وزارة التعليم العالي والبحث العلمي

Sétif 1 University-Ferhat ABBAS
Faculty of Sciences
Department of mathematics

COURSES AND EXERCISES

Maths1
Analysis and Algebra

For students of the common core sciences of matter

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Academic year 2024/2025

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Introduction

This handout corresponds to the program for the module: Mathematics 1, which is intended mainly for 1st year LMD students, in the Science of Matter specialty, and we have taken care to follow the official program.
The objective of this course is to allow the student to make a transition between the knowledge of mathematics acquired in secondary school and the bases of fundamental units such as Analysis (Continuity, Derivable function, etc.) and Algebra (Structure of a field, Vector space, Linear application, etc.),which will constitute one of the pillars in their training of the License. The chapters of this handout are designed as follows:
– The courses contain simple, precise and regular notions which allow the student to acquire a solid mathematical training necessary to profitably explore the vast field of Materials Science Field and they are also illustrated by examples.
– Each chapter of this course is equipped with solved exercises that allow you to go further in understanding and assimilating the mathematical concepts introduced. They help to provide a working method for learning mathematics and for obtaining a certain number of reflexes in solving problems. The learner must practice resolving the problem situation on their own without resorting to a proposed solution.
This document has five main chapters, where Sets, Relations, and Applications, Laws of Internal Composition, Structure of Real Number Fields R, Real Functions of a Real Variable, and Vector Spaces.

Chapter 1

Sets, relations and applications

Notations :

$:=$: means "define";	$\in$: means "belongs to" $a \in S$ means that "a is an element in S";
$\exists$: means "there exists";	$\exists!$: means "there exists a unique";
$\forall$: means "for all";	$\notin$: means "does not belong to";
$\subset$: means "contained in";	$\subseteq$: means "content or equal to";
$\not\subseteq$: means "is not contained in";	$\forall$: means "for all";
$\Rightarrow$: means "implies";	$\Longleftrightarrow$: means "if and only if".

Some famous sets :

· Set of Natural numbers is denoted by $\mathbb{N}$ ($\mathbb{N} = \{0, 1, 2, 3, ...\}$).

· Set of Integers is denoted by $\mathbb{Z}$ ($\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$).

· Set of Rational numbers is denoted by $\mathbb{Q}$ ($\mathbb{Q} = \{\frac{p}{q} \not\ p \in \mathbb{Z}, p \in (\mathbb{N}^*\}$).

· Set of Real numbers is denoted by $\mathbb{R}$ for example : 1, $\sqrt{3}$, π, $\ln 3$,...

· Set of Complex numbers $\mathbb{C}$ for example : $1 + 3i$,...

We will try to see the properties of sets, without focusing on a particular example. You will quickly realize that what is at least as important as sets are the relations between sets : this will be the notion of application (or function) between two sets.

1.1 Sets

1.1.1 Definition of sets

► A set is a collection of objects that verify certain properties. An object which satisfies the needed rules is called element of the set. If the set is denoted by A and x is an element of A, we say x belongs to A and we write $x \in A$

Example 1.1.1 *(i)* $A = \{0, 1\}$. *This means that the set* $\mathcal{A}$ *consists of two elements,* 0 *and* 1*.*

(ii) $B = \{x \in \mathbb{R} : -3 < x \leqslant 2\} =]-3, 2]$.

(iii) $C = \{0, \{1\}, \{0, 1\}\}$*. The set* C *contains three elements: the number* 0*; the set* $\{1\}$ *containing one element, namely the number* 1*; and the set containing two elements, the numbers* 0 *and* 1*.*

► The order in which the elements are listed is not important. Like this $\{0, 1\} = \{1, 0\}$. An element may occur more than once. So $\{1, 2, 1\} = \{1, 2\}$. But $\{1, 2, \{1\}\} \neq \{1, 2\}$!

A set can be also specified by an elementhood test.

1.1.2 Cardinality of a finite set

If a set A contains a finite number of elements it is said to be finite, otherwise it is said to be infinite. If A is finite and it contains $n \in \mathbb{N}$ elements, then n is called the cardinality of A we write $card\ A = n$ or $|A| = n$. If $n = 0$ the set A is called an empty set and is denoted by $\emptyset$ and we have $card\ A = 0$.

Definition 1.1.2 *The empty set is the set which contains no elements, and is denoted by* $\emptyset$*.*

In the previous example B is infinite set, $|A| = 2$ and $|C| = 3$.

1.1.3 Operations on sets

Now we introduce operations on sets. The main operations are: Inclusion, union, intersection, difference and symmetric difference.

Definition 1.1.3 *1. A set* A *is a subset of* B*,* $A \subset B$*, if every element of* A *is in* B*. Given* $A \subset B$*, if* $a \in A \Longrightarrow a \in B$*.*

2. *Two sets A and B are equal, $A = B$, if $A \subset B$ and $B \subset A$.*
3. *A set A is a proper subset of B, $A \subsetneq B$ if $A \subset B$ and$A \neq B$*

Thus, one way to show that two sets, A and B, coincide is to show that each element in A is contained in B and vice-versa.

Example 1.1.4 *We have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$*

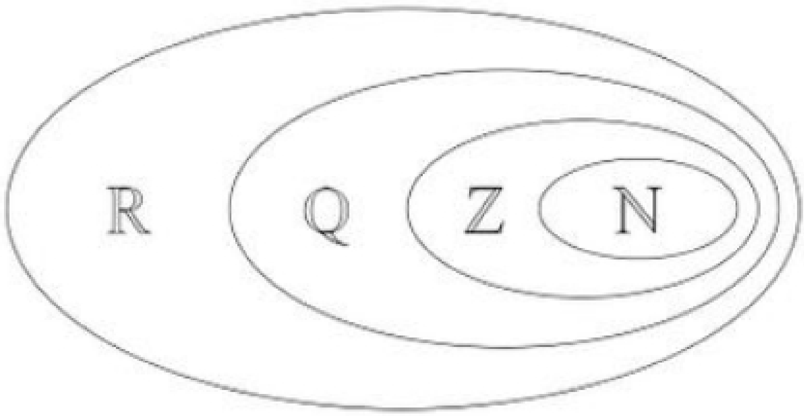

Figure 1: $\mathbb{N}\subset\mathbb{Z}\subset\mathbb{Q}\subset\mathbb{R}$

Definition 1.1.5 *The union of sets A and B is the set containing the elements of A and the elements of B, and no other elements.*

Notation 1 *We denote the union of A and B by $A \cup B$.*

Note: existence of the union for arbitrary A and B is accepted as an axiom.
For arbitrary x and arbitrary A and B the following proposition is true. $x \in A \cap B) \Leftrightarrow (x \in A) \wedge (x \in B)$.
$A \cap B = \{x : x \in A \text{ and } x \in B\}$.

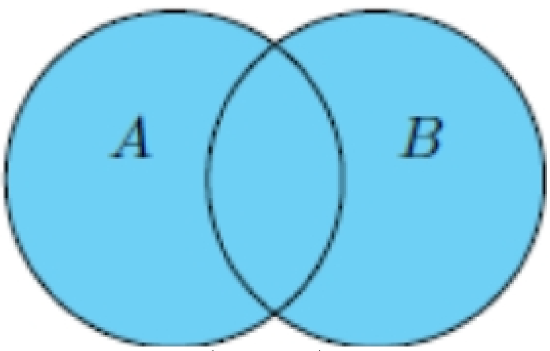

Figure 2: $A\cup B$

Definition 1.1.6 *The intersection of sets A and B is the set containing the elements which are elements of both A and B, and no other elements.*

We denote the intersection of A and B by $A \cap B$. Thus for arbitrary x and arbitrary A and B the following proposition is true. $x \in A \cap B) \Leftrightarrow (x \in A) \wedge (x \in B)$.
$A \cap B = \{x : x \in A \text{ and } x \in B\}$.

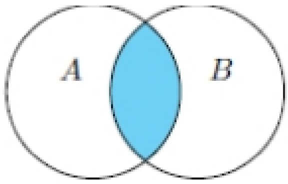

Figure 3: $A \cap B$

Note: When $A \cap B = \emptyset$, then A and B are said to be disjoint.

Definition 1.1.7 *The difference of sets A and B is the set containing the elements of A which do not belong to B.*

We use the notation $A - B$ or $A \backslash B$ for the difference or the complement of B with respect to A . The following is true for arbitrary x and arbitrary A and B : $x \in A - B) \Leftrightarrow [(x \in A) \wedge (x \not\in B)]$.
$A - B = \{x : x \in A \text{ and } x \notin B\}$

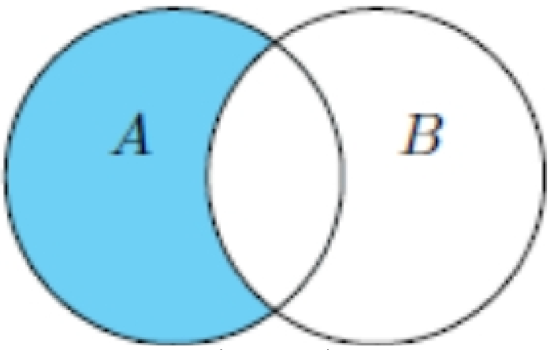

Figure 4: $A - B$

Definition 1.1.8 *The symmetric difference of the sets A and B is defined by:* $A \triangle B =$

$(A - B) \cup (B - A)$.

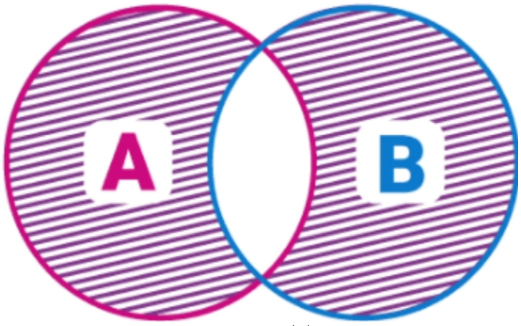

Figure 5: $A \triangle B$

Definition 1.1.9 *Suppose that* $A \subset U$. *The complement of the set* A *in* U *denoted by* A^c, $\complement_U(A)$ *or* $\bar{A}$, *is the set of all elements of* U *that are not in* A. *That is* $A^c = \{x \in U, x \notin A\}$.

Let us illustrate these operations with a simple example.

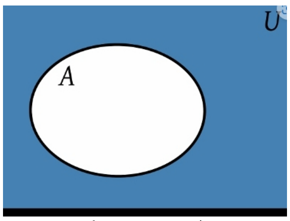

Figure 6: A^c

Example 1.1.10 *Let* $U = \mathbb{N}$, $A = \{0, 1, 2, 3, 4, 5\}$ *and* $B = \{1, 3, 5, 7, 9\}$. *Then*

$$\begin{aligned}
A \cup B &= \{0, 1, 2, 3, 4, 5, 7, 9\}. \\
A \cap B &= \{1, 3, 5\}. \\
A - B &= \{0, 2, 4\}. \\
B - A &= \{7, 9\}. \\
A \triangle B &= \{0, 2, 4, 7, 9\}. \\
A^c &= \{k : k \in \mathbb{N} \text{ and } k \geq 6\} = \{6, 7, ...\}
\end{aligned}$$

Note that

$$A \cup B = (A \cap B) \cup (A \triangle B)$$

1.1.4 Laws for operations on sets

Let A, B be subsets of an universal set U

Idempotent Laws	**(a)** $A \cup A = A$	**(b)** $A \cap A = A$
Associative Laws	**(a)** $(A \cup B) \cup C = A \cup (B \cup C)$	**(b)** $(A \cap B) \cap C = A \cap (B \cap C)$
Commutative Laws	**(a)** $A \cup B = B \cup A$	**(b)** $A \cap B = B \cap A$
Distributive Laws	**(a)** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	**(b)** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Morgan's Laws	**(a)** $(A \cup B)^c = A^c \cap B^c$	**(b)** $(A \cap B)^c = A^c \cup B^c$
Identity Laws	**(a)** $A \cup \emptyset = A$ **(b)** $A \cup U = U$	**(a)** $A \cap \emptyset = \emptyset$ **(b)** $A \cap U = A$
Complement Laws	**(a)** $A \cup A^c = U$ **(b)** $A \cap A^c = \emptyset$	**(a)** $U^c = \emptyset$ **(b)** $\emptyset^c = U$
Involution Law	**(a)** $(A^c)^c = A$	

A few demonstrations * $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$?

$x \in A \cap (B \cup C) \Leftrightarrow (x \in A \text{ and } x \in (B \cup C))$
$\Leftrightarrow (x \in A \text{ and } (x \in B \text{ or } x \in C))$
$\Leftrightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)$
$\Leftrightarrow (x \in A \cap B) \text{ or } (x \in A \cap C)$
$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$.

Then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$?

$x \in A \cup (B \cap C) \Leftrightarrow x \in A \text{ or } x \in B \cap C)$
$\Leftrightarrow x \in A \text{ or}(x \in B \text{ and } x \in C)$
$\Leftrightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)$
$\Leftrightarrow x \in A \cup B \text{ and } x \in A \cup C$
$\Leftrightarrow x \in (A \cup B) \cap (A \cup C)$.

Then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

* $\complement_U(A \cap B) = \complement_U(A) \cup \complement_U(B)$ and $\complement_U(A \cup B) = \complement_U(A) \cap \complement_U(B)$?

- $x \in \complement_U(A \cap B) \Leftrightarrow x \notin A \cap B$
$\Leftrightarrow x \notin A \text{ or } x \notin B$
$\Leftrightarrow x \in \complement_U(A) \text{ or } x \in \complement_U(B)$
$\Leftrightarrow x \in \complement_U(A) \cup \complement_U(B)$.

Therefore $\complement_U(A\cap B)=\complement_U(A)\cup\complement_U(B)$.

- $x\in\complement_U(A\cup B)\Leftrightarrow x\notin A\cup B$

$$\Leftrightarrow x\notin A \text{ and } x\notin B$$
$$\Leftrightarrow x\in\complement_U(A) \text{ and } x\in\complement_U(B)$$
$$\Leftrightarrow x\in\complement_U(A)\cap\complement_U(B)$$

Therefore $\complement_U(A\cup B)=\complement_U(A)\cap\complement_U(B)$.

* $\complement_U\left(\complement_U(A)\right)=A$?

$x\in\complement_U\left(\complement_U(A)\right)\Leftrightarrow x\notin\complement_U(A)$

$$\Leftrightarrow x\in A.$$

1.1.5 Set of parts.

Definition 1.1.11 *Let E be a set, we form a set called the set of parts of E, denoted $P(E)$ which is characterized by the following relation $P(E)=\{A:A\subset E\}$.*

Example 1.1.12 *Let $E=\{0,1,2\}$. Then*

$$P(E)=\{\emptyset,\{0\},\{1\},\{2\},\{0,1\},\{0,2\},\{1,2\},\{0,1,2\}\}.$$

Remark 1.1.13 *If card $(A)=n$ then card $(P(A))=2^n$.*

Example 1.1.14 - *If $E=\{a,b\}$,then*

$$P(E)=\{\emptyset,\{a\},\{b\},\{a,b\}\},$$

as card $(E)=2$, then card $(P(E))=2^2=4$.

- *If $E=\{a\}$, then $P(A)=\{\emptyset,\{a\}\}$.*

1.1.6 Cartesian product

Definition 1.1.15 *. Let A and B be sets. The Cartesian product of A and B, denoted by $A\times B$, is the set of all ordered pairs (a,b) in which $a\in A$ and $b\in B$, i.e.*

$A\times B=\{(a,b):(a\in A)\wedge(b\in B)\}$.

Thus

$$p\in A\times B\Leftrightarrow\exists a\in A,\ \exists b\in B\ /\ p=(a,b).$$

Example 1.1.16 *(i) If $A=\{red,green\}$ and $B=\{1,2,3\}$ then*

$$A\times B=\{(red,1),(red,2),(red,3),(green,1),(green,2),(green,3)\}.$$

(ii) $\mathbb{Z}\times\mathbb{Z}=\{(x,y)|\ x$ and y are integers$\}$. This is the set of integer coordinates points in the $x,y-$plane. The notation $\mathbb{Z}^2$ is usually used for this set.

Example 1.1.17 *If* $E = \{1, 2\}$ *and* $F = \{3, 5\}$, *then*

$$E \times F = \{(1, 3), (1, 5), (2, 3), (2, 5)\}$$
$$F \times E = \{(3, 1), (3, 2), (5, 1), (5, 2)\}$$

$E \times F \neq F \times E$

Example 1.1.18 *1)* $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : x, y \in \mathbb{R}\}$
2) $[0, 1] \times \mathbb{R} = \{(x, y) : 0 \leq x \leq 1,\ y \in \mathbb{R}\}$

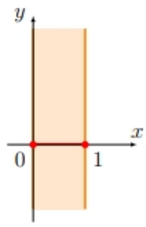

Figure 7: [0,1]×ℝ

Example 1.1.19 $[0, 1] \times [0, 1] \times [0, 1] = \{(x, y, z) : 0 \leq x, y, z \leq 1\}$

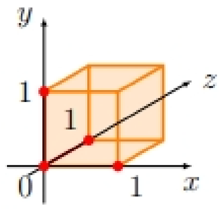

Figure 8: [0,1]×ℝ

Notation 2 *Let* E^2 *be the Cartesian square of* E. *More generally, we define the Cartesian product of* n *sets* E_1, E_2, ...,E_n *by*

$$E_1 \times E_2 \times ... \times E_n = \{(x_1, x_2, ..., x_n) : xi \in E_i \ ,\ for\ i = 1, ..., n\}.$$

Example 1.1.20 *If* $E = \{1, 2\}$, *then*

$E^2 = E \times E = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.
$E^3 = E \times E \times E = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 2, 1), (2, 1, 2), (2, 2, 2)\}$.

Proposition 1.1.21 *Let* E *and* F *be two finite sets.Then*
$card\ (E \times F) = card\ (E) \times card\ (F)$.

The following theorem provides some basic properties of the Cartesian product.

Theorem 1.1.22 *Let A, B, C and D be sets. Then*

a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$,

b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$,

c) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$,

d) $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$,

e) *If A or B are empty sets ($(A = \emptyset$ and $B \neq \emptyset)$ or $(B = \emptyset$ and $A \neq \emptyset)$ or $(A = \emptyset$ and $B = \emptyset)$), then $A \times B = \emptyset$.*

Proof. (**a**) $(\Rightarrow)$

Let $p \in A \times (B \cap C)$. Then

$$\exists a \in A,\ \exists x \in B \cap C \ / \ p = (a, x)$$

In particular,

$$(\exists a \in A,\ \exists x \in B \ / \ p = (a, x)) \text{ and } (\ \exists a \in A,\ \exists x \in C \ / \ p = (a, x)) \,.$$

So $p \in (A \times B) \cap (A \times C)$.

(**a**) $(\Leftarrow)$

Let $p \in (A \times B) \cap (A \times C)$. Then

$$p \in (A \times B) and p \in (A \times C).$$

So

$$(\exists a \in A,\ \exists b \in B \ / \ p = (a, b)) \text{ and } (\exists \grave{a} \in A,\ \exists c \in C \ / \ p = (\grave{a}, c)) \,.$$

But then $(a, b) = p = (\grave{a}, c)$ and hence $a = \grave{a}$ and $b = c$. Thus $p = (a, x)$ for some $a \in A$ and $x \in B \cap C$, i.e.$p \in A \times (B \cap C)$. This proves (**a**) **.** ■

The proof of (**b**), (**c**), (**d**) and (**e**) are left as exercises.

1.2 Relations

Definition 1.2.1 *We call a relation $\mathcal{R}$ from E to F any part of the Cartesian product $E \times F$. The domain of $\mathcal{R}$ is the set*

$$D(\mathcal{R}) = \{x \in E : \exists y \in F \ [(x, y) \in \mathcal{R}]\}.$$

The range of $\mathcal{R}$ is the set

$$Ran(\mathcal{R}) = \{y \in F : \exists x \in E[(x, y) \in \mathcal{R}]\}.$$

If $E = F$, we say that $\mathcal{R}$ is a binary relation on E.
The inverse of $\mathcal{R}$ is the relation $\mathcal{R}^{-1}$ from F to E defined as follows

$$\mathcal{R}^{-1} = \{(y, x) \in F \times E : (x, y) \in \mathcal{R}\}.$$

The graph of this relation is:

$$G_R = \{(x, y) \in E \times F : x \ \mathcal{R} \ y \ \}$$

Example 1.2.2 ***(i)*** *Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$. The set $\mathcal{R} = \{(1, 3), (1, 5), (3, 3)\}$ is a relation from A to B since $\mathcal{R} \sqsubseteq A \times B$.*

(ii) *$G = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x > y\}$ is a relation from $\mathbb{Z}$ to $\mathbb{Z}$.*

Example 1.2.3 *Let $A = \{1, 2, 3, 4, 5, 6\}$ be a set and the relation $\mathcal{R}$ is defined by*

$$x\mathcal{R}y \Leftrightarrow x \text{ divide } y \text{ (in } \mathbb{Z})$$

$$\begin{aligned} G_R &= \{(x, y) \in A \times A, \quad x \text{ divide } y \ \} \\ &= \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (5,5), (6,6)\}. \end{aligned}$$

Definition 1.2.4 *Let $\mathcal{R}$ be a binary relation over a set E. For all $x, y, z \in E$, we say that $\mathcal{R}$ is*

(1) Reflexive*, if each element is related to itself, i.e*

$$x\mathcal{R}x, \forall x \in E.$$

(2) Symmetric*, if for all $x, y \in E$, if x is related to y then y is related to x, i.e. $x\mathcal{R}y \Rightarrow y\mathcal{R}x, \forall x, y \in E$.*

(3) Transitive*, if for all $x, y, z \in E$, if x is in relation to y and y in relation to z then x is in relation to z, i.e. $(x\mathcal{R}y$ and $y\mathcal{R}z) \Rightarrow x\mathcal{R}z, \forall x, y, z \in E$.*

(4) Anti-symmetric*, if two elements are related to each other, then they are equal, i.e.*

$$(x\mathcal{R}y \text{ and } y\mathcal{R}x) \Rightarrow x = y, \forall x, y \in E.$$

A particularly important class of relations are equivalence relations.

1.2.1 Equivalence relation

Definition 1.2.5 *A relation $\mathcal{R}$ on E is called equivalence relation if it is reflexive, symmetric and transitive.*

Example 1.2.6 ***(i)*** *Let E be a set of students. A relation on $E \times E$: "to be friends". It is reflexive (I presume that everyone is a friend to himself / herself). It is symmetric. But it's not transitive.*

(ii) *Let $E = Z$,$a \in N$. Define $\mathcal{R} \subseteq E \times E$ as*

$$\mathcal{R} = \{(x, y) : |x - y| \leq a\}.$$

$\mathcal{R}$ is reflexive, symmetric, but not transitive.

(iii) *Let $E = Z$, $m \in N$. Define the congruence $\bmod m$ on $E \times E$ as follows:*

$$x \equiv y \text{ if } (\exists k \in \mathbb{Z} : x - y = km).$$

This is an equivalence relation on E.

Definition 1.2.7 *Let $\mathcal{R}$ be an equivalence relation on E.*

1. *The equivalence class of an element x in E is the set of all elements $y \in E$ that are in relation with x we denote this set by $\dot{x}$, $\bar{x}$ or $\mathcal{C}(x)$,and we write it as follow*

$$\dot{x} = \bar{x} = \mathcal{C}(x) = \{y \in E : y\mathcal{R}x\}.$$

2. *$\bar{x}$ is a representative of the equivalence class $\mathcal{C}(x)$.*

3. *The set of equivalence classes for all elements in E is called the "quotient set" of E for the equivalence relation $\mathcal{R}$. It is denoted as $E/\mathcal{R}$, and written as follows:*

$$E/\mathcal{R} = \{\mathcal{C}(x) : x \in E\}.$$

Example 1.2.8 *In $\mathbb{R}$ we define the relation $\mathcal{R}$ by:*

$$x\mathcal{R}y \Leftrightarrow x - y \in \mathbb{Z}.$$

This relation is indeed a relation of equivalence. Indeed,

- *For $x \in \mathbb{R} : x\mathcal{R}x \Leftrightarrow 0 \in \mathbb{Z}$, as $0 \in \mathbb{Z}$, then $x\mathcal{R}x$,$\forall x \in \mathcal{R}$, so $\mathcal{R}$ is a reflexive relation.*
- *For $x, y \in \mathbb{R}$, we have $(x\mathcal{R}y) \Leftrightarrow (x - y \in \mathbb{Z}) \Leftrightarrow (y - x \in \mathbb{Z}) \Rightarrow y\mathcal{R}x$, then $\mathcal{R}$ is a symmetric relation.*

- *For $x, y, z \in \mathbb{R}$, we have*

$$\begin{aligned}(x\mathcal{R}y \text{ and } y\mathcal{R}z) &\Rightarrow (x-y \in \mathbb{Z} \text{ and } y-z \in \mathbb{Z})\\ &\Rightarrow (x-y+y-z \in \mathbb{Z})\\ &\Rightarrow (x-z \in \mathbb{Z}) \Rightarrow (x\mathcal{R}z),\end{aligned}$$

then $\mathcal{R}$ is a transitive relation.
Therefore, the set of equivalence classes $\mathcal{C}(x)$ is the set

$$\begin{aligned}\mathcal{C}(x) &= \{y \in \mathbb{R} : y-x \in \mathbb{Z}\}\\ &= \{y \in \mathbb{R} : y \in x+\mathbb{Z}\}\\ &= \{y \in \mathbb{R} : y = k+x : k \in \mathbb{Z}\}\\ &= \{k+x : k \in \mathbb{Z}\},\end{aligned}$$

if $x \in \mathbb{Z}$, we have $\mathcal{C}(x) = \mathbb{Z}$.

Exercise 1.2.9 *Let us consider the relation $\mathcal{R}$ defined on $\mathbb{R}$ by :*

$$\forall x, y \in \mathbb{R},\ x\mathcal{R}y \Leftrightarrow xe^y = ye^x$$

Prove that $\mathcal{R}$ is an equivalence relation.

Solution 1.2.10 *We show that $\mathcal{R}$ is reflexive, symmetric and transitive..*
1.$\forall x \in \mathbb{R}$ *we have $xe^x = xe^x$. In other words, we have $x\mathcal{R}x$ and then $\mathcal{R}$ is reflexive.*
2. *$\mathcal{R}$ is symmetric . In fact, let x, $y \in \mathbb{R}$, such that $x\mathcal{R}y$, hence we have*

$$\begin{aligned}x\mathcal{R}y &\Rightarrow xe^y = ye^x,\\ &\Rightarrow ye^x = xe^y,\\ &\Rightarrow y\mathcal{R}x.\end{aligned}$$

3. *$\mathcal{R}$ is transitive because for all x, y, $z \in \mathbb{R}$, such that $(x\mathcal{R}y) \wedge (y\mathcal{R}z)$, we have :*

$$x\mathcal{R}y \Rightarrow xe^y = ye^x \qquad (1)$$
$$y\mathcal{R}z \Rightarrow ye^z = ze^y \qquad (2)$$

(2) *gives $y = \dfrac{ze^y}{e^z}$, moreover, using* (1) *and by substituting y we have $xe^y = \dfrac{ze^y}{e^z}e^x$ hence $xe^ye^z = ze^ye^x$. Since $e^y \neq 0$ Thus $xe^z = ze^x$, which implies $x\mathcal{R}z$.*
4. *$\mathcal{R}$ is reflexive, symmetric and transitive then it is an equivalence relation.*

1.2.2 Order relation

Definition 1.2.11 *A binary relation $\mathcal{R}$ over E is said to be an order relation if it is antisymmetric, transitive and reflexive.*

Example 1.2.12 *On $\mathbb{R}$ the relation $\leq$ is an order relation. In fact*
- *$\mathcal{R}$ is reflexive*

$$\forall x \in \mathbb{R}, x\mathcal{R}z \Leftrightarrow x = x.$$

- *$\mathcal{R}$ is antisymmetric, if only if:*

$$\forall (x, y) \in \mathbb{R} \times \mathbb{R}; (x\mathcal{R}y \Leftrightarrow x \leq y) \text{ and } (y\mathcal{R}x \Leftrightarrow y \leq x) \Leftrightarrow x = y.$$

-*$\mathcal{R}$ transitive, if only if :*

$$\forall (x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}; \; (x\mathcal{R}y \Leftrightarrow x \leq y) \text{ and } (y\mathcal{R}z \Leftrightarrow y \leq z) \Leftrightarrow x \leq z \Leftrightarrow x\mathcal{R}z.$$

- *In $\mathbb{R}$, the relation $<$ is not a relation of order (it is not reflexive.)*

Total order and partial order

Definition 1.2.13 *Let $\mathcal{R}$ be an order relation defined on a set E, we say the ordernis total, if for all $x, y \in E$, we have*

$$x\mathcal{R}y \text{ or } y\mathcal{R}x.$$

Otherwise, we say that $\mathcal{R}$ is a partial order relation, i.e.

$$\exists x, y \in E : \text{neither } x\mathcal{R}y \text{ nor } y\mathcal{R}x$$

Example 1.2.14 *$A = \{1, 2, 3, 4, 5, 6\}$ with*

$$a\mathcal{R}b \Leftrightarrow a \text{ divide } b$$

is a partial order relation (it is not total).
Indeed 2 and 3, for example, are not comparable : 2 does not divide 3 and 3 does not divide 2.

Example 1.2.15 *Let A be a non-empty set and $\mathcal{R}$ a relation on A defined by :*

$$\forall a, b \in A, a\mathcal{R}b \Leftrightarrow a = b.$$

$\mathcal{R}$ is a an order relation on A.

If A is a singleton, then the order is total, if not, the order is partial.

1.3 Applications

1.3.1 Definition of an application

Definition 1.3.1 *Let E and F be given sets, we call the application of E in F, any correspondence f between the elements of E and those of F which associates to any element of E one and only element of F, we write*

$$\begin{aligned} f: \quad & E \to F \\ & x \longrightarrow f(x) \end{aligned}$$

or $(f$ *is an application*$) \Leftrightarrow (\forall x \in E, \exists! y \in F / y = f(x))$.

The set E is said to be the starting set and F is said to be the end set.
The element x is said to be the antecedent and y is said to be the image of x by f.
The map f is said to be a function if, for each $x \in E$, there exists at most $y \in F$ such that $f(x) = y$.

