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CONGESTION GAMES AND APPLICATIONS, NASH EQUILIBRIUM AND ANARCHY PRICE

Presented by

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Dedication

In the name of Allah, the Most Gracious, the Most Merciful.

I dedicate this humble work to myself, in appreciation of the long journey and the patience, perseverance, and strength it required, and to my esteemed family, my first source of support and constant pillar throughout this journey

FATIMA

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KHANCHOUCHE FATIMA

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List of Acronyms

- **NE:** Nash equilibrium.
- **SNE:** Strong Nash equilibrium.
- **FIP:** Finite improvement property.
- **PO:** Pareto optimum.
- $\boldsymbol{P^\circ S}$**:** Price of stability.
- $\boldsymbol{P^\circ A}$**:** Price of anarchy.
- **CCV:** Corresponding congestion vector.
- **ANE:** All Nash equlibria.
- **OUF:** Ordinal utility function.

General introduction

Game theory constitutes one of the central analytical tools employed across a wide range of scientific and applied disciplines, including economics [45], biologyparticularly through the development of evolutionary game theory by Maynard Smith ([58],[59], [8]) as well as business and finance ([4], [60]), transportation management ([46],[62]), marketing ([29], [12]), political and social sciences ([40], [34]), environmental studies ([25], [13]), operations research ([56], [57], [11]), and other related fields. Its applications have further expanded into strategic domains such as military research ([36], [14]), control theory ([9], [63], [64]), law ([7], [10]), and telecommunications and information networks ([52], [53]). Game theory is considered a relatively recent discipline, with its formal establishment marked by the publication of the seminal work "*Theory of Games and Economic Behavior*" by John von Neumann and Oskar Morgenstern in 1944. The principal objective of game theory is to model the behavior of rational agents (players) within interactive frameworks characterized by competition or cooperation, emphasizing the analysis of decisions made by each agent based on their expectations regarding the actions of others.

A game represents a situation in which players are required to make strategic decisions from a set of available actions (strategies) within a predefined framework that stipulates the rules governing the interactions. The outcome of these decisions constitutes the result of the game, which is associated with gains or losses for each participant. In this context, various solution concepts can be considered depending on the type of game under study. One such solution concept is the Nash equilibrium, proposed by mathematician John Nash in 1950 ([42], [41], [43]). A Nash equilibrium is defined as a state in which no player has an incentive to unilaterally change their strategy, given the strategies adopted by the other players.

The concept of Nash equilibrium, in its essence, is straightforward and aligns with the foundational principles of non-cooperative games. This category of games pertains to interactive scenarios where individuals are free to make their own decisions and pursue personal and independent objectives. These individuals do not communicate prior to the game and, as such, do not necessarily possess the means to commit to a particular strategy. Within this framework, the Nash equilibrium seeks to identify outcomes that are stable in the face of unilateral deviations, that is, changes in strategy by a single player. The absence of communication implies the lack of explicit coordination and precludes the possibility of multilateral deviations.

Many real-world problems can be represented through non-cooperative game models, which have been analyzed since the pioneering research conducted by John Nash.

In this thesis, we focus on two specific types of non-cooperative games [45] the family of congestion games and the family of potential games. The study of congestion games has seen significant growth in academic literature, largely thanks to Rosenthal [49], who first formally introduced and analyzed them. Congestion games are those in which each player selects a specific combination of resources (or goods) from a shared set of resources. The reward associated with each good is a function of the number of players who include that good in their choice. A player's utility is the sum of the rewards associated with the goods they select.

To illustrate the applicability of congestion games in real-life situations, consider the example of a group of unemployed individuals who must decide whether to migrate in search of employment.Their choice of destination depends on the labor market conditions in each country. However, if too many individuals decide to migrate to the same country, the attractiveness of that country as a destination decreases.

Another example, though simpler, involves a group of students who must choose between different libraries in a city for study purposes. Their decision is influenced not only by the characteristics of the library, such as the availability of books or the quietness of the environment, but also by the number of other students studying there. Some students may prefer libraries with more of their peers, as it creates a sense of motivation and engagement. On the other hand, students seeking a quieter, less crowded environment might avoid libraries that are heavily frequented by others.

In both examples, the decisions made by individuals are not solely based on the characteristics of the resources they choose, but also on the actions of others, which constitutes a fundamental characteristic of congestion games.

In the initial study of congestion games, Rosenthal [49] proved the existence of Nash equilibrium using the concept of potential functions. His work played a pivotal role in the development of this field, which was later expanded by Monderer and Shapley [39]. These researchers generalized the class of congestion games to a broader category known as "potential games," where Nash equilibria can be represented through a potential function. This potential function is a real-valued function defined on the strategy space, and achieving the maximum value of this function corresponds to a Nash equilibrium of the game.

Monderer and Shapley not only demonstrated that every congestion game is, in fact, a potential game, but they also established that every exact potential game is isomorphic to a congestion game. This generalization has led to further studies on the topic. Several researchers have contributed to the study of congestion games, including Weber, Quint and Shubik, Le Breton, Konishi, and Milchtaich (see [31]), (see [48]),(see [37]). Their work focuses on various forms of congestion games, which typically lack a potential function, yet are still capable of supporting a Nash equilibrium in pure strategies.

In these models, the existence of a Nash equilibrium is typically established through the convergence of a successive improvement process known as the Nash mechanism. In this process, players are introduced into the game one by one. Each player selects an action that maximizes their utility given the actions chosen by the other players already in the game. Once a new player selects their action, the previous players are allowed to revise their strategies. This revision phase continues as long as there are players who wish to deviate from their current strategies. When all players are satisfied with their actions, a new player enters and selects an action, and the process continues until a Nash equilibrium is reached. The convergence time of this process has been the focus of numerous research studies (see, for example, [1], [28], [37]).

Several studies have focused on the analysis of specific classes of congestion games. Ieong et al. [28] examined the category of singleton congestion games, where each player can select only one resource at a time, rather than multiple resources simultaneously. They demonstrated that the improvement process converges to an equilibrium in polynomial time. Moreover, they demonstrated that this result also holds in the pursuit of Nash equilibria that are optimal in terms of maximizing the collective utility of all participants. Variations of singleton monotone congestion games have also been explored, particularly with respect to the convergence rate of best-response dynamics towards Nash equilibrium (Even-Dar et al. [19]), as well as the investigation of alternative solution concepts (Rozenfeld and Tennenholtz [51]).

In addition to equilibrium stability, the economic efficiency of outcomes constitutes a fundamental dimension in the study of strategic games. Among the key measures of efficiency is Pareto optimality, which guarantees that no player can improve their payoff without decreasing the payoff of at least one other player. While classical Nash equilibria can often result in inefficient outcomes and suboptimal resource allocations, strong Nash equilibria (SNE) tend to align more closely with Pareto-optimal solutions, as they inherently prevent coalitional deviations that would benefit a subset of players at the expense of others. This relationship between strong Nash equilibria and Pareto optimality plays a crucial role in congestion games, where the allocation of resources among players directly affects the overall system efficiency.

The existence and properties of strong Nash equilibria have been a central subject of research in the game theory literature. Holzman [27] and Voorneveld [61] analyzed monotone congestion games, and they were able to identify the set of strong Nash equilibria, while Rosenfeld et al. [51] investigated the concept of the correlated strong Nash equilibrium[1] in both decreasing and increasing monotone congestion games.

Nevertheless, computing strong Nash equilibria remains one of the most computationally challenging problems in game theory. Verifying the existence of an SNE is classified as NP-complete[2], implying that, in the worst case, its resolution requires exponential time complexity, although the verification of solutions can be performed in polynomial time. This computational hardness becomes particularly critical in large-scale games with many players or complex strategic spaces ([24], [15]).

To address these challenges, several algorithms have been proposed for computing strong Nash equilibria. For instance, [23] introduced the Aumann Crowding-Based Differential Evolution (A-CrDE) algorithm, specifically designed for computing strong Nash equilibria in multiplayer games. Other algorithms have been scientifically developed and optimized for pure strategy strong Nash equilibria in various classes of games, including continuous games [44], connection games [18], and congestion games [26].

Network congestion games represent a fundamental class of congestion games, where the focus is on modeling congestion issues in road networks consisting of n travelers and m links (roads). This type of game reflects the interactions between individuals who compete for limited resources, in this case, the roads. In this context, Koutsoupias and Papadimitriou [32] pioneered this area of research, with their work primarily focusing on calculating the "price of anarchy" for Nash equilibria in pure strategies.

The "price of anarchy" was initially defined by Koutsoupias and Papadimitriou [32] to capture the concept of worst-case performance in selfish behavior compared to the optimal equilibrium. This concept addresses the outcomes resulting from a lack of coordination between players, leading to suboptimal results when each

[1]The correlated strong Nash equilibrium is an extension of the strong Nash equilibrium, allowing players to coordinate their strategies based on a shared signal or correlation device.

[2]NP-complete describes problems that are hard to solve in polynomial time, although a proposed solution can be checked efficiently. Verification of solutions means confirming that a given strategy profile is a strong Nash equilibrium by checking that no coalition of players can jointly deviate to improve their outcomes.

player makes decisions based on their own individual interests.

Following this work, subsequent studies were carried out by Czumaj and Vöcking [16], Czumaj, Krysta, and Vöcking [17], Mavronicolas and Spirakis [35], and Koutsoupias, Mavronicolas, and Spirakis [33], among others, who further expanded this research, examining the impact of Nash equilibria on players' strategies in various contexts of network congestion games.

Ackermann et al [3] proposed a modification to Rosenthals classical congestion game model by introducing a variant in which each player is assigned an individual payment function. As a result, utility functions differ across players. Their analysis revealed that such games may fail to guarantee the existence of a Nash equilibrium.

Milchtaich [37] established that in congestion games where players are limited to choosing a single resource and the payment functions decrease as the number of users increases, the existence of at least one Nash equilibrium is always guaranteed.

In contrast to Rosenthal's proof, Milchtaich's result does not rely on the concept of potential functions. Instead, it uses a modified version of the improvement mechanism, which follows a sequence of successive improvements to reach an equilibrium.

In this thesis, we place particular emphasis on the works of Rosenthal, Milchtaich and Sbabou ([49],[37],[54]). Our primary focus is on congestion games with single-choice strategies, either with a common utility function or with specific payment functions.

Our research aims to address several key questions: First, can we identify algorithmic frameworks for analyzing Nash equilibria in this class of games, without relying on potential functions or any iterative optimization mechanisms? Second, rather than settling for a single equilibrium, can we formulate analytical algorithms that allow for the systematic exploration and characterization of the entire set of Nash equilibria? Third, Can we construct algorithms that allow for the ranking of Nash equilibria according to player-specific cost criteria, such that the most efficient equilibrium corresponds to the minimum cost, while the least efficient equilibrium corresponds to the maximum cost?

Furthermore, is it feasible to develop efficient algorithms for computing the possible equilibria in these games? Can we calculate the difference between the price of anarchy and the price of stability [3], thereby gaining insights into the impact of selfish behavior on system performance compared to the optimal equilibrium solution? Finally, how can we analyze the strong equilibria within this context? Can we identify the conditions that guarantee the existence of strong equilibria in different congestion game scenarios?

[3] Price of stability (PoS)[5] is a concept in game theory that measures the efficiency of the best Nash equilibrium relative to the socially optimal solution.

Lastly, in which cases can we reduce the computational time for determining one or more equilibria?

The answers to these questions form the core of our work in this thesis.
Chapter 1 presents the basic definitions and terminology in non-cooperative game theory.
Chapter 2, focuses on non-symmetric singleton congestion games, where players select only one resource, but they do not necessarily share the same utility function. We will first provide a brief description of Milchtaich's method [37] in this context, which is based on the finite best-reply property (FBRP) [4]. We will highlight the limitations of this approach when the goal is to identify the set of Nash equilibria for a specific game.

Next, we will focus on exploring the possibility of achieving this objective in a specific case where the resource set consists of only two elements.

In addition, we will present algorithms for calculating possible Nash equilibria in these games. We will also calculate the differences between the "price of anarchy" and the "price of stability", which will provide a better understanding of the impact of selfish behavior on the overall system performance compared to the ideal equilibrium. We will further study strong equilibria and focus on developing algorithms to determine these equilibria in different contexts.

Chapter 3 will be dedicated to the study of a specific class of non-symmetric congestion games with individual choice. After presenting the main steps of the method proposed by Milchtaich [37] for finding a Nash equilibrium, we will proceed to analyze the limitations of this method, which is based on the finite improvement property (FIP) [5]. We will then explore the possibility of adopting an alternative approach that leads to a complete description of the equilibria. The idea is to leverage this approach to construct an algorithm that reduces the time required to compute at least one Nash equilibrium.
Chapter 4 presents non-symmetric singleton congestion games. We first provide a brief overview of the work by Holzman and Law-Yone [27], as well as that of Voorneveld et al. [61] on monotonic congestion games and the characterization of strong Nash equilibria. A strong Nash equilibrium is a strategy profile in which no subset of players can simultaneously improve their payoffs by deviating from their current strategies. It has been shown that, in singleton monotone congestion games with player non-specific payoff functions, the set of strong equilibria coincides with

[4] FBRP: This is the same property as the finite improvement property (FIP), except that in each improvement, the player who changes their strategy deviates to their best reply.

[5] Improvement refers to transitioning from one state to another by changing the strategy of a single player to increase their utility. A game is said to have the finite improvement property (FIP) if every sequence of successive improvements terminates in a finite number of steps. Any game satisfying this property must necessarily have a Nash equilibrium.

the set of Nash equilibria, and with the set of profiles that maximize the potential function.
In this chapter, we will focus on analyzing strong equilibria in such games. We will demonstrate that, in non-symmetric singleton congestion games with player non-specific payoff functions (case of two resources), strong equilibria do not necessarily coincide with classical Nash equilibria. Moreover, we have developed efficient algorithms to compute these strong equilibria, thus providing practical tools for analyzing the strategic behavior in these games. Additionally, we will compute the differences between the "price of anarchy" and the "price of stability," enabling a deeper understanding of the impact of selfish behavior on the overall system performance compared to the socially optimal equilibrium.

Finally, we end this thesis by a general conclusion and suggestions for future work.

Chapter 1

Definitions and notations

In this introductory chapter, we explore key aspects of non-cooperative game theory, which provides a framework for modeling and analyzing competitive environments. We begin by defining the various types of equilibria in games, which represent states where no player has an incentive to deviate given the strategies of other players.

After presenting and discussing these equilibrium concepts, we will focus on Pareto optimality and strong Nash equilibrium, analyzing their relationship with Nash equilibrium.

Next, we will introduce fundamental tools used to examine the existence of Nash equilibrium, discussing the characteristics of congestion games and potential games. Through examples, we will illustrate how these two classes of games can exhibit multiple Nash equilibria.

Finally, we will provide a detailed explanation of the Finite Improvement Property (FIP) introduced by Milchtaich, highlighting its significance in the analysis of player strategies and equilibrium stability.

Introduction

Game theory has become an increasingly important analytical tool in the study of corporate strategies. It focuses on rational individual decision-making and the outcomes of interactions between different agents. Specifically, it also encompasses models from neoclassical economic theory, offering a versatile framework for analyzing a wide range of strategic situations, whether cooperation and communication are present or absent.

The first step in the analysis involves modeling a conflict situation between several decision-makers (players) in the form of a game. This includes defining the possible actions for each player, numerically evaluating the outcomes in terms of payoffs or gains, and analyzing the likelihood of these outcomes, especially in the presence of randomness.

The second step is to analyze the formal structure of the game and predict the possible outcomes based on the strategies adopted. However, it is rare for this analysis to yield a unique solution (equilibrium) without introducing additional assumptions or extensions, which may require revisiting and refining the initial game model. Within this refined framework, the concept of equilibrium becomes more precise and allows for a broad range of applications, particularly in economics.

In this context, we distinguish between cooperative solutions to games, as defined by von Neumann and Morgenstern in their 1944 book "Theory of Games and Economic Behavior", and non-cooperative solutions characterized by specific forms of Nash equilibrium. This distinction is illustrated through the well-known example of the "Battle of the Sexes", which first appeared in 1952. This classic game serves as an elegant model to explore issues of coordination and the determination of different types of equilibria.

1.1 Non-cooperative game

Non-cooperative game theory analyzes strategic interactions where each participant acts independently to optimize their own outcome, without considering the potential benefits to others. Such models are typically studied under the assumption of perfect and complete information, which means that all players are fully informed about the structure and rules governing the game.

A strategic game is a formal representation consisting of a set of strategies that define each player's actions in all possible game configurations, along with the corresponding payoffs each player receives when the strategies of all participants are specified.

Definition 1.1.0.1. A strategic game is defined by:

- N represents the finite number of n players.
- σ_i^* represents a player's strategy $i \in N$ (not necessarily a numerical value; it can also be a vector or a function).
- S_i represents the set of strategies of player i. This set defines all available strategies for the player i.
- $\sigma = (\sigma_1, \ldots, \sigma_i, \ldots, \sigma_n) \in S_1 \times \ldots \times S_i \times \ldots \times S_n \equiv S$, it represents an outcome of the game, that is, a combination of strategies, with each player selecting a single strategy. We denote by $\sigma_{-i} \in S_{-i}$ all the strategies chosen except for that of player i. Thus, σ can be referred to as the strategy profile of the set of players.
- $u_i(\sigma) \in \mathbb{R}$ represents the payoff function of the player $i \in N$. In other words, the "utility function" of the player i is influenced not only by their own strategy σ_i, but also by the strategies chosen by the other players, captured in σ_{-i}. If $u_i(\sigma) > u_i(\sigma')$, the player i strictly prefers the outcome σ over the outcome σ'. If $u_i(\sigma) = u_i(\sigma')$, the player is indifferent between the two outcomes.

Example 1.1.0.1. Let $N = \{1, 2\}$ et $S = \{X, Y, L, M\}$. The player 1 must choose a line from either the top or the bottom, $S_1 = \{X, Y\}$. The player 2 must choose the left column or the right column, $S_2 = \{L, M\}$. The utility functions u_1 et u_2 are as follows:

	L	M
X	(0,1)	(2,2)
Y	(5,0)	(2,5)

The cells of this matrix represent the elements of $S_1 \times S_2$.
Each cell contains a pair of real numbers:
The first component gives the payoff for player 1.
The second component gives the payoff for player 2.
For example, in the cell (X, L) we found the pair $(0, 1)$, which means that:

$$u_1(X, L) = 0 \text{ and } u_2(X, L) = 1.$$

In the cell (X, M) , we found the pair $(2, 2)$, meaning that:

$$u_1(X, M) = 2 \text{ and } u_2(X, M) = 2\text{,...etc.}$$

This matrix represents the payoff functions for each player.

1.1.1 Nash equilibrium in non-cooperative games

Building on the concept of Nash equilibrium in non-cooperative games, we now present the fundamental principles of Nash equilibrium itself.

1.1.1.1 Nash equilibrium

Nash equilibrium ([42], [43]) refers to the condition in a game where, given the strategies of the other players, a player has no incentive to unilaterally change their strategy. In other words, after all the players choose their strategies, no one player can do better by changing their strategy alone. This equilibrium happens when each player's chosen strategy is the optimal response to the strategies chosen by the other players, resulting in a strategy set that is stable within the game. One of the key concepts that facilitate the attainment of Nash equilibrium is the "best response", which refers to the strategy that yields the highest immediate payoff for a given player, given the strategies chosen by the other players. The concept of best response is fundamental to the definition of Nash equilibrium, as it relies on each player selecting, at every stage, the strategy that constitutes their optimal response to the strategies of others. In other words, in game theory, a best response is a strategy that maximizes a player's immediate payoff given the strategies of the other players. This concept is central to the definition of Nash equilibrium, where each player consistently selects the strategy that serves as their best response to the choices of others.

Definition 1.1.1.1. The strategy σ_i^* of the player i is called a *best response* for a given strategy σ_{-i} chosen by the other players if, for all $\sigma_i \in S_i$, the following inequality holds:

$$u_i(\sigma_i^*, \sigma_{-i}) \geq u_i(\sigma_i, \sigma_{-i})$$

Definition 1.1.1.2. A strategy profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_i^*, \ldots, \sigma_n^*)$, where $\sigma_i^* \in S_i$ for each player $i = 1, 2, \ldots, n$, is considered a *Nash equilibrium* if no player has an incentive to unilaterally deviate from their strategy σ_i^*, given that all other players adhere to the strategy profile σ_{-i}^*. This implies that for every player $i \in N$, for all $\sigma_{-i}^* \in S_{-i}$, and for any $\sigma_i \in S_i$, the following holds:

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) \tag{1.1.1}$$

Furthermore, the strategy profile σ^* is a *strict Nash equilibrium* if, for every player $i \in N$, for all $\sigma_{-i}^* \in S_{-i}$, and for any $\sigma_i \in S_i$, the following strict inequality holds:

$$u_i(\sigma_i^*, \sigma_{-i}^*) > u_i(\sigma_i, \sigma_{-i}^*) \tag{1.1.2}$$

If a Nash equilibrium is strict, any deviation from the strategy must come at a cost to the players. To determine whether σ^* constitutes a Nash equilibrium, we need to verify that no player has an incentive to adopt a different strategy. If no player finds it beneficial to deviate, then σ^* is indeed a Nash equilibrium.

Example 1.1.1.1. Two players (a man and a woman) are planning to go out together, but each has a different preference: The man prefers to attend a football match. The woman prefers to attend a cinema. However, both prefer to be together rather than going alone. Payoff Matrix:

Man

Woman

	Cinema	Football
Cinema	(3,2)	(0,0)
Football	(0,0)	(2,3)

If the man prefers to go to the cinema, the best response for the woman is to go to the cinema because $3 > 0$. If the man prefers to attend a football match, the best response for the woman is to attend the football match because $2 > 0$. If the woman prefers to go to the cinema, the best response for the man is to go to the cinema because $2 > 0$. If the woman prefers to attend a football match, the best response for the man is to attend the football match because $3 > 0$. Based on the best response for each of them we can conclude that there are two pure strategy Nash equilibria:

Both go to the cinema (C, C).
Both go to the football match (F, F).

1.1.1.2 Nash equilibrium through strategy dominant

A strategy is said to be dominant for a player if it guarantees a payoff that is at least equal to, if not greater than, the payoff from any alternative strategy, irrespective of the actions taken by the other participants in the game

Formally, a strategy profile

$$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$$

constitutes a dominant strategy equilibrium in the game

$$G(N, (S_i)_{i \in N}, (u_i)_{i \in N})$$

if, for every player $i \in N$, for all strategies $\sigma'_i \in S_i$, and for every strategy profile $\sigma_{-i} \in S_{-i}$, the following condition holds:

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i}).$$

To illustrate them, we will use the example of the "Battle of the Sexes", which was developed by mathematician R. Duncan Luce and economist Howard Raiffa. [21].

Example 1.1.1.2. Let $R = \{1, 2\}$, company A and company B, must decide whether to advertise or not advertise. Their profits depend on their choices, as shown in the payoff matrix below.

$CompanyA$

$CompanyB$

	Advertise	No Advertise
Advertise	(3,3)	(5,2)
No Advertise	(2,5)	(4,4)

The first number in each pair represents CompanyA's profit, and the second number represents CompanyB's profit.

A dominant strategy is one that always provides the highest payoff for a player, regardless of the opponents decision.

For Company A:

- If B advertises, A gets 3 by advertising or 2 by not advertising ⇒ A prefers to advertise.
- If B does not advertise, A gets 5 by advertising or 2 by not advertising ⇒ A still prefers to advertise.

So Advertising is a dominant strategy for A.

For Company B:

- If A advertises, B gets 3 by advertising or 2 by not advertising ⇒ B prefers to advertise.
- If A does not advertise, B gets 5 by advertising or 2 by not advertising ⇒ B still prefers to advertise.

So "Advertising" is a dominant strategy for B.

Since both companies have a dominant strategy to advertise, the Nash Equilibrium of this game is Advertise, Advertise with payoffs $(3, 3)$.

1.1.1.3 Equilibrium through iterated elimination of dominated strategies

Let $N = \{1, 2\}$ and $S = \{A, B, C, D, E, F\}$. The payoffs of both players are:

$P2$

$P1$		A	B	C
	D	(4,3)	(3,2)	(2,1)
	E	(3,4)	(2,3)	(1,2)
	F	(2,5)	(1,4)	(0,3)

Each cell (i, j) in the Matrix represents the payoffs of Player 1 and Player 2. We note that for player 2, the strategy C is a strategy dominated, because we have $(1, 2, 3)$ is strictly lower than $(2, 3, 4)$ and than $(3, 4, 5)$ The reduced game is as follows:

P_2

P_1

	A	B
D	(4,3)	(3,2)
E	(3,4)	(2,3)
F	(2,5)	(1,4)

We note that for player 1, strategy F is a dominated strategy, because the payoff from strategy F, namely $(2, 1)$, is strictly lower than both $(3, 2)$ and $(4, 3)$. The reduced game is as follows:

P_2

P_1

	A	B
D	(4,3)	(3,2)
E	(3,4)	(2,3)

The strategy B is a strategy dominated of player 2. The reduced game is as follows:

P_2

P_1

	A
D	(4,3)
E	(3,4)

Finally, strategy E is a dominated strategy for player 1. Therefore, the reduced game is as follows:

P_2

P_1

	A
D	(4,3)

Since the only remaining strategy pair is (D, A), this is the "Nash equilibrium", with payoffs:

$$(4, 3)$$

1.1.2 Pareto optimum in non-cooperative games

After examining Nash equilibrium in non-cooperative games, we now turn our attention to the concept of Pareto optimum in the same context. This concept is the most essential thing in game theory and economic efficiency. To make sure it works we have to insure Moving from strategy σ to $\widehat{\sigma}$ improves the outcome for at least one individual while keeping others at the same level of utility, or benefits someone without making anyone else worse off. Awareness of dominant strategies is essential for achieving Pareto optimal outcomes states in which no individual can be made better off without making at least one other individual worse off.

