People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research

Setif 1 University-Ferhat Abbas
Faculty of Sciences
Department of Mathematics

Doctoral thesis

Field: Mathematics and Computer Science
Option: Mathematics
Speciality: Optimization and Control

Theme:

Contribution of the Dynamic Programming Approach to Solving a War Game of Attrition and Attack

Presented by

Safa BENGHEBRID

Supervisor: **Pr. Touffik BOUREMANI**
Co-supervisor: **Pr. Djamel BENTERKI**

Thesis defended on 02/12/2026, in front of the jury composed of:

Mr.	Bachir MERIKHI	Prof	Setif 1 University Ferhat Abbas	President
Mr.	Touffik BOUREMANI	Prof	Setif 1 University Ferhat Abbas	Supervisor
Mr.	Djamel BENTERKI	Prof	Setif 1 University Ferhat Abbas	Co- supervisor
Mr.	Abdelkrim MERZOUGUI	Prof	Mohamed Boudiaf University of M'sila	Examiner
Mr.	Djillali BOUAGADA	Prof	Abdelhamid Ibn Badis University of Mostaganem	Examiner
Mme.	Rebiha SAFFIDINE	MCA	Setif 1 University Ferhat Abbas	Examiner
Mr.	Rachid ZITOUNI	Prof	Setif 1 University Ferhat Abbas	Guest
Mr.	Ahmed BENDJEDDOU	Prof	Setif 1 University Ferhat Abbas	Guest

2025/2026

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List of publication

- S. Benghebrid, T. Bouremani and D. Benterki, On Isaac's War Game of Attrition and Attack Using Dynamic Programming Approach. Games 2024, 15(6), 35. 1–18. `https://doi.org/10.3390/g15060035`

List of communications

International communications

- S. Benghebrid, T. Bouremani and D. Benterki, A numerical technique for solving a Warfare game model based on Dynamic Programming Method. Second International Conference on Mathematics and Applications (ICMA'23), 26-27 September 2023 Blida (Algeria).
- S. Benghebrid, T. Bouremani and D. Benterki, Dynamic Programming Solution of a War Game in generalized form. International Conference on Mathematics and its Applications in Science and Technology (ICMAST'2024), December 15-16, 2024, at the Univesity Setif 1 Ferhat Abbas, Algeria.

National communications

- S. Benghebrid, T. Bouremani and D. Benterki, Characterization of some admissible trajectories for a warfare differential game problem. Second National Conference on Mathematics and its Applications- cnma Bba2022, University Mohamed El Bachir El Ibrahimi of Bordj Bou Arraridj, September 17-18, 2022.
- S. Benghebrid, T. Bouremani and D. Benterki, Continuation of some admissible trajectories of warfare differential game problem. Nouvelles Tendances en Mathématiques Théoriques et Computationnelles-NTMTC22, Université Amine Okkal Hadj Moussa Eg Akhamouk- Tamanghasset, 08-09 novembre 2022.
- S. Benghebrid, T. Bouremani and D. Benterki, Value function and Feedback strategies of warfare differential game problem. Optimization and its Applications Day, OAD'24, April 30, 2024 Setif 1 University-Ferhat Abbas, Setif- Algeria.

Key Notations and Abbreviations

$\langle a, b\rangle = a^T b = a.b$	Scalar product of a and b;
$Int(X_i)$	Interior of the set X_i;
$Cl(X_i)$	Closure of the set X_i;
$co(X_i)$	Convex hull of X_i;
$B_{r_0}(x_i)$	Ball of radius r_0 centred at x_i;
$\overline{B}_{r_0}(x_i)$	Closed ball of radius r_0 centred at x_i;
$L(\mathbb{R}^n, \mathbb{R}^m)$	The vector space of all linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$;
$\mathcal{C}^k(X_i; Y_i)$	The space of k-times continuously differentiable mappings from X_i to Y_i;
$T_{x_i}Y$	the tangent space at $x_i \in Y_i$;
$AC(\mathbb{R}^n, \mathbb{R}^p)$	The space of absolutely continuous functions from $\mathbb{R}^n$ to $\mathbb{R}^p$;
$\mathcal{P}(A)$	The collection of all subsets of the set A;
$J_g(x_i)$	The Jacobian matrix of the function $g(.)$ at the point x_i;
$pr_1 Z$	projection of Z on the first coordinate (or first factor);
DG	Differential game;
DG_A	An autonomous differential game;
iff	Abbreviation for "if and only if";
resp	Abbreviation for "respectively";
$a.e.(I)$	Holds almost everywhere on the interval I;
e.g.,	Latin abbreviation "exempli gratia", meaning "for example";
i.e.	Latin abbreviation "id est," meaning "that is".

Introduction

Differential games, a branch of static game theory, provide a framework for analyzing strategic interactions between two or more players, in which each player's decisions, in turn, influence the dynamics of a system. Rufus Isaac's seminal book Differential Games [32], published in 1965, laid the foundation for the study of pursuit-evasion games and offered a comprehensive framework for analyzing differential games involving multiple players. These works laid the groundwork for addressing a diverse array of conflict scenarios and inspired researchers to delve further into this field. This exploration brought numerous criticisms of Isaac's heuristic approach to the forefront, including those highlighted in [4, 5, 10, 22, 33, 38, 60]. Ultimately, the research prompted by Isaac's missteps has left a lasting legacy, manifesting in improved methodologies, interdisciplinary collaborations, and a richer understanding of the nuances of the problem. Also, the divide between theoretical rigor and practical applicability has prompted researchers to seek a middle ground. Efforts have been made to develop theoretical frameworks that can accommodate the intricacies of concrete examples (e.g., [8, 16, 35, 63]). This involved refining existing mathematical methods and, in some cases, introducing novel approaches that maintain a balance between theoretical soundness and practical relevance. Since 1982, significant developments have occurred in the theory of viscosity solutions, initiated by [24], which have provided various characterizations of the value function as a solution to the Hamilton-Jacobi-Isaac's equation. However, the elaboration of the theory of viscosity solutions has not effectively contributed to the accurate and complete resolution of any concrete problems proposed in the literature. This prevailing anomaly in the study of differential games is about to be corrected due to the recent contributions presented in Mirică's work [54, 55, 56]. The fundamental content of this new dynamic programming approach consists of sufficient optimality conditions, illustrated by a main theoretical support consisting of seven verification theorems. It essentially extends to much more realistic cases. The elementary verification theorem is from Isaac's in [32], the only known previous one (applied, unjustifiably, to problems in which the value function is not differentiable). The constructive aspect of this approach also encompasses significant extensions and generalizations of the characteristics' method for nonsmooth Hamilton-Jacobi equations, providing a rigorous foundation for the heuristic procedures proposed by Isaac's 1965 and various other related works. Unlike previous methodologies, optimality is defined not within the framework of saddle points [5, 26], but rather in the more convenient, though ostensibly equivalent, class of relatively optimal feedback strategies. As noted in [55], the only realistic approach to engaging in a differential game (particularly in the context of an optimal control problem) is to employ feedback strategies that are calculated in advance. This perspective is, in fact, shared by [32, 35]. One of the classic problems in differential games is the War of Attrition, a compelling concept with applications across various fields, including military strategy and resource management. It is regarded as a well-known example within the domain of differential games. It models a scenario in which two or more players are engaged in a contest to capture a valuable resource, such as territory or prey. However, this contest incurs costs. Players must determine the duration of their involvement in the struggle, carefully weighing the potential benefits of victory against the cumulative costs of sustained conflict.

The aim of this thesis is to apply in a step-by-step manner the theoretical Dynamic Programming algorithm, described in chapter 2, and to integrate these results with numerical procedures in order to achieve a more rigorous and theoretically complete solution to the War of Attrition game, which was formulated and studied heuristically in [32] (Section 5.4, page 96). This model may be considered as a two-player zero-sum game, where the players have completely opposite interests; namely, a player's gain is an equivalent loss to the opposing player. Our approach was first used to address problems of exceptionally high complexity. Among these challenges was the well-known homicidal chauffeur game in [15]. Furthermore, in the explicit model illustrated in [13], it was shown that only the maximal value function is admissible and is associated with certain feedback strategies. Furthermore, within the realm of conflict problems, the model presented in [14] is particularly relevant as it provides a framework that can be directly compared with the results in [21]. The primary distinction between the two studies lies in the methodological approaches employed to analyze warfare dynamics: dynamic programming in our case, contrasted with the Lanchester equation in theirs. The conclusions drawn from this comparison suggest that our approach offers a broader and more realistic framework for modeling warfare dynamics, particularly in real-world scenarios where strategies and conditions evolve over time. While the Lanchester equation is useful, it is constrained by its static nature and limited scope, making our approach more applicable to complex, dynamic conflicts.

Using the dynamic programming method to solve this problem presents the advantage of allowing us to determine all admissible trajectories associated with the problem. Moreover, the hypotheses that need to be verified are significantly more natural and easier to establish, drawing upon elements of Hamilton-Jacobi theory, as well as recent findings in Nonsmooth Analysis as referenced in [2, 20, 52, 55].

Our thesis contains three chapters, organized as follows.

Chapter 1 is devoted to the exposition of the fundamental mathematical tools and theoretical concepts that underpin the developments carried out in the subsequent chapters. These elements establish an essential framework for the rigorous investigation of differential game problems. We begin by presenting the theory of differential mappings on differentiable submanifolds, together with the corresponding tangent spaces, which play a crucial role in describing the local geometry of domains where the dynamics are defined. Next, Stratified sets and mappings are then introduced, providing an analytical structure suitable for handling spaces that may contain singularities or exhibit varying degrees of smoothness. Subsequently, we revisit the monotonicity properties of real-valued functions, which serve as key tools in analyzing the qualitative behavior of dynamical systems, particularly in the context of stability and convergence. We also present the notion of generalized tangent directions to trajectories that proves instrumental in examining dynamics across nonsmooth strata. In the end, we provide a more comprehensive examination of smooth Hamiltonian equations and their corresponding characteristic flows.

Chapter 2 is dedicated to the synthesis of the new theory of differential games initiated by Mirică [55, 56]. Including, considerations on the vague and rigorous formulations of a differential game problem as well as the definitions of the concepts of admissible pairs and optimal feedback strategies using the well-known verification theorems considered as sufficient optimality conditions. This new theory is structured as an algorithmic framework that can be applied effectively to solving concrete differential games.

Our original contributions are presented in **Chapter 3**, which focuses on a detailed study of the War of Attrition and Attack game, first introduced by Isaacs [32]. Applying the algorithm of dynamic programming described in Section 2.4, we derive rigorous and complete solutions to this game. The main directions explored here include the following.

- Exploring feedback strategies that depend on the current state of the system and offer improved adaptability in real-time conflict scenarios. These strategies provide a more robust alternative to the static decision rules typically used in game theory.
- Employing a refined version of Cauchy's method of characteristics, adapted to stratified Hamilton-Jacobi equations, to construct the value function and corresponding optimal strategies.
- Showing the optimality of the constructed strategies using the Elementary Verification Theorem, a straightforward yet powerful result that ensures the value function meets the necessary conditions for optimality.
- Identify a feedback strategy providing a complete optimality solution using one of the verification theorems.

CHAPTER 1

Essential Tools and Theoretical Background in Nonsmooth Analysis

1.1 Introduction

This chapter provides a comprehensive overview of the essential mathematical tools and theoretical foundations of nonsmooth analysis, with a specific focus on their application to differential games. In classical analysis, the smoothness of functions is often a central assumption; however, in real-world modeling particularly, in game theory the value functions and optimal strategies frequently exhibit nonsmooth behavior due to various constraints. Nonsmooth analysis offers a rigorous framework to address these challenges by extending the concepts of derivatives and optimality to broader, less regular settings.

1.2 Differential mappings on submanifolds

As is well known, the value function in differential games is typically nondifferentiable, and its domain may not be open. Consequently, it becomes necessary to employ tools from nonsmooth analysis, which provide an appropriate mathematical framework for handling such irregularities. In this context, we begin by recalling the notion of Fréchet differentiability, that may be considered as the first level of differentiability mappings.

Definition 1.1. A function $g : X_1 \subseteq \mathbb{R}^n \to \mathbb{R}^p$ is said to be Fréchet differentiable at $x_i \in Int\,(X_1)$ if there exists a linear mapping $M \in L\,(\mathbb{R}^n, \mathbb{R}^p)$ such that:

$$\lim_{\mu \to 0} \frac{1}{\|\mu\|} \left[g\,(x_i + \mu) - g\,(x_i) - M.\mu\right] = 0 \in \mathbb{R}^p,\ \forall \mu \in \mathbb{R}^n, \tag{1.1}$$

as particular cases of the derivatives of the function $g\,(.)$ are illustrated below.

Proposition 1.2. *([55]) If the function g is defined on $X_1 \subseteq \mathbb{R}^n$ with values in $\mathbb{R}^p$, and if g is differentiable in the sense of Definition 1.1, then g is continuous at each point $x_i \in Int(X_1)$. Moreover, there exists $M \in L\,(\mathbb{R}^n, \mathbb{R}^p)$ such that, $Dg\,(x_i) = M$.*

- *If a coordinate system is chosen for both $\mathbb{R}^n$ and $\mathbb{R}^p$, then the derivative $Dg\,(x_i) \in L\,(\mathbb{R}^n, \mathbb{R}^p)$ can be represented by the matrix of partial derivatives:*

$$Dg\,(x_i) = \frac{\partial g_k}{\partial x_{i_j}}\,(x_{i_1}, \cdots, x_{i_n})\,,\ k = \overline{1,p},\ j = \overline{1,n} \tag{1.2}$$

- *If $p = 1$, the derivative is identified with the gradient of g:*

$$Dg\,(x_i) = \nabla g\,(x_i) \tag{1.3}$$

- *If $n = 1$, the derivative having the standard notation:*

$$g'\,(x_i) = \lim_{\delta \to 0} \frac{g\,(x_i + \delta) - g\,(x_i)}{\delta}. \tag{1.4}$$

A first extension of the classical differential calculus is made in the framework of differential geometry, by considering functions defined on sets which may not be open, but which admit certain differentiable structures. Such sets are called differentiable manifolds, and classical differential calculus extends to this framework without significant difficulty.

Definition 1.3. Let $k \geq 1$ be a given natural number. A nonempty subset $X_1 \subset \mathbb{R}^n$ is called a differentiable (sub-) manifold of dimension $m \in \{1, 2, \cdots, n-1\}$ of class $\mathcal{C}^k$ if, for each point $x_{i_0} \in X_1$, there exist an open neighborhood $X_{1_0} \subset \mathbb{R}^n$ of x_{i_0}, an open neighborhood $U \times V \subset \mathbb{R}^m \times \mathbb{R}^{n-m}$ containing the point $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$, and a diffeomorphism $\alpha\,(.,.) : U \times V \to X_{1_0}$ of class $\mathcal{C}^k$ such that:

$$\alpha\,(0, 0) = x_{i_0},\ \alpha\,(U \times \{0\}) = X_1 \cap X_{1_0}. \tag{1.5}$$

In this case the pair $(U \times V, \alpha(.,.))$(as well as $(X_{1_0}, \alpha^{-1}(.))$) is referred to be as a local coordinate chart, and the relationship:

$$x_i = \alpha(u, 0), \ u = \left(u^1, u^2, \cdots, u^m\right) \in U \subset \mathbb{R}^m, \tag{1.6}$$

defines a local parametric representation of X_1 in a neighborhood $X_1 \cap X_{1_0}$ of x_{i_0}.

To understand smooth structures and the relationships between differentiable manifolds, it is necessary to recall some fundamental definitions from differential geometry.

Proposition 1.4. *([55]) Given $k \in \mathbb{N}^*$, $m = \overline{1, n-1}$. Let $X_1 \subseteq \mathbb{R}^n$, $X_2 \subseteq \mathbb{R}^n$, and $X_3 \subseteq \mathbb{R}^m$ be differentiable manifolds of class $\mathcal{C}^k$ of dimension m.*

1. ***C^k-submersion***

   A differentiable mapping $g : X_1 \to \mathbb{R}^{n-m}$ is called $\mathcal{C}^k$-submersion if, for every point $x_i \in X_1$, the differential of g at x_i, denoted by $Dg(x_i)$, is surjective, where $Dg(x_i) \in L(\mathbb{R}^n, \mathbb{R}^{n-m})$.

2. ***C^k-immersion***

   The mapping $g : X_3 \to X_1$ is called $\mathcal{C}^k$-immersion if, $\forall x_i \in X_3$, the differential $Dg(x_i) \in L(\mathbb{R}^m, \mathbb{R}^n)$, is injective.

3. ***Diffeomorphism***

   The mapping $g : X_1 \to X_2$ is called diffeomorphism if, g is a differentiable mapping of class $\mathcal{C}^1$, and it is a bijection, and the inverse mapping $g^{-1} : X_2 \to X_1$ is differentiable and of class $\mathcal{C}^1$.

4. ***Homoemorphism***

   If $g : X_1 \to X_2$ is a bijective and continuous function, and its inverse $g^{-1} : X_2 \to X_1$ is also continuous, then g is a homeomorphism.

Remark 1.5. Let $A \subseteq \mathbb{R}$ and $x_i \in A$. The point x_i is called an **isolated point** of A, if there exists $\varepsilon > 0$ such that:

$$(x_i - \varepsilon, x_i + \varepsilon) \cap A = \{x_i\}.$$

In other words, x_i is isolated if there is a neighborhood around it that contains no other points of A.

It is also important to note, that the differentiable manifolds described in Definition 1.3 are locally closed subsets of $\mathbb{R}^n$ in our case, even locally compact.

Definition 1.6. A subset $X_1 \subseteq \mathbb{R}^n$ is said to be **locally closed**, if each point $x_i \in X_1$ has an open neighborhood X_{1_0}, with $X_{1_0} \in \mathcal{V}(x_i)$, such that $X_1 \cap X_{1_0}$ is relatively closed in X_{1_0}.

Many results in classical analysis, typically applied to open subsets, can be extended to mappings defined on differentiable manifolds, owing to the existence of a well-defined natural derivative.

Definition 1.7. If $X_1 \subseteq \mathbb{R}^n$ is a differentiable manifolds of class $\mathcal{C}^k$, $k \geq 1$, then a mapping $g : X_1 \to \mathbb{R}^p$ is said to be differentiable at $x_i \in X_1$ in any of the following cases:

(i) $dim\,(X_1) = 0$ (i.e., X_1 consists of isolated points).

(ii) $dim\,(X_1) = n$ ($X_1 \subseteq \mathbb{R}^n$ is open) and g is Fréchet differentiable (in the sense of Definition 1.1).

(iii) $dim\,(X_1) = m \in \{1, 2, \cdots, n-1\}$ and there exists a local chart $(U \times V, \alpha\,(.,.))$ at $x_i \in X_1$, where the local representative of g (relative to this chart) defined by:

$$g_\alpha\,(u) = g\,(\alpha\,(u, 0))\,,\ u \in U \subseteq \mathbb{R}^m\ \ (x_i = \alpha\,(0, 0))\,, \tag{1.7}$$

is differentiable at $u = 0 \in \mathbb{R}^m$, in this case the derivative $Dg\,(x_i) \in L\,(T_{x_i}X_1, \mathbb{R}^p)$ of g at x_i is defined by:

$$Dg\,(x_i)\,.w = \begin{cases} 0 & \text{if} \quad \dim\,(X_1) = 0,\, w \in T_{x_i}X_1 = \{0\}\,, \\ Dg\,(x_i)\,.w & \text{if} \quad \dim\,(X_1) = n,\, w \in T_{x_i}X_1 = \mathbb{R}^n, \\ Dg_\alpha\,(0)\,.v & \text{if} \quad m \neq 0, n,\, w = D_1\alpha\,(0, 0)\,v, \end{cases} \tag{1.8}$$

where, in the latter case, $(U \times V, \alpha\,(.,.))$ is a local chart at x_i, $g_\alpha\,(.)$ is the corresponding local representative in (1.7) and $v \in \mathbb{R}^m$ is the unique vector satisfying $w = D_1\alpha\,(0, 0)\,v$ if $w \in T_{x_i}X_1$.

$T_{x_i}X_1$ represent the tangent space of X_1 at x_i, which be defined below.

Now, we proceed to present a formal definition of the well-established notion of the tangent space in differentiale geometry. The tangent space at the point x_i on a differentiable manifold X_1, denoted $T_{x_i}X_1$, is a vector space that represents all possible directions in which one can move through x_i within the manifold X_1. More rigorously, it is defined as the vector space of partial derivative operators at x_i, associated with local coordinate charts of the manifold.

In particular, for a point $x_i = \alpha\left(u, 0\right) \in X_1 \cap X_{1_0}$, defined in (1.6), the tangent space is given by:

$$T_{x_i}X_1 = D\alpha\left(u, 0\right)\left(\mathbb{R}^m \times \{0\}\right) = D_1\alpha\left(u, 0\right)\left(\mathbb{R}^m\right) \text{ if } x_i = \alpha\left(u, 0\right), \tag{1.9}$$

where, $D_1\alpha$ denotes the derivative of α with respect to its first $m-$dimensional argument. Importantly, this definition can be shown to be independent of the choice of the local coordinate chart.

Furthermore, if $X_1 \subseteq \mathbb{R}^n$, is an open set, then X_1 is called n-dimensional (sub-)manifold. if X_1 consists only of isolated points, It is called a 0-dimensional (sub-) maniflod. That is, for any $x_i \in X_1$ there exists a neighborhood $X_{1_0} \subseteq \mathbb{R}^n$ of x_i such that $X_1 \cap X_{1_0} = \{x_i\}$.

In both cases,The set of tangent vectors at $x_i \in X_1$, defined by the space $T_{x_i}X_1$ as follows:

$$T_{x_i}X_1 = \begin{cases} \mathbb{R}^n & \text{if} \quad \dim\left(X_1\right) = n, \\ \{0\} & \text{if} \quad \dim\left(X_1\right) = 0. \end{cases} \tag{1.10}$$

It is well established (see [36]) that the definition of the tangent space in (1.9) remains invariant under changes in local coordinates.

Moreover, the tangent space (1.10) can be equivalently defined in a unified manner as:

$$T_{x_i}X_1 = \left\{v \in \mathbb{R}^n \;\middle|\; \exists c\left(.\right) \in \mathcal{C}^k\left(\left(-\epsilon, \epsilon\right), X\right),\ c\left(0\right) = x_i, c'\left(0\right) = v\right\}. \tag{1.11}$$

We note that, in certain special cases, it may be necessary to adopt more effective alternative characterizations, such as the immersion and submersion characterization presented in Proposition 1.4, which relies on the well-known theorem in differential geometry called the Implicit Function Theorem. In this sense, some equivalent forms of the tangent space are illustrated by the following result.

Lemma 1.8 ([55]). *If $k \geq 1$, $m = \{1, 2, \cdots, n-1\}$ are natural numbers and $X_1 \subset \mathbb{R}^n$ is a nonempty subset then the following statements are equivalent:*

1. *X_1 is a differentiable manifold of dimension m and of class $\mathcal{C}^k$.*

2. *For each* $x_{i_0} \in X_1$ *there exist* $X_{1_0} \in \mathcal{V}(x_{i_0})$*, and* $\mathcal{C}^k-$*submersion* $G(.) : X_{1_0} \to \mathbb{R}^{n-m}$*, with* $X_1 \cap X_{1_0} = \{x_i \in X_{1_0},\ G(x_i) = 0_{\mathbb{R}^{n-m}}\}$*, in this case, the space of the tangents at* x_i *is given by :*

$$T_{x_i}X = \left\{v \in \mathbb{R}^n,\ DG(x_i).v = 0 \in \mathbb{R}^{n-m}\right\},\ x_i \in X_1 \cap X_{1_0}, \tag{1.12}$$

   and X_1 *is said to be "Implicitly" defined by* $G(x_i) = 0$ *around* x_{i_0}*.*

3. *For each* $x_{i_0} \in X_1$ *there exist* $X_{1_0} \in \mathcal{V}(x_{i_0})$*,* $U \in \mathcal{V}(0)$ *in* $\mathbb{R}^m$ *and a* $\mathcal{C}^k-$*immersion* $\gamma(.) : U \to X_1 \cap X_{1_0}$ *that it is also a homeomorphism, such that:*

$$X_1 \cap X_{1_0} = \{\gamma(u),\ u = (u_1, \cdots, u_m) \in U \subseteq \mathbb{R}^m\}. \tag{1.13}$$

   Furthermore, the tangent space at x_i *is:*

$$T_{x_i}X_1 = \{D\gamma(u).\overline{u} \in \mathbb{R}^n,\ \overline{u} \in \mathbb{R}^m\},\ \textit{if}\ x_i = \gamma(u) \in X_1 \cap X_{1_0}, \tag{1.14}$$

   and X_1 *is said to be "Parametrically" defined by* $x_i = \gamma(u),\ u \in U$*, around* x_{i_0}*.*

1.3 Stratified sets and mappings

Many aspects of modern mathematics particularly those involving the solution of problems derived from mathematical models of real-world phenomena highlight the need to extend classical differential calculus. However, experience has shown that this extension alone is not sufficient. A further step involves extending classical differential calculus to applications defined on stratified sets, which are essentially unions of disjoint differentiable manifolds.

In recent years, various results have emerged that support such extensions to functions defined on arbitrary sets. Among the pioneers in this area is [72] and later extensively developed by Thom (1969) in [69], Mather (1971) [42]. Further, the concept of stratification is employed to model the evolution of game states and strategies. The game space is divided into distinct strata, each with its own dynamics that influence player behavior. A stratificatin describes how strategies drive transitions across these regions, reflecting the game's progression through different phases.

Definition 1.9. A countable partition of X_1 is a family of disjoint subsets $\left\{X_{1_j}\right\}_{j\in J}$ that satisfy:

1. $X_{1_j} \subset X_1, \forall j \in J$,
2. $X_{1_j} \cap X_{1_i} = \emptyset$ for all $i \neq j$ (disjoint subsets),
3. $\cup_{j \in J} X_{1_j} = X_1$ (the union of all subsets covers the entire set X_1),
4. The index set J is countable.

Definition 1.10. Consider a set $X_1 \subseteq \mathbb{R}^n$, which is called a weakly $\mathcal{C}^k-$stratified set of size m (with $k \in \mathbb{N}^*$ and $m \in \{0, 1, \cdots, n\}$) iff there exists a countable partition $\left\{X_{1_j}\right\}_{j \in J}$ of X_1 such that:

- Each X_{1_j}, for $j \in J$, is a differentiable submanifold of class $\mathcal{C}^k$ with dimension strictly less than m (i.e., the strata have dimension strictly less than m).
- There exists at least one stratum $X_{1_{j_0}}$ with dimension m (i.e., there is a stratum of dimension m).

This means that X_1 can be decomposed into a collection of smooth submanifolds, with the strata having smaller dimensions, and at least one stratum having the maximum dimension m.

Definition 1.11. Let $X_1 \subset \mathbb{R}^n$ be a $\mathcal{C}^k-$stratified set of size m, and let $x_i \in X_1$. The tangent space of X_1 in x_i (in the stratified sense) relative to the stratification $S_{X_1} = \left\{X_{1_j}\right\}_{j \in J}$ is defined as the vector space:

$$T_{x_i} X_1 = T_{x_i} X_{1_j}, \, x_i \in X_{1_j} \in S_{X_1}, \tag{1.15}$$

where, the tangent space $T_{x_i} X_{1_j}$ on the right-hand side is understood in the classical sense of differential geometry ([37], [41], [42] and [59]).

