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ANALYSIS 2

COURSE & EXERCISES

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Bachelor's Degree in Mathematics and Computer Science (LMD)

First Year, Semester 2

April 2025

Preface

This manuscript presents *Analysis 2*, which provides the fundamental tools and results of mathematical analysis across its three chapters. It is a continuation and complement of *Analysis 1*.

Chapter 1 explores limited developments, focusing on Taylor and Maclaurin developments for approximating functions. Chapter 2 introduces the Riemann integral, emphasizing its definition, properties, and applications to calculating areas and primitives. Finally, Chapter 3 addresses ordinary differential equations (ODEs), where we study methods for solving first and second order linear ODEs and more other types of differential equations modeling real world phenomena. Each chapter is followed by solved exercises to help students assimilate the course.

This course aims to provide technical skills and conceptual understanding, thus laying a solid foundation for further study and practical applications in mathematics and related fields.

The content of this course is primarily intended for first-year undergraduates in the Bachelor of Mathematics and Computer Science (LMD) program, as well as computer engineering students. However, this course may also be useful for students in various scientific and technical fields, such as physics, chemistry, biology, and economics, who require a solid foundation in mathematical analysis.

We hope this manuscript will serve as an effective teaching resource, supporting students in their regular attendance and diligent preparation for tutorials.

Dr. Khelladi Samia, April 2025

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Introduction

Mathematical analysis is a cornerstone of higher mathematics, providing the theoretical foundation for many areas of study in both pure and applied mathematics. In this course, we will focus on three key topics that are essential for a deeper understanding of analysis: limited developments, Riemann integrals, and ordinary differential equations (ODEs). These concepts not only form the basis for more advanced topics but also have wide-ranging applications in various fields, as physics, engineering, economics, and beyond.

In Chapter 1 we introduce the limited developments using Taylor and Maclaurin developments, which allow us to approximate functions locally around a point. This is fundamental for understanding how functions behave near specific values and for solving complex problems in areas such as optimization and numerical analysis.

Chapter 2 is devoted to the study of Riemann integrals, which will provide us with the tools to understand and compute areas under curves, an essential concept in calculus also how to determine primitives for different types of functions. We will cover the rigorous definition of the Riemann integral, its properties, and its applications, deepening our appreciation of the concept of integration.

Lastly, we will explore ordinary differential equations (ODEs) trough the Chapter 3, which describe the rates of change of quantities in many natural systems, from physics to biology to economics, ODEs are used to model dynamic systems. We will learn various techniques for solving different types of ODEs, in particular first and second order linear differential equations, as well as the Bernoulli and Riccati equations.

By the end of this course, students will have gained a solid understanding of these foundational concepts and the ability to apply them to a variety of problems in mathematics and science.

Chapter 1

Limited Developments

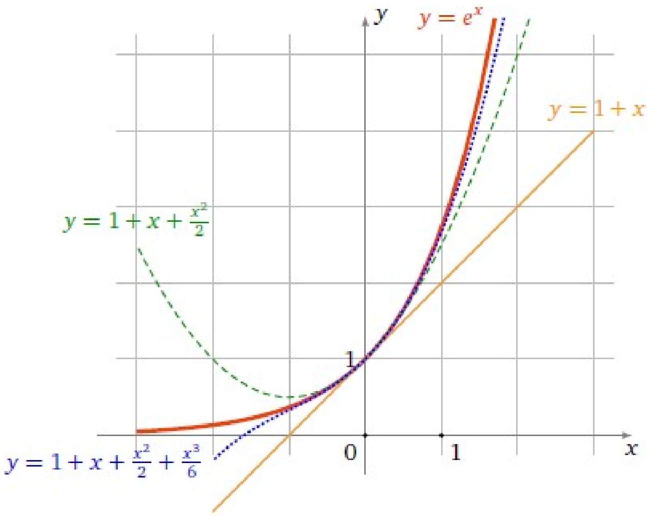
1.1 Introduction

Let us take the example of the exponential function

$$y = f(x) = e^x.$$

An idea of the behavior of the function f around the point $x = 0$ is given by its tangent $y = 1 + x$.

We approximated the graph of f by a straight line.



