definition 46

Let $f : I \to \mathbb{R}$ be a function and $x_0 \in I$,

* If $f"(x_0) \neq 0 = l$ we stop .

$l > 0$has a minimum extremum at x_0.

$l < 0$has a maximum extremum at x_0.

* If $f"(x_0) = 0$ we calculate $f^{(3)}(x_0)$ So:

* If $f^{(3)}(x_0) \neq 0 = l$ we stop, no extrema .

* If $f^{(3)}(x_0) = 0$ we calculate $f^{(4)}(x_0)$...

Theorem 5.20.1

Let f be a function n times derivable on $[a, b]$ if:

$f'(x_0) = f"(x_0) = f^{(3)}(x_0) = ... = f^{(n-1)}(x_0) = 0.$ and $f^{(n)}(x_0) \neq 0 = l.$

* If n even, we have:

$l > 0$ we have a minimum extrema.

$l < 0$ we have a maximum extrema.

* If n odd, no extrema

Example 115

$$f(x) = x^2 \begin{cases} f'(x) = 2x = 0 \Rightarrow x = 0 \\ f"(0) = 2 > 0 \quad \text{a minimum extrema} \end{cases}$$

5.21 Differential application

We call differential application f defined at point x, the form linear form $\mathbb{R}$ by:

$$h \mapsto A(h) = f'(x) \cdot h.$$

Noting dx the variable h and dy its image $A(h)$. and, we have:

$$dy = f'(x) \cdot dx \Rightarrow \frac{dy}{dx} = f'(x).$$

5.22 Solved exercises

Exercise 1:

Are the following functions derivable at $x_0 = 0$?

$f(x) = x|x|, g(x) = x^{\frac{3}{5}}, h(x) = \cos\sqrt{|x|}$
Solution:

We study the derivability at $x_0 = 0$

$$*f(x) = x|x| = \begin{cases} x^2, \text{ if } x > 0 \\ \\ -x^2, \text{ if } x \leq 0 \end{cases}$$

$$\lim_{x \overset{>}{\to} x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x \overset{>}{\to} 0} \frac{x^2-0}{x-0} = \lim_{x \overset{>}{\to} 0} \frac{x^2}{x} = \lim_{x \overset{>}{\to} 0} x = 0 = f'_+(0)$$

$$\lim_{x \overset{<}{\to} x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x \overset{<}{\to} 0} \frac{-x^2-0}{x-0} = \lim_{x \overset{<}{\to} 0} \frac{-x^2}{x} = \lim_{x \overset{<}{\to} 0} - x = 0 = f'_-(0)$$

We have:$f'_+(0) = f'_-(0)$, Then f is derivable at $x_0 = 0$.

$*g(x) = x^{\frac{3}{5}}$.

$$\lim_{x \to x_0} \frac{g(x)-g(x_0)}{x-x_0} = \lim_{x \to 0} \frac{x^{\frac{3}{5}}-0}{x-0} = \lim_{x \to 0} \frac{x^{\frac{3}{5}}}{x} = \lim_{x \to 0} x^{\frac{-2}{5}} = \lim_{x \to 0} \frac{1}{x^{\frac{2}{5}}} = +\infty,$$

so g is not derivable in $x_0 = 0$.

$$*h(x) = \cos\sqrt{|x|} = \begin{cases} \cos\sqrt{x}, \text{if } x > 0 \\ \\ \cos\sqrt{-x}, \text{ if } x \leq 0 \end{cases}$$

$$\lim_{x \overset{>}{\to} x_0} \frac{h(x)-h(x_0)}{x-x_0} = \lim_{x \overset{>}{\to} 0} \frac{\cos\sqrt{x}-1}{x-0} = \lim_{y=\sqrt{x} \overset{>}{\to} 0} \frac{\cos y - 1}{y^2} = -\frac{1}{2} = h'_+(0).$$

$$\lim_{x \overset{<}{\to} x_0} \frac{h(x)-h(x_0)}{x-x_0} = \lim_{x \overset{<}{\to} 0} \frac{\cos\sqrt{-x}-1}{x-0} = \lim_{y=\sqrt{-x} \overset{<}{\to} 0} \frac{\cos y - 1}{-y^2} = \frac{1}{2} = h'_-(0).$$

We have: $h'_+(0) \neq h'_-(0)$, So h is not derivable at $x_0 = 0$.

Exercise 2:

Let f be the function defined on $[0, 1]$ by:

$$\begin{cases} 0 & \text{if } x = 0 \\ x + \frac{x \ln x}{1-x} & \text{if } 0 < x < 1 \\ 0 & \text{if } x = 1. \end{cases}$$

1) Show that f is continuous on $[0, 1]$.
2) Show that there exists $c \in]0, 1[$ such that $f'(c) = 0$.

Solution:

1) Let's show that f is continuous on $[0, 1]$.

$x \mapsto x + \frac{x \ln x}{1-x}$ is continuous on $]0,1[$.

$*\lim\limits_{x \overset{>}{\to} 0} f(x) = \lim\limits_{x \overset{>}{\to} 0} \left(x + \frac{x \ln x}{1-x}\right) = 0 = f(0)$.

So f is continuous to the right of $x_0 = 0$.

$*\lim\limits_{x \overset{<}{\to} 1} f(x) = \lim\limits_{x \overset{<}{\to} 1} \left(x + \frac{x \ln x}{1-x}\right) = \lim\limits_{y \overset{<}{\to} 0} \left((y+1) + \frac{(y+1)\ln(y+1)}{-y}\right) = 0 = f(1)$.

So f is continuous to the left of $x_0 = 1$.

. Then: f is continuous on $[0,1]$.

2) Let's show that there exists $c \in]0,1[$ such that $f'(c) = 0$.

We apply Rolle's theorem to $]0,1[$.

*We have f is continuous on $[0,1]$.

*We have f is derivable on $]0,1[$.

$*f(0) = f(1)$.

According to Rolle, there exists $c \in]0,1[$ such that $f'(c) = 0$.

Exercise 3:

Let $f : \mathbb{R} \to \mathbb{R}$ the function defined by:

$$f(x) = \begin{cases} e^x & \text{if } x < 0 \\ ax^2 + bx + c & \text{if } x \geq 0 \end{cases}$$

Find a, b, c in $\mathbb{R}$ such that $f \in C^2(\mathbb{R})$. In this case $f \in C^3(\mathbb{R}.)$

Solution:

*We determine a, b, c in $\mathbb{R}$ such that $f \in C^2(\mathbb{R})$.

$$f \in C^2(\mathbb{R}) \Leftrightarrow \begin{cases} f, f' \text{are continuous and derivable on } \mathbb{R} \\ f\text{" is continuous on } \mathbb{R}. \end{cases}$$

$*f$ is continuous on $\mathbb{R} \Leftrightarrow f$ is continuous at "0".

$\Leftrightarrow \lim\limits_{x \overset{>}{\to} 0} f(x) = \lim\limits_{x \overset{<}{\to} 0} f(x)$.

$\Leftrightarrow \lim\limits_{x \overset{>}{\to} 0} ax^2 + bx + c = \lim\limits_{x \overset{<}{\to} 0} e^x$.

$\Leftrightarrow c = 1$

$$f'(x) = \begin{cases} e^x & \text{if } x < 0 \\ 2ax + b & \text{if } x \geq 0. \end{cases}$$

$*f'$ is continuous on $\mathbb{R} \Leftrightarrow f'$ is continuous at "0".

$\Leftrightarrow \lim\limits_{x \overset{>}{\to} 0} f'(x) = \lim\limits_{x \overset{<}{\to} 0} f'(x)$.

$\Leftrightarrow \lim\limits_{x \overset{>}{\to} 0} 2ax + b = \lim\limits_{x \overset{<}{\to} 0} e^x$.

$\Leftrightarrow b = 1$.

$$f"(x) = \begin{cases} e^x & \text{if } x < 0 \\ 2a & \text{if } x \geq 0 \end{cases}$$

$*f"$is continuous on $\mathbb{R} \Leftrightarrow f'$ is continuous at "0".

$\Leftrightarrow \lim\limits_{x \overset{>}{\to} 0} f"(x) = \lim\limits_{x \overset{<}{\to} 0} f"(x)$.

$\Leftrightarrow \lim\limits_{x \overset{>}{\to} 0} 2a = \lim\limits_{x \overset{<}{\to} 0} e^x$

$\Leftrightarrow a = \frac{1}{2}$.

In this case, is $f \in C^3(\mathbb{R})$

$$f'''(x) = \begin{cases} e^x & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$$

$*f'''$ is continuous on $\mathbb{R} \Leftrightarrow f'''$ is continuous at "0".

$\Leftrightarrow \lim\limits_{x \overset{>}{\to} 0} f'''(x) = \lim\limits_{x \overset{<}{\to} 0} f'''(x)$

$\Leftrightarrow \lim\limits_{x \overset{>}{\to} 0} 0 = \lim\limits_{x \overset{<}{\to} 0} e^x$.

$\Leftrightarrow 0 = 1 \Leftrightarrow f \notin C^3(\mathbb{R})$.

Exercise 4: .

Let $f : \mathbb{R} \to \mathbb{R}$ the function defined by:

$$f(x) = \begin{cases} \frac{3-x^2}{2} & \text{if } x \leq 1 \\ \frac{1}{x} & \text{if } x > 1 \end{cases}$$

Show that there exists $c \in]0,2[$ such that $f(2) - f(0) = 2f'(c)$.

Determine the possible values of c.

Solution:

Let's show that there exists $c \in]0,2[$ such that $f(2) - f(0) = 2f'(c)$.

We apply the finite increment theorem on f in $]0,2[$.

$*f$ is continuous on $\mathbb{R} - \{1\}$, in particular on $]0,1[\cup]1,2[$.

$\lim\limits_{x \overset{>}{\to} 1} f(x) = \lim\limits_{x \overset{>}{\to} 0} \frac{1}{x} = 1$.

$\underset{x \overset{<}{\to} 1}{lim} f(x) = \underset{x \overset{<}{\to} 1}{lim} \left(\frac{3-x^2}{2}\right) = 1.$

So f is continuous at $x_0 = 1$, so it is continuous on $]0,2[$.

$*f$ is derivable on $\mathbb{R} - \{1\}$, in particular on $]0,1[\cup]1,2[$.