The weakly $\mathcal{C}^\kappa$-stratified structure of a set with dimension n is closely related to the selected stratification S_{X_1}. Therefore, the tangent space of the stratified set depends directly on the specific stratification used.

Remark 1.12. The term "weak stratification" is used because this concept is much more general than "Whitney stratification", where the stratification S_{X_1} must satisfy more restrictive assumptions.

Definition 1.13. A function $g : X_1 \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is said to be weakly $\mathcal{C}^k$-stratified with respect to stratification $S_{X_1} = \left\{X_{1_j}, j \in J\right\}$ of the set X_1 if, for every stratum $X_{1_j} \in S_{X_1}$, the restriction $g_j : X_{1_j} \to \mathbb{R}^m$ belongs to the class $\mathcal{C}^k$, understood in the context of differentiable mapping (in the sense of Definition 1.7) on differentiable manifolds.

Definition 1.14. Let $g : X_1 \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a weakly $\mathcal{C}^k$-stratified function with respect to stratification S_{X_1}, and let $x_i \in X_1$. The stratified derivative of $g(.)$ at x_i is defined by the following formula:

$$Dg(x_i) = Dg_j(x_i) \in L\left(T_{x_i}X_1, \mathbb{R}^n\right), \text{ if } x_i \in X_{1_j} \in S_{X_1}. \tag{1.16}$$

Remark 1.15. If a stratum $X_{1_j} \in S_{X_1}$ has dimension n (the maximum dimension), then $X_{1_j} \subseteq \mathbb{R}^n$ is an open set, and the derivative defined in (1.16) coincides with the classical Fréchet derivative.

Example 1.16. Consider the functions $g_1(x_i) = |x_i|$ and $g_2(x_i) = x_i^3$, defined on $X_1 = \mathbb{R}$. Both functions are $\mathcal{C}^\infty$ (analytic) stratified functions with respect to the finite stratification $S_{X_1} = \{X_{1_1}, X_{1_2}, X_{1_3}\}$, where:

$$X_{1_1} = (-\infty, 0), X_{1_2} = (0, +\infty), X_{1_3} = \{0\},$$

thus, each stratum corresponds to a segment of the domain in which functions are characterized as differentiable in the sense of geometry.

The following result allows us to classify the transition points that occur when a trajectory moves from one stratum to another. Such a classification greatly simplifies both computation and analysis.

Definition 1.17. ([55]) The mapping $x_i(.) : [a, b] \to X_1$ is defined to be absolutely continuous when the following condition holds:

1. Continuity in strata: $x_i(.)$ is continuous within each stratum and lies in a finite sequence of strata.

2. Locally absolutely continuous in each stratum: For each subinterval $[a_i, b_i] \subset [a, b]$ where $x_i(.)$ lies entirely within a stratum $X_{1_{j_i}}$, there exists an integrable vector field $v \in L\left([a_i, b_i], T_{x_i}X_{1_{j_i}}\right)$ such that:

$$x_i'(t_i) = v(t_i) \text{ a.e } [a_i, b_i], \tag{1.17}$$

and

$$x_i(t_i) = x_i(a_i) + \int_{a_i}^{t_i} v(s)\, ds, \forall t_i \in [a_i, b_i]. \tag{1.18}$$

3. Strata transitions: At the points where $x_i(.)$ transitions between strata, continuity of $x_i(.)$ is required, but neither the existence nor the continuity of its derivative is necessary.

Definition 1.18. A function $u(.) : I \subseteq \mathbb{R} \to \mathbb{R}^m$ is defined as a regulated function when, the left and right limits exist at every point of its domain I. Thus, for every $t \in I$:

$$\exists u(t_i \pm) = \lim_{s \to t_i \pm} u(s),$$

where, $u(t_i+)$, $u(t_i-)$ represent the right-hand and left-hand limits of u at t_i respectively.

Equivalently, a regulated function is the uniform limit of a sequence of step functions. In this case, the set of discontinuities of u is at most countable, and all discontinuities are of the first kind, meaning the function value changes by a finite jump at these points (see for instance, [12, 25]).

Definition 1.19. A function $x_i(.) : I \subseteq \mathbb{R} \to \mathbb{R}^m$ is called **regular** if there exists a regulated function $u(.)$ and a countable set $J_{x_i} \subset I$ such that:

$$\exists x_i'(t_i) = u(t_i) \ \forall t_i \in I \backslash J_{x_i},$$

where $x_i'(t_i)$ denotes the derivative of x_i at t_i. The set J_{x_i} represents the points where the derivative $x_i'(t_i)$ may not be well-defined or continuous.

Additionally, if $x_i(.)$ is regular, the set J_{x_i} of discontinuities of its derivative is at most countable, meaning it contains a finite or countable number of points (see, [12, 20]).

Lemma 1.20. *Let $X_1 \subseteq \mathbb{R}^n$ be a $\mathcal{C}^1$-stratified by S_{X_1}. Assume that, $x_i(.) : [a,b] \to X_1$ is absolutely continuous (AC) mapping. Define the subsets $J_{x_i}^{is}$, $J_{x_i}^{d}$, $J_{x_i} \subset [a,b]$ as follows:*

$$\begin{aligned} J_{x_i}^{is} &= \left\{ t_i \in [a,b],\ t_i \text{ is isolated in } x_i^{-1}(S') \text{ if } x_i(t_i) \in S' \subset S_{X_1} \right\}, \\ J_{x_i}^{d} &= \left\{ t_i \in [a,b],\ x_i'(t_i),\ (x_i'(t_i) \text{ does not exist}) \right\}, \\ J_{x_i} &= J_{x_i}^{is} \cup J_{x_i}^{d}, \end{aligned} \tag{1.19}$$

where, $x_i^{-1}(S') = \{\tau \in [a,b],\ x_i(\tau) \in S'\}$. Then, the following properties hold:

- *$J_{x_i}^{is}$ is at most countable.*

- *J_{x_i} is a null subset in $[a,b]$ (i.e., it has zero Lebesgue measure).*

- *For all $t_i \in [a,b] \setminus J_{x_i}$, the derivative $x_i'(t_i)$ exists and satisfies:*

$$x_i'(t_i) \in T_{x_i(t_i)} X_1. \tag{1.20}$$

Additionally, if $x_i(.)$ is regular in the sense of Definition 1.18 then, J_{x_i} is at most countable.

We now recall the well-known concept of bilateral Dini derivatives, as introduced in [3], which are provide a fundamental analytical tool that will prove essential in our approach to studying the monotonicity properties of real-valued functions. The bilateral Dini derivatives are defined as follow:

$$\overline{D}k(t_i) = \limsup_{\xi \to 0} \frac{k(t_i+\xi) - k(t_i)}{\xi}, \ \underline{D}k(t_i) = \liminf_{\xi \to 0} \frac{k(t_i+\xi) - k(t_i)}{\xi}, \tag{1.21}$$

for a composite function of the form:

$$k(t_i) = K(x_i(t_i)), \ t_i \in I = [a,b]. \tag{1.22}$$

Lemma 1.21. *(**Chain rule lemma**, [55]) If $K(.) : X_1 \subseteq \mathbb{R}^n \to \mathbb{R}$ is stratified by S_K in the sense of Definition 1.13, $x_i(.) : [a,b] \to X_1$ is absolutely continuous (AC) and if J_{x_i}, $J_{x_i}^{is}$, $J_{x_i}^{d} \subset [a,b]$ are the subsets defined in (1.19) with the properties (1.20) then, the bilateral Dini derivatives in (1.21) of the composite function $k(.)$ (1.22) satisfy the relations:*

$$\overline{D}k(t_i) \geq DK(x_i(t_i)).x_i'(t_i) \geq \underline{D}k(t_i), \ \forall t_i \in I \backslash J_{x_i}, \tag{1.23}$$

In particular, if $J_{x_i}^d \subseteq I$ is the subset of non-differentiability points

$$J_{x_i}^d = \{t_i \in I, \ k'(t_i)\}, \tag{1.24}$$

then, the function $k(.)$ is derived as follows

$$k'(t_i) = DK(x_i(t_i)).x_i'(t_i), \forall t_i \in I \backslash (J_{x_i} \cup J_{x_i}^d). \tag{1.25}$$

The auxiliary results provided in Lemma 1.20 and Lemma 1.21 will play a central role below, where they form the basis for establishing a verification theorem concerning stratified value functions. Additionally, the notion of stratified mapping defined above, will be employed in the following sections to formally define stratified Hamiltonian systems and stratified Hamiltonian flows.

1.3.1 Monotonicity of real functions

The concept of monotonicity in real-valued functions can be rigorously characterized using Dini's derivatives, particularly in contexts where classical differentiability fails. The bilateral Dini derivatives defined as upper and lower limits of difference quotients from both sides provide generalized notions of directional behavior at points of non-differentiability. These derivatives are especially valuable in the analysis of value functions in dynamic programming, where continuity is often preserved, but smoothness is not guaranteed. In this setting, Dini derivatives enable the formulation of sufficient conditions for monotonicity, which in turn support the construction of comparison principles and verification theorems. This generalized framework proves essential for studying optimality conditions in stratified or nonsmooth domains, where standard derivative-based methods are inapplicable.

Theorem 1.22 (Elementary monotonicity theorem, [55]). *If* $w\left(.\right):I\rightarrow\mathbb{R}$ *is differentiable (at each point) and its direvative,* $w'\left(.\right)$ *is Riemann integrable (i.e., bounded and a.e continuous) then, it is increasing (non-decreasing) in the sense that:*

$$w\left(t_s\right)\leq w\left(t_f\right)\ \forall t_s<t_f,\ t_s,t_f\in I, \tag{1.26}$$

if and only if verifies

$$w'\left(t_i\right)\geq 0,\ \forall t_i\in I. \tag{1.27}$$

1.4 Generalized tangent directions to trajectories

In this section, we examine the notion of generalized tangent directions, a concept that facilitates the characterization of admissible trajectories for solutions to differential inclusions. These directions serve as a foundational tool for understanding how trajectories evolve within constrained dynamical systems, especially when classical smoothness assumptions are violated. The overarching goal is to establish constructive criteria for verifying value functions that may lack differentiability or fail to be locally Lipschitz continuous. By formulating the problem as a differential inclusion, we aim to capture the set of feasible solutions that respect the imposed constraints, thus providing a rigorous mathematical structure for the analysis of viable dynamics. This investigation extends the seminal work of Mirică [48] and is closely related to the contributions of notable researchers who have made significant advances in this area. We start by introducing the concept of a differential inclusion.

Definition 1.23. We consider the following constrained differential inclusion:

$$x_i'(t_i) \in F(x_i(t_i)), \text{a.e. } I_{Y_i}(x_i(.)), \ x_i(0) = y_i \in Y_i, \tag{1.28}$$

here, $F(.) : X_1 \subseteq \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a given orientor field, and $Y_i \subseteq X_1$ is the constraint set. The function $x_i(t_i) \in Y_i$ for all $t_i \in I_{Y_i}(x_i(.)) = dom(x_i(.)) \subseteq \mathbb{R}$. For any $y_i \in Y_i$, $S_{F,Y_i}(y_i)$ represent the set of all locally absolutely continuous functions $x_i(t_i)$ that satisfy the constrained differential inclusion given in (1.28).

According to the terminology introduced in [3], these solutions are referred to as viable trajectories. Specifically, if $Y_i = X_1$, the set $S_{F,Y_i}(y_i)$ is denoted by $S_F(y_i)$ for any $y_i \in X_1$. Furthermore, in this context, the subset $X_1 \subseteq \mathbb{R}^n$ is assumed to coincide with the effective domain of the multifunction $F(.)$, and it is implicitly assumed that $F(x_i) \neq \emptyset$ for all $x_i \in X_1$.

While, if $y_i \in Y_i$ and $S_{F,Y_i}(y_i)$ is the set of solutions of the constrained differential inclusion (1.28), then we associate the following set of bilateral tangent directions:

$$T_{F,Y_i}(y_i) = \{x_i'(0), \ x_i(.) \in S_{F,Y_i}(y_i)\}. \tag{1.29}$$

The subset $T_{F,Y_i}(y_i)$ as defined in above, encapsulates the bilateral tangent directions associated with the admissible trajectories originating from $y_i \in Y_i$. This set plays a crucial role in the local analysis of the dynamics governed by the differential inclusion (1.28). Specifically, it characterizes the instantaneous directions in which the system can evolve while remaining viable within the constraint set Y_i. As such $T_{F,Y_i}(y_i)$, provides essential geometric information that supports the verification of stratified value functions, especially when traditional differentiability assumptions are relaxed. Furthermore, this concept lays the groundwork for the development of tangential conditions and viability theorems in subsequent sections.

1.5 Smooth Hamiltonian and Characteristic flows

In this section, we will present some notions and results regarding the characteristics Hamiltonian flow. These results are used to obtain generalized solutions of the

Hamilton-Jacobi equations associated with classes of optimal control problems and autonomous differential games.

The following results are motivated by the fact that the value functions of autonomous problems in the calculus of variations and optimal control theory are shown to be generalized (sometimes even classical) solutions of the autonomous Hamilton-Jacobi equation of the form:

$$H\left(x_i, DW\left(x_i\right)\right)=0,\ x_i \in Y_s \subset \mathbb{R}^n, W\left(x_i\right)=g\left(x_i\right), x_i \in Y_f \subset Cl\left(Y_s\right), \tag{1.30}$$

defined by the Hamiltonian $H(.) : Z \subseteq \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$, and by the terminal function $g(.) : Y_f \longrightarrow \mathbb{R}$. Moreover, compared the so-called viscosity solutions (Bardi şi Capuzzo-Dolcetta (1997), Grandall şi Lions (1983), Elliot (1987),...,etc.), the Cauchy of characteristic, wherever it can be applied, provides not only the value function but also descriptions (characterizations) of the optimal trajectories needed in the solution of an optimal control problem.

A natural extension of the characteristic method is obtained in the case where the domain $Z \subseteq \mathbb{R}^n \times \mathbb{R}^n$ of the Hamiltonian $H(\cdot)$ is a differentiable manifold, and the classical Hamiltonian field is replaced by the orientor Hamiltonian field, defined for $z=(x_i, p_i) \in Z$ by:

$$d_S^{\#} H(z)=\{z' \in T_z Z,\ <x_i', \overline{p_i}>-<p_i', \overline{x_i}>=DH(z).\overline{z},\ \forall \overline{z} \in T_z Z\}, \tag{1.31}$$

where $T_z Z$ is the tangent space and $DH(z)$ is the derivative of the Hamiltonian in the sense of Differential Geometry.

A first extension of the method of characteristics is obtained under the following hypothesis:

Hypothesis 1. *The subset $Z \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is of class C^2 submaniflod, $H(.,.) : Z \rightarrow \mathbb{R}$ is of class $\mathcal{C}^2$ and there exists $h(.,.) : Z \rightarrow \mathbb{R}^n \times \mathbb{R}^n$, of class $\mathcal{C}^2$, a selection of the multifunction $d_S^{\#} H(.,.)$* in (1.31) *satisfying*:

$$h(z)=\left(h_1(z), h_2(z)\right) \in d_S^{\#} H(z),\ z \in Z, \tag{1.32}$$

or, equivalently, a smooth mapping satisfying the following condition:

$$\begin{aligned} < & \ h_1(z), \overline{p_i}>-<h_2(z), \overline{x_i}>=DH(z)\overline{z}, \\ \forall \overline{z} & \in T_z Z, h(z) \in T_z Z\ \forall z \in Z. \end{aligned} \tag{1.33}$$

Remark 1.24. As can be easily seen, if $z \in Z$ and $\dim(Z) = 2n$ (so, $Z \subset \mathbb{R}^n \times \mathbb{R}^n$ is an open set and the restriction $H(\cdot,\cdot) := H(\cdot,\cdot)\mid_Z$ is differentiable (Fréchet)) then, the orientor Hamiltonian field in (1.31) coincides with the classical Hamiltonian field:

$$d_S^{\#}H(z) = \left\{ \left(\frac{\partial H}{\partial p_i}(z), -\frac{\partial H}{\partial x_i}(z) \right) \right\}, \; z \in Z = Int(Z), \tag{1.34}$$

this holds since, in this case, the following relation is satisfied:

$$DH(z).\overline{z} =< \frac{\partial H}{\partial p_i}(z), \overline{p_i} > + < \frac{\partial H}{\partial x_i}(z), \overline{x}_i >, \; \forall \overline{z} \in \mathbb{R}^n \times \mathbb{R}^n. \tag{1.35}$$

Next, if $Z \subset \mathbb{R}^n \times \mathbb{R}^n$ is no more of maximal dimension $(2n)$, standard arguments show that at each point $z = (x_i, p_i) \in Z$, the value $d_S^{\#}H(z) \subset T_zZ$ is either an empty set or an affine manifold parallel to the linear subspace $d_S^{\#}H_0(z) \subseteq \mathbb{R}^n \times \mathbb{R}^n$, which is defined in a similar way using the null Hamiltonian:

$$\begin{aligned} H_0(z) &= 0, \; \forall z \in Z, \\ d_S^{\#}H(z) &= \left\{ z' \in T_zZ, \; < x_i', \overline{p_i} > - < p_i', \overline{x_i} >= 0, \forall \overline{z} \in T_zZ \right\}, \end{aligned} \tag{1.36}$$

in this context, the expression for $d_S^{\#}H(z)$ is as follows:

$$d_S^{\#}H(z) = \{z_0'\} + d_S^{\#}H_0(z) \;\; \forall z_0' \in d_S^{\#}H(z). \tag{1.37}$$

However, restrictive Hypothesis 1 may seem, it is nevertheless a significant generalization of the case of Hamilton-Jacobi equations on symplectic manifolds (e.g. Mirică [55]) (in particular, of classical equations defined on open subsets of $\mathbb{R}^n \times \mathbb{R}^n$). This was ascertained by analyzing some significant examples (e.g., Mirică [52, 55]), on the other hand, this approach suggests possibilities for significant generalizations of these classical cases.

As is known, in the classical case (i.e., $Z = Int(Z) \subseteq \mathbb{R}^n \times \mathbb{R}^n$) the essential tool of Cauchy's Method of Characteristics is the associated characteristic system:

$$\begin{cases} x_i' = \frac{\partial H}{\partial p_i}(z), \; x_i(0) = \xi \in pr_1 Z \\ p_i' = -\frac{\partial H}{\partial x_i}(z), \; p_i(0) = q \in Z(\xi) = \{q \in \mathbb{R}^n; (\xi, q) \in Z\} \\ v' =< p_i, \frac{\partial H}{\partial p_i}(z) >, \; v(0) = v_0 \in \mathbb{R}, \end{cases} \tag{1.38}$$

which, due to the particular form of the characteristic field:

$$c(z,v) = \left(\frac{\partial H}{\partial p_i}(z), -\frac{\partial H}{\partial x_i}(z), < p_i, \frac{\partial H}{\partial p_i}(z) > \right), \;\; (z,v) \in Z \times \mathbb{R},$$

is evidently reduced to the smooth Hamoltonian system:

$$\begin{cases} x_i' = \frac{\partial H}{\partial p_i}(z), \; x_i(0) = \xi \in pr_1 Z \\ p_i' = -\frac{\partial H}{\partial x_i}(z), \; p_i(0) = q \in Z(\xi), \end{cases} \tag{1.39}$$

which under Hypothesis 1, is replaced by the generalized Hamiltonian system:

$$z' = h(z), \; (x_i(0), p_i(0)) = z_0 = (\xi, q) \in Z, \tag{1.40}$$

defined by the smooth selection $h(.)$ in (1.32).

For the sake of completeness, we present the fundamental properties of smooth Hamiltonian flows, which can be regarded as the main technical foundation of **Cauchy's method of characteristics**. In the following, we assume that the Hypothesis is holds.

Hypothesis 2. The Hamiltonian $H(.,.) : Z \subseteq \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and the Hamiltonian vector field $h(.,.) : Z \subseteq \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ satisfy Hypothesis 1, the boundary set $Y_f \subset Cl(Y_s)$ is a differentiable manifold, the terminal function $g(.) : Y_f \to \mathbb{R}$ is a differentiable function and the set of terminal data $Z_1^s \subset Z$, defined by:

$$Z_1^s = \left\{ z_0 \in Z, \; \xi \in Y_f, \; H(z_0) = 0, \; < q, \overline{\xi} >= Dg(\xi)\overline{\xi}, \; \forall \overline{\xi} \in T_\xi Y_f \right\} \tag{1.41}$$

is a non-empty subset of $Z \subseteq \mathbb{R}^n \times \mathbb{R}^n$.

Lemma 1.25. ***(Properties of smooth Hamiltonian flows)*** *If Hypothesis 2 is satisfied and the Hamiltonian vector field $h(.,.)$ defined in (1.32) is of class C^1 on the differential manifold $Z \subseteq \mathbb{R}^n \times \mathbb{R}^n$, then there exists a unique relatively open submanifold $D_h \subseteq \mathbb{R} \times Z$ and a unique C^1mapping $X_h^*(.,.) = (X_i(.,.), P_i(.,.)) : D_h \to Z$ (smooth Hamiltonian flow) with the following properties:*

- *For any $z_0 = (\xi, q) \in Z$ the partial mapping $X_h^*(., z_0) : I_h(z_0) \subseteq \mathbb{R} \to Z$ is the unique maximal solution of problem (1.39) in the sense that its derivatives, $D_1 X_h^*(t_i, z_0) = (X_i'(t_i, z_0), P_i'(t_i, z_0))$ satisfies the identity:*

$$D_1 X_h^*(t_i, z_0) \equiv h((X_i(t_i, z_0), P_i(t_i, z_0))), X_h^*(0, z_0) = z_0. \tag{1.42}$$

- *The partial derivatives of the mapping $X_h^*(.,.)$ satisfy the commutativity relation:*

$$D_1 D_2 X_h^*(t_i, z_0) = D_2 D_1 X_h^*(t_i, z_0), \; \forall (t_i, z_0) \in D_h. \tag{1.43}$$

- *The Hamiltonian $H(.,.)$ is a first integral of $h(.,.)$, in the sense that:*

$$H\left(X_h^*\left(t_i, z_0\right)\right)=H\left(X_h^*\left(0, z_0\right)\right)=H\left(z_0\right), \ \forall\left(t_i, z_0\right) \in D_h. \tag{1.44}$$

- *The partial derivatives of the components $X_i(.,.)$, $P_i(.,.)$ of $X_h^*(.,.)$ satisfy:*

$$\begin{aligned}&\tfrac{\partial}{\partial z_0}[\langle P_i\left(s, z_0\right), X_i'\left(s, z_0\right)\rangle].\bar{z}=\tfrac{\partial}{\partial s}\langle P_i\left(s, z_0\right), D_2 X_i\left(s, z_0\right).\bar{z}\rangle+\\ &+DH(z).\bar{z}, \ \forall\left(s, z_0\right) \in D_h, \bar{z}=\left(\bar{x}_i, \bar{p}_i\right) \in T_{z_0} Z \subseteq \mathbb{R}^n \times \mathbb{R}^n.\end{aligned} \tag{1.45}$$

As in the classical theory of Hamilton–Jacobi equations, the Hamiltonian flow $X_h^*(.,.)$ described in Lemma 1.25 is extended to the characteristic flow $C_h^*(.,.)=\left(X_h^*(.,.), V_h^0(.,.)\right)$, where $V_h^0(.,.)$ is defined as follows:

$$V_h^0\left(t_i, z_0\right)=\int_0^{t_i}<P_i\left(s, z_0\right), X_i'\left(s, z_0\right)>ds, \ \left(t_i, z_0\right) \in D_h. \tag{1.46}$$

The role of the smooth characteristic flow is ullusrates in the following result.

Theorem 1.26. *(**Basic properties of the smooth characteristic flow**, [55]) If Hypothesis 1 is satisfied and $X_h^*(.,.)=\left(X_i(.,.), P_i(.,.)\right)$ represents the Hamiltonian flow defined in Lemma 1.25, then the function $V_h^0(.,.)$, defined in (1.46) is of class $\mathcal{C}^1$and its derivative is given by:*

$$\begin{aligned}&DV_h^0\left(t_i, z_0\right).\left(\overline{t_i}, \bar{z}\right)=<P_i\left(t_i, z_0\right), DX_i\left(t_i, z_0\right).\left(\bar{t}, \bar{z}\right)>-<q, \overline{x_i}>+\\ &+t_i.DH\left(z_0\right).\bar{z}, \ \textit{if } z_0=(\xi, q) \in Z, \bar{z}=\left(\overline{x}_i, \overline{p_i}\right) \in T_{z_0} Z.\end{aligned} \tag{1.47}$$

Expanding upon the findings of the preceding section, we develop both classical and generalized characteristic solutions to autonomous Hamilton-Jacobi equations as specified in (1.30), emphasizing their applications in optimal control. Our methodology enhances the classical Cauchy method of characteristics (see to Benton (1977), Hartman (1964), Miričă (1987), Subbotin (1994), Van et al. (2000), among others) in two principal aspects the Hamiltonian domain in (1.30) is not necessitated to be an open set and the first component of the Hamiltonian flow is not assumed to be invertible. These generalizations enhance the method's applicability to more complex and realistic systems.

First, suppose that the data $H(.,.)$ and $g(.)$ related to problem (1.30) satisfy the conditions given in Hypothesis 2. Under this assumption, the characteristic flow defined in (1.42), (1.46) represents the classical characteristic flow corresponding to problem (1.30), as described below.

Definition 1.27. The **classical characteristic flow** associated with the problem (1.30) is the pair

$$C^* (.,.) = (X_i^* (.,.), V (.,.)),$$

where:

$$\begin{aligned} & X_i^* (t_i, z_0) = X_h^* (t_i, z_0) = (X_i (t_i, z_0), P_i (t_i, z_0)), \ \ (t_i, z_0) \in B \subset \mathbb{R} \times Z_1^s \\ & V (t_i, z_0) = g (\xi) + V_h^0 (t_i, z_0) \text{ if } z_0 = (\xi, q) \in Z_1^s, \ \ (t_i, z_0) \in B, \end{aligned} \tag{1.48}$$

and the domain B is defined as

$$B = \{(t_i, z_0) \in D_h \,|z_0 \in Z_1^s, \ X_i (t_i) \in Y_s \text{ if } t_i \neq 0\}. \tag{1.49}$$

The fact that the Hamiltonian is equal to zero along all admissible trajectories, and can thus be regarded as a first integral of the Hamiltonian system in (3.39), is illustrated below.

