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A Study of Various Free Boundary Value Problems in Fluid Mechanics : The Asymptotic Behavior of Weak Solution

Presented by

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Dedication

*I dedicate this work to my **parents**, whose guidance and unwavering belief in me have been a constant source of inspiration and strength. Their values and lessons have shaped my journey in ways for which I will be forever grateful.*

*To my **sister** and my **brothers**, I owe my deepest thanks for their unconditional love, encouragement, and support, which have carried me through every challenge and triumph. Their presence in my life has been a true blessing in moments of both difficulty and joy.*

*Finally, I extend my heartfelt gratitude to my dear **friends** and **colleagues**, whose constant encouragement, understanding, and companionship have provided me with the resilience to pursue this path. Their invaluable friendship and unwavering belief in me have been a true pillar of strength throughout this remarkable journey.*

0.1 Introduction

Understanding the behavior of non-Newtonian fluids under various physical and geometrical constraints is crucial in numerous scientific, industrial, and engineering applications. Among the diverse classes of non-Newtonian fluids, Bingham and Herschel-Bulkley fluids are particularly notable due to their complex rheological properties, including yield stress and nonlinear shear-thinning or shear-thickening behavior. Such fluids are prevalent in many natural and industrial processes, ranging from the flow of drilling muds and cement slurries in petroleum engineering to the behavior of food products, blood flow, and geological formations such as lava.

Historically, the Bingham fluid model, introduced by Eugene C. Bingham in 1916, describes materials that behave as rigid bodies when the applied stress is below a critical threshold (yield stress) and flow as viscous fluids when this threshold is exceeded. This foundational model opened new research directions in the study of viscoplastic materials. Subsequently, the Herschel-Bulkley model generalized the Bingham model by introducing a nonlinear power-law relationship between the shear rate and stress, providing an even richer framework for describing real-world non-Newtonian fluids.[15]

Modeling and analyzing Bingham fluid flows have been sources of significant mathematical challenges for decades. A major advancement came with the formulation of variational inequalities by Duvaut and Lions [15], which allowed rigorous mathematical analysis of these flows. Subsequent works have extensively investigated the existence, uniqueness, and regularity of solutions for Bingham fluid models[2,19,22,27,29]. More recently, there has been a surge of interest in the asymptotic analysis of such models, particularly in thin domain settings, motivated by practical scenarios where one spatial dimension is significantly smaller than the others. Works such as those by Boukrouche et al.[4,5,6,7,8] and . Bayada et al[1,10,16,17]. have contributed significantly to this area.

This doctoral thesis is dedicated to studying both the asymptotic behavior of non-isothermal Bingham and Herschel-Bulkley fluid flows with nonlinear boundary conditions in thin three-dimensional domains. The primary objective is to analyze the limit behavior as the domain thickness, represented by a small parameter ε, tends to zero. By conducting asymptotic analysis, the original complex three-dimensional problems are reduced to lower-dimensional problems, which retain the essential physics while being more tractable mathematically. One of the essential goals of asymptotic analysis, as applied here, is to derive and describe a two-dimensional problem starting from a three-dimensional one by taking the limit on the thin domain thickness, eventually leading to a generalized Reynolds equation.

Chapter 1 introduces the necessary theoretical foundations. We present the constitutive laws governing Bingham and Herschel-Bulkley fluids, focusing on the nonlinear yield criteria:

$$|\mathbf{\Sigma}| \leq g(\theta), \quad \text{and if} \quad |\mathbf{\Sigma}| = g(\theta), \quad \mathbf{D}(\mathbf{w}) \neq 0.$$

We also discuss the frictional boundary conditions of Tresca type , which play crucial roles in the formulation of the coupled mechanical and thermal problems. In addition, fundamental tools from nonlinear functional analysis, such as Minty's Lemma and fixed point theorems, are reviewed.

Chapter 2 focuses on the stationary solution of non-isothermal Bingham flow with nonlinear boundary conditions in a thin domain. It begins with the introduction and positioning of the problem, followed by the variational formulation of the governing system. The domain is then changed through scaling to fix it onto a reference domain Ω, facilitating the derivation of uniform a priori estimates independent of ε. Convergence results are established, leading to the derivation of the limit problem, where a generalized Reynolds equation emerges naturally. The chapter concludes with proving the uniqueness

of the solutions to the limit problem.

Chapter 3 extends the asymptotic study to Herschel-Bulkley fluids, incorporating the additional nonlinearity into the model. After setting up the problem and its variational formulation, the change of domain is performed similarly to Chapter 2. A priori estimates are obtained, and convergence results are rigorously proven. The limit problem is then derived, revealing a generalized Reynolds equation adapted for Herschel-Bulkley fluids. Finally, the uniqueness of the solutions to this new limit problem is addressed.

The problems addressed in this thesis highlight the intricate interplay between nonlinear rheology, thermal effects, and complex boundary conditions. The asymptotic analysis not only deepens the theoretical understanding of non-Newtonian fluid flows in constrained geometries but also provides essential insights for practical applications where thin geometrical configurations are encountered. This includes applications in lubrication theory, biological fluid dynamics, thin film technologies, and enhanced oil recovery processes.

Chapter 1

Formulation of Boundary Problems and Preliminaries

This chapter establishes the theoretical foundation for the mechanical problems addressed in this dissertation, with a particular focus on the mathematical modeling and analysis of non-Newtonian fluid flows. It also provides a brief overview of selected tools from nonlinear analysis that will be essential for the developments in the subsequent chapters. The chapter is divided into two main sections.

The first section is devoted to the mechanical formulation of the steady-state thermal flow of non-Newtonian fluids. Two distinct problems are considered: the first concerns a Bingham fluid, while the second deals with a Herschel–Bulkley fluid. Both problems are subject to boundary conditions involving Tresca-type frictional contact. These formulations are motivated by realistic physical interactions observed in systems involving fluid–structure interaction.

The second section revisits the fundamental results of nonlinear functional analysis, which are essential for the mathematical treatment of the proposed problems. This includes a discussion on convex function theory, elliptic variational inequalities, multi-

valued operators, and the application of the Kakutani–Glicksberg fixed point theorem. The chapter concludes with a review of the basic concepts related to Sobolev spaces and vector-valued function spaces, which provide the appropriate functional framework for the analysis for the analysis.

1.1 Mathematical formulation of boundary value problems

In this section of the first chapter, we first present the constitutive behavior of both Bingham and Herschel–Bulkley fluids. Subsequently, we provide a detailed description of the Tresca-type frictional contact law. We then revisit fundamental thermodynamic principles pertinent to fluid mechanics, which form the basis for modeling the thermal flow behavior of these fluid types. The section concludes with the rigorous mathematical formulation of the mechanical problems associated with Bingham and Herschel–Bulkley fluids.

1.1.1 Constitutive Law of the Bingham Fluid

The constitutive laws govern the mechanical behavior of various types of continuous media. Although these laws are often established empirically through a series of physical experiments aimed at selecting the most appropriate model, they must also satisfy certain symmetry properties. In this section, we introduce the constitutive law of a viscoplastic fluid of Bingham type, drawing upon several previous studies. This model is distinguished by the fact that the material remains rigid until the applied stress surpasses a critical threshold, known as the yield stress. To formulate the model with precision, we begin by introducing some symbolic definitions.

Let w denote the velocity field and D the strain rate tensor, given by:

$$D(w) = \frac{1}{2}(\nabla w + \nabla^T w) \tag{1.1}$$

We also define the deviatoric part of this tensor:

$$\bar{D}(w) = D(w) - \frac{1}{n}\operatorname{tr}(D(w))\,\delta \tag{1.2}$$

where $\operatorname{tr}(D(w))$ denotes the trace of $D(w)$, and δ represents the identity tensor. Let Σ denote the Cauchy stress tensor, with its deviatoric part given by:

$$\bar{\Sigma} = \Sigma + \rho\,\delta \tag{1.3}$$

where $-\rho = \frac{1}{n}\operatorname{tr}(\Sigma)$ defines the spherical part of the stress, commonly interpreted as the pressuree.

In addition to $\bar{D}(w)$ and $\bar{\Sigma}$, we introduce the tensor S, which characterizes the plastic part of the stress. To model the material's behavior, we consider a family of smooth functions $t \mapsto (\bar{D}(t), \bar{\Sigma}(t), S(t))$, defined on the interval $t \in [0, T]$ with $T > 0$, representing the duration of the process.

The Bingham model is defined by the following set of equations:

$$\bar{\Sigma} = S + 2\Lambda\bar{D}(w) \tag{1.4}$$

$$f(S) = |S|^2 - g^2 \leq 0 \tag{1.5}$$

$$\bar{D} = 2\lambda S \tag{1.6}$$

Here, Λ is the dynamic viscosity and g is the yield shear stress in pure shear. The

function $\lambda(t)$ is defined as:

$$. \begin{cases} \lambda(t) = 0 & \text{if } f(S) < 0 \text{ or } (f(S) = 0 \text{ and } f'(S) < 0) \\ \lambda(t) > 0 & \text{if } f(S) = 0 \text{ and } f'(S) = 0 \end{cases} \tag{1.7}$$

Equation (1.5) expresses the von Mises yield condition, which requires that the second invariant of S, i.e., $S_{II} = \frac{1}{2}|S|^2$, does not exceed $\frac{g^2}{2}$.

From (1.6) and (1.7), it follows that the deviatoric strain rate tensor can evolve only if S lies on the yield surface $f(S) = 0$, which implies $\bar{D}(w) = 0$ in all other cases. This condition, referred to as the flow rule, is characterized by the relation $|S| = g$.

In the Bingham model, it is assumed that the fluid is incompressible:

$$\text{tr}(D(w)) = 0 \tag{1.8}$$

This assumption holds for any process and for all $T > 0$.

We can reformulate the model using only the tensors $D(w)$ and $\bar{\Sigma}$. From (1.4) and (1.6), we obtain:

$$\bar{\Sigma} = (1 + 4\Lambda\lambda)S \tag{1.9}$$

$$|\bar{\Sigma}| = (1 + 4\Lambda\lambda)|S| \tag{1.10}$$

If $|\bar{\Sigma}| > g$, then from (1.5) and (1.10) it is deduced that $\lambda > 0$, and from (1.7) it follows that $|S| = g > 0$. Furthermore, (1.10) implies that

$$\lambda = \frac{1}{4\Lambda}\left(\frac{|\bar{\Sigma}|}{g} - 1\right)$$

From (1.6) and (1.9), we obtain

$$\bar{D} = \frac{2\lambda}{1+4\Lambda\lambda}|\bar{\Sigma}| = \frac{1}{2\Lambda}\left(1 - \frac{g}{|\bar{\Sigma}|}\bar{\Sigma}\right)$$

Since $tr(D) = 0$, it follows from (1.2) that $\bar{D} = D$. Consequently,

$$\bar{D}(w) = \frac{1}{2\Lambda}\left(1 - \frac{g}{|\bar{\Sigma}|}\right)\bar{\Sigma}$$

Let us now assume that $|\bar{\Sigma}| \leq g$.Then, if $|S| = g$, we obtain $\lambda = 0$ from (1.10), and if $|S| < g$, it also follows from (1.7) that $\lambda = 0$. Hence, we finally have $D = 0$. Thus, we obtain the following Bingham fluid constitutive law::

$$D(w) = \begin{cases} \frac{1}{2\Lambda}\left(1 - \frac{g}{|\bar{\Sigma}|}\right)\bar{\Sigma} & \text{if } |\bar{\Sigma}| > g \\ 0 & \text{if } |\bar{\Sigma}| \leq g \end{cases} \tag{1.11}$$

The constitutive equation (1.11) can also be inverted. If $|D(\mathbf{w})| = 0$, then it follows from (1.11) that $|\bar{\Sigma}| \leq g$. Conversely, if $|D(\mathbf{w})| \neq 0$, then $|\bar{\Sigma}| > g$. Moreover, it is known that $|\bar{\Sigma}| = 2\Lambda|D(\mathbf{w})| + g$. Therefore, by combining this expression with (1.8), the constitutive law (1.11) can be written as:

$$\bar{\Sigma} = \begin{cases} 2\Lambda D(w) + g\frac{D(w)}{|D(w)|} & \text{if } |D(w)| \neq 0 \\ |\bar{\Sigma}| \leq g & \text{if } |D(w)| = 0 \end{cases} \tag{1.12}$$

Equations (1.11) and (1.12) are equivalent, although (1.12) is generally adopted as the standard form of the Bingham law.

Finally, experimental studies have shown that the two parameters μ and g, which govern the mechanical behavior of the Bingham fluid (viscosity and yield stress), depend on temperature, thus accounting for the thermodynamic behavior of Bingham-type fluids.

1.1.2 Constitutive Law of the Herschel–Bulkley Fluid

Following the previous presentation of the Bingham model, we now introduce the constitutive law governing the behavior of Herschel–Bulkley fluids. This model generalizes the Bingham model by incorporating a non-linear dependence of the strain rate tensor on the stress tensor.

The Herschel–Bulkley model is described by the following set of equations:

$$\bar{\Sigma} = \Lambda |D|^{\nu-1} \bar{D} + S, \tag{1.13}$$

$$f(S) = |S|^2 - g^2 \leq 0, \tag{1.14}$$

$$\bar{D} = 2\lambda S, \tag{1.15}$$

where the symbols $\Lambda > 0$,$g > 0$,$\bar{\Sigma}$ and $\bar{D}$ are defined as previously introduced .

The function $\lambda(t)$ is given by:

$$. \begin{cases} \lambda(t) = 0 & \text{if } f(S) < 0 \text{ or } (f(S) = 0 \text{ and } f'(S) < 0) \\ \lambda(t) > 0 & \text{if } f(S) = 0 \text{ and } f'(S) = 0 \end{cases}$$

Similarly to the Bingham model, the Herschel–Bulkley fluid satisfies the von Mises yield criterion, ensuring that the second invariant satisfies

$$S_{II} = \frac{1}{2}|S|^2 \leq \frac{g^2}{2}.$$

Assuming incompressibility, $\mathrm{tr}(D) = 0$, and proceeding analogously, the model can be reformulated in terms of D and $\bar{\Sigma}$. From the constitutive relations, it follows that:

$$\bar{\Sigma} = S(1 + (2\Lambda|D(w)|^{\nu-2}\lambda)), \tag{1.16}$$

$$|\bar{\Sigma}| = |S|(1 + 2\Lambda|D(w)|^{\nu-1}\lambda). \tag{1.17}$$

If $|\bar{\Sigma}| > g$, then $\lambda > 0$, and solving for μ yields:

$$\lambda = \frac{1}{2\Lambda|D(w)|^{\nu-2}} \left(\frac{|\bar{\Sigma}|}{g} - 1 \right) |\bar{\Sigma}|.$$

Substituting into the expression for $\bar{D}$, we obtain:

$$\bar{D} = \frac{1}{2\Lambda|D(w)|^{\nu-2}} \left(1 - \frac{g}{|\bar{\Sigma}|} \right) |\bar{\Sigma}|.$$

Thus, the final form of the constitutive law for the Herschel–Bulkley fluid is:

$$D = \begin{cases} \frac{1}{2\Lambda|D(w)|^{\nu-2}} \left(1 - \frac{g}{|\bar{\Sigma}|} \right) \bar{\Sigma}|, & \text{if } |\bar{\Sigma}| > g, \\ 0, & \text{if } |\bar{\Sigma}| \leq g. \end{cases}$$

Alternatively, the law can also be expressed as:

$$\bar{\Sigma} = \begin{cases} 2\Lambda|D(w)|^{\nu-2}D(w) + g\frac{D(w)}{|D(w)|}, & \text{if } D(w) \neq 0, \\ |\bar{\Sigma}| \leq g, & \text{if } D(w) = 0. \end{cases}$$

These relations generalize the Bingham law by allowing a non-linear relationship between the deviatoric stress and the strain rate.

1.1.3 Sliding with Tresca Friction Law

The contact between the fluid and the obstacle occurs with sliding. This property is mathematically expressed by:

$$w_\nu = 0, \tag{1.18}$$

meaning that the normal component of the velocity vector vanishes on the contact surface.

The Tresca friction law introduces a friction threshold $h(\theta)$, which depends on temperature. When the fluid and the obstacle are in contact, the tangential stress cannot exceed a certain limit:

$$|\Sigma_\tau| \leq k(\theta). \tag{1.19}$$

As long as the tangential stress does not reach the threshold, the fluid cannot move tangentially with respect to the obstacle, and sticking occurs:

$$|\Sigma_\tau| < k(\theta) \Longrightarrow w_\tau = 0. \tag{1.20}$$

Once the threshold is reached, the fluid can move tangentially along the obstacle, resulting in sliding. The tangential stress resists the motion. Consequently, there exists a positive scalar λ such that:

$$w_\tau = -\lambda \Sigma_\tau. \tag{1.21}$$

In conclusion, the boundary conditions corresponding to a sliding contact governed by the Tresca friction law are written as:

$$\begin{cases} w_\nu = 0, \quad |\Sigma_\tau| \leq k(\theta) & \\ |\Sigma_\tau| < k(\theta) \Rightarrow w_\tau = 0 & \text{on } \Gamma_t \\ |\Sigma_\tau| = k(\theta) \Rightarrow w_\tau = -\lambda \Sigma_\tau, \quad \lambda \geq 0 & \end{cases} \tag{I.1.14}$$

where $k(\theta)$ is the friction threshold.

1.1.4 Energy Conservation Equation and Fourier Boundary Conditions

Temperature variations within a fluid lead to its expansion or contraction, often giving rise to convective motions, wherein warmer (and thus lighter) fluid tends to ascend while cooler (denser) fluid descends to replace it. Conversely, mechanical processes such as internal friction or interactions at boundaries generate heat, thereby altering the temperature field. These phenomena underscore the strong coupling between the pressure, velocity, displacement, and temperature fields. To accurately capture this interaction, it becomes necessary to introduce additional constitutive relations involving the stress tensor, the heat flux vector, and the velocity and temperature fields

This approach is rooted in the thermodynamics of irreversible processes, specifically the principle stating that the rate of change of internal energy within an elemental control volume equals the sum of mechanical work done by external forces and the heat supplied to the system. In fluid mechanics, this principle is expressed in terms of power, that is, energy rates, leading to the energy conservation equation.

Assume that an incompressible fluid occupies a bounded domain $\Omega \subset \mathbb{R}^3$, with regular boundary $\partial\Omega = \Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_L \cup \bar{\varpi}$, where $\bar{\Gamma}_1$ and $\bar{\Gamma}$ are disjoint measurable parts. Let the fluid density be constant and equal to $\rho = 1$. The local form of the energy conservation equation is then given by:

$$\frac{De}{Dt} = \boldsymbol{\Sigma}.D(\mathbf{w}) - \nabla\mathbf{Q} - \alpha\theta, \tag{1.22}$$

where:

- $\frac{D}{Dt}$ is the material derivative;

- e represents the specific internal energy of the fluid, that is, the energy per unit mass,
- θ represents the specific internal energy, which coincides with the temperature under the assumption of constant specific heat;
- $\mathbf{\Sigma}.D(\mathbf{w})$ corresponds to the rate of mechanical energy dissipation resulting from internal stresses;
- $\mathbf{Q}$ denotes the heat flux vector;
- $\alpha\theta$ represents a volumetric heat source arising from external effects (such as Joule heating, radiation, or exothermic chemical reactions), with $\alpha > 0$.

According to Fourier's law of heat conduction, the heat flux is given by:

$$\mathbf{Q} = -\mathbf{K}\nabla\theta, \tag{1.23}$$

where $\mathbf{K}$ is the thermal conductivity tensor. In the isotropic case, common for fluids, this reduces to:

$$\mathbf{Q} = -k\nabla\theta, \tag{1.24}$$

with k denoting the thermal conductivity coefficient.

Assuming the specific heat C_θ is constant and equal to 1, the internal energy simplifies to:

$$e(\theta) = \theta. \tag{1.25}$$

Substituting into the conservation equation yields:

$$\frac{d\theta}{dt} - k\Delta\theta + \mathbf{w}\cdot\nabla\theta = \mathbf{\Sigma}.D(\mathbf{w}) - \alpha\theta \quad \text{in } \Omega. \tag{1.26}$$

In the case of a steady-state phenomenon, the time derivative vanishes, and the equation reduces to:

$$-k\Delta\theta + \mathbf{w} \cdot \nabla\theta = \mathbf{\Sigma}.D(\mathbf{w}) - \alpha\theta \quad \text{in } \Omega. \tag{1.27}$$

1.1.5 Steady-State Thermal Flow of a Bingham Fluid under Tresca-Type Frictional Contact

We consider the steady-state thermal flow of a Bingham fluid within a bounded and regular domain $\Omega \subset \mathbb{R}^3$. The flow is driven by a body force $\mathbf{f}$ and a heat source of intensity $\alpha\theta$. The governing system consists of the momentum balance, energy conservation, the Bingham constitutive law, and appropriate boundary conditions. The boundary $\partial\Omega$ is divided into three measurable, non-intersecting parts: $\bar{\Gamma_1}$, $\bar{\Gamma_L}$, and $\bar{\varpi}$, with $\text{meas}(\Gamma_1) > 0$. The velocity field vanishes on $\Gamma_1 \cup \Gamma_L$, while a Tresca-type friction law is imposed on $\bar{\varpi}$. For the thermal part, we consider a homogeneous Neumann boundary condition on $\bar{\varpi}$ and a Dirichlet condition on $\Gamma_1 \cup \Gamma_L$.

The problem can be formulated as follows: Find the velocity field $w = (w_i)_{1\leq i\leq 3} : \Omega \to \mathbb{R}^3$, the stress tensor $\Sigma = (\Sigma_{ij})_{1\leq i,j\leq 3} : \mathbb{Q} \to S_n$, and the temperature distribution $\theta : \Omega \to \mathbb{R}$ satisfing:

$$
\begin{cases}
\sum_{i=1}^{3} w_i \frac{\partial w}{\partial x_i} = \operatorname{div}(\Sigma) + f \quad \text{in } \Omega, \\
\operatorname{div}(w) = 0 \quad \text{in } \Omega, \\
\begin{cases}
\Sigma = 2\Lambda(\theta) D(w) + g(\theta) \dfrac{D(w)}{|D(w)|}, & \text{if } |D(w)| \neq 0, \\
|\Sigma| \leq g(\theta), & \text{if } |D(w)| = 0,
\end{cases} \\
-k\Delta\theta + w \cdot \nabla\theta = \Sigma . D(w) - \alpha\theta \quad \text{in } \Omega, \\
w = 0 \quad \text{on } \Gamma_1 \cup \Gamma_L, \\
wn = 0 \quad \text{on } \bar{\varpi}, \\
\begin{cases}
|\Sigma_\tau| < k \implies w_\tau = 0 \\
|\Sigma_\tau| = k \implies \exists \lambda \geq 0, w_\tau^\epsilon = -\lambda \Sigma_\tau
\end{cases} \quad \text{on } \varpi, \\
\theta = 0 \quad \text{on } \Gamma_1 \cup \Gamma_L, \\
\frac{\partial \theta}{\partial n} = 0 \quad \text{on } \bar{\varpi},
\end{cases}
$$

This coupled, nonlinear system is governed by a variational inequality for the velocity field associated with a variational inequality for the thermal equation. The existence of solutions has been established in Chapter 3 [27], using tools such as the Kakutani–Glicksberg fixed-point theorem, monotonicity arguments, and compactness techniques in L^1 spaces.

1.1.6 Steady-State Thermal Flow of a Herschel–Bulkley Fluid under Tresca-Type Frictional Contact

We now extend the study to the case of a Herschel–Bulkley fluid under the same thermal and mechanical constraints. The setting remains a bounded, regular domain $\Omega \subset \mathbb{R}^3$, with boundary decomposition $\partial\Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_L \cup \bar{\varpi}$, where $\bar{\Gamma}_1$ and $\bar{\Gamma}$, are subject to identical thermal and mechanical conditions.

The mechanical behavior is governed by a constitutive law that generalizes the Bingham model by incorporating a nonlinear dependence of the shear stress on the magnitude of the strain rate. The Herschel–Bulkley law combines yield stress and non-Newtonian viscous effects.

The governing system becomes:

$$
\begin{cases}
\sum_{i=1}^{3} w_i \frac{\partial w}{\partial x_i} = \operatorname{div}(\Sigma) + f \quad \text{in } \Omega, \\
\operatorname{div}(w) = 0 \quad \text{in } \Omega, \\
\begin{cases}
\Sigma = 2\Lambda(\theta)|D(w)|^{\nu-2}D(w) + g(\theta)\dfrac{D(w)}{|D(w)|}, & \text{if } |D(w)| \neq 0, \\
|\Sigma| \leq g(\theta), & \text{if } |D(w)| = 0,
\end{cases} \\
-k\Delta\theta + w \cdot \nabla\theta = \Sigma.D(w) - \alpha\theta \quad \text{in } \Omega, \\
w = 0 \quad \text{on } \Gamma_1 \cup \Gamma_L, \\
wn = 0 \quad \text{on } \Gamma_1, \\
\begin{cases}
|\Sigma_\tau| < k \implies w_\tau = 0 \\
|\Sigma_\tau| = k \implies \exists \lambda \geq 0, w_\tau^\epsilon = -\lambda\Sigma_\tau
\end{cases} \quad \text{on } \varpi, \\
\theta = 0 \ \text{ on } \Gamma_1 \cup \Gamma_L, \\
\frac{\partial\theta}{\partial n} = 0 \quad \text{on } \bar{\varpi},
\end{cases}
$$

As in the Bingham case, the problem is formulated as a nonlinear variational inequality coupled with a thermal variational inequality. Existence results have been obtained using fixed-point arguments, monotonicity techniques, and compactness methods adapted to the non-Newtonian framework of the Herschel–Bulkley model. This system has been rigorously analyzed in Chapter 2 [29].

1.2 Review of Mathematical Analysis

1.2.1 Sobolev Spaces and L^p Spaces

Definition 1.1. ***(The L^p Spaces)***

Let $\Omega \subset \mathbb{R}^n$ be a measurable set. For $1 \leq p \leq \infty$, the Lebesgue space $L^p(\Omega)$ is defined as:

- *If $1 \leq p < \infty$,*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ measurable} : \|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p} < \infty \right\}.$$

- *If $p = \infty$,*

$$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ measurable} : \|f\|_{L^\infty(\Omega)} = \ sup_{x \in \Omega} |f(x)| < \infty \right\}.$$

Proposition 1.1. ***(Comparison between L^p spaces)*** *Let (E) be a finite measure space, i.e., $m(E) < +\infty$. Let $p, q \in \mathbb{R}$ such that $1 \leq p < q \leq +\infty$. Then:*

$$L^q(E) \subset L^p(E).$$

Moreover, there exists a constant C, depending only on p, q, and $m(E)$, such that:

$$\|f\|_{L^p} \leq C \|f\|_{L^q} \quad \text{for all } f \in L^q(E).$$

This shows that the injection from L^q into L^p is continuous.

Definition 1.2. ***(Sobolev Spaces)***

Let $\Omega \subset \mathbb{R}^n$ be an open set. For an integer $k \geq 0$ and $1 \leq p \leq \infty$, the Sobolev space

$W^{k,p}(\Omega)$ is defined as:

$$W^{k,p}(\Omega) = \{w \in L^p(\Omega) : D^\alpha w \in L^p(\Omega),\ \forall |\alpha| \leq k\},$$

where α is a multi-index and $D^\alpha w$ denotes the weak derivative of order α.

This space is a Banach space endowed with the norm:

$$\|w\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha w\|^p_{L^p(\Omega)} \right)^{1/p} \quad \text{for } 1 \leq p < \infty.$$

Particular case: *When $p = 2$ and $k = 1$, the Sobolev space $W^{1,2}(\Omega)$ is denoted by $H^1(\Omega)$. It is a Hilbert space with inner product:*

$$(w_1, w_2)_{H^1(\Omega)} = \int_\Omega w_1 w_2 \, dx + \int_\Omega \nabla w_1 \cdot \nabla w_2 \, dx.$$

1.2.2 Minty's Lemma

Let E be a Banach space and E' its dual.