If we wanted to do better, we would search for a parabola of equation $y = a_0 + a_1x + a_2x^2$, which is the best approximation of the graph of f around $x = 0$. It is the parabola of equation

$$y = 1 + x + \frac{x^2}{2}.$$

If we set

$$g(x) = e^x - \left(1 + x + \frac{x^2}{2}\right),$$

then we have

$$g(0) = 0, \quad g'(0) = 0, \quad \text{and} \quad g''(0) = 0.$$

Finding the equation of this parabola is to make a limited development to order 2 of the function f. Of course, if we wanted to be more precise, we would continue with a third-degree curve, which would be

$$y = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

In this chapter, we will try to find a polynomial of degree n that best approximates a given function f. The results are only valid for x around a fixed value (this will often be around 0). This polynomial will be calculated from the successive derivatives at the considered point, as we will see now.

1.2 Taylor formulas

We will see three Taylor formulas. They will have the same polynomial part ($P_n(x)$), but provide more or less information about the remainder, denoted $R_n(x)$.

Around a point $x = a$, we have:

$$f(x) = P_n(x) + R_n(x).$$

We will start with the Taylor formula with **integral remainder**, which gives an exact expression for the remainder $R_n(x)$. Then, we present the Taylor formula with remainder involving $f^{(n+1)}(c)$, which provides an estimate for the remainder. Finally, we finish with the **Taylor-Young formula**, which is very practical when no information about the remainder is needed.

Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ a function.

- For $n \in \mathbb{N}^*$, we say that f is a function of class $\mathcal{C}^n$ if f is n times differentiable on

I and $f^{(n)}$ is continuous.

- f is of class $\mathcal{C}^0$ if f is continuous.
- f is of class $\mathcal{C}^\infty$ if f is of class $\mathcal{C}^n$ for all $n \in \mathbb{N}$.

1.2.1 Taylor formula with integral remainder

Theorem 1.2.1. *Let $f : I \to \mathbb{R}$ be a function of class $\mathcal{C}^{n+1}(I)$ $(n \in \mathbb{N})$ and let $a, x \in I$. Then:*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt.$$

Example 1.2.2. *The function $f(x) = e^x$ is of class $\mathcal{C}^\infty(\mathbb{R})$.*

Let $a \in \mathbb{R}$ (a fixed), we have $f'(x) = e^x$, $f''(x) = e^x$, $\ldots$, $f^{(n)}(x) = e^x$. Then, for x around a $(x \in V(a))$ we have:

$$f(x) = e^x = e^a + e^a(x-a) + e^a\frac{(x-a)^2}{2!} + \cdots + e^a\frac{(x-a)^n}{n!} + \int_a^x \frac{e^t}{n!}(x-t)^n\, dt.$$

If $a = 0$, we get:

$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \int_0^x \frac{e^t}{n!}(x-t)^n\, dt.$$

1.2.2 Taylor formula with remainder $f^{(n+1)}(c)$

Theorem 1.2.3. *Let $f : I \to \mathbb{R}$ be a function of class $\mathcal{C}^{n+1}(I)$ $(n \in \mathbb{N})$, and let $a \in I$. There exists a real number c between a and x ($c \in]x, a[$ or $c \in]a, x[$) such that:*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Example 1.2.4. *Let $a \in \mathbb{R}$. $\forall n \in \mathbb{N}$, $\exists c$ between x and a such that*

$$e^x = e^a + e^a(x-a) + \frac{e^a(x-a)^2}{2!} + \cdots + \frac{e^a(x-a)^n}{n!} + \frac{e^c(x-a)^{n+1}}{(n+1)!}.$$

In most cases, we will not know this "c". But the theorem allows us to bound the remainder. This is expressed by the following corollary:

Corollairy 1.2.5. *If in addition the function* $| f^{(n+1)} |$ *is bounded above on* I *by a real number* M *(i.e.,* $\forall x \in I, |f^{(n+1)}(x)| \leq M$*), then for all* $x, a \in I$*, we have*

$$|f(x) - P_n(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!}.$$

1.2.3 Taylor-Young formula

Theorem 1.2.6. *Let* $f : I \to \mathbb{R}$ *be a function of class* $\mathcal{C}^n(I)$ *(*$n \in \mathbb{N}$*) and* $a, x \in I$*. Then, we have*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \varepsilon_n(x)(x-a)^n,$$

where ε_n *is a function defined on* I *such that* $\lim_{x\to a} \varepsilon_n(x) = 0$.