Proposition 1.28. *Let $C^* (.,.) = (X_i^* (.,.), V (.,.))$ be the classical characteristic flow defined in Definition 1.27. If Hypothesis 2 is satisfied, then the following properties hold:*

$$H (X_i (t_i, z_0), P_i (t_i, z_0)) = 0, \ \forall (t_i, z_0) \in B = dom (C^* (.,.)). \tag{1.50}$$

Moreover, the function $V (.,.)$ defined in (1.48) is differentiable and its derivative (tangent) is given by:

$$DV (t_i, z_0). \left(\overline{t_i}, \overline{z}\right) =< P_i (t_i, z_0), X_i' (t_i, z_0). \left(\overline{t_i}, \overline{z}\right) >, \ \forall \left(\overline{t_i}, \overline{z}\right) \in T_{(t_i, z_0)} B. \tag{1.51}$$

Remark 1.29. The classical characteristic flow allows for the construction and characterization of characteristic solutions to the boundary value problem (BVP) in (1.30) by using appropriate subset

$$B_0 = \{(t_i, z_0) \in B, \ t_i \neq 0\}. \tag{1.52}$$

Theorem 1.30 ([55])**.** *Let Hypothesis 2 be satisfied, $C^* (.,.) = (X_i^* (.,.), V (.,.))$ be the associated characteristic flow in Definition 1.27 and $\widetilde{B_0} \subseteq B_0$ be a submanifold of dimension n of $\mathbb{R}^n$ such that $\widetilde{Y_s} = X_i \left(\widetilde{B_0}\right) \subseteq \mathbb{R}^n$ is open and the restriction $\widetilde{X_{i_0}} (.,.) = X_i (.,.) : \widetilde{B_0} \to \widetilde{Y_s} \subseteq Y_s$ is invertible with the differential inverse $\widehat{B}_0 (.) : \widetilde{Y_s} = Int \left(\widetilde{Y_s}\right) \to \widetilde{B_0}$, satisfying:*

$$X_i \left(\widehat{B}_0 (y_i)\right) = y_i, \ \forall y_i \in \widetilde{Y_s}, \ \widehat{B}_0 (X_i (t_i, z_0)) = (t_i, z_0), \forall (t_i, z_0) \in \widetilde{B_0}. \tag{1.53}$$

Then, the real function $\widetilde{W}(.)$ defined by:

$$\widetilde{W}(y_i) = \begin{cases} g(y_i)\,, \textit{ if } y_i \in Y_f \\ V\left(\widehat{B}_0(y_i)\right) \textit{ if } y_i \in \widetilde{Y}_s \subseteq Y_s, \end{cases} \tag{1.54}$$

is a solution of problem (1.30) *on the subset* $\widetilde{Y} = \widetilde{Y}_s \cup Y_f$.

This chapter presented the fundamental tools and theoretical background in nonsmooth analysis, which form the basis for the study of more complex dynamic systems. In the following chapter, these concepts will be applied to the analysis of autonomous differential games, highlighting their role in modeling strategic interactions within continuous-time frameworks.

CHAPTER 2

Autonomous Differential Games

Differential game theory represents a vast and significant domain within applied mathematics, offering powerful tools for modeling and analyzing dynamic interactions between multiple decision-makers with conflicting objectives. In this chapter, we focus on autonomous differential games in which the dynamic and cost function are independent of time, within the framework of dynamic programming. The main objective is to present a rigorous and robust version of the dynamic programming algorithm specifically adapted to this class of games, as developed by Mirică in [55, 56]. Such approach not only extends and generalizes existing methodologies but also enhances mathematical precision in the characterization of optimal strategies and value functions. One of the key contributions of Mirică's formulation is its capacity to establish the optimality of admissible control pairs through a refined version of one of the seven verification theorems, presented in [54]. Which play a pivotal role in linking the theoretical properties of the value functions with practical conditions for optimality, thereby providing a bridge between abstract mathematical constructs and their implementation in real-world decision problems.

2.1 Formulation of Differential game

An autonomous differential game denoted by (DG_A) is a type of differential game (DG) in which the system dynamic and cost function do not explicitly depend on time. More formally, such a game is defined as follows:

Problem 1. (DG_A) is formally defined as:

$$DG_A = \left(Y_s, Y_f, g\left(.\right), f_0\left(.,.,.\right), f\left(.,.,.\right), U, V, \mathcal{U}_\alpha, \mathcal{V}_\alpha, \Omega_\alpha\right),$$

where each component has the following interpretation:

- $Y_s \subset \mathbb{R}^n$ denotes the set of initial states, while $Y_f \subset Cl(Y_s)$ the set of terminal states.
- $g\left(.\right) : Y_f \to \mathbb{R}$ is the terminal payoff function.
- $f_0\left(.,.,.\right) : D \times U \times V \to \mathbb{R}$ represents the running payoff function.
- $f\left(.,.,.\right) : D \times U \times V \to \mathbb{R}^n$ defines the parametrized vector field.
- U and V are compact control spaces for the two players and the admissible control pairs belong to the product space $\mathcal{P}_\alpha = \mathcal{U}_\alpha \times \mathcal{V}_\alpha$ where, $\mathcal{U}_\alpha$ and $\mathcal{V}_\alpha$ represent the class of admissible control functions.
- $\Omega_\alpha \in \{\Omega_1, \Omega_\infty, \Omega_r\}$ is the class of admissible trajectories.
- $t_f = t_f(x_i(.)) > 0$ is the terminal time.
- $L^1\left([0, t_f]; \mathbb{R}\right)$ represents the space of Lebesgue integrable functions.

The problem (DG_A) consists in optimizing the cost functional defined by:

$$\inf_{u(.)\in\mathcal{U}_\alpha} \sup_{v(.)\in\mathcal{V}_\alpha} C(y_i; u(.), v(.)),\ \forall y_i \in Y_s, \tag{2.1}$$

where

$$C(y_i; u(.), v(.)) = g(x_i(t_f)) + \int_0^{t_f} f_0(x_i(t_i), u(t_i), v(t_i))dt_i, \forall y_i \in Y_s. \tag{2.2}$$

Subject to:

1. State equation:

$$x_i'(t_i) = f\left(x_i(t_i), u(t_i), v(t_i)\right)\ \text{a.e.}(0, t_f),\ x_i(0) = y_i. \tag{2.3}$$

2. Control constraints:

$$u(t_i) \in U,\ v(t_i) \in V\ a.e.(0, t_f),\ (u(.), v(.)) \in \mathcal{P}_\alpha. \tag{2.4}$$

3. Trajectory properties:

$$x_i\left(.\right) \in AC\left(\left[0,t_f\right];\mathbb{R}^n\right),\ f_0\left(x_i\left(.\right),u\left(.\right),v\left(.\right)\right) \in L^1\left(\left[0,t_f\right];\mathbb{R}\right). \tag{2.5}$$

4. Boundary conditions:

$$x_i(t) \in Y_s,\ \forall t_i \in [0,t_f),\ x_i(t_f) \in Y_f,\ Y_s \cap Y_f = \emptyset. \tag{2.6}$$

Remark 2.1. The class of admissible control pairs is taken as $\mathcal{P}_\alpha = \mathcal{U}_\alpha \times \mathcal{V}_\alpha$ where, the class $\mathcal{U}_\alpha = \{\mathcal{U}_c,\mathcal{U}_r,\mathcal{U}_1\}$ and $\mathcal{V}_\alpha = \{\mathcal{V}_c,\mathcal{V}_r,\mathcal{V}_1\}$ are defined by

$$\mathcal{U}_\alpha : \begin{cases} \mathcal{U}_c = \left\{u\left(.\right) : I \to U;\ u\left(.\right) \text{ is continuous}\right\}, \\ \mathcal{U}_r = \left\{u\left(.\right) : I \to U;\ u\left(.\right) \text{ is regulated}\right\}, \\ \mathcal{U}_1 = \left\{u\left(.\right) : I \to U;\ u\left(.\right) \text{ is measurable}\right\}, \end{cases}$$

analogous, definitions apply to the sets $\mathcal{V}_c, \mathcal{V}_r$ and $\mathcal{V}_1$ corresponding to the second player.

The associated class of admissible trajectories is denoted by $\Omega_\alpha = \{\Omega_1, \Omega_\infty, \Omega_r\}$, where each subset is given by:

$$\begin{aligned} &\Omega_1 = \left\{x_i(.) \in AC;\ \exists x_i'(.) \in L^1\right\}, \\ &\Omega_\infty = \left\{x_i(.) \in AC;\ \exists x_i'(.) \text{ piecewise continuous}\right\}, \\ &\Omega_r = \left\{x_i(.) \in AC;\ \exists x_i'(.) \text{ regulated}\right\}. \end{aligned}$$

Remark 2.2. From [55, 56], we take $\Omega_1 = AC\left(\left[0,t_f\right];\mathbb{R}^n\right)$ as the largest class of (AC) trajectories. The corresponding class $\mathcal{P}_1 = \mathcal{U}_1 \times \mathcal{V}_1$ refers to the measurable admissible controls $(u(.), v(.))$.

2.2 Admissible Feedback Strategies and Relative Optimality

In the study of differential games, feedback strategies constitute a central concept for guiding decision-making in dynamic environments. These strategies U and V are defined so that the controls $u(.)$ and $v(.)$ applied at each instant depend explicitly on the current state of the system. This feature provides flexibility and robustness, allowing players to adjust their actions as the game evolves over time. Admissible feedback strategies are restricted to satisfy well-defined measurability and regularity conditions, ensuring their mathematical consistency. Such admissibility guarantees that strategies remain implementable across a wide range of trajectories. The concept of optimality is essential in practice, since absolute optimality is rarely achievable in complex interactive settings. Classical contributions have emphasized this direction, most notably the works of Krasovskii and Subbotin [35], Mirică [56] and Subbotin [64], which established the theoretical foundations of admissible feedback strategies in differential games.

Hypothesis 3. The data of Problem 1 satisfy the following assumptions:

(i) The control sets U, V are compact and the functions $f_0(.,.,.), f(.,.,.)$ are continuous on the open domain $D \subset \mathbb{R}^n$.

(ii) The vector field $f\left(.,.,.\right) : D \times U \times V \rightarrow \mathbb{R}^n$ satisfies a uniform local Lipschitz condition with respect to its first argument. Specifically, for every compact subset $D_0 \subset D$, there exists a constant $L > 0$ such that:

$$\left\| f\left(x_{i_1}, u, v\right) - f\left(x_{i_2}, u, v\right)\right\| < L \left\| x_{i_1} - x_{i_2}\right\|, \ \forall x_{i_1},\ x_{i_2} \in D_0,\ u \in U,\ v \in V. \quad (2.7)$$

Remark 2.3. .

(i) Under Hypothesis 3 then for any admissible control pair $\left(u\left(.\right), v\left(.\right)\right) \in \mathcal{P}_\alpha$ and any initial state $y_i \in Y_s$, the initial value problem (IVP) in (2.3) has a unique Carathéodory (AC) solution

$$x_i\left(.\right) = x_{y_i}^{u,v}\left(.\right) : (0, \tau) \rightarrow \mathbb{R}^n. \quad (2.8)$$

(ii) For the trajectory $x_i\left(.\right)$ to be admissible, it must satisfy the control constraints in (2.4), as well as the state and terminal constraints in (2.6). Specifically, there exists $t_f \in (0, \tau)$ such that for all $t_i \in [0, t_f)$, the trajectory remains within the initial state set, i.e., $x_i\left(t_i\right) \in Y_s$ and at time t_f, it reaches the terminal state set, i.e., $x_i\left(t_f\right) \in Y_f$.

From the vague formulation of Problem (1) it is understood that there are two players $\mathbb{U}$ and $\mathbb{V}$, select admissible controls $u\left(.\right)$ and $v\left(.\right)$, respectively, which together generate an admissible trajectory $x_i\left(.\right)$. Player $\mathbb{U}$ aims to minimize the functional in (2.1) while, player $\mathbb{V}$ seeks to maximize it. However, such a formulation is unrealistic since each player must know the control chosen by the other. Accordingly, the pursuit of a realistic information structure necessitates distinct formulations of the playing rules and their corresponding value functions. (e.g., Bardi and Capuzzo-Dolcetta [4], Berkovitz [9], Clarke et al. [23], Elliot [26], Subbotin [64],..., etc.). From [56], we use the following notation to simplify later formulations, for any $y_i \in Y_s$:

$$\begin{aligned} \Omega_\alpha(y_i) &= \left\{x_i(.) \in \Omega_\alpha;\ \exists (u(.), v(.)) \in \mathcal{P}_\alpha : x_i(.) = x_{y_i}^{u,v}(.)\right\} \\ \mathcal{P}_\alpha(y_i) &= \left\{(u(.), v(.)) \in \mathcal{P}_\alpha;\ \exists x_i(.) = x_{y_i}^{u,v}(.) \in \Omega_\alpha(y_i)\right\}, \end{aligned} \tag{2.9}$$

where $x_{y_i}^{u,v}(.)$ defined in (2.8) is a solution of $(2.3 - 2.6)$. While, the set:

$$\Omega_\alpha(y_i, u, v) = \left\{x_{y_i}^{u,v}(.)\right\},$$

represents the admissible trajectories corresponding to $(u(.), v(.)) \in \mathcal{P}_\alpha$ and it is either an empty set or a singleton.

In view of Problem 1, we introduce additional notations for the sets of relatively admissible control pairs corresponding to $y_i \in Y_s$:

$$\begin{aligned} \mathcal{U}_\alpha(y_i) &= \left\{u(.) \in \mathcal{U}_\alpha;\ x_{y_i}^{u,v}(.) \in \Omega_\alpha(y_i)\right\},\ v \in \mathcal{V}_\alpha, \\ \mathcal{V}_\alpha(y_i) &= \left\{V(.) \in \mathcal{V}_\alpha;\ x_{y_i}^{u,v}(.) \in \Omega_\alpha(y_i)\right\},\ u \in \mathcal{U}_\alpha. \end{aligned} \tag{2.10}$$

Definition 2.4. Let $h(.,.,.) : E \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be a parametrized vector field and $H(.) : Y_i \subset \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^m)$ be a multifunction. Then we define the parametrized differential inclusion:

$$x_i' \in F(t_i, x_i) = h(t_i, x_i, H(x_i)),\ x_i(t_0) = x_0,\ \ (t_i, x_0) \in pr_1 E \times Y_i, \tag{2.11}$$

a mapping $x_i(.) \in AC([t_0, t_f], \mathbb{R}^n)$ is called a classical solution of problem (2.11) if there exists a measurable function $\xi(.) \in \mathcal{M}([t_0, t_f], \mathbb{R}^n)$ such that:

$$x_i'(t_i) = h(t_i, x_i(t_i), \xi(t_i)),\ \xi(t_i) \in H(x_i(t_i))\ \ a.e.\ [t_0, t_f], x_i(t_0) = x_0. \tag{2.12}$$

Next, we will provide a comprehensive review of relatively optimal feedback strategies, focusing on their definition, key characteristics and their role in achieving a relatively optimal balance, as well as their connection to the associated value function introduced in [53]. We begin by defining feedback strategies in alignment with the frameworks presented in [54] and [56]. Due to the concepts in Definition 2.4, we formalize the rules of the game, if player $\mathbb{U}$ (resp. player $\mathbb{V}$) chooses a feedback strategy $\widetilde{U}(.) : Y_s \to \mathcal{P}(U)$ (resp. $\widetilde{V}(.) : Y_s \to \mathcal{P}(V)$, where $\mathcal{P}(V)$ denotes the family of subsets of V) as follows:

Definition 2.5. (**The Rule Of The Game** , [54]) A feedback strategy of player $\mathbb{U}$ is a multifunction $\widetilde{U}(.) \subset U(.)$ (i.e., satisfying: $\widetilde{U}(x_i) \subset U(x_i)\ \forall x_i \in Y_s$) that defines the parametrized differential inclusions:

$$x_i' \in f\left(x_i, \widetilde{U}(x_i), v(t_i)\right),\ x_i(0) = y_i,\ v(.) \in \mathcal{V}_\alpha, \tag{2.13}$$

whose admissible trajectories in the sense of Definition 2.4, that satisfy $(2.3-2.6)$, are defined by:

$$\begin{aligned}\overline{\Omega}_v(y_i) = \ & \{\overline{x}_{y_i,v}(.),\ \exists \overline{u}_v(.) \in \mathcal{U}_v(y_i) : \overline{x}'_{y_i,v}(t_i) = f(\overline{x}_{y_i,v}(t_i), \overline{u}_v(t_i), v(t_i)),\\ & \overline{u}_v(t_i) \in \widetilde{U}(\overline{x}_{y_i,v}(t_i)),\ v(t_i) \in V(\overline{x}_{y_i,v}(t_i))\ a.e.\ \left(0, \overline{t_f}\right)\},\end{aligned} \tag{2.14}$$

while, the corresponding sets $\overline{\mathcal{U}}_v(y_i)$, of admissible controls that satisfy the conditions in (2.14) are denoted by:

$$\overline{\mathcal{U}}_v(y_i) = \{\overline{u}_v(.) \in \mathcal{U}_v(y_i),\ \exists \overline{x}_{y_i,v}(t_i) \text{ satisfying } (2.14)\},\ v(.) \in \mathcal{V}_\alpha \tag{2.15}$$

and the associated upper value function $\overline{W}_{\widetilde{U}}(.)$ (the best for player $\mathbb{V}$) is given by:

$$\overline{W}_{\widetilde{U}}(y_i) = \sup_{v(.) \in \mathcal{V}_\alpha, \overline{u}_v(.) \in \overline{\mathcal{U}}_v(y_i)} C(\overline{u}_v(.), v(.)),\ y_i \in Y_s. \tag{2.16}$$

Symetrically, a feedback strategy of player $\mathbb{V}$ is a multifunction $\widetilde{V}(.) \subset V(.)$ (i.e., satisfying: $\widetilde{V}(x_i) \subset V(x_i), \forall x_i \in Y_s$) that defines the parametrized differential inclusions:

$$x_i' \in f\left(x_i, U(x_i), \widetilde{V}(x_i)\right),\ x_i(0) = y_i,\ u(.) \in \mathcal{U}_\alpha \tag{2.17}$$

whose admissible trajectories satisfy $(2.3-2.6)$, are defined by:

$$\begin{aligned}\overline{\Omega}_u(y_i) = \ & \{\overline{x}_{y_i,u}(.),\ \exists \overline{v}_u(.) \in \mathcal{V}_u(y_i) :\ \overline{x}'_{y_i,u}(t_i) = f(\overline{x}_{y_i,u}(t_i), u(t_i), \overline{v}_u(t_i)),\\ & \overline{v}_u(t_i) \in \widetilde{V}(\overline{x}_{y_i,u}(t_i)),\ u(t_i) \in V(\overline{x}_{y_i,u}(t_i)) \text{ a.e. } \left(0, \overline{t_f}\right)\}\end{aligned} \tag{2.18}$$

while, the sets $\overline{\mathcal{V}}_u(y_i)$ of admissible controls that satisfy the conditions in (2.18) are denoted by:

$$\overline{\mathcal{V}}_u(y_i) = \{\overline{v}_u(.) \in \mathcal{V}_u(y_i);\ \exists \overline{x}_{y_i,u}(t_i) \text{ satisfying } (2.18)\}, v(.) \in \mathcal{V}_\alpha \tag{2.19}$$

and the associated lower value function $\underline{W}_{\widetilde{V}}(.)$ (the best for player $\mathbb{U}$) having as formula:

$$\underline{W}_{\widetilde{V}}(y_i) = \inf_{u(.) \in \mathcal{U}_\alpha, \overline{v}_u(.) \in \overline{\mathcal{V}}_u(y_i)} C(u(.), \overline{v}_u(.)),\ y_i \in Y_s. \tag{2.20}$$

Finally, a pair of feedback strategies is a multifunction $\left(\widetilde{U}(.),\widetilde{V}(.)\right)\subset(U(.),V(.))$ that defines the differential inclusion:

$$x_i'(.)\in f\left(x_i,\widetilde{U}(x_i),\widetilde{V}(x_i)\right),\ x_i(0)=y_i, \tag{2.21}$$

whose admissible trajectories satisfy (2.3 − 2.6) are defined by:

$$\begin{aligned}\overline{\Omega}(y_i)=\ &\{x_{y_i}(.),\ \exists\,(u_{y_i}(.),v_{y_i}(.))\in\mathcal{P}_\alpha(y_i),\ u_{y_i}(t_i)\in\widetilde{U}(x_{y_i}(t_i)),\\ &v_{y_i}(t_i)\in\widetilde{V}(x_{y_i}(t_i))\ \text{a.e.}\ (0,t_f(x_{y_i}(.))),\ y_i\in Y_s\},\end{aligned}\tag{2.22}$$

and also, the sets of pairs $\mathcal{P}_{\widetilde{U},\widetilde{V}}(y_i),\ y_i\in Y_s$ of corresponding admissible controls that satisfy the conditions in (2.22) are denoted by:

$$\mathcal{P}_{\widetilde{U},\widetilde{V}}(y_i)=\left\{(u_{y_i}(.),v_{y_i}(.)):\ \exists x_{y_i}(.)\in\overline{\Omega}(y_i)\ \text{satisfying (2.22)}\right\}, \tag{2.23}$$

where, the associated lower value functions $\underline{W}_{\widetilde{V}}(.)$ (resp, upper value functions $\overline{W}_{\widetilde{U}}(.)$) are defined by:

$$\begin{cases}\underline{W}_{\widetilde{U},\widetilde{V}}(y_i)=\inf_{\left(u_{y_i}(.),v_{y_i}(.)\right)\in\mathcal{P}_{\widetilde{U},\widetilde{V}}(y_i)}C(u_{y_i}(.),v_{y_i}(.)),\\ \overline{W}_{\widetilde{U},\widetilde{V}}(y_i)=\sup_{\left(u_{y_i}(.),v_{y_i}(.)\right)\in\mathcal{P}_{\widetilde{U},\widetilde{V}}(y_i)}C(u_{y_i}(.),v_{y_i}(.)).\end{cases}\tag{2.24}$$

The fundamental concepts introduced by Mirică in [53] are formally defined as follows.

Definition 2.6. Let $\left(\widetilde{U}(.),\widetilde{V}(.)\right):\widetilde{Y}_s\subset Y_s\to\mathcal{P}(U)\times\mathcal{P}(V)$ be a multifunction, where $\widetilde{U}(x_i)\subset U(x_i),\widetilde{V}(x_i)\subset V(x_i),\forall x_i\in\widetilde{Y}_s$. These multifunctions represent **an admissible pair of feedback strategies** for the restriction of the differential game $DG_A\Big|\widetilde{Y}_s$ to the subset $\widetilde{Y}_s\subset Y_s$ as described in (2.2 − 2.6) if the following conditions are satisfied:

(i) For each $y_i\in\widetilde{Y}_s$ the set $\widetilde{\Omega}(y_i)$ of absolutely continuous (AC) solutions $x_{y_i}(.):\left[0,\widetilde{t}_f\right]\to\widetilde{Y}_i$ of the parametrized differential inclusion in (2.21) that satisfy the constraints:

$$x_{y_i}(t_i)\in\widetilde{Y}_s,\ \forall t_i\in[0,t_f),\ x_{y_i}\left(\widetilde{t}_f\right)\in Y_f, \tag{2.25}$$

is non empty.

(ii) Let $\widetilde{\mathcal{P}}_\alpha(y_i)=\{(u_{y_i}(.),v_{y_i}(.))\},\ y_i\in\widetilde{Y}_s$ be the set of control pairs corresponding to (2.23). Then the cost function satisfies the consistency condition:

$$\begin{aligned}&C\left(u^1_{y_i}(.),v^1_{y_i}(.)\right)=C\left(u^2_{y_i}(.),v^2_{y_i}(.)\right),\\ &\forall\left(u^j_{y_i}(.),v^j_{y_i}(.)\right)\in\widetilde{\mathcal{P}}_\alpha(y_i),j=1,2,\ y_i\in\widetilde{Y}_s\subseteq Y_s.\end{aligned}\tag{2.26}$$

(iii) The associated value function $\widetilde{W}(.)$ is well-defined as:

$$\widetilde{W}(y_i) = \begin{cases} g(y_i), \text{ if } y_i \in \widetilde{Y}_f, \\ \widetilde{W}_0(y_i) = C(u_{y_i}(.), v_{y_i}(.)), \text{ if } (u_{y_i}(.), v_{y_i}(.)) \in \widetilde{\mathcal{P}}_\alpha(y_i), \ y_i \in \widetilde{Y}_s. \end{cases} \tag{2.27}$$

(iv) The subset $\widetilde{Y}_f \subseteq Y_f$, defined as:

$$\widetilde{Y}_f = \left\{ x_{y_i}\left(\widetilde{t}_f\right) ; \ x_{y_i}(.) \in \widetilde{\Omega}(y_i), \ y_i \in \widetilde{Y}_s \right\} \subseteq Y_f, \tag{2.28}$$

represents the set of terminal points of all trajectories in $\widetilde{\Omega}(y_i), y_i \in \widetilde{Y}_s$.

Furthermore, in different formulations of differential games, the existence of exact optimal strategies cannot be guaranteed, or their analytical or computational derivation may prove impractical. In such cases, it becomes essential to consider alternative solution concepts that are both theoretically justified and practically implementable. One such concept is that of relatively optimal strategies, which offer an approximate yet effective means of characterizing near-optimal behavior within admissible classes of controls.

Definition 2.7. (**Relatively Optimal Strategies**) The multifunction $\left(\widetilde{U}(.), \widetilde{V}(.)\right) : \widetilde{Y}_s \subset Y_s \to \mathcal{P}(U) \times \mathcal{P}(V)$ is said to define **a relatively optimal pair of feedback strategies** for the restriction $DG_A \Big| \widetilde{Y}_s$ to the subset $\widetilde{Y}_s \subset Y_s$ of the game described in $(2.2 - 2.6)$, if the following conditions are satisfied:

1. The multifunction is admissible in the sense of Definition 2.6.
2. The associated value function $\widetilde{W}(.)$ in (2.27) satisfies the following properties:
   (i) If player $\mathbb{V}$ chooses the feedback strategy $\widetilde{V}(.)$, then $\widetilde{U}(.)$ is the optimal response for player $\mathbb{U}$ in the sense that:

$$\widetilde{W}_0(y_i) = \underline{W}_{\widetilde{V}}(y_i) = \inf_{u(.)\in\mathcal{U}_\alpha(.), \overline{v}_u(.)\in\overline{\mathcal{V}}_u(.)} C(u(.), \overline{v}(.)) \, \forall y_i \in \widetilde{Y}_s. \tag{2.29}$$

   (ii) Similarly, if player $\mathbb{U}$ chooses the feedback strategy $\widetilde{U}(.)$, then $\widetilde{V}(.)$ is the optimal response for player $\mathbb{V}$ in the sense that:

$$\widetilde{W}_0(y_i) = \overline{W}_{\widetilde{U}}(y_i) = \sup_{v(.)\in\mathcal{V}_\alpha(.), \overline{u}_v(.)\in\overline{\mathcal{U}}_v(.)} C(\overline{u}(.), v(.)) \ \forall y_i \in \widetilde{Y}_s. \tag{2.30}$$

Remark 2.8. We note that:

1. Based on Definition 2.5 and Definition 2.7, a relatively optimal pair of feedback strategies $\left(\widetilde{U}\left(.\right),\widetilde{V}\left(.\right)\right)$ for the restriction $DG_A\left|\widetilde{Y}_s\right.$ of the game described in $(2.2-2.6)$, can be characterized as **a saddle point** satisfying the following inequality:

$$\overline{W}_{\widetilde{U},V_1}\left(y_i\right)\leq\widetilde{W}_0\left(y_i\right)=\overline{W}_{\widetilde{U}}\left(y_i\right)=\underline{W}_{\widetilde{V}}\left(y_i\right)<\underline{W}_{U_1,\widetilde{V}}\left(y_i\right),\forall y_i\in\widetilde{Y}_s, \tag{2.31}$$

for any other pair of feedback strategies $U_1\left(.\right),V_1\left(.\right)$.

