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The Analytical Study of Some Contact Problems with or without Friction

Presented by

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Chahira Guenoune

Contents

Notations

In all that follows, we will use the following notations.

Sets

$\mathbb{R}^d$	the d-dimensional Euclidean space,
Ω	an open, bounded, and connected subset of $\mathbb{R}^d$ with Lipschitz boundary,
$\overline{\Omega}$	the closure of Ω,
$\Gamma = \partial\Omega$	the boundary of Ω, decomposed as $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$,
Γ_1	the part of the boundary where the displacement vanishes,
Γ_2	the part of the boundary where the traction condition is specified,
Γ_3	the part of the boundary over which contact may take place,
$[0, T]$	the time interval of interest, $T > 0$.

Spaces

$\mathbb{S}^d$	the space of second-order symmetric tensors on $\mathbb{R}^d$, i.e., $\mathbb{S}^d = \mathbb{R}^{d\times d}_s$,
$\mathbb{L}^2$	the Lebesgue space of square-integrable functions,
H	the space $\mathbb{L}^2(\Omega)^d$,
$\mathcal{H}$	the space $\mathbb{L}^2(\Omega)^{d\times d}_s$,
H_1	the space $H^1(\Omega)^d$,
$\mathcal{H}_1$	$\{\sigma \in \mathcal{H} \mid \operatorname{Div}\sigma = (\partial_j \sigma_{ij}) \in H\}$,
H'_Γ	the dual of H_Γ,
$D(\Omega)^d$	$\{\varphi = (\varphi_i) \mid \varphi_i \in D(\Omega),\ i = 1, \ldots, d\}$,
$C_0^\infty(\Omega)^d$	$\{f \in C^\infty(\Omega)^d \mid f = 0 \text{ on } K\}$,
X	a Hilbert space with norm $\\|\cdot\\|_X$ and inner product $(\cdot,\cdot)_X$.

Time-dependent function spaces

$C(0,T;X)$	the space of continuous functions defined on $[0,T]$ with values in X,
$C^1(0,T;X)$	the space of continuously differentiable functions defined on $[0,T]$ with values in X,
$\mathbb{L}^p(0,T;X)$	the space of strongly measurable functions on $[0,T]$ with values in X,
$\|\cdot\|_{\mathbb{L}^p(0,T;H)}$	the norm of $\mathbb{L}^p(0,T;H)$,
$W^{k,p}(0,T;X)$	the Sobolev space with parameters $k \in \mathbb{N}$ and $1 \le p \le +\infty$.

For a function f

$\dot{f}, \ddot{f}$	the first and second derivatives of f with respect to time,
$\operatorname{supp} f$	the support of f,
$\partial_i f,\ f_{,i}$	the partial derivative of f with respect to the i-th component x_i,
∇f	the gradient of f,
$\varepsilon(f)$	the symmetric part of the gradient of f,
$\operatorname{div} f$	the divergence of the vector field f.

Other notations

ν	unit outward normal on $\partial\Omega$,
v_ν, v_τ	normal and tangential components of a vector v,
$\operatorname{Div}\sigma$	the tensor divergence of σ,
χ_K	the indicator function of the set K,
$\lvert\cdot\rvert$	the Euclidean norm on $\mathbb{R}^d$ or $\mathbb{S}^d$,
$(\cdot,\cdot)_{X'\times X}$	the duality pairing between X' and X,
$x_n \rightharpoonup x$	weak convergence of the sequence (x_n) to x in X,
$x_n \to x$	strong convergence of the sequence (x_n) to x in X,
$\liminf$	lower limit,
$\limsup$	upper limit,
Λ^n	nth power of the operator Λ,
$a.e.$	almost everywhere,
iff	if and only if.

Introduction

Contact mechanics is an essential discipline within mechanical engineering that merges the concepts of material mechanics and continuum mechanics. Its primary focus is on the mathematical and physical formulation of problems involving elastic, viscoelastic, and plastic bodies in static or dynamic contact situations. The principles of contact mechanics are applied in various fields, including locomotive wheel-rail interactions, coupling mechanisms, braking systems, tire engineering, combustion engine components, mechanical linkages, metal forming, electrical contact interfaces, and many additional areas. Much effort has been dedicated to modeling, analyzing, and numerically approximating the physical processes that occur during contact between deformable bodies or between a body and a rigid, deformable, or lubricated base. Consequently, a comprehensive mathematical theory of contact mechanics (MTCM) has developed. This theory focuses on the mathematical frameworks that underlie contact problems involving various constitutive laws (see [33, 37, 38]).

The subject of contact mechanics and friction is ultimately about our ability to control friction, adhesion, and wear and to mould them to our wishes. For that, a detailed understanding of the dependency of contact, friction and wear phenomena on the materials and system properties is necessary.

Friction is a phenomenon that people have been interested in for over hundreds and even thousands of years and still today remains in the middle of the development of new products and technologies. Friction leads to energy dissipation and in micro-contacts, where extreme stress is present, to micro-fractures and surface wear. We often try to minimize friction during design in an attempt to save energy. There are, however, many situations in which friction is necessary. Without friction, we can not enjoy even walk or drive. There are countless instances, in which friction should be maxi-mixed instead of minimized, for example between tires and the road during braking [32]. Well-known and common friction laws used in the Mathematica literature are Coulomb's law and Tresca's law.

Wear is an unavoidable phenomenon in tribology field, defined as the gradual loss of material from the surfaces of solids due to their contact interactions, resulting in gradual removal of one or both materials. Consequently, the dimensions and mass of the body decrease due to wear. This phenomenon is caused by chemical reactions between the material and its environment, leading to the degradation of the surface. Or under stress during friction processes, leading to wear at the contact. Understanding these different causes of wear is crucial for predicting the lifespan of components and for designing systems that minimize wear-related failure. The study of models that incorporate wear aims to reduce energy consumption in friction processes and minimize material loss due to wear, while also increasing the lifespan of components and industrial parts. Therefore, the development of effective models for predicting wear in industrial environments is essential for design engineers. Frequently used models are based on Archard's observationsn [3], where wear is identified as an increase in the gap between the body and its foundation. The inclusion of wear in mathematical models is a recent advancement, and there is a wealth of literature on this subject, [23, 29, 32, 35, 39].

Damage mechanics is vital in various engineering fields, particularly in assessing the durability and reliability of materials and structures under operational conditions. Damage mechanics provides a framework for understanding and predicting material degradation, which is essential for ensuring the safety and longevity of engineering structures. The subject holds critical importance in design engineering, as it directly impacts the service life of a designed structure or component. While there is extensive engineering literature on this topic, mathematical models that account for the effect of internal material damage on contact processes have only recently been explored. The damage function is expressed as:

$$\beta(x,t) = \frac{E_{eff}}{E_Y}$$

such that E_Y be the Young modulus of the original material and E_{eff} be the current modulus. It is evident from this definition that the damage function β is limited to values between zero and one. When $\beta = 1$ there is no damage in the material, when $\beta = 0$ the material is completely damaged, when $0 < \beta < 1$ there is a partial damage and the system has a reduced load carrying capacity. Mathematical analysis of one-dimensional problems can be found in [15,16]. The three-dimensional case has been investigated in [21]. Some problems in thermo-mechanics or electro-mechanics of contact with damage have been studied in [5,10,13,24].

There exist recent and rapidly growing mathematical literature on contact problems which include thermal effects. As will be mention in third and fourth chapter of this thesis.

Thermal effects can play a significant role in certain applications. For instance, the sudden application of a car's brakes leads to a rapid reduction in kinetic energy, releasing a substantial amount of heat generated by friction. This heat can lead to a rapid rise in temperature, which may affect the friction coefficient and could potentially cause the softening or even local melting of the surfaces in contact.

The rapid temperature increase resulting from this heat generation can alter the friction coefficient and, in severe cases, even lead to softening or localized melting of the contacting surfaces. Due to this heat, a rapid temperature surge can occur, impacting the friction coefficient and potentially causing the contacting surfaces to soften or melt in localized areas. This heat poses a risk of rapid temperature escalation, which can affect the friction coefficient and possibly degrade the contacting surfaces through softening or even localized melting.

The heat generated can induce a rapid thermal excursion, potentially modifying the friction coefficient and, in extreme conditions, causing softening or localized phase change (melting) of the contacting surfaces.

General models for thermodynamic contact problems can be found in [2,10,24,25].

The connection of mechanical and electrical properties is a characteristic of piezoelectric materials. When mechanical stress is present, this coupling causes an electric potential to develop, and when mechanical stress is present, an electric potential is created. In engineering control equipment, the first effect is utilized in mechanical sensors, while the reverse effect is employed in actuators. Electroelastic materials are piezoelectric materials with elastic mechanical properties, whereas electro-viscoelastic materials are those with viscoelastic mechanical properties. You can find general models for electro-viscoelastic materials in [4,12,36].

Anti-plane shear deformations are one of the simplest classes of deformations that solids can undergo. In recent years, considerable attention has been paid to the analysis of anti-plane shear deformations within the context of various constitutive theories (linear and non-linear) of solid mechanics. These studies were largely motivated by the relative analytical simplicity of anti-plane shear problems with a single linear or quasi-linear partial differential equation of second order compared to planar problems with systems of higher-order or coupled partial differential equations. Thus the anti-plane shear problem plays a useful role as a pilot problem, within which various aspects of solutions in solid mechanics. Considerable attention has been paid to the modeling of such kind of problems, see for instance [18,19] and the references therein. Currently there is a considerable interest in dynamic or quasistatique frictional contact problems involving piezoelectric materials, i-e. materials characterized by the coupling between the mechanical and electrical properties (see [7, 12, 26, 27]).

The thesis is motivated by problems of contact mechanics with friction. The objective of this dissertation is the study of some boundary problems in contact mechanics, we consider different behavior laws such as thermo-viscoelastic, thermo-visco-plastic and electro-viscoelastic. We have studied contact problems with friction in a dynamic process. With boundary conditions, for which we couple both material damage and wear or damage and thermal or electrical effects. For each of these problems, we give the variational formulation, then the existence and uniqueness of the weak solution.

The main chapters of this work can be summarized as follows.

The first chapter aims to present the necessary tools for a good understanding of the subsequent problems addressed. Includes the classical description of some mechanical models of contact, material constitutive relations, boundary conditions.

In the second chapter, we introduce the main notations and we are interested to recalling some basic definitions and theorems of functional analysis which allow us to better understand the content of this job. Including Sobolev's theorem, lower semi continuitly, convex basede definition, and strongly monotone and Lipschitz operators, parabolic variational evolution equation and inequalities. The basic tools presented in this chapter are standard and can be found in many functional analysis books. In the third chapter we study the process of a dynamic contact problem between a thermo-viscoelastic body and foundation. The contact is frictional and bilateral with a moving rigid foundation. We write the mechanical problem and specify the assumptions in order to obtain the variational formulation. Then we establish our result of existence and uniqueness of the weak solution. The proof is based on a theorem for the existence and uniqueness of the solution of linear and nonlinear first-order evolution equations, followed by a fixed point argument. The content of this chapter was the subject of the publication [11].

The fourth chapter is devoted to the mathematical study of a contact problem between an thermo-elastic-viscoplastic body with wear and damage and an obstacle. The contact is normal compliance condition. We derive a variational formulation in the form of a coupled system in terms of displacement fields u, the stress field σ, the damage field β, the wear ω and the temperature θ. Also, we establish an existence and uniqueness result of a weak solution for the model.

The last chapter is focused on an anti-plane electro-viscoelastic contact problem with long memory with friction between deformable cylindrical bodies and a foundation. We are interested in the case of anti-plane deformations, i-e, the displacement field is parallel to the generatrices of the cylinder and is independent of the axial coordinate. We start by describing anti-plane electro-viscoelastic contact problem with long memory with friction defined by versions of the Tresca law and after specifying the assumptions about the data, we present variational formulation of the problem posed for which we will demonstrate the existence and uniqueness of the solutions with respect to the data and parameters.

Finally, we close this thesis with a general conclusion, a perspective and a detailed bibliography.

Chapter 1

Modeling

This chapter represents a brief reminder with basic notion of continuum mechanics, where we begin with a description of the general physical setting and the corresponding mathematical models that will be employed in this thesis, then we describe the constitutive laws, boundary conditions and contact condition with friction laws. Finally, we list the modeling of anti-plane contact problems. for more details of the modeling aspects of contact mechanics treated in this chapter see, e.g., [31, 37] and references therein.

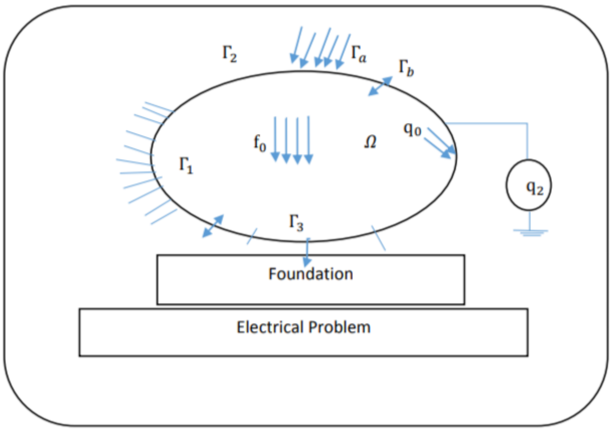
1.1 Physical settings - Mathematical models

In this section we will introduce the two physical frameworks and mathematical models of the mechanical problems involved in this thesis.

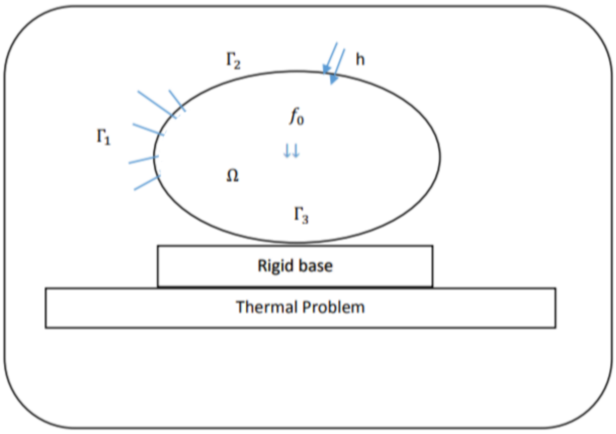
Physical setting $n^\circ 1$**:** We consider a material body occupies a bounded domain $\Omega \subset \mathbb{R}^d (d = 2, 3)$ with outer Lipschitz surface (regular surface) Γ, which is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, corresponding to the mechanical boundary conditions on one hand, and into two measurable parts Γ_a and Γ_b, corresponding to electrical boundary conditions on the other hand. We assume that meas $(\Gamma_1) > 0$, meas $(\Gamma_a) > 0$ and $\Gamma_3 \subset \Gamma_b$. The unit outward normal vector on Γ is given by ν. Let $[0, T]$ be the time interval of interest, where $T > 0$. The body is held fixed (clamped) on $\Gamma_1 \times [0, T]$ and the displacement vanishes there. Surface tractions of density h act on $\Gamma_2 \times [0, T]$ and a volume force of density f_0 is applied in $\Omega \times [0, T]$. We also assume that the electrical potential vanishes on $\Gamma_a \times [0, T]$

and a surface electric charge of density q_2 is prescribed on $\Gamma_b \times [0,T]$. The body may come in contact with an obstacle, the foundation, over the potential contact surface Γ_3 (see the figure below).

We study the evolution of these properties in the time interval $[0,T]$ under the hypothesis of small transformations. This assumption is reasonable for the piezoelectric materials usually used such as ceramics, polymers and piezocomposites.



Physical setting $n°2$**:** we consider a material body occupies the domain with surface $\Omega \subset \mathbb{R}^d (d = 2; 3)$ that is portioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $meas(\Gamma_1) > 0$. The body is clamped on $\Gamma_1 \times [0,T]$ and therefore the displacement field vanishes there. We also assume that a volume force of density f_0 acts in $\Omega \times [0,T]$ and that surface tractions of density f_2 act on $\Gamma_2 \times [0,T]$ and an external heat source given by the function q act in Ω (see the figure below). The body is in contact with or without friction with a deformable or lubricated obstacle on part Γ_3. We take into consideration the mechanical properties of the body. We are interested in the dynamic evolution of these properties over time, under the assumption of small transformations. We will use this physical setting in the third and fourth chapter of this thesis.



Before turning to the mathematical models that correspond to the physical settings presented, here are a few notations and conventions that will be used throughout this thesis.

We denote by $\mathbb{S}^d$ $(d = 2; 3)$ the space of second-order symmetric tensors on $\mathbb{R}$. $(.,.)$ and $|.|$ represent respectively the dot product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$. Thus, we have

$$u.v = u_i v_i \ , \quad |v| = (v,v)^{\frac{1}{2}} \ , \quad \forall u, v \in \mathbb{R}^d,$$
$$\sigma.\tau = \sigma_{ij}\tau_{ij}, \quad |\tau| = (\tau,\tau)^{\frac{1}{2}} \ , \quad \forall \sigma, \tau \in \mathbb{S}^d,$$

with the convention of the mute index. For a vector v, we use the notation v to denote the trace γv of v on Γ. We denote by v_ν and v_τ the normal and tangential components of v on the boundary given by

$$v_\nu = v.\nu, \ v_\tau = v - \ v_\nu \nu. \tag{1.1.1}$$

We denote by $u = u(x,t)$, $\sigma = \sigma(x,t)$, and $\varepsilon = \varepsilon(u)$ the displacement vector, the stress tensor, and the linearized strain tensor, respectively.

To simplify notations, we do not explicitly indicate the dependence of functions on $x \in \overline{\Omega}$ et $t \in [0,T]$.

For a stress field σ we denote by σ_ν and σ_τ the normal and tangential components at the boundary given by

$$\sigma_\nu = (\sigma\nu).\nu, \ \sigma_\tau = \sigma\nu - \sigma_\nu.\nu. \tag{1.1.2}$$

Using (1.1.1) and (1.1.2), we obtain the relation

$$(\sigma\nu).v = \sigma_\nu v_\nu + \sigma_\tau v_\tau, \tag{1.1.3}$$

who will be involved throughout this thesis, in the establishment of variational formulations of mechanical contact problems.

In addition, the points above a function represent the derivation with respect to time, for example

$$\dot{u} = \frac{du}{dt} = u_t,\ \ddot{u} = \frac{d^2u}{dt} = u_{tt},$$

where $\dot{u}$ denotes the velocity field and $\ddot{u}$ denotes the acceleration field. For the $\dot{u}$ velocity field, the notations $\dot{u}_\nu$ and $\dot{u}_\tau$ represent respectively the normal and tangential velocities at the boundary, that is to say (i.e).

$$\dot{u}_\nu = \dot{u}.\nu,\ \dot{u}_\tau = \ \dot{u} - \dot{u}_\nu\nu.$$

We denote partial derivatives and components by subscripts, e.g., the components of the linearized strain tensor $\varepsilon(u)$ are given by

$$\varepsilon(u) \ = (\varepsilon_{ij}(u)),\ \ \varepsilon_{ij}(u) = \frac{1}{2}\ (u_{i,j} + u_{j,i}) \text{ or } u_{i,j} = \frac{\partial u_i}{\partial x_j}, 1 \leq i, j \ \leq d. \qquad (1.1.4)$$

Note that here and throughout the thesis a subscript that follows a comma indicates a partial derivative with respect to the corresponding spatial variable, $u_{i,j} = \partial u_i / \partial x_j$.

Now Let us move on to a description of the mathematical models associated with the physical settings above.

Mathematical model $n°1$: The first mathematical model studied in this thesis describes the evolution of the body in the physical setting $n°2$.

The unknown functions of the problem are a displacement field $u : \Omega \times [0, T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0.T] \to \mathbb{S}^d$ and the temperature $\theta : \Omega \times [0, T] \to \mathbb{R}$.

We know that, the equations of motion that govern the evolution of the mechanical state of the body are

$$Div\sigma + f_0 = \rho\ddot{u},\ \text{in } \Omega \times [0, T], \qquad (1.1.5)$$

where $\rho : \Omega \to \mathbb{R}_+$ is the mass density and f_0 is the density (per unit volume) of applied forces, such as gravity or electromagnetic forces. "Div" is the divergence operator, and $(Div\sigma)i = (\sigma_{ij,j})$. When the external forces and tractions vary slowly with time, and the accelerations in the system are rather small and can be neglected, we omit the inertial terms in the equations of motion and obtain the equations of equilibrium.

$$Div\sigma + f_0 = 0, \text{ in } \Omega \times [0, T] \,. \tag{1.1.6}$$

Since the body is held fixed on Γ_1, the displacement field vanishes there

$$u = 0, \text{ on } \Gamma_1 \times [0, T] \,. \tag{1.1.7}$$

The traction boundary condition is

$$\sigma_\nu = h, \text{ on } \Gamma_2 \times [0, T] \,. \tag{1.1.8}$$

Processes modeled by the equations of motion (1.1.5) are called dynamic processes and those modeled by the equilibrium equations (1.1.6) are called quasistatic processes [37].

All our models in this thesis follow the dynamic process. We will later complete the mathematical model (1.1.5), (1.1.7), (1.1.8) with contact boundary conditions on the Γ_3 part.

Mathematical model $n°2$: This mathematical model describes the evolution of the body describes the evolution of the body in the physical setting $n°1$. It is an electro-mechanical model. The mechanical unknowns of the problem are the displacement field u, the stress field σ satisfying the equalities $(1.1.7) - (1.1.8)$.

To these are added the electrical unknowns of the problem, the electric displacement field $D : [0, T] \to \mathbb{R}^d$ and electric potential $\varphi : [0, T] \to \mathbb{R}$. The evolution of the piezoelectric body is described by the equilibrium equation for the electric displacement field.

$$\text{div } D = q_0, \text{ in } \Omega \times [0, T], \tag{1.1.9}$$

where "div" is the divergence operator for vectors, $\text{div } D = (D_{i,i})$ and q_0 represents the density of volumetric electric charges.

Recall that in physical setting $n°1$, the electric potential is vanishes on Γ_a.

$$\varphi = 0, \text{ on } \Gamma_a \times [0, T], \tag{1.1.10}$$

while on Γ_b an electric charge of density q_2 is prescribed

$$D\nu = q_2, \text{ on } \Gamma_b \times [0, T]. \tag{1.1.11}$$

This piezoelectric model $(1.1.5) - (1.1.11)$ will later be completed by boundary conditions on the contact surface Γ_3. The preceding equations are insufficient on their own to describe the movement of the material body considered. It is necessary to describe what is specific to the material itself, which is the subject of the constitutive laws that we will describe in the following section.

1.2 Constitutive laws

The relationship between the stresses σ and the strains ε and their derivatives that cause them is given by the constitutive equation, which characterizes a specific material. It describes the deformations of the body resulting from the action of forces and tractions.

It is a whole series of tests that it takes imaginer and realize to establish a behavior law. Physical experiments for one-dimensional materials constitute the starting point for establishing behavior laws. Here are four classic examples of solid tests: load tests-monotonous, charge-discharge tests, tests of uage (creep tests) and relaxation tests.

We consider, within the framework viscoelastic, viscoplastic and elasto-viscoplastic materials.

- Viscoelastic constitutive laws

  A general viscoelastic constitutive law is given by

$$\sigma = \mathcal{A}\varepsilon(\dot{u}) + \mathcal{B}\varepsilon(u), \tag{1.2.1}$$

  where $\mathcal{A}$ and $\mathcal{B}$ are non-linear constitutive functions, such that $\mathcal{A}$ represents the viscosity operator and $\mathcal{B}$ is the elasticity operator. For an elastic body, the law is reduced to

$$\sigma = \mathcal{B}\varepsilon(u). \tag{1.2.2}$$

  Recall that in linear viscoelasticity, the stress tensor $\sigma = (\sigma_{ij})$ is given by the Kelvin-

Voigt type of relation

$$\sigma_{ij} = a_{ijkl}\varepsilon_{kl}(\dot{u}) + b_{ijkl}\varepsilon_{kl}(u),$$

where $\mathcal{A} = (a_{ijkl})$ is the viscosity tensor and $\mathcal{B} = (b_{ijkl})$ is the elasticity tensor, for $i, j, k, l = 1, ..., d$.

In viscoelasticity with long-term memory, the body follows a Kelvin-Voigt law of the form

$$\sigma = \mathcal{A}\varepsilon\left(\dot{u}\right) + \mathcal{B}\varepsilon\left(u\right) + \int_0^t \mathcal{G}(t-s)\varepsilon(u(s))ds, \tag{1.2.3}$$

here, $\mathcal{G}$ is the fourth-order relaxation tensor that defines the behavior of the material with long memory. When $\mathcal{G} \approxeq 0$ we find short-term memory viscoelasticity (1.2.1) [37].

- Viscoplastic constitutive law

  For viscoplastic materials, the constitutive relation have the form

$$\sigma = \mathcal{A}\varepsilon(\dot{u}) + \mathcal{B}(\sigma, \varepsilon), \tag{1.2.5}$$

  where the $\mathcal{A}$ and $\mathcal{B}$ are the viscosity and plasticity operators, respectively.

- Elasto-viscoplastic constitutive law

  This constitutive relation

$$\sigma = \mathcal{A}\varepsilon\left(\dot{u}\right) + \mathcal{B}\varepsilon\left(u\right) + \int_0^t \mathcal{G}(\sigma(s) - \mathcal{A}\varepsilon\left(\dot{u}(s), \varepsilon(u(s))\right)ds, \tag{1.2.6}$$

  describe an elastic-viscoplastic material.
  Where the operators $\mathcal{A}$ and $\mathcal{B}$ are fourth-order and nonlinear tensors, their components a_{ijkl} and b_{ijkl} are called viscosity and elasticity coefficients respectively and $\mathcal{G}$ represents a nonlinear constitutive function that describes the viscoplastic material.

  This law slightly modifies the description of thermo-mechanical or electro-mechanical phenomena because here we must also take into consideration the temperature field θ, the electric displacement field $D = (D_i)$ as well as the electric field $E(\varphi) = -\bigtriangledown(\varphi) = -(\varphi_{,i})$. We then present the constitutive laws intervening in this thesis, they correspond to three particular categories of materials: electro-viscoelastic materials, thermo-viscoelastic materials and thermo-elasto-viscoplastic materials.

1.2.1 Electro-viscoelastic constitutive law

A general electro-viscoelastic constitutive law with short memory is given by

$$\sigma = \mathcal{A}\varepsilon(\dot{u}(t)) + \mathcal{B}\varepsilon\left(u(t)\right) - \xi^* E(\varphi), \text{ in } \Omega \times [0,T]\,, \tag{1.2.7}$$

$$D = E(\varphi) + \xi\varepsilon(u), \quad \text{in } \ \Omega \times [0,T]. \tag{1.2.8}$$

Equation (1.2.7) indicates that the mechanical properties of the material are described by a nonlinear viscoelastic constitutive law with short memory which takes into account the dependence of the stress field on the electric field. It has been employed in [4, 32] and the references therein. Note that in the case when ξ^* vanishes, equation (1.2.7) reduces to the purely viscoelastic constitutive law (1.2.1) [38].

1.2.2 Thermo-viscoelastic constitutive law with long memory

If we take into account the thermal effect of the material during contact, we arrive at a generalization of law (1.2.3), which is a law of thermo-viscoelastic behavior with long memory in the following form

$$\sigma = \mathcal{A}(\varepsilon(u(t))) + \mathcal{G}(\varepsilon(\dot{u}(t))) + \int_0^t \mathcal{B}(t-s)\varepsilon(u(s))ds - \theta(t)\mathcal{M}, \text{ in } \Omega \times [0,T], \tag{1.2.9}$$

the temperature θ is defined by a differential equation, which represents the conservation of energy as follows

$$\dot{\theta} - div(K\nabla\theta) = -\mathcal{M}.\nabla\dot{u} + q, \text{ in } \Omega \times [0,T], \tag{1.2.10}$$

where $q(x,t)$ the density of volume heat in Ω during the time interval, $K = (k_{ij})$ represents the conductivity tensor thermal, $div(K\nabla\theta) = (k_{ij}\theta_{,i})$ represents the volume density of the source temperature and $\mathcal{M}$ is an operator of thermal expansion. We will use this constitutive law (1.2.9) in the third chapter of this thesis.

1.2.3 Thermo-elasto-viscoplastic constitutive laws with damage

A thermo-elasto-viscoplastic relation, taking into account the thermal expansion of the material is given, generally, in the form

$$\sigma = \mathcal{A}(\varepsilon(\dot{u}(t))) + \mathcal{B}(\varepsilon(u(t)), \beta(t)) + \int_0^t \mathcal{G}(\sigma(s) - \mathcal{A}(\varepsilon\dot{u}(s))), \varepsilon(u(s)), \theta(s))ds. \quad (1.2.11)$$

The evolution of the temperature field is governed by the heat equation, obtained from the conservation of energy and defined by the following parabolic equation

$$\dot{\theta} - div(K_0 \triangle \theta) = \psi(\sigma, \varepsilon\dot{u}(s), \theta) + q \text{ in } \Omega \times [0, T], \quad (1.2.12)$$

where ψ is a nonlinear constituent function that represents the heat generated by the internal forces and $q(x,t)$ the density of volume heat in Ω during the time interval during the time interval.

Damage is an internal variable caused by viscoplastic deformation, the following differential inclusion will be used to describe the evolution of the damage field

$$\dot{\beta} - K_1 \triangle \beta + \partial\varphi_K(\beta) \ni S(\varepsilon(u), \beta), \text{ in } \Omega \times [0, T], \quad (1.2.13)$$

where K_1 and K_0 are strictly positive constants that represent the coefficients of microcracked diffusion, $\partial\varphi_K$ is the subdifferential of the indicator function φ_K and S is a given constitutive function that represents the source of damage in the system.

Finally, in order to complete the mathematical model that describes the evolution of the body, it is necessary to specify the boundary conditions on Γ_3. Moreover, we describe in the following the contact conditions and laws with friction.

1.3 Boundary conditions

We now turn to the boundary conditions. The surface Γ is assumed to be Lipschitz, and thus, at almost every point the outer unit normal vector ν is defined.

-Mechanical boundary conditions (displacement-traction)

Throughout the thesis we assume that the body is held fixed on Γ_1 and therefore,

$$u = 0, \text{ on } \Gamma_1 \times [0, T], \tag{1.3.1}$$

which represents the displacement boundary condition. Known tractions of density h act on the portion Γ_2, thus,

$$\sigma_\nu = h, \text{ on } \Gamma_2 \times [0, T]. \tag{1.3.2}$$

This condition is called the traction boundary condition. It means that the Cauchy stress vector σ_ν is imposed on the Γ_2 part of the Γ boundary, h representing the density of forces applied to the surface and constituting a given of the problem.

Finally, the body is possibly in contact with a foundation on $\Gamma_3 \times [0, T]$. This is where all the richness of the problems begins and where our interest lies, because the conditions on the potential contact surface Γ_3 can be very diverse and thus give rise to a variety of contact models with or without friction [37].

-Electrical boundary conditions

Next, since the electric potential vanishes on Γ_a during the process, we impose the boundary condition

$$\varphi = 0, \text{ on } \Gamma_a \times [0, T]. \tag{1.3.3}$$

Also, we recall that a surface electric charge of density q_2 is prescribed on Γ_b and therefore,

$$D_\nu = q_2, \text{ on } \Gamma_b \times [0, T]. \tag{1.3.4}$$

-The thermal conditions

The boundary condition associated with the temperature derivative of the Fourier condition

$$-k_{ij}\tfrac{\partial\theta}{\partial\nu}\nu_j = k_e(\theta - \theta_R) - h_\tau(|\dot{u}_\tau|), \text{ on } \Gamma_3 \times [0, T]\,, \tag{1.3.5}$$

$$\theta = 0, \text{ in } \Gamma_1 \cup \ \Gamma_2 \times [0, T]\,, \tag{1.3.6}$$

where θ_R is the foundation temperature, k_e represents the exchange of temperature between the deformable body and the foundation, h_τ the tangential function. (1.3.6) is the boundary condition of the temperature on $\Gamma_1\cup\,\Gamma_2$. We also consider the conditions

$$-k_0\frac{\partial\theta}{\partial\nu}+B\theta=0,\ \text{ on } \Gamma\times[0,T], \tag{1.3.7}$$

where (1.3.7) represents a Fourier boundary condition for temperature.

