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Title

قسم الرياضيات

Integral transforms in L^p spaces:
Course and corrected exercises

Intended mainly for students in the 3rd year of the
"Licence in Mathematics"

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بسم الله الرحمن الرحيم

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Introduction

This course document is primarily intended for students of the 3^{rd} year of the "Licence in Mathematics" and covers the module "**Integral transforms in L^p spaces**". It contains the essentials of the course with examples according to the program proposed by the Ministry of Higher Education and Scientific Research.

The particularity of this work is that it was designed to allow a student to acquire, understand, and dominate by himself all the concepts covered.

This work contains three chapters, written in an easy-to-read style and very detailed:

The first chapter is mainly devoted to the study of the L^p space of functions whose absolute value is of p^{th} integrable power and their properties. L^p spaces play a central role in many questions of mathematical analysis.

In the second chapter, we consider the Fourier transform of functions defined on any complex set $\mathbb{C}$. The Fourier analysis in this case yields exceptionally rich mathematical results, but these far exceed the aim of this course. In this chapter, we will focus on the Fourier transform of functions of type L^1 or L^2.

In the third chapter, we present the Laplace transform and some interesting properties for solving linear differential equations with constant coefficients.

Exercises with solutions are provided at the end of each chapter to allow students to test their knowledge and prepare for tests and final exams.

Finally, I hope that this work can help students who want to master the various concepts that have been well developed.

Chapter 1

L^p spaces

An L^p space is a vector space of classes of functions whose exponent power p is integrable in the Lebesgue sense, where p is a strictly positive real number. The passage to the limit of the exponent results in the construction of spaces L^∞ of bounded functions. The spaces L^p are called Lebesgue spaces.

This space constitutes a fundamental tool of functional analysis by allowing the resolution of equations by approximation with solutions that are not necessarily derivable or even continuous.

Throughout the following, Ω denotes an open set of $\mathbb{R}^n$ equipped with the Lebesgue measure dx. We denote by $L^1(\Omega)$ the space of integrable functions on Ω with values in $\mathbb{R}$ equipped with the norm:

$$\|f\|_{L^1} = \int_\Omega |f(x)|\, dx.$$

When there is no ambiguity we will often write L^1 instead of $L^1(\Omega)$ and $\int f$ instead of $\int_\Omega f(x)dx$. As usual we identify two functions of $L^1(\Omega)$ that coincide a.e. $=$ almost everywhere ($=$ except on a negligible set).

1.1 Some integration results

In this section, we recall some definitions and theorems, without demonstrations, which will be useful later.

Theorem 1.1.1 ***(Beppo Levi's monotone convergence theorem)***

Let (f_n) *be an increasing sequence of functions in* $L^1(\Omega)$ *such that* $\sup\limits_n \int f_n < \infty$.

Then $f_n(x)$ *converges a.e. on* Ω *to a finite limit denoted* $f(x)$, *moreover* $f \in L^1(\Omega)$ *and*

$$\|f_n - f\|_{L^1} \to 0.$$

Theorem 1.1.2 ***(Lebesgue's dominated convergence theorem)***

Let (f_n) *be a sequence of functions in* $L^1(\Omega)$ *that satisfy*

a) $f_n(x) \to f(x)$ *a.e. on* Ω.

b) There exists a function $g \in L^1(\Omega)$ *such that for all* n, $|f_n(x)| \leq g(x)$ *a.e. on* Ω.

Then $f \in L^1(\Omega)$ *and* $\|f_n - f\|_{L^1} \to 0$.

Lemma 1.1.1 ***(Fatou's lemma)***

Let (f_n) *be a sequence of functions in* $L^1(\Omega)$ *that satisfy*

1. For all n, $f_n(x) \geq 0$ *a.e. on* Ω.

2. $\sup\limits_n \int f_n < \infty$.

For allmost all $x \in \Omega$, *we set* $f(x) = \lim\limits_{n\to\infty} \inf f_n(x)$.

Then $f \in L^1(\Omega)$ *and*

$$\int f \leq \lim_{n\to\infty} \inf \int f_n.$$

Notation. We denote by $C_c(\Omega)$ the space of all continuous functions on Ω with compact support, i.e.

$$C_c(\Omega) = \left\{ f \in C(\Omega) : f(x) = 0, \forall x \in \Omega \backslash K \text{ where } K \text{ is a compact} \right\}.$$

Theorem 1.1.3 ***(Density theorem)***

The space $C_c(\Omega)$ *is dense in* $L^1(\Omega)$, *i.e.*

$$\forall f \in L^1(\Omega),\ \forall \varepsilon > 0,\ \exists f_1 \in C_c(\Omega),\ \textit{such that}\ \|f - f_1\|_{L^1} < \varepsilon.$$

Next, we present two theorems which are used in the following.

Let $\Omega_1 \in \mathbb{R}^n, \Omega_2 \in \mathbb{R}^n$ be open sets and let $F : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a measurable function.

Theorem 1.1.4 ***(Tonelli's theorem)***

Suppose that

$$\int_{\Omega_2} |F(x,y)|dy < \infty \text{ for almost all } x \in \Omega_1,$$

and that

$$\int_{\Omega_1} dx \int_{\Omega_2} |F(x,y)|dy < \infty.$$

Then $F \in L^1(\Omega_1 \times \Omega_2)$.

Theorem 1.1.5 ***(Fubini's theorem)***

Suppose that $F \in L^1(\Omega_1 \times \Omega_2)$. *Then for almost all* $x \in \Omega_1$

$$F(x,y) \in L^1_y(\Omega_2) \text{ and } \int_{\Omega_2} F(x,y)dy \in L^1_x(\Omega_1).$$

Similarly, for almost all $y \in \Omega_2$

$$F(x,y) \in L^1_x(\Omega_1) \text{ and } \int_{\Omega_1} F(x,y)dy \in L^1_y(\Omega_2).$$

Furthermore we have

$$\int_{\Omega_1} dx \int_{\Omega_2} F(x,y)dy = \int_{\Omega_2} dy \int_{\Omega_1} F(x,y)dx = \int_{\Omega_1 \times \Omega_2} F(x,y)dxdy.$$

1.2 L^p spaces: definitions and properties

In this section, we will present some definitions and properties of L^p spaces.

Definition 1.2.1 *Let* $p \in \mathbb{R}$ *with* $1 \leq p < \infty$. *We call the Lebesgue space* $L^p(\Omega)$ *the space*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R}, f \text{ measurable and } |f|^p \in L^1(\Omega) \right\}.$$

Definition 1.2.2 *For any function* $f \in L^p(\Omega)$, *we set*

$$\|f\|_{L^p} = \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

Definition 1.2.3 *The Lebesgue space* $L^\infty(\Omega)$ *is defined by*

$$L^\infty(\Omega) = \{ f : \Omega \to \mathbb{R}, f \text{ measurable and } \exists \text{ a constant } C \text{ sucht hat } |f(x)| \leq C \text{ a.e. on } \Omega \}.$$

For any function $f \in L^\infty(\Omega)$, we put

$$\|f\|_{L^\infty} = \inf\{C : |f(x)| \leq C \ \text{a.e.on } \Omega\}.$$

We call space $L^\infty(\Omega)$, the space of functions essentially bounded on Ω.

Remark 1.2.1 *If $f \in L^\infty(\Omega)$ we have*

$$|f(x)| \leq \|f\|_{L^\infty} \ \text{ a.e. on } \Omega.$$

Indeed there exists a sequence C_n such that $C_n \to \|f\|_{L^\infty}$ and for each $n, |f(x)| \leq C_n$ a.e. on Ω. Therefore $||f(x)| \leq C_n$ for all $x \in \Omega \backslash E_n$ with E_n negligible. We set $E = \bigcup\limits_n E_n$ so that E is negligible and we have $|f(x)| \leq C_n$ for all n and for all $x \in \Omega \backslash E$. Consequently $|f(x)| \leq \|f\|_{L^\infty}$ for all $x \in \Omega \backslash E$.

Notation. Let $1 \leq p \leq \infty$, we denote by q the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$ with $\frac{1}{\infty} = 0$.

Lemma 1.2.1 ***(Young's inequality)***

Let $a, b \in \mathbb{R}^+$ and $p, q \in [1, +\infty[$. Then

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q.$$

Proof. We use the definition of a concave function.

The function $x \longmapsto \ln(x)$ is a concave function on $]0, +\infty[$ i.e. $\forall x, y \in \,]0, +\infty[$

$$\ln(\alpha x + \beta y) \geq \alpha \ln(x) + \beta \ln(y) \text{ with } \alpha + \beta = 1.$$

So for $\alpha = \frac{1}{p}$ and $\beta = \frac{1}{q}$ we have $\alpha + \beta = \frac{1}{p} + \frac{1}{q} = 1$.

Let us put $x = a^p$ and $y = b^q$, so we have

$$\begin{aligned}
\ln\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) &\geq \frac{1}{p}\ln\left(a^p\right) + \frac{1}{q}\ln\left(b^q\right) \\
&\geq \ln\left(a^p\right)^{\frac{1}{p}} + \ln\left(b^q\right)^{\frac{1}{q}} \\
&= \ln\left(a\right) + \ln\left(b\right) \\
&= \ln\left(ab\right).
\end{aligned}$$

Since $x \longmapsto \exp(x)$ is an increasing function, we obtain

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q.$$

■

Theorem 1.2.1 ***(Hölder inequality)***

Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $1 \leq p, q \leq \infty$. Then $f.g \in L^p(\Omega)$ and

$$\int_\Omega |f(x)g(x)|\, dx = \|fg\|_{L^1} \leq \|f\|_{L^p}\, \|g\|_{L^q}\,.$$

Proof. Young's inequality gives to show Hölder inequality, we distinguish 3 cases:

Case 1. If $p = 1$ and $q = \infty$ we have

$$\begin{aligned}
\|fg\|_{L^1} &= \int_\Omega |f(x)g(x)|\, dx = \int_\Omega |f(x)|\, \|g\|_{L^\infty}\, dx \\
&\leq \|g\|_{L^\infty} \int_\Omega |f(x)|\, dx \\
&\leq \|f\|_{L^1}\, \|g\|_{L^\infty}\,.
\end{aligned}$$

Case 2. If $p = \infty, q = 1$ is similar.

Case 3. If $p \in]0, +\infty[$.

(i) If $\|f\|_{L^p} = 0$ or $\|g\|_{L^q} = 0$, then $f = 0$ a.e on Ω or $g = 0$ a.e on Ω. We deduce $fg = 0$ a.e on Ω, so $\|fg\|_{L^1} = 0$ and finally, Hölder's inequality reduces to the trivial inequality $0 \leq 0$ therefore it is verified.

We can therefore assume that $\|f\|_{L^p} \neq 0$ and $\|g\|_{L^q} \neq 0$.

(ii) If $\|f\|_{L^p} = 1$ and $\|g\|_{L^q} = 1$, then we have, with Hölder's inequality

$$\begin{aligned}
\|fg\|_{L^1} &= \int_\Omega |f(x)g(x)|\, dx = \int_\Omega |f(x)|\, |g|\, dx \\
&\leq \frac{1}{p}\, \|f\|_{L^p} + \frac{1}{q}\, \|g\|_{L^q} \\
&= \frac{1}{p} + \frac{1}{q} \\
&= 1 \\
&= \|f\|_{L^p}\, \|g\|_{L^q}\,.
\end{aligned}$$

(iii) If $\|f\|_{L^p} > 1$ and $\|g\|_{L^q} > 1$, define the following functions

$$f_1 = \frac{|f(x)|}{\|f\|_{L^p}} \text{ and } g_1 = \frac{|g(x)|}{\|g\|_{L^q}},$$

so that $\|f\|_{L^p} = 1$ and $\|g\|_{L^q} = 1$.

Case (ii) then gives

$$\begin{aligned} \|fg\|_{L^1} &= \|f\|_{L^p} \|g\|_{L^q} \left\| \frac{|f(x)|}{\|f\|_{L^p}} \; \frac{|g(x)|}{\|g\|_{L^q}} \right\|_{L^1} \\ &\leq \|f\|_{L^p} \|g\|_{L^q} .1 \\ &= \|f\|_{L^p} \|g\|_{L^q} . \end{aligned}$$

■

In general, let $1 \leq p, q \leq \infty$ and r defined by $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then $f.g \in L^r(\Omega)$ and

$$\left[\int_\Omega |f(x)g(x)|^r \, dx \right]^{1/r} = \|f\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q} .$$

Corollary 1.2.1 ***(Cauchy-Schwarz inequality)***

When $p = 2$, *we have* $q = 2$ *and the Hölder inequality then reduces to the Cauchy-Schwarz inequality*

$$\int_\Omega |f(x)g(x)| \, dx \leq \left(\int_\Omega |f(x)|^2 \, dx \right)^{\frac{1}{2}} \left(\int_\Omega |g(x)|^2 \, dx \right)^{\frac{1}{2}} .$$

Lemma 1.2.2 ***(Minkowisky inequality)***

Let $f, g \in L^p(\Omega)$ *with* $1 \leq p \leq +\infty$. *Then* $f + g \in L^p(\Omega)$ *and*

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p} .$$

Proof. We have

$$\begin{aligned} (f+g)^p &= (f+g)(f+g)^{p-1} \\ &= f(f+g)^{p-1} + g(f+g)^{p-1} \end{aligned}$$

$$\Longrightarrow |f+g|^p \leq |f| \, |f+g|^{p-1} + |g| \, |f+g|^{p-1} .$$

We apply Hölder inequality, we get

$$\begin{aligned} \int_\Omega |f(x)g(x)|^p \, dx &\leq \int_\Omega |f(x)| \, |f(x)+g(x)|^{p-1} \, dx + \int_\Omega |g(x)| \, |f(x)+g(x)|^{p-1} \, dx \\ &\leq \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}} \left(\int_\Omega |f(x)+g(x)|^{q(p-1)} \, dx \right)^{\frac{1}{q}} \\ &\quad + \left(\int_\Omega |g(x)|^p \, dx \right)^{\frac{1}{p}} \left(\int_\Omega |f(x)+g(x)|^{q(p-1)} \, dx \right)^{\frac{1}{q}} , \end{aligned}$$

with $q\,(p-1) = p$.

Finally we get

$$\left(\int_\Omega |f(x)g(x)|^p \, dx\right)^{\frac{1}{p}} = \left(\int_\Omega |f(x)g(x)|^p \, dx\right)^{1-\frac{1}{q}} \leq \left(\int_\Omega |f(x)|^p \, dx\right)^{\frac{1}{p}} + \left(\int_\Omega |g(x)|^p \, dx\right)^{\frac{1}{p}},$$

i.e.

$$\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}\,.$$

■

Proposition 1.2.1 *$L^p(\Omega)$ is a vector space and $\|.\|_{L^p}$ is a norm for all $1 \leq p \leq \infty$.*

Proof. 1) The space $L^p(\Omega)$ is a vector space. Indeed,

a) Let $\alpha \in \mathbb{R}$ and $f \in L^p(\Omega)$, then we have

$$\int_\Omega |\alpha f(x)|^p \, dx = |\alpha|^p \int_\Omega |f(x)|^p \, dx < +\infty,$$

therefore $\alpha\ f \in L^p(\Omega)$.

b) Let $f, g \in L^p(\Omega)$. According to Minkowisky's inequality, then we have

$$\|f+g\|_{L^p} \leq \underbrace{\|f\|_{L^p}}_{<+\infty} + \underbrace{\|g\|_{L^p}}_{<+\infty} < +\infty,$$

therefore $f + g \in L^p(\Omega)$.

We conclude that $L^p(\Omega)$ is a vector space.

2)) The application $\|.\|_p : f \rightarrow \|f\|_{L^p}$, is a norm on $L^p(\Omega)$. Indeed,

1)$\|f\|_{L^p} > 0$ for all $f \in L^p(\Omega)$.

2) $\|\alpha f\|_{L^p} = \alpha \|f\|_{L^p}$ for all $f \in L^p(\Omega)$ and $\alpha \in \mathbb{R}$.

3) According to Minkowski's inequality we have

$$\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p} \text{ for all } f, g \in L^p(\Omega).$$

4) $\|f\|_p = 0$ implies that $f = 0$ for all $f \in L^p(\Omega)$. ■

Theorem 1.2.2 ***(Fischer-Riesz)***

$L^p(\Omega)$ is a Banach space for all $1 \leq p \leq \infty$.

Proof. Let us first treat the case $p = \infty$.

Let (f_n) be a Cauchy sequence in L^∞. Given an integer $k \geq 1$ there exists N_k such that

$$\|f_m - f_n\|_{L^\infty} < \frac{1}{k} \text{ for } m, n \geq N_k.$$

This implies that there exists negligible E_k such that

$$|f_m(x) - f_n\ (x)| < \frac{1}{k}, \forall x \in \Omega \backslash E_k, \forall m, n \geq N_k. \tag{*}$$

Finally, let $E = \bigcup\limits_k E_k$ (E is negligible), we see that for all $x \in \Omega \backslash E$ the sequence $f_n(x)$ is Cauchy in $\mathbb{R}$. Let $f_n(x) \to f(x)$ for $x \in \Omega \backslash E$. Passing to the limit in (*) when $m \to +\infty$ we obtain

$$|f_m(x) - f_n\ (x)| < \frac{1}{k}, \forall x \in \Omega \backslash E_k, \forall n \geq N_k.$$

Therefore $f \in L^\infty \Omega)$ and

$$\|f - f_n\|_{L^\infty} < \frac{1}{k}\ \forall n \geq N_k,$$

and consequently $\|f - f_n\|_{L^\infty} \to 0$.

Now suppose that $1 \leq p < \infty$.

Let (f_n) be a Cauchy sequence in $L^p(\Omega)$. To conclude, it suffices to show that an extracted subsequence converges in $L^p(\Omega)$. We extract a subsequence (f_n) such that

$$\left\|f_{n_{k+1}} - f_{n_k}\right\|_{L^p} < \frac{1}{2^k}, \forall k \geq 1.$$

[We proceed as follows: there exists n_1 such that $\|f_m - f_n\|_{L^p} < \frac{1}{2}$ for $m, n \geq n_1$, we then take $n_2 \geq n_1$ such that $\|f_m - f_n\|_{L^p} < \frac{1}{2^2}$ for $m, n \geq n_2$, etc...]. We will show that f_{n_k} converges in $L^p(\Omega)$. To simplify the notations we write f_k instead of f_{n_k}, so that we have

$$\|f_{k+1} - f_k\|_{L^p} < \frac{1}{2^k}, \forall k \geq 1.$$

Let's put

$$g_n(x) = \sum_{k=1}^{n} |f_{k+1}(x) - f_k(x)|,$$

it comes

$$\|g_n\|_{L^p} < 1.$$

From the theorem of monotone convergence, we deduce that $g_n(x)$ converges to a finite limit denoted $g(x)$ with $g \in L^p(\Omega)$. On the other hand, we have for $m \geq n > 2$

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f_{m-1}(x)| \leq ... \leq |f_{n+1}(x) - f_n(x)| \leq |g(x) - g_{n-1}(x)| \, .$$

It results that a.e. on Ω, $(f_n(x))$ is Cauchy and converges to a limit denoted $f(x)$.

We have a.e. on Ω

$$|f(x) - f_n(x)| \leq g(x) \text{ for } n \geq 2.$$

It results that $f \in L^p$. Finally $\|f_n - f\|_{L^p} \to 0$, indeed we have $|f_n(x) - f(x)|^p \to 0$ a.e. and $|f(x) - f_n(x)| \leq g(x)$ is major integrable. We conclude thanks to Lebesgue's theorem, $L^p(\Omega)$ is a Banach space. ■

1.3 Reflexibility, Separability, and Duality of L^p

In this section, we present very important results concerning the reflexivity, separability, and duality of L^p spaces. For this, we will distinguish the study of the three cases

- $1 < p < \infty$.
- $p = 1$.
- $p = \infty$.

1.3.1 Reflexivity

Theorem 1.3.1 ***(Clarkson's inequality)***

Let $f, g \in L^p(\Omega)$, then

(H1) If $2 \leq p < \infty$, we have

$$\left\| \frac{f+g}{2} \right\|_{L^p}^p + \left\| \frac{f-g}{2} \right\|_{L^p}^p \leq \frac{1}{2} \left(\|f\|_{L^p}^p + \|g\|_{L^p}^p \right). \tag{1}$$

(H2) If $1 < p \leq 2$, we have

$$\left\| \frac{f+g}{2} \right\|_{L^p}^q + \left\| \frac{f-g}{2} \right\|_{L^p}^q \leq \frac{1}{2} \left(\|f\|_{L^p}^p + \|g\|_{L^p}^p \right)^{q-1}. \tag{2}$$

Definition 1.3.1 ***(Uniformly convex spaces)***

A Banach space E is said to be uniformly convex if $\forall \varepsilon > 0, \exists \delta > 0$ such as

$$\forall x, y \in E : \|x\| \leq 1, \|y\| \leq 1 \text{ and } \|x - y\| \geq \varepsilon \Longrightarrow \left\|\frac{x+y}{2}\right\| \leq 1 - \delta.$$

Theorem 1.3.2 ***(Milman-Pettis theorem)***

Any uniformly convex Banach space is reflexive.

Theorem 1.3.3 *The space $L^p(\Omega)$ is reflexive for all $1 < p < \infty$.*

Proof. For $2 \leq p < \infty$, let $\varepsilon > 0$, assume that $\|f - g\|_{L^p} > \varepsilon$, $\|f\|_{L^p} \leq 1$ and $\|g\|_p \leq 1$. Using inequality (1), we obtain that

$$\left\|\frac{f+g}{2}\right\|_{L^p}^p \leq 1 - \left(\frac{\varepsilon}{2}\right)^p,$$

and therefore

$$\left\|\frac{f+g}{2}\right\|_{L^p} \leq 1 - \delta,$$

with

$$\delta = 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}.$$

Thus $L^p(\Omega)$ is uniformly convex and therefore reflexive thanks to Theorem 1.3.2.

For $1 < p \leq 2$, we deduce from inequality (2) that

$$\left\|\frac{f+g}{2}\right\|_{L^p}^q \leq 1 - \left(\frac{\varepsilon}{2}\right)^q,$$

and therefore

$$\left\|\frac{f+g}{2}\right\|_{L^p} \leq 1 - \delta,$$

with

$$\delta = 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^q\right)^{\frac{1}{q}}.$$

Then $L^p(\Omega)$ is uniformly convex, we deduce that $L^p(\Omega)$ is reflexive for $1 < p < \infty$. ■

Theorem 1.3.4 *The spaces $L^1(\Omega)$ and $L^\infty(\Omega)$ are not reflexive.*

Proof. See [1]. ■

1.3.2 Separability

Definition 1.3.2 *A metric space is said to be separable if there exists a countable subset* $F \subset E$ *and dense in* E.

Theorem 1.3.5 *The space* $L^p(\Omega)$ *is separable for* $1 \leq p < \infty$.

Proof. We denote by $(R_i)_{i\in I}$ the countable family of blocks R of the form

$$R = \prod_{k=1}^{N}]a_k, b_k[\text{ with } a_k, b_k \in \mathbb{Q} \text{ and } R \subset \mathbb{Q}.$$

We denote by F the vector space generated by the functions $\mathbb{I}_{R_i}$ (the indicator function of $R_i, i \in I$), i.e

$$F = \left\{ \sum_{i=1}^{n} l_i \mathbb{I}_{]a_i,b_i[} \right\} \text{with } l_i, a_i, b_i \in \mathbb{Q} \text{ for all } i = 1, 2, ..., n,$$

so that F is countable. By Theorem 1.1.3, it suffices to show that F is dense in $L^p(\Omega)$.

Let $f \in L^p(\Omega)$ and $\varepsilon > 0$ be fixed. By density of $C_c(\Omega)$ in $L^p(\Omega)$, there exists $f_1 \in C_c(\Omega)$ such that

$$\|f - f_1\|_{L^p} < \varepsilon.$$

Consider Ω' a bounded open set such that $supp(f_1) \subset \Omega' \subset \Omega$. Since $f_1 \in Cc(\Omega')$ and using the uniform continuity of f_1, we easily construct a function $f_2 \in F$ such that $supp(f_2) \subset \Omega'$ and

$$\|f_2 - f_1\|_{L^p} < \frac{\varepsilon}{|\Omega'|}.$$

We start by covering $supp(f_1)$ with a finite number of blocks R_i on which the oscillation of f_1 is less than $\frac{\varepsilon}{|\Omega'|^{1/p}}$. It follows that $\|f_2 - f_1\|_{L^p} < \varepsilon$ and therefore we have

$$\begin{aligned} \|f - f_2\|_{L^p} &= \|f - f_1 + f_1 - f_2\|_{L^p} \\ &\leq \underbrace{\underbrace{\|f - f_1\|_{L^p}}_{<\varepsilon} + \underbrace{\|f_1 - f_2\|_{L^p}}_{<\varepsilon}}_{<2\varepsilon} \end{aligned}$$

So F is dense in $L^p(\Omega)$, finally according to Definition 1.3.2, we have $L^p(\Omega)$ is separable for $1 \leq p < \infty$. ∎

Lemma 1.3.1 *Let E be a Banach space. Assume that there exists a free family $(Oi)_{i\in I}$ such that*

1. *Oi is a nonempty open set of E for all $i \in I$.*
2. *$O_i \cap O_j = \varnothing$ if $i \neq j$.*
3. *I is not countable.*

Then E is not separable.

Proof. Let us reason by the absurd and suppose that there exists a sequence $(u_n)_{n\in\mathbb{N}}$ a dense sequence in E.

For each $i \in I, O_i \cap \{u_n, n \in \mathbb{N}\} = \varnothing$. We choose n_i such that $u_{n_i} \in O_i$, we have $n_i = n_j \Longrightarrow u_{n_i} = u_{n_j} \in O_i \cap O_j$ therefore $i = j$. We also have the application $i \longrightarrow n_i$ is injective, consequently I is countable, which contradicts the statement that I is uncountable. ■

Theorem 1.3.6 *The space $L^\infty(\Omega)$ is not separable.*

Proof. For all $a \in \Omega$, let us fix $r_a < dist(a, \Omega)$. We put $u_a = \mathbb{I}_{B(a,r_a)}$

$$O_a = \left\{ f \in L^\infty(\Omega) : \|f - u_a\|_\infty < \frac{1}{2} \right\}.$$

We easily verify that the family $(O_a)_{a\in\Omega}$ satisfies the hypotheses of Lemma 1.3.1. We therefore conclude that $L^\infty(\Omega)$ is not separable. ■

1.3.3 Duality

The following theorems will allow us to identify the topological dual of $L^p(\Omega)$ spaces.