Definition 1.1.2.1. A strategy profile σ^* is said Pareto-optimal if:
$\nexists\ \sigma$ such that:

$$u_i(\sigma) \geq u_i(\sigma^*),\ \forall i \in N$$

and $\exists j$ such that:

$$u_j(\sigma) > u_j(\sigma^*)$$

Example 1.1.2.1. Assume we have two players and five strategies $S = \{A, B, C, D, E\}$. The payoffs for both players are:

$P2$

$P1$

	A	B	C
D	(5,5)	(4,2)	(3,1)
E	(4,4)	(5,2)	(4,2)

A strategy pair (S_1, S_2) is Pareto optimal if there is no other strategy combination that can make one player better off without making the other player worse off. We note that $(D, A) = (5, 5)$ is Pareto optimal because no other cell has a strictly higher value for both players simultaneously.

1.1.2.1 The relation between NE and PO

The relationship between Nash equilibrium and Pareto optimality has been extensively studied in game theory and economics. It is well known that a Nash equilibrium is not necessarily Pareto optimal. Likewise, a Pareto-optimal strategy profile does not necessarily correspond to a Nash equilibrium.

Example 1.1.2.2. Assume we have two players and four strategies $S = (A, B, D, E)$ and the payoffs for both players are:

$P2$

$P1$

	A	B
D	(6,5)	(4,6)
E	(5,3)	(5,4)

We observe that the Nash equilibrium is $(E, B) = (5, 4)$ but is not the Pareto optimal. In contrast,$(D, A) = (6, 5)$ is a Pareto optimal but not a Nash equilibrium.

Example 1.1.2.3. Let $N = \{1, 2\}$ and $S = (E, F, G, H)$ and the payoffs of both players are:

$P2$

$P1$

	E	F
G	(7,7)	(5,4)
H	(3,5)	(6,3)

We observe that the Nash equilibrium is $(G, E) = (7, 7)$ and this cell is a Pareto optimum.

1.1.3 Strong Nash equilibrium in non-cooperative games

Strong Nash equilibrium [6]is a generalization of the Nash equilibrium in non-cooperative games considering coalition stability. While a classical Nash equilibrium guarantees that no single player has the incentive to deviate and increase their payoff, a strong Nash equilibrium instead guarantees that no coalition of players has the incentive to deviate and increase their payoff. This stricter stability criterion makes SNE particularly relevant in strategic scenarios where coordination among multiple players is feasible, such as economic markets, political negotiations, and multi-agent systems. However, the existence of strong Nash equilibria is highly restrictive, as it requires robustness against group deviations, making it significantly less prevalent than standard Nash equilibria in practical applications.

Definition 1.1.3.1. We say that a strategy profile $\sigma = (\sigma_1, \dots, \sigma_n)$ is a *strong Nash equilibrium* if for all coalition $D \subseteq N$ and $\forall y_D \in S_D$ and $S_D = \prod_{i \in D} S_i$, $\exists$ $i \in D$ such that: $u_i(y_D, \sigma_{-D}) \leq u_i(\sigma)$

In other words, if $D = \{i\}$, than:

$$u_i(y_i, \sigma_{-i}) \leq u_i(\sigma_i, \sigma_{-i})$$

If $D = \{i, j\}$ and $i \neq j$ and $\forall y_i, y_j \in S \times S$, we find:

$$u_i(y_i, y_j, \sigma_{-\{i,j\}}) \leq u_i(\sigma) \text{ or } u_j(y_i, y_j, \sigma_{-\{i,j\}}) \leq u_j(\sigma)$$

Example 1.1.3.1. Assume we have two players and four strategies $S = (A, B, E, F)$ and the payoffs of both players are:

$P2$

$P1$

	A	B
E	(9,9)	2,4)
F	(3,2)	(1,3)

The strong Nash equilibrium in this game is:

$$(E, A)$$

We notice that no matter how players 1 and 2 cooperate, they cannot increase their payoffs.

1.1.3.1 The relation between SNE and NE and PO

Unlike a standard Nash equilibrium, a strong Nash equilibrium accounts for situations where multiple players deviate simultaneously. In such cases, at least one of the deviating players must experience a decrease in utility. A strong Nash equilibrium generalizes the standard Nash equilibrium by allowing for deviations by coalitions of players. In the special case where each coalition consists of a single player (i.e., $D = i, \forall i \in N$), it reduces to the standard Nash equilibrium. The relationship between Nash equilibrium, Pareto optimal, and strong Nash equilibrium is summarized as follows:

- Every strong Nash equilibrium is a Nash equilibrium, but not every Nash equilibrium is strong Nash equilibrium.
- Every strong Nash equilibrium is a Pareto optimum, but not every Pareto optimum is strong Nash equilibrium.

Example 1.1.3.2. Assume we have two players and four strategies $S = \{A, B, C, D\}$, the payoffs of each player are:

$P2$

$P1$

	A	B
C	(8,8)	(3,5)
D	(1,2)	(4,6)

We observe that there are two Nash equilibria: (C, A) and (D, B), but only one of them (C, A) is a strong Nash equilibrium.

Example 1.1.3.3. Assume we have two players and four strategies $S = \{A, B, C, D\}$, the payoffs of each player are:

$P2$

$P1$

	A	B
C	(5,5)	(3,4)
D	(1,2)	(2,6)

We observe that: there are two Pareto optima (C, A) and (D, B), but only one of them (C, A) is a strong Nash equilibrium.

1.2 Congestion games

Rosenthal [49] introduced the framework of congestion games, in which each player selects a subset of elements from a shared set of fundamental resources. The utility or cost associated with each resource is influenced by the number of players utilizing it. As a result, a player's overall utility is calculated as the aggregate of the utilities corresponding to the resources they have chosen.

Every game in this category guarantees at least one Nash equilibrium in pure strategies, which is established through the existence of a potential function.

There enumerous specialized types of congestion games exist numerous specialized types of congestion games, such as weighted congestion games, matroid congestion games [2], and network congestion games, some of which will be discussed in the following sections.

1.2.1 Standard congestion games

The concept we present here is grounded in the model introduced by Rosenthal [49], who was the first to rigorously analyze this class of games. Congestion games are formulated as non-cooperative games [45], meaning that players are unable to engage in cooperative behavior. In other words, players do not have the option to coordinate or jointly plan their strategies before making individual decisions. This framework is particularly relevant for modeling a wide range of real-world problems, where strategic interactions among self-interested agents lead to outcomes that can be analyzed using game-theoretic principles. Non-cooperative games, including congestion games, have been extensively studied since the seminal work of Nash, who established the foundational concepts of equilibrium and strategic decision-making. These games are characterized by their focus on individual rationality, optimization of utility or cost functions, and the absence of binding agreements among players. The mathematical and economic implications of such models have made them a cornerstone in the study of strategic interactions in economics, applied mathematics, and related disciplines.

The games introduced by Konishi, Le Breton, and Weber [31], Milchtaich [37], and Quint and Shubik [48] are relatively similar to Rosenthal's model [49], in the sense that the players' utility functions are characterized by the effect of congestion. This shared feature underscores the centrality of congestion effects in shaping strategic behavior and equilibrium outcomes in such games. The interplay between resource usage and the resulting utility or cost experienced by players remains a critical aspect of these models, further highlighting their applicability to real-world economic systems involving shared resources and competitive interactions. The formal definition of congestion games is presented as follows:

Definition 1.2.1.1. A standard congestion game $G(N, R, (S_i)_{i\in N}, (P_r)_{r\in R})$ is formally structured as a non-cooperative game with the following elements:

- $N = \{1, \dots, i, \dots, n\}$ is a finite set of players participating in the game;
- $R = \{r_1, \dots, r_j, \dots, r_m\}$ is a collection of resources (which may represent goods, services, or alternatives);
- For each player $i \in N$, $S_i \subseteq 2^R \setminus \{\emptyset\}$ specifies the strategy set available to player i, where each strategy consists of a non-empty combination of resources;
- For each resource $r \in R$, $P_r : \mathbb{N} \to \mathbb{R}$ is the cost function associated with resource r, where $P_r(l)$ denotes the cost incurred when exactly l players utilize resource r.

- The number of players choosing resource r is given by

$$n_r(\sigma) = \Big|\{\, i \in N : \sigma_i = r \,\}\Big|$$

where $\sigma = (\sigma_i)_{i\in N} \in S$ and $r \in R$

- The utility of player i is defined as

$$u_i(\sigma) = \sum_{r\in\sigma_i} P_r\Big(n_r(\sigma)$$

This framework models scenarios where players compete for shared resources, and the cost of each resource dynamically depends on the level of congestion, i.e., the number of players using it. The strategic choices of players and the resulting resource utilization patterns are central to the analysis of such games.

Example 1.2.1.1. Let $G(N, R, (S_i)_{i\in N}, (P_r)_{r\in R})$ be a congestion game, such that: $N = \{1, 2, 3\}$, $R = \{r_1, r_2, r_3\}$, $S_i = 2^R \setminus \{\emptyset\}$, for all $i \in N$. Assume that the cost functions are defined as follows:

$$P_{r_1}(n_{r_1}(\sigma)) = 3 + n_{r_1}(\sigma).$$
$$P_{r_2}(n_{r_2}(\sigma)) = 2.$$
$$P_{r_3}(n_{r_3}(\sigma)) = 4 - n_{r_3}(\sigma).$$

Let σ be the strategy profile defined as follows:

$$\sigma = (\{r_3\}, \{r_2, r_3\}, \{r_1, r_3\}).$$

By applying the formula used to compute the utility of each player, we obtain:

$$u_1(\sigma) = P_{r_3}(n_{r_3}(\sigma)) = P_{r_3}(3) = 1.$$
$$u_2(\sigma) = P_{r_2}(n_{r_2}(\sigma)) + P_{r_3}(n_{r_3}(\sigma)) = P_{r_2}(1) + P_{r_3}(3) = 2 + 1 = 3.$$
$$u_3(\sigma) = P_{r_1}(n_{r_1}(\sigma)) + P_{r_3}(n_{r_3}(\sigma)) = P_{r_1}(1) + P_{r_3}(3) = 4 + 1 = 5.$$

1.2.2 Congestion games with specific payment functions

A well-established extension of standard congestion games is the class of congestion games with player-specific cost functions, commonly referred to as "Crowding

Games" Milchtaich ([37], [38]). Unlike the canonical framework, where cost functions are homogeneous across all players, congestion games with player-specific cost functions are characterized by the heterogeneity of individual cost structures.

In this framework, each player's cost is determined exclusively by their chosen strategy and the cardinality of the set of players selecting the same strategy. This heterogeneity introduces strategic interdependencies that are absent in traditional congestion models, making such games particularly relevant for analyzing scenarios involving non-symmetric agents and differentiated cost-sharing mechanisms.

Definition 1.2.2.1. A *congestion game with player-specific cost functions* is defined as a tuple

$$G(N, R, (S_i)_{i\in N}, (P_r^i)_{i\in N,\ r\in R})$$

where:

- $N = \{1, ..., n\}$ is a finite set of players;
- $R = \{r_1, \ldots, r_j, \ldots, r_m\}$ is a set of resources;
- For each player $i \in N$, $S_i \subseteq 2^R \setminus \{\emptyset\}$ represents the strategy space of player i;
- Each player $i \in N$ has *player-specific cost functions*, denoted by

  $$P_r^i : \mathbb{N} \to \mathbb{N}, \quad \forall r \in R,$$

  which are associated with resource r in R.

Let define $n_r(\sigma) = |\{i \in N : \sigma_i = r\}|$, the number of players sharing resource r in the profile σ, where: σ is a strategy profile in S and $\sigma = (\sigma_i)_{i\in N}$, for a r in R. In this case, the utility of each player i is defined as follows:

$$u_i(\sigma) = \sum_{r\in\sigma_i} P_r^i(n_r(\sigma)) \qquad (**)$$

Let us elucidate this definition through the following illustrative example:

Example 1.2.2.1. Let G be a congestion game, such that: $N = \{1, 2, 3, 4\}$, $R = \{r_1, r_2, r_3, r_4\}$, $S_i = 2^R \setminus \{\emptyset\}$, for all $i \in N$. Assume that the cost function are defined as follows:

$P_{r_1}^1(n_{r_1}(\sigma)) = n_{r_1}(\sigma)$, $P_{r_2}^1(n_{r_2}(\sigma)) = 5$

$P_{r_3}^1(n_{r_3}(\sigma)) = 2 + n_{r_3}(\sigma)$, $P_{r_4}^1(n_{r_4}(\sigma)) = 8 - n_{r_4}(\sigma)$.

$P_{r_1}^2(n_{r_1}(\sigma)) = 5 - n_{r_1}(\sigma)$, $P_{r_2}^2(n_{r_2}(\sigma)) = 3 + n_{r_2}(\sigma)$,

$P^2_{r_3}(n_{r_3}(\sigma)) = 4$, $P^2_{r_4}(n_{r_4}(\sigma)) = n_{r_4}(\sigma)$.

$P^3_{r_1}(n_{r_1}(\sigma)) = 1 + n_{r_1}(\sigma)$, $P^3_{r_2}(n_{r_2}(\sigma)) = n_{r_2}(\sigma)$,

$P^3_{r_3}(n_{r_3}(\sigma)) = 4 - n_{r_3}(\sigma)$, $P^3_{r_4}(n_{r_4}(\sigma)) = 4$.

$P^4_{r_1}(n_{r_1}(\sigma)) = 6$, $P^4_{r_2}(n_{r_2}(\sigma)) = 2 + n_{r_2}(\sigma)$,

$P^4_{r_3}(n_{r_3}(\sigma)) = n_{r_3}(\sigma)$, $P^4_{r_4}(n_{r_4}(\sigma)) = 2 + n_{r_4}(\sigma)$.

Let σ be a strategy profile defined as: $\sigma = (\{r_2, r_4\}, \{r_1, r_3, r_4\}, \{r_1, r_2, r_3, r_4\}, \{r_3\})$. The utility function of each player are defined as follows:

$u_1(\sigma) = P^1_{r_2}(n_{r_2}(\sigma)) + P^1_{r_4}(n_{r_4}(\sigma)) = P^1_{r_2}(2) + P^1_{r_4}(3) = 5 + 5 = 10$

$u_2(\sigma) = P^2_{r_1}(n_{r_1}(\sigma)) + P^2_{r_3}(n_{r_3}(\sigma)) + P^2_{r_4} = P^2_{r_1}(2) + P^2_{r_3}(3) + P^2_{r_4}(3) = 3 + 4 + 3 = 10$

$u_3(\sigma) = P^3_{r_1}(n_{r_1}(\sigma)) + P^3_{r_2}(n_{r_2}(\sigma)) + P^3_{r_3}(n_{r_3}(\sigma)) + P^3_{r_4}(n_{r_4}(\sigma)) = P^3_{r_1}(2) +$ $P^2_{r_2}(2) +$

$P^3_{r_3}(3) + P^3_{r_4}(3) = 3 + 2 + 1 + 4 = 10$
$u_4(\sigma) = P^4_{r_3}(n_{r_3}(\sigma)) = P^4_{r_3}(3) = 3$

In this class of games, the existence of a Nash equilibrium is not necessarily guaranteed [3]) as demonstrated by the following example:

Example 1.2.2.2. Let us examine a congestion game with specific payoff function for each player, where $N = \{1, 2\}$ and $R = \{r_1, r_2, r_3, r_4\}$. We define the strategy space for each player as follows: $S_1 = \{\{r_1, r_3\}, \{r_2, r_4\}\}$ and $S_2 = \{\{r_1\}, \{r_4\}\}$. Let us assume that the payoff functions of each player are given by:

$P^1_{r_1}(n_{r_1}(\sigma)) = -2 \times n_{r_1}(\sigma)$	$P^2_{r_1}(n_{r_1}(\sigma)) = n_{r_1}(\sigma)$
$P^1_{r_2}(n_{r_2}(\sigma)) = 2 \times n_{r_2}(\sigma)$	$P^2_{r_2}(n_{r_2}(\sigma)) = 0$
$P^1_{r_3}(n_{r_3}(\sigma)) = 5$	$P^2_{r_3}(n_{r_3}(\sigma)) = 0$
$P^1_{r_4}(n_{r_4}(\sigma)) = 1 - n_{r_4}(\sigma)$	$P^2_{r_4}(n_{r_4}(\sigma)) = n_{r_4}(\sigma)$

We note that: $\sigma(0)$, $\sigma(1)$, $\sigma(2)$ and $\sigma(3)$.

The four possible strategy profiles are as follows:

$$\sigma(0) = (\{r_1, r_3\}, \{r_4\}),\ \sigma(1) = (\{r_2, r_4\}, \{r_4\}),$$
$$\sigma(2) = (\{r_1, r_3\}, \{r_1\}),\ \sigma(3) = (\{r_2, r_4\}, \{r_1\}).$$

By definition (∗∗), the utility of each player is:

$u_1(\sigma(0)) = P^1_{r_1}(n_{r_1}(\sigma(2))) + P^1_{r_3}(n_{r_3}(\sigma(2))) = P^1_{r_1}(1) + P^1_{r_3}(1) = -2 \times 1 + 5 = 3;$

$u_2(\sigma(0)) = P^2_{r_4}(n_{r_4}(\sigma(2))) = P^2_{r_4}(1) = 1;$

$u_1(\sigma(1)) = P^1_{r_2}(n_{r_2}(\sigma(3))) + P^1_{r_4}(n_{r_4}(\sigma(3))) = P^1_{r_2}(1) + P^1_{r_4}(2) = 2 \times 1 + 1 - 2 = 1;$

$u_2(\sigma(1)) = P^2_{r_4}(n_{r4}(\sigma(3))) = P^2_{r_4}(2) = 2;$

$u_1(\sigma(2)) = P^1_{r_1}(n_{r_1}(\sigma(0))) + P^1_{r_3}(n_{r_3}(\sigma(0))) = P^1_{r_1}(2) + P^1_{r_3}(1) = -2 \times 2 + 5 = 1;$

$u_2(\sigma(2)) = P^2_{r_1}(n_{r_1}(\sigma(0))) = P^2_{r_1}(2) = 2;$

$u_1(\sigma(3)) = P^1_{r_2}(n_{r_2}(\sigma(1))) + P^{1\,2}_{r_4 r_4}(n_{r_4}(\sigma(1))) = P^1_{r_2}(1) + P^1_{r_4}(1) = 2 \times 1 + 1 - 1 = 2;$

$u_2(\sigma(3)) = P^2_{r_1}(n_{r_1}(\sigma(1))) = P^2_{r_1}(1) = 1;$

This game is then described by the following table:

	$\{r_1\}$	$\{r_4\}$
$\{r_1, r_3\}$	(1,$\underline{2}$)	($\underline{2}$,1)
$\{r_2, r_4\}$	($\underline{3}$,1)	(1,$\underline{2}$)

Table 2.1: Game with specific payoff functions

According to this table, there is no Nash equilibrium. Note that in this example, the payoff functions are not monotonic.

However,[37] Milchtaich and Fotakis [22] demonstrated that at least one Nash equilibrium exists for this class of games when the strategies are restricted to single choices (singleton) and the payoff functions are monotonically increasing, which we will analyze in detail in the following paragraph.

It is worth noting that this result has also been extended by Ackermann et al. [2] in the context of matroid-based congestion games [1] with specific payoff functions.

[1]Matroids are combinatorial structures that generalize the concept of linear independence in matrices. For further details, see Schrijver [55]

1.2.3 Singleton congestion games

Milchtaich [37] introduced singleton congestion games, a special class of congestion games characterized by non-uniform payoff functions. In this model, each player has a distinct payoff function, and they do not share a common payoff function. Instead, the payoffs depend on an individual-specific function for each resource and player. Thus, this class of games restricts the standard congestion games described in the first paragraph.

1. The first restriction concerns the possible strategies: players can choose only one resource at a time. Thus, the set of strategies for each player coincides with the set of resources R.

2. The second restriction concerns the nature of the payoff functions P_r^i, which must be non-increasing (in a broad sense) as the number of players using resource r increases.

Remark 1.1. Milchtaich [37] introduced this class of games under the term "congestion games with player-specific payoff functions". However, this terminology lacks precision as it does not explicitly account for the assumption that players' strategies are singleton sets. Consequently, we reserve this term for the games described in the above paragraph and instead adopt the term "singleton congestion games" to refer to congestion games in which players' strategies are singletons and the payoff functions are both decreasing and player-specific.

Definition 1.2.3.1. A *singleton congestion games* $G(N, R, (P_r^i)_{i \in N,\ r \in R})$ defined by the following characteristics:

- $N = \{1, \dots, i, \dots, n\}$ is the set of players;

- $R = \{r_1, \dots, r_j, \dots, r_m\}$ is the set of resources;

- For each player $i \in N$, $S_i = \{r \in 2^R : |r| = 1\} = R$, That is, each player selects exactly one resource.

- P_r^i is the payoff function specific to player i when he choosing the resource r, (this function decreasing)

  For each player $i \in N$ and each resource $r \in R$, the player-specific payoff function P_r^i is non-increasing as the number of players selecting resource r increases, though it is not necessarily strictly decreasing.

In this type of game, the utility of each player i depends on the payoff provided by the single resource they choose, as well as the number of players sharing that resource with them.

Unlike standard congestion games, where players' utilities are obtained by summing the payoffs generated from selecting multiple resources, we can simplify the notation here and express the utility of player i using a function of the following form:

$$u_i(\sigma) = P_r^i(n_r(\sigma))$$

Where,

$$n_r(\sigma) = \Big|\{ i \in N : \sigma_i = r \}\Big|$$

The study of the existence and characterization of Nash equilibria in singleton-choice congestion games will be the primary focus of the following chapters.

After examining the different types of congestion games, we will move on to potential games and explore the possible relationships between these two classes of games.

1.3 potential games

In this section, we will focus on the different types of potential games introduced by Monderer and Shapley [39].

These games are characterized by the existence of a potential function, a concept closely related to congestion games. In general, the potential function captures how players' utilities change when they make unilateral deviations in their strategies.

For games that admit a potential function, finding a Nash equilibrium becomes equivalent to identifying the local optima of this function.

These games are divided into several categories, such as exact potential games [20], ordinal potential games, and generalized potential games.

1.3.1 Exact potential games

Let $G(N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a strategic-form game, where:

- N is a finite set of players.
- S_i is a finite set of pure strategies for player i.

  The strategy profile space is defined as $S = \prod_{j \in N} S_j$.

- For each $i \in N$, the function $u_i : S \to \mathbb{R}$ represents the utility function of player i.
- We recall that $S_{-i} = \prod_{j \in N \setminus \{i\}} S_j$ denotes the set of strategy profiles excluding player i.

Thus, any strategy profile $\sigma = (\sigma_j)_{j \in N}$ in S can be written as:

$$\sigma = (\sigma_i, \sigma_{-i})$$

where $\sigma_i \in S_i$ and $\sigma_{-i} \in S_{-i}$

Definition 1.3.1.1. A strategic game $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ is called an *exact potential game* if there exists a function Po: $S \to \mathbb{R}$ such that for every player $i \in N$, for all strategies $\sigma_i, \sigma_i' \in S_i$, and for all $\sigma_{-i} \in S_{-i}$ the following equality holds:

$$u_i(\sigma_i, \sigma_{-i}) - u_i(\sigma_i', \sigma_{-i}) = Po(\sigma_i, \sigma_{-i}) - Po(\sigma_i', \sigma_{-i}).$$

The impact of all unilateral deviations by all players is captured by a single potential function Po, which is called the exact potential function.

Example 1.3.1.1. Let's consider the strategic game shown in the following table:

2

1		C	D
	A	(3,1)	(5,2)
	B	(4,4)	(2,1)

Table 1: Exact potential game

Let us prove that this game is an exact potential game, $Po : S \to \mathbb{R}$ defined by:

$$Po(A, C) = 2, \; Po(A, D) = 3.$$
$$Po(B, C) = 3, \; Po(B, D) = 0.$$

This information can be represented by the following matrix:

$$Po = \begin{pmatrix} 2 & 3 \\ 3 & 0 \end{pmatrix}$$

We have:

$$u_1(A,C) - u_1(B,C) = 3 - 4 = Po(A,C) - Po(B,C) = 2 - 3.$$
$$u_1(A,D) - u_1(B,D) = 5 - 2 = Po(A,D) - Po(B,D) = 3 - 0.$$
$$u_2(A,C) - u_2(A,D) = 1 - 2 = Po(A,C) - Po(A,D) = 2 - 3.$$
$$u_2(B,C) - u_2(B,D) = 4 - 1 = Po(B,C) - Po(B,D) = 3 - 0.$$

Based on the results obtained above, we observe that the game is an exact potential game.

1.3.2 Weighted potential function

Definition 1.3.2.1. A function Po: $S \to \mathbb{R}$ is a *weighted potential function* for the game $G(N,(S_i)_{i\in N},(u_i)_{i\in N})$ if there exist positive real numbers $w_1, w_2, \ldots, w_n$ (called the weights) such that for every player i in N, for all σ_i and σ_i' in S_i, and for all σ_{-i} in S_{-i}, the following holds:

$$u_i(\sigma_i,\sigma_{-i}) - u_i(\sigma_i',\sigma_{-i}) = w_i(Po(\sigma_i,\sigma_{-i}) - Po(\sigma_i',\sigma_{-i})).$$

Example 1.3.2.1. Lets consider the strategic game shown in the table G_1 and the potential game shown in the table Po_1:

2

1		r_1	r_2
	r_3	(4,5)	(4,3)
	r_4	(2,3)	(2,1)

G_1

2

1		
	2	1
	1	0

Po_1

Then we have:

$$u_1(r_3,r_1) - u_1(r_4,r_1) = 2 = 2(Po_1(r_3,r_1) - Po_1(r_4,r_1)).$$
$$u_1(r_3,r_2) - u_1(r_4,r_2) = 2 = 2(Po_1(r_3,r_2) - Po_1(r_4,r_2)).$$
$$u_2(r_3,r_1) - u_2(r_3,r_2) = 2 = 2(Po_1(r_3,r_1) - Po_1(r_3,r_2)).$$
$$u_2(r_4,r_1) - u_2(r_4,r_2) = 2 = 2(Po_1(r_4,r_1) - Po_1(r_4,r_2)).$$

We observe that the function P_1 is a weighted potential function for the game G_1 ($w_1 = 2$ and $w_2 = 2$).