$\underset{x \overset{>}{\to} x_0}{lim} \frac{f(x)-f(x_0)}{x-x_0} = \underset{x \overset{>}{\to} 1}{lim} \frac{\frac{1}{x}-1}{x-1} = \underset{x \overset{>}{\to} 1}{lim} \frac{1-x}{x-1} \times \frac{1}{x} = \underset{x \overset{>}{\to} 1}{lim} \frac{-1}{x} = -1 = f'_+(1).$

$\underset{x \overset{<}{\to} x_0}{lim} \frac{f(x)-f(x_0)}{x-x_0} = \underset{x \overset{<}{\to} 1}{lim} \frac{\frac{3-x^2}{2}-1}{x-1} = \underset{x \overset{<}{\to} 0}{lim} \frac{1-x^2}{2(x-1)} = \underset{x \overset{<}{\to} 0}{lim} \frac{1+x}{-2} = -1 = f'_-(1).$

Then f is derivable at $x_0 = 1$, in particular on $]0,2[$.

According to the finite increment theorem, $\exists c \in]0,2[$ such that $f(2) - f(0) = 2f'(c)$.

Exercise 5:

a) Using Mac-Laurin's formula for the function $x \to \ln(1+x)$.

Calculate $\lim\limits_{n\to+\infty} \left(1 - \frac{1}{2} + \frac{1}{3} - ..... + \frac{(-1)^{n+1}}{n}\right)$.

b) Show that if $x > 0$ so $x - \frac{x^2}{2} < \ln(1+x) < x$.

c) Find the limit of the sequence (U_n) defined by $U_n = \prod_{p=1}^{n} \left(1 + \frac{p}{n^2}\right), \forall n \in \mathbb{N}^*$.

Solution:

a) Using Mac-Laurin's formula for the function $x \to \ln(1+x)$.

calculate $\lim\limits_{n\to+\infty} \left(1 - \frac{1}{2} + \frac{1}{3} - ..... + \frac{(-1)^{n+1}}{n}\right)$.

* Using Mac-Laurin's formula for the function $x \mapsto \ln(1+x)$.

$f(x) = \ln(1+x) \in C^\infty(]-1,+\infty[)$, in particular on $C^{n+1}([0,1])$.

$f(1) = f(0) + \frac{f'(0)}{1!}(1) + \frac{f"(0)}{2!}(1)^2 + ... + \frac{f^{(n)}(0)}{n!}(1)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(1)^{n+1}.$

We have: $f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n}$.

$ln2 = 0 + \frac{1}{1!}(1) - \frac{1}{2!}(1)^2 + ... + \frac{(-1)^{n+1}}{n}(1)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(1)^{n+1}$

$ln2 = 1 - \frac{1}{2} + \frac{1}{3} - ... + \frac{(-1)^{n+1}}{n} + \frac{f^{(n+1)}(c)}{(n+1)!}.$

$\underset{n\to+\infty}{lim}\ ln2 = \underset{n\to+\infty}{lim} \left(1 - \frac{1}{2} + \frac{1}{3} - ... + \frac{(-1)^{n+1}}{n} + \frac{f^{(n+1)}(c)}{(n+1)!}\right).$

We have: $\frac{f^{(n+1)}(c)}{(n+1)!} = \frac{(-1)^{n+2}(n-1)!}{(1+c)^n(n+1)!} \to 0$ (because: $(-1)^{n+2}$ is bonded).

So:$ln2 = \underset{n\to+\infty}{lim} \left(1 - \frac{1}{2} + \frac{1}{3} - ..... + \frac{(-1)^{n+1}}{n}\right).$

b) Show that if $x>0$ So $x-\frac{x^2}{2}<\ln(1+x)<x$.

$$\begin{cases} \exists c_1\in\,]0,x[\,: f(x)=f(0)+\frac{f'(0)}{1!}(x)+\frac{f"(c_1)}{2!}(x)^2 \\ \\ \exists c_2\in\,]0,x[\,: f(x)=f(0)+\frac{f'(0)}{1!}(x)+\frac{f"(0)}{2!}(x)^2+\frac{f"(c_2)}{3!}(x)^3. \end{cases}$$

$$\begin{cases} \exists c_1\in\,]0,x[\,: \ln(1+x)=0+x-\frac{1}{2(1+c_1)^2}x^2 \\ \\ \exists c_2\in\,]0,x[\,: \ln(1+x)=0+x-\frac{1}{2}x^2+\frac{1}{3(1+c_2)^3}x^3. \end{cases}$$

$$\begin{cases} \exists c_1>0: -\frac{1}{2(1+c_1)^2}x^2<0\Rightarrow 0+x-\frac{1}{2(1+c_1)^2}x^2<x \\ \\ \exists c_2>0: \frac{1}{3(1+c_2)^3}x^3>0\Rightarrow 0+x-\frac{1}{2}x^2+\frac{1}{3(1+c_2)^3}x^3>x-\frac{1}{2}x^2. \end{cases}$$

$$\begin{cases} \exists c_1>0: -\frac{1}{2(1+c_1)^2}x^2<0\Rightarrow \ln(1+x)<x \\ \exists c_2>0: \frac{1}{3(1+c_2)^3}x^3>0\Rightarrow \ln(1+x)>x-\frac{1}{2}x^2. \end{cases}$$

Therefore: if $x>0$ so $x-\frac{x^2}{2}<\ln(1+x)<x$.

c) We calculate the limit of the sequence (U_n) defined by $U_n=\prod_{p=1}^{n}\left(1+\frac{p}{n^2}\right),\forall n\in\mathbb{N}^*$

We take: $V_n=\ln\left(\prod_{p=1}^{n}\left(1+\frac{p}{n^2}\right)\right)=\sum_{p=1}^{n}\ln\left(1+\frac{p}{n^2}\right)$.

From (b) :

$$\Rightarrow \frac{p}{n^2}-\frac{p^2}{2n^4}<\ln\left(1+\frac{p}{n^2}\right)<\frac{p}{n^2}$$

$$\Rightarrow \frac{1}{n^2}\sum_{p=1}^{n}p-\frac{1}{2n^4}\sum_{p=1}^{n}p^2<\sum_{p=1}^{n}\ln\left(1+\frac{p}{n^2}\right)<\frac{1}{n^2}\sum_{p=1}^{n}p$$

$$\sum_{p=1}^{n}p=\frac{n(n+1)}{2},\sum_{p=1}^{n}p^2=\frac{n(n+1)(2n+1)}{6}$$

$$\Rightarrow \frac{1}{n^2}\frac{n(n+1)}{2}-\frac{1}{2n^4}\frac{n(n+1)(2n+1)}{6}<\sum_{p=1}^{n}\ln\left(1+\frac{p}{n^2}\right)<\frac{1}{n^2}\frac{n(n+1)}{2}$$

$$\frac{1}{2}<\lim_{n\to+\infty}\sum_{p=1}^{n}\ln\left(1+\frac{p}{n^2}\right)<\frac{1}{2}.$$

$$Then: \lim_{n\to+\infty}V_n=\frac{1}{2}\Rightarrow\lim_{n\to+\infty}U_n=e^{\frac{1}{2}};$$

Chapter 6

Elementary functions and their reciprocals

6.1 Logarithm Function

Definition 47

The Neperien logarithm function is given with the following notation log : the function defined on $]0,+\infty[$ by

$$\log x = \int_0^x \frac{dt}{t}$$

this is equivalent to

$$(\log x)' = \frac{1}{x} \quad \log 1 = 0$$

log is defined on $]0,+\infty[\to \mathbb{R}$ is strictly increasing If $x, y > 0$:

1/ $\log (x \cdot y) = \log x + \log y$

2/ $\log \left(\frac{x}{y}\right) = \log x - \log y$

3/ $\log (x^r) = r \cdot \log x$

4/ $\lim\limits_{x \to +\infty} \log x = +\infty$

5/ $\lim\limits_{x \to 0} \log x = -\lim\limits_{x \to 0} \log \frac{1}{x} = -\infty.$

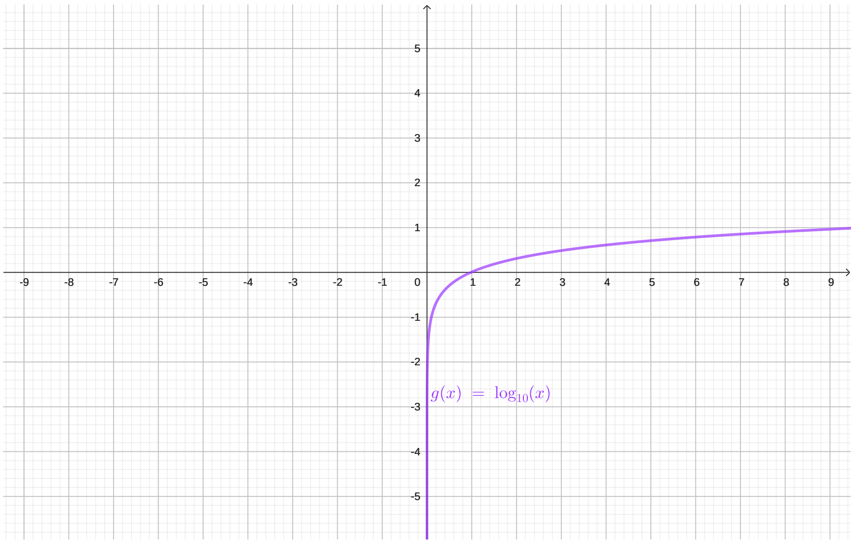


Figure 6.1: log(x)

The Variation table is given by:

x	0		1		e		$+\infty$
ln'(x)	‖			+			
lnx	‖ $-\infty$		0		1		$+\infty$
signe de lnx	‖	-	0		+		

Figure 6.2

6.2 The base "a" Logarithm Function

Definition 48

Let "a" being a positive number different from 1 ($a \neq 1$).

We call the logarithm with base "a" defined on $]0, +\infty[$ denoted by $\log_a$ defined by:

$$\log_a x = \frac{\log x}{\log a},$$

and we have: $\log_a a = 1$ log is defined:$]0, +\infty[\to \mathbb{R}$ and strictly increasing if $a > 1$, and strictly decreasing if $0 < a < 1$

If $x, y > 0$:

1/ $\log_a(x \cdot y) = \log_a x + \log_a y$

2/ $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

3/ $\log_a(x^r) = r \cdot \log_a x.$

6.3 Logarithmic derivative

Let U be a function derivable at x_0 with $U(x_0) \neq 0$.

The logarithmic derivative is the number $\dfrac{U'(x_0)}{U(x_0)}$.

6.4 Exponential Function "e"

The function log Neperien is a bijection (continuous and monotone) of: $]0, +\infty[\to \mathbb{R}$.