2. Furthermore, it can be shown that the upper and lower value functions, denoted as $W\left(.\right)\in\left\{\overline{W}\left(.\right),\underline{W}\left(.\right)\right\}$, satisfy the equation:

$$\inf_{U_1(.)}\sup_{V_1(.)}W_{U_1,V_1}\left(y\right)=\sup_{V_1(.)}\inf_{U_1(.)}W_{U_1,V_1}\left(y_i\right)\ \forall y_i\in\widetilde{Y}_s, \tag{2.32}$$

in the class of all pairs of feedback strategies $\left(U_1\left(.\right),V_1\left(.\right)\right)$.

2.3 Verification Theorems for Admissible Feedback Strategies

To deriving some weaker optimality conditions in terms of regularity for a pair of admissible feedback strategies $\left(\widetilde{U}\left(.\right),\widetilde{V}\left(.\right)\right)$ becomes necessary, since applying the saddle point condition (see for instance in, [5]) in (2.31) is particularly challenging, especially when the value function is defined implicitly. We note here that the concept of the value function was first introduced in control theory by W.H. Fleming and R.W. Rishel in 1975, but it had already been rigorously formulated by R. Isaacs [32] in game theory since 1950. Unfortunately, it has often been misapplied to problems whose value functions are not even differentiable. This anomaly was addressed by Mirică [54], who introduced a broad class of verification theorems with regularity conditions ranging from differentiability to semicontinuity. In total, seven such theorems were obtained, which may be regarded as weak sufficiency conditions for optimality when compared with other approximation approaches.

2.3.1 Abstract Verification Theorem

Given a pair of admissible feedback strategies $\left(\widetilde{U}\left(.\right),\widetilde{V}\left(.\right)\right)$ as defined in Definition 2.6, the objective is to establish criteria for their relative optimality with respect to properties (i) and (ii) stated in Definition 2.7. To address this problem, one may draw upon techniques from Optimal Control Theory, since the associated value function in (2.27) aligns with the value functions of two symmetric Bolza-type optimal control problems. As described in [55], the first one, $P\mathcal{B}_1$, cosists in the minimization of the functional:

$$\begin{aligned}&\underset{u\in\mathcal{U}_\alpha}{min}\, C_1\left(y_i,x_i\left(.\right)\right)\\ &\text{where, } C_1\left(y_i,x_i\left(.\right)\right)=g\left(x_i\left(t_f\right)\right)+\int_0^{t_f} f_0^1\left(x_i\left(t_i\right),x_i'\left(t_i\right)\right)dt_i,\ y_i\in\widetilde{Y}_s\\ &f_0^1\left(x_i,x_i'\right)=f_0\left(x_i,u,v\right)\text{ if } x_i'=f\left(x_i,u,v\right),\ u\in U\left(x_i\right),\ v\in\widetilde{V}\left(x_i\right)\end{aligned}\tag{2.33}$$

subject to:

$$x_i'\in F_{\widetilde{V}}\left(x_i\right)=f\left(x_i,U\left(x_i\right),\widetilde{V}\left(x_i\right)\right),\ x_i\left(0\right)=y_i,\tag{2.34}$$

with, the state space and end-point constraints are defined as in (2.6). On the other hand, $P\mathcal{B}_2$, consisting in the maximization of the functional:

$$\begin{aligned}&\underset{v\in\mathcal{V}_\alpha}{max}\, C_2\left(y_i,x_i\left(.\right)\right)\\ &\text{where, } C_2\left(y_i,x_i\left(.\right)\right)=g\left(x_i\left(t_f\right)\right)+\int_0^{t_f} f_0^2\left(x_i\left(t_i\right),x_i'\left(t_i\right)\right)dt_i,\ y_i\in\widetilde{Y}_s,\\ &f_0^2\left(x_i,x_i'\right)=f_0\left(x_i,u,v\right)\text{ if } x_i'=f\left(x_i,u,v\right),\ u\in\widetilde{U}\left(x_i\right),\ v\in V\left(x_i\right),\end{aligned}\tag{2.35}$$

subject to $(2.3-2.6)$ with the differential inclusion:

$$x_i'\in F_{\widetilde{U}}\left(x_i\right)=f\left(x_i,\widetilde{U}\left(x_i\right),V\left(x_i\right)\right),\ x_i\left(0\right)=y_i,\tag{2.36}$$

and also, the state space and end-point constraints are defined as in (2.6).

Thus, using the well-known fact that the value function $W(.)$ has certain monotonicity properties along admissible trajectories (e.g., [20], Lupulescu and Mirică [48, 49]), we obtain the following basic result.

Theorem 2.9 (Abstract Verification Theorem, [54]). *A pair* $\left(\widetilde{U}\left(.\right),\widetilde{V}\left(.\right)\right)$ *of admissible feedback strategies in the sense of Definition 2.6 is optimal in the sense of Definition 2.7 iff the associated value function* $W\left(.\right)$ *in (2.27) has the following monotonicity properties:*

(M_1) *For any* $y_i\in\widetilde{Y}_s$, $u\left(.\right)\in U$, $\overline{v}_u\left(.\right)\in\overline{\mathcal{V}}_u\left(y_i\right)$, *the real function* $\overline{w}_u\left(.\right)$ *defined by:*

$$\overline{w}_u\left(t_i\right)=\widetilde{W}\left(\overline{x}_u\left(t_i\right)\right)+\int_0^{t_i} f_0\left(\overline{x}_u\left(s\right),u\left(s\right),\overline{v}_u\left(s\right)\right)ds,\ t_i\in\left[0,t_f\right],\tag{2.37}$$

is increasing on the interval $[0, t_f]$.

(M_2) *For any* $y_i \in \widetilde{Y}_s$, $v(.) \in V$, $\overline{u}_v(.) \in \overline{\mathcal{U}}_v(y_i)$, *the real function* $\overline{w}_v(.)$ *defined by:*

$$\overline{w}_v(t_i) = \widetilde{W}(\overline{x}_v(t_i)) + \int_0^{t_i} f_0(\overline{x}_v(s), \overline{u_v}(s), v(s))\, ds,\ t_i \in [0, t_f], \tag{2.38}$$

is decreasing on the interval $[0, t_f]$.

Remark 2.10. The necessary monotonicity conditions presented in [49], formulated in terms of the Dini derivatives of the real functions $\overline{w}_u(t_i)$, $\overline{w}_v(t_i)$ in (2.37) and (2.38), respectively, and interpreted in the context of $(1.21 - 1.22)$, together with the appropriate chain rules for composite functions $\widetilde{W}(\overline{x_i}(.))$ (see Lemma 1.21), can lead to certain necessary optimality conditions of the Dynamic Programming type. However, in this work, we focus exclusively on sufficient optimality conditions of the same type.

2.3.2 Practical Verification Theorems

As in Optimal Control Theory (e.g., [20], [40], [48], [51],..., etc.), starting from the Abstract verification theorem described in Theorem 2.9, more practical verification theorems can be developed by making specific assumptions about the value function $\widetilde{W}(.)$ in (2.27). These theorems rely on monotonicity results for real-valued functions from Theorem 1.22 and the chain rules for composite functions, such as $\widetilde{W}(\overline{x}_u(.))$ and $\widetilde{W}(\overline{x}_v(.))$, as provided in Lemma 1.21, for (2.37) and (2.38).

Since, it is often difficult to verify the generalized differential inequalities needed to ensure the monotonicity properties (M_1) and (M_2) in Theorem 2.9, a practical approach is to use the weakest possible type of inequalities that match the regularity properties of the value function $\widetilde{W}(.)$ in (2.27). However, simple examples show that the value function may have weaker regularity on the terminal set $\widetilde{Y}_f$ in (2.28). In such cases, the differential inequalities may only need to involve the restricted value function $\widetilde{W}_0(.) = \widetilde{W}(.)\Big|_{\widetilde{Y}_s}$.

From this perspective, the simplest case is when the proper value function $\widetilde{W}_0(.)$ in (2.27) is differentiable on its open domain $\widetilde{Y}_s = Int\left(\widetilde{Y}_s\right) \subseteq Y_s$. This scenario aligns with the well-known Elementary verification theorem, described below, which is introduced in [32] then in, [64] and further, elaborated in [53, 54].

Theorem 2.11 (Elementary Verification Theorem, [54]). *Let* $\left(\widetilde{U}(.), \widetilde{V}(.)\right)$ *be a pair of admissible feedback strategies in the sense of Definition 2.6 and assume that the associated value function* $\widetilde{W}(.)$ *in (2.27) has the following properties:*

1. $\widetilde{W}(.)$ *is continuous at the terminal points in* $\widetilde{Y}_f$ *defined in (2.28).*
2. *The subset* $\widetilde{Y}_s \subseteq Y_s$ *is open, the restriction* $\widetilde{W}_0(.) = \widetilde{W}(.) \Big| \widetilde{Y}_s$ *is differential and satisfies* ***the basic differential inequalities :***

$$\inf_{u \in U(x_i),\, \overline{v} \in \widetilde{V}(x_i)} \left[D\widetilde{W}_0(x_i).f(x_i, u, \overline{v}) + f_0(x_i, u, \overline{v}) \right] \geq 0 \tag{2.39}$$

$$\sup_{\overline{u} \in \widetilde{U}(x_i),\, v \in V(x_i)} \left[D\widetilde{W}_0(x_i).f(x_i, \overline{u}, v) + f_0(x_i, \overline{u}, v) \right] \leq 0 \tag{2.40}$$

3. *Either the admissible controls are regulated (i.e.,* $\mathcal{U}_\alpha \times \mathcal{V}_\alpha = \mathcal{U}_r \times \mathcal{V}_r$*), or* $\widetilde{W}_0(.)$ *is locally Lipschitz (in particular, of class* $\mathcal{C}^1$*).*

Then $\left(\widetilde{U}(.), \widetilde{V}(.)\right)$ *are optimal in the sense of Definition 2.7.*

Remark 2.12. .

1. One may note that the differential inequalities in $(2.39 - 2.40)$ are equivalent with **Isaacs' basic equation** (see, [32]):

$$\begin{aligned}
&\min_{u \in U(x_i)} \max_{v \in V(x_i)} \left[D\widetilde{W}_0(x_i).f(x_i, u, v) + f_0(x_i, u, v) \right], \\
&= \max_{v \in V(x_i)} \min_{u \in U(x_i)} \left[D\widetilde{W}_0(x_i).f(x_i, u, v) + f_0(x_i, u, v) \right], \\
&= D\widetilde{W}_0(x_i).f(x_i, \overline{u}, \overline{v}) + f_0(x_i, \overline{u}, \overline{v}) = 0 \;\forall\, (\overline{u}, \overline{v}) \in \widetilde{U}(x_i) \times \widetilde{V}(x_i).
\end{aligned}$$

2. Theorem 2.9 can be viewed as a variant of Isaac's Elementary Verification Theorem 4.4.1 in [32]. However, Isaac's Theorem is often applied informally to problems where the value function does not fully meet its assumptions. For the examples in [32], where the domain $\widetilde{Y}_s$ in (2.27) of the proper value function $\widetilde{W}_0(.)$ is not open or the restriction $\widetilde{W}(.) \Big| \widetilde{Y}_s$ is not differentiable, rigorous solutions can be achieved. Achieving these solutions requires refinements and extensions of Theorem 2.11, which can be developed using concepts and results from Nonsmooth Analysis, as described in [55].

A natural and significant extension of Theorem 2.11 arises when the proper value function $\widetilde{W}_0(.) = \widetilde{W}(.)\Big|\widetilde{Y}_s$ is differentiably stratified in the sense of Definition 1.13. In this case, we adopt the following notations:

$$\begin{aligned} U_T(x_i, v) &= \left\{u \in U(x_i);\ f(x_i, u, v) \in T_{x_i}\widetilde{Y}_s\right\}, v \in V(x_i), && (2.41)\\ V_T(x_i, v) &= \left\{v \in V(x_i);\ f(x_i, u, v) \in T_{x_i}\widetilde{Y}_s\right\}, u \in U(x_i),\ x_i \in \widetilde{Y}_s, && (2.42)\end{aligned}$$

where $T_{x_i}\widetilde{Y}_s$ denotes the stratified tangent space as defined in Definition 1.11.

Using the properties of stratified sets and mappings described in Section 1.2, we derive the Verification theorem for stratified value functions, as stated in [54].

Theorem 2.13 (Verification Theorem for Stratified Value Functions, [54]). *Let $\left(\widetilde{U}(.), \widetilde{V}(.)\right)$ be a pair of admissible feedback strategies in the sense of Definition 2.6 and assume that the associated value function $\widetilde{W}(.)$ in (2.27) has the following properties:*

(i) $\widetilde{W}(.)$ is continuous at the terminal points in $\widetilde{Y}_f$ defined in (2.28).

(ii) The restriction $\widetilde{W}_0(.) = \widetilde{W}(.)\Big|\widetilde{Y}_s$ is differentiably stratified in the sense of Definition 1.13 and its stratified derivatives in Definition 1.14 satisfy the following pair of ***differential inequalities:***

$$\begin{aligned} \inf_{u\in U_T(x_i,\overline{v}),\ \overline{v}\in\widetilde{V}(x_i)} \left[D\widetilde{W}_0(x_i).f(x_i, u, \overline{v}) + f_0(x_i, u, \overline{v})\right] &\geq 0 && (2.43)\\ \sup_{\overline{u}\in\widetilde{U}(x_i),\ v\in V_T(x_i,\overline{u})} \left[D\widetilde{W}_0(x_i).f(x_i, \overline{u}, v) + f_0(x_i, \overline{u}, v)\right] &\leq 0 && (2.44)\end{aligned}$$

(iii) Either $\widetilde{W}_0(.)$ is continuous and the admissible controls are regulated (i.e., $\mathcal{P}_\alpha = \mathcal{P}_r$) or $\widetilde{W}_0(.)$ is locally Lipschitz.

Then $\left(\widetilde{U}(.), \widetilde{V}(.)\right)$ are optimal in the sense of Definition 2.7.

Another significant extension of Theorem 2.9 can be obtained when the proper value function $\widetilde{W}_0(.)$ is contingent differentiable. This leads to the verification theorem for contingent differentiable value functions. Similarly, in the case of locally Lipschitz value functions, applying the first part of Hypothesis (ii) from the previous theorem results in the verification theorem for locally Lipschitz value functions. Furthermore, additional cases align with the verification theorems for value functions that are lower semicontinuous or upper semicontinuous.

2.4 The general algorithm of Dynamic Programming Method

To facilitate the practical application of the comprehensive results developed in the preceding sections to the *autonomous differential game* (DG_A) formulated in Section 2.1, we summarize these findings in the form of a theoretical dynamic programming algorithm, as proposed in [55] and [56] as follows:

Step.1. Preliminary operations

1.1. Statement of the problem and identification of the data:

In this step, the problem is formulated in its standard form as defined in Problem 1, using the data and framework outlined in Section 2.1.

1.2. Compute the auxiliary Hamiltonians and the sets of extremal points:

$$\begin{aligned}
\mathcal{H}(z,u,v) &= \langle p_i, f(x_i,u,v)\rangle + f_0(x_i,u,v),\\
\mathcal{H}^+(z,u) &= \sup_{v\in V(x_i)} \mathcal{H}(x_i,p_i,u,v),\\
\mathcal{H}^-(z,v) &= \inf_{u\in U(x_i)} \mathcal{H}(z,u,v),\\
H^+(z) &= \inf_{u\in U(x_i)} \mathcal{H}^+(z,u),\\
H^-(z) &= \sup_{v\in V(x_i)} \mathcal{H}^-(z,v),\\
\widehat{U}^+(z) &= \left\{\overline{u}\in U(x_i) \mid \mathcal{H}^+(z,\overline{u}) = H^+(z)\right\},\\
\widehat{V}^-(z) &= \left\{\overline{v}\in V(x_i) \mid \mathcal{H}^-(z,\overline{v}) = H^-(z)\right\}, \quad z\in Y_s\times\mathbb{R}^n.
\end{aligned} \tag{2.45}$$

where, the domain Z of Isaac's Hamiltonian is specified as follows:

$$\begin{aligned}
Z = \quad &\{z\in Y_0\times\mathbb{R}^n,\ H^+(z) = H^-(z),\\
&\widehat{U}^+(z)\neq\emptyset,\ \widehat{V}^-(z)\neq\emptyset\},
\end{aligned} \tag{2.46}$$

Isaac's condition is described as follows:

$$\begin{aligned}
&H(z) = H^+(z) = H^-(z),\, z\in Z,\\
&\widehat{U}(z) = \widehat{U}^+(z),\ \widehat{V}(z) = \widehat{V}^-(z),
\end{aligned} \tag{2.47}$$

as well as, the Hamiltonian should be continued at the terminal points in the following way:

$$Z_1 = \left\{z_0\in Y_f\times\mathbb{R}^n,\ \exists H\left(z_{(0)}\right) = \lim_{Z\ni z\to z_0} H(z)\right\}. \tag{2.48}$$

Remark 2.14. Depending on the problem data, several cases may arise concerning the initial set Y_s:

- If the set Y_s is **not open**, In this case, the state constraints in (2.6) become active. Consequently, the admissible control sets $U\left(x_i\right)$ and $V\left(x_i\right)$, must be restricted by removing directions $f\left(x_i, u, v\right)$ in (2.3) that, would lead to trajectories exiting the set Y_s.
- If the set Y_s is **stratified**, in the sense of Definition 1.10. Here, the sets $U\left(x_i\right)$, and $V\left(x_i\right)$ must be replaced with the restricted sets defined in $(2.41 - 2.42)$. Substituting these into the expressions in $(2.45 - 2.47)$ leads to the construction of a geometric Isaac's Hamiltonian $H_T\left(.,.\right)$.
- As can be observed from optimal control theory (e.g., [55]), when state constraints in (2.6) are active, the use of the constrained Hamiltonian $H_T\left(.,.\right)$ typically leads to a more accurate and effective characterization of the solution.

In the subsequent analysis, one may utilize either Isaac's Hamiltonian $H\left(.,.\right)$ in $(2.46 - 2.47)$ which is preferable when $Y_s = Int\left(Y_s\right) \subseteq \mathbb{R}^n$, or the restricted Hamiltonian $H_T\left(.,.\right)$ as introduced in Remark 2.14, which is applicable when Y_s is not open. To simplify the notation, however, we shall consider only the first case in the following discussion.

Step.2. Construction of the generalized Hamiltonian and the characteristic flow

2.1. The set of terminal transversality points:

If the terminal function $g\left(.\right) : Y_f \to \mathbb{R}$ is stratified in the sense of Definition 1.13, specifically, if Y_f is a differentiable manifold and $g\left(.\right)$ is differentiable function, then the set of terminal transversality points is defined as:

$$Z_1^s = \{z_0 = \left(\xi, q\right) \in Z,\ \xi \in Y_f, H\left(z_0\right) = 0, < q, \bar{\xi} >= Dg\left(\xi\right).\bar{\xi},\ \forall \bar{\xi} \in T_\xi Y_f\}. \tag{2.49}$$

2.2. The Hamiltonian orientor field:

If Isaacs Hamiltonian $H(.,.) : Z \to \mathbb{R}$ is stratified then, the Hamiltonian orientor field $d_S^{\#} H\left(.,.\right)$ defined as:

$$\begin{aligned} d_S^{\#} H\left(z\right) = \{ & z' \in T_z Z,\ x_i' \in f\left(x_i, \widehat{U}\left(z\right), \widehat{V}\left(z\right)\right), \\ & < x_i', \bar{p}_i > - < p_i', \overline{x_i} >= DH\left(z\right).\bar{z}, \forall \bar{z} \in T_z Z\}. \end{aligned} \tag{2.50}$$

Furthermore, in the open strata $S \in S_H$ (i.e., $dim(s) = 2n$), the Hamiltonian orientor field described in (2.50) coincides with the classical Hamiltonian vector field given by:

$$d_S^{\#} H\left(z\right) = \left\{\left(\frac{\partial H}{\partial p_i}\left(z\right), -\frac{\partial H}{\partial x_i}\left(z\right)\right)\right\},\ \forall z \in S \in S_H. \tag{2.51}$$

2.3. Construction of a generalized Hamiltonian flows:

Given a Hamiltonian field $d^{\#}H\left(z\right)=d_{S}^{\#}H\left(z\right)$ and the corresponding set of terminal points $Z_{1}^{*}=Z_{1}^{s}$, the Hamiltonian differential inclusion integrated backward in time $\left(t_{i}\leq 0\right)$, is expressed as follows:

$$z^{\prime}\in d^{\#}H\left(z\right),\ z\left(0\right)=\left(\xi,q\right)=z_{0}\in Z_{1}^{*}. \tag{2.52}$$

For each terminal point $z_{0}=\left(\xi,q\right)\in Z_{1}^{*}$, we determine the set of maximal solutions $X_{i}^{*}\left(.,.\right)$ such that:

$$X_{i}^{*}\left(.,.\right)=\left(X_{i}\left(.,.\right),P_{i}\left(.,.\right)\right):I\left(z_{0}\right)=\left(t_{i}^{-}\left(z_{0}\right),0\right]\rightarrow\mathbb{R}^{n}\times\mathbb{R}^{n}. \tag{2.53}$$

If for the same terminal point $z_{0}\in Z_{1}^{*}$ there exists more than one solution, the set of solutions $X^{*}\left(.,.\right)$ should be parameterized in the following manner:

$$\begin{aligned}&X_{i}^{*}\left(.,a\right)=\left(X_{i}\left(.,a\right),P_{i}\left(.,a\right)\right):I\left(a\right)=\left(t_{i}^{-}\left(a\right),0\right]\rightarrow Z,\\&a=\left(z_{0},\lambda\right)\in A=Graph\left(\Lambda\right),\ z_{0}\in Z_{1}^{*},\ \lambda\in\Lambda\left(z_{0}\right),\end{aligned}$$

the solutions $X_{i}^{*}\left(.,a\right)=\left(X_{i}\left(.,a\right),P_{i}\left(.,a\right)\right)$ of differential inclusion (2.52) associated to Problem 1, satisfy the following conditions:

$$\begin{aligned}X_{i}^{*}\left(0,a\right)&=\left(\xi,q\right)=z_{0}\in Z_{1}^{*},if\ a=\left(z_{0},\lambda\right)\in A,\lambda\in\Lambda\left(z_{0}\right), && (2.54)\\X_{i}\left(t_{i},a\right)&\in Y_{s},\forall t_{i}\in I_{0}\left(a\right)=\left(t_{i}^{-}\left(a\right),0\right), && (2.55)\\H\left(X_{i}\left(t_{i},a\right),P_{i}\left(t_{i},a\right)\right)&=0,\forall t_{i}\in I\left(a\right)=\left(t_{i}^{-}\left(a\right),0\right], && (2.56)\\X_{i}^{\prime}\left(t_{i},a\right)&=f\left(X_{i}\left(t_{i},a\right),u_{a}\left(t_{i}\right),v_{a}\left(t_{i}\right)\right),a.e.\ \left(I_{0}\left(a\right)\right), && (2.57)\\u_{a}\left(t_{i}\right)&\in\widehat{U}\left(X_{i}^{*}\left(t_{i},a\right)\right),v_{a}\left(t_{i}\right)\in\widehat{V}\left(X_{i}^{*}\left(t_{i},a\right)\right),\forall t_{i}\in I_{0}\left(a\right). && (2.58)\end{aligned}$$

2.4. The generalized characteristic flow:

Characterize the following function $V\left(.,a\right)$, defined as:

$$V\left(t_{i},a\right)=g\left(\xi\right)+\int_{0}^{t_{i}}<P_{i}\left(s,a\right),X_{i}^{\prime}\left(s,a\right)>ds \tag{2.59}$$

where, $a=\left(\xi,q,\lambda\right)$, $X_{i}^{\prime}\left(.,a\right)$ is the derivative of the mapping $X_{i}\left(.,a\right)$ as well as, the mapping $C^{*}\left(.,.\right)=\left(X_{i}^{*}\left(.,.\right),V\left(.,.\right)\right)$ is the generalized characteristic flow associated to Problem 1. Note also that if the subset $B_{0}\subseteq\left(-\infty,0\right)\times A$ is defined as:

$$\begin{aligned}&B_{0}=\left\{\left(t_{i},a\right);\ a\in A,\ t_{i}\in I_{0}\left(a\right)\right\},\\&A=\left\{a=\left(z,\lambda\right),\ z\in Z_{1}^{*},\ \lambda\in\Lambda\left(z\right)\right\}.\end{aligned} \tag{2.60}$$

Then, it follows from the definitions above that for any $(t_i, a) \in B_0$ the mapping

$$\begin{aligned} u_{t,a}(s) &= u_a(t_i + s),\ v_{t,a}(s) = v_a(t_i + s), \\ x_{t,a}(s) &= X(t_i + s, a),\ s \in [0, -t_i], \end{aligned} \tag{2.61}$$

are admissible controls and respectively trajectories with respect to the initial point $y_i = X_i(t_i, a) \in Y_s$ and moreover, the corresponding value of the payoff functional in (2.1) is given by the formula:

$$C\left(y_i, u_{t_i,a}(.), v_{t_i,a}(.)\right) = V(t_i, a) \text{ if } y_i = X_i(t_i, a) \in B_0. \tag{2.62}$$

Step.3. Admissible Value Functions and Strategies

3.1. Admissible extreme proper value functions:

Compute the marginal characteristic value functions

$$\begin{aligned} & W_0^m(y_i) = \inf_{X_i(t_i,a)=y_i} V(t_i, a),\ W_0^M(y_i) = \sup_{X_i(t_i,a)=y_i} (t_i, a), \\ & \widehat{B}_m(y_i) = \{(t_i, a) \in B_0,\ X_i(t_i, a) = y_i,\ V(t_i, a) = W_0^m(y_i)\}, \\ & \widehat{B}_M(y_i) = \left\{(t_i, a) \in B_0,\ X_i(t_i, a) = y_i,\ V(t_i, a) = W_0^M(y_i)\right\}, \\ & Y_s^m = \left\{y_i \in X_i(B_0),\ \widehat{B}_m(y_i) \neq \emptyset\right\}, \\ & Y_s^M = \left\{y_i \in X_i(B_0),\ \widehat{B}_M(y_i) \neq \emptyset\right\}. \end{aligned} \tag{2.63}$$

Let us consider, $\widetilde{W}_0(y_i) \in \left\{W_0^m(y_i),\ W_0^M(y_i)\right\}$ with $\widehat{B}_0(y_i) \in \left\{\widehat{B}_m(y_i),\ \widehat{B}_M(y_i)\right\}$, $\forall y_i \in \widetilde{Y}_s \in \left\{Y_s^m,\ Y_s^M\right\}$, $\widetilde{W}_0(.)$ is admissible in the sense that the set $\widetilde{B}_0(.)$ is defined by:

$$\begin{aligned} \widetilde{B}_0(y_i) = \quad & \{(t_i, a) \in \widehat{B}_0(.), x_{t_i,a}(s) \in \widetilde{Y}_s, \\ & (t_i + s, a) \in \widehat{B}_0(x_{t_i,a}(s))\ \forall s \in [0, -t_i)\}, \end{aligned} \tag{2.64}$$

having as proprieties:

$$\widetilde{B}_0(y_i) \neq \emptyset,\ \forall y_i \in \widetilde{Y}_s \in \left\{Y_s^m,\ Y_s^M\right\}. \tag{2.65}$$

3.2. Admissible intermediate proper value functions:

If the marginale proper value functions $W_0^m(.)$ and $W_0^M(.)$ are not admissible in the sense of (2.65), then an "intermediate" proper value fuction can be introduced. may be taken as:

$$\widetilde{W}_0(.) \in \left[W_0^m(y_i), W_0^M(y_i)\right]\ \forall y_i \in \widetilde{Y}_s \subseteq Y_s^m \cup Y_s^M.$$

Furthermore, the corresponding multifunction $\widehat{B}_0(.)$, defined in (2.63) is expressed as:

$$\widehat{B}_0(y_i) = \left\{(t_i, a) \in B_0,\ V(t_i, a) = \widetilde{W}_0(y_i)\right\} \neq \emptyset, \tag{2.66}$$

where, $\widetilde{B}_0(.)$ defines in (2.64), which is admissible in the sense of (2.65).