1.3.3 Ordinal potential function

Definition 1.3.3.1. A function $Po\colon S \to \mathbb{R}$ is an *ordinal potential function* for the game $G(N, (S_i)_{i\in N}, (u_i)_{i\in N})$ if for every player $i \in N$, for all σ_i and $\sigma'_i \in S_i$, and for all $\sigma_{-i} \in S_{-i}$,

$$u_i(\sigma_i, \sigma_{-i}) - u_i(\sigma'_i, \sigma_{-i}) > 0 \text{ if and only if } Po(\sigma_i, \sigma_{-i}) - Po(\sigma'_i, \sigma_{-i}) > 0.$$

So, the game G is called a *ordinal potential game*.

Example 1.3.3.1. Lets consider the strategic game shown in the table G_2 and the potential game shown in the table Po_2:

2

1		r_3	r_4
	r_1	(3,2)	(2,3)
	r_2	(1,3)	(1,1)

G_2

2

1		
	5	6
	3	0

Po_2

We have:

$$u_1(r_1, r_3) - u_1(r_2, r_3) > 0 \text{ and } Po_2(r_1, r_3) - Po_2(r_2, r_3) > 0;$$
$$u_1(r_1, r_4) - u_1(r_2, r_4) > 0 \text{ and } Po_2(r_1, r_4) - Po_2(r_2, r_4) > 0;$$
$$u_2(r_1, r_4) - u_2(r_1, r_3) > 0 \text{ and } Po_2(r_1, r_4) - Po_2(r_1, r_3) > 0;$$
$$u_2(r_2, r_3) - u_2(r_2, r_4) > 0 \text{ and } Po_2(r_2, r_3) - Po_2(r_2, r_4) > 0.$$

By definition 11, the function Po_2 is an ordinal potential for the game G_2.

1.3.4 Generalized ordinal potential function

Definition 1.3.4.1. A function Po: $S \to \mathbb{R}$ is a *generalized ordinal potential function* for the game $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ if for every player $i \in N$, for all σ_i and $\sigma_i' \in S_i$, and for all σ_{-i} in S_{-i}, we have:

$$u_i(\sigma_i, \sigma_{-i}) - u_i(\sigma_i', \sigma_{-i}) > 0 \text{ implies } Po(\sigma_i, \sigma_{-i}) - Po(\sigma_i', \sigma_{-i}) > 0.$$

Then the game G is called a *generalized ordinal potential game.*

Example 1.3.4.1. Lets consider the strategic game shown in the table G_3 and the potential game shown in the table Po_3:

2

1		r_3	r_4
	r_3	(2,1)	(3,1)
	r_4	(3,1)	(1,2)

G_3

2

1		
	1	4
	2	3

Po_3

The function Po_3 is a generalized ordinal potential game for G_3. However, the function Po_3 is not a potential ordinal for the game G_3, since $Po_3(r_3, r_4) - Po_3(r_3, r_3) > 0$ does not involve $u_3(r_3, r_4) - u_3(r_3, r_3) > 0$.

Remark 1.2. The following relationships exist between different types of potential games:

1. Every ordinal potential function is a generalized ordinal potential function, but the converse is not always true. Consequently, every ordinal potential game is also a generalized ordinal potential game, but the reverse does not always hold.

2. Every exact potential function is an ordinal potential function (and therefore also a generalized ordinal potential function), but the converse is not always true. Thus, every exact potential game is an ordinal potential game (and consequently a generalized potential game as well).

3. Every exact potential game is a weighted potential game with $w_i = 1$.

1.4 Improvement path

Definition 1.4.0.1. Let l be a non-zero integer, a sequence of strategies of length $l+1$ is a finite sequence of strategies such that:

$$i(0), i(1), \ldots, i(l-1), i(l), \ldots, i(l+1)$$

Definition 1.4.0.2. A path is a sequence (finite or infinite) of strategy profiles:

$$\sigma(0), \sigma(1), \ldots, \sigma(l-1), \sigma(l), \ldots$$

The notation $\sigma(l) \to \sigma(l+1)$ indicates a transition from profile $\sigma(l)$ to profile $\sigma(l+1)$, in which exactly one player deviates by changing their strategy and experiences a loss as a result.

Definition 1.4.0.3. An improvement path is a path in which, at each step $\sigma(l-1) \to \sigma(l)$, the deviating player strictly improves their utility.

This concept of an improvement path allows to show that a Nash equilibrium exists for certain types of games. These games must satisfy the property stated below.

Definition 1.4.0.4. A game has the Finite Improvement Property (FIP) if every improvement path is finite.

It is evident that any game with the Finite Improvement Property must necessarily have a Nash equilibrium. Indeed, if:

$$\sigma(0) \to \sigma(1) \to \ldots \to \sigma(l)$$

It is a maximal improvement path, the profile $\sigma(l)$ It is a Nash equilibrium because no player can deviate further to strictly improve their utility.

Example 1.4.0.1. Let $N = \{1, 2, 3, 4\}$ and $S = \{r_1, r_2, r_3\}$. Let us assume a sequence of strategies defined as follows:

$$j(0) = r_3,\ j(1) = r_2$$
$$j(2) = r_1,\ j(3) = r_1.$$

The strategy profiles are defined as follows:

$$\sigma(0) = (r_2, r_1, r_3, r_1),\ \sigma(1) = (r_3, r_1, r_3, r_1) \text{ and } \sigma(2) = (r_3, r_1, r_1, r_1)\dots$$

The transition $\sigma(0) \to \sigma(1)$ it means that player 1 changed their strategy from r_2 to r_3.
The transition $\sigma(1) \to \sigma(2)$ means that player 3 changed their strategy from r_3 to r_1, etc...
Indeed, the utility of player 1 when playing r_3 is :

$$u_2(\sigma(1)) > u_2(\sigma(0))$$

This translates to the fact that Player 2 has changed their strategy to another one in order to strictly improve their utility. By the same method, we get that for player 3:

$$u_3(\sigma(2)) > u_3(\sigma(1))$$

1.5 Conclusion

In this chapter, we established the comprehensive theoretical foundations of non-cooperative game theory by defining key concepts, formulating formal definitions, and presenting the notations adopted in this study. We also provide definitions for Nash equilibrium, strong Nash equilibrium, and Pareto optimality, highlighting the fundamental differences between them. It was concluded that Nash equilibrium does not necessarily guarantee Pareto optimal, whereas strong Nash equilibrium is a specific case of classical Nash equilibrium, but the converse is not always true.

Furthermore, this chapter explores several fundamental aspects, including the primary methodological tools employed to prove the existence of Nash equilibrium, as well as the study of congestion games and the family of potential games, analyzing their structural properties and the associated equilibria.

Finally, the next chapter will provide an in-depth examination of the family of singleton congestion games, analyzing their structural properties and the corresponding equilibria.

Chapter 2

Nash equilibrium in singleton congestion games

The main objective of this chapter is to revisit the analysis conducted by Milchtaich on non-symmetric congestion games, with a particular focus on the case where each player has exactly two strategies. We begin by simplifying the original proofs provided by Milchtaich and presenting the key results obtained in this specific setting.

In addition, we introduce a novel approach for identifying Nash equilibria in such games, leveraging structural properties inherent to the non-symmetric framework. We further analyze the performance of several algorithms designed to compute Nash equilibria, with an emphasis on their computational complexity and practical applicability.

Finally, we quantitatively assess the efficiency of Nash equilibria compared to socially optimal outcomes by evaluating two fundamental metrics: the price of Anarchy (PoA) and the price of stability (PoS). This comparison aims to highlight the potential discrepancy between self-interested player behavior and globally efficient solutions.

2.1 Introduction

Congestion games represent a fundamental class of non-cooperative games that model scenarios where multiple agents compete for limited shared resources. These games have found widespread applications across diverse fields, including traffic networks, communication systems, economics, and ecology. In a typical congestion game, each player's payoff depends not only on their individual choice but also on the number of other players selecting the same resource.

One of the most influential extensions of congestion games was proposed by Milchtaich [37], who introduced non-symmetric congestion games a variant where players may have distinct utility functions for the same resource. In his seminal work (1996), Milchtaich [37] proved that pure-strategy Nash equilibria always exist in such games when each player chooses a single resource. His analysis focused on the convergence properties of best-response dynamics and introduced novel concepts to deal with the loss of potential functions in the non-symmetric setting.

While Milchtaich's [37] results ensure the existence of equilibrium, they also raise critical questions regarding the efficiency of equilibria reached by self-interested players. This brings into focus two central notions in algorithmic game theory: the price of anarchy (PoA) and the price of stability (PoS). Introduced by Koutsoupias [32] and Papadimitriou (1999) [47], the price of anarchy measures the ratio between the worst possible Nash equilibrium and the optimal social outcome. In contrast, the price of stability, coined by Anshelevich et al. [5] , evaluates the best Nash equilibrium in comparison to the optimum.

These metrics provide insight into how selfish behavior can lead to suboptimal system-wide performance. Subsequent research by Roughgarden [50] and Tardos and others has further deepened our understanding of how the structure of the gameparticularly its symmetry, the number of players, and the shape of cost functionsaffects these efficiency bounds.

This chapter aims to revisit Milchtaich's framework [37], offering simplified proofs of certain results, introducing alternative techniques for equilibrium computation, and analyzing the efficiency of the resulting equilibria through the lenses of PoA and PoS. In doing so, we not only highlight the theoretical underpinnings of asymmetric congestion games but also address their practical implications in systems governed by decentralized decision-making.

2.2 Singleton congestion games

In Chapter 1, we have already introduced the definition of *singleton congestion games*, originally proposed by Milchtaich [37]. To begin, let us recall this definition.

Definition 2.2.0.1. A *singleton congestion game* G is defined by the following components:

- $N = \{1, \ldots, i, \ldots, n\}$, where N is the set of players;
- $R = \{1, \ldots, r, \ldots, m\}$, where R is the set of resources;

- Each player selects exactly one resource. For each player $i \in N$, $S_i = \{r \in 2^R : |r| = 1\} = R$.
- P_r^i is a specific payoff function.
- For $\sigma = (\sigma_i)_{i \in N}$ in S, and for r in R,

$$n_r(\sigma) = |\{i \in N : \sigma_i = r\}|$$

  The number of players sharing resource r in profile σ.

It is recalled that a player's utility is determined by the payoff associated with the uniquely selected resource and the number of players concurrently utilizing that resource. Given a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$, we define:

$$u_i(\sigma) = P_r^i(n_r(\sigma)), \quad \text{where } \sigma_i = r.$$

This utility can be described by a function u as follows:

$$\begin{array}{rcl} u_i : R \times \{1, \ldots, n\} & \to & \mathbb{R} \\ (r, h) & \mapsto & u_i(r, h) \end{array}$$

where u_i is decreasing in h.

In this chapter and the following one, we focus primarily on the existence (and possibly the characterization) of Nash equilibria in single-choice congestion games. For this purpose, we assume that the cardinal information contained in the utility functions is secondary. Instead, utilities are defined through preference orders, and we adapt the notion of Nash equilibrium to this ordinal framework:

1. The utility of player i is represented by a *complete preorder* (a reflexive, transitive, and complete binary relation) $\preceq_i$ over $R \times \{1, \ldots, n\}$ such that: for all $r \in R$, and for all $h, h' \in \{1, \ldots, n\}$, $h \preceq_i h' \Rightarrow (r, h) \preceq_i (r, h')$. The indifference relation of player i is denoted by $\sim_i$.
2. A strategy profile is a Nash equilibrium if no player has an incentive to deviate by switching to another resourceeither one chosen by others or an unchosen one. Formally, a strategy profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ is a Nash equilibrium if:

$$(\sigma_i^*, n_{\sigma_i^*}(\sigma^*)) \succeq_i (r, n_r(\sigma^*) + 1).$$

Example 2.2.0.1. Let $N = \{1, 2, 3\}$ and $R = \{r_1, r_2, r_3\}$. We can rank the cardinal utility functions for each player in decreasing order as follows:

$$u_1(3r_2) < u_1(3r_3) = u_1(3r_1) < u_1(2r_1) < u_1(2r_3) = u_1(r_3) < u_1(2r_2) < u_1(r_2) < u_1(r_1);$$

$$u_2(3r_2) < u_2(3r_1) = u_2(2r_1) < u_2(3r_3) < u_2(2r_3) < u_2(r_3) < u_2(2r_2) < u_2(r_1) = u_2(r_2);$$

$$u_3(3r_2) < u_3(3r_3) < u_3(2r_3) < u_3(2r_2) < u_3(r_3) < u_3(r_2) < u_3(3r_1) < u_3(2r_1) < u_3(r_1).$$

To simplify the notation, we now employ ordinal utility functions. Specifically, we denote $u_i(r, h)$ using the shorthand $h.r$. Accordingly, the players preferences can be rewritten as follows:

$$3r_2 \prec_1 3r_3 \sim_1 3r_1 \prec_1 2r_1 \prec_1 2r_3 \sim_1 r_3 \prec_1 2r_2 \prec_1 r_2 \prec_1 r_1;$$

$$3r_2 \prec_2 3r_1 \sim_2 2r_1 \prec_2 3r_3 \prec_2 2r_3 \prec_2 r_3 \prec_2 2r_2 \prec_2 r_1 \sim_2 r_2;$$

$$3r_2 \prec_3 3r_3 \prec_3 2r_3 \prec_3 2r_2 \prec_3 r_3 \prec_3 r_2 \prec_3 3r_1 \prec_3 2r_1 \prec_3 r_1.$$

In this example, $2r_2$ represents the pair $(r_2, 2)$, indicating that two players selected resource r_2. Similarly, r_3 denotes the pair $(r_3, 1)$, meaning that only one player chose resource r_3, and so on.

It is easy to verify that the strategy profile $\sigma^* = (r_2, r_2, r_1)$ is a Nash equilibrium since:

$$2r_2 \succeq_1 r_3 \quad \text{and} \quad 2r_2 \succeq_1 2r_1;$$

$$2r_2 \succeq_2 r_3 \quad \text{and} \quad 2r_2 \succeq_2 2r_1;$$

$$r_1 \succeq_3 r_3 \quad \text{and} \quad r_1 \succeq_3 3r_2.$$

2.3 Nash equilibrium in non-symmetric games

In this section, we present the results established by Milchtaich [37], focusing on the case where players have different utility functions. He demonstrated that any congestion game with only two strategies always admits a Nash equilibrium. We will use the ordinal description of the players' utility functions.

2.3.1 Case of two strategies

Theorem 2.3.1. (Milchtaich [37]) Every congestion game with two strategies satisfies the finite improvement property.

Proof. Suppose we are given a non-symmetric singleton congestion game with $R = \{r_1, r_2\}$.
Suppose that this game does not satisfy the (FIP), i,e.,

$$\sigma(0) \to \sigma(1) \to \ldots \to \sigma(l) \to \sigma(l+1) \to \ldots \to \sigma(l-1) \to \sigma(l) = \sigma(0)$$

Let us denote the number of players choosing each resource in profile $\sigma(l)$ as: $n(\sigma(l)) = (n_{r_1}(l), n_{r_2}(l))$ By reordering the sequence if necessary, we can assume that: $n_{r_2}(1) = max_l n_{r_2}(l)$.
this implies

$$n_{r_2}(1) \succ_i n_{r_2}(0)$$

$$n_{r_2}(1) \succ_i n_{r_1}(1) + 1$$

Since n_{r_2} is the maximum number of players choosing over the cycle, and the total number of players is constant, it follows that: $n_{r_1}(1) = min_l n_{r_1}(l)$. Therefore, we have:

$$n_{r_2}(1) \succ_i n_{r_1}(1) + 1 \implies n_{r_2}(l) \succ_i n_{r_1}(l) + 1, \forall l \geq 1.$$

This implies that player i can never switch back from to in any later step, because such a move would mean:

$$n_{r_1}(l) + 1 \succ_i n_{r_2}(l)$$

Thus, we reach a contradiction: an infinite improvement path cannot exist. Therefore, every such game must satisfy the FIP.

2.3.2 Games without the finite improvement property

In this part, we will explain what Milchtaich[37] established about non-symmetric congestion games that do not satisfy the Finite Improvement Property (FIP), using simple examples.

Example 2.3.2.1. Let $G(N, R, (P_r^i)_{i \in N,\ r \in R})$ be a singleton congeston game where $N = \{1, 2\}$ and $R = \{r_1, r_2, r_3\}$. Suppose the ordinal utility functions of the two players are given by:

$$2r_3 \prec_1 2r_2 \prec_1 r_2 \prec_1 2r_1 \prec_1 r_1 \prec_1 r_3.$$
$$2r_1 \prec_2 2r_2 \prec_2 r_2 \prec_2 2r_3 \prec_2 r_3 \prec_2 r_1.$$

We consider the following improvement path:

$$(r_2, r_1) \to (r_1, r_1) \to (r_1, r_2) \to (r_3, r_2) \to (r_3, r_3) \to (r_2, r_3) \to (r_2, r_1) \to \dots$$

This sequence forms a closed loop of six strategy profiles, beginning and ending at , and repeating indefinitely. Such a recurring improvement path clearly demonstrates the absence of a generalized ordinal potential function in this game. Nevertheless, it is straightforward to verify that both (r_3, r_1) and $(r_1, r_3)$are Nash equilibria.

In this context, we introduce a refined version of the Finite Improvement Property (FIP), known as the Best-Response Improvement Property. This concept, employed by Milchtaich [37] in his analysis of congestion games, is based on the idea that players only deviate from their current strategies when there exists another strategy that serves as a best response to the strategies chosen by the other players. A sequence of such deviations is referred to as a best-response path.

In other words, a player will only change their strategy if the current one is not optimal given the actions of the others. Any improvement path that follows this logic is called a best-response improvement path.

It is important to note that while the FIP includes both best-response paths and the Finite Best-Reply Property (FBRP), the converse does not always hold. That is, the existence of one of these properties does not necessarily imply the existence of the other. As Milchtaich [37] pointed out, congestion games involving more than two players, which may include infinite improvement paths based on best responses, still guarantee the existence of at least one Nash equilibrium. In such games, when a single player changes their strategy to another option, this shift negatively affects another player, whose payoff would decrease if they also chose the same strategy. This observation implies that no player can continuously increase their payoff indefinitely, and eventually, the system settles into a state of equilibrium.

2.4 Price of anarchy and price of stability

The inefficiency of game-theoretic equilibria has been a significant area of study. Two widely recognized metrics for assessing this inefficiency are the price of anarchy (PoA) ([32]) and the price of stability (PoS) ([5]). Both metrics compare the social cost incurred in a Nash equilibrium with the optimal social cost achievable under central control. The PoA focuses on the worst possible Nash equilibrium, which makes it a pessimistic measure, whereas the PoS examines the best possible Nash equilibrium, making it an optimistic measure. Consequently, PoA serves as a worst-case scenario guarantee in situations where equilibrium selection is beyond our control. On the other hand, PoS provides a more favorable estimate of Nash equilibrium, for instance, in scenarios where players coordinate to identify the optimal equilibrium or when a trusted mediator proposes this solution to them.

2.4.1 Price of anarchy

The price of anarchy, or PoA is a means of measuring inefficiencies in systems that ensue when people only look out for number one by game theorists and economists who practise. It is calculated as the ratio of the worst possible social outcome (when individuals act selfishly) to the optimal social outcome (when centralized coordination is implemented).

$$\text{PoA} = \frac{\text{Cost of the worst Nash equilibrium}}{\text{Cost of the social optimum}}$$

Example 2.4.1.1. Consider a game G with $R = \{r_1, r_2, r_3\}$ and $N = \{1, 2\}$. Suppose that the ordinal preferences of those players are:

$$\begin{aligned} \text{Player}_1: &\quad 2r_1 \prec 2r_3 \prec r_3 \prec 2r_2 \prec r_2 \prec r_1 \\ \text{Player}_2: &\quad 2r_3 \prec r_3 \prec 2r_1 \prec 2r_2 \prec r_1 \prec r_2 \end{aligned}$$

Now we suppose the cardinal utilities of those Players are given by:

n(r)\ r	r_1	r_2	r_3
1	1	2	9
2	13	6	11

Table 2.1: The utility cardinal of player 1

n(r)\ r	r_1	r_2	r_3
1	3	2	13
2	10	9	15

Table 2.2: The utility cardinal of player 2

It is easy to note that, the worst Nash equilibrium: (r_2, r_1).

- The cost of (r_2, r_1) is:

$$T_{(r_2,r_1)} = 5$$

- The cost of the social optimal:

$$T_{\text{optimal}} = 3$$

Then, the price of anarchy is:

$$\text{PoA} = \frac{T_{\text{worst equilibrium}}}{T_{\text{optimal}}} = \frac{5}{3} \approx 1.67$$

2.4.2 Price of stability

The concept of the price of stability (PoS) is a key measure in algorithmic game theory that evaluates the efficiency of the best possible Nash equilibrium compared to the socially optimal solution. It is formally defined as follows:

$$\text{PoS} = \frac{\text{Cost of the best Nash equilibrium}}{\text{Cost of the social optimum}}$$

Unlike the price of anarchy, which considers the worst-case equilibrium, the PoS reflects the best-case efficiency loss due to selfish behavior.

Example 2.4.2.1. Consider a game G with $R = \{r_1, r_2, r_3\}$ and $N = \{1, 2\}$. Suppose that the ordinal preferences of those players are:

$$\begin{aligned} \text{Player}_1 : &\quad 2r_1 \prec 2r_3 \prec r_3 \prec r_1 \prec 2r_2 \prec r_2 \\ \text{Player}_2 : &\quad 2r_3 \prec r_3 \prec 2r_2 \prec r_2 \prec 2r_1 \prec r_1 \end{aligned}$$

Now we suppose the cardinal utilities of those Players are given by:

n(r)\ r	r_1	r_2	r_3
1	6	2	9
2	11	4	13

Table 2.3: The utility cardinal of player 1

n(r)\ r	r_1	r_2	r_3
1	2	5	13
2	3	9	15

Table 2.4: The utility cardinal of player 2

It is easy to note that, the best Nash equilibrium: (r_2, r_1).

- The cost of (r_2, r_1) is:
$$T_{(r_2,r_1)} = 4$$
- The cost of the social optimal:
$$T_{\text{optimal}} = 4$$

Then, the price of stability is:

$$\text{PoS} = \frac{T_{\text{best equilibrium}}}{T_{\text{optimal}}} = \frac{4}{4} = 1$$

2.5 Results: Novel proofs and characterization of Nash equilibria

In the following, we will identify Nash equilibria (at least one) in congestion games with a single choice, where we will present a simplified method for constructing the general structure of Nash equilibria, with a particular focus on studying specific cases of these games. We aim to identify cases where it is possible to compile a comprehensive list of all possible Nash equilibria. Unlike the traditional approach commonly found in the literature, our method will not rely on improvement mechanisms. Instead, we will simplify the analysis by focusing on the sequences of the last terms in ordinal utility functions. Additionally, we will explore the possibility of providing a simpler proof of Milchtaich's theorem without resorting to recursive reasoning, thus reducing the complexity of finding Nash equilibria in singleton congestion games. Finally, we will propose a set of algorithms for identifying Nash equilibria in these games, facilitating their practical applications.

2.5.1 The case of two ressources ($m = (r_1, r_2)$)

This section presents two fundamental results in the context where the set of resources is restricted to two resources only. The first result pertains to the case of strict preferences, while the second addresses situations in which indifference is allowed.

Case 1: Strict order of preferences

We present below some essential notation that we will rely on in developing our approach: For all players $i \in N$, we define two integers as follows:

$$b_i = \max_{b \in \{0,1,\ldots,n\}} \{b : (r_1, b) \succ_i (r_2, n+1-b)\}$$
$$d_i = \max_{d \in \{0,1,\ldots,n\}} \{d : (r_2, d) \succ_i (r_1, n+1-d)\}$$

Taking into account that:

$$(r_1, 0) \succ_i (r_2, n+1) \quad \text{if} \quad (r_1, 1) \prec_i (r_2, n).$$
$$(r_2, 0) \succ_i (r_1, n+1) \quad \text{if} \quad (r_2, 1) \prec_i (r_1, n).$$

The integer b_i represents the maximum number of players who can choose alternative r_1 in a given strategy profile, where player i can be part of this group. Beyond this number, player i will choose alternative r_2. According to the definition, we have.

$$b_i \cdot r_1 \succ_i (n+1-b_i) \cdot r_2$$

If player i exceeds this number, they will choose alternative r_2, as shown in the following relationship:

$$(b_i + 1) \cdot r_1 \prec_i (n - b_i) \cdot r_2$$

In the same way as the integer b_i is defined, we define the integer d_i by replacing r_1 with r_2.
To facilitate the identification of the congestion vector corresponding to a Nash equilibrium in the game, we define two additional integers using the integers b_i and d_i.

$$n(r_1) = \max \{b \in \{0,1,\ldots,n\} :\mid \{i \in N : b_i \geq b\} \mid\geq b\}$$
$$n(r_2) = \max \{d \in \{0,1,\ldots,n\} :\mid \{i \in N : d_i \geq d\} \mid\geq d\}$$

The number $n(r_1)$ represents the maximum number of players who can choose

resource r_1 without any of them having an incentive to deviate from their strategy. The number $n(r_2)$ has the same definition as $n(r_1)$, with replaced r_1 by r_2. Accordingly, it necessarily follows that $n(r_1) + n(r_2) = n$, with the CCV being $v = (n(r_1), n(r_2))$.
We define the following sets that help identify the resources corresponding to each player, which in turn assist in describing all possible equilibria $\forall i \in N$:

$$E(G) = \{b_i > n(r_1)\}$$
$$F(G) = \{b_i < n(r_1)\}$$
$$H(G) = \{b_i = n(r_1)\}$$

The union of these sets constitutes the entire set N.

Theorem 2.5.1. Khanchouche et al[30]
Let $R = \{r_1, r_2\}$ and $G(N, R, (\prec)_{i \in N})$ be a singleton congestion game where all preference orderings are strict.

1. G admits at least one Nash equilibrium. All equilibria correspond to the same congestion vector: $v = (n(r_1), n(r_2))$.
2. Each Nash equilibrium of G, $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$, is characterized by a unique subset C (possibly empty) of $H(G)$, of cardinal $n(r_1) - |E(G)|$, such that: For all $i \in N$, $\sigma_i^* = r_1$ if $i \in E(G) \cup C$ and $\sigma_i^* = r_2$ if $i \in F(G) \cup (H \setminus C)$.
3. The game admits exactly $C_{|H(G)|}^{n(r_1)-|E(G)|}$ equilibria. If $n(r_1) = |E(G)|$ G admits a unique Nash equilibrium.