We call the reciprocal function denoted (exp) of the the function log Neperien : $]-\infty, +\infty[\to]0, +\infty[$ and we will have:

$$y = \underset{x \in \mathbb{R}}{\exp x} \Leftrightarrow x = \underset{y>0}{\log y}.$$

$\exp x = e^x$ is a strictly increasing continuous function on $\mathbb{R}$ and infinitely differentiable.

Some properties

$\forall x, y \in \mathbb{R} : e^{x+y} = e^x \cdot e^y.$

$\lim\limits_{x \to -\infty} e^x = 0, \lim\limits_{x \to +\infty} e^x = +\infty$

the varition table is given by:

x	$-\infty$		0		$+\infty$
$\exp'(x)$		$+$		$+$	
$\exp(x)$	0	↗	1	↗	$+\infty$

Figure 6.3

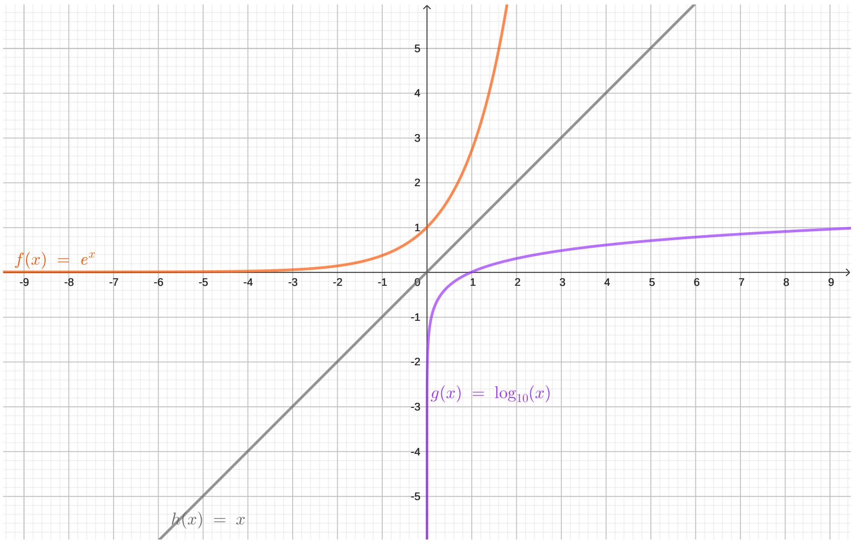


Figure 6.4: log and exp

6.4.1 The base "a" $(a > 0)$ Exponential Function,

Definition 49

$$\exp_a x = a^* = \exp(\log a^*) = \exp(x \log a)\ (a > 0, x \in \mathbb{R})$$

*If $a \neq 1$:the exponential is strictly increasing.

*If $a > 1$: the exponential is strictly decreasing.

*If $0 < a < 1$: it realizes a bijection of $]-\infty, +\infty[\to]0, +\infty[$.

* The function $\exp_a x$ is only the reciprocal of the function and $\log_a x$ we get:

$$y = a^x = \exp(x \log a) \Leftrightarrow \log_a y = \frac{\log y}{\log a} = \frac{x \log a}{\log a} = x$$
$$y = \underset{x\in\mathbb{R}}{a^x} = \Leftrightarrow \underset{y>0}{\log_a y} = x$$

The function a^* is continuous on $\mathbb{R}$, monotonic and indefinitely differentiable.

1/ $\forall x, y \in \mathbb{R} : (a^x)^y = \left(e^{x \log a}\right)^y = \log a \cdot a^*$

2/ $\forall x, y \in \mathbb{R} : a^{x+y} = a^x \cdot a^y$

3/ $\forall x, y \in \mathbb{R} : (a^x)^y = a^{x \cdot y}$

4/ $\forall a, b \in \mathbb{R} : (a \cdot b)^x == a^x \cdot b^x$

6.5 Power functions

Definition 50

$\forall n \in \mathbb{N}$, the function $x \mapsto x^n$ defined on $\mathbb{R}$ is monotone if "n" is odd, continuous and derivable.

The function $x \mapsto x^n$ realizes a bijection of $\mathbb{R} \to \mathbb{R}$ therefore, it admits a relation noted:

$$x^{\frac{1}{n}} = \sqrt[n]{x}.$$

"n" is odd: $y = \underset{x\in\mathbb{R}}{x^n} \Leftrightarrow x = \underset{y\in\mathbb{R}}{\sqrt[n]{y}}$ is constant, monotonic and derivable.

$$\forall x > 0 : (x^n)' = \left(e^{n\log x}\right)' = \frac{n}{x}e^{n\log x} = \frac{n}{x}x^n = nx^{n-1}$$

6.6 Circular functions

6.6.1 cosinus and sinus functions

The essential relation is

$$\forall x \in \mathbb{R} : \cos^2 x + \sin^2 x = 1$$

	Definition domain	Parity	Period	continuity	derivability	derivative
$sinx$	$\mathbb{R}$	odd	2π	$\mathbb{R}$	$\mathbb{R}$	$\cos x$
$cosx$	$\mathbb{R}$	even	2π	$\mathbb{R}$	$\mathbb{R}$	$-\sin x$

6.6.2 The Variation table

cosinus and sinus functions are each continuous and derivable on $\mathbb{R}$, periodic.

The two functions are studied on a period $T = 2\pi$.

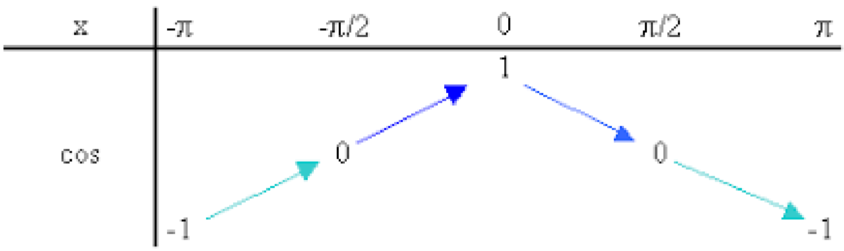


Figure 6.5: Varition Table of cos

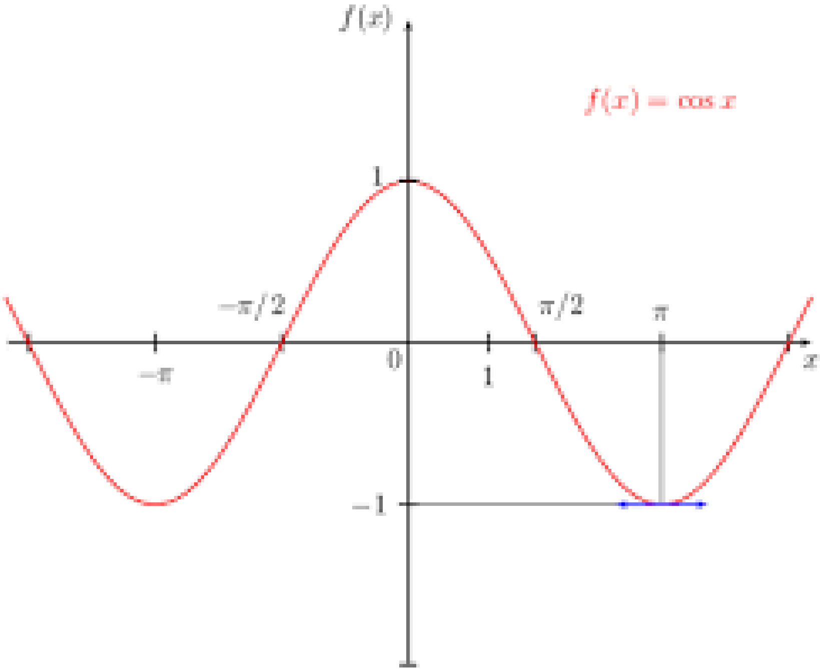


Figure 6.6: cos

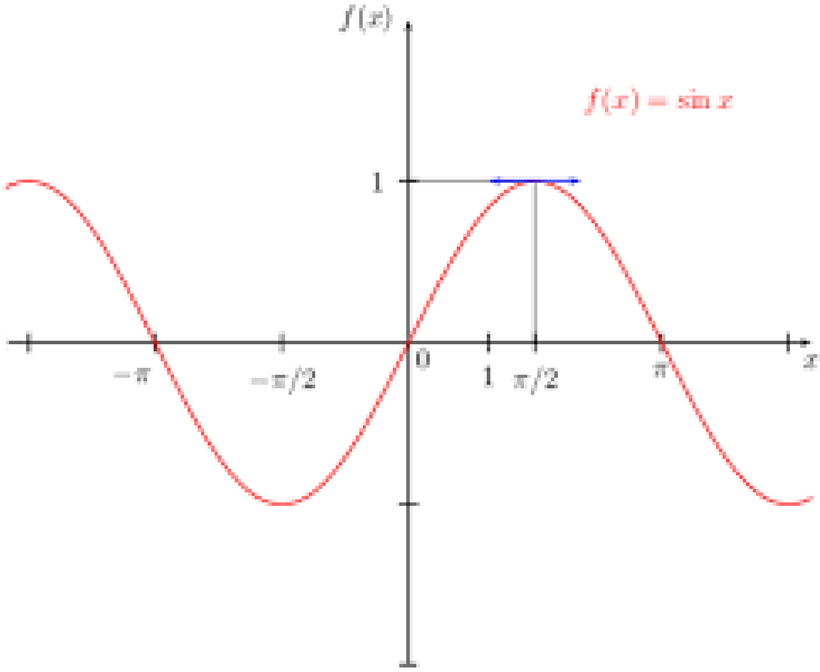


Figure 6.7: sin x

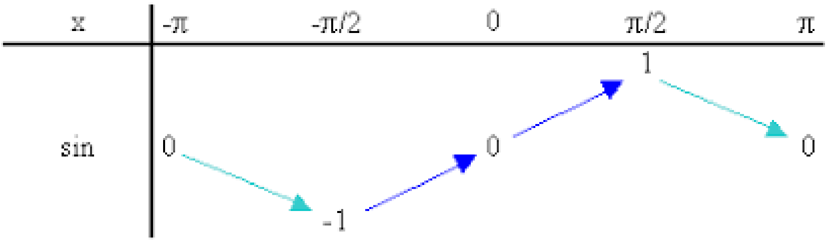


Figure 6.8: Varition Table of sin

6.6.3 Tangent and cotangent functions

Definition 51

*The tangent function is defined by:

$$\forall x \in \mathbb{R} \backslash \left\{ \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \right\}$$

$$x \mapsto tgx = \frac{\sin x}{\cos x}$$

*The cotangent function is defined by:

$$\forall x \in \mathbb{R} \backslash \left\{ k\pi, k \in \mathbb{Z} \right\}$$

$$x \mapsto \cot gx = \frac{\cos x}{\sin x}$$

*We have:

$$\forall x \in \mathbb{R} \backslash \left\{ \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \right\} \cup \left\{ k\pi, k \in \mathbb{Z} \right\}$$
$$x \mapsto \cot gx = \frac{1}{tgx}$$

Some relations.