1.4 Contact conditions and friction laws

-Bilateral contact law [33, 37]

We now turn to a description of various contact conditions and cite the main laws of friction on the contact surface Γ_3 , which is where our main interest lies. These are divided naturally into the conditions in the normal direction and those in the tangential directions. First, consider the so-called bilateral contact, i.e., the contact between the body and the foundation is maintained at all times. This is generally the case in many machines and between moving parts and components in equipment or machinery. Since there is no gap, $g = 0$, and no separation, we have

$$u_\nu=0,\ \text{ on } \Gamma_3\times[0,T]. \tag{1.4.1}$$

The bilateral contact condition (1.4.1) has been used in a number of documents, for more details see for example [17, 31]. We use this contact type in the third chapter of this thesis.

-Signorini's contact law

Since the base is rigid, it will therefore not undergo deformations. So the body will not be able to penetrate there. This property translates mathematically into the following equality

$$\sigma_\tau=0,\ \text{ on } \Gamma_3\times[0,T]. \tag{1.4.2}$$

Since the base is rigid, it will not be deformed. The body cannot penetrate it. This property translates mathematically into the following inequality

$$u_\nu=u.\nu\leq 0,\ \text{ on } \Gamma_3\times[0,T]. \tag{1.4.3}$$

At points on Γ_3 where $u_\nu < 0$, the deformable body leaves the rigid base. Normal are zero. Consequently, we have

$$u_\nu < 0 \Rightarrow \sigma_\nu = 0, \text{ on } \Gamma_3 \times [0, T]. \tag{1.4.4}$$

At points of Γ_3 where $u_\nu < 0$, the deformable body leaves the rigid base. The normal constraints are then zero there Ω. Therefore, we have

$$u_\nu = 0 \Rightarrow \sigma_\nu \leq 0, \text{ on } \Gamma_3 \times [0, T]. \tag{1.4.5}$$

To summarize, the contact conditions (1.4.2) − (1.4.5) can be written in the following combined way

$$u_\nu \leq 0, \sigma_\nu \leq 0, \sigma_\tau = 0, \sigma_\nu u_\nu = 0, \text{ on } \Gamma_3 \times [0, T]. \tag{1.4.6}$$

The contact boundary conditions of the form (1.4.6) are also called Signorini contact conditions or unilateral contact condition.

Signorini's contact conditions (1.4.6) model the contact of a deformable body with a rigid base. We can therefore consider contact conditions for a deformable body with a deformable base [6].

-Contact with normal compliance

Condition describes a reactive foundation. It assigns a reactive normal traction or pressure that depends on the interpenetration of the asperities on the body's surface and those of the foundation. A general expression for the normal reactive traction on Γ_3 is

$$\sigma_\nu = p_\nu(u_\nu - g), \tag{1.4.7}$$

where $p_\nu(.)$ is a nonnegative prescribed function which vanishes when its argument is non-positive. Indeed, when $u_\nu < g$ there is no contact and the normal pressure vanishes, and when contact takes place $(u_\nu - g \geq 0)$ is a measure of the interpenetration of the surface asperities [33, 37]. A commonly used example of the normal compliance function p_ν is

$$p_\nu(r) = c_\nu r_+,$$

or, more generally,

$$p_\nu(r) = c_\nu(r_+)^m.$$

Here, $c_\nu > 0$ is the surface stiffness coefficient and $m \geq 1$ is the normal compliance exponent. We recall that $r_+ = \max\{r, 0\}$ denotes the positive part of r. We can also consider the following truncated normal compliance function

$$\begin{cases} p_\nu(r) = c_\nu r_+ & \text{if } r \leq \alpha, \\ p_\nu(r) = c_\nu \alpha & \text{if } r > \alpha, \end{cases} \tag{1.4.8}$$

where α is a positive coefficient related to the wear and hardness of the surface. In this case the contact condition (1.4.7) means that when the penetration is too large, i.e., when it exceeds α, the obstacle offers no additional resistance to penetration [37]. The normal compliance contact condition used in many publications, see, e.g., [20, 23, 30] and references therein.

1.5 Boundary conditions for bilateral contact with friction and wear

Let us introduce the wear function $\omega = \omega(x,t) : \Gamma_3 \times [0,T] \longrightarrow \mathbb{R}_+$ which measures the wear of the contacting surfaces. The wear is introduced, measuring the depth, in the normal direction, of the removed material. Therefore, it measures the change in the surface geometry and represents the cumulative amount of material removed, per unit surface area, in the neighborhood of the point x up to time t [33]. Since the body is in bilateral contact with the foundation, it results in that

$$u_\nu = -\omega \text{ on } \Gamma_3. \tag{1.5.1}$$

Thus, the location of the contact evolves with wear. We remind that the effect of wear is the recession on Γ_3, it is normal to expect that $u_\nu \leq 0$ on Γ_3, which implies $\omega \geq 0$ on Γ_3. The evolution of the contact surface wear is governed by a simplified version of Archard's law [39]. It is usually assumed that the rate of wear of the surface is proportional to the contact pressure and to the relative slip speed, that is to the dissipated frictional power. This leads to the rate form of Archard's law of surface wear

$$\dot{\omega} = -K\sigma_\nu|\dot{u}_\tau - v^*|, \tag{1.5.2}$$

where $k > 0$ is the wear coefficient, v^* is the tangential velocity of the foundation and $|\dot{u}_\tau - v^*|$ is the sliding velocity between the contact surface and the foundation.

We can see that wear evolution is assumed to be proportional to contact stress and sliding speed. For the sake of simplicity, we assume in the remainder of the that foundation motion is uniform, i.e. v^* does not vary with time. We note $v^* = |v^*| > 0$.
We assume that $v^*$is large, so that we can neglect $\dot{u}_\tau$ with respect to v^* in the following. in order to obtain the following version of Archard's law

$$\dot{\omega} = -Kv^*\sigma_\nu. \tag{1.5.3}$$

The use of the simplified law (1.5.3) for the wear evolution allows us to avoid certain mathematical difficulties in the study of contact problems. We can now eliminate the unknown function ω from the problem. in this way the problem is decoupled, and once the solution to the friction contact problem has been obtained, surface wear can be obtained by integrating (1.5.3). Let $\zeta = Kv^*$and $\alpha = \dfrac{1}{\zeta}$. Using (1.5.1) and (1.5.3) we have

$$\sigma_\nu = \alpha\dot{u}_\nu. \tag{1.5.4}$$

We model the frictional contact between the body and the foundation using Coulomb's law of dry friction see [4, 6]. Since there is only one sliding contact, therefore

$$|\sigma_\tau| = \mu|\sigma_\nu|, \quad \sigma_\tau = -\lambda(\dot{u}_\tau - v^*), \quad \lambda \geq 0, \tag{1.5.5}$$

where $\mu > 0$ is the friction coefficient. These relations define the evolution of stresses, particularly the tangential stress. This stress is in the opposite direction to the relative sliding velocity $|\dot{u}_\tau - v^*|$.
Naturally, the wear increases over time, i.e., $\dot{\omega} \geq 0$. Consequently, it follows from (1.5.1)

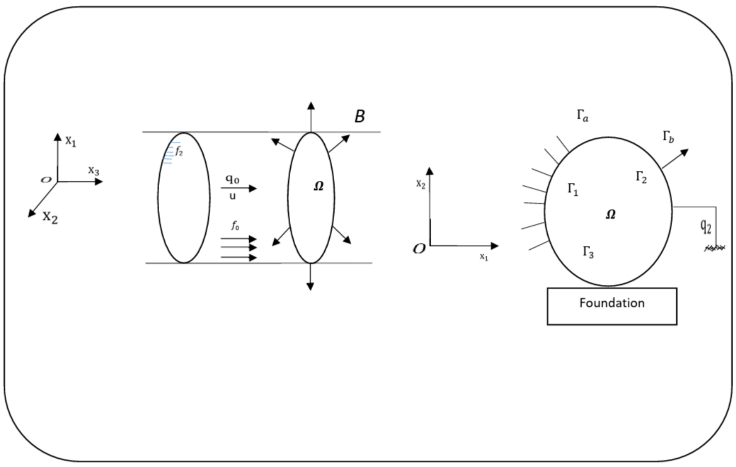
and (1.5.3) that $\dot{u}_\nu \leq 0$ and $\sigma_\nu \leq 0$ on Γ_3. Then, conditions (1.5.4) and (1.5.5) imply

$$\sigma_\nu = \alpha|\dot{u}_\nu|, \quad |\sigma_\tau| = -\mu\sigma_\nu, \quad \sigma_\tau = -\lambda(\dot{u}_\tau - v^*), \ \lambda \geq 0. \tag{1.5.6}$$

1.6 Modeling of anti-plane contact problems

1.6.1 Physical setting

We consider a piezoelectric body $\mathbb{B}$ identified with a region in $\mathbb{R}^3$ it occupies in a fixed and undistorted reference configuration. We assume that $\mathbb{B}$ is a cylinder with generators parallel to the x_3-axes with a cross-section which is a regular region Ω in the x_1, x_2-plane, $Ox_1x_2x_3$ being a Cartesian coordinate system. The cylinder is assumed to be sufficiently long so that the end effects in the axial direction are negligible. Thus, $\mathbb{B} = \Omega\times]-\infty, +\infty[$. The cylinder is acted upon by body forces of density f_0 and has volume free electric charges of density q_0. It is also constrained mechanically and electrically on the boundary. To describe the boundary conditions we denote by $\partial\Omega = \Gamma$ the boundary of Ω and we assume a partition of Γ into three open disjoint parts Γ_1, Γ_2 and Γ_3, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand. We assume that the one-dimensional measure of Γ_1 and Γ_a, denoted meas Γ_1 and meas Γ_a are positive. The cylinder is clamped on $\Gamma_1\times]-\infty, +\infty[$ and therefore the displacement field vanishes there. Surface tractions of density f_2 act on $\Gamma_2\times]-\infty, +\infty[$. We also assume that the electrical potential vanishes on $\Gamma_a\times]-\infty, +\infty[$ and a surface electrical charge of density q_2 is prescribed on $\Gamma_b\times]-\infty, +\infty[$, as shown in the figure below.



The cylinder is in contact over $\Gamma_3\times]-\infty,+\infty[$ with a conductive obstacle, the so called foundation. The contact is frictional and is modelled with Tresca's law.

The indices i and j denote components of vectors and tensors and run from 1 to 3, summation over two repeated indices is implied, and the index that follows a comma represents the partial derivative with respect to the corresponding spatial variable also, a dot above represents the time derivative. We use $\mathbb{S}^3$ for the linear space of second order symmetric tensors on $\mathbb{R}^3$ or, equivalently, the space of symmetric matrices of order 3, and ".", $\|.\|$ will represent the inner

$$u.v = u_i v_i, \;\; \|v\| = (v,v)^{\frac{1}{2}} \quad \text{for all } u = (u_i), \;\; v = (v_i) \in \mathbb{R}^3,$$

and

$$\sigma.\tau = \sigma_{i,j}\tau_{i,j}, \;\; \|\tau\| = (\tau.\tau)^{\frac{1}{2}} \quad \text{for all } \sigma = (\sigma_{i,j}), \;\; \tau = (\tau_{i,j}) \in \mathbb{S}^3.$$

We assume that

$$f_0 = (0,0,f_0) \text{ with } f_0 = f_0(x_1,x_2,t) : \Omega\times[0,T] \to \mathbb{R}, \tag{1.6.1}$$

$$f_2 = (0,0,f_2) \text{ with } f_2 = f_2(x_1,x_2,t) : \Omega\times[0,T] \to \mathbb{R}, \tag{1.6.2}$$

$$q_0 = q_0(x_1,x_2,t) : \Omega\times[0,T] \to \mathbb{R}, \tag{1.6.3}$$

$$q_2 = q_2(x_1,x_2,t) : \Omega\times[0,T] \to \mathbb{R}. \tag{1.6.4}$$

The forces (1.6.1), (1.6.2) and the electric charges (1.6.3), (1.6.4) would be expected to give rise to deformations and to electric charges of the piezoelectric cylinder corresponding to a displacement u and to an electric potential field φ which are independent on x_3 and have the form

$$u = (0,0,u) \text{ with } u = u(x_1,x_2,t) : \Omega\times[0,T] \to \mathbb{R}, \tag{1.6.5}$$

$$\varphi = \varphi(x_1,x_2,t) : \Omega\times[0,T] \to \mathbb{R}. \tag{1.6.6}$$

Such kind of deformation, associated to a displacement field of the form (1.6.5), is called an anti-plan shear.

The infinitesimal strain tensor is denoted by $\varepsilon(u) = (\varepsilon_{ij}(u))$ and the stress field by $\sigma =$

(σ_{ij}). We also denote by $E(\varphi) = (E_i(\varphi)) = -\bigtriangledown\varphi$ the electric field and by $D = (D_i)$the electrical displacement field.

Here and below, in order to simplify the notation, we do not indicate the dependence of various functions on x_1, x_2, x_3 or t and we recall that

$$\varepsilon_{i,j}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad E_i(\varphi) = -\varphi_{,i} = -\begin{pmatrix} \bigtriangledown\varphi_{,1} \\ \bigtriangledown\varphi_{,2} \\ 0 \end{pmatrix}, \tag{1.6.8}$$

$$\mathcal{E}\varepsilon = \begin{pmatrix} e(\varepsilon_{1,3} + \varepsilon_{3,1}) \\ e(\varepsilon_{2,3} + \varepsilon_{3,2}) \\ 0 \end{pmatrix}, \ \forall\varepsilon = (\varepsilon_{ij}) \in \mathbb{S}^3, \tag{1.6.9}$$

where e is a piezoelectric coefficient. We also assume that the coefficients μ, β and e depend on the spatial variables x_1, x_2, but are independent on the spatial variable x_3. Since $\mathcal{E}\varepsilon.v = \varepsilon.\mathcal{E}^T v$ for all $\varepsilon \in \mathbb{S}^3$ $v \in \mathbb{R}^3$ it follows from (1.6.9) that

$$\mathcal{E}^* v = \begin{pmatrix} 0 & 0 & \mu u_{,1} \\ 0 & 0 & \mu u_{,2} \\ \mu u_{,1} & \mu u_{,1} & 0 \end{pmatrix}, \ \forall v = (v_i) \in \mathbb{S}^3. \tag{1.6.10}$$

We assume that the process is mechanically dynamic and electrically static and there-fore is governed by the equilibrium equations

$$Div\sigma + f_0 = \rho\ddot{u}, \qquad\qquad divD = q_0$$

$$\sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} = 0 \text{ in } \Omega \times [0,T],$$

$$\sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} = 0 \text{ in } \Omega \times [0,T],$$

$$\sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} = \rho\ddot{u} \text{ in } \Omega \times [0,T].$$

Then and after simple calculations, we obtain

$$D_{1,1} + D_{2,2} + D_{3,3} - q_0 = 0.$$

1.6.2 Modeling of electro-viscoelastic contact problem with long memory

The material is modeled by the following electro-viscoelastic constitutive law with long-term memory. We have

$$\sigma = \lambda\left(tr\varepsilon u\right)I + 2\mu\varepsilon u + \gamma\left(tr\varepsilon(\dot{u})\right)I + 2\varsigma\varepsilon(\dot{u}) + 2\int_0^t \mathcal{M}\left(t-s\right)\varepsilon(u)ds - \xi^* E(\varphi), \quad (1.6.11)$$

$$D = \xi\varepsilon u + \beta E(\varphi), \quad (1.6.12)$$

where λ and μ are the Lamé coefficients, $\mathcal{M} : [0,T] \to \mathbb{R}$ is the viscosity coefficient, $tr\varepsilon(u) = \varepsilon_{ii}(u)$, I is the unit tensor in $\mathbb{R}^3$, β is the electric permittivity constant, ξ represents the third-order piezoelectric tensor and ξ^* is its transpose. In the anti-plan context (1.6.5), (1.6.6), using the constitutive equations (1.6.11), (1.6.12) it follows that the stress field and the electric displacement field are given by

$$\sigma = \begin{pmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & 0 \end{pmatrix}, \quad (1.6.13)$$

$$D = \begin{pmatrix} eu_{,1} - \beta\varphi_{,1} \\ eu_{,2} - \beta\varphi_{,2} \\ 0 \end{pmatrix}, \quad (1.6.14)$$

where

$$\sigma_{13} = \sigma_{31} = \varsigma\partial_{x_1}\dot{u} + \mu\partial_{x_1}u + \int_0^t \mathcal{M}\left(t-s\right)\partial_{x_1}u\left(s\right)ds,$$

and

$$\sigma_{23} = \sigma_{32} = \varsigma\partial_{x_2}\dot{u} + \mu\partial_{x_2}u + \int_0^t \mathcal{M}\left(t-s\right)\partial_{x_2}u\left(s\right)ds.$$

We assume that

$$\xi\varepsilon = \begin{pmatrix} e\left(\varepsilon_{13} + \varepsilon_{31}\right) \\ e\left(\varepsilon_{23} + \varepsilon_{32}\right) \\ e\varepsilon_{33} \end{pmatrix}, \ \forall\varepsilon = (\varepsilon_{ij}) \in \mathbb{S}^3, \quad (1.6.15)$$

where e is a piezoelectric coefficient. We also assume that the coefficients $\mathcal{M}, \mu$, β and e

depend on the spatial variables x_1, x_2, but are independent on the spatial variable x_3. Since $\xi\varepsilon.\mathbf{v} = \varepsilon.\xi^*\mathbf{v}$ for all $\varepsilon \in \mathbb{S}^3$, $\mathbf{v} \in \mathbb{R}^3$, it follows from (1.6.15) that

$$\xi^*\mathbf{v} = \begin{pmatrix} 0 & 0 & ev_1 \\ 0 & 0 & ev_2 \\ ev_1 & ev_2 & ev_3 \end{pmatrix}, \ \mathbf{v} = (v_i) \in \mathbb{R}^3. \tag{1.6.16}$$

We further assume that the process is mechanically dynamic and electrically static and therefore is governed by the motion and equilibrium equations

$$\rho\ddot{u} = Div\ \sigma + f_0, \quad divD - q_0 = 0, \text{ in } \mathbb{B} \times [0,T]\,. \tag{1.6.17}$$

Taking into account (1.6.1), (1.6.3), (1.6.5), (1.6.6), (1.6.13) and (1.6.14), the motion and equilibrium equations above reduce to the scalar equations

$$Div\,(\varsigma\nabla\dot{u} + \mu\nabla u) + \int_0^t \mathcal{M}\,(t-s)\,Div(\nabla u\,(s))ds + Div\,(e\nabla\varphi) + f_0 = \rho\ddot{u},$$
$$\text{in } \Omega \times [0,T]\,, \tag{1.6.18}$$

$$div\,(e\nabla u - \beta\nabla\varphi) = q_0. \tag{1.6.19}$$

Here and below we use the notation

$$div\tau = \tau_{1,1} + \tau_{1,2} \quad \text{ for } \quad \tau = (\tau_1\,(x_1,x_2,t)\,, \tau_2\,(x_1,x_2,t))\,,$$
$$\nabla v = (v_{,1}, v_{,2})\,, \quad \partial_\nu v = v_{,1}\nu_1 + v_{,2}\nu_2 \ \text{ for } \ v = v\,(x_1,x_2,t)\,.$$

We now describe the boundary conditions. During the process the cylinder is clamped on $\Gamma_1\times]-\infty,+\infty[$, and the electric potential vanishes on $\Gamma_a\times]-\infty,+\infty[$. Thus, (1.6.5) and (1.6.6) imply that

$$u = 0, \text{ on } \Gamma_1 \times [0,T], \tag{1.6.20}$$

$$\varphi = 0, \text{ on } \Gamma_a \times [0,T]. \tag{1.6.21}$$

Let ν denote the unit normal on $\Gamma\times]-\infty,+\infty[$. We have

$$\nu = (\nu_1, \nu_2, 0), \text{ with } \nu_i = \nu_i(x_1, x_2) : \Gamma \to \mathbb{R}, \ i = 1, 2. \tag{1.6.22}$$

For a vector v we denote by v_ν and v_τ its normal and tangential components on the boundary, given by

$$v_\nu = v.\nu, \ v_\tau = v - v_\nu \nu. \tag{1.6.23}$$

For a stress field σ we have

$$\sigma_\nu = (\sigma\nu).\nu, \ \sigma_\tau = \sigma\nu - \sigma_\nu \nu. \tag{1.6.24}$$

From (1.6.13), (1.6.14) and (1.6.22) we deduce that the Cauchy stress vector and the normal component of the electric displacement field are given by

$$\begin{cases} \sigma\nu = \left(0, 0, \varsigma\partial_\nu \dot{u} + \mu\partial_\nu u + \int_0^t \mathcal{M}(t-s)\,\partial_\nu u ds + e\partial_\nu\varphi\right), \\ D.\nu = e\partial_\nu u - \beta\partial_\nu\varphi. \end{cases} \tag{1.6.25}$$

Taking into account relations (1.6.2), (1.6.4) and (1.6.25), the traction condition on $\Gamma_2\times]-\infty, +\infty[$ and the electric conditions on $\Gamma_b\times]-\infty, +\infty[$ are given by

$$\varsigma\partial_\nu \dot{u} + \mu\partial_\nu u + \int_0^t \mathcal{M}(t-s)\,\partial_\nu u ds + e\partial_\nu\varphi = h, \tag{1.6.26}$$

$$e\partial_\nu u - \beta\partial_\nu\varphi = q_2. \tag{1.6.27}$$

We now describe the frictional contact condition and the electric conditions on $\Gamma_3\times]-\infty, +\infty[$. First, from (1.6.5) and (1.6.22) we infer that the normal displacement vanishes ($u_\nu = 0$), which shows that the contact is bilateral, that is, the contact is kept during all the process. Using now (1.6.5) and (1.6.22), (1.6.24) we conclude that

$$u_\tau = (0, 0, u_\tau), \ \sigma_\tau = (0, 0, \sigma_\nu), \tag{1.6.28}$$

where

$$\sigma_\tau = \left(0, 0, \varsigma\partial_\nu \dot{u} + \mu\partial_\nu u + \int_0^t \mathcal{M}(t-s)\,\partial_\nu u ds + e\partial_\nu\varphi\right). \tag{1.6.29}$$

We assume that the friction is invariant with respect to the x_3 axis and is modeled with

Tresca's friction law, that is

$$\begin{cases} |\sigma_\tau(t)| \le g\left(\int_0^t |\dot{u}_\tau(s)|\, ds\right), \\ |\sigma_\tau(t)| < g\left(\int_0^t |\dot{u}_\tau(s)|\, ds\right) \Rightarrow \dot{u}_\tau(t) = 0, \\ |\sigma_\tau(t)| = g\left(\int_0^t |\dot{u}_\tau(s)|\, ds\right) \Rightarrow \exists \beta \ge 0, \text{ such that } \sigma_\tau = -\beta \dot{u}_\tau. \end{cases} \quad \text{on } \Gamma_3 \times [0,T]. \quad (1.6.30)$$

Here $g : \Gamma_3 \to \mathbb{R}_+$ is a given function, the friction bound, and $\dot{u}_\tau$ represents the tangential velocity on the contact boundary. Using now (1.6.28) it is straight forward to see that the friction law (1.6.30) implies

$$\begin{cases} \left|\varsigma\partial_\nu \dot{u} + \mu\partial_\nu u + \int_0^t \mathcal{M}(t-s)\,\partial_\nu u ds + e\partial_\nu\varphi\right| \le g\left(\int_0^t |\dot{u}_\tau(s)|\, ds\right), \\ \left|\varsigma\partial_\nu \dot{u} + \mu\partial_\nu u + \int_0^t \mathcal{M}(t-s)\,\partial_\nu u ds + e\partial_\nu\varphi\right| < g\left(\int_0^t |\dot{u}_\tau(s)|\, ds\right) \Rightarrow \dot{u}_\tau(t) = 0, \\ \left|\varsigma\partial_\nu \dot{u} + \mu\partial_\nu u + \int_0^t \mathcal{M}(t-s)\,\partial_\nu u ds + e\partial_\nu\varphi\right| = g\left(\int_0^t |\dot{u}_\tau(s)|\, ds\right) \Rightarrow \exists \beta \ge 0, \\ \text{such that } \varsigma\partial_\nu \dot{u} + \mu\partial_\nu u + \int_0^t \mathcal{M}(t-s)\,\partial_\nu u ds + e\partial_\nu\varphi = -\beta\dot{u}_\tau. \end{cases} \quad \text{on } \Gamma_3 \times [0,T]. \quad (1.6.31)$$

Finally, we prescribe the initial displacement,

$$u(0) = u_0, \ \dot{u}(0) = u_1, \quad \text{in } \Omega, \quad (1.6.32)$$

where u_0 is a given function on Ω.

Chapter 2

Mathematical tools

This chapter presents preliminary material from functional analysis which will be used in the following chapters of this thesis. We start with a review of definitions and properties of several function spaces, including $\mathbb{L}^p$ spaces, Hilbert spaces, Sobolev spaces, and spaces of vector-valued functions. We then recall the Banach fixed point theorem and some standard results on variational inequalities and evolution equations that will be applied repeatedly in proving existence and uniqueness results for the contact problems. Finally, we list several types of inequalities that will be used repeatedly.

2.1 Functional spaces

We introduce in this chapter a summary of functional analysis, and some results that help us study the problems addressed in this thesis. Let Ω an open from $\mathbb{R}^d$. We must define some necessary spaces in Ω.

2.1.1 Important definitions about $\mathbb{L}^p$ spaces

These notes will briefly review some basic concepts related to the theory of $\mathbb{L}^p$ spaces. We are not trying to give a complete development, but rather review the basic definitions and theorems, mostly without proof.

Definition 2.1.1. *Let $p \in \mathbb{R}$, with $1 \leq p < \infty$, the Lebesgue space $\mathbb{L}^p(\Omega)$ is defined to be*

$$\mathbb{L}^p(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^p dx < \infty \right\}.$$

The space $\mathbb{L}^p(\Omega)$ is equipped with the norm

$$\|u\|_{\mathbb{L}^p(\Omega)} = \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}}.$$

If $p = \infty$ and $u : \Omega \longrightarrow \mathbb{R}$ is measurable then we define the $\mathbb{L}^\infty(\Omega)$

$$\mathbb{L}^\infty(\Omega) = \{ u : \Omega \longrightarrow \mathbb{R} \text{ is measurable and } \exists\ C \text{ such that } |u(x)| < C \text{ in } \Omega \},$$

its norm is

$$\|u\|_{\mathbb{L}^\infty(\Omega)} = inf\{C : |u(x)| < C, \ \text{in } \Omega\}.$$

Theorem 2.1.1. ***(Separability)*** *$\mathbb{L}^p(\Omega)$ is separable space for any $p \in [1, \infty[$.*

Theorem 2.1.2. ***(Reflxivity)*** *$\mathbb{L}^p(\Omega)$ is reflxive space for any $p \in]1, \infty[$.*
In particular, when p=2 the space $\mathbb{L}^2(\Omega)$ is a space of measurable functions of squared summable in Ω, provided with the inner product

$$(u, v)_{\mathbb{L}^2(\Omega)} = \int_\Omega u(x)v(x)dx,$$

$\mathbb{L}^2(\Omega)$ is a Hilbert space, we notice

$$\|u\|_{\mathbb{L}^2(\Omega)} = \left(\int_\Omega |u(x)|^2 \right)^{\frac{1}{2}}.$$

2.1.2 Sobolev's spaces

Definition 2.1.2. *Let Ω an open from $\mathbb{R}^d$. The Sobolev space $H^1(\Omega)$ is defined by*

$$H^1(\Omega) = \left\{ u \in \mathbb{L}^2(\Omega) \text{ such that } \frac{\partial u}{\partial x_i} \in \mathbb{L}^2(\Omega), \forall i \in \{1, ..., d\} \right\},$$

where $\dfrac{\partial u}{\partial x_i}$ is the partial derivative of u in the meaning of distributions.

Proposition 2.1.1. *The Sobolev $H^1(\Omega)$ endowed with the scalar product*

$$(u,v)_{H^1(\Omega)} = \int_\Omega (u(x)v(x) + \nabla u(x).\nabla v(x))\,dx,$$

and the standard norme

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega |u(x)|^2 + |\nabla u(x)|^2\right)^{\frac{1}{2}},$$

$$\| u \|_{H^1(\Omega)} = (u,u)^{\frac{1}{2}}_{H^1(\Omega)} \text{ and we write } \| u \|^2_{H^1(\Omega)} = \| u \|^2_{\mathbb{L}^2(\Omega)} + \| \nabla u \|^2_{\mathbb{L}^2(\Omega)^d}.$$

$H^1(\Omega)$ is a Hilbert space.

Theorem 2.1.3. *$H^1(\Omega) \subset L^2(\Omega)$ with compact injection.*

Theorem 2.1.4. ***(Trace of Sobolev)*** *There is a linear and continuous application $\gamma : H^1(\Omega) \to \mathbb{L}^2(\Gamma)$ such that $\gamma u = u|_\Gamma$ for $u \in C^1(\overline{\Omega})$.*

- **$\mathbf{H^1_0(\Omega)}$ Space**

Definition 2.1.3. *$H^1_0(\Omega)$ denote the vector subspace of the functions of $H^1(\Omega)$ zero on Γ*

$$H^1_0(\Omega) = \{u \in H^1(\Omega), u = 0 \text{ in } \Gamma\}.$$

The norm of $H^1_0(\Omega)$ is defined by

$$\|u\|_{H^1_0(\Omega)} = \left(\int_\Omega |\nabla u(x)|^2\right)^{\frac{1}{2}} = \|\nabla u(x)\|_{L^2(\Omega)}$$

- **$\mathbf{W^{1.p}(\Omega)}$ Space**

Definition 2.1.4. *Let $p \in \mathbb{R}$, with $1 \le p \le \infty$, the Sobolev space $W^{1.p}(\Omega)$ is defined to be*

$$W^{1.p}(\Omega) = \left\{u \in \mathbb{L}^p(\Omega), \exists g_i \in \mathbb{L}^p(\Omega) \text{ such that } \int_\Omega u\frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_i\varphi,\ \forall \varphi \in D(\Omega),\ i = 1,2,3\right\}.$$

for $u \in W^{1.p}(\Omega)$ we have

$$\frac{\partial u}{\partial x_i} = g_i \text{ and } \nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}\right).$$

The space $W^{1,p}(\Omega)$ *is equipped with the norm*

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \Sigma_{i=1}^{3}\|\frac{\partial u}{\partial x_i}\|_{L^p(\Omega)}$$

Proposition 2.1.2. *The space* $W^{1,p}(\Omega)$ *is Banach space for* $1 \leq p \leq \infty$*. It is reflxive for* $1 < p < \infty$*, and separable for* $1 \leq p < \infty$*.*

Definition 2.1.5. *For* $1 \leq p < \infty$*, the space* $W_0^{1,p}(\Omega)$ *desing the closure of* $C_c^1(\Omega)$ *in* $W^{1,p}$ *The space* $W_0^{1,p}(\Omega)$ *equipped the same norm of* $W^{1,p}(\Omega)$ *and is separable, Banach space and it reflxive for* $1 < p < \infty$*.*

Definition 2.1.6. *The dual space of* $W_0^{1,p}(\Omega)$ *is denoted by* $W^{-1,p}$*.*

See [8, 9] for more details about $\mathbb{L}^p$ and Sobolev's spaces.