Remark 2.15. Note that if $X_i(.,.)$ is invertible at $(t_i, a) \in B_0$ with the inverse:

$$\widehat{B}_0(y_i) = X_i(.,.)^{-1}(y_i) = \widehat{B}_m(y_i) = \widehat{B}_M(y_i),$$

then,

$$W_0^m(y_i) = W_0^M(y_i) = V\left(\widehat{B}_0(y_i)\right). \tag{2.67}$$

Moreover, it follows from (2.60) that, if the function $\widetilde{W}_0(.) = W_0^m(.) = W_0^M(.)$ is differentiable at the point $y_i \in Int\left(\widetilde{Y}_s\right)$, then its derivative is given by:

$$D\widetilde{W}_0(y_i) = \widetilde{P}_i(y_i) = P_i\left(\widehat{B}_0(y_i)\right), \tag{2.68}$$

and verifies the equation:

$$D\widetilde{W}_0(y_i).f(x_i, \overline{u}, \overline{v}) + f_0(x_i, \overline{u}, \overline{v}) = 0,\ \forall \overline{u} \in \widetilde{U}(y_i),\ \overline{v} \in \widetilde{V}(y_i), \tag{2.69}$$

where

$$\widetilde{U}(y_i) = \widehat{U}\left(y_i, \widetilde{P}_i(y_i)\right), \widetilde{V}(y_i) = \widehat{V}\left(y_i, \widetilde{P}_i(y_i)\right),$$

are the corresponding candidates for optimal feedback strategies. From (2.45), it follows that, the value function $\widetilde{W}_0(.)$ verifies **Isaac's basic equation** :

$$\begin{aligned} &\min_{u\in U(y_i)}\max_{v\in V(y_i)}\left[D\widetilde{W}_0(y_i).f(x_i, u, v) + f_0(x_i, u, v)\right] = \\ &= \max_{v\in V(y_i)}\min_{u\in U(y_i)}\left[D\widetilde{W}_0(y_i).f(x_i, u, v) + f_0(x_i, u, v)\right] = 0. \end{aligned} \tag{2.70}$$

Due to these relations, the computations and the arguments to follow may be significantly simplified if the characteristic flow may be split into a finite collection of smooth invertible characteristic flows, $C_j^*(.,.)$, $j = \overline{1, k}$, so, that the marginal proper value functions in (2.63) may be represented by:

$$\begin{aligned} W_0^m(y_i) &= \min_{1\leq j\leq k} W_0^j(y_i), \\ W_0^M(y_i) &= \max_{1\leq j\leq k} W_0^j(y_i),\ \forall y_i \in \widetilde{Y}_s, \end{aligned} \tag{2.71}$$

where, $W_0^j(.)$ are differentiable functions of the form in (2.65) satisfying relations of the forms in $(2.66) - (2.68)$.

3.3. Selections of admissibles controls and trajectories

For each propre value function $\widetilde{W}_0\left(.\right)$ identified in Steps $\left(3.1-3.2\right)$ which is admissible in the sense of (2.66), identify the set of terminal points:

$$\widetilde{Y}_f=\left\{x_{t_i,a}\left(-t_i\right)=\xi\in Y_f,\ \left(t_i,a\right)\in\widetilde{B}_0\left(y_i\right),\ a=\left(\xi,q,\lambda\right)\in A\right\},\tag{2.72}$$

the corresponging selections of admissible controls and trajectories are given as:

$$\begin{aligned}\widetilde{\mathcal{P}}_\alpha&=\left\{\left(u_{t_i,a}\left(.\right),v_{t_i,a}\left(.\right)\right),\ \left(t_i,a\right)\in\widetilde{B}_0\left(y_i\right)\right\},\\ \widetilde{\Omega}_\alpha&=\left\{x_{t_i,a}\left(.\right),\ \left(t_i,a\right)\in\widetilde{B}_0\left(y_i\right)\right\},\end{aligned}\tag{2.73}$$

where $u_{t_i,a}\left(.\right),v_{t_i,a}\left(.\right)$ and $x_{t_i,a}\left(.\right)$ are the mapping in (2.61). In addition, the corresponding value function is defined as:

$$\widetilde{W}\left(y_i\right)=\begin{cases}g\left(y_i\right) & if\ y_i\in\widetilde{Y}_f,\\ \widetilde{W}_0\left(y_i\right) & if\ y_i\in\widetilde{Y}_s,\end{cases}\tag{2.74}$$

and, together with a suitable pair of feedback strategies, should satisfy the admissibility conditions in (2.21) and (2.25).

3.4. Admissible feedback strategies:

As stated in [56], feedback strategies play an important role in analyzing and solving differential games and optimal control problems. These strategies help balance theoretical solutions with practical applications and are essential for identifying optimal or acceptable solutions from the perspective of all players. However, retrieving and defining suitable feedback strategies remains a mathematical challenge that has not been fully resolved in general contexts. In some cases, this may require relying on specific arguments and computations. Based on these findings, systematic approaches can be developed to determine optimal strategies using theoretical foundations and practical experience in solving differential problems.

For each $a=\left(\xi,q,\lambda\right)\in A$ identify the sets $\mathcal{U}\left(a\right)$ and $\mathcal{V}\left(a\right)$ of the controls $u_a\left(.\right)$ and $v_a\left(.\right)$ respectively, that satisfy the conditions in $\left(2.57-2.58\right)$. Additionally, compute the corresponding multifunctions:

$$\begin{aligned}\overline{U}\left(t_i,a\right)&=\left\{u_a\left(t_i\right),\ u_a\left(t_i\right)\in\mathcal{U}\left(a\right)\right\},\\ \overline{V}\left(t_i,a\right)&=\left\{v_a\left(t_i\right),\ v_a\left(t_i\right)\in\mathcal{V}\left(a\right)\right\}.\end{aligned}\tag{2.75}$$

Finally, for each proper value function $\widetilde{W}\left(.\right)$ identified in Steps $\left(3.1-3.2\right)$, that is admissible, use the corresponding multifunction $\widetilde{B}_0\left(.\right)$ in (2.66) and the multifunctions in (2.75) to define the corresponding pairs of feedback strategies:

$$\widetilde{U}\left(y_i\right)=\overline{U}\left(\widetilde{B}_0\left(y_i\right)\right),\ \widetilde{V}\left(y_i\right)=\overline{V}\left(\widetilde{B}_0\left(y_i\right)\right),\ y_i\in\widetilde{Y}_s, \tag{2.76}$$

that, obviously, are admissible in the sense of Defintion 2.6, since one may take $\widetilde{\Omega}_\alpha\left(y_i\right)$, $y_i\in\widetilde{Y}_s$ in (2.73) as the (multi-)selection of admissible (possibly optimal) trajectories that satisfy the conditions in $(2.3-2.27)$.

Note that at some points $y_i\in\widetilde{Y}_s$, these feedback strategies can be expressed in a simpler form:

$$\begin{aligned}\widetilde{U}\left(y_i\right)&=\widehat{U}\left(y_i,\widetilde{P}_i\left(y_i\right)\right),\\ \widetilde{V}\left(y_i\right)&=\widehat{V}\left(y_i,\widetilde{P}_i\left(y_i\right)\right),\\ \widetilde{P}_i\left(y_i\right)&=P_i\left(\widehat{B}_0\left(y_i\right)\right),\end{aligned} \tag{2.77}$$

where, $\widehat{U}\left(.,.\right)$ and $\widehat{V}\left(.,.\right)$ are the marginal multifunctions in $(2.46-2.47)$ and of the component $P_i\left(.,.\right)$ of the Hamiltonian flow in $(2.54-2.58)$.

Step.4.. Proof of Relative Optimality

Select one of the admissible value functions $\widetilde{W}\left(.\right)$ from equation (2.74), giving preference to the one with the best regularity properties. Then, identify the corresponding admissible feedback strategies $\widetilde{U}\left(.\right)$ and $\widetilde{V}\left(.\right)$ from equation (2.76), potentially considering suitable sub-multifunctions. Utilize one of the verification theorems from Section 2.3 (or another relevant theorem from [54]) to demonstrate the relative optimality of the solution, as defined in Definition 2.7, for the restricted problem $DG_A\Big|\widetilde{Y}_s$.

It should be noted that the value function satisfying the "optimality" conditions of Definition 2.7 (if it exists) is uniquely determined under certain assumptions. However, the associated pair of optimal feedback strategies, $\left(\widetilde{U}\left(.\right),\widetilde{V}\left(.\right)\right)$, is not necessarily unique. As highlighted in step 3.4, the original feedback strategies described in (2.77) may be reduced to multifunctions that satisfy the relevant differential inequalities required by the verification theorems. Nonetheless, it is prudent to identify the largest multifunctions satisfying these properties, as recommended in [51].

The complex task of computing (or at least estimating) the generalized derivatives of the value function $\widetilde{W}\left(.\right)$ in equation (2.74), as well as verifying the differential inequalities, can be significantly simplified by employing the following techniques:

- Applying the fundamental differential property, which involves smooth partial characteristic flows.

- Utilizing appropriate chain rules to compute the generalized derivatives of the marginal functions $W_0^m(.)$ and $W_0^M(.)$, as given in equation (2.63).
- Representing the value function using one of the forms in equation (2.71), expressed through smooth partial value functions $W_0^j(.)$, where $j = 1, \cdots, k$. Each of these functions satisfies specific conditions given in equations $(2.67 - 2.70)$.

4.1. End-point regularity propreties:

Check whether the value function is continuous at the end-point in (2.72) in this sense that:

$$\exists \lim_{\widetilde{Y}_s \ni y_i \to \xi} \widetilde{W}_0(.) = \widetilde{W}(\xi) = g(\xi) \ \forall \xi \in \widetilde{Y}_f. \tag{2.78}$$

4.2. Stratified proper value functions

Check the following hypothesis:

1. The value function $\widetilde{W}(.)$ in(3.48) is continuous at the terminal points in $\widetilde{Y}_f$ as defined in (2.78). Moreover, its restriction $\widetilde{W}_0(.) = \widetilde{W}(.)\Big|\widetilde{Y}_s$ is both continuous and stratified, in the sense of Definition1.13.
2. Compute the stratified derivatives of $\widetilde{W}_0(.)$ according to Definition1.13. Identify the multifunctions $U_T(.)$ and $V_T(.)$, which are defined in $(2.41) - (2.42)$. Verify that the multifunctions $\widetilde{U}(.)$ and $\widetilde{V}(.)$, as given in equation (3.50), satisfy the differential inequalities stated in (2.43) and (2.44).

Next, apply Theroem 2.13 from Section 2.3 to conclude that $\widetilde{U}(.)$, $\widetilde{V}(.)$ are relatively optimal within the restricted class Ω_r of regular trajectories, which are generated by the control class $\mathcal{P}_r = \mathcal{U}_r \times \mathcal{V}_r$. Furthermore, if $\widetilde{W}_0(.)$ is also locally Lipschitz, then $\widetilde{U}(.)$, $\widetilde{V}(.)$ remain relatively optimal in the broader class Ω_1 of absolutely continuous (AC) trajectories.

Chapter 3 will focus on the study and analysis of the War of Attrition and Attack Differential Game, a problem originally introduced in Isaacs's seminal work *Differential Games* [32]. The main objective is to apply the dynamic programming approach discussed in the current chapter to characterize all optimal trajectories associated with this game.

Chapter 3

On Isaac's War Game of Attrition and Attack using dynamic programming approach

3.1 Introduction

In this chapter, we study a well-known differential game called the War of Attrition and Attack, first introduced by Isaac in 1965. We use the dynamic programming method developed by Mirică in 2004 to find optimal strategies for this game. This modern approach helps use extend the classical model and gives a complete and solid solution within the framework of differential games.

One of the main goals of this work is to find and study feedback strategies. These are special types of strategies that change based on the current state of the system. Unlike fixed strategies, feedback strategies are more flexible and better suited for real-time decision-making, especially in conflict situations like war. They represent an important improvement compared to other types of strategies used in game theory.

To build these strategies and the related value function, we use a refined version of Cauchy's Method of Characteristics, which is adapted to work with stratified Hamilton-Jacobi equations. Then, we prove that the strategies are truly optimal using a simple but effective result called the Elementary Verification Theorem. This result helps use confirm that the value function satisfies the conditions needed for optimality.

Overall, this chapter provides a new contribution to the study of differential games by offering a clear and practical way to solve a classic problem using feedback strategies.

It also shows the benefits of using dynamic programming and feedback control in modeling strategic situations like war.

The set of results obtained in this chapter has been the subject of an article published in "Journal of Games, MDPI " [6].

3.2 Formulation of the problem

In [32], it has been considered a warfare game model between two nations $\mathbb{U}$ and $\mathbb{V}$, engaged in a protracted war, that consists in optimizing the cost functional given by:

$$\mathcal{C}\left(u\left(.\right), v\left(.\right)\right) = \int_0^{t_f} \left[(1 - v\left(t_i\right))x_{i_2}\left(t_i\right) - (1 - u\left(t_i\right))x_{i_1}\left(t_i\right)\right] dt_i, \tag{3.1}$$

and defined by the warfare dynamic system:

$$\begin{cases} x_i' = \left(m_1 - c_1 v\left(t_i\right) x_{i_2}, m_2 - c_2 u\left(t_i\right) x_{i_1}, -1\right), \ x_i\left(0\right) = x_{i_0} \\ \left(u\left(t_i\right), v\left(t_i\right)\right) \in [0,1] \times [0,1], \ t \in [0, t_f] \\ x_{i_0} \in \mathbb{R}_+^3, \ c_1 > c_2 > 0, \end{cases} \tag{3.2}$$

the involved functions have the following significance:

- $x_{i_1}\left(t_i\right), x_{i_2}\left(t_i\right)$: represent the force of the nations $\mathbb{U}$ and $\mathbb{V}$ respectively at $t_i \in [0, t_f]$;
- m_1, m_2 : the weapon production rate of the nations $\mathbb{U}$ and $\mathbb{V}$ respectively;
- c_1, c_2 : are the measure of weapon effectiveness of $\mathbb{V}$ versus $\mathbb{U}$ and $\mathbb{U}$ versus $\mathbb{V}$ respectively;
- $u\left(t_i\right)$ and $v\left(t_i\right)$: represent the strategies of the two nations (or players) involved in the game. Specifically, $u\left(t_i\right)$ is the strategy chosen by nation $\mathbb{U}$ (the attacker), which determines the intensity or allocation of its military efforts over time t_i. The value of $u\left(t_i\right)$ is constrained within the interval $[0, 1]$, where $u(t_i) = 0$ represents no attack, and $u(t_i) = 1$ represents the maximum possible attack effort. In relation to $v\left(t_i\right)$ it is the strategy chosen by nation $\mathbb{V}$ (the defender), representing how much effort it allocates to defending itself at any time t_i. Similar to $u(t_i)$, $v(t_i)$ lies within $[0, 1]$, with $v(t_i) = 0$ indicating no defense effort and $v(t_i) = 1$ representing the maximum defense effort.

- The first and second equation of system (3.2) represent, the rate of change of nation $\mathbb{U}$'s military forces over time (respectively, the rate of change of nation $\mathbb{V}$'s military forces). While, The third equation models the time evolution within the game. It implies that time is decreasing uniformly, as the conflict proceeds, from t_f to 0. This negative time progression is a standard feature in differential games to reflect the countdown toward the end of the game.

From the intuitive formulation of the problem in (3.1) and (3.2), it understood that, there are two nations (players), $\mathbb{U}$ and respectivel, $\mathbb{V}$ they can choose, an optimal strategy $\widetilde{u}\,(.)$ respectively, $\widetilde{v}\,(.)$ for which the dynamic system in (3.2) generate a trajectory $\widetilde{x}_i\,(.) = x_{i_{\widetilde{u},\widetilde{v}}}\,(.)$ and such that, the player $\mathbb{U}$ tries to minimize the cost functional $\mathcal{C}\,(.,\widetilde{v}\,(.))$, while the player $\mathbb{V}$ tries to maximize the cost functional $\mathcal{C}\,(\widetilde{u}\,(.)\,,.)$.

3.2.1 Dynamic Programming Formulation

In order to use the Dynamic Programming approach in [55, 56] , we reformulate the problem (3.1) and (3.2) using standard notations in game theory and embedding this problem in a set of problems associated to each initial point in the phase-space as in [13, 14, 15]. We obtain the following standard Lagrangian autonomous differential game problem which, in a rather vague formulation, may be stated as:

Problem 2. Given $m_1, m_2 > 0$, $c_1 > c_2 > 0$. Find:

$$\inf_{u(.)}\sup_{v(.)} C(y_i; u(.), v(.)),\ \forall y_i \in Y_s, \tag{3.3}$$

subject to:

$$\begin{aligned}
&\mathcal{C}(y_i; u(.), v(.)) = g(x_i(t_f)) + \textstyle\int_0^{t_f} f_0(x_i(t_i), u(t_i), v(t_i))dt_i,\ y_i \in Y_s,\\
&x_i'(t_i) = f(x_i(t_i), u(t_i), v(t_i))\ a.e.\ (0, t_f),\ x_i(0) = y_i,\\
&u(t_i) \in U(x_i(t_i)),\ v(t_i) \in V(x_i(t_i))\ a.e.\ (0, t_f),\\
&x_i(.) \in \Omega,\ (u(.), v(.)) \in \mathcal{P},\ f_0(x_i(.), u(.), v(.)) \in L^1\left([0, t_f], \mathbb{R}\right),\\
&x_i(t_i) \in Y_s,\ \forall\, t_i \in [0, t_f),\ x_i(t_f) \in Y_f,
\end{aligned} \tag{3.4}$$

defined by the following data:

$$\begin{aligned}
&f(x_i, u, v) = (m_1 - c_1 v x_{i_2}, m_2 - c_2 u x_{i_1}, -1)\\
&f_0(x_i, u, v) = (1 - v)x_{i_2} - (1 - u)x_{i_1}\\
&U(x_i) = U = [0, 1],\ V(x_i) = V = [0, 1],\ g(\xi) = 0,\ \forall \xi \in Y_f\\
&Y_s = \mathbb{R}^+ \times \mathbb{R}^+ \times [0, t_f),\ Y_f = \mathbb{R}^+ \times \mathbb{R}^+ \times \{0\}.
\end{aligned} \tag{3.5}$$

where $\mathcal{P} = \mathcal{U} \times \mathcal{V}$ is the (largest) class of measurable admissible control functions $(u(.), v(.))$ and Ω is the corresponding class of absolutely continuous admissible trajectories.

3.2.2 The Hamiltonian and the set of transversely terminal points

The pseudo-Hamiltonian $\mathcal{H}(x_i, p_i, u, v) = \langle p_i, f(x_i, u, v)\rangle + f_0(x_i, u, v)$ is given in our case by:

$$\mathcal{H}(x_i, p_i, u, v) = p_{i_1} m_1 + p_{i_2} m_2 - p_{i_3} + x_{i_2} - x_{i_1} + x_{i_1}(1 - c_2 p_{i_2})u - x_{i_2}(1 + c_1 p_{i_1})v, \quad (3.6)$$

where, p_i represents Lagrange multipliers, using the fact that:

$$\min\nolimits_{u\in U}\left[(1 - c_2 p_{i_2})u\right] = \begin{cases} 0 & \text{if } \; p_{i_2} \leq \frac{1}{c_2} \\ 1 - c_2 p_{i_2} & \text{if } \; p_{i_2} > \frac{1}{c_2} \end{cases}$$
$$\max\nolimits_{v\in V}\left[-(1 + c_1 p_{i_1})v\right] = \begin{cases} 0 & \text{if } \; p_{i_1} \geq -\frac{1}{c_1} \\ -(1 + c_1 p_{i_1}) & \text{if } \; p_{i_1} < -\frac{1}{c_1}, \end{cases}$$

hence, the corresponding extrem value of the control parameters is given by the formulas:

$$\begin{aligned} \widehat{U}(x_i, p_i) = \widehat{U}(p_i) &= \begin{cases} \{0\} & \text{if } \; p_{i_2} < \frac{1}{c_2} \\ \{1\} & \text{if } \; p_{i_2} > \frac{1}{c_2} \\ U = [0,1] & \text{if } \; p_{i_2} = \frac{1}{c_2} \end{cases} \\ \widehat{V}(x_i, p_i) = \widehat{V}(p_i) &= \begin{cases} \{0\} & \text{if } \; p_{i_1} > -\frac{1}{c_1} \\ \{1\} & \text{if } \; p_{i_1} < -\frac{1}{c_1} \\ V = [0,1] & \text{if } \; p_{i_1} = -\frac{1}{c_1}. \end{cases} \end{aligned} \quad (3.7)$$

The Isaac's Hamiltonian:

$$H(x_i, p_i) = \min_{u\in U}\max_{v\in V} \mathcal{H}(x_i, p_i, u, v) = \max_{v\in V}\min_{u\in U} \mathcal{H}(x_i, p_i, u, v), (x_i, p_i) \in Z = dom\left(H(.,.)\right),$$

as well as its domain Z are stratified by the stratification $S_H = \{Z^{\pm,\pm}$, $Z^{\pm,\mp}$, $Z^{0,\pm}$, $Z^{\pm,0}$,

$Z^{0,0}\}$ defined by:

$$\begin{aligned}
Z^{+,+} &= \{(x_i,p_i)\in Z : p_{i_1} > -\tfrac{1}{c_1}, p_{i_2} > \tfrac{1}{c_2}\}\\
Z^{+,-} &= \{(x_i,p_i)\in Z : p_{i_1} > -\tfrac{1}{c_1}, p_{i_2} < \tfrac{1}{c_2}\}\\
Z^{+,0} &= \{(x_i,p_i)\in Z : p_{i_1} > -\tfrac{1}{c_1}, p_{i_2} = \tfrac{1}{c_2}\}\\
Z^{-,+} &= \{(x_i,p_i)\in Z : p_{i_1} < -\tfrac{1}{c_1}, p_{i_2} > \tfrac{1}{c_2}\}\\
Z^{-,-} &= \{(x_i,p_i)\in Z : p_{i_1} < -\tfrac{1}{c_1}, p_{i_2} < \tfrac{1}{c_2}\}\\
Z^{-,0} &= \{(x_i,p_i)\in Z : p_{i_1} < -\tfrac{1}{c_1}, p_{i_2} = \tfrac{1}{c_2}\}\\
Z^{0,+} &= \{(x_i,p_i)\in Z : p_{i_1} = -\tfrac{1}{c_1}, p_{i_2} > \tfrac{1}{c_2}\}\\
Z^{0,-} &= \{(x_i,p_i)\in Z : p_{i_1} = -\tfrac{1}{c_1}, p_{i_2} < \tfrac{1}{c_2}\}\\
Z^{0,0} &= \{(x_i,p_i)\in Z : p_{i_1} = -\tfrac{1}{c_1}, p_{i_2} = \tfrac{1}{c_2}\}.
\end{aligned} \tag{3.8}$$

If we denote by $H^{\pm,\pm}(.,.) = H(.,.)\mid_{Z^{\pm,\pm}}$, $H^{\pm,\mp}(.,.) = H(.,.)\mid_{Z^{\pm,\mp}}$, $H^{\pm,0}(.,.) = H(.,.)\mid_{Z^{\pm,0}}$, $H^{0,\pm}(.,.) = H(.,.)\mid_{Z^{0,\pm}}$, $H^{0,0}(.,.) = H(.,.)\mid_{Z^{0,0}}$ we get:

$$\begin{aligned}
H^{+,+}(x_i,p_i) &= p_{i_1}m_1 + p_{i_2}(m_2 - c_2x_{i_1}) - p_{i_3} + x_{i_2}\\
H^{+,-}(x_i,p_i) &= p_{i_1}m_1 + p_{i_2}m_2 - p_{i_3} + x_{i_2} - x_{i_1}\\
H^{+,0}(x_i,p_i) &= p_{i_1}m_1 + \tfrac{m_2}{c_2} - p_{i_3} + x_{i_2} - x_{i_1}\\
H^{-,+}(x_i,p_i) &= p_{i_1}(m_1 - c_1x_{i_2}) + p_{i_2}(m_2 - c_2x_{i_1}) - p_{i_3}\\
H^{-,-}(x_i,p_i) &= p_{i_1}(m_1 - c_1x_{i_2}) + p_{i_2}m_2 - p_{i_3} - x_{i_1}\\
H^{-,0}(x_i,p_i) &= p_{i_1}(m_1 - c_1x_{i_2}) + \tfrac{m_2}{c_2} - p_{i_3} - x_{i_1}\\
H^{0,+}(x_i,p_i) &= -\tfrac{m_1}{c_1} + p_{i_2}(m_2 - c_2x_{i_1}) - p_{i_3} + x_{i_2}\\
H^{0,-}(x_i,p_i) &= -\tfrac{m_1}{c_1} + p_{i_2}m_2 - p_{i_3} + x_{i_2} - x_{i_1}\\
H^{0,0}(x_i,p_i) &= -\tfrac{m_1}{c_1} + \tfrac{m_2}{c_2} - p_{i_3} + x_{i_2} - x_{i_1}.
\end{aligned} \tag{3.9}$$

Next, we need to compute the set of terminal transversality values defined in the general case by:

$$\begin{aligned}
&Z^{*}_{+,-} = \{(\xi,q)\in Y_f\times\mathbb{R}^3 : H(\xi,q)=0,\ \langle q,\bar{\xi}\rangle = Dg(\xi)\bar{\xi},\ \forall\bar{\xi}\in T_\xi Y_f\}\\
&Dg(\xi)\bar{\xi} = \tfrac{\partial g}{\partial\xi}(\xi)\bar{\xi}.
\end{aligned} \tag{3.10}$$