Remark 1.3.2 ***(1)*** *The application from E to F if every element x of E has a unique image in F.*

(2) *If f is an application from E to F, then the element y of F can have more than one antecedent in E.*

(3) *We must differentiate between $f(x)$ and f : we have $f(x) \in F$, while f represents the application as a whole, and it belongs to the space of applications defined from E to F.*

Example 1.3.3 *We have $A = \{1, 2, 3\}$ and $B = \{7, 9, 13\}$.*

- *We have $f(3) = 9$, $f(2) = 9$; $f(1) = 7$.*
- *f is an application from A to B. Every element x of A has a unique image in B.*

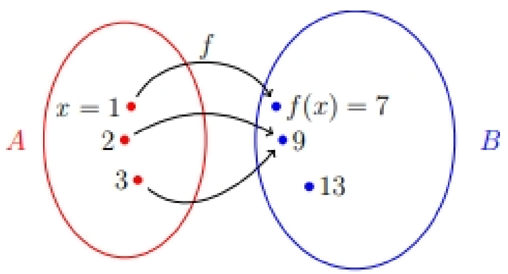

Figure 9 : *$f : A \longrightarrow B$*

- *This element 13 has no precedent according to the application.*
- *This element 9 has two precedent : 2 and 3.*

Definition 1.3.4 *(Graph). Let E and F be given sets. The graph of a map $f : E \to F$ is*

$$G_f := \{(x, f(x)) : x \in E\} \subset E \times F.$$

Definition 1.3.5 *(Equality). Let $f, g : E \to F$ be two applications. We say that f and g are equal if and only if for all $x \in E : f(x) = g(x)$. We then write $f = g$.*

Definition 1.3.6 *(Composition). Let E, F and G be three sets and f and g two maps such as*

$$E \xrightarrow{f} F \xrightarrow{g} G$$

We can deduce from this a map of E in G denoted $g \circ f$ and called a map composed of f and g, by

$$(g \circ f)(x) = g(f(x)), \qquad \text{for all} \quad x \in E.$$

Figure 10 : $g \circ f$

Exercise 1.3.7 *Let $f : Z \to Z$, $g : Z \to Z$,*

$$f(x) = x^2 + 2, \;\; g(x) = 2x - 1.$$

Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

Solution 1.3.8 *we have*

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) = g(x)^2 + 2 = 4x^2 - 4x + 3, \\ (g \circ f)(x) &= g(f(x)) = 2f(x) - 1 = 2x^2 + 3.\end{aligned}$$

As you clearly see from the above,$f \circ g \neq g \circ f$ in general.

Definition 1.3.9 *Let E be a set, we call an identity map, denoted $Id_E : E \to E$, the map that verifies $Id_E(x) = x$, $\forall x \in E$.*

Definition 1.3.10 *Let $f : E \to F$ be a function. The domain of definition of f, denoted D_f, is the set of elements $x \in E$ in which there exists a single element $y \in F$, such that $y = f(x)$.*

Example 1.3.11 *Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sqrt{x+1}$, then*

$$D_f = \{x \in \mathbb{R} : x + 1 \geq 0\} = [-1, +\infty[.$$

1.3.2 Restricting and extending an application

Definition 1.3.12 *Let $A \subset E$ and $f : E \to F$ be an application. We call the restriction from f to A, the map $f_{/A} : A \to F$ defined by*

$$f_{/A}(x) = f(x), \text{ for all } x \in A.$$

Definition 1.3.13 *Let $E \subset G$ and $f : E \to F$ a map. We call an extension from f to G, any map g from G to F whose restriction to E is f.*

Example 1.3.14 *Given the application :*

$$\begin{array}{ll} f : \mathbb{R}_+^* & \to R \\ \quad x & \to \ln x \end{array},$$

then

$$\begin{array}{ll} g : \mathbb{R}^* & \to R \\ \quad x & \to \ln |x| \end{array}, \qquad \begin{array}{ll} h : \mathbb{R}^* & \to R \\ \quad x & \to \ln (|3x| - 2x) \end{array},$$

are two different extensions of f to $\mathbb{R}^*$.

1.3.3 Direct image and inverse image

Definition 1.3.15 *Let A and B be non-empty sets. Let E be a subset of A, and $f : A \to B$ be an application. The direct image of the set E is defined by :*

$$f(E) = \{f(x) : x \in E\}$$

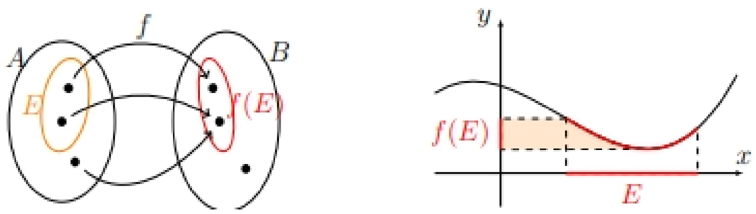

Figure 11 : $f(E)$

Example 1.3.16 *Let $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^2$. Let*

$$A = \{x \in \mathbb{Z} : 0 \leq x \leq 2\}.$$

Then $f(A) = \{0, 1, 4\}$.

Exercise 1.3.17 *We consider the application $f : [0, +\infty[\to \mathbb{R}$ defined by $f(x) = x^2 + 2$.*

Show that

$$f([0, +\infty[) = [2, +\infty[\,.$$

Solution 1.3.18 *It suffices to demonstrate the double inequality.*

1) That is, first, showing that $f([0,+\infty[) \subset [2,+\infty[$

Let

$$\begin{aligned} x \in [0,+\infty[\ &\Rightarrow x^2+2 \geq 2 \\ &\Rightarrow f(x) \geq 2 \\ &\Rightarrow f(x) \in [2,+\infty[\,. \end{aligned}$$

Therfore, $f([0,+\infty[) \subset [2,+\infty[\,.$

2) Now let's show the opposite $(f([0,+\infty[) \supset [2,+\infty[)$.

It means tha $\forall y \in [2,+\infty[$, $\exists ? x \in [0,+\infty[$ *such as* $f(x)=y$.

We solve the equation

$$\begin{aligned} y = x^2+2 \ &\Leftrightarrow y-2 = x^2 \\ &\Rightarrow x = \pm\sqrt{y-2}, \end{aligned}$$

or $x = \sqrt{y-2} \geq 0$ *So*

$$\forall y \in [2,+\infty[\,, \exists x \left(x = \sqrt{y-2}\right) \in [0,+\infty[/ f(x) = y$$

finally $f([0,+\infty[) = [2,+\infty[\,.$

Definition 1.3.19 *Let* A *and* B *be non-empty sets, let* F *be a subset of* B*, and* $f : A \to B$ *be an application. The inverse image of the set* F *is defined by :*

$$f^{-1}(F) = \{x \in A : \ f(x) \in F\}$$

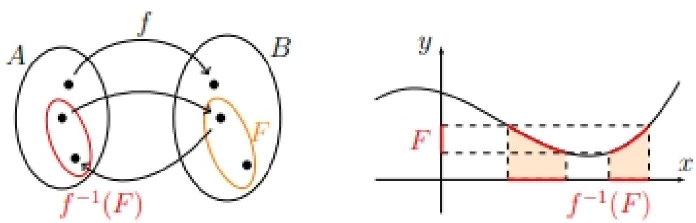

Figure 12 : $f^{-1}(F)$

Example 1.3.20 *Let* $f : Z \to Z$ *defined by* $f(x) = x^2$*, let* $B = \{y \in Z : y \leq 10\}$*. Then* $f^{-1}(B) = \{-3,-2,-1,0,1,2,3\}$.

Theorem 1.3.21 *Let* $f : X \to Y$ *and* $A_1 \subset X$, $A_2 \subset X$, $B_1 \subset Y$, $B_2 \subset Y$*. Then*

(i) *)* $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$ *and* $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$.

(ii) $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$ *and* $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

(iii) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$ *and* $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.

(iv) $A_1 \subset f^{-1}(f(A_1))$ *and* $f\left(f^{-1}(B_1)\right) \subset B_1$.

1.3.4 Injective, surjective and bijective application

Definition 1.3.22 *Let $f : E \to F$. f is said to be injective if and only if :*

$$\forall (x_1, x_2) \in E^2 : f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

Example 1.3.23

$$\begin{aligned} f : \mathbb{R}^+ / \{2\} &\longrightarrow \mathbb{R} \\ x &\longrightarrow \frac{1}{x^2 - 4} \end{aligned}$$

is an injective application because we have :

$$\forall (x_1, x_2) \in \left(\mathbb{R}^+ / \{2\}\right)^2 : \ f(x_1) = f(x_2) \Leftrightarrow \frac{1}{x_1^2 - 4} = \frac{1}{x_2^2 - 4} \Leftrightarrow x_1^2 = x_2^2 \Leftrightarrow x_1 = \pm x_2,$$

but as x_1, $x_2 \in \mathbb{R}^+ / \{2\}$ then $x_1 = x_2$.

Definition 1.3.24 *Let $f : E \to F$. We say that f is surjective if and only if: for all $y \in F$, there exists $x \in E$ such that $f(x) = y$, i.e.*

$$\forall y \in F, \exists x \in E : \ y = f(x).$$

Example 1.3.25 *Let $f : \mathbb{Z} \to \mathbb{N}$, be the map defined by $f(x) = |x|$, then f is surjective. Indeed, let $y \in \mathbb{N}$, for $x = y$ or $x = -y$, we have $x \in \mathbb{Z}$ and $f(x) = |x| = y$, so there exists $x \in \mathbb{Z}$ such that $y = f(x)$.*

Definition 1.3.26 *Let $f : E \to F$. f is said to be bijective if and only if: f is both injective and surjective. This is equivalent to : for all $y \in F$ there exists a unique $x \in E$ such that $y = f(x)$. In other words:*

$$\forall y \in F, \exists ! x \in E : y = f(x).$$

Example 1.3.27 *Let $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x + 1$, then f is bijective. Indeed, let $y \in \mathbb{R}$, such that $f(x) = y$, then $x = y - 1$, so there exists a unique x in $\mathbb{R}$ such that $y = f(x)$.*

Remark 1.3.28 *If the application f is bijective, then to every $y \in F$ we match a single element $x \in E$.*

Definition 1.3.29 *Let $f : E \to F$ be a bijective function. We define the function $f^{-1} : F \to E$, called the reciprocal function of f, given by $f^{-1}(x) = y$ if and only if $f(y) = x$.*

Example 1.3.30 *Let f be the map defined by $f(x) = x^2 + 1$ of $\mathbb{R}^+ \to [1, +\infty[$, then f is bijective, because for all $y \in [1, \infty[$, the equation $y = f(x)$ admits a single solution $x = \sqrt{y-1}$. The reciprocal bijection is $f^{-1} : [1, +\infty[\to \mathbb{R}+$ defined by:*

$$f^{-1}(x) = \sqrt{x-1} \textit{ for all } x \in [1, +\infty[.$$

Proposition 1.3.31 *Let E, F be sets and $f : E \to F$ an application.*

- *The map f is bijective if and only if there is a map $g : F \to E$ such that*

$$f \circ g = Id_F \textit{ and } g \circ f = Id_E.$$

- *Let $f : E \to F$ and $g : F \to G$ be bijective maps. The map $g \circ f$ is bijective and its reciprocal bijection is*

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

1.4 Some methods of proof

1. First we discuss a couple of widely used methods of proof: contrapositive proof and proof by contradiction.

   The idea of contrapositive proof is the following equivalence

$$(A \Rightarrow B) \Leftrightarrow (\bar{B} \Rightarrow \bar{A}).$$

So to prove that$A \Rightarrow B$ is true is the same as to prove that $\bar{B} \Rightarrow \bar{A}$ is true.

Exercise 1.4.1 *For integers m and n, if mn is odd then so are m and n.*

Solution 1.4.2 *We have to prove that*

$$(\forall m, n \in \mathbb{Z}_+)(mn \textit{ is odd}) \Rightarrow [(m \textit{ is odd}) \wedge (n \textit{ is odd})],$$

which is the same as to prove that

$$[(m \textit{ is even}) \vee (n \textit{ is even})] \Rightarrow (mn \textit{ is even})$$

The latter is evident.

The idea of proof by contradiction is the following equivalence

$$(A \Rightarrow B) \Leftrightarrow (\bar{A} \vee B) \Leftrightarrow \overline{(A \wedge \bar{B})}$$

So to prove that $A \Rightarrow B$ is true is the same as to prove that $\bar{A} \vee B$ is true or elsethat $A \wedge \bar{B}$ is false.

2 The Principle of Mathematical Induction is often used when one needs to prove statements of the form

$$(\forall n \in \mathbb{N})\ P(n).$$

Thus one can show that 1 has property P and that whenever one adds 1 to a number that has property P, the resulting number also has property P.

Principle of Mathematical Induction. If for a statement $P(n)$

(i) $P(1)$is true,

(ii) $[P(n) \Rightarrow P(n+1)]$ is true,

then $(\forall n \in N)\ P(n)$ is true.

Part **(i)** is called the basic case; **(ii)** is called the induction step.

Example 1.4.3 *Prove that*

$$\forall n \in N : 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution: *Basic case: $n = 1$. $1^2 = \frac{1 \cdot 2 \cdot 3}{6}$ is true.*

Induction step: Suppose that the statement is true for $n = k (k \geq 1)$. We have to prove that it is true for $n = k + 1$. So our assumption is

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Therefore we have

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6},$$

which proves the statement for $n = k + 1$. By the principle of mathematical induction the statement is true for all $n \in \mathbb{N}$.

Chapter 2

Structure of real numbers field $\mathbb{R}$

The aim of this chapter is to introduce axiomatically the set of Real numbers

2.1 Set of rational numbers $\mathbb{Q}$.

2.1.1 Integers numbers

We take for granted the system $\mathbb{N}$ of natural numbers $\mathbb{N} = \{1, 2, 3, 4...\}$. In general the equation $x + a = 0$ is not solvable in $\mathbb{N}$ whose case or a is positive. In order to make this equation solvable, we must enlarge the set $\mathbb{N} = \mathbb{Z}_+$ by introducing negative integers as unique solutions of the equations $a + x = 0$ (existence of the additive inverse) for each $a \in \mathbb{N}$. Our extended system, which is denoted by $\mathbb{Z}$, now contains all integers and can be arranged in order

$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\} = \mathbb{N} \cup \{0\} \cup \{-a \ : a \in \mathbb{N}\}.$$

Theorem 2.1.1 (Fundamental theorem of arithmetic) *Every positive integer except* 1 *can be expressed uniquely as a product of primes.*

2.1.2 Rational numbers

Let $a \in \mathbb{Z}$, $b \in \mathbb{Z}$. The equation

$$ax = b \tag{1}$$

need not have a solution $x \in \mathbb{Z}$. In order to solve (1) (for $a \neq 0$) we have to enlarge our system of numbers again so that it includes fractions $\frac{b}{a}$ (existence of multiplicative inverse in $\mathbb{Z} - \{0\}$.This motivates the following definition.

Definition 2.1.2 *The set of rational numbers (or rationals) $\mathbb{Q}$ is the set*

$$\mathbb{Q} = \{r = \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}, hcf(p,q) = 1\}.$$

Here $hcf(p,q)$ stands for the highest common factor of p and q, so when writing $\frac{p}{q}$ for a rational we often assume that the numbers p and q have no common factor greater than 1.

Definition 2.1.3 *Let $b \in \mathbb{N}$, $d \in \mathbb{N}$. Then*

$$\left(\frac{a}{b} > \frac{c}{d}\right) \Leftrightarrow (ad > bc)$$

The following theorem provides a very important property of rationals.

Theorem 2.1.4 *Between any two rational numbers there is another (and, hence, infinitely many others).*

Proof. Let $b \in \mathbb{N}$, $d \in \mathbb{N}$, and $\frac{a}{b} > \frac{c}{d}$.
Notice that

$$\forall m \in \mathbb{N} : \frac{a}{b} > \frac{a+mc}{b+md} > \frac{c}{d}.$$

Indeed, since b, d and m are positive we have

$$[a(b+md) > b(a+mc)] \Leftrightarrow [mad > mbc] \Leftrightarrow (ad > bc),$$

and

$$[d(a+mc) > c(b+md)] \Leftrightarrow (ad > bc).$$

■

2.2 Irrational numbers

Suppose that $a \in Q^+$ and consider the equation

$$x^2 = a. \tag{2}$$

In general (2) does not have rational solutions. For example, the following theorem holds.

Theorem 2.2.1 *No rational number has square* 2.

Proof. Suppose for a contradiction that the rational number $\frac{p}{q}(p \in \mathbb{Z},\ q \in \mathbb{N}$, in lowest terms) is such that $(\frac{p}{q})^2 = 2$. Then $p^2 = 2q^2$.

Hence, appealing to the Fundamental Theorem of Arithmetic, p^2 is even, and hence p is even. Thus $(\exists k \in \mathbb{Z})\ [p = 2k]$. This implies that

$$2k^2 = q^2,$$

and therefore q is also even. The last statement contradicts our assumption that p and q have no common factor. ■

The last theorem provides an example of a number which is not rational. We call such numbers irrational.

We leave the following as an exercise.

Exercise 2.2.2 *No rational x satisfies the equation*

- $x^3 = x + 7$.
- $x^5 = x + 4$.

2.3 Real numbers

Real numbers can be defined as the union of both rational and irrational numbers. They can be both positive or negative and are denoted by the symbol "$\mathbb{R}$". All the natural numbers, decimals and fractions come under this category.In this course we postulate the existence of the set of real numbers $\mathbb{R}$ as well as basic properties summarized in a collection of axioms.Will find that axioms $A.1 - A.11$ characterize $\mathbb{R}$ as an algebraic field.

2.3.1 Axiomatic definition

$A.1$ $\forall a, b \in \mathbb{R} : (a + b) \in \mathbb{R}$ (closed under addition).

$A.2$ $\forall a, b \in \mathbb{R} : [a + b = b + a]$ (commutativity of addition).

$A.3$ $\forall a, b, c \in R : [(a + b) + c = a + (b + c)]$ (associativity of addition).

$A.4$ $\exists 0 \in \mathbb{R}, \forall a \in \mathbb{R} : [0 + a = a]$ (existence of additive identitie).

$A.5$ $\forall a \in \mathbb{R}, \exists! x \in \mathbb{R} : [a + x = 0]$ (existence of additive inverse). We write $x = -a$.

Axioms $A.6 - A.10$ are analogues of $A.1 - A.5$ for the operation of multiplication.

$A.6$ $\forall a, b \in \mathbb{R} : [ab \in \mathbb{R}]$ (closed under multiplication).

$A.7$ $\forall a, b \in \mathbb{R} : [ab = ba]$ (commutativity of multiplication).

$A.8$ $\forall a, b, c \in \mathbb{R} : [(ab)c = a(bc)]$ (associativity of multiplication).

$A.9$ $\exists 1 \in \mathbb{R}, \forall a \in \mathbb{R} : [1{\cdot}a = a]$ (existence of multiplicative identitie).

$A.10$ $\forall a \in \mathbb{R} - \{0\}, \exists! y \in \mathbb{R} : [ay = 1]$ (existence of multiplicative inverse). We write $y = \dfrac{1}{a}$.

The last axiom links the operations of summation and multiplication.

A.11 $\forall a, b, c \in \mathbb{R} : [(a + b)c = ac + bc]$ (distributive law).

Example 2.3.1 $\forall a \in \mathbb{R} : 0a = 0$.

Indeed, we have

$$\begin{aligned} a + 0a &= 1a + 0a \ \textit{(byA.9)} \\ &= (1 + 0)a \textit{(byA.11)} \\ &= 1a \ \textit{(byA.2 and A.4)} \\ &= a \ \textit{(byA.9)} \end{aligned}$$

Now add $-a$ to both sides.

$$\begin{aligned} -a + (a + 0a) &= -a + a \\ &\Rightarrow (-a + a) + 0a = 0 \ \textit{(by A.3 and A.5)} \\ &\Rightarrow 0 + 0a = 0 \ \textit{(by A.5)} \\ &\Rightarrow 0a = 0 \ \textit{(byA.4).} \end{aligned}$$

Remark 2.3.2 *The set of rationals $\mathbb{Q}$ also forms an algebraic field (that is, the rational numbers satisfy axioms **A.1 - A.11**.*

Now we add axioms of order.

O.1 $\forall a, b \in \mathbb{R} : [(a = b) \vee (a < b) \vee (a > b)]$

$$\equiv \forall a, b \in \mathbb{R} : [(a \geq b) \wedge (b \geq a) \Rightarrow (a = b)] \text{ (trichotomy law).}$$

O.2 $\forall a, b, c \in \mathbb{R} : [(a > b) \wedge (b > c) \Rightarrow (a > c)]$ (transitive law).

O.3 $\forall a, b, c \in \mathbb{R} : [(a > b) \Rightarrow (a + c > b + c)]$ (compatibility with addition).

O.4 $\forall a, b, c \in \mathbb{R} : [(a > b) \wedge (c > 0) \Rightarrow (ac > bc)]$ (compatibility with multiplication).

Remark 2.3.3 *Note that*

$$\forall a, b \in \mathbb{R} : \{(a > b) \Leftrightarrow (a - b > 0)\}.$$

*This follows from (**O.3**) by adding $-b$.*

Axioms **A.1-A.11** and **O.1 - O.4** define $\mathbb{R}$ to be an ordered field. Observe that the rational numbers also satisfy axioms **A.1 - A.11** and **O.1 - O.4**, so $\mathbb{Q}$ is also an ordered field.

2.3.2 Absolute value

Definition 2.3.4 *We define the maximum and the minimum of two real a and b by:*

$$\max(a, b) = \begin{cases} a & \text{if } a \geq b \\ b & \text{if } b > a \end{cases}, \ \min(a, b) \begin{cases} a & \text{if } a \leq b \\ b & \text{if } b < a \end{cases}$$

Definition 2.3.5 *The absolute value $|x|$ of x is defined by*

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0, \\ -x & \text{if} \quad x < 0. \end{cases}$$

Theorem 2.3.6 *We can prove a bunch of theorems about the absolute value function that we usually take for granted:*

1) $|x| \geq 0$ *and* $(|x| = 0 \Leftrightarrow x = 0)$.

2) $\forall x \in \mathbb{R}$, $|-x| = |x|$.

3) $\forall x, y \in \mathbb{R}$, $|xy| = |x||y|$.

4) $|x^2| = x^2 = |x|^2$.

5) If $x, y \in \mathbb{R}$*, then* $|x| \leq y \Leftrightarrow -y \leq x \leq y$.

6) $\forall x \in \mathbb{R}$, $x \leq |x|$.

Proof. :

1) If $x \geq 0$ then $|x| = x \geq 0$. If $x \leq 0$, then $-x \geq 0 \Rightarrow |x| = -x \geq 0$. Thus, $|x| \geq 0$. Now suppose $x = 0$.Then, $|x| = x = 0$. For the other direction, suppose $|x| = 0$. Then, if $x \geq 0 \Rightarrow x = |x| = 0$. If $x \leq 0$, then $-x = |x| = 0$. Therefore, $x = 0 \Leftrightarrow |x| = 0$.

2) If $x \geq 0$ then $-x \leq 0$. Thus, $|x| = x = -(-x) = |-x|$. If $x \leq 0$ then $-x \geq 0$ and thus $|-x| = |-(-x)| = |x|$.

3) a) If $x \geq 0$ and $y \geq 0$, then $xy \geq 0$ and $|xy| = xy = |x||y|$.
b) If $x \leq 0$ and $y \leq 0$, then $xy \geq 0 \Rightarrow |xy| = xy = (-x)(-y) = |x||y|$.
c) If $x \leq 0$ and $y \geq 0$, then $xy \leq 0 \Rightarrow |xy| = -xy = (-x)(y) = |x||y|$.
d) If $x \geq 0$ and $y \leq 0$, then $xy \leq 0 \Rightarrow |xy| = -xy = (x)(-y) = |x||y|$.
4) Take $x = y$ in 3). Then, $|x^2| = |x|^2$. Since $x^2 \geq 0$, it follows that $|x^2| = x^2$.
5) Suppose $|x| \leq y$. If $x \geq 0$, then $-y \leq 0 \leq x = |x| \leq y$. Therefore, $-y \leq x \leq y$.
If $x \leq 0$, then $-x \geq 0$ and $|x| = -x \leq y$. Hence, $-y \leq -x \leq y \Rightarrow -y \leq x \leq y$.
6) If $x \geq 0$ then $x = |x|$, if $x \leq 0$ then $x \leq |x|$ and thus $x \leq |x|$. ■

Theorem 2.3.7 (triangle inequality)

$$\forall a, b \in \mathbb{R} : |a+b| \leq |a| + |b|.$$

Proof. We split the proof into two cases. We use the fact that $a \leq |a|$ for all $a \in \mathbb{R}$.
Case $a + b \geq 0$. Then

$$|a+b| = a+b \leq |a| + |b|.$$

Case $a + b < 0$. Then

$$|a+b| = -(a+b) = (-a) + (-b) \leq |a| + |b|.$$

■

Exercise 2.3.8 *Prove that*

1) $(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})[a^2 + b^2 \geq 2ab]$.

2) $(\forall a \in \mathbb{R}^+)(\forall b \in \mathbb{R}^+)[\frac{a+b}{2} \geq \sqrt{ab}]$.

3) $(\forall a \in \mathbb{R}^+)(\forall b \in \mathbb{R}^+)(\forall c \in \mathbb{R}^+)(\forall d \in \mathbb{R}^+)\left[\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}\right]$.

4) Let $n \geq 2$ be a natural number. Prove that

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{1}{2}.$$

Recall that $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$.

Proof. 1) The result is equivalent to $a^2 + b^2 - 2ab \geq 0$. But,

$$a^2 + b^2 - 2ab = (a-b)^2 \geq 0.$$

Note that the equality holds if and only if $a = b$.

2) As above, let us prove that the difference between the left-hand side (LHS) and the right-hand side (RHS) is non-negative:

$$\frac{a+b}{2} - \sqrt{ab} = \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2} \geq 0.$$

The equality holds if and only if $a = b$.
3)By (**2**) and by (**O.2**) we have

$$\frac{a+b+c+d}{4} \geq \frac{\sqrt{ab}+\sqrt{cd}}{2} \geq \sqrt{\sqrt{ab}\sqrt{cd}} = \sqrt[4]{abcd}.$$

The equality holds if and only if $a = b = c = d$.
4)$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \underbrace{\frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n}}_{n} = n\frac{1}{2} = \frac{1}{2}.$ ■

Theorem 2.3.9 (Bernoulli's inequality) $(\forall n \in \mathbb{N})(\forall x > -1) \quad [(1+x)^n \geq 1 + xn].$

Proof. Basic case. The inequality holds for $n = 0, 1$.
Induction step. Suppose that the inequality is true for $n = k$ with $k \geq 1$; that is,

$$(1+x)^k \geq 1 + kx.$$

We have to prove that it is true for $n = k+1$; in other words,

$$(1+x)^{k+1} \geq 1 + (k+1)x.$$

Now,

$$\begin{aligned} (1+x)^{k+1} &= (1+x)^k(1+x) \\ &\geq (1+kx)(1+x) \\ &= 1 + (k+1)x + kx^2 \\ &\geq 1 + (k+1)x. \end{aligned}$$

This concludes the induction step. By the principle of mathematical induction, the result is true for all $n \in \mathbb{N}$. ■

2.3.3 Bounded sets of $\mathbb{R}$

Definition 2.3.10 *Let A be a subset of $\mathbb{R}$ and non-empty .*
We say that A is bounded from above if and only if :

$$\exists M \in \mathbb{R}; \forall x \in A : x \leq M$$

We say that A is bounded from below if and only if

$$\exists m \in \mathbb{R}; \forall x \in A : x \geq m$$

A is bounded if and only if it is bounded from above and below.

Proposition 2.3.11 *The three following conditions are equivalent*

1) A is bounded set,

2) $\exists m \in \mathbb{R}, \exists M \in \mathbb{R}; \forall x \in A : m \leq x \leq M$.

3) $\exists M \in \mathbb{R}_+^*; \forall x \in A : |x| \leq M$.

Definition 2.3.12 *Let* $A \subseteq \mathbb{R}$. *We say that* $M \in \mathbb{R}$ *is the supremum of A, written* $\sup A$, *if*

(i) $\forall x \in A : x \leq M$ *for all* $x \in A$; *(M is an upper bound of A)*

(ii) if $x \leq \grave{M}$ *for all* $x \in A$ *then* $M \leq \grave{M}$ *(M is the least upper bound of A).*

Definition 2.3.13 *Let* $A \subseteq \mathbb{R}$. *We say that* $m \in \mathbb{R}$ *is the infimum of A, written* $\inf A$, *if*

(i) $\forall x \in A : x \geq m$ *for all* $x \in A$; *(m is a lower bound of A)*

(ii) if $x \geq \grave{m}$ *for all* $x \in A$ *then* $m \geq \grave{m}$ *(m is the greatest lower bound of A).*

Definition 2.3.14 *If* $\sup A \in A$, *it is called* $\max A$.

If $\inf A \in A$, *it is called* $\min A$.

Notation 3 *If A is infinite from above (from below, respectively) in* $\mathbb{R}$ *we write* $\sup A = +\infty$ *(*$\inf A = -\infty$*,respectively).*

Remark 2.3.15 *If A has a supremum (an infimum, respectively), then* $\sup A$ *(*$\inf A$*) is unique.*

Example 2.3.16 • *Let* $A = [1, 2)$. *Then 2 is an upper bound, and is the least upper bound: if* $\grave{M} < 2$ *then* $\grave{M}$ *is not an upper bound because* $\max(1, 1 + \frac{\grave{M}}{2}) \in A$ *and* $\max(1, 1 + \frac{\grave{M}}{2}) > \grave{M}$. *Note that in this case* $\sup A \notin A$, *so* $\nexists \max A$.

• *Let* $A = (1, 2]$. *Then we again have* $\sup A = 2$, *and this time* $\sup A \in A$. *The supremum is the least upper bound of a set. There's an analogous definition for lower bounds.*

Axiom 2.3.17 (supermum and infimum) *Let A be a non-empty subset of* $\mathbb{R}$ *that is bounded above (below, respectively). Then A has a supremum (an infimum, respectively).*

Let's explore some useful properties of sup and inf.

Proposition 2.3.18 ***(i)*** *Let A, B be non-empty subsets of $\mathbb{R}$, with $A \subseteq B$ and with B bounded above. Then A is bounded above, and* $\sup A \le \sup B$.