2.1.3 Functional spaces in solid mechanics

In studying mechanical problems, we frequently use function spaces involving the deformation and divergence operators. We therefore introduce the following Hilbert spaces, associated with the mechanical unknowns u and σ

$$\begin{aligned} H &= \{u = (u_i) \text{ such that } u_i \in \mathbb{L}^2(\Omega)\} = \left(\mathbb{L}^2(\Omega)\right)^d, \\ \mathcal{H} &= \{\sigma = (\sigma_{ij}) \text{ such that } \sigma_{ij} = \sigma_{ji} \in \mathbb{L}^2(\Omega)\} = \left(\mathbb{L}^2(\Omega)\right)_s^{d\times d}, \\ H_1 &= \{u = (u_i) \text{ such that } u_i \in H^1(\Omega)\} = H^1(\Omega)^d, \\ \mathcal{H}_1 &= \{\sigma \in \mathcal{H} \text{ such that } \sigma_{ij,j} \in H\}. \end{aligned}$$

The spaces $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with canonical inner products given respectively by

$$\begin{aligned} (u,v)_H &= \int_\Omega u_i.v_i dx, \\ (\sigma,\tau)_{\mathcal{H}} &= \int_\Omega \sigma_{ij}.\tau_{ij} dx, \\ (u,v)_{H_1} &= (u,v)_H + (\varepsilon(u),\varepsilon(v))_{\mathcal{H}}, \\ (\sigma,\tau)_{\mathcal{H}_1} &= (\sigma,\tau)_{\mathcal{H}} + (Div\sigma, Div\tau)_H, \end{aligned}$$

where $\varepsilon : H_1 \to \mathcal{H}$ and $Div : \mathcal{H}_1 \to H$ are respectively the deformation and the divergence operators, defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2}\ (u_{i,j} + u_{j,i}), \quad Div\ \sigma = (\ \sigma_{ij,j}).$$

Let X be a Banach space, we denote by $\|\ .\ \|_X$ the associated norm, in particular the norms associated with the spaces H, $\mathcal{H}, H_1$ and $\mathcal{H}_1$.

Since the Γ boundary is Lipschitzian, the exterior normal vector to the boundary is defined *a.e.* For any vector field $v \in H_1$ we use the notation v to designate the γv trace of v on Γ. Recall that the trace map $\gamma : H_1 \to H_\Gamma$ is linear and continuous, but not surjective. The image of H_1 by this map is denoted by H_Γ, and this subspace is continuously injected into $\mathbb{L}^2(\Gamma)^d$. We denote by H'_Γ the dual of H_Γ, and $(.,.)$ the product of duality between H'_Γ and H_Γ. $\sigma\nu \in \mathcal{H}_1$, there exists an element $\sigma\nu \in H'_\Gamma$ such that

$$(\sigma\nu, \gamma v)_{H'_\Gamma \times H_\Gamma} = (\sigma, \varepsilon(v))_{\mathcal{H}} + (Div\ \sigma, v)_H\ , \forall v \in H_1.$$

Furthermore, if σ is sufficiently regular (for example C^1), we have the formula

$$(\sigma\nu, \gamma v) = \int_\Gamma \sigma\nu.vda\ , \forall v \in H_1.$$

So, for a sufficiently regular σ we have the following formula (Green's formula)

$$(\sigma, \varepsilon(v))_{\mathcal{H}} + (Div\sigma, v)_H = \int_\Gamma \sigma\nu.vda, \quad \forall v \in H_1. \tag{2.1.1}$$

We define the closed subspaces of $\mathbb{L}^2(\Omega)$ and H^1

$$Y = \left\{v \in \mathbb{L}^2(\Omega)/\varepsilon_{ij}(v) \in \mathbb{L}^2(\Omega)\right\} = H^1(\Omega), \quad V = \{v \in H_1\ /v = 0 \text{ on } \Gamma_1\}.$$

Since $\Gamma_1 > 0$, Korn's inequality applies on V, then, there exists a constant $C_K > 0$ depending only on Ω and Γ_1 such that

$$|\varepsilon(v)|_{\mathcal{H}} \geq C_K\, |v|_{H_1}\,, \quad \forall v \in H_1.$$

On the space V, we consider the scalar product given by

$$(u,v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \quad \forall u, v \in V, \tag{2.1.2}$$

and let $|.|_V$ be the associated norm

$$|v|_V = |\varepsilon(v)|_{\mathcal{H}}, \ \forall v \in V. \tag{2.1.3}$$

By Korn's inequality,it follows that $|.|_{H^1(\Omega)}$ and $|.|_V$ are equivalent norms on V and therefore $(V, |.|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant $\widetilde{c}_0$, depending only on Ω, Γ_1 and Γ_3 such that

$$|v|_{L^2(\Gamma_3)} \leq \widetilde{c}_0 \, |v|_W \quad \forall v \in V. \tag{2.1.4}$$

Furthermore, the dual space of V is denoted by V' and that of Y by Y'.

We use the notations $(.,.)_{V' \times V}$ and $(.,.)_{Y' \times Y}$ to represent the duality between V', V and Y', Y respectively.

In the following, we define the Sobolev spaces associated with the electrical unknowns (electric displacement field D and electric potential φ) of the electro-mechanical problems that will be introduced in the last chapter of the this dissertation.

Let the spaces

$$\begin{aligned} \mathcal{W} &= \{D = D_i \text{ such that } D_i \in \mathbb{L}^2, \ divD \in \mathbb{L}^2(\Omega)\}, \\ W &= \{\zeta \in H^1(\Omega) \text{ such that } \zeta = 0 \text{ on } \Gamma_a\}, \end{aligned}$$

where $divD = (D_{i,i})$. These spaces $\mathcal{W}$ and W are real Hilbert spaces with the scalar products given by

$$(D, E)_{\mathcal{W}} = \int_\Omega D_i E_i dx, \quad (D, E)_{\mathcal{W}_1} = (D, E)_{\mathcal{W}} + (divD, divE)_{\mathbb{L}^2(\Omega)}.$$

And their associated norm $|.|_{\mathcal{W}}$ qnd $|.|_{\mathcal{W}_1}$, respectively

$$|D|^2_{\mathcal{W}} = |D|^2_{\mathbb{L}^2(\Omega)^d} + |divD|^2_{\mathbb{L}^2(\Omega)}, \quad |\varphi|^2_W = |\bigtriangledown \varphi|^2_{\mathbb{L}^2(\Omega)^d}.$$

Since $meas\ (\Gamma_a) > 0$, the following Friedrichs-Poincaré's inequality holds, thus

$$|\nabla\psi|_{\mathcal{W}} \geq c_F\, |\psi|_{H^1(\Omega)} \quad \forall\psi \in W, \tag{2.1.5}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a. On W, we use the inner product given by

$$(\varphi, \psi)_W = (\nabla\varphi, \nabla\psi)_{\mathcal{W}}\,,$$

and let $|.|_W$ be the associated norm. It follows from (2.1.5) that $|.|_{H^1(\Omega)}$ and $|.|_W$ are equivalent norms on W and therefore $(W, |.|_W)$ is a real Hilbert space.

Moreover, by the Sobolev trace theorem, there exists a constant $\widetilde{c}_0$, depending only on Ω, Γ_a and Γ_3 such that

$$|\psi|_{L^2(\Gamma_3)} \leq \widetilde{c}_0\, |\psi|_W \quad \forall\psi \in W. \tag{2.1.6}$$

We recall that when $D \in \mathcal{W}_1$ is a sufficiently regular function, the Green's type formula holds

$$(D, \nabla\psi)_{\mathcal{W}} + (divD, \psi)_{\mathbb{L}^2(\Omega)} = \int_\Gamma D\nu.\psi da. \tag{2.1.7}$$

We introduce the real Hilbert space of the temperature denoted by

$$E = \left\{\gamma \in H^1(\Omega)\,/\gamma = 0 \text{ in } \Gamma_1 \cup \Gamma_2\right\}. \tag{2.1.8}$$

And we consider the inner product and the corresponding norm given by

$$(\theta, \gamma)_E = (\theta, \gamma)_{H^1(\Omega)}, \qquad |\gamma|_E = |\gamma|_{H^1(\Omega)} \quad \forall\theta, \gamma \in E.$$

By Sobolev's trace theorem, there exists a constant $C_1 > 0$ which depends only on Ω and Γ such that

$$|\gamma|_{\mathbb{L}^2(\Gamma_3)} \leq |\gamma|_E \qquad \forall\gamma \in E. \tag{2.1.9}$$

The following Friedrichs-Poincaré inequality holds on E is

$$|\bigtriangledown\gamma|_{\mathbb{L}^2(\Omega)^d} \geq \tilde{C}_F|\gamma|_E \qquad \forall\gamma \in E. \tag{2.1.10}$$

$\mathbb{L}^2(\Omega)$ is identified with its dual and with a subspace of the dual E' of E, i-e, $E \subset \mathbb{L}^2(\Omega) \subset$

E' and we say that the inclusions above define a Gelfand triple.

2.1.4 Vector-valued functions spaces

We shall mention the spaces of vector valued functions for studying time-dependent variational problems.

Let $0 < T < \infty$ and let $(X; |.|_X)$ be a real Banach space. We denote by $C_c(0,T;X)$ the set of continuous functions with compact support in $[0,T]$ with values in X.

Definition 2.1.7. *A function $f : [0,T] \longrightarrow X$ is said to be measurable if there exists a subset $E \subset [0,T]$ of measure zero and a sequence $(f_n)_{n\in\mathbb{N}}$, of functions belonging to $C_c(0,T;X)$ such that $|f_n(t) - f(t)|_X \longrightarrow 0$ when $n \longrightarrow \infty$, for all $t \in [0,T]\backslash E$.*

Definition 2.1.8. *A function $f : [0,T] \longrightarrow X$ is said to be strongly derivable at t_0 if there exists an element $\dfrac{df}{dt}(t_0) \in X$ called the strong derivative of f at t_0, such that*

$$\lim_{h\to 0} |\frac{1}{h}(f(t_0+h)) - f(t_0) - \frac{df}{dt}(t_0)|_X = 0,$$

where $t_0,\ t_0+h \in [0,T]$.

Definition 2.1.9. *A function $f : [0,T] \longrightarrow X$ is said to be integrable if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of functions belonging to $C_c(0,T;X)$ such that*

$$\lim_{n\to\infty} \int_0^T |f_n(t) - f(t)|_X \, \mathrm{d}t = 0.$$

Theorem 2.1.5. ***(Bochni)*** *A function $f : [0,T] \longrightarrow X$ is said to be measurable and integrable if and only if $x \longrightarrow |f(x)|_X : [0,T] \longrightarrow \mathbb{R}_+$ is integrable. In this case, we have*

$$|\int_0^T f dt| \leq \int_0^T |f|_X dt.$$

$\mathbf{C(0,T;X)}$ spaces Let T > 0. We denote by $C(0,T;X)$ the space of continuous functions defined on $[0,T]$ with values in X. It is well known that $C(0,T;X)$ is a Banach space with the norm

$$\| x \|_{C(0,T;X)} = \max_{t\in[0,T]} \| x(t) \|_X .$$

We also denote by $C^1(0,T;X)$ the spaces of continuously differentiable functions on $[0,T]$ with values in X, and we recall that this is a Banach space with the norm

$$\| x \|_{C^1(0,T;X)} = \max_{t\in[0,T]} \| x(t) \|_X + \max_{t\in[0,T]} \| \dot{x}(t) \|_X .$$

$\mathbb{L}^p(0,T;X)$ spaces Let $1 \leq p \leq \infty$. The Lebesgue space $\mathbb{L}^p(0,T;X)$ is the set of classes of measurable functions $f:[0,T] \to X$ such that the application $t \to |f|_X$ belongs to $\mathbb{L}^p[0,T]$. It is known that $\mathbb{L}^p(0,T;X)$ is a normed vector space with the norm given by

$$\| f \|_{\mathbb{L}^p(0.T,X)} = \left(\int_0^T |f|_X^p \, dt \right)^{\frac{1}{p}} \qquad \text{if } 1 \leq p \leq \infty,$$

on the other hand, we have the following results.

Proposition 2.1.3. (1) $\mathbb{L}^p(0,T;X)\,(1 \leq p \leq \infty)$ *is a Banach space.*
(2) $\mathbb{L}^r(0,T;X) \subset \mathbb{L}^q(0,T;X)$ *with continuous injection* $1 \leq q \leq r \leq \infty$.
(3) *if X is a Hilbert space, then*

$$\begin{aligned} &\mathbb{L}^p(0,T;X)' = \mathbb{L}^q(0,T;X), \qquad \textit{if } 1 < q,p < \infty, \ \tfrac{1}{p} + \tfrac{1}{q} = 1, \\ &\mathbb{L}^1(0,T;X)' = \mathbb{L}^\infty(0,T;X), \end{aligned}$$

where $\mathbb{L}^p(0,T;X)'$ *represents the dual of space* $\mathbb{L}^p(0,T;X)$, $1 \leq p \leq \infty$.

$\mathbf{W}^{k.p}(0,T;X)$ spaces Given an integer $k \geq 2$ and a real $1 \leq p \leq \infty$, we define by recurrence the space

$$W^{k.p}(0,T;X) = \left\{ u \in \mathbb{L}^p(0,T;X), u^{(s)} \in \mathbb{L}^p(0,T;X), s \leq k \right\},$$

it is easily verified that $u \in W^{k.p}(0,T;X)$ iff there exist k functions $g_1,...,g_k \in \mathbb{L}^\infty(0,T;X)$ such that

$$\int_0^T u(t)\varphi^{(j)}(t)\,dt = (-1)^j \int_0^T g_i(t)\varphi(t)\,dt, \ \forall\varphi \in C_c^\infty(I), \forall j = 1,2,...,k,$$

where $\varphi^{(j)}$ denotes the j^{th} order derivative of φ. We can consider the successive derivatives $\dot{u} = g_1, u^{(2)} = g_2, ..., u^{(k)} = g_k$.

The space $W^{k.p}(0,T;X)$ is a Banach space equipped with the norm

$$\| u \|_{W^{k.p}(0,T;X)} = \| u \|_{\mathbb{L}^p(0,T;X)} + \sum_{\alpha=1}^{k} \| u^{(\alpha)} \|_{\mathbb{L}^p(0,T;X)} .$$

More details can be found in [8], [38].

2.2 Gelfand triple

In this section we recall the definition of a Gelfand triplet. To do this, we will start with a reminder of the Riesz-Fréchet representation theorem.

Theorem 2.2.1. *Let H be a Hilbert space and let H' be its dual space. Then, for any $\varphi \in H'$, there exists $f \in H$ unique such that*

$$(\varphi, v)_{H' \times H} = (f, v), \forall v \in H.$$

Moreover,

$$\| \varphi \|_{H'} = \| f \|_H .$$

The importance of this theorem is that any continuous linear form on H can be represented using the dot product. The mapping $\varphi \longrightarrow f$ is an isometric isomorphism that allows the identification of H and H'.

Now let H be a real Hilbert space such that V dense in H and the injection $V \subset H$ is continuous. We identify H and H'. Let V' be the dual of V. We can then extend H in V' using the following procedure: given $f \in H$, the application $v \in V \longrightarrow (f, v)_H$ is a continuous linear form on H and a fortiori on V, it is denoted $Tf \in V'$ so that

$$(Tf, v)_{H' \times H} = (f, v), \forall f \in H, \forall v \in H.$$

We check that $T : H \longrightarrow V'$ has the following properties

- $\| Tf \|_{V'} \leq C \| f \|_H \ \forall f \in H,$
- T is injective,

- $T(H)$ is dense in V.

In general, T is not surjective of H on V'. By means of T we extend H into V' and we have the diagram

$$V \subset H = H' \subset V', \tag{2.2.1}$$

where the canonical injections are continuous and dense. This triplet is called the Gelfand triplet. We say that H is the pivot space.

2.3 Elements of nonlinear analysis

In this section, we recall some elements of analysis in Hilbert spaces and some results concerning first-order nonlinear evolution variational inequalities and first-order parabolic evolution variational equations involved in the study of mechanical problems.

2.3.1 Linear and nonlinear operators

We begin this section with a brief reminder of the strongly monotone and Lipschitz operators. For this purpose, we consider a Hilbert space V with the scalar product $(.,.)_V$ and the associated norm $|.|_V$ and $V\ '$ the dual space of V, denoted by $(.,.)_{V'\times V}$ the duality product between V and V'.

Linear continuous operators Recall that an operator $L : X \longrightarrow Y$ is linear if

$$L(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 L(v_1) + \alpha_2 L(v_2) \ \ \forall v_1, v_2 \in X, \ \alpha_1, \alpha_2 \in \mathbb{R}.$$

A linear operator is continuous iff it is bounded, i.e., there exists $M > 0$ such that

$$\| L(v) \|_Y \leq M \| v \|_X \ \ \forall v \in X.$$

Let X and Y be linear spaces a mapping a: $X \times X \longrightarrow \mathbb{R}$ is called a bilinear form if it is linear in each argument that is for any $u_1, u_2, u \in X$, $v_1, v_2, v \in Y$ and $\alpha_1, \alpha \in \mathbb{R}$,

$$a(\alpha_1 u_1 + \alpha_2 u_2, v) = \alpha_1 a(u_1, v) + \alpha_2 a(u_2, v)$$

$$a(u, \alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 a(u, v_1) + \alpha_2 a(u, v_2)$$

Definition 2.3.1. *Let X be Hilbert space. A scalar product, a(u,v) is a bilinear from $X \times X$ with values in $\mathbb{R}$ such that*

$$a(u, v) = a(v, u), \forall u, v \in X \textit{(symmetric)},$$

$$a(u, v) \geq 0, \forall u \in X \textit{(positive)},$$

$$a(u, v) \neq 0, \forall u \neq 0 \ \textit{(definite)}.$$

A bilinear form a is said to be

Continuous: if there exists a constant $C > 0$ such that

$$|a(u, v)| \leq C|u||v|, \ \forall u, v \in X.$$

Coercive: if there exists a constant $\gamma > 0$ such that

$$|a(u, u)| \geq \gamma |u|^2, \ \forall u, v \in X.$$

Non linear operators An important class of nonlinear operators defined in a Hilbert space is provided by the following result.

Definition 2.3.2. *The operator $A : V \to V'$ is said to be*

(a) *monotome if*

$$(Au - Av, u - v)_{V' \times V} \geq 0, \ \forall u, v \in V,$$

(b) *strictly monotone if*

$$(Au - Av, u - v)_{V' \times V} > 0, \ \forall u, v \in V,$$

(c) *strongly monotone if there exists a constant $m > 0$ such that*

$$(Au - Av, u - v)_{V' \times V} \geq m \left| u - v \right|_V^2, \forall u, v \in V,$$

(d) *Lipschitz continuous if there exists* $L_A > 0$

$$|Au - Av|_{V'} \leq L_A |u - v|_V , \quad \forall u, v \in V,$$

(e) *hemicontinuous if, for any numerical sequence* $(\lambda_n) \subset \mathbb{R}$ *such that* $\lambda_n \to \lambda$, *when* $n \to +\infty$ *we have*

$$(A(u + \lambda_n v), w)_{V' \times V} \to (A(u + \lambda v), w)_{V' \times V} \ \text{ when } n \to +\infty. \ \forall w \in V.$$

Using the previous definition, we have the following result.

Proposition 2.3.1. *Every Lipschitz operator is hemicontinuous.*

Corollary 2.3.1. *Let* X *be a Hilbert space equipped with the inner product* $(.,.)_X$ *and the associated norm* $|.|_X$. *Let* $A : X \to X$ *an operator that is strongly monotone and Lipschitz. Then, for every* $f \in X$, *there exists a unique element* $u \in X$ *such that* $Au = f$.

2.3.2 Convex lower semicontinuous function

Convex lower semicontinuous functions represent a crucial ingredient in the study of variational inequalities. To introduce them, we give some following definitions.

-Convex function

Let X be a linear space. In the study of convex functions, it is convenient to consider functions that take on values on the extended real line $\bar{\mathbb{R}} = [-\infty, +\infty]$.

Definition 2.3.3. *Let* $f : X \longrightarrow \bar{\mathbb{R}}$. *The function f is said to be proper if* $f(v) > -\infty$ *for all* $v \in X$ *and* $f(u) < \infty$ *for some* $u \in X$. *The function* $f : X \longrightarrow \bar{\mathbb{R}}$ *is convex if*

$$f(tu + (1-t)v) \leq tf(u) + (1-t)f(v), \quad \forall u, v \in X, \ t \in]0,1[.$$

The function f is strictly convex if the later inequality is strict for $u \neq v$ *and all* $t \in]0,1[$

-Lower semi-continuity

Definition 2.3.4. *A function* $f : X \to]-\infty, +\infty]$ *is said to be lower semicontinuous written* (**l.s.c.**) *at* $u \in X$ *if*

$$\lim_{u \to u_0} \inf f(u) \geq f(u_0).$$

For each sequence $\{u_n\} \subset X$ *converging to* u *in* X*. The function* f *is l.s.c. on a subset of* X *if it is l.s.c. at each point of the subset. We say that* f *is l.s.c. if it is l.s.c. on* X*.*

2.4 Evolutionary variational inequalities

In this section, an extension of the existence and uniqueness results for elliptic and parabolic variational inequalities, as well as for ordinary differential equations in abstract spaces. We use these results when we study contact problems in the following chapters of this thesis.

2.4.1 Elliptic variational inequality of first and second kind

Theorem 2.4.1. *Let* $(V, \|\ .\ \|)$ *be a reflexive real Banach space with its dual* $(V', \|\ .\ \|)$ *and* $K \subset V$ *a non-empty convex closed set. Then, for each* $f \in V$ *there exists a unique solution to the elliptic variational inequality of the first kind,*

$$u \in K, \qquad (Au, v-u)_V \geq (f, v-u)_V\,, \quad \forall v \in K. \tag{2.4.1}$$

Given a set $K \subset X$, an operator $A : K \longrightarrow X$, a function $j : K \longrightarrow \mathbb{R}$ and an element $f \in X$, in this section we consider the problem of finding an element u such that

$$(Au, v-u)_X + j(v) - j(u) \geq (f, v-u)_X\,, \quad \forall v \in K. \tag{2.4.2}$$

A variational inequality of the form (2.4.2) is called an elliptic variational inequality of the second kind.
In the study of (2.4.2) we consider the following assumptions

(a) K is a nonempty closed convex subset of X,

(b) $j : K \longrightarrow R$ is a convex l.s.c. function,

(c) $A : K \longrightarrow X$ is a strongly monotone Lipschitz continuous operator,
i.e. it satisfies those conditions ,

$$\begin{cases} \exists\alpha > 0 \text{ such that } (Au - Av, u - v)_X \geq \alpha\, |u - v|_X^2\,, \quad \forall u, v \in X, \\ \forall u, v \in X, \text{l'application } t \in [0,1] \to (A\,(1-t)\,u + tv, u - v)_X\,, \\ \text{ is continuous.} \end{cases}$$

The main result of this section is the following.

Theorem 2.4.2. *Let X be a Hilbert space and assume that $(a) - (c)$ hold. Then, for each $f \in X$ the variational inequality* (2.4.2) *has a unique solution.*

The proof of this theorem will be carried out in two steps you can find it in [38].

2.4.2 Parabolic variational inequalities.

Theorem 2.4.3. *Let $V \subset H \subset V'$ be a Gelfand triple. Let K be a nonempty, closed, and convex set of V. Assume that $a(.,.) : V \times V \longrightarrow \mathbb{R}$ is a continuous and symmetric bilinear form such that for some constants $\alpha > 0$ and c_0,*

$$a(v, v) + c_0|v|_H^2 \geq \alpha|v|_V^2, \quad \forall v \in V. \tag{2.4.3}$$

Then, for every $u_0 \in K$ and $f \in \mathbb{L}^2(0, T; H)$, there exists a unique function $u \in H^1(0, T; H) \cap \mathbb{L}^2(0, T; V)$ such that $u(0) = u_0, u(t) \in K$ for all $t \in [0, T]$, and for almost all $t \in [0, T]$,

$$(\dot{u}(t), v - u(t))_{V' \times V} + a(u(t), v - u(t)) \geq (f(t), v - u(t))_H \tag{2.4.4}$$

2.4.3 Ordinary differential equations in abstract spaces

Theorem 2.4.4. *Let $V \subset H \subset V'$ a Gelfand triple, we consider an operator A monoton and hemicontinus which verify the properties*

$$i) (Av, v)_{V' \times V} \geq \omega\, |v|_V^2 + \lambda\ , \forall v \in V,$$
$$ii)\, |Av|_{V'} \leq (C_1\, |v|_V + 1), \ \forall v \in V.$$

where $\omega > 0$, $C_1 > 0$ are constants and $\lambda \in \mathbb{R}$, $u_0 \in H$ and $f \in \mathbb{L}^2(0, T; V')$. Then there exists a unique function u such that

$$u \in \mathbb{L}^2(0.T, V) \cap C(0.T, H) \text{ and } \dot{u} \in \mathbb{L}^2(0, T; V').$$

$$\begin{cases} \dot{u}(t) + Au(t) = f(t), \ a.e \ t \in [0, T], \\ u(0) = u_0. \end{cases}$$

Theorem 2.4.5. *Let H be a Hilbert space with the scalar product $(.,.)$ and the norm $\|\,.\,\|$. We identify H and its dual. Let V be another Hilbert space of norm $|.|$. Assume that*

$$V \subset H \subset V'.$$

Let $T > 0$ be fixed, for almost any $t \in [0, T]$ we give a bilinear form $a(t; u, v) : V \times V \to \mathbb{R}$ which verify the properties

$$\begin{aligned} &i) \text{ the function } t \mapsto a(t; u, v) \text{ is measurable } \forall u, v \in V, \\ &ii) \ a(t; u, v) \leq M |u| |v| \ a.e. \ t \in [0, T], \forall u, v \in V, \\ &iii) \ a(t; v, v) > \alpha \|v\|^2 - C |v|^2, \ \forall v \in V, \ a.e. \ t \in [0, T]. \end{aligned} \tag{2.4.5}$$

where $\alpha > 0$, M and C are constants, $u_0 \in H$ and $f \in \mathbb{L}^2(0, T; V')$. Then there exists a unique function u such that

$$u \in \mathbb{L}^2(0, T; V) \cap C(0, T; H) \text{ and } \frac{du}{dt} \in \mathbb{L}^2(0, T; V'). \tag{2.4.6}$$

$$\begin{cases} \left(\frac{du}{dt}(t), v\right) + a(t; u(t), v) = (f(t), v), \ a.e \ t \in [0, T], \quad \forall v \in V, \\ u(0) = u_0. \end{cases} \tag{2.4.7}$$

Theorem 2.4.6. *$(X, |.|_X)$ be a real Banach space and let $F(t,.) : X \to X$ be an operator defined a.e. on $[0, T]$ which satisfies the following properties*

$$\begin{cases} \text{there exists } L_F > 0 \text{ such that} \\ |F(t, x) - F(t, y)|_X \leq L_F |x - y|_X, \ \forall x, y \in X, \ a.e. \ t \in [0.T], \\ \text{there exists } 1 \leq p \leq \infty \text{ such that } F(., x) \in \mathbb{L}^p(0, T; X) \ \forall x \in X. \end{cases} \tag{2.4.8}$$

Then, for all $x_0 \in X$, there exists a unique function $x \in W^{1.p}(0, T; X)$ such that

$$\begin{cases} \dot{x}(t) = F(t, x(t)), \quad a.e. \ t \in [0.T], \\ x(0) = 0. \end{cases} \tag{2.4.9}$$

2.5 Miscellaneous complements

2.5.1 Banach fixed point theorem

Banach fixed point theorem will be used later in this thesis to prove the existence and uniqueness of the solution. Let X be a Banach space equipped with the norm $|.|_X, K \subset X$ a part of X and let $\Lambda : K \longrightarrow K$ an operator defined on K we are interested in the existence of a solution of the equation

$$\Lambda(u) = u, \ u \in K.$$

A solution of $\Lambda(u) = u$ is called a fixed point of Λ in K.

Theorem 2.5.1. ***(The Banach fixed point)*** *Let K be non-empty closed subset of a Banach space $(X, \| \ . \ \|)_X$. Assume that $\Lambda : K \longrightarrow K$ is a contraction, i.e, $\exists K \in]0,1[$ such that*

$$|\Lambda u - \Lambda v|_X \leq K|u - v|_X, \ \forall u, v \in K.$$

Then there exists a unique element $u \in K$ such that $\Lambda(u) = u$, i.e, Λ has a unique fixed point in K.

Recall that the powers of the operator are defined recursively by

$$\Lambda^n = \Lambda(\Lambda^{n-1}) \text{ for } n \geq 2.$$

Theorem 2.5.2. *Assume that K is a nonempty closed subset of a Banach space X, and that $\Lambda : K \longrightarrow K$. Suppose that Λ^n is a contraction mapping for a positive integer $n \geq 2$, then Λ has a unique fixed point in K.*

The proofs of those theorems are found in [38].

2.5.2 Subdifferentiability

Definition 2.5.1. *Let $(X, (\cdot, \cdot)_X)$ be an inner product space, $\varphi : X \to \mathbb{R}$ and $u \in X$. Then φ is* Gâteaux differentiable *at u if there exists an element $\nabla\varphi(u) \in X$ such that*

$$\lim_{t \to 0} \frac{\varphi(u + tv) - \varphi(u)}{t} = (\nabla\varphi(u), v)_X \quad \forall v \in X. \tag{2.5.1}$$

The element $\nabla\varphi(u)$ *which satisfies* (2.5.1) *is unique and is called the* gradient *of* φ *at* u*. The function* $\varphi : X \to \mathbb{R}$ *is said to be* Gâteaux differentiable *if it is Gâteaux differentiable at every point of* X*. In this case the operator* $\nabla\varphi : X \to X$ *which maps every element* $u \in X$ *into the element* $\nabla\varphi(u)$ *is called the* gradient operator *of* φ*.*

Proposition 2.5.1. *Let* $(X, (\cdot,\cdot)_X)$ *be an inner product space and let* $\varphi : X \to \mathbb{R}$ *be a Gâteaux differentiable function. Then the following statements are equivalent:*

(i) φ *is a convex function;*

(ii) φ *satisfies the inequality*

$$\varphi(v) - \varphi(u) \geq (\nabla\varphi(u), v - u)_X \quad \forall u, v \in X,$$

(iii) the gradient of φ *is a monotone operator, that is*

$$(\nabla\varphi(u) - \nabla\varphi(v), u - v)_X \geq 0 \quad \forall u, v \in X.$$

Definition 2.5.2. *Let* $K \subset X$*. We call indicator function of* K*, the function* χ_K *defined by*

$$\chi_K = \begin{cases} 0 & \text{if} \quad x \in K, \\ +\infty & \text{if} \quad x \notin K. \end{cases}$$

It is important to introduce concepts such as sub-differentiability and the sub-gradient of a function. The notion of sub-differentiability frequently used in mechanics, particularly in contact mechanics.

Definition 2.5.3. *Let* $j : X \longrightarrow \mathbb{R}$ *be a function and* u *an element of* X *such that* $j(u) \neq \pm\infty$*. The sub-differential of* j *in* u *denoted* $\partial j(u)$ *is the set of* X' *defined by*

$$\partial j(u) = \{u' \in X' : j(v) \geq j(u) + (u', v - u)_{X' \times X}, \forall v \in X\}. \tag{2.5.2}$$

Every element u' *of the set* $\partial j(u)$ *is called a sub-gradient of* j *at* u*. The function* j *is said to be sub-differentiable at* u *if* $\partial j(u) \neq \emptyset$ *and it is said to be sub-differentiable if it is so at every point* $u \in X$*.*

In the case of a Hilbert space X, the sub-differential of j at u can also be written as

$$\partial j(u) = \{u' \in X : j(v) \geq j(u) + (u', v-u)_X, \forall v \in X\}. \tag{2.5.3}$$

Now we assume that K is a non-empty convex set.