Proof 1 (Proof of theorem 2.5.1)**.** 1) As defined by $n(r_1)$, there are at least $n(r_1)$ players $i \in N$ such that $b_i \geq n(r_1)$. We construct a subset $E \subseteq N$ containing exactly $n(r_1)$ such players, ensuring that all players with $b_i > n(r_1)$ are included in E. Let $F = N \setminus E$ be the remaining set of players. For each $i \in F$, we must then have $b_i \leq n(r_1)$, which implies $d_i \geq n(r_2)$ due to the relationship between b_i and d_i.
Now, consider the strategy profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ defined by:

$$\sigma_i^* = \begin{cases} r_1 & \text{if } i \in E \\ r_2 & \text{if } i \in F \end{cases}$$

We can verify that this profile forms a Nash equilibrium, based on the definitions of b_i and d_i.

Let σ^* be a Nash equilibrium of the game G, and let (x, y) be the congestion vector induced by σ^*, where x (resp. y) is the number of players choosing r_1 (resp. r_2). Assume, for contradiction, that $x > n(r_1)$. Since σ^* is a Nash equilibrium, there must exist x players such that $b_i \geq x$, contradicting the maximality of $n(r_1)$. Therefore, we must have $x \leq n(r_1)$.

By a symmetric argument, $y \leq n(r_2)$. Given that $x+y = n$ and $n(r_1)+n(r_2) = n$, we conclude that $x = n(r_1)$ and $y = n(r_2)$.

2)Let $C \subseteq H(G)$ be any (possibly empty) subset such that $|C| = n(r_1) - |E(G)|$. Define the strategy profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ by:

$$\sigma_i^* = \begin{cases} r_1 & \text{if } i \in E(G) \cup C, \\ r_2 & \text{if } i \in F(G) \cup (H(G) \setminus C). \end{cases}$$

We claim that σ^* is a Nash equilibrium.

Indeed, for any player $i \in E(G) \cup C$, it follows from the definitions of $E(G)$ and C that $b_i \geq n(r_1)$. By the definition of b_i and under the assumption of monotonic preferences, we obtain:

$$n(r_1) \cdot r_1 \succsim_i (n(r_2) + 1) \cdot r_2.$$

Similarly, for all $i \in F(G) \cup (H(G) \setminus C)$, we have:

$$n(r_2) \cdot r_2 \succsim_i (n(r_1) + 1) \cdot r_1.$$

Conversely, suppose that $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ is a Nash equilibrium of the game G. From result (1), we know the associated congestion vector must be $(n(r_1), n(r_2))$. This implies that:

$$\sigma_i^* = \begin{cases} r_1 & \text{for all } i \in E(G), \\ r_2 & \text{for all } i \in F(G). \end{cases}$$

The remaining players assigned to r_1 can be grouped into a set defined by:

$$C = \{i \in N : \sigma_i^* = r_1 \text{ and } i \notin E(G)\}.$$

Clearly, $C \subseteq H(G)$ and $|C| = n(r_1) - |E(G)|$, completing the proof.

3) By performing a simple calculation from 2, we can obtain the result.

Example 2.5.1.1. Let us define a player set $N = \{1, 2, 3, 4, 5, 6\}$, and consider a choice set $R = \{r_1, r_2\}$ comprising two distinct alternatives. The individual preferences of the players are assumed to follow strict, transitive orderings as specified below:

$\text{Player}_1: \quad 6r_2 \prec 5r_2 \prec 4r_2 \prec 6r_1 \prec 5r_1 \prec 3r_2 \prec 4r_1 \prec 3r_1 \prec 2r_1 \prec 2r_2 \prec r_2 \prec r_1$

$\text{Player}_2: \quad 6r_1 \prec 6r_2 \prec 5r_2 \prec 4r_2 \prec 5r_1 \prec 4r_1 \prec 3r_1 \prec 3r_2 \prec 2r_1 \prec 2r_2 \prec r_1 \prec r_2$

$\text{Player}_3: \quad 6r_2 \prec 6r_1 \prec 5r_1 \prec 4r_1 \prec 3r_1 \prec 2r_1 \prec 5r_2 \prec 4r_2 \prec 3r_2 \prec 2r_2 \prec r_1 \prec r_2$

$\text{Player}_4: \quad 6r_1 \prec 6r_2 \prec 5r_2 \prec 4r_2 \prec 5r_1 \prec 4r_1 \prec 3r_1 \prec 3r_2 \prec 2r_2 \prec 2r_1 \prec r_2 \prec r_1$

$\text{Player}_5: \quad 6r_1 \prec 6r_2 \prec 5r_2 \prec 4r_2 \prec 5r_1 \prec 4r_1 \prec 3r_1 \prec 2r_1 \prec r_1 \prec 3r_2 \prec 2r_2 \prec r_2$

$\text{Player}_6: \quad 6r_2 \prec 5r_2 \prec 4r_2 \prec 6r_1 \prec 5r_1 \prec 4r_1 \prec 3r_1 \prec 2r_1 \prec 3r_2 \prec 2r_2 \prec r_1 \prec r_2$

To simplify the presentation, the player indices have been omitted from the preference orderings. For a given player i, we aim to determine the largest integer b_i such that the inequality $b \cdot r_1 \succ_i (n+1-b) \cdot r_2$, while simultaneously satisfying $(n-b) \cdot r_2 \succ_i (b+1) \cdot r_1$.

$$\begin{array}{lccc}
b_1 = 4: & 4r_1 \succ_1 3r_2 & \text{and} & 2r_2 \succ_1 5r_1 \\
b_2 = 3: & 3r_1 \succ_2 4r_2 & \text{and} & 3r_2 \succ_2 4r_1 \\
b_3 = 1: & 1r_1 \succ_3 6r_2 & \text{and} & 5r_2 \succ_3 2r_1 \\
b_4 = 3: & 3r_1 \succ_4 4r_2 & \text{and} & 3r_2 \succ_4 4r_1 \\
b_5 = 3: & 3r_1 \succ_5 4r_2 & \text{and} & 3r_2 \succ_5 4r_1 \\
b_6 = 3: & 3r_1 \succ_5 4r_2 & \text{and} & 3r_2 \succ_5 4r_1
\end{array}$$

Accordingly, we can conclude that:

$$n(r_1) = 3 \text{ and } n(r_2) = 3.$$

The only CCV is:

$$(3r_1, 3r_2).$$

Hence, we get:

$$E(G) = \{1\}$$
$$F(G) = \{3\}$$
$$H(G) = \{2, 4, 5, 6\}$$

By applying theorem 2.5.1, we obtain exactly C_4^2= 6 different equilibria. All these equilibria characterized as follows:

$$(r_1, r_1, r_2, r_1, r_2, r_2),\ (r_1, r_1, r_2, r_2, r_1, r_2),\ (r_1, r_1, r_2, r_2, r_2, r_1),$$
$$(r_1, r_2, r_2, r_1, r_2, r_1),\ (r_1, r_2, r_2, r_1, r_1, r_2),\ (r_1, r_2, r_2, r_2, r_1, r_1).$$

Remark 2.1. By this method, we are able to identify all equilibria without resorting to the repeated application of the finite improvement property, a process that may potentially never terminate due to infinite repetition.

Case 2: Order of preferences with ties

We now define:

$$b_i = \max_{b \in \{0,1,\ldots,n\}} \{b : (r_1, b) \succsim_i (r_2, n + 1 - b)\}$$
$$d_i = \max_{d \in \{0,1,\ldots,n\}} \{d : (r_2, d) \succsim_i (r_1, n + 1 - d)\}$$

The integers b_i and d_i, for all $i \in N$, retain the same definition as in the previous case. However, the condition $n(r_1) + n(r_2) = n$ does not necessarily hold here due to the potential occurrence of ties, which may affect this equality. Hence, $b_i + d_i \geq n$, for all $i \in N$. Based on that, it is possible for $b_i + d_i > n$ In this case, there may be more than one congestion vector corresponding to a Nash equilibrium for each player $i \in N$
We will use $n(r_1)$, $n(r_2)$ defined in the first case. Since we have $n(r_1) + n(r_2) \geq n$, the CCV is $v = (x, y)$, where $x \leq n(r_1)$, $y \leq n(r_2)$ and $x + y = n$.
To describe all Nash equilibria, we define three sets that help us identify the alternatives associated with each player:
$\forall i \in N$

$$E(G, v) = \{b_i \geq x \ \&\ d_i < y\}$$
$$F(G, v) = \{b_i < x \ \&\ d_i \geq y\},$$
$$H(G, v) = \{b_i \geq x \ \&\ d_i \geq y\}$$

The union of these sets constitutes the entire set N.

N is the disjoint union of these three sets, and each of these sets may be empty. We do not examine the case in which $b_i < x$ and $d_i < y$, as $b_i + d_i \geq n$ and $x + y = n$.

Theorem 2.5.2. Khanchouche et al [30]

Let $R = \{r_1, r_2\}$ and $G(N, R, (\precsim)_{i \in N})$ be a singleton congestion game where the order of preferences may include ties.

1. Each congestion vector $v = (x, y)$ such that $x \leq n(r_1)$, $y \leq n(r_2)$ and $x + y = n$, corresponds to (at least) one Nash equilibrium of G.
2. Each of the Nash equilibria of G corresponding to the vector v, namely each $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$, is characterized by a unique subset C (possibly empty) $H(G, v)$, of cardinal $x - |E(G, v)|$, so that: For all $i \in N$, $\sigma_i^* = r_1$ if $i \in E(G, v) \cup C$ and $\sigma_i^* = r_2$ if $i \in F(G, v) \cup (H(G, v) \setminus C)$.

Proof 2 (Proof of theorem 2.5.2)**.** It is sufficient to prove statement (2), since statement (1) follows as a direct consequence.

Let $v = (x, y)$ be a congestion vector satisfying:

$$x \leq n(r_1), \quad y \leq n(r_2), \quad \text{and} \quad x + y = n.$$

Let $C \subseteq H(G, v)$ be any (possibly empty) subset such that:

$$|C| = x - |E(G, v)|.$$

Define the strategy profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ as follows:

$$\sigma_i^* = \begin{cases} r_1 & \text{if } i \in E(G, v) \cup C, \\ r_2 & \text{if } i \in F(G, v) \cup (H(G, v) \setminus C)\,. \end{cases}$$

We claim that σ^* is a Nash equilibrium. For any player $i \in E(G, v) \cup C$, we have $b_i \geq x$ by construction. Then, by monotonicity and the definition of b_i, we obtain:

$$x \cdot r_1 \succsim_i (y + 1) \cdot r_2.$$

Similarly, for every player $i \in F(G, v) \cup (H(G, v) \setminus C)$, we have:

$$y \cdot r_2 \succsim_i (x + 1) \cdot r_1.$$

Conversely, suppose that $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ is a Nash equilibrium of the game G, and let $v = (x, y)$ be the CCV. If $x > n(r_1)$, then there must exist at least x

players such that $b_i \geq x$, contradicting the definition of $n(r_1)$. Hence, $x \leq n(r_1)$. A similar argument shows that $y \leq n(r_2)$, and since $x + y = n$, all three conditions on v hold.

Since σ^* is a Nash equilibrium, we must have:

$$\sigma_i^* = \begin{cases} r_1 & \text{if } i \in E(G, v), \\ r_2 & \text{if } i \in F(G, v). \end{cases}$$

Define the set:

$$C = \{i \in N : \sigma_i^* = r_1 \text{ and } i \notin E(G, v)\}.$$

Then $C \subseteq H(G, v)$ and $|C| = x - |E(G, v)|$. Moreover, the scenario where a player satisfies both $b_i < x$ and $d_i < y$ cannot occur, completing the proof.

Example 2.5.1.2. Let us suppose we have two resources,r_1, r_2 and five players whose preferences are as follows:

$$\text{Player}_1: \quad 5r_1 \prec 5r_2 \prec 4r_2 \prec 4r_1 \prec 3r_2 \sim 3r_1 \sim 2r_1 \prec 2r_2 \sim r_1 \prec r_2$$

$$\text{Player}_2: \quad 5r_2 \sim 4r_2 \sim 5r_1 \sim 4r_1 \sim 3r_2 \sim 3r_1 \sim 2r_1 \sim 2r_2 \sim r_1 \sim r_2$$

$$\text{Player}_3: \quad 5r_1 \prec 5r_2 \prec 4r_2 \prec 4r_1 \sim 3r_2 \sim 3r_1 \sim 2r_2 \prec 2r_1 \prec r_1 \prec r_2$$

$$\text{Player}_4: \quad 5r_2 \prec 4r_2 \prec 5r_1 \prec 4r_1 \sim 3r_2 \sim 3r_1 \sim 2r_2 \prec 2r_1 \prec r_2 \prec r_1$$

$$\text{Player}_5: \quad 5r_2 \sim 4r_2 \sim 5r_1 \sim 4r_1 \sim 3r_2 \sim 3r_1 \sim 2r_1 \sim 2r_2 \sim r_1 \sim r_2$$

it is easy to see that:

$$\begin{gathered} b_1 = 3,\ d_1 = 3, \\ b_2 = 5,\ d_2 = 5, \\ b_3 = 4,\ d_3 = 3, \\ b_4 = 4,\ d_4 = 3, \\ b_5 = 5,\ d_5 = 5. \\ n(r_1) = 4,\ n(r_2) = 3. \end{gathered}$$

By applying theorem 2.5.2, the possible CCV:

$$\begin{gathered} v_1 = (4r_1, r_2), \\ v_2 = (3r_1, 2r_2), \\ v_3 = (2r_1, 3r_2). \end{gathered}$$

As $v_1 = (4r_1, r_2)$, we get

$$E(G, v_1) = \emptyset$$
$$F(G, v_1) = \{1\}$$
$$H(G, v_1) = \{2, 3, 4, 5\}.$$

The unique Ne corresponding to v_1, which is given by the profile $(r_2, r_1, r_1, r_1, r_1)$.
As $v_2 = (3r_1, 2r_2)$ we get

$$E(G, v_2) = \emptyset$$
$$F(G, v_2) = \emptyset$$
$$H(G, v_3) = \{1, 2, 3, 4, 5\}.$$

The set of all NE corrisponding to v_2 are:

$$(r_2, r_2, r_1, r_1, r_1),\ (r_2, r_1, r_2, r_1, r_1),\ (r_2, r_1, r_1, r_2, r_1),\ (r_2, r_1, r_1, r_1, r_2),$$
$$(r_1, r_2, r_1, r_1, r_2),\ (r_1, r_1, r_2, r_1, r_2),\ (r_1, r_1, r_1, r_2, r_2),\ (r_1, r_2, r_1, r_2, r_1),$$
$$(r_1, r_2, r_2, r_1, r_1),\ (r_1, r_1, r_2, r_2, r_1).$$

For $v_3 = (2r_1, 3r_2)$ we have

$$E(G, v_3) = \emptyset$$
$$F(G, v_3) = \emptyset$$
$$\text{and } H(G, v_3) = \{1, 2, 3, 4, 5\}.$$

The set of all Nash equilibria corresponding to v_3 are:

$$(r_2, r_2, r_2, r_1, r_1),\ (r_2, r_2, r_1, r_2, r_1),\ (r_2, r_2, r_1, r_1, r_2),\ (r_2, r_1, r_1, r_2, r_2),$$
$$(r_1, r_2, r_2, r_2, r_1),\ (r_1, r_1, r_2, r_2, r_2),\ (r_2, r_1, r_2, r_2, r_1),\ (r_2, r_1, r_2, r_1, r_2),$$
$$(r_1, r_2, r_2, r_1, r_2),\ (r_1, r_2, r_1, r_2, r_2).$$

Remark 2.2. If player's preferences are represented by a strict order, there is a single congestion vector; otherwise we can find at least one congestion vector.

2.6 Numerical experiments

We propose two algorithms, the first one for calculating all Nash equilibria of the game, and the second one for calculating the best Nash equilibrium (equilibrium with the minimum social cost) and the worst Nash equilibrium (with the maximum social cost). We give the time spent by the algorithm to find the best Nash equilibrium when the number of players increases in both cases (the case when the order of preferences is strict and the case when the order of preferences with ties). For this end we need the cardinal representation of the utility functions. We need to put the cardinal utilities of resource $'r_1'$ in the matrix A of size $(N \times N)$

(N is the number of players), and we need to put the cardinal utilities of resource $'r_2'$ in the matrix of B (inverted: from the greatest utility to the smallest utility).

Algorithm1 : Find All Nash Equilibria

Input: Congestion game $\mathcal{G} = (N, R, \prec_i)$, cardinal utilities of each players

Let $i := 1$

repeat

  Increase i by one

  Find the integers (p_i, q_i)

  Find the integers $n(a), n(b)$

  Find $A(G), B(G), C(G), D$

until $i = N$

Output: σ_i^* (all Nash equilibria).

Algorithm 2 : Find the best Nash equilibrium and the worst equilibrium

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Input: σ_i^* (all Nash equilibria)

Let $i := 1$

repeat

  Increase i by one

  Find the minimum sum

  Find the maximum sum

until $i = N$

Output: σ_i^* (the best and worst Nash equilibria).

Example 2.6.0.1. Let the game be defined by 5 players and the ordinal preferences given by:

$\text{Player}_1: \quad 5r_2 \prec 5r_1 \prec 4r_1 \prec 4r_2 \prec 3r_2 \sim 3r_1 \sim 2r_1 \prec 2r_2 \sim r_1 \prec r_2$

$\text{Player}_2: \quad 5r_2 \sim 5r_1 \prec 4r_2 \sim 4r_1 \sim 3r_1 \prec 3r_2 \prec 2r_1 \sim 2r_2 \prec r_1 \sim r_2$

$\text{Player}_3: \quad 5r_1 \prec 5r_2 \prec 4r_2 \prec 4r_1 \sim 3r_2 \sim 3r_1 \sim 2r_2 \prec 2r_1 \prec r_1 \sim r_2$

$\text{Player}_4: \quad 5r_2 \prec 5r_1 \prec 4r_2 \prec 4r_1 \prec 3r_2 \sim 3r_1 \prec 2r_2 \sim 2r_1 \sim r_1 \prec r_2$

$\text{Player}_5: \quad 5r_2 \sim 5r_1 \prec 4r_2 \sim 4r_1 \prec 3r_2 \prec 3r_1 \prec 2r_1 \sim 2r_2 \prec r_1 \sim r_2$

Suppose that the cardinal preferences are given by:

$$A=\begin{bmatrix}2 & 3 & 3 & 5 & 6\\ 1 & 2 & 5 & 5 & 6\\ 1 & 2 & 7 & 7 & 11\\ 3 & 3 & 4 & 6 & 10\\ 1 & 2 & 3 & 7 & 9\end{bmatrix}$$

$$B=\begin{bmatrix}7 & 4 & 3 & 2 & 1\\ 6 & 5 & 4 & 2 & 1\\ 10 & 9 & 7 & 7 & 1\\ 11 & 7 & 4 & 3 & 1\\ 9 & 7 & 4 & 2 & 1\end{bmatrix}$$

A represents the cardinal utilities of each person when he takes resource r_1.
2 is the cost of person 1 when he is the only one who takes resource r_1.
3 is the cost of person 1 when he and another person take resource r_1....ect
B represents the cardinal utilities of each person when he takes resource r_2.
7 is the cost of person 1 when he and four other persons take resource r_2,.
4 is the cost of person 1 when he and three other persons take resource r_2.
The matrix A_1 is the matrix A but (from the smallest cost to the biggest cost).
The matrix B_1 is the matrix B but (from the biggest cost to the smallest cost).
According to the first algorithm, we find

$b_1 = 3$	$d_1 = 3$
$b_2 = 2$	$d_2 = 3$
$b_3 = 4$	$d_3 = 3$
$b_4 = 3$	$d_4 = 3$
$b_5 = 3$	$d_5 = 2$

$$E(G)=\begin{bmatrix}v_1 & v_2\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 5\end{bmatrix} F(G)=\begin{bmatrix}v_1 & v_2\\ 0 & 0\\ 2 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix} H(G)=\begin{bmatrix}v_1 & v_2\\ 1 & 1\\ 0 & 2\\ 3 & 3\\ 4 & 4\\ 5 & 0\end{bmatrix}$$

$|C| = 3$

All equilibria corresponding to the congestion vector v_1 are:

$$\sigma_1 = \{r_1, r_2, r_1, r_1, r_2\}, \sigma_2 = \{r_1, r_2, r_1, r_2, r_1\}, \sigma_3 = \{r_1, r_2, r_2, r_1, r_1\}, \sigma_4 = \{r_2, r_2, r_1, r_1, r_1\}.$$

All equilibria corresponding to the congestion vector v_1 are:

$$\sigma_1 = \{r_2, r_2, r_2, r_1, r_1\}, \sigma_2 = \{r_2, r_2, r_1, r_2, r_1\}, \sigma_3 = \{r_2, r_1, r_2, r_2, r_1\}, \sigma_4 = \{r_1, r_2, r_2, r_2, r_1\}.$$

According to Algorithm 2 for the first vector $(3r_1, 2r_2)$, we find the best Nash equilibirium:

$$\sigma = (r_1, r_2, r_1, r_2, r_1)$$

and for the vector $(2r_1, 3r_2)$

$$\sigma = (r_2, r_2, r_1, r_2, r_1)$$

The best Nash equilibrium between them is:

$$\sigma = (r_2, r_2, r_1, r_2, r_1).$$

Example 2.6.0.2. Let the game be defined by 10 players and the cardinal preferences given by :

$$A=\begin{bmatrix} 1 & 3 & 4 & 7 & 9 & 12 & 17 & 19 & 22 & 27 \\ 2 & 5 & 6 & 9 & 10 & 11 & 13 & 14 & 35 & 39 \\ 1 & 3 & 4 & 6 & 8 & 12 & 13 & 15 & 17 & 23 \\ 2 & 28 & 29 & 30 & 33 & 35 & 37 & 38 & 39 & 42 \\ 1 & 4 & 5 & 7 & 9 & 13 & 14 & 16 & 18 & 24 \\ 1 & 2 & 5 & 6 & 9 & 13 & 14 & 16 & 18 & 25 \\ 2 & 4 & 5 & 6 & 9 & 10 & 11 & 16 & 19 & 35 \\ 1 & 4 & 5 & 8 & 10 & 13 & 18 & 20 & 23 & 28 \\ 1 & 3 & 7 & 13 & 15 & 18 & 22 & 23 & 27 & 30 \\ 1 & 3 & 4 & 5 & 7 & 8 & 11 & 14 & 16 & 19 \end{bmatrix}$$

$$B=\begin{bmatrix} 20 & 18 & 16 & 15 & 13 & 11 & 10 & 8 & 6 & 5 \\ 33 & 27 & 25 & 23 & 22 & 20 & 19 & 15 & 3 & 1 \\ 30 & 27 & 25 & 20 & 18 & 10 & 9 & 7 & 5 & 2 \\ 27 & 23 & 21 & 20 & 17 & 16 & 13 & 8 & 5 & 3 \\ 33 & 28 & 26 & 21 & 19 & 11 & 10 & 8 & 6 & 3 \\ 31 & 28 & 26 & 22 & 19 & 12 & 10 & 8 & 7 & 4 \\ 30 & 27 & 23 & 22 & 18 & 15 & 13 & 7 & 3 & 1 \\ 26 & 17 & 16 & 14 & 12 & 11 & 9 & 7 & 6 & 2 \\ 35 & 19 & 16 & 11 & 10 & 9 & 6 & 5 & 4 & 2 \\ 41 & 37 & 35 & 33 & 30 & 28 & 27 & 25 & 23 & 20 \end{bmatrix}$$

According to the first algorithm, we find

$$\begin{aligned} b_1 &= 5 & d_1 &= 5 \\ \mathrm{b}_2 &= 8 & d_2 &= 2 \\ \mathrm{b}_3 &= 5 & d_3 &= 5 \\ \mathrm{b}_4 &= 1 & d_4 &= 9 \\ \mathrm{b}_5 &= 5 & d_5 &= 5 \\ \mathrm{b}_6 &= 5 & d_6 &= 5 \\ \mathrm{b}_7 &= 7 & d_7 &= 3 \\ \mathrm{b}_8 &= 5 & d_8 &= 5 \\ \mathrm{b}_9 &= 3 & d_9 &= 7 \\ \mathrm{b}_{10} &= 10 & d_{10} &= 0 \end{aligned}$$

$$E(G) = \begin{bmatrix} v_1 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 7 \\ 0 \\ 0 \\ 10 \end{bmatrix} F(G) = \begin{bmatrix} v_1 \\ 0 \\ 0 \\ 0 \\ 4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 9 \\ 0 \end{bmatrix} H(G) = \begin{bmatrix} v_1 \\ 1 \\ 0 \\ 3 \\ 0 \\ 5 \\ 6 \\ 0 \\ 8 \\ 9 \\ 10 \end{bmatrix}$$

$|C| = 2$

According to Algorithm 2 for the congestion vector $(5r_1, 5r_2)$, all Nash equilibria corresponding to this vector are:

$$\begin{aligned} \sigma_1 &= (r_1, r_1, r_1, r_2, r_2, r_2, r_1, r_2, r_2, r_1) & \sigma_2 &= (r_1, r_1, r_2, r_2, r_1, r_2, r_1, r_2, r_2, r_1) \\ \sigma_3 &= (r_1, r_1, r_2, r_2, r_2, r_1, r_1, r_2, r_2, r_1) & \sigma_4 &= (r_2, r_1, r_1, r_2, r_2, r_1, r_1, r_2, r_2, r_1) \\ \sigma_5 &= (r_2, r_1, r_1, r_2, r_1, r_2, r_1, r_2, r_2, r_1) & \sigma_6 &= (r_2, r_1, r_2, r_2, r_1, r_1, r_1, r_2, r_2, r_1) \\ \sigma_7 &= (r_1, r_1, r_2, r_2, r_2, r_2, r_1, r_1, r_2, r_1) & \sigma_8 &= (r_2, r_1, r_1, r_2, r_2, r_2, r_1, r_1, r_2, r_1) \\ \sigma_9 &= (r_2, r_1, r_2, r_2, r_1, r_2, r_1, r_1, r_2, r_1) & \sigma_{10} &= (r_2, r_1, r_2, r_2, r_2, r_1, r_1, r_1, r_2, r_1) \end{aligned}$$

The best Nash equilibrium between them is:

$$\sigma = (r_1, r_1, r_1, r_2, r_2, r_2, r_1, r_2, r_2, r_1)$$

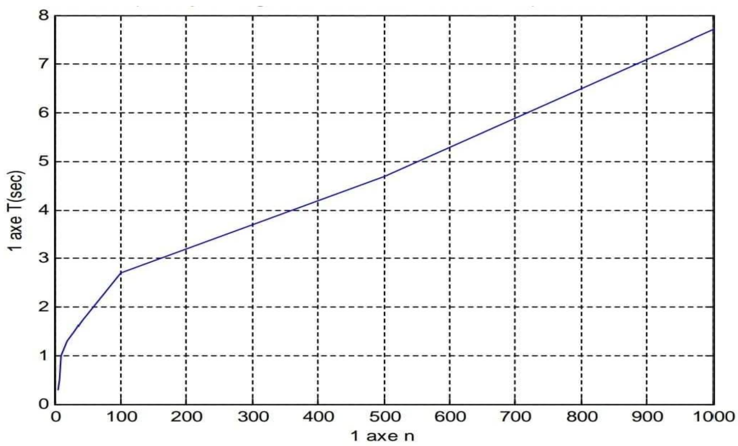

Figure 2.1: The time spent by the algorithm to find the best Nash equilibrium in the strict preferences case when n increases.