Lemma 3.1. *the set of terminal transversality values Z^* in our case is given by:*

$$Z^* = \{((s_1,s_2,0),(0,0,s_2-s_1))\,;\ s_1,\ s_2\geq 0\}\subset Z^{+,-}. \tag{3.11}$$

proof of Lemma 3.1. Since, $g(\xi) = 0$ and the tangent space $T_\xi Y_f = \mathbb{R}\times\mathbb{R}\times\{0\}$ then, it follows from (3.10) that, $q_1\bar{\xi}_1 + q_2\bar{\xi}_2 + q_3\bar{\xi}_3 = 0$, $\forall\bar{\xi}_1,\ \bar{\xi}_2\in\mathbb{R}$, $\bar{\xi}_3 = 0$ and therefore:

$$q_1 = q_2 = 0,\ q_3 \in \mathbb{R}.$$

Starting from the fact that, for $\xi = (s_1, s_2, 0) \in Y_1$, $q = (0, 0, q_3)$, $q_3 \in \mathbb{R}$. If $z = (\xi, q) \in Z^{+,+} \cup Z^{-,\pm}$ then, we obtain the following contradictions, $q_2 = 0\frac{1}{c_2}$, $q_1 = 0 - \frac{1}{c_1}$, and if $z = (\xi, q) \in Z^{\pm,0} \cup Z^{0,\pm} \cup Z^{0,0}$ we get $q_2 = \frac{1}{c_2} \neq 0$ and $q_1 = -\frac{1}{c_1} \neq 0$. Therefore, the only admissible trajectories are the ones which have segments on the stratum $Z^{+,-}$ because, $q_1 = 0 > -\frac{1}{c_1}$, $q_2 = 0 < \frac{1}{c_2}$. Besides, using the fact that, $H^{+,-}(\xi, q) = -q_3 + s_2 - s_1 = 0$ hence, $q_3 = s_2 - s_1$. □

3.3 Generalized Hamiltonian and characteristic flow

The first main computational operation consists in the backward integration for $t_i \leq 0$, of the Hamiltonian inclusion:

$$(\acute{x_i}, \acute{p_i}) \in d_S^\sharp H(x_i, p_i),\ (x_i(0), p_i(0)) = z_0 = (\xi, q) \in Z^*, \tag{3.12}$$

defined by the generalized Hamiltonian orientor field $d_S^\sharp H(.,.)$:

$$\begin{aligned} d_S^\sharp H(x_i, p_i) = \Big\{ &(\acute{x_i}, \acute{p_i}) \in T_{(x_i,p_i)}Z;\ \acute{x_i} \in f(x_i, \widehat{U}(x_i, p_i), \widehat{V}(x_i, p_i)), \\ &\langle \acute{x_i}, \bar{p_i} \rangle - \langle \acute{p_i}, \bar{x_i} \rangle = DH(x_i, p_i)(\bar{x_i}, \bar{p_i}),\ \forall\ (\bar{x_i}, \bar{p_i}) \in T_{(x_i,p_i)}Z \Big\}, \end{aligned} \tag{3.13}$$

where, $DH\left(x_i, p_i\right)\left(\overline{x_i}, \overline{p_i}\right)$ denotes the directional derivative of Hamiltonian function $H\left(.,.\right)$ at the point $(x_i, p_i) \in Z$ in the direction $(\overline{x_i}, \overline{p_i}) \in T_{(x_i,p_i)}Z$ and is described as follows:

$$DH\left(x_i, p_i\right)\left(\overline{x_i}, \overline{p_i}\right) = \frac{\partial H}{\partial x_i}\left(x_i, p_i\right)\overline{x_i} + \frac{\partial H}{\partial p_i}\left(x_i, p_i\right)\overline{p_i}$$

As it is specified in the Algorithm described in [55, 56], for each terminal point $z_0 = (\xi, q) \in Z^*$ one should identify the maximal solutions $X_i^*(.) = (X_i(.), P_i(.)) : I(z_0) = (t_i^-(z_0), 0] \rightarrow Z$, of the Hamiltonian inclusion in (3.12) that satisfy the following conditions:

$$\begin{aligned} &X_i(t_i) \in Y_s\ \forall t_i \in I_0(z_0) = (t_i^-(z_0), 0), \\ &X_i'(t_i) = f(X_i(t_i), u(t_i), v(t_i)) \text{ a.e. } I_0(z_0), \\ &u(t_i) \in \widehat{U}(X_i^*(t_i)), v(t_i) \in \widehat{V}(X_i^*(t_i)),\ \text{a.e. } I_0(z_0). \end{aligned} \tag{3.14}$$

If there are several solutions for the same terminal point $z_0 = (\xi, q) \in Z^*$, it is necessary to parameterize all these solutions by $\lambda \in \Lambda\left(z_0\right)$ in order to obtain the generalized Hamiltonian flow $X_i^*(.,.) = (X_i(.,.), P_i(.,.)) : B = \left\{(t_i, a), t_i \in I\left(a\right)\ a \in A\right\} \rightarrow Z;$

$A = graph\,(\Lambda\,(.))$, $a = (z_0, \lambda)$. We recall also the fact that, for each $(t_i, a) \in B_0 = \{(t_i, a) \in B, t_i \neq 0\}$ the Hamiltonian flow $X_i^*(.,.)$ defines the controls and respectively the trajectories:

$$\begin{aligned} & u_{t_i,a}\,(s) = u_a\,(t_i + s)\,,\ v_{t_i,a}\,(s) = v_a\,(t_i + s)\,,\ s \in [0, -t_i] \\ & x_{i_{t_i,a}}\,(s) = X_i\,(t_i + s, a)\,, \end{aligned} \tag{3.15}$$

which are admissible with respect to the initial point $y_i = X_i\,(t_i, a) \in Y_s$, and for which, the value of the cost functional in (3.4) is given by the function $V\,(.,.)$ defined by:

$$V\,(t_i, a) = g(\xi) + \int\limits_0^{t_i} \langle P_i\,(\sigma, a)\,, X_i^{'}\,(\sigma, a)\rangle d\sigma,\ a = (z_0, \lambda)\,, \tag{3.16}$$

and which, together with the Hamiltonian flow $X_i^*\,(.,.) = (X_i\,(.,.)\,, P_i\,(.,.))$ defines the generalized characteristic flow $C^*\,(.,.) = (X_i^*\,(.,.)\,, V\,(.,.))$, using the definition of the Hamiltonian $H\,(.,.)$ and the second condition in (3.14) one has $< P_i\,(\sigma, a)\,, X_i^{'}\,(\sigma, a) >$ $= -f_0\,(X_i\,(\sigma, a)\,, \widehat{u}\,(X_i^*\,(\sigma, a))\,, \widehat{v}\,(X_i^*\,(\sigma, a)))$, it follows from (3.3) that, the function $V(.,.)$ having as formula:

$$V\,(t_i, a) = \int_0^{t_i} ((1 - \widehat{u}\,(X_i^*\,(\sigma, a)))\,X_{i_1}\,(\sigma, a) - (1 - \widehat{v}\,(X_i^*\,(\sigma, a)))\,X_{i_2}\,(\sigma, a))\,d\sigma, \tag{3.17}$$

therefore, it follows from (3.8) and (3.9) that, the Hamiltonian orientor field $d_S^\sharp H(.,.)$ is given by the formulas:

$$d_S^\sharp H(x_i, p_i) = \begin{cases} d_S^\sharp H^{\pm,\pm}(x_i, p_i) & \text{if } (x_i, p_i) \in Z^{\pm,\pm} \\ d_S^\sharp H^{\pm,\mp}(x_i, p_i) & \text{if } (x_i, p_i) \in Z^{\pm,\mp} \\ d_S^\sharp H^{\pm,0}(x_i, p_i) & \text{if } (x_i, p_i) \in Z^{\pm,0} \\ d_S^\sharp H^{0,\pm}(x_i, p_i) & \text{if } (x_i, p_i) \in Z^{0,\pm} \\ d_S^\sharp H^{0,0}(x_i, p_i) & \text{if } (x_i, p_i) \in Z^{0,0}. \end{cases} \tag{3.18}$$

Since the manifolds $Z^{\pm,\pm}$, $Z^{\pm,\mp} \subset Z$ are open subsets, the Hamiltonian orientor fields $d_S^\sharp H^{\pm,\pm}(.,.)$ and $d_S^\sharp H^{\pm,\mp}(.,.)$ in (3.13) coincide with classical Hamiltonian vector fields:

$$\begin{aligned} d_S^\sharp H^{\pm,\pm}(x_i, p_i) &= \left\{ (\tfrac{\partial H^{\pm,\pm}}{\partial p_i}(x_i, p_i), -\tfrac{\partial H^{\pm,\pm}}{\partial x_i}(x_i, p_i) \right\}, \\ d_S^\sharp H^{\pm,\mp}(x_i, p_i) &= \left\{ (\tfrac{\partial H^{\pm,\mp}}{\partial p_i}(x_i, p_i), -\tfrac{\partial H^{\pm,\mp}}{\partial x_i}(x_i, p_i) \right\}, \end{aligned} \tag{3.19}$$

which are easy to calculate and will be described and studied later. While, on the singular stratum $\widetilde{Z} \in \{Z^{\pm,0}, Z^{0,\pm}, Z^{0,0}\}$ the corresponding Hamiltonian field $d_S^\sharp \widetilde{H}(.,.) \in \left\{ d_S^\sharp H^{\pm,0}(.,.),\ d_S^\sharp H^{0,\pm}(.,.),\ d_S^\sharp H^{0,0}(.,.) \right\}$ is characterized by the following result.

Lemma 3.2. *For any* $(x_i, p_i) \in \widetilde{Z}$ *one has:*

$$d_S^\sharp \widetilde{H}(x_i, p_i) = \emptyset. \tag{3.20}$$

proof of Lemma 3.2. If $(x_i, p_i) \in Z^{\pm,0}$, in order to compute the generalized Hamiltonian field $d_S^\sharp H^{\pm,0}(.,.)$, we note first that, according to some classical results as in [55], the tangent space to the 5-dimensional manifolds $Z^{\pm,0}$ is given by:

$$T_{(x_i,p_i)}Z^{\pm,0} = \{(\bar{x}_i, \bar{p}_i) \in \mathbb{R}^3 \times \mathbb{R}^3;\ \bar{p_{i_2}} = 0\}, \tag{3.21}$$

and $DH^{+,0}(x_i, p_i)(\bar{x}_i, \bar{p}_i) = -\bar{x}_{i_1} + \bar{x}_{i_2} + m_1\bar{p}_{i_1} - \bar{p}_{i_3}$. Therefore, the condition $\langle \acute{x}_i, \bar{p}_i \rangle - \langle \acute{p}_i, \bar{x}_i \rangle = DH^{+,0}(x_i, p_i)(\bar{x}_i, \bar{p}_i)$ is fully characterized by the expression:

$$\begin{aligned} &(p'_{i_1} - 1)\bar{x}_{i_1} + (1 + p'_{i_2})\bar{x}_{i_2} + p'_{i_3}\bar{x}_{i_3} + (m_1 - x'_{i_1})\bar{p}_{i_1} - x'_{i_2}\bar{p}_{i_2} - (x'_{i_3} + 1)\bar{p}_{i_3} = 0, \\ &\forall \bar{x}_{i_j},\ \bar{p}_{i_j} \in \mathbb{R}\ j = 1,\ 2,\ 3. \end{aligned} \tag{3.22}$$

It follows that, at each point $(x_i, p_i) \in Z^{+,0}$ one has:

$$x'_{i_1} = m_1,\ x'_{i_2} = 0,\ x'_{i_3} = -1,\ p'_{i_1} = 1,\ p'_{i_2} = -1,\ p'_{i_3} = 0, \tag{3.23}$$

since $(x'_i, p'_i) \in T_{(x_i,p_i)}Z^{+,0}$ then, $p'_{i_2} = 0$, this contradicts the fact that $p'_{i_2} = -1$.

Symmetrically, on the stratum $Z^{-,0}$ working as in the previous case we obtain:

$$x'_{i_1} = m_1 - c_1 x_{i_2},\ x'_{i_2} = 0,\ x'_{i_3} = -1,\ p'_{i_1} = 1,\ p'_{i_2} = c_1 p_{i_1},\ p'_{i_3} = 0, \tag{3.24}$$

since $(x'_i, p'_i) \in T_{(x_i,p_i)}Z^{-,0}$ then, $p'_{i_2} = 0$. While, from (3.24) it follows that, $p_{i_1} = 0$ that contradicts the fact that $(x_i, p_i) \in Z^{-,0}$. Concerning the strata $Z^{0,\pm}$, the proof is done in the same way as in the previous cases.

Next, on the stratum $Z^{0,0}$, using the same type of computations and arguments as in above, we get:

$$\begin{aligned} &T_{(x_i,p_i)}Z^{0,0} = \{(\overline{x_i}, \overline{p_i}) \in \mathbb{R}^2 \times \mathbb{R}^2;\ \left(\overline{p}_{i_1}, \overline{p}_{i_2}\right) = (0, 0)\}, \\ &DH^{0,0}\left(x_i, p_i\right)\left(\overline{x_i}, \overline{p_i}\right) = -\overline{x}_{i_1} + \overline{x}_{i_2} - \overline{p}_{i_3}. \end{aligned} \tag{3.25}$$

While, the condition $\langle x'_i, \overline{p_i} \rangle - \langle p'_i, \overline{x_i} \rangle = DH^{0,0}\left(x_i, p_i\right)\left(\overline{x_i}, \overline{p_i}\right)$ is characterized by the expression:

$$\left(p'_{i_1} - 1\right)\overline{x}_{i_1} + \left(p'_{i_2} + 1\right)\overline{x}_{i_2} - \left(x'_{i_3} + 1\right)\overline{p}_{i_3} + p'_{i_3}\overline{x}_{i_3} = 0,\ \forall\, \overline{x}_{i_j},\ \overline{p}_{i_j} \in \mathbb{R},\ j = 1, 2, 3$$

from here we deduce that, at each point $(x_i, p_i) \in Z^{0,0}$ one has:

$$\left(x'_{i_1}, x'_{i_2}\right) \in \mathbb{R}^2,\ x'_{i_3} = -1,\ p'_{i_1} = 1,\ p'_{i_2} = -1,\ p'_{i_3} = 0$$

the fact that, $(x'_i, p'_i) \in T_{(x_i,p_i)}Z^{0,0}$ gives $p'_{i_1} = 0 \neq 1$ and $p'_{i_2} = 0 \neq -1$, which leads to a contradiction. □

3.3.1 The Hamiltonian system on the open stratum $Z^{+,+}$

On the open stratum $Z^{+,+}$ for which, $p_{i_1} > -\frac{1}{c_1}$ and $p_{i_2} > \frac{1}{c_2}$ the differential inclusion in (3.19) coincides with the smooth Hamiltonian system:

$$\begin{cases} x_i^{'} = (m_1, m_2 - c_2 x_{i_1}, -1) \\ p_i^{'} = (p_{i_2} c_2, -1, 0). \end{cases} \tag{3.26}$$

Standard results from differential equations theory show that the general solution of system (3.26) is described by the formulas:

$$\begin{cases} x^{+,+}(t_i) = (m_1 t_i + k_1, -\frac{c_2 m_1}{2} t_i^2 + (m_2 - c_2 k_1) t_i + k_2, -t_i + k_3),\ t_i < 0 \\ p^{+,+}(t_i) = (-\frac{c_2}{2} t_i^2 + k_4 c_2 t_i + k_5, -t_i + k_4, k_6),\ k_j \in \mathbb{R},\ j = 1, ..., 6. \end{cases} \tag{3.27}$$

3.3.2 The Hamiltonian system on the open stratum $Z^{+,-}$

On the open stratum $Z^{+,-}$ for which $p_{i_1} > -\frac{1}{c_1}$ and $p_{i_2} < \frac{1}{c_2}$ the differential inclusion in (3.19) coincides with the Hamiltonian system:

$$\begin{cases} x_i^{'} = (m_1, m_2, -1) \\ p_i^{'} = (1, -1, 0), \end{cases} \tag{3.28}$$

its general solution is described by:

$$\begin{cases} x_i^{+,-}(t_i) = (m_1 t_i + k_1, m_2 t_i + k_2, -t_i + k_3),\ t_i < 0 \\ p_i^{+,-}(t_i) = (t_i + k_4, -t_i + k_5, k_6),\ k_j \in \mathbb{R},\ j = 1, ..., 6. \end{cases} \tag{3.29}$$

3.3.3 The Hamiltonian system on the open stratum $Z^{-,+}$

On the stratum $Z^{-,+}$ for which $p_{i_1} < -\frac{1}{c_1}$ and $p_{i_2} > \frac{1}{c_2}$ differential inclusion (3.19) coincides with Hamiltonian system:

$$\begin{cases} x_i^{'} = (-c_1 x_{i_2} + m_1, -c_2 x_{i_1} + m_2, -1) \\ p_i^{'} = (c_2 p_{i_2}, c_1 p_{i_1}, 0), \end{cases} \tag{3.30}$$

which has as a general solution:

$$\begin{cases} x_{i_1}^{-,+}(t) = k_1 e^{-\sqrt{c_1 c_2} t_i} + k_2 e^{\sqrt{c_1 c_2} t_i} + \frac{m_2}{c_2},\ t_i < 0 \\ x_{i_2}^{-,+}(t_i) = k_1 \sqrt{\frac{c_2}{c_1}} e^{-\sqrt{c_1 c_2} t_i} - k_2 \sqrt{\frac{c_2}{c_1}} e^{\sqrt{c_1 c_2} t_i} + \frac{m_1}{c_1} \\ x_{i_3}^{-,+}(t_i) = -t_i + k_3, \\ p_{i_1}^{-,+}(t_i) = k_4 e^{-\sqrt{c_1 c_2} t_i} + k_5 e^{\sqrt{c_1 c_2} t_i} \\ p_{i_2}^{-,+}(t_i) = -k_4 \sqrt{\frac{c_1}{c_2}} e^{-\sqrt{c_1 c_2} t_i} + k_5 \sqrt{\frac{c_1}{c_2}} e^{\sqrt{c_1 c_2} t_i} \\ p_{i_3}^{-,+}(t_i) = k_6,\ k_j \in \mathbb{R},\ j = 1, ..., 6. \end{cases} \tag{3.31}$$

3.3.4 The Hamiltonian system on the open stratum $Z^{-,-}$

On the open stratum $Z^{-,-}$ for which $p_{i_1} < -\frac{1}{c_1}$ and $p_{i_2} < \frac{1}{c_2}$ the differential inclusion in (3.19) coincides with the smooth Hamiltonian system:

$$\begin{cases} x_i^{'} = (m_1 - c_1 x_{i_2}, m_2, -1) \\ p_i^{'} = (1, c_1 p_{i_1}, 0), \end{cases} \tag{3.32}$$

which, in turn has the general solution:

$$\begin{cases} x_i^{-,-}(t_i) = (-\frac{1}{2} c_1 m_2 t_i^2 + (m_1 - c_1 k_1) t_i + k_2, m_2 t_i + k_1, -t_i + k_3),\ t_i < 0 \\ p_i^{-,-}(t_i) = (t_i + k_4, \frac{1}{2} c_1 t_i^2 + c_1 k_4 t_i + k_5, k_6),\ k_j \in \mathbb{R},\ j = 1, ..., 6. \end{cases} \tag{3.33}$$

3.4 Construction of Hamiltonian flows

3.4.1 The Hamiltonian flow ending on the stratum $Z^{+,-}$

In this section, we describe the partial Hamiltonian flow which his trajectories have terminal segments on the stratum $Z^{+,-}$. Consider the general solution in (3.29) an admissible trajectory $X_{i_{+,-}}^{*}(., z) = (X_i^{+,-}(., z), P_i^{+,-}(., z)), z \in Z^*$ of system (3.28). This trajectory should satisfy the terminal conditions from the set of transversality terminal points Z^* and $X_{i_{+,-}}^{*}(t_i, z) \in Z^{+,-}$, $\forall t_i < 0$. From the terminal condition in (3.11) it follows that, $k_1 = s_1$, $k_2 = s_2$, $k_3 = k_4 = k_5 = 0$ and $k_6 = s_2 - s_1$. Therefore, we obtain the solution of the differential system in (3.28) in the form of maximal flow, with its components are given by the formulas:

$$\begin{cases} X_i^{+,-}(t_i, s_1, s_2) = (m_1 t_i + s_1, m_2 t_i + s_2, -t_i),\ t_i < 0 \\ P_i^{+,-}(t_i, s_1, s_2) = (t_i, -t_i, s_2 - s_1),\ s_1,\ s_2 \geq 0. \end{cases} \tag{3.34}$$

From the Dynamic Programming algorithm in [55, 56], it follows that we must retain only the trajectories $X^*_{i_{+,-}}(., s_1, s_2)$, s_1, $s_2 \geq 0$ that satisfy the conditions in (3.14). We note that the second condition in (3.14) is automatically satisfied since $H^{+,-}(.,.)$ defined in (3.9) is a first integral of the differential system in (3.28) , hence:

$$\begin{aligned} &P^{+,-}_{i_1}(t_i) > -\tfrac{1}{c_1}, P^{+,-}_{i_2}(t_i) < \tfrac{1}{c_2}, \\ &H^{+,-}(X^*_{i_{+,-}}(t_i, s_1, s_2)) = 0, \ \forall t_i < 0. \end{aligned} \tag{3.35}$$

The admissible trajectories $X^*_{i_{+,-}}(., s_1, s_2)$, s_1, $s_2 \geq 0$ must satisfy also the conditions:

$$\begin{aligned} &X^*_{i_{+,-}}(t_i, s_1, s_2) = (X^{+,-}_i(t_i, s_1, s_2), P^{+,-}_i(t_i, s_1, s_2)) \in Z^{+,-} \\ &X^{+,-}_i(t_i, s_1, s_2) \in Y_s = \mathbb{R}^+ \times \mathbb{R}^+ \times (0, t_f), \end{aligned} \tag{3.36}$$

on the maximal intervals $I^{+,-}(s_1, s_2) = (\tau^{+,-}(s_1, s_2), 0)$, s_1, $s_2 \geq 0$, hence, the extremity $\tau^{+,-}(.,.)$ is defined by:

$$\begin{aligned} &\tau^{+,-}(s_1, s_2) = \max\{\tau^{+,-}_1(s_1, s_2), \tau^{+,-}_2\} \\ &\tau^{+,-}_1(s_1, s_2) = \inf\{\tau < 0; \ X^{+,-}_i(t_i, s_1, s_2) \in Y_s, \ \forall t_i \in (\tau, 0)\}, \\ &\tau^{+,-}_2 = \inf\{\tau < 0; \ P^{+,-}_{i_1}(t_i) > -\tfrac{1}{c_1}, \ P^{+,-}_{i_2}(t_i) < \tfrac{1}{c_2}, \ \forall t_i \in (\tau, 0)\}. \end{aligned} \tag{3.37}$$

Trying to obtain an explicit formula for the extremity $\tau^{+,-}(.,.)$. To this end, it follows from (3.34) as well as from the fact that, $c_1 > c_2$ then, $P^{+,-}_{i_1}(t_i) = t_i > -\frac{1}{c_1} > -\frac{1}{c_2}$ and therefore:

$$\tau^{+,-}_2 = -\frac{1}{c_1}. \tag{3.38}$$

From here together with (3.34) and (3.37) , we deduce that, the extremity $\tau^{+,-}(.,.)$ may be characterized as follows:

$$\tau^{+,-}(s_1, s_2) = \max\left\{-\frac{s_1}{m_1}, -\frac{s_2}{m_2}, -\frac{1}{c_1}\right\}, \ s_1, s_2 \geq 0, \tag{3.39}$$

which, in turn can be expressed as follows:

$$\tau^{+,-}(s_1, s_2) = \begin{cases} \tau^{+,-}_2 = -\frac{1}{c_1} & \text{if} \quad s_1 > \frac{m_1}{c_1}, \ s_2 > \frac{m_2}{c_1} \\ -\frac{s_1}{m_1} & \text{if} \quad s_1 < \frac{m_1}{c_1}, \ s_2 > \frac{s_1 m_2}{m_1} \\ -\frac{s_2}{m_2} & \text{if} \quad s_1 > \frac{s_2 m_1}{m_2}, \ s_2 < \frac{m_2}{c_1}. \end{cases} \tag{3.40}$$

Geometrically, the trajectories $X^{+,-}_i(.,.)$ are the curves in Figures 1 and cover the domain $Y^{+,-} = Y_s^{+,-} \cup Y_f^{+,-}$ defined by:

$$\begin{aligned}
Y_s^{+,-} &= X_i^{+,-}(B^{+,-}) = \{X_i^{+,-}(t_i, s)\,;\ (t_i, s) \in B^{+,-}\} \\
Y_f^{+,-} &= (0, +\infty) \times (0, +\infty) \times \{0\} \\
B^{+,-} &= \begin{cases} (-\frac{1}{c_1}, 0) \times (\frac{m_1}{c_1}, +\infty) \times (\frac{m_2}{c_1}, +\infty) \\ \Big\{(t_i, s_1, s_2)\,;\ t_i \in (-\frac{s_1}{m_1}, 0),\ s_1 < \frac{m_1}{c_1},\ s_2 > \frac{m_2}{m_1} s_1\Big\}, \\ \Big\{(t_i, s_1, s_2)\,;\ t_i \in (-\frac{s_2}{m_2}, 0),\ s_1 > \frac{m_1}{m_2} s_2,\ s_2 < \frac{m_2}{c_1}\Big\}. \end{cases}
\end{aligned} \tag{3.41}$$

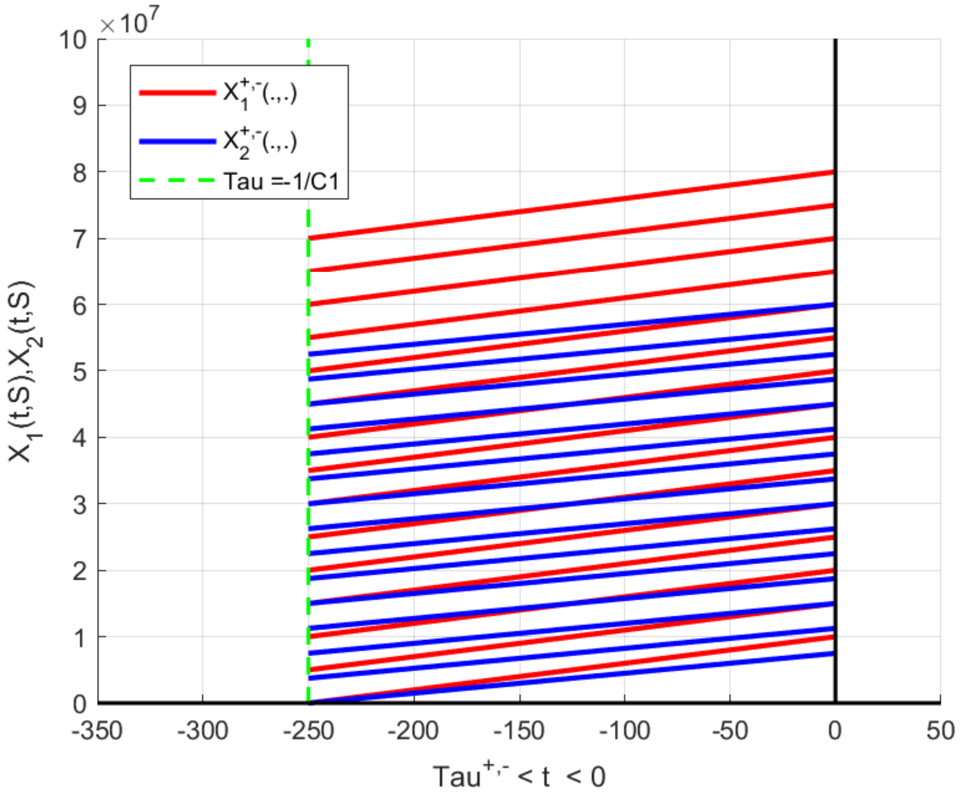

Figure 3.1: Admissible trajectories $X_i^{+,-}(.,.)$ for $\tau^{+,-}(s_1, s_2) = \tau_2^{+,-} = -\frac{1}{c_1}$