We consider the indicator function χ_K of the set K. If $u \notin K$, then $\partial\chi_K(u) = \emptyset$. Suppose that $u \in K$. It follows that if $u' \in \partial\chi_K$, then

$$(u', v-u)_{X'\times X} \leq 0, \quad \forall v \in K.$$

We can thus characterize the sub-differential $\partial\chi_K$ of an indicator function χ_K of a non-empty convex set

$$\partial\chi_K = \{u' \in X' : (u', v-u)_{X'\times X} \leq 0, \ \forall v \in X\}. \tag{2.5.4}$$

2.5.3 Some further inequalities

Let us mention some very useful inequalities involving the Sobolev norms

Lemma 2.5.1. ***(Gronwall inequality)*** *Let $m, n \in C(0,T;\mathbb{R})$ such that $m(t) \geq 0$ and $n(t) \geq 0$ for all $t \in [0,T]$, $a \geq 0$ a constant and $\phi \in C(0,T;\mathbb{R})$ is a function.*

1. if

$$\phi(t) \leq a + \int_0^t m(s)\,ds + \int_0^t n(s)\,\phi(s)\,ds, \quad \forall\ t \in [0,T],$$

so

$$\phi(t) \leq \left(a + \int_0^t m(s)\,ds\right) \exp\left(\int_0^t n(s)\,ds\right), \quad \forall\ t \in [0,T].$$

2. if

$$\phi(t) \leq m(t) + a.\int_0^t \phi(s)\,ds, \quad \forall\ t \in [0,T],$$

so

$$\int_0^t \phi(s)\,ds \leq e^{aT}\int_0^t m(s)\,ds, \quad \forall\ t \in [0,T].$$

For the particular case $m = 0$, part (1) of this lemma becomes

Let $n \in C(0,T;\mathbb{R})$ be such that $n(t) \geq 0$ for all $t \in [0,T]$, $a \geq 0$ be a constant, and $\phi \in C(0,T;\mathbb{R})$ be a function such that

Corollary 2.5.1. *Let $n \in C(0,T;\mathbb{R})$ such that $n(t) \geq 0$ for all $t \in [0,T]$, $a \geq 0$ a constant and $\phi \in C(0,T;\mathbb{R})$ a function such that*

$$\phi(t) \leq a + \int_0^t n(s)\,\phi(s)\,ds,\ \forall t \in [0,T],$$

so

$$\phi(t) \leq a \exp\left(\int_0^t n(s)\,ds\right),\ \forall t \in [0,T].$$

The corollary 2.5.1 is often used to prove the uniqueness of the solution in the following way. One assumes there are two solutions, denoted by ϕ as the norm of the difference between these solutions, and then tries to major ϕ under the form

$$\phi(t) \leq \int_0^t n(s)\,\phi(s)\,ds,\ \forall t \in [0,T],$$

with a certain function $n \geq 0$. The corollary 2.5.1 immediately gives the nullity of ϕ.

Theorem 2.5.3. ***(Hölder inequality)*** *If $p \in]1,+\infty[$, the conjugate exponent of p is the only one $q \in]1,+\infty[$ such as $\frac{1}{p}+\frac{1}{q}=1$, for any $u \in \mathbb{L}^p$ and $v \in \mathbb{L}^q$, we have*

$$uv \in \mathbb{L}^1(\Omega) \text{ and } \| uv \|_{\mathbb{L}^1(\Omega)} \leq \| u \|_{\mathbb{L}^p(\Omega)} . \| v \|_{\mathbb{L}^q(\Omega)} .$$

In particular, for $p = q = 2$ this reduces to the Cauchy–Schwarz inequality.

Theorem 2.5.4. ***(Young inequality)*** *Let $p > 1, q < \infty$ satisfy $\frac{1}{p}+\frac{1}{q}=1$ then*

$$ab < \frac{a^p}{p} + \frac{b^q}{q},\quad \forall (a,b) \in \mathbb{R}_+^2.$$

Proposition 2.5.2. ***(Poincaré inequality)*** *Let Ω a bounded open of $\mathbb{R}^d(\Omega)$ and let mes $\Gamma_a > 0$, then there is a constant $C_F > 0$ only depends on Ω and Γ_a such as for any function $\varphi \in W$*

$$\| \nabla\varphi \| \geqslant C_F \| \varphi \|_{H^1(\Omega)},\ \forall \varphi \in W, \quad \nabla\varphi = \frac{\partial \varphi_i}{\partial x_j}.$$

Chapter 3

A dynamic problem with wear involving thermo-viscoelastic materials with long memory

In this chapter, we consider a thermodynamic problem of a viscoelastic body with long memory. Bilateral contact with friction on a rigid moving base with wear. We write the mechanical problem and specify the appropriate assumptions on the data to obtain the variational formulation. Then we establish our result of existence and uniqueness of the weak solution. The proof is based on a theorem for the existence and uniqueness of the solution of first order nonlinear evolution inequalities variational equations of the first-order parabolic type, followed by a fixed-point argument. The contents of this chapter have been published in [11].

3.1 Problem statement

The problem studied in this chapter falls within the physical setting $n°2$ presented in the first chapter of the thesis and consequently we use the first mathematical model, for the study to be complete, let us specify that the behavior law is thermo-viscoelastic with long memory of the type $(1, 2, 9), (1, 2, 10)$ and the contact condition with friction and wear on the contact surface Γ_3 is specified in $(1, 5, 6)$. The classical model for this process is as follows

Problem P_1 : Find a displacement field $u:\Omega\times[0,T]\to\mathbb{R}^d$, a stress field $\sigma:\Omega\times[0,T]\to\mathbb{S}^d$, the temperature $\theta:\Omega\times[0,T]\to\mathbb{R}_+$ and the wear $\omega:\Gamma_3\times[0,T]\to\mathbb{R}_+$ such that

$$\sigma=\mathcal{A}(\varepsilon(u(t)))+\mathcal{G}(\varepsilon(\dot{u}(t)))+\int_0^t\mathcal{B}(t-s)\varepsilon(u(s))ds-\theta(t)\mathcal{M},\ \text{ in }\Omega\times[0,T], \tag{3.1.1}$$

$$\rho\ddot{u}=Div\ \sigma+f_0,\text{in }\Omega\times[0,T], \tag{3.1.2}$$

$$\dot{\theta}-div(K\nabla\theta)=-\mathcal{M}.\nabla\dot{u}+q,\ \text{ in }\Omega\times[0,T], \tag{3.1.3}$$

$$u=0,\text{on }\Gamma_1\times[0,T], \tag{3.1.4}$$

$$\sigma\nu=h,\text{on }\Gamma_2\times[0,T], \tag{3.1.5}$$

$$\begin{cases}\sigma_\nu=-\alpha\left|\dot{u}_\nu\right|,\ \ |\sigma_\tau|=-\mu\sigma_\nu,\\ \sigma_\tau=-\lambda\left(\dot{u}_\tau-v^*\right),\lambda\geq 0,\ \dot{\omega}=-kv^*\sigma_\nu,\ k>0.\text{ on }\Gamma_3\ \times[0,T],\end{cases} \tag{3.1.6}$$

$$-k_{ij}\frac{\partial\theta}{\partial\nu}\nu_j=k_e(\theta-\theta_R)-h_\tau(|\dot{u}_\tau|),\ \text{ on }\Gamma_3\times[0,T], \tag{3.1.7}$$

$$\theta=0,\ \text{ in }\Gamma_1\cup\Gamma_2\times[0,T], \tag{3.1.8}$$

$$u(0)=u_0,v(0)=v_0,\theta(0)=\theta_0,\omega(0)=\omega_0,\ \text{ in }\Omega. \tag{3.1.9}$$

where (3.1.1) is the thermo-visco-elastic constitutive law with long memory, we denote $\varepsilon(u),\mathcal{A},\mathcal{G}$ the linearized strain tensor, the elasticity operator, the viscosity nonlinear operator, respectively, (3.1.2) represents the equation of motion, where ρ represents the mass density, we mention that $Div\sigma$ is the divergence operator, (3.1.3) represents the evolution equation of the heat field, (3.1.4) and (3.1.5) are the displacement and traction boundary conditions, (3.1.6) describes the bilateral frictional contact with wear described above on the potential contact surface Γ_3, (3.1.7) is the boundary condition related to the temperature, derived from Fourier's condition, where k_{ij} are the components of the thermal conductivity tensor, v_j are the normal components of the outward unit normal v, k_e is the heat exchange coefficient and θ_R is the known temperature of the foundation and h is the tangential function. (3.1.8) represents the temperature boundary conditions. Finally, (3.1.9) represents the initial conditions.

3.2 Variational formulation

In what follows, we assume the following assumptions for the problem P_1.
The elasticity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} (a)\, \exists\ L_{\mathcal{A}} > 0 \text{ such that: } \left|\mathcal{A}\left(x, \varepsilon_1\right) - \mathcal{A}\left(x, \varepsilon_2\right)\right| \leq L_{\mathcal{A}} \left|\varepsilon_1 - \varepsilon_2\right| \\ \forall\ \varepsilon_1\, , \varepsilon_2 \in \mathbb{S}^d,\ a.\ e.\ x \in \Omega, \\ (b)\ \text{ The mapping } x \to \mathcal{A}\left(x, \varepsilon\right) \text{ is Lebesgue measurable in } \Omega \text{ for all } \varepsilon \in \mathbb{S}^d, \\ (c)\ \text{ The mapping } x \to \mathcal{A}\left(x, 0\right) \in \mathcal{H}. \end{cases} \tag{3.2.1}$$

The viscosity operator $\mathcal{G} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} (a)\ \exists\ L_{\mathcal{G}} > 0 : |\mathcal{G}(x, \varepsilon_1) - \mathcal{G}(x, \varepsilon_2)| \leq L_{\mathcal{G}} |\varepsilon_1 - \varepsilon_1|, \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d,\ a.e.\ x \in \Omega, \\ (b)\ \exists m_{\mathcal{G}} > 0 : \left(\mathcal{G}\left(x, \varepsilon_1\right) - \mathcal{G}\left(x, \varepsilon_2\right), \varepsilon_1 - \varepsilon_2\right) \geq m_{\mathcal{G}} \left|\varepsilon_1 - \varepsilon_2\right|^2, \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \\ (c)\ \text{the mapping } x \to \mathcal{G}\left(x, \varepsilon\right) \text{ is Lebesgue measurable in } \Omega \text{ for all } \varepsilon \in \mathbb{S}^d, \\ (d)\ \text{the mapping } x \mapsto \mathcal{G}(x, 0) \in\ \mathcal{H}. \end{cases} \tag{3.2.2}$$

The relaxation tensor $\mathcal{B} : [0, T] \times \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ such that $(t, x, \tau) \mapsto \left(\mathcal{B}_{ijkh}\left(t, x\right) \tau_{kh}\right)$ satisfies

$$\begin{cases} (a)\ \mathcal{B}_{ijkh} \in W^{1.\infty}(0, T; \mathbb{L}^\infty(\Omega)), \\ (b)\ \mathcal{B}\left(t\right) \sigma \cdot \tau = \sigma \cdot \mathcal{B}\left(t\right) \tau, \forall \sigma, \tau \in \mathbb{S}^d,\ a.e.t \in \left[0, T\right], a.e. \text{ in } \Omega. \end{cases} \tag{3.2.3}$$

The function $h_\tau : \Gamma_3\ \times\ \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfies

$$\begin{cases} (a)\ \exists\ \ L_\tau > 0\ :\ \ |h_\tau(x, r_1) -\ \ h_\tau(x, r_2)| \leq L_h |r_1 - r_2|, \quad \forall r_1, r_2 \in \mathbb{R}_+, \\ a.e.x \in \Gamma_3, \\ (b)\ \ x \longrightarrow\ h_\tau\left(x, r\right) \in \mathbb{L}^2(\Gamma_3)\ \text{ is Lebesgue measurable in } \Gamma_3, \forall r \in \mathbb{R}_+. \end{cases} \tag{3.2.4}$$

The mass density ρ satisfies

$$\rho \in \mathbb{L}^\infty(\Omega) \text{ there exists } \rho^* > 0 \text{ such that } \rho(x) \geq \rho^*, a.e.x \in \Omega. \tag{3.2.5}$$

The body forces, surface tractions, the densities of electric charges, and the functions α and μ satisfy

$$f_0 \in \mathbb{L}^2(0,T;H),\ h \in \mathbb{L}^2(0,T;\mathbb{L}^2(\Gamma_2)^d), \tag{3.2.6}$$

$$q \in W^{1.\infty}(0,T;\mathbb{L}^2(\Omega)),\ \theta_R \in W^{1.\infty}(0,T;\mathbb{L}^2(\Gamma_3)),\ k_e \in \mathbb{L}^\infty(\Omega,\mathbb{R}_+), \tag{3.2.7}$$

$$\mathcal{M} = (m_{ij}),\ m_{ij} = m_{ji} \in \mathbb{L}^\infty(\Omega), \tag{3.2.8}$$

$$\alpha \in \mathbb{L}^\infty(\Gamma_3), \alpha(x) \geq \alpha^* > 0, a.e. \text{ on } \Gamma_3, \tag{3.2.9}$$

$$\mu \in \mathbb{L}^\infty(\Gamma_3),\ \mu(x) > 0, a.e. \text{ on } \Gamma_3, \tag{3.2.10}$$

$$\begin{cases} K = (k_{i,j}), \quad k_{ij} = k_{ji} \in \mathbb{L}^\infty(\Omega), \\ \exists c_k > 0, \forall(\xi_i) \in \mathbb{R}^d,\ k_{ij}\xi_i\xi_j \geq c_k\xi_i\xi_j. \end{cases} \tag{3.2.11}$$

The initial data satisfies

$$u_0 \in V,\ u_1 \in \mathbb{L}^2(\Omega),\ \theta_0 \in \mathbb{L}^2(\Omega),\ \omega_0 \in \mathbb{L}^\infty(\Gamma_3). \tag{3.2.12}$$

We use a modified inner product on $H = \mathbb{L}^2(\Omega)^d$ given by

$$((u,v)) = (\rho u, v)_{\mathbb{L}^2(\Omega)^d}, \forall u.v \in H.$$

That is, it is weighted with ρ. We let H be with the associated norm

$$\|v\|_H = (\rho v, v)^{\frac{1}{2}}_{\mathbb{L}^2(\Omega)^d},\ \forall v \in H.$$

It follows from assumption (3.2.5) that $\|.\|_H$ and $|.|_H$ are equivalent norms on H, and also, the inclusion mapping of $(V, |.|_V)$ into $(H, \|.\|_H)$ is continuous and dense. We denote by V' the dual space of V. Identifying H with its own dual, we can write the Gelfand triple

$$V \subset H = H' \subset V'.$$

We use the notation $(.,.)_{V'\times V}$ to represent the duality pairing between V' and V. Then we have

$$(u,v)_{V'\times V} = ((u,v)),\ \forall u \in H, \forall v \in V.$$

We define the space

$$E = \{\gamma \in H^1(\Omega)/\gamma = 0 \text{ on } \Gamma_1 \cup \Gamma_2\}. \tag{3.2.13}$$

We define the function $f(t) \in V'$ by

$$(f(t), v)_V = \int_\Omega f_0(t) v dx + \int_{\Gamma_2} h(t) v da, \ \forall v \in V, \ t \in [0,T],$$

for all u, $v \in V$,, $\psi \in W$ and $t \in [0,T]$, and note that condition (3.2.6) implies that

$$f \in \mathbb{L}^2(0,T; V'). \tag{3.2.14}$$

We consider the wear functional $j : V \times V \to \mathbb{R}$,

$$j(u,v) = \int_{\Gamma_3} \alpha \left| u_\nu \right| (\mu \left| v_\tau - v^* \right|) da. \tag{3.2.15}$$

Finally, we consider $\phi : V \times V \to \mathbb{R}$,

$$\phi(u,v) = \int_{\Gamma_3} \alpha \left| u_\nu \right| v_\nu da, \forall v \in V. \tag{3.2.16}$$

We define $Q : [0,T] \to E^{'}$, $\ K : E \to E^{'}$ and $R : V \to E^{'}$ by

$$(Q(t), \mu)_{E' \times E} = \int_{\Gamma_3} k_e \theta_R(t) \mu ds + \int_\Omega q \mu dx, \ \forall \mu \in E \tag{3.2.17}$$

$$(K\tau, \mu)_{E' \times E} = \sum_{i,j=1}^{d} \int_\Omega k_{ij} \frac{\partial \tau}{\partial x_j} \frac{\partial \mu}{\partial x_i} dx + \int_{\Gamma_3} k_e \tau \mu ds, \ \forall \mu \in E \tag{3.2.18}$$

$$(Rv, \mu)_{E' \times E} = \int_{\Gamma_3} h_\tau(|v_\tau|) \mu dx - \int_\Omega (\mathcal{M}.\nabla v) \mu dx, \ \forall v \in V, \tau, \mu \in E. \tag{3.2.19}$$

Using Green's formula (2.1.1), it follows that if $(u, \sigma, \theta, \omega)$ are sufficiently regular functions that satisfy $(3.1.1) - (3.1.9)$, then we have the following variational formulation

Problem PV_1 : Find a displacement field $u : \Omega \times [0,T] \to V$, a stress field $\sigma : \Omega \times [0,T] \to \mathbb{S}^d$, the temperature $\theta : \Omega \times [0,T] \to \mathbb{R}_+$ and the wear $\omega : \Gamma_3 \times [0,T] \to \mathbb{R}_+$ such that

$$(\ddot{u}(t), w - \dot{u}(t)) + \left(\mathcal{A}\varepsilon\left(u\left(t\right)\right) + \mathcal{G}(\varepsilon(\dot{u}(t))) + \int_0^t \mathcal{B}(t-s)\varepsilon(u(s))ds - \theta\left(t\right)\mathcal{M}, \varepsilon\left(w - \dot{u}\left(t\right)\right)\right)$$
$$+ j\left(u, w\right) - j\left(u, \dot{u}\right) + \phi(\dot{u}, w) - \phi(\dot{u}, \dot{u}(t)) \geq \left(f(t), w - \dot{u}(t)\right), \ \forall u, w \in V, \tag{3.2.20}$$

$$\dot{\theta}\left(t\right) + k\theta\left(t\right) = R\dot{u}\left(t\right) + Q\left(t\right), \text{ on } E', \tag{3.2.21}$$

$$\dot{\omega} = -k\upsilon^{*}\sigma_{\nu}, \text{ on } \Gamma_3 \times [0, T]. \tag{3.2.22}$$

3.3 Existence and uniqueness of the solution

The existence of a unique solution of the variational problem PV_1 is given by the following theorem

Theorem 3.3.1. *Let the assumptions* (3.2.1) – (3.2.19) *hold. Then, Problem PV_1 has a unique solution*$\{u, \sigma, \theta, \omega\}$ *which satisfies*

$$u \in C^1(0, T; H) \cap W^{1.2}(0, T; V) \cap W^{2.2}(0, T; V') \tag{3.3.1}$$

$$\sigma \in \mathbb{L}^2(0, T; \mathcal{H}_1), \ Div\sigma \in \mathbb{L}^2(0, T; V') \tag{3.3.2}$$

$$\theta \in W^{1.2}(0, T; E') \cap \mathbb{L}^2(0, T; E) \cap C(0, T; \mathbb{L}^2) \tag{3.3.3}$$

$$\omega \in C^1(0, T; \mathbb{L}^2(\Gamma_3)) \tag{3.3.4}$$

We conclude that under the assumptions (3.2.1) – (3.2.19), the mechanical problem (3.1.1) – (3.1.9) has a unique weak solution with the regularity (3.3.1) – (3.3.4).

The proof of this theorem will be carried out in several steps. It is based on arguments of first order evolution nonlinear inequalities, evolution equations, a parabolic variational inequality, and fixed point arguments.

First step: Let $g \in \mathbb{L}^2(0, T; V)$ and $\eta \in \mathbb{L}^2(0, T; V')$ be given, we consider the following auxiliary problem

Problem $PV_1^{g\eta}$: Find a displacement field $u_{g\eta} : [0,T] \to V$ such that

$$\begin{cases} u_{g\eta}(t) \in V : (\ddot{u}_{g\eta}(t), w - \dot{u}_{g\eta}(t))_{V'\times V} + ((\mathcal{G}(\varepsilon(u_{g\eta})(t), \varepsilon(w - \dot{u}_{g\eta}(t)))_{\mathcal{H}} \\ +(\eta, w - \dot{u}_{g\eta}(t))_{V'\times V} + j(g,w) - j(g, \dot{u}_{g\eta}(t)) \geq (f(t), w - \dot{u}_{g\eta}(t)))_{V'\times V}, \\ \quad \forall w \in V, \end{cases} \tag{3.3.5}$$

$$\dot{u}_{g\eta}(0) = v(0) = v_0. \tag{3.3.6}$$

We use the Riesz Fréchet representation theorem to define the function $f_\eta(t) \in V'$ For $a.e.\ t \in [0,T]$ by

$$(f_\eta(t), w)_{V'\times V} = (f(t) - \eta(t), w)_{V'\times V}, \ \forall w \in V. \tag{3.3.7}$$

From (3.2.14), we deduce that

$$f_\eta \in \mathbb{L}^2(0,T;V'). \tag{3.3.8}$$

Let now $u_{g\eta} : [0,T] \to V$ be the function defined by

$$u_{g\eta}(t) = \int_0^t v_{g\eta}(s)ds + u_0, \ \forall t \in [0,T]. \tag{3.3.9}$$

We define the operator $G : V' \to V$ by

$$(Gv, w)_{V'\times V} = (\mathcal{G}\varepsilon(v), \varepsilon(w))_{\mathcal{H}}, \ \forall v, w \in V. \tag{3.3.10}$$

The following variational inequality is obtained

Problem $PV^{g\eta}$: Find a displacement field $v_{g\eta} : \Omega \times [0,T] \to V$ such that

$$\begin{cases} (\dot{v}_{g\eta}(t), w - v_{g\eta}(t))_{V'\times V} + (Gv_{g\eta}(t), w - v_{g\eta}(t)))_{\mathcal{H}} \\ +j(g,w) - j(g, v_{g\eta}(t)) \geq (f_\eta(t), w - \dot{u}_{g\eta}(t)))_{V'\times V}, \ \forall w \in V, \end{cases} \tag{3.3.11}$$

$$v_{g\eta}(0) = v_0. \tag{3.3.12}$$

Lemma 3.3.1 For all $g \in \mathbb{L}^2(0,T;V)$ and $\eta \in \mathbb{L}^2(0,T;V')$, $PV_1^{g\eta}$ has a unique solution with the regularity

$$v_{g\eta} \in C(0,T;H) \cap \mathbb{L}^2(0,T;V) \ \text{ and } \ \dot{v}_{g\eta} \in \mathbb{L}^2(0,T;V')$$

Proof. Using the continuous injection of V into $\mathbb{L}^2(\Gamma_3)^d$ then j is continuous and convex. We define the sequence

$$j_\varepsilon(g,v) = \int_{\Gamma_3} \alpha\, |g_\nu| \left(\mu\sqrt{|v_\tau - v^*|^2 + \varepsilon^2}\right) da, \forall v \in V.$$

Its Fréchet derivative is given by

$$j'_\varepsilon(g,v).w = \int_{\Gamma_3} \alpha\mu\, |g_\nu| \frac{(v_\tau - v^*, w_\tau)}{\sqrt{|v_\tau - v^*|^2 + \varepsilon^2}} da, \forall v \in V,\ \forall \varepsilon > 0.$$

So j_ε of class C^1, algebraic calculations show that for all $\alpha \geq 0, \beta \geq 0$ such that $\alpha + \beta = 1$, and for all real numbers x and y, $n \geq 1$

$$\sqrt{(\alpha x + \beta y)^2 + \frac{1}{n}} \leq \alpha\sqrt{x^2 + \frac{1}{n}} + \beta\sqrt{y^2 + \frac{1}{n}}.$$

Therefore, j'_ε is convex $\forall \varepsilon > 0$. We have

$$\exists C > 0, \forall w \in V,\ |j'_\varepsilon(g,w)|_{V'} \leq C|g|_{\mathbb{L}^2(\Gamma_3)}. \tag{3.3.13}$$

According to (3.2.2)(b) and the monotonicity of j'_ε so $G + j'_\varepsilon$ is a monotone operator. Assumption (3.2.2)(a) implies that $G : V \longrightarrow V'$ is a continuous Lipschitz operator. Hence the application $t \longrightarrow G(u + tv)$ is continuous and then G is a hemicontinuous operator. Since j'_ε is continuous, then $G + j'_\varepsilon$ is a hemicontinuous operator.

Now, from (3.2.2)(b) and the monotonicity of j'_ε we find

$$((G + j'_\varepsilon)u - (G + j'_\varepsilon)v, u - v)_{V' \times V} \geq m_G|u - v|_V^2,\ \forall u, v \text{ in } V. \tag{3.3.14}$$

So, $G + j'_\varepsilon$ is a monotone operator.

We choose $v = 0_V$ in (3.3.14) and using the inequality $2\alpha\beta \leq m_{\mathcal{G}}\alpha^2 + \frac{1}{m_{\mathcal{G}}}\beta^2$, we obtain $\forall u, v \in V$

$$((G + j'_\varepsilon)u, u)_{V' \times V} \geq m_{\mathcal{G}}\, |u|_V^2 - |G0_V|_{V'}\ \ |u|_V \geq \frac{1}{2}m_{\mathcal{G}}\, |u|_V^2 - \frac{1}{2m_{\mathcal{G}}}\, |G0_V|_{V'}^2.$$

So the condition (theo 2.4.4)(i) is checked for $\omega = \frac{1}{2}m_{\mathcal{G}}$, $\lambda = \frac{1}{2m_{\mathcal{G}}}\, |G0_V|_{V'}^2 \in \mathbb{R}$.

Then we use (3.2.2)(a) and (3.3.13) we obtain

$$|(G+j_\varepsilon')u-(G+j_\varepsilon')v|_{V'}\leq L_{\mathcal{G}}\,|u-v|_V+C.$$

Choosing $v=0_V$ we find

$$|(G+j_\varepsilon')u|_{V'}\leq C(|u|_V+1),\ \forall u\in V.$$

Then the condition (theo 2.4.4)(ii) is verified.

Finally, we recall that by (3.2.12) and (3.3.8) it then follows from theorem (2.4.4) that there exists

$$v_{g\eta}^\varepsilon\in C(0,T;H)\cap\mathbb{L}^2(0,T;V)\ \text{ and }\ \dot{v}_{g\eta}^\varepsilon\in\mathbb{L}^2(0,T;V')$$

such that:

$$\begin{cases}\dot{v}_{g\eta}^\varepsilon(t)+Gv_{g\eta}^\varepsilon(t)+j_\varepsilon^{'}(g,v_{g\eta}^\varepsilon)=f_\eta(t)\text{ in }V',a.e.\ t\in[0,T]\\ v_{g\eta}^\varepsilon(0)=v_0\end{cases}\tag{3.3.15}$$

So $v_{g\eta}^\varepsilon\in\mathbb{L}^2(0,T;V)\cap W^{1.2}(0,T;V')$ and we have

$$\begin{cases}(\dot{v}_{g\eta}^\varepsilon(t),w-v_{g\eta}^\varepsilon)_{V'\times V}+(Gv_{g\eta}^\varepsilon(t)),w-v_{g\eta}^\varepsilon(t))_{V'\times V}+j_\varepsilon(g,w)-j_\varepsilon(g,v_{g\eta}^\varepsilon(t))\\ \geq(f_\eta(t),w-v_{g\eta}^\varepsilon(t))_{V'\times V},\ \forall w\in V,\ a.e.\ t\in[0,T].\end{cases}\tag{3.3.16}$$

From (3.3.15) we have

$$(\dot{v}_{g\eta}^\varepsilon(t),v_{g\eta}^\varepsilon)_{V'\times V}+(Gv_{g\eta}^\varepsilon(t)),v_{g\eta}^\varepsilon(t))_{V'\times V}+j_\varepsilon'(g,v_{g\eta}^\varepsilon(t))_{V'\times V}=$$
$$(f_\eta(t),v_{g\eta}^\varepsilon(t))_{V'\times V}\quad a.e.\ t\in[0,T].$$

Using (3.2.2), the monotony of j_ε' and (3.3.15)

$$\exists c>0,\ \forall t\in[0,T]:\left|v_{g\eta}^\varepsilon(t)\right|_H\leq c.\ \int_0^T\left|v_{g\eta}^\varepsilon(t)\right|_V^2\leq c.\ \int_0^T\left|\dot{v}_{g\eta}^\varepsilon(t)\right|_{V'}^2\leq c.$$

So there exists a sub-sequence $(v_{g\eta})$ such that

$$\begin{aligned}&v_{g\eta}^\varepsilon\rightharpoonup v_{g\eta}\text{ in }\mathbb{L}^2(0,T;V)\ \text{ and in }\mathbb{L}^\infty(0,T;H),\\ &\dot{v}_{g\eta}^\varepsilon\rightharpoonup\dot{v}_{g\eta}\ \text{ weakly star in }\mathbb{L}^2(0,T;V').\end{aligned}\tag{3.3.17}$$

It comes that

$$v_{g\eta} \in C(0,T;H) \text{ and } v^{\varepsilon}_{g\eta}(t) \rightharpoonup v_{g\eta}(t) \text{ in } H,\ \forall t \in [0,T]\,. \tag{3.3.18}$$

By integrating (3.3.16) , we get $\forall w \in \mathbb{L}^2(0,T;V)$

$$\begin{cases} \int\limits_0^T \left(\dot{v}^{\varepsilon}_{g\eta}(t)\,,w\right)_{V'\times V} dt + \int\limits_0^T \left(Gv^{\varepsilon}_{g\eta}(t)),w\right)_{V'\times V} dt + \int\limits_0^T j_{\varepsilon}(g,w)dt \geq \\ \int\limits_0^T \left(f_{\eta}(t), w - v^{\varepsilon}_{g\eta}(t)\right)_{V'\times V} dt + \int\limits_0^T \left(\dot{v}^{\varepsilon}_{g\eta}(t)\,, v^{\varepsilon}_{g\eta}(t)\right)_{V'\times V} dt + \int\limits_0^T \left(Gv^{\varepsilon}_{g\eta}(t)), v^{\varepsilon}_{g\eta}(t)\right)_{V'\times V} dt \\ + \int\limits_0^T j_{\varepsilon}(g, v^{\varepsilon}_{g\eta}(t))dt \geq \int\limits_0^T \left(f_{\eta}(t), w - v^{\varepsilon}_{g\eta}(t)\right)_{V'\times V} dt \ + \frac{1}{2}\left|v^{\varepsilon}_{g\eta}(T)\right|^2_H - \frac{1}{2}\left|v^{\varepsilon}_{g\eta}(0)\right|^2_H \\ + \int\limits_0^T \left(Gv^{\varepsilon}_{g\eta}(t)), v^{\varepsilon}_{g\eta}(t)\right)_{V'\times V} dt + \int\limits_0^T j_{\varepsilon}(g, v^{\varepsilon}_{g\eta}(t))dt. \end{cases} \tag{3.3.19}$$

From (3.3.17), (3.3.18) and the lower semi-continuity, we have $\forall w \in \mathbb{L}^2(0,T;V)$

$$\begin{cases} \int\limits_0^T (\dot{v}_{g\eta}(t)\,, w - v_{g\eta})_{V'\times V} dt + \int\limits_0^T (Gv_{g\eta}(t)), w - v_{g\eta})_{V'\times V} dt + \int\limits_0^T j(g,w)dt \\ - \int\limits_0^T j(g, v_{g\eta})dt \geq \int\limits_0^T (f_{\eta}(t), w - v_{g\eta})_{V'\times V} dt. \end{cases}$$

Which implies that

$$\begin{cases} (\dot{v}_{g\eta}(t)\,, w - v_{g\eta})_{V'\times V} + (Gv_{g\eta}(t)), w - v_{g\eta})_{V'\times V} + j(g,w) - j(g, v_{g\eta}) \\ \geq (f_{\eta}(t), w - v_{g\eta})_{V'\times V}, \forall w \in V,\ a.e.\ t \in [0,T]\,. \end{cases}$$

So the problem $PV^{g\eta}$ admits at least solution $v_{g\eta} \in C(0.T,H) \cap \mathbb{L}^2(0,T;V)$ and $\dot{v}_{g\eta} \in \mathbb{L}^2(0,T;V')$.