Definition 1.3. *Let A be an operator from $E \to E'$.*

- *A is said to be* hemicontinuous *if and only if for every sequence $(\lambda_n)_n$ converging to λ, we have:*

$$(A(w_1 + \lambda_n w_2), w_3) \to (A(w_1 + \lambda w_2), w_3) \quad \forall (w_1, w_2, w_3) \in E^3.$$

- *A is said to be* monotone *if and only if:*

$$(A(w_1) - A(w_2), w_1 - w_2) \geq 0 \quad \forall (w_1, w_2) \in E^2.$$

Lemma 1.1. *Let E be a Banach space, and let $A : E \to E'$ be a hemicontinuous and monotone operator; let $J : E \to]-\infty, +\infty]$ be a lower semicontinuous and proper convex functional.*

Then, for any $w_1 \in E$ and $f \in E'$, the following two inequalities are equivalent:

$$\langle A(w_1 - w_2), w_2 \rangle_{EE'} + J(w_2) - J(w_1) \geq \langle f, w_2 - w_1 \rangle_{EE'} \quad \forall w_2 \in E,$$
$$\langle A(w_2 - w_1), w_1 \rangle_{EE'} + J(w_1) - J(w_2) \geq \langle f, w_1 - w_2 \rangle_{EE'} \quad \forall w_2 \in E.$$

1.2.3 Fundamental Inequalities

Proposition 1.2. *If $w_1, w_2 \in L^2(\Omega)$, then we have the **Cauchy–Schwarz inequality**:*

$$|(w_1, w_2)| \leq |w_1||w_2|.$$

Lemma 1.2. ***(Young's Inequality)** If $a, b \in \mathbb{R}_+$ and $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then:*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Lemma 1.3. ***(Hölder Inequality)** If $w_1 \in L^p(\Omega)$, $w_2 \in L^q(\Omega)$ with $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then $w_1 w_2 \in L^1(\Omega)$, and we have:*

$$\|w_1 w_2\|_{L^1(\Omega)} \leq \|w_1\|_{L^p(\Omega)} \|w_2\|_{L^q(\Omega)}.$$

Moreover, if $\frac{1}{p} + \frac{1}{q} + \frac{1}{n} = 1$, then:

$$\|w_1 w_2 w_3\|_{L^1(\Omega)} \leq \|w_1\|_{L^p(\Omega)} \|w_2\|_{L^q(\Omega)} \|w_3\|_{L^n(\Omega)}.$$

Lemma 1.4. ***(Poincaré Inequality)***

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let $w \in W_0^{1,p}(\Omega)$, with $1 \leq p < \infty$. Then, there exists a constant $C > 0$ such that:

$$\|w\|_{L^p(\Omega)} \leq C\|\nabla w\|_{L^p(\Omega)}.$$

This inequality expresses that the L^p-norm of a function vanishing on the boundary can be controlled by the L^p-norm of its gradient. It plays a crucial role in the theory of Sobolev spaces and partial differential equations.

Lemma 1.5. ***(Korn Inequality)*** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. For any vector field $w \in (W^{1,p}(\Omega))^n$, with $1 < p < \infty$, and satisfying appropriate boundary conditions (e.g., $w = 0$ on a portion of the boundary), there exists a constant $C > 0$ such that:*

$$\|\nabla w\|_{L^p(\Omega)} \leq C\|D(w)\|_{L^p(\Omega)},$$

where $D(w) = \frac{1}{2}(\nabla w + \nabla w^T)$ is the symmetric part of the gradient. This inequality ensures that the full gradient of a displacement field can be bounded by its symmetric part, which is essential in elasticity and continuum mechanics.

Chapter 2

Stationary Solutions of Non-Isothermal Bingham Flow with Nonlinear Boundary Conditions in a Thin Domain

2.1 Introduction and Problem Statement

We consider the incompressible, isothermal, and viscous flow of a non-Newtonian fluid in a thin film Ω^ϵ, where $0 < \epsilon < 1$ is a positive real number that approaches zero. The boundary of Ω^ϵ will be denoted as $\Gamma^\epsilon = \bar{\Gamma}_1^\epsilon \cup \bar{\Gamma}_L^\epsilon \cup \bar{\varpi}$,with

- Γ_1^ϵ is the upper boundary of equation $s_3 = \epsilon h(s)$;
- $\Gamma_L^\epsilon$is the lateral boundary;
- is a bounded domain in $\mathbb{R}^2$ of equation $s_3 = 0$ which constitutes the lower bound of the domain Ω^ϵ.

We assume that h is a function of class C^1 defined on ϖ such that :

$$0 < h_* < h(s) < h^* \forall (s,0) \in \varpi$$

The domain Ω^ϵ is given by

$$\Omega^\epsilon = \{(s, s_3) \in \mathbb{R}^3 : (s,0) \in \varpi, 0 < s_3 < \epsilon h(s)\}$$

Let $w^\epsilon = (w_i^\epsilon)_{1 \leq i \leq 3}$ the velocity field, we denote by $\Sigma^\epsilon = (\Sigma_{ij}^\epsilon)_{1 \leq i,j \leq 3}$ the Cauchy stress tensor and by $D = (d_{ij})$ the strain rate tensor :

$$d_{i,j}(w^\epsilon) = \frac{1}{2}\left(\frac{\partial w_i^\epsilon}{\partial s_j} + \frac{\partial w_j^\epsilon}{\partial s_i}\right), 1 \leq i,j \leq 3$$

It is assumed that the constitutive law follows the power law so the relation ship between Σ_{ij}^ϵ and $D(w^\epsilon)$ is given by

$$\begin{cases} \Sigma_{ij}^\epsilon = \tilde{\Sigma}_{ij}^\epsilon - \rho^\epsilon \delta_{ij}, \\ \tilde{\Sigma}^\epsilon = g^\epsilon(\theta^\epsilon) \dfrac{D(w^\epsilon)}{|D_{II}(w^\epsilon)|} + 2\Lambda^\epsilon(\theta^\epsilon) D(w^\epsilon) \text{ if } D(w^\epsilon) \neq 0, \\ |\tilde{\Sigma}^\epsilon| \leq g^\epsilon(\theta^\epsilon) \text{ if } D(w^\epsilon) = 0. \end{cases}$$

where

- Λ^ϵ is the viscosity of the fluid;
- ρ^ϵ is its pressure;
- θ^ϵ is its temperature..

We assume that the fluid flow is governed by the following equations :

• The low of conservation of momentum

$$w^\epsilon \nabla w^\epsilon - \operatorname{div} \Sigma^\epsilon = f^\epsilon \text{ in } \Omega^\epsilon. \tag{2.1}$$

Where $f^\epsilon = (f_i^\epsilon)_{1<i<3}$ denotes the body forces.

• The equation of the heat energy

$$w^\epsilon \nabla \theta^\epsilon - \frac{\partial}{\partial s_i}\left(K^\epsilon \frac{\partial \theta^\epsilon}{\partial s_i}\right) = 2\Lambda^\epsilon(\theta^\epsilon)\, d_{ij}(w^\epsilon) d_{ij}(w^\epsilon) + \sqrt{2} g^\epsilon(\theta^\epsilon)\, |D(w^\epsilon)| - \alpha^\epsilon(\theta^\epsilon) \text{ in } \Omega^\epsilon, \tag{2.2}$$

where $K^\epsilon > 0$ is the thermal conductivity and the term $-\alpha^\epsilon \theta^\epsilon$ represents the external heat source with $\alpha^\epsilon > 0$.

• the fluid is assumed to be incompressible

$$\operatorname{div}(w^\epsilon) = 0 \text{ in } \Omega^\epsilon. \tag{2.3}$$

The boundary conditions for velocity and temperature is described as follows:

For the vitesse we assume:

$$\begin{cases} w^\epsilon = 0 \ on\ \Gamma_L^\epsilon, \\ w^\epsilon n = 0, \text{ on } \Gamma_1^\epsilon \cup . \end{cases} \tag{2.4}$$

where $n = (n_1, n_2, n_3)$ denotes the unit vector normal to Γ_L^ϵ outside Ω^ϵ.

We also assume the existence of friction on ϖ, this friction is modeled by the non-linear law of Tresca

$$\begin{cases} |\Sigma_\tau^\epsilon| < k^\epsilon \Rightarrow w_\tau^\epsilon = 0, \\ |\Sigma_\tau^\epsilon| = k^\epsilon \Rightarrow \exists \lambda \geq 0 : w_\tau^\epsilon = -\lambda \Sigma_\tau^\epsilon. \end{cases} \tag{2.5}$$

Here $|.|$ is the Euclidean norm in $\mathbb{R}^2$,we consider the following usual symbols

$$w_n^\epsilon = w^\epsilon \cdot n = w_i^\epsilon \cdot n_i, \quad w_{\tau_i}^\epsilon = w_i^\epsilon - w_n^\epsilon n_i,$$
$$\Sigma_n^\epsilon = (\Sigma^\epsilon \cdot n)n = \Sigma_{ij}^\epsilon n_i n_j, \quad \Sigma_{\tau_i}^\epsilon = \Sigma_{ij}^\epsilon n_j - \Sigma_n^\epsilon n_i.$$

respectively, the normal velocity, the tangential velocity, the normal component and the tangential component of the tensor

For the temperature θ^ϵ we assume the following conditions :

$$\theta^\epsilon = 0 \ \text{ on } \Gamma_1^\epsilon \cup \Gamma_L^\epsilon, \quad \textit{(homogeneous Dirichlet condition)} \tag{2.6}$$

$$\frac{\partial \theta^\epsilon}{\partial n} = 0 \ \text{ on } . \ \text{ (homogeneous Neumann condition)} \tag{2.7}$$

The complete problem consists in finding the velocity field w^ϵ , the pressure ρ^ϵ and the temperature θ^ϵ, which satisfy the following equations and boundary conditions:

$$
\begin{cases}
w^\epsilon \nabla w^\epsilon - div\left(\Sigma^\epsilon\right) = f^\epsilon \text{ in } \Omega^\epsilon, \\
\Sigma_{ij}^\epsilon = \sqrt{2} g^\epsilon\left(\theta^\epsilon\right) \dfrac{D(w^\epsilon)}{|D_{II}(w^\epsilon)|} + 2\Lambda^\epsilon\left(\theta^\epsilon\right) D(w^\epsilon) - \rho^\epsilon \delta_{ij}, \\
div(w^\epsilon) = 0 \text{ in } \Omega^\epsilon, \\
w^\epsilon \nabla \theta^\epsilon - \nabla.\left(K^\epsilon \nabla \theta^\epsilon\right) = \Lambda^\epsilon\left(\theta^\epsilon\right) |D(w^\epsilon)|^2 + \sqrt{2} g^\epsilon\left(\theta^\epsilon\right) |D(w^\epsilon)| - \alpha^\epsilon \theta^\epsilon \text{ in } \Omega, \\
w^\epsilon = 0 \text{ on } \Gamma_1 \cup \Gamma_L, \\
w^\epsilon n = 0 \text{ on } \varpi, \\
\begin{cases}
|\Sigma_\tau^\epsilon| < k^\epsilon \implies w_\tau^\epsilon = 0 \\
|\Sigma_\tau^\epsilon| = k^\epsilon \implies \exists \lambda \geq 0, w_\tau^\epsilon = -\lambda \Sigma_\tau^\epsilon
\end{cases} \text{ on } \varpi, \\
\theta^\epsilon = 0 \text{ on } \Gamma_1 \cup \Gamma_L, \\
\frac{\partial \theta^\epsilon}{\partial n} = 0 \text{ on } \varpi.
\end{cases}
$$

2.2 Variational Problem and Existence Results

in this section, we define the variational problem and establish an existence result using Kakutan Glicksberg fixed point theorem.

Lemma 2.1. *The Tresca condition is equivalent to the following relation*

$$w_\tau^\epsilon \Sigma_\tau^\epsilon + k^\epsilon |w_\tau^\epsilon| = 0 \quad on\ .$$

Proof. Assume that $w_\tau^\epsilon \Sigma_\tau^\epsilon + k^\epsilon |w_\tau^\epsilon| = 0$.

- **If** $|\Sigma_\tau^\epsilon| = k^\epsilon$**, then**

$$w_\tau^\epsilon \Sigma_\tau^\epsilon = -|\Sigma^\epsilon||w_\tau^\epsilon|,$$

which implies the existence of $\beta \geq 0$ such that

$$w_\tau^\epsilon = -\beta \Sigma_\tau^\epsilon.$$

- **If** $|\Sigma^\epsilon| < k^\epsilon$**, then**

$$w_\tau^\epsilon \Sigma_\tau^\epsilon + k^\epsilon |w_\tau^\epsilon| = 0 \geq -|w_\tau^\epsilon| \cdot |\Sigma_\tau^\epsilon| + k^\epsilon |w_\tau^\epsilon| \geq |w_\tau^\epsilon| \cdot (-|\Sigma_\tau^\epsilon| + k^\epsilon).$$

Since $-|\Sigma_\tau^\epsilon| + k^\epsilon > 0$, it follows that $w_\tau^\epsilon = 0$.

Conversely, assume that w^ϵ satisfies the Tresca boundary condition:

- **If** $|\Sigma_\tau^\epsilon| < k^\epsilon$**, then** $w_\tau^\epsilon = 0$, and

$$w_\tau^\epsilon \Sigma_\tau^\epsilon + k^\epsilon |w_\tau^\epsilon| = 0.$$

- **If** $|\Sigma_\tau^\epsilon| = k^\epsilon$**, then there exists** $\beta \geq 0$ **such that** $w_\tau^\epsilon = -\beta \Sigma_\tau^\epsilon$, and

$$w_\tau^\epsilon \Sigma_\tau^\epsilon + k^\epsilon |w_\tau^\epsilon| = -\beta |\Sigma_\tau^\epsilon|^2 + \beta |\Sigma_\tau^\epsilon|^2 = 0. \qquad \square$$

2.2.1 Functional Framework and Variational Formulation

To establish a weak formulation of problem (2.1)-(2.7), we consider the functional framework on Ω^{ε}

$$V^{\epsilon}(\Omega^{\epsilon}) = \left\{\varphi \in H^1\left(\Omega^{\epsilon}\right)^3 : \varphi = 0 \text{ on } \Gamma_L^{\epsilon}, \varphi \cdot n = 0 \text{ on } \cup\Gamma_1^{\epsilon}\right\},$$
$$V_{div}^{\epsilon}(\Omega^{\epsilon}) = \left\{\varphi \in V^{\epsilon}(\Omega^{\epsilon}) : \operatorname{div}(\varphi) = 0\right\},$$
$$L_0^2\left(\Omega^{\epsilon}\right) = \left\{q \in L^2\left(\Omega^{\epsilon}\right) : \int_{\Omega^{\epsilon}} q ds ds_3 = 0\right\},$$

and

$$H^1_{\Gamma_1^{\epsilon}\cup\Gamma_L^{\epsilon}}\left(\Omega^{\epsilon}\right) = \left\{\Phi \in H^1(\Omega^{\epsilon}) : \Phi = 0 \text{ on } \Gamma_1^{\epsilon} \cup \Gamma_L^{\epsilon}\right\}.$$

we also introduce the following notations

$$a(\theta^{\epsilon}, w^{\epsilon}, \varphi) = 2\int_{\Omega^{\epsilon}} \Lambda^{\epsilon}\left(\theta^{\epsilon}\right) D(w^{\epsilon}) D\left(\varphi\right) ds ds_3,$$

$$B(w^{\epsilon}, w^{\epsilon}, \varphi) = \int_{\Omega^{\epsilon}} w^{\epsilon} \nabla w^{\epsilon} \varphi ds ds_3,$$

$$\left(\rho^{\epsilon}, div\, \varphi\right) = \int_{\Omega^{\epsilon}} \rho^{\epsilon}\, div\, \varphi ds ds_3,$$

$$j(\theta^{\epsilon}, \varphi) = \int_{\varpi} k^{\epsilon} |\varphi| \; ds + \sqrt{2}\int_{\Omega^{\epsilon}} g^{\epsilon}\left(\theta^{\epsilon}\right) |D\left(\varphi\right)| ds ds_3,$$

$$\left(f^{\epsilon}, \varphi\right) = \int_{\Omega^{\epsilon}} f^{\epsilon} \varphi ds ds_3 = \sum_{i=1}^{3} \int_{\Omega^{\epsilon}} f_i^{\epsilon} \varphi_i ds ds_3,$$

$$E\left(w^{\epsilon}, \theta^{\epsilon}, \Phi\right) = \int_{\Omega^{\epsilon}} \theta^{\epsilon} \nabla \Phi w^{\epsilon} ds ds_3$$

$$C\left(\theta^{\epsilon}, \Phi\right) = \int_{\Omega^{\epsilon}} K^{\epsilon} \nabla \theta^{\epsilon} \nabla \Phi ds ds_3,$$

$$F\left(w^{\epsilon}, \theta^{\epsilon}, \Phi\right) = 2\int_{\Omega^{\epsilon}} \Lambda^{\epsilon}\left(\theta^{\epsilon}\right) |D(w^{\epsilon})|^2 \Phi ds ds_3 + 2\int_{\Omega^{\epsilon}} g^{\epsilon}\left(\theta^{\epsilon}\right) |D(w^{\epsilon})| \Phi ds ds_3 + \int_{\Omega^{\epsilon}} \alpha^{\epsilon}\left(\theta^{\epsilon}\right) \Phi ds ds_3.$$

Problem 1

Find $w^\epsilon \in V^\epsilon_{div}(\Omega^\epsilon)$, $\rho^\epsilon \in L^2_0(\Omega^\epsilon)$ and $\theta^\epsilon \in W^1_{\Gamma^\epsilon_1 \cup \Gamma^\epsilon_L}(\Omega^\epsilon)$ such that

$$B\left(w^\epsilon, w^\epsilon, \varphi - w^\epsilon\right) + a(\theta^\epsilon, w^\epsilon, \varphi - w^\epsilon) - \left(\rho^\epsilon, div\,\varphi\right) + j(\theta^\epsilon, \varphi) - j\left(\theta^\epsilon, w^\epsilon\right) \geq \left(f^\epsilon, \varphi - w^\epsilon\right), \forall \varphi \in V^\epsilon\left(\Omega^\epsilon\right), \tag{2.8}$$

$$-E\left(w^\epsilon, \theta^\epsilon, \Phi\right) + C\left(\theta^\epsilon, \Phi\right) = F\left(w^\epsilon, \theta^\epsilon, \Phi\right), \quad \forall \Phi \in H^1_{\Gamma^\epsilon_1 \cup \Gamma^\epsilon_L}\left(\Omega^\epsilon\right) \tag{2.9}$$

Proof. Multiplying equation (2.1) by $\varphi - w^\epsilon$, where $\varphi \in V^\epsilon$, and using Green's formula, we obtain:

$$\int_{\Omega^\epsilon} w^\epsilon \nabla w^\epsilon \left(\varphi - w^\epsilon\right) dsds_3 + \int_{\Omega^\epsilon} \Sigma^\epsilon_{ij} \frac{\partial}{\partial s_j}\left(\varphi - w^\epsilon\right) dsds_3 - \int_{\Gamma^\epsilon} \Sigma^\varepsilon_{ij} n_j \left(\varphi - w^\epsilon\right) ds = \int_{\Omega^\epsilon} f^\epsilon \left(\varphi - w^\epsilon\right) dsds_3 \tag{2.10}$$

From the boundary conditions, we find:

$$\int_{\Gamma^\epsilon} \Sigma^\varepsilon_{ij} n_j \left(\varphi_i - w^\epsilon_i\right) ds = \int_{\varpi} \Sigma^\epsilon_{ij} n_j \left(\varphi_i - w^\epsilon_i\right) ds$$

On the other hand, $\Sigma^\epsilon_{ij} n'_j = \Sigma^\epsilon_{\tau_i} + \Sigma^\epsilon_n n_i$ and $\left(\varphi_i - w^\epsilon_i\right)^i n_i = 0$ sur $\dot{}$, then

$$\int_{\Gamma^\epsilon} \Sigma^\varepsilon_{ij} n_j \left(\varphi_i - w^\epsilon_i\right) ds = \int_{\varpi} \Sigma^\epsilon_\tau \left(\varphi - w^\epsilon\right) ds$$

So from (2.10), we obtain

$$\begin{aligned} \int_{\Omega^\epsilon} w^\epsilon \nabla w^\epsilon \left(\varphi - w^\varepsilon\right) dsds_3 + \int_{\Omega^\epsilon} \Sigma^\epsilon_{ij} \frac{\partial}{\partial s_j}\left(\varphi - w^\epsilon\right) dsds_3 \\ - \int_{\varpi} \Sigma^\epsilon_\tau \left(\varphi - w^\varepsilon\right) ds = \int_{\Omega^\epsilon} f^\epsilon \left(\varphi - w^\epsilon\right) dsds_3 \end{aligned}$$

By adding and subtracting the term $\int_{\varpi} k^\epsilon \left(|\varphi| - |w^\epsilon|\right) ds - \int_{\varpi} \Sigma^\epsilon_\tau \cdot \left(\varphi - w^\epsilon\right) ds$,

we have

$$\int_{\Omega^\epsilon} w^\epsilon \nabla w^\epsilon \left(\varphi - w^\epsilon\right) dsds_3 + \int_{\Omega^\epsilon} \Sigma_{ij}^\epsilon \frac{\partial}{\partial s_j} \left(\varphi - w^\epsilon\right) dsds_3 - \int_\varpi k^\epsilon \left(|\varphi| - |\dot{w}^\varepsilon|\right) ds$$
$$- \int_\varpi \Sigma_\tau^\epsilon \cdot \left(\varphi - w^\epsilon\right) ds - \int_\varpi k^\epsilon \left(|\varphi| - |w^\varepsilon|\right) ds = \int_{\Omega^\varepsilon} f^\epsilon \left(\varphi - w^\epsilon\right) dsds_3$$

Let $\beta = \int_\varpi \Sigma_\tau^\varepsilon \cdot \left(\varphi - w^\epsilon\right) ds + \int_\varpi k^\epsilon \left(|\varphi| - |w^\epsilon|\right) ds$

using Lemma 2.1, we show that we prove that

$$\beta = \int_\varpi \left(\Sigma_\tau^\epsilon \cdot (\varphi) + k^\epsilon |\varphi|\right) ds$$

or

$$\Sigma_\tau^\epsilon \cdot (\varphi) \geq |\Sigma_\tau^\epsilon| \cdot |\varphi| \geq: -k^\epsilon |\varphi| \text{ on },$$

so β is positive. we deduce that

$$\int_{\Omega^\epsilon} w^\epsilon \nabla w^\epsilon \left(\varphi - w^\epsilon\right) dsds_3 + \int_{\Omega^\epsilon} \Sigma_{ij}^\epsilon \frac{\partial}{\partial s_j} \left(\varphi - w^\epsilon\right) dsds_3$$
$$+ \int_\varpi k^\epsilon \left(|\varphi| - |w^\epsilon|\right) ds \geq \int_\varpi f^\epsilon \left(\varphi - w^\epsilon\right) dsds_3$$

Replacing Σ_{ij}^ϵ ij with its value, we obtain

$$\begin{aligned} & B\left(w^\epsilon, w^\epsilon, \varphi - w^\epsilon\right) + a(\theta^\epsilon, w^\epsilon, \varphi - w^\epsilon) - \left(\rho^\epsilon, div\, \varphi\right) \\ & + g^\epsilon \left(\theta^\epsilon\right) \int_{\Omega^\varepsilon} \frac{D(w^\epsilon)}{|D_{II}(w^\epsilon)|} d_{ij} \left(\varphi - w^\epsilon\right) dsds_3 \\ & + \int_\varpi k^\epsilon \left(|\varphi| - |w^\epsilon|\right) ds \geq \int_{\Omega^\epsilon} f^\epsilon \left(\varphi - w^\epsilon\right) dsds_3 \end{aligned} \tag{2.11}$$

According to the Cauchy-Schwarz inequality, we deduce :

$$d_{ij}\left(w^\epsilon\right) d_{ij}(\varphi) \leq \left|D\left(w^\epsilon\right)\right| |D(\varphi)| \tag{2.12}$$

Substituting the inequality (2.12) into (2.11), we obtain (2.8)

To obtain (2.9) , multiplying equation (2.2) by $\Phi \in H^1_{\Gamma_1^\epsilon \cup \Gamma_L^\epsilon}(\Omega^\epsilon)$ and using Green's formula, we obtain

$$\begin{aligned}
&\int_{\Omega^\epsilon} \theta^\epsilon \nabla \Phi w^\epsilon ds ds_3 + \int_{\Omega^\epsilon} K^\epsilon \nabla \theta^\epsilon \nabla \Phi ds ds_3 \\
&= 2 \int_{\Omega^\epsilon} \Lambda^\epsilon(\theta^\epsilon) \left| D(w^\epsilon) \right|^2 \Phi ds ds_3 \\
&+ 2 \int_{\Omega^\epsilon} g^\epsilon(\theta^\epsilon) \left| D(w^\epsilon) \right| \Phi ds ds_3 + \int_{\Omega^\epsilon} \alpha^\epsilon(\theta^\epsilon) \Phi ds ds_3.
\end{aligned}$$

□

Theorem 2.1. *Suppose that $f^\epsilon \in L^2(\Omega^\epsilon)^3$ and $k^\epsilon \in L^\infty()$ such that $k^\epsilon \geq 0$. There exists a unique solution $w^\epsilon \in V^\epsilon_{div}(\Omega^\epsilon), \rho^\epsilon \in L^2_0(\Omega^\epsilon)$ and $\theta^\epsilon \in H^1_{\Gamma_1^\epsilon \cup \Gamma_L^\epsilon}(\Omega^\epsilon)$ to problem (2.8) − (2.9).*

Proof. The proof of the theorem is based on the application of Kakutan Glicksberg fixed point theorem, see for more details [27]. □

We suppose that there exist $\Lambda_*,\ \Lambda^*,\ g^*,\ K_*,\ K^*, \alpha_*, \alpha^*$ in $\mathbb{R}$ such that

$$0 \leq \Lambda_* \leq \Lambda^\epsilon \leq \Lambda^*,\ 0 \leq g^\epsilon \leq g^* \tag{2.13}$$

and

$$0 \leq K_* \leq K^\epsilon \leq K^*,\ 0 \leq \alpha^\epsilon \leq \alpha^* \tag{2.14}$$

2.3 Domain Transformation and A Priori Estimates

2.3.1 Change of Reference Domain

We follow the approach in [1], where the original problem, initially defined in the domain Ω^ϵ, is reformulated into an equivalent problem in a fixed domain Ω that is independent of ϵ. To achieve this asymptotic analysis, we introduce the scaling transformation $\kappa = \frac{s_3}{\epsilon}$, where

$$\Omega = \left\{(s,\kappa) \in \mathbb{R}^3 : (s,0) \in, 0 < \kappa < h(s)\right\}.$$

We denote by $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_L \cup^-$ its boundary, then we define the following functions in Ω

$$\hat{w}_i^\epsilon(s,\kappa) = w_i^\epsilon(s,s_3), i = 1,2,\ \hat{w}_3^\epsilon(s,\kappa) = \epsilon^{-1} w_3^\epsilon(s,s_3) \text{ and } \hat{\rho}^\epsilon(s,\kappa) = \epsilon^2 \rho^\epsilon(s,s_3). \quad (2.15)$$

Let us assume that

$$\left.\begin{array}{c} \hat{K}(s,\kappa) = K^\epsilon(s,s_3), \hat{g} = \epsilon g^\epsilon,\ \hat{\Lambda} = \Lambda^\epsilon,\ \hat{\theta}^\epsilon(s,\kappa) = \theta^\epsilon(s,s_3) \\ \hat{f}(s,\kappa) = \epsilon^2 f^\epsilon(s,s_3),\quad \hat{\alpha}(s,\kappa) = \epsilon^2 \alpha^\epsilon(s,s_3),\ \hat{k} = \epsilon k^\epsilon \end{array}\right\}. \quad (2.16)$$

2.3.2 Variational Formulation on Ω

Now we introduce the functional framework on Ω. For this, we write

$$
\begin{aligned}
V(\Omega) &= \left\{ \hat{\varphi} \in H^1(\Omega)^3 : \ \hat{\varphi} = 0 \text{ on } \Gamma_L, \ \hat{\varphi} \cdot n = 0 \text{ on } \cup \Gamma_1 \right\}, \\
V_{div}(\Omega) &= \left\{ \hat{\varphi} \in V(\Omega) : \operatorname{div}(\hat{\varphi}) = 0 \right\}, \\
H^1_{\Gamma_1 \cup \Gamma_L}(\Omega) &= \left\{ \hat{\Phi} \in H^1(\Omega) : \ \hat{\Phi} = 0 \text{ on } \Gamma_1 \cup \Gamma_L \right\}, \\
V_\kappa &= \left\{ \hat{v} \in \left(L^2(\Omega)\right)^2 ; \frac{\partial \hat{v}_i}{\partial \kappa} \in L^2(\Omega) : \hat{v} = 0 \ \text{ on } \Gamma_L \right\},
\end{aligned}
$$

and V_κ is the Banach space with the norm

$$
\|v\|_{V_\kappa} = \left(\sum_{i=1}^{2} \left(\|v_i\|^2_{L^2(\Omega)} + \left\| \frac{\partial v_i}{\partial \kappa} \right\|^2_{L^2(\Omega)} \right) \right)^{\frac{1}{2}}.
$$