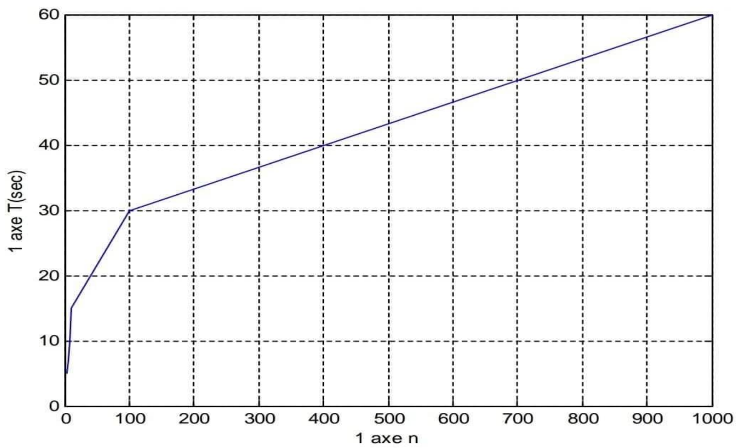

Figure 2.2: The time spent by the algorithm to find the best Nash equilibrium in the preferences with ties case when n increases.

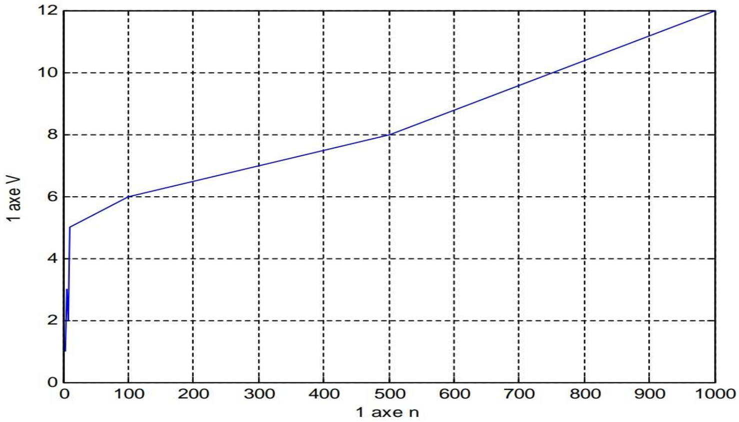

Figure 2.3: The number of congestion vectors in equilibrium when 'n' increases in the strict preferences case

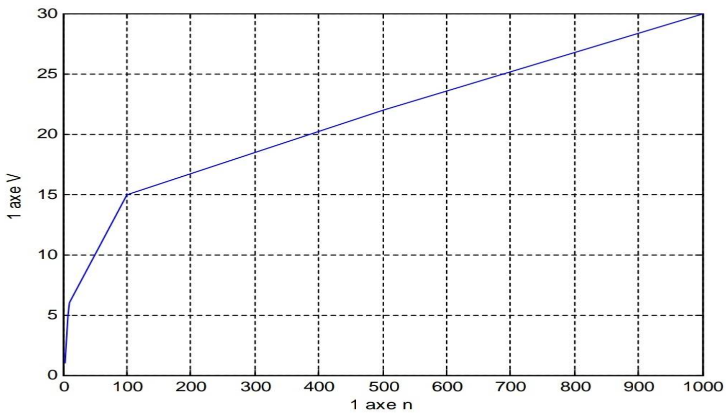

Figure 2.4: The number of congestion vectors in equilibrium when 'n' increases in the preferences with ties case

In this subsection, we analyze the ratio between the price of anarchy (PoA) and the price of stability (PoS) as a function of the number of players n. Our results indicate that this ratio $\frac{\text{PoA}}{\text{PoS}}$ increases with n. Specifically, when $n = 1000$, the ratio is approximately 3.5, implying that the social cost at the worst Nash equilibrium is 3.5 times higher than that at the best Nash equilibrium. This finding highlights the importance of *coordination among players* to achieve a significant reduction in social cost. The observed behavior holds under both strict or with ties.

The social cost of a pure strategy profile s is the sum of the player's costs: $SC(s) = \sum_{i \in N} C_i(s)$, where the C_i is the individual cost. Let us denote by

$OPT = Min_s SC(s)$ the optimum social cost over all strategy profiles.

The price of anarchy (PoA) is a term used in game theory and economics to describe how the selfish actions of a system's agents cause the system's efficiency to decline.

$$PoA = max_{s \in NE} \frac{Cost(s)}{OPT}$$

where NE is the set of Nash equilibruim profiles,

A related notion is that of the price of stability (PoS) which measures the ratio of the best equilibrium to the social optimum

$$PoS = min_{s \in NE} \frac{Cost(s)}{OPT}$$

In this section we will calculate the ratio of PoA to PoS, $\frac{PoA}{PoS}$, when the number of players increases.

$$\frac{PoA}{PoS}(n) = \frac{max_{s \in NE} Cost(s)}{min_{s \in NE} Cost(s)}$$

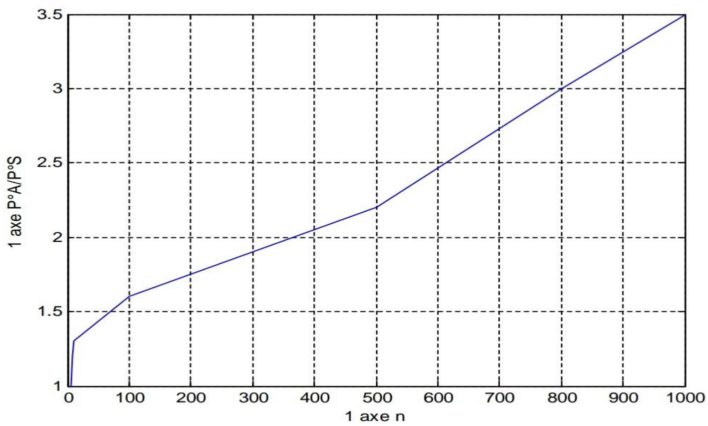

Figure 2.5: The quotient of the price of anarchy and the price of stability in the strict preferences case

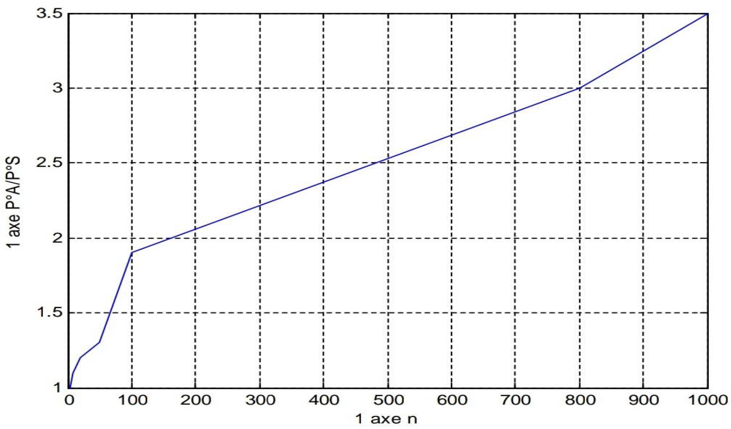

Figure 2.6: The quotient of the price of anarchy and the price of stability in the preferences with ties case

2.7 Conclusion

In this chapter, we have extended the theoretical framework of non-symmetric singleton congestion games by proposing a novel methodology that enables the identification of multiple Nash equilibria, rather than limiting the analysis to the existence of a single solution. This advancement allows for a deeper understanding of the strategic structure of such games and highlights the variety of equilibrium outcomes that can emerge depending on players heterogeneous preferences.

Additionally, we introduced and analyzed specific algorithms tailored to compute Nash equilibria efficiently in this class of games.

Finally, we evaluated the overall efficiency of equilibrium outcomes by calculating the price of anarchy (PoA) and the price of stability (PoS). Through comparative analysis, we highlighted the gap between the worst-case and best-case equilibria, thereby shedding light on the trade-off between individual rationality and collective efficiency.

These contributions not only deepen the understanding of equilibrium dynamics in non-symmetric congestion games but also pave the way for future work focused on algorithmic design and efficiency optimization in decentralized systems.

Chapter 3

Computation of Nash equilibrium in congestion game with player specific payoff functions

In this chapter, we focus on a special subclass of singleton congestion games introduced by Ziad et al. [65], which exhibits unique structural properties. We demonstrate that, despite the potential asymmetry in player preferences, a pure Nash equilibrium always exists in this class of games. Moreover, we present simple and efficient methods for identifying such equilibria, based on the inherent characteristics of the utility functions involved. These methods contribute to simplifying the analysis and enhancing our understanding of equilibrium behavior in non-symmetric singleton congestion games.

3.1 Introduction

Singleton congestion games represent a fundamental subclass of congestion games in which each player selects exactly one resource from a set of available resources. These games provide a simplified yet powerful framework for modeling competitive interactions in shared environments, where the cost or payoff associated with each resource depends on the number of players utilizing it.

Introduced by Milchtaich (1996), singleton congestion games are particularly relevant in contexts such as traffic routing, network bandwidth allocation, wireless communication, and load balancing. In such settings, each user (player) makes a discrete choice of a single option (e.g., a route, server, or channel), and their experience is influenced by how many others make the same choice. Instead of following the traditional approach used by Milchtaich [37], which relies on the Finite Improvement Property (FIP) through successive improvements of players' strategies, we propose an alternative methodology that directly utilizes a common utility function shared among all players. This method aims to identify Nash equilibria in a more direct and efficient way, based on a well-defined mathematical structure that relies on a complete preorder, as outlined in the following paragraph.

3.2 Congestion game with exact partition

In this section, we define specific configurations from which we can find ANE. We begin by generalizing the concept of n-sequence of a congestion game with player-specific ordinal utilities

Definition 3.2.0.1. Ziad et al [65] An n-sequence (or sequence of the n last terms) extracted from $\precsim$ is a set Sq of $R \times N$ such that:

- $|Sq| = n$ (1)
- $(r, h) \in Sq \Rightarrow ((r, h') \in Sq, \forall h' < h)$ (2)
- $((r, h) \in Sq$ and $(r', h') \notin Sq) \Rightarrow (r, h) \succsim (r', h')$ (3)

Let give an example

Example 3.2.0.1. Let $N = \{1, 2, 3\}$ and $R = \{r_1, r_2, r_3\}$. Suppose that the OUF of $\forall i \in N$ is given by:

$$3r_2 \prec_1 3r_1 \prec_1 3r_3 \prec_1 2r_3 \prec_1 2r_2 \prec_1 r_2 \prec_1 \underbrace{2r_1 \prec_1 r_3 \prec_1 r_1}.$$
$$3r_3 \prec_2 3r_2 \prec_2 3r_1 \prec_2 2r_3 \prec_2 2r_1 \prec_2 r_1 \prec_2 \underbrace{2r_2 \prec_2 r_2 \prec_2 r_3}.$$
$$3r_3 \prec_3 3r_1 \prec_3 3r_2 \prec_3 2r_1 \prec_3 2r_3 \prec_3 r_1 \prec_3 \underbrace{r_3 \prec_3 2r_2 \prec_3 r_2}.$$

By above definition , the unique 3-sequence for each player are:

$$(Sq)_1 = \{2r_1, r_3, r_1\}$$
$$(Sq)_2 = \{2r_2, r_2, r_3\}$$
$$(Sq)_3 = \{r_3, 2r_2, r_2\}$$

When indifference is allowed, the number of n-sequences increases, and consequently, the number of Nash equilibria also increases. For each n-sequence, we are able to determine all associated Nash equilibria. Therefore, if our goal is to identify all Nash equilibria, we must consider all n-sequences. We conduct our study with strict preferences and omit the case of preferences with indifference, as the exposition and computation process would become much more complicated.

Definition 3.2.0.2. Ziad et al [65] Let $G(N, R, (\preceq_i)_{i \in N})$ be a non-symmetric singleton congestion games. A configuration of G is a choice of an n-sequence (ordered) for each player $i \in N$.

Thus, a configuration consists of an n-sequence for each player. There are various potential configurations for a game, and we can represent a configuration using an array with n rows and n columns. Each row corresponds to a player, with the first row containing $(Sq)_1$, the second row containing $(Sq)_2$, and so on.

Definition 3.2.0.3. Ziad et al [65] Let $G(N, R, (\preceq_i)_{i \in N})$ be a non-symmetric singleton congestion games and $((Sq)_1, (Sq)_2, \dots, (Sq)_n)$ a configuration (ordered). For each resource $r \in R$, we note $\alpha_i(r)$ the number defined by:

$$\alpha_i(r) = \max \{q : (r, q) \in (Sq)_i\}$$

and $\alpha(r)$ the number defined by:

$$\alpha(r) = \max \{q : |i \in N : (r, q) \in (Sq)_i| \geq q\}.$$

Remark 3.1. $\alpha(r)$ represents the maximum number of players that can choose the resource r in a strategy profile.

To illustrate the above definitions, we give an example:

Example 3.2.0.2. Suppose we have 6 players and 3 resources (r_1, r_2, r_3). Let the n last terms be given by:

$$\dots 3r_1 \prec_1 2r_1 \prec_1 r_1 \prec_1 2r_3 \prec_1 r_3 \prec_1 r_2.$$
$$\dots r_2 \prec_2 r_3 \prec_2 4r_1 \prec_2 3r_1 \prec_2 2r_1 \prec_2 r_1\ .$$
$$\dots 4r_1 \prec_3 2r_3 \prec_3 r_3 \prec_3 3r_1 \prec_3 2r_1 \prec_3 r_1.$$
$$\dots 2r_3 \prec_4 3r_1 \prec_4 r_3 \prec_4 2r_1 \prec_4 r_2 \prec_4 r_1.$$
$$\dots 4r_3 \prec_5 2r_2 \prec_5 3r_3 \prec_5 2r_3 \prec_5 r_2 \prec_5 r_3.$$
$$\dots 3r_3 \prec_6 2r_1 \prec_6 2r_3 \prec_6 r_2 \prec_6 r_1 \prec_6 r_3.$$

Now, we can represent the individual preferences $(Sq)_i$ using the following table:

$(Sq)_1$	$\underline{3r_1}$	$2r_1$	r_1	$\underline{2r_3}$	r_3	$\underline{r_2}$
$(Sq)_2$	$\underline{r_2}$	r_3	$4r_1$	$\underline{3r_1}$	$2r_1$	r_1
$(Sq)_3$	$4r_1$	$\underline{2r_3}$	r_3	$\underline{3r_1}$	$2r_1$	r_1
$(Sq)_4$	$\underline{2r_3}$	$\underline{3r_1}$	r_3	$2r_1$	$\underline{r_2}$	r_1
$(Sq)_5$	$4r_3$	$2r_2$	$3r_3$	$\underline{2r_3}$	$\underline{r_2}$	r_3
$(Sq)_6$	$3r_3$	$2r_1$	$\underline{2r_3}$	$\underline{r_2}$	r_1	r_3

By definition 3.2.0.3, we find:

$$\alpha(r) = max\{q : |i \in N : (r_1, q) \in (Sq)_i| \geq q\} = 3.$$

Hence,

$$\begin{gathered} \alpha(r_1) = 3. \\ \alpha(r_2) = 1 \\ \alpha(r_3) = 2. \end{gathered}$$

Definition 3.2.0.4. A configuration $((Sq)_1, (Sq)_2, \dots, (Sq)_n)$ is said to be *an exact partition* if:

$$\sum_{r=1}^{m} \alpha(r) = n$$

In the example 3.2.0.2, we have:

$$\sum_{r \in R} \alpha(r) = \alpha(r_1) + \alpha(r_2) + \alpha(r_3) = 6 = n$$

So, the given configuration $((Sq)_1, (Sq)_2, \dots, (Sq)_6)$ is an exact partition.

3.2.1 Nash Equilibrium in congestion games with precise resource allocation

The following theorem allows us to prove the existence of a Nash equilibrium in singleton congestion game with exact partition.

Theorem 3.2.1. Ziad et al[65] Any singleton congestion game that satisfies the exact partition condition admits at least one Nash equilibrium.

Proof. First, we observe that there exists at least one ressources $r \in R$ such that the subset $B_0(r) = \{i \in N_0 : (r, \alpha(r)) \succeq_i (r', \alpha(r')), \forall r' \in R_0\}$ satisfies $|B_0(r)| \geq \alpha(r)$. This is true because $\sum_{r \in R} |B_0(r)| \geq n$ and $\sum_{r=1}^{m} \alpha(r) = n$. And this argument is true in the differents interations for the same reason. So in iteration k, we have also there exists at least one resource $r \in R_k$ such that $B_k(r) \geq \alpha(r)$ as $\sum_{r \in R_k} \alpha(r) = n - (\alpha(r_1) + \ldots + \alpha(r_k))$ and $\sum_{r \in R_k} |B_k(r)| \geq n - (\alpha(r_1) + \ldots + \alpha(r_k))$. We will compare two following ressources for witch the algorithm reachs step 6. So the arguement is reccursive. Let r_1 (respectiveley r_2) be the first ressources for which step 6 is reached. We have for all $i \in A_0(r_1)$, $\sigma_i^\star = r_1$ and by definition of $B_0(r_1)$: $\{(r_1, \alpha(r_1)) \succeq_i (r', \alpha(r')), \forall r' \in R_0 = R$ (i.e., $(r', \alpha(r')) \succeq_i (r', \alpha(r') + 1))\}$. Thus: $(r_1, \alpha(r_1)) \succeq_i (r', \alpha(r') + 1)$ $\forall r' \in R$. For all $i \in A_1(r_2)$, $s_i^\star = r_2$ and by definition of $B_1(r_2)$: $\{(r_2, \alpha(r_2)) \succeq_i (r', \alpha(r')), \forall r' \in R_1 = R \backslash \{r_1\}\}$——$(*)$. Thus, $(r_2, \alpha(r_2)) \succeq_i (r', \alpha(r') + 1)$ $\forall r' \in R \backslash \{r_1\}$. Remains to show that: $(r_2, \alpha(r_2)) \succeq_i (r_1, \alpha(r_1) + 1)$. Suppose the contrary. That is to say: $(r_1, \alpha(r_1) + 1) \succ_i (r_2, \alpha(r_2))$——$(**)$. We will have: $(r_1, \alpha(r_1)) \succ_i (r_2, \alpha(r_2))$ and using $(*)$, we obtain : $(r_1, \alpha(r_1)) \succeq_i (r', \alpha(r'))$, $\forall r' \in R_0 = R$. So, i would be in $B_0(r_1)$. Accordingly to $(**)$, $(r_1, \alpha(r_1) + 1) \succ_i (r_2, \alpha(r_2))$, always using $(*)$, we will have: $(r_1, \alpha(r_1) + 1)$ appears into $(Sq)_i$ (because $(r_2, \alpha(r_2))$ appears into $(Sq)_i$). i was not retained into $A_0(r_1)$. It follows that all players j of $A_0(r_1)$ are such that $(r_1, \alpha(r_1) + 1)$ appears into $(Sq)_j$. This is absurd because otherwise $\alpha(r_1) \geq \alpha(r_1) + 1$ (that is to say that there will be at least $\alpha(r_1) + 1$ players j having $(r_1, \alpha(r_1) + 1)$ in $(Sq)_j$), which is impossible. So, $(r_2, \alpha(r_2)) \succeq_i (r_1, \alpha(r_1) + 1)$. Therefore, we have: for all $i \in A_1(r_2)$, $(r_2, \alpha(r_2)) \succeq_i (r', \alpha(r') + 1)$, $\forall r' \in R$.

3.3 Nash Equilibrium in congestion games with imprecise resource allocation

In this section, we present several definitions that involve the concept of the n-sequence of a congestion game with player-specific ordinal utilities.

Definition 3.3.0.1. Ziad et al[65] A configuration $((Sq)_1, (Sq)_2, \ldots, (Sq)_n)$ is said to be *no-exact partition* if:

$$\sum_{r=1}^{m} \alpha(r) > n$$

Example 3.3.0.1. Consider $N = \{1, 2, 3, 4\}$ and $R = \{r_1, r_2, r_3\}$. Suppose the sequences of the n last terms are given by:

$$\begin{gathered} \ldots r_3 \prec_1 3r_1 \prec_1 2r_1 \prec_1 r_1. \\ \ldots r_1 \prec_2 2r_3 \prec_2 r_3 \prec_2 r_2. \\ \ldots r_3 \prec_3 2r_1 \prec_3 r_2 \prec_3 r_1. \\ \ldots r_3 \prec_4 3r_1 \prec_4 2r_1 \prec_4 r_1 \ . \end{gathered}$$

By definition 3.3.0.1 we find:

$$\begin{gathered} \alpha(r_1) = 2 \\ \alpha(r_2) = 1 \text{ and } \alpha(r_3) = 2 \end{gathered}$$

In the example 3.3.0.1, we have:

$$\sum_{r \in R} \alpha(r) = \alpha(r_1) + \alpha(r_2) + \alpha(r_3) = 5 > 4$$

So, the given configuration $((Sq)_1, (Sq)_2, \ldots, (Sq)_4)$ is a non-exact partition.

We will define some variables as $B(r_j), A(r_j)$.

Definition 3.3.0.2. $B(r_j) = \{i \in N : (r', \alpha(r')) \prec_i (r_j, \alpha(r_j)), \forall r_j, r' \in R\}$ is the set of players who have $\alpha(r)$ better than $\alpha(r')$ $\forall r' \in R$ according to their individual preference order, the number of players in $B(r_j)$ can be greater than $\alpha(r)$.

Let define the subset $A(r_j)$:

Definition 3.3.0.3. $A(r_j)$ is the set of players who are in $B(r_j)$ where the number of players in $A(r_j)$ must be equal to or less than $\alpha(r)$ i.e:

$$A(r_j) : \{|A(r_j)| \leq \alpha(r_j), \ \forall r_j \in R\}$$

Example 3.3.0.2. Let N={1 2 3 4 5} and $R = \{r_1, r_2, r_3\}$ we suppose that:

$$\alpha(r_1) = 2, \alpha(r_2) = 2, \alpha(r_3) = 2$$

The preferences of each player are as follows:

$$\begin{aligned}
&\ldots 2r_3 \prec_1 2r_2 \prec_1 2r_1.\\
&\ldots 2r_1 \prec_2 2r_3 \prec_2 2r_2.\\
&\ldots 2r_3 \prec_3 2r_1 \prec_3 2r_2.\\
&\ldots 2r_1 \prec_4 2r_3 \prec_4 2r_2.\\
&\ldots 2r_3 \prec_5 2r_2 \prec_5 2r_1.
\end{aligned}$$

By definition of $B(r_j)$ and $A(r_j)$, we find:

$B(r_1) = \{1, 5\}$ so:

$$A(r_1) = \{1, 5\}$$

Because $\alpha(r_1) = 2$

$B(r_2) = \{2, 3, 4\}$, we will find:

$$A(r_2) = \{2, 3\} \text{ or } A(r_2) = \{2, 4\} \text{ or } A(r_2) = \{4, 3\}.$$

Because $\alpha(r_2) = 2$.

If $A(r_2) = \{2, 3\}$ then:

$$B(r_3) = \{4\} \text{ so } A(r_3) = \{4\}$$

If $A(r_2) = \{2, 4\}$ then:

$$B(r_3) = \{3\} \text{ so } A(r_3) = \{3\}$$

If $A(r_2) = \{4, 3\}$ then:

$$A(r_3) = \{2\}, \text{ because } \alpha(r_3) = 2.$$

Let us establish an algorithm designed to calculate the congestion vector by processing specific groups, denoted as $A(r_j), B(r_j)$. The algorithm systematically analyzes these groups, applying specific operations to extract and compute the congestion vactor. By leveraging the structure of $A(r_j)$ and $B(r_j)$, the algorithm efficiently determines the congestion vector, providing valuable insights for further analysis.

3.3.1 Existence of Nash equilibirium in singleton congestion game with exact non-partition

Theorem 3.3.1. Ziad et al[65] Every singleton congestion game, with exact partition or not, guarantees the existence of at least one Nash equilibrium.

Proof. We use mathematical induction to prove the result based on the number of players in the game. We begin with the base case where there is only one player $n = 1$. In this case, a Nash equilibrium clearly exists, as the player simply chooses their preferred resource without any competition. Next, we assume that the result holds true for n players that is, every singleleton congestion game involving players has at least one Nash equilibrium. This assumption is known as the induction hypothesis. We now move on to proving that the property holds for the case of $n + 1$ players. Consider a congestion game with $n + 1$ players and m resources. To simplify the analysis, we remove the last player, resulting in a subgame with only n players. Now we have congestion game with n players. By the induction hypothesis, this game admits a Nash equilibrium, denoted by:

$$\sigma = (\sigma_1, \ldots, \sigma_n).$$

The congestion vector associated with this equilibrium is given by:

$$v = (\alpha_1, \ldots, \alpha_m).$$

which represents the number of players choosing each resource.