(ii) *Let $B \subseteq \mathbb{R}$ be non-empty and bounded below. Let $A = \{-x : x \in B\}$. Then A is non-empty and bounded above. Furthermore,* $\inf B$ *exists, and* $\inf B = -\sup A$.

Proof. **(i)** Since B is bounded above, it has an upper bound, say M. Then $x \le M$ for all $x \in B$, so certainly $x \le M$ for all $x \in A$, so M is an upper bound for A. Now A, B are non-empty and bounded above, so by Axiom of supermum .

Note that $\sup B$ is an upper bound for B and hence also for A, so $\sup B \ge \sup A$ (since $\sup A$ is the least upper bound for A).

(ii) Since B is non-empty, so is A.

Let m be a lower bound for B, so $x \ge m$ for all $x \in B$. Then $-x \le -m$ for all $x \in B$, so $y \le -m$ for all $y \in A$, so $-m$ is an upper bound for A.

Now A is non-empty and bounded above, so by Axiom of supremum. Then $y \le \sup A$ for all $y \in A$, so $x \ge -\sup A$ for all $x \in B$, so $-\sup A$ is a lower bound for B. Also, we saw before that if m is a lower bound for B then $-m$ is an upper bound for A. Then $-m \ge \sup A$ (since $\sup A$ is the least upper bound), so $m \le -\sup A$.

So $-\sup A$ is the greatest lower bound.

So $\inf B$ exists and $\inf B = -\sup A$. ■

Proposition 2.3.19 (Approximation property) ***1)*** *Let $A \subseteq \mathbb{R}$ be non-empty and bounded above, then*

$$M = \sup A \Leftrightarrow \begin{cases} \forall x \in A : x \le M \\ \textit{and} \\ \forall \epsilon ; \exists a_\epsilon \in A : M - \epsilon < a_\epsilon \end{cases}$$

2) *Let A be bounded from below, then*

$$m = \inf A \Leftrightarrow \begin{cases} \forall x \in A : x \ge m \\ \textit{and} \\ \forall \epsilon ; \exists b_\epsilon \in A : b_\epsilon < m + \epsilon \end{cases}$$

Proof. **1)** Take $\epsilon > 0$. Note that by definition of the supremum we have $x \le \sup A$ for all $x \in A$. Suppose, for a contradiction, that $\sup A - \epsilon \ge x$ for all $x \in A$. Then $\sup A - \epsilon$ is an upper bound for A, but $\sup A - \epsilon < \sup A$. Contradiction.

So there is $a_\epsilon \in A$ with $\sup A - \epsilon < a_\epsilon$.

2) In the same way we prove the second case. ■

Axiom 2.3.20 (of Archimedes) *$\forall x > 0$; $\forall y \in \mathbb{R}$; $\exists n \in \mathbb{N}^* : y < nx$.*

Proof. We suppose that: $\exists x > 0$; $\exists y \in \mathbb{R}$; $\forall n \in \mathbb{N}^* : y \geq nx$ or $\exists x > 0$; $\exists y \in \mathbb{R}$; $\forall n \in \mathbb{N}^* : n \leq \frac{y}{x}$, that's mean the set $\mathbb{N}^*$ is limited from above it accepts an upper limit in $\mathbb{R}$ called M. so

$$\forall \epsilon; \exists n_\epsilon \in \mathbb{N}^* : M - \epsilon < n_\epsilon.$$

Putting $\epsilon = 1$, we get :

$$\exists n_\epsilon \in \mathbb{N}^* : M - 1 < n_\epsilon \text{ or } \exists n_\epsilon \in \mathbb{N}^* : M < n_\epsilon + 1$$

but $n_\epsilon + 1 \in \mathbb{N}^*$, this is a contradiction with $\sup \mathbb{N}^* = M$. ■

Example 2.3.21 $A = [1, 2[$; $\sup A = 2 \notin A$, *then* $\nexists \max A$; $\inf A = 1 = \min A$
$B = \left\{ \frac{1}{n}; n \in \mathbb{N}^* \right\}$; $\forall n \in \mathbb{N}^* : n \geq 1 \Rightarrow 0 < \frac{1}{n} \leq 1$, *then* $\sup B = \max B = 1 \in B$.
Let we proof that $\inf B = 0$ *i.e.*

$$0 = \inf B \Leftrightarrow \begin{cases} \forall x \in B : x \geq 0 \\ \textit{and} \\ \forall \epsilon; \exists b_\epsilon \in B : b_\epsilon < 0 + \epsilon. \end{cases}$$

On the other side we have

$$\forall \epsilon; \exists b_\epsilon \in B : b_\epsilon < 0 + \epsilon \Leftrightarrow \forall \epsilon; \exists n \in \mathbb{N}^* : \frac{1}{n} < \epsilon.$$

and this proposition is true and its according to Archimedes' Axiom

$$\forall \epsilon; \exists n \in \mathbb{N}^* : n\epsilon > 1$$

$\min B$ *is unavailable, because* $0 \notin B$.

Definition 2.3.22 *Let $x \in \mathbb{R}$, there exists a unique relative integer, the integer part denoted $E(x)$, such that $E(x) \leq x < E(x) + 1$. We also note $E(x) = [x]$.*

Example 2.3.23 *1)* $E(3,5) = 3$ *since* $3 \leq 3,5 < 3 + 1$.
2) $E(-3,5) = -4$ *since* $-4 \leq -3,5 < -4 + 1$.
3) $\forall n \in \mathbb{N}^* : E\left(\frac{1}{n+1}\right) = 0$ *since* $\forall n \in \mathbb{N}^* : 0 \leq \frac{1}{n+1} < 0 + 1$.

2.3.4 Dense groups in $\mathbb{R}$

Theorem 2.3.24 *Between every two different real numbers there is at least one rational number.*

Proof. Let y and x be two real numbers where $x < y$. According to Archimedean axiom

$$\exists n \in \mathbb{N}^* : 1 < n(y - x) \text{ or } nx + 1 < ny.$$

On the other hand we have

$$\begin{aligned} E(nx) &\leq nx < E(nx) + 1 \\ \text{or } nx &< E(nx) + 1 \leq nx + 1 < ny. \end{aligned}$$

So

$$x < \frac{E(nx) + 1}{n} < y$$

Well the rational number $\dfrac{E(nx)+1}{n}$ is bounded between the two real numbers x and y. ■

Theorem 2.3.25 *between every two different real numbers there is at least one irrational number.*

To prove this theory we need the following proposition.

Proposition 2.3.26 *if $x \in I$ (irrational number)and $r \in \mathbb{Q}^*$ then $rx \in I$.*

Proof. We assume $x \in I$ and $r \in \mathbb{Q}^*$and that $rx \in \mathbb{Q}$, then

$$\begin{aligned} \left(\frac{1}{r} \in \mathbb{Q}^* \text{or } rx \in \mathbb{Q}\right) &\Rightarrow \frac{1}{r}.rx \in \mathbb{Q} \\ &\Rightarrow x \in \mathbb{Q}. \end{aligned}$$

This is a contradiction because $x \in I$. ■

Proof (Theorem). Let x, y be two real numbers, where $x < y$, according to the theorem , there sexist a rational number r $(r \neq 0)$ such that:

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}} \text{ or } x < r\sqrt{2} < y$$

and according to proposition we conclude that $r\sqrt{2}$ is a irrational number. ■

Corollary 2.3.27 *The two sets $\mathbb{Q}$ and I is dense in $\mathbb{R}$.*

2.3.5 Intervals in $\mathbb{R}$

Definition 2.3.28 *An interval is a subset of the real numbers that contains all real numbers lying between any two numbers of the subset.*

Let a, b be real numbers, where $a < b$, we define

- $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ is called closed interval.
- $]a,b[= \{x \in \mathbb{R} : a < x < b\}$ is called open interval.
- $[a,b[= \{x \in \mathbb{R} : a \leq x < b\}$ is called half open interval.
- $]a,b] = \{x \in \mathbb{R} : a < x \leq b\}$" " " " " " " " " " " " " " " " " ".
- $[a,+\infty[= \{x \in \mathbb{R} : x \geq a\}$ is unbounded closed interval.
- $]-\infty,b] = \{x \in \mathbb{R} : x \leq b\}$" " " " " " " " " " " " " " " " " ".
- $]a,+\infty[= \{x \in \mathbb{R} : x > a\}$ is unbounded open interval.
- $]-\infty,b[= \{x \in \mathbb{R} : x < b\}$" " " " " " " " " " " " " " " " " ".
- $]-\infty,+\infty[$" " " " " " " " " " " " " " " ".

Exercise 2.3.29 *Let*

$$S = \left\{1 - \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}.$$

Find $\inf S$ *and* $\sup S$ *and prove your answers.*

Solution 2.3.30 *We claim that* $\inf S = \frac{1}{2}$ *and* $\sup S = 2$. *Note that, if n is odd,* $1 - \frac{(-1)^n}{n} = 1 + \frac{1}{n}$,*while if n is even,* $1 - \frac{(-1)^n}{n} = 1 - \frac{1}{n}$.

It follows, if n is odd, that

$$1 - \frac{(-1)^n}{n} > 1 > \frac{1}{2}.$$

If $n \geq 2$ is even,

$$1 - \frac{(-1)^n}{n} = 1 - \frac{1}{n} \geq 1 - \frac{1}{2} = \frac{1}{2}.$$

Arguing similarly, $1 - \frac{(-1)^n}{n} \leq 2$ *and so* $\frac{1}{2}$ *and* 2 *are, respectively, lower and upper bounds for S. Since* $\frac{1}{2} \in S$, *there cannot be a lower bound* $m > \frac{1}{2}$ *and so* $\frac{1}{2}$*is the greatest lower bound for S, i.e.* $\inf S = \frac{1}{2}$. *Since* $2 \in S$, *there cannot be a upper bound* $M < 2$ *and so* 2*is the least upper bound for S, i.e.* $\sup S = 2$.

Exercise 2.3.31 *Let* $A = \left\{ \frac{1}{n} : \quad n = 1, 2, 3, ... \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ... \right\}$

1) *Show that A is a non-empty set, both bounded above and below.*

2) *Show that* $\sup(A) \ = \max(A) = 1$.

3) *Show that* $\inf(A) \ = 0$.

4) *Show that* $\min(A)$ *does not exist..*

Example 2.3.32 ***Solution 2.3.33*** *Let* $A = \left\{ \frac{1}{n} : \quad n = 1, 2, 3, ... \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ... \right\}$

1) $1 \in A \Rightarrow A \neq \emptyset$, $\forall n : n \geq 1$ *we have* $0 < \frac{1}{n} \leq 1 \Rightarrow 1$ *is an upper bound of A and* 0 *is a lower bound of A.*

2) $\sup A$ *and* $\inf A$ *exist, according to the axiom of the upper bound : Let's show that* $\sup A = 1$. *Let $\varepsilon > 0$, we show that $\exists x_0 \in A/\ x_0 > 1 - \varepsilon$. In fact, let's take $x_0 = 1$. First of all $x_0 = 1$ verifies the precedent relation, since : $\forall \varepsilon > 0$, $1 > 1 - \varepsilon$, morever $1 \in A$ then :* $\sup A = \max A = 1$.

3) $\inf A = 0$? *Let $\varepsilon > 0$, we show that $\exists x_0 \in A/ \quad 0 + \varepsilon > x_0$, the elements of A are of the form $\frac{1}{n}$ we must find $n \in \mathbb{N}^*/\ \frac{1}{n} < \varepsilon$ or $n > \frac{1}{\varepsilon}$. For $\varepsilon > 0$ if we take $x_0 = \frac{1}{n}$ with $n > \frac{1}{\varepsilon}$ we obtain $x_0 \in A$ and $0 + \varepsilon > x_0$ then* $\inf A = 0$.

4) *We have* $\forall n \geq 1, \frac{1}{n} > 0 \Rightarrow 0 \notin A \Rightarrow \nexists \min A$.

Chapter 3

Real functions of a real variable

3.1 Introduction

In this chapter the key notion of a continuous function is introduced, followed by several important theorems about continuous functions. We deal exclusively with functions taking values in the set of real numbers (that is, real-valued functions).

3.1.1 Bounded functions, monotonic functions

Definition 3.1.1 *Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function. It is said that*

a) f is said to be bounded above on D if

$$\exists M \in \mathbb{R}, \forall x \in D : f(x) \leq M.$$

b) f is said to be bounded below on D if

$$\exists m \in \mathbb{R}, \forall x \in D : f(x) \geq m.$$

c) f is bounded on D if f is both bounded above and below on D, i.e. if

$$\exists M > 0, \forall x \in D : |f(x)| \leq M \text{ or } \exists M, \exists m \in \mathbb{R}, \forall x \in D : m \leq f(x) \leq M.$$

Example 3.1.2 *1/ $f(x) = \cos(x)$ is bounded because $\forall x \in \mathbb{R} : -1 \leq \cos(x) \leq 1$.*

2/ $f(x) = e^x$ is bounded below because $\forall x \in \mathbb{R} : e^x > 0$.

3/ $f(x) = x^2$ is not bounded.

Definition 3.1.3 *Let $f : D \to \mathbb{R}$ be a function. We say that:*

a) f is increasing over D if

$$\forall x, y \in D, x < y \Rightarrow f(x) \leq f(y).$$

b) f is strictly increasing over D if

$$\forall x,\, y \in D,\, x < y \Rightarrow f(x) < f(y).$$

c)f is decreasing over D if

$$\forall x,\, y \in D,\, x < y \Rightarrow f(x) \geq f(y).$$

d) f is strictly decreasing over D if

$$\forall x,\, y \in D,\, x < y \Rightarrow f(x) > f(y).$$

e) f is monotonic (or strictly monotonic) on D if f is increasing or decreasing (or strictly increasing or decreasing) on D.

Example 3.1.4 ***i)*** *The exponential function : $exp : \mathbb{R} \to \mathbb{R}$ is strictly increasing.*
ii) *The function absolute value : $x \to |x|$ defined on $\mathbb{R}$ is not monotonic.*

3.1.2 Odd, even, periodic function

Definition 3.1.5 (Parity) *Let I be be a symmetric interval with respect to 0 in $\mathbb{R}$. Let $f : I \to \mathbb{R}$ be a function. We say that:*
i) *f is even if $\forall x \in I : f(-x) = f(x)$.*
ii) *f is odd if $\forall x \in I : f(-x) = -f(x)$.*

Remark 3.1.6 *f is even if and only if its graph is symmetric with respect to on the y-axis and f is odd if and only if its graph is symmetric with respect to at the origin.*

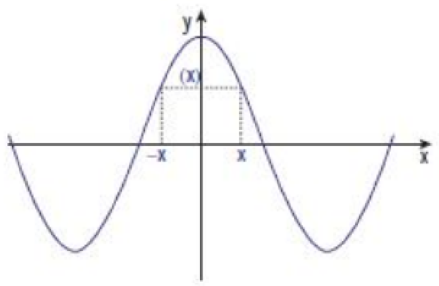

Figure 13: Even function

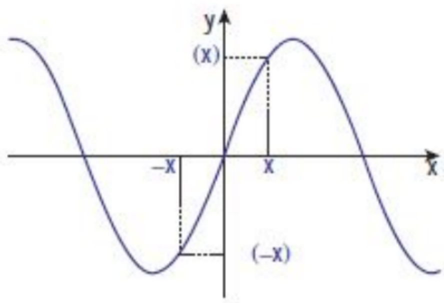

Figure 14 : Odd function

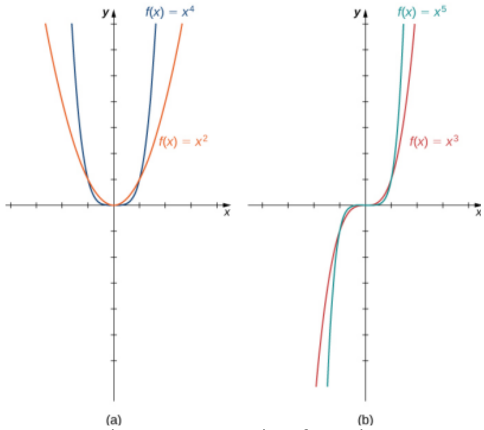

Figure15 : Parity function

(a) For any even integer n, $f(x) = ax^n$ is an even function,

(b) For any odd integer n, $f(x) = ax^n$, is an odd function.

Definition 3.1.7 (Periodicity) *Let $f : \mathbb{R} \to \mathbb{R}$ be a function and T be a real number, $T > 0$. The function f is called periodic of period T if $\forall x \in \mathbb{R}$, $f(x+T) = f(x)$.*

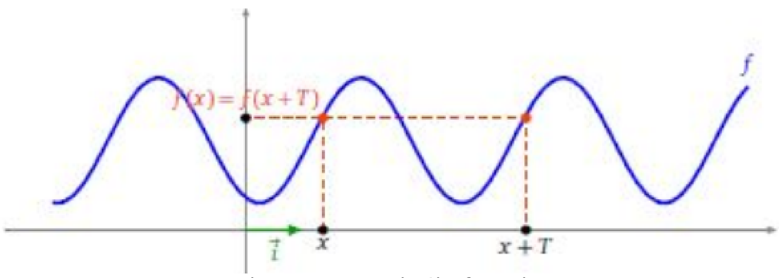

Figure 16: Periodic function

Example 3.1.8 *The functions* sin *and* cos *are 2π-periodic. The tangent function is π-periodic.*

3.1.3 Algebraic operations on functions

The set of functions of $D \subset \mathbb{R}$ in $\mathbb{R}$, is denoted $\mathcal{F}(D, \mathbb{R})$.

Definition 3.1.9 *Let f and $g \in \mathcal{F}(D, \mathbb{R})$ and $\lambda \in \mathbb{R}$. We define*

- *Sum of two functions $f + g : x \to (f+g)(x) = f(x) + g(x)$.*
- *For $\lambda \in \mathbb{R}$, $\lambda f : x \to (\lambda f)(x) = \lambda f(x)$.*
- *Product of two functions $fg : x \to (fg)(x) = f(x)g(x)$.*

Remark 3.1.10 *The functions $f + g$, λf and fg are functions belonging to $\mathcal{F}(D, \mathbb{R})$.*

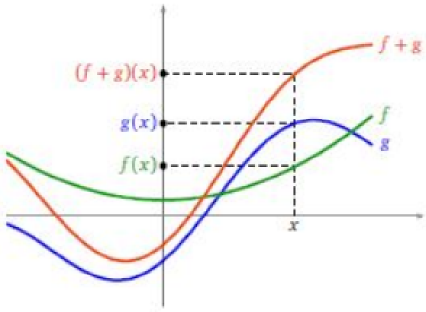

Figure 17: Sum of functions

Definition 3.1.11 *Let f and $g \in \mathcal{F}(D, \mathbb{R})$ and $\lambda \in \mathbb{R}$. We say that*

- $f \leq g$ *if* $\forall x \in D,\ f(x) \leq g(x)$.
- $f < g$ *if* $\forall x \in D, f(x) < g(x)$.

Example 3.1.12 *Let f and g be two functions defined on $]0,1[$ by $f(x) = x$, $g(x) = x^2$. We have $g < f$, because $\forall x \in]0,1[\ ,\ x^2 < x$.*

3.1.4 Limit of a function

General definitions

Let $f : I \to \mathbb{R}$, be a function defined on the interval I of $\mathbb{R}$. Let $x_0 \in \mathbb{R}$ be a point of I or an end of I.

Definition 3.1.13 *Let $l \in \mathbb{R}$. We say that f has l for limit in x_0 if,*

$$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall x \in I,\ |x - x_0| < \delta \Rightarrow |f(x) - l| < \epsilon.$$

In this case, we write $\lim\limits_{x \to x_0} f(x) = l$.

Example 3.1.14 *Consider the function $f(x) = 2x - 1$ which is defined on $\mathbb{R}$. At the point $x = 1$, we have $\lim\limits_{x \to 1} f(x) = 1$. Indeed, for all $\epsilon > 0$, we have $|f(x) - 1| = 2|x - 1| < \epsilon$, if we have $|x - 1| < \frac{\epsilon}{2}$. The right choice will then be to take $\delta = \frac{\epsilon}{2}$.*

Uniqueness of the limit

Proposition 3.1.15 *If f admits a limit at the point x_0, this limit is unique.*

Proof. If f admits two limits l_1 and l_2 at the point x_0, then we have, by definition, $\forall \epsilon > 0$,

$$\begin{aligned} \exists \delta_1 &> 0, \forall x \in I, \text{ if } |x - x_0| < \delta_1 \Rightarrow |f(x) - l_1| < \frac{\epsilon}{2}. \\ \exists \delta_2 &> 0, \forall x \in I, \text{ if } |x - x_0| < \delta_2 \Rightarrow |f(x) - l_2| < \frac{\epsilon}{2}. \end{aligned}$$

Let $\delta = \min(\delta_1, \delta_2) > 0$, then

$$|l_1 - l_2| \leq |f(x) - l_1| + |f(x) - l_2| \leq \epsilon$$

Since ϵ is any positive value, for $\epsilon = \dfrac{|l_1 - l_2|}{2}$ results in $l_1 = l_2$. ■

Limit to the right, limit to the left.

Definition 3.1.16 *We say that the function f admits l as the limit to the right of x_0, or when x tends to x_0^+, if for all $\epsilon > 0$ there exists a $\delta > 0$, such that: $x_0 < x < x_0 + \delta$, results in $|f(x) - l| \leq \epsilon$. In this case, we will write:*

$$\lim_{x \to x_0^+} f(x) = l \textit{ or } \lim_{x \overset{>}{\to} x_0} f(x) = l.$$

We say that the function f admits l as the limit to the left of x_0, or when x tends to x_0^-, if for all $\epsilon > 0$ there exists a $\delta > 0$, such that: $x_0 - \delta < x < x_0$, results in $|f(x) - l| \leq \epsilon$. In this case, we will write:

$$\lim_{x \to x_0^-} f(x) = l \textit{ or } \lim_{x \overset{<}{\to} x_0} f(x) = l.$$

Example 3.1.17 *The function $\sqrt{x}$ tends to 0 when $x \to 0^+$.*

Remark 3.1.18 *If the function f admits a limit l to the left of the point x_0 and a limit $l'$to the right of x_0, then for f to have a limit at the point x_0, it is necessary and sufficient that $l = l'$.*

Example 3.1.19 *Consider the function defined by*

$$f(x) = \begin{cases} 1, & \textit{if } x \geq 0, \\ -1, & \textit{if } x < 0. \end{cases}$$

It admits 1 as the limit to the right of 0 and -1 as the limit to the left of 0. But it does not admit any limit to point 0.

Cases where x becomes infinite

We will pose by definition

a) $\lim_{x \to +\infty} f(x) = l$, if

$$\forall \epsilon > 0,\ \exists A > 0, \text{ such that } x > A \Rightarrow |f(x) - l| < \epsilon.$$

b) $\lim_{x \to -\infty} f(x) = l$, if

$$\forall \epsilon > 0,\ \exists A > 0, \text{ such that } x < -A \Rightarrow |f(x) - l| < \epsilon.$$

Infinite limit

Let $x_0 \in \mathbb{R}$, we have

a) $\lim_{x \to x_0} f(x) = +\infty$,

$$\forall A > 0,\ \exists \delta > 0, \text{ such that } |x - x_0| < \delta \Rightarrow f(x) > A.$$

b) $\lim_{x \to x_0} f(x) = -\infty$, if

$$\forall A > 0,\ \exists \delta > 0, \text{ such that } |x - x_0| < \delta \Rightarrow f(x) < -A.$$

If $x_0 = +\infty$ or $x_0 = -\infty$, we put

a) $\lim_{x \to +\infty} f(x) = +\infty$,

$$\forall A > 0,\ \exists B > 0, \text{ such that } x > B \Rightarrow f(x) > A.$$

b) $\lim_{x \to -\infty} f(x) = +\infty$,

$$\forall A > 0,\ \exists B > 0, \text{ such that } x < -B \Rightarrow f(x) > A.$$

c) $\lim_{x \to +\infty} f(x) = -\infty$,

$$\forall A > 0,\ \exists B > 0, \text{ such that } x > B \Rightarrow f(x) < -A.$$

d) $\lim_{x \to -\infty} f(x) = -\infty$,

$$\forall A > 0,\ \exists B > 0, \text{ such that } x < -B \Rightarrow f(x) < -A.$$

3.1.5 Limit theorems

Theorem 3.1.20 *Let $f : [a,b] \to \mathbb{R}$ and $x_0 \in]a,b[$. The following two properties are equivalent:*

(1) $\lim\limits_{x \to x_0} f(x) = l$,

(2) For any sequence $(x_n)_{n \in \mathbb{N}}$, $x_n \in]a,b[$ such that $\lim\limits_{n \to +\infty} x_n = x_0$, then $\lim\limits_{n \to +\infty} f(x_n) = l$.

Exercise 3.1.21 *1)* $\lim\limits_{x \to 0} \sin\left(\frac{1}{x}\right)$ *does not exist, and 2)* $\lim\limits_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$.

Solution 3.1.22 *1)* $x_n = \frac{1}{(2n-1)\frac{\pi}{2}}$. *Then, $x_n \neq 0$, and $x_n \to 0$. But,*

$$\sin\left(\frac{1}{x_n}\right) = \sin\left((2n-1)\frac{\pi}{2}\right) = (-1)^{n+1},$$

for all n. However, this sequence does not converge (i.e. the limit does not exist).

2) Suppose $x_n \neq 0$ and $x_n \to 0$. Then

$$0 \leq \left| x_n \sin\left(\frac{1}{x_n}\right) \right| = |x_n| \left| \sin\left(\frac{1}{x_n}\right) \right| \leq |x_n|.$$

By the Gendarmes Theorem, $\lim\limits_{n \to +\infty} \left| x_n \sin\left(\frac{1}{x_n}\right) \right| = 0$.

3.1.6 Operations of limits

Theorem 3.1.23 *Let $f, g : [a,b] \to \mathbb{R}$ and $x_0 \in]a,b[$, such that $\lim\limits_{x \to x_0} f(x) = l$ and $\lim\limits_{x \to x_0} g(x) = l'$, Then*

a) $\lim\limits_{x \to x_0} [f(x) + g(x)] = l + l'$.

b) $\lim\limits_{x \to x_0} (\lambda f(x)) = \lambda l$ *for any* $\lambda \in \mathbb{R}$.

c) $\lim\limits_{x \to x_0} f(x).g(x) = ll'$.

d) $\lim\limits_{x \to x_0} |f(x)| = |l|$.

e) $\lim\limits_{x \to x_0} |f(x) - l| = 0$.

f) $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{l}{l'}$, *if* $l' \neq 0$.

Theorem 3.1.24 *Let $f : [a,b] \to [c,d]$ and $g : [c,d] \to \mathbb{R}$ and $x_0 \in]a,b[$, $y_0 \in [c,d]$, such that $\lim\limits_{x \to x_0} f(x) = y_0$ and $\lim\limits_{y \to y_0} g(y) = l$ Then $\lim\limits_{x \to x_0} (g \circ f)(x) = l$.*

Proposition 3.1.25 *Let $f, g : [a, b] \to \mathbb{R}$ and $x_0 \in]a, b[$, we have*

a) If $\lim\limits_{x\to x_0} f(x) = +\infty$, *then* $\lim\limits_{x\to x_0} \dfrac{1}{f(x)} = 0$.

b) If $\lim\limits_{x\to x_0} f(x) = -\infty$, *then* $\lim\limits_{x\to x_0} \dfrac{1}{f(x)} = 0$.

c) If $f \le g$, *and* $\lim\limits_{x\to x_0} f(x) = l$, *then* $\lim\limits_{x\to x_0} g(x) = l'$ *and* $l \le l'$.

d) If $f \le g$, *and* $\lim\limits_{x\to x_0} f(x) = +\infty$, *then* $\lim\limits_{x\to x_0} g(x) = +\infty$.

Theorem 3.1.26 *Let $f, g, h : [a, b] \to \mathbb{R}$ and $x_0 \in]a, b[$, we have*

i) $f(x) \le g(x) \le h(x)$, *for all* $x \in]a, b[$,

ii) $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} h(x) = l \in \mathbb{R}$.

Then $\lim\limits_{x\to x_0} g(x) = l$.

Indeterminate forms

$+\infty - \infty$, $0 \times \infty$, $\dfrac{\infty}{\infty}$, $\dfrac{0}{0}$, 1^{∞}, ∞^{0}.

Example 3.1.27 $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = \dfrac{0}{0}$, $\lim\limits_{x\to\infty} x^{\frac{1}{x}} = \infty^{0}$.

Proposition 3.1.28 *Let f and g be two functions if :*

1) f is a bounded function in the neighbourhood of x_0 ($\exists D$ a neighbourhood of x_0) s.t

$$\exists m, M \in \mathbb{R}, \forall x \in D : m \le f(x) \le M$$

2) $\lim\limits_{x\to x_0} g(x) = 0$.

Then $\lim\limits_{x\to x_0} f(x) \times g(x) = 0$

Example 3.1.29 *Calculate the limit* $\lim\limits_{x\to+\infty} \dfrac{\sin x}{x}$.

Indeed $\sin(\infty)$ *is not defined but it is bounded because* $|\sin x| \le 1$ *and* $\lim\limits_{x\to+\infty} \dfrac{1}{x} = 0$, *so* $\lim\limits_{x\to+\infty} \dfrac{\sin x}{x} = 0$.

Equivalent functions

Definition 3.1.30 *Let f and g be two functions defined in a neighborhood of a point x_0 ($x_0 \in \mathbb{R}$ or $x_0 = \pm\infty$).*

We assume, moreover, that g does not cancel in a neighborhood of x_0, except perhaps in x_0 where we can have $g(x_0) = 0$.

We say that f is equivalent to g in a neighborhood of x_0 if, and only if: $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = 1$

We denote this by $f \underset{x_0}{\sim} g$. *We also say that* f *and* g *are equivalent to the neighborhood of* x_0 *or in* x_0.

Example 3.1.31 *1) the functions* $f(x) = \ln(x+1)$ *and* $g(x) = x$ *are equivalent since* $\lim_{x \to 0} \frac{\ln(x+1)}{x} = 1$. *We note* $\ln(x+1) \underset{0}{\sim} x$.