By multiplying inequality (2.8)-(2.9) by ϵ and transforming to the fixed domain Ω while injecting the new data and unknown factors in (2.8)–(2.9) , we demonstrate that the problem is equivalent to the problem

$$
\begin{aligned}
B_0(\hat{w}^\epsilon, \hat{w}^\epsilon, \hat{\varphi} - \hat{w}^\epsilon) + a_0(\hat{\theta}^\epsilon, \hat{w}^\epsilon, \hat{\varphi} - \hat{w}^\epsilon) - (\rho^\epsilon, div\ \hat{\varphi}) + j_0(\hat{\theta}^\epsilon, \hat{\varphi}) \\
- j_0\left(\hat{\theta}^\epsilon, \hat{w}^\epsilon\right) \geq (\hat{f}^\epsilon, \hat{\varphi} - \hat{w}^\epsilon), \ \forall \hat{\varphi} \in V(\Omega),
\end{aligned}
\tag{2.17}
$$

$$
-E_0\left(\hat{w}^\epsilon, \hat{\theta}^\epsilon, \hat{\Phi}\right) + C_0\left(\hat{\theta}^\epsilon, \hat{\Phi}\right) = F_0\left(\hat{w}^\epsilon, \hat{\theta}^\epsilon, \hat{\Phi}\right), \quad \forall \hat{\Phi} \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega), \tag{2.18}
$$

where

$$a_0(\hat{\theta}^\epsilon, \hat{w}^\epsilon, \hat{\varphi} - \hat{w}^\epsilon) = \sum_{i,j=1}^{2} \int_\Omega \left[\epsilon^2 \hat{\Lambda} \left(\hat{\theta}^\epsilon \right) \left(\frac{\partial \hat{w}_i^\epsilon}{\partial s_j} + \frac{\partial \hat{w}_j^\epsilon}{\partial s_i} \right) \right] \frac{\partial (\hat{\varphi}_i - \hat{w}_i^\epsilon)}{\partial s_j} ds d\kappa$$
$$+ \sum_{i=1}^{2} \int_\Omega \hat{\Lambda} \left(\hat{\theta}^\epsilon \right) \left(\frac{\partial \hat{w}_i^\epsilon}{\partial \kappa} + \epsilon^2 \frac{\partial \hat{w}_3^\epsilon}{\partial s_i} \right) \frac{\partial (\hat{\varphi}_i - \hat{w}_i^\epsilon)}{\partial \kappa} ds d\kappa$$
$$+ \int_\Omega \left(2\hat{\Lambda} \left(\hat{\theta}^\epsilon \right) \epsilon^2 \frac{\partial \hat{w}_3^\epsilon}{\partial \kappa} \right) \frac{\partial (\hat{\varphi}_3 - \hat{w}_3^\epsilon)}{\partial \kappa} ds d\kappa +$$
$$+ \sum_{j=1}^{2} \int_\Omega \epsilon^2 \hat{\Lambda} \left(\hat{\theta}^\epsilon \right) \left(\epsilon^2 \frac{\partial \hat{w}_3^\epsilon}{\partial s_j} + \frac{\partial \hat{w}_j^\epsilon}{\partial \kappa} \right) \frac{\partial (\hat{\varphi}_3 - \hat{w}_3^\epsilon)}{\partial s_j} ds d\kappa,$$

$$B_0 \left(\hat{w}^\epsilon, \hat{w}^\epsilon, \hat{\varphi} - \hat{w}^\epsilon \right) = \sum_{i,j=1}^{2} \int_\Omega \epsilon^2 \hat{w}_i^\epsilon \frac{\partial \hat{w}_j^\epsilon}{\partial s_i} \left(\hat{\varphi} - \hat{w}^\epsilon \right) ds d\kappa$$
$$+ \sum_{i=1}^{2} \int_\Omega \epsilon^4 \hat{w}_i^\epsilon \frac{\partial \hat{w}_3^\epsilon}{\partial s_i} \left(\hat{\varphi} - \hat{w}^\epsilon \right) ds d\kappa + \sum_{i=1}^{2} \int_\Omega \epsilon^2 \hat{w}_3^\epsilon \frac{\partial \hat{w}_i^\epsilon}{\partial \kappa} \left(\hat{\varphi}_i - \hat{w}_i^\epsilon \right) ds d\kappa$$
$$+ \int_\Omega \epsilon^4 \hat{w}_3^\epsilon \frac{\partial \hat{w}_3^\epsilon}{\partial \kappa} \left(\hat{\varphi}_3 - \hat{w}_3^\epsilon \right) ds d\kappa,$$

$$\left(\hat{\rho}^\epsilon, div \left(\hat{\varphi} - \hat{w}^\epsilon \right) \right) = \int_\Omega \hat{\rho}^\epsilon div \left(\hat{\varphi} - \hat{w}^\epsilon \right) ds d\kappa,$$

$$j_0(\hat{\theta}^\epsilon, \hat{\varphi}) = \sqrt{2} \int_\Omega \hat{g} \left(\hat{\theta}^\epsilon \right) \left| \tilde{D} \left(\hat{\varphi} \right) \right| ds d\kappa + \int \hat{k} |\hat{\varphi}| ds,$$

$$\left(\hat{f}^\epsilon, \hat{\varphi} - \hat{w}^\epsilon \right) = \sum_{j=1}^{2} \int_\Omega \hat{f}_i (\hat{\varphi}_i - \hat{w}_i^\epsilon) ds d\kappa + \int_\Omega \epsilon \hat{f}_3 (\hat{\varphi}_3 - \hat{w}_3^\epsilon) ds d\kappa,$$

$$C_0\left(\hat{\theta}^\epsilon,\hat{\Phi}\right)=\int_\Omega \epsilon^2\hat{K}(\hat{\theta}^\epsilon)\nabla\hat{\theta}^\epsilon\nabla\hat{\Phi}dsd\kappa=\sum_{i=1}^{2}\int_\Omega \epsilon^2\hat{K}(\hat{\theta}^\epsilon)\frac{\partial\hat{\theta}^\epsilon}{\partial s_i}\frac{\partial\hat{\Phi}}{\partial s_i}dsd\kappa+\int_\Omega \hat{K}(\hat{\theta}^\epsilon)\frac{\partial\hat{\theta}^\epsilon}{\partial\kappa}\frac{\partial\hat{\Phi}}{\partial\kappa}dsd\kappa,$$

$$E_0(\hat{w}^\epsilon,\hat{\theta}^\epsilon,\hat{\Phi}^\epsilon)=\int_\Omega \epsilon^2\hat{\theta}^\epsilon\nabla\hat{\Phi}\hat{w}^\epsilon dsd\kappa=\sum_{i=1}^{2}\int_\Omega \epsilon^2\hat{\theta}^\epsilon\frac{\partial\hat{\Phi}}{\partial s_i}\hat{w_i}^\epsilon dsd\kappa+\int_\Omega \epsilon\hat{\theta}^\epsilon\frac{\partial\hat{\Phi}}{\partial\kappa}(\epsilon\hat{w}_3^\epsilon)dsd\kappa,$$

$$F_0\left(\hat{w}^\epsilon,\hat{\theta}^\epsilon,\hat{\Phi}\right)=2\int_\Omega \hat{\Lambda}\left(\hat{\theta}^\epsilon\right)\left|\tilde{D}(\hat{w}^\epsilon)\right|^2\hat{\Phi}dsd\kappa+\sqrt{2}\int_\Omega \hat{g}\left(\hat{\theta}^\epsilon\right)\left|\tilde{D}(\hat{w}^\epsilon)\right|\hat{\Phi}dsd\kappa-\int_\Omega \hat{\alpha}\left(\hat{\theta}^\epsilon\right)\hat{\Phi}dsd\kappa,$$

with

$$\left|\widetilde{D}\left(\hat{w}^\epsilon\right)\right|=\left[\frac{1}{4}\sum_{i,j=1}^{2}\epsilon^2\left(\frac{\partial\hat{w}_i^\epsilon}{\partial s_j}+\frac{\partial\hat{w}_j^\epsilon}{\partial s_i}\right)^2+\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial\hat{w}_i^\epsilon}{\partial\kappa}+\epsilon^2\frac{\partial\hat{w}_3^\epsilon}{\partial s_i}\right)^2+\epsilon^2\left(\frac{\partial\hat{w}_3^\epsilon}{\partial\kappa}\right)^2\right]^{\frac{1}{2}}.$$

2.3.3 A Priori Estimates for Velocity and Pressure

Theorem 2.2. *Let the assumptions of theorem* 2.1 *and* (2.13)-(2.14) *hold, then there exists a constant* $C>0$ *independent of* ϵ *such that*

$$\epsilon^2\sum_{i,j=1}^{2}\left\|\frac{\partial\hat{w}_i^\epsilon}{\partial s_j}\right\|_{L^2(\Omega)}^2+\sum_{i=1}^{2}\left\|\frac{\partial\hat{w}_i^\epsilon}{\partial\kappa}\right\|_{L^2(\Omega)}^2+\epsilon^2\left\|\frac{\partial\hat{w}_3^\epsilon}{\partial\kappa}\right\|_{L^2(\Omega)}^2+\epsilon^4\sum_{i=1}^{2}\left\|\frac{\partial\hat{w}_3^\epsilon}{\partial s_i}\right\|_{L^2(\Omega)}^2\leq C. \quad (2.19)$$

$$\|\hat{w_i}^\epsilon\|_{L^2(\Omega)}^2\leq C \quad \textit{for } i=1,2 \quad (2.20)$$

$$\|\epsilon\hat{w_3}^\epsilon\|_{L^2(\Omega)}^2\leq C, \quad (2.21)$$

$$\left\|\frac{\partial\hat{\rho}^\epsilon}{\partial s_i}\right\|_{H^{-1}(\Omega)}\leq C \quad \textit{for } i=1,2 \quad (2.22)$$

$$\left\|\frac{\partial\hat{\rho}^\epsilon}{\partial\kappa}\right\|_{H^{-1}(\Omega)}\leq \epsilon C. \quad (2.23)$$

Proof. Choosing $\varphi = 0$ in inequality (2.8), we find

$$B\left(w^{\epsilon}, w^{\epsilon}, w^{\epsilon}\right) + a(\theta^{\epsilon}, w^{\epsilon}, w^{\epsilon}) + \int_{\varpi} k^{\epsilon} \left|w^{\epsilon}\right| \, ds + \sqrt{2} \int_{\Omega^{\epsilon}} g^{\epsilon}\left(\theta^{\epsilon}\right) \left|D\left(w^{\epsilon}\right)\right| ds ds_3 \leq (f^{\epsilon}, w^{\epsilon}), \tag{2.24}$$

as $B\left(w^{\epsilon}, w^{\epsilon}, w^{\epsilon}\right) = 0$, we obtain

$$a(\theta^{\epsilon}, w^{\epsilon}, w^{\epsilon}) + \int_{\varpi} k^{\epsilon} \left|w^{\epsilon}\right| \, ds + \sqrt{2} \int_{\Omega^{\epsilon}} g^{\epsilon}\left(\theta^{\epsilon}\right) \left|D\left(w^{\epsilon}\right)\right| ds ds_3 \leq (f^{\epsilon}, w^{\epsilon}). \tag{2.25}$$

By Cauchy-Schwarz and Young's inequalities, we obtain

$$\begin{aligned} (f^{\epsilon}, w^{\epsilon}) &\leq \epsilon h^{*} \left\|w^{\epsilon}\right\|_{L^2(\Omega)} \left\|f^{\epsilon}\right\|_{L^2(\Omega^{\epsilon})} \\ &\leq \left((\Lambda^{*} C_k)^{\frac{1}{2}} \left\|\nabla w^{\epsilon}\right\|_{L^2(\Omega^{\epsilon})} \right) \left(\frac{\epsilon h^{*}}{(\Lambda^{*} C_k)^{\frac{1}{2}}} \left\|f^{\epsilon}\right\|_{L^2(\Omega^{\epsilon})} \right). \end{aligned} \tag{2.26}$$

then

$$(f^{\epsilon}, w^{\epsilon}) \leq \frac{\Lambda^{*} C_k}{2} \left\|\nabla w^{\epsilon}\right\|^2_{L^2(\Omega^{\epsilon})} + \frac{(\epsilon \bar{h})^2}{2\left(\Lambda^{*} C_k\right)} \left\|f^{\epsilon}\right\|^2_{L^2(\Omega^{\epsilon})}. \tag{2.27}$$

Then from Korn's inequality, there exist a constant C_k independent of ϵ, such that

$$\Lambda_{*} C_k \left\|\nabla w^{\epsilon}\right\|^2_{L^2(\Omega^{\epsilon})} \leq a(\theta^{\epsilon}, w^{\epsilon}, w^{\epsilon}) \tag{2.28}$$

From (2.27)(2.28),we deduce

$$\begin{aligned} &\Lambda_{*} C_k \left\|\nabla w^{\epsilon}\right\|^2_{L^2(\Omega^{\epsilon})} + \int_{\varpi} k^{\epsilon} \left|w^{\epsilon}\right| \, ds + \sqrt{2} \int_{\Omega^{\epsilon}} g^{\epsilon}\left(\theta^{\epsilon}\right) \left|D\left(w^{\epsilon}\right)\right| ds ds_3 \\ &\leq \frac{\Lambda^{*} C_k}{2} \left\|\nabla w^{\epsilon}\right\|^2_{L^2(\Omega^{\epsilon})} + \frac{(\epsilon \bar{h})^2}{2\left(\Lambda^{*} C_k\right)} \left\|f^{\epsilon}\right\|^2_{L^2(\Omega^{\epsilon})}. \end{aligned} \tag{2.29}$$

$$\begin{aligned}&\Lambda_* C_k \left\|\nabla w^\epsilon\right\|^2_{L^2(\Omega^\epsilon)} + \int_\varpi k^\epsilon |w^\epsilon|\, ds + \sqrt{2}\int_{\Omega^\epsilon} g^\epsilon(\theta^\epsilon)|D(w^\epsilon)|\, dx \\ &\leq \frac{\Lambda^* C_k}{2}\left\|\nabla w^\epsilon\right\|^2_{L^2(\Omega^\epsilon)} + \frac{(\epsilon\bar{h})^2}{2\Lambda^* C_k}\left\|f^\epsilon\right\|^2_{L^2(\Omega^\epsilon)}.\end{aligned} \tag{2.30}$$

Multiplying (2.29) by ε, and as $\varepsilon^2 \left\|f^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)} = \varepsilon^{-1}\|\hat{f}\|^2_{L^2(\Omega)}$, we have

$$\begin{aligned}&\Lambda_* C_k \epsilon \left\|\nabla w^\epsilon\right\|^2_{L^2(\Omega^\epsilon)} + \int_\varpi k^\epsilon \left|\hat{w}^\epsilon\right|\ ds + \sqrt{2}\int_{\Omega^\epsilon} g^\epsilon\left(\theta^\epsilon\right)\left|D\left(\hat{w}^\epsilon\right)\right| ds d\kappa \\ &\leq \frac{\Lambda^* C_k}{2}\epsilon\left\|\nabla w^\epsilon\right\|^2_{L^2(\Omega^\epsilon)} + \frac{(\bar{h})^2}{4\left(\Lambda^* C_k\right)}\left\|\hat{f}\right\|^2_{L^2(\Omega)}.\end{aligned} \tag{2.31}$$

So, from (2.30) we deduce (2.19), (2.20) and (2.21), with $C = \left(\frac{\bar{h}}{2\Lambda_* C_k}\right)^2 \|\hat{f}\|^2_{L^2(\Omega)}$.

To obtain the pressure estimates (2.22) – (2.23), we choose $\hat{\varphi} = \hat{w^\epsilon} + \psi, \psi \in H^1(\Omega)$ in (2.8) we get

$$\begin{aligned}&B_0\left(\hat{w}^\epsilon, \hat{w}^\epsilon, \psi\right) + a_0\left(\hat{\theta}^\epsilon, \hat{w}^\epsilon, \psi\right) - \left(\hat{\rho}^\epsilon, div\psi\right) + \sqrt{2}\int_\Omega \hat{g}\left(\hat{\theta}^\epsilon\right)\left|\tilde{D}\left(\hat{w^\epsilon} + \psi\right)\right| ds d\kappa \\ &-\sqrt{2}\int_\Omega \hat{g}\left(\hat{\theta}^\epsilon\right)\left|\tilde{D}\left(\hat{w^\epsilon}\right)\right| ds d\kappa \geq \left(\hat{f}^\epsilon, \psi\right),\end{aligned} \tag{2.32}$$

then

$$\begin{aligned}\left(\hat{\rho}^\epsilon, div\psi\right) \leq\ & B_0\left(\hat{w}^\epsilon, \hat{w}^\epsilon, \psi\right) + a_0\left(\hat{\theta}^\epsilon, \hat{w}^\epsilon, \psi\right) + \sqrt{2}\int_\Omega \hat{g}\left(\hat{\theta}^\epsilon\right)\left|\tilde{D}\left(\hat{w^\epsilon} + \psi\right)\right| ds d\kappa \\ &-\sqrt{2}\int_\Omega \hat{g}\left(\hat{\theta}^\epsilon\right)\left|\tilde{D}\left(\hat{w^\epsilon}\right)\right| ds d\kappa - \left(\hat{f}^\epsilon, \psi\right),\end{aligned} \tag{2.33}$$

As $\sqrt{2}\left|\tilde{D}\left(\hat{w^\epsilon} + \psi\right)\right| \leq 2\left|\tilde{D}\left(\hat{w^\epsilon}\right)\right| + \left|\tilde{D}\left(\psi\right)\right|$ we obtain

$$\begin{aligned}\left(\hat{\rho}^\epsilon, div\psi\right) \leq\ & B_0\left(\hat{w}^\epsilon, \hat{w}^\epsilon, \psi\right) + a_0\left(\hat{\theta}^\epsilon, \hat{w}^\epsilon, \psi\right) + 2\int_\Omega \hat{g}\left(\hat{\theta}^\epsilon\right)\left|\tilde{D}\left(\psi\right)\right| ds d\kappa \\ &+\left(2-\sqrt{2}\right)\int_\Omega \hat{g}\left(\hat{\theta}^\epsilon\right)\left|\tilde{D}\left(\hat{w^\epsilon}\right)\right| ds d\kappa - \left(\hat{f}^\epsilon, \psi\right),\end{aligned} \tag{2.34}$$

Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}(\hat{\rho}^{\epsilon}, \operatorname{div}\psi) \leq \Lambda^* \|D(\tilde{w}^{\varepsilon})\|_{L^2(\Omega)} \|\psi\|_{H^1(\Omega)} + \|\tilde{w}^{\varepsilon}\|^2_{L^2(\Omega)} \|\psi\|_{H^1(\Omega)} \\ + g^* \sqrt{2} |\Omega|^{\frac{1}{2}} \|\psi\|_{H^1(\Omega)} + (2-\sqrt{2}) g^* \|D(\tilde{w}^{\varepsilon})\|_{L^2(\Omega)} + \|\tilde{f}\|_{L^2(\Omega)} \|\psi\|_{H^1(\Omega)}.\end{aligned} \tag{2.35}$$

In the same manner, we choose $\hat{\varphi} = \hat{w}^{\epsilon} - \psi, \psi \in H^1(\Omega)$ in (2.8) then we find

$$\begin{aligned}-(\hat{\rho}^{\epsilon}, \operatorname{div}\psi) \leq \Lambda^* \|D(\tilde{w}^{\varepsilon})\|_{L^2(\Omega)} \|\psi\|_{H^1(\Omega)} + \|\tilde{w}^{\varepsilon}\|^2_{L^2(\Omega)} \|\psi\|_{H^1(\Omega)} + g^* \sqrt{2} |\Omega|^{\frac{1}{2}} \|\psi\|_{H^1(\Omega)} \\ + (2-\sqrt{2}) g^* \|D(\tilde{w}^{\varepsilon})\|_{L^2(\Omega)} + \|\tilde{f}\|_{L^2(\Omega)} \|\psi\|_{H^1(\Omega)}.\end{aligned} \tag{2.36}$$

We combine now (2.34) and (2.35) we obtain .

$$\begin{aligned}|(\hat{\rho}^{\epsilon}, \operatorname{div}\psi)| \leq \Lambda^* \|D(\tilde{w}^{\varepsilon})\|_{L^2(\Omega)} \|\psi\|_{H^1(\Omega)} + \|\tilde{w}^{\varepsilon}\|^2_{L^2(\Omega)} \|\psi\|_{H^1(\Omega)} + g^* \sqrt{2} |\Omega|^{\frac{1}{2}} \|\psi\|_{H^1(\Omega)} \\ + (2-\sqrt{2}) g^* \|D(\tilde{w}^{\varepsilon})\|_{L^2(\Omega)} + \|\tilde{f}\|_{L^2(\Omega)} \|\psi\|_{H^1(\Omega)}.\end{aligned} \tag{2.37}$$

When $i = 1, 2$, we take $\psi = (\psi_1, 0, 0)$ then $\psi = (0, \psi_2, 0)$ in the inequality (2.36) we have

$$\left| \int_{\Omega} \frac{\partial \hat{\rho}^{\epsilon}}{\partial s_i} \psi ds d\kappa \right| \leq \left(C_1 + C^2 + g^* \sqrt{2} |\Omega|^{\frac{1}{2}} + (2-\sqrt{2}) g^* C + \|\tilde{f}_i\|_{L^2(\Omega)} \right) \|\psi\|_{H^1(\Omega)} \tag{2.38}$$

where $C_1 = \Lambda^* C$ Then (2.22) follows for $i = 1, 2$. For getting (2.23),we take in the inequality (2.37),$\psi = (0, 0, \psi_3)$ we find :

$$\frac{1}{\varepsilon} \left| \int_{\Omega} \frac{\partial \tilde{p}^{\varepsilon}}{\partial s_i} ds d\kappa \right| \leq \left(C_1 + C^2 + g^* \sqrt{2} |\Omega|^{\frac{1}{2}} + (2-\sqrt{2}) g^* C + \|\tilde{f}_3\|_{L^2(\Omega)} \right) \|\psi_3\|_{H^1(\Omega)} \tag{2.39}$$

This ends the proof of Theorem 2.2

□

2.3.4 A Priori Estimates for Temperature

Theorem 2.3. *Assume that the assumptions of Theorem* 2.2 *are satisfied, assume also there exist three positive constants* K^*, K_*, C_4 , *such that*

$$0 < C_4 < K_* \leq \hat{K} \leq K^*, \text{ where } C_4 \text{ are determined in the proof .} \tag{2.40}$$

Then, there exists a positive constant C_1 *independent of* ϵ, *such that*

$$\sum_{i=1}^{2} \left\| \epsilon \frac{\partial \hat{\theta}^\epsilon}{\partial s_i} \right\|^2_{L^2(\Omega)} \leq C_1, \tag{2.41}$$

$$\left\| \frac{\partial \hat{\theta}^\epsilon}{\partial \kappa} \right\|^2_{L^2(\Omega)} \leq C_1, \tag{2.42}$$

Proof. choosing $\hat{\Phi} = \hat{\theta}^\epsilon$ in (2.9), we obtain

$$\begin{aligned}\int_\Omega \epsilon^2 \hat{K}(\hat{\theta}^\epsilon) \nabla \hat{\theta}^\epsilon \nabla \hat{\theta}^\epsilon ds d\kappa = 2 \int_\Omega \hat{\Lambda}\left(\hat{\theta}^\epsilon\right) \left|\tilde{D}(\hat{w}^\epsilon)\right|^2 \hat{\theta}^\epsilon ds d\kappa \\ + \sqrt{2} \int_\Omega \hat{g}\left(\hat{\theta}^\epsilon\right) \left|\tilde{D}(\hat{w}^\epsilon)\right| \hat{\theta}^\epsilon ds d\kappa - \int_\Omega \hat{\alpha}\left(\hat{\theta}^\epsilon\right) \hat{\Phi} ds d\kappa + \int_\Omega \epsilon^2 \hat{\theta}^\epsilon \nabla \hat{\Phi} \hat{w}^\epsilon ds d\kappa\end{aligned}$$

then

$$\int_\Omega \epsilon^2 \hat{K} \nabla \hat{\theta}^\epsilon \nabla \hat{\theta}^\epsilon ds d\kappa = \sum_{i=1}^{4} I_i. \tag{2.43}$$

Where

$$\begin{aligned} I_1 = 2 \textstyle\int_\Omega \hat{\Lambda}\left(\hat{\theta}^\epsilon\right) \left|\widetilde{D}\left(\hat{w}^\epsilon\right)\right|^2 \hat{\theta}^\epsilon ds d\kappa, \qquad I_2 = \sqrt{2} \textstyle\int_\Omega \hat{g}\left(\hat{\theta}^\epsilon\right) \left|\widetilde{D}\left(\hat{w}^\epsilon\right)\right| \hat{\theta}^e ds d\kappa, \\ I_3 = \textstyle\int_\Omega \hat{\alpha}\left(\hat{\theta}^\epsilon\right) \hat{\theta}^\epsilon ds d\kappa, \qquad I_4 = \textstyle\int_\Omega \epsilon^2 \hat{\theta}^\epsilon \nabla \hat{\theta}^\epsilon w^\epsilon ds d\kappa. \end{aligned}$$

From (2.39), we have

$$\int_\Omega \epsilon^2 \hat{K}(\hat{\theta}^\epsilon)\nabla\hat{\theta}^\epsilon\nabla\hat{\theta}^\epsilon ds d\kappa \geqslant K_* \epsilon^2 \left\|\nabla\hat{\theta}^\epsilon\right\|^2_{L^2(\Omega)} \geqslant K_*\epsilon^2 \left\|\frac{\partial\hat{\theta}^\epsilon}{\partial s_i}\right\|^2_{L^2(\Omega)} + K_* \left\|\frac{\partial\hat{\theta}^\epsilon}{\partial\kappa}\right\|^2_{L^2(\Omega)}. \quad (2.44)$$

For I_1 by the cauchy-schwarz inequality, we give

$$|I_1| \leq \Lambda^* \left[\sum_{i,j=1}^{2} \frac{\epsilon^2}{2} \left\|\frac{\partial\hat{w}_i^\epsilon}{\partial s_j} + \frac{\partial\hat{w}_j^\epsilon}{\partial s_i}\right\|^2_{L^4(\Omega)} + \sum_{i=1}^{2} \left\|\frac{\partial\hat{w}_i^\epsilon}{\partial\kappa} + \epsilon^2\frac{\partial\hat{w}_3^\epsilon}{\partial s_i}\right\|^2_{L^4(\Omega)} + 2\epsilon^2 \left\|\frac{\partial\hat{w}_3^\epsilon}{\partial\kappa}\right\|^2_{L^4(\Omega)}\right] \left\|\hat{\theta}^\epsilon\right\|_{L^2(\Omega)}.$$