To extend this to a game with $n + 1$ players, we aim to construct a new Nash equilibrium by examining the available strategies for all $n + 1$ players, including the additional one. We compare the strategies formed by adding one player to each resource, namely:

$$(\alpha_1 + 1)r_1,\ (\alpha_2 + 1)r_2,\ \ldots,\ (\alpha_m + 1)r_m.$$

Since there are $n + 1$ first-choice positions, at least one of the strategies $(\alpha_j + 1)r_j$ appears at least $\alpha_j + 1$ times in the first position.

We will use the notation $q(i)$ to denote the resources (goods) assigned to each player i in the set $\{1, \ldots, n + 1\}$. Subsequently, we assign the $n + 1$ players to the

m (goods) in order to obtain the profile σ^*, following the procedure outlined below.

1. For player $n+1$, we set $\sigma^*_{n+1} = q(n+1)$. For every player $i \in N = \{1, \ldots, n\}$, if $q(i) = \sigma_i$, then we set $\sigma^*_i = q(i)$.

2. Among the players not involved in the previous step, we identify an initial basis of a cycle: $F = \{i_1, \ldots, i_t\}$. We then define the allocation as follows: $\sigma^*_{i_s} = \sigma_{i_{s+1}}$ for all $1 \leq s \leq t-1$, and $\sigma^*_{i_t} = \sigma_{i_1}$. Note that in this case, the strategies of the players in F in the profile σ^* are obtained by a simple permutation of their strategies in σ. We then search for another cycle basis among the remaining players and repeat the process until no cycles remain.

3. If all players have already been addressed in one of the two previous steps, the process terminates, and the only good whose congestion level increases is $q(n+1)$. Otherwise, only maximal chain bases remain. In this case, two scenarios arise:
   a) If the good $q(n+1)$ does not appear in any of these chains, or appears only as a terminal node, then for every player i involved in a chain, we set $\sigma^*_{i_s} = \sigma_{i_s}$. Once again, the only good whose congestion increases is $q(n+1)$.

   b) If $q(n+1)$ appears in one or more chains as an interior point, we choose one such chain, say $F = \{i_1, \ldots, i_t\}$, where $\sigma(i_k) = q(n+1)$ for some $1 \leq k \leq t$. In this case:

   For all $s < k$, we set $\sigma^*_{i_s} = \sigma_{i_s}$, and for all $s \geq k$, we set $\sigma^*_{i_s} = q(i_s)$.

   For all players involved in other chains, we define $\sigma^*_{i_s} = \sigma_{i_s}$. In this situation, the only good whose congestion increases is the terminal good of chain F.

We thus obtain a strategy profile σ^* whose associated congestion vector has the same components as the vector V, except for one component that increases by a single unit. Suppose that this component is k (corresponding to good k), which changes from α_k to $\alpha_k + 1$. We aim to show that σ^* constitutes a Nash equilibrium for the game with $n+1$ players.

Let i be an arbitrary player in the set $\{1, \ldots, n+1\}$. If player i is among those selecting strategy r_k, then $(\alpha_k + 1)r_k$ appears as the most preferred option in this player's preference ordering. Consequently, we have:

$$(\alpha_k + 1)r_k \succeq_i (\alpha_j + 1)r_j, \quad \forall r_j \in R.$$

Now suppose that player i selects a good r_t different from r_k. According to the process described previously, $(\alpha_t + 1)r_t$ appears at the top of player is preference ranking (above all other $(\alpha_j+1)r_j$), or r_t was player i's strategy in the equilibrium σ of the n player game. In either case, utilizing the assumption that player preferences are decreasing with respect to the number of players selecting a given good, we obtain:

$$(\alpha_t)r_t \succeq_i (\alpha_j + 1)r_j, \quad \forall r_j \in R, \quad \text{and} \quad (\alpha_t)r_t \succeq_i (\alpha_k + 2)r_k.$$

These relations imply that no player has an incentive to deviate, confirming that σ^* is a Nash equilibrium.

To illustrate this approach, we present three examples that demonstrate the proposed method for identifying (at least) one Nash equilibrium in singleton congestion games with asymmetric player preferences. The first two examples aim to highlight the procedure for transitioning from an equilibrium in an n-player game to one in an $(n + 1)$-player game. The final, more comprehensive example starts from an asymmetric congestion game and details all steps leading to a Nash equilibrium for that game.

3.4 The results: description of Nash equilibria

Algorithm 1: Find equilibrium congestion vector of configuration with non exacte partition

Input: $\mathcal{G} = (N, R, \prec_i)$, $((Sq)_1, (Sq)_2, \ldots, (Sq)_n)$ and $(\alpha(r))_{r\in R}$ be given such that: $\sum_{r=1}^{m} \alpha(r) > n$.

1: If $\sum_{r=1}^{m} \alpha(r) = n$, stop: consider the congestion vector $(r(1).1, r(2).2, \ldots, r(m).m)$.
2: for j=1 to m
3: Construct $B(r_j) = \{i \in N : (r', \alpha(r')) \prec_i (r_j, \alpha(r_j)), \forall r_j, r' \in R\}$
4: Construct $A(r_j) : |A(r_j)| \leq \alpha(r_j) \forall r_j \in R$
5: $N = N - A(r_j), R = R - r_j$
6: $\sigma = (|A(r_1)|r_1, ..., |A(r_m)|r_m)$
Output: (Congestion vector at equilibrium).

Remark 3.2. The output of algorithm 1 can be more than one congestion vector in the case where $|B(r_j)| > \alpha(r_j)$.

Proposition 3.4.1. Any output congestion vector of algorithm 1 is a congestion vector of equilibrium.

Proof: Let $\mathcal{G} = (N, R, \prec_i)$ a congestion game, $((Sq)_1, (Sq)_2, \ldots, (Sq)_n)$ a non-exact partition configuration and $(\alpha(r))_{r\in R}$ be given such that $\sum_{r=1}^{m} \alpha(r) > n$ i.e : $\alpha(r_1) + \alpha(r_2) + ... + \alpha(r_m) > n$, let $N_1 = N$, $R_1 = R$ and $B(r_1) = \{i \in N_1 : (r', \alpha(r')) \prec_i (r_1, \alpha(r_1)), \forall r_1, r' \in R - 1\}$, $B(r_1)$ is a set of Players who prefer to choose the pair$(r_1, \alpha(r_1))$. Let $A(r_1)$ is a subset of $B(r_1)$ such that: $|A(r_1)| \leq \alpha(r_1)$, $\forall r_1 \in R_1$, according to the definition of $A(r_1)$, there may be many possible choices for $A(r_1)$.
Let $R_2 = R_1 - (r_1), N_2 = N_1 - A(r_1), B(r_2) = \{i \in N_2 : (r', \alpha(r')) \prec_i (r_2, \alpha(r_2)), \forall r_2, r' \in R_2\}$, $B(r_2)$ is a set of Players who prefer to choose the pair$(r_2, \alpha(r_2))$. we construct $A(r_2) : i \in B(r_2), |A(r_2)| \leq \alpha(r_2)$.
Let $R_3 = R_2 - (r_2), N_3 = N_2 - (A(r_2)), \forall r_3 \in R_3$ we construct $A(r_3) : i \in B(r_3), |A(r_3)| \leq \alpha(r_3)$. By the same method we cunstruct all the subset of $A(r_j) : j = (1, ..m)$.
By definition of $B(r_j)$, $\forall j : (1, ..., m)$ each player i chose the best resource. we necessarily have: $|A(r_1)| + ... + |A(r_m)| = n$.
From what we proof above the process automaticly substract '1' for $\alpha(r)$ if the number of players who will choose the resource r isn't $\alpha(r)$ player. By this algorithm we find the congestion vector of equilibrium in any configuration non-exact.

Example 3.4.0.1. Let the set of players be defined as $N = \{1, 2, 3, 4, 5, 6\}$ and the set of resources as $R = \{r_1, r_2, r_3\}$. It is assumed that for each player i and each resource r, the utility derived from selecting a resource decreases strictly as more players choose the same resource. When applying definition (3.2.1), it is not necessary to have full knowledge of the utility functions. Instead, it suffices to understand the relative ordering of the utility values corresponding to the scenarios with a higher number of users. Accordingly, only the final segments of these utility sequences are required, and it is assumed that such segments are already provided:

$$\ldots 2r_3 \prec_1 r_3 \prec_1 r_2 \prec_1 2r_1 \prec_1 r_1.$$
$$\ldots 2r_3 \prec_2 2r_1 \prec_2 r_1 \prec_2 r_3 \prec_2 r_2.$$
$$\ldots 3r_2 \prec_3 2r_2 \prec_3 2r_1 \prec_3 r_1 \prec_3 r_2\ .$$
$$\ldots 3r_2 \prec_4 2r_2 \prec_4 2r_3 \prec_4 r_3 \prec_4 r_2.$$
$$\ldots 3r_2 \prec_5 2r_2 \prec_5 r_2 \prec_5 2r_3 \prec_5 r_3.$$

At this point, the individual preference profiles $(Sq)^i$ can be conveniently illustrated using the following table:

$(Sq)_1$	$2r_3$	r_3	r_2	$2r_1$	r_1
$(Sq)_2$	$2r_3$	$2r_1$	r_1	r_3	r_2
$(Sq)_3$	$3r_2$	$2r_2$	$2r_1$	r_1	r_2
$(Sq)_4$	$3r_2$	$2r_2$	$2r_3$	r_3	r_2
$(Sq)_5$	$3r_2$	$2r_2$	r_2	$2r_3$	r_3

We have:

$$\alpha(r_1) = 2,\ \alpha(r_2) = 3,\ \alpha(r_3) = 2$$

and

$$\sum_{r=1}^{m} \alpha(r) = 7 > 5.$$

If we organize the three terms $2r_1, 3r_2, 2r_3$from $(Sq)_i$previously given, we find:

$$\ldots \prec_1 3r_2 \prec_1 2r_3 \prec_1 2r_1.$$
$$\ldots \prec_2 3r_2 \prec_2 2r_3 \prec_2 2r_1.$$
$$\ldots \prec_3 2r_3 \prec_3 3r_2 \prec_3 2r_1\ .$$
$$\ldots \prec_4 2r_1 \prec_4 3r_2 \prec_4 2r_3.$$
$$\ldots \prec_5 2r_1 \prec_5 3r_2 \prec_5 2r_3.$$

We have:

$B(r_1) = \{1,2,3\}$, as $\alpha(r_1) = 2$ then we must choose only two players from them, so $A(r_1) = \{1,2\}$, or $A(r_1) = \{1,3\}$, or $A(r_1) = \{2,3\}$.
$B(r_3) = \{4,5\}$, then $A(r_3) = \{4,5\}$.

If we choose that $A(r_1) = \{1,2\}$ then $B(r_2) = \{3\}$, i.e $A(r_2) = \{3\}$.
If we choose that $A(r_1) = \{1,3\}$ then $B(r_2) = \{2\}$, i.e $A(r_2) = \{2\}$.
If we choose that $A(r_1) = \{2,3\}$ then $B(r_2) = \{1\}$, i.e $A(r_2) = \{1\}$.
So, the congestion vector of this game is:

$$(2r_1, r_2, 2r_3).$$

Example 3.4.0.2. Consider $N = \{1,2,3,4\}$ and $R = \{r_1, r_2, r_3\}$. Suppose that the n last terms given by:

$(Sq)_1$	r_3	r_2	$2r_1$	r_1
$(Sq)_2$	r_1	$2r_3$	r_3	r_2
$(Sq)_3$	r_2	$2r_1$	r_1	r_3
$(Sq)_4$	$2r_2$	r_2	$2r_3$	r_3

In this example we have:

$$\alpha(r_1) = 2,\ \alpha(r_2) = 1,\ \alpha(r_3) = 2$$

Therefore,

$$\sum_{r=1}^{m} \alpha(r) = 5 > 4.$$

If we organize the three terms $2r_1, r_2, 2r_3$ from $(Sq)_i$previously given, we find :

$$\begin{array}{c} \ldots \prec_1 2r_3 \prec_1 r_2 \prec_1 2r_1. \\ \ldots \prec_2 2r_1 \prec_2 2r_3 \prec_2 r_2. \\ \ldots \prec_3 2r_3 \prec_3 r_2 \prec_3 2r_1\ . \\ \ldots \prec_4 2r_1 \prec_4 r_2 \prec_4 2r_3. \end{array}$$

We have:

$$\begin{array}{c} B(r_1) = \{1,3\}, \text{ then } A(r_1) = \{1,3\}. \\ B(r_2) = \{2\}, \text{ then } A(r_2) = \{2\}. \\ B(r_3) = \{4\}, \text{ then } A(r_3) = \{4\}. \end{array}$$

Therefore,

$$\alpha(r_1) = 2,\ \alpha(r_2) = 1,\ \alpha(r_3) = 1$$

Thus,

$$\sum_{r=1}^{m} \alpha(r) = 4.$$

The congestion game of this game is:

$$(2r_1, r_2, r_3).$$

Algorithm 2: From a configuration with exact partition to the subsets helping in the determination of Nash equilibria

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Input: Congestion game $\mathcal{G} = (N, R, \prec_i)$, $R = \{1, 2, \ldots, m\}$ and configuration with exact partition $\sum_{r=1}^{m} \alpha(r) = n$, $m \geq 2$.

1: Initiate $i = 1$,

2: Examine the rankings of $(\alpha(r)r)_{r\in R}$ in individual preferences.

3: Construct the subset $BR_i = \{r \in R : \alpha(r)r \succsim_i (\alpha(r') + 1)r', \forall r' \in R\}$,

4: If $i > n$ Stop

5: $i = i + 1$

6: Go step 3

Output: The subset $(BR_i)_{i\in N}$.

Algorithm 3: From the subsets $(BR_i)_{i\in N}$ to Nash equilibria

Input: Congestion game $\mathcal{G} = (N, R, \prec_i)$, $R = \{1, 2, \dots, m\}$, configuration with exact partition $\sum_{r=1}^{m} \alpha(r) = n$, $m \geq 2$ and $(BR_i)_{i\in N}$.

1: Initiate i=1,
2: Initiate j=1,k = 1
3: Construct $AR_k(r_j)$: $\{r \in BR_i, |AR(r_j)| = \alpha(r_j)\}$,
4: $s_i = AR(rj)$,
5: If $i = n$ stop,
6: If $j = m$ stop.

Output: The subset of Nash equilibrium.

Proposition 3.4.2. The out put of algorithm 2 and 3 are the subset of Nash equilibrium.

Proof: We have that $BR_i = \{r \in R : \alpha(r)r \succsim_i (\alpha(r') + 1)r', \forall r' \in R\}$ is the set of all pairs preferred by each player $i \in N$, by definition of Nash equilibrium we conclude that each resources in BR_i will acheive us the equilibrium.
We suppose that there is another resource that achieves equiibrium other than those in BR_i. We assume that r_1 achieves equilibrium for player i, i.e: $\sigma_i = r_1$.
By definition of Nash equilibrium we find that $u_i(r_1) > u_i(r')\forall r' \in R$ i.e $\{\alpha(r_1)r_1 \succsim_i (\alpha(r'))r', \forall r' \in R\}$. Contradition because $r_1 \notin BR_i$.
This example serves to provide additional clarity on the methodology presented in Algorithm 2, demonstrating the process of identifying the subset of Nash equilibria.

Example 3.4.0.3. Let the set of players be defined as $N = \{1, 2, 3, 4, 5, 6\}$ and the set of resources as $R = \{r_1, r_2, r_3\}$. It is assumed that for each player i and each resource r, the utility derived from selecting a resource decreases strictly as more players choose the same resource. When applying definition (3.2.1), it is not necessary to have full knowledge of the utility functions. Instead, it suffices to understand the relative ordering of the utility values corresponding to the scenarios with a higher number of users. Accordingly, only the final segments of these utility sequences are required, and it is assumed that such segments are already provided:

$$\dots 3r_1 \prec_1 r_3 \prec_1 2r_2 \prec_1 r_2 \prec_1 2r_1 \prec_1 r_1.$$
$$\dots 3r_3 \prec_2 2r_1 \prec_2 r_1 \prec_2 r_2 \prec_2 2r_3 \prec_2 r_3.$$
$$\dots 3r_2 \prec_3 2r_2 \prec_3 r_2 \prec_3 2r_3 \prec_3 r_1 \prec_3 r_3 .$$
$$\dots 3r_3 \prec_4 2r_3 \prec_4 2r_2 \prec_4 r_1 \prec_4 r_3 \prec_4 r_2.$$

$$\ldots 3r_1 \prec_5 2r_3 \prec_5 r_3 \prec_5 r_2 \prec_5 2r_1 \prec_5 r_1.$$
$$\ldots 3r_2 \prec_6 2r_2 \prec_6 2r_1 \prec_6 r_2 \prec_6 r_1 \prec_6 r_3.$$

At this point, the individual preference profiles $(Sq)_i$ can be conveniently illustrated using the following table:

$(Sq)_1$	$3r_1$	r_3	$2r_2$	r_2	$2r_1$	r_1
$(Sq)_2$	$3r_3$	$2r_1$	r_1	r_2	$2r_3$	r_3
$(Sq)_3$	$3r_2$	$2r_2$	r_2	$2r_3$	r_1	r_3
$(Sq)_4$	$3r_3$	$2r_3$	$2r_2$	r_1	r_3	r_2
$(Sq)_5$	$3r_1$	$2r_3$	r_3	r_2	$2r_1$	r_1
$(Sq)_6$	$3r_2$	$2r_2$	$2r_1$	r_2	r_1	r_3

In this example, we observe that:

$$\alpha(r_1) = 2, \alpha(r_2) = 2, \alpha(r_3) = 2$$

As a result, we obtain the following:

$$\alpha(r_1) + \alpha(r_2) + \alpha(r_3) = 6$$

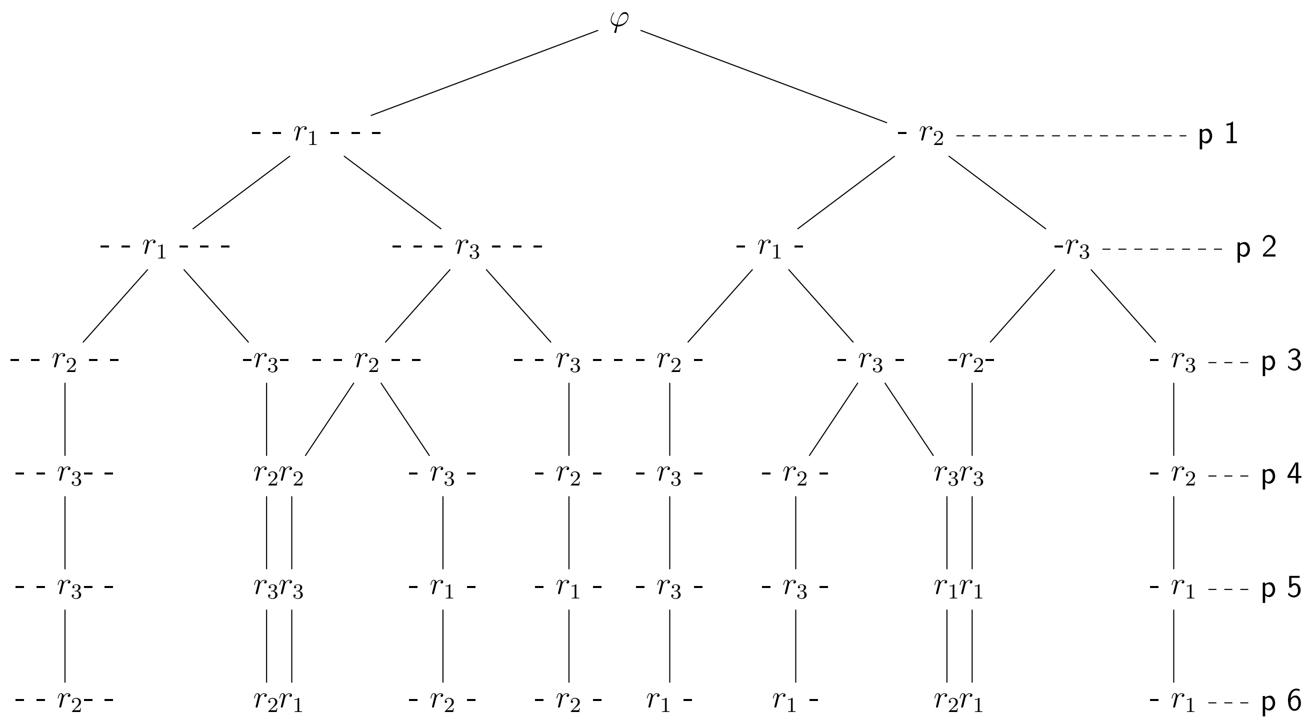
This corresponds to a configuration with an exact partition. Each Nash equilibrium in this game is characterized by the congestion vector:

$$(2r_1, 2r_2, 2r_3)$$

This common congestion vector simplifies the identification of all Nash equilibria in the game. To begin the analysis. Applying algorithm 2 and 3 facilitate the identification of the congestion vector and all Nash equilibrium, we focus on the ranking of pairs $(r_1, 2)$, $(r_2, 2)$, and $(r_3, 2)$, represented as $2r_1$, $2r_2$, and $2r_3$ in the preferences of each player:

$(Sq)_1$: $2r_1 \prec_1 2r_2 \prec_1 2r_3$ (player 1 may choose r_1 or r_2 as $n_{r_1} \succ_1 n_{r_2} + 1$, $n_{r_1} \succ_1 n_{r_3} + 1$, $n_{r_2} \succ_1 n_{r_1} + 1$ and $n_{r_2} \succ_1 n_{r_3} + 1$).
$(Sq)_2$: $2r_3 \prec_2 2r_1 \prec_2 2r_2 \rightsquigarrow$ player 2 may choose r_3 or r_1.
$(Sq)_3$: $2r_3 \prec_3 2r_2 \prec_3 2r_2 \rightsquigarrow$ player 3 choose only r_2.or r_3
$(Sq)_4$: $2r_2 \prec_4 2r_3 \prec_4 2r_1 \rightsquigarrow$ player 4 may choose r_2 or r_3.
$(Sq)_5$: $2r_1 \prec_5 2r_3 \prec_5 2r_2 \rightsquigarrow$ player 5 may choose r_1 or r_3.
$(Sq)_6$: $2r_1 \prec_6 2r_2 \prec_6 2r_3 \rightsquigarrow$ player 6 may choose $r_1$or r_2.

We can build the next tree to detect all Nash equilibria:



We can read the above tree as follows:

Player 1 has two possible choices: r_1 and r_2.
If player 1 chooses r_1, player 2 has two possible choices r_1 and r_3.
If player 2 chooses r_1, player 3 has two possible choices r_2 and r_3.
If player 3 chooses r_2, player 4 have to choose r_3, player 5 have to choose r_3.
Player 6 must choose only the alternative r_2.

Remark 3.3. The selection of values must comply with the constraint on the number of available resources. ($\alpha(r_1) = 2$, $\alpha(r_2) = 2$ and $\alpha(r_3) = 2$).

Thus, the Nash equilibria in this example are:

$$\sigma_1 = (r_1, r_1, r_2, r_3, r_3, r_2),\ \sigma_2 = (r_1, r_1, r_3, r_2, r_3, r_2),\ \sigma_3 = (r_1, r_3, r_2, r_2, r_3, r_1),$$
$$\sigma_4 = (r_1, r_3, r_2, r_3, r_1, r_2),\ \sigma_5 = (r_1, r_3, r_3, r_2, r_1, r_2),\ \sigma_6 = (r_2, r_1, r_2, r_3, r_3, r_1),$$
$$\sigma_7 = (r_2, r_1, r_3, r_2, r_3, r_1),\ \sigma_8 = (r_2, r_1, r_3, r_3, r_1, r_2),\ \sigma_9 = (r_2, r_3, r_2, r_3, r_1, r_1),$$
$$\sigma_{10} = (r_2, r_3, r_3, r_2, r_1, r_1).$$

Example 3.4.0.4. Consider $N = \{1, 2, 3, 4, 5, 6\}$ and $R = \{r_1, r_2, r_3\}$. Suppose the sequences of the n last terms are given by:

$$\begin{array}{c} \ldots 3r_2 \prec_1 r_1 \prec_1 2r_3 \prec_1 r_3 \prec_1 2r_2 \prec_1 r_2. \\ \ldots 2r_3 \prec_2 2r_2 \prec_2 r_2 \prec_2 r_3 \prec_2 2r_1 \prec_2 r_1. \\ \ldots 3r_3 \prec_3 2r_3 \prec_3 r_3 \prec_3 2r_1 \prec_3 r_2 \prec_3 r_1 \ . \\ \ldots 3r_1 \prec_4 2r_1 \prec_4 2r_3 \prec_4 r_2 \prec_4 r_1 \prec_4 r_3. \\ \ldots r_3 \prec_5 2r_1 \prec_5 r_1 \prec_5 3r_2 \prec_5 2r_2 \prec_5 r_2. \\ \ldots 2r_3 \prec_6 3r_2 \prec_6 2r_2 \prec_6 r_3 \prec_6 r_2 \prec_6 r_1. \end{array}$$

Now we can represent the individual preferences $(Sq)_i$ by the following table:

$(Sq)_1$	$3r_2$	r_1	$2r_3$	r_3	$2r_2$	r_2
$(Sq)_2$	$2r_3$	$2r_2$	r_2	r_3	$2r_1$	r_1
$(Sq)_3$	$3r_3$	$2r_3$	r_3	$2r_1$	r_2	r_1
$(Sq)_4$	$3r_1$	$2r_1$	$2r_3$	r_2	r_1	r_3
$(Sq)_5$	r_3	$2r_1$	r_1	$3r_2$	$2r_2$	r_2
$(Sq)_6$	$2r_3$	$3r_2$	$2r_2$	r_3	r_2	r_1

We observe $\alpha(r_1) = 2, \alpha(r_2) = 3$, and $\alpha(r_3) = 2$, resulting in $\alpha(r_1)+\alpha(r_2)+\alpha(r_3) = 7 > 6$, thus the partition non-exact. If we organize the three terms $2r_1, 3r_2, 2r_3$ from $(Sq)_i$previously given, we find :

$$\begin{array}{c} \ldots \prec_1 2r_2 \prec_1 3r_2 \prec_1 2r_3. \\ \ldots \prec_2 3r_2 \prec_2 2r_3 \prec_2 2r_1. \\ \ldots \prec_3 3r_2 \prec_3 2r_3 \prec_3 2r_1 \ . \\ \ldots \prec_4 3r_2 \prec_4 2r_1 \prec_4 2r_3. \\ \ldots \prec_5 2r_3 \prec_5 2r_1 \prec_5 3r_2 \ . \\ \ldots \prec_6 2r_1 \prec_6 2r_3 \prec_6 3r_2. \end{array}$$

By applying definitions (3.3.0.2) and (3.3.0.3) we find:

$B(r_1) = \{2, 3\}$ then $(r_1) = \{2, 3\}$.
$B(r_3) = \{1, 4\}$ then $(r_3) = \{1, 4\}$.
$B(r_2) = \{5, 6\}$ then $(r_2) = \{5, 6\}$.