2) Always, in the vicinity of zero, $\sin x \sim x$ *because* $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

3.2 Continuity of a function

3.2.1 General definitions

Definition 3.2.1 *Let us consider a function* $f : I \to \mathbb{R}$, *where* I *is an interval of* $\mathbb{R}$. *We say that* f *is continuous at the point* $x_0 \in I$ *if* $\lim_{x \to x_0} f(x) = f(x_0)$, *i.e. if*

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in I, |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon.$$

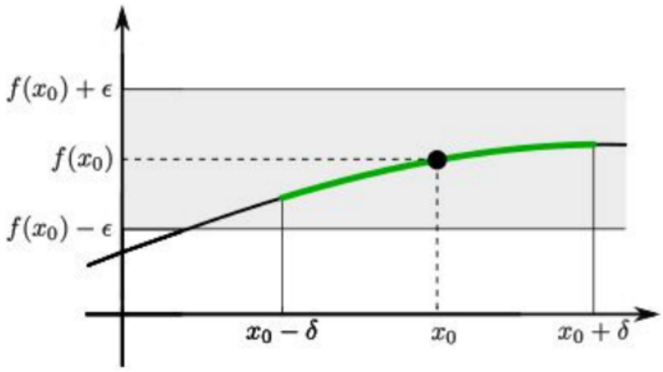

Figure 18: Continuity at the point x_0

Example 3.2.2 *Let the real function* f *be defined by*

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

At the point $x_0 = 0$, *we have*

$$|f(x) - f(x_0)| = |x \sin\left(\frac{1}{x}\right)| \leq |x| .$$

For $\epsilon > 0$, *we will choose* $\delta = \epsilon$. *Thus*

$$|x| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$$

So f *is continuous at the point* $x_0 = 0$.

Definition 3.2.3 *A function defined on an interval I is continuous on I if it is continuous at any point of I. The set of continuous functions on I is denoted by $\mathcal{C}(I)$.*

Continuity on the left, continuity on the right

Definition 3.2.4 *Let us consider a function $f : I \to R$, where I is an interval of $\mathbb{R}$.*

(1) The function f is said to be continuous on the left at x_0 if $\lim\limits_{x \to x_0^-} f(x) = f(x_0)$, i.e. if

$$\forall \epsilon > 0,\ \exists \delta > 0, \forall x \in I,\ \text{if } 0 < x_0 - x < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon.$$

(2) The function f is said to be continuous on the right at x_0 if $\lim\limits_{x \to x_0^+} f(x) = f(x_0)$, i.e. if

$$\forall \epsilon > 0,\ \exists \delta > 0, \forall x \in I,\ \text{if } 0 < x - x_0 < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon.$$

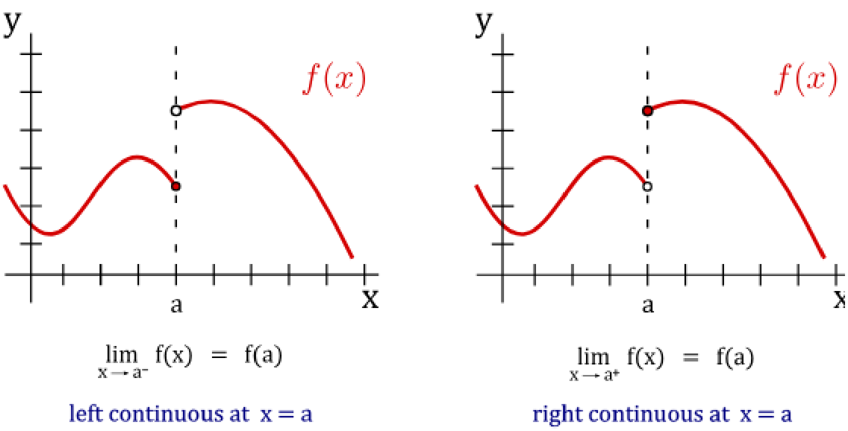

Figure 19 : Left (right) Continuous at x=a

Note. - The function f is continuous at x_0 if and only if f is continuous at the left and right of the point x_0.

- f is continuous at $x_0 \Leftrightarrow \lim\limits_{x \to x_0^-} f(x) = \lim\limits_{x \to x_0^+} f(x) = f(x_0)$.

Summary of discontinuities

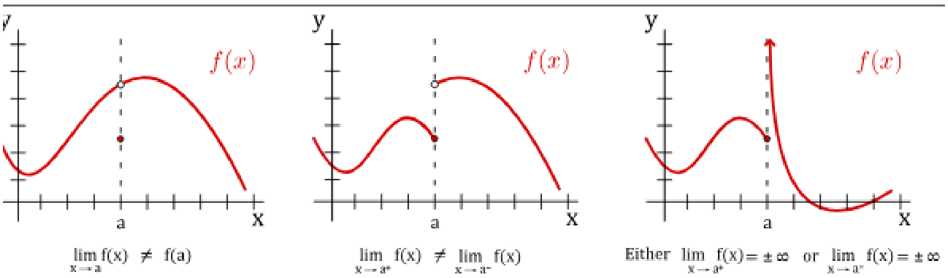

Figure 20:discontinuity at point a

Example 3.2.5 *The function defined by*

$$f(x) = \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{if } x \leq 0. \end{cases}$$

is continuous on $\mathbb{R}^*$*. At the point* $x_0 = 0$*, the function* f *is continuous on the left, but it is not continuous on the right because*

$$\lim_{x \to 0^-} f(x) = f(0) = -1 \text{ and } \lim_{x \to 0^+} f(x) = 1 \neq f(0)$$

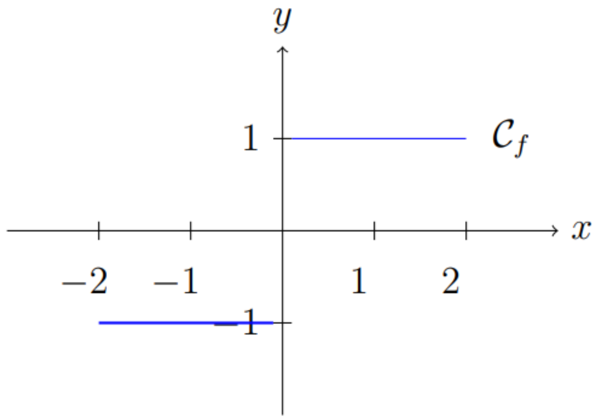

***Figure 21: discontinuity at point* 0**

Definition 3.2.6 (Continuity on a closed iterval.) *A function* f *is continuous on the closed interval* $[a, b]$ *if:*

1. *it is continuous on the open interval* (a, b);
2. *it is right continuous at point* a :

$$\lim_{x \to a^+} f(x) = f(a);$$

and

3. it is left continuous at point b:

$$\lim_{x \to b-} f(x) = f(b).$$

Example 3.2.7 *The function* $f(x) = \sqrt{x}$ *is continuous on the (closed) interval* $[0, +\infty)$.
The function $f(x) = \sqrt{4 - x}$ *is continuous on the (closed) interval* $(-\infty, 4]$.

Continuity extension

Definition 3.2.8 *Let* I *be an interval,* x_0 *a point of* I. *If the function* f *is not defined at the point* $x_0 \in I$ *and admits at this point a finite limit denoted* l, *the function defined by*

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } x \neq x_0, \\ l, & \text{if } x = x_0. \end{cases}$$

is said to be a continuity extension of f *at the point* x_0.

Example 3.2.9 *The function*

$$f(x) = x \sin \frac{1}{x}$$

is defined and continues on $\mathbb{R}^*$. *Now, for all* $x \in \mathbb{R}^*$ *we have*

$$|f(x)| = \left| x \sin \frac{1}{x} \right| \leq |x|$$

So $\lim_{x \to 0} f(x) = 0$. *The continuity extension of* f *to the point* 0 *is therefore the function* $\tilde{f}$ *defined by:*

$$\tilde{f}(x) = \begin{cases} x \sin \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

3.2.2 Operations on continuous functions

Definition 3.2.10 *Let* I *be an interval, and* f *and* g *functions defined on* I *and continuous at* $x_0 \in I$. *Then*

(1) λf *is continuous at* x_0, $(\lambda \in \mathbb{R})$.

(2) $f + g$ *is continuous at* x_0.

(3) $f.g$ *is continuous at* x_0.

(4) $\frac{f}{g}$ *(if* $g(x_0) \neq 0$*) is continuous at* x_0.

3.2.3 Continuity of composition functions

Theorem 3.2.11 *If g is continuous at x_0 and f is continuous at $g(x_0)$, then the composition function $f \circ g$ is continuous at x_0.*

Theorem 3.2.12 *If $f : [a, b] \to \mathbb{R}$ is continuous then f is bounded.*

Definition 3.2.13 (Absolute Minimum / Maximum) *Let $f : I \to \mathbb{R}$. Then, f achieves an absolute minimum at $c \in I$,if $\forall x \in I,\ f(x) \geq f(c)$. Similarly, f achieves an absolute maximum at $d \in I$,if $\forall x \in I,\ f(x) \leq f(d)$.*

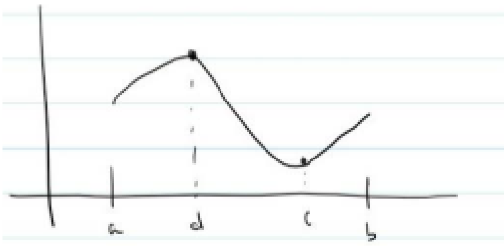

Figure 22: Maximum and minimum

3.2.4 The Intermediate Value Theorem

Whether or not an equation has a solution is an important question in mathematics.

Theorem 3.2.14 (Intermediate Value Theorem) *If f is continuous on the interval $[a, b]$ and N is between $f(a)$ and $f(b)$,where $f(a) \neq f(b)$, then there is a number c in (a, b) such that $f(c) = N$*

The Intermediate Value Theorem guarantees that if f is continuous and $f(a) < N < f(b)$, the line $y = N$ intersects the function at some point $x = c$. Such a number c is between a and b and has the property that $f(c) = N$ (see **Figure 23**)

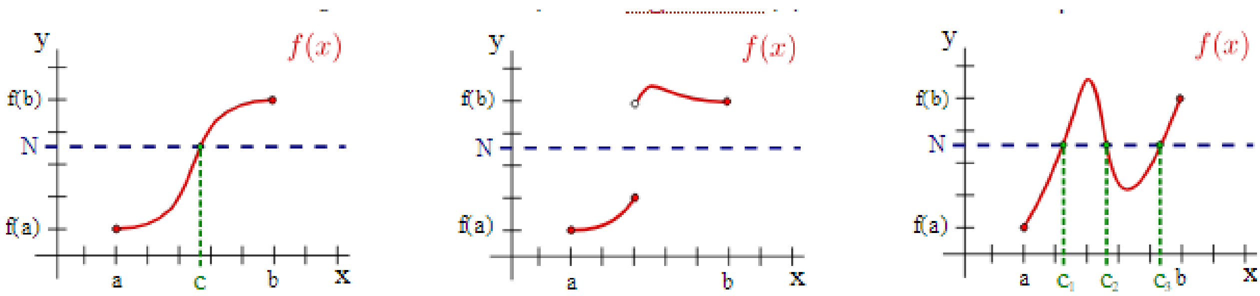

Figure 23: IntermediateValueTheorem

(a) A continuous function where IVT holds for a single value c.

(b) A discontinuous function where IVT fails to hold.

(c) A continuous function where IVT holds for multiple values in (a, b).

The Intermediate Value Theorem is most frequently used for $N = 0$.

Exercise 3.2.15 *Show that there is a solution of* $\sqrt[3]{x} + x = 1$ *in the interval* $(0, 8)$.

Solution 3.2.16 *Let* $f(x) = \sqrt[3]{x} + x - 1$, $a = 0$, *and* $b = 8$, *Since* $\sqrt[3]{x}$, x *and* -1 *are continuous on* $\mathbb{R}$, *and the sum of continuous functions is again continuous, we have that* f *is continuous on* $\mathbb{R}$, *thus in particular,* f *is continuous on* $[0, 8]$. *We have* $f(a) = f(0) = \sqrt[3]{0} + 0 - 1 = -1$ *and* $f(b) = f(8) = \sqrt[3]{8} + 8 - 1 = 9$. *Thus* $N = 0$ *lies between* $f(a) = -1$ *and* $f(b) = 9$, *so the conditions of the Intermediate Value Theorem are satisfied. So, there exists a number* c *in* $(0, 8)$ *such that* $f(c) = 0$. *This means that* c *satisfies*$\sqrt[3]{c} + c - 1 = 0$, *in otherwords, is a solution for the equation given.*

Alternatively we can let $f(x) = \sqrt[3]{x} + x$, $N = 1$, $a = 0$ *and* $b = 8$. *Then as before* f *is the sum of two continuous functions, so is also continuous everywhere, in particular, continuous on the interval* $[0, 8]$, $f(a) = f(0) = \sqrt[3]{0} + 0 = 0$ *and* $f(b) = f(8) = \sqrt[3]{8} + 8 = 10$. *Thus* $N = 1$ *lies between* $f(a) = 0$ *and* $f(b) = 10$, *so the conditions of the Intermediate Value Theorem are satisfied. So, there exists a number* c *in* $(0, 8)$ *such that* $f(c) = 1$. *This means that* c *satisfies* $\sqrt[3]{c} + c = 1$, *in otherwords, is a solution for the equation given.*

Proposition 3.2.17 *Let* f *be a continuous function on interval* $[a, b]$, *such that* $f(a).f(b) < 0$, *there exists* $c \in]a, b[$ *such that* $f(c) = 0$.

3.2.5 Uniform continuity

Recall the definition of continuity : $f : I \to \mathbb{R}$ is continuous on I if $\forall x_0 \in I$ and $\forall \epsilon > 0$, $\exists \delta(\epsilon, x_0) > 0$, such that $\forall x \in I, |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$.
Here, $\delta(\epsilon, x_0)$ denote the fact that δ can depend on ϵ and x_0.

Exercise 3.2.18 *Let* $f(x) = \dfrac{1}{x}$, *defined on the interval* $(0, 1)$. *Justify that* f *is continuous on* $(0, 1)$?

Solution 3.2.19 *We want to show that if* $|x - x_0| < \delta$, *then* $|\dfrac{1}{x} - \dfrac{1}{x_0}| < \epsilon$.*Specifically, we can choose* $\delta = \min\left\{\dfrac{x_0}{2}, \dfrac{x_0^2}{2}\epsilon\right\}$. *suppose* $|x - x_0| < \delta$. *Then,*

$$|x - x_0| < \frac{x_0}{2} \Rightarrow |x| > x_0 - |x - x_0| > \frac{x_0}{2}.$$

Thus, $\frac{1}{|x|} < \frac{2}{x_0}$*. Therefore,*

$$\begin{aligned} |\frac{1}{x} - \frac{1}{x_0}| &= \frac{|x - x_0|}{|xx_0|} \\ &< \frac{\delta}{|x||x_0|} \\ &< \frac{2\delta}{x_0^2} \\ &< \frac{2\frac{x_0^2}{2}\epsilon}{x_0^2} = \epsilon. \end{aligned}$$

Definition 3.2.20 (Uniformly Continuous) *Let* $f : I \to \mathbb{R}$*. Then* f *is uniformly continuous on* I *if* $\forall\epsilon > 0$, $\exists\delta\,(\epsilon) > 0$*, such that* $\forall x, y \in I, |x-y| < \delta \Rightarrow |f(x)-f(y)| < \epsilon$*.*

Remark 3.2.21 *Thus, in the definition of uniform continuity,* δ *only depends on* ϵ*!.*

Example 3.2.22 *The function* $f(x) = x^2$ *is uniformly continuous on* $[0, 1]$*.*
Indeed : *Let* $\epsilon > 0$*. Choose* $\delta = \frac{\epsilon}{2}$*. Then, if* $x, y \in [0, 1]$ *then* $|x - y| < \delta$ *implies that*

$$|x^2 - y^2| = |x - y||x + y| \leq 2|x - y| < 2\delta = \epsilon.$$

However, there are of course continuous functions that are not uniformly continuous. For example, we will show that

Exercise 3.2.23 *Show that the function* $f\,(x) = \frac{1}{x}$ *is not uniformly continuous on* $(0, 1)$*?*

Negation (Not uniformly continuous)

Let $f : I \to \mathbb{R}$. Then, f is not uniformly continuous on I if $\exists\epsilon_0 > 0$, $\forall\delta > 0$, such that $\exists x, y \in I$ with $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon_0$.

Solution 3.2.24 *Let* $\delta > 0$*, choose* $\epsilon_0 = 2$*,* $y = \min\left\{\delta, \frac{1}{2}\right\}$ *and* $x = \frac{y}{2}$*. Then* $|x - y| = \frac{y}{2} \leq \frac{\delta}{2} < \delta$ *and*

$$|\frac{1}{x} - \frac{1}{y}| = |\frac{2}{y} - \frac{1}{y}| = \frac{1}{y} \geq 2$$

Theorem 3.2.25 *Let $f : [a, b] \to \mathbb{R}$. Then, f is continuous if and only if f is uniformly continuous.*

The following procedure is a practical method of showing that a function is uniformly continuous.

Definition 3.2.26 *A function f definite of $I \subset \mathbb{R}$ in $\mathbb{R}$ is said to be k-Lipschitzian over I if:*

$$\exists k \geq 0, \forall x, y \in I : |f(x) - f(y)| \leq k|x - y|$$

Remark 3.2.27 *A k-Lipschitzian function on I is uniformly continuous on I.*

Indeed; for $\epsilon > 0$, we just need to take $\delta = \frac{\epsilon}{k}$, such that

$$\forall x, y \in I : |x - y| < \delta \Rightarrow |f(x) - f(y)| \leq k|x - y| < \epsilon.$$

Definition 3.2.28 *A function f is said to be contracting on I if f is k-Lipschitzian with $0 \leq k < 1$.*

Conclusion 3.2.29 *A contracting function on I is uniformly continuous on I.*

Here is a theorem very used in practice to show that a function is bijective.

Theorem 3.2.30 *Let $f : I \to \mathbb{R}$ be a function defined on an interval I of $\mathbb{R}$. If f is continuous and strictly monotonic on I, so*

1. f establishes a bijection of the interval I in the image interval $J = f(I)$,
2. The inverse function $f^{-1} : J \to I$ is continuous and strictly monotonic on J and it has the same direction of variation as f.

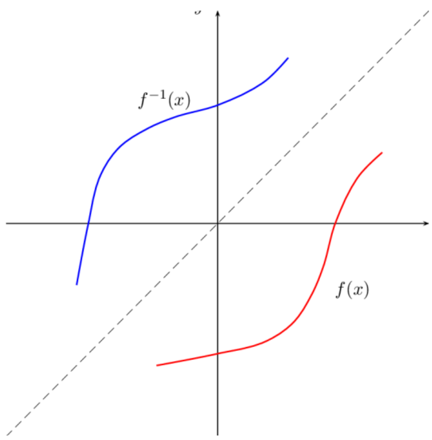

Figure 24: inverse function

3.3 Derivable function

3.3.1 Definition and properties

Definition 3.3.1 *Let f be defined in a δ-neighbourhood $(x_0-\delta, x_0+\delta)$ of $x_0 \in \mathbb{R}$ $(\delta > 0)$. We say that f is differentiable at x_0 if the limit*

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$

exists in $\mathbb{R}$. This limit, denoted by $f'(x_0)$, is called the derivative of f at x_0 .

Furthermore, if f is differentiable at every $x_0 \in I$ (an interval), we write f' or $\dfrac{df}{dx}$ for the function f'.

Example 3.3.2 *1)* $f(x) = c \Rightarrow f'(x_0) = \lim\limits_{x\to x_0} \dfrac{c-c}{x-x_0} = 0 \Rightarrow f'(x_0) = 0,\ \forall x_0 \in \mathbb{R}.$

2) $f(x) = x^2 \Rightarrow f'(x_0) = \lim\limits_{x\to x_0} \dfrac{x^2-x_0^2}{x-x_0} = \lim\limits_{x\to x_0} (x+x_0) = 2x_0 \Rightarrow f'(x_0) = 2x_0.$

3) $f(x) = \sqrt{x} \Rightarrow f'(x_0) = \lim\limits_{x\to x_0} \dfrac{\sqrt{x}-\sqrt{x_0}}{x-x_0} = \lim\limits_{x\to x_0} \dfrac{1}{\sqrt{x}+\sqrt{x_0}} = \dfrac{1}{2\sqrt{x_0}} \Rightarrow f'(x_0) = \dfrac{1}{2\sqrt{x_0}}.$

Remark 3.3.3 *By substituting $x - x_0 = h$, we find:*

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \text{ exists and is finite} \Leftrightarrow (\ f \text{ is derivative at } x_0)$$

Example 3.3.4 *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. The derivative of f at a point $x_0 \in \mathbb{R}$ is*

$$\begin{aligned} f'(x_0) &= \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0} \frac{(x_0+h)^2-x_0^2}{h} \\ &= \lim_{h\to 0} \frac{h^2+2hx_0}{h} = \lim_{h\to 0} (h+2x_0) = 2x_0 \end{aligned}$$

Theorem 3.3.5 *If $f : I \to \mathbb{R}$ is differentiable at $x_0 \in I$, then f is continuous at x_0.*

Proof.

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}.$$

Hence

$$\begin{aligned} \lim_{h\to 0} (f(x)-f(x_0)) &= \lim_{h\to 0} (f(x_0+h)-f(x_0)) \\ &= \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} h \\ &= f'(x_0) \lim_{h\to 0} h = 0. \end{aligned}$$

Therefore

$$\lim_{h\to 0} f(x) = f(x_0)$$

■

Example 3.3.6 *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x|$. Then f is differentiable at any $x \in \mathbb{R} - \{0\}$. But f is not differentiable at 0.*

In fact, we have : if $x > 0$, then

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = 1.$$

If $x < 0$, then

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = -1.$$

Therefore, the derivative does not exist at 0, as

$$\lim_{h\to 0^+} \frac{f(x+h) - f(x)}{h} \neq \lim_{h\to 0^-} \frac{f(x+h) - f(x)}{h}.$$

Note that the function f in the above example is continuous at 0 : thus, continuity does not imply differentiability. However, the converse is true.

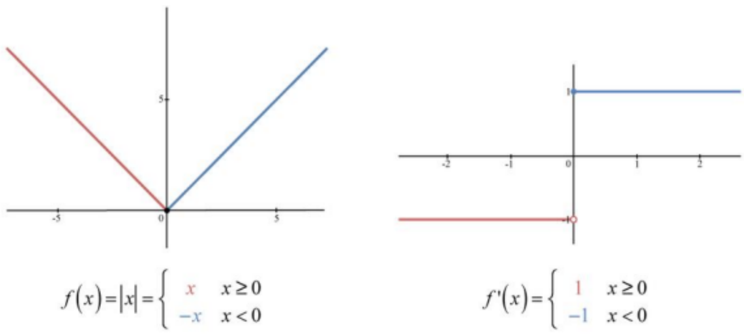

Figure25 : f and f' s.t $f(x)=|x|$.

3.3.2 One-sided derivatives

1) In a manner similar to the definition of the one-sided limit, we may also define the left and right derivatives of f at x_0 via

$$f'_-(x_0) = \lim_{h\to 0^-} \frac{f(x+h) - f(x)}{h},\ f'_+(x_0) = \lim_{h\to 0^+} \frac{f(x+h) - f(x)}{h}$$

2) $\begin{pmatrix} f \text{ is derivative on the right and left at } x_0 \\ \text{and } f'_-(x_0) = f'_+(x_0) \end{pmatrix} \Leftrightarrow \begin{pmatrix} f \text{ is derivative at } x_0 \\ f'(x_0) = f'_-(x_0) = f'_+(x_0) \end{pmatrix}$

3) If $f'_-(x_0) \neq f'_+(x_0)$, then f is not differentiable at x_0 and we say that x_0 is an angular point.

Remark 3.3.7 *If f is differentiable at $x_0 \in \mathbb{R}$ then there exists a function $\varepsilon(x)$ such that $\lim_{x\to 0}\varepsilon(x) = 0$ and*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \varepsilon(x)(x - x_0)$$

Indeed, define

$$\varepsilon(x) := \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)$$

Then $\varepsilon(x) \to 0$ as $x \to 0$ and $f(x) = f(x_0) + f'(x_0)(x - x_0) + \varepsilon(x)(x - x_0)$

This enables one to re-interpret the formula in the above Remark as follows. If f is differentiable at $x_0 \in \mathbb{R}$, then one can write for the value of $f(x = x_0 + h)$, that is "near" x_0:

$$f(x_0 + h) = f(x_0) + f'(x_0)h + o(h),$$

where the notation $o(h)$ reads as "little o of h", and denotes any function which has the following property: $\lim_{h\to 0}\frac{o(h)}{h} = 0$.

3.3.3 Operations on derivative functions

Theorem 3.3.8 *Let $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$ be differentiable at $x_0 \in I$. Then,*

1. *(Linearity)* $\forall\alpha \in \mathbb{R}$, $(\alpha f + g)'(x_0) = \alpha f'(x_0) + g'(x_0)$.
2. *(Product rule)* $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.
3. *(Quotient rule) If $g(x) \neq 0$ for all $x \in I$, then* $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}$.

Proof.

1.

$$\begin{aligned}\lim_{x\to x_0}\frac{(\alpha f + g)(x) - (\alpha f + g)(x_0)}{x - x_0} &= \lim_{x\to x_0}\left(\alpha\frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0}\right)\\ &= \alpha f'(x) + g'(x).\end{aligned}$$

2. We first write

$$\frac{(fg)(x) - (fg)(x_0)}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0}g(x) + f(x_0)\frac{g(x) - g(x_0)}{x - x_0}$$

then

$$\begin{aligned}\lim_{x\to x_0}\frac{(fg)(x) - (fg)(x_0)}{x - x_0} &= \lim_{x\to x_0}\left[\frac{f(x) - f(x_0)}{x - x_0}g(x) + f(x_0)\frac{g(x) - g(x_0)}{x - x_0}\right]\\ &= f'(x_0)g(x_0) + f(x_0)g'(x_0)\end{aligned}$$

3.The result follows from

$$\begin{aligned}\frac{\left(\dfrac{f}{g}\right)(x)-\left(\dfrac{f}{g}\right)(x_0)}{x-x_0} &= \frac{f(x)\,g(x_0)-g(x)\,f(x_0)}{g(x)\,g(x_0)\,(x-x_0)}\\ &= \frac{f(x)-f(x_0)}{g(x)\,g(x_0)\,(x-x_0)}g(x_0)-\frac{g(x)-g(x_0)}{g(x)\,g(x_0)\,(x-x_0)}f(x_0),\end{aligned}$$

then

$$\begin{aligned}\lim_{x\to x_0}\frac{\left(\dfrac{f}{g}\right)(x)-\left(\dfrac{f}{g}\right)(x_0)}{x-x_0} &= \lim_{x\to x_0}\frac{f(x)\,g(x_0)-g(x)\,f(x_0)}{g(x)\,g(x_0)\,(x-x_0)}\\ &= \lim_{x\to x_0}\left(\frac{f(x)-f(x_0)}{g(x)\,g(x_0)\,(x-x_0)}g(x_0)-\frac{g(x)-g(x_0)}{g(x)\,g(x_0)\,(x-x_0)}f(x_0)\right)\\ &= \frac{f\prime(x_0)g(x_0)-f(x_0)g\prime(x_0)}{g^2(x_0)}.\end{aligned}$$

■

Theorem 3.3.9 *If g is differentiable at $x_0 \in \mathbb{R}$ and f is differentiable at $g(x_0)$, then $f \circ g$ is differentiable at x_0 and*

$$(f\circ g)'(x_0) = f'(g(x_0))\,g'(x_0)$$

Proof. By definition of the derivative and Remark 3.3.7, we have

$$f(y)-f(y_0) = f'(y_0)(y-y_0)+\varepsilon(y)(y-y_0),$$

where $\varepsilon(y)\to 0$ as $y \to y_0$. Replace y and y_0 in the above equality by $y = g(x)$ and $y_0 = g(x_0)$, and divide both sides by $x - x_0$, to obtain

$$\frac{f(g(x))-f(g(x_0))}{x-x_0} = f'(g(x_0))\frac{g(x)-g(x_0)}{x-x_0}+\varepsilon(g(x))\frac{g(x)-g(x_0)}{x-x_0}.$$

By Theorem 3.3.5, g is continuous at x_0. Hence $y = g(x) \to g(x_0) = y_0$ as $x \to x_0$, and $\varepsilon(g(x)) \to 0$ as $x \to x_0$.Passing to limit $x \to x_0$ in the above equality yields the required result. ■

Theorem 3.3.10 *Let f be continuous and strictly increasing on (a, b). Suppose that, for some $x_0 \in (a, b)$, f is differentiable at x_0 and $f'(x_0) \neq 0$. Then the inverse function $g = f^{-1}$is differentiable at $y_0 = f(x_0)$ and*

$$g'(y_0) = \frac{1}{f'(x_0)},$$

and we write $f'(x_0)$ as a function of y_0.