For uniqueness, let $v^1_{g\eta}, v^2_{g\eta}$ be two solutions of $PV^{g\eta}$. Using (3.3.11) we obtain for all $t \in [0,T]$

$$\left(\dot{v}^2_{g\eta}(t) - \dot{v}^1_{g\eta}(t)\,, v^2_{g\eta}(t) - v^1_{g\eta}(t)\right)_{V'\times V} + \left(Gv^2_{g\eta\xi}(t) - Gv^1_{g\eta}(t), v^2_{g\eta}(t) - v^1_{g\eta}(t)\right)_{V'\times V} \leq 0.$$

We integrate the previous inequality and using (3.2.2) and (3.3.11) we get

$$\frac{1}{2}\left|v_{g\eta}^{2}(t)-v_{g\eta}^{1}(t)\right|_{V}^{2}+m_{\mathcal{G}}\int_{0}^{t}\left|v_{g\eta}^{2}(s)-v_{g\eta}^{1}(s)\right|_{V}^{2}ds\leq 0,\forall t\in[0,T].$$

Which implies

$$v_{g\eta}^{1}=v_{g\eta}^{2}.$$

In the study of the problem $PV_1^{g\eta}$, we have the following result

Lemma 3.3.2. The $PV_1^{g\eta}$ problem has a unique solution

$$u_{g\eta}\in W^{1.2}(0,T;V)\cap C^{1}(0,T;H)\cap W^{2.2}(0,T;V').$$

Proof. The proof of lemma 3.3.2 is a consequence of lemma 3.3.1 and the relation (3.3.9).

Second step: Let us consider now the operator

$$\Lambda_{\eta}:\mathbb{L}^{2}(0,T;V)\rightarrow\mathbb{L}^{2}(0,T;V)\quad\text{defined by}$$

$$\Lambda_{\eta}(g)=v_{g\eta}. \tag{3.3.20}$$

We have the following lemma.

lemma 3.3.3 The operator Λ_η has a unique fixed point $g\in\mathbb{L}^2(0,T;V)$ such that $\Lambda_\eta(g^*)=g^*_\eta$.

proof. Let $g_1,g_2\in\mathbb{L}^2(0,T;V)$. Using similar arguments as those in (3.3.11), we find

$$\begin{aligned}&(\dot{v}_1(t)-\dot{v}_2(t),v_1(t)-v_2(t))+(\mathcal{G}\varepsilon(v_1(t))-\mathcal{G}\varepsilon(v_2(t)),\varepsilon(v_1(t))-\varepsilon(v_2(t)))\\&+j(g_1,v_1(t))-j(g_1,v_2(t))-j(g_2,v_1(t))+j(g_2,v_2(t))\leq 0\end{aligned} \tag{3.3.21}$$

From the definition of the functional j given by (3.2.15)

$$\begin{gathered}j(g_1,v_2(t))-j(g_1,v_1(t))-j(g_2,v_2(t))+j(g_2,v_1(t))=\\ \int_{\Gamma_3}(\alpha\left|g_{1\nu}\right|-\alpha\left|g_{2\nu}\right|)(\mu\left|v_{1\tau}-v^*\right|-\mu\left|v_{2\tau}-v^*\right|)\,da\end{gathered} \tag{3.3.22}$$

And using (2.1.4), (3.2.9) and (3.2.10), we have

$$j(g_2, v_2(t)) - j(g_2, v_1(t)) - j(g_1, v_2(t)) + j(g_1, v_1(t)) \leq C\, |g_1 - g_2|_V\, |v_1 - v_2|_V\,. \tag{3.3.23}$$

Integrating inequality (3.3.21) with respect to time, using the initial conditions $v_2(0) = v_1(0) = v_0$, using (2.1.3), (3.2.2), we find

$$\begin{aligned} &\tfrac{1}{2}\left|v_1(t) - v_2(t)\right|_V^2 + m_{\mathcal{G}} \int_0^t \left|v_1(s) - v_2(s)\right|_V^2 ds \leq \\ &C \int_0^t \left|g_1(s) - g_2(s)\right|_V \left|v_1(s) - v_2(s)\right|_V ds. \end{aligned} \tag{3.3.24}$$

And we use the inequality $2ab \leq \frac{1}{2m_{\mathcal{G}}} a^2 + 2m_{\mathcal{G}} b^2$, we obtain

$$\begin{aligned} &\tfrac{1}{2}\left|v_1(t) - v_2(t)\right|_V^2 + \tfrac{m_{\mathcal{G}}}{2} \int_0^t \left|v_1(s) - v_2(s)\right|_V^2 ds \leq \\ &C \times \tfrac{C}{2m_{\mathcal{G}}} \int_0^t \left|g_1(s) - g_2(s)\right|_V^2 ds + C \times \frac{m_{\mathcal{G}}}{2C} \int_0^t \left|v_1(s) - v_2(s)\right|_V^2 ds. \end{aligned} \tag{3.3.25}$$

So (3.3.25) becomes

$$|v_1(t) - v_2(t)|_V^2 \leq C \int_0^t |g_1(s) - g_2(s)|_V^2 ds. \tag{3.3.26}$$

we conclude from (3.3.26) that the operator Λ_η is a contraction in Banach space $\mathbb{L}^2(0, T; V)$ then there exists a unique fixed point $g_\eta^* \in \mathbb{L}^2(0, T; V)$ such that

$$\Lambda_\eta(g^*) = g_\eta^*. \tag{3.3.27}$$

Third step: We consider the following variational problem.

Problem $P_{1\theta\eta}$: Find $\theta_\eta \in E$ such that

$$\dot{\theta}_\eta(t) + K\theta_\eta(t) = R\dot{u}_\eta(t) + Q(t), \ \text{ on } \ E'. \tag{3.3.28}$$

Lemma 3.3.4 Under the assumptions (3.2.1) – (3.2.19), the problem $P_{1\theta\eta}$ has a unique solution

$$\theta_\eta \in W^{1.2}(0, T; E') \cap \mathbb{L}^2(0, T; E) \cap C(0, T; \mathbb{L}^2(\Omega)).$$

proof Since we have the Gelfand triple $E \subset \mathbb{L}^2(\Omega) \subset E'$, we use a classical result for first order evolution equations given in [37] to prove the unique solvability of (3.3.28). Now, we have $\theta_0 \in \mathbb{L}^2(\Omega)$. The operator K is linear and continuous, so $a(\tau,\mu) = (K\tau,\mu)_{E'\times E}$ is bilinear, continuous and coercive, we use the continuity of $a(.,.)$ and from (3.2.7), (3.2.11), we deduce that

$$\begin{aligned}
&a(\tau,\mu) = (K\tau,\mu)_{E'\times E} \leq |k|_{\mathbb{L}^\infty(\Omega)^{d\times d}} |\nabla\tau|_E |\nabla\mu|_E + |k_e|_{\mathbb{L}^\infty(\Gamma_3)} |\tau|_{\mathbb{L}^2(\Gamma_3)} |\mu|_{\mathbb{L}^2(\Gamma_3)} \\
&\leq C|\tau|_E |\mu|_E .
\end{aligned}$$

We have

$$a(\tau,\tau) = (K\tau,\tau)_{E'\times E} = \sum_{i,j=1}^{d} \int_\Omega k_{ij} \frac{\partial\tau}{\partial x_j}\frac{\partial\tau}{\partial x_i} dx + \int_{\Gamma_3} k_e \tau^2 ds.$$

By (3.2.7) and (3.2.11) there exists a constants $C > 0$ such that

$$(K\tau,\tau)_{E'\times E} \geq C|\tau|_E^2 .$$

Where is condition (iii) of theorem 2.4.5 satisfied We have $\theta_0 \in \mathbb{L}^2(\Omega)$. Let

$$F(t) \in E' : \ (F(t),\tau)_{E'\times E} = (R\dot{u}_\eta(t) + Q(t),\tau) \quad \forall\tau \in E.$$

Under the assumptions (3.2.4), (3.2.7), we have

$$\int_0^T |R\dot{u}|_{E'}^2 dt < \infty, \quad \int_0^T |Q(t)|_{\mathbb{E}'}^2 dt < \infty, \quad \int_0^T |F|_{E'}^2 dt < \infty.$$

We find

$$F \in \mathbb{L}^2(0,T;E^{'}).$$

By a classical result for first order evolution equations (theo 2.4.5),

$$\exists!\theta_\eta \in W^{1.2}(0,T;E^{'}) \cap \mathbb{L}^2(0,T;E) \cap C(0,T;\mathbb{L}^2(\Omega)).$$

Consider the operator

$$\begin{aligned}
&\Lambda : \mathbb{L}^2(0,T;V') \to \mathbb{L}^2(0,T;V') \\
&\Lambda(\eta) = \Lambda_\eta, \forall\eta \in \mathbb{L}^2(0,T;V'), \\
&(\Lambda(\eta),w)_{V'} = (\mathcal{A}(\varepsilon(u(t))) + \int_0^t \mathcal{B}(t-s)\varepsilon(u(s))ds - \theta(t)\mathcal{M}, \varepsilon(w))_{\mathcal{H}} + \phi(v,w).
\end{aligned} \tag{3.3.29}$$

We have the following result.

Lemma 3.3.5 The mapping $\Lambda : \mathbb{L}^2(0,T;V') \to \mathbb{L}^2(0,T;V')$ has a unique element $\eta^* \in \mathbb{L}^2(0,T;V')$ such that

$$\Lambda(\eta^*) = \eta^*. \tag{3.3.30}$$

proof. Let , $\eta_i \in \mathbb{L}^2(0,T;V')$. We use the notation (u_i, θ_i). For $\eta = \eta_i, i = 1.2$, let $t \in [0,T]$. We have

$$(\Lambda(\eta), w)_{V'\times V} = (\mathcal{A}(\varepsilon(u(t))) + \int_0^T \mathcal{B}(t-s)\varepsilon(u(s))ds - \theta(t)\mathcal{M}, \varepsilon(w))_{\mathcal{H}} + \phi(v,w). \tag{3.3.31}$$

From the definition of the functional ϕ given by (3.2.16),

$$\phi(v_1, v_2(t)) - \phi(v_1, v_1(t)) - \phi(v_2, v_2(t)) + \phi(v_2, v_1(t)) = \int_{\Gamma_3} (\alpha |v_{1\nu}| - \alpha |v_{2\nu}|)(v_{1\nu} - v_{2\nu})\, da. \tag{3.3.32}$$

And using (2.1.4), (3.2.9), we have

$$\phi(v_1, v_2(t)) - \phi(v_1, v_1(t)) - \phi(v_2, v_2(t)) + \phi(v_2, v_1(t)) \le C\, |v_1 - v_2|_V^2 . \tag{3.3.33}$$

So

$$|\eta_1(t) - \eta_2(t)|_{V'}^2 \le C(|u_1(t) - u_2(t)|_V^2 + \int_0^t |u_1(s) - u_2(s)|_V^2\, ds + |\theta_1(t) - \theta_2(t)|_{\mathbb{L}^2(\Omega)}^2 + |v_1(t) - v_2(t)|_V^2). \tag{3.3.34}$$

Using the inequalities $2ab \le \frac{2C}{m_{\mathcal{G}}} a^2 + \frac{m_{\mathcal{G}}}{2C} b^2$ and $2ab \le \frac{2}{m_{\mathcal{G}}} a^2 + \frac{m_{\mathcal{G}}}{2} b^2$, we find

$$\frac{1}{2} |v_1(t) - v_2(t)|_V^2 + \frac{m_{\mathcal{G}}}{4} \int_0^t |v_1(s) - v_2(s)|_V^2\, ds \le \frac{1}{m_{\mathcal{G}}} \int_0^t |\eta_1(s) - \eta_2(s)|_{V'}^2\, ds + + \frac{m_{\mathcal{G}}}{4} \int_0^t |v_1(s) - v_2(s)|_V^2\, ds + C \int_0^t |v_1(s) - v_2(s)|_V^2\, ds. \tag{3.3.35}$$

So

$$|v_1(t) - v_2(t)|_V^2 \le C \int_0^t |\eta_1(s) - \eta_2(s)|_{V'}^2\, ds. \tag{3.3.36}$$

By (3.3.28), we find

$$\begin{aligned}&\left(\dot{\theta}_1(t)-\dot{\theta}_2(t),\theta_1(t)-\theta_2(t)\right)_{E'\times E}+\left(K(\theta_1)-K(\theta_2),\theta_1(t)-\theta_2(t)\right)_{E'\times E}\\&=\left(R(v_1)-R(v_2),\theta_1(t)-\theta_2(t)\right)_{E'\times E}.\end{aligned}\tag{3.3.37}$$

We integrate (3.3.37) over $[0,T]$, we use the initial conditions $\theta_1(0)=\theta_2(0)=\theta_0$, and we use the coercive of K and the Lipschitz continuity of R to deduce that

$$\begin{aligned}&\tfrac{1}{2}\left|\theta_1(t)-\theta_2(t)\right|^2_{\mathbb{L}^2(\Omega)}+C\int_0^t\left|\theta_1(s)-\theta_2(s)\right|^2_{\mathbb{L}^2(\Omega)}ds\leq\\&C\left(\int_0^t\left|v_1(s)-v_2(s)\right|_V\left|\theta_1(s)-\theta_2(s)\right|_{\mathbb{L}^2(\Omega)}ds\right).\end{aligned}$$

Using the inequality $2ab\leq\frac{1}{2}a^2+2b^2$, we find

$$\begin{aligned}&\tfrac{1}{2}\left|\theta_1(t)-\theta_2(t)\right|^2_{\mathbb{L}^2(\Omega)}+C\int_0^t\left|\theta_1(s)-\theta_2(s)\right|^2_{\mathbb{L}^2(\Omega)}ds\leq\\&\tfrac{C}{4}\int_0^t\left|v_1(s)-v_2(s)\right|^2_V ds+C\left|\theta_1(s)-\theta_2(s)\right|^2_{\mathbb{L}^2(\Omega)}ds.\end{aligned}$$

Also

$$\left|\theta_1(t)-\theta_2(t)\right|^2_{\mathbb{L}^2(\Omega)}\leq C\int_0^t\left|v_1(s)-v_2(s)\right|^2_V ds.\tag{3.3.38}$$

By (3.3.36) , we find

$$\left|\theta_1(t)-\theta_2(t)\right|^2_{\mathbb{L}^2(\Omega)}\leq C\int_0^t\left|\eta_1(s)-\eta_2(s)\right|^2_{V'} ds.\tag{3.3.39}$$

So

$$\left|\eta_1(t)-\eta_2(t)\right|^2_{V'}\leq C\int_0^t\left|\eta_1(s)-\eta_2(s)\right|^2_{V'} ds.\tag{3.3.40}$$

From $u_i=u_0+\displaystyle\int_0^t v_i(s)\,ds$, we have

$$\left|u_1(t)-u_2(t)\right|^2_V\leq C\int_0^t\left|v_1(s)-v_2(s)\right|^2_V ds.\tag{3.3.41}$$

$$\left|u_1(t)-u_2(t)\right|^2_V+\int_0^t\left|u_1(s)-u_2(s)\right|^2_V ds\leq C\begin{pmatrix}\displaystyle\int_0^t\left|v_1(s)-v_2(s)\right|^2_V ds\\ \displaystyle+\int_0^t\left|u_1(s)-u_2(s)\right|^2_V ds.\end{pmatrix}\tag{3.3.42}$$

But

$$|u_1(t) - u_2(t)|_V^2 \geq 0.$$

$$\int_0^t \int_0^s |u_1(r) - u_2(r)|_V^2 \, drds \geq 0.$$

So

$$\begin{aligned}
|u_1 - u_2|_V^2 + \int_0^t |u_1 - u_2|_V^2 \, ds \leq C \int_0^t (\, |v_1(s) - v_2(s)|_V^2 + |u_1(s) - u_2(s)|_V^2) ds + \\
+ \int_0^t \int_0^s |u_1 - u_2|_V^2 \, drds \\
|u_1 - u_2|_V^2 + \int_0^t |u_1 - u_2|_V^2 \, ds \leq C \int_0^t (\, |v_1(s) - v_2(s)|_V^2 + |u_1(s) - u_2(s)|_V^2 \\
+ \int_0^s |u_1(r) - u_2(r)|_V^2 \, dr) ds
\end{aligned}$$

According to Gronwall inequality, we have

$$|u_1 - u_2|_V^2 + \int_0^t |u_1 - u_2|_V^2 \, ds \leq C \int_0^t \; |v_1(s) - v_2(s)|_V^2 \, ds. \tag{3.3.43}$$

By (3.3.36), on obtain

$$|u_1 - u_2|_V^2 + \int_0^t |u_1 - u_2|_V^2 \, ds \leq C \; \textstyle\int_0^t |\eta_1(s) - \eta_2(s)|_{V'}^2 \, ds \tag{3.3.44}$$

And using (3.3.30) and (3.3.34), we find

$$|\Lambda(\eta_1) - \Lambda(\eta_2)|_{\mathbb{L}^2(0,T;V')}^2 \leq C \int_0^t |\eta_1 - \eta_2|_{V'}^2 \, ds. \tag{3.3.45}$$

By induction, with Λ^n the n^{th} power of the operator Λ, we have

$$|\Lambda^n(\eta_1) - \Lambda^n(\eta_2)|_{\mathbb{L}^2(0,T;V')}^2 \leq \frac{(Ct)^n}{n!} |\eta_1 - \eta_2|_{\mathbb{L}^2(0,T;V')}^2 . \tag{3.3.46}$$

We now that $\left(\frac{(Ct)^n}{n!}\right)$ converge to 0, so for n big enough $\frac{(Ct)^p}{n!} < 1$. That is to say that the operator Λ^n is a contraction on the Banach space $\mathbb{L}^2(0,T;V')$. So, there is a unique $\eta^* \in \mathbb{L}^2(0,T;V')$ such that

$$\Lambda\eta^* = \ \eta^*. \tag{3.3.47}$$

We consider the operator $\mathcal{L} : C(0,T;\mathbb{L}^2(\Gamma_3)) \to C(0,T;\mathbb{L}^2(\Gamma_3))$,

$$\mathcal{L}\omega(t) = -kv^* \int_0^t \sigma_\nu(s)ds,\ \forall t \in [0,T]. \tag{3.3.48}$$

The operator $\mathcal{L} : C(0,T;\mathbb{L}^2(\Gamma_3)) \to C(0,T;\mathbb{L}^2(\Gamma_3))$ has a unique element $\omega^* \in C(0,T;\mathbb{L}^2(\Gamma_3))$ such that

$$\mathcal{L}\omega^* = \omega^*.$$

Proof. using (3.3.48), we have

$$|\mathcal{L}\omega_1(t) - \mathcal{L}\omega_2(t)|^2_{\mathbb{L}^2(\Gamma_3)} \leq kv^* \int_0^t |\sigma_1(s) - \sigma_2(s)|^2_{\mathcal{H}}\, ds. \tag{3.3.49}$$

From (3.1.1), we have

$$|\mathcal{L}\omega_1(t) - \mathcal{L}\omega_2(t)|^2_{\mathbb{L}^2(\Gamma_3)} \leq C \int_0^t \left(\begin{array}{c} |u_1(t) - u_2(t)|^2_V + \int_0^t |u_1(s) - u_2(s)|^2_V\, ds \\ + |\theta_1(t) - \theta_2(t)|^2_{\mathbb{L}^2(\Omega)} \end{array} \right) ds. \tag{3.3.50}$$

By (3.3.38) and (3.3.42), we find

$$|u_1 - u_2|^2_V + \int_0^t |u_1 - u_2|^2_V\, ds + |\theta_1(t) - \theta_2(t)|^2_{\mathbb{L}^2(\Omega)} \leq \int_0^t |v_1(s) - v_2(s)|^2_V\, ds. \tag{3.3.51}$$

So

$$|u_1 - u_2|^2_V + \int_0^t |u_1 - u_2|^2_V\, ds + |\theta_1(t) - \theta_2(t)|^2_{\mathbb{L}^2(\Omega)} \leq C \left(\begin{array}{c} \int_0^t |v_1(s) - v_2(s)|^2_V\, ds \\ + |\omega_1(t) - \omega_2(t)|^2_{\mathbb{L}^2(\Gamma_3)} \end{array} \right).$$

So, we have

$$|u_1 - u_2|^2_V + \int_0^t |u_1 - u_2|^2_V\, ds + |\theta_1(t) - \theta_2(t)|^2_{\mathbb{L}^2(\Omega)} \leq C\, |\omega_1(t) - \omega_2(t)|^2_{\mathbb{L}^2(\Gamma_3)}\,. \tag{3.3.52}$$

By (3.3.50), we find

$$|\mathcal{L}\omega_1(t) - \mathcal{L}\omega_2(t)|_{\mathbb{L}^2(\Gamma_3)} \leq C \int_0^t |\omega_1(s) - \omega_2(s)|_{\mathbb{L}^2(\Gamma_3)} \, ds.$$

Thus, for m sufficiently large, $\mathcal{L}^m$ is a contraction on $C(0.T, \mathbb{L}^2(\Gamma_3))$ and so $\mathcal{L}$ has a unique fixed point in this Banach space.

Now, we have all the ingredients to prove theorem 3.3.1.

Existence Let $g^* \in \mathbb{L}^2(0,T;V)$ be the fixed point of Λ_η defined by (3.3.20), let $\eta^* \in \mathbb{L}^2(0,T;V')$ be the fixed point of Λ defined by (3.3.29), let $\omega^* \in C(0,T;\mathbb{L}^2(\Gamma_3))$ be the fixed point of $\mathcal{L}\omega^*$ defined by (3.3.48), and let $(u,\theta) = (u_{g^*\eta^*}, \theta_{\eta^*})$ be the solutions of Problems $PV_1^{g^*\eta^*}$ and $P_{1\theta\eta}$. It results from (3.3.5), (3.3.28) that $(u_{g^*\eta^*}, \theta_{\eta^*})$ is the solution of Problem PV_1. Properties (3.3.1) – (3.3.4) follow from Lemmas 3.3.1 and 3.3.2.

Uniqueness The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operators $\Lambda_\eta, \Lambda, \mathcal{L}$ defined by (3.3.20), (3.3.29), (3.3.48), and the unique solvability of the Problems $PV_1^{g\eta}$ and $P_{1\theta\eta}$. This completes the proof.

Chapter 4

A problem with wear and damage involving thermo-elastic-viscoplastic materials with friction

This chapter presents a mathematical model for a contact problem involving thermo-elastic-viscoplastic materials with damage. The body is in contact with an obstacle. The contact is frictional and normal compliance with a moving rigid foundation which results in the wear of the contacting surface. The problem is represented by a system of first-order inequalities, a parabolic variational inequality with respect to the damage field, and a parabolic variational equation respect to the temperature field.

We establish a variational formulation for the model and we prove the existence of a unique weak solution to the problem. The proof is based on a classical existence and uniqueness result on parabolic inequalities, differential equations and fixed point arguments.

4.1 Problem statement

The problem studied in this chapter falls within the physical setting $n°2$ presented in the first chapter of this thesis. And consequently we use the first mathematical model for the study to be complete, specifying that the behavioral law is thermo-elastic viscoplastic of type $(1.2.11), (1.2.12), (1.2.13)$. The classical formulation of the mechanical problem is as follows

Problem P_2: Find a displacement field $u : \Omega \times [0,T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0,T] \to \mathbb{S}^d$, the damage field $\beta : \Omega \times [0,T] \to \mathbb{R}^d$, the temperature $\theta : \Omega \times [0,T] \to \mathbb{R}$ and the wear $\omega : \Gamma_3 \times [0,T] \to \mathbb{R}_+$ such that

$$\sigma(t) = \mathcal{A}\varepsilon(\dot{u}(t)) + \mathcal{B}(\varepsilon(u(t)), \beta(t)) + \int_0^t \mathcal{G}(\sigma(s)\mathcal{A}(\varepsilon(\dot{u}(s))), \varepsilon(u(s)), \theta(s))ds, \quad \text{in } \Omega \times [0,T], \tag{4.1.1}$$

$$\rho\ddot{u} = Div\sigma + f_0, \text{ in } \Omega \times [0,T], \tag{4.1.2}$$

$$\dot{\theta} - K_0\Delta\theta = \psi(\sigma, \varepsilon(\dot{u}), \theta) + q, \text{ in } \Omega \times [0,T], \tag{4.1.3}$$

$$\dot{\beta} - K_1\Delta\beta + \partial\varphi_K(\beta) \ni S(\varepsilon(u), \beta), \text{ in } \Omega \times [0,T], \tag{4.1.4}$$

$$u = 0, \text{ on } \Gamma_1 \times [0,T], \tag{4.1.5}$$

$$\sigma_\nu = h, \text{ on } \Gamma_2 \times [0,T], \tag{4.1.6}$$

$$\begin{cases} -\sigma_\nu = p_\nu, \\ |\sigma_\tau| \le \mu p_\nu, \\ \sigma_\tau = -\mu p_\nu \dfrac{(\dot{u}_\tau - v^*)}{|\dot{u}_\tau - v^*|}, \\ \dot{\omega} = k_2 \mu p_\nu R^*(|\dot{u}_\tau - v^*|). \end{cases} \quad \text{on } \Gamma_3 \times [0,T], \tag{4.1.7}$$

$$k_0\frac{\partial\theta}{\partial\nu} + B\theta = 0, \quad \text{on } \Gamma \times [0,T], \tag{4.1.8}$$

$$\frac{\partial\beta}{\partial\nu} = 0, \text{ on } \Gamma \times [0,T], \tag{4.1.9}$$

$$u(0) = u_0, v(0) = v_0, \theta(0) = \theta_0, \beta(0) = \beta_0, \omega(0) = \omega_0, \text{ in } \Omega. \tag{4.1.10}$$

Here, $\mu = \mu(\dot{\omega}, |\dot{u}_\tau - v^*|)$ and $p_\nu = p_\nu(u_\nu - h - \omega)$. Where, equation (4.1.1) is the thermo-elastic-viscoplastic constitutive law with damage. (4.1.2) represents the equation of motion where ρ represents the mass density, we mention that $Div\sigma$ is the divergence operator. Equation (4.1.3) represents the energy conservation where ψ is a nonlinear constitutive function which represents the heat generated by the work of internal forces and q is a given volume heat source. Inclusion (4.1.4) describes the evolution of damage field, governed by the source damage function φ, where $\partial\varphi_K(\zeta)$ is the sub-differential of indicator function of the set K of admissible damage functions. Equalities (4.1.5) and (4.1.6) are the displacement-traction boundary conditions, respectively. (4.1.7) represent the condition with normal compliance, friction and wear. The wear function ω which measures the wear accumulated of the surface. The evolution of the wear of the contacting surface is governed by the differential form of Archard's law (see, e.g., [1,19,24,25]), where $k_2 > 0$ is a wear coefficient, p_ν is a prescribed function of the normal compliance, $\mu \geq 0$ is the coefficient of friction, h represents the gap in direction of ν, v^* is the tangential velocity of the foundation, $|\dot{u}_\tau - v^*|$ represents the slip rate between the contact surface and the foundation, and $R^* : \mathbb{R}_+ \to \mathbb{R}_+$ is the truncation operator $R^* = r$ if $r \leq R$ and $R^* = R$ if $r > R$, R being a fixed positive constant. We recall that in the case without wear, a general version of normal compliance is given by $-\sigma_\nu = p_\nu(u_\nu - h)$. The difference $(u_\nu - h)$, when positive, represents the penetration of the surface asperities into those of the foundation. (4.1.8), (4.1.9) represent, respectively, a Fourier boundary condition for the temperature and a homogeneous Neumann boundary condition for the damage field on Γ. The functions u_0, v_0, θ_0, β_0 and ω_0 in (4.1.10) are the initial data.