Therefore, $\alpha(r_1) = 2, \alpha(r_2) = 2$, $\alpha(r_3) = 2$ and $\alpha(r_1) + \alpha(r_2) + \alpha(r_3) = 6$.

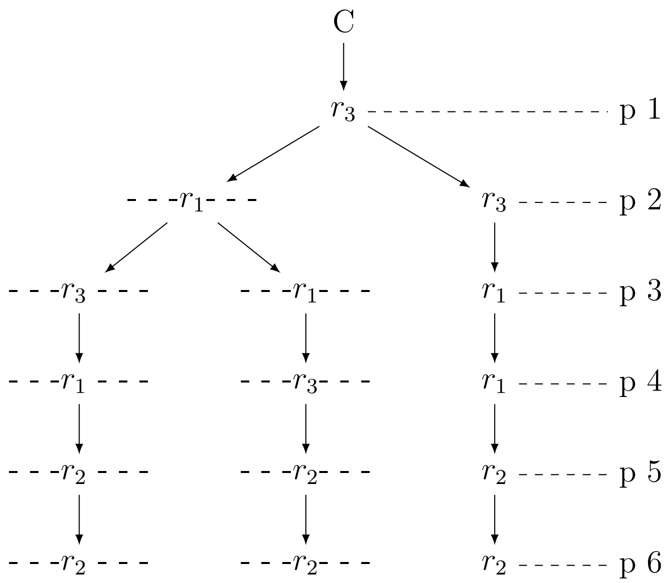
The new congestion vector is:

$$v = (2r_1, 2r_2, 2r_3)$$

Each Nash equilibrium in this game is characterized by the congestion vector $(2r_1, 2r_2, 2r_3)$. This common congestion vector simplifies the identification of all Nash equilibria in the game. To begin the analysis. Applying Algorithm 2 and 3 facilitate the identification of the congestion vector and all Nash equilibrium, we focus on the ranking of pairs $(r_1, 2)$, $(r_2, 2)$, and $(r_3, 2)$, represented as $2r_1$, $2r_2$, and $2r_3$ in the preferences of each player:

$(Sq)_1$: $2r_1 \prec_1 2r_2 \prec_1 2r_3$ (player 1 may choose r_2 or r_3 as $n_{r_2} \succ_1 n_{r_1} + 1$, $n_{r_2} \succ_1 n_{r_3} + 1$, $n_{r_3} \succ_1 n_{r_1} + 1$ and $n_{r_3} \succ_1 n_{r_2} + 1$).
$(Sq)_2$: $2r_3 \prec_2 2r_1 \prec_2 2r_2 \rightsquigarrow$ player 2 may choose r_1 or r_2 or r_3.
$(Sq)_3$: $2r_3 \prec_3 2r_2 \prec_3 2r_2 \rightsquigarrow$ player 3 choose only r_1.or r_3
$(Sq)_4$: $2r_2 \prec_4 2r_3 \prec_4 2r_1 \rightsquigarrow$ player 4 may choose r_1 or r_3.
$(Sq)_5$: $2r_1 \prec_5 2r_3 \prec_5 2r_2 \rightsquigarrow$ player 5 may choose r_2.
$(Sq)_6$: $2r_1 \prec_6 2r_2 \prec_6 2r_3 \rightsquigarrow$ player 6 may choose r_2.

We can build the next tree to detect all Nash equilibria:



We can read the above tree as follows:
Player 1 must choose the alternative r_3.
Player 2 has two possible choices r_1 and r_3.

If player 2 chooses r_1, player 3 has two possible choices r_1 and r_3.
If player 3 chooses r_1, player 4 have to choose r_3, player 5 must choose r_2.
Player 6 must choose only the alternative r_2.
Thus, the Nash equilibria in this example are:

$$\sigma_1 = (r_3, r_1, r_1, r_3, r_2, r_2)$$
$$\sigma_2 = (r_3, r_1, r_3, r_1, r_2, r_2)$$
$$\sigma_3 = (r_3, r_3, r_1, r_1, r_2, r_1)$$

3.5 Numerical experiments

In this section, we aim to conduct a series of experiments to investigate Nash equilibria in both ordinal and cardinal games. In the existing literature, cardinal disutility functions (cost functions) are typically used, where each player seeks to minimize their cost by selecting a specific strategy or resource. Specifically, we focus on cardinal disutility games with varying numbers of available resources: three resources (R_3) , four resources (R_4), five resources (R_5), and extending up to ten resources (R_{10}). For these experiments, we assume strict preferences, meaning each player ranks their strategies in a distinct, non-equal order of preference. The results of our experiments indicate that as the number of players increases, the PoA/PoS ratio also rises. For a player count of $n = 1000$, we observed that this ratio is approximately $PoA/PoS = 5$. This implies that the social cost of selecting the worst Nash equilibrium is 5 times greater than the social cost when players choose the best Nash equilibrium. Therefore, coordination among players may be necessary to significantly reduce this cost.

Although we cannot definitively verify that our algorithms operate in polynomial time, practical applications have demonstrated their efficiency in the cases we examined. For example, with players, the execution time remains reasonable, taking only a few seconds under strict preferences. We utilized the algorithms presented in paper of Khanchouche et al[30] to analyze and determine the best and worst Nash equilibria based on the evaluation criteria applied.

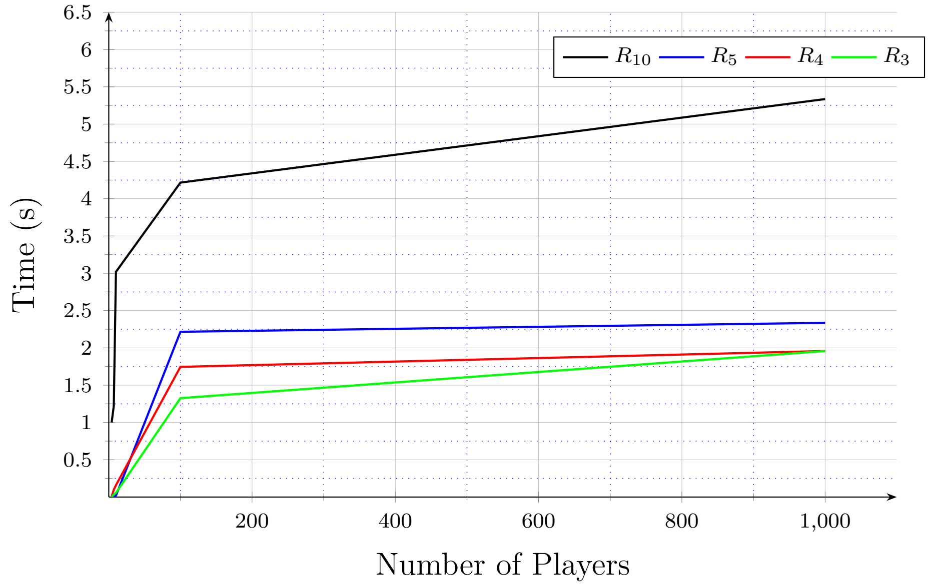


Figure 3.1: The time spent by above algorithms to find Nash equilibrium when the partition is exact.

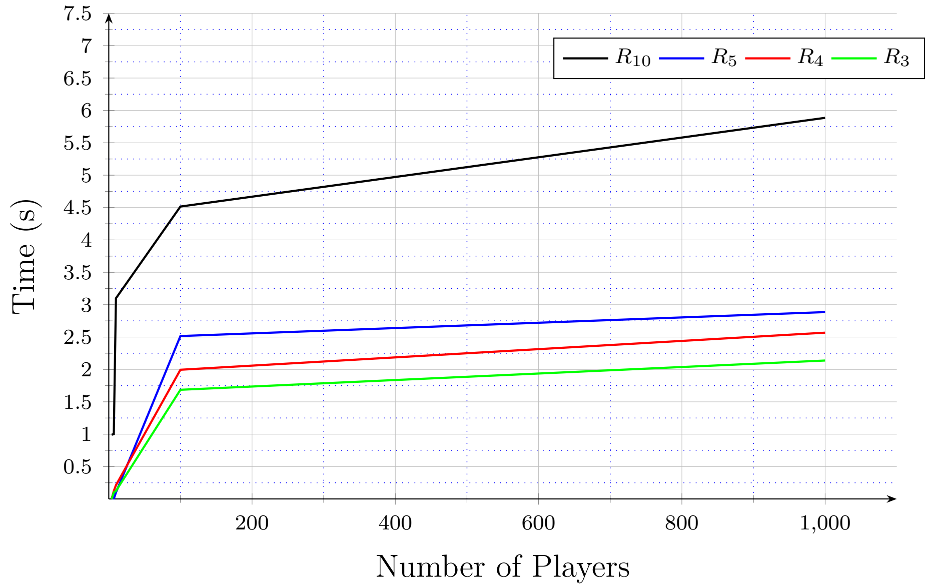


Figure 3.2: The time spent by above algorithms to find Nash equilibrium when the partition is non-exact.

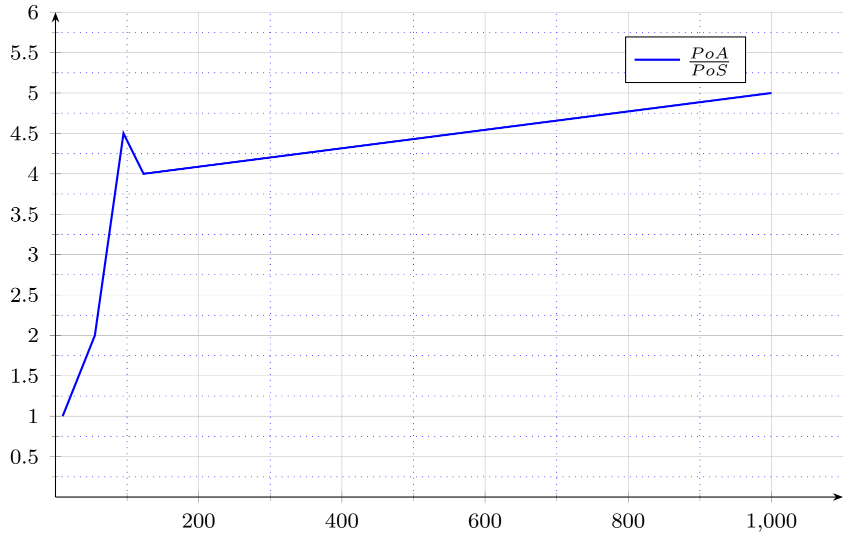


Figure 3.3: The quotient of the price of anarchy and the price of stability.

3.6 Conclusion

In this chapter, we explored non-symmetric singleton congestion games and proposed a novel approach based solely on ordinal preferences, without relying on potential functions or the finite improvement property. Through this method, we demonstrated the possibility of systematically identifying all Nash equilibria with high computational efficiency, paving the way for simpler and more effective algorithms.

We also developed an algorithm capable of determining all Nash equilibria by analyzing the stability of strategy profiles based on players relative preferences. This makes it a powerful tool for studying this class of games. These findings contribute to both the theoretical understanding and practical analysis of non-symmetric congestion games, providing a solid foundation for future research in the field.

Chapter 4

Strong Nash equilibrium in singleton congestion games

The concept of Nash Equilibrium is one of the fundamental pillars of game theory, representing a state in which no player has an incentive to unilaterally deviate from their strategy, assuming other players maintain theirs. However, this concept does not account for the possibility of cooperation among players. This is where the concept of strong Nash equilibrium emerges as a more robust and comprehensive extension, defined as a state where no group of players regardless of size can jointly deviate in a way that benefits all its members.

This concept holds particular significance in environments where coordination or coalition formation among players is feasible, such as in economic systems, networks, and competitive markets. Nevertheless, identifying and analyzing Strong Nash Equilibria poses significant challenges due to their computational complexity and limited existence in certain types of games.

In this chapter, we explore the theoretical foundations of strong Nash equilibrium, examining its properties, existence, and applications, with a focus on specific classes of games that allow for a systematic analysis of this equilibrium, such as singleton congestion games. We also highlight ongoing research efforts aimed at developing efficient algorithms to detect Strong Nash Equilibria in such contexts.

4.1 Introduction

The strong Nash equilibrium (SNE), introduced by Aumann [6], is a refinement of the classical Nash Equilibrium (NE) that accounts for group deviations. While NE ensures individual stability, SNE strengthens this by requiring that no coalition of players can jointly deviate to improve the payoff of every member. This added robustness makes SNE particularly relevant in strategic environments where collaboration is possible.

Moreover, SNE is closely tied to Pareto optimality, often resulting in more efficient outcomes compared to NE. This is especially significant in congestion games, where resource allocation directly impacts system performance. Despite its theoretical appeal, computing SNE is computationally complex classified as NP-complete and remains a challenging task in large or structured games.

This study builds on previous work Khanchouche et-al [30], which focused on classical Nash equilibria, by investigating the structure and computation of Strong Nash Equilibria in congestion games with player-specific payoff function, and by proposing efficient algorithmic solutions tailored to this setting.

4.2 Singleton congestion games with player non-specific payoff functions

Definition 4.2.0.1. A singleton congestion game is a congestion game where each player selects exactly one resource from a common set of resources.

If the game has player non-specific payoff functions, it means that the cost or payoff of each resource depends only on how many players choose it, not on who those players are.

Example 4.2.0.1. Let $G(N, R, (\prec)_{i \in N})$ be a singleton congestion game with player non-specific payoff functions where $R = \{r_1, r_2\}$ and $N = 3$. Suppose that players' preferences are given by the following weak orderings:

$$\begin{aligned} \text{Player}_1: &\quad 3r_2 \prec 2r_2 \prec r_2 \\ \text{Player}_2: &\quad 2r_1 \prec r_1 \prec r_2 \\ \text{Player}_3: &\quad 2r_1 \prec r_2 \prec r_1 \end{aligned}$$

The cost of resource r_1 is defined as follows:

$$\begin{aligned} &\text{cost} = 1 \text{ if one player chooses } r_1. \\ &\text{cost} = 3 \text{ if two players choose } r_1. \\ &\text{cost} = 5 \text{ if three players choose } r_1. \end{aligned}$$

The cost of resource r_2 is defined as follows:

$$\begin{aligned} &\text{cost} = 2 \text{ if one player chooses } r_2. \\ &\text{cost} = 4 \text{ if two players choose } r_2. \\ &\text{cost} = 5 \text{ if three players choose } r_2. \end{aligned}$$

All players are affected in the same way by the number of users (i.e., the payoff functions are player non-specific).

For example, if Player 1 and Player 2 choose resource r_1, and Player 3 chooses resource r_2, then:

Player 1 and Player 2 each pay a cost of 3

Player 3 pays a cost of 2

4.2.1 Existence of SNE in singleton congestion game with player non-specific payoff function

Congestion games are among the fundamental models in game theory, where players compete over limited resources whose usage levels impact the payoff received by each player. Singleton congestion games represent a particular subclass of these games, in which each player selects exactly one resource. This model becomes especially relevant when considering the case of player non-specific payoff functions, where the congestion effect on the payoff is identical for all players, regardless of their identity. Such a simplification provides a tractable framework for analyzing Nash equilibria and enables the development of efficient algorithms for equilibrium computation. Notably, the work of Holzman and Law-yone on this class of games demonstrates that a strong Nash equilibrium always exists and, in this specific setting, coincides with the classical Nash equilibrium.

Theorem 4.2.1. Holzman and Low-yone[27] Let G be a monotone symmetric cogestion game in which all strategies are singletons. Then $SE(G) = NE(G)$

Proof 3. We aim to prove that every Nash equilibrium in the game is also a strong equilibrium, i.e.,

$$\mathrm{NE}(G) \subseteq \mathrm{SE}(G)$$

Assume that σ^* is a Nash equilibrium for game G. Suppose, for contradiction, that there exists a coalition $C \subseteq N$, with $|C| \neq \emptyset$, that can achieve a strictly better outcome by jointly deviating to a strategy profile y_C, while the rest of the players maintain their strategies in σ^{-S}.

We claim that the congestion vectors V_r associated with the profiles σ^* and (y_C, σ_{-C}) are identical. If not, then:

$$\sum_{r \in M} V_r(\sigma) = n = \sum_{r \in M} V_r(y_C, \sigma_{-C})$$

This implies that there exists a resource $r \in M$ such that:

$$V_r(y_c, \sigma_{-c}) \geq V_r(\sigma) + 1$$

Let $i \in C$ be a player for which $r \in \sigma_y \setminus \sigma^*$, i.e., player i is the cause of the increased congestion on resource r.

Since σ^* is a Nash Equilibrium and the utility function u_r is non-increasing in the number of users (due to monotonicity), the utility of player i after the deviation is:

$$u_r(V_r(\sigma_y, \sigma_{-C})) \leq u_r(V_r(\sigma) + 1)$$

This means that player i does not benefit from the deviation and might even be worse off, contradicting the assumption that the coalition C benefits from the joint deviation.

Therefore, the congestion vectors must be equal:

$$V(\sigma) = V(\sigma_y, \sigma_{-C})$$

This implies that the deviation is merely a reshuffling of the strategies among the players in C, without changing the congestion. Hence, all players in C experience the same utilities as before, and no one strictly benefits.

Thus, every Nash Equilibrium is also a Strong Equilibrium.

Example 4.2.1.1. Let G be a singleton congestion game with player non-specific function, defined as follows:

- **Players:** Two players P_1 and P_2.
- **Resources:** Two resources r_1 and r_2.
- **Strategies:** Each player selects exactly one resource (singleton strategy).
- **Cost functions (non-player-specific):**

$$c_{r_1}(1) = 1, \quad c_{r_1}(2) = 3$$
$$c_{r_2}(1) = 2, \quad c_{r_2}(2) = 4$$

The cost functions depend solely on the number of players using the resource, not on player identity, thus they are non-specific.

P_1 selects	P_2 selects	Costs	Notes
r_1	r_1	(3, 3)	Both players choose r_1
r_2	r_2	(4, 4)	Both players choose r_2
r_1	r_2	(1, 2)	Optimal and efficient case
r_2	r_1	(2, 1)	Optimal and efficient case

We see that the profiles (r_1, r_2) and (r_2, r_1) are pure strategy Nash equilibria. No player has an incentive to deviate unilaterally. It is straightforward the same profiles(r_1, r_2) and (r_2, r_1) are also strong Nash equilibria, as no coalition of players can jointly deviate in a way that benefits all members. In this example, the strong Nash equilibria coincide with the Nash Equilibria due to the alignment of individual and collective optimality in the context of player non-specific cost functions.

4.3 The results: computation of strong Nash equilibrium in singleton congestion games with player specific payoff function

To clarify the results we have obtained, we will use some of the sets previously introduced in the paper [30], such that $N = \{1, ..., i, ..., n\}$, and $R = \{r_1, r_2\}$.

Case 1: Strict order of preferences

$$b_i = \max_{b \in \{0,1,\ldots,n\}} \{b : (r_1, b) \succ_i (r_2, n+1-b)\}$$

where, the integer b_i represents the largest number of players selecting the alternative r_1, in which the player i can belong. If this size is exceeded, the player i will choose the resource r_2. Indeed, by definition, we have $b_i \cdot r_1 \succ_i (n+1-b_i) \cdot r_2$ and $(b_i + 1) \cdot r_1 \prec_i (n - b_i) \cdot r_2$.

$$d_i = \max_{d \in \{0,1,\ldots,n\}} \{d : (r_2, d) \succ_i (r_1, n+1-d)\}$$

The integer d_i is interpreted in the same way as b_i; we replace r_1 by r_2. We mention that $b_i + d_i = n$, $\forall i \in N$.

$$n(r_1) = \max_{b \in \{0,1,\ldots,n\}} \{b : |\{i \in N : b_i \geq b\}| \geq b\}$$

Note that $n(r_1)$ refers to the maximum number of players that can select the resource r_1 with no member of this group wishing to deviate from their strategy. The latter will serve to identify the congestion vector that may correspond to a Nash equilibrium of the game.

$$n(r_2) = \max_{d \in \{0,1,\ldots,n\}} \{d : |\{i \in N : d_i \geq d\}| \geq d\}$$

(It has the same definition as $n(r_1)$, except that r_1 is replaced by r_2. In this case, we necessarily have $n(r_1) + n(r_2) = n$. To describe the set of all SNe, for each congestion vector $v = (x, y)$ satisfying $x = n(r_1)$ and $y = n(r_2)$, we introduce the three following three sets:

$$\begin{aligned} E(G, v(x, y)) &= \{i \in N : b_i > x\} \\ F(G, v(x, y)) &= \{i \in N : b_i < x\}, \\ H(G, v(x, y) &= \{i \in N : b_i = x\} \end{aligned}$$

<u>Case 2: Order of preferences with ties</u>

$$\begin{aligned} b_i &= \max_{b \in \{0,1,\ldots,n\}} \{b : (r_1, b) \succsim_i (r_2, n + 1 - b)\} \\ d_i &= \max_{d \in \{0,1,\ldots,n\}} \{d : (r_2, d) \succsim_i (r_1, n + 1 - d)\} \end{aligned}$$

Where b_i and d_i, $\forall i \in N$, have the same meaning as in the above case. We get $b_i + d_i = n$ because of the possible presence of ties. Hence, $b_i + d_i \geq n$, for all $i \in N$. we define $n(r_1)$ and $n(r_2)$ in the same way as in case 1. In order to describe the set of all SNE, for each congestion vector $v = (x, y)$ satisfying $x \leq n(r_1)$, $y \leq n(r_2)$ and $x + y = n$, we introduce the three following three sets:

$$\begin{aligned} E(G, v(x, y)) &= \{i \in N : b_i \geq x \text{ and } d_i < y\} \\ F(G, v(x, y)) &= \{i \in N : b_i < x \text{ and } d_i \geq y\}, \\ H(G, v(x, y) &= \{i \in N : b_i \geq x \text{ and } d_i \geq y\} \end{aligned}$$

Now, we define two sets that will aid in the analysis of our new results:

$$\begin{aligned} A(r_1) &= \{i \in N : x.r_1 \succ_i y.r_2\} \\ A(r_2) &= \{i \in N : x.r_1 \prec_i y.r_2\} \end{aligned}$$

Theorem 4.3.1. Let $G(N, R, \prec_i)$ be a singleton congestion game with player specific payoff function, with $R = \{r_1, r_2\}$, the set of strong Nash equilibria satisfies the following conditions:

- if $A(r_1) = x$ then there exists a unique strong Nash equilibrium belonging to the set of Nash equilibria, i.e., $\exists! SNE \subseteq NE$
- if $A(r_1) \neq x$ the set of SNE is identical to the set of Nash equilibria, i.e., $SNE = NE$

To clarify the previous theorems, we introduce a proposition to determine how to find strong Nash equilibria.

Proposition 4.3.1. Every strong Nash equilibria $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ of G associated with the vector $v = (x, y)$, is characterized by a subset $D \subseteq H(G, v(x, y))$, where $D = H(G, v(x, y)) \cap A(r_1)$ which may be empty, such that: $|D| \leq x - |A(r_1)|$ and a subset $W \subseteq H(G, v(x, y))$, where $W = H(G, v(x, y)) \cap A(r_2))$ which may be empty, such that: $|W| \leq y - |A(r_2)|$ to ensure that:
For all $i \in N$

- $\sigma_i^* = r_1$ if $i \in (E \cup D \cup H \setminus W)$
- $\sigma_i^* = r_2$ if $i \in (F \cup W \cup H \setminus D))$.

Proof of theorem 4.3.1

1)By definition of $A(r_1)$, We necessarily have that:
$u_i(r_1) > u_i(r_2)$, $\forall i \in A(r_1)$ and as defined, x refer to the maximum size of a group of players that can choose the resource r_1 without any member of this group having interest in deviating from his strategy. Since we have $x + y = n$, hence we must find $y = |A(r_2)|$ and by definition of $A(r_2)$ we get that: $u_i(r_2) > u_i(r_1)$, $\forall i \in A(r_2)$, as defined, y refer to the maximum size of a group of players that can choose the resource r_2 without any member of this group having interest in deviating from his strategy. Hence, no group of players can consider deviating from another resource, as their payoffs will surely decrease. Therefore, there will be a unique strong Nash equilibrium belonging to the set of Nash equilibria, where all players in $A(r_1)$ choose r_1, and all players in $A(r_2)$ choose r_2.
2) Since $x \neq |A(r_1)|$, it follows that:

$$x > |A(r_1)| \Rightarrow y < |A(r_2)|$$

or

$$x < |A(r_1)| \Rightarrow y > |A(r_2)|$$

Given that $x + y = n$, this implies that at least k players will necessarily be compelled to select a different resource.

Meanwhile, we have:

$$xr_1 \succ_i yr_2, \quad \forall i \in A(r_1)$$

and also,

$$xr_1 \prec_i yr_2, \quad \forall i \in A(r_2).$$

For these k players, we obtain:

$$x \cdot r_1 \succsim_j (y+1) \cdot r_2, \quad \forall j \notin A(r_1).$$

Using the same approach, we derive the result for the case where $y \neq A(r_2)$.

Since no player in $A(r_1)$ can deviate to another choice without reducing their payoff (by definition of $A(r_1)$), it follows that every Strong Nash Equilibrium coincides with a Nash Equilibrium.