1/ $\cos^2 x + \sin^2 x = 1$.

2/ $\cos^2 x = \frac{1}{1+tg^2 x}$.

3/ $\sin^2 x = \frac{tg^2 x}{1+tg^2 x}$.

4/ $tg\left(a+b\right) = \frac{tga+tgb}{1-tga \cdot tgb}$.

5/ $tg\left(a-b\right) = \frac{tga-tgb}{1+tga \cdot tgb}$.

6/ If $a = b = x$: $\begin{cases} tg2x = \frac{2tgx}{1-tg^2 x} \\ \sin 2x = \frac{2tgx}{1+tg^2 x} \\ \cos 2x = \frac{1-tg^2 x}{1+tg^2 x} \end{cases}$

The tangent and cotangent functions each continuous and derivable on its domain of definition:

$\forall x \in \mathbb{R} \backslash \left\{ \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \right\}$

$(tgx)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \cdot \cos x - (-\sin x) \cos x}{\cos^2 x} = \frac{1}{\cos^2 x} = 1 + tg^2 x$

$\forall x \in D_f (tg) : (tgx)' > 0.$

*The tangent function is strictly increasing:

$\forall x \in \mathbb{R} \backslash \{k\pi, k \in \mathbb{Z}\}.$

$(\cot gx)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{(-\sin x) \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\left(1 + \cot g^2 x\right).$

$\forall x \in D_f (\cot g) : (tgx)' < 0.$

*The cotangent function is strictly decreasing:

*tg is a periodic function of period $\pi, 2\pi$ we see its variations on $\left] \frac{-\pi}{2}, \frac{\pi}{2} \right[.$

*cotg is a periodic function of period $\pi, 2\pi$ we see its variations on $]0, \pi[.$

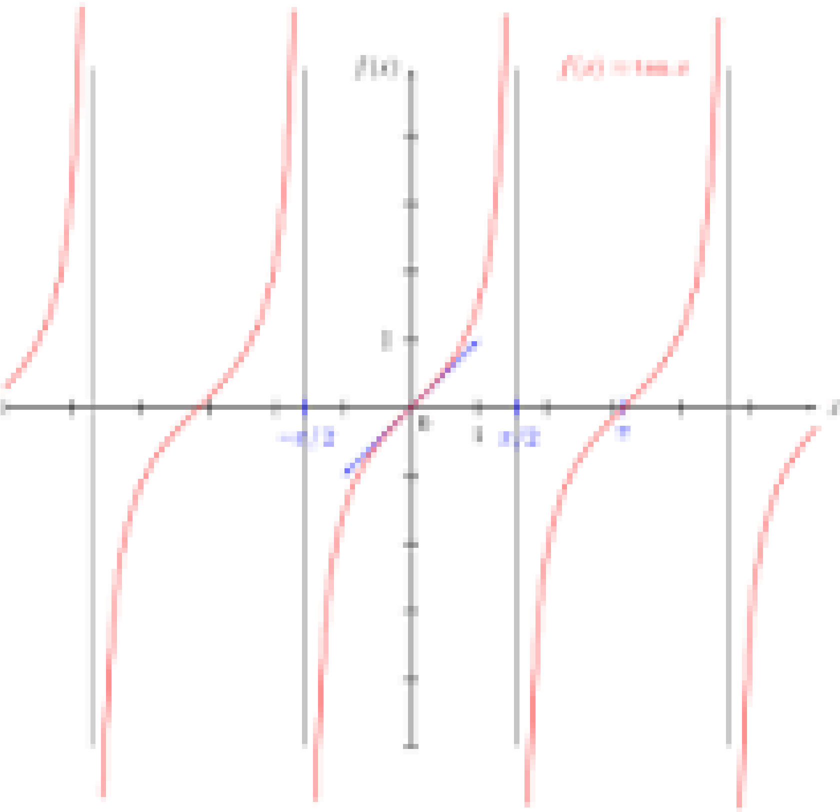


Figure 6.9: tgx

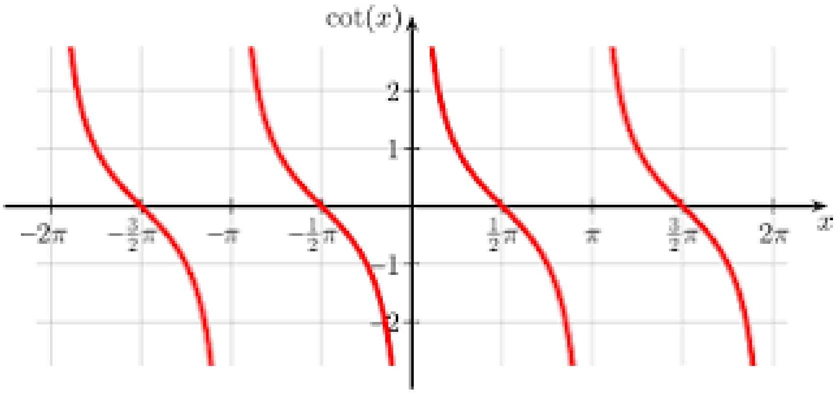


Figure 6.10: cotg

6.7 Reciprocal circular functions

6.7.1 Arcsinus Function

Definition 52

$f(x) = sin_{\left[\frac{-\pi}{2},\frac{\pi}{2}\right]}(x)$ is the restriction of the sinus function on $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ it is continuous, strictly increasing, it realizes a bijection of $\left[\frac{-\pi}{2},\frac{\pi}{2}\right] \to [-1,1]$ and it then admits a reciprocal function

$$f^{-1} : Arc\sin : [-1,1] \to \left[\frac{-\pi}{2},\frac{\pi}{2}\right]$$

$$x \mapsto y = Arc\sin x$$

We get:

$$\underset{-1\leq x\leq 1}{Arc\sin x} \Leftrightarrow x = \underset{\frac{-\pi}{2}\leq y\leq\frac{\pi}{2}}{\sin y}$$

$Arc\sin$ is continuous on $[-1,1]$, strictly increasing and indefinitely derivable, in fact:

$$(\sin y)' = \cos y > 0 \text{ sur } \left[\frac{-\pi}{2},\frac{\pi}{2}\right]$$

$$(Arc\sin x)' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

We have: $\cos^2 y + \sin^2 y = 1 \Leftrightarrow \cos^2 y = 1 - \sin^2 y$

$$\Leftrightarrow \cos y = \sqrt{1-\sin^2 y} \Leftrightarrow \cos y = \sqrt{1-x^2}$$

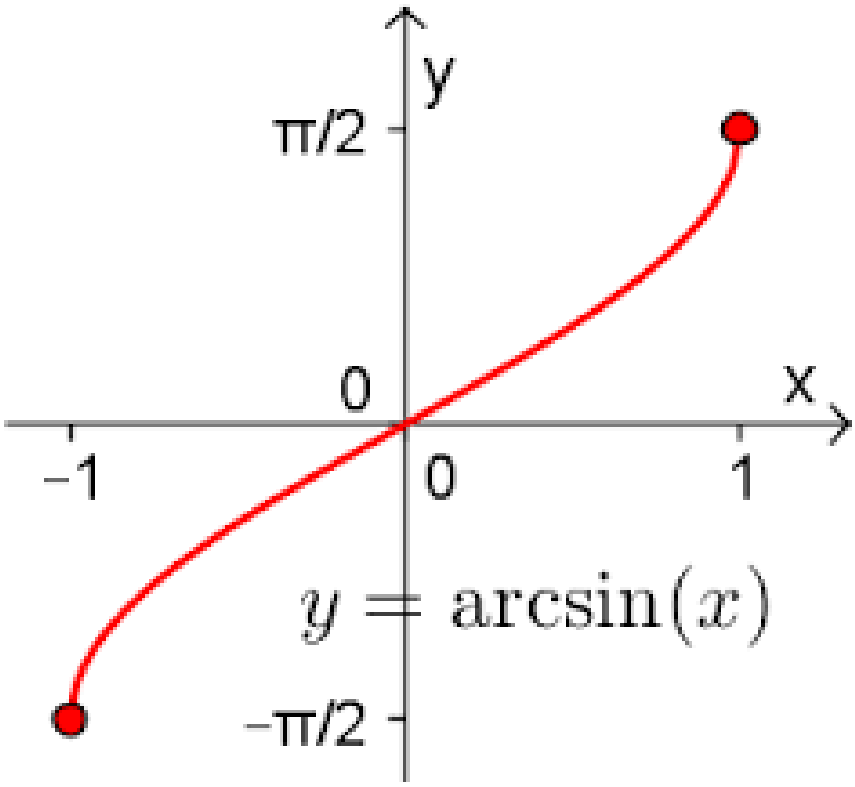


Figure 6.11: arcsin

6.7.2 Arccos Function

Definition 53

The function $f(x) = cos_{[0,\pi]}(x)$ is the restriction of the function cos on $[0, \pi]$, it is continuous, strictly decreasing, so it realizes a bijection of $[0, \pi] \to [-1, 1]$ and consequently it admits a relation $Arc\cos x$

$$f^{-1} : Arc\cos : [-1, 1] \to [0, \pi]$$

$$x \mapsto y = Arc\cos x$$

We get:

$$\underset{-1\leq x\leq 1}{Arc\cos x} \Leftrightarrow x = \underset{0\leq y\leq \pi}{\cos y}.$$

$Arsin$ is continuous on $[-1, 1]$, strictly increasing and indefinitely derivable, in fact:

$(\cos y)' = -\sin y < 0$ on $[0, \pi]$

$(Arc\cos x)' = \dfrac{1}{(\cos x)'}(y) = \dfrac{1}{-\sin y} = \frac{-1}{\sqrt{1-x^2}}$

We have: $\cos^2 y + \sin^2 y = 1 \Leftrightarrow \sin^2 y = 1 - \cos^2 y$

$\Leftrightarrow \sin y = -\sqrt{1-\cos^2 y} \Leftrightarrow \sin y = -\sqrt{1-x^2}$