Using Young's inequality and the compact injection $H_1(\Omega)$ in $L^4(\Omega)$, there exists a constant $C_1(\Omega)$ independent of ϵ, such that

$$|I_1| \leqslant 2\Lambda^* C_1(\Omega) \left[\begin{array}{l} \sum_{i,j=1}^{2} \epsilon^2 \left\|\frac{\partial\hat{w}_i^\epsilon}{\partial s_j}\right\|^2_{H^1(\Omega)} + \sum_{i=1}^{2} \left\|\frac{\partial\hat{w}_i^\epsilon}{\partial\kappa}\right\|^2_{H^1(\Omega)} + \\ + \sum_{i=1}^{2} \epsilon^4 \left\|\frac{\partial\hat{w}_3^\epsilon}{\partial s_i}\right\|^2_{H^1(\Omega)} + \epsilon^2 \left\|\frac{\partial\hat{w}_3^\epsilon}{\partial\kappa}\right\|^2_{H^1(\Omega)} \end{array}\right] \left\|\hat{\theta}^\epsilon\right\|_{L^2(\Omega)},$$

also, from (2.19), we get: $|I_1| \leqslant 2\Lambda^* C_1(\Omega) C \left\|\hat{\theta}^\epsilon\right\|_{L^2(\Omega)}$. Similarly, applying the Cauchy-Schwarz inequality we obtain,

$$|I_2| \leqslant \sqrt{2} \int_\Omega \hat{g}\left(\hat{\theta}^\epsilon\right) \left|\widetilde{D}\left(\hat{w}^\epsilon\right)\right| \hat{\theta}^e ds d\kappa,,$$

$$|I_2| \leqslant \sqrt{2} g^* \left\|\widetilde{D}\left(\hat{w}^\epsilon\right)\right\|_{L^2(\Omega)} \left\|\mid \hat{\theta}^\epsilon\right\|_{L^2(\Omega)} \leqslant \sqrt{2} g^* C \left\|\hat{\theta}^\epsilon\right\|_{L^2(\Omega)} \quad (2.45)$$

$$|I_3| \leqslant \alpha^* \bar{h} \left\|\mid \hat{\theta}^\epsilon\right\|_{L^2(\Omega)} \quad (2.46)$$

The analog of the last inequality gives

$$\begin{aligned}
|I_4| &\leqslant \epsilon^2 \left\|\hat{\theta}^\epsilon\right\|_{L^4(\Omega)} \|\hat{w}^\epsilon\|_{L^4(\Omega)} \left\|\nabla\hat{\theta}^\epsilon\right\|_{L^2(\Omega)}, \\
&\leqslant \epsilon^2 C_2 \left\|\hat{\theta}^\epsilon\right\|^{\frac{1}{2}}_{H^1_0(\Omega)} \left\|\hat{\theta}^\epsilon\right\|^{\frac{1}{2}}_{L^2(\Omega)} \|\hat{w}^\epsilon\|^{\frac{1}{2}}_{H^1_0(\Omega)} \|\hat{w}^\epsilon\|^{\frac{1}{2}}_{L^2(\Omega)} \left\|\nabla\hat{\theta}^\epsilon\right\|_{L^2(\Omega)}, \\
&\leqslant \epsilon^2 C_2 \left\|\hat{\theta}^\epsilon\right\|^{\frac{1}{2}}_{L^2(\Omega)} \|\nabla\hat{w}^\epsilon\|^{\frac{1}{2}}_{L^2(\Omega)} \|\hat{w}^\epsilon\|^{\frac{1}{2}}_{L^2(\Omega)} \left\|\nabla\hat{\theta}^\epsilon\right\|^{\frac{3}{2}}_{L^2(\Omega)}.
\end{aligned}$$

By Young's inquality and from (2.19) we find

$$\begin{aligned}
|I_4| &\leqslant \frac{3}{4} C_2 \epsilon^2 \left\|\nabla\hat{\theta}^\epsilon\right\|^2_{L^2(\Omega)} + \frac{1}{4}\epsilon^2 \left\|\hat{\theta}^\epsilon\right\|^2_{L^2(\Omega)} \|\nabla\hat{w}^\epsilon\|^2_{L^2(\Omega)} \|\hat{w}^\epsilon\|^2_{L^2(\Omega)}, \\
&\leqslant \frac{3}{4} C_2 \epsilon^2 \left\|\nabla\hat{\theta}^\epsilon\right\|^2_{L^2(\Omega)} + \frac{1}{4}\epsilon^2 C_3 \left\|\nabla\hat{\theta}^\epsilon\right\|^2_{L^2(\Omega)} \|\nabla\hat{w}^\epsilon\|^2_{L^2(\Omega)} \|\hat{w}^\epsilon\|^2_{L^2(\Omega)}, \\
&\leqslant (\frac{3}{4} C_2 + \frac{1}{4} C_3 C^2)\epsilon^2 \left\|\nabla\hat{\theta}^\epsilon\right\|^2_{L^2(\Omega)}.
\end{aligned}$$

So,

$$|I_4| \leqslant C_4 \epsilon^2 \left\|\frac{\partial\hat{\theta}^\epsilon}{\partial s_i}\right\|^2_{L^2(\Omega)} + C_4 \left\|\frac{\partial\hat{\theta}^\epsilon}{\partial\kappa}\right\|^2_{L^2(\Omega)}, \tag{2.47}$$

where $C_4 = \frac{3}{4}C_2 + \frac{1}{4}C_3C^2$.

By injecting (2.44) − (2.45) and (2.46) in (2.42), and using (2.39), it becomes

$$(K_* - C_4)\epsilon^2 \left\|\frac{\partial\hat{\theta}^\epsilon}{\partial s_i}\right\|^2_{L^2(\Omega)} + (K_* - C_4) \left\|\frac{\partial\hat{\theta}^\epsilon}{\partial\kappa}\right\|^2_{L^2(\Omega)} \leqslant (2\Lambda^* C_1(\Omega) C + \sqrt{2} g^* C + \alpha^* \bar{h}) \left\|\hat{\theta}^\epsilon\right\|_{L^2(\Omega)}.$$

As: $\left\|\hat{\theta}^\epsilon\right\|_{L^2(\Omega)} \leqslant \bar{h} \left\|\frac{\partial\hat{\theta}^\epsilon}{\partial\kappa}\right\|_{L^2(\Omega)}$, we find

$$(K_* - C_4)\epsilon^2 \left\|\frac{\partial\hat{\theta}^\epsilon}{\partial s_i}\right\|^2_{L^2(\Omega)} + (K_* - C_4) \left\|\frac{\partial\hat{\theta}^\epsilon}{\partial\kappa}\right\|^2_{L^2(\Omega)} \leqslant C_5 \left\|\frac{\partial\hat{\theta}^\epsilon}{\partial\kappa}\right\|_{L^2(\Omega)}, \tag{2.48}$$

where $C_5 = (2\Lambda^* C_4(\Omega)C + \sqrt{2}g^*C + \alpha^*\bar{h})\bar{h}$.

According to (2.47) we deduce that : $\left\| \frac{\partial \hat{\theta}^\epsilon}{\partial \kappa} \right\|_{L^2(\Omega)} \leqslant C_5(K_* - C_4)^{-1}$.

By injecting this last estimate in (2.47), we deduce (2.40) and (2.41).

□

Theorem 2.4. *Under the same assumptions as in Theorem* 2.2 *and Theorem* 2.3*, there exist* $w^\star = (w_1^\star, w_2^\star) \in V_\kappa$, $\rho^\star \in L_0^2(\Omega)$ *and* $\theta^* \in V_\kappa$ *such that:*

$$\begin{cases} \hat{w}_i^\epsilon \rightharpoonup w_i^\star, \ i = 1,2 \quad \textit{weakly in } V_\kappa, \\ \epsilon \dfrac{\partial \hat{w}_i^\epsilon}{\partial s_j} \rightharpoonup 0, \ i,j = 1,2 \ \textit{weakly in} \ \ L^2(\Omega) \\ \epsilon \dfrac{\partial \hat{w}_3^\epsilon}{\partial \kappa} \rightharpoonup 0, \quad \textit{weakly in} \ \ L^2(\Omega) \\ \epsilon^2 \dfrac{\partial \hat{w}_3^\epsilon}{\partial s_i} \rightharpoonup 0, \ i = 1,2 \ \textit{weakly in} \ \ L^2(\Omega) \end{cases} \tag{2.49}$$

$$\epsilon \hat{w}_3^\epsilon \rightharpoonup 0, \quad \textit{weakly in} \ \ L^2(\Omega), \tag{2.50}$$

$$\hat{\rho}^\epsilon \rightharpoonup \rho^\star, \textit{weakly in} \ \ L^2(\Omega), \ \rho^\star \ \textit{depend only of } s, \tag{2.51}$$

$$\begin{cases} \hat{\theta}^\epsilon \rightharpoonup \theta^* \ \textit{weakly in } V_\kappa, \\ \frac{\partial \hat{\theta}^\epsilon}{\partial s_i} \rightharpoonup 0, \ i = 1,2 \ \textit{weakly in } L^2(\Omega). \end{cases} \tag{2.52}$$

Proof. From the inequality (2.19) − (2.20) we find directly the convergence of (2.48), to prove (2.49) we use (2.19) and (2.21) Since $div(\hat{w}^\epsilon) = 0$, from (2.22) and (2.23) by choosing a particular test function, we get (2.50).

By inequality (2.41), we have

$$\left\|\hat{\theta}^{\epsilon}\right\|_{L^2(\Omega)} \leqslant \bar{h}\left\|\frac{\partial\hat{\theta}^{\epsilon}}{\partial\kappa}\right\|_{L^2(\Omega)} \leqslant \bar{h}C$$

So, $\hat{\theta}^{\epsilon}$ is bounded in V_{κ}, which implies the existence of an element $\hat{\theta}^{\epsilon}$ in V_{κ}, such that $\hat{\theta}^{\epsilon}$ converges weakly to $\hat{\theta}^{*}$ in V_{κ}.

Moreover, inequality (2.40) shows that $\epsilon\left\|\frac{\partial\hat{\theta}^{\epsilon}}{\partial s_i}\right\|_{L^2(\Omega)} \leqslant C$, therefore $\epsilon\frac{\partial\hat{\theta}^{\epsilon}}{\partial s_i}$ converge to $\frac{\partial\hat{\theta}^{*}}{\partial s_i}$, and since $\hat{\theta}^{\epsilon}$ converge to $\hat{\theta}^{*}$ in V_{κ}, we have that $\epsilon\frac{\partial\hat{\theta}^{*}}{\partial s_i}$ converges to zero in V_{κ}.

□

2.4 Study of the Limit Problem

Theorem 2.5. *With the same assumptions of Theorem* 2.4*, the solution* $(w^{*},\rho^{*},\theta^{*})$ *satisfying the following relations*

$$\sum_{i=1}^{2}\int_{\Omega}\hat{\Lambda}\left(\theta^{*}\right)\frac{\partial\left(w_i^{*}\right)}{\partial\kappa}\frac{\partial\left(\hat{\varphi}_i-w_i^{*}\right)}{\partial\kappa}dsd\kappa-\int_{\Omega}\rho^{*}(s)\left(\frac{\partial\hat{\varphi}_1}{\partial s_1}+\frac{\partial\hat{\varphi}_2}{\partial s_2}\right)dsd\kappa$$
$$+\int_{\Omega}\hat{g}\left(\theta^{*}\right)\left(\left|\frac{\partial\hat{\varphi}}{\partial\kappa}\right|-\left|\frac{\partial w^{*}}{\partial\kappa}\right|\right)dsd\kappa+\int_{\varpi}\hat{k}\left(|\hat{\varphi}|-|w^{*}|\right)ds\geq\sum_{i=1}^{2}\int_{\Omega}\hat{f}_i\left(\hat{\varphi}_i-w_i^{*}\right)dsd\kappa,\forall\hat{\varphi}\in\Pi(V) \tag{2.53}$$

$$-\frac{\partial}{\partial\kappa}\left(\widehat{K}\frac{\partial\theta^{*}}{\partial\kappa}\right)=\sum_{i=1}^{2}\hat{\Lambda}\left(\theta^{*}\right)\left(\frac{\partial w_i^{*}}{\partial\kappa}\right)^{2}+\sqrt{2}\hat{g}\left|\frac{\partial w^{*}}{\partial\kappa}\right|-\hat{\alpha}\left(\theta^{*}\right),\ \textit{in } L^2(\Omega). \tag{2.54}$$

Moreover if

$$\int_{\Omega}\left(\left(\hat{\varphi}_1\left(s,\kappa\right)\frac{\partial\Psi}{\partial s_1}\left(s\right)+\hat{\varphi}_2\left(s,\kappa\right)\right)\frac{\partial\Psi}{\partial s_2}\left(s\right)\right)dsd\kappa=0,\quad\forall\Psi\in C_0^1() \tag{2.55}$$

Then

$$\sum_{i=1}^{2}\int_{\Omega}\hat{\Lambda}\left(\theta^{*}\right)\frac{\partial w_{i}^{\star}}{\partial\kappa}\frac{\partial\left(\hat{\varphi}_{i}-w_{i}^{\star}\right)}{\partial\kappa}dsd\kappa+\hat{g}\int_{\Omega}\left(\left|\frac{\partial\hat{\varphi}}{\partial\kappa}\right|-\left|\frac{\partial w^{\star}}{\partial\kappa}\right|\right)dsd\kappa$$
$$+\int_{\varpi}\hat{k}\left(\left|\hat{\varphi}\right|-\left|w^{\star}\right|\right)ds\geqslant\sum_{j=1}^{2}\left(\hat{f},\hat{\varphi}-w^{\star}\right). \tag{2.56}$$

Where $\Pi(V)=\{\bar{\varphi}=(\hat{\varphi}_1,\hat{\varphi}_2)\in H^1(\Omega)^2:\exists\hat{\varphi}_3 \text{ such that } \hat{\varphi}=(\hat{\varphi}_1,\hat{\varphi}_2,\hat{\varphi}_3)\in V\}$.

Proof. We apply the $\lim_{\epsilon\to 0}$ on the variational inequality (2.17) and using the convergence results of the Theorem (2.4), we deduce

$$\sum_{i=1}^{2}\int_{\Omega}\hat{\Lambda}(\theta^{*})\frac{\partial w_{i}^{*}}{\partial\kappa}\frac{\partial w_{i}^{*}}{\partial\kappa}dsd\kappa+\hat{g}\int_{\Omega}\left|\frac{\partial w_{i}^{*}}{\partial\kappa}\right|dsd\kappa+\int\hat{k}\left|u^{*}\right|ds\geqslant\sum_{i=1}^{2}\int_{\Omega}\hat{\Lambda}(\theta^{*})\frac{\partial w_{i}^{*}}{\partial\kappa}\frac{\partial\hat{\varphi}}{\partial\kappa}dsd\kappa$$
$$+\hat{g}\int_{\Omega}\left|\frac{\partial\hat{\varphi}}{\partial\kappa}\right|dsd\kappa+\int_{\varpi}\hat{k}\left|\hat{\varphi}\right|ds+\sum_{i=1}^{2}\int_{\Omega}\hat{f}_{i}(\hat{\varphi}_{i}-w_{i}^{*})dsd\kappa+\sum_{i=1}^{2}\int_{\Omega}\rho^{*}\frac{\partial\hat{\varphi}_{i}}{\partial s_{i}}dsd\kappa+\int_{\Omega}\rho^{*}\frac{\partial\hat{\varphi}_{3}}{\partial\kappa}dsd\kappa, \tag{2.57}$$

as $\int_{\Omega}\rho^{*}\frac{\partial\hat{\varphi}_{3}}{\partial\kappa}dsd\kappa=0$, because ρ^{*} independent of κ, we find (2.52) and if $\hat{\varphi}$ verifies the conition (2.54) we deduce directly the relation (2.52).

The same for (2.53) we apply the $\lim_{\epsilon\to 0}$ on (2.18), we get

$$\int_{\Omega}\widehat{K}\frac{\partial\theta^{*}}{\partial\kappa}\frac{\partial\hat{\Phi}}{\partial\kappa}dsd\kappa=\sum_{i=1}^{2}\int_{\Omega}\hat{\Lambda}\left(\theta^{*}\right)\left(\frac{\partial w_{i}^{*}}{\partial\kappa}\right)^{2}dsd\kappa+\sqrt{2}\hat{g}\int_{\Omega}\left|\frac{\partial w^{*}}{\partial\kappa}\right|dsd\kappa-\int_{\Omega}\hat{\alpha}\left(\theta^{*}\right)dsd\kappa, \tag{2.58}$$

by Green's formula, we obtain

$$-\frac{\partial}{\partial\kappa}\left(\widehat{K}\frac{\partial\theta^{*}}{\partial\kappa}\right)=\sum_{i=1}^{2}\hat{\Lambda}\left(\theta^{*}\right)\left(\frac{\partial w_{i}^{*}}{\partial\kappa}\right)^{2}+\sqrt{2}\hat{g}\left|\frac{\partial w^{*}}{\partial\kappa}\right|-\hat{\alpha}\left(\theta^{*}\right),\ \text{in } L^{2}(\Omega) \tag{2.59}$$

□

Theorem 2.6. *The variational inequality* (2.55) *is equivalent to the following system*

$$\int_{\Omega}\hat{\Lambda}\left(\theta^{*}\right)\left|\frac{\partial w^{*}}{\partial\kappa}\right|^{2}dsd\kappa+\int_{\Omega}\hat{g}\left(\theta^{*}\right)\left|\frac{\partial w^{*}}{\partial\kappa}\right|dsd\kappa+\int_{\varpi}\hat{k}\left|w^{*}\right|ds=\int_{\Omega}\hat{f}w^{*}dsd\kappa, \tag{2.60}$$

and

$$\int_{\Omega}\hat{\Lambda}\left(\theta^{*}\right)\frac{\partial w^{*}}{\partial\kappa}\frac{\partial\hat{\Phi}}{\partial\kappa}dsd\kappa+\int_{\Omega}\hat{g}\left(\theta^{*}\right)\left|\frac{\partial\hat{\Phi}}{\partial\kappa}\right|dsd\kappa+\int_{\varpi}\hat{k}\left|\hat{\Phi}\right|ds\geq\int_{\Omega}\hat{f}\hat{\Phi}dsd\kappa,\ \forall\hat{\Phi}\in\Sigma\left(V\right), \tag{2.61}$$

where

$$\Sigma\left(V\right)=\left\{\hat{\Phi}\in\Pi(V):\hat{\varphi}\ satisfy\ (2.54)\right\}.$$

Proof. By replacing the function $\hat{\varphi}$ with $\hat{\varphi}=2w^{*}$ and then with $\hat{\varphi}=0$ in (2.55), we obtain:

$$\begin{aligned}&\sum_{i,j=1}^{2}\int_{\Omega}\hat{\Lambda}\left(\theta^{*}\right)\frac{\partial w_{i}^{*}}{\partial\kappa}\frac{\partial w_{j}^{*}}{\partial\kappa}\,ds\,d\kappa+\sum_{i=1}^{2}\int_{\varpi}w_{i}^{*}(s,h(s))w_{i}^{*}(s,h(s))\,ds\\&+\hat{\alpha}\int_{\Omega}\left|\frac{\partial w^{*}}{\partial\kappa}\right|\,ds\,d\kappa+\int_{\varpi}\hat{k}|w^{*}|\,ds\geq\sum_{j=1}^{2}\int_{\Omega^{c}}(\hat{f}_{i},w^{*})\,ds\,d\kappa,\quad\forall\hat{\varphi}\in\Sigma(V)\end{aligned} \tag{2.62}$$

$$\begin{aligned}&\sum_{i,j=1}^{2}\int_{\Omega}\hat{\Lambda}\left(\theta^{*}\right)\frac{\partial w_{i}^{*}}{\partial\kappa}\frac{\partial w_{j}^{*}}{\partial\kappa}\,ds\,d\kappa+\sum_{i=1}^{2}\int_{\varpi}w_{i}^{*}(s,h(s))w_{i}^{*}(s,h(s))\,ds\\&+\hat{\alpha}\int_{\Omega}\left|\frac{\partial w^{*}}{\partial\kappa}\right|\,ds\,d\kappa+\int_{\varpi}\hat{k}|w^{*}|\,ds\leq\sum_{j=1}^{2}\int_{\Omega^{c}}(\hat{f}_{i},w^{*})\,ds\,d\kappa,\quad\forall w^{*}\in\Sigma(V)\end{aligned} \tag{2.63}$$

From (2.61) and (2.62), we deduce (2.59).

For (2.60), we choose $\hat{\Phi}=\hat{\varphi}-w^{*}$ for all $\hat{\Phi}\in\Sigma\left(V\right)$. □

Theorem 2.7. *Let us set*

$$\Sigma^* = \tilde{\Sigma}^* - \nabla\rho^* \text{ and } \tilde{\Sigma}^* = \hat{\Lambda}(\theta^*)\frac{\partial w^*}{\partial \kappa} + \hat{g}(\theta^*)\pi, \tag{2.64}$$

then

$$-\frac{\partial}{\partial \kappa}\left[\hat{\Lambda}(\theta^*)\frac{\partial w^*}{\partial \kappa} + \hat{g}(\theta^*)\frac{\frac{\partial w^*}{\partial \kappa}}{\left|\frac{\partial w^*}{\partial \kappa}\right|}\right] = \hat{f} - \nabla\rho^* \ , \ in\ L^2(\Omega)^2. \tag{2.65}$$

Where $\pi \in L^{\infty}(\Omega)^2$ *and* $\|\pi\|_{\Omega,\infty} \leq 1$.

Proof. If $\dfrac{\partial w^*}{\partial \kappa} = 0$, from (2.63) we find $\left|\tilde{\Sigma}^*\right| < \hat{g}(\theta^*)$. For all $\hat{\Phi} \in \Sigma(K)$, choosing $\hat{\Phi} = \hat{\Phi}$, then $\hat{\Phi} = -\hat{\Phi}$ in (2.60), we obtain

$$\left|G\left(\hat{k}\hat{\Phi}, \frac{\partial \hat{\Phi}}{\partial \kappa}\right)\right| \leq \int_{\varpi} \hat{k}\left|\hat{\Phi}\right| ds + \int_{\Omega} \hat{g}(\theta^*)\left|\frac{\partial \hat{\Phi}}{\partial \kappa}\right| ds d\kappa,$$

where

$$G\left(\hat{k}\hat{\Phi}, \frac{\partial \hat{\Phi}}{\partial \kappa}\right) = \int_{\Omega} \hat{\Lambda}(\theta^*)\frac{\partial w^*}{\partial \kappa}\frac{\partial \hat{\Phi}}{\partial \kappa} ds d\kappa - \int_{\Omega} \hat{f}\hat{\Phi} ds d\kappa. \tag{2.66}$$

Now, by the Hanh-Banach theorem, then, $\exists(\chi, \pi) \in L^{\infty}()^2 L^{\infty}(\Omega)^2$, with $\|\chi\|_{,\infty} \leq 1$ $\|\pi\|_{\Omega,\infty} \leq 1$, such that

$$G\left(\hat{k}\hat{\Phi}, \frac{\partial \hat{\Phi}}{\partial \kappa}\right) = -\int_{\varpi} \chi\hat{k}\hat{\Phi} ds - \int_{\Omega} \pi\hat{g}(\theta^*)\frac{\partial \hat{\Phi}}{\partial \kappa} ds d\kappa. \tag{2.67}$$

In particular, from (2.59) and (2.60) we find

$$\int_{\varpi} \hat{k}|w^*| ds + \int_{\Omega} \hat{g}(\theta^*)\left|\frac{\partial w^*}{\partial \kappa}\right| ds d\kappa = \int_{\varpi} \chi\hat{k}w^* ds + \int_{\Omega} \pi\hat{g}(\theta^*)\frac{\partial w^*}{\partial \kappa} ds d\kappa. \tag{2.68}$$

Moreover, from (2.65) and (2.66), we have

$$\int_{\Omega}\hat{\Lambda}\left(\theta^{*}\right)\frac{\partial w^{*}}{\partial\kappa}\frac{\partial\hat{\Phi}}{\partial\kappa}dsd\kappa+\int_{\varpi}\chi\hat{k}\hat{\Phi}ds+\int_{\Omega}\pi\hat{g}\left(\theta^{*}\right)\frac{\partial\hat{\Phi}}{\partial\kappa}dsd\kappa-\int_{\Omega}\hat{f}\hat{\Phi}dsd\kappa=0. \tag{2.69}$$

Next using (2.67), we have

$$\int_{\varpi}\hat{k}\left(\left|w^{*}\right|-\chi w^{*}\right)ds+\int_{\frac{\partial w^{*}}{\partial\kappa}\neq0}\hat{g}\left(\theta^{*}\right)\left(\left|\frac{\partial w^{*}}{\partial\kappa}\right|-\pi\frac{\partial w^{*}}{\partial\kappa}\right)dsd\kappa=0.$$

As $\left\|\chi\right\|_{,\infty}\leq1\,,\ \left\|\pi\right\|_{\Omega,\infty}\leq1$, we deduce

$$\left|\frac{\partial w^{*}}{\partial\kappa}\right|=\pi\frac{\partial w^{*}}{\partial\kappa}\ \text{and}\ \left|w^{*}\right|-\chi w^{*}.$$

So, if $\left|\frac{\partial w^{*}}{\partial\kappa}\right|\neq0$, by (2.63), we get

$$\tilde{\Sigma}^{*}=\hat{\Lambda}\left(\theta^{*}\right)\frac{\partial w^{*}}{\partial\kappa}+\hat{g}\left(\theta^{*}\right)\frac{\partial w^{*}/\partial\kappa}{\left|\partial w^{*}/\partial\kappa\right|}.$$

In this case, $\left|\tilde{\Sigma}^{*}\right|=\hat{\Lambda}\left(\theta^{*}\right)\left|\frac{\partial w^{*}}{\partial\kappa}\right|+\hat{g}\left(\theta^{*}\right)>\hat{g}\left(\theta^{*}\right),$
therefore, we can write

$$\hat{\Lambda}\left(\theta^{*}\right)\frac{\partial w^{*}}{\partial\kappa}=\begin{cases}0, & \text{if}\ \left|\tilde{\Sigma}^{*}\right|\leq\hat{g}\\ \tilde{\Sigma}^{*}-\hat{g}\left(\theta^{*}\right)\dfrac{\partial w^{*}/\partial\kappa}{\left|\partial w^{*}/\partial\kappa\right|}, & \text{if}\ \left|\tilde{\Sigma}^{*}\right|>\hat{g},\end{cases} \tag{2.70}$$

Besides, from (2.68), there exist $\rho^{*}\in L^{2}\left(\Omega\right)^{2}$ such that

$$\int_{\Omega}\hat{\Lambda}\left(\theta^{*}\right)\frac{\partial w^{*}}{\partial\kappa}\frac{\partial\hat{\Phi}}{\partial\kappa}dsd\kappa+\int_{\varpi}\chi\hat{k}\hat{\Phi}ds+\int_{\Omega}\pi\hat{g}\left(\theta^{*}\right)\frac{\partial\hat{\Phi}}{\partial\kappa}dsd\kappa-\int_{\Omega}\hat{f}\hat{\Phi}dsd\kappa=-\int_{\Omega}\nabla\rho^{*}\hat{\Phi}dsd\kappa. \tag{2.71}$$

Using (2.69) and (2.70) becomes

$$\int_{\Omega}\tilde{\Sigma}^{*}\frac{\partial\hat{\Phi}}{\partial\kappa}dsd\kappa+\int_{\varpi}\chi\hat{k}\hat{\Phi}ds=\int_{\Omega}\hat{f}\hat{\Phi}dsd\kappa-\int_{\Omega}\nabla\rho^{*}\hat{\Phi}dsd\kappa, \tag{2.72}$$

from which (2.64) follows if we take $\hat{\Phi}\in H_{0}^{1}\left(\Omega\right)^{2}$ in (2.71). □

Theorem 2.8. *Under the assumptions of preceding theorems, w^{*} and ρ^{*} satisfy the following equality*

$$\begin{aligned}\int_{\varpi}\Bigg[\frac{h^{3}}{12}\nabla\rho^{*}+\tilde{F}+\int_{0}^{h}\int_{0}^{y}\hat{\Lambda}\left(\theta^{*}\left(s,\zeta\right)\right)\frac{\partial w^{*}\left(s,\xi\right)}{\partial\xi}d\xi dy+\hat{g}\int_{0}^{h}\int_{0}^{y}\frac{\partial w^{*}/\partial\xi}{|\partial w^{*}/\partial\xi|}(s,\xi)d\xi dy\\-\frac{h}{2}\int_{0}^{h}\hat{\Lambda}\left(\theta^{*}\left(s,\zeta\right)\right)\frac{\partial w^{*}\left(s,\xi\right)}{\partial\xi}d\xi+\frac{\hat{g}h}{2}\int_{0}^{h}\frac{\partial w^{*}/\partial\xi}{|\partial w^{*}/\partial\xi|}(s,\xi)d\xi\Bigg].\nabla\varphi\left(s\right)ds=0,\end{aligned} \tag{2.73}$$

for all $\varphi\in H^{1}\left(\right)$ where

$$\tilde{F}\left(s\right)=\int_{0}^{h}F\left(s,y\right)dy-\frac{h}{2}F\left(s,h\right),\ F\left(s,y\right)=\int_{0}^{h}\int_{0}^{\xi}\hat{f}\left(s,\theta\right)dtd\xi.$$

Proof. Integrating (2.64) from 0 to κ, we obtain:

$$-\hat{\Lambda}\left(T^{*}(s,\kappa)\right)\frac{\partial w^{*}}{\partial\kappa}(s,\kappa)-\sqrt{2}\hat{\alpha}\frac{\partial w^{*}/\partial\kappa}{|\partial w^{*}/\partial\kappa|}+\hat{\Lambda}\left(\zeta^{*}(s)\right)\tau^{*}(s)+\sqrt{2}\hat{\alpha}\frac{\tau^{*}}{|\tau^{*}|}=\int_{0}^{\kappa}\hat{f}(s,\xi)\,d\xi-\kappa\nabla\rho^{*}, \tag{2.74}$$

with $\tau^{*}(s)=\frac{\partial w^{*}}{\partial\kappa}(s,0)$ and $\zeta^{*}(s)=\theta^{*}(s,0)$.