[***Proof of Proposition 4.3.1***]

Let D be a subset where: $D = A(r_1) \cap H$ and let W be a subset where: $W = A(r_2) \cap H$, by definifions of $A(r_1)$ and $A(r_2)$, we nessecarily have: $xr_1 \succ_i yr_2\ \forall i \in D$ and $yr_2 \succ_i xr_1\ \forall i \in W$. By definition of $E(G, v(x, y))$ and $H(G, v(x, y))$ we have $b_i \geq n(r_1)$ and by the assumption of monotonicity, we find: $xr_1 \succsim_i (y+1)r_2\ \forall i \in E$. Similarly, we show that for all i in $F(G, v(x, y)) \cup (H(G, v(x, y)),\ y \cdot r_2 \succsim_i (x+1) \cdot r_1$. Let $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ be the strategy profile defined by: For all $i \in N$, $\sigma_i^* = r_1$ if $i \in E(G, v(x, y)) \cup H(G, v) \cup D \backslash W$ and $\sigma_i^* = r_2$ if $i \in F(G, v(x, y)) \cup H(G, v) \cup W \backslash D)$. Based on the previous results, we conclude that σ^* is a Strong Nash equilibrium, meaning that no group of players can jointly deviate in a way that benefits all its members. Reciprocally, $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ be a strong Nash equilibrium of G, (by proof of theorem 1) $v = (x, y)$ is the congestion vector associated with this equilibrium, where $(x, y) = (n(r_1), n(r_2))$. By definition of a congestion vector, we also have $x + y = n$. As σ^* is a strong Nash equilibrium, for any $i \in N$, we must have: $\sigma_i^* = r_1$ if $i \in (E \cup D \cup H \setminus W)$. $\sigma_i^* = r_2$ if $i \in (F \cup W \cup H \setminus D)$.

4.4 Numerical experiments

This section aims to provide a detailed explanation and theoretical justification of the results established in this study. To achieve this, we propose two algorithms, Algorithm 1 and Algorithm 2, designed to compute all strong Nash equilibria and identify Pareto-optimum outcomes, followed by a comprehensive comparison between them.

Both algorithms are grounded in fundamental game-theoretic principles in this specific class of games. The first algorithm identifies all strong Nash equilibria, ensuring that no coalition of players can collectively deviate to improve their outcomes without harming at least one member. In contrast, the second algorithm seeks to determine Pareto-optimum outcomes.

By analyzing the results obtained from these algorithms, we conduct an in-depth comparison between strong Nash equilibria and Pareto-optimum outcomes to examine the interplay between collective stability and social efficiency. This analysis provides insights into the extent to which these two solution concepts align or diverge within the studied applications.

Algorithm 1: Find all strong Nash equilibria

Input: Congestion game $\mathcal{G} = (N, R, \precsim_i)$, cardinal utilities of each players
Let $i := 1$
    Increase i by one
    Find p_i, q_i, $n(a)$, $n(b)$
    If $(n(a) <_i n(b))$, then $i \in S(a)$
    If $(n(a) >_i n(b))$, then $i \in S(b)$
    If $|S(a)| = n(a)$, then $\exists! \sigma^*$
    Else
    Find $A(G), B(G), C(G)$
    End if
Output: all strong Nash equilibria.

Algorithm 2: Find outcomes Pareto-optimum

Input: Congestion game $\mathcal{G} = (N, R, \precsim_i)$, cardinal utilities of each players
Let $i := 1$
repeat
    Increase i by one
    Find the Pareto optima
until $i = N$
Output: Pareto-optimum outcomes.

Example 4.4.0.1. Let $G(N, R, (\prec)_{i\in N})$ be a singleton congestion game, where: $N = 6$ players and $R = 2$ resources. Suppose that players' preferences are given by the following weak orderings:

$$P_1: \quad 6r_2 \prec 5r_2 \prec 4r_2 \prec 6r_1 \prec 5r_1 \prec 3r_2 \prec 2r_2 \prec 4r_1 \prec r_2 \prec 3r_1 \prec 2r_1 \prec r_1$$

$$P_2: \quad 6r_1 \prec 6r_2 \prec 5r_2 \prec 4r_2 \prec 5r_1 \prec 4r_1 \prec 3r_1 \prec 2r_1 \prec 3r_2 \prec r_1 \prec 2r_2 \prec r_2$$

$$P_3: \quad 6r_2 \prec 6r_1 \prec 5r_1 \prec 4r_1 \prec 3r_1 \prec 2r_1 \prec 5r_2 \prec r_1 \prec 4r_2 \prec 3r_2 \prec 2r_2 \prec r_2$$

$$P_4: \quad 6r_1 \prec 6r_2 \prec 5r_2 \prec 4r_2 \prec 5r_1 \prec 4r_1 \prec 3r_2 \prec 2r_2 \prec r_2 \prec 3r_1 \prec 2r_1 \prec r_1$$

$$P_5: \quad 6r_1 \prec 6r_2 \prec 5r_2 \prec 4r_2 \prec 5r_1 \prec 4r_1 \prec 3r_1 \prec 2r_1 \prec r_1 \prec 3r_2 \prec 2r_2 \prec r_2$$

$$P_6: \quad 6r_2 \prec 5r_2 \prec 4r_2 \prec 6r_1 \prec 5r_1 \prec 4r_1 \prec 3r_2 \prec 3r_1 \prec 2r_1 \prec 2r_2 \prec r_1 \prec r_2$$

Based on the aforementioned groups, we can conclude that:

$$\begin{aligned} b_1 &= 4, d_1 = 2 \\ b_2 &= 3, d_2 = 3 \\ b_3 &= 1, d_3 = 5 \\ b_4 &= 3, d_4 = 3 \\ b_5 &= 3, d_5 = 3 \\ b_6 &= 3, d_6 = 3 \end{aligned}$$

So, we can verify that:

$$n(r_1) = 3 \ , \ n(r_2) = 3.$$

According to theorem (2.5.1), we noted that the only congestion vector corresponding to a Nash equilibrium is:
$(x, y) = (3r_1, 3r_2)$. Furthermore, we have:

$$E(G(x, y)) = \{4\}$$

$$F(G, (x, y)) = \{3\}$$

$$H(G, v(x, y)) = \{1, 2, 5, 6\}$$

The list of Nash equilibria of this game are:

$$\sigma_1 = (r_1, r_1, r_2, r_1, r_2, r_2), \sigma_2 = (r_1, r_2, r_2, r_1, r_1, r_2), \sigma_3 = (r_1, r_2, r_2, r_1, r_2, r_1),$$
$$\sigma_4 = (r_2, r_2, r_2, r_1, r_1, r_1), \sigma_5 = (r_2, r_1, r_2, r_1, r_1, r_2), \sigma_6 = (r_2, r_1, r_2, r_1, r_2, r_1).$$

We now aim to identify the strong Nash equilibrium.
We have:

$$A(r_1) = \{1, 4, 6\}, \text{ i.e., } |A(r_1)| = x.$$

Therefore, the unique strong Nash equilibrium in this game is given by $(r_1, r_2, r_2, r_1, r_2, r_1)$.

Example 4.4.0.2. Let $G(N, R, (\prec)_{i \in N})$ be a singleton congestion game, where: $N = 5$ players and $R = 2$ resources. Suppose that players' preferences are given by the following weak orderings:

$$\text{P}_1: \quad 5r_2 \prec 4r_2 \prec 5r_1 \prec 4r_1 \prec 3r_2 \prec 3r_1 \prec 2r_2 \prec r_2 \prec 2r_1 \prec r_1$$

$$\text{P}_2: \quad 5r_2 \prec 4r_2 \prec 5r_1 \prec 3r_2 \prec 4r_1 \prec 3r_1 \prec 2r_1 \prec 2r_2 \prec r_1 \prec r_2$$

$$\text{P}_3: \quad 5r_1 \prec 4r_1 \prec 5r_2 \prec 4r_2 \prec 3r_1 \prec 2r_1 \prec r_1 \prec 3r_2 \prec 2r_2 \prec r_2$$

$$\text{P}_4: \quad 5r_2 \prec 4r_2 \prec 5r_1 \prec 3r_1 \prec 2r_2 \prec 4r_1 \prec 3r_1 \prec 2r_1 \prec r_2 \prec r_1$$

$$\text{P}_5: \quad 5r_2 \prec 4r_2 \prec 5r_1 \prec 4r_1 \prec 3r_2 \prec 3r_1 \prec 2r_1 \prec 2r_2 \prec r_2 \prec r_1$$

It easy to find:

$$(n(r_1), n(r_2)) = (x, y) = (3, 2)$$
$$E(G(x, y)) = \{4\}$$
$$F(G, (x, y)) = \{3\}$$
$$H(G, v(x, y)) = \{1, 2, 5\}$$

The list of Nash equilibria are:

$$\sigma_1 = (r_2, r_1, r_2, r_1, r_1)$$
$$\sigma_2 = (r_1, r_2, r_2, r_1, r_1)$$
$$\sigma_3 = (r_1, r_1, r_2, r_1, r_2)$$

We have:

$$A(r_1) = \{\emptyset\}, A(r_2) = \{1, 2, 5\}$$

As $|A(r_1)| \neq x$, $SNE = NE$.

The list of strong Nash equilibria are:

$$\sigma_1 = (r_2, r_1, r_2, r_1, r_1)$$
$$\sigma_2 = (r_1, r_2, r_2, r_1, r_1)$$
$$\sigma_3 = (r_1, r_1, r_2, r_1, r_2)$$

Example 4.4.0.3. Let $G(N, R, (\precsim)_{i \in N})$ be a singleton congestion game, where: $N = 5$ players and $R = 2$ resources. Suppose that players' preferences are given by the following weak orderings:

$$\text{P}_1: \quad 5r_1 \sim 4r_1 \sim 3r_1 \sim 2r_1 \sim 1r_1 \sim 5r_2 \sim 4r_2 \sim 3r_2 \sim 2r_2 \sim r_2$$

$$\text{P}_2: \quad 5r_1 \prec 5r_2 \prec 4r_2 \prec 3r_2 \sim 2r_2 \sim 4r_1 \sim 3r_1 \prec 2r_1 \prec r_1 \prec r_2$$

$$\text{P}_3: \quad 5r_2 \prec 4r_2 \prec 5r_1 \prec 4r_1 \sim 3r_1 \sim 3r_2 \sim 2r_2 \prec 2r_1 \prec r_2 \prec r_1$$

$$\text{P}_4: \quad 5r_2 \sim 4r_2 \sim 5r_1 \sim 4r_1 \sim 3r_2 \sim 3r_1 \sim 2r_1 \sim 2r_2 \sim r_1 \sim r_2$$

$$\text{P}_5: \quad 5r_1 \prec 5r_2 \prec 4r_2 \prec 4r_1 \prec 3r_1 \sim 2r_1 \sim 3r_2 \prec 2r_2 \sim r_1 \prec r_2$$

Based on the aforementioned groups, we can conclude that:

$$b_1 = 5,\ d_1 = 5$$
$$b_2 = 4,\ d_2 = 3$$
$$b_3 = 4,\ d_3 = 3$$
$$b_4 = 5,\ d_4 = 5$$
$$b_5 = 3,\ d_5 = 3$$

Hence,

$$n(r_1) = 4,\ n(r_2) = 3$$

According to theorem (2.5.2), the possible congestion vectors are:

$$v_1(x, y) = (3r_1, 2r_2)$$
$$v_2(x, y) = (2r_1, 3r_2)$$
$$v_3 = (x, y) = (4r_1, r_2)$$

Since, $v_1(x, y) = (3r_1, 2r_2)$ we have:

$$E(G, v_1(x, y)) = \{\emptyset\}$$
$$F(G, v_1(x, y)) = \{\emptyset\}$$
$$H(G, v_1(x, y)) = \{1, 2, 3, 4, 5\}$$

The list of Nash equilibria are:

$$\sigma_1 = (r_2, r_1, r_1, r_1, r_2), \sigma_2 = (r_1, r_2, r_1, r_1, r_2), \sigma_3 = (r_1, r_1, r_2, r_1, r_2),$$
$$\sigma_4 = (r_1, r_1, r_1, r_2, r_2), \sigma_5 = (r_2, r_1, r_1, r_2, r_1), \sigma_6 = (r_1, r_2, r_1, r_2, r_1),$$
$$\sigma_7 = (r_1, r_1, r_2, r_2, r_1), \sigma_8 = (r_2, r_1, r_2, r_1, r_1), \sigma_9 = (r_2, r_2, r_1, r_1, r_1),$$
$$\sigma_{10} = (r_1, r_2, r_2, r_1, r_1).$$

We have: $A(r_1) = \{\emptyset\}, A(r_2) = \{5\}$.
As $|A(r_1)| \neq x$, $SNE = NE$.
The list of strong Nash equilibria corresponding to $v_1(x, y)$ are:

$$\sigma_1 = (r_2, r_1, r_1, r_1, r_2), \sigma_2 = (r_1, r_2, r_1, r_1, r_2), \sigma_3 = (r_1, r_1, r_2, r_1, r_2),$$
$$\sigma_4 = (r_1, r_1, r_1, r_2, r_2), \sigma_5 = (r_2, r_1, r_1, r_2, r_1), \sigma_6 = (r_1, r_2, r_1, r_2, r_1),$$
$$\sigma_7 = (r_1, r_1, r_2, r_2, r_1), \sigma_8 = (r_2, r_1, r_2, r_1, r_1), \sigma_9 = (r_2, r_2, r_1, r_1, r_1),$$
$$\sigma_{10} = (r_1, r_2, r_2, r_1, r_1).$$

Since, for $v_2(x, y) = (2r_1, 3r_2)$, we have:

$$E(G, v_2(x, y)) = \{\emptyset\}$$
$$F(G, v_2(x, y)) = \{\emptyset\}$$
$$H(G, v_2(x, y)) = \{1; 2, 3, 4, 5\}$$

The list of Nash equilibria are:

$$\sigma_1 = (r_2, r_2, r_1, r_1, r_2), \sigma_2 = (r_2, r_1, r_2, r_1, r_2), \sigma_3 = (r_2, r_1, r_1, r_2, r_2),$$
$$\sigma_4 = (r_1, r_1, r_2, r_2, r_2), \sigma_5 = (r_2, r_2, r_2, r_1, r_1), \sigma_6 = (r_1, r_2, r_2, r_2, r_1),$$
$$\sigma_7 = (r_1, r_2, r_2, r_1, r_2), \sigma_8 = (r_1, r_2, r_1, r_2, r_2), \sigma_9 = (r_2, r_2, r_1, r_2, r_1),$$
$$\sigma_{10} = (r_2, r_1, r_2, r_2, r_1).$$

We have: $A(r_1) = \{2, 3\}, A(r_2) = \{\emptyset\}$, As $|A(r_1)| = x$, $\exists!\ SNE$.
The unique strong Nash equilibrium corresponding to $v_2(x, y)$ is:

$$\sigma_1 = (r_2, r_1, r_1, r_2, r_2).$$

Finally, for $v_3(x, y) = (4r_1, r_2)$, we have:

$$E(G, v_3(x, y)) = \{\emptyset\}$$
$$F(G, v_3(x, y)) = \{5\}$$
$$H(G, v_3(x, y)) = \{1, 2, 3, 4\},\ A(r_1) = \{\emptyset\}, A(r_2) = \{2, 3, 5\}$$

The unique Nash equilibrium corresponding to $v_3(x, y)$ is:

$$\sigma_1 = (r_1, r_1, r_1, r_1, r_2)$$

We have:

$$A(r_1) = \{\emptyset\}, A(r_2) = \{2, 3, 5\}$$

As $|A(r_1)| \neq x$, $SNE = NE$.
The unique strong Nash equilibrium corresponding to $v_3(x, y)$ is:

$$\sigma_1 = (r_1, r_1, r_1, r_1, r_2).$$

The following graph illustrate some of the obtained results:

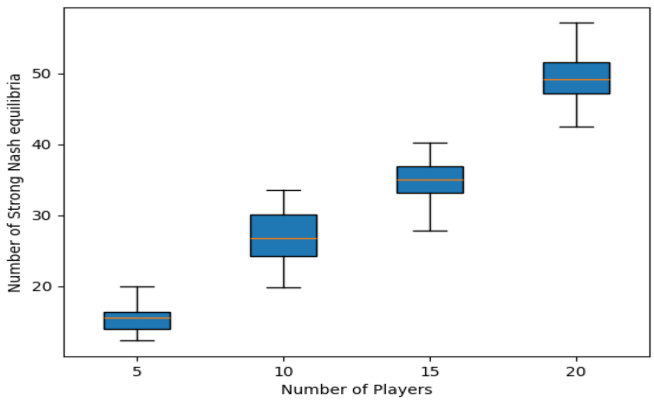


Figure 4.1: The convergence of strong Nash equilibria.

Each time we change the input data, new values are obtained, leading to variations in the number of strong Nash equilibria and Pareto-optimal outcomes, even when the number of players remains the same.

This is clearly illustrated in the figure above, where:
When the number of players is 5, the number of equilibria converges to approximately 12 strong equilibria. When the number of players is 10, the number of strong equilibria converges to approximately 30. When the number of players is 15, the number of strong equilibria converges to approximately 35. When the number of players is 20, the number of strong equilibria converges to approximately 50.

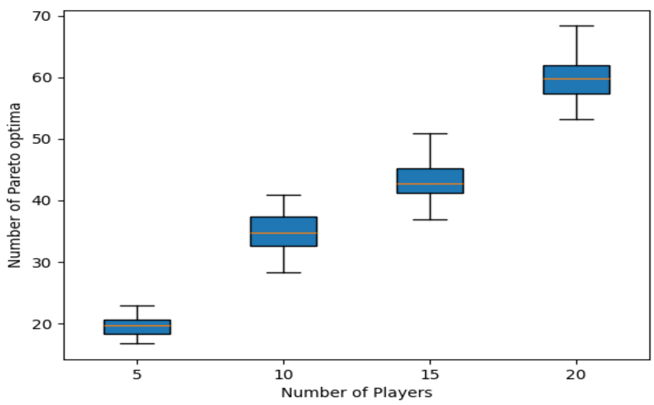


Figure 4.2: The convergence of Pareto optima.

4.5 Conclusion

In this chapter, we addressed the existence of strong Nash equilibrium in singleton congestion games with player non-specific payoff functions. We reviewed the foundational results established by Holzman and Low-yone [27], which demonstrated the existence of strong equilibria under specific conditions particularly in the context of monotone games, thus providing a solid theoretical basis for analyzing collective stability in competitive environments.

Building upon this theoretical foundation, we introduced a novel framework for characterizing multiple strong Nash equilibria in non-symmetric congestion games, focusing on scenarios involving exactly two resources. Our approach leveraged both ordinal and cardinal representations of player preferences, enabling a more flexible and realistic analysis of strategic behavior.

Moreover, we translated this theoretical framework into practical tools by developing efficient algorithmic implementations. These algorithms proved effective in accurately identifying strong Nash equilibria and computing Pareto-optimal outcomes. The integration of theory with practical computation highlights the applicability of our results and offers new directions for analyzing equilibrium structures in more complex congestion models.

General Conclusion

The study of congestion games and their applications constitutes a highly active field of research today. This domain was initiated through the foundational contributions of Rosenthal, Konishi, Le Breton, Weber, and Milchtaich, and continues to attract significant scholarly interest. Within this framework, our research focused specifically on a particular class of congestion games, namely, non-symmetric singleton congestion games. The primary objective of this dissertation was to contribute to the study and analysis of both classical and strong Nash equilibria within these games. It is important to note that our work does not claim to exhaustively address all facets of the subject, nor to provide definitive answers to the complex issues involved. Nonetheless, we strived to go beyond a mere synthesis of existing literature by proposing original insights and solutions wherever possible. The main contributions of this dissertation can be summarized in three key points:

Our first contribution involved a thorough re-examination of the results obtained by Milchtaich [37] concerning non-symmetric singleton congestion games. In the specific case involving only two resources, our analysis was comprehensive: we successfully characterized all possible Nash equilibria and proposed explicit formulas to describe and categorize these equilibria systematically.

Beyond this, we developed efficient algorithms capable of computing all possible Nash equilibria within this framework. These algorithms enabled an exhaustive exploration of the solution space, identifying both optimal and suboptimal equilibria from a social welfare perspective.

Leveraging these computational tools, we were able to precisely determine the gap between the price of anarchy and the price of stability. This quantitative analysis provided key insights into the potential degradation of system efficiency caused by decentralized, self-interested behavior (leading to the worst-case equilibria) as compared to the best achievable decentralized outcomes.

Our findings offer a deeper understanding of how asymmetries among players affect overall system performance and highlight how mechanism designers can exploit the structure of possible equilibria to minimize efficiency loss. Consequently, our work goes beyond theoretical characterization, offering practical tools for evaluating and improving social performance in non-symmetric congestion games.

Our second contribution lies in the study of a specialized class of singleton congestion games characterized by structured resource partitions. We demonstrated that, under suitable conditions, it is possible to identify not only a single Nash equilibrium but multiple distinct equilibria. Through a refined structural analysis, we developed methods that enable the systematic computation of these equilibria.

To operationalize our theoretical results, we proposed efficient algorithms designed to enumerate all Nash equilibria in these games, whenever feasible. These computational tools allowed us to conduct a detailed evaluation of system performance by explicitly calculating the price of anarchy and the price of stability, thus providing a quantitative measure of the efficiency loss due to decentralized, self-interested behavior.

Overall, our work bridges the gap between theoretical existence proofs and practical computation of equilibria, while offering important insights into the social efficiency of congestion games with structured partitions.

Our final contribution focuses on the study and computation of strong Nash equilibria in singleton congestion games in the case of two resources. Strong Nash equilibria, which are more robust than classical Nash equilibria by accounting for potential deviations by coalitions of players, represent a particularly relevant solution concept in many practical contexts where collective behavior can arise.

In this framework, we successfully characterized the structure of strong Nash equilibria when players choose between two resources, providing a complete description of the equilibrium profiles. Furthermore, we developed specialized algorithms capable of efficiently computing all strong Nash equilibria in this setting.

In addition to identifying these equilibria, we also addressed the problem of social efficiency by computing Pareto-optimal outcomes among the set of equilibria. Our algorithms allow not only the detection of strong Nash equilibria but also the selection of those equilibria that maximize the collective welfare of the players.

Through these developments, we extended the analysis of singleton congestion games beyond individual stability to coalition-proof stability, enriching the theoretical understanding of these games and offering practical computational tools for real-world applications.

Despite the significant contributions achieved through this research in addressing several of the questions initially posed, numerous aspects remain open and require further investigation. Our study has demonstrated that, in specific cases, the analysis of Nash equilibria can be comprehensive and precise. However, generalizing these results to more complex frameworks still demands considerable theoretical effort. In this regard, future research is encouraged to address these challenges by developing new analytical tools capable of overcoming the current limitations of the studied models.

Moreover, all the results presented in this dissertation are restricted to the framework of singleton congestion games, where players are limited to selecting only one resource among multiple alternatives. Although this assumption greatly simplifies the analysis, many real-world applications require considering more complex models in which players may select multiple resources or form combinations of resources.

Accordingly, our future research aims to extend the analysis to standard congestion games, where players are no longer restricted to a single choice. Identifying all Nash equilibria in such settings constitutes a significant theoretical challenge due to the increased complexity of the strategy space and the difficulty of characterizing the possible equilibrium structures. Addressing this challenge will require the development of novel analytical methods and the design of more efficient computational algorithms capable of handling the high combinatorial complexity inherent in these models, potentially opening new avenues for applications in economics, transportation, network management, and decentralized systems.

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العنوان: ألعاب الازدحام وتطبيقاتها، توازن ناش وسعر الفوضى .

ملخص: تتناول هذه الأطروحة ألعاب الازدحام أحادية المورد في الحالة غير المتماثلة، حيث اقترحنا صيغًا تحليلية مبسطة تسمح بوصف دقيق لمجموعات التوازنات الممكنة، بما في ذلك توازن ناش الكلاسيكي وتوازن ناش القوي في حالة موردين. علاوة على ذلك، طورنا منهجية عامة لحساب توازن ناش في الحالة العامة، مدعومة بخوارزميات عملية وفعالة. وقد مكنت هذه الخوارزميات من تقييم أداء الحلول المقترحة وحساب الفرق بين سعر الفوضى وسعر الاستقرار، مما يبرز القيمة النظرية والتطبيقية لهذا العمل.

كلمات مفتاحية: ألعاب الازدحام الأحادية، توازن ناش الكلاسيكي، توازن ناش القوي، سعر الفوضى، سعر الاستقرار.

Title: CONGESTION GAMES AND APPLICATIONS, NASH EQUILIBRIUM AND ANARCHY PRICE.

Abstract: This thesis examines singleton congestion games in the non-symmetric case. We proposed simplified analytical formulations that allow for an accurate description of the possible sets of equilibria, including the classical Nash equilibrium and the strong Nash equilibrium in the case of two resources. Furthermore, we developed a general methodology for computing the Nash equilibrium in the general case, supported by practical and efficient algorithms. These algorithms enabled the evaluation of the proposed solutions' performance and the calculation of the difference between the Price of Anarchy and the Price of Stability, thereby highlighting the theoretical and practical value of this work.

Keywords: Singleton congestion games, Classical Nash equilibrium, Strong Nash equilibrium, Price of Anarchy, Price of Stability.

Titre : JEUX DE CONGESTION ET APPLICATIONS, ÉQUILIBRE DE NASH ET PRIX DE L'ANARCHIE.

Résumé : Cette thèse examine les jeux de congestion à choix unique dans le cas non-symétrique. Nous avons proposé des formulations analytiques simples permettant une description précise des ensembles d'équilibres possibles, notamment l'équilibre de Nash classique et l'équilibre de Nash fort, dans le cas de deux ressources. Par la suite, nous avons développé une méthodologie générale pour le calcul de l'équilibre de Nash classique dans le cas général, accompagnée d'algorithmes pratiques et efficaces. Ces algorithmes ont permis d'évaluer la performance des approches proposées et de calculer la différence entre le prix de l'anarchie et le prix de la stabilité, mettant ainsi en évidence l'apport théorique et pratique de ce travail.

Mots-clés : Jeux de congestion à choix unique, Équilibre de Nash classique, Équilibre de Nash fort, Prix de l'anarchie, Prix de la stabilité.