Example 3.3.11 *Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Then $f'(0) = 0$ and f is not invertible on any neighborhood of the origin, because the function is non-monotonic. On the other hand, if $f :]0, +\infty[\to]0, +\infty[$ is defined by $f(x) = x^2$, then$f'(x) = 2x \neq 0$ and the inverse function $f^{-1} :]0, +\infty[\to]0, +\infty[$ is given by*

$$f^{-1}(y) = \sqrt{y}.$$

The formula for the inverse of the derivative gives

$$\left(f^{-1}\right)'\left(x^2\right) = \frac{1}{f'(x)} = \frac{1}{2x}$$

or, writing $x = f^{-1}(y)$,

$$\left(f^{-1}\right)'(y) = \frac{1}{2\sqrt{y}}.$$

Example 3.3.12 *Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$. Then f is strictly increasing. The inverse function $f^{-1} : \mathbb{R} \to \mathbb{R}$ is given by*

$$f^{-1}(y) = y^{\frac{1}{3}}.$$

Then $f'(0) = 0$ and f^{-1} is not differentiable at $f(0) = 0$, On the other hand, $f^{-1}$is differentiable at non-zero points of $\mathbb{R}$, with

$$\left(f^{-1}\right)'\left(x^3\right) = \frac{1}{f'(x)} = \frac{1}{3x^2},$$

or, writing $x = y^{\frac{1}{3}}$,

$$\left(f^{-1}\right)'(y) = \frac{1}{3y^{\frac{2}{3}}}.$$

3.3.4 Derivative of usual functions

μ represents a function $x \to \mu(x)$.

function	derivative
x^n	nx^{n-1} $(n \in \mathbb{Z})$
$\frac{1}{x}$	$-\frac{1}{x^2}$
$\sqrt{x}$	$\frac{1}{2}\frac{1}{\sqrt{x}}$
x^α	$\alpha x^{\alpha-1}$ $(\alpha \in \mathbb{R})$
e^x	e^x
$\ln x$	$\frac{1}{x}$
$\cos x$	$-\sin x$
$\sin x$	$\cos x$
$\tan x$	$1+\tan^2 x = \frac{1}{\cos^2 x}$

function	derivative
μ^n	$n\mu'\mu^{n-1}$, $(n \in \mathbb{Z})$
$\frac{1}{\mu}$	$-\frac{\mu'}{\mu^2}$
$\sqrt{\mu}$	$\frac{1}{2}\frac{\mu'}{\sqrt{\mu}}$
μ^α	$\alpha\mu'\mu^{\alpha-1}$, $(\alpha \in \mathbb{R})$
e^μ	$\mu' e^\mu$
$\ln \mu$	$\frac{\mu'}{\mu}$
$\cos \mu$	$-\mu' \sin \mu$
$\sin \mu$	$\mu' \cos \mu$
$\tan \mu$	$\left(1+\tan^2 \mu\right)\mu' = \frac{\mu'}{\cos^2 \mu}$

3.3.5 The n^{th} derivative

Definition 3.3.13 *Let $f : I \to \mathbb{R}$ be a differentiable function and let f' be its derivative. If the function $f' : I \to \mathbb{R}$ is also differentiable, we denote $f'' = (f')'$ the second derivative of f. More generally we note:*

$$f^{(0)} = f,\ f^{(1)} = f',\ f^{(2)} = f'' \text{ and } f^{(n+1)} = \left(f^{(n)}\right)'$$

If the n^{th} derivative $f^{(n)}$ exists, we say that f is n times differentiable.

- *If f is n times differentiable on I and $f^{(n)}$ is continuous on I, we say that f belongs to class C^n, and we write $f \in C^n(I, \mathbb{R})$.*

- *If f is differentiable an infinite number of times, i.e., $\forall n \in \mathbb{N}$, $f^{(n)}$ exists and is continuous, we say that f belongs to class C^∞, and we write $f \in C^\infty\ (I, \mathbb{R})$*

- *If f is continuous but not differentiable, we say that f belongs to class C^0, and we write $f \in C^0(I, \mathbb{R})$.*

Example 3.3.14 *Polynomial functions, $\cos x$, $\sin x$, e^x are functions belonging to class C^∞ on $\mathbb{R}$.*

Exercise 3.3.15 *Computing the n^{th} derivative of the function $f(x) = \ln x$.*

Solution 3.3.16 *The first derivatives of $\ln x$ are*

$$f'(x) = \frac{1}{x},\ f''(x) = \frac{(-1).1}{x^2},\ f^{(3)}(x) = \frac{(-1)^2.1.2}{x^3}.$$

The n^{th} derivative of $\ln x$, denoted $f^{(n)}$, is given by the following formula for : $n \geq 1$

$$f^{(n)}(x) = \frac{(-1)^{n-1} . (n-1)!}{x^n}.$$

We assume that the relation is true for some integer n. In other words, we assume $P(n)$is true. We must then prove that $P(n+1)$ is true.

$$\begin{aligned} f^{(n+1)}(x) &= \left(f^{(n)}\right)'(x) = \left(\frac{(-1)^{n-1} . (n-1)!}{x^n}\right)' \\ &= \frac{(-1)^{n-1} . (n-1)! (-n) x^{n-1}}{x^{2n}} = \frac{(-1)^n .n!}{x^{n+1}}. \end{aligned}$$

Exercise 3.3.17 *Using the same method, prove that.*

$$\begin{aligned} \sin^{(n)} x &= \sin\left(x + n\frac{\pi}{2}\right). \\ \cos^{(n)} x &= \cos\left(x + n\frac{\pi}{2}\right). \end{aligned}$$

Solution 3.3.18 *For $\sin^{(n)} x$: we have*

$$\begin{aligned} \sin x^{(1)} &= \cos x = \sin\left(x + \frac{\pi}{2}\right) \\ \sin^{(2)} x &= \sin\left(x + \frac{\pi}{2}\right)' = \cos\left(x + \frac{\pi}{2}\right) = \sin\left(x + 2\frac{\pi}{2}\right) \\ \sin^{(3)} x &= \sin\left(x + 2\frac{\pi}{2}\right)' = \cos\left(x + 2\frac{\pi}{2}\right) = \sin\left(x + 3\frac{\pi}{2}\right) \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \sin^{(n)} x &= \sin\left(x + n\frac{\pi}{2}\right). \end{aligned}$$

In the same way we demonstrate the second.

Leibniz's rule:

Let f and g be two functions belonging to class $C^n(I, \mathbb{R})$. Then $f.g$ is also a function in class $C^n(I, \mathbb{R})$, and we have:

$$(f.g)^{(n)} = \sum_{k=0}^{n} C_n^k f^{(k)} g^{(n-k)},$$

where $C_n^k = \dfrac{n!}{(n-k)!k!}$.

Poof of Leibniz Rule. The Leibniz rule can be proved with the help of mathematical induction. Let $f(x)$ and $g(x)$ be n times differentiable functions. Applying the initial case of mathematical induction for $n = 1$ we have the following expression.

$$(f(x).g(x))' = f'(x).g(x) + f(x).g'(x).$$

Which is the simple product rule and it holds true for $n = 1$. Let us assume that this statement is true for all $n > 1$, and we have the below expression.

$$\begin{aligned}
(f.g)^{(n)} &= \sum_{k=0}^{n} C_n^k f^{(k)} g^{(n-k)} = f^{(0)} g^{(n)} + ... + C_n^k f^{(k)} g^{(n-k)} + ... + f^{(n)} g^{(0)} \\
(f.g)^{(n+1)} &= \left((f.g)^{(n)}\right)' = \sum_{k=0}^{n} C_n^k \left(f^{(k)} g^{(n-k)}\right)' \\
&= \sum_{k=0}^{n} C_n^k \left(f^{(k+1)} g^{(n-k)} + f^{(k)} g^{(n+1-k)}\right) \\
&= \sum_{k=0}^{n} C_n^k f^{(k+1)} g^{(n-k)} + \sum_{k=0}^{n} C_n^k f^{(k)} g^{(n+1-k)}.
\end{aligned}$$

We change the variable in the first sum: $p = k + 1$

$$\sum_{k=0}^{n} C_n^k f^{(k+1)} g^{(n-k)} = \sum_{p=1}^{n+1} C_n^{p-1} f^{(p)} g^{(n+1-p)}.$$

Therefore:

$$(f.g)^{(n+1)} = \sum_{k=1}^{n+1} C_n^{k-1} f^{(k)} g^{(n+1-k)} + \sum_{k=0}^{n} C_n^k f^{(k)} g^{(n+1-k)},$$

consequently

$$(f.g)^{(n+1)} = \left(\sum_{k=1}^{n} \left(C_n^{k-1} + C_n^k\right) f^{(k)} g^{(n+1-k)}\right) + C_n^n f^{(n+1)} g^{(0)} + C_n^0 f^{(0)} g^{(n+1)}.$$

Note that $C_n^n = C_n^0 = 1$ and $C_n^{k-1} + C_n^k = C_{n+1}^k$ then

$$(f.g)^{(n+1)} = \left(\sum_{k=1}^{n} C_{n+1}^k f^{(k)} g^{(n+1-k)}\right) + f^{(n+1)} g^{(0)} + f^{(0)} g^{(n+1)}.$$

Note that we can include the last two terms in the sum

$$\begin{aligned}
&C_{n+1}^0 f^{(0)} g^{(n+1)} = f^{(0)} g^{(n+1)} \quad \text{and} \\
&C_{n+1}^{n+1} f^{(n+1)} g^{(n+1-n+1)} = f^{(n+1)} g^{(0)},
\end{aligned}$$

then

$$(f.g)^{(n+1)} = \sum_{k=0}^{n+1} C_{n+1}^{k} f^{(k)} g^{(n+1-k)}.$$

So, according to the proof by induction

$$(\forall n \in \mathbb{N})\,(\forall x \in I) : (f.g)^{(n)}(x) = \sum_{k=0}^{n} C_n^k f^{(k)}(x)\, g^{(n-k)}(x)$$

■

Exercise 3.3.19 *Calculate the n^{th} derivative of the function: $f(x) = \left(x^2 + x\right)\ln x$ and $g(x) = e^x \sin x$.*

Solution 3.3.20 *1)* $f(x) = \left(x^2 + x\right)\ln x = f_1(x).g_1(x)$ *or* $f_1(x) = x^2 + x$ *and* $g_1(x) = \ln x$

$f_1^{(0)}(x) = x^2 + x \Rightarrow f_1^{(1)}(x) = 2x + 1,\ f_1^{(2)}(x) = 2,\ f_1^{(k)}(x) = 0$ *for all* $k \geq 3\,(k \in \mathbb{N})$,

$g_1^{(n)}(x) = \dfrac{(-1)^{n-1}.(n-1)!}{x^n}$,

$$\begin{aligned} f^{(n)}(x) &= (f_1.g_1)^{(n)}(x) = \sum_{k=0}^{n} C_n^k f_1^{(k)} g_1^{(n-k)}(x) \\ &= C_n^0 f_1^{(0)} g_1^{(n)}(x) + C_n^1 f_1^{(1)} g_1^{(n-1)}(x) + C_n^2 f_1^{(2)} g_1^{(n-2)}(x) + 0 \\ &= \left(x^2 + x\right)\frac{(-1)^{n-1}.(n-1)!}{x^n} + n(2x+1)\frac{(-1)^{n-2}.(n-2)!}{x^{n-1}} + \frac{n(n-1)}{2}.2.\frac{(-1)^{n-3}.(n-3)!}{x^{n-2}}. \end{aligned}$$

2) $f(x) = e^x \sin x = f_2(x).g_2(x)$ *or* $f_2(x) = e^x$ *and* $g_2(x) = \sin x$.

$f_2^{(n)}(x) = e^x$ *and* $g_2^{(n)}(x) = \sin\left(x + n\dfrac{\pi}{2}\right)$

$$\begin{aligned} f^{(n)}(x) &= (f_2.g_2)^{(n)}(x) = \sum_{k=0}^{n} C_n^k f_2^{(k)} g_2^{(n-k)}(x) \\ &= \sum_{k=0}^{n} C_n^k e^x \sin\left(x + (n-k)\frac{\pi}{2}\right). \end{aligned}$$

Definition 3.3.21 (Critical Points) *Let c be an interior point in the domain of f. We say that c is a critical point of f if $f'(c) = 0$, or $f'(c)$ is undefined.*

Theorem 3.3.22 (Fermat's Theorem) *If f has a local extremum at c and f is differentiable at c, then $f'(c) = 0$.*

Exercise 3.3.23 *Find the local extremum (maximum and minimum) over the specified interval*

$$f(x) = -x^2 + 3x - 2 \quad over\ [1,3].$$

Solution 3.3.24 ***Step 1.*** *Evaluate f at the endpoints $x = 1$ and $x = 3$.*

$$f(1) = 0 \text{ and } f(3) = -2.$$

Step 2 *Since $f'(x) = -2x + 3 = 0$ at $x = \frac{3}{2}$ and $\frac{3}{2}$ is in the interval $[1, 3]$, $f\left(\frac{3}{2}\right) = \frac{1}{4}$ is a candidate for a local extremum of f over $[1, 3]$.*

Step 3. *We compare the values found in **steps 1** and **2**. We find that the local extremum minimum of f is -2, and it occurs at $x = 3$. The local extremum maximum of f is $\frac{1}{4}$, and it occurs at $x = \frac{3}{2}$ as shown in Figure*

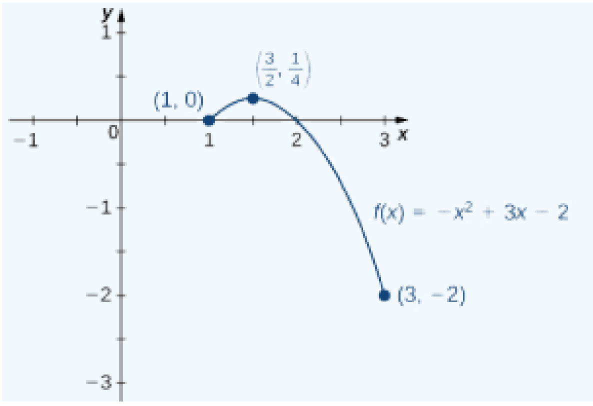

Figure 26 : *This function has both local extremum maximum and minimum*

Method of finding points where the function f possesses extreme values:

Theorem 3.3.25 *Let $f \in F(D, \mathbb{R})$ be differentiable on D, assuming that f'' exists, let $x_0 \in D$ then :*

$$\begin{cases} f'(x_0) = 0 \\ f''(x_0) > 0 \end{cases} \implies x_0 \text{ is a local minimum point of } f$$

$$\begin{cases} f'(x_0) = 0 \\ f''(x_0) < 0 \end{cases} \implies x_0 \text{ is a local maximum point of } f$$

Example 3.3.26 *Let the function $f(x) = \cos x$ and $x_0 = 0$, $x_1 = \pi$.*

$$f'(x) = -\sin x \Rightarrow \begin{cases} f'(0) = 0 \\ f'(\pi) = 0 \end{cases} \Rightarrow x_0 \text{ and } x_1 \text{ are critical points.}$$

$$f''(x) = -\cos x \Rightarrow \begin{cases} f''(0) = -1 < 0 \rightarrow x_0 = 0 \text{ is a local maximum point of } f. \\ f''(\pi) = 1 > 0 \ \rightarrow x_1 = \pi \text{ is a local minimum point of } f. \end{cases}.$$

In general : Let $f \in C^{(n)}(D, \mathbb{R})$, where:

$$f'(x_0) = f''(x_0) = ... = f^{(n-1)}(x_0) = 0,\ f^{(n)}(x_0) \neq 0$$

Case1: If n is even

$$\begin{aligned} f^{(n)}(x_0) &> 0 \Rightarrow x_0 \text{ is a local minimum point of } f. \\ f^{(n)}(x_0) &< 0 \Rightarrow x_0 \text{ is a local maximum point of } f \end{aligned}$$

Case2: If n is odd

$$f^{(n)}(x_0) \neq 0 \Rightarrow x_0 \text{ is not extreme point but rather an inflection point.}$$

Example 3.3.27 *$x_0 = 0$ and $f(x) = x^3$*

$$\begin{aligned} f'(x) &= 3x^2 \Rightarrow f'(0) = 0 \rightarrow f \textit{ has a critical point at } x_0 = 0 \\ f''(x) &= 6x \Rightarrow f''(0) = 0 \\ f'''(x) &= 6 \Rightarrow f'''(0) \neq 0 \end{aligned}$$

With $n = 3$ being an odd number and $f'''(x) \neq 0$, hence $x_0 = 0$ is an inflection point, and f does not possess an extreme value at $x_0 = 0$.

Example 3.3.28 *Let the function $f : \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = 6\ln x - 2x^3 + 9x^2 - 18x$. Does f have an extreme value at $x_0 = 0$?*

$$\begin{aligned} f'(x) &= \frac{6}{x} - 6x^2 + 18x - 18 \Rightarrow f'(1) = 0 \\ f''(x) &= -\frac{6}{x^2} - 12x + 18 \Rightarrow f''(1) = 0 \\ f'''(x) &= \frac{12}{x^3} - 12 \Rightarrow f'''(1) = 0 \\ f^{(4)}(x) &= -\frac{36}{x^4} \Rightarrow f^{(4)}(1) \neq 0. \end{aligned}$$

Since $n = 4$ is even number and $f^{(4)}(1) < 0$, then $x_0 = 1$is a local maximum point of f and $f(1) = -11$ is the local maximum value of f.

Theorem 3.3.29 (Rolle's Theorem) *Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function over the closed interval $[a, b]$ and differentiable over the open interval $]a, b[$ such that*

$$f(a) = f(b).$$

There then exists at least one $c \in]a, b[$ such that $f'c) = 0$.

Proof. - If f is constant over $[a, b]$ then it is obvious ($f' = 0$).

-Otherwise; since f is continuous on $[a,b]$ then it is bounded on $[a,b]$, so $\sup\limits_{x\in]a,b[} f(x) = M$ exists, we then have $\forall x \in]a,b[$: $f(x) \leq M$, we can assume that M is different from $f(a) = f(b)$ and therefore there exists c in $]a,b[$ such that $M = f(c)$, therefore

$$\forall x \in]a,b[: f(x) \leq f(c),$$

then c is a local maximum of f so according to Fermat's theorem $f'(c) = 0$. ■

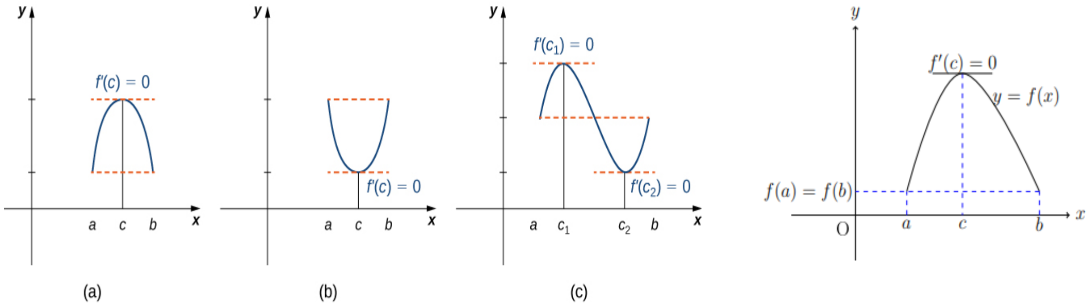

Figure 27 : Illustration of Rolle's Theorem

- If a differentiable function f satisfies $f(a) = f(b)$, then its derivative must be zero at some point (c) between a and b

- This means that the curve at the point $(c, f(c))$ accepts a tangent parallel to the x-axis.

Example 3.3.30 *Can Rolle's Theorem be applied to the function* $f(x) = x^2 + 1$ *in the interval* $[-1,1]$*?*

We have f is continuous in the interval $[-1,1]$, differentiable on $]-1,1[$, and $f(1) = f(-1)$. Therefore, Rolle's Theorem can be applied.

Example 3.3.31 *Can Rolle's Theorem be applied to the function* $f(x) = (|x| - 1)$ *on* $[-1,1]$.

We have f is continuous over $[-1,1]$ and $f(1) = f(-1) = 0$, but $f'(c) \neq 0$ for any $c \in]-1,1[$ because f is not differentiable at $x = 0$, the conditions of Rolle's theorem are not satisfied. In fact, the conclusion does not hold here; there is no $c \in]-1,1[$, such *that* $f'(c) = 0$.

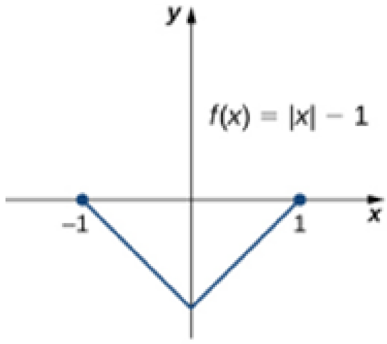

Figure 28 : No **c** ***such that*** $f'(c)$=0

Example 3.3.32 *Consider* $f(x) = x^3 + 1$. *The function is continuous on* $[-1, 1]$*and differentiable on* $]-1, 1[$*, with*

$$f'(x) = 3x^2.$$

Thus, there exists a poin $c \in]-1, 1[$ *such that* $f'(c) = 0$.

However,

$$f(-1) = 0 \neq 2 = f(1),$$

so the condition $f(a) = f(b)$ *is not satisfied. Therefore, the existence of a point where* $f'(c) = 0$ ***does not guarantee*** *that Rolle's Theorem applies.*

Theorem 3.3.33 (Finite Increment Theorem or Mean Value Theorem) *Let* f *be continuous over the closed interval* $[a, b]$ *and differentiable over the open interval* $]a, b[$. *Then, there exists at least one point* $c \in]a, b[$ *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. We put

$$g(x) := f(x) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a).$$

Then g is continuous on $[a, b]$ and differentiable on $]a, b[$, and

$$g'(x) := f'(x) - \left[\frac{f(b) - f(a)}{b - a}\right].$$

Moreover, $g(a) = f(a)$, and

$$g(b) = f(b) - \left[\frac{f(b) - f(a)}{b - a}\right](b - a) = f(a).$$

Therefore, by Rolle's theorem,

$$(\exists c \in \,]a, b[)\left[g'(c)\right] = 0.$$

■

Corollary 3.3.34 *If f is differentiable on an interval I and $f'(x) = 0$ for all $x \in I$, then f is constant on I.*

Proof. Let a and b be any two points in the interval with $a \neq b$. Then, by the Mean Value Theorem, there is a point x in $]a, b[$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

But $f'(x) = 0$ for all x in the interval, so

$$0 = \frac{f(b) - f(a)}{b - a},$$

and consequently, $f(b) = f(a)$. Thus the value of f at any two points is the same and f is constant on the interval. ■

Corollary 3.3.35 *If f and g are defined on the same interval and $f'(x) = g'(x)$ there, then $f = g + c$ for some number $c \in \mathbb{R}$.*

The proof is left as an exercise.

Corollary 3.3.36 *If $f'(x) > 0$ (resp. $f'(x) < 0$) for all x in some interval, then f is increasing (resp. decreasing) on this interval.*

Proof. Consider the case $f'(x) > 0$. Let a and b be any two points in the interval, with $a < b$. By the Mean Value Theorem, there is a point x in $]a, b[$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

But $f'(x) > 0$ for all x in the interval, so that

$$\frac{f(b) - f(a)}{b - a} > 0.$$

Since $b - a > 0$, it follows that $f(b) > f(a)$, which proves that f is increasing on the interval. ■

The case $f'(x) < 0$ is left as an exercise.

Exercise 3.3.37 *Using the Finite Increments Theorem on the function* $f(x) = \sin x$ *, we prove that*

$$\forall x > 0 : |\sin x| \leq |x|.$$

Solution 3.3.38 *The function* f *is continuous on* $\mathbb{R}$ *and differentiable on* $\mathbb{R}$*, so it is continuous on* $[0, x]$ *and differentiable on* $]0, x[$*, according to the Finite Increments Theorem:*

$$\exists c \in]0, x[: f(x) - f(0) = (x - 0)f'(c)$$

So:

$$\begin{aligned} \sin x = x \cos c &\Rightarrow |\sin x| = |x| \, |\cos c| \\ &\Rightarrow |\sin x| \leq |x| \quad (\forall x \in \mathbb{R} : |\cos x| \leq 1) \end{aligned}$$

Hence:

$$\forall x > 0 : |\sin x| \leq |x| \, .$$

Exercise 3.3.39 *Prove that* $\forall x > 0 : \dfrac{x}{x+1} < \ln(1+x) < x.$

Solution 3.3.40 *We set :* $f(t) = \ln(1+t) \Rightarrow f'(t) = \dfrac{1}{t+1}$ *is continuous and differentiable on*$]-1, +\infty[$*. Thus,* f *is continuous on* $[0, x]$ *and differentiable on* $]0, x[$*. According to the Finite Increments Theorem:*

$$\exists c \in]0, x[: f(x) - f(0) = (x - 0)f'(c)$$

So,

$$\ln(1+x) = x.\frac{1}{c+1}$$

And we have :

$$0 < c < x \Rightarrow 1 < 1 + c < 1 + x$$

Which implies:

$$\text{for } x > 0, \frac{x}{1+x} < \frac{x}{1+c} < x,$$

and $\ln(1+x) = \dfrac{x}{1+c}$.
Therefore,

$$\text{for } x > 0, \frac{x}{1+x} < \ln(1+x) < x$$

The next theorem is a generalization of the Mean Value Theorem. It is of interest because of its use in applications.

Theorem 3.3.41 (Cauchy Mean Value Theorem) *If f and g are continuous on $[a,b]$ and differentiable on $]a,b[$, then*

$$\exists c \in \,]a,b[\, : [f(b)-f(a)]\, g'(c) = [g(b)-g(a)]\, f'(c).$$

(If $g(b) \neq g(a)$, and $g'(c) \neq 0$, the above equality can be rewritten as

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Note that if $g(x) = x$, we obtain the Mean Value Theorem.)

Proof. Let $h : [a,b] \to \mathbb{R}$ be defined by

$$h(x) = [f(b)-f(a)]\, g(x) - [g(b)-g(a)]\, f(x).$$

Then

$$h(a) = f(b)g(a) - f(a)g(b) = h(b),$$

so that h satisfies Rolle's theorem. Therefore,

$$\exists c \in \,]a,b[\, : h'(c) = 0 = [f(b)-f(a)]\, g'(c) - [g(b)-g(a)]\, f'(c).$$

■

3.3.6 Hôpital's rule:

Eliminate cases of indeterminacy in the form $(\infty - \infty)\,(0 \times \infty)$

It is used to remove cases of indeterminacy in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Theorem 3.3.42 *Let f and g be differentiable functions near x_0 in domain D:*

Where:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}.$$

Therefore: $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = l$ *(supposing l is a defined limit, it could be ∞),*

$$\Rightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} = l.$$

Proof. By the Cauchy Mean Value Theorem,

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+th)}{g'(a+th)}$$

for some $0 < t < 1$. Now, move to the limit $h \to 0$ to obtain the result. ■

Example 3.3.43 *1-* $\lim\limits_{x\to0}\dfrac{e^x-1}{\sin x}=\left(\dfrac{0}{0}\right)\overset{H}{\to}\lim\limits_{x\to0}\dfrac{e^x}{\cos x}=1\Rightarrow\lim\limits_{x\to0}\dfrac{e^x-1}{\sin x}=1.$

2- $\lim\limits_{x\to\infty}\dfrac{\ln x}{e^x}=\left(\dfrac{\infty}{\infty}\right)\overset{H}{\to}\lim\limits_{x\to\infty}\dfrac{\frac{1}{x}}{e^x}=\lim\limits_{x\to\infty}\dfrac{1}{xe^x}=0\Rightarrow\lim\limits_{x\to\infty}\dfrac{\ln x}{e^x}=0.$

Remark 3.3.44 *The converse of Hôpital's Rule is not true. It is possible for* $\lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)}$ *to exist while* $\lim\limits_{x\to x_0}\dfrac{f'(x)}{g'(x)}$ *does not exist (where either f or g is not differentiable at x_0).*

Example 3.3.45

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to0}\frac{x^2\sin\frac{1}{x}}{\sin x}=\lim_{x\to0}\frac{x\sin\frac{1}{x}}{\left(\frac{\sin x}{x}\right)}=\frac{0}{1}=0$$

$$\begin{aligned}\lim_{x\to x_0}\frac{f'(x)}{g'(x)}&=\lim_{x\to0}\frac{\left[2x\sin\frac{1}{x}-\frac{\cos\frac{1}{x}}{x^2}\left(x^2\right)\right]}{\cos x}\\&=\lim_{x\to0}\frac{\left[2x\sin\frac{1}{x}-\cos\frac{1}{x}\right]}{\cos x}\left(\lim_{x\to0}\cos\frac{1}{x}\ \textit{does not exist}\right)\end{aligned}$$

So $\lim\limits_{x\to x_0}\dfrac{f'(x)}{g'(x)}$ *doesn't exist*

Eliminate cases of indeterminacy in the form $(\infty-\infty)$ **or** $(0\times\infty)$

To remove cases of indeterminacy in the form $(0\times\infty)$, we apply Hospital's rule, we write it in
the form $\frac{0}{0}$or $\frac{\infty}{\infty}$.

* $(0\times\infty)=\lim\limits_{x\to x_0}f(x)g(x)=\lim\limits_{x\to x_0}\dfrac{f(x)}{\frac{1}{g(x)}}=\dfrac{0}{\frac{1}{\infty}}=\dfrac{0}{0}\to H.$

* $(\infty\times0)=\lim\limits_{x\to x_0}f(x)g(x)=\lim\limits_{x\to x_0}\dfrac{f(x)}{\frac{1}{g(x)}}=\dfrac{\infty}{\frac{1}{0}}=\dfrac{\infty}{\infty}\to H.$

To remove cases of indeterminacy in the form $\infty-\infty$ we use:

* $(\infty-\infty)=\lim\limits_{x\to x_0}(f(x)-g(x))=\lim\limits_{x\to x_0}f(x)\left[1-\dfrac{f(x)}{g(x)}\right].$

Applying Hôpital's rule to $\frac{f(x)}{g(x)}$, which is of the form $\frac{\infty}{\infty}$, we have two cases:

a) $\lim_{x\to x_0} \frac{f(x)}{g(x)} \neq 1$. Hence, $\lim_{x\to x_0} f(x)\left[1-\frac{f(x)}{g(x)}\right]=\infty$.

b) $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$.It becomes the indeterminacy of the form $\infty\times 0$.

$$\lim_{x\to x_0}(f(x)-g(x)) = \lim_{x\to x_0}\frac{1-\frac{g(x)}{f(x)}}{\frac{1}{f(x)}} = \frac{1-1}{\frac{1}{\infty}} = \frac{0}{0}$$

Or

$$\lim_{x\to x_0}\frac{\frac{f(x)}{1}}{\frac{1}{1-\frac{g(x)}{f(x)}}} = \frac{\infty}{\frac{1}{1-1}}\frac{\infty}{\infty} \to H.$$

Example 3.3.46 *a)* $\lim_{x\to+\infty} e^{-x}\ln x = (0\times\infty)$.

$$\lim_{x\to+\infty}\frac{\ln x}{e^x} \overset{H}{\to} \lim_{x\to+\infty}\frac{\lim_{x\to+\infty}\frac{1}{x}}{e^x} = \lim_{x\to+\infty}\frac{1}{xe^x} = 0.$$

b) $\lim_{x\to 0^+} \ln x + \frac{1}{x} = (-\infty+\infty)$.

$$\lim_{x\to 0^+}\ln x+\frac{1}{x} = \lim_{x\to 0^+}\ln x\left(1+\frac{\frac{1}{x}}{\ln x}\right).$$

$$\lim_{x\to 0^+}\frac{\frac{1}{x}}{\ln x} = \frac{+\infty}{-\infty} \overset{H}{\to} \lim_{x\to 0^+}\frac{-\frac{1}{x^2}}{\frac{1}{x}} = \lim_{x\to 0^+} -\frac{1}{x} = -\infty.$$

Therefore $\lim_{x\to 0^+}\ln x+\frac{1}{x} = \lim_{x\to 0^+}\ln x\left(1+\frac{\frac{1}{x}}{\ln x}\right) = (-\infty)(-\infty) = +\infty.$

3.4 Elementary functions

We now use power series to strictly define the Exponential, Logarithmic, and Trigonometric functions and describe their properties.