Topological dual

The topological dual of a normed vector space E over the field K is by definition the set of continuous linear forms of E, i.e. continuous linear applications of E into K. We denote this set by E^*. Thus, $E^* = L(E, K)$ and by the above E equipped with the norm $\|.\|$ defined by

$$\|f\| = \sup_{x\in E, \|x\|=1} |f(x)| = \sup_{x\in E, \|x\|\leq 1} |f(x)| = \sup_{x\in E, \|x\|\neq 0} \frac{|f(x)|}{\|x\|},$$

is a Banach space since K is complete.

Theorem 1.3.7 ***(Riez representation theorem)***

Let $1 < p < \infty$ *and let* $\varphi \in (L^p(\Omega))^*$ *then there exists a unique* $u \in L^q(\Omega)$ *such that*

$$\langle \varphi, f \rangle = \int_\Omega uf, \forall f \in L^p(\Omega).$$

This theorem is important because it allows us to represent all the continuous linear forms on $L^p(\Omega)$ with $1 < p < \infty$, using a function of $L^q(\Omega)$ the application

$$\begin{aligned} T \quad &: \quad L^q(\Omega) \to (L^p(\Omega))^* \\ u \quad &\to \quad Tu = \varphi, \end{aligned}$$

is a continuous isometric and bijective linear operator which allows us to identify the dual of $L^p(\Omega)$ with $L^q(\Omega)$ and we have

$$\|Tu\|_{(L^p(\Omega))^*} = \|u\|_{L^q(\Omega)}.$$

Theorem 1.3.8 *Let* $\varphi \in (L^1(\Omega))^*$, *then there exists a unique* $u \in L^\infty(\Omega)$ *such that*

$$\langle \varphi, f \rangle = \int_\Omega uf, \forall f \in L^1(\Omega),$$

plus we have

$$\|\varphi\|_{(L^1(\Omega))^*} = \|u\|_{L^\infty(\Omega)}.$$

This theorem allows us to identify the dual of $L^1(\Omega)$ with $L^\infty(\Omega)$.

The following table summarizes the main properties of the spaces $L^p(\Omega)$.

	Reflexive	Separable	Dual space
$L^p(\Omega)$ $(1 < p < \infty)$	Oui	Oui	$L^q(\Omega)$
$L^1(\Omega)$	Non	Oui	$L^\infty(\Omega)$
$L^\infty(\Omega)$	Non	Non	Contains strictly $L^1(\Omega)$

1.4 Convolution and regularization on $\mathbb{R}^n$

In this section, we present the convolution product and their property, the regularization, and the supports in the convolution, we take in all this part $\Omega = \mathbb{R}^n$.

Definition 1.4.1 *Let f and g be two measurable functions on $\mathbb{R}^n$. We say that the convolution product of f by g exists at the point $x \in \mathbb{R}^n$ and we denote it by $(f * g)(x)$, if the integral $\int_{\mathbb{R}^n} f(x-y)g(y)dy$ has a meaning. In this case, we put*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

Proposition 1.4.1 *Let f and g be two measurable functions on $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. If $(f * g)(x)$ exists, then $(g * f)(x)$ also exists and*

$$(f * g)(x) = (g * f)(x).$$

Proof. Suppose that $(f * g)(x)$ exists, then the function $y \mapsto f(x-y)g(y)$ is integrable on $\mathbb{R}^n$, i.e. $\int_{\mathbb{R}^n} f(x-y)g(y)dy < \infty$.

We put $\varphi : u \mapsto x - u = y$ which is a diffeomorphism of $\mathbb{R}^n \to \mathbb{R}^n$. Then by applying the change of variable theorem to the function h defined on $\mathbb{R}^n$ by $h(y) = f(x-y)g(y)$, we obtain

$$\int_{\mathbb{R}^n} f(x-y)g(y)dy = \int_{\mathbb{R}^n} h(\varphi(u))du = \int_{\mathbb{R}^n} f(u)g(x-u)du.$$

Therefore

$$(f * g)(x) = (g * f)(x), \forall x \in \mathbb{R}^n.$$

■

Remark 1.4.1 *Let f, g and h be three positive measurable functions on $\mathbb{R}^n$, for all $x \in \mathbb{R}^n$ we have*

$$[(f * g) * h](x) = [f * (g * h)](x).$$

Theorem 1.4.1 *Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$. Then for almost all $x \in \mathbb{R}^n$ the function $y \mapsto f(x-y)g(y)$ is integrable on $\mathbb{R}^n$ and $(f * g) \in L^p(\mathbb{R}^n)$, moreover we have*

$$\|f * g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}.$$

Proof. - For $p = \infty$.

Let $L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$. Let us show that the measurable function $y \mapsto f(x-y)g(y)$ is integrable on $\mathbb{R}^n$.

We have $L^1(\mathbb{R}^n)$ then $\|f\|_{L^1} = \int_{\mathbb{R}^n} |f|dx < \infty, g \in L^\infty$ so $\exists C > 0$ such that $|g(x)| \leq C$ a.e, therefore

$$\int_{\mathbb{R}^n} |f(x-y)||g(y)|dy \leq C \int_{\mathbb{R}^n} |f(x-y)|dy.$$

Since the Lebesgue measure is stable by translation, then $\int_{\mathbb{R}^n} |f(x-y)|dy = \|f\|_{L^1}$.

Thus

$$\int_{\mathbb{R}^n} |f(x-y)||g(y)|dy \leq C \|f\|_{L^1} < \infty,$$

which gives $y \mapsto f(x-y)g(y)$ is integrable on $\mathbb{R}^n$, consequently $(f * g)(x)$ exists almost everywhere.

Let us show the inequality, we have

$$|(f * g)(x)| = \left| \int_{\mathbb{R}^n} f(x-y)g(y)dy \right| \leq \int_{\mathbb{R}^n} |f(x-y)||g(y)|dy,$$

and we know that $|g(y)| \leq \|g\|_{L^\infty}$ we then obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |f(x-y)||g(y)|dy &\leq \int_{\mathbb{R}^n} |f(x-y)| \|g\|_{L^\infty} dy \\ &\leq \|g\|_{L^\infty} \int_{\mathbb{R}^n} |f(x-y)|dy \\ &\leq \|g\|_{L^\infty} \|f\|_{L^1}, \end{aligned}$$

and consequently $|(f * g)(x)| \leq \|g\|_{L^\infty} \|f\|_{L^1}$ a.e. So by definition of the norm $\|.\|_{L^\infty}$, we have

$$\|f * g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}.$$

- For $p = 1$.

We put $F(x,y) = f(x-y)g(y)$, then F is measurable and for almost all $y \in \mathbb{R}^n$ we have

$$\int_{\mathbb{R}^n} |F(x,y)|dx = |g(y)| \int_{\mathbb{R}^n} |f(x-y)|dx = |g(y)| \|f\|_{L^1}.$$

As $f \in L^1(\mathbb{R}^n)$ then $\|f\|_{L^1} < \infty$, and $g \in L^1(\mathbb{R}^n)$ therefore $\lambda(\{g = +\infty\}) < \infty$, consequently $|g(y)| \|f\|_{L^1} < \infty$ a.e. Therefore $\int_{\mathbb{R}^n} |F(x,y)|dx < \infty$ for almost all $y \in L^1(\mathbb{R}^n)$.

By Fubini's theorem, we have

$$\begin{aligned} \int_{\mathbb{R}^n} dy \int_{\mathbb{R}^n} |F(x,y)|dx &= \int_{\mathbb{R}^n} |g(y)|dy \int_{\mathbb{R}^n} |f(x-y)|dx \\ &= \|g\|_{L^1} \|f\|_{L^1} < \infty, \end{aligned}$$

and by application of Tonelli's theorem, we see that $F \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$ and by Fubini's theorem, we obtain $\int_{\mathbb{R}^n} |F(x,y)|dy < \infty$. Therefore $y \mapsto f(x-y)g(y)$ is integrable on $\mathbb{R}^n$ and

$$\begin{aligned} \|f * g\|_{L^1} &\leq \int_{\mathbb{R}^n} |f * g(x)|\, dx \\ &\leq \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} F(x,y)|dy \\ &\leq \|f\|_{L^1} \|g\|_{L^1}. \end{aligned}$$

- For $1 < p < \infty$.

Suppose that $L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, then $|g|^p \in L^1(\mathbb{R}^n)$ and consequently for almost all $x \in \mathbb{R}^n$, the function $y \mapsto |f(x-y)|g(y)|^p$ is integrable on $\mathbb{R}^n$. We have $|f(x-y)||g(y)|^p \in L^1_y(\mathbb{R}^n)$ therefore $|f(x-y)|^{\frac{1}{p}}|g(y)| \in L^p_y(\mathbb{R}^n)$.

Let $q \in]1,+\infty[$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, then $f \in L^1(\mathbb{R}^n)$ implies that $|f|^{\frac{1}{q}} \in L^q(\mathbb{R}^n)$, hence $|f(x-y)|^{\frac{1}{q}} \in L^q_y(\mathbb{R}^n)$ and therefore

$$\begin{aligned} |f(x-y)||g(y)| &= |f(x-y)|^{\frac{1}{p}+\frac{1}{q}} |g(y)| \\ &= |f(x-y)|^{\frac{1}{p}}|g(y)||f(x-y)|^{\frac{1}{q}}. \end{aligned}$$

Using Hölder's inequality we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |f(x-y)g(y)|dy &\leq \left(\int_{\mathbb{R}^n} \left(|f(x-y)|^{\frac{1}{p}}|g(y)|\right)^p dy\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |f(x-y)|^{\frac{q}{q}} dy\right)^{\frac{1}{q}} \\ &\leq \left(\int_{\mathbb{R}^n} |f(x-y)||g(y)|^p dy\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |f(x-y)|dy\right)^{\frac{1}{q}}, \end{aligned}$$

then

$$|(f * g)(x)| \leq [(|f| * |g|^p(x))]^{\frac{1}{p}} \|f\|_{L^1}^{\frac{1}{q}},$$

which gives

$$|(f * g)(x)|^p \leq (|f| * |g|^p(x)) \|f\|_{L^1}^{\frac{p}{q}} \in L^p(\mathbb{R}^n).$$

So $f * g \in L^p(\mathbb{R}^n)$ and

$$\begin{aligned} \int_{\mathbb{R}^n} |(f * g)(x)|^p dx &\leq \|f\|_{L^1}^{\frac{p}{q}} \int_{\mathbb{R}^n} (|f| * |g|^p(x))\, dx \\ &\leq \|f\|_{L^1}^{\frac{p}{q}} \|f\|_{L^1} \|g\|_{L^1}^p \\ &\leq \|f\|_{L^1} \|g\|_{L^p}, \end{aligned}$$

which gives

$$\|f * g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}.$$

■

Theorem 1.4.2 ***(Young's inequality).***

Let $1 \leq p, q \leq +\infty$ *be such that* $\frac{1}{p} + \frac{1}{q} \geq 1$. *Let* $1 \leq r \leq +\infty$ *be defined by the equality* $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. *Let* $f \in L^p(\mathbb{R}^n)$ *and* $g \in L^q(\mathbb{R}^n)$, *then*

i) The convolution product $f * g$ *is defined almost everywhere and defines a measurable Lebesgue function.*

ii) We have $f * g \in L^r(\mathbb{R}^n)$ *and*

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

iii) If $\frac{1}{p} + \frac{1}{q} = 1$ *(and therefore* $r = \infty$*), then we have the following stronger conclusions* $f * g$ *is defined at every point and*

$$|(f * g)(x)| \leq \|f\|_{L^p} \|g\|_{L^q}, \forall x \in \mathbb{R}^n.$$

Proof. See [1]. ■

1.4.1 Supports in convolution

The notion of support of a continuous function is well known: it is the complement of the largest open set on which f is zero or it is the adherence of the set $\{x, f(x) \neq 0\}$. When working with measurable functions, one must be more careful, since these functions are only defined almost everywhere. The appropriate definition is the following:

Proposition 1.4.2 ***(and definition of support)***

Let $\Omega \subset \mathbb{R}^n$ *be an open set and let* f *be a function defined on* Ω *with values in* $\mathbb{R}$. *Consider the family of all open sets* $(\omega_i)_{i \in I}, \omega_i \subset \Omega$ *such that for each* $i \in I, f = 0$ *a.e. on* ω_i. *We put* $\omega = \bigcup\limits_{i \in I} \omega_i$, *then* $f = 0$ *a.e on* ω *and by definition, supp*$f = \Omega \backslash \omega$.

Proof. The family of indices I is arbitrary, to reduce to the countable case we proceed as follows.

Let $(Ki)_{i\in\mathbb{N}^*}$ be a sequence of compacts such that $\omega = \bigcup\limits_{i\in\mathbb{N}} K_i$, we can take

$$K = \left\{ x \in \omega, d(x, \mathbb{R}^n \backslash \omega) \geq \frac{1}{i} \text{ and } \|x\| \leq i \right\},$$

we have $K_i \subset \omega = \bigcup\limits_{i\in I} \omega_i$, since K_i is compact there exists $I_i \subset I$ finite such that $K_i \subset \bigcup\limits_{j\in I_i} \omega_j$.

We put $J = \bigcup\limits_{i\in\mathbb{N}} I_i$, so J is countable and since $K_i \subset \bigcup\limits_{j\in I_i} \omega_j$, then

$$\bigcup_{i\in\mathbb{N}} K_i \subset \bigcup_{i\in\mathbb{N}} \bigcup_{j\in I_i} \omega_j = \bigcup_{j\in\cup_{i\in\mathbb{N}}} \omega_j.$$

So $\omega \subset \bigcup\limits_{j\in J} \omega_j$ and since $J \subset I$, then

$$\bigcup_{j\in J} \omega_j \subset \omega.$$

As $f = 0$ a.e. on ω_j, then $\exists N_j \subset \omega_j$ such that N_j is negligible and $\forall x \in N^c : f(x) = 0$, we take $N = \bigcup\limits_{j\in J} N_j$ and we have

$$\lambda(N) = \lambda\left(\bigcup_{j\in J} N_j\right) \leq \sum_{j\in J} \lambda(N_j) = 0.$$

So N is negligible and consequently $\exists N \subset \omega$ such that $\forall x \in N^c : f(x) = 0$, so $f = 0$ a.e. on ω. ■

Remark 1.4.2 *1. If f is continuous on we easily verify that this definition coincides with the usual definition.*

2. If f_1 and f_2 are two functions such that $f_1 = f_2$ a.e. on Ω, then supp$(f_1) =$ supp(f_2), we can therefore speak of the support of a function $f \in L^p(\Omega)$.

Proposition 1.4.3 *Let $f \in L^1(\Omega)$ and $g \in L^p(\Omega)$, then*

$$supp(f * g) \subset \overline{supp(f) + supp(g)}.$$

Proof. Let $x \in \mathbb{R}^n$ be fixed such that the function $y \mapsto f(x-y)g(y)$ is integrable. We can write

$$\begin{aligned} (f * g)(x) &= \int_{\mathbb{R}^n} f(x-y)g(y)dy \\ &= \int_{S_x} f(x-y)g(y)dy + \int_{S_x^c} f(x-y)g(y)dy, \end{aligned}$$

where

$$S_x = \{y \in \mathbb{R}^n, y \in supp(g) \text{ and } x - y \in supp(f)\}.$$

We then have

$$\int_{S_x^c} f(x-y)g(y)dy = 0,$$

which gives

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy = \int_{S_x} f(x-y)g(y)dy.$$

If x is not in $supp(f)+supp(g)$, then $(x-supp(f))\cap supp(g) = \varnothing$ otherwise $\exists y$ such that $y \in (x - supp(f))$ and $y \in supp(g)$, so $x - y \in supp(f)$ and $y \in supp(g)$ and consequently $x = x-y+y \in supp(f)+supp(g)$, which contradicts the fact that x is not in $supp(f)+supp(g)$ and therefore $(f * g)(x) = 0$ a.e. on $(supp(f) + supp(g))^c$, in particular on $int((supp(f) + supp(g))^c) = \overline{supp(f) + supp(g)^c}$ where $int(B)$ is the interior of B, and consequently $upp(f * g) \subset \overline{supp(f) + supp(g)}$. ■

Remark 1.4.3 *Of course, if both f and g are compactly supported, then $f * g$ is compactly supported. In general, if only one of the supports is compact, then $f * g$ is not compactly supported.*

Definition 1.4.2 *Let $1 \leq p \leq \infty$, We say that a function $f \in L^p_{loc}(\Omega)$ if for all compact $K \subset \Omega$, $\int_K |f(x)|^p\, dx < +\infty$.*

Proposition 1.4.4 ***(Regularization)***

Let $f \in C_c(\mathbb{R}^n)$ and $g \in L^1_{loc}(\mathbb{R}^n)$, then

$$f * g \in C(\mathbb{R}^n).$$

Proof. Let us first note that for all $x \in \mathbb{R}^n$ the function $y \longmapsto f(x-y)g(y)$ is integrable on Rn and therefore $(f * g)(x)$ has a meaning for all $x \in \mathbb{R}^n$. Let $x_n \longmapsto x$ and let's put

$$F_n(y) = f(x_n - y)g(y),$$

$$F(y) = f(x-y)g(y),$$

so that $F_n(y) \to F(y)$ a.e. on $\mathbb{R}^n$. On the other hand, let K be a fixed compact such that $(x_n - Suppf) \subset K$ for all n, therefore $f(x_n - y) = 0$ for y which is not in K and consequently $|F_n(y)| \leq \|f\|_{L^\infty} \mathbb{I}_K(y) g(y)$ integrable majorante. We deduce from Lebesgue's theorem that

$$(f * g)(x_n) = \int F_n(y)dy \to \int F(y)dy = (f * g)(x).$$

■

Notations

$C^k(\Omega)$ denotes the space of k times continuously differentiable functions on Ω,

$$\begin{aligned} C^\infty(\Omega) &= \bigcap_k C^k(\Omega), \\ C_c^k(\Omega) &= C^k(\Omega) \cup C_c(\Omega), \\ C_c^\infty(\Omega) &= C^\infty(\Omega) \cup C_c(\Omega). \end{aligned}$$

Theorem 1.4.3 *Let* $f \in C_c^m(\mathbb{R}^n)$ *and* $g \in L^1_{loc}(\mathbb{R}^n)$ $(m \in \mathbb{N})$, *then*

$$f * g \in C^m(\mathbb{R}^n) \text{ and } D^\alpha(f * g) = D^\alpha f * g \text{ with } |\alpha| \leq m.$$

In particular, if $f \in C_c^\infty(\mathbb{R}^n)$ *and* $g \in L^1_{loc}(\mathbb{R}^n)$, *then* $f * g \in C^\infty(\mathbb{R}^n)$.

Proof. Let $x_0 \in \mathbb{R}^n, r > 0$ and h be the function defined on $(x_0, r) \times K'$ by: $h(x, y) = f(x - y)g(y)$ where $K' = \{a - b/a \in B_f(0, \eta) \text{ and } b \in K\}$ with $K = suppf$ then K' is a compact because $K' = \tau(B_f(0, \eta) \times K)$ where τ is the continuous application defined by $\tau(a, b) = a - b$. For all $x \in B(x_0, \eta)$, the function $y \longmapsto h(x, y)$ is integrable on K'. For almost all $y \in K', x \mapsto h(x, y)$ is of class C^m on $B(x_0, \eta)$ and for all multi-index α with $|\alpha| \leq m$, we have for all $(x, y) \in B(x_0, \eta) \times (K' \backslash N)$ where N is a negligible set on which g is not defined and

$$\begin{aligned} |D^\alpha h(x, y)| &= |D^\alpha f(x - y) g(y)| \\ &\leq \sup_{z \in \mathbb{R}^n} |D^\alpha f(z)| \, |g(y)| \\ &\leq \max_{z \in K} |D^\alpha f(z)| \, |g(y)| \, . \end{aligned}$$

Since $g \in L^1_{loc}(\mathbb{R}^n)$ then g/K' is integrable and by the derivation theorem under the integral sign, the function $x \mapsto \int_{K'} h(x, y) dy = (f * g)(x)$ admits partial derivatives on

$B(x_0, \eta)$ and $D^\alpha(f * g) = D^\alpha f * g$ on $B(x_0, \eta)$ for all $|\alpha| \leq m$, which gives that $f * g$ is of class C^m on $B(x_0, \eta)$ from which the result since x_0 and η are arbitrary. In particular if $f \in C_c^\infty(\mathbb{R}^n)$ and $g \in L^1_{loc}(\mathbb{R}^n)$ then $f * g \in C^\infty(\mathbb{R}^n)$. ■

1.4.2 Regularizing sequences

Definition 1.4.3 *We call a regularizing sequence any sequence $(\rho_n)_{n\geq 1}$ of functions such that*

1. $\rho_n \in C_c^\infty(\mathbb{R}^n)$,
2. $supp_n \subset B\left(0, \frac{1}{n}\right)$,
3. $\int \rho_n = 1$,
4. $\rho_n \geq 0$ on $\mathbb{R}^n$.

Remark 1.4.4 *From now on, we will systematically use $(\rho_n)_{n\geq 1}$ to designate a regularizing sequence.*

An application to the regularizing sequence is presented by the following example.

Example 1.4.1 *Let $u : \mathbb{R} \to \mathbb{R}$ be the function defined by*

$$x \mapsto f(x) = \begin{cases} \exp\left(\frac{1}{x}\right) & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases},$$

then $u \in C^\infty(\mathbb{R}^+), u^{(k)}(x) = P_k(\frac{1}{x}) \exp \frac{1}{x}$ if $x < 0$ where P_k are polynomials defined by recurrence

$$\begin{aligned} P_0(x) &= 1, \\ P_{k+1}(x) &= -x^2\left(P_k'(x) + P_k(x)\right), \forall k \in \mathbb{N}. \end{aligned}$$

Moreover

$$\lim_{x \to 0^-} u^{(k)}(x) = 0,$$

so $u \in C^\infty(\mathbb{R})$ and $supp(u) = \mathbb{R}^-$. We put

$$\rho(x) = \begin{cases} \exp\left(\frac{1}{\|x\|^2 - 1}\right) & \text{if } \|x\| < 1 \\ 0 & \text{if } \|x\| \geq 1 \end{cases},$$

where $x \in \mathbb{R}^n$ *and*

$$\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2} \ (\textit{Euclidean norm}).$$

Then

$$\rho : x \in \mathbb{R}^n \mapsto \|x\|^2 - 1 \in \mathbb{R} \mapsto \rho(x) \in \mathbb{R},$$

is of class C^∞ *on* $\mathbb{R}^n$, *supp*$(\rho) = B(0,1), \rho(x) \geq 0, \forall x \in \mathbb{R}^n, \rho > 0$ *on* $B(0,1)$ *so*

$$\int \rho(x)dx > 0.$$

We put then $c = \left(\int_{\mathbb{R}^n} \rho(x)dx\right)^{-1}$ *and take* $\rho_k(x) = ck^n\rho(kx), k \in \mathbb{N}$, *then* $(\rho_k)_{k\in\mathbb{N}}$ *is a regularizing sequence.*

Proposition 1.4.5 *Let* $f \in C(\mathbb{R}^n)$ *then* $\rho_n * f$ *converges uniformly to* f *on any compact of* $\mathbb{R}^n$.

Proof. Let K be a compact of $\mathbb{R}^n$, f being continuous on $\mathbb{R}^n$ it is uniformly continuous on K according to Heine's theorem. Consequently

$$\forall \varepsilon > 0, \exists \eta > 0, \forall x, y \in K : \|x - y\| < \eta \Longrightarrow \|f(x) - f(y)\| < \varepsilon.$$

For $y \in B(0, \eta)$ and $x \in K$, we have

$$\|y\| = \|x - y + y\| < \eta \Longrightarrow \|f(x - y) - f(y)\| < \varepsilon,$$

and therefore

$$\begin{aligned} \left|(\rho_n * f)(x) - f(x)\right| &= \left|\int_{\mathbb{R}^n} f(x-y)\rho_n(y)dy - f(x)\right| \\ &= \left|\int_{\mathbb{R}^n} f(x-y)\rho_n(y)dy - \int_{\mathbb{R}^n} f(x)\rho_n(y)dy\right| \\ &= \int_{B\left(0,\frac{1}{n}\right)} \left|f(x-y) - f(x)\right| \rho_n(y)dy. \end{aligned}$$

Finally, for $\varepsilon > 0$ and for $E\left(\frac{1}{\eta}\right)$ we have for all $n > \frac{1}{\eta} > E(\frac{1}{\eta})$ and for all $x \in K$ we have

$$\left|(\rho_n * f)(x) - f(x)\right| < \varepsilon.$$

■

Theorem 1.4.4 *Let $f \in L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$. Then $\rho_n * f$ converges to f in $L^p(\mathbb{R}^n)$.*

Proof. Let $f \in L^p(\mathbb{R}^n)$ and $\varepsilon > 0$. Since $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, then $\exists f_\varepsilon \in C_c(\mathbb{R}^n)$ such that $\|f - f_\varepsilon\| < \frac{3}{\varepsilon}$.

According to the previous proposition $\rho_n * f_\varepsilon$ converges uniformly to f_ε on any compact.