4.2 Variational formulation

We now give hypotheses about the data of the problem
The viscosity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$

$$\begin{cases} (a) \quad \exists M_{\mathcal{A}} > 0 \text{ such that} |\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)| \leq M_{\mathcal{A}}|\varepsilon_1 - \varepsilon_2|, \\ \qquad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, a.e.\ x \in \Omega, \\ (b) \quad \exists m_{\mathcal{A}} > 0 \text{ such that} |\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2), \varepsilon_1 - \varepsilon_2| \geq m_{\mathcal{A}}|\varepsilon_1 - \varepsilon_2|, \\ \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d,\ a.e.\ x \in \Omega, \\ (c) \quad \text{The mapping } x \to \mathcal{A}(x,\varepsilon) \text{ is lebesgue measurable in } \Omega \text{ For all } \varepsilon \in \mathbb{S}^d, \\ (d) \quad \text{The mapping } x \to \mathcal{A}(x,0) \in \mathcal{H}. \end{cases} \tag{4.2.1}$$

The elasticity operator $\mathcal{B}$: $\Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies

$$\begin{cases} (a) \quad \exists L_{\mathcal{B}} > 0 \text{ such that } |\mathcal{B}(x,\varepsilon_1,\alpha_1) - \mathcal{B}(x,\varepsilon_2,\alpha_2)| \leq L_{\mathcal{B}}(|\varepsilon_1-\varepsilon_2| \\ +|\alpha_1-\alpha_2|), \quad \forall \varepsilon_1,\varepsilon_2 \in \mathbb{S}^d, \forall \alpha_1,\alpha_2 \in \mathbb{R}, \ a.e. \ x \in \Omega, \\ (b) \quad \text{The mapping } x \to \mathcal{B}(x,\varepsilon,\alpha) \text{ is lebesgue measurable in } \Omega, \\ \text{For all } \varepsilon \in \mathbb{S}^d \text{ and } \alpha \in \mathbb{R}, \\ (c) \quad \text{The mapping } x \to \mathcal{B}(x,0,0) \in \mathcal{H}. \end{cases} \tag{4.2.2}$$

The viscoplasticity operator $\mathcal{G}$: $\Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies

$$\begin{cases} (a) \quad \exists L_{\mathcal{G}} > 0 \text{ such that} |\mathcal{G}(x,\sigma_1,\varepsilon_1,\theta_1) - \mathcal{G}(x,\sigma_2,\varepsilon_2,\theta_2)| \leq L_{\mathcal{G}}(|\sigma_1-\sigma_2| \\ +|\varepsilon_1-\varepsilon_2| + |\theta_1-\theta_2|), \quad \forall \sigma_1,\sigma_2 \in \mathbb{S}^d, \forall \varepsilon_1,\varepsilon_2 \in \mathbb{S}^d, \forall \theta_1,\theta_2 \in \mathbb{R}, a.e. \ x \in \Omega, \\ (b) \quad \text{The mapping } x \to \mathcal{G}(x,\sigma,\varepsilon,\theta) \ \text{ is lebesgue measurable in } \Omega \\ \text{For all } \sigma,\varepsilon \in \mathbb{S}^d \ \text{ and } \ \theta \in \mathbb{R}, \\ (c) \quad \text{The mapping } x \to \mathcal{G}(x,0,0,0) \in \mathcal{H}. \end{cases} \tag{4.2.3}$$

The nonlinear constitutive function ψ:$\Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies

$$\begin{cases} (a) \quad \exists L_{\psi} > 0 \text{ such that } |\psi(x,\sigma_1,\varepsilon_1,\theta_1) - \psi(x,\sigma_2,\varepsilon_2,\theta_2)| \leq L_{\psi}(|\sigma_1-\sigma_2| \\ + |\varepsilon_1-\varepsilon_2| + |\theta_1-\theta_2|), \quad \forall \sigma_1,\sigma_2 \in \mathbb{S}^d, \forall \varepsilon_1,\varepsilon_2 \in \mathbb{S}^d, \forall \theta_1,\theta_2 \in \mathbb{R}, \ a.e. \ x \in \Omega, \\ (b) \quad \text{The mapping } x \to \psi(x,\sigma,\varepsilon,\theta) \text{ is lebesgue measurable in } \Omega \\ \text{For all } \sigma,\varepsilon \in \mathbb{S}^d \text{ and } \theta \in \mathbb{R}, \\ (c) \quad \text{The mapping } x \to \psi(x,0,0,0) \in \mathbb{L}^2(\Omega). \end{cases} \tag{4.2.4}$$

The damage source function S: $\Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies

$$\begin{cases} (a) \quad \exists M_S > 0 \text{ such that } |S(x,\varepsilon_1,\alpha_1) - S(x,\varepsilon_2,\alpha_2)| \leq M_S(|\varepsilon_1 - \varepsilon_2| \\ +|\alpha_1 - \alpha_2|), \quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \forall \alpha_1, \alpha_2 \in \mathbb{R}, a.e.\ x \in \Omega, \\ (b) \quad \text{The mapping } x \to S(x,\varepsilon,\alpha) \text{ is lebesgue measurable in } \Omega \\ \text{for all } \varepsilon \in \mathbb{S}^d \text{ and } \alpha \in \mathbb{R}, \\ (c) \quad \text{The mapping } x \to S(x,0,0) \in \mathbb{L}^2(\Omega). \end{cases} \tag{4.2.5}$$

The function of the normal compliance p_ν satisfies

$$\begin{cases} (a) \quad p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+, \\ (b) \quad |p_\nu(x,u_1) - p_\nu(x,u_2)| \leq L_\nu \, |u_1 - u_2| \quad \forall u_1, u_2 \in \mathbb{R},\ a.e.\ x \in \Gamma_3, \\ (c) \quad \text{For any } u \in \mathbb{R}\ x \to p_\nu(x,u) \text{ is Lebesgue measurable on } \Gamma_3, \\ (d) \quad p_\nu(x,u) = 0 \text{ for all } u \leq 0,\ a.e.\ x \in \Gamma_3, \\ (e) \quad p_\nu(x,u) \leq p_\nu^* \text{ for all } u \in \mathbb{R},\ a.e.\ x \in \Gamma_3. \end{cases} \tag{4.2.6}$$

The coefficient of friction μ, satisfies

$$\begin{cases} (a) \quad \mu : \Gamma_3 \times \mathbb{R}^2 \to \mathbb{R}, \\ (b) \quad |\mu(x,a_1,b_1) - \mu(x,a_2,b_2)| \leq L_\mu(|a_1 - a_2| + |b_1 - b_2|), \\ \forall a_1, a_2, b_1, b_2 \in \mathbb{R},\ a.e.\ x \in \Gamma_3, \\ (c) \quad \text{The mapping} \quad x \to \mu(x,a,b) \text{ is Lebesgue measurable on } \Gamma_3, \\ \forall a, b \in \mathbb{R}, \\ (d) \quad \mu(x,a,b) \leq \mu^*, \quad \forall a, b \in \mathbb{R}, a.e.\ x \in \Gamma_3. \end{cases} \tag{4.2.7}$$

The mass density ρ satisfies

$$\rho \in \mathbb{L}^\infty(\Omega), \text{there exists } \rho^* > 0, \text{ such that } \rho(x) \geq \rho^*,\ a.e.\ x \in \Omega. \tag{4.2.8}$$

The body forces, surface tractions and volume heat source; satisfy

$$\begin{cases} f_0 \in \mathbb{L}^2(0,T;H), h \in \mathbb{L}^2(0,T;\mathbb{L}^2(\Gamma_2)^d), q \in \mathbb{L}^2(0,T;\mathbb{L}^2(\Omega)), \\ B > 0, k_i > 0, i = 0,1. \end{cases} \tag{4.2.9}$$

The set K of admissible damage functions defined by

$$K = \left\{ \beta \in H^1(\Omega)/0 \leq \beta \leq 1 \; a.e \text{ in } \Omega \right\}. \tag{4.2.10}$$

The initial data satisfy

$$u_0 \in V, \;\; \theta_0 \in \mathbb{L}^2(\Omega), \;\; \beta_0 \in K, \;\; \omega_0 \in \mathbb{L}^\infty(\Gamma_3). \tag{4.2.11}$$

We use a modified inner product on $H = \mathbb{L}^2(\Omega)^d$ given by

$$((u,v)) = (\rho u, v)_{\mathbb{L}^2(\Omega)^d}, \forall u, v \in H.$$

That is, it is weighted with ρ. We let H be the associated norm

$$\|v\|_H = (\rho u, v)^{\frac{1}{2}}_{\mathbb{L}^2(\Omega)^d}, \forall v \in H.$$

It follows from assumption (4.2.8) that $\|.\|_H$ and $|.|_H$ are equivalent norms on H, and also the inclusion mapping of $(V, |.|_V)$ into $(H, \|.\|_H)$ is continuous and dense. We denote by V' the dual space of V. Identifying H with its own dual, we can write the Gelfand triple $V \subset H = H' \subset V'$

We use the notation $(\cdot,\cdot)_{V'\times V}$ to represent the duality pairing between V' and V. Then, we have

$$(u,v)_{V'\times V} = ((u,v)), \forall u \in H, \forall v \in V.$$

We denote by $f(t) \in V$ the following element

$$(f(t), v)_{V'\times V} = \int_\Omega f_0(t) v dx + \int_{\Gamma_2} h(t) v da, \; \forall v \in V, t \in [0,T]. \tag{4.2.12}$$

The use of (4.2.9) allows us to verify that

$$f \in \mathbb{L}^2(0,T;V'). \tag{4.2.13}$$

We introduce the following bilinear forms a_i satisfy

$$a_0 : \mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega) \to \mathbb{R} : \quad a_0(\zeta, \xi) = k_0 \int_\Omega \nabla\zeta \cdot \nabla\xi \, dx + B \int_\Gamma \zeta\xi, \quad \forall \zeta, \xi \in \mathbb{L}^2(\Omega), \tag{4.2.14}$$

$$a_1 : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R} : \quad a_1(\zeta, \xi) = k_1 \int_\Omega \zeta \cdot \xi \, dx, \quad \forall \zeta, \xi \in H^1(\Omega). \tag{4.2.15}$$

We consider the wear functional $j : V \times V \to \mathbb{R}$,

$$j(u, v) = \int_{\Gamma_3} \mu p_\nu(u_\nu - h - \omega)(|v_\tau - v^*|)da. \tag{4.2.16}$$

Finally, We consider $\phi : V \times V \to \mathbb{R}$,

$$\phi(u, v) = \int_{\Gamma_3} p_\nu(u_\nu - h - \omega)(v_\tau - u_\tau)da, \ \forall v \in V. \tag{4.2.17}$$

Using Green's formula (2.1.1), it follows that if $(u, \sigma, \theta, \beta, \omega)$ are sufficiently regular functions that satisfy (4.1.1) – (4.1.10), then we have the following variational formulation
Problem PV_2: Find a displacement field $u : \Omega \times [0, T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0, T] \to \mathbb{S}^d$,the damage field $\beta : \Omega \times [0, T] \to \mathbb{R}^d$, the temperature $\theta : \Omega \times [0, T] \to \mathbb{R}$ and the wear $\omega : \Gamma_3 \times [0, T] \to \mathbb{R}_+$ such that

$$\begin{aligned} &(\ddot{u}(t), w - \dot{u}(t))_{V' \times V} + (\sigma(t), \varepsilon(w - \dot{u}(t)))_{\mathcal{H}} + j(u(t), w) - j(u, \dot{u}(t)) \\ &+ \phi(u, w) - \phi(u, \dot{u}(t)) \geq (f(t), w - \dot{u}(t)), \ \forall u, w \in V, \end{aligned} \tag{4.2.18}$$

$$\begin{aligned} &(\dot{\theta}(t), w) + a_0(\theta(t), w) = (\psi(\sigma(t), \varepsilon(\dot{u}(t)), \theta(t)), w)_{V' \times V} \\ &+ (q(t), w)_{\mathbb{L}^2(\Omega)}, \forall w \in V, \ a.e. \ t \in [0, T], \end{aligned} \tag{4.2.19}$$

$$\begin{aligned} &(\dot{\beta}(t), \zeta - \beta(t))_{\mathbb{L}^2(\Omega)} + a_1(\beta(t), \zeta - \beta(t)) \geq (S(\varepsilon(u(t)), \beta), \zeta - \beta(t))_{\mathbb{L}^2(\Omega)}, \\ &\forall \zeta \in K, \ a.e. \ t \in [0, T], \end{aligned} \tag{4.2.20}$$

$$\dot{\omega} = k_2 \mu(\dot{\omega}, |\dot{u}_\tau - v^*|) p_\nu(u_\nu - h - \omega) R^*(|\dot{u}_\tau - v^*|). \tag{4.2.21}$$

$$u(0) = u_0, v(0) = v_0, \theta(0) = \theta_0, \beta(0) = \beta_0, \omega(0) = \omega_0, \ \text{in } \Omega. \tag{4.2.22}$$

4.3 Existence and uniqueness of the solution

The existence of a unique solution of the PV_2 variational problem is given by the following theorem

Theorem 4.3.1. *Let the assumptions* (4.2.1) − (4.2.17) *hold. Then, Problem* PV_2 *has a unique solution*$\{u, \sigma, \theta, \beta, \omega\}$ *which satisfies*

$$u \in C^1(0,T;H) \cap W^{1.2}(0,T;V) \cap W^{2.2}(0,T;V') \tag{4.3.1}$$

$$\sigma \in \mathbb{L}^2(0,T;\mathcal{H}_1), Div\sigma \in \mathbb{L}^2(0,T;V') \tag{4.3.2}$$

$$\theta \in W^{1.2}(0,T;(H^1(\Omega))') \cap \mathbb{L}^2(0,T;H^1(\Omega)) \tag{4.3.3}$$

$$\beta \in W^{1.2}(0,T;\mathbb{L}^2(\Omega)) \cap \mathbb{L}^2(0,T;H^1(\Omega)) \tag{4.3.4}$$

$$\omega \in C^1(0,T;\mathbb{L}^2(\Gamma_3)) \tag{4.3.5}$$

We conclude that under the assumptions (4.2.1) − (4.2.17), the mechanical problem (4.1.1) − (4.1.10) has a unique weak solution with the regularity (4.3.1) − (4.3.5).

The proof of this theorem will be carried out in several steps. It is based on arguments of first order evolution nonlinear inequalities, evolution equations, a parabolic variational inequality and equations, and fixed point arguments.

First step: Let $g \in \mathbb{L}^2(0,T;V), \xi \in C(0,T;\mathbb{L}^2(\Gamma_3))$ and $\eta \in \mathbb{L}^2(0,T;V')$ be given, we consider the following auxiliary problem

Problem $PV_2^{g\eta\xi}$: Find a displacement field $u_{g\eta\xi} : [0,T] \to V$ such that

$$\begin{cases} u_{g\eta\xi}(t) \in V : (\ddot{u}_{g\eta\xi}(t), w - \dot{u}_{g\eta\xi}(t))_{V'\times V} + \\ +(\mathcal{A}\varepsilon(\dot{u}_{g\eta\xi})(t), \varepsilon(w - \dot{u}_{g\eta\xi}(t)))_{\mathcal{H}} + (\eta, w - \dot{u}_{g\eta\xi}(t))_{V'\times V} + j(g,w) \\ -j(g, \dot{u}_{g\eta\xi}(t)) \geq (f(t), w - \dot{u}_{g\eta\xi}(t)))_{V'\times V}, \quad \forall w \in V. \end{cases} \tag{4.3.6}$$

$$\dot{u}_{g\eta\xi}(0) = v(0) = v_0. \tag{4.3.7}$$

We use the Riesz Fréchet representation theorem to define the function $f_\eta(t) \in V'$ For $a.e.t \in [0,T]$ by

$$(f_\eta(t), w)_{V'\times V} = (f(t) - \eta(t), w)_{V'\times V}, \forall w \in V. \tag{4.3.8}$$

From (4.2.12), we deduce that

$$f_\eta \in \mathbb{L}^2(0,T;V'). \tag{4.3.9}$$

Let now $u_{g\eta\xi} : [0,T] \to V$ be the function defined by

$$u_{g\eta\xi}(t) = \int_0^t v_{g\eta\xi}(s)ds + u_0, \forall t \in [0,T]. \tag{4.3.10}$$

We define the operator $A : V' \to V$ by

$$(Av,w)_{V'\times V} = (\mathcal{A}\varepsilon(v), \varepsilon(w))_{\mathcal{H}}, \forall v,w \in V. \tag{4.3.11}$$

The following variational inequality is obtained

Problem $PV^{g\eta\xi}$: Find a displacement field $v_{g\eta\xi} : \Omega \times [0,T] \to V$ such that

$$\begin{cases} (\dot{v}_{g\eta\xi}(t), w - v_{g\eta\xi}(t))_{V'\times V} + (Av_{g\eta\xi}(t), w - v_{g\eta\xi}(t)))_{\mathcal{H}} \\ +j(g,w) - j(g, v_{g\eta\xi}(t)) \geq (f_\eta(t), w - \dot{u}_{g\eta\xi}(t)))_{V'\times V}, \ \forall w \in V. \end{cases} \tag{4.3.12}$$

$$v_{g\eta\xi}(0) = v_0. \tag{4.3.13}$$

Lemma 4.3.1 For all $g \in \mathbb{L}^2(0,T;V), \xi \in C(0,T;\mathbb{L}^2(\Gamma_3))$ and $\eta \in \mathbb{L}^2(0,T;V')$, $PV_2^{g\eta\xi}$ has a unique solution with the regularity

$$v_{g\eta\xi} \in C(0,T;H) \cap \mathbb{L}^2(0,T;V) \text{ and } \dot{v}_{g\eta\xi} \in \mathbb{L}^2(0,T;V').$$

Proof. Using the continuous injection of V into $\mathbb{L}^2(\Gamma_3)^d$ then j is continuous and convex. We define the sequence

$$j_\varepsilon(g,v) = \textstyle\int_{\Gamma_3} \mu p_\nu(g_\nu - h - \omega)(\sqrt{|v_\tau - v^*|^2 + \varepsilon^2})da, \ \forall v \in V.$$

Its Fréchet derivative is given by

$$j'_\varepsilon(g,v).w = \textstyle\int_{\Gamma_3} \mu p_\nu(g_\nu - h - \omega)\dfrac{(v_\tau - v^*, w_\tau)}{\sqrt{|v_\tau - v^*|^2 + \varepsilon^2}}da, \ \forall v \in V.$$

So j_ε of class C^1, algebraic calculations show that for all $\alpha \geq 0, \beta \geq 0$ such that $\alpha + \beta = 1$, and for all real numbers x and y, $n \geq 1$

$$\sqrt{(\alpha x+\beta y)^2+\frac{1}{n}} \leq \alpha\sqrt{x^2+\frac{1}{n}}+\beta\sqrt{y^2+\frac{1}{n}}.$$

Therefore, j'_ε is convex $\forall\varepsilon > 0$. On a

$$\exists C > 0, \forall w \in V,\ |j'_\varepsilon(g,w)|_{V'} \leq C|g|_{\mathbb{L}^2(\Gamma_3)}. \tag{4.3.14}$$

According to (4.2.1)(b) and the monotonicity of j'_ε so $A + j'_\varepsilon$ is a monotone operator. Assumption (4.2.1) (a) implies that $A : V \longrightarrow V'$ is a continuous Lipschitz operator. Hence the application $t \longrightarrow A(u+tv)$ is continuous and then A is a hemicontinuous operator. Since j'_ε is continuous, then $A + j'_\varepsilon$ is a hemicontinuous operator.

Now, from (4.2.1)(b) and the monotonicity of j'_ε we find

$$((A+j'_\varepsilon)u-(A+j'_\varepsilon)v, u-v)_{V'\times V} \geq m_A|u-v|^2_V, \forall u, v \text{ in } V. \tag{4.3.15}$$

So $A + j'_\varepsilon$ is a monotone operator.

We choose $v = 0_V$ in (4.3.15) and using the inequality $2\alpha\beta \leq m_{\mathcal{A}}\alpha^2 + \frac{1}{m_{\mathcal{A}}}\beta^2$, we obtain $\forall u, v \in V$

$$((A+j'_\varepsilon)u, u)_{V'\times V} \geq m_{\mathcal{A}}\left|u\right|^2_V - \left|A0_V\right|_{V'}\ \left|u\right|_V \geq \frac{1}{2}m_{\mathcal{A}}\left|u\right|^2_V - \frac{1}{2m_{\mathcal{A}}}\left|A0_V\right|^2_{V'}.$$

So the condition (theo 2.4.4)(i) is checked for $\omega = \frac{1}{2}m_{\mathcal{A}}$, $\lambda = \frac{1}{2m_{\mathcal{A}}}\left|A0_V\right|^2_{V'} \in \mathbb{R}$.

Then we use (4.2.1)(a) and (4.3.14) we obtained

$$|(A+j'_\varepsilon)u-(A+j'_\varepsilon)v|_{V'} \leq L_{\mathcal{A}}\left|u-v\right|_V + C.$$

Choosing $v = 0_V$ we find

$$|(A+j'_\varepsilon)u|_{V'} \leq C(|u|_V+1),\ \forall u \in V.$$

Then the condition (theo 2.4.4) (ii) is verified.

Finally, we recall that by (4.2.11) and (4.3.9) it then follows from theorem (2.4.4) that there exists $v^{\varepsilon}_{g\eta\xi} \in C(0,T;H) \cap \mathbb{L}^2(0,T;V)$ and $\dot{v}^{\varepsilon}_{g\eta\xi} \in \mathbb{L}^2(0,T;V')$, such that

$$\begin{cases} \dot{v}^{\varepsilon}_{g\eta\xi}(t) + Av^{\varepsilon}_{g\eta\xi}(t) + j^{'}_{\varepsilon}(g, v^{\varepsilon}_{g\eta\xi}) = f_\eta(t) \text{ in } V',\ a.e.\ t \in [0,T], \\ v^{\varepsilon}_{g\eta}(0) = v_0. \end{cases} \tag{4.3.16}$$

So $v^{\varepsilon}_{g\eta\xi} \in \mathbb{L}^2(0,T;V) \cap W^{1.2}(0,T;V')$ and we have

$$\begin{cases} (\dot{v}^{\varepsilon}_{g\eta\xi}(t), w - v^{\varepsilon}_{g\eta\xi})_{V'\times V} + (Av^{\varepsilon}_{g\eta}(t)), w - v^{\varepsilon}_{g\eta\xi}(t))_{V'\times V} + j_\varepsilon(g,w) - \\ j_\varepsilon(g, v^{\varepsilon}_{g\eta\xi}(t)) \geq (f_\eta(t), w - v^{\varepsilon}_{g\eta\xi}(t))_{V'\times V} \quad \forall w \in V\ ,\ a.e. t \in [0,T], \end{cases} \tag{4.3.17}$$

From (4.3.16) we have

$$(\dot{v}^{\varepsilon}_{g\eta\xi}(t), v^{\varepsilon}_{g\eta\xi})_{V'\times V} + (Av^{\varepsilon}_{g\eta\xi}(t)), v^{\varepsilon}_{g\eta\xi}(t))_{V'\times V} + j'_\varepsilon(g, v^{\varepsilon}_{g\eta\xi}(t))_{V'\times V} = \\ (f_\eta(t), v^{\varepsilon}_{g\eta\xi}(t))_{V'\times V} \quad a.e. t \in [0,T].$$

Using (4.2.1), the monotony of j'_ε and (4.3.16)

$$\exists C > 0,\ \forall t \in [0,T] : \left| v^{\varepsilon}_{g\eta\xi}(t) \right|_H \leq C. \int_0^T \left| v^{\varepsilon}_{g\eta\xi}(t) \right|^2_V \leq C. \int_0^T \left| \dot{v}^{\varepsilon}_{g\eta\xi}(t) \right|^2_{V'} \leq C.$$

So there exists a sub-sequence $(v_{g\eta\xi})$ such that

$$\begin{aligned} & v^{\varepsilon}_{g\eta\xi} \rightharpoonup v_{g\eta\xi} \text{ in } \mathbb{L}^2(0,T;V) \text{ and in } \mathbb{L}^\infty(0,T;H). \\ & \dot{v}^{\varepsilon}_{g\eta} \rightharpoonup \dot{v}_{g\eta\xi} \text{ weakly star in } \mathbb{L}^2(0,T;V'). \end{aligned} \tag{4.3.18}$$

It comes that

$$v_{g\eta\xi} \in C(0,T;H) \text{ and } v^{\varepsilon}_{g\eta\xi}(t) \rightharpoonup v_{g\eta\xi}(t) \text{ in } H,\ \forall t \in [0,T]. \tag{4.3.19}$$

By integrating (4.3.17), we get $\forall w \in \mathbb{L}^2(0,T;V)$

$$\begin{cases} \int\limits_0^T \left(\dot{v}^{\varepsilon}_{g\eta\xi}(t), w\right)_{V'\times V} dt + \int\limits_0^T \left(Av^{\varepsilon}_{g\eta\xi}(t)), w\right)_{V'\times V} dt + \int\limits_0^T j_{\varepsilon}(g,w)dt \geq \\ \int\limits_0^T \left(f_\eta(t), w - v^{\varepsilon}_{g\eta\xi}(t)\right)_{V'\times V} dt + \int\limits_0^T \left(\dot{v}^{\varepsilon}_{g\eta\xi}(t), v^{\varepsilon}_{g\eta\xi}(t)\right)_{V'\times V} dt + \int\limits_0^T \left(Av^{\varepsilon}_{g\eta\xi}(t)), v^{\varepsilon}_{g\eta\xi}(t)\right)_{V'\times V} dt \\ + \int\limits_0^T j_{\varepsilon}(g, v^{\varepsilon}_{g\eta\xi}(t))dt \geq \int\limits_0^T \left(f_\eta(t), w - v^{\varepsilon}_{g\eta\xi}(t)\right)_{V'\times V} dt + \frac{1}{2}\left|v^{\varepsilon}_{g\eta\xi}(T)\right|^2_H - \frac{1}{2}\left|v^{\varepsilon}_{g\eta\xi}(0)\right|^2_H \\ + \int\limits_0^T \left(Av^{\varepsilon}_{g\eta\xi}(t)), v^{\varepsilon}_{g\eta\xi}(t)\right)_{V'\times V} dt + \int\limits_0^T j_{\varepsilon}(g, v^{\varepsilon}_{g\eta\xi}(t))dt. \end{cases} \tag{4.3.20}$$

From (4.3.18), (4.3.19) and the lower semi-continuity, we have $\forall w \in \mathbb{L}^2(0,T;V)$

$$\begin{cases} \int\limits_0^T (\dot{v}_{g\eta\xi}(t), w - v_{g\eta\xi})_{V'\times V} dt + \int\limits_0^T (Av_{g\eta\xi}(t)), w - v_{g\eta\xi})_{V'\times V} dt + \int\limits_0^T j(g,w)dt \\ - \int\limits_0^T j(g, v_{g\eta\xi})dt \geq \int\limits_0^T (f_\eta(t), w - v_{g\eta\xi})_{V'\times V} dt. \end{cases}$$

Which implies that

$$\begin{cases} (\dot{v}_{g\eta\xi}(t), w - v_{g\eta\xi})_{V'\times V} + (v_{g\eta\xi}(t)), w - v_{g\eta\xi})_{V'\times V} + j(g,w) - j(g, v_{G\eta}) \\ \geq (f_\eta(t), w - v_{g\eta\xi})_{V'\times V}, \ \forall w \in V, a.e.\ t \in [0,T]. \end{cases}$$

So the problem $PV^{g\eta\xi}$ admits at least solution $v_{g\eta\xi} \in C(0,T;H) \cap \mathbb{L}^2(0,T;V)$ and $\dot{v}_{g\eta\xi} \in \mathbb{L}^2(0,T;V')$.

For uniqueness, let $v^1_{g\eta\xi}, v^2_{g\eta\xi}$ be two solutions of $PV^{g\eta\xi}$. Using (4.3.12) we obtain for all $t \in [0,T]$,
$\left(\dot{v}^2_{g\eta\xi}(t) - \dot{v}^1_{g\eta\xi}(t), v^2_{g\eta\xi}(t) - v^1_{g\eta\xi}(t)\right)_{V'\times V} + \left(Av^2_{g\eta\xi}(t) - Av^1_{g\eta\xi}(t), v^2_{g\eta\xi}(t) - v^1_{g\eta\xi}(t)\right)_{V'\times V} \leq 0.$
We integrate the previous inequality and using (4.2.1) and (4.3.11) we get

$$\frac{1}{2}\left|v^2_{g\eta\xi}(t) - v^1_{g\eta\xi}(t)\right|^2_V + m_{\mathcal{A}} \int\limits_0^t \left|v^2_{g\eta\xi}(s) - v^1_{g\eta\xi}(s)\right|^2_V ds \leq 0, \ \forall t \in [0,T].$$

Which implies

$$v^1_{g\eta\xi} = v^2_{g\eta\xi}.$$

In the study of the problem $PV_2^{g\eta\xi}$, we have the following result

Lemma 4.3.2. The $PV_2^{g\eta\xi}$ problem has a unique solution

$$u_{g\eta\xi} \in W^{1.2}(0,T;V) \cap C^1(0,T;H) \cap W^{2.2}(0,T;V').$$

Proof. The proof of lemma 4.3.2 is a consequence of lemma 4.3.1 and the relation(4.3.10).

Second step: We use the displacement field $u_{g\eta\xi}$ to consider the following variational problem

Let us consider now the operator

$$\Lambda_{\eta\xi} : \mathbb{L}^2(0,T;V) \to \mathbb{L}^2(0,T;V) \quad \text{defined by}$$

$$\Lambda_{\eta\xi}(g) = v_{g\eta\xi}. \tag{4.3.21}$$

We have the following lemma.

lemma 4.3.3. The operator $\Lambda_{\eta\xi}$ has a unique fixed point $g \in \mathbb{L}^2(0,T;V)$.

proof. Let $g_1, g_2 \in \mathbb{L}^2(0,T;V)$, Using similar arguments as those in (4.3.12), we find

$$\begin{aligned} &(\dot{v}_1(t) - \dot{v}_2(t), v_1(t) - v_2(t)) + (\mathcal{A}\varepsilon(v_1(t)) - \mathcal{A}\varepsilon(v_2(t)), \varepsilon(v_1(t)) - \varepsilon(v_2(t))) \\ &+ j(g_1, v_1(t)) - j(g_1, v_2(t)) - j(g_2, v_1(t)) + j(g_2, v_2(t)) \leq 0. \end{aligned} \tag{4.3.22}$$

From the definition of the functional j given by (4.2.16), we have

$$\begin{aligned} &j(g_1, v_2(t)) - j(g_1, v_1(t)) - j(g_2, v_2(t)) + j(g_2, v_1(t)) = \\ &\int_{\Gamma_3} \Big(\mu(\dot{\xi}, |v_{2\tau} - v^*|)\, p_\nu(g_{1\nu} - h - \xi)\, |v_{2\tau} - v^*| - \mu(\dot{\xi}, |v_{1\tau} - v^*|)\, p_\nu(g_{1\nu} - h - \xi)\, |v_{1\tau} - v^*| \\ &- \mu(\dot{\xi}, |v_{2\tau} - v^*|)\, p_\nu(g_{2\nu} - h - \xi)\, |v_{2\tau} - v^*| + \mu(\dot{\xi}, |v_{1\tau} - v^*|)\, p_\nu(g_{2\nu} - h - \xi)\, |v_{1\tau} - v^*| \Big)\, da. \end{aligned} \tag{4.3.23}$$

Using (4.2.7) and (2.1.4), we find

$$j(g_1, v_2(t)) - j(g_1, v_1(t)) - j(g_2, v_2(t)) + j(g_2, v_1(t)) \leq C|v_1 - v_2|_V |g_1 - g_2|_V. \tag{4.3.24}$$

Integrating the (4.3.22) inequality with respect to time, using the initial conditions $v_1(0) = v_2(0) = v_0$, using (4.3.24), we find

$$\begin{aligned}&\tfrac{1}{2}\left|v_1(t)-v_2(t)\right|_V^2 + m_{\mathcal{A}}\int_0^t \left|v_1(s)-v_2(s)\right|_V^2 ds \leq \\ &C\int_0^t \left|g_1(s)-g_2(s)\right|_V \left|v_1(s)-v_2(s)\right|_V ds.\end{aligned} \tag{4.3.25}$$

We use this inequality, $2ab \leq \frac{1}{2m_{\mathcal{A}}}a^2 + 2m_{\mathcal{A}}b^2$, we find

$$\begin{aligned}&\tfrac{1}{2}\left|v_1(t)-v_2(t)\right|_V^2 + \tfrac{m_{\mathcal{A}}}{2}\int_0^t \left|v_1(s)-v_2(s)\right|_V^2 ds \leq \\ &C\times\tfrac{C}{2m_{\mathcal{A}}}\int_0^t \left|g_1(s)-g_2(s)\right|_V^2 ds + C\times\frac{m_{\mathcal{A}}}{2C}\int_0^t \left|v_1(s)-v_2(s)\right|_V^2 ds.\end{aligned} \tag{4.3.26}$$

So (4.3.26) becomes

$$|v_1(t)-v_2(t)|_V^2 \leq C\int_0^t |g_1(s)-g_2(s)|_V^2 ds. \tag{4.3.27}$$

We conclude from (4.3.27) that the operator $\Lambda_{\eta\xi}^n$ is a contraction in Banach space $\mathbb{L}^2(0.T,V)$ then there exists a unique fixed point $g_{\eta\xi}^* \in \mathbb{L}^2(0,T;V)$ such that

$$\Lambda_{\eta\xi}(g^*) = g_{\eta\xi}^*. \tag{4.3.28}$$

The third step: For $\chi \in C(0,T;(H^1(\Omega))')$ given, we use the displacement $u_{g^*\eta\xi}$ obtained.

Consider the following $PV_{2\chi}$ problem

Problem $PV_{2\chi}$: Find the temperature field $\theta_\chi : [0,T] \to H^1(\Omega)$ such that

$$\left(\dot{\theta}_\chi(t), w\right)_{(H^1(\Omega))'\times H^1(\Omega)} + a_0\left(\theta_\chi(t), w\right) = \left(\chi(t)+q(t), w\right)_{(H^1(\Omega))'\times H^1(\Omega)}, \tag{4.3.29}$$

$$\theta_\chi(0) = \theta_0 \text{ in } \Omega. \tag{4.3.30}$$

We have the following result

Lemma 4.3.4 For any $\chi \in C(0,T;(H^1(\Omega))')$, there is a unique solution θ_χ of the problem $PV_{2\chi}$ satisfies (4.3.3).

Proof. We apply the Friedrichs-Poincaré inequality, we can find a constant, $B' > 0$ such that

$$\int_\Omega |\nabla\zeta|^2 dx + \frac{B}{k_0}\int_\Gamma |\zeta|^2 d\gamma \geq B'\int_\Omega |\zeta|^2 dx, \ \forall\zeta \in \mathbb{L}^2(\Omega).$$

Thus, we obtain that

$$a_0(\zeta,\zeta) \geq c_0 \left|\zeta\right|^2_{H^1(\Omega)}, \quad \forall \zeta \in \mathbb{L}^2(\Omega), \tag{4.3.31}$$

with $c_0 = k_0 min(1, B')/2$, which implies that a_0 is $\mathbb{L}^2$-elliptic. By applying the arguments of classical functional analysis to parabolic equations, the problem PV_χ admits a unique solution θ_χ that satisfies (4.3.3).

Fourth step: For $\phi \in C(0,T;\mathbb{L}^2(\Omega))$ we consider the following variational problem.

Problem $PV_{2\phi}$: Find the damage field $\beta_\phi : [0,T] \to K$ such that

$$\begin{aligned} &(\dot{\beta}_\phi(t), \zeta - \beta(t))_{\mathbb{L}^2(\Omega)} + a_1(\beta_\phi(t), \zeta - \beta(t)) \geq (\phi, \zeta - \beta(t))_{\mathbb{L}^2(\Omega)}, \\ &\forall \zeta \in K,\ a.e.\ t \in [0,T], \end{aligned} \tag{4.3.32}$$

$$\beta_\phi(0) = \beta_0. \tag{4.3.33}$$

Lemma 4.3.5 There exists a unique solution β_ϕ to the auxiliary problem $PV_{2\phi}$ such that

$$\beta_\phi \in W^{1.2}(0,T;\mathbb{L}^2(\Omega)) \cap \mathbb{L}^2(0,T;H^1(\Omega)). \tag{4.3.34}$$

Proof. The inclusion mapping of $(H^1(\Omega), |.|_{H^1(\Omega)})$ into $(\mathbb{L}^2(\Omega), |.|_{\mathbb{L}^2(\Omega)})$ is continuous and its range is dense.We denote by $(H^1(\Omega))'$ the dual space of $H^1(\Omega)$ and, identifying the dual of $\mathbb{L}^2(\Omega)$ with itself, we can write the Gelfand triple

$$H^1(\Omega) \subset \mathbb{L}^2(\Omega) \subset (H^1(\Omega))'$$

We use the notation $(\cdot,\cdot)_{(H^1(\Omega))' \times H^1(\Omega)}$ to represent the duality pairing between$(H^1(\Omega))'$ and $H^1(\Omega)$, we have

$$(\beta,\xi)_{(H^1(\Omega))' \times H^1(\Omega)} = (\beta,\xi)_{\mathbb{L}^2(\Omega)}, \forall \beta \in \mathbb{L}^2(\Omega),\ \xi \in H^1(\Omega).$$

Moreover, K is a nonempty closed convex set in $H^1(\Omega)$, a_1 defined by (4.2.15) is a bilinear form, and $\beta_\phi \in K$.