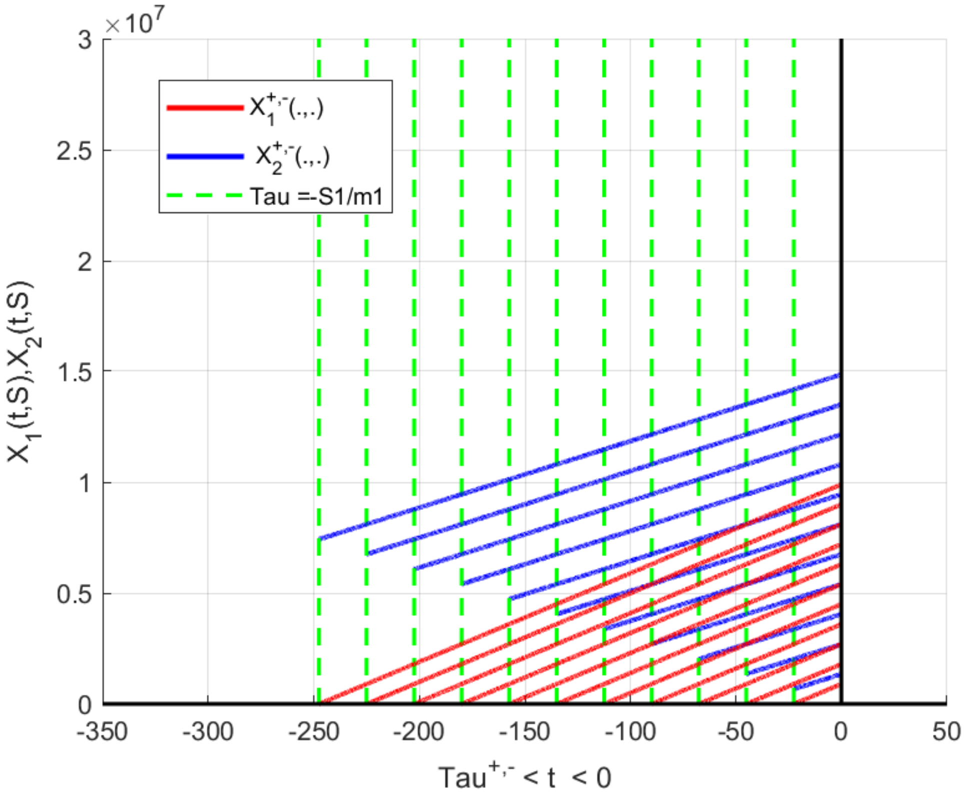

Figure 3.2: Admissible trajectories $X_i^{+,-}(.,.)$ for $\tau^{+,-}(s_1, s_2) = -\frac{s_1}{m_1}$

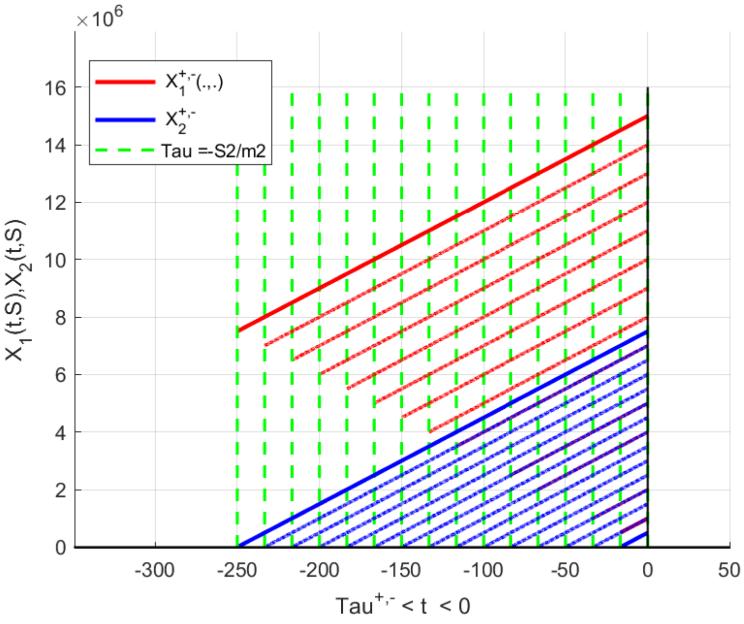

Figure 3.3: Admissible trajectories $X_i^{+,-}(.,.)$ for $\tau^{+,-} = -\frac{s_2}{m_2}$

3.4.2 Continuation of the trajectories on the stratum $Z^{-,-}$

The trajectories $X^*_{i_{+,-}}(t_i, s_1, s_2)$, $(t_i, s_1, s_2) \in B^{+,-}$ in (3.24) my be continuated for $t_i < \tau_2^{+,-} = -\frac{1}{c_1}$, for $s_1 > \frac{m_1}{c_1}$, $s_2 > \frac{m_2}{c_1}$ since, the extremity:

$$\begin{aligned} z^{+,-}(s_1, s_2) &= X^*_{i_{+,-}}(\tau_2^{+,-}, s_1, s_2) \\ &= \Big((-\tfrac{m_1}{c_1} + s_1, -\tfrac{m_2}{c_1} + s_2, \tfrac{1}{c_1}), (-\tfrac{1}{c_1}, \tfrac{1}{c_1}, s_2 - s_1) \Big) \end{aligned} \tag{3.42}$$

belongs to the open stratum $Z^{+,-}$ and also to the boundary of the stratum $Z^{-,-}$ therefore, the continuation of the trajectories $X^*_{i_{+,-}}(., s_1, s_2)$ for $s_1 > \frac{m_1}{c_1}$, $s_2 > \frac{m_2}{c_1}$, is possible only on the open stratum $Z^{-,-}$.

Starting from the fact that, $P_{i_1}^{-,-}(\tau_2^{+,-}) = -\frac{1}{C_1}$, the possibility of continuation of the trajectories $X^*_{i_{+,-}}(., s_1, s_2)$ in (3.34) for $t_i < \tau_2^{+,-}$, on the stratum $Z^{-,-} \subset Z$ (for which, $p_{i_1} < -\frac{1}{c_1}$, $p_{i_2} < \frac{1}{c_2}$) is guaranteed firstly by the condition $\frac{d}{dt_i}(P_{i_1}^{-,-}(\tau_2^{+,-})) = 1$ since in this case, the component $P_{i_1}^{-,-}(.)$ is strictly increasing on an interval of the form $(\tau_2^{+,-} - \delta, \tau_2^{+,-})$, $\delta > 0$. Hence, for any $t_i < \tau_2^{+,-}$, $P_{i_1}^{-,-}(t_i) < P_{i_1}^{-,-}(\tau_2^{+,-}) = -\frac{1}{C_1}$.

In this case the trajectories in (3.34) may be continuated by the trajectories $X^*_{i_{-,-}}(., s_1, s_2) = (X_i^{-,-}(., s_1, s_2), P_i^{-,-}(., s_1, s_2))$ which are solution of the Hamiltonian system in (3.32) with the property, $X^*_{i_{-,-}}(\tau_2^{+,-}, s_1, s_2) = z^{+,-}(s_1, s_2)$ and there exists $\tau^{-,-}(s_1, s_2) < \tau_2^{+,-}$ for $s_1 > \frac{m_1}{c_1}$, $s_2 > \frac{m_2}{c_1}$ such that:

$$P_{i_1}^{-,-}(t_i) < -\frac{1}{c_1},\ P_{i_2}^{-,-}(t_i) < \frac{1}{c_2},\ H^{-,-}\Big(X^*_{i_{-,-}}(t_i, s_1, s_2)\Big) = 0,\ t_i < \tau_2^{+,-}. \tag{3.43}$$

From the general solution in (3.33) on the open stratum $Z^{-,-}$ together with terminal condition in (3.42), the trajectories $X^*_{i_{-,-}}(., s_1, s_2)$ having as components:

$$\begin{cases} X_{i_1}^{-,-}(t_i, s_1, s_2) = -\frac{1}{2}c_1 m_2 t_i^2 + (m_1 - c_1 s_2)t_i + \frac{m_2}{2C_1} + s_1 - s_2 \\ X_{i_2}^{-,-}(t_i, s_1, s_2) = m_2 t_i + s_2 \\ X_{i_3}^{-,-}(t_i, s_1, s_2) = -t_i \\ P_i^{-,-}(t_i) = \Big(t_i, \frac{1}{2}c_1 t_i^2 + \frac{1}{2c_1}, s_2 - s_1\Big), \end{cases} \tag{3.44}$$

at which one must also satisfy the following admissibility conditions:

$$\begin{cases} X^*_{i_{-,-}}(t_i, s_1, s_2) = (X_i^{-,-}(t_i, s_1, s_2), P_i^{-,-}(t_i, s_1, s_2)) \in Z^{-,-} \\ X_i^{-,-}(t_i, s_1, s_2) \in Y_s = \mathbb{R}^+ \times \mathbb{R}^+ \times (0, t_f), \end{cases} \tag{3.45}$$

on the maximal intervals $I^{-,-}(s_1, s_2) = (\tau^{-,-}(s_1, s_2), \tau^{+,-}(s_1, s_2))$, $s_1 > \frac{m_1}{c_1}$, $s_2 > \frac{m_2}{c_1}$.

Hence, the extremity $\tau^{-,-}(.)$ is defined by:

$$\begin{aligned}
&\tau^{-,-}(s_1,s_2)=\max\{\tau_1^{-,-}(s_1,s_2),\tau_2^{-,-}\},\ s_1>\tfrac{m_1}{c_1},\ s_2>\tfrac{m_2}{c_1}\\
&\tau_1^{-,-}(s_1,s_2)=\inf\{\tau<\tau^{+,-};\ X_i^{-,-}(t_i,s_1,s_2)\in Y_s,\ \forall t_i\in(\tau,\tau^{+,-})\},\\
&\tau_2^{-,-}=\inf\{\tau<\tau^{+,-};\ P_{i_1}^{-,-}(t_i)<-\tfrac{1}{c_1},\ P_{i_2}^{-,-}(t_i)<\tfrac{1}{c_2},\ \forall t_i\in(\tau,\tau^{+,-})\}.
\end{aligned} \tag{3.46}$$

Trying to obtain an explicit formula for the extremity $\tau^{-,-}(.,.)$. To this end, it follows from (3.44) that:

$$P_{i_2}^{-,-}(t_i)-\frac{1}{c_2}=\frac{c_1}{2}\left[t_i^2-\frac{1}{c_1^2}\left(\frac{2c_1}{c_2}-1\right)\right],$$

since, $\tau_2^{-,-}<0$ then:

$$\tau_2^{-,-}=-\frac{1}{c_1}\sqrt{\frac{2c_1}{c_2}-1}, \tag{3.47}$$

and therefore, $P_{i_2}^{-,-}(t_i)<\frac{1}{c_2}$ $\forall t_i\in\left(\tau_2^{-,-},0\right)\supset\left(\tau_2^{-,-},\tau_2^{+,-}\right)$. Next, let $\tau_0(.)$ denotes the extremity defined by:

$$\tau_0(s_2)=-\frac{s_2}{m_2}, \tag{3.48}$$

for which, $X_{i_2}^{-,-}(t_i,s_1,s_2)>0, \forall t\in(\tau_0(s_2),0)$ then, from (3.47) we deduce that:

$$\tau_2^{-,-}-\tau_0(s_2)=\frac{c_1}{m_2^2}\frac{\left[s_2^2-\left(\frac{m_2}{c_1}\right)^2\left(\frac{2c_1}{c_2}-1\right)\right]}{\left[\frac{c_1s_2}{m_2}+\sqrt{\frac{2c_1}{c_2}-1}\right]}, \tag{3.49}$$

and we get:

$$\begin{cases}
\tau_2^{-,-}=\tau_0(s_2), \text{ for } s_2=\frac{m_2}{c_1}\sqrt{\frac{2c_1}{c_2}-1}\\
\tau_2^{-,-}>\tau_0(s_2),\ \forall s_2\in\left(\frac{m_2}{c_1}\sqrt{\frac{2c_1}{c_2}-1},+\infty\right)\\
\tau_2^{-,-}<\tau_0(s_2),\ \forall s_2\in\left(\frac{m_2}{c_1},\frac{m_2}{c_1}\sqrt{\frac{2c_1}{c_2}-1}\right).
\end{cases} \tag{3.50}$$

In order to characterize the extremity $\tau_1^{-,-}(.,.)$, the second-order equation, $X_{i_1}^{-,-}(t_i,s_1,s_2)=0$ is characterized by $\Delta(s_1,s_2)=(m_1-c_1s_2)^2+m_2[m_2+2c_1(s_1-s_2)]$. If we assume that, $\Delta(s_1,s_2)=0$ then, $\Delta'_{\Delta(s_1,s_2)}=2m_2c_1^2(m_1-c_1s_1)<0, \forall s_1>\frac{m_1}{c_1}$ and therefore, $\Delta(s_1,s_2)>0, \forall s_1>\frac{m_1}{c_1}, s_2>\frac{m_2}{c_1}$. From here, we deduce that, the second order equation has as a negative root:

$$t_1(s_1,s_2)=\frac{m_1-c_1s_2-\sqrt{\Delta(s_1,s_2)}}{c_1m_2}. \tag{3.51}$$

To describe $\tau^{-,-}(.,.)$, we use the same type of computations as in previous case and we obtain:

$$\begin{aligned}
&t_1(s_1,s_2) - \tau_0(s_2) = -\tfrac{\Delta(s_1,s_2)-m_1^2}{c_1\left[m_1+\sqrt{\Delta(s_1,s_2)}\right]}\\
&\Delta(s_{1,}s_2) - m_1^2 = c_1^2 s_2^2 - 2c_1(m_1+m_2)s_2 + m_2^2 + 2m_2c_1s_1\\
&\Delta'_{\Delta(s_1,s_2)}(s_2) = c_1^2\left[m_1^2 + 2m_2(m_1 - c_1s_1)\right],
\end{aligned} \tag{3.52}$$

and we can extract two cases:

Case 1: $\Delta'_{\Delta(s_1,s_2)}(s_2) < 0, \forall s_1 > \frac{m_1}{c_1}\left[1+\frac{m_1}{2m_2}\right]$ hence, $\Delta(s_{1,}s_2) - m_1^2 > 0$, which leads that, $t_1(s_1,s_2) < \tau_0(s_2)\ \forall s_1 > \frac{m_1}{c_1}\left[1+\frac{m_1}{2m_2}\right]$. It follows from (3.50) that:

$$\tau_2^{-,-} > \tau_0(s_2) > t_1(s_1,s_2),\ \forall s_1 > \frac{m_1}{c_1}\left[1+\frac{m_1}{2m_2}\right],\ s_2 > \frac{m_2}{c_1}\sqrt{\frac{2c_1}{c_2}-1}, \tag{3.53}$$

and therefore:

$$\tau^{-,-}(s_1,s_2) = \tau_2^{-,-} = -\frac{1}{c_1}\sqrt{\frac{2c_1}{c_2}-1}. \tag{3.54}$$

Case 2: $\Delta'_{\Delta(s_1,s_2)}(s_2) > 0, \forall s_1 \in \left(\frac{m_1}{c_1}, \frac{m_1}{c_1}\left[1+\frac{m_1}{2m_2}\right]\right)$ in this case, the second order equation $\Delta(s_{1,}s_2) - m_1^2 = 0$ has also two roots:

$$\begin{aligned}
\tilde{s}_2 &= \tfrac{1}{c_1}\left[m_1+m_2-\sqrt{m_1^2+2m_2(m_1-c_1s_1)}\right] > \tfrac{m_2}{c_1}\\
\tilde{\tilde{s}}_2 &= \tfrac{1}{c_1}\left[m_1+m_2+\sqrt{m_1^2+2m_2(m_1-c_1s_1)}\right] > \tilde{s}_2.
\end{aligned} \tag{3.55}$$

Therefore, $\Delta(s_{1,}s_2) - m_1^2 < 0, \forall s_2 \in \left(\tilde{s}_2, \tilde{\tilde{s}}_2\right)$ hence, $t_1(s_1,s_2) > \tau_0(s_2)$. From here as well as from (3.50), we deduce that:

$$t_1(s_1,s_2) > \tau_0(s_2) > \tau_2^{-,-}, \forall s_1 \in \left(\frac{m_1}{c_1}, \frac{m_1}{c_1}\left[1+\frac{m_1}{2m_2}\right]\right), s_2 \in \left(\tilde{s}_2, \tilde{\tilde{s}}_2\right), \tag{3.56}$$

and also:

$$\tau^{-,-}(s_1,s_2) = t_1(s_1,s_2) = \frac{m_1 - c_1s_2 - \sqrt{\Delta(s_{1,}s_2)}}{c_1m_2}, \tag{3.57}$$

while, in the case $s_2 \in \left(\frac{m_2}{c_1}, \tilde{s}_2\right) \cup \left(\tilde{\tilde{s}}_2, +\infty\right)$ then, $\Delta(s_{1,}s_2) - m_1^2 > 0$ which implies that, $t_1(s_1,s_2) < \tau_0(s_2)$.

We mention here that, condition (3.54) is the only case studied in [32]. On the other hand, the complexity of the extremities $t_1(.,.)$ and $\tau_2^{-,-}$ does not allow an explicit expression for the extremity $\tau^{-,-}(s_1,s_2)$ in the case $s_1 \in \left(\frac{m_1}{c_1}, \frac{m_1}{c_1}\left[1+\frac{m_1}{2m_2}\right]\right)$, $s_2 \in \left(\frac{m_2}{c_1}, \tilde{s}_2\right) \cup \left(\tilde{\tilde{s}}_2, +\infty\right)$. However, certain information on the admissible conditions may be obtained from numerical tests and the images of the trajectories $X_i^{-,-}(.,.)$. We develop an implementation using *Matlab 2018 language* and we present some simulations on the graphs of these trajectories for different parameter values.

Geometrically, the trajectories $X_i^{-,-}(.,.)$ given in (3.44) are the curves in Figures 3.4, 3.6, 3.7, 3.8 and cover the domain $Y_i^{-,-} = Y_s^{-,-} \cup Y_f^{-,-}$, $Y_s^{-,-} = X_i^{-,-}(B^{-,-})$ defined by:

$$\begin{aligned} B^{-,-} &= \{(t_i, s_1, s_2)\,,\ t_i \in I^{-,-}(s_1, s_2)\,,\ s_1 > \tfrac{m_1}{c_1}, s_2 > \tfrac{m_2}{c_1}\} \\ Y_f^{-,-} &= (0, +\infty) \times (0, +\infty) \times \{0\}. \end{aligned} \tag{3.58}$$

Finally, the trajectories $X_{i_{+,-}}^*(.,.)$ in (3.34) together with $X_{i_{-,-}}^*(.,.)$ in (3.44) may be concatenated to obtain a new extended Hamiltonian flow, described by the formula:

$$X_{i_{\oplus,\ominus}}^*(t_i, s_1, s_2) = \begin{cases} X_{i_{+,-}}^*(t_i, s_1, s_2),\ t_i \in [\tau_2^{+,-}, 0) \\ X_{i_{-,-}}^*(t_i, s_1, s_2),\ t_i \in \left(\tau^{-,-}(s_1, s_2)\,, \tau_2^{+,-}\right), \end{cases}$$

where its trajectories are illustrated below in Figures.

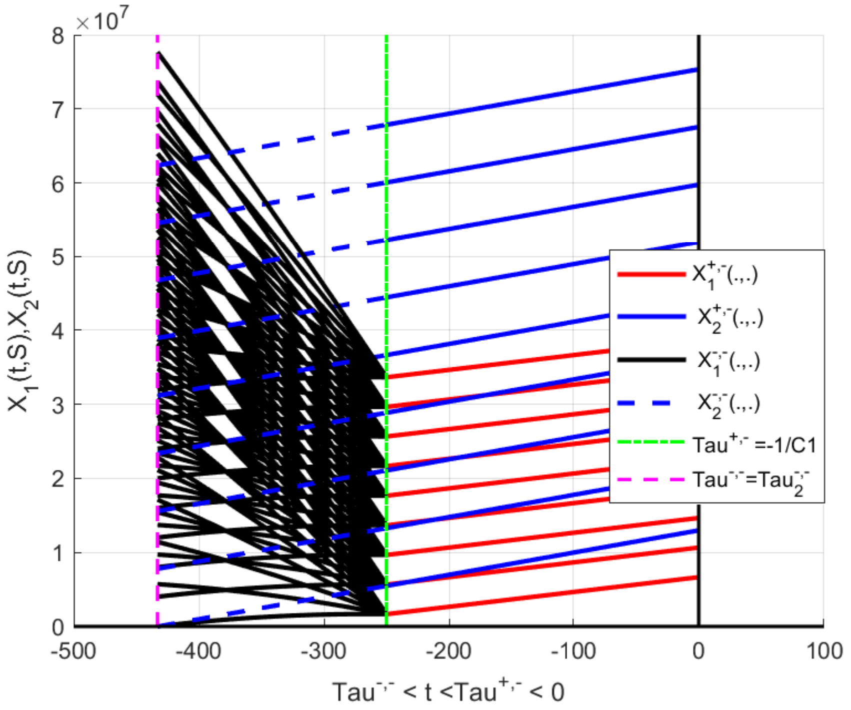

Figure 3.4: Admissible trajectories $X_i^{\oplus,\ominus}(.,.)$ for $\tau^{-,-}(s_1, s_2) = -\frac{1}{c_1}\sqrt{\frac{2c_1}{c_2} - 1}$.

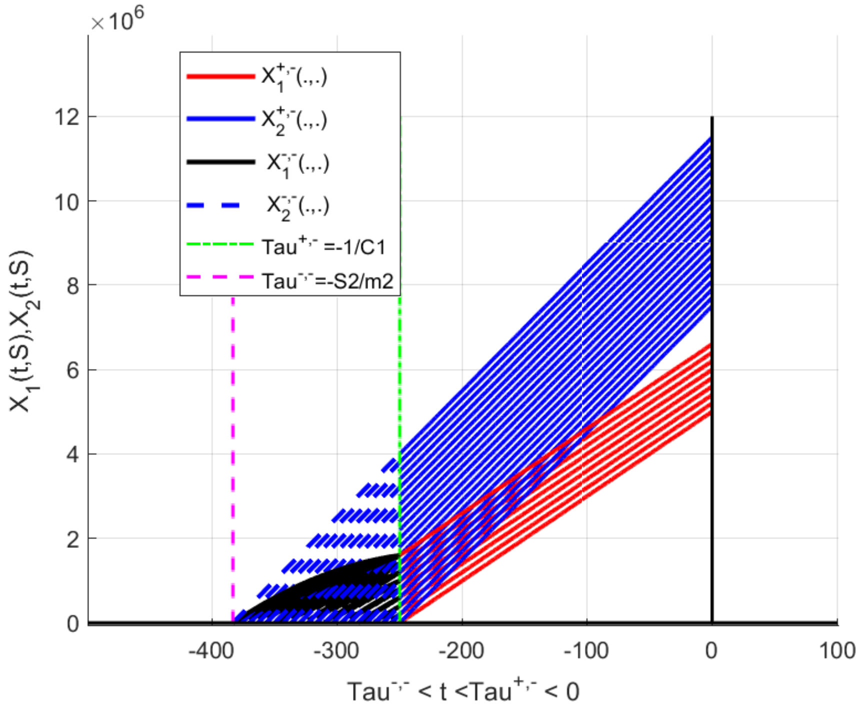

Figure 3.5: Admissible trajectories $X_i^{\oplus,\ominus}(.,.)$ for $\tau^{-,-}(s_1,s_2) = -\frac{s_2}{m_2}$.

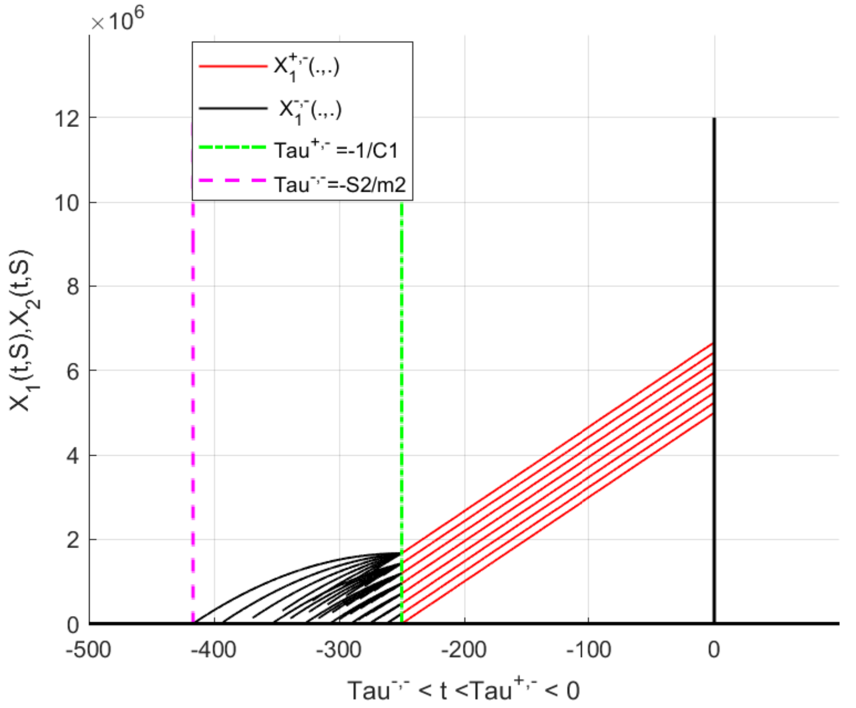

Figure 3.6: Admissible trajectories $X_{i_1}^{\oplus,\ominus}(.,.)$ for $\tau^{-,-}(s_1,s_2) = -\frac{s_2}{m_2}$.