On the other hand we have

$$supp(\rho_n * f_\varepsilon) \subset \overline{supp(\rho_n) + supp(f_\varepsilon)},$$

and

$$supp(\rho_n) \subset B\left(0, \frac{1}{\eta}\right),$$

so

$$supp(\rho_n * f_\varepsilon) \subset \overline{B\left(0, \frac{1}{\eta}\right) + supp(f_\varepsilon)} \subset K,$$

where K is a fixed compact. K being compact therefore bounded and measurable therefore $\lambda(K) < \infty$ in addition $\rho_n * f_\varepsilon$ converges uniformly towards f_ε so

$$\|\rho_n * f_\varepsilon - f_\varepsilon\|_p \text{ tends to wards 0 with } \frac{1}{\eta}.$$

We have

$$\begin{aligned}(\rho_n * f)(x) - f(x) &= (\rho_n * f)(x) - (\rho_n * f_\varepsilon)(x) + (\rho_n * f_\varepsilon)(x) - f_\varepsilon(x) + f_\varepsilon(x) - f(x)\\ &= \rho_n * (f - f_\varepsilon)(x) + (\rho_n * f_\varepsilon)(x) - f_\varepsilon(x) + f_\varepsilon(x) - f(x).\end{aligned}$$

So

$$\|\rho_n * f - f\|_p \leq \|\rho_n * (f - f_\varepsilon)\|_p + \|\rho_n * f_\varepsilon - f_\varepsilon\|_p + \|f_\varepsilon - f\|_p.$$

Now according to Theorem 1.4.1, we have

$$\|\rho_n * f - f\|_p \leq \|\rho_n\|_1 \|f - f_\varepsilon\|_p = \|f - f_\varepsilon\|_p.$$

hence

$$\begin{aligned}\|\rho_n * f - f\|_p &\leq \|f - f_\varepsilon\|_p + \|\rho_n * f_\varepsilon - f_\varepsilon\|_p + \|f_\varepsilon - f\|_p\\ &\leq 2\|f - f_\varepsilon\|_p + \|\rho_n * f_\varepsilon - f_\varepsilon\|_p\\ &\leq 2\frac{\varepsilon}{3}\|\rho_n * f_\varepsilon - f_\varepsilon\|_p.\end{aligned}$$

Let $\varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} : \forall n \geq N_\varepsilon$, we have $\|\rho_n * f_\varepsilon - f_\varepsilon\|_p < \frac{\varepsilon}{3}$. By sequence $\forall n \geq N_\varepsilon$, we have $\|\rho_n * f - f\|_p < \varepsilon$, i.e.

$$\lim_{n \to +\infty} \|\rho_n * f - f\|_p = 0.$$

■

Corollary 1.4.1 *Let $\Omega \subset \mathbb{R}^n$ be any open set, then $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for $1 \leq p < \infty$.*

Proof. Let $f \in L^p(\Omega)$, let us show that there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset C_c^\infty(\Omega)$ such that $(f_n)_{n\in\mathbb{N}}$ converges to f in $L^p(\Omega)$. We put

$$\overline{f}(x) = \begin{cases} f(x), & \text{if } x \in \Omega \\ 0, & \text{if } x \in \mathbb{R}^n \backslash \Omega \end{cases},$$

then $\overline{f} \in L^p(\mathbb{R}^n)$. Let $f_n = n * f$ with $(\rho_n)_{n\geq 1}$ is a regularizing sequence. By Theorem 1.4.2, we have $f_n \in C_c^\infty(\mathbb{R}^n), \forall n \geq 1$ and by Theorem 1.4.3, the sequence $(f_n)_{n\geq 1}$ converges to $\overline{f}$ in $L^p(\mathbb{R}^n)$, hence $f_{n/\Omega}$ converges to $\overline{f}_{/\Omega} = f$ in $L^p(\Omega)$. Therefore $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$.

■

1.5 Corrected exercises

Exercise 1.5.1 *Study the membership in $L^1(\mathbb{R})$ and in $L^2(\mathbb{R})$ of the following functions:*

$$f_1(x) = \sin(x) 1_{[-\pi,\pi]}\ ,\quad f_2(x) = \frac{\sin(x)}{x} 1_{[1+\infty[}\ ,\ f_3(x) = e^{-a|x|}(a > 0).$$

Solution:

$f_1 \in L^1(\mathbb{R})$

$$\begin{aligned} \int_{-\infty}^{+\infty} |f_1(x)|\, dx &= \int_{-\infty}^{+\infty} \left|\sin(x) 1_{[-\pi,\pi]}\right| dx \\ &= \int_{-\pi}^{+\pi} |\sin(x)|\, dx \\ &= 2\int_0^{+\pi} \sin(x) dx \\ &= 2\left[\cos(x)\right]_0^\pi \\ &= 1 < +\infty. \end{aligned}$$

Then $f_1 \in L^1(\mathbb{R})$.

$f_1 \in L^2(\mathbb{R})$

$$\begin{aligned}
\int_{-\infty}^{+\infty} |f_1(x)|^2\, dx &= \int_{-\infty}^{+\infty} \left|\sin(x)1_{[-\pi,\pi]}\right|^2 dx \\
&= \int_{-\pi}^{+\pi} \sin^2(x)dx \\
&= \frac{1}{2}\int_{-\pi}^{+\pi} (1-\cos(2x))\, dx \\
&= \frac{1}{2}\left[x - \frac{1}{2}\sin(2x)\right]_{-\pi}^{+\pi} \\
&= \pi < +\infty.
\end{aligned}$$

Then $f_1 \in L^2(\mathbb{R})$.

$f_2 \in L^1(\mathbb{R})$

$$\begin{aligned}
\int_{-\infty}^{+\infty} |f_2(x)|\, dx &= \int_{-\infty}^{+\infty} \left|\frac{\sin(x)}{x}1_{[1+\infty[}\right| dx \\
&= \int_{1}^{+\infty} \left|\frac{\sin(x)}{x}\right| dx.
\end{aligned}$$

We integrate by parts, we obtain

$$\begin{aligned}
\int_{1}^{+\infty} \frac{\sin(x)}{x}dx &= \left[-\frac{\cos(x)}{x}\right]_1^{*\infty} - \int_1^{+\infty}\frac{\cos(x)}{x^2}dx \\
&= \cos 1 - \int_1^{+\infty}\frac{\cos(x)}{x^2}dx.
\end{aligned}$$

In addition, we have

$$\left|\frac{\cos(x)}{x^2}\right| \leq \frac{1}{x^2}.$$

So

$$\begin{aligned}
\int_{1}^{+\infty} \frac{\sin(x)}{x}dx &\leq \cos 1 - \int_1^{+\infty}\frac{1}{x^2}dx \\
&\leq \cos 1 - \left[-\frac{1}{x}\right]_1^{*\infty} \\
&\leq \cos 1 - 1 < +\infty.
\end{aligned}$$

Then $f_1 \in L^1(\mathbb{R})$.

$f_2 \in L^2(\mathbb{R})$

$$\begin{array}{rcl}\displaystyle\int_{-\infty}^{+\infty} |f_1(x)|^2\, dx & = & \displaystyle\int_{-\infty}^{+\infty} \left|\frac{\sin(x)}{x} 1_{[1+\infty[}\right|^2 dx \\ & = & \displaystyle\int_{1}^{+\infty} \frac{\sin^2(x)}{x^2} dx \\ & \leq & \displaystyle\int_{1}^{+\infty} \frac{1}{x^2} dx \\ & \leq & \displaystyle\left[-\frac{1}{x}\right]_1^{+\infty} \\ & = & 1 < +\infty.\end{array}$$

Then $f_2 \in L^2(\mathbb{R})$.

$f_3 \in L^1(\mathbb{R})$

$$\begin{array}{rcl}\displaystyle\int_{-\infty}^{+\infty} |f_3(x)|\, dx & = & \displaystyle\int_{-\infty}^{+\infty} \left|e^{-a|x|}\right| dx \\ & = & \displaystyle\int_{-\infty}^{0} e^{ax} dx + \int_{0}^{+\infty} e^{-ax} dx \\ & = & \displaystyle\left[\frac{1}{a}e^{ax}\right]_{-\infty}^{0} + \left[-\frac{1}{a}e^{-ax}\right]_{0}^{+\infty} \\ & = & \displaystyle\frac{2}{a} < +\infty \ (a > 0).\end{array}$$

Then $f_3 \in L^1(\mathbb{R})$.

$f_3 \in L^2(\mathbb{R})$

$$\begin{array}{rcl}\displaystyle\int_{-\infty}^{+\infty} |f_3(x)|^2\, dx & = & \displaystyle\int_{-\infty}^{+\infty} \left|e^{-a|x|}\right|^2 dx \\ & = & \displaystyle\int_{-\infty}^{0} e^{2ax} dx + \int_{0}^{+\infty} e^{-2ax} dx \\ & = & \displaystyle\left[\frac{1}{2a}e^{2ax}\right]_{-\infty}^{0} + \left[-\frac{1}{2a}e^{-2ax}\right]_{0}^{+\infty} \\ & = & \displaystyle\frac{1}{a} < +\infty \ (a > 0).\end{array}$$

Then $f_3 \in L^2(\mathbb{R})$.

Exercise 1.5.2 *Let* $f \in C^2(\mathbb{R})$ *assume that* f *and* f'' *belong to* $L^2(\mathbb{R})$.

Show that $f\ f''$ *belongs to* $L^1(\mathbb{R})$.

Solution:

We deduce from the elementary inequality

$$|ab| \leq \frac{1}{2}\left(a^2+b^2\right), \forall a, b \in \mathbb{R},$$

that

$$|f(x)f''(x)| \leq \frac{1}{2}\left((f(x))^2+(f''(x))^2\right),$$

and therefore

$$\begin{aligned}\int_{\mathbb{R}} |f(x)f''(x)|\,dx &\leq \frac{1}{2}\left(\int_{\mathbb{R}} (f(x))^2\,dx + \int_{\mathbb{R}} (f''(x))^2\,dx\right)\\ &\leq \frac{1}{2}\left(\|f\|_{L^2}^2 + \|f''\|_{L^2}^2\right) < +\infty,\end{aligned}$$

because

$$f \in L^2(\mathbb{R}) \text{ and } f'' \in L^2(\mathbb{R}) \Longrightarrow ff'' \in L^1(\mathbb{R}).$$

Exercise 1.5.3 *Suppose that* Ω *is of finite measure* ($|\Omega| < \infty$) *and let* $1 \leq p \leq q \leq \infty$.

Prove that $L^q(\Omega) \subset L^p(\Omega)$ *with continuous injection. More precisely, show that*

$$\|f\|_{L^p} \leq |\Omega|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q}, \forall f \in L^q(\Omega).$$

*[**Hint:** Use Hölder inequality]*

Solution:

Suppose $1 \leq p < q$ (Because for $p = q$ the inequality is trivial).

We apply Hölder's inequality with exponents

$$r = \frac{q}{p} \text{ (because } 1 \leq p < q),$$

and

$$r' = \frac{q}{q-p},$$

with

$$\frac{1}{r} + \frac{1}{r'} = 1.$$

Then, we have

$$\begin{aligned}\|f\|_p^p &= \int_\Omega |f(x)|^p\,dx = \int_\Omega |f(x)|^p\,.1dx \\ &\leq \left(\int_\Omega |f(x)|^{p.\frac{q}{p}}\,dx\right)^{\frac{p}{q}} \left(\int_\Omega 1^{\frac{q}{q-p}}dx\right)^{\frac{q-p}{q}} \\ &\leq \left(\int_\Omega |f(x)|^q\,dx\right)^{\frac{p}{q}} |\Omega|^{1-\frac{p}{q}}\end{aligned}$$

$$\Longrightarrow \|f\|_{L^p} \leq \left(\int_\Omega |f(x)|^q\,dx\right)^{\frac{1}{q}} |\Omega|^{\frac{1}{p}-\frac{1}{q}} \leq \|f\|_{L^q}\,|\Omega|^{\frac{1}{p}-\frac{1}{q}}\,.$$

As $1 \leq p < q$, then $\frac{1}{p} - \frac{1}{q} > 1$, finally $\|f\|_{L^p} < \infty$.

Furthermore the inequality below shows that the inclusion of $L^q(\Omega)$ in $L^p(\Omega)$ is continuous.

Exercise 1.5.4 *Let $C([0,1])$ be the space of all continuous functions on $[0,1]$.*

1- Prove that if $C([0,1])$ is equipped with the uniform norm

$$\|f\|_{L^\infty} = \max_{x\in[0,1]} |f(x)|\,, \forall f \in C([0,1])\,,$$

then $C([0,1])$ is a Banach space (i.e. a complete normed vector space).

2- Give an example showing that $C([0,1])$ equipped with the norm of $L^1([0,1])$

$$\|f\|_{L^1} = \int_0^1 |f(x)|\,dx, \forall f \in C([0,1])\,,$$

is not a Banach space.

Solution:

1- We show that $C([0,1])$ is a Banach space over $[0,1]$ equipped with the norm $\|f\|_{L^\infty} = \max_{x\in[0,1]} |f(x)|$. It suffices to show that $C([0,1])$ is a complete space. To do this, consider a Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in $C([0,1])$ and show that it converges uniformly in $C([0,1])$ to a function $f \in C([0,1])$.

According to Cauchy's criterion, for all $\varepsilon > 0$, there exists a rank $N \in \mathbb{N}$, such that for all $n, m \geq N$ we have

$$\|f_n - f_m\|_{L^\infty} \leq \varepsilon.$$

So for all $x \in [0,1]$, we have

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_{L^\infty} \leq \varepsilon.$$

It follows that the numerical sequence $(f_n(x))_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$ complete, which ensures that there exists a real number $f(x) \in \mathbb{R}$ such that

$$f_n(x) \underset{n\to\infty}{\to} f(x) \text{ in } \mathbb{R}.$$

So, we fix $n \in \mathbb{N}$ and by passing to the limit $m \to +\infty$, it comes that

$$\|f_n - f\|_{L^\infty} \leq \varepsilon,$$

which ensures that f_n converges uniformly towards f on $[0,1]$ so that f is continuous i.e. $f \in C([0,1])$ then

$$\lim_{n\to+\infty} \|f_n - f\|_{L^\infty} = 0.$$

2- We give an example showing that $C([0,1])$ equipped with the norm of $L^1([0,1])$ defined by $\|f\|_{L^1} = \int_0^1 |f(x)|\,dx$, is not a Banach space.

For each $n \in \mathbb{N}$, we consider the function

$$f_n(x) = \begin{cases} nx, & \text{if } 0 \leq x < \frac{1}{n} \\ 1, & \text{if } \frac{1}{n} < x \leq 1 \end{cases}.$$

We have the sequence (f_n) is Cauchy with respect to the norm of $L^1([0,1])$ but it converges towards

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } 0 < x \leq 1 \end{cases},$$

which is not continuous i.e. $f \notin C([0,1])$, so the space $C([0,1])$ equipped with the norm of $L^1([0,1])$ is not a Banach space.

Exercise 1.5.5 *Let $f \in L^2(\mathbb{R}^+)$. We put for $x > 0$,*

$$F(x) = \frac{1}{x}\int_0^x f(t)dt.$$

1- Prove that

$$\left(\int_0^x f(t)dt\right)^2 \leq 2\sqrt{x}\int_0^x \sqrt{t}\,|f(t)|^2\,dt.$$

2- Deduce that $F \in L^2(\mathbb{R}^+)$.

Solution:

Let $f \in L^2(\mathbb{R}^+)$. We put for $x > 0$

$$F(x) = \frac{1}{x}\int_0^x f(t)dt.$$

1- We demonstrate that

$$\left(\int_0^x f(t)dt\right)^2 \leq 2\sqrt{x}\int_0^x \sqrt{t}\,|f(t)|^2\,dt.$$

We use the Cauchy Schwarz inequality

$$\int_0^x |g(t)h(t)|\,dt \leq \left(\int_0^x |g(t)|^2\,dt\right)^{\frac{1}{2}}\left(\int_0^x |h(t)|^2\,dt\right)^{\frac{1}{2}}.$$

To do this, we must write

$$f(t) = g(t)h(t) \text{ with } h^2(t) = \sqrt{t}\,|f(t)|^2\,.$$

Therefore, we have

$$\begin{aligned}
\int_0^x f(t)dt &= \int_0^x t^{-\frac{1}{4}}t^{\frac{1}{4}}f(t)dt \\
&\leq \left[\int_0^x t^{-\frac{1}{2}}dt\right]^{\frac{1}{2}}\left[\int_0^x t^{\frac{1}{2}}\,|f(t)|^2\,dt\right]^{\frac{1}{2}} \\
&\leq \sqrt{2}x^{\frac{1}{4}}\left[\int_0^x t^{\frac{1}{2}}\,|f(t)|^2\,dt\right]^{\frac{1}{2}}.
\end{aligned}$$

$$\Longrightarrow \left(\int_0^x f(t)dt\right)^2 \leq 2\sqrt{x}\int_0^x \sqrt{t}\,|f(t)|^2\,dt.$$

2- We deduce that $F \in L^2(\mathbb{R}^+)$ i.e. $\int_0^{+\infty}|F(x)|^2\,dx < \infty$.

For this we have

$$\begin{aligned}
\int_0^{+\infty}|F(x)|^2\,dx &= \int_0^{+\infty}\frac{1}{x^2}\left(\int_0^x f(t)dt\right)^2 dx \\
&\leq 2\int_0^{+\infty}\frac{1}{x^{\frac{3}{2}}}\left(\int_0^x \sqrt{t}\,|f(t)|^2\,dt\right)^2 dx.
\end{aligned}$$

We apply the Fubini-Tonelli theorem. The condition $x \geq 0$ and $t \in [0, x]$ translates into $t \geq 0$ and $x \in [t, +\infty]$, we find

$$\begin{aligned} \int_0^{+\infty} |F(x)|^2 \, dx &\leq 2 \int_0^{+\infty} \sqrt{t} \, |f(t)|^2 \left(\int_t^{+\infty} \frac{1}{x^{\frac{3}{2}}} dx \right)^2 dt \\ &\leq 2 \int_0^{+\infty} t^{\frac{1}{2}} \, |f(t)|^2 \times 2t^{-\frac{1}{2}} dt \\ &\leq 4 \int_0^{+\infty} |f(t)|^2 \, dt < \infty. \end{aligned}$$

We deduce that

$$F \in L^2(\mathbb{R}^+) \text{ and } \|F\|_2 \leq 2 \|f\|_2 .$$

Exercise 1.5.6 *1- Calculate*

$$I = \int\int_D \frac{1}{(1+x^2)(1+y^2)} dxdy,$$

such that $D = \{(x, y) : 0 \leq x \leq y\}$.

2- Show that we cannot apply Fubini's theorem for the calculation of

$$\int\int_D \frac{x^2 - y^2}{(x^2 + y^2)^2} dxdy \text{ with } D =]0, 1]^2.$$

Solution:

1- The function $(x; y) \mapsto \frac{1}{(1+x^2)(1+y^2)}$ is continuous and positive on D, therefore measurable, we can therefore apply Fubini's theorem. Then x varies from 0 to y with y a positive real number.

$$\begin{aligned} \int\int_D \frac{1}{(1+x^2)(1+y^2)} dxdy &= \int_0^{+\infty} \left(\int_0^y \frac{1}{(1+x^2)(1+y^2)} dx \right) dy \\ &= \int_0^{+\infty} \left(\frac{1}{(1+y^2)} \left[\arctan x \right]_0^y \right) dy \\ &= \int_0^{+\infty} \frac{\arctan y}{(1+y^2)} dy \\ &= \left[\frac{(\arctan y)^2}{2} \right]_0^{+\infty} \\ &= \frac{\pi^2}{8}. \end{aligned}$$

So

$$I = \int\int_D \frac{1}{(1+x^2)(1+y^2)} dxdy = \frac{\pi^2}{8}.$$

2- We put $f(x,y)=\dfrac{x^2-y^2}{(x^2+y^2)^2}$ and we calculate the two integrals

$$\int_0^1\left(\int_0^1 f(x,y)dx\right)dy \text{ and } \int_0^1\left(\int_0^1 f(x,y)dy\right)dx.$$

So

$$\begin{aligned}\int_0^1 f(x,y)dx &= \int_0^1 \frac{2x^2}{(x^2+y^2)^2}dx-\int_0^1\frac{1}{x^2+y^2}dx\\ &= \left[\frac{-x}{x^2+y^2}\right]_0^1+\int_0^1\frac{1}{x^2+y^2}dx \;-\int_0^1\frac{1}{x^2+y^2}dx\\ &= -\frac{1}{1+y^2}.\end{aligned}$$

So

$$\int_0^1\left(\int_0^1 f(x,y)dx\right)dy=\int_0^1-\frac{1}{1+y^2}dy=-\frac{\pi}{4}.$$

In the same way we calculate the other integral and we will find

$$\int_0^1\left(\int_0^1 f(x,y)dy\right)dx=\int_0^1\frac{1}{1+x^2}dx=\frac{\pi}{4}.$$

These results show us that Fubini's theorem does not apply to this integral, because the function $f\notin L^1(]0,1]^2)$. Indeed, by passing to polar coordinates we find

$$\int\int_D f(x,y)dxdy=\int_0^{2\pi}\int_0^1\frac{|\cos 2\theta|}{r}drd\theta.$$

1.6 Suggested exercises

Exercise 1.6.1 *Let $f:\mathbb{R}\to\mathbb{R}$ be defined by*

$$x\mapsto f(x)=\begin{cases} x^\alpha, & \text{if } 0\le x\le 1\\ 0, & \text{if } x<0\end{cases}.$$

1- Find the value of α such that $f\in L^{\frac{3}{2}}(]-\infty,1])$.

2- Find the value of α such that $f\notin L^1(]-\infty,1])$.

Exercise 1.6.2 *Let (Ω,A,μ) be a measured space, $p\in[1,+\infty[$ and let $g\in L^q(\Omega)$, where q is the conjugate exponent of p.*

Let $T : L^p(\Omega) \rightarrow \mathbb{C}$ *defined by* $T(f) = \int_\Omega f\overline{g}d\mu$.

1- Show that T *is defined and continuous. Show that* $\|T\| \leq \|g\|_{L^q}$.

2- Let

$$f(x) = \begin{cases} g(x)\,|g(x)|^{q-2}, & \text{if } g(x) \neq 0 \\ 0, & \text{if } g(x) = 0 \end{cases}.$$

Show that $\|f\|_{L^p}^p = \|g\|_{L^q}^q$, *Calculate* $T(f)$, *deduce that* $\|T\| = \|g\|_{L^q}$.

Exercise 1.6.3 *Let* f *be a complex function, measurable on* $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$.

1- Let $1 \leq \alpha < \beta < \infty$. *Assume that* $f \in L^\alpha(\mathbb{R}) \cap L^\beta(\mathbb{R})$. *Show that for all* $p \in [\alpha, \beta]$, *we have* $|f|^p \leq |f|^\alpha + |f|^\beta$. *Deduce that* $\{p \in [1, +\infty[, f \in L^p(\mathbb{R})\}$ *is an interval of* $\mathbb{R}$.

2- Show that the application $p \longmapsto \| f \|_{L^q}$ *is continuous on its domain of definition.*

Exercise 1.6.4 *Let* $f \in L^p(\mathbb{R})$ *with* $1 < p < +\infty$.

1- Show that we can define $\forall\, x \geq 0, F(x) = \int_0^x f(t)dt$. *Justify that* $F(x) =_{+\infty} O(x^{(p-1)/p})$.

2- Let $\varepsilon > 0$. *Demonstrate that there exists* $a > 0$ *such that* $\left(\int_a^{+\infty} |f(t)|^p dt\right)^{1/p} \leq \varepsilon$.

3- Deduce that $F(x) =_{+\infty} o(x^{(p-1)/p})$.

Exercise 1.6.5 *Let* a, b *be two real numbers with* $a < b$ *and let* $f \in L^\infty([a, b])$. *Prove that*

$$\lim_{p \to +\infty} \| f \|_{L^p} = \| f \|_{L^\infty}.$$

Exercise 1.6.6 *1- For* $x > 0$, *calculate the integral*

$$\int_0^{+\infty} \frac{1}{(1+y)(1+x^2y)}dy,$$

and deduce the value of

$$\int_0^{+\infty} \frac{\ln x}{x^2 - 1}dx.$$

2- For $x > 0$, *calculate the integral*

$$\int_0^1 \cos(xy)dy,$$

and deduce the value of

$$\int_0^{+\infty} \frac{\sin x}{x} e^{-tx}dx,$$

for all $t > 0$.

Chapter 2

Fourier transform

The Fourier transform is an extension for non-periodic functions of the Fourier series expansion of periodic functions. The Fourier transform associates with an integrable function defined on the set of real numbers or that of complex numbers, a function called the Fourier transform whose independent variable can be interpreted in physics as the frequency or the pulsation.

When a function represents a physical phenomenon such as the state of the electromagnetic field or the acoustic field at a point, it is called a signal and its Fourier transform is called its spectrum.

2.1 Fourier transform for integrable functions

Definition 2.1.1 *Let* $f \in L^1(\mathbb{R})$. *We call the Fourier transform of the function* f, *the function* $\mathcal{F}(f) = \widehat{f} : \mathbb{R} \to \mathbb{C}$ *defined by*

$$\omega \mapsto \mathcal{F}(f)(\omega) = \widehat{f}(\omega) = \int_{-\infty}^{+\infty} f(x)e^{-2\pi i\omega x}dx, \omega \in \mathbb{R}.$$

This integral is well defined because

$$\left|f(x)e^{-2\pi i\omega x}\right| = |f(x)| \text{ and } f \in L^1(\mathbb{R}).$$

The previous definition of the Fourier transform is not always the one chosen. For example, some authors define the Fourier transform of f by

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-i\omega x}dx,$$

$$\widehat{f}(\omega) = \int_{-\infty}^{+\infty} f(x)e^{-i\omega x}dx,$$

$$\widehat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{-i\omega x}dx.$$

The definition chosen may lead to changes in certain properties simply because, for example

$$\widehat{f}(\omega) = \widehat{f}\left(\frac{\omega}{2\pi}\right).$$

It is therefore always necessary to check the definition used before applying a formula.

Proposition 2.1.1 *If* $f \in L^1(\mathbb{R})$, *then we have*

1) $\widehat{f}$ *is a continuous and bounded function on* $\mathbb{R}$.

2) $\left\|\widehat{f}\right\|_\infty \leq \|f\|_1$.

3) $\lim\limits_{\omega\to\infty} \widehat{f}(\omega) = 0$.

Proof. We have $\widehat{f}(\omega) = \int_{-\infty}^{+\infty} f(x)e^{-2\pi i\omega x}dx, \omega \in \mathbb{R}$.

1) The function under the integral sign is continuous for almost all $x \in \mathbb{R}$ and is measurable for all $\omega \in \mathbb{R}$. Furthermore we have $|f(x)e^{-2\pi i\omega x}| = |f(x)|, \forall\omega \in \mathbb{R}$.