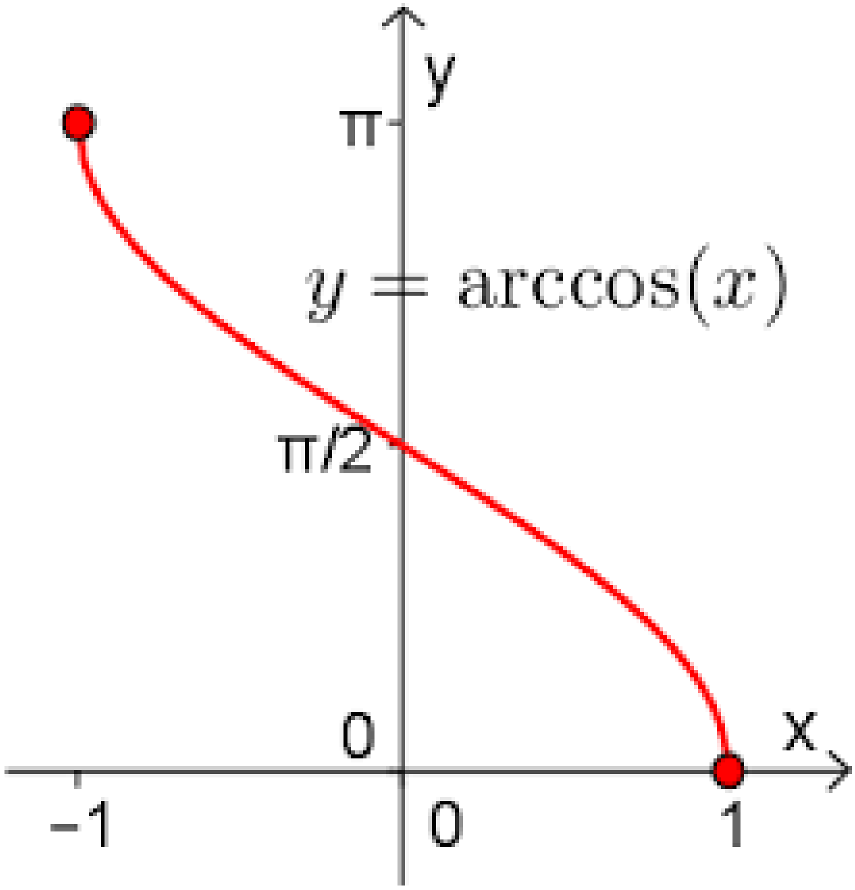


Figure 6.12: arccos

6.7.3 Arctg Function

Definition 54

$f(x) = tg_{\left]\frac{-\pi}{2}, \frac{\pi}{2}\right[}(x)$ is the restriction of the sine function sinus on $\left]\frac{-\pi}{2}, \frac{\pi}{2}\right[$ it is

continuous, strictly increasing, it realizes a bijection of $\left]\frac{-\pi}{2}, \frac{\pi}{2}\right[\to \mathbb{R}$ and it then admits a reciprocal function.

$$f^{-1}: Arctg : \mathbb{R} \to \left]\frac{-\pi}{2}, \frac{\pi}{2}\right[$$

$$x \mapsto y = Arctgx$$

$$\underset{x\in\mathbb{R}}{Arctgx} \Leftrightarrow x = \underset{\frac{-\pi}{2}<y<\frac{\pi}{2}}{tgy}$$

$Arctg$ is continuous on $\mathbb{R}$, strictly increasing and indefinitely derivable, in fact:

$$(tgy)' = \frac{1}{\cos^2 y} = 1 + tg^2 y \text{ on } \left]\frac{-\pi}{2}, \frac{\pi}{2}\right[$$

$$(Arctgx)' = \frac{1}{1+tg^2 y} = \frac{1}{1+x^2}$$

We have: $\cos^2 y + \sin^2 y = 1 \Leftrightarrow \cos^2 y = 1 - \sin^2 y$

$\Leftrightarrow \cos y = \sqrt{1 - \sin^2 y} \Leftrightarrow \cos y = \sqrt{1 - x^2}$

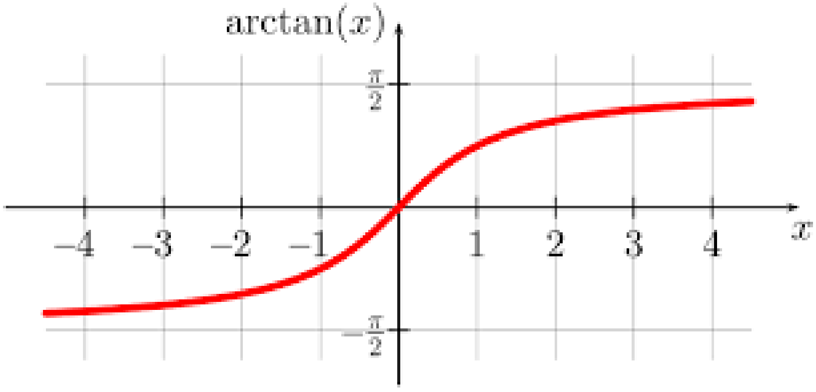


Figure 6.13: arctg

6.7.4 Arccotg Function

Definition 55

The function $f(x) = \cot g_{]0,\pi[}(x)$ is the restriction of the function $\cot g$ on $]0, \pi[$, it is

continuous, strictly decreasing, so it realizes a bijection of $]0, \pi[\to \mathbb{R}$ and

consequently it admits a reciprocal function $Arc\cot gx$

$$f^{-1}: Arc\cot g : \mathbb{R} \to]0, \pi[$$

$$x \mapsto y = Arc\cot gx.$$

we get:

$$\underset{x\in\mathbb{R}}{Arc\cot gx} \Leftrightarrow x = \underset{0<y<\pi}{\cot gy}$$

Arcsin is continuous on $\mathbb{R}$, strictly increasing and indefinitely derivable, in fact:

$$(\cot gy)' = \frac{-1}{\sin^2 y} = -\left(1+\cot g^2 y\right)$$

$$(Arc\cot gx)' = \frac{-1}{1+\cot g^2 y} = \frac{-1}{1+x^2} < 0.$$

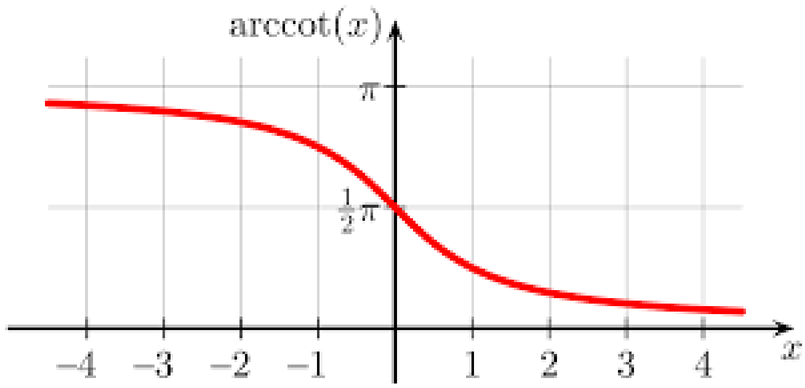


Figure 6.14: arccotg

Some relations

$*\sin\left(Arc\sin x\right) = x$

$*\cos\left(Arc\cos x\right) = x$

$*tg\left(Arctgx\right) = x$

$\cos\left(Arc\sin x\right) = \sqrt{1-x^2}$ because

$\cos y = \sqrt{1-\sin^2 y} \Leftrightarrow \cos y = \sqrt{1-x^2}$ on $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$\cos\left(Arc\sin x\right) = \sqrt{1-\sin^2\left(Arc\sin x\right)} \Leftrightarrow \cos\left(Arc\sin x\right) = \sqrt{1-x^2}.$

$*\sin\left(Arc\cos x\right) = \sqrt{1-x^2}.$

$*tg\left(Arc\sin x\right) = \frac{\sin(Arc\sin x)}{\cos(Arc\sin x)} = \frac{x}{\sqrt{1-x^2}}.$

6.8 Hyperbolic functions and their inverses

6.8.1 Hyperbolic Sinus

Definition 56

The hyperbolic sinus noted shx is the function defined by

$$shx = \frac{1}{2}\left(e^{x} - e^{-x}\right) \;\; / \; \forall x \in \mathbb{R}$$