3.4.1 Trigonometric functions

Arcsine function

$$\begin{aligned} f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] &\to [-1, 1] \\ x &\to f(x) = \sin x \end{aligned}$$

f is continuous, strictly increasing over $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then f is bijective and therefore f^{-1} exists, is continuous and strictly increasing, and we have $f\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right) = [-1, 1]$ and

$$\begin{aligned} f^{-1} : [-1, 1] &\to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ y &\to f^{-1}(y) = \arcsin y \end{aligned}$$

from where we have

$$\left(\begin{array}{l} \arcsin y = x \\ -1 \leq y \leq 1 \end{array}\right) \Leftrightarrow \left(\begin{array}{l} \sin x = y \\ -\frac{\pi}{2} \leq x \leq -\frac{\pi}{2} \end{array}\right).$$

Forthermore, the arcsine function is:

- Differentiable on $]-1, +1[$ and

$$\forall y \in \,]-1, 1[\,, (\arcsin y)' = \frac{1}{\sqrt{1 - y^2}},$$

in fact

$$\begin{aligned} &y \in \,]-1, 1[\, : \arcsin y = x \Leftrightarrow y = \sin x \text{ and} \\ &(\arcsin y)' = \frac{1}{(\sin x)'} = \frac{1}{\cos x}. \end{aligned}$$

But we have

$$\begin{aligned} \cos^2 x + \sin^2 x = 1 \quad &\Leftrightarrow \cos x = \pm\sqrt{1 - \sin^2 x} \\ &\Leftrightarrow \cos x = \sqrt{1 - \sin^2 (\arcsin y)} \quad (\cos x > 0, \text{ on } \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[) \\ &\Leftrightarrow \cos x = \sqrt{1 - y^2}. \end{aligned}$$

So

$$(\arcsin y)' = \frac{1}{\sqrt{1 - y^2}}, \forall y \in \,]-1, 1[\,.$$

See Figure 29

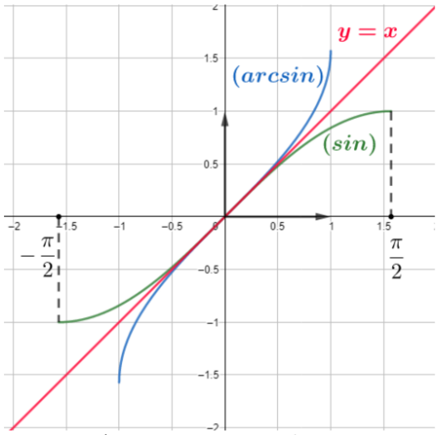

Figure 29 : sin **and** arcsin

Note

$$\begin{aligned} \sin(\arcsin y) &= y \quad \forall y \in [-1,1] \\ \arcsin(\sin x) &= x \quad \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \end{aligned}$$

In other words

$$\sin x = y \Leftrightarrow x = \arcsin y \quad \text{if } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Arccosine function

$$\begin{aligned} f : [0,\pi] &\to [-1,1] \\ x &\to f(x) = \cos x \end{aligned}$$

f is continuous, strictly decreasing over $[0,\pi]$, then f is bijective and therefore f^{-1} exists, is continuous and strictly decreasing , and we have $f([0,\pi]) = [-1,1]$ and

$$\begin{aligned} f^{-1} : [-1,1] &\to [0,\pi] \\ y &\to f^{-1}(y) = \arccos y, \end{aligned}$$

from where we have

$$\begin{pmatrix} \arccos y = x \\ -1 \le y \le 1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \cos x = y \\ 0 \le x \le \pi \end{pmatrix}.$$

Forthermore, the arccosine function is:

- Differentiable on $]-1,+1[$ and

$$\forall y \in]-1,1[, (\arccos y)' = \frac{-1}{\sqrt{1-y^2}},$$

in fact

$$\forall y \in]-1,1[: \arccos y = x \Leftrightarrow y = \cos x,$$

and

$$\begin{aligned}(\arccos y)' &= \frac{1}{(\cos x)'}\\ &= \frac{-1}{\sin x} \quad (\sin x > 0, \text{ on }]0,\pi[)\\ &= \frac{-1}{\sqrt{1-\cos^2 x}} = \frac{-1}{\sqrt{1-y^2}}.\end{aligned}$$

See Figure 30

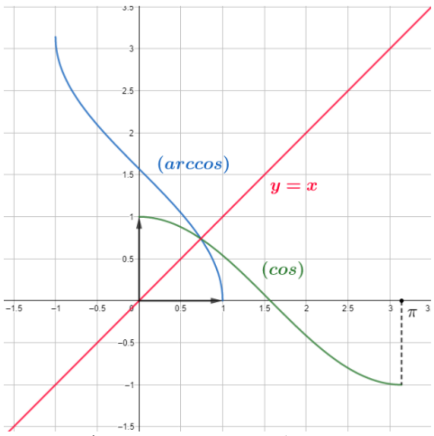

Figure 30 : cos **and** arccos

Note

$$\begin{aligned}\cos(\arccos y) &= y \quad \forall y \in [-1,1]\\ \arccos(\cos x) &= x \quad \forall x \in [0,\pi]\end{aligned}$$

Arctangent function

$$\begin{aligned}f : \left]-\frac{\pi}{2},\frac{\pi}{2}\right[&\to]-\infty,+\infty[\\ x &\to f(x) = \tan x = \frac{\sin x}{\cos x}\end{aligned}$$

f is continuous, strictly increasing on $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, then f is bijective and therefore f^{-1} exists, is continuous and strictly increasing and we have $f\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right) =]-\infty,+\infty[$ and

$$\begin{aligned}f^{-1} :]-\infty,+\infty[&\to \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\\ y &\to f^{-1}(y) = \arctan y,\end{aligned}$$

from which we have

$$\begin{pmatrix}\arctan y = x\\ y \in \mathbb{R}\end{pmatrix} \Leftrightarrow \begin{pmatrix}\tan x = y\\ -\frac{\pi}{2} < x < \frac{\pi}{2}\end{pmatrix}.$$

Forthermore, the arctangent function is:

- Differentiable on $\mathbb{R}$ and

$$\forall y \in \mathbb{R}, (\arctan y)' = \frac{1}{1+y^2},$$

in fact

$$\forall y \in \mathbb{R} : \arctan y = x \Leftrightarrow y = \tan x$$

and

$$\begin{aligned}(\arctan y)' &= \frac{1}{(\tan x)'} \\ &= \frac{1}{1+\tan^2 x} \\ &= \frac{1}{1+y^2}.\end{aligned}$$

See Figure 31

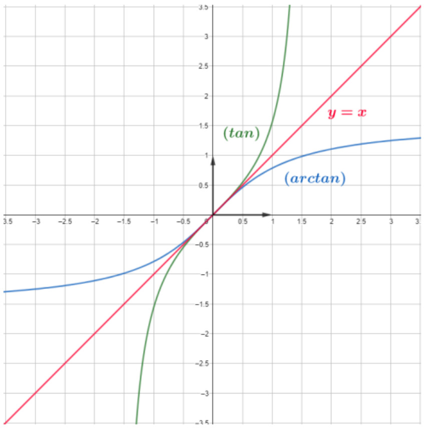

Figure 31 : tan and arctan

Example 3.4.1 *1)*

$$\begin{aligned}\arctan 0 &= \alpha : \alpha \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\\ &\Rightarrow \tan(\arctan 0) = \tan \alpha \\ &\Rightarrow 0 = \tan \alpha : \alpha \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\\ &\Rightarrow \alpha = 0.\end{aligned}$$

2)

$$\lim_{y \to -\infty} \arctan y = -\frac{\pi}{2} \textit{ and } \lim_{y \to \infty} \arctan y = \frac{\pi}{2}.$$

Arccotangent function

$$\begin{aligned} f :]0,\pi[&\to]-\infty,+\infty[\\ x &\to f(x) = \cot x = \frac{\cos x}{\sin x} \end{aligned}$$

f is continuous, strictly decreasing on $]0,\pi[$, then f is bijective and therefore f^{-1} exists, is continuous and strictly decreasing and we have $f(]0,\pi[) =]-\infty,+\infty[$ and

$$\begin{aligned} f^{-1} :]-\infty,+\infty[&\to]0,\pi[\\ y &\to f^{-1}(y) = \operatorname{arccot} y, \end{aligned}$$

from which we have

$$\begin{pmatrix} \operatorname{arccot} y = x \\ y \in \mathbb{R} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \cot x = y \\ 0 < x < \pi \end{pmatrix}.$$

Forthermore, the arccotangent function is:

- Differentiable on $\mathbb{R}$ and

$$\forall y \in \mathbb{R}, (\operatorname{arccot} y)' = \frac{1}{1+y^2},$$

in fact

$$\forall y \in \mathbb{R} : \operatorname{arccot} y = x \Leftrightarrow y = \cot x$$

and

$$\begin{aligned} (\operatorname{arccot} y)' &= \frac{1}{(\cot x)'} \\ &= \frac{1}{-1-\cot^2 x} \\ &= \frac{-1}{1+y^2}. \end{aligned}$$

- Class C^∞ on $\mathbb{R}$.

See Figure 32

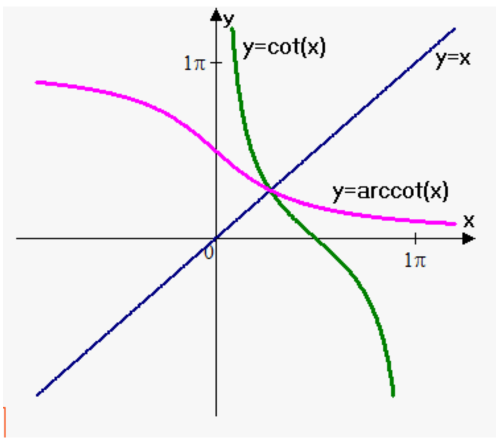

Figure 32 : cot and arccot

Example 3.4.2 *1. Show that:* $2\arctan x = \arccos \dfrac{1-x^2}{1+x^2}$

2. Deduce a simplified expression of $\cos(4\arctan x)$.

3. Solve the equation

$$\arctan x + \arctan 4x = \frac{\pi}{4} - \arctan \frac{1}{5}$$

Solution 3.4.3 *1. Let's assume*

$$\alpha = \arctan x \Leftrightarrow x = \tan\alpha \text{ with } \alpha \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[,$$

and determine:

$$\cos(2\arctan x) = \cos 2\alpha = 2\cos^2\alpha - 1,$$

hence

$$\cos^2\alpha = \frac{1}{1+\tan^2 x} = \frac{1}{1+x^2},$$

where from

$$\cos 2\alpha = \frac{2}{1+x^2} - 1 = \frac{1-x^2}{1+x^2}$$

and

$$2\alpha = \arccos\frac{1-x^2}{1+x^2} = 2\arctan x.$$

2.Relationship

$$\begin{aligned}
\cos 4\alpha &= 2\cos^2(2\alpha) - 1 \\
&= 2\cos^2\left(\arccos\frac{1-x^2}{1+x^2}\right) - 1 \\
&= 2\left(\frac{1-x^2}{1+x^2}\right)^2 - 1 \\
&= \frac{2\left(1-x^2\right)^2 - \left(1+x^2\right)^2}{\left(1+x^2\right)^2} \\
&= \frac{x^4 - 6x^2 + 1}{\left(1+x^2\right)^2}.
\end{aligned}$$

$\cos(4\arctan x) = \dfrac{x^4 - 6x^2 + 1}{\left(1+x^2\right)^2}.$

3.We consider the equation

$$\arctan x + \arctan 4x = \frac{\pi}{4} - \arctan\frac{1}{5}.$$

Since

$$\left(\frac{\pi}{4} - \arctan\frac{1}{5}\right) \in \left[0; \frac{\pi}{4}\right],$$

the values of x we are looking for must satisfy

$$0 \leq \arctan x + \arctan 4x \leq \frac{\pi}{4}.$$

Thus, we may safely take the tangent of both sides. Using the identity

$$\tan(A+B) \quad \frac{\tan A + \tan B}{1 - \tan A \tan B} \ ,$$

we obtain

$$\tan(\arctan x + \arctan 4x) = \frac{x+4x}{1-4x^2} = \frac{5x}{1-4x^2}.$$

On the right-hand side,

$$\tan\left(\frac{\pi}{4} - \arctan\frac{1}{5}\right) == \frac{1-\frac{1}{5}}{1+\frac{1}{5}} = \frac{2}{3}.$$

We are therefore led to the equation.

$$\frac{5x}{1-4x^2} = \frac{2}{3} \Leftrightarrow 2 - 8x^2 = 15x \Leftrightarrow 8x^2 + 15x - 2 = 0.$$

This quadratic equation has discriminant

$$\Delta = (15)^2 - 4 \times (-2) \times 8 = (17)^2 \,.$$

Hence,

$$x = \frac{-15 \pm 17}{16} = \begin{cases} x_1 = \frac{1}{8} \\ \textit{and} \\ x_2 = -2, \textit{ rejected} \end{cases}$$

However, only $x_1 = \frac{1}{8}$ satisfies the inequality

$$0 \leq \arctan x + \arctan 4x \leq \frac{\pi}{4}.$$

3.4.2 Exponential function

Definition 3.4.4 *The exponential function denoted* exp *is the only differentiable function on $\mathbb{R}$, equal to its derivative and verifying:* $\exp(0) = 1$.

Properties

1. $\forall x \in \mathbb{R} : \exp(x) > 0.$

2. $\forall x, y \in \mathbb{R} : \exp(x+y) = \exp(x)\exp(y)$.

3. Euler's notation: We set $\exp(x) = e^x$; where $e^1 = e \simeq 2.718$, whence $\forall x, y \in \mathbb{R} : e^{x+y} = e^x e^y$, $e^{-x} = \dfrac{1}{e^x}$, $e^{x-y} = \dfrac{e^x}{e^{-y}}$, $(e^x)^n = e^{nx}$.

4. The exp function is strictly increasing on $\mathbb{R}$.

5. $\forall x, y \in \mathbb{R} : \begin{cases} e^x = e^y \Leftrightarrow x = y. \\ e^x < e^y \Leftrightarrow x < y. \end{cases}$

6. The function $x \to e^x$ is a bijection of $\mathbb{R}$ in $\mathbb{R}_+^*$.

Some reference limits:

1. $\lim\limits_{x\to-\infty} e^x = 0$, $\lim\limits_{x\to\infty} e^x = +\infty$, $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = 1$,

2. $\lim\limits_{x\to 0} \dfrac{e^x}{x^n} = +\infty$, $\lim\limits_{x\to-\infty} x^n e^x = 0$, for all $n \in \mathbb{N}$.

3.4.3 Logarithm function

We call the natural logarithm function denoted ln, the reciprocal function of the exponential function, defined from $]0, +\infty[$ on $\mathbb{R}$ such as

$$\forall x > 0 : x = e^y \Leftrightarrow y = \ln x.$$

Note: The graphs of the natural logarithm function and the exponential function are symmetric with respect to the first bisector, i.e. the line of equation $y = x$, see Figure 33

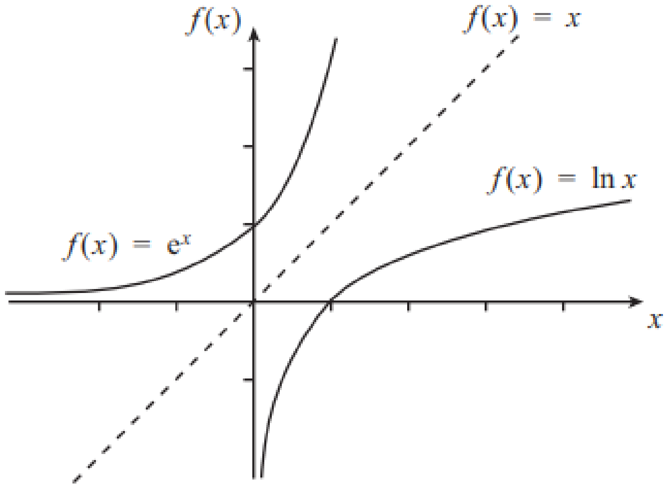

Figure 33 : e^x **and** $\ln x$

Properties

1. $\ln 1 = 0$, $\ln e = 1$.

2. $\forall x \in \mathbb{R} : \ln e^x = x$ and $\forall x \in]0, +\infty[: e^{\ln x} = x$.

3. The function ln is strictly increasing on $]0, +\infty[$.

4. $\forall x, y \in]0, +\infty[: \ln x = \ln y \Leftrightarrow x = y$.

5. $\forall x, y \in]0, +\infty[: \ln (xy) = \ln x + \ln y$.

6. $\forall x, y \in]0, +\infty[: \ln \frac{1}{x} = -\ln x; \ln \frac{y}{x} = \ln y - \ln x$.

7. $\forall x \in]0, +\infty[, \forall n \in \mathbb{N} : \ln x^n = n \ln x$.

Some reference limits:

1. $\lim\limits_{x \to 0^+} \ln x = -\infty$, $\lim\limits_{x \to +\infty} \ln x = +\infty$, $\lim\limits_{x \to 0} \frac{\ln (x+1)}{x} = 1$,

2. $\lim\limits_{x \to +\infty} \frac{\ln x}{x^n} = 0$, $\lim\limits_{x \to -\infty} x^n \ln x = 0$, for all $n \in \mathbb{N}$.

3.4.4 Logarithm function of any base

Definition 3.4.5 *Let a be a strictly positive real number different from 1. The logarithm function with base a is the real function denoted* $\log_a$*, defined on* $]0, +\infty[$ *by*

$$f(x) = \log_a (x) = \frac{\ln (x)}{\ln a},$$

where ln *denotes the natural logarithm.*

• *For $a = e$, we obtain the natural logarithm function* ln, *since* $\ln e = 1$.

• *For $a = 10$, the logarithm is called the decimal logarithm, denoted* log. *Since* $\ln 10 \simeq 2,302$, *it is commonly used in chemistry.*

• *Another frequently used logarithm is the base-2 logarithm, denoted* $\log_2$*, defined by* $\log_2 x = \frac{\ln x}{\ln 2}$, *and widely used in computer science.*

Properties Let a and b be strictly positive real numbers different from 1. For all x, $y \in (0, +\infty)$, we have:

1. $\log_a 1 = 0, \log_a a = 1, \log_{\frac{1}{a}} (x) = -\log_a (x)$.

2. $\log_a x = \frac{\ln b}{\ln a} \log_b x$.
   In particular, for $a = e$ and $b = 10$, we have $\ln x = \ln 10 \log x$.

3. $\log_a x = \log_a y \Leftrightarrow x = y$.

4. $\log_a (xy) = \log_a x + \log_a y$.

5. $\log_a \left(\frac{1}{y}\right) = -\log_a y$, $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$.

6. For all $x \in (0, +\infty)$ and $n \in \mathbb{N}$, $\log_a (x^n) = n \log_a x$.

7. The function $\log_a$ is strictly increasing on $]0, +\infty[$ if $a > 1$ and strictly decreasing on $]0, +\infty[$ if $0 < a < 1$.

3.4.5 Power (Exponential) function

Definition 3.4.6 *Let a be a strictly positive real number different from* 1*, and let $x \in \mathbb{R}$. The exponential function with base a is defined by*

$$a^x = e^{x \ln a}$$

This function is the inverse function of the logarithm function $\log_a$

Properties Let a and b be two strictly positive real numbers, x, $y \in \mathbb{R}$.

1. $a^x > 0$, $\ln a^x = x \ln a$.

2. $1^x = 1$, $a^{x+y} = a^x a^y$, $a^{-x} = \frac{1}{a^x}$, $a^{y-x} = \frac{a^y}{a^x}$.

3. $(ab)^x = a^x b^x$, $(a^x)^y = a^{xy}$.

4. The exponential function a^x is :

   - strictly increasing on $\mathbb{R}$ if $a > 1$,

   - strictly decreasing on $\mathbb{R}$ if $0 < a < 1$.

5. There also exists the power function x^a, defined for $a \in \mathbb{R}_+^*$.

3.4.6 Hyperbolic functions and their inverses

Hyperbolic sine and cosine

Definition 3.4.7 *The functions hyperbolic sine denoted* sinh *or sh and hyperbolic cosine denoted* cosh *or ch are defined on $\mathbb{R}$ by*

$$\begin{aligned} \cosh : \mathbb{R} &\to [1, +\infty[\\ x &\to \frac{e^x + e^{-x}}{2} \end{aligned}, \qquad \begin{aligned} \sinh : \mathbb{R} &\to \mathbb{R} \\ x &\to \frac{e^x - e^{-x}}{2} \end{aligned}$$

Remark 3.4.8 *Any function $f : I \subset \mathbb{R} - \to \mathbb{R}$ decomposes uniquely into the sum of an even function and of an odd function*

$$\forall x \in I,\ f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}.$$

Indeed, $\frac{f(x) + f(-x)}{2}$ *is even and* $\frac{f(x) - f(-x)}{2}$ *is odd. The hyperbolic cosine and hyperbolic sine functions are respectively the even part and the odd part of the exponential function in this decomposition.*

Proposition 3.4.9 *The functions* cosh *and* sinh *are differentiable on $\mathbb{R}$, for all $x \in \mathbb{R}$*

$$(\cosh x)' = \sinh x, \quad (\sinh x)' = \cosh x.$$

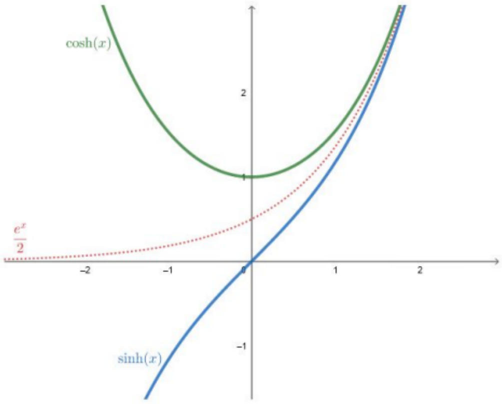

Figure 34 *:* sinh ***and*** cosh

Hyperbolic tangent

Definition 3.4.10 *The hyperbolic tangent function, denoted* tanh *(or sometimes th), is defined on $\mathbb{R}$ by*

$$\begin{aligned} \tanh : \mathbb{R} &\to \mathbb{R} \\ x &\to \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \end{aligned}$$

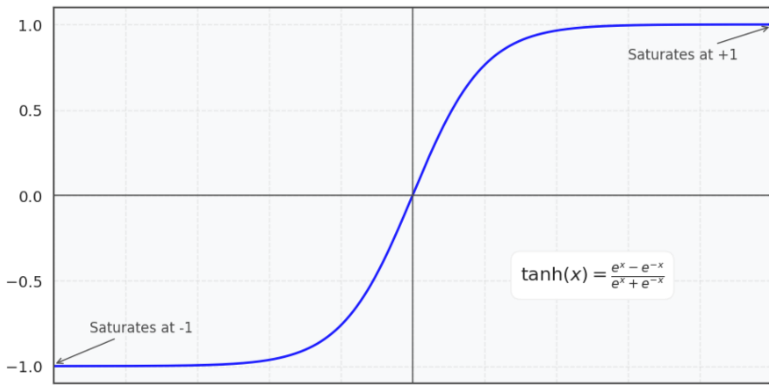

Figure 35 : tanh

Proposition 3.4.11 *The hyperbolic tangent function* tanh *is odd function and is differentiable on* $\mathbb{R}$*. For all* $x \in \mathbb{R}$*, its derivative is*

$$(\tanh x)' = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}.$$

Consequently, since $(\tanh x)' > 0$*, for all* $x \in \mathbb{R}$*, the function* tanh *is strictly increasing on* $\mathbb{R}$ *.*

Hyperbolic cotangent

Definition 3.4.12 *The hyperbolic cotangent function, denoted* coth*, is defined on* $\mathbb{R}^*$ *by*

$$\begin{aligned} \coth : \mathbb{R}^* &\longrightarrow]-\infty, -1[\cup]1, +\infty[\\ x &\longrightarrow \coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \end{aligned}$$

Proposition 3.4.13 *The hyperbolic cotangent function is odd function and is differentiable on* $\mathbb{R}^*$*, for all* $x \in \mathbb{R}^*$*, its derivative is*

$$(\coth x)' = 1 - \coth^2 x = \frac{-1}{\sinh^2 x}.$$

Consequently, since $(\coth x)' < 0$*, for all* $x \in \mathbb{R}^*$*, the function* coth *is strictly decreasing* *on each interval* $]-\infty, 0[$ *and* $]0, +\infty[$.

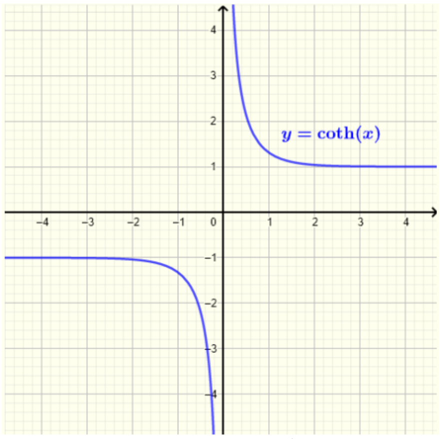

Figure 36 : coth

Proposition 3.4.14 *For all* $x \in \mathbb{R}$

1. $chx + shx = e^x$
2. $chx - shx = e^{-x}$
3. $ch^2x - sh^2x = 1$
4. $ch(x+y) = chxchy + shxshy$
5. $ch(x-y) = chxchy - shxshy$
6. $sh(x+y) = shxchy + chxshy$
7. $sh(x-y) = shxchy - chxshy$
8. $th(x+y) = \dfrac{th(x)+th(y)}{1+th(x)th(y)}$
9. $th(x-y) = \dfrac{th(x)-th(y)}{1-th(x)th(y)}$

Hyperbolic sine argument function

Proposition 3.4.15 *The hyperbolic sine function* $\sinh : \mathbb{R} \to \mathbb{R}$ *is continuous and strictly increasing. Therefore, it admits an inverse function, denoted* $\arg\sin$ *or* $\arg sh$*, defined from* $\mathbb{R}$ *to* $\mathbb{R}$.

Hence, for all x*,* $y \in \mathbb{R}$*,*

$$\begin{pmatrix} \arg\sinh y = x \\ y \in \mathbb{R} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \sinh x = y \\ x \in \mathbb{R} \end{pmatrix}$$

See Figure 37

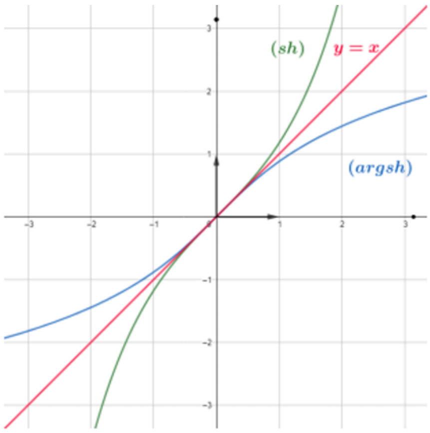

Figure 37 : *sh and* arg *sh*

Forthermore, the arg sinh *function is:*

- *Differentiable on* $\mathbb{R}$ *and*

$$(\arg\sinh y)' = \frac{1}{\sqrt{y^2+1}}$$

- *For all* $y \in \mathbb{R}$, $\arg\sinh y = \ln\left(y + \sqrt{y^2+1}\right)$.

Proof. in fact

$$\arg\sinh y = x \Leftrightarrow y = \sinh x$$

and

$$\begin{aligned}(\arg\sinh y)' &= \frac{1}{(\sinh x)'} \\ &= \frac{1}{\cosh x} \\ &= \frac{1}{\sqrt{\sinh^2 x + 1}} \\ &= \frac{1}{\sqrt{y^2+1}}.\end{aligned}$$

For all $y \in \mathbb{R}$,

$$\arg\sinh y = \ln\left(y + \sqrt{y^2+1}\right).$$

Let

$$x = \arg\sinh y \Leftrightarrow y = \sinh x.$$

Using the hyperbolic identity

$$\cosh^2 x - \sinh^2 x = 1,$$

we obtain

$$\cosh x = \sqrt{\sinh^2 x + 1} = \sqrt{y^2 + 1},$$

where we take the positive square root since $\cosh x > 0$ for all $x \in \mathbb{R}$. Now recall the exponential representation:

$$e^x = \sinh x + \cosh x = y + \sqrt{y^2 + 1}$$
$$x = \ln\left(y + \sqrt{y^2 + 1}\right).$$

■

Hyperbolic cosine argument function

Proposition 3.4.16 *The hyperbolic cosine function* $\cosh : [0, +\infty[\to [1, +\infty[$ *is continuous and strictly increasing. Therefore, it admits an inverse function, denoted* $\arg\cosh$ *or* $\arg ch$*, defined from* $[1, +\infty[$ *to* $[0, +\infty[$.

Hence, for $x \geq 0$ *and* $y \geq 1$*, we have the equivalence :*

$$\arg\cosh y = x \Leftrightarrow y = \cosh x$$

See Figure 38

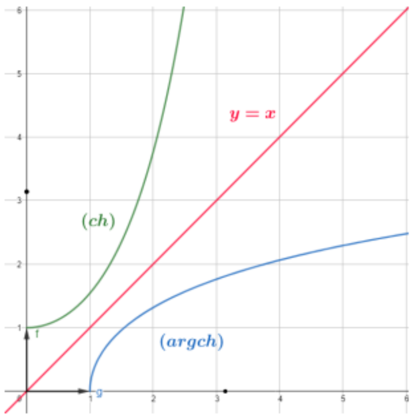

Figure 38 : *ch and* $\arg ch$

Forthermore, the $\arg\cosh$ *function is:*

- *Differentiable on* $]1, +\infty[$ *and*

$$(\arg\cosh y)' = \frac{1}{\sqrt{y^2 - 1}}.$$

- *For all* $y \in [1, +\infty[$, $\arg\cosh y = \ln\left(y + \sqrt{y^2 - 1}\right)$.