We know that $|f(x)| \in L^1(\mathbb{R})$ and according to the continuity theorem for functions by an integral the function $\widehat{f}$ is continuous. Moreover we have

$$\left|\widehat{f}(\omega)\right| \leq \int_{\mathbb{R}} |f(x)|\, dx = \|f\|_1, \forall\omega \in \mathbb{R},$$

which shows that $\widehat{f}$ is bounded.

2)

$$\left\|\widehat{f}\right\|_\infty = \sup_{\omega\in\mathbb{R}} \left|\widehat{f}(\omega)\right| \leq \int_{\mathbb{R}} |f(x)|\, dx = \|f\|_1.$$

3) We now show that $\lim\limits_{\omega\to\infty} \widehat{f}(\omega) = 0$.

We know that the space of stepped functions with bounded support on $\mathbb{R}$ is dense in $L^1(\mathbb{R})$, i.e. for any function $f \in L^1(\mathbb{R})$ and $\forall\varepsilon > 0$, we can find a stepped function φ such as

$$\|f - \varphi\|_1 < \frac{\varepsilon}{2},$$

with

$$\varphi(x) = \sum_{k=1}^{n} C_k 1_{[\alpha_{k-1},\alpha_k]}(x), \forall x \in \mathbb{R},$$

where $C_1, C_2, ..., C_k \in \mathbb{R}$ and $[\alpha_{k-1}, \alpha_k]\,, k = 1, 2, ..., n$ are intervals.

Let us first calculate the Fourier transform of φ

$$\begin{aligned}\widehat{\varphi}(\omega) &= \int_{\mathbb{R}} \sum_{k=1}^{n} C_k 1_{[\alpha_{k-1},\alpha_k]}(x) e^{-2\pi i \omega x} dx \\ &= \sum_{k=1}^{n} C_k \int_{\alpha_{k-1}}^{\alpha_k} e^{-2\pi i \omega x} dx \\ &= \sum_{k=1}^{n} C_k \left[\frac{-1}{2\pi i \omega} e^{-2\pi i \omega x} \right]_{\alpha_{k-1}}^{\alpha_k} \\ &= \sum_{k=1}^{n} \frac{i C_k}{2\pi\omega} \left(e^{-2\pi i \omega \alpha_k} - e^{-2\pi i \omega \alpha_{k-1}} \right).\end{aligned}$$

Which implies that

$$\begin{aligned}|\widehat{\varphi}(\omega)| &\leq \sum_{k=1}^{n} \left| \frac{C_k}{2\pi\omega} \right| \left(\left| e^{-2\pi i \omega \alpha_k} \right| + \left| e^{-2\pi i \omega \alpha_{k-1}} \right| \right) \\ &\leq \frac{1}{\pi} \left(\sum_{k=1}^{n} |C_k| \right) \frac{1}{|\omega|}.\end{aligned}$$

So

$$\lim_{\omega \longrightarrow \infty} \widehat{\varphi}(\omega) = 0,$$

i.e. for $\omega \longrightarrow \infty$, we have

$$|\widehat{\varphi}(\omega)| < \frac{\varepsilon}{2}.$$

Therefore

$$\begin{aligned}\left| \widehat{f}(\omega) \right| &\leq \left| \widehat{f}(\omega) - \widehat{\varphi}(\omega) \right| + |\widehat{\varphi}(\omega)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon,\end{aligned}$$

i.e.

$$\lim_{\omega \longrightarrow \infty} \widehat{f}(\omega) = 0.$$

■

Example 2.1.1 ***Gate function***

Let $a > 0$, the gate function is defined by

$$\Pi_a(x) = \begin{cases} 1, & \textit{if} \quad |x| \leq a \\ 0, & \textit{if} \quad |x| > a \end{cases}.$$

Let's calculate its Fourier transform

$$\begin{aligned} \widehat{\Pi}_a(\omega) &= \int_{-\infty}^{+\infty} \Pi_a(x) e^{-2\pi i \omega x} dx \\ &= \int_{-a}^{+a} e^{-2\pi i \omega x} dx \\ &= -\frac{1}{2\pi i \omega} \left[e^{-2\pi i \omega x} \right]_{-a}^{+a} \\ &= \frac{1}{\pi \omega} \left[\frac{e^{2\pi i \omega a} - e^{-2\pi i \omega a}}{2i} \right] \\ &= \frac{\sin(2\pi \omega a)}{\pi \omega}, \omega \neq 0. \end{aligned}$$

If $\omega = 0$, we have

$$\widehat{\Pi}_a(0) = \int_{-a}^{+a} dx = 2a.$$

So

$$\widehat{\Pi}_a(\omega) = \begin{cases} \frac{\sin(2\pi\omega a)}{\pi\omega}, & \textit{if} \;\; \omega \neq 0 \\ 2a, & \textit{if} \;\; \omega = 0 \end{cases}.$$

Example 2.1.2 ***Exponential function***

Let $a > 0$, the exponential function be defined by

$$f(x) = e^{-a|x|}.$$

Let's calculate its Fourier transform

$$\begin{aligned} \widehat{f}(\omega) &= \int_{-\infty}^{+\infty} f(x) e^{-2\pi i \omega x} dx \\ &= \int_{-\infty}^{+\infty} e^{-a|x|} e^{-2\pi i \omega x} dx \\ &= \int_{-\infty}^{0} e^{(a-2\pi i \omega)x} dx + \int_{0}^{+\infty} e^{-(a+2\pi i \omega)x} dx \\ &= \frac{1}{a - 2\pi i \omega} \left[e^{(a-2\pi i \omega)x} \right]_{-\infty}^{0} - \frac{1}{a + 2\pi i \omega} \left[e^{-(a+2\pi i \omega)x} \right]_{0}^{+\infty} \\ &= \frac{1}{a - 2\pi i \omega} + \frac{1}{a + 2\pi i \omega} \\ &= \frac{2a}{a^2 + 4\pi^2 \omega^2}. \end{aligned}$$

2.2 Cosine and Sine-Fourier transforms

Definition 2.2.1 *If the function $f(x)$ is real and even, the Fourier transform of $f(x)$is a real and even function. We call the cosine-Fourier transform of $f(x)$, the function*

$$\widehat{f}(\omega) = 2\int_0^{+\infty} f(x)\cos(2\pi\omega x)dx.$$

Indeed, by separating the domain of integration into $\mathbb{R} = \mathbb{R}^- \cup \mathbb{R}^+$, it comes by making the change of variable $t = -x$ in the first integral

$$\begin{aligned}
\widehat{f}(\omega) &= \int_{-\infty}^0 f(x)e^{-2\pi i\omega x}dx + \int_0^{+\infty} f(x)e^{-2\pi i\omega x}dx \\
&= -\int_{+\infty}^0 f(-t)e^{2\pi i\omega t}dt + \int_0^{+\infty} f(x)e^{-2\pi i\omega x}dx \\
&= \int_0^{+\infty} f(x)e^{2\pi i\omega x}dx + \int_0^{+\infty} f(x)e^{-2\pi i\omega x}dx \\
&= 2\int_0^{+\infty} f(x)\left(\frac{e^{2\pi i\omega x} + e^{-2\pi i\omega x}}{2}\right)dx \\
&= 2\int_0^{+\infty} f(x)\cos(2\pi\omega x)dx.
\end{aligned}$$

Since the cosine function is even, we have $\widehat{f}(-\omega) = \widehat{f}(\omega)$, and obviously $\widehat{f}(\omega)$ is real.

A similar reasoning allows us to conclude that if $f(x)$ is imaginary and even, then $\widehat{f}(\omega)$ will be imaginary and even.

Definition 2.2.2 *Similarly, if $f(x)$ is real and odd, then the Fourier transform of $f(x)$ is imaginary and even. We call the sine-Fourier transform of $f(x)$, the function*

$$\widehat{f}(\omega) = -2i\int_0^{+\infty} f(x)\sin(2\pi\omega x)dx.$$

Indeed

$$\begin{aligned}
\widehat{f}(\omega) &= \int_{-\infty}^0 f(x)e^{-2\pi i\omega x}dx + \int_0^{+\infty} f(x)e^{-2\pi i\omega x}dx \\
&= -\int_{+\infty}^0 f(-t)e^{2\pi i\omega t}dt + \int_0^{+\infty} f(x)e^{-2\pi i\omega x}dx \\
&= -\int_0^{+\infty} f(x)e^{2\pi i\omega x}dx + \int_0^{+\infty} f(x)e^{-2\pi i\omega x}dx \\
&= -2i\int_0^{+\infty} f(x)\left(\frac{e^{2\pi i\omega x} - e^{-2\pi i\omega x}}{2i}\right)dx \\
&= -2i\int_0^{+\infty} f(x)\sin(2\pi\omega x)dx.
\end{aligned}$$

Since the sine function is odd, we have $\widehat{f}(-\omega) = \widehat{f}(\omega)$, and $\widehat{f}(\omega)$ is clearly imaginary.

A similar reasoning allows us to conclude that if $f(x)$ is imaginary and odd, then $\widehat{f}(\omega)$ will be real and even.

2.3 Properties of Fourier transform

2.3.1 Linearity

Theorem 2.3.1 *Let f and g be two functions that admit a Fourier transform, then for all real numbers α and β, we have*

$$\mathcal{F}(\alpha f + \beta g)(\omega) = \alpha \mathcal{F}(f)(\omega) + \beta \mathcal{F}(g)(\omega).$$

Proof. According to the definition of the Fourier transform, we have

$$\begin{aligned}
\mathcal{F}(\alpha f + \beta g)(\omega) &= \int_{-\infty}^{+\infty} (\alpha f + \beta g)(x) e^{-2\pi i \omega x} dx \\
&= \alpha \int_{-\infty}^{+\infty} f(x) e^{-2\pi i \omega x} dx + \beta \int_{-\infty}^{+\infty} g(x) e^{-2\pi i \omega x} dx \\
&= \alpha \mathcal{F}(f)(\omega) + \beta \mathcal{F}(g)(\omega).
\end{aligned}$$

■

2.3.2 Complex conjugation

Theorem 2.3.2 *Let f be a function that admits a Fourier transform. Then $\overline{f} : x \to \overline{f(x)}$ also admits a Fourier transform and*

$$\mathcal{F}(\overline{f(x)})(\omega) = \overline{\widehat{f}(-\omega)}.$$

Proof. We have

$$\begin{aligned}
\mathcal{F}(\overline{f(x)})(\omega) &= \int_{-\infty}^{+\infty} \overline{f(x)} e^{-2\pi i \omega x} dx \\
&= \int_{-\infty}^{+\infty} \overline{f(x) e^{2\pi i \omega x}} dx \\
&= \overline{\widehat{f}(-\omega)}.
\end{aligned}$$

■

2.3.3 Translation

Theorem 2.3.3 *Let f be a function that admits a Fourier transform and $a \in \mathbb{R}$. Then the Fourier transform of a translated function of a is given by*

$$\mathcal{F}((\varphi_a f)(x))(\omega) = \mathcal{F}(f(x-a))(\omega) = e^{-2\pi i\omega a}\widehat{f}(\omega).$$

where $(\varphi_a f)(x) = f(x-a)$.

Proof. With the change of variable $y = x - a$, we find

$$\begin{aligned}\mathcal{F}(f(x-a))(\omega) &= \int_{-\infty}^{+\infty} f(x-a)e^{-2\pi i\omega x}dx \\ &= \int_{-\infty}^{+\infty} f(y)e^{-2\pi i\omega(y+a)}dy \\ &= e^{-2\pi i\omega a}\int_{-\infty}^{+\infty} f(y)e^{-2\pi i\omega y}dy \\ &= e^{-2\pi i\omega a}\widehat{f}(\omega).\end{aligned}$$

■

2.3.4 Change of scale

Theorem 2.3.4 *Let f be a function that admits a Fourier transform and $\lambda \in \mathbb{R}$. Then the Fourier transform of a function whose variable undergoes a change of scale of ratio $\lambda > 0$, is given by*

$$\mathcal{F}((h_\lambda f)(x))(\omega) = \mathcal{F}(f(\lambda x))(\omega) = \frac{1}{|\lambda|}\widehat{f}\left(\frac{\omega}{\lambda}\right),$$

where $(h_\lambda f)(x) = f(\lambda x)$.

Proof. We have

$$\mathcal{F}(f(\lambda x))(\omega) = \int_{-\infty}^{+\infty} f(\lambda x)e^{-2\pi i\omega x}dx.$$

There are two cases

1) If $\lambda > 0$ we put $y = \lambda x \Longrightarrow dy = \lambda dx$, then

$$\begin{aligned}\mathcal{F}(f(\lambda x))(\omega) &= \int_{-\infty}^{+\infty} f(y)e^{-2\pi i\omega\frac{y}{\lambda}}\frac{dy}{\lambda} \\ &= \frac{1}{\lambda}\int_{-\infty}^{+\infty} f(y)e^{-2\pi iy\frac{\omega}{\lambda}}dy \\ &= \frac{1}{\lambda}\widehat{f}\left(\frac{\omega}{\lambda}\right).\end{aligned}$$

2) If $\lambda < 0$, then

$$\begin{aligned}\mathcal{F}(f(\lambda x))(\omega) &= \int_{+\infty}^{-\infty} f(y)e^{-2\pi i\omega\frac{y}{\lambda}}\frac{dy}{\lambda}\\ &= -\frac{1}{\lambda}\int_{-\infty}^{+\infty} f(y)e^{-2\pi i y\frac{\omega}{\lambda}}dy\\ &= -\frac{1}{\lambda}\widehat{f}\left(\frac{\omega}{\lambda}\right).\end{aligned}$$

Finally, we have

$$\mathcal{F}(f(\lambda x))(\omega) = \frac{1}{|\lambda|}\widehat{f}\left(\frac{\omega}{\lambda}\right).$$

■

2.3.5 Modulation

Theorem 2.3.5 *Let f be a function that admits a Fourier transform and $\omega_0 \in \mathbb{R}$. Then the Fourier transform of a signal $f(x)$ modulated by $e^{-2\pi i\omega_0 x}$ is given by*

$$\mathcal{F}(e^{-2\pi i\omega_0 x}f(x))(\omega) = \widehat{f}\,(\omega - \omega_0)\,.$$

This property is shown by recognizing the Fourier transform of $f(x)$ where ω is replaced by $\omega - \omega_0$.

Proof. We have

$$\begin{aligned}\mathcal{F}(e^{-2\pi i\omega_0 x}f(x))(\omega) &= \int_{-\infty}^{+\infty} e^{-2\pi i\omega_0 x}f(x)e^{-2\pi i\omega x}dx\\ &= \int_{-\infty}^{+\infty} f(x)e^{-2\pi i\omega x}e^{-2\pi i\omega_0 x}dx\\ &= \int_{-\infty}^{+\infty} f(x)e^{-2\pi i(\omega-\omega_0)x}dx\\ &= \widehat{f}\,(\omega-\omega_0)\,.\end{aligned}$$

■

2.3.6 Convolution

Theorem 2.3.6 *Let f and g be two functions that admit a Fourier transform, then the Fourier transform of a convolution product of two functions $f * g$ is the usual product of the*

Fourier transforms of each function and we have

$$\mathcal{F}((f * g)(x))(\omega) = \widehat{f}(\omega)\,\widehat{g}(\omega).$$

Proof. The definition of the convolution product of f and g is given by

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(t)g(x-t)dt.$$

Then

$$\begin{aligned}\mathcal{F}((f * g)(x))(\omega) &= \int_{-\infty}^{+\infty} (f * g)(x))e^{-2\pi i\omega x}dx \\ &= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(t)g(x-t)dt\right) e^{-2\pi i\omega x}dx.\end{aligned}$$

Applying Fubini's theorem, we obtain

$$\mathcal{F}((f * g)(x))(\omega) = \int_{-\infty}^{+\infty} f(t) \left(\int_{-\infty}^{+\infty} g(x-t)e^{-2\pi i\omega x}dx\right) dt.$$

We put $y = x - t \Longrightarrow dy = dx$, then we have

$$\begin{aligned}\mathcal{F}((f * g)(x))(\omega) &= \int_{-\infty}^{+\infty} f(t) \left(\int_{-\infty}^{+\infty} g(y)e^{-2\pi i\omega(y+t)}dy\right) dt \\ &= \int_{-\infty}^{+\infty} f(t)e^{-2\pi i\omega t}dt \times \int_{-\infty}^{+\infty} g(y)e^{-2\pi i\omega y}dy \\ &= \widehat{f}(\omega)\,\widehat{g}(\omega).\end{aligned}$$

■

2.3.7 Fourier transform of the derivative of a function

Theorem 2.3.7 *Let f be a function of class C^1 that admits a Fourier transform. Further assume that $\lim\limits_{x\pm\infty} f(x) = 0$. Then the Fourier transform of the derivative $f'(x)$ of the function $f(x)$ is given by the following relation*

$$\mathcal{F}(f'(x))(\omega) = 2\pi i\omega\widehat{f}(\omega).$$

Proof. We have

$$\begin{aligned}\mathcal{F}(f'(x))(\omega) &= \int_{-\infty}^{+\infty} f'(x)e^{-2\pi i\omega x}dx \\ &= \lim_{A\to\infty}\left[f(x)e^{-2\pi i\omega x}\right]_{-A}^{+A} + 2\pi i\omega\int_{-\infty}^{+\infty} f(x)e^{-2\pi i\omega x}dx \text{ (Integration by part)} \\ &= 0 + 2\pi i\omega\int_{-\infty}^{+\infty} f(x)e^{-2\pi i\omega x}dx \\ &= 2\pi i\omega\widehat{f}(\omega)\,.\end{aligned}$$

■

Corollary 2.3.1 *Let f be a function of class C^k and admitting a Fourier transform. Suppose that $\lim_{x\to\pm\infty} f^{(k)}(x) = 0$. for $0 \le k \le n$. Then the Fourier transform of the derivative $f^{(n)}(x)$ of the function $f(x)$ is given by the following relation*

$$\mathcal{F}(f^{(n)}(x))(\omega) = (2\pi i\omega)^n\,\widehat{f}(\omega)\,.$$

Proof. This relation can be demonstrated by recurrence. ■

2.3.8 Derivation of the Fourier transform

Theorem 2.3.8 *Let $f \in L^1(\mathbb{R})$. If $xf(x) \in L^1(\mathbb{R})$, then $\widehat{f}$ is differentiable and the derivation of $\widehat{f}(\omega)$ with respect to the variable ω gives*

$$\frac{d}{d\omega}\widehat{f}(\omega) = \mathcal{F}(-2\pi ixf(x))(\omega).$$

Also, if $x^n f(x) \in L^1(\mathbb{R})$, then

$$\frac{d^n}{d\omega^n}\widehat{f}(\omega) = \mathcal{F}((-2\pi ix)^n\,f(x))(\omega).$$

Proof. We have

$$\frac{d}{d\omega}\widehat{f}(\omega) = \frac{d}{d\omega}\int_{-\infty}^{+\infty} f(x)e^{-2\pi i\omega x}dx.$$

As $|-2\pi ixf(x)e^{-2\pi i\omega x}| = 2\pi\,|xf(x)|$ and by hypothesis $xf(x) \in L^1(\mathbb{R})$, then according to the theorem of derivation under the integral sign we have

$$\begin{aligned}\frac{d}{d\omega}\widehat{f}(\omega) &= \int_{-\infty}^{+\infty} \frac{d}{d\omega}f(x)e^{-2\pi i\omega x}dx \\ &= \int_{-\infty}^{+\infty} -2\pi ixf(x)e^{-2\pi i\omega x}dx \\ &\mathcal{F}(-2\pi ixf(x))(\omega).\end{aligned}$$

More generally $x^n f(x) \in L^1(\mathbb{R})$, we can show by recurrence that

$$\frac{d^n}{d\omega^n} \widehat{f}(\omega) = \mathcal{F}((-2\pi i x)^n f(x))(\omega).$$

■

2.4 Inverse Fourier transform

Definition 2.4.1 *Let f be a function in $L^1(\mathbb{R})$. We call the inverse Fourier transform of f denoted $\mathcal{F}^{-1}$ such that if $\widehat{f}(\omega) = \mathcal{F}(f(x))(\omega)$, then $f(x)$ is the inverse Fourier transform of $\widehat{f}(\omega)$. In other words, we have the equivalence*

$$\widehat{f}(\omega) = \mathcal{F}(f(x))(\omega) \Leftrightarrow f(x) = \mathcal{F}^{-1}(\widehat{f}(\omega)).$$

The inverse Fourier transform is defined by

$$f(x) = \mathcal{F}^{-1}(\widehat{f}(\omega)) = \int_{-\infty}^{+\infty} \widehat{f}(\omega)\, e^{2\pi i \omega x} d\omega.$$

More generally if f is not continuous at x_0 we have

$$\int_{-\infty}^{+\infty} \widehat{f}(\omega)\, e^{2\pi i \omega x} d\omega = \frac{f(x_0+0) + f(x_0-0)}{2},$$

where

$$f(x_0+0) = \lim_{h \overset{\geq}{\to} 0} f(x_0+h) \text{ and } f(x_0-0) = \lim_{h \overset{\leq}{\to} 0} f(x_0+h),$$

are the limits to the right and left of $f(x)$.

Exercise 2.4.1 *1) Find the Fourier transform of the function f defined by*

$$f(x) = \begin{cases} 1, & \text{if } |x| \leq a \\ 0, & \text{if } |x| > a \end{cases}.$$

2) Using the inverse Fourier transform, calculate the integral

$$\int_0^{+\infty} \frac{\cos(2\pi\omega x)\sin(2\pi\omega a)}{\omega} d\omega.$$

3) Deduce the value of the integral

$$\int_0^{+\infty} \frac{\sin(x)}{x} dx.$$

Solution: 1) The Fourier transform of f is

$$\widehat{f}(\omega)=\int_{-\infty}^{+\infty} f(x)e^{-2\pi i\omega x}dx=\int_{-a}^{+a} e^{-2\pi i\omega x}dx=\frac{\sin(2\pi\omega a)}{\pi\omega}, \omega\neq 0.$$

2) We have

$$\mathcal{F}^{-1}(\widehat{f}(\omega))=f(x).$$

$$\Leftrightarrow \int_{-\infty}^{+\infty} e^{2\pi i\omega x}\widehat{f}(\omega)\,d\omega=f(x)$$

$$\Leftrightarrow \int_{-\infty}^{+\infty} (\cos(2\pi\omega x)+i\sin(2\pi\omega x))\frac{\sin(2\pi\omega a)}{\pi\omega}d\omega=\begin{cases}1, & \text{if } |x|\leq a\\ 0, & \text{if } |x|>a\end{cases}.$$

Which implies that

$$\Longrightarrow \int_{-\infty}^{+\infty}\frac{\cos(2\pi\omega x)\sin(2\pi\omega a)}{\pi\omega}d\omega+i\underbrace{\int_{-\infty}^{+\infty}\frac{\sin(2\pi\omega x)\sin(2\pi\omega a)}{\pi\omega}d\omega}_{=0}=\begin{cases}1, & \text{if } |x|\leq a\\ 0, & \text{if } |x|>a\end{cases}$$

$$\Longrightarrow \int_{-\infty}^{+\infty}\frac{\cos(2\pi\omega x)\sin(2\pi\omega a)}{\omega}d\omega=\begin{cases}\pi, & \text{if } |x|\leq a\\ 0, & \text{if } |x|>a\end{cases}.$$

3) For $x=0$ and $a=\frac{1}{2\pi}$, then we have

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{\sin(\omega)}{\omega}d\omega &= \pi\Longrightarrow 2\int_{0}^{+\infty}\frac{\sin(\omega)}{\omega}d\omega=\pi\\ &\Longrightarrow \int_{0}^{+\infty}\frac{\sin(\omega)}{\omega}d\omega=\frac{\pi}{2}.\end{aligned}$$

We put $\omega=x$, we get

$$\int_{0}^{+\infty}\frac{\sin(x)}{x}dx=\frac{\pi}{2}.$$

Proposition 2.4.1 *Let $f\in L^1(\mathbb{R})$ such that $\widehat{f}\in L^1(\mathbb{R})$, then for all $x\in\mathbb{R}$ we have*

$$\mathcal{F}\left(\mathcal{F}(f\,(x))\right)(\omega)=\widehat{\widehat{f}}(\omega)=f(-x).$$

Proof. We have

$$\widehat{f}(\omega)=\int_{-\infty}^{+\infty} f(x)e^{-2\pi i\omega x}dx,$$

and

$$\mathcal{F}^{-1}(\widehat{f}\,(\omega)) = f(x) = \int_{-\infty}^{+\infty} \widehat{f}\,(\omega)\, e^{2\pi i\omega x} d\omega.$$

So

$$\begin{aligned} f(-x) &= \int_{-\infty}^{+\infty} \widehat{f}\,(\omega)\, e^{-2\pi i\omega x} d\omega \\ &= \mathcal{F}(\widehat{f}\,(\omega)). \end{aligned}$$

Finally, we have

$$f(-x) = \mathcal{F}\left(\mathcal{F}(f\,(x))\right)(\omega) = \widehat{\widehat{f}}(\omega).$$

■

2.5 Fourier transform for square-summable functions

Proposition 2.5.1 *Let f and g be two functions of $L^1(\mathbb{R})$. We have $f\widehat{g}$ and $\widehat{f}g$ are in $L^1(\mathbb{R})$ and*

$$\int_{\mathbb{R}} f(x)\widehat{g}(x)dx = \int_{\mathbb{R}} \widehat{f}(\omega)g(\omega)d\omega.$$

Proof. We have $f, g \in L^1(\mathbb{R})$ and according to Proposition 2.1.1 we have $\widehat{g}$ is bounded, therefore $f\widehat{g} \in L^1(\mathbb{R})$. Similarly $\widehat{f}g \in L^1(\mathbb{R})$ (by hypothesis).