The variation Table is given by

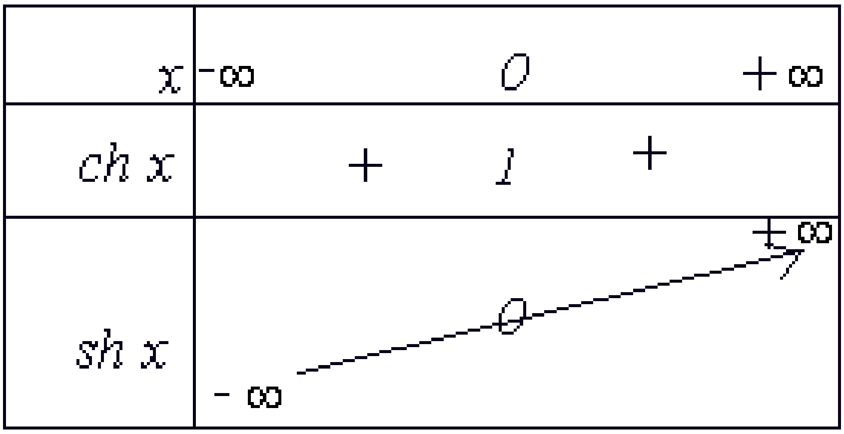


Figure 6.15

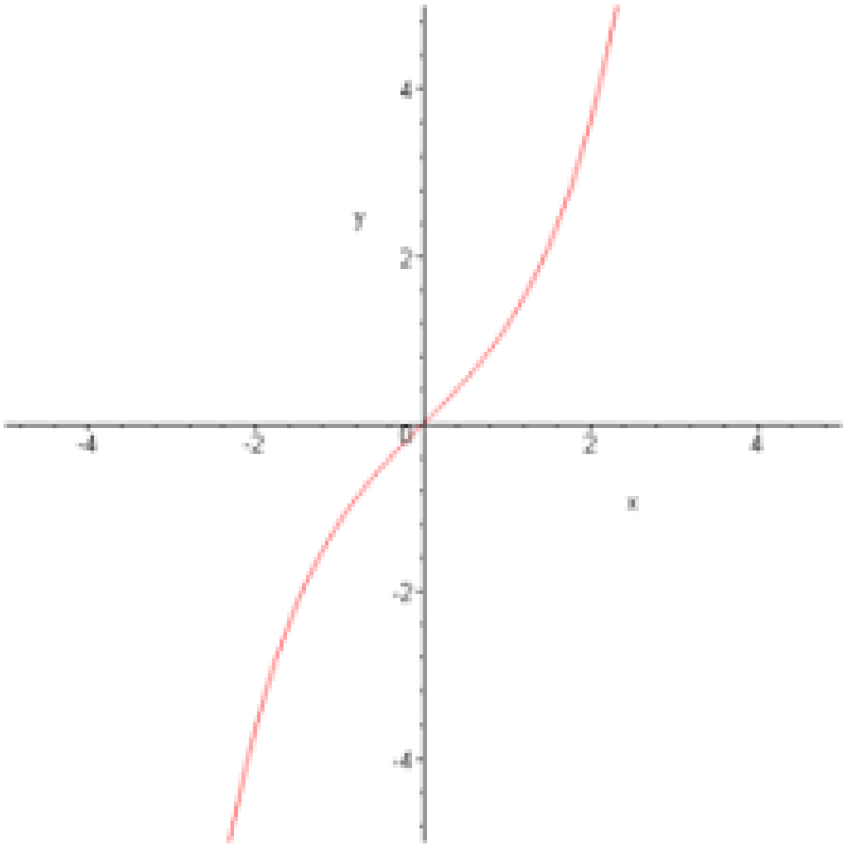


Figure 6.16: shx

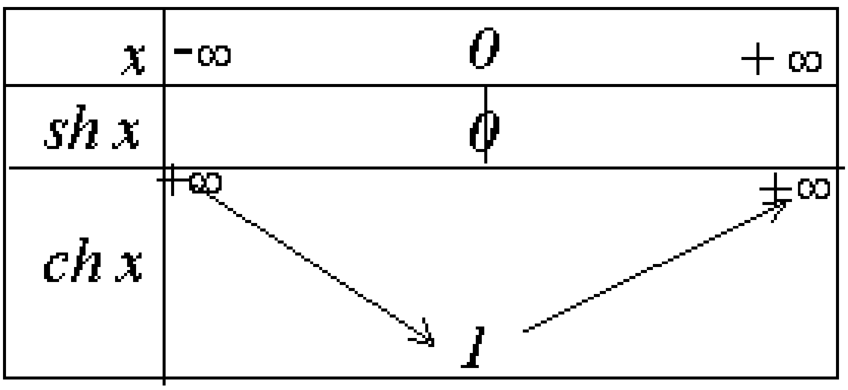
6.8.2 Hyperbolic Cosinus

Definition 57

The hyperbolic Cosinus noted chx is the function defined by

$$chx = \frac{1}{2}\left(e^{x} + e^{-x}\right) \;\; / \; \forall x \in \mathbb{R}$$

The variation Table is given by



x	$-\infty$	0	$+\infty$
$sh\,x$		0	
$ch\,x$	$+\infty$ ↘	1	↗ $+\infty$

Figure 6.17

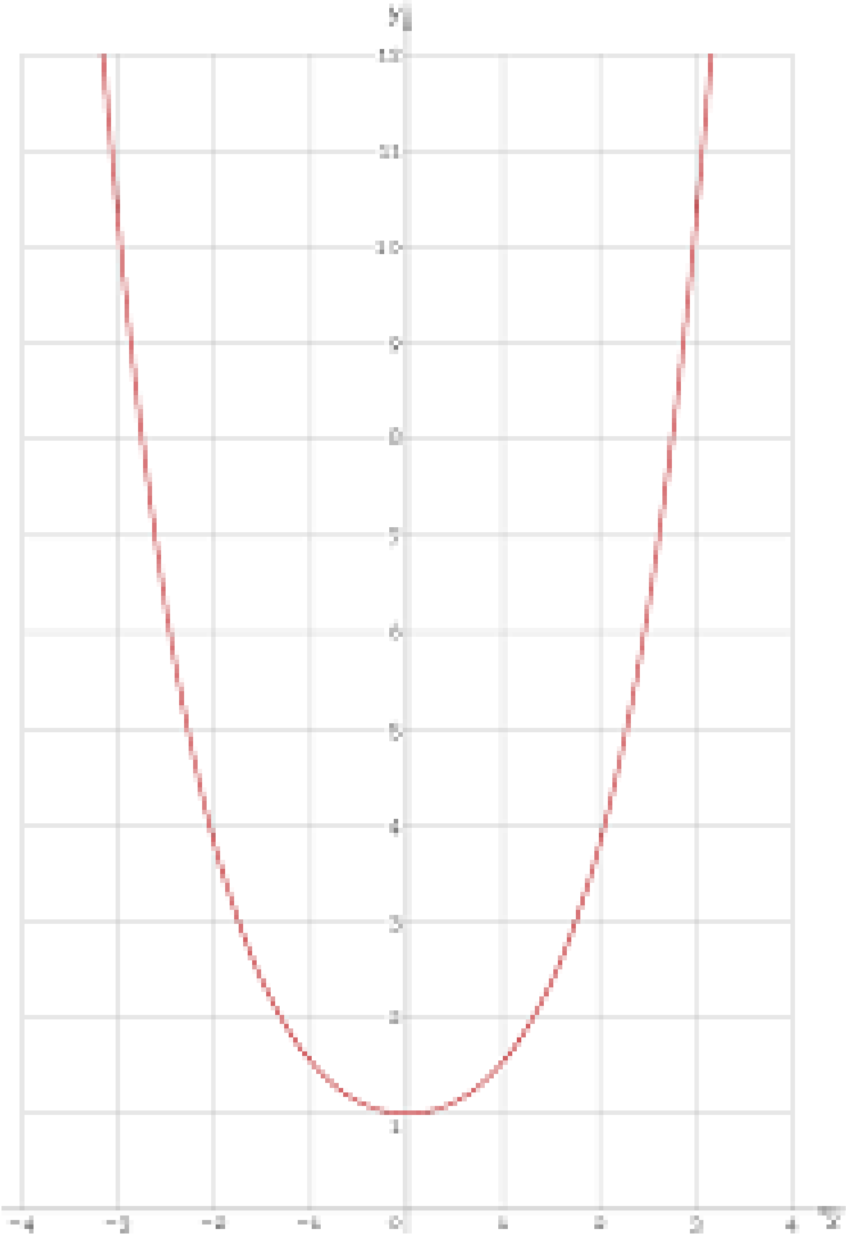


Figure 6.18: chx

$\forall x \in \mathbb{R}$

$e^x = chx + shx$

$e^{-x} = chx - shx$

$e^x \cdot e^{-x} = (chx + shx)\,(chx - shx)$

$1 = ch^2x - sh^2x$

Some properties

$\forall x \in \mathbb{R}$

$*sh\,(0) = 0 \qquad ch\,(0) = 1$

$*\ sh\,(-x) = -shx \qquad ch\,(-x) = chx$

$*shx$ is a continuous odd function on $\mathbb{R}$

$*chx$ is a continuous even function on $\mathbb{R}$

$*\forall x \in \mathbb{R} : (shx)' = chx \qquad (chx)' = shx$

$*sh\,(x+y) = \frac{1}{2}\left(e^{x+y} - e^{-(x+y)}\right)$

$*sh\,(x+y) = shx \cdot chy + chx \cdot shy$

$*ch\,(x+y) = chx \cdot chy + shx \cdot shy$

If $x = y$

$*sh\,(2x) = 2shx \cdot chy$

$*ch\,(2x) = ch^2x + sh^2x$

The limits

$*\lim\limits_{x\to-\infty} chx = \lim\limits_{x\to-\infty} \left(\frac{1}{2}\left(e^x + e^{-x}\right)\right) = +\infty.$

$*\lim\limits_{x\to+\infty} chx = \lim\limits_{x\to+\infty} \left(\frac{1}{2}\left(e^x + e^{-x}\right)\right) = +\infty.$

$*\lim\limits_{x\to-\infty} shx = \lim\limits_{x\to-\infty} \left(\frac{1}{2}\left(e^x - e^{-x}\right)\right) = -\infty.$

$*\lim\limits_{x\to+\infty} shx = \lim\limits_{x\to+\infty} \left(\frac{1}{2}\left(e^x - e^{-x}\right)\right) = +\infty$

6.8.3 Hyperbolic Tangent

Definition 58

The hyperbolic tangent denoted thx is the function defined by

$$thx = \frac{shx}{chx} = \frac{\dfrac{1}{2}\left(e^x - e^{-x}\right)}{\dfrac{1}{2}\left(e^x + e^{-x}\right)} \ / \ \forall x \in \mathbb{R}$$

$$thx = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1 - e^{-2x}}{1 + e^{-2x}}$$

The variation Table is giveb by

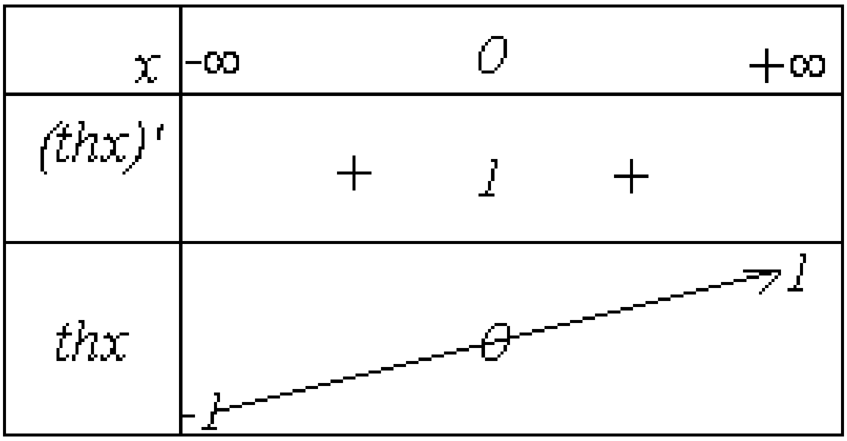


Figure 6.19

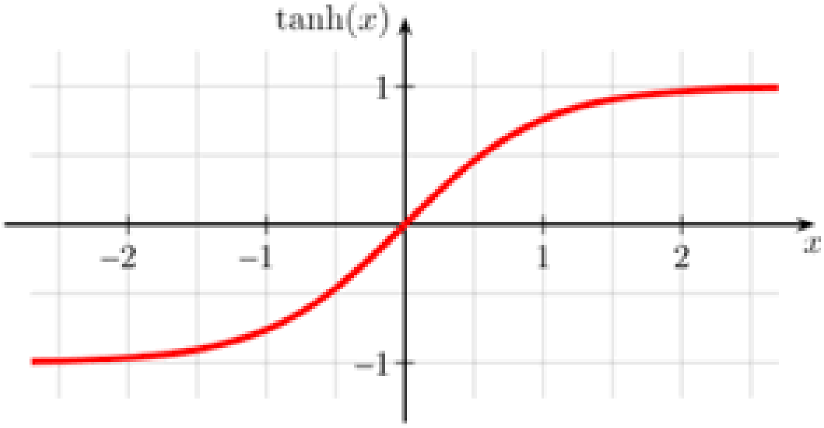


Figure 6.20: thx

The limits

$* \lim_{x \to -\infty} thx = -1 \qquad * \lim_{x \to +\infty} thx = 1$

$*th\,(0) = 0$

$*(thx)' = \left(\frac{shx}{chx}\right)' = \frac{ch^2x - sh^2x}{ch^2x} = \frac{1}{ch^2x} = 1 - th^2x$

$*thx$ is an odd function, continuous and strictly increasing.