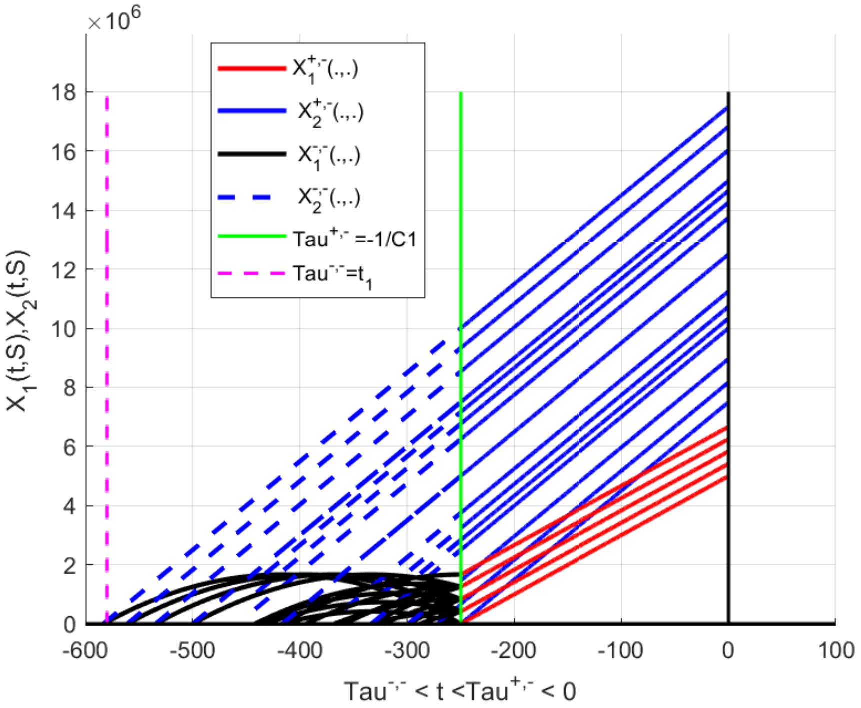

Figure 3.7: Admissible trajectories $X_i^{\oplus,\ominus}(.,.)$ for $\tau^{-,-}(s_1,s_2) = \frac{m_1 - c_1 s_2 - \sqrt{\Delta(s_1,s_2)}}{c_1 m_2}$

Thus, the Hamiltonian systems in (3.28) and (3.32) generate the generalized characteristic flows $C^*_{+,-}(.,.) = \left(X^*_{i+,-}(.,.), V(.,.)\right)$ and $C^*_{-,-}(.,.) = (X^*_{i-,-}(.,.), V(.,.))$ described explicitly in (3.34) and (3.44) and which, according to the well known classical results as in [55] satisfy the basic differential relation:

$$DV(t_i, s)(\bar{t}_i, \bar{s}) = \langle P_i^{+,-}(t_i, s), DX_i^*(t_i, s)(\bar{t}_i, \bar{s})\rangle \; \forall (\bar{t}_i, \bar{s}) \in T_{(t_i,s)} B^{+,-}, \tag{3.59}$$

where, $T_{(t_i,s)}B^{+,-}$ denotes the tangent space at the point $(t_i, s) \in B^{+,-}$.

3.5 Partial value functions and feedback strategies

As indicated in the theoretical algorithm in [55, 56] and presented in Section 2.4, the natural candidates for value functions and optimal strategies of Problem 2 are the extreme ones, defined by:

$$\begin{aligned}
&W^m = \begin{cases} g\left(x_i\right), & \text{if } x_i \in Y_f \\ W_0^m(x_i) = \inf_{X_i(t_i,s)=x_i} V\left(t_i, s\right), & \text{if } x_i \in Y_s \end{cases}, \\
&W^M = \begin{cases} g\left(x_i\right), & \text{if } x_i \in Y_f \\ W_0^M\left(x_i\right) = \sup_{X_i(t_i,s)=x_i} V\left(t_i, s\right), & \text{if } x_i \in Y_s \end{cases}, \\
&\widehat{B}_m\left(x_i\right) = \{(t_i, s) \in B : X_i\left(t_i, s\right) = x_i,\ W_0^m(x_i) = V\left(t_i, s\right)\}, \\
&\widehat{B}_M\left(x_i\right) = \{(t_i, s) \in B : X_i\left(t_i, s\right) = x_i,\ W_0^M(x_i) = V\left(t_i, s\right)\}, \\
&\widetilde{U}_m\left(x_i\right) = \overline{U}\left(\widehat{B}_m\left(x_i\right)\right),\ \widetilde{V}_m\left(x_i\right) = \overline{V}\left(\widehat{B}_m\left(x_i\right)\right), \\
&\widetilde{U}_M\left(x_i\right) = \overline{U}\left(\widehat{B}_M\left(x_i\right)\right),\ \widetilde{V}_M\left(x_i\right) = \overline{V}\left(\widehat{B}_M\left(x_i\right)\right), \\
&\overline{U}\left(t_i, s\right) = \{u_s\left(t_i\right),\ u_s\left(t_i\right) \in \overline{\mathcal{U}}\left(s\right)\},\ \overline{V}\left(t_i, s\right) = \{v_s\left(t_i\right),\ v_s\left(t_i\right) \in \overline{\mathcal{V}}\left(s\right)\},
\end{aligned} \tag{3.60}$$

where, $\overline{\mathcal{U}}\left(s\right) = \{u_s\left(.\right)\}$, $\overline{\mathcal{V}}\left(s\right) = \{v_s\left(.\right)\}$, denote the sets of control mapping that satisfy (3.14), one may note that:

$$\overline{U}\left(t_i, s\right) \in \widehat{U}\left(X_i^*\left(t_i\right)\right),\ \overline{V}\left(t_i, s\right) \in \widehat{V}\left(X^*\left(t_i\right)\right),\ \forall \left(t_i, s\right) \in B, \tag{3.61}$$

and also that, if the component $X_i\left(.,.\right)$ is invertible at $(t_i, s) \in B$ with inverse:

$$\widehat{B}\left(x_i\right) = \left(X_i\left(.,.\right)\right)^{-1}(x_i),$$

then one has:

$$W_0^M\left(x_i\right) = W_0^m(x_i) = V(\widehat{B}\left(x_i\right)) = W_0\left(x_i\right), \widehat{B}_M\left(x_i\right) = \widehat{B}_m\left(x_i\right) = \widehat{B}\left(x_i\right). \tag{3.62}$$

Moreover, it follows from (3.59) that if, in addition the function $W_0\left(.\right)$ is differentiable at the point $x_i \in Int\left(Y_s\right)$ then, its derivative is given by:

$$DW_0\left(x_i\right) = \widetilde{P}_i\left(x_i\right) = P_i(\widehat{B}\left(x_i\right)), \tag{3.63}$$

and also satisfies the following relations:

$$\begin{aligned}
&DW_0\left(x_i\right).f\left(x_i, \overline{u}, \overline{v}\right) + f_0\left(x_i, \overline{u}, \overline{v}\right) = 0\ \forall \overline{u} \in \widetilde{U}\left(x_i\right),\ \overline{v} \in \widetilde{V}\left(x_i\right) \\
&\widetilde{U}\left(x_i\right) = \widehat{U}\left(x_i, \widetilde{P}_i\left(x_i\right)\right),\ \widetilde{V}\left(x_i\right) = \widehat{V}\left(x_i, \widetilde{P}_i\left(x_i\right)\right),
\end{aligned} \tag{3.64}$$

where, $\widetilde{U}(.)$, $\widetilde{V}(.)$, are the corresponding candidates for optimal feedback strategies, and also from (3.9) and (3.14) it follows that, $W_0(.)$ verifies Isaac's bacis equations:

$$\begin{aligned}&\min\nolimits_{u\in U(x_i)}\max\nolimits_{v\in V(x_i)}\left[DW_0(x_i).f(x_i,u,v)+f_0(x_i,u,v)\right]=\\&\max\nolimits_{v\in V(x_i)}\min\nolimits_{u\in U(x_i)}\left[DW_0(x_i).f(x_i,u,v)+f_0(x_i,u,v)\right]=0.\end{aligned}\tag{3.65}$$

Next, we shall prove that, the extreme solutions in (3.60) may be expressed as in (3.62). The main ingredient is the following quasi-elementary result:

Lemma 3.3. *The following statements are true:*

(1) *The mapping $X_i^{+,-}(.,.) : B^{+,-} \to Y_s^{+,-}$ defined in (3.34) is a $\mathcal{C}^1-$stratified diffeomorphism with its inverse $\hat{B}^{+,-}(.)$ is described by:*

$$\begin{aligned}&\widehat{B}^{+,-}(x_i)=\left(\widehat{t_i}^{+,-}(x_i),\widehat{s_1}^{+,-}(x_i),\widehat{s_2}^{+,-}(x_i)\right),\ x_i\in Y_s^{+,-},\\&\widehat{t_i}^{+,-}(x_i)=-x_{i_3},\\&\widehat{s_1}^{+,-}(x_i)=x_{i_1}+m_1x_{i_3},\\&\widehat{s_2}^{+,-}(x_i)=x_{i_2}+m_2x_{i_3}.\end{aligned}\tag{3.66}$$

(2) *Symmetrically, the mapping $X_i^{-,-}(.,.) : B^{-,-} \to Y_s^{-,-}$ defined in (3.44) is a $\mathcal{C}^1-$stratified diffeomorphism, with its inverse $\hat{B}^{-,-}(.)$ is described by:*

$$\begin{aligned}&\widehat{B}^{-,-}(x_i)=\left(\widehat{t_i}^{-,-}(x_i),\widehat{s_1}^{-,-}(x_i),\widehat{s_2}^{-,-}(x_i)\right),\ x_i\in Y_s^{-,-},\\&\widehat{t_i}^{-,-}(x_i)=-x_{i_3},\\&\widehat{s_1}^{-,-}(x_i)=x_{i_1}+x_{i_2}+(m_1+m_2)x_{i_3}-c_1x_{i_2}x_{i_3}-\tfrac{1}{2}c_1m_2x_{i_3}^2-\tfrac{m_2}{2c_1},\\&\widehat{s_2}^{-,-}(x_i)=x_{i_2}+m_2x_{i_3}.\end{aligned}\tag{3.67}$$

Proof of Lemma 3.3. (1) If $x_i=(x_{i_1},x_{i_2},x_{i_3})\in Y_s^{+,-}$ then, it follows from (3.34) that, a point $(t_i,s_1,s_2)\in B^{+,-}$ for which $X_i^{+,-}(t_i,s_1,s_2)=x_i$ is characterized by the expressions, $x_{i_1}=m_1t_i+s_1$, $x_{i_2}=m_2t_i+s_2$, $x_{i_3}=-t_i$. Hence, the existence and uniqueness of the functions $t_i=\widehat{t_i}^{+,-}(x_i)<0$, $s_1=\widehat{s_1}^{+,-}(x_i)$ and $s_2=\widehat{s_2}^{+,-}(x_i)$ having the formulas as in (3.66).

(2) In order to prove the second statement, using the same type of computation and arguments as in previous case. Thus, it follows easly from (3.44) that, there exist $t_i=\widehat{t_i}^{-,-}(x_i)$, $s_1=\widehat{s_1}^{-,-}(x_i)$ and $s_2=\widehat{s_2}^{-,-}(x_i)$ of the form as in (3.67). □

The results in Lemma 3.3 show that, the characteristic flows

$$C^*_{+,-}\left(.,.\right)=\left(X^*_{i_{+,-}}\left(.,.\right),V\left(.,.\right)\right)\text{ and }C^*_{-,-}\left(.,.\right)=\left(X^*_{i_{-,-}}\left(.,.\right),V\left(.,.\right)\right),$$

described respectively, in (3.17) , (3.34) and (3.44) are invertible in the sense of (3.62) and generate the smooth partial proper value function:

$$W_0\left(x_i\right)=\begin{cases}W_0^{+,-}\left(x_i\right)=\frac{1}{2}\left(m_2-m_1\right)x_{i_3}^2+\left(x_{i_2}-x_{i_1}\right)x_{i_3},\, x_i\in Y_s^{+,-}\\ W_0^{-,-}\left(x_i\right)=\frac{1}{6}c_1m_2x_{i_3}^3-\frac{1}{2}m_1x_{i_3}^2+\frac{1}{2}c_1x_{i_2}x_{i_3}^2-x_{i_1}x_{i_3},\, x_i\in Y_s^{-,-},\end{cases}\tag{3.68}$$

which is of class $\mathcal{C}^1$ and may benaturally extended by , $W\left(\xi\right)=g\left(\xi\right)=0,\forall\xi\in Y_f$.

While, from (3.7) and (3.64), we deduce that the corresponding admissible feedback strategies are given by:

$$\widetilde{U}\left(x_i\right)\times\widetilde{V}\left(x_i\right)=\begin{cases}\{(\widetilde{u}^{+,-}\left(x_i\right),\widetilde{v}^{+,-}\left(x_i\right))\}=\{(0,0)\},\ x_i\in Y_s^{+,-}\\ \{(\widetilde{u}^{-,-}\left(x_i\right),\widetilde{v}^{-,-}\left(x_i\right))\}=\{(0,1)\},\ x_i\in Y_s^{-,-}.\end{cases}\tag{3.69}$$

The main result in this section is the following.

Theorem 3.4. *The following statements are true:*

1. *The function $W_0\left(.\right)$ defined in (3.68) is a solution of Isaac's equation defined in (3.65) on the corresponding open domain $Y_s^{+,-}\cup Y_s^{-,-}$. Moreover, each of them is the value function in the sense of (3.60) of the corresponding admissible feedback strategies given in (3.69).*
2. *The feedback strategies $\left(\widetilde{U}\left(.\right),\widetilde{V}\left(.\right)\right)$ described in (3.69) are optimal for the restriction on their open domain $Y_s^{+,-}\cup Y_s^{-,-}$.*

Proof of Theorem 3.4. For (1), from (3.9), (3.14), (3.63), (3.65) and (3.69) it follows that, if $x_i\in Y_s^{+,-}$ then:

$$\begin{aligned}&\min\nolimits_{u\in U(x_i)}\max\nolimits_{v\in V(x_i)}\left[DW_0^{+,-}\left(x_i\right).f\left(x_i,u,v\right)+f_0\left(x_i,u,v\right)\right]\\&=\min_{u\in U(x_i)}\max_{v\in V(x_i)}\mathcal{H}(x_i,\widetilde{P}_i^{+,-}\left(x_i\right),u,v)=\mathcal{H}(x_i,\widetilde{P}_i^{+,-}\left(x_i\right),\widetilde{u}^{+,-}\left(x_i\right),\widetilde{v}^{+,-}\left(x_i\right))\\&=H^{+,-}(X_i^{+,-}(\widehat{B}^{+,-}\left(x_i\right)),\widetilde{P}_i^{+,-}\left(x_i\right))=0,\end{aligned}$$

while, if $x_i\in Y_s^{-,-}$ we obtain:

$$\begin{aligned}&\min\nolimits_{u\in U(x_i)}\max\nolimits_{v\in V(x_i)}\left[DW_0^{-,-}\left(x_i\right).f\left(x_i,u,v\right)+f_0\left(x_i,u,v\right)\right]\\&=\min_{u\in U(x_i)}\max_{v\in V(x_i)}\mathcal{H}(x_i,\widetilde{P}_i^{-,-}\left(x_i\right),u,v)=\mathcal{H}(x_i,\widetilde{P}_i^{-,-}\left(x_i\right),\widetilde{u}^{-,-}\left(x_i\right),\widetilde{v}^{-,-}\left(x_i\right))\\&=H^{-,-}(X_i^{-,-}(\widehat{B}^{-,-}\left(x_i\right)),\widetilde{P}_i^{-,-}\left(x_i\right))=0,\end{aligned}$$

hence, $W_0(.)$ defined in (3.68) is a solution of Isaac's equation (3.65).

(2) . Since the value function $W_0\left(.\right)$ in (3.68) is of class $\mathcal{C}^1$ then, in order to prove the optimality of the pair of feedback strategies in (3.69), we use the well known *Elementary Verification Theorem* (Theorem 2.11) according to which, a sufficient optimality condition, for the admissible feedback strategies $\left(\widetilde{U}\left(.\right), \widetilde{V}\left(.\right)\right)$ is checking the following differential inequalities:

$$\begin{aligned}
&\inf\nolimits_{u\in U,\ \bar{v}\in \tilde{V}(x_i)}\left[DW_0\left(x_i\right)f\left(x_i,u,\bar{v}\right)+f_0\left(x_i,u,\bar{v}\right)\right]\geq 0\\
&\sup\nolimits_{\bar{u}\in \tilde{U}(x_i), v\in V}\left[DW_0\left(x_i\right)f\left(x_i,\bar{u},v\right)+f_0\left(x_i,\bar{u},v\right)\right]\leq 0.
\end{aligned} \tag{3.70}$$

First, if $x_i\in Y_s^{+,-}$ then, it follows from (3.5) , (3.68) and (3.69) that:

$$\begin{aligned}
&DW_0^{+,-}\left(x_i\right)=\left(-x_{i_3},x_{i_3},-x_{i_1}+x_{i_2}+m_2x_{i_3}-m_1x_{i_3}\right)\\
&f(x_i,u,\bar{v})=\left(m_1,m_2-c_2ux_{i_1},-1\right),\ f_0\left(x_i,u,\bar{v}\right)=x_{i_2}-(1-u)x_{i_1}\\
&f(x_i,\bar{u},v)=\left(m_1-c_1vx_{i_2},m_2,-1\right), f_0(x_i,\bar{u},v)=(1-v)x_{i_2}-x_{i_1}\\
&\overline{u}=\widetilde{u}^{+,-}\left(x_i\right)=0,\ \overline{v}=\widetilde{v}^{+,-}\left(x_i\right)=0,
\end{aligned} \tag{3.71}$$

and therefore:

$$\inf_{u\in U,\ \bar{v}\in \tilde{V}^{+,-}(x_i)}\left[DW_0^{+,-}\left(x_i\right)f\left(x_i,u,\bar{v}\right)+f_0\left(x_i,u,\bar{v}\right)\right]=\inf_{u\in U}\left[(1-c_2x_{i_3})ux_{i_1}\right],$$

since $c_1>c_2$ and $x_{i_3}\in[0,\frac{1}{c_1})$ we deduce that:

$$\inf_{u\in U}\left[(1-c_2x_{i_3})ux_{i_1}\right]=0,$$

for the second inequality, it follows from (3.71) that:

$$\sup_{\bar{u}\in \tilde{U}^{+,-}(x_i), v\in V}\left[DW_0^{+,-}\left(x_i\right)f\left(x_i,\bar{u},v\right)+f_0\left(x_i,\bar{u},v\right)\right]=\sup_{v\in V}\left[\left(-1+c_1x_{i_3}\right)x_{i_2}v\right]=0.$$

Next, if $x_i\in Y_s^{-,-}$ we use the same type of computation and arguments as in previous case, thus, it follows from (3.68) and (3.69) that:

$$\begin{aligned}
&DW_0^{-,-}\left(x_i\right)=\left(-x_{i_3},\tfrac{1}{2}c_1x_{i_3}^2,-x_{i_1}+c_1x_{i_2}x_{i_3}-m_1x_{i_3}+\tfrac{1}{2}c_1m_2x_{i_3}^2\right)\\
&f(x_i,u,\bar{v})=\left(m_1-c_1x_{i_2},m_2-c_2ux_{i_1},-1\right),\ f_0\left(x_i,u,\bar{v}\right)=(u-1)x_{i_1}\\
&f(x_i,\bar{u},v)=\left(m_1-c_1vx_{i_2},m_2,-1\right), f_0(x_i,\bar{u},v)=(1-v)x_{i_2}-x_{i_1}\\
&\overline{u}=\widetilde{u}^{-,-}\left(x_i\right)=0,\ \overline{v}=\widetilde{v}^{-,-}\left(x_i\right)=1,
\end{aligned} \tag{3.72}$$

and we also find:

$$\inf_{u\in U,\ \bar{v}\in \tilde{V}^{-,-}(x_i)}\left[DW_0^{-,-}\left(x_i\right)f\left(x_i,u,\bar{v}\right)+f_0\left(x_i,u,\bar{v}\right)\right]=\inf_{u\in U}\left[(1-\frac{1}{2}c_1c_2x_{i_3}^2)x_{i_1}u\right],$$

from here, we can extract two cases:

Case 1 : If $x_{i_3} \in \left(\frac{1}{c_1}, \frac{1}{c_1}\sqrt{\frac{2c_1}{c_2}-1}\right)$ then, $1-\frac{1}{2}c_1c_2x_{i_3}^2 > 0$ hence:

$$\inf_{u\in U}\left[\left(1-\frac{1}{2}c_1c_2x_{i_3}^2\right)x_{i_1}u\right]=0.$$

Case 2 : If $x_{i_3} \in \left(\frac{1}{c_1}, -t_1\left(s_1,s_2\right)\right)$, $s_1 \in \left(\frac{m_1}{c_1}, \frac{m_1}{c_1}\left[1+\frac{m_1}{2m_2}\right]\right)$, $s_2 \in \left(\tilde{s}_2, \widetilde{\tilde{s}}_2\right)$ then, from (3.56) we get, $x_{i_3} < -t_1\left(s_1,s_2\right) < \frac{1}{c_1}\sqrt{\frac{2c_1}{c_2}-1}$ and the rest of the proof is done in the same way as in the previous case; therefore, the first inequality in (3.70) is verified in both cases.

For the second inequality, one has:

$$\sup_{\bar{u}\in\tilde{U}^{-,-}(x_i),v\in V}\left[DW_0^{-,-}\left(x_i\right)f\left(x_i,\bar{u},v\right)+f_0\left(x_i,\bar{u},v\right)\right]=\sup_{v\in V}\left[\left(1-c_1x_{i_3}\right)x_{i_2}\left(1-v\right)\right]=0,$$

which proves inequalities (3.70) and hence the optimality of the admissible feedback strategies $\left(\widetilde{U}\left(.\right),\widetilde{V}\left(.\right)\right)$ holds. □

Conclusions and Comments

Finally, we are able to draw some conclusions, among which we mention the following:

1. Our analysis builds upon Isaac's foundational framework for the war game of attrition and attack, expanding the understanding of the strategic interactions between nations. We demonstrated that the extremities of the maximal interval of trajectories presented a more nuanced view than what had previously been outlined, revealing multiple pathways for conflict resolution beyond Isaac's original single-path analysis.

2. We showed that the introduction of newly extended trajectories significantly altered the dynamics of the attrition model. Unlike Isaac's initial considerations, our findings indicated that nations adopted strategies aimed at eroding the opponent's resources over time, compelling them to extend the duration of conflict. This shift emphasized the importance of considering various strategic responses in prolonged engagements.

3. Our results highlighted that specific choices regarding the extremity of trajectories, particularly those described in Equations (3.40), led to fixed durations of warfare. This insight underscored the critical role of strategic decision-making in shaping the outcome of conflicts, where the absence of attrition could occur under certain conditions, ultimately influencing the war's trajectory.

4. The analysis revealed distinct differences in how nations, represented by $\mathbb{U}$ and $\mathbb{V}$, managed their military engagements. Our findings indicated that while one nation experienced diminishing returns in strength, the other maintained a gradual increase in power. This asymmetry suggested that strategic advantages shifted over time, favoring the nation that effectively managed attrition.

5. We demonstrated that while nation $\mathbb{V}$ could sustain prolonged action with a gradual increase in strength, nation $\mathbb{U}$ faced a sharp decline in resources, ultimately leading to its disadvantage. This asymmetry suggested that the balance of power shifted in favor of $\mathbb{V}$ over time, with implications for optimal strategy and conflict outcomes. By comparing our results to Isaac's original findings, we highlighted previously unexplored possibilities in the structure of the maximal interval of trajectories and their role in shaping the course of warfare. The conclusions offer a refined perspective on how strategic choices impact attrition and warfare duration, pointing to $\mathbb{V}$'s eventual victory over $\mathbb{U}$.

However, working heuristically, Isaac's in [32] tries to identify certain geometric concepts such as, the dispersal line, equivocal line and singular surface,..., etc. Unfortunately, the significance of optimality was not specified. To address this aspect, in works like [5, 8, 26, 63], optimality is examined through the saddle point condition for the cost function $\mathcal{C}(.,.)$, in the sense that:

$$\mathcal{C}\left(\widetilde{U_1}(.)\,, V(.)\right) \leq \mathcal{C}\left(\widetilde{U_1}(.)\,, \widetilde{V_1}(.)\right) \leq \mathcal{C}\left(U(.)\,, \widetilde{V_1}(.)\right), \; \forall \left(U(.)\,, V(.)\right) \in \mathcal{P}. \tag{3.73}$$

In relation with our approach, the optimality of a pair of admissible feedback strategies $\left(\widetilde{U_1}(.)\,, \widetilde{V_1}(.)\right)$ is confirmed through the verification of the weaker conditions, in (3.70) which are easier to verify and much more efficient because they do not require the presence of all pairs of admissible strategies, $\left(\widetilde{U_1}(.)\,, V(.)\right)$ and $\left(U(.)\,, \widetilde{V_1}(.)\right)$ which in return, are necessary when checking the saddle point optimality condition in (3.73).

In summary, the current study includes contributions from the authors in the following directions:

1. The use of some recent concepts and results from Nonsmooth Analysis and relevant applications in the differential games theory, as well as employing the synthesis of the very recent theory in [54, 55, 56] regarding the rigorous approach and constructive of differential game problems.

2. The identification of a pair of feedback strategies, as well as the corresponding complete solution and the rigorous demonstration of its optimality.

3. The development of the implementation with *MATLAB 2018-software* , has traced the evolutions of the states constraints considered in the problem. The results

found show that Dynamic Programming is the most effective tool for the complete resolution of concrete problems and provides accurate results.

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ملخـــص :

في هذه الأطروحة، نهتم بتطبيق طريقة البرمجة الديناميكية لـ Ș. Mirică لحل لعبة حرب الهجوم و الاستنزاف التفاضلية التي قدمها R. Isaacs. نقترح استراتيجيات التغذية الراجعة كإسهام جديد يوفر التكيف، الفعالية والبساطة مع تقليل تكلفة الخوارزمية. تعتمد هذه المقاربة على طريقة محسنة لخصائص كوشي لمعالجة معادلات هاميلتون–جاكوبي الطبقية مع ضمان وجود دالة القيمة. يتم التحقق بدقة من مثالية استراتيجيات التغذية الراجعة باستخدام مبرهنة التحقق لدالة القيمة من الفئة $\mathcal{C}^1$، كما يتم تعزيزها بتجارب عددية مثبتة.

الكلمات المفتاحية: لعبة تفاضلية، الاحتواء التفاضلي، استراتيجيات التغذية الراجعة، البرمجة الديناميكية، التدفق الهاميلتوني، دالة القيمة، مبرهنة التحقق.

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Abstract:

In this thesis, we focus on applying Ș. Mirică's Dynamic Programming method to solve the Attack and attrition warfare differential game introduced by R. Isaacs. We propose feedback strategies as a new contribution that offers adaptability, efficiency, and simplicity while reducing algorithmic complexity. This approach employs a refined Cauchy characteristics method to handle stratified Hamilton-Jacobi equations while ensuring the existence of the value function. The optimality of the feedback strategies is rigorously validated using the Verification Theorem for value functions of class $\mathcal{C}^1$ and further supported by established numerical tests.

Key words : Differential game, Differential inclusion, Feedback strategies, Dynamic programming, Hamiltonian flow, Value function, Verification theorem

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Résumé :

Dans cette thèse, on s’intéresse à l'application de la méthode de Programmation Dynamique de Ș. Mirică pour résoudre le jeu différentiel de guerre d'attaque et d'attrition introduit par R. Isaacs. On propose les stratégies de rétroaction comme nouvelle contribution qui offre l’adaptabilité, l’efficacité et la simplicité tout en réduisant la complexité algorithmique. Cette approche utilise une méthode raffinée des caractéristiques de Cauchy pour traiter les équations de Hamilton-Jacobi stratifiées tout en garantissant l'existence de la fonction de valeur. L'optimalité des stratégies de rétroaction est rigoureusement validée à l'aide du Théorème de vérification pour les fonctions du valeurs de class $\mathcal{C}^1$ et consolidée par des tests numériques établis.

Mots-clés : Jeu différentiel, Inclusion différentielle, Stratégies de rétroaction, Programmation dynamique, Flux Hamiltonien, Fonction de valeur, Théorème de vérification.