Integrating again from 0 to κ, we find:

$$\begin{aligned}-\int_{0}^{\kappa}\hat{\Lambda}\left(\theta^{*}(s,\xi)\right)\frac{\partial w^{*}}{\partial\xi}(s,\xi)\,d\xi-\sqrt{2}\hat{\alpha}\int_{0}^{\kappa}\frac{\partial w^{*}/\partial\xi}{|\partial w^{*}/\partial\xi|}\,d\xi\\+\hat{\Lambda}\left(\zeta^{*}(s)\right)\tau^{*}(s)\kappa+\sqrt{2}\hat{\alpha}\frac{\tau^{*}}{|\tau^{*}|}\kappa=\int_{0}^{\kappa}\int_{0}^{\xi}\hat{f}(s,y)\,dyd\xi-\frac{\kappa^{2}}{2}\nabla\rho^{*}.\end{aligned} \tag{2.75}$$

Substituting κ with h, we get:

$$\begin{aligned}\int_0^h \hat{\Lambda}\left(\theta^*(s,\xi)\right)\frac{\partial w^*}{\partial \xi}(s,\xi)\,d\xi + \sqrt{2}\hat{\alpha}\int_0^h \frac{\partial w^*/\partial \xi}{|\partial w^*/\partial \xi|}\,d\xi \\ +\hat{\Lambda}\left(\zeta^*(x)\right)\tau^*(x)h + \sqrt{2}\hat{\alpha}\frac{\tau^*}{|\tau^*|}h = \int_0^h\int_0^\xi \hat{f}(s,y)\,dyd\xi - \frac{h^2}{2}\nabla\rho^*.\end{aligned} \tag{2.76}$$

Integrating (2.74) from 0 to h, we obtain:

$$\begin{aligned}-\int_0^h\int_0^y \hat{\Lambda}\left(\theta^*(s,\xi)\right)\frac{\partial w^*}{\partial \kappa}(s,\xi)\,d\xi dy - \sqrt{2}\hat{\alpha}\int_0^h\int_0^y \frac{\partial w^*/\partial \xi}{|\partial w^*/\partial \xi|}\,d\xi dy \\ +\hat{\Lambda}\left(\zeta^*(x)\right)\tau^*(s)\frac{h^2}{2} + \sqrt{2}\hat{\alpha}\frac{\tau^*}{|\tau^*|}\frac{h^2}{2} = \int_0^h\int_0^y\int_0^\xi \hat{f}(s,t)\,dtd\xi dy - \frac{h^3}{6}\nabla\rho^*.\end{aligned} \tag{2.77}$$

From (2.75), we deduce:

$$\begin{aligned}\hat{\Lambda}\left(\zeta^*(s)\right)\tau^*(s)h + \sqrt{2}\hat{\alpha}\frac{\tau^*}{|\tau^*|}h = \int_0^h \hat{\Lambda}\left(T^*(s,\xi)\right)\frac{\partial w^*}{\partial \xi}(s,\xi)\,d\xi \\ +\sqrt{2}\hat{\alpha}\int_0^h \frac{\partial w^*/\partial \xi}{|\partial w^*/\partial \xi|}\,d\xi + \int_0^h\int_0^\xi \hat{f}(s,y)\,dyd\xi - \frac{h^2}{2}\nabla\rho^*.\end{aligned} \tag{2.78}$$

From (2.76) and (2.77), we obtain:

$$\begin{aligned}\int_\varpi \frac{h^3}{12}\nabla\rho^* + \bar{F} + \int_0^h\int_0^y \Lambda\left(\theta^*(s,\xi)\right)\frac{\partial w^*(s,\xi)}{\partial \xi}\,d\xi dy + \sqrt{2}\alpha\int_0^h\int_0^y \frac{\partial w^*/\partial \xi}{|\partial w^*/\partial \xi|}(s,\xi)\,d\xi dy \\ -\frac{h}{2}\int_0^h \Lambda\left(\theta^*(s,\xi)\right)\frac{\partial w^*(s,\xi)}{\partial \xi}\,d\xi - \frac{\sqrt{2}\hat{\alpha}h}{2}\int_0^h \frac{\partial w^*/\partial \xi}{|\partial w^*/\partial \xi|}(s,\xi) = 0.\end{aligned} \tag{2.79}$$

Thus, for all $\varphi \in H^1()$, we have:

$$\begin{aligned}\int_\varpi \left[\frac{h^3}{12}\nabla\rho^* + \tilde{F} + \int_0^h\int_0^y \hat{\Lambda}\left(\theta^*\left(s,\zeta\right)\right)\frac{\partial w^*\left(s,\xi\right)}{\partial \xi}d\xi dy + \hat{g}\int_0^h\int_0^y \frac{\partial w^*/\partial \xi}{|\partial w^*/\partial \xi|}(s,\xi)d\xi dy\right. \\ \left. -\frac{h}{2}\int_0^h \hat{\Lambda}\left(\theta^*\left(s,\zeta\right)\right)\frac{\partial w^*\left(s,\xi\right)}{\partial \xi}d\xi + \frac{\hat{g}h}{2}\int_0^h \frac{\partial w^*/\partial \xi}{|\partial w^*/\partial \xi|}(s,\xi)d\xi\right].\nabla\varphi\left(s\right)ds = 0,\end{aligned} \tag{2.80}$$

This completes the proof. □

For the uniqueness of the limit velocity and temperature, we let:

$$\mathcal{W}_\kappa = \left\{ w \in V_\kappa : \tfrac{\partial^2 w}{\partial \kappa^2} \in L^2(\Omega) \right\},$$
$$\mathcal{B}_c = \left\{ w \in \mathcal{W}_\kappa \mathcal{W}_\kappa : \left\| \tfrac{\partial w}{\partial \kappa} \right\|_{V_\kappa} \leqslant c \right\},$$
$$\tilde{\mathcal{W}}_\kappa = \{ w \in \mathcal{W}_\kappa \mathcal{W}_\kappa : w \text{ satisfies condition } (2.54) \}.$$

Theorem 2.9. *Under the assumptions* $(2.13) - (2.14)$ *and if* K_* *is sufficiently large such that*

$$K_* > \left[1 + (\bar{h})^2\right] C_{\hat{\alpha}}.$$

Then, the solution (w^*, ρ^*, θ^*) *of the limit problem* (2.53) *and* $(2.59) - (2.60)$ *is unique* $\left(\tilde{\mathcal{W}}_\kappa \cap \mathcal{B}_c\right) \mathcal{W}_\kappa$, *For all*

$$0 < c < c_0 = \left(2 C_{\hat{\Lambda}} \beta^4\right)^{-\frac{1}{2}} \left[K \left[1 + (\bar{h})^2\right]^{(-1)} - C_{\hat{\alpha}} \right]^{\frac{1}{2}}.$$

Where $\beta > 0, C_{\hat{\Lambda}} > 0, C_{\hat{\alpha}} > 0$.

Proof. We use the same techniques as in [4] to prove this theorem, Let $(w^{*,1}, p^{*,1}, \theta^{*,1})$ and $(w^{*,2}, \rho^{*,2}, \theta^{*,2})$ be two solutions of (2.53) and $(2.59) - (2.60)$

$$\int_\Omega \hat{K} \frac{\partial \theta^{*,1}}{\partial \kappa} \frac{\partial \hat{\Phi}}{\partial \kappa} ds d\kappa = \sum_{i=1}^{2} \int_\Omega \hat{\Lambda}\left(\theta^{*,1}\right) \left(\frac{\partial w_i^{*,1}}{\partial \kappa} \right)^2 \hat{\Phi} ds d\kappa + \sqrt{2} \hat{g} \int_\Omega \left| \frac{\partial w_i^{*,1}}{\partial \kappa} \right| \hat{\Phi} ds d\kappa + \int_\Omega \hat{\alpha}\left(\theta^{*,1}\right) \hat{\Phi} ds d\kappa, \tag{2.81}$$

$$\int_\Omega \hat{K} \frac{\partial \theta^{*,2}}{\partial \kappa} \frac{\partial \hat{\Phi}}{\partial \kappa} ds d\kappa = \sum_{i=1}^{2} \int_\Omega \hat{\Lambda}\left(\theta^{*,2}\right) \left(\frac{\partial w_i^{*,2}}{\partial \kappa} \right)^2 \hat{\Phi} ds d\kappa + \sqrt{2} \hat{g} \int_\Omega \left| \frac{\partial w_i^{*,2}}{\partial \kappa} \right| \hat{\Phi} ds d\kappa + \int_\Omega \hat{\alpha}\left(\theta^{*,2}\right) \hat{\Phi} ds d\kappa. \tag{2.82}$$

By subtraction and choosing $\hat{\Phi} = (\theta^{*,1} - \theta^{*,2}) \in H^1_{\Gamma_L \cup \Gamma_1} \in \Omega$, we get

$$\int_{\Omega} \hat{K} \left| \frac{\partial}{\partial \kappa} (\theta^{*,1} - \theta^{*,2}) \right|^2 ds d\kappa = \sum_{i=1}^{4} N_i. \tag{2.83}$$

Where

$$N_1 = \sum_{j=1}^{2} N_1^j, N_1^j = \int_{\Omega} \hat{\Lambda} \left(\theta^{*,1}\right) \frac{\partial}{\partial \kappa} \left(w_i^{*,1} + w_i^{*,2}\right) \frac{\partial}{\partial \kappa} \left(w_i^{*,1} - w_i^{*,2}\right) \left(\theta^{*,1} - \theta^{*,2}\right) ds d\kappa,$$
$$N_2 = \sum_{j=1}^{2} N_2^j, N_2^j = \int_{\Omega} \left[\hat{\Lambda} \left(\theta^{*,1}\right) - \hat{\Lambda} \left(\theta^{*,2}\right)\right] \left(\frac{\partial w_i^{*,2}}{\partial \kappa}\right)^2 \left(\theta^{*,1} - \theta^{*,2}\right) ds dz,$$
$$N_3 = \int_{\Omega} \left(\hat{\alpha} \left(\theta^{*,1}\right) - \hat{\alpha} \left(\theta^{*,2}\right)\right) \left(\theta^{*,1} - \theta^{*,2}\right) ds dz,$$
$$N_4 = \sqrt{2} \hat{g} \int_{\Omega} \left(\left(\frac{\partial w_i^{*,1}}{\partial \kappa}\right)^2 - \left(\frac{\partial w_i^{*,2}}{\partial \kappa}\right)^2 \right) \left(\theta^{*,1} - \theta^{*,2}\right) dx' dz.$$

The increases of $N_i, i = 1, 2, 3$ are given by [10] as follows

$$N_1 = \left| \sum_{j=1}^{2} N_1^j \right| \leqslant 2\sqrt{2} \Lambda^* \beta^2 c \left\| w_i^{*,1} - w_i^{*,2} \right\|_{V_\kappa V_\kappa} \left\| \theta^{*,1} - \theta^{*,2} \right\|_{V_\kappa}, \tag{2.84}$$

$$\left| N_2^i \right| \leqslant C_{\hat{\Lambda}} \beta^4 \left\| \theta^{*,1} - \theta^{*,2} \right\|^2_{V_\kappa} \left\| w_2^2 \right\|^2_{W_\kappa}, \tag{2.85}$$

$$|N_3| \leqslant C_{\hat{\tau}} \left\| \theta^{*,1} - \theta^{*,2} \right\|^2_{V_\kappa}, \tag{2.86}$$

where $C_{\hat{\alpha}} > 0$ deducted from the assumption $\hat{\alpha}$ is $C_{\hat{\alpha}}$-Lipschitz continuous function on $\mathbb{R}$. Utilising the Cauchy-Schwartz inequality, we get:

$$|N_4| \leqslant 2g^* \left\| \theta^{*,1} - \theta^{*,2} \right\|^2_{V_\kappa} \left\| w_i^{*,1} - w_i^{*,2} \right\|_{V_\kappa V_\kappa}. \tag{2.87}$$

Injecting (2.83) − (2.86) in (2.82), we find:

$$K_* \left[1+(h)^2\right]^{-1} \left\|\theta^{*,1}-\theta^{*,2}\right\|_{V_\kappa}^2 \leqslant 2\sqrt{2}\Lambda^*\beta^2 c \left\|w_i^{*,1}-w_i^{*,2}\right\|_{V_\kappa V_\kappa} \left\|\theta^{*,1}-\theta^{*,2}\right\|_{V_\kappa}$$
$$+ C_{\hat{\Lambda}}\beta^4 c^2 \left\|\theta^{*,1}-\theta^{*,2}\right\|_{V_\kappa}^2 + C_{\hat{\tau}} \left\|\theta^{*,1}-\theta^{*,2}\right\|_{V_\kappa}^2 + 2g^* \left\|\theta^{*,1}-\theta^{*,2}\right\|_{V_\kappa}^2 \left\|w^{*,2}-w^{*,1}\right\|_{V_\kappa V_\kappa},$$

so,

$$\left\|\theta^{*,1}-\theta^{*,2}\right\|_{V_\kappa} \leqslant \left[K_* \left[1+(\bar{h})^2\right]^{-1} - 2C_{\Lambda^*}\beta^4 c^2 - C_{\hat{\alpha}}\right]^{-1} \left[2\sqrt{2}\Lambda^*\beta^2 c + 2g^*\right] \left\|w^{*,2}-w^{*,1}\right\|_{V_\kappa V_\kappa}.$$

We assumed that:

$$0 < c < c_0 = \left(2C_{\hat{\Lambda}}\beta^4\right)^{-\frac{1}{2}} \left[K_* \left[1+(\bar{h})^2\right]^{(-1)} - C_{\hat{\alpha}}\right]^{\frac{1}{2}}.$$

Thus

$$\left\|\theta^{*,1}-\theta^{*,2}\right\|_{V_\kappa} \leqslant \left[2\sqrt{2}\bar{\Lambda^*}\beta^2 c + 2g^*\right] \left(c_0^2 - c^2\right)^{-1} \left\|w^{*,2}-w^{*,1}\right\|_{V_\kappa V_\kappa}. \tag{2.88}$$

We have also the two inequalities

$$\begin{aligned} &\sum_{i=1}^{2} \int_\Omega \hat{\Lambda}\left(\theta^{*,1}\right) \frac{\partial w_i^{*,1}}{\partial \kappa} \frac{\partial\left(\hat{\varphi}_i - w_i^{*,1}\right)}{\partial \kappa} ds d\kappa + \hat{g} \int_\Omega \left(\left|\frac{\partial \hat{\varphi}}{\partial \kappa}\right| - \left|\frac{\partial w^{*,1}}{\partial \kappa}\right|\right) ds d\kappa \\ &+ \int_\varpi \hat{k}\left(|\hat{\varphi}| - \left|w^{*,1}\right|\right) ds \geqslant \sum_{j=1}^{2} \left(\hat{f}, \hat{\varphi} - w^{*,1}\right). \end{aligned} \tag{2.89}$$

$$\sum_{i=1}^{2}\int_{\Omega}\hat{\Lambda}\left(\theta^{*,2}\right)\frac{\partial w_i^{*,2}}{\partial\kappa}\frac{\partial\left(\hat{\varphi}_i-w_i^{*,2}\right)}{\partial\kappa}dsd\kappa+\hat{g}\int_{\Omega}\left(\left|\frac{\partial\hat{\varphi}}{\partial\kappa}\right|-\left|\frac{\partial w^{*,2}}{\partial\kappa}\right|\right)dsd\kappa$$
$$+\int_{\varpi}\hat{k}\left(|\hat{\varphi}|-\left|w^{*,2}\right|\right)ds\geqslant\sum_{j=1}^{2}\left(\hat{f},\hat{\varphi}-w^{*,2}\right). \tag{2.90}$$

We can take $\hat{\varphi}=w^{*,2}$ in (2.88) and $\hat{\varphi}=w^{*,1}$ in (2.89). Hence, we obtain

$$\sum_{i=1}^{2}\int_{\Omega}\left(\hat{\Lambda}\left(\theta^{*,1}\right)\frac{\partial w_i^{*,1}}{\partial\kappa}\frac{\partial\left(\hat{\varphi}_i-w_i^{*,1}\right)}{\partial\kappa}+\hat{\Lambda}\left(\theta^{*,2}\right)\frac{\partial w_i^{*,2}}{\partial\kappa}\frac{\partial\left(\hat{\varphi}_i-w_i^{*,2}\right)}{\partial\kappa}\right)dsd\kappa\geqslant 0,$$

then

$$\begin{aligned}&\sum_{i=1}^{2}\int_{\Omega}\hat{\Lambda}\left(\theta^{*,1}\right)\left|\frac{\partial}{\partial\kappa}\left(w_i^{*,1}-w_i^{*,1}\right)\right|^2dsd\kappa\\&\leqslant\sum_{i=1}^{2}\int_{\Omega}[\hat{\Lambda}\left(\theta^{*,1}\right)-\hat{\Lambda}\left(\theta^{*,2}\right)]\frac{\partial w_i^{*,1}}{\partial\kappa}\frac{\partial\left(w_i^{*,1}-w_i^{*,2}\right)}{\partial\kappa}dsd\kappa.\end{aligned} \tag{2.91}$$

As: $\hat{\Lambda}\geqslant\Lambda_*>0$ and by Poincaré's inequality, we find

$$\sum_{i=1}^{2}\int_{\Omega}\hat{\Lambda}\left(\theta^{*,1}\right)\left|\frac{\partial}{\partial\kappa}\left(w_i^{*,1}-w_i^{*,1}\right)\right|^2dsd\kappa\geqslant\Lambda_*\left[1+(\bar{h})^2\right]^{-1}\left\|w^{*,2}-w^{*,1}\right\|_{V_\kappa}^2. \tag{2.92}$$

Now, the analogous results of [28], is given by

$$\left|\sum_{i=1}^{2}\int_{\Omega}[\hat{\Lambda}\left(\theta^{*,1}\right)-\hat{\Lambda}\left(\theta^{*,2}\right)]\frac{\partial w_i^{*,1}}{\partial\kappa}\frac{\partial\left(w_i^{*,1}-w_i^{*,2}\right)}{\partial\kappa}dsd\kappa\right|\leqslant\sqrt{2}\beta^2C_{\hat{\Lambda}}c\left\|\theta^{*,2}-\theta^{*,1}\right\|_{V_\kappa}\left\|w^{*,2}-w^{*,1}\right\|_{V_\kappa}.$$

Where, $\beta>0, C_{\hat{\Lambda}}>0$ and $c>0$ are respectively deduced from, the embedding of V_κ in $L^4()$, the assumption $\hat{\Lambda}$ is $C_{\hat{\Lambda}}$-Lipschitz continuous function on $\mathbb{R}$, and $w^{*,i}\in B_c$.

Therefore

$$\left\|w^{*,2}-w^{*,1}\right\|_{V_\kappa V_\kappa} \leqslant \sqrt{2}\beta^2 C_{\hat{\Lambda}}\Lambda_*^{-1}\left[1+(\bar{h})^2\right]c\left\|\theta^{*,1}-\theta^{*,2}\right\|_{V_\kappa}. \tag{2.93}$$

And from (2.87), we deduce

$$\left(1-\left(2\sqrt{2}\bar{\Lambda}\beta^2 c+2g^*\right)(c_0^2-c^2)^{-1}\sqrt{2}\beta^2 C_{\hat{\Lambda}}\Lambda_*^{-1}\left[1+\bar{(}h)^2\right]c\right)\left\|\theta^{*,1}-\theta^{*,2}\right\|_{V_\kappa}\leqslant 0.$$

Assuming that

$$\left(1-\left(2\sqrt{2}\Lambda^*\beta^2 c+2g^*\right)(c_0^2-c^2)^{-1}\sqrt{2}\beta^2 C_{\hat{\Lambda}}\Lambda_*^{-1}\left[1+\bar{(}h)^2\right]c\right)>0.$$

We have

$$\left\|\theta^{*,1}-\theta^{*,2}\right\|_{V_\kappa}=0.$$

Then $\theta^{*,1}=\theta^{*,2}$ a.e in V_κ. From (2.92), we conclure $w^{*,2}=w^{*,1}$ a.e on $V_\kappa V_\kappa$, of uniqueness (w^*,θ^*) implies that of ρ^*.

□

Chapter 3

Asymptotic Behavior of Solutions for a Non-Isothermal Coupled Problem With Nonlinear Boundary Conditions in a Thin Domain

3.1 Basic Equations

Consider a fixed region ϖ in the plane defined by $s = (s_1, s_2) \in \mathbb{R}^2$. We assume that ϖ has a Lipschitz-continuous boundary and represents the base of a fluid domain. The upper surface Γ_1^ϵ is described by $s_3 = \epsilon h(s)$, where ϵ is a small positive parameter $(0 < \epsilon < 1)$ approaching zero. The function h is smooth, bounded, and satisfies $0 < h_* < h(s) < h^*$ for all $(s, 0) \in \varpi$. We denote the fluid domain by Ω^ϵ, which is defined as follows:

$$\Omega^\epsilon = \{(s, s_3) \in \mathbb{R}^3 : (s, 0) \in \varpi, 0 < s_3 < \epsilon h(s)\}$$

Let Γ^ϵ be the boundary of Ω^ϵ, where $\Gamma^\epsilon = \bar{\Gamma}_1^\epsilon \cup \bar{\Gamma}_L^\epsilon \cup \bar{\varpi}$, and $\bar{\Gamma}_L^\epsilon$ is the lateral boundary.

We denoteby $w^\epsilon = (w_i^\epsilon)_{1\leq i\leq 3}$ the velocity field,$\Sigma^\epsilon = (\Sigma_{ij}^\epsilon)_{1\leq i,j\leq 3}$ the Cauchy stress tensor given by the power law:

$$\begin{cases} \Sigma_{ij}^\epsilon = \tilde{\Sigma}_{ij}^\epsilon - \rho^\epsilon \delta_{ij}, \\ \tilde{\Sigma}^\epsilon = \sqrt{2} g^\epsilon(\theta^\epsilon) \dfrac{D(w^\epsilon)}{|D_{II}(w^\epsilon)|} + \Lambda^\epsilon(\theta^\epsilon) |D(w^\epsilon)|^{\nu-2} D(w^\epsilon) \text{ if } D(w^\epsilon) \neq 0, \\ |\tilde{\Sigma}^\epsilon| \leq g^\epsilon(\theta^\epsilon) \text{ if } D(w^\epsilon) = 0. \end{cases}$$

where ρ^ϵ the pressure , and $D(w^\epsilon) = (d_{i,j}(w^\epsilon))_{1\leq i,j\leq 3}$ the strain rate tensor with

$$d_{i,j}(w^\epsilon) = \frac{1}{2}\left(\frac{\partial w_i^\epsilon}{\partial s_j} + \frac{\partial w_j^\epsilon}{\partial s_i}\right)$$

Problem 1. Find a velocity field $w^\epsilon : \Omega^\epsilon \to \mathbb{R}^3$, the pressure ρ^ϵ and a temperature: $\theta^\epsilon : \Omega^\epsilon \to \mathbb{R}$ such that

$$w^\epsilon \nabla w^\epsilon - div(\Sigma^\epsilon) = f^\epsilon \text{ in } \Omega^\epsilon, \tag{3.1}$$

$$div(w^\epsilon) = 0 \text{ in } \Omega^\epsilon, \tag{3.2}$$

$$w^\epsilon \nabla \theta^\epsilon - \nabla.(K^\epsilon \nabla \theta^\epsilon) = \Lambda^\epsilon(\theta^\epsilon) |D(w^\epsilon)|^\nu + \sqrt{2} g^\epsilon(\theta^\epsilon) |D(w^\epsilon)| - \alpha^\epsilon \theta^\epsilon \text{ in } \Omega^\epsilon, \tag{3.3}$$

$$w^\epsilon = 0 \text{ on } \Gamma_1^\epsilon \cup \Gamma_L^\epsilon, \tag{3.4}$$

$$w^\epsilon n = 0 \text{ on } \varpi, \tag{3.5}$$

$$\begin{cases} |\Sigma_\tau^\epsilon| < k^\epsilon \implies w_\tau^\epsilon = 0 \\ |\Sigma_\tau^\epsilon| = k^\epsilon \implies \exists \lambda \geq 0, w_\tau^\epsilon = -\lambda \Sigma_\tau^\epsilon \end{cases} \text{ on } \varpi, \tag{3.6}$$

$$\theta^\epsilon = 0 \text{ on } \Gamma_1 \cup \Gamma_L, \tag{3.7}$$

$$\frac{\partial \theta^\epsilon}{\partial n} = 0 \text{ on } \varpi. \tag{3.8}$$

The flow is described by equation (3.1), assuming a density value of one, Equation (3.2) outlines the condition of incompressibility. In equation (3.3), the principle of energy conservation is depicted, where the specific heat is set to one, $K^\epsilon > 0$ denotes thermal conductivity, and the term $-\alpha^\epsilon \theta^\epsilon$ accounts for the external heat source, with $\alpha^\epsilon > 0$. . Equation (3.4) specifies the velocity on $\Gamma_1^\epsilon \cup \Gamma_L^\epsilon$. Given the absence of flux across ϖ, equation (3.5) applies. Meanwhile, condition (3.6) describes a Tresca thermal friction law on ϖ where k^ϵ is the coefficient for frictional yield. Let $n = (n_1, n_2, n_3)$ oundary Ω^ϵ represent the unit outward normal vector on the boundary Γ^ϵ

The normal and tangential components of velocity on the boundary Ω^ϵ are defined as $w_n^\epsilon = w^\epsilon.n$ and $w_\tau^\epsilon = w^\epsilon - w_n^\epsilon.n$ respectively. Additionally, Σ_n^ϵ represents a regular stress tensor field, where the normal and tangential components of Σ_n^ϵ on the boundary ϖ are expressed as $\Sigma_n^\epsilon = (\Sigma^\epsilon.n)\,.n$, $\Sigma_\tau^\epsilon = \Sigma^\epsilon.n - \Sigma_n^\epsilon.n$. respectively

3.2 Weak formulation

We consider the following functional framework on Ω^ϵ

$$V^\epsilon = \left\{\varphi \in W^{1,\nu}(\Omega^\epsilon)^3 : \varphi = 0 \text{ on } \Gamma_1^\epsilon \cup \Gamma_L^\epsilon,\ \varphi.n = 0 \text{ on } \varpi\right\},$$
$$V_{div}^\epsilon = \left\{\varphi \in V^\epsilon : div(\varphi) = 0\right\},$$
$$L_0^{\nu'}(\Omega^\epsilon) = \left\{\varphi \in L^{\nu'}(\Omega^\epsilon) : \int_{\Omega^\epsilon} \varphi\ dsds_3 = 0\right\},$$

and

$$W^{1,q}_{\Gamma_1^\epsilon \cup \Gamma_L^\epsilon}(\Omega^\epsilon) = \left\{\Phi \in W^{1,q}(\Omega^\epsilon)^3 : \Phi = 0 \text{ on } \Gamma_1^\epsilon \cup \Gamma_L^\epsilon\right\}.$$

Problem 2 Find $(w^\epsilon, \rho^\epsilon, \theta^\epsilon) \in V^\epsilon_{div} L_0^{\nu'}(\Omega^\epsilon)(\varpi) W^{1,q}_{\Gamma_1^\epsilon \cup \Gamma_L^\epsilon}(\Omega^\epsilon)$, $\left(1 < q < \frac{3}{2}\right)$ such that

$$\begin{aligned} B(w^\epsilon, w^\epsilon, \varphi - w^\epsilon) + a(\theta^\epsilon, w^\epsilon, \varphi - w^\epsilon) - (\rho^\epsilon, div\ (\varphi - w^\epsilon)) + j(\theta^\epsilon, \varphi) \\ -j(\theta^\epsilon, w^\epsilon) \geq (f, \varphi - w^\epsilon),\ \forall \varphi \in V(\Omega^\epsilon) \end{aligned} \tag{3.9}$$

$$C(\theta^\epsilon, \Phi) - E(w^\epsilon, \theta^\epsilon, \Phi) = F(w^\epsilon, \theta^\epsilon, \Phi), \quad \forall \Phi \in W^{1,q'}_{\Gamma_1^\epsilon \cup \Gamma_L^\epsilon}(\Omega^\epsilon), \tag{3.10}$$

where

$$a(\theta^\epsilon, w^\epsilon, \varphi) = 2\int_{\Omega^\epsilon} \Lambda^\epsilon(\theta^\epsilon) |D(w^\epsilon)|^{\nu-2} D(w^\epsilon) D(\varphi)\, dsds_3,$$

$$B(w^\epsilon, w^\epsilon, \varphi) = \int_{\Omega^\epsilon} w^\epsilon \nabla w^\epsilon \varphi dsds_3,$$

$$(\rho^\epsilon, div\, \varphi) = \int_{\Omega^\epsilon} \rho^\epsilon\, div\, \varphi dsds_3,$$

$$j(\theta^\epsilon, \varphi) = \int_{\varpi} k^\epsilon |\varphi|\ ds + \sqrt{2}\int_{\Omega^\epsilon} g^\epsilon(\theta^\epsilon) |D(\varphi)|\, dsds_3,$$

$$(f^\epsilon, \varphi) = \int_{\Omega^\epsilon} f^\epsilon \varphi dsds_3 = \sum_{i=1}^{3} \int_{\Omega^\epsilon} f_i^\epsilon \varphi_i dsds_3,$$

$$E(\theta^\epsilon, \Phi, w^\epsilon) = \int_{\Omega^\epsilon} \theta^\epsilon \nabla \Phi w^\epsilon dsds_3$$

$$C(\theta^\epsilon, \Phi) = \int_{\Omega^\epsilon} K^\epsilon \nabla \theta^\epsilon \nabla \Phi dsds_3,$$

$$F(\theta^\epsilon, w^\epsilon, \Phi) = 2\int_{\Omega^\epsilon} \Lambda^\epsilon(\theta^\epsilon) |D(w^\epsilon)|^2 \Phi dsds_3 + 2\int_{\Omega^\epsilon} g^\epsilon(\theta^\epsilon) |D(w^\epsilon)| \Phi dsds_3 + \int_{\Omega^\epsilon} \alpha^\epsilon(\theta^\epsilon) \Phi dsds_3.$$

This variational problem is known to have a unique solution; see [29] for further details.