We apply Fubini's theorem and we get

$$\begin{aligned} \int_{\mathbb{R}} f(x)\widehat{g}(x)dx &= \int_{\mathbb{R}} f(x) \left(\int_{\mathbb{R}} g\,(\omega)\, e^{-2\pi i\omega x} d\omega \right) dx \\ &= \int_{\mathbb{R}} g\,(\omega) \left(\int_{\mathbb{R}} f(x) e^{-2\pi i\omega x} dx \right) d\omega \\ &= \int_{\mathbb{R}} g\,(\omega)\, \widehat{f}(\omega) d\omega. \end{aligned}$$

■

Proposition 2.5.2 ***(Parseval-Plancherel Equality)***

Let f and g be two piecewise C^1 functions, absolutely integrable and square summable over $\mathbb{R}$. Then we have Parseval's equality

$$\int_{\mathbb{R}} f(x)\overline{g(x)}dt = \int_{\mathbb{R}} \widehat{f}(\omega)\overline{\widehat{g}(\omega)}d\omega.$$

Proposition 2.5.3 *In particular (taking $f = g$), we obtain Plancherel's identity*

$$\int_{\mathbb{R}} |f(x)|^2 \, dt = \int_{\mathbb{R}} \left|\widehat{f}(\omega)\right|^2 d\omega.$$

Proof. Let us apply Proposition 2.5.1, we put $h(\omega) = \overline{\widehat{g}(\omega)}$, so we have

$$\int_{\mathbb{R}} \widehat{f}(\omega)\overline{\widehat{g}(\omega)}d\omega = \int_{\mathbb{R}} \widehat{f}(\omega)h(\omega)d\omega = \int_{\mathbb{R}} f(x)\widehat{h}(x)dx.$$

But

$$\begin{aligned} h(\omega) &= \overline{\widehat{g}(\omega)} \\ &= \int_{\mathbb{R}} \overline{g\,(x)\, e^{-2\pi i\omega x}} dx \\ &= \int_{\mathbb{R}} \overline{g\,(x)} e^{2\pi i\omega x} dx \\ &= \overline{\widehat{g(-x)}}. \end{aligned}$$

According to Proposition 2.4.1 we have

$$\widehat{h}(x) = \overline{\widehat{\widehat{g(-x)}}} = \overline{g(x)},$$

which implies that

$$\int_{\mathbb{R}} \widehat{f}(\omega)\overline{\widehat{g}(\omega)}d\omega = \int_{\mathbb{R}} f(x)\overline{g(x)}dx.$$

■

Exercise 2.5.1 *We recall that the Fourier transform of function*

$$f(x) = \left(\frac{\sin x}{x}\right)^2,$$

is function

$$\widehat{f}(\omega) = \pi(1 - \pi\,|\omega|)\mathcal{X}_{\left[-\frac{1}{\pi}, \frac{1}{\pi}\right]}(\omega).$$

From the Parseval-Plancherel equality, deduce the value of the integral

$$I = \int_{-\infty}^{+\infty} \left(\frac{\sin x}{x}\right)^4 dx.$$

Solution: We have

$$I=\int_{-\infty}^{+\infty}\left(\frac{\sin x}{x}\right)^4 dx=\int_{-\infty}^{+\infty}\left(\left(\frac{\sin x}{x}\right)^2\right)^2 dx.$$

From the Parseval-Plancherel equality we have

$$\int_{-\infty}^{+\infty}|f(x)|^2\,dt=\int_{-\infty}^{+\infty}\left|\widehat{f}(\omega)\right|^2 d\omega.$$

So it is enough to calculate

$$\int_{-\infty}^{+\infty}\left|\widehat{f}(\omega)\right|^2 d\omega=?$$

We have

$$\begin{aligned}\int_{-\infty}^{+\infty}\left|\widehat{f}(\omega)\right|^2 d\omega &= \pi^2\int_{-\frac{1}{\pi}}^{\frac{1}{\pi}}(1-\pi\,|\omega|)^2 d\omega\\ &= 2\pi^2\int_0^{\frac{1}{\pi}}(1-\pi\omega)^2 d\omega\\ &= 2\pi^2\int_0^{\frac{1}{\pi}}(1-2\pi\omega+\pi^2\omega^2)d\omega\\ &= 2\pi^2\left[\omega-\pi\omega^2+\pi^2\frac{\omega^3}{3}\right]_0^{\frac{1}{\pi}}\\ &= 2\pi^2\left[\frac{1}{\pi}-\pi\frac{1}{\pi^2}+\pi^2\frac{1}{3\pi^3}\right]\\ &= \frac{2\pi}{3}.\end{aligned}$$

So

$$I=\int_{-\infty}^{+\infty}\left(\frac{\sin x}{x}\right)^4 dx=\frac{2\pi}{3}.$$

2.6 Applications

2.6.1 Ordinary differential equations

The Fourier transform allows you to explicitly solve a linear differential equation by transforming it into a simpler equation. For example, if the initial equation has constant coefficients, the Fourier transform of this equation is an algebraic equation. The Fourier transform is used when the equation is set over all $\mathbb{R}$. If the data in the equation are periodic, Fourier series are used instead. If the equation is set over a half-line with initial conditions for the desired solution, the Laplace transform is used instead.

To solve linear differential equations with constant coefficients, the following properties of the Fourier transform are used.

Proposition 2.6.1 *Let* $f, g : \mathbb{R} \to \mathbb{C}$ *be integrable* $f \xrightarrow{\mathcal{F}} \widehat{f}$ *and* $g \xrightarrow{\mathcal{G}} \widehat{g}$*. Then*

$$\begin{aligned}
&\frac{df}{dx} \xrightarrow{\mathcal{F}} (2\pi i\omega)\, \widehat{f}\,(\omega)\,,\\
&\frac{d^2 f}{dx^2} \xrightarrow{\mathcal{F}} (2\pi i\omega)^2\, \widehat{f}\,(\omega) = -4\pi^2\omega^2 \widehat{f}\,(\omega)\,,\\
&\frac{d^2 f}{dx^2} \xrightarrow{\mathcal{F}} (2\pi i\omega)^3\, \widehat{f}\,(\omega) = -8\pi^3 i\omega^3 \widehat{f}\,(\omega)\,,\\
&\vdots
\end{aligned}$$

Proof. Use the definition of $\mathcal{F}$ and the integration by parts formula. ■

Proposition 2.6.2 *Let* $f : \mathbb{R} \to \mathbb{C}$ *integrable* $f \xrightarrow{\mathcal{F}} \widehat{f}$ *. Then*

$$\begin{aligned}
xf(x) &\xrightarrow{\mathcal{F}} \frac{i}{2\pi}\frac{d\widehat{f}\,(\omega)}{d\omega}\\
x^2 f(x) &\xrightarrow{\mathcal{F}} \left(\frac{i}{2\pi}\right)^2 \frac{d^2\widehat{f}\,(\omega)}{d\omega^2}\\
&= \frac{-1}{4\pi^2}\frac{d^2\widehat{f}\,(\omega)}{d\omega^2}.
\end{aligned}$$

Proof. Use the definition of $\mathcal{F}$ and the formula $xe^{-2\pi i\omega x} = \frac{i}{2\pi}\frac{\partial}{\partial\omega}\left(e^{-2\pi i\omega x}\right)$, and the properties of integrals depending on a parameter. ■

Example 2.6.1 *Find a function* $f : \mathbb{R} \to \mathbb{C}$ *integrable on* $\mathbb{R}$ *such that*

$$-\frac{d^2 f(x)}{dx^2} + f(x) = e^{-x^2}, x \in \mathbb{R}.$$

The Fourier transform of the above equation is written as

$$4\pi^2\omega^2 \widehat{f}\,(\omega) + \widehat{f}\,(\omega) = \mathcal{F}\left(e^{-x^2}\right),$$

where the unknown is $\widehat{f} = \mathcal{F}(f)$*. We deduce that*

$$\begin{aligned}
\widehat{f}\,(\omega) &= \frac{1}{4\pi^2\omega^2 + 1}\mathcal{F}\left(e^{-x^2}\right)\\
&= \frac{1}{2}\mathcal{F}\left(e^{-|x|}\right)\mathcal{F}\left(e^{-x^2}\right).
\end{aligned}$$

The last equality being a consequence of $e^{-|x|} \xrightarrow{\mathcal{F}} \frac{2}{1+(2\pi\omega)^2}$ *(see Example 2.1.2).*

Therefore

$$\widehat{f}\,(\omega) = \frac{1}{2}\mathcal{F}\left(e^{-|x|} * e^{-x^2}\right) \Longrightarrow f(x) = \frac{1}{2}\left(e^{-|x|} * e^{-x^2}\right).$$

since $\mathcal{F}$ *is injective.*

Example 2.6.2 *Using the Fourier transform, find a particular solution to the following second-order differential equation*

$$y''(x) + 2\pi t y'(x) + 2\pi y(x) = 0. \tag{E}$$

Suppose that $\mathcal{F}(y(x)) = \widehat{y}(\omega)$ *is the Fourier transform of* $y(x)$*. Then the Fourier transform of equation (E) is written*

$$\begin{aligned}
\mathcal{F}((E)) \quad &= \quad 0 \Longleftrightarrow (2\pi i\omega)^2\,\widehat{y}\,(\omega) + 2\pi\left(\frac{-1}{2i\pi}\right)[2i\pi\omega\widehat{y}\,(\omega)]' + 2\pi\widehat{y}\,(\omega) = 0 \\
&\Longleftrightarrow \quad \left(-4\pi^2\omega^2 + 2\pi\right)\widehat{y}\,(\omega) - 2\pi\omega\widehat{y}'\,(\omega) = 0 \\
&\Longleftrightarrow \quad -\omega\widehat{y}'\,(\omega) + \left(-2\pi\omega^2 + 1\right)\widehat{y}\,(\omega) = 0.
\end{aligned}$$

This last differential equation is linear without a right-hand side. The solution is

$$\begin{aligned}
\widehat{y}\,(\omega) \quad &= \quad k\omega e^{-\pi\omega^2} \Longrightarrow y(x) = \mathcal{F}^{-1}(\widehat{y}\,(\omega)) = k\left(e^{-\pi x^2}\right)' = -2k\pi x e^{-\pi x^2} \\
&\Longrightarrow \quad y(x) = Cxe^{-\pi x^2} \text{ with } C = -2k\pi \in \mathbb{R}.
\end{aligned}$$

2.6.2 Partial differential equations

Example 2.6.3 *Using the Fourier transform, determine the solution* $u = u(x,t)$ *to the heat equation*

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}, \ a > 0.$$

with the initial condition: $u(x,0) = u_0(x)$ *where* $u_0(x)$ *is the temperature at time* $t = 0$*.*

We put

$$\mathcal{F}(u(x,t))(\omega) = \widehat{u}(\omega,t) = \int_{-\infty}^{+\infty} u(x,t)e^{-2\pi i\omega x}dx.$$

Then we have

$$\frac{\partial\widehat{u}}{\partial t}(\omega,t) = \int_{-\infty}^{+\infty}\frac{\partial u}{\partial t}(x,t)e^{-2\pi i\omega x}dx = a\int_{-\infty}^{+\infty}\frac{\partial^2 u}{\partial x^2}(x,t)e^{-2\pi i\omega x}dx.$$

Taking into account Proposition 2.6.1, we obtain

$$\mathcal{F}\left(\frac{\partial u}{\partial x}(x,t)\right)(\omega) = \frac{\partial \widehat{u}}{\partial \omega}(\omega,t) = \int_{-\infty}^{+\infty} \frac{\partial u}{\partial x}(x,t)e^{-2\pi i\omega x}dx = (2\pi i\omega)\,\widehat{u}\,(\omega,t)\,,$$
$$\mathcal{F}\left(\frac{\partial^2 u}{\partial x^2}(x,t)\right)(\omega) = \frac{\partial^2 \widehat{u}}{\partial \omega^2}(\omega,t) = \int_{-\infty}^{+\infty} \frac{\partial^2 u}{\partial x^2}(x,t)e^{-2\pi i\omega x}dx = -4\pi^2\omega^2\widehat{u}\,(\omega,t)\,.$$

Therefore, the previous equation is written in the form

$$\frac{\partial \widehat{u}}{\partial t}(\omega,t) = -4a\pi^2\omega^2\widehat{u}\,(\omega,t)\,.$$

By integrating this equation into the unknown $\widehat{u}\,(\omega,t)$ of the variable t, we obtain

$$\widehat{u}\,(\omega,t) = Ce^{-4a\pi^2\omega^2 t},$$

where

$$\widehat{u}\,(\omega,0) = C = \int_{-\infty}^{+\infty} u(x,0)e^{-2\pi i\omega x}dx = \int_{-\infty}^{+\infty} u_0(x)e^{-2\pi i\omega x}dx.$$

So

$$\widehat{u}\,(\omega,t) = e^{-4a\pi^2\omega^2 t}\int_{-\infty}^{+\infty} u_0(x)e^{-2\pi i\omega x}dx = e^{-4a\pi^2\omega^2 t}\widehat{u_0}(\omega),$$

i.e.

$$\mathcal{F}\left(u(x,t)\right)(\omega) = e^{-4a\pi^2\omega^2 t}\mathcal{F}\left(u_0(x)\right)(\omega).$$

Since $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$, we can put $u(x,t) = f * g$, with $f = u_0(x)$ given and g such that

$$\mathcal{F}(g) = \widehat{g}(\omega) = e^{-4a\pi^2\omega^2 t},$$

we have

$$g = \int_{-\infty}^{+\infty} \widehat{g}(\omega)e^{2\pi i\omega x}d\omega = \frac{1}{4\pi\sqrt{a\pi t}}e^{-\frac{x^2}{16a\pi^2 t}}.$$

Finally, we get the solution

$$\begin{aligned} u(x,t) &= u_0 * \frac{1}{4\pi\sqrt{a\pi t}}e^{-\frac{x^2}{16a\pi^2 t}} \\ &= \frac{1}{4\pi\sqrt{a\pi t}}\int_{-\infty}^{+\infty} u_0(y)e^{-\frac{(x-y)^2}{16a\pi^2 t}}dy. \end{aligned}$$

2.7 Fourier transform of usuel functions

The following table shows the expression of Fourier transforms for some usuel functions.

Function	Fourier Transform
$e^{-ax^2}, a > 0$	$\sqrt{\frac{\pi}{a}} \exp\left(-\frac{\pi^2\omega^2}{a}\right)$
$e^{-a\|x\|}, a > 0$	$\frac{2a}{a^2+4\pi^2\omega^2}$
$e^{-a\|x\|}\text{sign}(x), a > 0$	$a^2 + 4\pi^2\omega$
$e^{-ax}H(x), a > 0$	$\frac{1}{a^2+2i\pi\omega}$
$\frac{1}{a^2+x^2}$	$\frac{\pi}{\|a\|} e^{-2\pi a\|\omega\|}$
$\Pi(x)$	$\frac{1}{\|a\|} \sin c\left(\frac{\omega}{a}\right)$
$\sin c\,(ax)$	$\frac{1}{\|a\|}\Pi\left(\frac{\omega}{a}\right)$
$\sin c^2\,(ax)$	$\frac{1}{\|a\|}\text{tri}\left(\frac{\omega}{a}\right)$

Table 2.1: Table of usuel Fourier transforms

2.8 Corrected exercises

Exercise 2.8.1 *Let $\Pi(x)$ be the rectangular (or gate) function and $\Delta(x)$ the triangle function, defined by*

$$\Pi(x) = \begin{cases} 1, & \text{if} \quad |x| \leq \frac{1}{2} \\ 0, & \text{if} \quad |x| > \frac{1}{2} \end{cases} \quad \text{and } \Delta(x) = \begin{cases} 1 - |x|, & \text{if} \quad |x| < 1 \\ 0, & \text{if} \quad |x| \geq 1 \end{cases}.$$

1- Compute the Fourier transforms $\Pi(x)$ and $\Delta(x)$.

2- Deduce the Fourier transforms of the following functions

$$a)\ \Pi\left(\frac{x+1}{2}\right),\ b)\ (x+1)\,\Pi(x),\ c)\ \frac{\sin(\pi x)}{\pi x},\ d)\ \left(\frac{\sin(\pi x)}{\pi x}\right)^2,$$

3- Using Parsevals identity, deduce the values of the integrals

$$\int_{\mathbb{R}} \left(\frac{\sin(\pi x)}{\pi x}\right)^n dx,\ \text{for } n = 2, 4.$$

Solution:

1) We compute the Fourier transform of $\Pi(x)$ and $\Delta(x)$, for this we have

$$\begin{aligned}
\widehat{\Pi}(\omega) &= \int_{-\infty}^{+\infty} \Pi(x) e^{-2\pi i\omega x} dx \\
&= \int_{-\frac{1}{2}}^{+\frac{1}{2}} e^{-2\pi i\omega x} dx \\
&= \left[-\frac{1}{2\pi i\omega} e^{-2\pi i\omega x}\right]_{-\frac{1}{2}}^{\frac{1}{2}} \\
&= -\frac{1}{2\pi i\omega}\left[e^{-\pi i\omega} - e^{\pi i\omega}\right] \\
&= \frac{1}{\pi\omega}\left[\frac{e^{\pi i\omega} - e^{*\pi i\omega}}{2i}\right] \\
&= \frac{\sin(\pi\omega)}{\pi\omega}, \quad \text{if } \omega \neq 0.
\end{aligned}$$

For $\omega = 0$, we have $\widehat{\Pi}(0) = 1$.

Since $\Delta(x)$ is even then

$$\begin{aligned}
\widehat{\Delta}(\omega) &= 2\int_{0}^{+\infty} \Delta(x)\cos(2\pi\omega x) dx \\
&= 2\int_{0}^{1} \underbrace{(1-x)}_{u}\underbrace{\cos(2\pi\omega x) dx}_{dv}.
\end{aligned}$$

Integration by parts gives

$$\begin{aligned}
\widehat{\Delta}(\omega) &= 2\int_{0}^{+\infty} \Delta(x)\cos(2\pi\omega x) dx \\
&= \left[\frac{2(1-x)\sin(2\pi\omega x)}{2\pi\omega}\right]_{0}^{1} + \frac{2}{2\pi\omega}\int_{0}^{1} \sin(2\pi\omega x) dx \\
&= \frac{1}{\pi\omega}\left[-\frac{1}{2\pi\omega}\cos(2\pi\omega x)\right]_{0}^{1} \\
&= \frac{1}{\pi^2\omega^2}\left(\frac{1-\cos(2\pi\omega)}{2}\right) \\
&= \frac{\sin^2(\pi\omega)}{\pi^2\omega^2} \\
&= \left(\frac{\sin(\pi\omega)}{\pi\omega}\right)^2, \quad \text{if } \omega \neq 0.
\end{aligned}$$

For $\omega = 0$, we have $\widehat{\Delta}(0) = 1$.

2) We deduce the Fourier transform of

a) $\Pi\left(\dfrac{x+1}{2}\right)$, for this we have

$$\Pi\left(\frac{x+1}{2}\right)=\Pi\left(\frac{1}{2}\left(x-(-1)\right)\right),$$

then

$$\begin{aligned}
\mathcal{F}\left(\Pi\left(\frac{x+1}{2}\right)\right) &= \mathcal{F}\left(\Pi\left(\frac{1}{2}\left(x-(-1)\right)\right)\right)\\
&= \mathcal{F}\left(h_{\frac{1}{2}}\left(\varphi_{-1}\Pi\right)(x)\right)(\omega)\\
&= 2\left(\widehat{\varphi_{-1}\Pi}\right)(2\omega)\\
&= 2e^{-2\pi i(2\omega)(-1)}\widehat{\Pi}(2\omega)\\
&= 2e^{4\pi i\omega}\frac{\sin(2\pi\omega)}{2\pi\omega}\\
&= e^{4\pi i\omega}\frac{\sin(2\pi\omega)}{\pi\omega}.
\end{aligned}$$

b) $(x+1)\,\Pi(x)$, for this we have

$$(x+1)\,\Pi(x)=x\Pi(x)+\Pi(x),$$

then

$$\begin{aligned}
\mathcal{F}\left((x+1)\,\Pi(x)\right) &= \mathcal{F}\left(x\Pi(x)+\Pi(x)\right)\\
&= \mathcal{F}\left(x\Pi(x)\right)+\mathcal{F}\left(\Pi(x)\right)\\
&= -\frac{1}{2\pi i}\frac{d}{d\omega}\widehat{\Pi}(\omega)+\widehat{\Pi}(\omega)\\
&= -\frac{1}{2\pi i}\frac{d}{d\omega}\frac{\sin(\pi\omega)}{\pi\omega}+\frac{\sin(\pi\omega)}{\pi\omega}\\
&= \frac{\sin(\pi\omega)}{2i\pi^2\omega^2}-\frac{\cos(\pi\omega)}{2i\pi\omega}+\frac{\sin(\pi\omega)}{\pi\omega}.
\end{aligned}$$

c) $\dfrac{\sin(\pi x)}{\pi x}$, we have the property

$$\mathcal{F}\left(\mathcal{F}\left(\Pi(x)\right)\right)(\omega)=\widehat{\widehat{\Pi}}(\omega),$$

then

$$\mathcal{F}\left(\frac{\sin(\pi x)}{\pi x}\right)(\omega)=\Pi(-x)=\Pi(x),$$

because $\Pi(x)$ is an even function.

So the Fourier transform of $\dfrac{\sin(\pi x)}{\pi x}$, is ginen by

$$\mathcal{F}\left(\frac{\sin(\pi x)}{\pi x}\right)(\omega) = \Pi(\omega) = \begin{cases} 1, & \text{if} \quad |\omega| \leq \frac{1}{2} \\ 0, & \text{if} \quad |\omega| > \frac{1}{2} \end{cases}.$$

d) $\left(\dfrac{\sin(\pi x)}{\pi x}\right)^2$ we have the property

$$\mathcal{F}\left(\mathcal{F}\left(\Delta(x)\right)\right)(\omega) = \widehat{\widehat{\Delta}}(\omega),$$

then

$$\mathcal{F}\left(\left(\frac{\sin(\pi x)}{\pi x}\right)^2\right)(\omega) = \Delta(-x) = \Delta(x),$$

because $\Delta(x)$ is an even function.

So the Fourier transform of $\left(\dfrac{\sin(\pi x)}{\pi x}\right)^2$, is ginen by

$$\mathcal{F}\left(\left(\frac{\sin(\pi x)}{\pi x}\right)^2\right)(\omega) = \Delta(\omega) = \begin{cases} 1 - |\omega|, & \text{if} \quad |\omega| < 1 \\ 0, & \text{if} \quad |\omega| \geq 1 \end{cases}.$$

3) We use the Parseval identity and have

- For $n = 2$

$$\begin{aligned} \int_{\mathbb{R}} \left(\frac{\sin(\pi x)}{\pi x}\right)^2 dx &= \int_{\mathbb{R}} \left(\Pi(\omega)\right)^2 d\omega \\ &= \int_{-\frac{1}{2}}^{+\frac{1}{2}} 1 d\omega \\ &= [\omega]_{-\frac{1}{2}}^{\frac{1}{2}} \\ &= 1. \end{aligned}$$

- For $n = 4$

$$\begin{aligned}
\int_{\mathbb{R}} \left(\frac{\sin(\pi x)}{\pi x}\right)^4 dx &= \int_{\mathbb{R}} \left(\left(\frac{\sin(\pi x)}{\pi x}\right)^2\right)^2 dx \\
&= \int_{\mathbb{R}} (\Delta(\omega))^2 \, d\omega \\
&= \int_{-1}^{0} (1+\omega)^2 \, d\omega + \int_{0}^{1} (1-\omega)^2 \, d\omega \\
&= \int_{-1}^{0} \left(1+2\omega+\omega^2\right) d\omega + \int_{0}^{1} \left(1-2\omega+\omega^2\right) d\omega \\
&= \left[\omega + 2\frac{\omega^2}{2} + \frac{\omega^3}{3}\right]_{-1}^{0} + \left[\omega - 2\frac{\omega^2}{2} + \frac{\omega^3}{3}\right]_{0}^{1} \\
&= \frac{2}{3}.
\end{aligned}$$

Exercise 2.8.2 *Let* $f \in L^1(\mathbb{R})$.

1- Using the Fourier transform, show that there does not exist any function $g \in L^1(\mathbb{R})$ *such that, for all* $f \in L^1(\mathbb{R})$

$$g * f = f.$$

2- Solve in $L^1(\mathbb{R})$ *the convolution equation*

$$f * f = f.$$

Solution:

1- By the absurd, we suppose that there exists $g \in L^1(\mathbb{R})$ such that $g * f = f$.

We apply the Fourier transform we obtain

$$\mathcal{F}(g * f)(\omega) = \mathcal{F}(f)(\omega)$$

$$\begin{aligned}
&\Longrightarrow \widehat{g}(\omega)\widehat{f}(\omega) = \widehat{f}(\omega) \\
&\Longrightarrow \widehat{f}(\omega)\left[\widehat{g}(\omega) - 1\right] = 0 \\
&\Longrightarrow \begin{cases} \widehat{f}(\omega) = 0 \\ \quad\vee \\ \widehat{g}(\omega) = 1 \end{cases}.
\end{aligned}$$

$\widehat{g}(\omega) = 1$ is a contradiction, because $g \in L^1(\mathbb{R})$ and $\lim\limits_{\omega \longrightarrow \pm\infty} \widehat{g}(\omega) = 0$, so there is no $g \in L^1(\mathbb{R})$ so that we have $g * f = f$.

2- We solve the convolution equation $f * f = f$.

By applying the Fourier transform we find

$$\mathcal{F}\left(f * f\right)(\omega) = \mathcal{F}\left(f\right)(\omega)$$

$$\begin{aligned}
&\Longrightarrow \widehat{f}(\omega)\widehat{f}(\omega) = \widehat{f}(\omega)\\
&\Longrightarrow \widehat{f}(\omega)\left[\widehat{f}(\omega) - 1\right] = 0\\
&\Longrightarrow \begin{cases} \widehat{f}(\omega) = 0 \\ \qquad\vee \\ \widehat{f}(\omega) = 1 \end{cases},
\end{aligned}$$

$\widehat{f}(\omega) = 1$ is impossible, because $f \in L^1(\mathbb{R})$ and $\lim\limits_{\omega \longrightarrow \pm\infty} \widehat{f}(\omega) = 0$, then

$$\widehat{f}(\omega) = 0 \Longrightarrow f = 0 \text{ a.e on } \mathbb{R}.$$

Exercise 2.8.3 *Let f be the function defined on $\mathbb{R}$ by $f(x) = e^{-\pi x^2}$*

1- Verify that f is the solution to the differential equation

$$f'(x) + 2\pi x f(x) = 0. \tag{1}$$

2- By applying the Fourier transform to equation (1); show that $\widehat{f}(\omega)$ satisfies a first-order differential equation, which you should determine.

3- Solve this di erential equation and determine $\widehat{f}(\omega)$ knowing that $\int_{\mathbb{R}} e^{-\pi x^2} x = 1$.