6.8.4 Hyperbolic Cotangent

Definition 59

The hyperbolic Cotangent $\cot hx$ is the function defined by

$$\cot hx = \frac{chx}{shx} = \frac{\frac{1}{2}\left(e^{x}+e^{-x}\right)}{\frac{1}{2}\left(e^{x}-e^{-x}\right)} \ / \ \forall x \in \mathbb{R}$$

$$\cot hx = \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}} = \frac{e^{2x}+1}{e^{2x}-1} = \frac{1+e^{-2x}}{1-e^{-2x}}$$

The variation Table is giveb by

x	0		$+\infty$
coth'x	$-\infty$	$-$	0^{-}
coth x	$+\infty$	$\searrow$	1

Figure 6.21

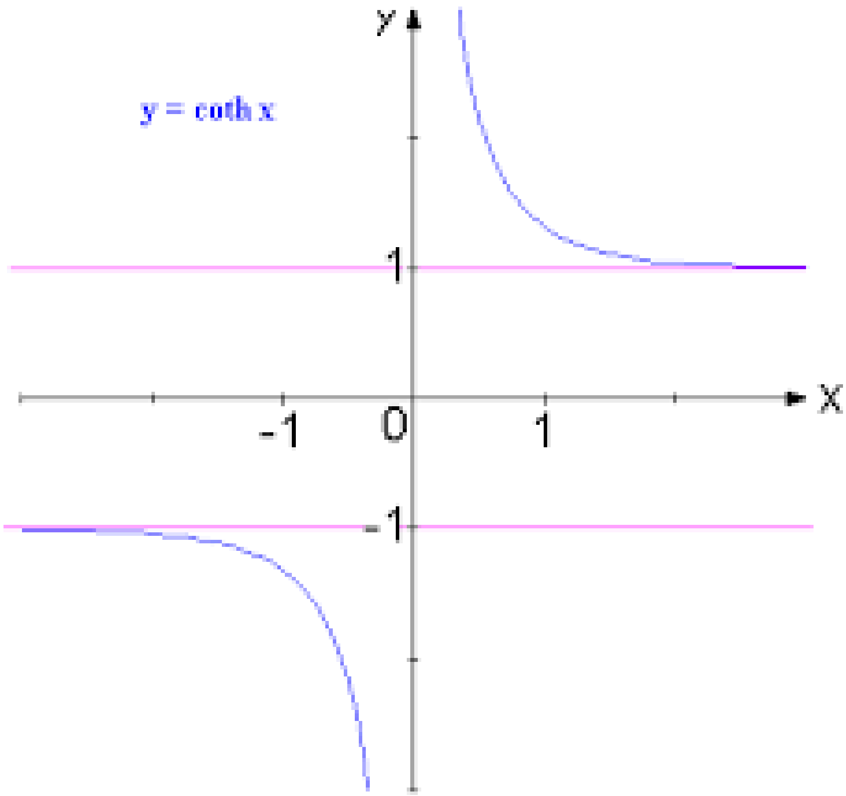


Figure 6.22: cothx

The limits

* $\lim\limits_{x\to-\infty} \cot hx = \lim\limits_{x\to-\infty} \dfrac{e^{2x}+1}{e^{2x}-1} = -1$

* $\lim\limits_{x\to+\infty} \cot hx = \lim\limits_{x\to+\infty} \dfrac{1+e^{-2x}}{1-e^{-2x}} = 1$

* $\lim\limits_{x\to0^-} \cot hx = \lim\limits_{x\to0^-} \frac{1+e^{-2x}}{1-e^{-2x}} = -\infty$

* $\lim\limits_{x\to0^+} \cot hx = \lim\limits_{x\to0^+} \dfrac{1+e^{-2x}}{1-e^{-2x}} = +\infty$

*$(\cot hx)' = \left(\dfrac{chx}{shx}\right)' = \dfrac{sh^2x - ch^2x}{sh^2x} = \dfrac{-1}{sh^2x} = 1 - \cot h^2x$

*$\cot hx$ is an even function, continuous and strictly decreasing.

6.8.5 Reciprocal hyperbolic functions

6.8.5.1 Hyperbolic sinus-argument function

Definition 60

The hyperbolic *sinus* function is continuous and strictly increasing on $\mathbb{R}$, it realizes a bijection of $\mathbb{R} \to \mathbb{R}$, it then admits an *Argsh*-Reciprocal and we will have:

$$y = \underset{x\in\mathbb{R}}{shx} \Leftrightarrow x = \underset{x\in\mathbb{R}}{Argshy}$$

This function can be expressed as follows

$$ch^2x = 1 + sh^2x \qquad chx > 0$$

$$chx = \sqrt{1+sh^2x} \Leftrightarrow chx = \sqrt{1+y^2}$$

$$e^x = chx + shx \Leftrightarrow e^x = \sqrt{1+y^2} + y$$

$$x = \ln\left(\sqrt{1+y^2} + y\right)$$

Argsh is continuous, strictly increasing and indefinitely derivable

$$(Argshy)' = \frac{1}{chx} = \frac{1}{\sqrt{1+sh^2x}} = \frac{1}{\sqrt{1+y^2}}$$

$$(Argshy)' = \left(\ln\left(\sqrt{1+y^2}+y\right)\right)' = \frac{1 + \dfrac{2y}{2\sqrt{1+y^2}}}{\sqrt{1+y^2}+y} = \frac{1}{\sqrt{1+y^2}}$$

6.8.5.2 Hyperbolic cosinus-argument function

Definition 61

The hyperbolic *cosinus* function is continuous and strictly increasing on $[0, +\infty[$, it realizes a bijection of $[0, +\infty[\to [1, +\infty[$, it then admits $Argch$-reciprocal and we will have:

$$y = \underset{x \geq 0}{chx} \Leftrightarrow x = \underset{y \geq 1}{Argchy}$$

This function can be expressed as follows

$$sh^2x = ch^2x - 1 \qquad shx > 0$$

$$shx = \sqrt{ch^2x - 1} \Leftrightarrow shx = \sqrt{y^2 - 1}$$

$$e^x = chx + shx \Leftrightarrow e^x = x + \sqrt{y^2 - 1}$$

$$x = \ln\left(y + \sqrt{y^2 - 1}\right)$$

$Argch$ is continuous, strictly increasing on $[1, +\infty[$ and indefinitely derivable

$$(Argchy)' = \frac{1}{shx} = \frac{1}{\sqrt{ch^2x - 1}} = \frac{1}{\sqrt{y^2 - 1}} > 0 \quad \forall y > 1$$

$$(Argchy)' = \left(\ln\left(\sqrt{y^2 - 1} + y\right)\right)' = \frac{1 + \dfrac{2y}{2\sqrt{y^2 - 1}}}{\sqrt{x^2 - 1} + x} = \frac{1}{\sqrt{y^2 - 1}}$$

6.8.5.3 Hyperbolic tangent-argument function

Definition 62

The tangente hyperbolic function is continuous and strictly increasing on$\mathbb{R}$, it realizes a bijection of $\mathbb{R} \to]-1, 1[$, then admits a reciprocal $Argth$ and we get:

$$y = \underset{x \in \mathbb{R}}{thx} \Leftrightarrow x = \underset{y \in]-1,1[}{Argthy}$$

This function can be expressed as follows

$$y = thx = \frac{e^{2x}-1}{e^{2x}+1} \quad \Leftrightarrow x\left(e^{2x}+1\right) = e^{2x}-1$$

$$e^{2x}(y-1) = (y+1) \Leftrightarrow e^{2x} = \frac{y+1}{y-1}$$

$$2x = \ln\left(\frac{y+1}{y-1}\right) \Leftrightarrow x = \frac{1}{2}\ln\left(\frac{y+1}{y-1}\right)$$

$$x = Argthy = \frac{1}{2}\ln\left(\frac{y+1}{y-1}\right) \qquad |y| < 1$$

$Argth$ is continuous, strictly increasing and indefinitely derivable

$$(Argthy)' = \frac{1}{1-th^2x} = \frac{1}{1-y^2} \qquad\qquad |y| < 1$$

$$(Argthy)' = \left(\frac{1}{2}\ln\left(\frac{y+1}{y-1}\right)\right)' = \frac{1}{2}\frac{\frac{y-1-y-1}{(y-1)^2}}{\frac{y+1}{y-1}} = \frac{1}{1-y^2}$$

6.8.5.4 Hyperbolic cotangent argument function

Definition 63

The cotangente hyperbolic function is continuous and strictly increasing on $\mathbb{R}$, it realizes a bijection of $\mathbb{R} \to]-1,1[$, it then admits a reciprocal $Arg\cot h$ and we get:

$$y = \underset{x\in\mathbb{R}}{\cot hx} \Leftrightarrow x = \underset{y\in]-1,1[}{Arg\cot hy}$$

This function can be expressed as follows

$$y = \cot hx = \frac{e^{2x}+1}{e^{2x}-1} \quad \Leftrightarrow x\left(e^{2x}-1\right) = e^{2x}+1$$

$$e^{2x}(y-1) = (y+1) \Leftrightarrow e^{2x} = \frac{y+1}{y-1}$$

$$2x = \ln\left(\frac{y+1}{y-1}\right) \Leftrightarrow x = \frac{1}{2}\ln\left(\frac{y+1}{y-1}\right)$$

$$x = Argthy = \frac{1}{2}\ln\left(\frac{y+1}{y-1}\right) \qquad |y| < 1$$

6.9 Solved exercises

Exercise 1.