It is easy to see that Lemma 4.3.5 is a consequence of theorem 2.4.3.

Taking into account the above results and the properties of the operators $\mathcal{B}$ and $\mathcal{G}$ and of the functions ψ and S, we can consider the operator Λ such that

$$\begin{aligned}&\Lambda : C(0,T;V'\times(H^1(\Omega))'\times\mathbb{L}^2(\Omega))\to C(0,T;V'\times(H^1(\Omega))'\times\mathbb{L}^2(\Omega)),\\&\Lambda(\eta,\chi,\phi)(t)=(\Lambda_1(\eta)(t),\Lambda_2(\chi)(t),\Lambda_3(\phi)(t))\end{aligned}\tag{4.3.35}$$

$$\begin{aligned}&(\Lambda_1(\eta),w)_{V'\times V}=(\mathcal{B}(\varepsilon(u_\eta)(t),\beta_\phi(t)),w)+\\&(\int_0^t\mathcal{G}(\sigma_\eta(s),\mathcal{A}(\varepsilon(\dot{u}_\eta(s))),\varepsilon(u_\eta(s)),\theta_\chi(s))ds,w)+\phi(u_\eta(t),w),\ \forall w\in V,\end{aligned}\tag{4.3.36}$$

$$(\Lambda_2(\chi),w)_{(H^1(\Omega))'\times H^1(\Omega)}=(\psi(\sigma_\eta(t),\varepsilon(\dot{u}_\eta)(t),\theta_\chi(t)),w),\ \forall w\in H^1(\Omega),\tag{4.3.37}$$

$$\Lambda_3(\phi)(t)=S(\varepsilon(u_\eta)(t),\beta_\phi(t),\theta_\chi(t)).\tag{4.3.38}$$

We have the following result.

Lemma 4.3.6 The mapping Λ has a unique element $(\eta^*,\chi^*,\phi^*)\in C(0,T;V'\times(H^1(\Omega))'\times\mathbb{L}^2(\Omega))$ such that $\Lambda(\eta^*,\chi^*,\phi^*)=(\eta^*,\chi^*,\phi^*)$.

proof. Let $(\eta_1,\chi_1,\phi_1),(\eta_2,\chi_2,\phi_2)\in C(0,T;V'\times(H^1(\Omega))'\times\mathbb{L}^2(\Omega))$ and $t\in[0,T]$. We use the notation $u_{\eta i}=u_i,\dot{u}_{\eta i}=v_{\eta i}=v_i,\beta_{\phi i}=\beta_i,\theta_{\chi i}=\theta_i,\sigma_{\eta i}=\sigma_i$ for $i=1,2$.
Using (4.3.35) and the relations (4.2.1) – (4.2.4), we obtain

$$\begin{aligned}&|\eta_1(t)-\eta_2(t)|^2_{V'}\le C(|\beta_1(t)-\beta_2(t)|^2_{\mathbb{L}^2(\Omega)}+|u_1(t)-u_2(t)|^2_V\\&+|v_1(t)-v_2(t)|^2_V+\int_0^t(|\sigma_1(s)-\sigma_2(s)|^2_{\mathcal{H}_1}+|v_1(s)-v_2(s)|^2_V\\&+|u_1(s)-u_2(s)|^2_V+|\theta_1(s)-\theta_2(s)|^2_{\mathbb{L}^2(\Omega)})ds)+\phi(u_1,v_2(t))\\&-\phi(u_1,v_1(t))-\phi(u_2,v_2(t))+\phi(u_2,v_1(t)).\end{aligned}\tag{4.3.39}$$

From the definition of the functional ϕ given by (4.2.17)

$$\begin{aligned}&\phi(u_1,v_2(t))-\phi(u_1,v_1(t))-\phi(u_2,v_2(t))+\phi(u_2,v_1(t))=\\&\int_{\Gamma_3}(p_\nu(u_{1\nu}-h-\omega)v_{2\nu}-p_\nu(u_{1\nu}-h-\omega)v_{1\nu}-p_\nu(u_{2\nu}-h-\omega)v_{2\nu}+\\&p_\nu(u_{2\nu}-h-\omega)v_{1\nu})da.\end{aligned}\tag{4.3.40}$$

And using (2.1.4), (4.2.6) we have

$$\begin{aligned}&\phi(u_1,v_2(t))-\phi(u_1,v_1(t))-\phi(u_2,v_2(t))+\phi(u_2,v_1(t))\le C|v_1(t)-v_2(t)|_V\\&|u_1(t)-u_2(t)|_V.\end{aligned}\tag{4.3.41}$$

Since

$$u_i\,(t) = \int\limits_0^t v_i\,(s)\,ds + u_0,\;\; \forall t \in [0,T]\,. \tag{4.3.42}$$

We have

$$|u_1(t) - u_2(t)|_V^2 \leq C \int_0^t |v_1(s) - v_2(s)|_V^2\,ds. \tag{4.3.43}$$

Applying (4.3.41), (4.3.43), the relation (4.3.39) becomes

$$\begin{aligned}
&|\eta_1\,(t) - \eta_2\,(t)|_{V'}^2 \leq C(|\beta_1\,(t) - \beta_2\,(t)|_{\mathbb{L}^2(\Omega)}^2 + |v_1\,(t) - v_2\,(t)|_V^2 \\
&+ \int_0^t (|\sigma_1\,(s) - \sigma_2\,(s)|_{\mathcal{H}_1}^2 + |v_1\,(s) - v_2\,(s)|_V^2 + |u_1\,(s) - u_2\,(s)|_V^2 \\
&+ |\theta_1\,(s) - \theta_2\,(s)|_{H^1(\Omega)}^2)ds.
\end{aligned} \tag{4.3.44}$$

Taking into account that

$$\sigma_i(t) = \mathcal{A}(\varepsilon(\dot{u}_i(t)) + \eta_i(t), \forall t \in [0,T]. \tag{4.3.45}$$

By (4.1.1), and using (4.2.1), (4.2.3) we find

$$|\sigma_1(t) - \sigma_2(t)|_{\mathcal{H}_1}^2 \leq C(|v_1(t) - v_2(t)|_V^2 + |\eta_1(t) - \eta_2(t)|_{V'}^2). \tag{4.3.46}$$

(4.3.12) it follows that

$$\begin{aligned}
&(\dot{v}_1(t) - \dot{v}_2(t), v_1(t) - v_2(t)) + (\mathcal{A}\varepsilon(v_1(t)) - \mathcal{A}\varepsilon(v_2(t)), \varepsilon(v_1(t)) - \varepsilon(v_2(t))) \\
&+(\eta_1(t) - \eta_2(t), v_1(t) - v_2(t)) \leq j(u_1, v_2(t)) - j(u_1, v_1(t)) - j(u_2, v_2(t)) \\
&+j(u_2, v_1(t)).
\end{aligned} \tag{4.3.47}$$

From the definition of the functional j given by (4.2.16)

$$\begin{aligned}
&j(u_1, v_2(t)) - j(u_1, v_1(t)) - j(u_2, v_2(t)) + j(u_2, v_1(t)) = \\
&\int_{\Gamma_3} \mu p_\nu(u_{1\nu} - h - \omega)(|v_{2\tau} - v^*|)da - \int_{\Gamma_3} \mu p_\nu(u_{1\nu} - h - \omega)(|v_{1\tau} - v^*|)da \\
&- \int_{\Gamma_3} \mu p_\nu(u_{2\nu} - h - \omega)(|v_{2\tau} - v^*|)da + \int_{\Gamma_3} \mu p_\nu(u_{2\nu} - h - \omega)(|v_{1\tau} - v^*|)da.
\end{aligned} \tag{4.3.48}$$

And using (2.1.5), (4.2.7) we have

$$j(u_1, v_2(t)) - j(u_1, v_1(t)) - j(u_2, v_2(t)) + j(u_2, v_1(t)) \leq C|v_1(t) - v_2(t)|_V \\ |u_1(t) - u_2(t)|_V. \tag{4.3.49}$$

Integrating the (4.3.49) inequality with respect to time, using the initial conditions $v_2(0) = v_1(0) = v_0$, using (4.1.1), (4.3.47), and Cauchy-Schwartz's inequality (4.3.47) becomes

$$\tfrac{1}{2}\left|v_1(t) - v_2(t)\right|_V^2 + m_{\mathcal{A}} \int_0^t \left|v_1(s) - v_2(s)\right|_V^2 ds \leq C \int_0^t \left|v_1(s) - v_2(s)\right|_V^2 ds \\ + \int_0^t \left|\eta_1(s) - \eta_2(s)\right|_{V'} \left|v_1(s) - v_2(s)\right|_V ds.$$

Using the inequality $2ab \leq m_{\mathcal{A}} a^2 + \frac{1}{m_{\mathcal{A}}} b^2$ we obtain

$$\tfrac{1}{2}\left|v_1(t) - v_2(t)\right|_V^2 + \tfrac{m_{\mathcal{A}}}{2} \int_0^t \left|v_1(s) - v_2(s)\right|_V^2 ds \leq C \int_0^t \left|v_1(s) - v_2(s)\right|_V^2 ds \\ + \tfrac{1}{2m_{\mathcal{A}}} \int_0^t \left|\eta_1(s) - \eta_2(s)\right|_{V'}^2 ds + \frac{m_{\mathcal{A}}}{2} \int_0^t \left|v_1(s) - v_2(s)\right|_V^2 ds,$$

this inequality combined with the Gronwall inequality, gives

$$|v_1(t) - v_2(t)|_V^2 \leq C \int_0^t |\eta_1(s) - \eta_2(s)|_{V'}^2 ds. \tag{4.3.50}$$

So

$$\int_0^t \left|v_1(s) - v_2(s)\right|_V^2 ds \leq C \int_0^t \int_0^s \left|\eta_1(r) - \eta_2(r)\right|_{V'}^2 dr ds \\ \leq C \int_0^t \left|\eta_1(s) - \eta_2(s)\right|^2 ds. \tag{4.3.51}$$

In addition, by (4.3.43) and (4.3.51) , we find

$$\left|u_1(t) - u_2(t)\right|_V^2 \leq C \int_0^t \left|\eta_1(s) - \eta_2(s)\right|^2 ds. \tag{4.3.52}$$

Which implies

$$|u_1(t) - u_2(t)|_V^2 \leq C \int_0^t \int_0^s |\eta_1(r) - \eta_2(r)|_{V'}^2 dr ds \leq C \int_0^t |\eta_1(s) - \eta_2(s)|_{V'}^2 ds. \tag{4.3.53}$$

For the temperature, if we take the substitution $\chi = \chi_1, \chi = \chi_2$ in (4.3.29) and subtracting the two obtained equations, we deduce by choosing $v = \theta_1 - \theta_2$ as test function

$$\begin{array}{l}|\theta_1(t) - \theta_2(t)|^2_{H^1(\Omega)} + C_0 \int_0^t |\theta_1(s) - \theta_2(s)|^2_{H^1(\Omega)} ds \leq \\ \int_0^t |\chi_1(s) - \chi_2(s)|_{(H^1(\Omega))'} |\theta_1(s) - \theta_2(s)|_{H^1(\Omega)} ds, \ \forall t \in [0, T].\end{array}$$

Employing inequality $2ab \leq 2C_0 a^2 + \frac{1}{2C_0} b^2$

$$\begin{array}{l}|\theta_1(t) - \theta_2(t)|^2_{H^1(\Omega)} + C_0 \int_0^t |\theta_1(s) - \theta_2(s)|^2_{H^1(\Omega)} ds \leq \\ \frac{1}{4C_0} \int_0^t |\chi_1(s) - \chi_2(s)|^2_{(H^1(\Omega))'} ds + C_0 \int_0^t |\theta_1(s) - \theta_2(s)|^2_{H^1(\Omega)} , \ \forall t \in [0, T].\end{array}$$

We get

$$|\theta_1(t) - \theta_2(t)|^2_{H^1(\Omega)} \leq C \int_0^t |\chi_1(s) - \chi_2(s)|^2_{(H^1(\Omega))'} ds. \tag{4.3.54}$$

For the damage field, from (4.3.32) we deduce that

$$(\dot{\beta}_1 - \dot{\beta}_2, \beta_1 - \beta_2)_{\mathbb{L}^2(\Omega)}) + a_1(\beta_1 - \beta_2, \beta_1 - \beta_2) \leq (\phi_1 - \phi_2, \beta_1 - \beta_2)_{\mathbb{L}^2(\Omega)}, a.e.t \in [0, T].$$

Integrating the previous inequality with respect to time, using the initial conditions $\beta_1(0) = \beta_2(0) = \beta_0$ and the inequality $a_1(\beta_1 - \beta_2, \beta_1 - \beta_2) \geq 0$, we find

$$\frac{1}{2} |\beta_1(t) - \beta_2(t)|^2_{\mathbb{L}^2(\Omega)} \leq \int_0^t (\phi_1(s) - \phi_2(s), \beta_1(s) - \beta_2(s))_{\mathbb{L}^2(\Omega)} ds, \tag{4.3.55}$$

which implies

$$\frac{1}{2} |\beta_1(t) - \beta_2(t)|^2_{\mathbb{L}^2(\Omega)} \leq \int_0^t |\phi_1(s) - \phi_2(s)|_{\mathbb{L}^2(\Omega)} |\beta_1(s) - \beta_2(s)|_{\mathbb{L}^2(\Omega)} ds,$$

We apply the inequality $2ab \leq a^2 + b^2$

$$|\beta_1(t) - \beta_2(t)|^2_{\mathbb{L}^2(\Omega)} \leq C \left(\begin{array}{c} \int_0^t |\phi_1(s) - \phi_2(s)|^2_{\mathbb{L}^2(\Omega)} ds \\ + \int_0^t |\beta_1(s) - \beta_2(s)|^2_{\mathbb{L}^2(\Omega)} ds \end{array} \right),$$

this inequality, combined with the Gronwall inequality, gives

$$|\beta_1(t) - \beta_2(t)|^2_{\mathbb{L}^2(\Omega)} \leq C \int_0^t |\phi_1(s) - \phi_2(s)|^2_{\mathbb{L}^2(\Omega)} ds, \forall t \in [0, T]. \tag{4.3.56}$$

From (4.3.46), (4.3.50), (4.3.51), (4.3.53), (4.3.54), (4.3.56), the relationship (4.3.44) becomes

$$|\eta_1(t)-\eta_2(t)|_{V'}^2 \leq C\begin{pmatrix} \int_0^t |\phi_1(s)-\phi_2(s)|_{\mathbb{L}^2(\Omega)}^2\,ds \\ +\int_0^t |\chi_1(s)-\chi_2(s)|_{(H^1(\Omega))'}^2\,ds \\ +\int_0^t |\eta_1(s)-\eta_2(s)|_{V'}^2\,ds \end{pmatrix}. \tag{4.3.57}$$

We use (4.2.4), (4.3.37), we get

$$|\chi_1(t)-\chi_2(t)|_{(H^1(\Omega))'}^2 \leq L_\psi \begin{pmatrix} (|\sigma_1(t)-\sigma_2(t)|_{\mathcal{H}_1}^2 \\ +|v_1(t)-v_2(t)|_V^2 \\ +|\theta_1(t)-\theta_2(t)|_{H^1(\Omega)}^2 \end{pmatrix}. \tag{4.3.58}$$

From (4.3.46), (4.3.50), (4.3.54) and (4.3.58) we find

$$|\chi_1(t)-\chi_2(t)|_{(H^1(\Omega))'}^2 \leq C\begin{pmatrix} \int_0^t |\phi_1(s)-\phi_2(s)|_{\mathbb{L}^2(\Omega)}^2\,ds \\ +\int_0^t |\chi_1(s)-\chi_2(s)|_{(H^1(\Omega))'}^2\,ds \\ +\int_0^t |\eta_1(s)-\eta_2(s)|_{V'}^2\,ds \end{pmatrix}. \tag{4.3.59}$$

Using (4.2.5), (4.3.38), we obtain

$$|\phi_1(t)-\phi_2(t)|_{\mathbb{L}^2(\Omega)}^2 \leq C(|\beta_1(t)-\beta_2(t)|_{\mathbb{L}^2(\Omega)}^2 + |u_1(t)-u_2(t)|_V^2). \tag{4.3.60}$$

From (4.3.52) and (4.3.56), we obtain

$$|\phi_1(t)-\phi_2(t)|_{\mathbb{L}^2(\Omega)}^2 \leq C\begin{pmatrix} \int_0^t |\phi_1(s)-\phi_2(s)|_{\mathbb{L}^2(\Omega)}^2\,ds \\ +\int_0^t |\eta_1(s)-\eta_2(s)|_{V'}^2\,ds \end{pmatrix}. \tag{4.3.61}$$

Applying the previous inequalities, the estimates (4.3.57) – (4.3.61), and sub- stituting (4.3.35), we obtain

$$|\Lambda(\eta_2,\chi_2,\phi_2)(t)-\Lambda(\eta_1,\chi_1,\phi_1)(t)|_{V'\times(H^1(\Omega))'\times H^1(\Omega)} \leq C\int_0^t |(\eta_2,\chi_2,\phi_2)(s)-(\eta_1,\chi_1,\phi_1)(s)|\,ds.$$

Thus, for m sufficiently large, Λ^m is a contraction on $C(0,T;V'\times(H^1(\Omega))'\times H^1(\Omega))$ and so Λ has a unique fixed point in this Banach space.

We consider the operator $L_\xi : C(0,T;\mathbb{L}^2(\Gamma_3)) \to C(0,T;\mathbb{L}^2(\Gamma_3))$

$$L_\xi(t) = k_2 \int_0^t \mu(\dot{\xi}, |\dot{u}_\tau - v^*|) p_\nu(u_\nu - h - \xi) R^*(|\dot{u}_\tau - v^*|) ds, \ \forall t \in [0,T]. \tag{4.3.62}$$

Lemma 4.3.7 The operator $L\xi : C(0,T;\mathbb{L}^2(\Gamma_3)) \to C(0,T;\mathbb{L}^2(\Gamma_3))$ has a unique element $\xi^* \in C(0.T, \mathbb{L}^2(\Gamma_3))$, such that $L\xi^* = \xi^*$.

Using (4.3.62), we have

$$\begin{aligned} &|L_{\xi_1}(t) - L_{\xi_2}(t)|^2_{\mathbb{L}^2(K_3)} \leq k_2 \int_0^t |v_1(s) - v_2(s)|^2_V + |p_1(s) - p_2(s)|^2_V \\ &+|\xi_1(s) - \xi_2(s)|^2_{\mathbb{L}^2(K_3)} + |\dot{\xi}_1(s) - \dot{\xi}_2(s)|^2_{\mathbb{L}^2(K_3)} ds. \end{aligned} \tag{4.3.63}$$

We have

$$|\xi_1(t) - \xi_2(t)|^2_{\mathbb{L}^2(\Gamma_3)} \leq C \int_0^t (|\dot{\xi}_1(s) - \dot{\xi}_2(s)|^2_{\mathbb{L}^2(\Gamma_3)} ds. \tag{4.3.64}$$

And

$$\textstyle\int_0^t |v_1(s) - v_2(s)|^2_V + |u_1(s) - u_2(s)|^2_V \leq C \displaystyle\int_0^t |v_1(s) - v_2(s)|^2 ds.$$

So, we have

$$\begin{aligned} &\textstyle\int_0^t |v_1(s) - v_2(s)|^2_V + |u_1(s) - u_2(s)|^2_V \leq C(\displaystyle\int_0^t |v_1(s) - v_2(s)|^2 ds \\ &+ \int_0^t |\xi_1(t) - \xi_2(t)|^2_{\mathbb{L}^2(\Gamma_3)}. \end{aligned} \tag{4.3.65}$$

By Gronwall's inequality we find

$$\int_0^t |v_1(s) - v_2(s)|^2_V + |u_1(s) - u_2(s)|^2_V \leq C \int_0^t |\xi_1(t) - \xi_2(t)|^2_{\mathbb{L}^2(\Gamma_3)}.$$

Using (4.3.62), we find

$$|L\xi_1(t) - L\xi_2(t)|_{\mathbb{L}^2(\Gamma_3)} \leq \int_0^t |\xi_1(s) - \xi_2(s)|_{\mathbb{L}^2(\Gamma_3)} ds.$$

By induction, with L^m the m^{th} power of the operator L, we have

$$|L^m \xi_1(t) - L^m \xi_2(t)|_{\mathbb{L}^2(\Gamma_3)} \leq \frac{(Ct)^m}{m!} |\xi_1(t) - \xi_2(t)|_{\mathbb{L}^2(\Gamma_3)},$$

we know that $\left(\frac{(Ct)^m}{m!}\right)$ converge to 0 so for m big enough $\frac{(Ct)^m}{m!} < 1$. That is to-to say that the operator L^m is a contraction on the space of Banach $C(0,T;\mathbb{L}^2(\Gamma_3))$. So, there is a unique $\xi^* \in C(0,T;\mathbb{L}^2(\Gamma_3))$ such that

$$L\xi^* = \xi^*.$$

Now, we have all the ingredients to prove theorem 4.3.1.

Existence Let $g^* \in \mathbb{L}^2(0,T;V)$ be the fixed point of $\Lambda_{\eta^*\xi^*}$ defined by (4.3.21), let (η^*, χ^*, ϕ^*) $\in C(0,T;V' \times (H^1(\Omega))' \times \mathbb{L}^2(\Omega))$ be the fixed point of Λ defined by $(4.3.35) - (4.3.38)$,let $\xi^* \in C(0,T;\mathbb{L}^2(\Gamma_3))$ be the fixed point of $L\xi^*$ defined by (4.3.62), and let $(u, \theta, \beta) = (u_{g^*\eta^*\xi^*}, \theta_{\chi^*}, \beta_{\phi^*})$ be the solutions of Problems $PV_2^{g^*\eta^*\xi^*}, PV_{2\chi^*}$ and respectively, $PV_{2\phi^*}$. It results from $(4.3.6), (4.3.7), (4.3.29)$,$(4.3.30), (4.3.32)$ and $(4.3.33)$ that $(u_{g^*\eta^*\xi^*}, \theta_{\chi^*}, \beta_{\phi^*})$ is the solutions of Problems PV_2. Properties $(4.3.1) - (4.3.5)$ follow from Lemmas 4.3.2, 4.3.4 and 4.3.5.

Uniqueness The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operators$\Lambda_{\eta\xi}, \Lambda, L$ defined by (4.3.21), $(4.3.35)-(4.3.38)$, (4.3.62) and the unique solvability of the Problem $PV_2^{g\eta\xi}, PV_{2\chi}$ and $PV_{2\phi}$ which completes the proof.

Chapter 5

Variational analysis of electro-viscoelastic anti-plane contact problem with long memory

In this chapter we consider a mathematical model which describes the anti-plane shear deformation of a cylinder in frictional contact with a rigid foundation. We consider a dynamic contact problem with friction in electro-viscoelasticity with long memory. The body is in contact with an obstacle. The problem is formulated as a coupled system of an elliptic variational inequality for the displacement, variational equation for the electric potential. We establish a variational formulation for the model and we prove the existence of a unique weak solution to the problem. The proof is based on a classical existence and uniqueness result on parabolic inequalities, differential equations and fixed point arguments.

5.1 Problem statement

Our electro-visco-elastic anti-plane problem can be formulated as follows

Problem P_3: Find a displacement field $u : \Omega \times [0,T] \to V$, a stress field $\sigma : \Omega \times [0,T] \to \mathbb{S}^d$, an electric potential field $\varphi : \Omega \times [0,T] \to \mathbb{R}$ and the an electric displacement field D :

$\Omega \times [0.T] \to \mathbb{R}^d$ such that

$$\sigma = \lambda \left(tr\varepsilon(u) \right) I + 2\mu\varepsilon(u) + \gamma \left(tr\varepsilon(\dot{u}) \right) I + 2\varsigma\varepsilon(\dot{u}) + 2\int_0^t \mathcal{M}\left(t-s\right)\varepsilon(u)ds \\ -\xi^* E(\varphi), \quad \text{in } \Omega \times [0,T], \tag{5.1.1}$$

$$D = \xi\varepsilon(u) + \beta E(\varphi), \quad \text{in } \Omega \times [0,T], \tag{5.1.2}$$

$$Div\left(\varsigma\partial_\nu \dot{u} + \mu\nabla u\right) + \int_0^t \mathcal{M}\left(t-s\right) Div(\nabla u\left(s\right))ds + Div\left(e\nabla\varphi\right) + f_0 = \rho\ddot{u}, \\ \text{in } \Omega \times [0,T], \tag{5.1.3}$$

$$div\left(e\nabla u - \beta\nabla\varphi\right) = q_0, \quad \text{in } \Omega \times [0,T], \tag{5.1.4}$$

$$u = 0, \quad \text{on } \Gamma_1 \times [0,T], \tag{5.1.5}$$

$$\varsigma\partial_\nu \dot{u} + \mu\partial_\nu u + \int_0^t \mathcal{M}\left(t-s\right)\partial_\nu u ds + e\partial_\nu\varphi = h, \quad \text{on } \Gamma_2 \times [0,T] \tag{5.1.6}$$

$$\begin{cases} \left|\varsigma\partial_\nu \dot{u} + \mu\partial_\nu u + \int_0^t \mathcal{M}\left(t-s\right)\partial_\nu u ds + e\partial_\nu\varphi\right| \leq g\left(\int_0^t \left|\dot{u}_\tau\left(s\right)\right| ds\right) \\ \left|\varsigma\partial_\nu \dot{u} + \mu\partial_\nu u + \int_0^t \mathcal{M}\left(t-s\right)\partial_\nu u ds + e\partial_\nu\varphi\right| < g\left(\int_0^t \left|\dot{u}_\tau\left(s\right)\right| ds\right) \Rightarrow \dot{u}_\tau\left(t\right) = 0 \\ \left|\varsigma\partial_\nu \dot{u} + \mu\partial_\nu u + \int_0^t \mathcal{M}\left(t-s\right)\partial_\nu u ds + e\partial_\nu\varphi\right| = g\left(\int_0^t \left|\dot{u}_\tau\left(s\right)\right| ds\right) \Rightarrow \exists\beta \geq 0, \\ \text{such that } \varsigma\partial_\nu \dot{u} + \mu\partial_\nu u + \int_0^t \mathcal{M}\left(t-s\right)\partial_\nu u ds + e\partial_\nu\varphi = -\beta\dot{u}_\tau \end{cases} \text{on } \Gamma_3 \times [0,T], \tag{5.1.7}$$

$$\varphi = 0, \quad \text{on } \Gamma_a \times [0,T], \tag{5.1.8}$$

$$e\partial_\nu u - \beta\partial_\nu\varphi = q_2, \quad \text{on } \Gamma_b \times [0,T], \tag{5.1.9}$$

$$u\left(0\right) = u_0, \ \dot{u}\left(0\right) = u_1, \quad \text{in } \Omega. \tag{5.1.10}$$

5.2 Variational formulation

In what follows, we assume the following assumptions on the problem P_3.

$$\mathcal{M} \in W^{1.2}\left(0,T;\mathbb{R}\right), \tag{5.2.1}$$

and the permittivity coefficient satisfies

$$\varsigma \in \mathbb{L}^\infty\left(\Omega\right) \text{ and there exists } \varsigma^* > 0 \text{ such that } \varsigma\left(x\right) \geq \varsigma^*, \ a.e.\ x \in \Omega, \tag{5.2.2}$$

$$\alpha \in \mathbb{L}^\infty\left(\Omega\right) \text{ and there exists } \alpha^* > 0 \text{ such that } \alpha\left(x\right) \geq \alpha^*, \ a.e.\ x \in \Omega. \tag{5.2.3}$$

We also assume that the Lamé coefficient and the piezoelectric coefficient satisfy

$$\mu \in \mathbb{L}^{\infty}(\Omega) \text{ and } \mu(x) > 0, \; a.e. \; x \in \Omega, \tag{5.2.4}$$

$$e \in \mathbb{L}^{\infty}(\Omega). \tag{5.2.5}$$

The friction bound function g satisfy the following property

$$\begin{cases} (a): g : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+, \\ (b): \exists L_g \geq 0 \text{ such that } |g(x,r_1) - g(x,r_2)| \leq L_g |r_1 - r_2|, \\ \forall\, r_1, r_2 \in \mathbb{R}, \; a.e. \; x \in \Gamma_3, \\ (c): \forall\, r \in \mathbb{R}, g(.,r) \text{ is lebegue mesurable on } \Gamma_3, \\ (d): g(.,0) \in \mathbb{L}^2(\Gamma_3). \end{cases} \tag{5.2.6}$$

The body forces, surface tractions and the densities of electric charges satisfy

$$\begin{cases} f_0 \in \mathbb{L}^2(0,T;H), \quad h \in \mathbb{L}^2(0,T;\mathbb{L}^2(\Gamma_2)^d), \\ q_0 \in L^2(0,T;\mathbb{L}^2(\Omega)), \quad q_2 \in L^2(0,T;\mathbb{L}^2(\Gamma_b)). \end{cases} \tag{5.2.7}$$

The initial data satisfy

$$u_0 \in V, \quad u_1 \in \mathbb{L}^2(\Omega). \tag{5.2.8}$$

We use a modified inner product on $H = \mathbb{L}^2(\Omega)^d$ given by

$$((u,v)) = (\rho u, v)_{\mathbb{L}^2(\Omega)^d}, \forall u, v \in H.$$

That is, it is weighted with ρ. We let H be the associated norm

$$||v||_H = (\rho v, v)^{\frac{1}{2}}_{\mathbb{L}^2(\Omega)^d}, \; \forall v \in H.$$

We use the notation $(.,.)_{V' \times V}$ to represent the duality pairing between V' and V. Then, we have

$$(u,v)_{V' \times V} = ((u,v)), \forall u \in H, \forall v \in V. \tag{5.2.9}$$

It follows from assumption (5.2.9) that $\|.\|_H$ and $|.|_H$ are equivalent norms on H, and also the inclusion mapping of $(V, |.|_V)$ into $(H, \|.\|_H)$ is continuous and dense. We denote by

V' the dual space of V. Identifying H with its own dual, we can write the Gelfand triple $V \subset H = H' \subset V'$.