Solution:

1- Let $f(x) = e^{-\pi x^2}$, we have

$$f'(x) + 2\pi x f(x) = 0$$

$$\Longrightarrow \left(e^{-\pi x^2}\right)' + 2\pi x e^{-\pi x^2} = -2\pi x e^{-\pi x^2} + 2\pi x e^{-\pi x^2} = 0.$$

2- Applying the Fourier transform to the equation (1), we get

$$\mathcal{F}\left(f'(x) + 2\pi x f(x)\right)(\omega) = 0$$

$$\Longrightarrow \mathcal{F}\left(f'(x)\right)(\omega) + 2\pi \mathcal{F}\left(x f(x)\right)(\omega) = 0$$

$$\Longrightarrow (2\pi i\omega)\,\widehat{f}(\omega) + 2\pi\left(-\frac{1}{2\pi i}\frac{d}{d\omega}\widehat{f}(\omega)\right) = 0$$

$$\Longrightarrow \frac{d}{d\omega}\widehat{f}(\omega) + 2\pi\omega\widehat{f}(\omega) = 0. \qquad (\widehat{1})$$

3- We solve equation $(\widehat{1})$. For this, we have

$$\frac{d}{d\omega}\widehat{f}(\omega) + 2\pi\omega\widehat{f}(\omega) = 0$$

$$\Longrightarrow \frac{d}{d\omega}\widehat{f}(\omega) = -2\pi\omega\widehat{f}(\omega)$$

$$\Longrightarrow \int_{\mathbb{R}} \frac{d\widehat{f}(\omega)}{\widehat{f}(\omega)} = -2\pi\int_{\mathbb{R}} \omega d\omega$$

$$\Longrightarrow \ln\left|\widehat{f}(\omega)\right| = -\pi\omega^2 + C / C \in \mathbb{R}.$$

$$\Longrightarrow \widehat{f}(\omega) = Ke^{-\pi\omega^2} / K = e^C \in \mathbb{R}.$$

We have $\widehat{f}(0) = K$ and

$$\widehat{f}(0) = \int_{\mathbb{R}} f(x)e^{-2\pi i\omega(0)}dx = \int_{\mathbb{R}} f(x)dx$$

$$\Longrightarrow K = \int_{\mathbb{R}} f(x)dx = \int_{\mathbb{R}} e^{-\pi x^2}dx = 1.$$

So, the solution of equation $(\widehat{1})$, is

$$\widehat{f}(\omega) = e^{-\pi\omega^2}.$$

Exercise 2.8.4 *Let* $f(x) = e^{-a|x|}$ *for* $a > 0$

1- Compute its Fourier transform.

2- Deduce the Fourier transform of the function

$$g : x \rightarrow \frac{1}{1+x^2}.$$

3- Compute the convolution $f * f$ *and deduce the Fourier transform of the function*

$$x \rightarrow \frac{1}{(1+x^2)^2}.$$

4- Determine the Fourier transform of the function

$$x \longrightarrow \frac{x}{(1+x^2)^2}.$$

Solution:

1- We compute the Fourier transform

$$f(x) = e^{-a|x|} = \begin{cases} e^{-ax}, & \text{if } \ x > 0, \\ e^{ax}, & \text{if } \ x < 0. \end{cases}$$

Using the definition of the Fourier transform, we get

$$\begin{aligned} \widehat{f}(\omega) &= \int_{\mathbb{R}} f(x) e^{-2\pi i \omega x} dx \\ &= \int_{-\infty}^{0} e^{ax} e^{-2\pi i \omega x} dx + \int_{0}^{+\infty} e^{-ax} e^{-2\pi i \omega x} dx \\ &= \int_{-\infty}^{0} e^{(a-2\pi i \omega)x} dx + \int_{0}^{+\infty} e^{-(a+2\pi i \omega)} dx \\ &= \left[\frac{1}{a-2\pi i \omega} e^{(a-2\pi i \omega)x} \right]_{-\infty}^{0} + \left[-\frac{1}{a-2\pi i \omega} e^{-(a+2\pi i \omega)x} \right]_{-\infty}^{0} \\ &= \frac{1}{a-2\pi i \omega} + \frac{1}{a+2\pi i \omega} \\ &= \frac{2a}{a^2+4\pi^2\omega^2}. \end{aligned}$$

So

$$\widehat{f}(\omega) = \frac{2a}{a^2+4\pi^2\omega^2}.$$

2- For $a = 2\pi$, we have

$$\begin{aligned} \widehat{f}(\omega) &= \frac{4\pi}{4\pi^2+4\pi^2\omega^2} \\ &= \frac{1}{\pi} \frac{1}{1+\omega^2}. \end{aligned}$$

So its Fourier transform is $f(-x) = e^{-2\pi|x|}$.

Thus the Fourier transform of function $x \longrightarrow \dfrac{1}{1+x^2}$ is function

$$\omega \longrightarrow \pi e^{-2\pi|\omega|}.$$

3- We calculate the convolution product

$$\begin{aligned}(f*f)(x) &= \int_{-\infty}^{+\infty} f(x-y)f(y)dy\\ &= \int_{-\infty}^{+\infty} e^{-a|x-y|}e^{-a|y|}dy\\ &= \int_{-\infty}^{+\infty} e^{-a(|x-y|+|y|)}dy\\ &= \int_{-\infty}^{0} e^{-a(|x-y|-|y|)}dy + \int_{0}^{+\infty} e^{-a(|x-y|+|y|)}dy.\end{aligned}$$

If $x > 0$, we have

$$(f*f)(x) = \int_{-\infty}^{0} e^{-a(x-2y)}dy + \int_{0}^{x} e^{ax}dy + \int_{x}^{+\infty} e^{-a(2y-x)}dy.$$

Because

$$|x-y| = \begin{cases} x-y, & \text{if } \ y < x,\\ y-x, & \text{if } \ y > x.\end{cases}$$

$$\begin{aligned}(f*f)(x) &= \left[\frac{1}{2a}e^{-a(x-2y)}\right]_{-\infty}^{0} + \left[ye^{ax}\right]_0^x + \left[-\frac{1}{2a}e^{-a(2y-x)}\right]_x^{+\infty}\\ &= \frac{e^{-ax}}{2a} + xe^{ax} + \frac{e^{-ax}}{2a}\\ &= e^{-ax}\left(x+\frac{1}{a}\right), \quad a > 0.\end{aligned}$$

The function f is even, so is $f*f$.

Indeed,

$$\begin{aligned}(f*f)(-x) &= \int_{-\infty}^{+\infty} f(-(x-y))f(-y)dy\\ &= \int_{-\infty}^{+\infty} f(x-y)f(y)dy\\ &= (f*f)(x).\end{aligned}$$

We deduce that

$$(f*f)(x) = e^{-a|x|}\left(|x|+\frac{1}{a}\right).$$

The Fourier transform of $f * f$ is

$$\begin{aligned}\left(\widehat{f * f}\right)(\omega) &= \widehat{f}(\omega)\widehat{f}(\omega)\\ &= \left(\widehat{f}(\omega)\right)^2\\ &= \frac{4a^2}{\left(a^2+4\pi^2\omega^2\right)^2}.\end{aligned}$$

For $a = 2\pi$, we have

$$\begin{aligned}\left(\widehat{f * f}\right)(\omega) &= \frac{16\pi^2}{\left(4\pi^2+4\pi^2\omega^2\right)^2}\\ &= \frac{16\pi^2}{16\pi^4\left(1+\omega^2\right)^2}\\ &= \frac{1}{\pi^2}\frac{1}{\left(1+\omega^2\right)^2}.\end{aligned}$$

We apply the inverse Fourier transform, we obtain the Fourier transform of the function $x \longrightarrow \dfrac{1}{\left(1+x^2\right)^2}$ is the function

$$\omega \longrightarrow (f * f)(\omega) = e^{-2\pi|\omega|}\left(|\omega| + \frac{1}{2\pi}\right).$$

4- we determine the Fourier transform of $x \longrightarrow \dfrac{x}{\left(1+x^2\right)^2}$.

Note that the derivative of $x \longrightarrow \dfrac{1}{1+x^2}$ is $x \longrightarrow \dfrac{-2x}{\left(1+x^2\right)^2}$.

Then

$$\begin{aligned}\mathcal{F}\left(\frac{x}{\left(1+x^2\right)^2}\right)(\omega) &= -\frac{1}{2}\mathcal{F}\left(\left(\frac{1}{1+x^2}\right)'\right)(\omega)\\ &= -\frac{1}{2}(2\pi i\omega)\mathcal{F}\left(\frac{1}{1+x^2}\right)(\omega)\\ &= -\frac{1}{2}(2\pi i\omega)\pi e^{-2\pi|\omega|}\\ &= -\pi^2 i\omega\pi e^{-2\pi|\omega|}.\end{aligned}$$

Exercise 2.8.5 *Consider the integral equation*

$$\int_{-\infty}^{+\infty}\frac{f(t)}{(x-t)^2+a^2}dt = \frac{1}{x^2+b^2} \text{ for } 0 < a < b,$$

with f is assumed to be absolutely integrable and bounded.

Determine the Fourier transform of f and then deduce f.

Solution:

Let the equation

$$\int_{-\infty}^{+\infty} \frac{f(t)}{(x-t)^2+a^2}dt = \frac{1}{x^2+b^2}, \tag{*}$$

for $0 < a < b$.

We express (*) as a convolution equation.

Let's set

$$g_c(t) = \frac{1}{t^2+c^2},$$

so we have

$$(f * g_a)(t) = g_b(t).$$

Indeed

$$\begin{aligned} (f * g_a)(t) &= \int_{-\infty}^{+\infty} f(t) g_a(x-t)dt \\ & \int_{-\infty}^{+\infty} f(t)\frac{1}{(x-t)^2+a^2}dt \\ &= \frac{1}{x^2+b^2} \\ &= g_b(t). \end{aligned}$$

We determine the Fourier transform of f and then we deduce f. For this we have

$$\begin{aligned} (f * g_a)(t) &= g_b(t) \Longrightarrow \left(\widehat{f * g_a}\right)(\omega) = \widehat{g_b}(\omega) \\ &\Longrightarrow \widehat{f}(\omega)\widehat{g_a}(\omega) = \widehat{g_b}(\omega) \\ &\Longrightarrow \widehat{f}(\omega) = \frac{\widehat{g_b}(\omega)}{\widehat{g_a}(\omega)}. \end{aligned}$$

We compute $\widehat{g_a}(\omega)$ and $\widehat{g_b}(\omega)$.

We have

$$g_c(t) = \frac{1}{c^2+t^2} = \frac{1}{c^2\left(1+\left(\frac{t}{c}\right)^2\right)}.$$

According to the exercise 2.8.4, we have

$$\mathcal{F}\left(\frac{1}{1+t^2}\right)(\omega) = \pi e^{-2\pi|\omega|}.$$

So

$$\widehat{g_c}(\omega) = \mathcal{F}\left(\frac{1}{c^2\left(1+\left(\frac{t}{c}\right)^2\right)}\right)(\omega) = \frac{\pi}{c}e^{-2\pi c|\omega|}.$$

Then we have

$$\widehat{f}(\omega) = \frac{\widehat{g_b}(\omega)}{\widehat{g_a}(\omega)} = \frac{\frac{\pi}{b}e^{-2\pi b|\omega|}}{\frac{\pi}{a}e^{-2\pi b|\omega|}} = \frac{a}{b}e^{-2\pi(b-a)|\omega|}.$$

We apply the inverse Fourier transform, we obtain

$$\widehat{f}(\omega) = \mathcal{F}\left(f(t)\right)(\omega) = \frac{a}{b}e^{-2\pi(b-a)|\omega|}$$

$$\begin{aligned} \Longrightarrow \quad f(t) &= \mathcal{F}^{-1}\left(\frac{a}{b}e^{-2\pi(b-a)|\omega|}\right) \\ &= \frac{a}{b}\frac{(b-a)}{\pi}\mathcal{F}^{-1}\left(\frac{\pi}{b-a}e^{-2\pi(b-a)|\omega|}\right) \\ &= \frac{a}{b}\frac{(b-a)}{\pi}\frac{1}{\left[(b-a)^2+t^2\right]}. \end{aligned}$$

Hence

$$f(t) = \frac{a(b-a)}{b\pi\left[t^2+(b-a)^2\right]}.$$

Exercise 2.8.6 *Let* $a \in]0,+\infty[$. *Consider the functions defined on* $\mathbb{R}$ *by*

$$f_a(x) = e^{-ax}\mathcal{X}_{]0,+\infty[}(x) \ \textit{and} \ \varphi_k(x) = \frac{x^k}{k!}f_a(x).$$

1- Compute $\mathcal{F}(f_a(x))(\omega)$ *and deduce* $\mathcal{F}(\varphi_k(x))(\omega)$.

2- Let g_a *be the function defined by*

$$g_a(x) = f_a(x) + f_a(-x).$$

Determine $\mathcal{F}(g_a(x))(\omega)$ *and deduce the value of the integral*

$$\int_0^{+\infty} \frac{\cos(\xi x)}{1+x^2}dx, \ \textit{where} \ \xi \in \mathbb{R}.$$

Solution:

1- We compute $\mathcal{F}(f_a(x))(\omega)$.

We have

$$f_a(x) = e^{-ax}\mathcal{X}_{]0,+\infty[}(x) = \begin{cases} e^{-ax}, & \text{if } x \in]0,+\infty[, \\ 0, & \text{if } x \notin]0,+\infty[. \end{cases}$$

Then, we have

$$\begin{aligned}\mathcal{F}(f_a(x))(\omega) &= \int_0^{+\infty} e^{-ax}e^{-2\pi i\omega x}dx\\ &= \int_0^{+\infty} e^{-(a+2\pi i\omega)x}dx\\ &= \left[-\frac{1}{a+2\pi i\omega}e^{-(a+2\pi i\omega)x}\right]_0^{+\infty}\\ &= \frac{1}{a+2\pi i\omega}.\end{aligned}$$

So

$$\widehat{f_a}(\omega) = \frac{1}{a+2\pi i\omega}.$$

We deduce $\mathcal{F}(\varphi_k(x))(\omega)$

We have

$$\frac{d^{(k)}}{d\omega^k}\widehat{f_a}(\omega) = \mathcal{F}((-2\pi ix)^k f_a(x))(\omega)$$

$$\Longrightarrow \frac{1}{k!}\frac{d^{(k)}}{d\omega^k}\widehat{f_a}(\omega) = (-2\pi i)^k\mathcal{F}\left(\frac{x^k}{k!}f_a(x)\right)(\omega)$$

$$\Longrightarrow \mathcal{F}\left(\frac{x^k}{k!}f_a(x)\right)(\omega) = \mathcal{F}(\varphi_k(x))(\omega) = \frac{1}{(-2\pi i)^k k!}\frac{d^{(k)}}{d\omega^k}\widehat{f_a}(\omega).$$

We use the successive derivative, we obtain

$$\begin{aligned}\frac{d^{(1)}}{d\omega}\widehat{f_a}(\omega) &= \frac{(-2\pi i)^1\,1!}{(a+2\pi i\omega)^2},\\ \frac{d^{(2)}}{d\omega^2}\widehat{f_a}(\omega) &= \frac{(-2\pi i)^2\,2!}{(a+2\pi i\omega)^3},\\ \frac{d^{(3)}}{d\omega^3}\widehat{f_a}(\omega) &= \frac{(-2\pi i)^3\,3!}{(a+2\pi i\omega)^4},\\ &\vdots\\ \frac{d^{(k)}}{d\omega^k}\widehat{f_a}(\omega) &= \frac{(-2\pi i)^k\,k!}{(a+2\pi i\omega)^{k+1}}.\end{aligned}$$

So we have

$$\begin{aligned}\mathcal{F}(\varphi_k(x))(\omega) &= \frac{1}{(-2\pi i)^k k!}\frac{d^{(k)}}{d\omega^k}\widehat{f_a}(\omega)\\ &= \frac{1}{(-2\pi i)^k k!}\frac{(-2\pi i)^k\,k!}{(a+2\pi i\omega)^{k+1}}\\ &= \frac{1}{(a+2\pi i\omega)^{k+1}}.\end{aligned}$$

2- We determine $\mathcal{F}(g_a(x))(\omega)$ and deduce the value of the integral $\displaystyle\int_0^{+\infty} \frac{\cos(\xi x)}{1+x^2}dx$, where $\xi \in \mathbb{R}$.

For this, we have

$$\begin{aligned}
\mathcal{F}(g_a(x))(\omega) &= \mathcal{F}(f_a(x))(\omega) + \mathcal{F}(f_a(-x))(\omega)\\
&= \widehat{f}_a(\omega) + \widehat{f}_a(-\omega)\\
&= \frac{1}{a+2\pi i\omega} + \frac{1}{a-2\pi i\omega}\\
&= \frac{2a}{a^2+4\pi^2\omega^2}.
\end{aligned}$$

To deduce the value of the integral, we use the inverse Fourier transform, so we have

$$\begin{aligned}
g_a(x) &= \mathcal{F}^{-1}(\widehat{g}_a(\omega))\\
&= \int_{\mathbb{R}} e^{2\pi i\omega x}\widehat{g}_a(\omega)d\omega
\end{aligned}$$

$$\Longrightarrow \int_{\mathbb{R}} (\cos(2\pi\omega x) + i\sin(2\pi\omega x))\,\widehat{g}_a(\omega)d\omega = g_a(x) = e^{-a|x|}.$$

Because

$$\begin{aligned}
g_a(x) &= f_a(x) + f_a(-x)\\
&= \begin{cases} e^{-ax}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases} + \begin{cases} e^{ax}, & \text{if } x \leq 0 \\ 0, & \text{if } x > 0 \end{cases} = e^{-a|x|}.
\end{aligned}$$

Because the function $\cos(2\pi\omega x)$ is even and $\sin(2\pi\omega x)$ is odd, then we have

$$2\int_0^{+\infty} \cos(2\pi\omega x)\widehat{g}_a(\omega)d\omega = e^{-a|x|}$$

$$\begin{aligned}
&\Longrightarrow 2\int_0^{+\infty} \cos(2\pi\omega x)\frac{2a}{a^2+4\pi^2\omega^2}d\omega = e^{-a|x|}\\
&\Longrightarrow 4a\int_0^{+\infty} \frac{\cos(2\pi\omega x)}{a^2+4\pi^2\omega^2}d\omega = e^{-a|x|}.
\end{aligned}$$

Let's choose $a = 2\pi$ and $\xi = 2\pi x$, we obtain

$$8\pi\int_0^{+\infty} \frac{\cos(\xi\omega)}{4\pi^2+4\pi^2\omega^2}d\omega = e^{-2\pi|x|} = e^{-|\xi|}$$

$$\Longrightarrow \frac{8\pi}{4\pi^2}\int_0^{+\infty}\frac{\cos(\xi\omega)}{1+\omega^2}d\omega = e^{-|\xi|}$$

$$\Longrightarrow \int_0^{+\infty}\frac{\cos(\xi\omega)}{1+\omega^2}d\omega = \frac{\pi}{2}e^{-|\xi|}.$$

We put $\omega = x$, we get

$$\int_0^{+\infty}\frac{\cos(\xi x)}{1+x^2}dx = \frac{\pi}{2}e^{-|\xi|},\ \xi \in \mathbb{R}.$$

2.9 Suggested exercises

Exercise 2.9.1 *Determine the Fourier transforms of the following functions*

1- $x \mapsto \mathbb{I}_{[-T,T]}(x)$.

2- $x \mapsto \dfrac{\sin(x)}{x}$.

3- $x \mapsto \exp\left(\dfrac{-|x|}{T}\right)$.

4- $x \mapsto \dfrac{1}{\pi}\dfrac{1}{1+x^2}$.

Exercise 2.9.2 *Let* $f, g \in L^1(\mathbb{R}^n)$, *we denote by* $\widehat{f}$ *the Fourier transform defined by*

$$\widehat{f}(y) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i(y,x)}dx,$$

where $(.,.)$ *denotes the scalar product of* $\mathbb{R}^n$. *Show that*

1- $\int_{\mathbb{R}^n} f(x)\widehat{g}(x)dx = \int_{\mathbb{R}^n} \widehat{f}(x)g(x)dx$.

2- $\widehat{f * g} = \widehat{f}\widehat{g}$.

Exercise 2.9.3 *Consider the following functional equation*

$$f(x) + A(f(x-1) + f(x+1)) = u(x),$$

where $u(x)$ *is a known function absolutely integrable over* $\mathbb{R}$ *and* A *is a constant.*

Assume that the Fourier transform of the function f *and its inverse exist.*

Determine $f(x)$ *in integral form. What condition must* A *satisfy?*

Exercise 2.9.4 *Let f and g be two absolutely square-integrable functions on $\mathbb{R}$. The correlation function $\varphi_{fg}(x)$ of $f(x)$ and $g(x)$ is defined by*

$$\varphi_{fg}(x) = \int_{\mathbb{R}} f(y)g(x+y)dy.$$

1- Show that

$$\varphi_{fg}(x) = \int_{-\infty}^{+\infty} \overline{\widehat{f}}(\omega)\widehat{g}(\omega)e^{2\pi i\omega x}d\omega.$$

2- Deduce Parseval's identity

$$\int_{-\infty}^{+\infty} f^2(x)dx = \int_{-\infty}^{+\infty} \left|\widehat{f}(\omega)\right|^2 d\omega.$$

3- Calculate the integral

$$\int_{-\infty}^{+\infty} \frac{\sin^2 t}{t^2}dt.$$

Exercise 2.9.5 *Use the Fourier transform with respect to x to solve the Laplace equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

where $y > 0$ with $\dfrac{\partial u}{\partial x} \to 0$ and $u \to 0$ when $\|(x,y)\| \to +\infty$,

$$u(x,0) = \begin{cases} 1, & \text{if } \ |x| \leq 1 \\ 0, & \text{if } \ |x| > 1 \end{cases}.$$

Exercise 2.9.6 *Using the Fourier transform, solve the following integral equation*

$$f(x) = g(x) + \int_{-\infty}^{+\infty} k(x-y)f(y)dy,$$

where g and k are known functions and their Fourier transforms $\mathcal{G}$ and $\mathcal{K}$, respectively.

Chapter 3

Laplace transform

Mathematicians have invented many integral transformations of a function, among which we have seen the Fourier transforms. Another widely used transformation is the Laplace transform. Laplace transforms are cousins of the Fourier transforms. Their relationship is that of the exponential function and the sine or cosine function. The most interesting property of the Laplace transform is that integration and differentiation become divisions and multiplications. The Laplace transform, for example, allows us to reduce the resolution of linear differential equations with constant coefficients to the resolution of affine equations. This transformation is widely used to solve equations and differential systems, particularly in electricity, electronics, heat theory, and signal theory. In this chapter, we present the Laplace transform and some interesting characteristics.

3.1 Definition and existence of the Laplace transform

Definition 3.1.1 *Let $f(t)$ be a function of t, defined for $t > 0$. We define and denote by $\mathcal{L}(f)(p) = F(p)$ the Laplace transform of $f(t)$ as follows*

$$\mathcal{L}(f)(p) = F(p) = \int_0^{+\infty} f(t)e^{-pt}dt,$$

where p is called the Laplace variable, f is called the origin of F and F is the image of f (or Laplace transform of f).

Remark 3.1.1 *1) The new variable* p *belongs to* $\mathbb{C}$.

2) The Laplace transform $F(p)$ *does not always exist. For example, it is easy to show that the integral* $f(t) = e^{t^2}$ *is undefined.*

Now, we will give conditions that guarantee the existence of $\int_0^{+\infty} f(t)e^{-pt}dt$.

Definition 3.1.2 *We say that* f *is of exponential order* α, *if there exist constants* $M > 0$ *such that*

$$|f(t)| \leq Me^{\alpha t}, \text{ for all } t \geq 0.$$

Intuitively, this means that from a certain value of $t(= N)$, $f(t)$ *cannot grow faster than an exponential* $Me^{\alpha t}$, *therefore*

$$\lim_{t\to+\infty} e^{-\alpha t}f(t) = 0.$$

Example 3.1.1 - $f(t) = t^2$ *is of exponential order* $1, 2, 3, ...$ *since*

$$t^2 \leq e^t \leq e^{t^2} \leq e^{t^3} \leq ... \text{ for } t > 0.$$

- $f(t) = e^{t^3}$ *is not of exponential order since*

$$\left|e^{t^3}e^{-\alpha t}\right| = e^{t^3-\alpha t},$$

can become as large as we want when t *increases (*t^3 *grows faster than* t *when* $t \to \infty$*).*

- $f(t) = e^{at}$ *is of exponential order* $\forall t \in \mathbb{R}$ *since*

$$e^{at} < e^{\alpha t}, \text{ for } \alpha > a.$$

Theorem 3.1.1 ***(Existence of the Laplace transform)***

If $f(t)$ *is piecewise continuous on each finite interval* $0 \leq t \leq N$ *and if* $f(t)$ *is of exponential order* α *for* $t > N$, *then the Laplace transform* $F(p)$ *exists for all* $Re(p) > \alpha$.