Proof. Multiplying equation (3.1) by $\varphi - w^{\epsilon}$, where $\varphi \in V^{\epsilon}$, and using Green's formula, we obtain:

$$\begin{aligned}\int_{\Omega^{\epsilon}} w^{\epsilon}\nabla w^{\epsilon}\left(\varphi - w^{\varepsilon}\right)dsds_3 + \int_{\Omega^{\varepsilon}} \Sigma_{ij}^{\varepsilon}\frac{\partial}{\partial s_j}\left(\varphi - w^{\varepsilon}\right)dsds_3 \\ - \int_{\Gamma^{\varepsilon}} \Sigma_{ij}^{\varepsilon} n_j\left(\varphi - w^{\varepsilon}\right)ds = \int_{\Omega^{\varepsilon}} f^{\varepsilon}\left(\varphi - w^{\varepsilon}\right)dsds_3\end{aligned} \tag{3.11}$$

From the boundary conditions, we find:

$$\int_{\Gamma^{\varepsilon}} \Sigma_{ij}^{\varepsilon} n_j\left(\varphi_i - w_i^{\varepsilon}\right)ds = \int_{\varpi} \Sigma_{ij}^{\varepsilon} n_j\left(\varphi_i - w_i^{\varepsilon}\right)ds$$

On the other hand, $\Sigma_{ij}^{\varepsilon} n_j' = \Sigma_{\tau_i}^{\varepsilon} + \Sigma_n^{\varepsilon} n_i$ and $\left(\varphi_i - w_i^{\varepsilon}\right)^i n_i = 0$ sur $\dot{}$, then

$$\int_{\Gamma^{\varepsilon}} \Sigma_{ij}^{\varepsilon} n_j\left(\varphi_i - w_i^{\varepsilon}\right)ds = \int_{\varpi} \Sigma_{\tau}^{\varepsilon}\left(\varphi - w^{\varepsilon}\right)ds$$

So from (3.11), we obtain

$$\begin{aligned}\int_{\Omega^{\epsilon}} w^{\epsilon}\nabla w^{\epsilon}\left(\varphi - w^{\varepsilon}\right)dsds_3 + \int_{\Omega^{\varepsilon}} \Sigma_{ij}^{\varepsilon}\frac{\partial}{\partial s_j}\left(\varphi - w^{\varepsilon}\right)dsds_3 \\ - \int_{\varpi} \Sigma_T^{\varepsilon}\left(\varphi - w^{\varepsilon}\right)ds = \int_{\Omega^{\varepsilon}} f^{\varepsilon}\left(\varphi - w^{\varepsilon}\right)dsds_3\end{aligned}$$

By adding and subtracting the term $\int_{\varpi} k^{\varepsilon}\left(|\varphi| - |w^{\varepsilon}|\right)ds - \int_{\varpi} \Sigma_{\tau}^{\varepsilon}\cdot\left(\varphi - w^{\varepsilon}\right)ds$, we have

$$\begin{aligned}&\int_{\Omega^{\epsilon}} w^{\epsilon}\nabla w^{\epsilon}\left(\varphi - w^{\varepsilon}\right)dsds_3 + \int_{\Omega^{\varepsilon}} \Sigma_{ij}^{\varepsilon}\frac{\partial}{\partial s_j}\left(\varphi - w^{\varepsilon}\right)dsds_3 - \int_{\varpi} k^{\varepsilon}\left(|\varphi| - |\dot{w}^{\varepsilon}|\right)ds \\ &- \int_{\varpi} \Sigma_{\tau}^{\varepsilon}\cdot\left(\varphi - w^{\varepsilon}\right)ds - \int_{\varpi} k^{\varepsilon}\left(|\varphi| - |w^{\varepsilon}|\right)ds = \int_{\Omega^{\varepsilon}} f^{\varepsilon}\left(\varphi - w^{\varepsilon}\right)dsds_3\end{aligned}$$

Let $\beta = \int_{\varpi} \Sigma_{\tau}^{\varepsilon}\cdot\left(\varphi - w^{\varepsilon}\right)ds + \int_{\varpi} k^{\varepsilon}\left(|\varphi| - |w^{\varepsilon}|\right)ds$

using Lemma 2.1, we show that we prove that

$$\beta = \int_{\varpi} \left(\Sigma_{\tau}^{\varepsilon} \cdot (\varphi) + k^{\varepsilon} |\varphi|\right) ds$$

or

$$\Sigma_{\tau}^{\varepsilon} \cdot (\varphi) \geq |\Sigma_{T}^{\varepsilon}| \cdot |\varphi| \geq: -k^{\varepsilon} |\varphi| \text{ sur },$$

so β is positive. we deduce that

$$\begin{aligned} \int_{\Omega^{\epsilon}} w^{\epsilon} \nabla w^{\epsilon} \left(\varphi - w^{\varepsilon}\right) ds ds_3 + \int_{\Omega^{\varepsilon}} \Sigma_{ij}^{\varepsilon} \frac{\partial}{\partial s_j} \left(\varphi - w^{\varepsilon}\right) ds ds_3 \\ + \int_{\varpi} k^{\varepsilon} \left(|\varphi| + |\dot{w}^{\varepsilon}|\right) ds \geq \int_{\Omega^{\varepsilon}} f^{\varepsilon} \left(\varphi - w^{\varepsilon}\right) ds ds_3 \end{aligned}$$

Replacing $\Sigma_{ij}^{\varepsilon}$ ij with its value, we obtain

$$\begin{aligned} & B\left(w^{\epsilon}, w^{\epsilon}, \varphi - w^{\epsilon}\right) + a(\theta^{\epsilon}, w^{\epsilon}, \varphi - w^{\epsilon}) - \left(\rho^{\epsilon}, div\, \varphi - w^{\epsilon}\right) \\ & + g^{\epsilon}\left(\theta^{\epsilon}\right) \int_{\Omega^{\varepsilon}} \frac{D(w^{\epsilon})}{|D_{II}(w^{\epsilon})|} d_{ij}\left(\varphi - w^{\varepsilon}\right) ds ds_3 \\ & + \int_{\varpi} k^{\varepsilon} \left(|\varphi| + |\dot{w}^{\varepsilon}|\right) ds \geq \int_{\Omega^{\varepsilon}} f^{\varepsilon} \left(\varphi - w^{\varepsilon}\right) ds ds_3 \end{aligned} \tag{3.12}$$

According to the Cauchy-Schwarz inequality, we deduce :

$$d_{ij}\left(w^{\varepsilon}\right) d_{ij}(\varphi) \leq \left|D\left(w^{\varepsilon}\right)\right| |D(\varphi)| \tag{3.13}$$

Substituting the inequality (3.13) into (3.12), we obtain (3.9)

To obtain (3.10) , multiplying equation (3.3) by $\Phi \in W^{1,q'}_{\Gamma_1^\epsilon \cup \Gamma_L^\epsilon}(\Omega^\epsilon)$ and using Green's formula, we obtain

$$\begin{aligned}
&\int_{\Omega^\epsilon} \theta^\epsilon \nabla \Phi w^\epsilon ds ds_3 + \int_{\Omega^\epsilon} K^\epsilon \nabla \theta^\epsilon \nabla \Phi ds ds_3 \\
&= 2 \int_{\Omega^\epsilon} \Lambda^\epsilon(\theta^\epsilon) \left| D(w^\epsilon) \right|^2 \Phi ds ds_3 \\
&+ 2 \int_{\Omega^\epsilon} g^\epsilon(\theta^\epsilon) \left| D(w^\epsilon) \right| \Phi ds ds_3 + \int_{\Omega^\epsilon} \alpha^\epsilon(\theta^\epsilon) \Phi ds ds_3.
\end{aligned}$$

□

We suppose the existence of $K^\epsilon_*,\ K^*_\epsilon,\ \nu^\epsilon_*,\ \nu^*_\epsilon \Lambda_*, \Lambda^*,\ g^*$ in $\mathbb{R}$ such that

$$\frac{1}{2} \leq K^\epsilon_* \leq K^\epsilon \leq K^*_\epsilon, \quad 0 \leq \nu^\epsilon_* \leq \nu^\epsilon \leq \nu^*_\epsilon. \tag{3.14}$$

and

$$0 \leq \Lambda_* \leq \Lambda^\epsilon \leq \Lambda^*,\ 0 \leq g^\epsilon \leq g^*, \quad f^\epsilon \in W^{1,\nu'}(\Omega^\epsilon)^3, \tag{3.15}$$

Based on previous results that will be valuable in the next sections

$$(a+b)^p \leq (2)^{p-1}\left(a^p + b^p\right), \quad \forall (a,b) \in \mathbb{R}^*_+, \quad \forall p > 1. \tag{3.16}$$

$$(a+b)^p \leq \left(a^p + b^p\right), \quad \forall (a,b) \in \mathbb{R}^*_+, \quad 0 < p < 1. \tag{3.17}$$

3.3 Change of the domain and study of convergence

Here, we will apply the technique of scaling in Ω^ϵ on the coordinate s_3, by introducing the change of the variables $\kappa = \dfrac{s_3}{\epsilon}$. we get

$$\Omega = \left\{(s,\, \kappa) \in \mathbb{R}^3 :\, (s,0) \in \varpi, \quad 0 < \kappa < h(s\right\}.$$

The boundary is defined as $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_L \cup \overline{\varpi}$,Additionally, we have.

$$\hat{w}_i^\epsilon(s,\kappa) = w_i^\epsilon(s,s_3), i = 1,2,\ \hat{w}_3^\epsilon(s,\kappa) = \epsilon^{-1} w_3^\epsilon(s,s_3) \text{ and } \hat{\rho}^\epsilon(s,\kappa) = \epsilon^\nu \rho^\epsilon(s,s_3). \quad (3.18)$$

Suppose that

$$\left.\begin{array}{c} \hat{K}(s,\kappa) = \epsilon^{\beta+\nu-2} K^\epsilon(s,s_3),\ \hat{\alpha}(s,\kappa) = \epsilon^{\beta+\nu}\alpha^\epsilon(s,s_3), \\ \hat{\Lambda} = \Lambda^\epsilon,\ \hat{f}(s,\kappa) = \epsilon^\nu f^\epsilon(s,s_3),\ \hat{g} = \epsilon^{\nu-1} g^\epsilon,\ \hat{k} = \epsilon^{\nu-1} k^\epsilon, \end{array}\right\}, \quad (3.19)$$

with

$$\beta = \frac{3\,(2-\nu)}{3-\nu}.$$

Let

$$\begin{aligned} V(\Omega) &= \left\{\hat{\varphi} \in \left(W^{1,\nu}(\Omega)\right)^3 : \hat{\varphi} = 0 \text{ on } \Gamma_1 \cup \Gamma_L;\ \hat{\varphi}.n = 0 \text{ on } \varpi\right\}, \\ V_{\text{div}}(\Omega) &= \left\{\hat{\varphi} \in V(\Omega) :\ \text{div}\hat{\varphi} = 0\right\}, \\ L_0^{\nu'}(\Omega) &= \left\{\varphi \in L^{\nu'}(\Omega) : \int_{\Omega^\epsilon} \varphi\ dsds_3 = 0\right\} \\ V_\kappa &= \left\{\hat{\varphi} \in (L^r(\Omega))^2 ; \frac{\partial \hat{\varphi}_i}{\partial \kappa} \in L^\nu(\Omega) : \hat{\varphi} = 0 \ \text{ on } \Gamma_1 \cup \Gamma_L\right\}, \\ \tilde{V}_\kappa &= \left\{\hat{\varphi} \in V_\kappa :\ \hat{\varphi} \text{ satisfy } (\mathrm{D}')\right\}, \\ \Pi_\kappa &= \left\{\hat{\varphi} \in (L^q(\Omega))^2 ; \frac{\partial \hat{\phi}_i}{\partial \kappa} \in L^q(\Omega)\right\}, \end{aligned}$$

where the condition (D′) is given by

$$\int_\varpi \left(\hat{\varphi}_1 \frac{\partial \Psi}{\partial s_1} + \hat{\varphi}_2 \frac{\partial \Psi}{\partial s_2}\right) dsd\kappa = 0, \text{ for all } (\hat{\varphi}_1, \hat{\varphi}_2) \in (L^\nu(\Omega))^2 \text{ and } \Psi \in C_0^\infty(\Omega). \quad (\mathrm{D}')$$

By injecting the new data and unknown variables into equations $(3,9)$-(3.10),, we demonstrate that $\left(\hat{w}^{\epsilon},\hat{\rho}^{\epsilon},\hat{\theta}^{\epsilon}\right)\in V_{\text{div}}\left(\Omega\right)L_{0}^{\nu'}(\Omega)W_{\Gamma_1\cup\Gamma_L}^{1,q'}\left(\Omega\right)$ form a solution to the following problem

$$\begin{aligned} B_0\left(\hat{w}^{\epsilon},\hat{w}^{\epsilon},\hat{\varphi}-\hat{w}^{\epsilon}\right)+a_0(\hat{\theta}^{\epsilon},\hat{w}^{\epsilon},\hat{\varphi}-\hat{w}^{\epsilon})-\left(\hat{\rho}^{\epsilon},div\ \left(\hat{\varphi}-\hat{w}^{\epsilon}\right)\right)+j_0(\hat{\theta}^{\epsilon},\hat{\varphi})\\ -j_0\left(\hat{\theta}^{\epsilon},\hat{w}^{\epsilon}\right)\geq\left(\hat{f},\hat{\varphi}-\hat{w}^{\epsilon}\right),\ \forall\hat{\varphi}\in V\left(\Omega\right) \end{aligned} \tag{3.20}$$

$$C_0\left(\hat{\theta}^{\epsilon},\hat{\Phi}\right)-E_0\left(\hat{w}^{\epsilon},\hat{\theta}^{\epsilon},\hat{\Phi}\right)=F_0\left(\hat{w}^{\epsilon},\hat{\theta}^{\epsilon},\hat{\Phi}\right),\quad\forall\hat{\Phi}\in W_{\Gamma_1\cup\Gamma_L}^{1,q'}\left(\Omega\right), \tag{3.21}$$

where

$$\begin{aligned} a_0(\hat{\theta}^{\epsilon},\hat{w}^{\epsilon},\hat{\varphi}-\hat{w}^{\epsilon})=&\sum_{i,j=1}^{2}\int_{\Omega}\left[\epsilon^2\hat{\Lambda}\left(\hat{\theta}^{\epsilon}\right)\left|\tilde{D}\left(\hat{w}^{\epsilon}\right)\right|^{\nu-2}\left(\frac{1}{2}\left(\frac{\partial\hat{w}_i^{\epsilon}}{\partial s_j}+\frac{\partial\hat{w}_j^{\epsilon}}{\partial s_i}\right)\right)\right]\frac{\partial(\hat{\varphi}_i-\hat{w}_i^{\epsilon})}{\partial s_j}dsd\kappa\\ &+\sum_{i=1}^{2}\int_{\Omega}\hat{\Lambda}\left(\hat{\theta}^{\epsilon}\right)\left|\tilde{D}\left(\hat{w}^{\epsilon}\right)\right|^{r-2}\left(\frac{1}{2}\left(\frac{\partial\hat{w}_i^{\epsilon}}{\partial\kappa}+\epsilon^2\frac{\partial\hat{w}_3^{\epsilon}}{\partial s_i}\right)\right)\frac{\partial(\hat{\varphi}_i-\hat{w}_i^{\epsilon})}{\partial\kappa}dsd\kappa\\ &+\int_{\Omega}\left(\hat{\Lambda}\left(\hat{\theta}^{\epsilon}\right)\left|\tilde{D}\left(\hat{w}^{\epsilon}\right)\right|^{r-2}\epsilon^2\frac{\partial\hat{w}_3^{\epsilon}}{\partial\kappa}\right)\frac{\partial(\hat{\varphi}_3-\hat{w}_3^{\epsilon})}{\partial\kappa}dsd\kappa+\\ &+\sum_{j=1}^{2}\int_{\Omega}\epsilon^2\hat{\Lambda}\left(\hat{\theta}^{\epsilon}\right)\left|\tilde{D}\left(\hat{w}^{\epsilon}\right)\right|^{\nu-2}\left(\frac{1}{2}\left(\epsilon^2\frac{\partial\hat{w}_3^{\epsilon}}{\partial s_j}+\frac{\partial\hat{w}_j^{\epsilon}}{\partial\kappa}\right)\right)\frac{\partial(\hat{\varphi}_3-\hat{w}_3^{\epsilon})}{\partial s_j}dsd\kappa, \end{aligned}$$

$$\begin{aligned} B_0\left(\hat{w}^{\epsilon},\hat{w}^{\epsilon},\hat{\varphi}-\hat{w}^{\epsilon}\right)=&\sum_{i,j=1}^{2}\int_{\Omega}\epsilon^2\hat{w}_i^{\epsilon}\frac{\partial\hat{w}_j^{\epsilon}}{\partial s_i}\left(\hat{\varphi}-\hat{w}^{\epsilon}\right)dsd\kappa\\ &+\sum_{i=1}^{2}\int_{\Omega}\epsilon^4\hat{w}_i^{\epsilon}\frac{\partial\hat{w}_3^{\epsilon}}{\partial s_i}\left(\hat{\varphi}-\hat{w}^{\epsilon}\right)dsd\kappa+\sum_{i=1}^{2}\int_{\Omega}\epsilon^2\hat{w}_3^{\epsilon}\frac{\partial\hat{w}_i^{\epsilon}}{\partial\kappa}\left(\hat{\varphi}_i-\hat{w}_i^{\epsilon}\right)dsd\kappa\\ &+\int_{\Omega}\epsilon^4\hat{w}_3^{\epsilon}\frac{\partial\hat{w}_3^{\epsilon}}{\partial\kappa}\left(\hat{\varphi}_3-\hat{w}_3^{\epsilon}\right)dsd\kappa, \end{aligned}$$

$$(\hat{\rho}^{\epsilon}, div\,(\hat{\varphi}-\hat{w}^{\epsilon}))=\int_{\Omega^{\epsilon}}\hat{\rho}^{\epsilon}div\,(\hat{\varphi}-\hat{w}^{\epsilon})\,dsd\kappa,$$

$$j_0(\hat{\theta}^{\epsilon},\hat{\varphi})=\sqrt{2}\int_{\Omega}\hat{g}\left(\hat{\theta}^{\epsilon}\right)\left|\tilde{D}\,(\hat{\varphi})\right|dsd\kappa+\int_{\varpi}\hat{k}|\hat{\varphi}|ds,$$

$$(\hat{f}^{\epsilon},\hat{\varphi}-\hat{w}^{\epsilon})=\sum_{j=1}^{2}\int_{\Omega}\hat{f}_i(\hat{\varphi}_i-\hat{w}_i^{\epsilon})dsd\kappa+\int_{\Omega}\epsilon\hat{f}_3(\hat{\vartheta}_3-\hat{w}_3^{\epsilon})dsd\kappa,$$

$$b_0\left(\hat{\theta}^{\epsilon},\hat{\Phi}\right)=\int_{\Omega}\epsilon^2\nabla_{\epsilon}\hat{\theta}^{\epsilon}\nabla_{\epsilon}\hat{\Phi}dsd\kappa=\sum_{i=1}^{2}\int_{\Omega}\epsilon^2\hat{K}\frac{\partial\hat{\theta}^{\epsilon}}{\partial s_i}\frac{\partial\hat{\Phi}}{\partial s_i}dsd\kappa+\int_{\Omega}\hat{K}\frac{\partial\hat{\theta}^{\epsilon}}{\partial\kappa}\frac{\partial\hat{\Phi}}{\partial\kappa}dsd\kappa,$$

$$E_0\left(\hat{w}^{\epsilon},\hat{\theta}^{\epsilon},\hat{\Phi}\right)=\int_{\Omega^{\epsilon}}\epsilon^{\beta+\nu}\hat{\theta}^{\epsilon}\nabla\hat{\Phi}\hat{w}^{\epsilon}dsd\kappa$$

$$C_0\left(\hat{\theta}^{\epsilon},\hat{\Phi}\right)=\int_{\Omega}\epsilon^2\nabla_{\epsilon}\hat{\theta}^{\epsilon}\nabla_{\epsilon}\hat{\Phi}dsd\kappa=\sum_{i=1}^{2}\int_{\Omega}\epsilon^2\hat{K}\frac{\partial\hat{\theta}^{\epsilon}}{\partial s_i}\frac{\partial\hat{\Phi}}{\partial s_i}dsd\kappa+\int_{\Omega}\hat{K}\frac{\partial\hat{\theta}^{\epsilon}}{\partial\kappa}\frac{\partial\hat{\Phi}}{\partial\kappa}dsd\kappa,$$

$$F_0\left(\hat{w}^{\epsilon},\hat{\theta}^{\epsilon},\hat{\Phi}\right)=\int_{\Omega}\epsilon^{\beta}\hat{\Lambda}\left(\hat{\theta}^{\epsilon}\right)\left|\tilde{D}(\hat{w}^{\epsilon})\right|^{\nu}\hat{\Phi}dsd\kappa+\sqrt{2}\int_{\Omega}\epsilon^{\beta}\hat{g}\left(\hat{\theta}^{\epsilon}\right)\left|\tilde{D}(\hat{w}^{\epsilon})\right|\hat{\Phi}dsd\kappa-\int_{\Omega}\hat{\alpha}\hat{\theta}^{\epsilon}\hat{\Phi}dsd\kappa,$$

$$\left|\widetilde{D}\,(\hat{w}^{\epsilon})\right|=\left(\frac{1}{4}\sum_{i,j=1}^{2}\epsilon^2\left(\frac{\partial\hat{w}_i^{\epsilon}}{\partial s_j}+\frac{\partial\hat{w}_j^{\epsilon}}{\partial s_i}\right)^2+\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial\hat{w}_i^{\epsilon}}{\partial\kappa}+\epsilon^2\frac{\partial\hat{w}_3^{\epsilon}}{\partial s_i}\right)^2+\epsilon^2\left(\frac{\partial\hat{w}_3^{\epsilon}}{\partial\kappa}\right)^2\right)^{\frac{1}{2}}.$$

3.3.1 A priori estimates on the velocity and the pressure

Theorem 3.1. *For all* $1<\nu<2$ *and under assumptions* $(3.14)-(3.15)$ *and* (3.19), *there exists a constant* $C>0$ *independent of* ϵ *such that*

$$\sum_{i,j=1}^{2}\left\|\epsilon\frac{\partial\hat{w}_i^\epsilon}{\partial s_j}\right\|_{L^\nu(\Omega)}^{\nu}+\sum_{i=1}^{2}\left(\left\|\frac{\partial\hat{w}_i^\epsilon}{\partial\kappa}\right\|_{L^\nu(\Omega)}^{\nu}+\left\|\epsilon^2\frac{\partial\hat{w}_3^\epsilon}{\partial s_i}\right\|_{L^\nu(\Omega)}^{\nu}\right)+\left\|\epsilon\frac{\partial\hat{w}_3^\epsilon}{\partial\kappa}\right\|_{L^\nu(\Omega)}^{\nu}\leq C. \tag{3.22}$$

$$\left\|\frac{\partial\hat{\rho}^\epsilon}{\partial s_i}\right\|_{W^{-1,\nu'}(\Omega)}\leq C\quad \textit{for } i=1,2 \tag{3.23}$$

$$\left\|\frac{\partial\hat{\rho}^\epsilon}{\partial\kappa}\right\|_{W^{-1,\nu'}(\Omega)}\leq \epsilon C. \tag{3.24}$$

Proof. Choosing $\varphi=0$ in inequality (3.9), we find

$$\begin{aligned}B\left(w^\epsilon,w^\epsilon,w^\epsilon\right)+a(\theta^\epsilon,w^\epsilon,w^\epsilon)+\int_{\varpi}k^\epsilon\left|w^\epsilon\right|\ ds\\ +\sqrt{2}\int_{\Omega^\epsilon}g^\epsilon\left(\theta^\epsilon\right)\left|D\left(w^\epsilon\right)\right|dsds_3\leq(f^\epsilon,w^\epsilon),\end{aligned} \tag{3.25}$$

as $B\left(w^\epsilon,w^\epsilon,w^\epsilon\right)=0$, we obtain

$$a(\theta^\epsilon,w^\epsilon,w^\epsilon)+\int_{\varpi}k^\epsilon\left|w^\epsilon\right|\ ds+\sqrt{2}\int_{\Omega^\epsilon}g^\epsilon\left(\theta^\epsilon\right)\left|D\left(w^\epsilon\right)\right|dsds_3\leq(f^\epsilon,w^\epsilon).$$

By Poincare inequality and Young inequality, we have

$$\begin{aligned}a(\theta^\epsilon,w^\epsilon,w^\epsilon)+\int_{\varpi}k^\epsilon\left|w^\epsilon\right|\ ds+\sqrt{2}\int_{\Omega^\epsilon}g^\epsilon\left(\theta^\epsilon\right)\left|D\left(w^\epsilon\right)\right|dsds_3\leq\frac{1}{2}\Lambda_*C_k\left\|\nabla w^\epsilon\right\|_{L^\nu(\Omega^\epsilon)}^{\nu}\\ +\frac{\left(\epsilon h^*\right)^{\nu'}}{\nu'\left(\frac{1}{2}\Lambda_*\nu C_k\right)^{\frac{\nu'}{\nu}}}\left\|f^\epsilon\right\|_{L^{\nu'}(\Omega^\epsilon)}^{\nu'}.\end{aligned} \tag{3.26}$$

Multiplying (3.26) by $\epsilon^{\nu-1}$,, and as $\epsilon^{\nu'}\left\|f^\epsilon\right\|_{L^{\nu'}(\Omega^\epsilon)}^{\nu'}=\epsilon^{1-\nu}\left\|\hat{f}\right\|_{L^{\nu'}(\Omega)}^{\nu'}$, we have

$$\epsilon^{\nu-1}a\left(w^{\epsilon},w^{\epsilon}\right)+\int_{\varpi}\hat{k}\left|\hat{w}^{\epsilon}\right|ds+\sqrt{2}\int_{\Omega}\hat{g}\left(\hat{\theta}^{\epsilon}\right)\left|\tilde{D}\left(\hat{w}^{\epsilon}\right)\right|dsd\kappa\leq\frac{1}{2}\Lambda_{*}C_{k}\epsilon^{\nu-1}\left\|\nabla w^{\epsilon}\right\|_{L^{\nu}(\Omega^{\epsilon})}^{\nu}+\frac{\left(h^{*}\right)^{\nu'}}{\nu'\left(\frac{1}{2}\Lambda_{*}\nu C_{k}\right)^{\frac{r'}{r}}}\left\|\hat{f}\right\|_{L^{\nu'}(\Omega)}^{\nu'}.$$

According to Korn's inequality, a constant C_k exists, independent of ϵ, such that.

$$\frac{1}{2}\Lambda_{*}C_{k}\epsilon^{\nu-1}\left\|\nabla w^{\epsilon}\right\|_{L^{\nu}(\Omega^{\epsilon})}^{\nu}+\sqrt{2}\int_{\Omega}\hat{g}\left(\hat{\theta}\right)\left|\tilde{D}\left(\hat{w}^{\epsilon}\right)\right|dsd\kappa+\int_{\varpi}\hat{k}\left|\hat{w}^{\epsilon}\right|ds\leq\frac{\left(h^{*}\right)^{\nu'}}{\nu'\left(\frac{1}{2}\Lambda_{*}\nu C_{k}\right)^{\frac{\nu'}{\nu}}}\left\|\hat{f}\right\|_{L^{\nu'}(\Omega)}^{\nu'}\tag{3.27}$$

So, from (3.27) we deduce (3.22), with $C=(\frac{1}{2}\Lambda_{*}C_{k})^{-1}\dfrac{\left(h^{*}\right)^{\nu'}}{\nu'\left(\frac{1}{2}\Lambda_{*}\nu C_{k}\right)^{\frac{\nu'}{\nu}}}\left\|\hat{f}\right\|_{L^{\nu'}(\Omega)}^{\nu'}$.