Write in algebraic form
a) $\cos(arctgx)$ b) $\sin(arctgx)$.

c) $\cos(arcsinx)$ d) $\sin(arccosx)$.

e) $tan(arcsinx)$ f) $tan(arccosx)$.
Solution:

a) We start from the relation

$$tg(arctgx) = x$$

true for all real x.

First, we have:

$$cos^2(arctgx) = \frac{1}{1 + tg^2(arctgx)} = \frac{1}{1 + x^2},$$

but $arctgx$ belongs to $]-\frac{\pi}{2}, \frac{\pi}{2}[$, and so $cos(arctgx)$ is positive.

So

$$cos(arctgx) = \frac{1}{\sqrt{1 + x^2}}.$$

b) We write

$$sin(arctgx) = (cos(arctgx))(tg(arctgx)),$$

and using a), we obtain

$$sin(arctgx) = \frac{x}{\sqrt{1 + x^2}}.$$

c) We start from the relation

$$sin(arcsinx) = x,$$

true for all x in the interval $[-1, 1]$. First, we have:

$$cos^2(arcsinx) = 1 - sin^2(arcsinx) = 1 - x^2,$$

but $arcsinx$ belongs to $]-\frac{\pi}{2}, \frac{\pi}{2}[$, and so $cos(arcsinx)$ is positive.

So

$$cos(arcsinx) = \sqrt{1 - x^2}.$$

d) We start from the relation

$$cos(arccosx) = x,$$

true for all x in the interval $[0, \pi]$. First, we have

$$sin^2(arccosx) = 1 - cos^2(arccosx) = 1 - x^2,$$

but $arccosx$ belongs to $[0, \pi]$, and so $sin(arccosx)$ is positive.

So

$$sin(arccosx) = \sqrt{1 - x^2}.$$

e) Using c)

$$tg(arcsinx) = \frac{sin(arcsinx)}{cos(arcsinx)} = \frac{x}{\sqrt{1 - x^2}}.$$

f) Using d)

$$tg(arccosx) = \frac{sin(arccosx)}{cos(arccosx)} = \frac{\sqrt{1 - x^2}}{x}.$$

Exercise 2.

Solve the following equations:
a) $arctg(2x) + arctgx = \frac{\pi}{4}$.

b) $arcsinx + arccos(x) = \frac{\pi}{2}$.

c) $(arcsinx - 5) \arcsin x = -4$.

Solution:

Taking the tangent of the two members, this implies:

$$tg(arctg(2x) + arctgx) = 1,$$

using the formula giving the tangent of a sum

$tg(a + b) = \frac{tg(a)+tg(b)}{1-tg(a)tg(a)}$.

Hence,

$$\frac{2x + x}{1 - 2xx} = 1$$

Finally, we obtain the equation

$$2x^2 + 3x - 1 = 0,$$

which has a unique positive solution

$$x_0 = \frac{3 + \sqrt{17}}{4}.$$

b) For all $x \in [-1,1]$,Let:$\alpha = arcsinx$, $\beta = arccosx$.

By definition: $\beta \in [0,\pi]$,$\alpha \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, So, $\left(\frac{\pi}{2}-\beta\right) \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

We have:

$$sin\alpha = x, sin\left(\frac{\pi}{2}-\beta\right) = \cos\beta = x.$$

Hence:

$$sin\alpha = sin\left(\frac{\pi}{2}-\beta.\right)$$

Or the function *sinus* is a bijective of $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ on $[-1,1]$.

Therefore: $\alpha = \frac{\pi}{2}-\beta$. i.e., $\alpha+\beta = \frac{\pi}{2}$.

And therefore:

$$arcsinx + arccosx = \frac{\pi}{2}$$

c) The equation

$$(arcsinx - 5)arcsinx = -4$$

is equivalent to the system

$$\begin{cases} U = arcsinx \\ \\ U^2 - 5U + 4 = 0 \end{cases}$$

The second-degree equation has two roots $U_1 = 1$ and $U_2 = 4$.

but $arcsinx$ lies between $\frac{\pi}{2}$ and $-\frac{\pi}{2}$, the equation $arcsinx = 4$ doesn't have a solution.

By on the other hand, the equation $arcsinx = 1$ has a solution $x = sin1$,which is the only solution of the initial equation.

Exercise 3.

Show that we have the following relation, if and only if $ab < 1$:

$arctga + arctgb = arctg\frac{a+b}{1-ab}$

Application: calculate $S = 2arctg\frac{1}{4} + arctg\frac{1}{7} + 2arctg\frac{1}{13}$

Solution:

Let's put

$A = arctga + arctgb$ and $B = arctg\frac{a+b}{1-ab}$.

applying the formula for the tangent of a sum

$$tgA = \frac{a+b}{1-ab}$$

but also

$$tgB = \frac{a+b}{1-ab}$$

The numbers A and B therefore have the same tangent.

Moreover B belongs to the interval $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, and A belongs to the interval $]-\pi, \pi[$.

So A and B will be equal if and only if $cosA > 0$. But, using the second exercise,

$$cosA = cos(arctga)cos(arctgb) - sin(arctga)sin(arctgb) = \frac{1-ab}{\sqrt{1+a^2}\sqrt{1+b^2}}$$

and this expression is positive if and only if $ab < 1$.

We can applying particular the formula if $a = b$ when $a^2 < 1$, which gives

$$2arctga = arctg\frac{2a}{1-a^2}$$

Application: in the following calculations, the numbers used are between 0 and 1, so the condition $ab < 1$ will be verified. We obtain successively

$arctg\frac{1}{4} + arctg\frac{1}{13} = \text{arctg}\frac{1}{3}$

$2arctg\frac{1}{3} = arctg\frac{3}{4}$

$arctg\frac{3}{4}+ arctg\frac{1}{7} = arctg1 = \frac{\pi}{4}$

Finally: $S = \frac{\pi}{4}$

Exercise 4.

Determine the domain of definition of the following functions

$1)f(x) = th\frac{x^2-1}{x^2+1}$ $\quad 2)f(x) = Arctg\sqrt{\frac{x-1}{x+1}}$

$3)f(x) = Argch(3x+2)$ $\quad 4)f(x) = Argth\frac{2x}{1-x^2}$

Solution:

1)The function th is defined on $\mathbb{R}$, so $f(x)$ exists if$\frac{x^2-1}{x^2+1}$ is defined, which verifies for all $x \in \mathbb{R}$, so $D_f = \mathbb{R}$.

2) the function $arctg$ is defined on $\mathbb{R}$. Or $\sqrt{\frac{x-1}{x+1}}$ belongs to $\mathbb{R}$ only if $x \neq -1$.

and $(x-1)(x+1) \geq 0$, i.e., $x \in]-\infty, -1[\cup[1, +\infty[$.

Hence

$$D_f =]-\infty, -1[\cup[1, +\infty[$$

3)We know that the function $argch$ is defined on $[1, +\infty[$.

So $f(x)$ exists if

$3x + 2 \geq 1$

Then

$$D_f = [-\frac{1}{3}, +\infty[.$$

4) The function $argth$ is defined on] - 1, 1[so $f(x)$ is defined if:

$\frac{2x}{1-x^2} \in]-1, 1[$ and $x \neq 1$.

Hence:

$$D_f =]-\infty, 2-\sqrt{3}[\cup[2+\sqrt{3}, +\infty.[$$

Exercise 5.

Let x be real. Assume $t = arctgshx$. Show that:

$tgt = shx$, $sint = thx$, $cost = \frac{1}{chx}$.

Solution:

If we put $t = arctgshx$, we deduce that

$$tgt = tg(arctgshx) = shx.$$

We know that chx is positive. Moreover t belongs to the interval $]-\frac{\pi}{2}, \frac{\pi}{2}[$,

so $cost$ is positive. But

$$\frac{1}{cos^2 t} = 1 + tg^2 t = 1 + sh^2 x = ch^2 x,$$

we therefore deduce

$$\frac{1}{cost} = chx.$$

So

$$sint = tgtcost = shx\frac{1}{chx} = thx.$$

Exercise 6.

Calculate the following limits

a) $2ch^2x - sh^2x(x \to +\infty)$.

b) $exp2x(2ch^2x - sh^2x)(x \to -\infty)$.

Solution:

a) Writing the function using exponentials, we obtain

$$2ch^2x - sh^2x = 2(\frac{e^x + e^{-x}}{2})^2 - (\frac{e^x + e^{-x}}{2})^2 = 1 + e^{-2x}$$

and this tends to 1 when x tends to $+\infty$.
b) So

$$e^{2x}(2ch^2x - sh^2x) = e^{2x} + 1$$

and this tends to 1 when x tends to -∞.

Exercise 7.

Solving the system

$$\begin{cases} Argshy = 2Argshx \\ Argchy = 3Argchx \end{cases}$$

Solution:

The system is defined on $D = [1, +\infty[\times[1, +\infty[$

Using formulas:

$sh2a = 2shacha$ and $cha = \sqrt{sh^2a + 1}$,

$ch(a + b) = chachb + shashb$ and $ch(2a) = 2ch^2a - 1$.

$sh(argcha) = \sqrt{a^2 - 1}$ and $ch(argsha) = \sqrt{a^2 + 1}$.

and taking the sh from the two members, the first equation of the system becomes:

$$y = 2x\sqrt{x^2 + 1}......(1)$$

Then by calculating $ch3t$ as a function of cht, we find:$ch3t = 4ch^3t - 3cht$

and taking the ch of the two members of the second equation

$$y = 4x^3 - 3x.......(2)$$

Relations 1 and 2 lead to

$$2x\sqrt{x^2+1} = 4x^3 - 3x.....(3)$$

We then look for the solutions of (3) for $x \geq 1$(so $x \neq 0$),which gives:

$2x\sqrt{x^2+1} = 4x^3 - 3x$ or again $16x^4 - 28x^2 + 5 = 0$

Posing $x^2 = X$, it follows that

$X_1 = \frac{7+\sqrt{29}}{8}$ where $X_2 = \frac{7-\sqrt{29}}{8}$

So

$x^2 = \frac{7+\sqrt{29}}{8}$ or again $x = \sqrt{\frac{7+\sqrt{29}}{8}}$

As a result:

$$S = \left\{\sqrt{\frac{7+\sqrt{29}}{8}}, 2\sqrt{\frac{7+\sqrt{29}}{8}}\sqrt{\frac{7+\sqrt{29}}{8}+1}\right\}.$$

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