Proof. Let $p = x + iy \in \mathbb{C}$, we have

$$\begin{aligned}
|F(p)| &= \left|\int_0^{+\infty} f(t)e^{-pt}dt\right| \\
&\leq \int_0^{+\infty} \left|f(t)e^{-(x+iy)t}\right| dt \\
&\leq \int_0^{+\infty} \left|f(t)e^{-xt}\right| \left|e^{-iyt}\right| dt \\
&\leq \int_0^{+\infty} |f(t)|\, e^{-xt}dt \\
&\leq \int_0^{+\infty} Me^{\alpha t}e^{-xt}dt \\
&\leq M\int_0^{+\infty} e^{-(x-\alpha)t}dt \\
&\leq M\left[-\frac{1}{x-\alpha}e^{-(x-\alpha)t}\right]_0^{+\infty} \\
&\leq \frac{M}{x-\alpha}.
\end{aligned}$$

Hence, the integral exists for $x = Re(p) > \alpha$. ■

3.2 Examples of Laplace transform of elementary functions

1) Heaviside function: Let the function $H(t)$ be defined by the following

$$H(t) = \begin{cases} 1, & \text{if } \ t \geq 0, \\ 0, & \text{if } \ t < 0. \end{cases}$$

The Laplace transform of $H(t)$, is compute as follows

$$\begin{aligned}
\mathcal{L}(H)(p) &= \int_0^{+\infty} H(t)e^{-pt}dt \\
&= \int_0^{+\infty} e^{-pt}dt \\
&= \left[-\frac{1}{p}e^{-pt}\right]_0^{+\infty} \\
&= \frac{1}{p}.
\end{aligned}$$

2) Exponential function: Let

$$f(t) = \begin{cases} e^{\alpha t}, & \text{if } \ t > 0, \\ 0, & \text{if } \ t < 0. \end{cases}$$

The Laplace transform of $f(t)$, is compute as follows

$$\begin{aligned} \mathcal{L}(f)(p) &= \int_0^{+\infty} f(t)e^{-pt}dt \\ &= \int_0^{+\infty} e^{\alpha t}e^{-pt}dt \\ &= \int_0^{+\infty} e^{-(p-\alpha)t}dt \\ &= \left[-\frac{1}{p-\alpha}e^{-(p-\alpha)t}\right]_0^{+\infty} \\ &= \frac{1}{p-\alpha}. \end{aligned}$$

3.3 Properties of the Laplace transform

3.3.1 Linearity

Theorem 3.3.1 *Let $f(t)$ and $g(t)$ be two functions admitting Laplace transforms $F(p)$ and $G(p)$, then*

$$\mathcal{L}(af + bg)(p) = aF(p) + bG(p), \ \forall a, b \in \mathbb{R}.$$

Proof. For $a, b \in \mathbb{R}$, we have

$$\begin{aligned} \mathcal{L}(af + bg)(p) &= \int_0^{+\infty} (af + bg)\,(t)e^{-pt}dt \\ &= \int_0^{+\infty} \left(af(t)e^{-pt} + bg(t)e^{-pt}\right) dt \\ &= a\int_0^{+\infty} f(t)e^{-pt}dt + b\int_0^{+\infty} g(t)e^{-pt}dt \\ &= aF(p) + bG(p). \end{aligned}$$

■

Using this property, we can determine the sine and cosine transform

3) Sine function: Let

$$f(t) = \sin(\alpha t).$$

The Laplace transform of $f(t)$, is compute as follows

$$\mathcal{L}(\sin(\alpha t))(p) = \mathcal{L}\left(\frac{e^{i\alpha t} - e^{-i\alpha t}}{2i}\right)(p).$$

We know that

$$\mathcal{L}(e^{\alpha t})(p) = \frac{1}{p-\alpha}.$$

Then

$$\begin{aligned}
\mathcal{L}(\sin(\alpha t))(p) &= \frac{1}{2i}\mathcal{L}\left(e^{i\alpha t} - e^{-i\alpha t}\right)(p) \\
&= \frac{1}{2i}\left(\frac{1}{p-i\alpha} - \frac{1}{p+i\alpha}\right) \\
&= \frac{1}{2i}\left(\frac{2i\alpha}{p^2+\alpha^2}\right) \\
&= \frac{\alpha}{p^2+\alpha^2}.
\end{aligned}$$

4) Consinis function: Let

$$f(t) = \cos(\alpha t).$$

The Laplace transform of $f(t)$, is compute as follows

$$\mathcal{L}(\cos(\alpha t))(p) = \mathcal{L}\left(\frac{e^{i\alpha t} + e^{-i\alpha t}}{2}\right)(p).$$

We know that

$$\mathcal{L}(e^{\alpha t})(p) = \frac{1}{p-\alpha}.$$

Then

$$\begin{aligned}
\mathcal{L}(\cos(\alpha t))(p) &= \frac{1}{2}\mathcal{L}\left(e^{i\alpha t} + e^{-i\alpha t}\right)(p) \\
&= \frac{1}{2}\left(\frac{1}{p-i\alpha} + \frac{1}{p+i\alpha}\right) \\
&= \frac{1}{2}\left(\frac{2p}{p^2+\alpha^2}\right) \\
&= \frac{p}{p^2+\alpha^2}.
\end{aligned}$$

Example 3.3.1 *Compute the Laplace transform of the following function*

$$f(t) = 4e^{5t} + 3\sin(4t) - 2\cos(2t) + 6.$$

Using the linearity of the Laplace transform, it comes

$$\begin{aligned}\mathcal{L}(f)(p) &= \mathcal{L}\left(4e^{5t} + 3\sin(4t) - 2\cos(2t) + 6\right)\\ &= 4\mathcal{L}\left(e^{5t}\right) + 3\mathcal{L}(\sin(4t)) - 2\mathcal{L}(\cos(2t)) + 6\mathcal{L}(1)\\ &= \frac{4}{p-5} + \frac{12}{p^2+16} - \frac{2p}{p^2+4} + \frac{6}{p}.\end{aligned}$$

3.3.2 Translation of the variable t (Delay Theorem)

Theorem 3.3.2 *Let $a \in \mathbb{R}$. If $\mathcal{L}(f)(p) = F(p)$ and*

$$(\varphi_a f)(t) \begin{cases} f(t-a), & \text{if } t > a,\\ 0, & \text{if } t < a,\end{cases}$$

then

$$\begin{aligned}\mathcal{L}(\varphi_a f(t))(p) &= \mathcal{L}(f(t-a))(p)\\ &= e^{-ap}F(p), \ \ \mathrm{Re}(p) > a.\end{aligned}$$

Proof. Using the definition of the Laplace transform, we get

$$\begin{aligned}\mathcal{L}(\varphi_a f(t))(p) &= \int_0^{+\infty} \varphi_a f(t)e^{-pt}dt\\ &= \int_0^{a} \varphi_a f(t)e^{-pt}dt + \int_a^{+\infty} \varphi_a f(t)e^{-pt}dt\\ &= 0 + \int_a^{+\infty} f(t-a)e^{-pt}dt.\end{aligned}$$

We put $u = t - a \Longrightarrow du = dt$, then

$$\begin{aligned}\mathcal{L}(\varphi_a f(t))(p) &= \int_a^{+\infty} f(u)e^{-p(u+a)}du\\ &= \int_a^{+\infty} f(u)e^{-pu}e^{-pa}du\\ &= e^{-pa}\int_a^{+\infty} f(u)e^{-pu}du\\ &= e^{-ap}F(p).\end{aligned}$$

■

Example 3.3.2 *Let us find the Laplace transform of*

$$g(t)\begin{cases}\sin(t-\frac{\pi}{3}), & \text{if } t>\frac{\pi}{3},\\ 0, & \text{if } t<\frac{\pi}{3}.\end{cases}$$

By the property of the translation of the variable t, we have

$$\begin{aligned}\mathcal{L}(\varphi_{\frac{\pi}{3}}\sin(t))(p) &= \mathcal{L}(\sin(t-\frac{\pi}{3}))(p)\\ &= e^{-\frac{\pi}{3}p}\mathcal{L}(\sin(t))\\ &= \frac{e^{-\frac{\pi}{3}p}}{p^2+1}.\end{aligned}$$

3.3.3 Translation of the variable p (Modulation)

Theorem 3.3.3 *Let $\delta\in\mathbb{C}$. If $\mathcal{L}(f)(p)=F(p)$, then we have*

$$\mathcal{L}(e^{\delta t}f(t))(p)=F(p-\delta).$$

Proof. Using the definition of the Laplace transform, we get

$$\begin{aligned}\mathcal{L}(e^{\delta t}f(t))(p) &= \int_0^{+\infty} e^{\delta t}f(t)e^{-pt}dt\\ &= \int_0^{+\infty} f(t)e^{-pt+\delta t}dt\\ &= \int_0^{+\infty} f(t)e^{-(p-\delta)t}dt\\ &= F(p-\delta).\end{aligned}$$

■

Example 3.3.3 *As*

$$\mathcal{L}(\cos(2t))(p)=\frac{p}{p^2+4}=F(p),$$

then we have

$$\begin{aligned}\mathcal{L}(e^{-t}\cos(2t))(p) &= F(p+1)\\ &= \frac{p+1}{(p+1)^2+4}\\ &= \frac{p+1}{p^2+2p+5}.\end{aligned}$$

3.3.4 Change of scale

Theorem 3.3.4 *For all* $\lambda > 0$. *If* $\mathcal{L}(f)(p) = F(p)$, *then we have*

$$\begin{aligned} \mathcal{L}(h_\lambda f(t))(p) &= \mathcal{L}(f(\lambda t))(p) \\ &= \frac{1}{\lambda} F\left(\frac{p}{\lambda}\right). \end{aligned}$$

Proof. Using the definition of the Laplace transform, we get

$$\begin{aligned} \mathcal{L}(h_\lambda f(t))(p) &= \mathcal{L}(f(\lambda t))(p) \\ &= \int_0^{+\infty} f(\lambda t) e^{-pt} dt. \end{aligned}$$

We put $u = \lambda t \Longrightarrow du = \lambda dt$, then

$$\begin{aligned} \mathcal{L}(h_\lambda f(t))(p) &= \int_0^{+\infty} f(\lambda t) e^{-pt} dt \\ &= \int_0^{+\infty} f(u) e^{-p\frac{u}{\lambda}} \frac{1}{\lambda} du \\ &= \frac{1}{\lambda} \int_0^{+\infty} f(u) e^{-\frac{p}{\lambda} u} du \\ &= \frac{1}{\lambda} F\left(\frac{p}{\lambda}\right). \end{aligned}$$

■

Example 3.3.4 *As*

$$\mathcal{L}(\sin(t))(p) = \frac{1}{p^2+1} = F(p),$$

then we have

$$\begin{aligned} \mathcal{L}(\sin(3t))(p) &= \frac{1}{3} F\left(\frac{p}{3}\right) \\ &= \frac{1}{3} \frac{1}{\left(\frac{p}{3}\right)^2 + 1} \\ &= \frac{3}{p^2+9}. \end{aligned}$$

3.3.5 Complex conjugation

Theorem 3.3.5 *If* $\mathcal{L}(f)(p) = F(p)$, *then we have*

$$\mathcal{L}(\overline{f(t)})(p) = \overline{F(\overline{p})}.$$

Proof. Using the definition of the Laplace transform, we get

$$\begin{aligned}\mathcal{L}(\overline{f(t)})(p) &= \int_0^{+\infty} \overline{f(t)} e^{-pt} dt \\ &= \overline{\int_0^{+\infty} f(t) e^{-\overline{p}t} dt} \\ &= \overline{F\left(\overline{p}\right)}.\end{aligned}$$

■

3.3.6 Laplace transform of a derivative

Laplace transform of the first derivative

Theorem 3.3.6 *Let f be a continuously differentiable function. If $\mathcal{L}\left(f\right)(p) = F(p)$, then*

$$\mathcal{L}(f'(t))(p) = pF(p) - f(0).$$

Proof. Using the definition of the Laplace transform, we get

$$\mathcal{L}(f'(t))(p) = \int\limits_0^{+\infty} f'(t) e^{-pt} dt.$$

Let us integrate by part, we put

$$\begin{aligned}u &= e^{-pt} \Longrightarrow du = -pe^{-pt} \\ dv &= f'(t)dt \Longrightarrow v = f(t).\end{aligned}$$

We find

$$\begin{aligned}\mathcal{L}(f'(t))(p) &= \int_0^{+\infty} f'(t) e^{-pt} dt \\ &= \left[e^{-pt} f(t)\right]_0^{+\infty} + \int_0^{+\infty} f(t) e^{-pt} dt \\ &= -f(0) + p \int_0^{+\infty} f(t) e^{-pt} dt \\ &= pF(p) - f(0).\end{aligned}$$

■

Remark 3.3.1 *If in Theorem 3.3.6, $f(t)$ is not continuous at $t = 0$ but $\lim\limits_{t \to 0^+} f(t) = f(0^+)$ exists (but is not equal to $f(0)$ which may or may not exist), then*

$$\mathcal{L}(f'(t))(p) = pF(p) - f(0^+).$$

3.3.7 Generalization to higher derivatives

If f'' verifies the hypotheses of Theorem 3.3.6, then we have

$$\begin{aligned}
\mathcal{L}(f''(t))(p) &= \mathcal{L}((f'(t))')(p)\\
&= p\mathcal{L}(f'(t)) - f'(0^+)\\
&= p(pF(p) - f(0^+)) - f'(0).
\end{aligned}$$

Hence,

$$\mathcal{L}(f''(t))(p) = p^2F(p) - pf(0^+) - f'(0^+).$$

We can demonstrate by recurrence that

$$\begin{aligned}
\mathcal{L}(f^{(n)}(t))(p) &= p^nF(p) - p^{n-1}f(0^+) - p^{n-2}f'(0^+) - ... - pf^{(n-2)}(0^+) - f^{(n-1)}(0^+)\\
&= p^nF(p) - \sum_{k=0}^{n-1} p^{n-1-k}f^{(k)}(0^+).
\end{aligned}$$

Special case, if

$$f(0^+) = f'(0^+) = f''(0^+) = ... = f^{(n-1)}(0^+) = 0,$$

then

$$\mathcal{L}(f^{(n)}(t))(p) = p^nF(p).$$

3.3.8 Derivation of the Laplace transform

Proposition 3.3.1 *The Laplace transform of a locally summable function is a holomorphic function in the summability domain $\{p \in \mathbb{C}, \operatorname{Re}(p) > \alpha\}$ and we have the formula*

$$\begin{aligned}
F^{(n)}(p) &= \int_0^{+\infty} (-t)^n f(t)e^{-pt}dt\\
&= (-1)^n\mathcal{L}(t^nf(t))(p).
\end{aligned}$$

Example 3.3.5 *Compute*

$$\mathcal{L}\left(t^ne^{2t}\right), \text{ for } n = 1, 2.$$

As

$$\mathcal{L}\left(e^{2t}\right) = \frac{1}{p-2},$$

then we have

$$\begin{aligned}\mathcal{L}(te^{2t})(p) &= -\frac{d}{dp}\left(\frac{1}{p-2}\right)\\ &= \frac{1}{(p-2)^2},\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}(t^2e^{2t})(p) &= \frac{d^2}{dp^2}\left(\frac{1}{p-2}\right)\\ &= \frac{2}{(p-2)^3}.\end{aligned}$$

Example 3.3.6 *We determine the Laplace transform of t^n.*

According to the previous proposition we have

$$\mathcal{L}(t^n f(t))(p) = (-1)^n F^{(n)}(p).$$

Here $f(t) = 1$.

So

$$\mathcal{L}(t^n)(p) = (-1)^n \left(\frac{1}{p}\right)^{(n)}, \mathrm{Re}(p) > 0.$$

We have

$$\left(\frac{1}{p}\right)^{(n)} = \frac{n!}{(-1)^n p^{n+1}}.$$

Then

$$\mathcal{L}(t^n)(p) = (-1)^n \frac{n!}{(-1)^n p^{n+1}}.$$

Hence,

$$\mathcal{L}(t^n)(p) = \frac{n!}{p^{n+1}}, \mathrm{Re}(p) > 0.$$

As special cases, we have

- For $n = 0$, we have

$$\mathcal{L}(1) = \frac{1}{p^1}.$$

- For $n = 1$, we have

$$\mathcal{L}(t) = \frac{1}{p^2}.$$

- For $n = 2$, we have

$$\mathcal{L}(t^2) = \frac{2!}{p^3}.$$

- For $n = 3$, we have

$$\mathcal{L}(t^3) = \frac{3!}{p^4}.$$

3.3.9 Laplace transform of integrals

Theorem 3.3.7 *If* $\mathcal{L}\left(f\right)(p) = F(p)$, *then we have*

$$\mathcal{L}\left(\int_0^t f(x)dx\right)(p) = \frac{F(p)}{p},$$

with $\mathrm{Re}(p) > \max(0, \alpha)$.

Proof. We put

$$G(t) = \int_0^t f(x)dx.$$

Then

$$G'(t) = f(t) \text{ and } G(0) = 0.$$

As

$$\mathcal{L}(G'(t))(p) = p\mathcal{L}\left(G(t)\right)(p) - G(0),$$

which implies that

$$\begin{aligned} \mathcal{L}(G(t))(p) &= \frac{1}{p}\mathcal{L}(G'(t))(p) \\ &= \frac{1}{p}\mathcal{L}(f(t))(p) \\ &= \frac{1}{p}F(p). \end{aligned}$$

Hence, we have

$$\mathcal{L}\left(\int_0^t f(x)dx\right)(p) = \frac{F(p)}{p}.$$

■

Example 3.3.7 *Compute*

$$\mathcal{L}\left(\int_0^t \sin(2x)dx\right).$$

As

$$\mathcal{L}\left(\sin(2t)\right) = \frac{2}{p^2+4},$$

then we have

$$\begin{aligned}\mathcal{L}\left(\int_0^t \sin(2x)dx\right) &= \frac{1}{p}\left(\frac{2}{p^2+4}\right)\\ &= \frac{2}{p\left(p^2+4\right)}.\end{aligned}$$

3.3.10 Laplace transform of a periodic function

Theorem 3.3.8 *Let f be a periodic function of period T (i.e. such that $f(t+T)=f(t)$), then*

$$\mathcal{L}\left(f(t)\right)(p) = \frac{1}{1-e^{-pT}}\int_0^T e^{-pt}f(t)dt.$$

Proof. We can write

$$\mathcal{L}\left(f(t)\right)(p) = \int_0^T e^{-pt}f(t)dt + \int_T^{2T} e^{-pt}f(t)dt + \int_{2T}^{3T} e^{-pt}f(t)dt + ... + \int_{nT}^{(n+1)T} e^{-pt}f(t)dt.$$

In the second integral, let $t=u+T$, in the third integral, let $t=u+2T$, etc. we obtain

$$\begin{aligned}\mathcal{L}\left(f(t)\right)(p) &= \int_0^T e^{-pt}f(t)dt + e^{-pT}\int_0^T e^{-pu}f(u)du + e^{-2pT}\int_0^T e^{-pu}f(u)du\\ &\quad +... + e^{-npT}\int_0^T e^{-pu}f(u)du\\ &= \left(1+e^{-pT}+e^{-2pT}+...+e^{-npT}\right)\int_0^T e^{-pt}f(u)du\\ &= \frac{1}{1-e^{-pT}}\int_0^T e^{-pt}f(t)dt,\end{aligned}$$

where we used the sum of the geometric series

$$1+r+r^2+...+r^n = \frac{1}{1-r}, \text{ if } |r|<1.$$

■

3.3.11 Laplace transform of convolution

Theorem 3.3.9 *If $\mathcal{L}(f)(p)=F(p)$ and $\mathcal{L}(g)(p)=G(p)$, then*

$$\mathcal{L}\left(\left(f*g\right)(t)\right)(p) = F(p)G(p).$$

Proof. Using the definition of the Laplace transform, we get

$$\begin{aligned}\mathcal{L}\left(\left(f*g\right)(t)\right)(p) &= \int_0^{+\infty}\left(f*g\right)(t)e^{-pt}dt\\ &= \int_0^{+\infty}\left(\int_0^{+\infty} f(\tau)g(t-\tau)d\tau\right)e^{-pt}dt.\end{aligned}$$

By setting the change of variable $v = t-\tau$, therefore $dv = dt$, and by separating the variables into two integrals, we get

$$\begin{aligned}\mathcal{L}\left(\left(f*g\right)(t)\right)(p) &= \int_0^{+\infty}\int_0^{+\infty} f(\tau)g(v)e^{-p(v+\tau)}d\tau dt\\ &= \int_0^{+\infty}\int_0^{+\infty} f(\tau)g(v)e^{-p\tau}e^{-pv}d\tau dv\\ &= \left(\int_0^{+\infty} f(\tau)e^{-p\tau}d\tau\right)\left(\int_0^{+\infty} g(v)e^{-pv}dv\right)\\ &= F(p)G(p).\end{aligned}$$

■

3.4 Inverse Laplace transform

Definition 3.4.1 *Let $F(p)$ be the Laplace transform of a function $f(t)$. We call the inverse Laplace transform (or the original of $F(p)$), the function $f(t)$ denote by*

$$f(t) = \mathcal{L}^{-1}\left(F(p)\right),$$

and is given by

$$f(t) = \frac{1}{2i\pi}\int_{-\infty}^{+\infty} F(p)e^{pt}dp.$$

Example 3.4.1 *Find the inverse Laplace transform (or original) of the following function*

$$F(p) = \frac{p+1}{p^2(p^2+4)}.$$

The decomposition of $F(p)$ is written as

$$F(p) = \frac{A}{p} + \frac{B}{p^2} + \frac{Cp+D}{p^2+4}.$$

By a simple calculation we obtain

$$\begin{aligned} A &= \frac{1}{4},\ B = \frac{1}{4}, \\ C &= -\frac{1}{4},\ D = -\frac{1}{4}. \end{aligned}$$

So, we have

$$\mathcal{L}\left(f(t)\right)(p) = F(p) = \frac{1}{4p} + \frac{1}{4p^2} - \frac{1}{4}\left(\frac{p+1}{p^2+4}\right).$$

Hence,

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left(F(p)\right) \\ &= \mathcal{L}^{-1}\left(\frac{1}{4p} + \frac{1}{4p^2} - \frac{1}{4}\left(\frac{p+1}{p^2+4}\right)\right) \\ &= \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{p}\right) + \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{p^2}\right) - \frac{1}{4}\mathcal{L}^{-1}\left(\frac{p}{p^2+4}\right) - \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{p^2+4}\right) \\ &= \frac{1}{4} + \frac{1}{4}t - \frac{1}{4}\cos(2t) - \frac{1}{8}\sin(2t). \end{aligned}$$

3.5 Laplace transform of usuel functions

The following table shows the expression of Laplace transforms for some usuel functions.

Function	Laplace transform
1	$\frac{1}{p}$
t	$\frac{1}{p^2}$
t^n, n positive integer	$\frac{n!}{p^{n+1}}$
t^n, $n \in \mathbb{R}$, $n > -1$	$\frac{\Gamma(n+1)}{p^{n+1}}$
e^{-at}	$\frac{1}{p+a}$
$te^{-\alpha t}$	$\frac{1}{(p+\alpha)^2}$
$\sin(\alpha t)$	$\frac{\alpha}{p^2+\alpha^2}$
$\cos(\alpha t)$	$\frac{p}{p^2+\alpha^2}$
$t\sin(\alpha t)$	$\frac{2\alpha p}{(p^2+\alpha^2)^2}$
$t\cos(\alpha t)$	$\frac{p^2-\alpha^2}{(p^2+\alpha^2)^2}$
$e^{-at}\sin(\alpha t)$	$\frac{p}{(p+a)^2+\alpha^2}$
$e^{-at}\cos(\alpha t)$	$\frac{p+\alpha}{(p+a)^2+\alpha^2}$
$H(t-a)$	$\frac{e^{-ap}}{p}$
$\delta(t)$	1
$\delta(t-a)$	e^{-ap}

Table 3.1: Table of usuel Laplace transforms

3.6 Application to solving of differential equations

3.6.1 Solving linear differential equations with constant coefficients

The Laplace transform is useful for solving linear differential equations with constant coefficients

$$a_0 y^{(n)}(t) + a_1 y^{(n-1)}(t) + a_2 y^{(n-2)}(t) + ... + a_n y(t) = f(t). \quad (1)$$

We want to find the solution to this equation $y = y(t)$ for $t \geq 0$ and verifying the initial conditions

$$y(0) = y_0,\ y'(0) = y_1,\ y''(0) = y_2, ...,\ y^{(n-1)}(0) = y_{n-1}. \tag{2}$$

The solution method consists of

- Taking the Laplace transform of both sides of the equation.

- Using the initial conditions to calculate the derivatives.

- Obtaining an algebraic equation to obtain $\mathcal{L}(y(t)) = Y(p)$.

- The desired solution is obtained by taking the inverse Laplace transform of $Y(p)$.

We will now present a simpler solution method by introducing the Laplace transform.

We apply the Laplace transform to both sides of equation (1)

$$\mathcal{L}\left(a_0 y^{(n)}(t) + a_1 y^{(n-1)}(t) + a_2 y^{(n-2)}(t) + ... + a_n y(t)\right) = \mathcal{L}\left(f(t)\right)(p). \tag{3}$$

Using the properties of linearity, equation (3) becomes

$$a_0 \mathcal{L}\left(y^{(n)}(t)\right) + a_1 \mathcal{L}\left(y^{(n-1)}(t)\right) + a_2 \mathcal{L}\left(y^{(n-2)}(t)\right) + ... + a_n \mathcal{L}(y(t)\big) = \mathcal{L}\left(f(t)\right)(p). \tag{4}$$

Knowing that

$$\mathcal{L}(y^{(n)}(t))(p) = p^n \mathcal{L}(y(t)) - \sum_{k=0}^{n-1} p^{n-1-k} y^{(k)}(0^+).$$

We substitute these expressions into equation (4) to obtain an algebraic equation of the type

$$\mathcal{L}(y(t))(p)(\Phi_n(p)) = \mathcal{L}\left(f(t)\right)(p) + \Psi_{n-1}(p),$$

with Φ_n a polynomial of degree n and Ψ_{n-1} a polynomial of degree $n-1$.

Then, the algebraic equation can be written in the following form

$$\mathcal{L}(y(t))(p) = \frac{\mathcal{L}\left(f(t)\right)(p)}{\Phi_n(p)} + \frac{\Psi_{n-1}(p)}{\Phi_n(p)}. \tag{5}$$

Finally, we use the inverse Laplace transform to equation (5) to determine the solution $y(t)$ of equation (1) with the initial condtions (2).

Example 3.6.1 *Solve the following differential equations*

1) $y'(t) + y(t) = 1$, *with the initial condition* $y(0) = 0$.

2) $y''(t) - 3y'(t) + 2y(t) = 4e^{3t}$, *with the initial conditions* $y(0) = 4$, $y'(0) = 9$.