For (3.23) and (3.24), we prove as in [19].

□

3.3.2 A priori estimates on the temperature

Theorem 3.2. *suppose that the hypotheses of Theorem* 3.1 *are verified. Then a positive constant* C_1 *that is independent of* ϵ *exists, such that*

$$\left\|\frac{\partial\hat{\theta}^{\epsilon}}{\partial\kappa}\right\|_{W^{1,q}(\Omega)}\leq C_{1}\tag{3.28}$$

$$\sum_{i=1}^{2}\left\|\epsilon\frac{\partial\hat{\theta}^{\epsilon}}{\partial s_{i}}\right\|_{W^{1,q}(\Omega)}\leq C_{1}\tag{3.29}$$

Proof. Choosing $\Phi=\vartheta\left(\theta^{\epsilon}\right)$ in (3.21) as ϑ is given by

$$\vartheta\left(\theta\right)=\zeta sign\left(\theta\right)\int_{0}^{|\theta|}\frac{d\tau}{\left(1+|\theta|\right)^{\zeta+1}}=sign\left(\theta\right)\left[1-\frac{1}{\left(1+|\theta|\right)^{\zeta}}\right],$$

we obtain

$$\zeta K_* \int_\Omega \frac{\left|\nabla\hat{\theta}^\epsilon\right|^2}{\left(1+\left|\hat{\theta}^\epsilon\right|\right)^{\zeta+1}} \leq \Lambda^* \epsilon^{\beta-2} \int_\Omega |\tilde{D}\left(\hat{w}^\epsilon\right)|^\nu ds d\kappa + \epsilon^{\beta-2}\sqrt{2}g^* \int_\Omega |\tilde{D}\left(\hat{w}^\epsilon\right)| ds d\kappa + \epsilon^{\beta+\nu-2}\zeta \int_\Omega \frac{\hat{\theta}^\epsilon \left|\nabla\hat{\theta}^\epsilon\right|}{\left(1+\left|\hat{\theta}^\epsilon\right|\right)^{\zeta+1}} \hat{w}^\epsilon ds d\kappa \tag{3.30}$$

Using Young inequality with for $q < \frac{3}{2} < p$, we obtain

$$\epsilon^{\beta+\nu-2} \int_\Omega \frac{\hat{\theta}^\epsilon \left|\nabla\hat{\theta}^\epsilon\right|}{\left(1+\left|\hat{\theta}^\epsilon\right|\right)^{\zeta+1}} \hat{w}^\epsilon ds d\kappa \leq \epsilon^{\beta+\nu-2}\frac{1}{q}\int_\Omega \left(\frac{\hat{\theta}^\epsilon \left|\nabla\hat{\theta}^\epsilon\right|}{\left(1+\left|\hat{\theta}^\epsilon\right|\right)^{\zeta+1}}\right)^q ds d\kappa + \epsilon^{\beta+\nu-2}\frac{1}{p}\int_\Omega (\hat{w}^\epsilon)^p \, ds d\kappa \tag{3.31}$$

On the other hand

$$\int_\Omega |\tilde{D}\left(\hat{w}^\epsilon\right)|^\nu ds d\kappa \leq C\left(\nu\right) \left[\sum_{i,j=1}^{2} \left\|\epsilon\frac{\partial\hat{w}_i^\epsilon}{\partial s_j}\right\|_{L^\nu(\Omega)}^\nu + \sum_{i=1}^{2}\left(\left\|\frac{\partial\hat{w}_i^\epsilon}{\partial\kappa}\right\|_{L^\nu(\Omega)}^\nu + \left\|\epsilon^2\frac{\partial\hat{w}_3^\epsilon}{\partial s_i}\right\|_{L^\nu(\Omega)}^\nu\right)^2 + \left\|\epsilon\frac{\partial\hat{w}_3^\epsilon}{\partial\kappa}\right\|_{L^\nu(\Omega)}^\nu\right.$$

as $C\left(\nu\right) > 0$ depends just on ν. Since $\nu > 1$ and $0 < \epsilon < 1$ it follows that $\epsilon^{\nu-1} \leq 1$, Applying this inequality along with (3.22), we conclude that

$$\begin{aligned}\zeta K_* \int_\Omega \frac{\left|\nabla_\epsilon\hat{\theta}^\epsilon\right|^2}{\left(1+\left|\hat{\theta}^\epsilon\right|\right)^{\zeta+1}} \leq & \left(\Lambda^* C\left(\nu\right) C + \sqrt{2}g^* C\right)\epsilon^{\beta-2} \\ & + \epsilon^{\beta+\nu-2}\zeta\frac{1}{q}\int_\Omega \left(\frac{\hat{\theta}^\epsilon \left|\nabla\hat{\theta}^\epsilon\right|}{\left(1+\left|\hat{\theta}^\epsilon\right|\right)^{\zeta+1}}\right)^q ds d\kappa \\ & + \epsilon^{\beta+\nu-2}\zeta\frac{1}{p}\int_\Omega (\hat{w}^\epsilon)^p \, ds d\kappa\end{aligned} \tag{3.32}$$

Applying Young's inequality with exponents $\frac{2}{q}$ and $\frac{2}{2-q}$, for $q < \frac{3}{2}$, we get

$$\begin{aligned}\epsilon^{\beta+\nu-2}\left(\frac{\hat{\theta}^{\epsilon}\left|\nabla\hat{\theta}^{\epsilon}\right|}{\left(1+\left|\hat{\theta}^{\epsilon}\right|\right)^{\zeta+1}}\right)^{q} &\leq \frac{q}{2}\frac{\left|\nabla_{\epsilon}\hat{\theta}^{\epsilon}\right|^{2}}{\left(1+\left|\hat{\theta}^{\epsilon}\right|\right)^{\zeta+1}} \\ &+ \frac{2-q}{2}\epsilon^{(\beta+\nu-2)\frac{2}{2-q}}\frac{\left(\hat{\theta}^{\epsilon}\right)^{\frac{2q}{2-q}}}{\left(1+\left|\hat{\theta}^{\epsilon}\right|\right)^{(\frac{\zeta+1}{2})\frac{2q}{2-q}}}\end{aligned} \tag{3.33}$$

As $\epsilon^{(\beta+\nu-2)^{\frac{2q}{2-q}}} \leq \epsilon^{(\beta-2)}$ and $\frac{(\hat{\theta}^{\epsilon})^{\frac{2q}{2-q}}}{\left(1+\left|\hat{\theta}^{\epsilon}\right|\right)^{(\frac{\zeta+1}{2})\frac{2q}{2-q}}} \leq 1$, we deduce

$$\zeta\left(K_{*}-\frac{1}{2}\right)\int_{\Omega}\frac{\left|\nabla_{\epsilon}\hat{\theta}^{\epsilon}\right|^{2}}{\left(1+\left|\hat{\theta}^{\epsilon}\right|\right)^{\zeta+1}} \leq \left(\Lambda^{*}C\left(\nu\right)C+\sqrt{2}g^{*}C+\frac{2-q}{2q}\zeta\Omega+\frac{1}{p}\zeta C\right)\epsilon^{\beta-2} \tag{3.34}$$

Applying Hölder's inequality with exponents $\frac{2}{q}$ and $\frac{2}{2-q}$, for $q < \frac{3}{2}$, we find

$$\int_{\Omega}\left|\nabla_{\epsilon}\hat{\theta}^{\epsilon}\right|^{q}dsd\kappa \leq \left(\int_{\Omega}\frac{\left|\nabla_{\epsilon}\hat{\theta}^{\epsilon}\right|^{2}}{\left(1+\left|\hat{\theta}^{\epsilon}\right|\right)^{\zeta+1}}\right)^{\frac{q}{2}}\left(\int_{\Omega}\left(1+\left|\hat{\theta}^{\epsilon}\right|\right)^{\frac{(\zeta+1)q}{2-q}}\right)^{\frac{2-q}{2}},$$

using (3.34), we get

$$\int_{\Omega}|\nabla_{\epsilon}\hat{\theta}^{\epsilon^{q}}dsd\kappa \leq$$
$$\left(\frac{1}{\zeta\left(K_{*}-\frac{1}{2}\right)}\left(\Lambda^{*}C\left(\nu\right)C+\sqrt{2}g^{*}C+\frac{2-q}{2q}\zeta\Omega+\frac{1}{p}\zeta C\right)\epsilon^{\beta-2}\right)^{\frac{q}{2}}\left(\int_{\Omega}\left(1+\left|\hat{\theta}^{\epsilon}\right|\right)^{q*}\right)^{\frac{2-q}{2}},$$

where $q^* = \frac{3q}{3-q} \geq \frac{(\zeta+1)q}{2-q}$. ,by (3.16) and (3.17) , we obtain

$$\int_\Omega \left|\nabla_\epsilon \hat{\theta}^\epsilon\right|^q dsd\kappa \leq \left(\frac{1}{\zeta\left(K_* - \frac{1}{2}\right)}\left(\Lambda^* C\left(\nu\right) C + \sqrt{2} g^* C + \frac{2-q}{2q}\zeta\Omega + \frac{1}{p}\zeta C\right)\epsilon^{\beta-2}\right)^{\frac{q}{2}} 2^{(q^*-1)\frac{2-q}{2}} \left(|\Omega|^{\frac{2-q}{2}} + \left(\int_\Omega \left|\hat{\theta}^\epsilon\right|^{q*}\right)^{\frac{2-q}{2}}\right). \tag{3.35}$$

Now applying the Poincaré inequality, we get

$$\begin{aligned}\left(\int_\Omega \left|\hat{\theta}^\epsilon\right|^{q*} dsd\kappa\right)^{\frac{1}{q^*}} &\leq C' \left\|\nabla_\epsilon \hat{\theta}^\epsilon\right\|_{L^q(\Omega)} \\ &\leq \left(\frac{1}{\zeta\left(K_* - \frac{1}{2}\right)}\left(\Lambda^* C\left(\nu\right) C + \sqrt{2} g^* C + \frac{2-q}{2q}\zeta\Omega + \frac{1}{p}\zeta C\right)\right)^{\frac{1}{2}} \epsilon^{\frac{\beta}{2}-1} 2^{(q^*-1)\frac{2-q}{2q}} \\ &\quad C'\left(|\Omega|^{\frac{2-q}{2q}} + \left(\int_\Omega \left|\hat{\theta}^\epsilon\right|^{q*}\right)^{\frac{2-q}{2q}}\right). \end{aligned} \tag{3.36}$$

Moreover, for all $x > 0,\ y > 0,\ z > 0$ and $0 < n < m$, it follows that:

$$\text{If } x^m \leq y + zx^n \text{ then } x \leq \max\left\{1, (y+z)^{\frac{1}{m-n}}\right\}. \tag{3.37}$$

Thus, from (3.36) − (3.37) and the fact that $\frac{2-q}{2} < 1$, we deduce

$$\int_\Omega \left|\hat{\theta}^\epsilon\right|^{q*} dsd\kappa \leq \xi_\epsilon, \tag{3.38}$$

where

$$\xi = \max\left[1,\ \gamma\epsilon^{\left(\frac{\beta}{2}-1\right)\left(\frac{1}{q^*}-\frac{2-q}{2q}\right)^{-1}\frac{2-q}{2}}\right] = \max\left[1,\ \gamma\epsilon^{3\left(\frac{\beta}{2}-1\right)(2-q)}\right],$$

and

$$\gamma = \left[\left(\frac{1}{\zeta\left(K_* - \frac{1}{2}\right)}\left(\Lambda^* C\left(\nu\right) C + \sqrt{2} g^* C + \frac{2-q}{2q}\zeta\Omega + \frac{1}{p}\zeta C\right)\right)^{\frac{1}{2}} 2^{(q^*-1)\frac{2-q}{2q}} C'\left(|\Omega|^{\frac{2-q}{2q}} + 1\right)\right]^{\left(\frac{1}{q^*} - \frac{2-q}{2q}\right.}$$

$$= \left[\left(\frac{1}{\zeta\left(K_* - \frac{1}{2}\right)}\left(\Lambda^* C\left(\nu\right) C + \sqrt{2} g^* C + \frac{2-q}{2q}\zeta\Omega + \frac{1}{p}\zeta C\right)\right)^{\frac{1}{2}} 2^{(q^*-1)\frac{2-q}{2q}} C'\left(|\Omega|^{\frac{2-q}{2q}} + 1\right)\right]^{6}.$$

As $\beta = 3\frac{2-q}{3-q}$, then $3\left(2-q\right)\left(\frac{\beta}{2} - 1\right) < 0$. Thus for $\epsilon \leq \beta^{\left[-3(2-q)\left(\frac{\beta}{2}-1\right)\right]^{-1}}$, We find

$$\xi = \gamma \epsilon^{3\left(\frac{\beta}{2}-1\right)(2-q)} \geq 1.$$

From (3.35) and (3.38), we obtain

$$\epsilon^q \int_\Omega \left|\nabla_\epsilon \hat{\theta}^\epsilon\right|^q ds d\kappa \leq \left(\frac{1}{\zeta\left(K_* - \frac{1}{2}\right)}\left(\Lambda^* C\left(\nu\right) C + \sqrt{2} g^* C + \frac{2-q}{2q}\zeta\Omega + \frac{1}{p}\zeta C\right)\right)^{\frac{q}{2}}$$

$$2^{(q^*-1)\frac{2-q}{2}}$$

$$\left(|\Omega|^{\frac{2-q}{2}} \epsilon^{\frac{\beta}{2}q} + \gamma \epsilon^{3\left(\frac{\beta}{2}-1\right)(2-q)+\frac{\beta}{2}q}\right),$$

as $3\left(\frac{\beta}{2} - 1\right)(2-q) + \frac{\beta}{2}q = 0$ and $\frac{\beta}{2}q > 0$, we get

$$\epsilon^q \int_\Omega \left|\nabla_\epsilon \hat{\theta}^\epsilon\right|^q ds d\kappa \leq C_1,$$

where

$$C_1 = \left(\frac{1}{\zeta\left(K_* - \frac{1}{2}\right)}\left(\Lambda^* C\left(\nu\right) C + \sqrt{2} g^* C + \frac{2-q}{2q}\zeta\Omega + \frac{1}{p}\zeta C\right)\right)^{\frac{q}{2}} 2^{(q^*-1)\frac{2-q}{2}}\left(|\Omega|^{\frac{2-q}{2}} + \gamma\right).$$

Where C_1 is a constant independent of ϵ.Thus, we obtain (3.28) and (3.29)

□

3.3.3 Theorem of Convergence

Theorem 3.3. *According to the same hypotheses in Theorem* 3.1 *and* (3.2)*, there exist* $w^\star = (w_1^\star, w_2^\star) \in \tilde{V}_\kappa$, $\rho^\star \in L_0^{\nu'}(\Omega)$ *and* $\theta^* \in \Pi_\kappa$ *such that:*

$$\hat{w}_i^\epsilon \rightharpoonup w_i^\star, \ i = 1, 2 \quad \textit{weakly in } \tilde{V}_\kappa, \tag{3.39}$$

$$\epsilon \frac{\partial \hat{w}_i^\epsilon}{\partial s_j} \rightharpoonup 0, \ i, j = 1, 2 \ \textit{weakly in} \ \ L^\nu(\Omega) \tag{3.40}$$

$$\epsilon \frac{\partial \hat{w}_3^\epsilon}{\partial \kappa} \rightharpoonup 0, \quad \textit{weakly in} \ \ L^\nu(\Omega) \tag{3.41}$$

$$\epsilon^2 \frac{\partial \hat{w}_3^\epsilon}{\partial s_i} \rightharpoonup 0, \ i = 1, 2 \ \textit{weakly in} \ \ L^\nu(\Omega) \tag{3.42}$$

$$\epsilon \hat{w}_3^\epsilon \rightharpoonup 0, \quad \textit{weakly in} \ \ L^\nu(\Omega), \tag{3.43}$$

$$\hat{\rho}^\epsilon \rightharpoonup \rho^\star, \textit{weakly in} \ \ L^{\nu'}(\Omega), \ \rho^\star \ \textit{depend only of } s, \tag{3.44}$$

$$\begin{cases} \hat{\theta}^\epsilon \rightharpoonup \theta^* \ \textit{weakly in } \Pi_\kappa, \\ \\ \frac{\partial \hat{\theta}^\epsilon}{\partial s_i} \rightharpoonup 0, \ i = 1, 2 \ \textit{weakly in } L^q(\Omega). \end{cases} \tag{3.45}$$

Proof. From (3.22), there exists a constant C independent of ϵ such that

$$\left\| \frac{\partial \hat{w}_i^\epsilon}{\partial \kappa} \right\|_{L^r(\Omega)} \leq C \quad (1 \leq i \leq 2).$$

Using this estimate and the Poincaré inequality, we deduce that the sequence $(\hat{w}_1^\epsilon, \hat{w}_2^\epsilon)_\epsilon$ is bounded in V_z. Since this space is reflexive, we obtain the weak convergence result (3.39) in V_κ.

We now show that $w^\star \in \tilde{V}_z$. Indeed, since $\mathrm{div}(\hat{w}^\epsilon) = 0$ in Ω, for all $q \in \mathcal{C}_0^\infty()$:

$$\int_\Omega q(s) \left(\frac{\partial \hat{w}_1^\epsilon}{\partial s_1} + \frac{\partial \hat{w}_2^\epsilon}{\partial s_2} + \frac{\partial \hat{w}_3^\epsilon}{\partial \kappa} \right) ds d\kappa = - \int_\Omega q(s) \left(\frac{\partial \hat{w}_1^\epsilon}{\partial s_1} + \frac{\partial \hat{w}_2^\epsilon}{\partial s_2} \right) ds d\kappa = 0,$$

because $\hat{w}^\epsilon \cdot n = 0$ on . Since $\hat{w}_i^\epsilon \rightharpoonup w_i^\star$ (for $i = 1, 2$) in V_κ, $w^\star$ satisfies condition (D'):

$$\int_\Omega q(s) \left(\frac{\partial w_1^\star}{\partial s_1} + \frac{\partial w_2^\star}{\partial s_2} \right) ds d\kappa = 0, \quad \forall q \in \mathcal{C}_0^\infty().$$

The convergences $(3.40) - (3.42)$ follow directly from (3.22). To prove (3.43), we first use the Poincaré inequality, which guarantees the existence of a constant $C > 0$ independent of ϵ such that

$$\epsilon \|\hat{w}_3^\epsilon\|_{L^r(\Omega)} \leq \epsilon h_{\max} \left\| \frac{\partial \hat{w}_3^\epsilon}{\partial \kappa} \right\|_{L^r(\Omega)} \leq C.$$

Since $L^r(\Omega)$ is reflexive, there exists a function $l \in L^r(\Omega)$ such that

$$\varepsilon \hat{w}_3^\varepsilon \rightharpoonup l \quad \text{weakly in} \quad L^r(\Omega).$$

We first show that this limit l depends only on s, and then prove that $l = 0$.

From $\mathrm{div}(\hat{w}^\epsilon) = 0$ in Ω, we have

$$\sum_{i=1}^{2} \int_\Omega q(s) \frac{\partial \hat{w}_i^\epsilon}{\partial s_i} ds d\kappa + \int_\Omega q(s) \frac{\partial \hat{w}_3^\epsilon}{\partial \kappa} ds d\kappa = 0, \quad \forall q \in D(\Omega).$$

Applying Green's formula yields

$$\sum_{i=1}^{2} \int_\Omega \epsilon \hat{w}_i^\epsilon \frac{\partial q}{\partial s_i} ds \kappa + \int_\Omega \epsilon w_3^\epsilon \frac{\partial q}{\partial \kappa} ds d\kappa = 0, \quad \forall q \in D(\Omega).$$

Passing to the limit as $\epsilon \to 0$ and using (3.39), we deduce that

$$\int_\Omega l \frac{\partial q}{\partial \kappa} ds d\kappa = 0, \quad \forall q \in D(\Omega),$$

which implies

$$\frac{\partial l}{\partial \kappa} = 0 \quad \text{a.e. in} \quad \Omega.$$

Following [1], for example, we choose $\Psi(s,\kappa) = \kappa\Psi(s) - \gamma$, where $\Psi \in \mathcal{C}_0^\infty()$ and

$$\gamma = |\Omega|^{-1} \int_\Omega \kappa\Psi ds d\kappa.$$

Using the fact that $\text{div}(\hat{w}^\epsilon) = 0$ in Ω, we obtain

$$\sum_{i=1}^{2} \int_\Omega \kappa\Psi \frac{\partial \hat{w}_i^\epsilon}{\partial s_i} ds d\kappa + \int_\Omega \kappa\Psi \frac{\partial \hat{w}_3^\epsilon}{\partial \kappa} ds d\kappa = 0.$$

Applying Green's formula again gives

$$\sum_{i=1}^{2} \int_\Omega \kappa \hat{w}_i^\epsilon \frac{\partial \Psi}{\partial s_i} ds d\kappa + \int_\Omega \hat{w}_3^\epsilon \Psi ds d\kappa = 0.$$

Since $\hat{w}_i^\epsilon \rightharpoonup w_i^*$ (for $i = 1, 2$) in V_κ and $\epsilon\hat{w}_i^\epsilon \rightharpoonup 0$ (for $i = 1, 2$) in $L^r(\Omega)$, it follows that

$$\int_\Omega \theta l ds d\kappa = \int_\varpi l h \Psi ds = 0, \quad \forall \Psi \in \mathcal{C}_0^\infty().$$

By the density of $\mathcal{C}_0^\infty()$ in $L^r()$, we conclude that $l = 0$ almost everywhere in Ω.

The proof of (3.44) is straightforward. Indeed, from (3.23) – (3.24), there exists a constant $C' > 0$ such that

$$\|\nabla \rho^\epsilon\|_{W^{-1,r'}(\Omega)} \leq C'.$$

Since $\rho^\epsilon \in L_0^{r'}(\Omega)$, there exists a constant $C'' > 0$ such that

$$\|\rho^\epsilon\|_{L^r(\Omega)} \leq C''\|\nabla\rho^\epsilon\|_{W^{-1,r'}(\Omega)}.$$

Thus,

$$\|\rho^\varepsilon\|_{L^r(\Omega)} \leq C'C'',$$

and consequently, (3.44) follows. □

3.4 Study of the Limit Problem

3.4.1 Limit Problem and the Generalized Reynolds Equation

This section introduces the equations that are verified by w^* and ρ^* within the domain Ω along with the inequalities governing the trace of the velocity the stress $\frac{\partial w^*}{\partial \kappa}(s,0)$ and $w^*(s,0)$ on the boundary ϖ

Lemma 3.1. *Under the same hypotheses as Theorem 3.3, the limit solutions w^*, p^* satisfy the following relations:*

$$w_i^* \to w_i^* \text{ strongly in } V_\kappa \quad \text{for} \quad i = 1,2, \quad \forall 1 < r < 2, \tag{3.46}$$

$$\rho^*(s,\kappa) = \rho^*(s) \text{ almost everywhere in } \Omega, \tag{3.47}$$

$$\int_\Omega \rho^* \left(\frac{\partial w_1^*}{\partial s_1} + \frac{\partial w_2^*}{\partial s_2} \right) ds d\kappa = 0 \tag{3.48}$$

Proof. We choose $\hat{\phi}$ in (3.20) such that $\hat{\phi}_i = \hat{w}_i^\epsilon$ for $i = 1,2$, and $\hat{\phi}_3 = \hat{w}_3^\epsilon \pm \varphi$ with $\varphi \in W_0^{1,r}(\Omega)$. Thus,

$$
\begin{aligned}
(\hat{\rho}^{\epsilon}, \frac{\partial \varphi}{\partial \kappa}) &= \Lambda\left(\theta^{*}\right) \int_{\Omega} \left(\epsilon^{2} |\hat{D}(\hat{w}^{\epsilon})|^{r-2} \frac{\partial \hat{w}_{3}^{\epsilon}}{\partial \kappa} \frac{\partial \varphi}{\partial \kappa} \right. \\
&\quad + \left. \frac{1}{2} \epsilon^{2} |\hat{D}(\hat{w}^{\epsilon})|^{r-2} \sum_{i=1}^{2} \left(\frac{\partial \hat{w}_{i}^{\epsilon}}{\partial \kappa} + \epsilon^{2} \frac{\partial \hat{w}_{3}^{\epsilon}}{\partial s_{i}} \right) \frac{\partial \varphi}{\partial s_{i}} \right) ds d\kappa \\
&\quad - \epsilon(\hat{f}_{3}, \varphi) - \sum_{i=1}^{2} \int_{\Omega} \epsilon^{4} \hat{w}_{i}^{\epsilon} \frac{\partial \hat{w}_{3}^{\epsilon}}{\partial s_{i}} (\varphi) \, ds d\kappa + \sum_{i=1}^{2} \int_{\Omega} \epsilon^{2} \hat{w}_{3}^{\epsilon} \frac{\partial \hat{w}_{i}^{\epsilon}}{\partial \kappa} (\varphi) \, ds d\kappa \\
&= \Lambda\left(\theta^{*}\right) A_{\epsilon}(\varphi) - \epsilon(\hat{f}_{3}, \varphi),
\end{aligned}
$$

where

$$
\begin{aligned}
A_{\epsilon}(\varphi) = \Lambda\left(\theta^{*}\right) \int_{\Omega} \left(\varepsilon^{2} |\hat{D}(\hat{w}^{\epsilon})|^{r-2} \frac{\partial \hat{w}_{3}^{\epsilon}}{\partial \kappa} \frac{\partial \varphi}{\partial \kappa} \right. \\
+ \left. \frac{1}{2} \epsilon^{2} |\hat{D}(\hat{w}^{\epsilon})|^{r-2} \sum_{i=1}^{2} \left(\frac{\partial \hat{w}_{i}^{\epsilon}}{\partial \kappa} + \epsilon^{2} \frac{\partial \hat{w}_{3}^{\epsilon}}{\partial s_{i}} \right) \frac{\partial \varphi}{\partial s_{i}} \right) ds d\kappa.
\end{aligned}
$$

We have $\lim_{\epsilon \to 0} \varepsilon(\hat{f}_{3}, \varphi) = 0$. Now, we must show that $\lim_{\epsilon \to 0} A_{\epsilon}(\varphi) = 0$. Indeed, we have

$$
\left| \epsilon \frac{\partial \hat{w}_{3}^{\epsilon}}{\partial \kappa} \right| \leq |\hat{D}(\hat{w}^{\epsilon})|,
$$

$$
\left| \frac{\partial \hat{w}_{i}^{\epsilon}}{\partial \kappa} + \epsilon^{2} \frac{\partial \hat{w}_{3}^{\epsilon}}{\partial s_{i}} \right| \leq |\hat{D}(\hat{w}^{\epsilon})|.
$$

Using Hölder's inequality, we obtain:

$$|A_\epsilon(\varphi)| \leq \epsilon \left(\int_\Omega |\hat{D}(\hat{w}^\epsilon)|^r ds d\kappa \right)^{\frac{r-1}{r}} \left(\int_\Omega \left| \frac{\partial \varphi}{\partial \kappa} \right|^r ds d\kappa \right)^{\frac{1}{r}} + \epsilon^2 \left(\int_\Omega |\hat{D}(\hat{w}^\epsilon)|^r ds d\kappa \right)^{\frac{r-1}{r}} \sum_{i=1}^{2} \left(\int_\Omega \left| \frac{\partial \varphi}{\partial s_i} \right|^r ds d\kappa \right)^{\frac{1}{r}}. \tag{3.49}$$

On the other hand, from the discrete Hölder inequality, we know that for any $(a_i)_{1\leq i\leq n} \in \mathbb{R}^n$,

$$\left(\sum_{i=1}^{n} |a_i| \right)^r \leq n^{r-1} \left(\sum_{i=1}^{n} |a_i|^r \right) \quad \forall 1 < r < 2.$$

Thus, there exists a constant $C(r) > 0$, depending only on r, such that:

$$|\hat{D}(\hat{w}^\epsilon)|^r \leq C(r) \left(\sum_{i,j=1}^{2} \left| \epsilon \frac{\partial \hat{w}_i^\epsilon}{\partial s_j} \right|^r + \left| \epsilon \frac{\partial \hat{w}_3^\epsilon}{\partial \kappa} \right|^r + \sum_{i=1}^{2} \left(\left| \frac{\partial \hat{w}_i^\epsilon}{\partial \kappa} \right|^r + \left| \epsilon^2 \frac{\partial \hat{w}_3^\epsilon}{\partial s_i} \right|^r \right) \right). \tag{3.50}$$

From (3.22), :

$$\sum_{i,j=1}^{2} \left\| \epsilon \frac{\partial \hat{w}_i^\epsilon}{\partial s_j} \right\|_{L^r(\Omega)}^r + \left\| \epsilon \frac{\partial \hat{w}_3^\epsilon}{\partial \kappa} \right\|_{L^r(\Omega)}^r + \sum_{i=1}^{2} \left(\left\| \frac{\partial \hat{w}_i^\epsilon}{\partial \kappa} \right\|_{L^r(\Omega)}^r + \left\| \epsilon^2 \frac{\partial \hat{w}_3^\epsilon}{\partial s_i} \right\|_{L^r(\Omega)}^r \right) \leq C.$$

Using (3.49)-(3.50), we deduce that:

$$|A_\varepsilon(\varphi)| \leq (C(r)C)^{\frac{r-1}{r}} \left[\epsilon \left\| \frac{\partial \varphi}{\partial \kappa} \right\|_{L^r(\Omega)} + \epsilon^2 \sum_{i=1}^{2} \left\| \frac{\partial \varphi}{\partial s_i} \right\|_{L^r(\Omega)} \right] \quad \forall \varphi \in W_0^{1,r}(\Omega),$$

hence:

$$\lim_{\epsilon \to 0} A_\epsilon(\varphi) = 0,$$

and consequently:

$$\lim_{\epsilon \to 0} \left(\hat{\rho}^\epsilon, \frac{\partial \varphi}{\partial \kappa} \right) = - \lim_{\epsilon \to 0} \left(\frac{\partial \hat{\rho}^\epsilon}{\partial \kappa}, \varphi \right) = 0 \quad \forall \varphi \in W_0^{1,r}(\Omega).$$

From (3.44), we deduce (3.47).