We define the function $f(t) \in V$ and $q : [0,T] \to W$ by

$$(f(t), v)_V = \int_\Omega f_0(t)\, v dx + \int_{\Gamma_2} h(t)\, v da, \quad \forall v \in V, \; t \in [0.T], \tag{5.2.10}$$

$$(q(t), \psi)_W = -\int_\Omega q_0(t)\, \psi dx + \int_{\Gamma_b} q_2(t)\, \psi da, \quad \forall \psi \in W, \; t \in [0,T]. \tag{5.2.11}$$

for all u, $v \in V$, $\psi \in W$ and $t \in [0,T]$ and note that condition (5.2.10), (5.2.11) imply that

$$f \in \mathbb{L}^2(0,T;V'), \quad q \in \mathbb{L}^2(0,T;W).$$

Finally, We consider the functional $j : V \times V \to \mathbb{R}$,

$$j(u,v) = \int_{\Gamma_3} g\left(\int_0^t |u_\tau(s)|\, ds\right) |v_\tau|\, da. \tag{5.2.12}$$

Next, we consider the bilinear forms $l_\varsigma : V \times V \to \mathbb{R}$, $l_\mu : V \times V \to \mathbb{R}$, $l_e : V \times W \to \mathbb{R}$, $l_e^* : W \times V \to \mathbb{R}$ and $l_\alpha : W \times W \to \mathbb{R}$ given by equalities

$$l_\mu(u,v) = \int_\Omega \mu \nabla u . \nabla v dx, \tag{5.2.13}$$

$$l_\varsigma(u,v) = \int_\Omega \varsigma \nabla u . \nabla v dx, \tag{5.2.14}$$

$$l_e(u,\varphi) = \int_\Omega e \nabla u . \nabla \varphi dx = l_e^*(\varphi, u), \tag{5.2.15}$$

$$l_\alpha(\varphi,\psi) = \int_\Omega \beta \nabla \varphi . \nabla \psi dx. \tag{5.2.16}$$

for all $u, v \in V$, $\varphi, w \in W$. Assumptions $(5.2.2) - (5.2.5)$ imply that the integrals above are well defined and,using $(5.2.13) - (5.2.16)$ it follows that the forms l_μ, l_ς, l_e and l_e^* are continuous. Moreover, the forms l_μ , l_ς and l_α are symmetric and in addition, the form l_α is W-elliptic and l_ς is V-elliptic since

$$l_\varsigma(v,v) \geq \varsigma^* |v|_V^2, \forall v \in V, \tag{5.2.17}$$

$$l_\alpha(\varphi,\varphi) \geq \alpha^* |\varphi|_W^2, \forall \varphi \in W. \tag{5.2.18}$$

Using Green's formulas (2.1.1), (2.1.6), it follows that if (u, σ, φ, D) are sufficiently regular functions that satisfy (5.1.1) – (5.1.10), then we have the following variational formulation

Problem PV_3 : Find a displacement field $u : \Omega \times [0, T] \to V$ and the an electric potential field $\varphi : \Omega \times [0, T] \to \mathbb{R}$ such that

$$
\begin{aligned}
&(\ddot{u}(t), w - \dot{u}(t))_{V' \times V} + l_\varsigma(\dot{u}(t), w - \dot{u}(t)))_{\mathcal{H}} + l_\mu(u(t), w - \dot{u}(t)))_{\mathcal{H}} \\
&+ \left(\int_0^t \mathcal{M}\,(t-s)\, u\,(s)\, ds, w - \dot{u}(t) \right)_V + j(\dot{u}, w) - j(\dot{u}, \dot{u}(t)) \qquad (5.2.19) \\
&+ l_e^*\,(\varphi\,(t)\,, w - \dot{u}(t)) \geq (f(t), w - \dot{u}(t)),\ \forall u, w \in V,
\end{aligned}
$$

$$
l_\alpha\,(\varphi(t), \psi) - l_e\,(u(t), \psi) = (q\,(t)\,, \psi)_W\,, \forall \psi \in W,\ a.e.\ t \in [0, T]\,, \qquad (5.2.20)
$$

$$
u\,(0) = u_0, v\,(0) = v_0,\ \text{in } \Omega. \qquad (5.2.21)
$$

5.3 Existence and uniqueness of the solution

Our main result which states the unique solvability of Problem are the following.

Theorem 5.3.1. *Let the assumptions* (5.2.1) – (5.2.12) *hold. Then, Problem* PV_3 *has a unique solution* (u, σ, φ, D) *which satisfies*

$$
u \in C^1\,(0, T; H) \cap W^{1.2}\,(0, T; V) \cap W^{2.2}\,(0, T; V') \qquad (5.3.1)
$$

$$
\sigma \in \mathbb{L}^2(0, T; \mathcal{H}_1), Div\sigma \in \mathbb{L}^2(0, T; V') \qquad (5.3.2)
$$

$$
\varphi \in W^{1.2}\,(0, T; W) \qquad (5.3.3)
$$

$$
D \in W^{1.2}(0, T; \mathcal{W}_1) \qquad (5.3.4)
$$

We conclude that under the assumptions (5.2.1) – (5.2.12), the mechanical problem (5.1.1) – (5.1.10) has a unique weak solution with the regularity (5.3.1) – (5.3.4).

The proof of this theorem will be carried out in several steps. It is based on arguments of first order evolution nonlinear inequalities, evolution equations, a parabolic variational inequality, and fixed point arguments.

First step: Let $\lambda \in \mathbb{L}^2(0, T; V)$, $\eta \in \mathbb{L}^2(0, T; V')$ be given, we consider the following auxiliary problem

Problem $PV_3^{\lambda\eta}$: Find a displacement field $u_{\lambda\eta} : [0,T] \to V$ such that

$$\begin{cases} u_{\lambda\eta}(t) \in V \quad (\ddot{u}_{\lambda\eta}(t), w - \dot{u}_{\lambda\eta}(t))_{V'\times V} + l_\varsigma(\dot{u}(t), w - \dot{u}_{\lambda\eta}(t))_{\mathcal{H}} \\ + (\eta, w - \dot{u}_{\lambda\eta}(t))_{V'\times V} + j(\lambda, w) - j(\lambda, \dot{u}_{\lambda\eta}(t)) \geq (f(t), w - \dot{u}_{\lambda\eta}(t)), \\ \forall w \in V, \end{cases} \tag{5.3.5}$$

$$\dot{u}_{\lambda\eta}(0) = v(0) = v_0. \tag{5.3.6}$$

We define $f_\eta(t) \in V$ for *a.e.* $t \in [0,T]$ by

$$(f_\eta(t), w)_{V'\times V} = (f(t) - \eta(t), w)_{V'\times V}, \forall w \in V. \tag{5.3.7}$$

From (5.2.7), we deduce that

$$f_\eta \in \mathbb{L}^2(0,T;V'). \tag{5.3.8}$$

Let now $u_{\lambda\eta} : [0,T] \to V$ be the function defined by

$$u_{\lambda\eta}(t) = \int_0^t v_{\lambda\eta}(s)\,ds + u_0, \quad \forall t \in [0,T]. \tag{5.3.9}$$

We define the operator $A : V' \to V$ by

$$(Av, w)_{V'\times V} = l_\varsigma(v(t), w)_{\mathcal{H}}, \ \forall v, w \in V. \tag{5.3.10}$$

Problem $PV^{\lambda\eta}$: Find a displacement field $v_{\lambda\eta}$: $\Omega \times [0,T] \to V$, such that

$$\begin{aligned} &(\dot{v}_{\lambda\eta}, w - v)_{V'\times V} + (Av_{\lambda\eta}(t)), w - v_{G\eta}(t))_{V'\times V} + j(\lambda, w) - j(\lambda, v_{\lambda\eta}(t)) \geq \\ &(f_\eta(t), w - v_{\lambda\eta}(t))_{V'\times V}, \ \forall w \in V, \end{aligned} \tag{5.3.11}$$

$$v_{\lambda\eta}(0) = v_0. \tag{5.3.12}$$

In the study of Problem $PV^{\lambda\eta}$, we have the following result.

Lemma 5.3.1 For all $\lambda \in \mathbb{L}^2(0,T;V)$ and $\eta \in \mathbb{L}^2(0,T;V')$, $PV_3^{\lambda\eta}$ has a unique solution with the regularity

$$v_{\lambda\eta} \in C(0,T;H) \cap \mathbb{L}^2(0,T;V) \text{ and } \dot{v}_{\lambda\eta} \in \mathbb{L}^2(0,T;V').$$

Proof. We begin by the step of regularisation and for all $\varepsilon > 0$ we define

$$j_\varepsilon(\lambda, w) = \int_{\Gamma_3} g\left(\lambda\right) \sqrt{|w_\tau|^2 + \varepsilon^2} da, \forall w \in V.$$

Its Fréchet derivative is given by

$$j'_\varepsilon(\lambda, w).v = \int_{\Gamma_3} g(\lambda) \frac{(w_\tau, v_\tau)}{\sqrt{|w_\tau|^2 + \varepsilon^2}} da, \ \forall w \in V.$$

So j_ε of class C^1, algebraic calculations show that for all $\alpha \geq 0, \beta \geq 0$ such that $\alpha + \beta = 1$, and for all real numbers x and y, $n \geq 1$

$$\sqrt{(\alpha x + \beta y)^2 + \frac{1}{n}} \leq \alpha \sqrt{x^2 + \frac{1}{n}} + \beta \sqrt{y^2 + \frac{1}{n}}.$$

Therefore, j'_ε is convex $\forall \varepsilon > 0$. We have

$$\exists C > 0, \ \forall w \in V, \ |j'_\varepsilon(\lambda, w)|_{V'} \leq C \left|\lambda\right|_{\mathbb{L}^2(\Gamma_3)}. \tag{5.3.13}$$

According to (5.2.17) and the monotonicity of j'_ε so $A + j'_\varepsilon$ is a monotone operator. $A : V \longrightarrow V'$ is a continuous Lipschitz operator. Hence the application $t \longrightarrow A(u + tv)$ is continuous and then A is a hemicontinuous operator. Since j'_ε is continuous, then $A + j'_\varepsilon$ is a hemicontinuous operator. Now, from (5.2.17) and the monotonicity of j'_ε we find

$$((A + j'_\varepsilon)u - (A + j'_\varepsilon)v, u - v)_{V' \times V} \geq m_A |u - v|_V^2, \ \forall u, v \text{ in } V. \tag{5.3.14}$$

So $A + j'_\varepsilon$ is a monotone operator.

We choose $v = 0_V$ in (5.3.14) and using the inequality $2\alpha\beta \leq m_{\mathcal{A}}\alpha^2 + \frac{1}{m_{\mathcal{A}}}\beta^2$, we obtain $\forall u, v \in V$

$$((A + j'_\varepsilon)u, u)_{V' \times V} \geq m_{\mathcal{A}} \left|u\right|_V^2 - \left|A0_V\right|_{V'} \ \left|u\right|_V \geq \frac{1}{2} m_{\mathcal{A}} \left|u\right|_V^2 - \frac{1}{2m_{\mathcal{A}}} \left|A0_V\right|_{V'}^2.$$

So the condition (theo 2.4.4)(i) is checked for $\omega = \frac{1}{2}m_{\mathcal{A}}$, $\lambda = \frac{1}{2m_{\mathcal{A}}}\left|A0_V\right|_{V'}^2 \in \mathbb{R}$.

Then we use (5.2.17) and (5.3.13) we obtain

$$|(A + j'_\varepsilon)u - (A + j'_\varepsilon)v|_{V'} \leq L_{\mathcal{A}}\left|u - v\right|_V + C.$$

Choosing $v = 0_V$ we find

$$|(A + j'_\varepsilon)u|_{V'} \leq C(|u|_V + 1), \ \forall u \in V.$$

Then the condition (theo 2.4.4) (ii) is verified.

From (5.2.17) and the monotonicity of j'_ε, it follows from classical first order evolution equation that $\forall \varepsilon > 0$, $\exists! v^\varepsilon_{\lambda\eta} \in \mathbb{L}^2\left(0, T; V\right) \cap W^{1.2}\left(0, T; V'\right)$ such that

$$\begin{cases} \dot{v}^\varepsilon_{\lambda\eta}\left(t\right) + Av^\varepsilon_{\lambda\eta}(t) + j^{'}_\varepsilon(\lambda, v^\varepsilon_{\lambda\eta}) = f_\eta(t) \ \text{ in } V', \ a.e. \ t \in [0,T]\,. \\ v^\varepsilon_{\lambda\eta}\left(0\right) = v_0, \end{cases} \tag{5.3.15}$$

Then, we obtain

$$\begin{cases} (\dot{v}^\varepsilon_{\lambda\eta}\left(t\right), w - v^\varepsilon_{\lambda\eta})_{V'\times V} + (Av^\varepsilon_{\lambda\eta}(t)), w - v^\varepsilon_{\lambda\eta}(t))_{V'\times V} + j_\varepsilon(\lambda, w) - j_\varepsilon(\lambda, v^\varepsilon_{\lambda\eta}(t)) \\ \geq (f_\eta(t), w - v^\varepsilon_{\lambda\eta}(t))_{V'\times V}, \ \forall w \in V, \ a.e. \ t \in [0,T]\,. \end{cases} \tag{5.3.16}$$

From (5.3.15) we have

$$\begin{aligned} &(\dot{v}^\varepsilon_{\lambda\eta}\left(t\right), v^\varepsilon_{\lambda\eta})_{V'\times V} + (Av^\varepsilon_{\lambda\eta}(t)), v^\varepsilon_{\lambda\eta}(t))_{V'\times V} + j'_\varepsilon(\lambda, v^\varepsilon_{\lambda\eta}(t))_{V'\times V} = \\ &(f_\eta(t), v^\varepsilon_{\lambda\eta}(t))_{V'\times V}, \quad a.e. \ t \in [0,T] \end{aligned}$$

Using (5.2.17) the monotony of j'_ε and (5.3.15) to deduce that

$$\exists c > 0, \ \forall t \in [0,T] : \left|v^\varepsilon_{\lambda\eta}\left(t\right)\right|_H \leq c. \ \int_0^T \left|v^\varepsilon_{\lambda\eta}(t)\right|_V^2 \leq c. \ \int_0^T \left|\dot{v}^\varepsilon_{\lambda\eta}(t)\right|_{V'}^2 \leq c.$$

Using a sub-sequence $v_{\lambda\eta}$ to find that

$$\begin{cases} v^\varepsilon_{\lambda\eta} \rightharpoonup v_{\lambda\eta} \text{ weakly in } \mathbb{L}^2\left(0, T; V\right) \text{ and weakly in } \mathbb{L}^\infty\left(0, T; H\right) \\ \dot{v}^\varepsilon_{\lambda\eta} \rightharpoonup \dot{v}_{\lambda\eta} \text{ weakly star in } \mathbb{L}^2\left(0, T; V'\right) \end{cases} \tag{5.3.17}$$

It follows that

$$v_{\lambda\eta} \in C\left(0,T;H\right), \text{and } v^{\varepsilon}_{\lambda\eta} \rightharpoonup v_{\lambda\eta} \text{ weakly in } H, \forall t \in [0,T]. \tag{5.3.18}$$

Integrating (5.3.16) we have $\forall w \in \mathbb{L}^2(0,T;V)$

$$\begin{cases} \int_0^T (\dot{v}^{\varepsilon}_{\lambda\eta}(t), w)_{V'\times V} dt + \int_0^T (Av^{\varepsilon}_{\lambda\eta}(t)), w)_{V'\times V} dt + \int_0^T j_{\varepsilon}(\lambda, w) dt \geq \\ \int_0^T (f_\eta(t), w - v_{\lambda\eta})_{V'\times V} dt + \int_0^T (\dot{v}^{\varepsilon}_{\lambda\eta}(t), v^{\varepsilon}_{\lambda\eta})_{V'\times V} dt + \int_0^T (Av^{\varepsilon}_{\lambda\eta}(t)), v^{\varepsilon}_{\lambda\eta}(t))_{V'\times V} dt \\ + \int_0^T j_{\varepsilon}(\lambda, v^{\varepsilon}_{\lambda\eta}(t)) dt \geq \int_0^T (f_\eta(t), w - v^{\varepsilon}_{\lambda\eta})_{V'\times V} dt \ + \frac{1}{2}\left|v^{\varepsilon}_{\lambda\eta}(T)\right|^2_H - \frac{1}{2}\left|v^{\varepsilon}_{\lambda\eta}(0)\right|^2_H \\ + \int_0^T (Av^{\varepsilon}_{\lambda\eta}(t), v^{\varepsilon}_{\lambda\eta}(t))_{V'\times V} dt + \int_0^T j_{\varepsilon}(\lambda, v^{\varepsilon}_{\lambda\eta}(t)) dt. \end{cases} \tag{5.3.19}$$

From (5.3.17), (5.3.18) and the lower-semi-continuity, we obtain that $\forall w \in \mathbb{L}^2(0,T;V)$,

$$\begin{cases} \int_0^T (\dot{v}_{\lambda\eta}(t), w - v_{\lambda\eta})_{V'\times V} dt + \int_0^T (Av_{\lambda\eta}(t)), w - v_{\lambda\eta})_{V'\times V} dt + \int_0^T j(\lambda, w) dt \\ -\int_0^T j(\lambda, v_{\lambda\eta}) dt \geq \int_0^T (f_\eta(t), w - v_{\lambda\eta})_{V'\times V} dt. \end{cases}$$

The previous inequality implies that

$$\begin{cases} (\dot{v}_{\lambda\eta}(t), w - v_{\lambda\eta})_{V'\times V} + (Av_{\lambda\eta}(t)), w - v_{\lambda\eta})_{V'\times V} + j(\lambda, w) - j(\lambda, v_{\lambda\eta}) \\ \geq (f_\eta(t), w - v_{\lambda\eta})_{V'\times V}, \ \forall w \in V, \ a.e. \ t \in [0,T]. \end{cases}$$

We conclude that Problem $PV^{\lambda\eta}$ has at least a solution $v_{\lambda\eta} \in C(0,T;H) \cap \mathbb{L}^2(0,T;V) \cap W^{1.2}(0,T;V')$.

For uniqueness, let $v^1_{\lambda\eta}, v^2_{\lambda\eta}$ be two solutions of $PV^{\lambda\eta}$. We use (5.3.11) to obtain for $a.e.\ t \in [0,T]$,

$$(\dot{v}^2_{\lambda\eta}(t) - \dot{v}^1_{\lambda\eta}(t), v^2_{\lambda\eta}(t) - v^1_{\lambda\eta}(t))_{V'\times V} + (Av^2_{\lambda\eta}(t) - Av^1_{\lambda\eta}(t), v^2_{\lambda\eta}(t) - v^1_{\lambda\eta}(t))_{V'\times V} \leq 0.$$

Integrating the previous inequality, using (5.2.17) and (5.3.10), we find

$$\frac{1}{2}\left|v_{\lambda\eta}^{2}(t)-v_{\lambda\eta}^{1}(t)\right|_{H}^{2}+\varsigma\int_{0}^{t}\left|v_{\lambda\eta}^{2}(s)-v_{\lambda\eta}^{1}(s)\right|_{V}^{2}ds\leq 0,\ \forall t\in[0,T],$$

which implies

$$v_{\lambda\eta}^{1}=v_{\lambda\eta}^{2}.$$

Let now $u_{\lambda\eta}:[0,T]\to V$ be the function defined by

$$u_{\lambda\eta}(t)=\int_{0}^{t}v_{\lambda\eta}(s)\,ds+u_{0},\forall t\in[0,T]. \tag{5.3.20}$$

In the study of Problem $PV_3^{\lambda\eta}$, we have the following result.

Lemma 5.3.2 $PV_3^{\lambda\eta}$ has a unique solution satisfying the regularity expressed in (5.3.1).

Proof. The proof of Lemma 5.3.2 is a consequence of Lemma 5.3.1 and the relation (5.3.20).

Second step: we use the displacement field $u_{\lambda\eta}$ to consider the following variational problem.

Let us consider now the operator Λ_η: $\mathbb{L}^2(0,T;V)\to\mathbb{L}^2(0,T;V)$, defined by

$$\Lambda_\eta\lambda=\lambda. \tag{5.3.21}$$

We have the following lemma.

Lemma 5.3.2 The operator Λ_η has a unique fixed point $\lambda\in\mathbb{L}^2(0,T;V)$

Proof. Let $\lambda_1,\lambda_2\in\mathbb{L}^2(0,T;V)$ and let $\eta\in\mathbb{L}^2(0,T;V')$. Using similar arguments as those in (5.3.7), (5.3.11) we find

$$\begin{aligned}&(\dot{v}_1(t)-\dot{v}_2(t),v_1(t)-v_2(t))+(Av_1(t)-Av_2(t),v_1(t)-v_2(t))+\\&+j(\lambda_1,v_1(t))-j(\lambda_1,v_2(t))-j(\lambda_2,v_1(t))+j(\lambda_2,v_2(t))\leq 0.\end{aligned} \tag{5.3.22}$$

From the definition of the functional j given by (5.2.12), we have

$$\begin{aligned} j(\lambda_1, v_2(t)) - j(\lambda_1, v_1(t)) - j(\lambda_2, v_2(t)) + j(\lambda_2, v_1(t)) = \\ = \int_{\Gamma_3} g(\lambda_1 - \lambda_2)(|v_{\tau 1}| - |v_{\tau 2}|)\, da. \end{aligned} \tag{5.3.23}$$

From (2.1.4) and (5.2.6), we find

$$j(\lambda_1, v_2(t)) - j(\lambda_1, v_1(t)) - j(\lambda_2, v_2(t)) + j(\lambda_2, v_1(t)) \le C\,|\lambda_1 - \lambda_2|_V\,|v_1 - v_2|_V\,. \tag{5.3.24}$$

Integrating the inequality (5.3.22) with respect to time, using the initial conditions $v_2(0) = v_1(0) = v_0$, using (5.3.24) and the inequality $2ab \le \frac{C}{m_{\mathcal{A}}}a^2 + \frac{m_{\mathcal{A}}}{C}b^2$, we find

$$|v_2(t) - v_1(t)|_V^2 \le C\int_0^t |\lambda_2(s) - \lambda_1(s)|_V^2\, ds. \tag{5.3.25}$$

Thus, for m sufficiently large, Λ_λ^m is a contraction on $\mathbb{L}^2(0,T;V)$ and so Λ_λ has a unique fixed point in this Banach space.

Third step: we use the displacement field $u_{\lambda\eta}$ to consider the following variational problem.

Problem $PV_{3\eta}^{\varphi}$: Find an electric potential field $\varphi_\eta : \Omega \times [0.T] \to W$ such that

$$l_\alpha(\varphi_\eta(t), \psi) - l_e(u_\eta(t), \psi) = (q(t), \psi)_W\,,\ \ \forall\psi \in W, t \in [0,T]\,. \tag{5.3.26}$$

We have the following result for $PV_{3\eta}^{\varphi}$

Lemma 5.3.3 There exists a unique solution $\varphi_\eta \in W^{1.2}(0,T;W)$ satisfies (5.3.26), moreover if φ_1 and φ_2 are two solutions to (5.3.26). Then, there exists a constants $c > 0$ such that

$$|\varphi_1(t) - \varphi_2(t)|_W \le c\,|u_1(t) - u_2(t)|_V\,,\ \ \forall t \in [0,T]\,. \tag{5.3.27}$$

Proof. Let $t \in [0,T]$. We use the properties of the bilinear form l_β and the Lax-Milgram lemma to see that there exists a unique element $\varphi_\eta \in W$ which solves (5.3.26) at any moment $t \in [0,T]$. Consider now $t_1, t_2 \in [0,T]$. From (5.3.26), we get

$$l_\alpha(\varphi_\eta(t_1), \psi) - l_e(u_\eta(t_1), \psi) = (q(t_1), \psi)_W\ \ \ \forall\psi \in W, t_1 \in [0,T]\,. \tag{5.3.28}$$

And

$$l_\alpha(\varphi_\eta(t_2), \psi) - l_e(u_\eta(t_2), \psi) = (q(t_2), \psi)_W, \quad \forall \psi \in W, \ t_2 \in [0, T]. \tag{5.3.29}$$

Let $\varphi_{\lambda\eta}(t_i) = \varphi_i(t), u_{\lambda\eta}(t_i) = u_i(t_i)$, we use (5.3.26), (5.2.4), (5.2.5) and (5.2.15)-(5.2.16) we find that

$$\alpha^* |\varphi_1(t) - \varphi_2(t)|_W^2 \leq \left(|e|_{\mathbb{L}^\infty(\Omega)} |u_1(t) - u_2(t)|_V + |q_1(t) - q_2(t)|_V\right) |\varphi_1(t) - \varphi_2(t)|_W,$$

and it follows from the previous inequality that

$$|\varphi_1(t) - \varphi_2(t)|_W \leq c\left(|u_1(t) - u_2(t)|_V + |q_1(t) - q_2(t)|_V\right).$$

Then, the regularity $u_\eta \in W^{1.2}(0, T; V)$ combined with (5.2.7) imply that $\varphi_\eta \in W^{1.2}(0, T; W)$ which concludes the proof.

We consider the operator

$$\begin{aligned} &\Lambda : \mathbb{L}^2(0, T; V') \to \mathbb{L}^2(0, T; V') \ \text{ define by} \\ &(\Lambda(\eta), w)_{V' \times V} = l_\mu(u(t), w))_{\mathcal{H}} + \left(\int_0^t \mathcal{M}(t-s)\, u(s)\, ds, w\right)_V + l_e^*(\varphi(t), w). \end{aligned} \tag{5.3.30}$$

We have the following result.

Lemma 5.3.4 The mapping $\Lambda : \mathbb{L}^2(0, T; V') \to \mathbb{L}^2(0, T; V')$ has a unique element $\eta^* \in \mathbb{L}^2(0, T; V')$, such that

$$\Lambda(\eta^*) = \eta^*.$$

Proof. Let $\eta_i \in \mathbb{L}^2(0, T; V')$ We use the notation $(u_i\ , \varphi_i)$. For $(\eta_i), i = 1.2$. Let $t \in [0.T]$.

We have

$$(\Lambda(\eta), w)_{V' \times V} = l_\mu(u(t), w))_{\mathcal{H}} + \left(\int_0^t \mathcal{M}(t-s)\, u(s)\, ds, w\right)_V + l_e^*(\varphi(t), w).$$

So

$$|\eta_1(t) - \eta_2(t)|_{V'}^2 \leq C\left(\int_0^t |u_1(s) - u_2(s)|_V^2\, ds + |\varphi_1(t) - \varphi_2(t)|_W^2\right). \tag{5.3.31}$$

We use (5.3.27), we find

$$|\eta_1(t) - \eta_2(t)|_{V'}^2 \leq C\left(\int_0^t |u_1(s) - u_2(s)|_V^2\, ds + |u_1(t) - u_2(t)|_V^2\right). \tag{5.3.32}$$

Using similar arguments as those in (5.3.11) we find

$$\begin{aligned}&(\dot{v}_1(t) - \dot{v}_2(t), v_1(t) - v_2(t)) + (Av_1(t) - Av_2(t), v_1(t) - v_2(t)) +\\ &+ j(v_1, v_1(t)) - j(v_1, v_2(t)) - j(v_2, v_1(t)) + j(v_2, v_2(t))\\ &\leq (\eta_1(t) - \eta_2(t), v_1(t) - v_2(t)).\end{aligned} \tag{5.3.33}$$

From the definition of the functional j given by (5.2.12), we have

$$\begin{aligned}&j(v_1, v_2(t)) - j(v_1, v_1(t)) - j(v_2, v_2(t)) + j(v_2, v_1(t)) =\\ &= \int_{\Gamma_3} g\left(\int_0^t |v_1(s) - v_2(s)|_V^2\, ds\right)(|v_{\tau 1}| - |v_{\tau 2}|)\, da.\end{aligned} \tag{5.3.34}$$

From (5.2.6), we find

$$j(v_1, v_2(t)) - j(v_1, v_1(t)) - j(v_2, v_2(t)) + j(v_2, v_1(t)) \leq C\,|v_1 - v_2|_V^2. \tag{5.3.35}$$

Integrating the inequality (5.3.33) with respect to time, using the initial conditions $v_2(0) = v_1(0) = v_0$, using (5.3.35) and the inequality $2ab \leq \frac{C}{m_{\mathcal{A}}} a^2 + \frac{m_{\mathcal{A}}}{C} b^2$, we find

$$|v_2(t) - v_1(t)|_V^2 \leq C\left(\int_0^t |v_2(s) - v_1(s)|_V^2\, ds + \int_0^t |\eta_1(s) - \eta_2(s)|^2\, ds\right). \tag{5.3.36}$$

By Gronwall,

$$|v_2(t) - v_1(t)|_V^2 \leq C \int_0^t |\eta_1(s) - \eta_2(s)|^2\, ds. \tag{5.3.37}$$

We find

$$|\Lambda\eta_1(t) - \Lambda\eta_2(t)|_{V'}^2 \leq C \int_0^t |\eta_1(s) - \eta_2(s)|^2\, ds. \tag{5.3.38}$$

Thus, for m sufficiently large, Λ^m is a contraction on $\mathbb{L}^2(0.T, V')$ and so Λ has a unique fixed point in this Banach space.

Now, we have all the ingredients to prove theorem 5.3.1.

Existence Let $(\lambda^*,\ \eta^*) \in L^2(0, T; V \times V')$ be the fixed point of $PV_3^{\lambda\eta}$ and let $(u^*,\ \varphi^*)$ be the solution to Problems $PV_3^{\lambda\eta}$, $PV_{3\eta}^{\varphi}$, that is, $u^* = u_{\lambda^*\eta^*}$and $\varphi^* = \varphi_{\eta^*}$. It results from (5.3.1) and (5.3.3) that $(u^*,\ \varphi^*)$ is a solution of Problem PV_3.

Uniqueness The uniqueness of the solution is a consequence of the uniqueness of the fixed point of operator defined by (5.3.21) – (5.3.30).

Conclusion

This thesis is motivated by problems of contact mechanics with friction. We consider some boundary problems with different behavior laws such as thermo-viscoelastic, thermo-visco-plastic with long memory and electro-viscoelastic. We have studied contact problems with friction in a dynamic process, for which we couple both damage and wear or damage and thermal or electrical effects. For each of these problems after specifying the assumptions about the data, we drive a variational formulation in the form of a coupled system in terms of displacement field, the stress field, the damage, the wear and temperature or electric potential. Then we establish our result of existence and uniqueness of the weak solution. The proofs were carried out in several steps. They are based on argument for first order nonlinear evolution inequalities, evolution equations, a parabolic variational inequality and fixed point arguments. The numerical analysis of these problems remains to be done in order to complete the mathematical study.

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ملخص

يهدف العمل الذي تم إنجازه في هذه الأطروحة إلى الدراسة الرياضية لبعض مشاكل ميكانيكا الاتصال مع الاحتكاك في سياق ديناميكي مع شروط حدودية. نعتبر القوانين التأسيسية المدروسة اللدونة الحرارية المرنة اللزجة، اللدونة الحرارية المرنة واللدونة الكهربائية اللزجة. لكل مشكلة، نحصل على الصيغة المتغيرة ثم نقوم بإثبات نتائج وجود و تفرد الحل الضعيف. تستند الأدلة إلى حجج تتعلق بالمتراجحات التطورية المتغيرة، متراجحات التباين المكافئة، المعادلات التفاضلية و نظرية النقطة الثابتة.

الكلمات المفتاحية: اللدونة الحرارية المرنة، اللدونة الحرارية المرنة اللزجة، اللدونة الكهربائية اللزجة، التآكل، التلف، ذاكرة طويلة المدى، احتكاك كولومب، احتكاك تريسكا، المتراجحات المتغيرة، حل ضعيف، نقطة ثابتة.

Résumé

Le travail réalisé dans cette thèse vise à l'étude mathématique de certains problèmes de mécanique de contact avec frottement dans un contexte dynamique avec des conditions aux limites. On considère des lois de comportement thermo-viscoélastiques, thermo-elasto-viscoplastique et électro-viscoélastiques. Pour chaque problème, nous obtenons la formulation variationnelle, puis nous établissons les résultats d'existence et d'unicité de la solution faible. Les preuves sont basées sur les arguments d'inéquations variationnelles d'évolution, d'inéquations paraboliques, d'équations différentielles et de théorème du point fixe.

Mots-clés: thermo-viscoélastique, thermo-élasto-viscoplastique, électro-viscoélastique, usure, endommagement, mémoire à long terme, frottement de Coulomb, frottement de Tresca, inéquation variationnelle, solution faible, point fixe.

Abstract

The work carried out in this thesis is devoted to the mathematical study of some problems of contact mechanics with friction in a dynamic context with boundary conditions. Thermo-elasto-viscoplastic, thermo-viscoelastic, and electro-viscoelastic constitutive laws are considered. For each problem, we derive a variational formulation and then establish existence and uniqueness results for a weak solution. The proofs are based on arguments of evolutionary variational inequalities, parabolic inequalities, differential equations, and a fixed point theorem.

Key words: thermo-viscoelastic, thermo-elasto-viscoplastic, electro-viscoelastic, wear, damage, long-term memory, Coulomb's friction, Tresca's friction, variational inequality, weak solution, fixed point.