1) To solve the equation $y'(t) + y(t) = 1$ *with* $y(0) = 0$, *follow the next step.*

Let us take the Laplace transform of both members of equation and using the linearity property, we get

$$\mathcal{L}(y'(t)) + \mathcal{L}\left(y(t)\right) = \mathcal{L}\left(1\right).$$

According to the Laplace transform of the first derivative, we have

$$p\mathcal{L}(y(t)) - y(0) + \mathcal{L}\left(y(t)\right) = \mathcal{L}\left(1\right).$$

Replacing the initial condition and by a simple calculation, we obtain

$$(p+1)\,\mathcal{L}(y(t)) = \frac{1}{p}$$

$$\begin{array}{rl} \Longrightarrow & Y(p) = \mathcal{L}(y(t)) \\ = & \dfrac{1}{p\,(p+1)}. \end{array}$$

Applying the inverse Laplace transform we find

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}(Y(p)) \\ &= \mathcal{L}^{-1}\left(\frac{1}{p\,(p+1)}\right) \\ &= \mathcal{L}^{-1}\left(\frac{1}{p} - \frac{1}{p+1}\right) \\ &= 1 - e^{-t}. \end{aligned}$$

So the solution of the differential equation $y'(t) + y(t) = 1$ *with the initial condition* $y(0) = 0$, *is the function*

$$y(t) = 1 - e^{-t}.$$

2) To solve the equation $y''(t) - 3y'(t) + 2y(t) = 4e^{3t}$ *with* $y(0) = 4$, $y'(0) = 9$, *follow the next step.*

Let us take the Laplace transform of both members of equation and using the linearity property, we get

$$\mathcal{L}(y''(t)) - 3\mathcal{L}\left(y'(t)\right) + 2\mathcal{L}\left(y(t)\right) = 4\mathcal{L}\left(e^{3t}\right).$$

According to the Laplace transform of the first and second derivatives, we have

$$p^2\mathcal{L}(y(t)) - py(0) - y'(0) - 3(p\mathcal{L}(y(t) - y(0)) + 2\mathcal{L}(y(t)) = 4\mathcal{L}\left(e^{3t}\right).$$

Replacing the initial condition and by a simple calculation, we obtain

$$p^2\mathcal{L}(y(t)) - 4p - 9 - 3p\mathcal{L}(y(t)) + 12 + 2\mathcal{L}(y(t)) = \frac{4}{p-3}$$

$$\begin{aligned} \Longrightarrow & \ Y(p) = \mathcal{L}(y(t)) \\ = & \ \frac{4p^2 - 15p + 13}{p^3 - 3 + 2} \\ = & \ \frac{1}{p-1} + \frac{1}{p-2} + \frac{2}{p-3}. \end{aligned}$$

Applying the inverse Laplace transform we find

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}(Y(p)) \\ &= \mathcal{L}^{-1}\left(\frac{1}{p-1} + \frac{1}{p-2} + \frac{2}{p-3}\right) \\ &= e^t + e^{2t} + 2e^{3t}. \end{aligned}$$

So the solution of the differential equation $y''(t) - 3y'(t) + 2y(t) = 4e^{3t}$ *with the initial conditions* $y(0) = 4$, $y'(0) = 9$, *is the function*

$$y(t) = e^t + e^{2t} + 2e^{3t}.$$

3.7 Corrected exercises

Exercise 3.7.1 *Let the Heaviside function be defined by*

$$H(t) = \begin{cases} 1, & \textit{if} \ \ t \geq 0, \\ 0, & \textit{if} \ \ t < 0. \end{cases}$$

Compute the Laplace transforms of the following functions

1- $H(t-1) - H(t-2)$.

2- $(t-2)^2 H(t-2)$.

3- $\sum\limits_{n=0}^{\infty} \left(H(t-2n) - H(t-(2n+1))\right).$

Solution:

1- $H(t-1)-H(t-2)$.

There are three distinct cases:

- If $t<1$, then $H(t-1)=0$ and $H(t-2)=0$, and the function is zero.

- If $t\in[1,2]$, then $H(t-1)=1$ and $H(t-2)=0$, and the function is equal to 1.

- If $t>2$, then $H(t-1)=0$ and $H(t-2)=0$, and the function is zero.

Therefore

$$\begin{aligned} F(p) &= \int_1^2 e^{-pt}dt \\ &= \left[-\frac{1}{p}e^{-pt}\right]_1^2 \\ &= -\frac{1}{p}e^{-2p}+\frac{1}{p}e^{-p} \\ &= \frac{e^{-p}-e^{-2}}{p}. \end{aligned}$$

2- $(t-2)^2H(t-2)$.

We know that

$$\mathcal{L}\left(t^2\right)=F(p)=\frac{2}{p^3}.$$

Using the translation property, we obtain

$$\begin{aligned} \mathcal{L}\left((t-2)^2H(t-2)\right) &= e^{-2p}F(p) \\ &= \frac{2e^{-2p}}{p^3}. \end{aligned}$$

3- $\sum\limits_{n=0}^{\infty}\left(H(t-2n)-H(t-(2n+1))\right).$

With the same reasoning as for the first function, we see that the function is equal to 1 on the interval of type $[2n,2n+1]$ and is equal to 0 elsewhere, the calculation of the Laplace transform then gives

$$\begin{aligned} F(p) &= \sum_{n=0}^{\infty} \int_{2n}^{2n+1} e^{-pt} dt \\ &= \sum_{n=0}^{\infty} \left[-\frac{1}{p} e^{-pt} \right]_{2n}^{2n+1} \\ &= \sum_{n=0}^{\infty} \left(-\frac{1}{p} e^{-(2n+1)p} + \frac{1}{p} e^{-2np} \right) \\ &= \frac{1}{p} \left(\sum_{n=0}^{\infty} e^{-2np} - \sum_{n=0}^{\infty} e^{-(2n+1)p} \right) \\ &= \frac{1}{p} \left(\frac{1}{1-e^{-2p}} - \frac{e^{-p}}{1-e^{-2p}} \right) \\ &= \frac{1-e^{p}}{p\left(1-e^{-2p}\right)}. \end{aligned}$$

Exercise 3.7.2 *Let* $a \in \mathbb{R}^+$

1- Using two different methods, compute the Laplace transform of

$$f(t) = te^{at}.$$

2- For each $k \in \mathbb{N}$, *define the function* $f_k : \mathbb{R}^+ \longrightarrow \mathbb{R}$ *by*

$$f_k(t) = \frac{t^k}{k!} e^{at}.$$

Show by induction that

$$\mathcal{L}\left(f_k(t)\right)(p) = \frac{1}{(p-1)^{k+1}}.$$

Solution:

1- By two different methods we calculate the Laplace transform of

$$f(t) = te^{at}.$$

- 1$^{\text{st}}$ method:

We have

$$\begin{aligned} \mathcal{L}\left(f(t)\right)(p) &= \int_0^{+\infty} f(t) e^{-pt} dt \\ &= \int_0^{+\infty} te^{at} e^{-pt} dt \\ &= \int_0^{+\infty} te^{-(p-a)t} dt. \end{aligned}$$

We integrate by part, we have

$$\begin{aligned} u &= t \longrightarrow du = 1, \\ dv &= e^{-(p-a)t} \longrightarrow v = -\frac{1}{p-a}e^{-(p-a)t}. \end{aligned}$$

So

$$\begin{aligned} \mathcal{L}\left(f(t)\right)(p) &= \underbrace{\left[-\frac{t}{p-a}e^{-(p-a)t}\right]_0^{+\infty}}_{=0} + \int_0^{+\infty} \frac{1}{p-a}e^{-(p-a)t}dt \\ &= \frac{1}{p-a}\left[-\frac{1}{p-a}e^{-(p-a)t}\right]_0^{+\infty} \\ &= \frac{1}{(p-a)^2}. \end{aligned}$$

- 2$^{\text{nd}}$ method:

Let

$$g(t) = t \Longrightarrow G(p) = \frac{1}{p^2}.$$

So

$$\begin{aligned} \mathcal{L}\left(te^{at}\right)(p) &= \mathcal{L}\left(e^{at}g(t)\right)(p) \\ &= G(p-a) \\ &= \frac{1}{(p-a)^2}. \end{aligned}$$

2- Let

$$\begin{aligned} f_k &: \mathbb{R}^+ \to \mathbb{R} \\ t &\to f_k(t) = \frac{t^k}{k!}e^{at}. \end{aligned}$$

we show by recurrence that

$$\mathcal{L}\left(f_k(t)\right)(p) = \frac{1}{(p-1)^{k+1}}. \qquad (*)$$

- For $k = 0$, we have

$$f_0(t) = e^{at} \Longrightarrow \mathcal{L}\left(f_0(t)\right) = \mathcal{L}\left(e^{at}\right) = \frac{1}{p-1}.$$

- For $k=1$, we have

$$f_1(t)=te^{at}\Longrightarrow \mathcal{L}\left(f_1(t)\right)=\mathcal{L}\left(te^{at}\right)=\frac{1}{(p-1)^2}.$$

Then the relation $(*)$ is true for $k=0$ and $k=0$.

Let's suppose that $(*)$ is true for k and we show that it is true for $k+1$, so

$$f_{k+1}(t)=\frac{t^{k+1}}{(k+1)!}e^{at},$$

which implies that

$$\begin{aligned}
\mathcal{L}\left(f_{k+1}(t)\right) &= \mathcal{L}\left(\frac{t^{k+1}}{(k+1)!}e^{at}\right)\\
&= \frac{1}{(k+1)!}\int_0^{+\infty} t^{k+1}e^{at}e^{-pt}dt\\
&= \frac{1}{(k+1)!}\int_0^{+\infty} \underbrace{t^{k+1}}_{u}\underbrace{e^{-(p-a)t}}_{dv}dt\\
&\overset{I.P}{=} \frac{1}{(k+1)!}\left(\underbrace{\left[-\frac{t^{k+1}}{p-a}e^{-(p-a)t}\right]_0^{+\infty}}_{=0}+\frac{1}{p-a}\int_0^{+\infty}(k+1)t^ke^{-(p-a)t}dt\right)\\
&= \frac{1}{(k+1)!}(k+1)\frac{1}{p-a}\int_0^{+\infty} t^ke^{-(p-a)t}dt\\
&= \frac{1}{p-a}\int_0^{+\infty}\frac{t^k}{k!}e^{-(p-a)t}dt\\
&= \frac{1}{p-a}\frac{1}{(p-a)^k}\\
&= \frac{1}{(p-a)^{k+1}}.
\end{aligned}$$

Exercise 3.7.3 *Let*

$$f(t)=(1-\cos(t))\ \textit{and}\ g(t)=e^{-t}f(t).$$

1- Show that

$$\mathcal{L}\left(f(t)\right)(p)=\frac{1}{p(p^2+1)}.$$

2- Deduce that

$$\mathcal{L}\left(e^tg''(t)\right)(p)=\frac{(p-1)^2}{p(p^2+1)}.$$

Solution:

Let

$$f(t) = (1 - \cos(t)) \text{ and } g(t) = e^{-t} f(t).$$

We show that

$$\mathcal{L}(f(t))(p) = \frac{1}{p(p^2+1)}.$$

For this we have

$$\begin{aligned}
\mathcal{L}(f(t))(p) &= \mathcal{L}(1 - \cos(t))(p) \\
&= \mathcal{L}(1)(p) - \mathcal{L}(\cos(t))(p) \\
&= \frac{1}{p} - \frac{p}{p^2+1} \\
&= \frac{1}{p(p^2+1)}.
\end{aligned}$$

2) We deduce that

$$\mathcal{L}\left(e^t g''(t)\right)(p) = \frac{(p-1)^2}{p(p^2+1)}.$$

We have

$$\mathcal{L}(g''(t))(p) = p^2 \mathcal{L}(g''(t)) - pg(0^+) - g'(0^+),$$

and

$$\begin{aligned}
g(0^+) &= e^{-0} f(0^+) \\
&= 1 - \cos(0^+) \\
&= 0,
\end{aligned}$$

and

$$\begin{gathered}
g'(t) = -e^{-t} f(t) + e^{-t} f'(t) \\
\Longrightarrow g'(0^+) = 0.
\end{gathered}$$

Then

$$\begin{aligned}
\mathcal{L}(g''(t))(p) &= p^2 \mathcal{L}(g''(t)) \\
&= p^2 \mathcal{L}\left(e^{-t} f(t)\right) \\
&= p^2 F(p+1) \\
&= \frac{p^2}{(p+1)((p+1)^2+1)}.
\end{aligned}$$

Which implies that

$$\mathcal{L}\left(e^t g''(t)\right)(p) = \frac{(p-1)^2}{p(p^2+1)}.$$

Exercise 3.7.4 *Solve the following integral equations using the Laplace transform*

$$(1) : x(t) = t + \int_0^t x(s)\sin(t-s)ds,\ t \geq 0.$$

$$(2) : e^{-t}x(t) - \int_0^t e^{-s}x(s)ds = \sin(t),\ t \geq 0.$$

Solution:

We use the Laplace transform to solve

$$(1) : x(t) = t + \int_0^t x(s)\sin(t-s)ds,\ t \geq 0.$$

We apply the Laplace transform to equation (1), we find

$$\begin{aligned} \mathcal{L}(x(t)) &= \mathcal{L}\left(t + \int_0^t x(s)\sin(t-s)ds\right) \\ &= \mathcal{L}(t) + \mathcal{L}\left(\int_0^t x(s)\sin(t-s)ds\right) \\ &= \mathcal{L}(t) + \mathcal{L}(x(s) * \sin(t)) \\ &= \mathcal{L}(t) + \mathcal{L}(x(s))\,\mathcal{L}(\sin(t)). \end{aligned}$$

Which implies that

$$\begin{aligned} \mathcal{L}(x(t)) &= \frac{\mathcal{L}(t)}{1 - \mathcal{L}(\sin(t))} \\ &= \frac{\frac{1}{p^2}}{1 - \frac{1}{p^2+1}} \\ &= \frac{p^2+1}{p^4} \\ &= \frac{1}{p^2} + \frac{1}{p^4}. \end{aligned}$$

We apply the inverse Laplace transform, we find

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left(\frac{1}{p^2}\right) + \mathcal{L}^{-1}\left(\frac{1}{p^4}\right) \\ &= t^2 + \frac{1}{6}t^3. \end{aligned}$$

We use the Laplace transform to solve

$$(2) : e^{-t}x(t) - \int_0^t e^{-s}x(s)ds = \sin(t),\ t \geq 0.$$

Multiple equation (2) by e^t, we find

$$(2') : x(t) - \int_0^t e^{t-s}x(s)ds = e^t \sin(t).$$

We apply the Laplace transform to equation $(2')$, we find

$$\begin{aligned}
\mathcal{L}\left(x(t) - \int_0^t e^{t-s}x(s)ds\right) &= \mathcal{L}\left(e^t \sin(t)\right) \\
&= \mathcal{L}(t) + \mathcal{L}\left(\int_0^t x(s)\sin(t-s)ds\right) \\
&= \mathcal{L}(t) + \mathcal{L}(x(s) * \sin(t)) \\
&= \mathcal{L}(t) + \mathcal{L}(x(s))\,\mathcal{L}(\sin(t)).
\end{aligned}$$

$$\Longrightarrow \mathcal{L}(x(t)) - \mathcal{L}\left(\int_0^t e^{t-s}x(s)ds\right) = \mathcal{L}\left(e^t \sin(t)\right),$$

$$\Longrightarrow \mathcal{L}(x(t)) - \mathcal{L}\left(e^t * x(t)\right) = \mathcal{L}\left(e^t \sin(t)\right),$$

$$\Longrightarrow \mathcal{L}(x(t)) - \mathcal{L}\left(e^t\right)\mathcal{L}(x(t)) = \mathcal{L}\left(e^t \sin(t)\right).$$

Which implies that

$$\begin{aligned}
\mathcal{L}(x(t)) &= \frac{\mathcal{L}(e^t \sin(t))}{1 - \mathcal{L}(e^t)} \\
&= \frac{\frac{1}{(p-1)^2+1}}{1 - \frac{1}{p-1}} \\
&= \frac{p-1}{(p-1)\left((p-1)^2+1\right)}.
\end{aligned}$$

We perform a decomposition into simple elements, we have

$$\begin{aligned}
\mathcal{L}(x(t)) &= \frac{p-1}{(p-1)\left((p-1)^2+1\right)} \\
&= \frac{1}{2}\frac{1}{p-1} - \frac{1}{2}\frac{p-1}{(p-1)^2+1} + \frac{1}{2}\frac{1}{(p-1)^2+1} \\
&.
\end{aligned}$$

We apply the inverse Laplace transform, we find

$$\begin{aligned}
x(t) &= \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{p-1}\right) - \frac{1}{2}\mathcal{L}^{-1}\left(\frac{p-1}{(p-1)^2+1}\right) + \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{(p-1)^2+1}\right) \\
&= \frac{1}{2}e^t - \frac{1}{2}e^t\sin(t) + \frac{1}{2}e^t\cos(t) \\
&= \frac{(1-\sin(t)+\cos(t))\,e^t}{2}.
\end{aligned}$$

Exercise 3.7.5 *Find the inverse Laplace transforms of the following functions*

$$F(P) = \frac{1}{p^2-3p+2},\ G(P) = \frac{p+1}{p(p^2+4)},\ K(P) = \frac{5}{(p+1)\,(p^2+4p+8)}.$$

Solution:

We find the inverse Laplace transforms of the following functions

1)

$$\begin{aligned}
F(P) &= \frac{1}{p^2-3p+2} \\
&= \frac{1}{(p-1)\,(p-2)} \\
&= \frac{a}{p-1} + \frac{b}{p-2}.
\end{aligned}$$

By a simple calculation, we find

$$a = -1 \text{ and } b = 1.$$

Then

$$F(p) = \frac{1}{p-2} - \frac{1}{p-1}.$$

So

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}\left(F(p)\right) \\
&= \mathcal{L}^{-1}\left(\frac{1}{p-2} - \frac{1}{p-1}\right) \\
&= e^{2t} - e^t
\end{aligned}$$

2)

$$\begin{aligned}
G(P) &= \frac{p+1}{p(p^2+4)} \\
&= \frac{a}{p} + \frac{bp+c}{p^2+4}.
\end{aligned}$$

By a simple calculation we find

$$a = \frac{1}{4},\ b = -\frac{1}{4} \text{ and } c = 1.$$

Then

$$\begin{aligned} G(p) &= \frac{1}{4}\frac{1}{p} + \frac{-\frac{1}{4}p + 1}{p^2 + 4} \\ &= \frac{1}{4}\left(\frac{1}{p} + \frac{-p + 4}{p^2 + 4}\right). \end{aligned}$$

So

$$\begin{aligned} g(t) &= \mathcal{L}^{-1}\left(G(p)\right) \\ &= \mathcal{L}^{-1}\left(\frac{1}{4}\left(\frac{1}{p} + \frac{-p+4}{p^2+4}\right)\right) \\ &= \frac{1}{4}\left(\mathcal{L}^{-1}\left(\frac{1}{p}\right) - \mathcal{L}^{-1}\left(\frac{p}{p^2+4}\right)\right) + \frac{1}{2}\mathcal{L}^{-1}\left(\frac{2}{p^2+4}\right) \\ &= \frac{1}{4} - \frac{1}{4}\cos(2t) + \frac{1}{2}\sin(2t). \end{aligned}$$

3)

$$\begin{aligned} K(P) &= \frac{5}{(p+1)\left(p^2 + 4p + 8\right)} \\ &= \frac{a}{p+1} + \frac{bp + c}{p^2 + 4p + 8}. \end{aligned}$$

By a simple calculation we find

$$a = 1,\ b = -1 \text{ and } c = -3.$$

Then

$$K(p) = \frac{1}{p+1} - \left(\frac{p+3}{p^2 + 4p + 8}\right).$$

So

$$\begin{aligned}
k(t) &= \mathcal{L}^{-1}\left(K(p)\right) \\
&= \mathcal{L}^{-1}\left(\frac{1}{p+1}\right) - \mathcal{L}^{-1}\left(\frac{p+3}{p^2+4p+8}\right) \\
&= \mathcal{L}^{-1}\left(\frac{1}{p+1}\right) - \mathcal{L}^{-1}\left(\frac{p+2+1}{(p+2)^2+4}\right) \\
&= \mathcal{L}^{-1}\left(\frac{1}{p+1}\right) - \mathcal{L}^{-1}\left(\frac{p+2}{(p+2)^2+2^2}\right) - \frac{1}{2}\mathcal{L}^{-1}\left(\frac{2}{(p+2)^2+2^2}\right) \\
&= e^{t} - e^{-2t}\cos(2t) - \frac{1}{2}e^{-2t}\sin(2t) \\
&= e^{t} - e^{-2t}\left(\cos(2t) - \frac{1}{2}\sin(2t)\right).
\end{aligned}$$

Exercise 3.7.6 *Use the Laplace transform to solve the following differential equation*

$$y''(t) - y(t) = 3e^{-2t} + t + 1,$$

with the initial conditions

$$y(0) = y'(0) = 0.$$

Solution:

Using the Laplace transform to solve the following differential equation

$$y''(t) - y(t) = 3e^{-2t} + t + 1, \tag{1}$$

with the initial conditions

$$y(0) = y'(0) = 0. \tag{2}$$

We apply the Laplace transform to equation (1) and use the property of linearity, we find

$$\mathcal{L}\left(y''(t)\right) - \mathcal{L}\left(y(t)\right) = 3\mathcal{L}\left(e^{-2t}\right) + \mathcal{L}\left(t\right) + \mathcal{L}\left(1\right).$$

We put $Y(p) = \mathcal{L}\left(y(t)\right)$ and we use the Laplace transform of the second derivative, we obtain

$$pY(p) - py'(0) - y(0) - Y(p) = 3\frac{1}{p+2} + \frac{1}{p^2} + \frac{1}{p}.$$

We replace the initial conditions and by a simple calculation we get

$$\begin{aligned} Y(p) &= \frac{4p^2+3p+2}{p^2(p+2)(p^2-1)} \\ &= \frac{a}{p}+\frac{b}{p^2}+\frac{c}{p+2}+\frac{d}{p-1}+\frac{e}{p+1} \\ &= \frac{-1}{p}-\frac{1}{p^2}+\frac{1}{p+2}+\frac{3}{2}\left(\frac{1}{p-1}\right)-\frac{3}{2}\left(\frac{1}{p+1}\right). \end{aligned}$$

We apply the inverse Laplace transform, we get

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}(Y(p)) \\ &= -\mathcal{L}^{-1}\left(\frac{1}{p}\right)-\mathcal{L}^{-1}\left(\frac{1}{p^2}\right)+\mathcal{L}^{-1}\left(\frac{1}{p+2}\right)+\frac{3}{2}\mathcal{L}^{-1}\left(\frac{1}{p-1}\right)-\frac{3}{2}\mathcal{L}^{-1}\left(\frac{1}{p+1}\right) \\ &= -1-t+e^{-2t}+\frac{3}{2}e^{t}-\frac{3}{2}e^{-t}. \end{aligned}$$

Finally, the solution of equations (1) and (2) is

$$y(t) = -1-t+e^{-2t}+\frac{3}{2}e^{t}-\frac{3}{2}e^{-t}.$$

3.8 Suggested exercises

Exercise 3.8.1 *Determine the Laplace transforms of the functions*

1- $x \longrightarrow \sin(\sqrt{x})$.

2- $x \longrightarrow \dfrac{\cos(\sqrt{x})}{\sqrt{x}}$.

Exercise 3.8.2 *From the definition of the Laplace transform, compute* $\mathcal{L}[f(t)]$ *and* $\mathcal{L}[g(t)]$ *for*

$$f(t) = \begin{cases} 0, & \text{if } 0 \le t < 1, \\ t-1, & \text{if } 1 \le t < 2, \\ 0, & \text{if } t \ge 2. \end{cases}$$

and

$$g(t) = \begin{cases} 0, & \text{if } 0 \le t < 1, \\ t-1, & \text{if } t \ge 1. \end{cases}$$

Exercise 3.8.3 *1- Determine the inverse Laplace transform of the function*

$$F(p) = \frac{p}{(p^2+1)^2},$$

using two different methods.

2- Determine the inverse Laplace transform of the function

$$G(p) = e^{-\pi p}\frac{p}{p^2+4}$$

Exercise 3.8.4 *Use Laplace transform technique to solve the initial value problem*

$$\begin{cases} y'(t) + 4y(t) = g(t), \\ y(0) = 2. \end{cases}$$

where

$$g(t) = \begin{cases} 0, & \textit{if } \ 0 \le t < 1, \\ 12, & \textit{if } \ 1 \le t < 3, \\ 0, & \textit{if } \ t \ge 3. \end{cases}$$

Exercise 3.8.5 *Determine the solution to the integral equation*

$$f''(t) + \int_0^x e^{2(x-t)} f'(t)dt = e^{2x},$$

satisfying the initial conditions

$$f(0) = 0, \ f'(0) = 1.$$

Exercise 3.8.6 *Use the Laplace transform to solve the following differential system*

$$\begin{cases} x'(t) = 4x(t) - 6y(t) + 1 \\ y'(t) = 12x(t) + 10y(t) \end{cases},$$

with the initial conditions

$$\begin{cases} x(0) = 0 \\ y(0) = 0 \end{cases}.$$

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ملخص:

هذه المطبوعة البيداغوجية هي المادة الدراسية لمقرر "التحويلات التكاملية في فضاءات L^p "، الذي يُدرّسه الدكتور خلوطة علي لطلاب السنة الثالثة ليسانس في الرياضيات بجامعة فرحات عباس سطيف 1. وقد طُوّرت هذه المادة على مدار العام، وهي لا تُغني عن المحاضرات. تتبع هذه الوثيقة المحاضرات بدقة، ومع تعديلات طفيفة فقط، تُعيد إنتاج المحاضرة كما قُدّمت لجميع الطلبة.

كلمات مفتاحية: فضاءات L^p ، تحويل فورييه، تحويل لابلاس.

Résumé:

Ce polycopié pédagogique est le support du cours «Transformations intégrales dans les espaces L^p», enseigné par le Dr. KHALOUTA Ali pour les étudiants de troisième année licence en mathématiques à l'université Ferhat ABBAS Setif 1. Élaboré tout au long de l'année, il ne remplace pas le cours magistral. Ce document est très proche du cours enseigné, et excepté quelques infimes modifications, il retranscrit le cours tel qu'il a été donné à tous les étudiants.

Mots clés: Les espaces L^p, Transformation de Fourier, Transformation de Laplace.

Abstract :

This pedagogical handout is the course material for "Integral transforms in L^p Spaces," taught by Dr. Ali KHALOUTA to third-year students licence in mathematics at Ferhat ABBAS University Setif 1. Developed throughout the year, this document does not replace the course. It closely follows the lectures, with only minor modifications, it reproduces the course exactly as it was given to all students.

Key words: L^p Spaces, Fourier transform, Laplace transform.