By (3.47), ρ^* belongs to $L^{r'}()$. Therefore, there exists a sequence (Ψ_m) in $\mathcal{C}_0^\infty()$ such that $\Psi_m \to \rho^*$ in $L^{r'}()$. Using (D‘) with $q = \Psi_m$, we obtain (3.48) as $m \to +\infty$. □

Theorem 3.4. *Under the same assumptions of Theorem* 3.3*,the solution* (w^*, ρ^*, θ^*) *satisfies the following inequalities.*

$$\begin{aligned} &\sum_{i=1}^{2} \int_\Omega \hat{\Lambda}(\theta^*) \left(\frac{1}{2} \right)^{\frac{\nu}{2}} \left(\sum_{i=1}^{2} \left(\frac{\partial w_i^*}{\partial \kappa} \right)^2 \right)^{\frac{\nu-2}{2}} \frac{\partial (w_i^*)}{\partial \kappa} \frac{\partial (\hat{\varphi}_i - w_i^*)}{\partial \kappa} ds d\kappa \\ &- \int_\Omega \rho^*(s) \left(\frac{\partial \hat{\varphi}_1}{\partial s_1} + \frac{\partial \hat{\varphi}_2}{\partial s_2} \right) ds d\kappa + \int_\Omega \hat{g}(\theta^*) \left(\left| \frac{\partial \hat{\varphi}}{\partial \kappa} \right| - \left| \frac{\partial w^*}{\partial \kappa} \right| \right) ds d\kappa \\ &+ \int_\varpi \hat{k} \left(|\hat{\varphi}| - |w^*| \right) ds \geq \sum_{i=1}^{2} \int_\Omega \hat{f}_i (\hat{\varphi}_i - w_i^*) ds d\kappa, \ \forall \hat{\varphi} \in W_{\Gamma_1 \cup \Gamma_L}, \end{aligned} \tag{3.51}$$

$$-\frac{\partial}{\partial \kappa} \left(K \frac{\partial \theta^*}{\partial \kappa} \right) + \hat{\alpha} \theta^* = 0 \quad in \ L^q(\Omega), \tag{3.52}$$

$$\theta^* = 0 \quad in \ \Gamma_1 \cup \Gamma_L, \tag{3.53}$$

$$\frac{\partial \theta^*}{\partial n} = 0 \quad in\ \varpi, \tag{3.54}$$

where

$$W_{\Gamma_1 \cup \Gamma_L} = \left\{ \hat{\varpi} = (\hat{\varpi}_1, \hat{\varpi}_2) \in W^{1,\nu}(\Omega)^2, \hat{\varpi} = 0 \ on\ \Gamma_1 \cup \Gamma_L \right\},$$

The demonstration of this theorem relies on the application of Minty's Lemma:

Proof. According to Minty's lemma and since $\operatorname{div} \hat{w}^\varepsilon = 0$, the problem (3.20) is equivalent to the following problem:

$$B_0(\ \hat{w}^\epsilon,\ \hat{w}^\epsilon, \hat{\varphi} - \hat{w}^\epsilon) + \hat{a}(\hat{\theta}^\epsilon; \hat{\varphi}, \hat{\varphi} - \hat{w}^\epsilon) - (\hat{\rho}^\epsilon, \operatorname{div}(\hat{\phi})) + \hat{j}(\hat{\varphi}) - \hat{j}(\hat{w}^\epsilon) \geq \sum_{i=1}^{2} (\hat{f}_i, \hat{\varphi}_i - \hat{w}_i^\epsilon) + \epsilon(\hat{f}_3, \hat{\varphi}_3 - \hat{w}_3^\epsilon) \quad \forall \hat{\varphi} \in W_{\Gamma_1 \cup \Gamma_L}. \tag{3.55}$$

According to (3.28)-(3.29), there exists a subsequence $\hat{\theta}^\epsilon$ that converges almost everywhere to θ^*. Consequently, $\hat{\Lambda}(\theta^\epsilon)$ converges almost everywhere to $\hat{\Lambda}(\theta^*)$ since $\hat{\Lambda}$ is continuous. Taking the limit in (3.55) while considering the convergence results from Theorem 3.3 and the fact that $\hat{j}$ is weakly lower semicontinuous, we obtain:

$$\sum_{i=1}^{2} \int_\Omega \hat{\Lambda}(\theta^*) \left(\frac{1}{2} \sum_{i=1}^{2} \left(\frac{\partial \hat{\varphi}_i}{\partial \kappa} \right)^2 \right)^{\frac{r-2}{2}} \frac{\partial \hat{\varphi}_i}{\partial \kappa} \frac{\partial}{\partial \kappa} (\hat{\varphi}_i - w_i^*) ds d\kappa - \sum_{i=1}^{2} (\rho^*, \frac{\partial \hat{\varphi}_i}{\partial s_i}) - (\rho^*, \frac{\partial \hat{\varphi}_3}{\partial \kappa}) + \hat{j}(\hat{\phi}) - \hat{j}(w^*) \geq \sum_{i=1}^{2} (\hat{f}_i, \hat{\varphi}_i - w_i^*), \quad \forall \hat{\varphi} \in W_{\Gamma_1 \cup \Gamma_L}. \tag{3.56}$$

As in Lemma 3.1, we have:

$$\int_\Omega \rho^* \left(\frac{\partial w_1^*}{\partial s_1} + \frac{\partial w_2^*}{\partial s_2} \right) ds d\kappa = 0, \tag{3.57}$$

therefore:

$$\sum_{i=1}^{2}\int_{\Omega}\hat{\Lambda}(\theta^*)\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial\hat{\varphi}_i}{\partial\kappa}\right)^2\right)^{\frac{r-2}{2}}\frac{\partial\hat{\varphi}_i}{\partial\kappa}\frac{\partial}{\partial\kappa}(\hat{\varphi}_i-w_i^*)dsd\kappa$$
$$-\sum_{i=1}^{2}(\rho^*,\frac{\partial}{\partial s_i}(\hat{\varphi}_i-w_i^*))+\hat{j}(\hat{\varphi})-\hat{j}(w^*)\geq\sum_{i=1}^{2}(\hat{f}_i,\hat{\varphi}_i-w_i^*),\quad\forall\hat{\varphi}\in W_{\Gamma_1\cup\Gamma_L}.\quad(3.58)$$

Applying Minty's lemma again, we conclude that inequality (3.58) yields (3.51).

On the other hand, by passing to the limit in (3.21), we obtain:

$$\int_{\Omega}\hat{K}\frac{\partial\theta^*}{\partial\kappa}\frac{\partial\hat{\Phi}}{\partial\kappa}\,ds\,d\kappa+\int_{\Omega}\hat{g}\theta^*\hat{\Phi}\,ds\,d\kappa=0\quad\forall\hat{\Phi}\in W^{1,q'}_{L_1,L_2}(\Omega).$$

In particular, this equality remains valid for all $\hat{\Phi}\in W_0^{1,q'}(\Omega)$. Using Green's formula, it follows that:

$$-\frac{\partial}{\partial\kappa}\left(\hat{K}\frac{\partial\theta^*}{\partial\kappa}\right)+\hat{g}\theta^*=0\quad\text{in}\quad W^{-1,q}(\Omega).$$

Since $\hat{K}\in C^1(\mathbb{R})$ with $(\hat{K})'\in L^\infty(\mathbb{R})$, we deduce that $\frac{\partial^2\theta^*}{\partial\kappa^2}\in L^q(\Omega)$. Hence:

$$-\frac{\partial}{\partial\kappa}\left(\hat{K}\frac{\partial\theta^*}{\partial\kappa}\right)+\hat{g}\theta^*=0\quad\text{in}\quad L^q(\Omega).$$

□

Theorem 3.5. *The variational inequality* (3.51) *is equivalent to the following system*

$$\int_{\Omega}\hat{\Lambda}\left(\theta^*\right)\left(\frac{1}{2}\right)^{\frac{\nu}{2}}\left|\frac{\partial w^*}{\partial\kappa}\right|^{\nu}dsd\kappa+\int_{\Omega}\hat{g}\left(\theta^*\right)\left|\frac{\partial w^*}{\partial\kappa}\right|dsd\kappa+\int_{\varpi}\hat{k}\left|w^*\right|ds=\int_{\Omega}\hat{f}w^*dsd\kappa\quad(3.59)$$

and

$$\int_{\Omega}\hat{\Lambda}\left(\theta^{*}\right)\left(\frac{1}{2}\right)^{\frac{\nu}{2}}\left|\frac{\partial w^{*}}{\partial\kappa}\right|^{\nu-2}\frac{\partial w^{*}}{\partial\kappa}\frac{\partial\hat{\Phi}}{\partial\kappa}dsd\kappa+\int_{\Omega}\hat{g}\left(\theta^{*}\right)\left|\frac{\partial\hat{\Phi}}{\partial\kappa}\right|dsd\kappa$$
$$+\int_{\varpi}\hat{k}\left|\hat{\Phi}\right|ds\geq\int_{\Omega}\hat{f}\hat{\Phi}dsd\kappa,\ \forall\hat{\Phi}\in\Sigma\left(V\right),\tag{3.60}$$

where

$$\tilde{\Sigma}\left(V\right)=\left\{\hat{\Phi}=\left(\hat{\Phi}_{1},\hat{\Phi}_{2}\right)\in W^{1,\nu}\left(\Omega\right)^{2}:\hat{\Phi}\ satisfy\ \left(D'\right)\right\}.$$

Proof. taking $\hat{\varphi}=2w^{*}$ and $\hat{\varphi}=0$ in (3.51), respectively. we find (5.59).

For (5.60), we take $\hat{\varphi}=\hat{\Phi}+w^{*}$ for all $\hat{\Phi}\in\Sigma\left(V\right)$. □

Theorem 3.6. *Let us set*

$$\Sigma^{*}=\tilde{\Sigma}^{*}-\nabla\rho^{*}\ and\ \tilde{\Sigma}^{*}=\left(\frac{1}{2}\right)^{\frac{\nu}{2}}\hat{\Lambda}\left(\theta^{*}\right)\left|\frac{\partial w^{*}}{\partial\kappa}\right|^{\nu-2}\frac{\partial w^{*}}{\partial\kappa}+\hat{g}\left(\theta^{*}\right)\pi,\tag{3.61}$$

then

$$-\frac{\partial}{\partial\kappa}\left[\frac{1}{2}\hat{\Lambda}\left(\theta^{*}\right)\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial w_{i}^{*}}{\partial\kappa}\right)^{2}\right)^{\frac{\nu-2}{2}}\frac{\partial w^{*}}{\partial\kappa}+\hat{g}\left(\theta^{*}\right)\frac{\dfrac{\partial w^{*}}{\partial\kappa}}{\left|\dfrac{\partial w^{*}}{\partial\kappa}\right|}\right]=\hat{f}-\nabla\rho^{*}\ ,\ in\ W^{-1,\nu'}\left(\Omega\right)^{2}.\tag{3.62}$$

with $\pi\in L^{\infty}\left(\Omega\right)^{2}$ *and* $\left\|\pi\right\|_{L^{\infty}(\Omega)^{2}}\leq1$.

In the next theorem, we prove that our problem converges to the Reynolds equation:

Theorem 3.7. *Under the same assumptions as Theorem 3.6*

$$\int_{\varpi}\left[\frac{h^{3}}{12}\nabla\rho^{*}+\tilde{F}+\Lambda\left(\theta^{*}\right)\int_{0}^{h}\int_{0}^{y}\hat{\Lambda}A^{*}\left(s,\zeta\right)\frac{\partial w^{*}\left(s,\xi\right)}{\partial\xi}d\xi dy+\hat{a}\int_{0}^{h}\int_{0}^{y}\frac{\partial w^{*}/\partial\kappa}{|\partial w^{*}/\partial\kappa|}d\xi dy.\right.$$

$$\left.-\frac{h\Lambda}{2}\int_{0}^{h}A^{*}(s,\xi)\frac{\partial w^{*}}{\partial\xi}(s,\xi)d\xi-\frac{h\hat{g}}{2}\int_{0}^{h}\frac{\partial w^{*}/\partial\kappa}{|\partial w^{*}/\partial\kappa|}d\xi\right].\nabla\vartheta\left(s\right)ds=0\tag{3.63}$$

for all $\vartheta \in W^{1,\nu}(\varpi)$ *where*

$$\tilde{F}(s) = \int_0^h F(s,y)\,dy - \frac{h}{2}F(s,h)\,,\ \ F(s,y) = \int_0^h \int_0^\xi \hat{f}(s,\theta)\,dtd\xi$$
$$A^*(s,\xi) = \frac{1}{2}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial w^*}{\partial \kappa}(s,\xi)\right)^2\right)^{\frac{\nu-2}{2}}.$$

Proof. By integrating (3.62) twice between 0 and z, we obtain:

$$-\int_0^z \Lambda(\theta^*)\,A^*(s,\xi)\frac{\partial w^*}{\partial \xi}(s,\xi)d\xi - \hat{g}\int_0^\kappa \frac{\partial w^*/\partial\kappa}{|\partial w^*/\partial\kappa|}d\xi + \Lambda r^*(s)\kappa + \hat{g}\frac{\rho^*}{|\rho^*|}\kappa + \frac{\kappa^2}{2}\nabla\rho^* = \int_0^\kappa\int_0^\xi f(s,y)dyd\xi \tag{3.64}$$

In particular, for $\kappa = h$, we have:

$$-\int_0^h \Lambda(\theta^*)\,A^*(s,\xi)\frac{\partial w^*}{\partial \xi}(s,\xi)d\xi - \hat{g}\int_0^h \frac{\partial w^*/\partial\kappa}{|\partial w^*/\partial\kappa|}d\xi + \Lambda r^*(s)h + \hat{g}\frac{\rho^*}{|\rho^*|}h + \frac{h^2}{2}\nabla\rho^* = \int_0^h\int_0^\xi f(s,y)dyd\xi \tag{3.65}$$

Integrating (3.64) between 0 and h, we obtain:

$$\frac{h^3}{6}\nabla p^* - \int_0^h\int_0^y \Lambda(\theta^*)\,A^*(s,\xi)\frac{\partial w^*}{\partial\kappa}(s,\xi)d\xi dy - \hat{a}\int_0^h\int_0^y \frac{\partial w^*/\partial\kappa}{|\partial w^*/\partial\kappa|}d\xi dy$$
$$+\Lambda(\theta^*)\,r^*(s)\frac{h^2}{2} + \hat{g}\frac{\rho^*}{|\rho^*|}\frac{h^2}{2} = \int_0^h\int_0^y\int_0^\xi f(s,t)dtd\xi dy \tag{3.66}$$

From (3.65), we deduce that:

$$-\frac{h\Lambda(\theta^*)}{2}\int_0^h A^*(s,\xi)\frac{\partial w^*}{\partial\xi}(s,\xi)d\xi - \frac{h\hat{g}}{2}\int_0^h \frac{\partial w^*/\partial\kappa}{|\partial w^*/\partial\kappa|}d\xi$$
$$+\Lambda(\theta^*)\,r^*(s)\frac{h^2}{2} + \hat{g}\frac{\rho^*}{|\rho^*|}\frac{h^2}{2} + \frac{h^3}{4}\nabla\rho^* = \frac{h}{2}\int_0^h\int_0^\xi f(s,y)dyd\xi \tag{3.67}$$

From (3.65)-(3.66), we obtain

$$\frac{h^3}{12}\nabla\rho^* + \tilde{F} + \Lambda(\theta^*)\int_0^h\int_0^y A^*(s,\zeta)\frac{\partial w^*(s,\xi)}{\partial\xi}d\xi dy + \hat{a}\int_0^h\int_0^y \frac{\partial w^*/\partial\kappa}{|\partial w^*/\partial\kappa|}d\xi dy.$$
$$-\frac{h\Lambda(\theta^*)}{2}\int_0^h A^*(s,\xi)\frac{\partial w^*}{\partial\xi}(s,\xi)d\xi - \frac{h\hat{g}}{2}\int_0^h \frac{\partial w^*/\partial\kappa}{|\partial w^*/\partial\kappa|}d\xi = 0 \tag{3.68}$$

Hence, for all $\vartheta \in W^{1,\nu}(\varpi)$, we have

$$\int_\varpi \left[\frac{h^3}{12}\nabla\rho^* + \tilde{F} + \Lambda(\theta^*)\int_0^h\int_0^y A^*(s,\zeta)\frac{\partial w^*(s,\xi)}{\partial\xi}d\xi dy + \hat{g}\int_0^h\int_0^y \frac{\partial w^*/\partial\kappa}{|\partial w^*/\partial\kappa|}d\xi dy.\right.$$

$$\left.-\frac{h\Lambda}{2}\int_0^h A^*(s,\xi)\frac{\partial w^*}{\partial\xi}(s,\xi)d\xi - \frac{h\hat{g}}{2}\int_0^h \frac{\partial w^*/\partial\kappa}{|\partial w^*/\partial\kappa|}d\xi\right].\nabla\vartheta(s)\,ds = 0 \tag{3.69}$$

This completes the proof. □

3.4.2 Uniqueness of Solutions to the Limit Problem

Theorem 3.8. *The solution (w^*, ρ^*, θ^*) of the limiting variational inequality (3.51) and (3.63) is unique in $V_\kappa L_0^{\nu'}(\varpi)\, W^{-1,q}(\Omega)$.*

Proof. Let $(w^{*,1}, \theta^{*,1}, \rho^{*,1})$ and $(w^{*,2}, \theta^{*,2}, \rho^{*,2})$ be two solutions of (3.51)−(3.54) and (3.63); then $\theta^{*,1}$ and $\theta^{*,2}$ solve (3.52) − (3.54), so $\theta = \theta^{*,1} - \theta^{*,2}$ satisfies the problem

$$-\frac{\partial}{\partial\kappa}\left(K\frac{\partial\theta}{\partial\kappa}\right) = -\hat{g}\theta,\ \theta = 0 \text{ in } \Gamma_1 \cup \Gamma_L,\ \frac{\partial\theta}{\partial\kappa} = 0 \text{ on } \varpi,$$

so $\theta = 0$, thus $\theta^{*,1} = \theta^{*,2}$. Taking $\vartheta = w^{*,2}$ and $\vartheta = w^{*,1}$ respectively, as test function in

(3.51) we get

$$\sum_{i=1}^{2}\int_{\Omega}\hat{\Lambda}\left(\theta^{*}\right)\left(\frac{1}{2}\right)^{\frac{\nu}{2}}\left(\sum_{i=1}^{2}\left(\frac{\partial w_{i}^{*,1}}{\partial\kappa}\right)^{2}\right)^{\frac{\nu-2}{2}}\frac{\partial w_{i}^{*,1}}{\partial\kappa}\frac{\partial}{\partial\kappa}(w_{i}^{*,1}-w_{i}^{*,2})dsd\kappa$$
$$\leq\sum_{i=1}^{2}\int_{\Omega}\hat{\Lambda}\left(\theta^{*}\right)\left(\frac{1}{2}\right)^{\frac{\nu}{2}}\left(\sum_{i=1}^{2}\left(\frac{\partial w_{i}^{*,2}}{\partial\kappa}\right)^{2}\right)^{\frac{\nu-2}{2}}\frac{\partial w_{i}^{*,2}}{\partial\kappa}\frac{\partial}{\partial\kappa}(w_{i}^{*,1}-w_{i}^{*,2})dsd\kappa.$$

Observe that for every $s,y\in\mathbb{R}^{n}$

$$\left(|s|^{\nu-2}s-|y|^{\nu-2}y,s-y\right)\geq(\nu-1)\left(|s|+|y|\right)^{\nu-2}|s-y|^{2},\ \forall\ 1<\nu\leq 2$$

we obtain

$$\int_{\Omega}\left[\left|\frac{\partial w^{*,1}}{\partial\kappa}\right|+\left|\frac{\partial w^{*,2}}{\partial\kappa}\right|\right]^{\nu-2}\left|\frac{\partial w^{*,1}}{\partial\kappa}-\frac{\partial w^{*,2}}{\partial\kappa}\right|^{2}dsd\kappa=0,\tag{3.70}$$

where $\left|\frac{\partial w^{*,j}}{\partial\kappa}\right|=\left(\sum_{i=1}^{2}\left(\frac{\partial w_{i}^{*,j}}{\partial\kappa}\right)^{2}\right)^{\frac{1}{2}}$, $j=1,2$. Using Hölder's inequality, we deduce

$$\int_{\Omega}\left[\frac{\partial}{\partial\kappa}\left(w^{*,1}-w^{*,2}\right)\right]^{\nu}dsd\kappa\leq C\left(\int_{\Omega}\left[\left|\frac{\partial w^{*,1}}{\partial\kappa}\right|+\left|\frac{\partial w^{*,2}}{\partial\kappa}\right|\right]^{\nu-2}\left|\frac{\partial w^{*,1}}{\partial\kappa}-\frac{\partial w^{*,2}}{\partial z}\right|^{2}dsd\kappa\right)^{\frac{\nu}{2}}$$
$$\left(\int_{\Omega}\left[\left|\frac{\partial w^{*,1}}{\partial\kappa}\right|+\left|\frac{\partial w^{*,2}}{\partial z}\right|\right]^{\nu}dsd\kappa\right)^{\frac{2-\nu}{2}}.\tag{3.71}$$

From (3.70) and (3.71), we obtain

$$\left\|\frac{\partial}{\partial\kappa}\left(w^{*,1}-w^{*,2}\right)\right\|_{L^{r}(\Omega)}=0,$$

using Poincare's inequality, we deduce

$$\left\|w^{*,1}-w^{*,2}\right\|_{V_{\kappa}}=0.$$

Finally, to prove the uniqueness of the pressure, we use equation (3.63) with the two pressures $\rho^{*,1}$ and $\rho^{*,2}$, we find

$$\int_{\varpi} \frac{h^3}{12} \nabla \left(\rho^{*,1} - \rho^{*,2} \right) \nabla \vartheta ds = 0.$$

Taking $\vartheta = \rho^{*,1} - \rho^{*,2}$, and by Poincare's inequality, we deduce
$\|\rho^{*,1} - \rho^{*,2}\|_{L^{r'}(\varpi)} = 0$. So $\rho^{*,1} = \rho^{*,2}$. □

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ملخص

تتناول هذه الأطروحة السلوك التقاربي للجريان غير متساوي الحرارة للموائع اللزوجة اللدنة (بنغهام وهيرشل-بولكلي) في المجالات الرفيعة تحت شروط احتكاك من نوع ترسكا. تم اشتقاق نموذج حدّي فعال يجمع بين متباينة اختيارية من نوع رينولدز ومعادلة حرارة مبسطة. يثبت التحليل رياضياًا وجود الحلول ووحدانيتها وتقاربها. تؤسس هذه النتائج إطاراً رياضياًا متيناًا لمحاكاة الجريان المحصور، مع تطبيقات مهمة في هندسة التزييت وتصميم العمليات.

الكلمات المفتاحية: :مقاربة تقاربية، مائع بنغهام، مائع هيرشل-بولكلي، درجة الحرارة، معادلة رينولدز.

Abstract

This thesis investigates the asymptotic behavior of non-isothermal flows of viscoplastic fluids (Bingham and Herschel-Bulkley) in thin domains with Tresca-type friction conditions. An effective limit model is derived, combining a Reynolds-type variational inequality with a simplified heat equation. The analysis rigorously proves the existence, uniqueness, and convergence of solutions. These results establish a robust mathematical framework for simulating confined flows, with significant applications in lubrication engineering and process design.

Keywords: Asymptotic approach, Bingham fluid,Herschel-Bulkley fluid, Temperature, Reynolds equation.

Résumé

Cette thèse étudie le comportement asymptotique des écoulements non isothermes de fluides viscoplastiques (Bingham et Herschel-Bulkley) dans des domaines minces avec conditions de frottement de Tresca. Un modèle limite effectif est dérivé, combinant une inégalité variationnelle de type Reynolds et une équation de chaleur simplifiée. L'analyse démontre l'existence, l'unicité et la convergence des solutions. Ces résultats offrent un cadre mathématique rigoureux pour la simulation d'écoulements confinés, avec applications en ingénierie et génie des procédés.

Mots Clée : Approche asymptotique, Fluide de Bingham, Fluide de Herschel-Bulkley, Température, Équation de Reynolds.