Proof. in fact

$$\arg\cosh y = x \Leftrightarrow y = \cosh x$$

and

$$\begin{aligned}(\arg\cosh y)' &= \frac{1}{(\cosh x)'}\\ &= \frac{1}{\sinh x}\\ &= \frac{1}{\sqrt{\cosh^2 x - 1}}\\ &= \frac{1}{\sqrt{y^2-1}}.\end{aligned}$$

For all $y \geq 1$,

$$\arg\cosh y = \ln\left(y + \sqrt{y^2-1}\right).$$

Let

$$x = \arg\cosh y \Leftrightarrow y = \cosh x.$$

Since

$$e^x = \sinh x + \cosh x = y + \sqrt{y^2+1}$$

we obtain

$$e^x = y + \sqrt{y^2+1},$$

Taking the natural logarithm gives

$$x = \ln\left(y + \sqrt{y^2+1}\right).$$

■

Hyperbolic tangent argument function

Proposition 3.4.17 *The hyperbolic tangent function* $\tanh : \mathbb{R} \to]-1, 1[$*, is continuous and strictly increasing. Therefore, it admits an inverse function, called the inverse hyperbolic tangent, denoted* $\arg\tanh$ *or* $\arg th$*, defined by*

$$\arg\tanh :]-1,1[\to \mathbb{R}.$$

Hence, for $x \in \mathbb{R}$ *and* $y \in]-1,1[$*, we have the equivalence :*

$$\arg\tanh y = x \Leftrightarrow \tanh x = y$$

See Figure 39

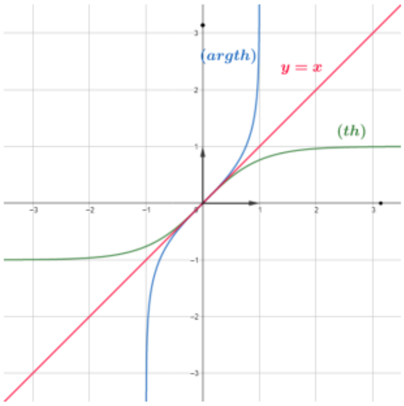

Figure 39 : *th and* arg *th*

Forthermore, the arg tanh *function is:*

- *Differentiable on* $]1,1[$, *and its derivative is*

$$(\arg\tanh y)' = \frac{1}{1-y^2}.$$

- *For all* $y \in]1,1[$,

$$\arg\tanh y = \frac{1}{2}\ln\frac{1+y}{1-y}.$$

Proof. in fact, for all $y \in]-1,1[$ and $x \in \mathbb{R}$:

$$\arg\tanh y = x \Leftrightarrow y = \tanh x$$

and

$$\begin{aligned}(\arg\tanh y)' &= \frac{1}{(\tanh)'} \\ &= \frac{1}{1-\tanh^2 x} \\ &= \frac{1}{1-y^2}.\end{aligned}$$

- For all $y \in]-1,1[$

$$\arg\tanh y = \frac{1}{2}\ln\frac{1+y}{1-y}.$$

Indeed, $\forall\,(y,x) \in]-1,1[\times\mathbb{R}$,

$$\begin{aligned}x = \arg\tanh y \Leftrightarrow y &= \tanh x \\ &\Leftrightarrow y = \frac{e^x+e^{-x}}{e^x-e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1} \\ &\Leftrightarrow y\left(e^{2x}+1\right) = e^{2x}-1 \\ &\Leftrightarrow e^{2x} = \frac{1+y}{1-y} \\ &\Leftrightarrow x = \frac{1}{2}\ln\frac{1+y}{1-y}.\end{aligned}$$

■

Hyperbolic cotangent argument function

Proposition 3.4.18 *The hyperbolic cotangent function* $\coth : \mathbb{R}^* \to]-\infty, -1[\cup]1, +\infty[$, *is continuous and strictly decreasing on each interval* $]-\infty, 0[$ *and* $]0, +\infty[$. *Therefore, it admits an inverse function, called the inverse hyperbolic cotangent, denoted*

$$\arg\coth \ \textit{or} \ \arg cth,$$

defined by

$$\arg\coth :]-\infty, -1[\cup]1, +\infty[\to \mathbb{R}^*.$$

Hence, for $x \in \mathbb{R}^*$ *and* $y \in]-\infty, -1[\cup]1, +\infty[$

$$\arg\coth y = x \Leftrightarrow y = \coth x$$

See Figure 40

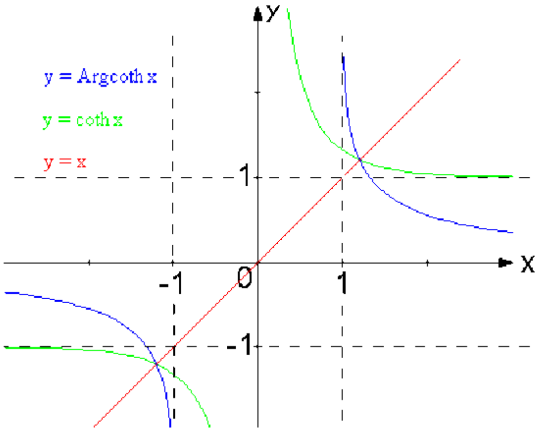

Figure 40 **:** $\coth$ *and* $\arg\coth$

Forthermore, the $\arg\coth$ *function is:*

- *Differentiable on* $]-\infty, -1[\cup]1, +\infty[$, *and its derivative is*

$$(\arg\coth y)' = \frac{1}{1-y^2}.$$

- *For all* $y \in]-\infty, -1[\cup]1, +\infty[$,

$$\arg\tanh y = \frac{1}{2}\ln\frac{y+1}{y-1}.$$

Proof. in fact, $\forall y \in]-\infty, -1[\cup]1, +\infty[$ and $x \in \mathbb{R}^*$:

$$\arg\coth y = x \Leftrightarrow y = \coth x$$

and

$$\begin{aligned}(\arg\coth y)' &= \frac{1}{(\coth x)'}\\ &= \frac{1}{1-\coth^2 x}\\ &= \frac{1}{1-y^2}.\end{aligned}$$

- For all $y \in]-\infty, -1[\cup]1, +\infty[$

$$\arg\coth y = \frac{1}{2}\ln\frac{y+1}{y-1}.$$

Indeed,

$$\begin{aligned}x = \arg\coth y \Leftrightarrow y = \coth x &= \frac{e^x + e^{-x}}{e^x - e^{-x}}\\ \Leftrightarrow y &= \frac{e^{2x}+1}{e^{2x}-1}\\ \Leftrightarrow e^{2x} &= \frac{y+1}{y-1}\\ \Leftrightarrow x &= \frac{1}{2}\ln\frac{y+1}{y-1}.\end{aligned}$$

■

Chapter 4

Internal composition laws

Definition 4.0.19 *Let E be a set. An internal composition law (ICL) on E is a map*

$$\begin{aligned} * : E \times \ E \ &\rightarrow E \\ (a, b) \ &\rightarrow a * b, \end{aligned}$$

and we say that $a * b$ is the composite of a and b for the law $*$. A set E provided with an internal composition law constitutes an algebraic structure and denoted $(E, *)$.

Example 4.0.20 ***1.*** *The addition defined by $(a, b) \rightarrow a + b$ is an internal composition law in $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$.*

2. *The multiplication defined by $(a, b) \rightarrow a \times \ b$ is an internal composition law in $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$.*

3. *The composition defined by $(f, g) \rightarrow f \circ g$ is an internal composition law on the sets of applications from E to E.*

4. *$(a, b) \rightarrow a - b$ isnt an internal composition law in N.*

Definition 4.0.21 *(Usual properties of internal laws). Let $*$ be an internal law on a set E. We say that*

- *The law $*$ is commutative if*

$$\forall a, b \in E : a * b = b * a.$$

- *The law $*$ is said to be associative if*

$$\forall a, b, c \in E : a * (b * c) = (a * b) * c.$$

- *The law* $*$ *admits a neutral element* $e \in E$ *if*

$$\forall a \in E : a * e = e * a = a.$$

- *An element* $\grave{a} \in E$ *is the symmetric of* a *in* E *if*

$$a * \grave{a} = e = \grave{a} * a.$$

$\grave{a}$ *is the inverse of* a *and is denoted* a^{-1}*for the law* $\times$*, (*$\grave{a}$ *is the opposite of* a *and is denoted* $-a$ *for the law* $+$*).*

Example 4.0.22 *In* $\mathbb{R}-\left\{\frac{1}{2}\right\}$ *we define the internal law* $*$ *by :*

$$x * y = x + y - 2xy.$$

1. Closure (internal law): *In fact, let* $x, y \in \mathbb{R}-\left\{\frac{1}{2}\right\}$*, let's show that* $x * y \in \mathbb{R}-\left\{\frac{1}{2}\right\}$,

$$\begin{aligned} x * y = \frac{1}{2} & \Leftrightarrow x + y - 2xy = \frac{1}{2} \\ & \Leftrightarrow x(1-2y) - \frac{1}{2}(1-2y) = 0 \\ & \Leftrightarrow (1-2y)\left(x - \frac{1}{2}\right) = 0 \\ & \Leftrightarrow \left(y - \frac{1}{2}\right)\left(x - \frac{1}{2}\right) = 0 \\ & \Leftrightarrow y = \frac{1}{2} \text{ or } x = \frac{1}{2}. \end{aligned}$$

Hence $x, y \in \mathbb{R}-\left\{\frac{1}{2}\right\}$ *and then* $*$ *is an internal law.*

2. Commutativity : *Let* $x, y \in \mathbb{R}-\left\{\frac{1}{2}\right\}$*, we have*

$$x * y = x + y - 2xy = y + x - 2yx = y * x,$$

so the law $*$ *is commutative.*

3. Associativity :

$$\begin{aligned} (x * y) * z = (x + y - 2xy) * z & = (x + y - 2xy) + z - 2(x + y - 2xy)z \\ & = x + y + z - 2xy - 2xz - 2yz + 4xyz \\ & = x + (y + z - 2yz) - 2x(y + z - 2yz) \\ & = x + (y + z - 2yz) - 2x(y + z - 2yz) \\ & = x + (y * z) - 2x(y * z) = x * (y * z), \end{aligned}$$

so the law $*$ is associative.

4. Neutral element : *Let $e \in \mathbb{R}-\left\{\frac{1}{2}\right\}$, such that $x*e = e*x = x$, then*

$$x + e - 2xe = e + x - 2ex = x \Leftrightarrow e(1-2x) = 0 \Leftrightarrow e = 0 \in \mathbb{R}-\left\{\frac{1}{2}\right\}.$$

Thus, the law $*$ admits as neutral element the element $e = 0$.

5. Symmetric element (Inverse) : *Let $x \in \mathbb{R}-\left\{\frac{1}{2}\right\}$, such that $x*\grave{x} = \grave{x}*x = e$, then*

$$x + \grave{x} - 2x\grave{x} = 0 \Leftrightarrow \grave{x}(1-2x) = -x \Leftrightarrow \grave{x} = \frac{x}{2x-1},$$

Therefore, the symmetric element of x is

$$\grave{x} = \frac{x}{2x-1}, \text{ for all } x \in \mathbb{R}-\left\{\frac{1}{2}\right\}.$$

Let's show that

$$\grave{x} \in \mathbb{R}-\left\{\frac{1}{2}\right\}.$$

Indeed, we must check:

$$\grave{x} = \frac{x}{2x-1} \neq \frac{1}{2}$$

Assume

$$\frac{x}{2x-1} = \frac{1}{2} \Leftrightarrow 2x = 2x - 1 \Leftrightarrow -1 = 0.$$

Impossible, hence. $\grave{x} \in \mathbb{R}-\left\{\frac{1}{2}\right\}$.

Definition 4.0.23 *Let G be a set with two internal laws of composition, denoted Δ and $*$ law is said to be distributive with respect to Δ if $\forall x, y, z \in G$:*

$$x*(y\Delta z) = (x*y)\Delta(x*z)$$

and

$$(y\Delta z)*x = (y*x)\Delta(z*x).$$

4.1 Group, Subgroups

Definition 4.1.1 *Let G be a nonempty set with an internal composition law*

$$* : G \times G \to G$$

The pair $(G, *)$ is called a group if the following conditions are satisfied :

(1) $*$ *is associative.*

(2) $*$ *admits a neutral element(identity elements)* e.

(3) Each element of G *admits a symmetric (inverse) element with respect to* $*$.

If, moreover, the law $*$ is commutative, then the group is said to be commutative or abelian, (named after the mathematician Abel).

Proposition 4.1.2 • *The neutral element of any commutative group is unique.*

• *Let* $(G, *)$ *be a commutative group. For each* $g \in G$, *the symmetric of* g *(denoted* g' *is unique.*

Proof. • Suppose e and θ are any neutral elements of a commutative group $(G, *)$ Then

$$\begin{aligned} e &= e * \theta \quad (\theta \text{ is an neutral element }) \\ &= \theta * e \quad (* \text{ is commutative}) \\ &= \theta \quad (e \text{ is an neutral element }) \end{aligned}$$

Since e and θ are arbitrary neutral elements of $(G, *)$, this implies that all neutral elements are equal to each other, so the neutral element is unique (there is only one of them).

• Suppose g' and h are any symmetric of g. Then

$$\begin{aligned} g' &= g' * e \quad (e \text{ is an neutral element }) \\ &= g' * (g * h) \quad (h \text{ is a symmetric of } g) \\ &= \big(g' * g\big) * h \ (* \text{ is associative }) \\ &= \big(g * g'\big) * h \ (* \text{ is commutative }) \\ &= e\ * h \quad (g' \text{ is a symmetric of } g) \\ &= h \quad (e \text{ is an neutral element }) \end{aligned}$$

Therefore, all symmetric of g are equal, so the symmetric is unique. ■

Example 4.1.3 *(1)* $(\mathbb{Z}, +)$ *is a commutative group.*

(2) $(\mathbb{R}, \times)$ *is not a group because* 0 *does not admit a symmetric element.*

(3) $(\mathbb{R}^*, \times)$ *is a commutative group.*

Definition 4.1.4 *Let* $(G, *)$ *be a group. A part* $H \subset G$ *(non-empty) is a subgroup of* G *if, the restriction of the operation* $*$ *to* H *gives it the group structure.*

Proposition 4.1.5 *Let H be a non-empty part of the group G. Then, H is a subgroup of G if, and only if*

*(**i**) for all $a, b \in H$, we have $a * b \in H$;*

*(**ii**) for all $a \in H$, we have $a' \in H$, where a' is the symmetry of a.*

Example 4.1.6 $(\mathbb{R}_+^*, \times)$ *is a subgroup of* $(\mathbb{R}^*, \times)$. *Indeed*

- *If* $x, y \in \mathbb{R}_+^*$ *then* $x \times y \in \mathbb{R}_+^*$;
- *If* $x \in \mathbb{R}_+^*$ *then* $x' = x^{-1} = \frac{1}{x} \in \mathbb{R}_+^*$.

Example 4.1.7 *We set* $2\mathbb{Z} = \{2z : z \in \mathbb{Z}\}, (2\mathbb{Z}, +)$ *is a subgroup of* $\mathbb{Z}$. *In fact:*

- *If* $x, y \in 2\mathbb{Z}$, *there exists* x_1, $y_1 \in \mathbb{Z}$ *such that* $x = 2x_1$ *and* $y = 2y_1$, *then*

$$x + y = 2x_1 + 2y_1 = 2(x_1 + y_1) \in 2\mathbb{Z},$$

- *If* $x \in 2\mathbb{Z}$, *there exists* $x_1 \in \mathbb{Z}$ *such that* $x = 2x_1$ *then*

$$x' = -x = -2x_1 = 2(-x_1) \in 2\mathbb{Z}.$$

Proposition 4.1.8 *If H is a subgroup of $(G, *)$ then the neutral element $e \in H$.*

Exercise 4.1.9 *We define the internal composition law $*$ by:*

$$\forall x, y \in \mathbb{R}, \quad x * y = xy + \left(x^2 - 1\right)\left(y^2 - 1\right)$$

1. Show that $*$ is commutative, non-associative, and that 1 is neutral element.

2. We define the internal composition law $*$ on $\mathbb{R}^{+*}$ by:

$$\forall x, y \in \mathbb{R}^{+*}, \quad x * y = \sqrt{x^2 + y^2}$$

Show that $*$ is commutative, associative, and that 0 is neutral element. Show that no element of $\mathbb{R}^{+*}$ has a symmetric with respect to $*$.

Solution 4.1.10 *1.*

$$x * y = xy + \left(x^2 - 1\right)\left(y^2 - 1\right) = yx + \left(y^2 - 1\right)\left(x^2 - 1\right) = y * x.$$

The law is commutative.

To show that the law is not associative, it is sufficient to find x, y and z such that:

$$x * (y * z) \neq (x * y) * z.$$

Take, for example : $x = 0$, $y = 2$ *and* $z = 3$,

$$\begin{aligned} x * (y * z) &= 0 * (2 * 3) = 0 * \left(2 \times 3 + \left(2^2 - 1\right)\left(3^2 - 1\right)\right) \\ &= 0 * (6 + 3 \times 8) = 0 * 30 \\ &= 0 + (-1)(900 - 1) = -899. \end{aligned}$$

$$\begin{aligned} (x * y) * z &= (0 * 2) * 3 = (0 + (-1)(3)) * 3 \\ &= -3 * 3 = -3 \times 3 + \left((-3)^2 - 1\right)\left(3^2 - 1\right) \\ &= -9 + 8 \times 8 = 55. \end{aligned}$$

The law $*$ *is not associative.*

$$1 * x = x + (1 - 1)\left(x^2 - 1\right) = x.$$

Moreover, since the law is commutative $1 * x = x * 1$.

We have $1 * x = x * 1 = x$, 1 *is the neutral element.*

2.$\forall x, y \in \mathbb{R}^{+*}$

$$x * y = \sqrt{x^2 + y^2} = \sqrt{y^2 + x^2} = y * x.$$

The law $*$ *is commutative.*

$$\begin{aligned} (x * y) * z &= \sqrt{x^2 + y^2} * z = \sqrt{\left(\sqrt{x^2 + y^2}\right)^2 + z^2} = \sqrt{x^2 + y^2 + z^2}. \\ x * (y * z) &= x * \sqrt{y^2 + z^2} = \sqrt{x^2 + \left(\sqrt{y^2 + z^2}\right)^2} = \sqrt{x^2 + y^2 + z^2}. \end{aligned}$$

The law $*$ *is associative.*

$$0 * x = \sqrt{0^2 + x^2} = \sqrt{x^2} = |x| = x \text{ because } x \geq 0$$

As $*$ *is commutative*

$$0 * x = x * 0 = x$$

0 *is the neutral element.*

Suppose that x *admits a symmetric* y

$$x * y = 0 \Leftrightarrow \sqrt{x^2 + y^2} = 0 \Leftrightarrow x^2 + y^2 = 0 \Leftrightarrow x = y = 0$$

However, if $x > 0$ *and* $y > 0$ *then* $x * y = 0$ *is impossible.*

Therefore, for any $x > 0$, x *does not have a symmetric element with respect to* $*$.

4.2 Ring Structure

Definition 4.2.1 *Let A be a set with two internal composition laws that we will denote $*$ and Δ. $(A, *, \Delta)$ is said to be a ring if the following conditions are met:*

1) *$(A, *)$ is a commutative group.*

2) *The Δ law is associative.*

3) *The Δ law is distributive in relation to the $*$ law, i.e :*

$$\forall a \in A, \forall b \in A, \forall c \in A : (a * b)\, \Delta c = a\Delta c * b\Delta c.$$

and

$$c\Delta\,(a * b) = c\Delta a * c\Delta b.$$

If the Δ law is commutative, the ring $(A, *, \Delta)$ is said to be commutative. If the Δ law admits a neutral element, we say that the ring $(A, *, \Delta)$ is unitary.

Example 4.2.2 $(\mathbb{Z}, +, \times)$ *is a commutative and unitary ring.*

Definition 4.2.3 *If $(A, *, \Delta)$ is a ring and B is a part of A, we say that B is a subring of A if, provided with the laws induced by A, is itself a ring, i.e. $(B, *, \Delta)$ is a ring.*

In the following, A will denote the ring $(A, +, \times)$ with 0 the neutral element of $+$ and if it is unitary, 1 would be its unit.

Proposition 4.2.4 *(characterization of the subrings). A part B of ring A is a subring of A if and only if:*

(i) *for all a, $b \in B$, $a - b \in B$*

(ii) *for all $a, b \in B$, $a \times b \in B$.*

Example 4.2.5 *The set $2\mathbb{Z} = \{2z : z \in \mathbb{Z}\}$is a subring of the ring ,$(\mathbb{Z}, +, \times)$. In fact, let $x, y \in 2\mathbb{Z}$, there exists $n, m \in \mathbb{Z}$, such that $x = 2n$ and $y = 2m$, and we have*

$$x - y = 2(n - m) \in 2\mathbb{Z} \text{ and } x \times y = 2(2nm) \in 2\mathbb{Z}$$

4.3 Structure of a field (body)

Definition 4.3.1 *Let K be a set with two internal composition laws always denoted $*$ and Δ. $(K, *, \Delta)$ is said to be a field if the following conditions are met:*

1) *$(K, *, \Delta)$ is a ring.*

2) *$(K - \{e\}, \Delta)$ is a group, where e is the neutral element of $*$.*

If Δ is commutative, we say that $(K, *, \Delta)$ is a commutative field.

Example 4.3.2 $(\mathbb{R}, +, \times)$ *is a commutative field (body).*

Definition 4.3.3 *If K is a field and H a non-empty part of K then, H is said to be a subfield of K if the restrictions of the two operations of K give H the structure of a field.*

The following result characterizes any subfield H of a given field :

Proposition 4.3.4 *If H is a non-empty part of a field K then, H is a subfield of K if, and only if,*

(1) $a \in H$ *and* $b \in H \Rightarrow a - b \in H$,

(2) $a \in H$ *and* $b \in H - \{0\} \Rightarrow a.b^{-1} \in H$.

Example 4.3.5 • *The set $(\mathbb{R}, +, \times)$ of real numbers is a subfield of the field $(\mathbb{C}, +, \times)$.*

• *The set $(\mathbb{Q}, +, \times)$ of rationals is a subfield of the field $(\mathbb{R}, +, \times)$ and therefore of $(\mathbb{C}, +, \times)$.*

Chapter 5

Vector spaces

In this chapter $\mathbb{K}$ represents a field.

5.1 Vector space

Definition 5.1.1 *Let $\mathbb{K}$ be a commutative field (usually it is $\mathbb{R}$ or $\mathbb{C}$) and let E be a non-empty set with an* ***internal composition law*** *called addition and denoted "+"*

$$\begin{aligned} +: \quad & E \times E \to E \\ & (x, y) \mapsto x + y \end{aligned}$$

and an ***external composition law*** *called multiplication by a scalar and denoted by "·"*

$$\begin{aligned} \cdot: \quad & \mathbb{K} \times E \to E \\ & (\lambda, x) \mapsto \lambda \cdot y \end{aligned}$$

Definition 5.1.2 *A vector space on the field $\mathbb{K}$ or a $\mathbb{K}$- vector space is a triplet $(E, +, .)$ such that:*

1. *(E, +) is a commutative group, where the neutral element is denoted by 0_E and the symmetric of an element x of E will be denoted $-x$.*

2.$\forall \alpha, \beta \in \mathbb{K}$, $\forall x \in E$,

$$\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$$

3. $\forall \alpha, \beta \in \mathbb{K}$, $\forall x \in E$,

$$(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$$

4. $\forall \alpha \in \mathbb{K}$, $\forall x, y \in E$,

$$\alpha \cdot (x + y) = (\alpha \cdot x) + (\alpha \cdot y)$$

5. $1_K \cdot x = x$.

Remark 5.1.3 *1. The elements of E are called vectors and those of $\mathbb{K}$ scalars.*
2. "vector space over $\mathbb{K}$", means $\mathbb{K}$-vector space.

Example 5.1.4 - $(\mathbb{R},+,\cdot)$ *is an $\mathbb{R}$- vector space,*
- $(\mathbb{C},+,\cdot)$ *is an $\mathbb{C}$- vector space,*
- *If we consider $\mathbb{R}^n$ with the following two operations*

$$\begin{array}{lll}(+): & \mathbb{R}^n \times \mathbb{R}^n & \to \mathbb{R}^n \\ & ((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)) & \to (x_1+y_1, x_2+y_2, ..., x_n+y_n)\end{array}$$

and

$$\begin{array}{lll}(\cdot): & \mathbb{R} \times \mathbb{R}^n & \to \mathbb{R}^n \\ & (\lambda, (x_1, x_2, ..., x_n)) & \to (\lambda x_1, \lambda x_2, ..., \lambda x_n)\end{array}$$

we can easily show that $(\mathbb{R}^n,+,.)$ is an $\mathbb{R}$- vector space.

Example 5.1.5 *The set $E = F(\mathbb{R},\mathbb{R})$ of functions from $\mathbb{R}$ to $\mathbb{R}$ endowed with the usual laws, addition of fuctions and multiplication of the functions by a real number:*

$$(f+g)(x) = f(x) + g(x)$$

and

$$(\alpha.f)(x) = \alpha.f(x),$$

is a $\mathbb{R}$ - vector space.

Proposition 5.1.6 *If E is $\mathbb{K}$- vector space, then we have the following properties:*
(1) $\forall x \in E,\ 0_{\mathbb{K}}.x = 0_E$,
(2) $\forall x \in E,\ (-1_{\mathbb{K}}).x = -x$
(3) $\forall \lambda \in \mathbb{K},\ \lambda 0_E = 0_E$
(4) $\forall \lambda \in \mathbb{K},\ \forall x, y \in E,\ \lambda.(x-y) = \lambda.x - \lambda.y$
(5) $\forall \lambda \in \mathbb{K},\ \forall x \in E,\ \lambda.x = 0_E \Leftrightarrow \lambda = 0_{\mathbb{K}}$ *or* $x = 0_E$.

5.1.1 Vector subspace

In this part, E will denote a $\mathbb{K}$-vector space.

Definition 5.1.7 *A subset F of E is called a vector subspace of E if*
(i) $\emptyset \neq F \subset E$,
(ii) F is a $\mathbb{K}$-vector space with respect to the same laws.

Remark 5.1.8 *1) When* $(F,+,.)$ *is a vector subspace of* E *then* $0_E \in F$.

2) If $0_E \notin F$. *then* $(F,+,.)$ *cannot be a vector subspace of* E.

Theorem 5.1.9 *Let* F *be a nonempty subset of* E*, the following assertions are equivalent :*

(1) F *is a vector subspace of over* $\mathbb{K}$,

(2) F *is stable for addition and for multiplication by a scalar .i.e*

$$\forall \lambda \in \mathbb{K},\ \forall x, y \in F,\ \ \lambda x \in F \text{ and } x + y \in F.$$

(3) $\forall \lambda, \mu \in \mathbb{K},\ \forall x, y \in F,\ \ \lambda x + \mu y \in F.$

Theorem 5.1.10 *A subset* F *of* E *is called a vector subspace of* E *if the following condition hold :*

(i) $0_E \in F$,

(ii) $\forall x, y \in F,\ x + y \in F$,

(ii) $\forall \alpha \in \mathbb{K},\ \forall x \in F,\ \alpha x \in F$.

Example 5.1.11 *(1)* E *and* 0_E *are vector subspaces of* E.

(2) $F = \left\{(x, y) \in \mathbb{R}^2 / x + y = 0\right\}$ *is a vector subspace of* $\mathbb{R}^2$ *over* $\mathbb{R}$ *because ,*

- $0_E = 0_{\mathbb{R}^2} = (0, 0) \in F \Rightarrow F \neq \emptyset$

- $\forall\,(x, y),\ (x', y') \in F,\ \forall \alpha, \beta \in \mathbb{R} : \alpha\,(x, y) + \beta\,(x', y') \in F$, *i.e* $(\alpha x + \beta x', \alpha y + \beta y') \in F$, *we have*

$$\begin{aligned} &(x, y) \in F \Rightarrow x + y = 0 \text{ and } (x', y') \in F \Rightarrow x' + y' = 0 \\ &\alpha x + \beta x' + \alpha y + \beta y' = \alpha\,(x + y) + \beta\,(x' + y') = \alpha\,(0) + \beta\,(0) = 0 \end{aligned}$$

Then $\alpha\,(x, y) + \beta\,(x', y') \in F$*, so* F *is vector subspace of* $\mathbb{R}^2$.

(3).The set $F = \left\{(x, y) \in \mathbb{R}^2 / x - y + 1 = 0\right\}$ *is not a vector subspace of* $\mathbb{R}^2$ *because the zero vector* $0_{\mathbb{R}^2}$ *does not belong to* F.

5.1.2 Intersection and union of vector subspaces

Proposition 5.1.12 *The intersection of two vector sub-spaces is a vector subspace.*

Proof. Consider F_1 and F_2 two vector subspaces of E. First $0_E \in F_1$, because F_1 is a vector subspace of E. Similarly, $0_E \in F_2$. Thus, $0_E \in F_1 \cap F_2$ and $F_1 \cap F_2$ is therefore not empty. Given x, $y \in F_1 \cap F_2$ and $\alpha, \beta \in \mathbb{R}$, then we have $\alpha x + \beta y \in F_1$ since F_1 is a vector subspace of E. Similarly, $\alpha x + \beta y \in F_2$. Thus, $\alpha x + \beta y \in F_1 \cap F_2$. It follows that $F_1 \cap F_2$ is a vector subspace of E. ■

Lemma 5.1.13 *The intersection* $\cap_{i=1}^{n} F_i$ *of* n *vector subspaces of a vector space* E *(*$n \geq 2$*,* $n \in \mathbb{N}$*) is a vector subspace of* E*.*

Remark 5.1.14 *The union of two vector subspaces is not necessarily a vector subspace.*

Example 5.1.15 *Let* $F_1 = \{(x, y) \in \mathbb{R}^2, x = 0\}$ *and* $F_2 = \{(x, y) \in \mathbb{R}^2, y = 0\}$ *two vector subspaces in* $\mathbb{R}^2$ *,* $F_1 \cup F_2$ *is not a vector subspace, because* $u_1 = (0, 1) \in F_1$*,* $u_2 = (1, 0) \in F_2$ *and* $u_1 + u_2 = (1, 1) \notin F_1 \cup F_2$*.*

5.1.3 Sum of two vector subspaces

Definition 5.1.16 *Let* E_1*,* E_2 *be two vector subspaces of a* $\mathbb{K}$*-vector space* E*, we call the sum of the two vector subspaces* E_1 *and* E_2 *that we denote* $E_1 + E_2$ *the following set:*

$$E_1 + E_2 = \{x \in E : \exists x_1 \in E_1, \exists x_2 \in E_2 \text{ such that } x = x_1 + x_2\}.$$

Example 5.1.17 *Let* $E_1 = \{(x, y) \in \mathbb{R}^2, x = 0\}$ *and* $E_2 = \{(x, y) \in \mathbb{R}^2, y = 0\}$ *vector subspaces in* $\mathbb{R}^2$*, if* $(x, y) \in \mathbb{R}^2$*, then*

$$(x, y) = \underset{\in E_1}{(x, 0)} + \underset{E_2}{(0, y)},$$

so $(x, y) \in E_1 + E_2$*, hence* $E_1 + E_2 = \mathbb{R}^2$ *.*

Proposition 5.1.18 *The sum of two vector subspaces* E_1 *and* E_2 *(of the same* $\mathbb{K}$ *-vector space) is a vector subspace of* E *containing* $E_1 \cup E_2$*, i.e.,* $E_1 \cup E_2 \subset E_1 + E_2$*.*

5.1.4 Direct sum of two vector subspaces

Definition 5.1.19 *Let* E_1 *and* E_2 *be two vector subspaces of the same* $\mathbb{K}$*-vector space* E*. We will say that the sum:* $E_1 + E_2$ *of two vector subspaces is direct if* $E_1 \cap E_2 = \{0_E\}$*. We write* $E_1 \oplus E_2$*.*

Proposition 5.1.20 *Let* E_1 *and* E_2 *be two vector subspaces of the same* $\mathbb{K}$*-vector space* E*. The sum* $E_1 + E_2$ *is direct if* $\forall x \in E_1 + E_2$*, there exists a single vector* $x_1 \in E_1$*, a single vector* $x_2 \in E_2$*, such that* $x = x_1 + x_2$*.*

Example 5.1.21 *Let* $F_1 = \{(x, y, z) \in \mathbb{R}^3 : x = 0\}$ *and* $F_2 = \{(x, y, z) \in \mathbb{R}^3 : y = z = 0\}$ *be two vector subspaces in* $\mathbb{R}^3$ *.*

- Let $(x, y, z) \in \mathbb{R}^3$*, then* $(x, y, z) = \underset{\in F_1}{(0, y, z)} + \underset{\in F_2}{(x, 0, 0)}$*, so* $(x, y, z) \in F_1 + F_2$*, hence* $F_1 + F_2 = \mathbb{R}^3$*.*

- Let $(x, y, z) \in F_1 \cap F_2$, then $(x, y, z) \in F_1$ and $(x, y, z) \in F_2$, this means that $x = 0$ and $y = z = 0$, then $(x, y, z) = 0_{\mathbb{R}^3}$, i.e. $F_1 \cap F_2 = \{0_{\mathbb{R}^3}\}$.

Finally, we conclude that $\mathbb{R}^3 = F_1 \oplus F_2$.

5.1.5 Generating, free, and basis families

Linear combination

Definition 5.1.22 *For $n \in \mathbb{N}^*$, A linear combination of vectors $u_1, u_2, ..., u_n$ of a $\mathbb{K}$-vector space E, is a vector which can be written $V = \sum\limits_{i=1}^{n} \lambda_i u_i$. The elements $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K}$ are called coefficients of the linear combination.*

Example 5.1.23 *In $\mathbb{R}^2$, the vector $U = (9, 8)$ is a linear combination of vectors $(1, 2)$ and $(3, 1)$ because*

$$U = (9, 8) = 3(1, 2) + 2(3, 1)$$

Remark 5.1.24 • *If F is a vector subspace of E, and $u_1, u_2, ..., u_n \in F$, then any linear combination of $u_1, u_2, ..., u_n$ is in F.*

• *Let $u_1, u_2, ..., u_n$, n vectors of a $\mathbb{K}$-vector space E. One can always write 0_E as a linear combination of these vectors, because it suffices to take all zero coefficients of the linear combination.*

• *If $n = 1$, then $V = \lambda_1 u_1$ we say that V is collinear with u_1..*

Generating (Spanning) family

Definition 5.1.25 *We consider a nonempty family $A = (u_1, u_2, ..., u_n)$ of vectors of a $\mathbb{K}$-vector space E with $n \in \mathbb{N}^*$. We say that A generates (spans) E, or that it is generator of E if and only if*

$$Span\{u_1, u_2, ..., u_n\} = E.$$

In other words, any vector of E is a linear combination of the elements of A.

Notation 4 *Given the vectors $u_1, u_2, ..., u_n$ of $\mathbb{K}$-vector space E, we denote $Span(u_1, u_2, ..., u_n)$ or $\langle u_1, u_2, ..., u_n \rangle$ the set of linear combination of $u_1, u_2, ..., u_n$. So we write :*

$$\langle u_1, u_2, ..., u_n \rangle = Span\{u_1, u_2, ..., u_n\} = \left\{ u \in E / \exists \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K}; u = \sum_{i=1}^{n} \lambda_i u_i \right\}.$$

Example 5.1.26 *$A = \{u_1 = (1, 0, 0), u_2 = (0, 1, 0), u_3 = (0, 0, 1)\}$ generates $\mathbb{R}^3$, because for all $U = (x, y, z) \in \mathbb{R}^3$ we have:*

$$(x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1).$$

Example 5.1.27 *In $\mathbb{R}^2$, we consider the vectors $u_1 = (1,1)$, $u_2 = (1,0)$ and $u_3 = (0,-1)$. Let us check that the family (u_1, u_2, u_3) generates $\mathbb{R}^2$. Let $X = (x,y) \in \mathbb{R}^2$, we seek if there exists $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^2$ such that $X = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3$.*

$$X = (x,y) = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \quad \Leftrightarrow \left\{ \begin{array}{l} x = \lambda_1 + \lambda_2 \\ y = \lambda_1 - \lambda_3 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \lambda_2 = x - \lambda_1 \\ \lambda_3 = \lambda_1 - y. \end{array} \right.$$

We therefore obtain $X = \lambda_1 u_1 + (x - \lambda_1) u_2 + (\lambda_1 - y) u_3$, with $\lambda_1 \in \mathbb{R}$. So (u_1, u_2, u_3) is a generating family of $\mathbb{R}^2$.

Free families

Definition 5.1.28 *We consider a nonempty family $A = (u_1, u_2, ..., u_n)$ of E with $n \in \mathbb{N}^*$. We say that A is free if and only if the null vector 0_E is a linear combination of elements of A unique way. In other words:*

$$\forall \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K}, \ \sum_{i=1}^{n} \lambda_i u_i = 0_E \Rightarrow \lambda_1 = \lambda_2 = ... = \lambda_i = 0_{\mathbb{K}}.$$

Example 5.1.29 *The set $A = \{u_1 = (1,0,1), u_2 = (0,2,2), u_3 = (3,7,1)\}$ is free.*

Indeed, let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, we have

$$\lambda_1(1,0,1) + \lambda_2(0,2,2) + \lambda_3(3,7,1) = 0_{\mathbb{R}^3} \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0_{\mathbb{R}}.$$

Remark 5.1.30 *We can use the following expressions:*

- *If A is free then we also say that the vectors $u_1, u_2, ..., u_n$ are linearly independent.*
- *If A is not free, we say that A is linked.*
- *A family of a single vector is free if and only if this vector is non-zero.*

Example 5.1.31 *In $\mathbb{R}^2$, the vector $u = (2,1)$ is not collinear with $v = (1,1)$, that is to say is free.*

Indeed: let $(\lambda_1, \lambda_2) \in \mathbb{R}^2$, such that

$$\lambda_1 u + \lambda_2 v = 0_{\mathbb{R}^2} \Leftrightarrow \left\{ \begin{array}{l} 2\lambda_1 + \lambda_2 = 0 \\ \lambda_1 + \lambda_2 = 0 \end{array} \right. \Rightarrow \lambda_1 = \lambda_2 = 0_{\mathbb{R}}.$$

The unique solution found is the trivial solution $(0,0)$, the family (u,v) is therefore free.

Example 5.1.32 *In* $\mathbb{R}^2$, *the vectors* $u = (1, 2)$, $v = (3, 4)$ *and* $w = (5, 6)$ *are linearly dependent.*

Indeed: let $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$, *such that*

$$\lambda_1 u + \lambda_2 v + \lambda_3 w = 0_{\mathbb{R}^2} \Leftrightarrow \begin{cases} \lambda_1 + 3\lambda_2 + 5\lambda_3 = 0 \\ 2\lambda_1 + 4\lambda_2 + 6\lambda_3 = 0 \end{cases}$$
$$\Longleftrightarrow \begin{cases} \lambda_1 + 3\lambda_2 + 5\lambda_3 = 0 \\ \lambda_1 + 2\lambda_2 + 3\lambda_3 = 0 \end{cases}$$
$$\Longleftrightarrow \begin{cases} \lambda_2 + 2\lambda_3 = 0 \\ \lambda_1 + 2\lambda_2 + 3\lambda_3 = 0 \end{cases}$$
$$\Longleftrightarrow \begin{cases} \lambda_2 = -2\lambda_3 \\ \lambda_1 = \lambda_3. \end{cases}$$

So, this system admits at least one non-trivial solution, for example:

$$\lambda_1 = 1,\ \lambda_2 = -2 \text{ and } \lambda_3 = 1.$$

Since $u - 2v + w = 0_{\mathbb{R}^2}$, *the family* $\{u, v, w\}$*is linearly dependent*

Basis

Definition 5.1.33 *Let* E *be a vector space over a field* $\mathbb{K}$. *A family*

$$A = (u_1, u_2, ..., u_n)$$

is called a basis of E *if it is linearly independent and generating.*

Equivalently, A *is a basis of* E *if and only if every vector* $u \in E$ *can be written in a unique way as a linear combination of the vectors in* A *:*

$$\forall u \in E,\ \exists! \left(\lambda_1, \lambda_2, ..., \lambda_n\right) \in \mathbb{K}^n, \text{ such that } u = \sum_{i=1}^{n} \lambda_i u_i.$$

The scalars $\lambda_1, \lambda_2, ..., \lambda_n$ *are called the coordinates of* u *in the basis* A.

Example 5.1.34
- $B_1 = \{(1, 0), (0.1)\}$ *is the canonical basis of* $\mathbb{R}^2$.
- $B_2 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ *is the canonical basis of* $\mathbb{R}^3$.
- $B_3 = \{(1, 0, ..., 0), (0, 1, 0, ..., 0), ..., (0, ..., 0, 1)\}$ *is the canonical basis of* $\mathbb{R}^n$.

Example 5.1.35 *Consider the vector space of real polynomials of degree less than or equal to* 2.

$$\mathbb{R}_2[x] = \left\{P\left(x\right) = a + bx + cx^2\ /\ a, b, c \in \mathbb{R}\right\}$$

We claim that the family

$$\mathcal{B} = \left\{P_1(x) = 1,\ P_2(x) = x,\ P_3(x) = x^2\right\}$$

is a basis of $\mathbb{R}_2[x]$.

In fact,

i) *Linear independence*

Let $\alpha,\ \beta,\ \gamma \in \mathbb{R}$, *suppose that*

$$\forall x \in \mathbb{R},\ \alpha P_1(x) + \beta P_2(x) + \gamma P_3(x) = 0.$$

This is equivalent to

$$\forall x \in \mathbb{R},\ \alpha + \beta x + \gamma x^2 = 0.$$

(Since a polynomial that is identically zero must have all coefficients equal to zero, we obtain.

$$\alpha = \beta = \gamma = 0$$

Hence, $\left\{1,\ x,\ x^2\right\}$ *is a linearly independent (free) family.*

ii) Generating property

Let $P \in \mathbb{R}_2[x]$, *by definition, there exist* $a,\ b,\ c \in \mathbb{R}$, *such that*

$$\forall x \in \mathbb{R},\ P(x) = a + bx + cx^2 = aP_1(x) + bP_2(x) + cP_3(x),$$

or equivalently,

$$P = aP_1 + bP_2 + cP_3.$$

Therefore, $\left\{1,\ x,\ x^2\right\}$ *generates* $\mathbb{R}_2[x]$.

Example 5.1.36 *Let*

$$u_1 = (1,1),\ u_2 = (1,0),\ u_3 = (0,-1)$$

be vectors in $\mathbb{R}^2$. *As seen in the previous example, the family* (u_1, u_2, u_3) *is a generating family of* $\mathbb{R}^2$ *However, this family is linearly dependent (linked), since*

$$u_1 + u_3 = u_2,$$

which yields a non-trivial linear relation between the vectors. Therefore, (u_1, u_2, u_3) *is not a basis of* $\mathbb{R}^2$. *On the other hand, the family* (u_1, u_2) *is both linearly independent and generating in* $\mathbb{R}^2$. *Consequently,* (u_1, u_2) *is a basis of* $\mathbb{R}^2$.

Example 5.1.37 *Let F be the subset of $\mathbb{R}^3$ defined by:*

$$F = \left\{(x, y, z) \in \mathbb{R}^3 \ / \ x = -2y + z\right\}$$

F is therefore a vector subspace of $\mathbb{R}^3$ generated by the vectors

$$u = (-2, 1, 0) \ \text{ and } v = (1, 0, 1)\,.$$

Indeed

$$\begin{aligned} F = \left\{(x, y, z) \in \mathbb{R}^3 \ / \ x = -2y + z\right\} &= \left\{(-2y + z, y, z) \ / \ (y, z) \in \mathbb{R}^2\right\} \\ &= \left\{y\,(-2, 1, 0) + z\,(1, 0, 1) \ / \ (y, z) \in \mathbb{R}^2\right\} \\ &= Span\left\{(-2, 1, 0), (1, 0, 1)\right\} = \langle(-2, 1, 0), (1, 0, 1)\rangle \end{aligned}$$

Furthermore, these vectors form a free family so (u, v) is a basis of F.

Proposition 5.1.38 *Let E be a vector space. If*

$$\{e_1, \, e_2, \, ..., \, e_n\} \ \text{ and } \ \{u_1, \, u_2, \, ..., \, u_m\}$$

are two bases of E, then $n = m$.

Remark 5.1.39 *If a vector space E admits a basis, then all the bases of E have the same number of elements, this number does not depend on the basis but it only depends on the space E. This common number is called the dimension of E.*

5.1.6 Dimension of vector spaces

Definition 5.1.40 *Let E be a vector space over a field $\mathbb{K}$, and let $\mathcal{B} = \{e_1, \, e_2, \, ..., \, e_n\}$ be a basis of E, The dimension of E, denoted $\dim(E)$, is defined as*

$$\dim(E) = Card(B),$$

that is, the number of elements of the basis $\mathcal{B}$.

Example 5.1.41 *Let*

$$e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1).$$

The family $\{e_1, \, e_2, \, e_3\}$ is a basis of $\mathbb{R}^3$, called the canonical (standard) basis.

Therefore,

$$\dim(\mathbb{R}^3) = Card\left(\{e_1, \, e_2, \, e_3\}\right) = 3.$$

Example 5.1.42 *In the vector space* $\mathbb{R}_2[x]$*, the family* $\left\{1,\ x,\ x^2\right\}$ *is a basis. Therefore,*

$$\dim\left(\mathbb{R}_2[x]\right) = Card\left\{1,\ x,\ x^2\right\} = 3.$$

Theorem 5.1.43 *Let* E *be a vector space of dimension* n*, then :*

1) Characterization of a basis: A family $\{e_1,\ e_2,\ \ldots,e_n\}$ *of* n *vectors in* E *is the basis of* E *if and only if it is either: generating, or linearly independent (free). That is,*

$$\{e_1,\ e_2,\ \ldots,e_n\} \text{ is abasis} \Leftrightarrow \text{it is generating } \Leftrightarrow \text{it is free.}$$

2) Families with more than n vectors: Let $\{e_1,\ e_2,\ \ldots,\ e_p\}$ *be* p *vectors in* E*, with* $p > n$*, then :*

- *The family cannot be free (it is linearly dependent).*
- *If the family is generating, then there exists a subset of* n *vectors among them that forms a basis of* E*.*

3) Families with fewer than n vectors: Let $\{e_1, e_2, \ldots, ep\}$ *be* p *vector in* E*, with* $p < n$ *, then :*

- *The family cannot be generating (it does not span* E*).*
- *If the family is free, it is possible to find* $(n-p)$ *additional vectors* $\{e_{p+1},\ e_{p+2},\ \ldots,e_n\}$ *in* E *such that* $\{e_1,\ e_2,\ \ldots,\ e_{p+1},\ \ldots,e_n\}$ *forms a basis for* E*.*

4) If F *is a vector subspace of* E *: then* $\dim F \leq n$*, and moreover* $\dim F = n \Leftrightarrow F = E$*.*

Proposition 5.1.44 *Let* E *be a finite-dimensional vector space, and let* F_1*,* F_2 *be subspaces of* E*, then:*

$$\dim(F_1 + F_2) = \dim F_1 + \dim F_2 - \dim(F_1 \cap F_2),$$

where $F_1 + F_2 = \{u + v\ /\ u \in F_1,\ v \in F_2\}$ *is the sum of subspaces and* $F_1 \cap F_2$ *is their intersection.*

Exercise 5.1.45 *Consider the subsets of* $\mathbb{R}^3$*:*

$$E = \left\{(x, y, 0) \in \mathbb{R}^3\ /\ x,\ y \in \mathbb{R}\right\} \text{ and } F = \{(x, 0, x)\ /\ x \in \mathbb{R}\}.$$

1. *Show that* E *and* F *are vector subspaces of* $\mathbb{R}^3$ *over* $\mathbb{R}$*.*
2. *Calculate* $\dim(E)$ *and* $\dim(F)$*.*
3. *Determine* $E \cap F$*.*
4. *Is* $\mathbb{R}^3 = E \oplus F$*?*

Solution 5.1.46 *1. Show that E and F are vector subspaces of $\mathbb{R}^3$ over $\mathbb{R}$.*

- *To show that E is a vector subspace, we verify the following conditions:*

(a) Non-empty: The zero vector $(0,0,0) \in E$ (taking $x = 0, y = 0$).

(b) Closed under addition: Let $u = (x_1, y_1, 0)$ and $v = (x_2, y_2, 0)$ be in E. Then:

$$u + v = (x_1 + x_2, y_1 + y_2, 0) \in E.$$

(c) Closed under scalar multiplication: Let $u = (x, y, 0) \in E$ and $\lambda \in \mathbb{R}$. Then:

$$\lambda u = (\lambda x, \lambda y, 0) \in E.$$

Thus, E is a vector subspace.

- *For F :*

(a) Non-empty: The zero vector $(0,0,0) \in F$ (taking $x = 0$).

(b) Closed under addition: Let $u = (x_1, 0, x_1)$ and $v = (x_2, 0, x_2)$ be in F. Then:

$$u + v = (x_1 + x_2, 0, x_1 + x_2) \in F.$$

(c) Closed under scalar multiplication: Let $u = (x, 0, x) \in F$ and $\lambda \in \mathbb{R}$. Then:

$$\lambda u = (\lambda x, 0, \lambda x) \in F.$$

Thus, $F$is also a vector subspace.

2. Calculate $\dim(E)$, $\dim(F)$

(a) To find the dimension of E, we have:

$$\begin{aligned} E &= \{(x, y, 0) : x, y \in \mathbb{R}\} \\ &= \{x(1,0,0) + y(0,1,0) : x, y \in \mathbb{R}\} \\ &= span\{(1,0,0), (0,1,0)\}. \end{aligned}$$

The vectors $(1,0,0)$ and $(0,1,0)$ are linearly independent and therefore form a basis for E. Thus, we conclude that the dimension of E is: $\dim(E) = 2$.

(b) To find the dimension of F:The vector$(1,0,1)$ spans F since any vector in F can be expressed as $x(1,0,1)$ for some x. Thus, we have:

$$\dim(F) = 1.$$

3. To find $E \cap F$, we note that:

$$(x, y, z) \in E \cap F \Rightarrow (x, y, z) \in E \text{ and } (x, y, z) \in F.$$

This implies:

$$(x, y, z) \in E \Rightarrow z = 0, (x, y, z) \in F \Rightarrow y = 0 \text{ and } z = x.$$

Thus, the intersection is:

$$E \cap F = \{(0, 0, 0)\}.$$

4. The dimension of $E + F$ can be calculated using the formula:

$$\dim(E + F) = \dim(E) + \dim(F) - \dim(E \cap F).$$

Substituting the dimensions, we find:

$$\dim(E + F) = 2 + 1 - 0 = 3.$$

Since $\dim\left(\mathbb{R}^3\right) = 3$ *and* $\dim(E + F) = 3$*, we conclude that* $E + F = \mathbb{R}^3$*. Furthermore, since* $E + F = \mathbb{R}^3$ *and from Question 3 we have* $E \cap F = \{(0, 0, 0)\}$*, we conclude that* $\mathbb{R}^3 = E \oplus F$*.*

5.2 Linear applications

5.2.1 Definitions and examples

Definition 5.2.1 (linear map) *Let E and F be two vector spaces over a field $\mathbb{K}$.*

A map $f : E \to F$ is called linear if it satisfies both of the following conditions:

$$\forall x,\, y \in E,\ f(x+y) = f(x) + f(y),$$
$$\forall x \in E,\ \forall \lambda \in \mathbb{K},\ f(\lambda x) = \lambda f(x),$$

Equivalently,

$$\forall x,\, y \in E,\ \lambda \in \mathbb{K},\ f(\lambda x + y) = \lambda f(x) + f(y).$$

Remark 5.2.2 *The set of linear maps of E to F is denoted by $\mathcal{L}(E,\, F)$.*

Example 5.2.3 *The map f defined by*

$$\begin{array}{l} f : \mathbb{R}^3 \to \mathbb{R}^2, \\ (x, y, z) \to f(x, y, z) = (2x + y, y - z) \end{array},$$

is a linear map.

Indeed, let (x, y, z), $(\acute{x}, \acute{y}, \acute{z}) \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} f\left[(x, y, z) + (\acute{x}, \acute{y}, \acute{z})\right] &= f\left(x + \acute{x}, y + \acute{y}, z + \acute{z}\right) \\ &= \left(2\left(x + \acute{x}\right) + \left(y + \acute{y}\right), \left(y + \acute{y}\right) - \left(z + \acute{z}\right)\right) \\ &= \left(2x + 2\acute{x} + y + \acute{y}, y + \acute{y} - z - \acute{z}\right) \\ &= \left(\left(2x + y\right) + \left(2\acute{x} + \acute{y}\right), \left(y - z\right) + \left(\acute{y} - \acute{z}\right)\right) \\ &= \left(2x + y, y - z\right) + \left(2\acute{x} + \acute{y}, \acute{y} - \acute{z}\right) \\ &= f\left(x, y, z\right) + f\left(\acute{x}, \acute{y}, \acute{z}\right), \end{aligned}$$

and

$$\begin{aligned} f\left(\lambda\left(x, y, z\right)\right) &= f\left(\lambda x, \lambda y, \lambda z\right) \\ &= \left(2\lambda x + \lambda y, \lambda y - \lambda z\right) \\ &= \left(\lambda\left(2x + y\right), \lambda\left(y - z\right)\right) \\ &= \lambda\left(2x + y, y - z\right) \\ &= \lambda f\left(x, y, z\right). \end{aligned}$$

Example 5.2.4 *The map* $f : \mathbb{R}^2 \to \mathbb{R}^3$ *defined by*

$$f(x, y) = \left(x^2, x + y, 1\right)$$

is not linear.

Indeed,

$$f\left((1, 0) + (0, 0)\right) = f\left(1, 0\right) = \left(1, 1, 1\right),$$

whereas

$$f(1, 0) + f(0, 0) = (1, 1, 1) + (0, 0, 1) = (1, 1, 2).$$

hence,

$$f\left((1, 0) + (0, 0)\right) \neq f(1, 0) + f(0, 0).$$

Proposition 5.2.5 *If* f *is a linear map from* E *to* F*, then :*

1.$f(0_E) = 0_F$*.*

2. $f(-x) = -f(x)$*.*

3. If V_1 *is a subspace of* E *, then* $f(V_1)$ *is a subspace of* F*.*

4. If W_1 *is a subspace of* F*, then* $f^{-1}(W_1)$ *is a subspace of* E*.*

5. The composition of two linear maps is a linear map.

Proposition 5.2.6 *Let* E *and* F *be vector spaces over* K*, and let* $f,\ g \in \mathcal{L}\left(E,\ F\right)$*. If* E *is finite-dimensional of dimension* n *and* $\{e_1,\ e_2,\ ...,\ e_n\}$ *is basis of* E*, then*

$$\forall k \in \{1, 2, ..., n\},\ f(e_k) = g(e_k) \Leftrightarrow \forall x \in E,\ f(x) = g(x).$$

Proof. The implication ($\Leftarrow$) is obvious.

For ($\Rightarrow$), since $\{e_1,\ e_2,\ ...,\ e_n\}$ generates E, for any $x \in E$ there exist scalars $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K}$ such that

$$x = \sum_{i=1}^{n} \lambda_i e.$$

Since f and g are linear maps,

$$f\left(x\right) = \sum_{i=1}^{n} \lambda_i f\left(e_i\right),\ g\left(x\right) = \sum_{i=1}^{n} \lambda_i g\left(e_i\right).$$

If $f\left(e_i\right) = g\left(e_i\right)$ for all i, then $f\left(x\right) = g\left(x\right)$for all $x \in E$. ■

5.2.2 Linear maps and dimension

Let $f : E \to F$ be a linear map.

The kernel of a linear map

Definition 5.2.7 *The kernel (or null space) of* f*, denoted by* $\ker f$*, is the set of all vectors* $x \in E$ *such that* $f(x) = 0_F$ *(the zero vector of* F *):*

$$\ker f = \{x \in E \ \ / \ f(x) = 0_F\} = f^{-1}(\{0_F\})$$

The image of a linear map

Definition 5.2.8 *The image of* f*, denoted by* $\operatorname{Im} f$*, is the set of all vectors in* F *of the form* $f(x)$ *for some* $x \in E$*:*

$$\operatorname{Im} f = \{f(x) \ \ / \ x \in E\} = f(E)$$

Proposition 5.2.9 *Let* $f : E \to F$ *be a linear map. Then:*

1. $\ker f$ *is a subspace of* E*.*
2. $\operatorname{Im} f$ *is a subspace of* F*.*
3. f *is injective if and only if* $\ker f = \{0_E\}$*.*
4. f *is surjective if and only if* $\operatorname{Im} f = F$*.*

Example 5.2.10 *Consider the map* $f : \mathbb{R}^3 \to \mathbb{R}^2$ *defined by*

$$f(x, y, z) = (x + y, z).$$

This map is not injective but is surjective.

- *Injectivity*

$$\begin{aligned}
\ker f &= \{(x, y, z) \in \mathbb{R}^3 \ \ / \ (x + y, z) = (0, 0)\} \\
&= \{(x, y, z) \in \mathbb{R}^3 \ \ / \ x + y = 0, \ z = 0\} \\
&= \{(x, y, z) \in \mathbb{R}^3 \ \ / \ y = -x, \ z = 0\} \\
&= \{(x, -x, 0) \ / \ x \in \mathbb{R}\}
\end{aligned}$$

Since

$$(1, -1, 0) \in \ker f \Rightarrow \ker f \neq \{0_{\mathbb{R}^3}\}.$$

Hence, f *is not injective.*

- *Surjectivity.*

$$\begin{aligned}
\operatorname{Im} f &= \left\{(x + y, z) \ / \ (x, y, z) \in \mathbb{R}^3\right\} \\
&= \{x(1, 0) + y(1, 0) + z(0, 1) \ / \ x, y, z \in \mathbb{R}\}.
\end{aligned}$$

Thus,

$$\operatorname{Im} f = span\left\{(1,0),\ (0,1)\right\} = \mathbb{R}^2,$$

and f is surjective.

Proposition 5.2.11 *Let $f : E \to F$ be a linear map, with E of finite dimension. Then:*

$$\dim E = \dim \ker f + \dim \operatorname{Im} f$$

The rank of a linear map

Definition 5.2.12 *The rank of a linear map f is the dimension of its image :*

$$rankf = \dim \operatorname{Im} f$$

Example 5.2.13 *Find* $\ker f$, $\operatorname{Im} f$ *and* $rankf$ *for the map* $f : \mathbb{R}^4 \to \mathbb{R}^3$ *defined by*

$$f(x,y,z,t) = (x-y, z+t, x-y+z)$$

Kernel

$$\ker f = \left\{(x,y,z,t) \in \mathbb{R}^4 \ / \ (x-y, z+t, x-y+z) = (0,0,0)\right\}$$

From $x - y = 0$, we get $x = y$.

From $x - y + z = 0$, we get $z = 0$, hence $t = 0$.

Thus,

$$\ker f = \{(x,x,0,0) \ / \ x \in \mathbb{R}\} = span\{(1,1,0,0)\}.$$

Image

$$\begin{aligned} \operatorname{Im} f &= \{(x-y, z+t, x-y+z) \ / \ x,y,z,t \in \mathbb{R}\} \\ &= \{(x-y).(1,0,1) + t.(0,1,0) + z(0,1,1) \ / \ x,y,z,t \in \mathbb{R}\} \\ &= span\left\{(1,0,1),\ (0,1,0),\ z(0,1,1)\right\}. \end{aligned}$$

To check linear independence, let $\lambda_1,\ \lambda_2,\ \lambda_3 \in \mathbb{R}$:

$$\begin{aligned} &\lambda_1(1,0,1) + \lambda_2(0,1,0) + \lambda_3(0,1,1) = (0,0,0) \\ &\qquad \Rightarrow (\lambda_1, \lambda_2 + \lambda_3, \lambda_1 + \lambda_3) = (0,0,0) \\ &\qquad \Rightarrow \lambda_1 = \lambda_1 = \lambda_1 = 0. \end{aligned}$$

Hence, the vectors are linearly independent and form a basis of $\operatorname{Im} f$.

$$rankf = \dim \operatorname{Im} f = 3.$$

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