Example 1.2.7. *Let* $f : I =]-1, +\infty[\to \mathbb{R}$*, with* $f(x) = \ln(1+x)$*. We have* $f \in \mathcal{C}^\infty(I)$*. We take* $a = 0$.

We have:

$$\begin{aligned}
&f(0) = \ln(1+0) = 0,\\
&f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = 1,\\
&f''(x) = -\frac{1}{(1+x)^2} \Rightarrow f''(0) = -1,\\
&f^{(3)}(x) = 2 \cdot \frac{1}{(1+x)^3} \Rightarrow f^{(3)}(0) = 2.
\end{aligned}$$

By recurrence, we show that:

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^n.$$

Then,

$$f^{(n)}(0) = (-1)^{n-1}(n-1)!$$

Thus for $n > 0$, we have

$$\frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n-1}(n-1)!}{n!} = \frac{(-1)^{n-1}}{n}.$$

So, we get

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + x^n \cdot \varepsilon_n(x), \quad \lim_{x \to 0} \varepsilon_n(x) = 0$$

$$\Rightarrow f(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + x^n \cdot \varepsilon_n(x), \quad \lim_{x \to 0} \varepsilon_n(x) = 0.$$

Notation

The term $(x-a)^n \varepsilon_n(x)$ where $\varepsilon_n(x) \to 0$ is often abbreviated to the "little o" of $(x-a)^n$ and is denoted $o((x-a)^n)$. So, $o((x-a)^n)$ is a function such that:

$$\lim_{x \to a} \frac{o((x-a)^n)}{(x-a)^n} = 0.$$

It is more practical to use this notation as it simplifies writing, but we must always keep in mind what it means.

Remark 1.2.8. *In the special case where $a = 0$, the Taylor formulas are called **Maclaurin formulas**, i.e., we have a limited development in the vicinity of 0 ($\mathcal{V}(0)$).*

1.2.4 Maclaurin-Young formula

It is the special case of Taylor-Young formula, with $a = 0$. Then we have:

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \cdots + f^{(n)}(0)\frac{x^n}{n!} + x^n \varepsilon(x), \quad \lim_{x \to 0} \varepsilon(x) = 0.$$

And with the "little o" notation, this gives:

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \cdots + f^{(n)}(0)\frac{x^n}{n!} + o(x^n), \quad \lim_{x \to 0} \frac{o(x^n)}{x^n} = 0.$$

Example 1.2.9. $f(x) = \sin x$. *Give the limited development of Maclaurin-Young for* $n = 4$.

We have:

$$f(x) = \sin x \Rightarrow f'(x) = \cos x \Rightarrow f''(x) = -\sin x \Rightarrow f^{(3)}(x) = -\cos x \Rightarrow f^{(4)}(x) = \sin x$$

So,

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f^{(3)}(0) = -1, \quad f^{(4)}(0) = 0$$

The Maclaurin-Young formula of f is:

$$f(x) = \sin x = 0 + 1 \cdot x + 0 \cdot \frac{x^2}{2!} + (-1) \cdot \frac{x^3}{3!} + 0 \cdot \frac{x^4}{4!} + x^4 \cdot \varepsilon(x), \quad \lim_{x \to 0} \varepsilon(x) = 0$$

$$\Rightarrow f(x) = \sin x = x - \frac{x^3}{3!} + o(x^4), \quad \forall x \in \mathcal{V}(0).$$

1.3 Limited developments in the vicinity of a point

1.3.1 Definition and existence of limited development

Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ a function. Let $a \in I$, f be defined on the vicinity of the point a, except perhaps at a.

Definition 1.3.1. *For $a \in I$ and $n \in \mathbb{N}$, we say that f admits a* ***limited development (LD)*** *at the point a of order n, if there exist real numbers $c_0, c_1, \ldots, c_n$ and a function $\varepsilon : I \to \mathbb{R}$ such that $\lim_{x \to a} \varepsilon(x) = 0$, so that for all $x \in I$:*

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots + c_n(x - a)^n + (x - a)^n \varepsilon(x).$$

- The term $c_0 + c_1(x - a) + \cdots + c_n(x - a)^n$ is called the polynomial part of the LD (or regular part).

- The term $(x - a)^n \varepsilon(x)$ is called the remainder (rest) of the LD.

Using the Taylor-Young formula, we can immediately obtain limited developments by setting $c_k = \dfrac{f^{(k)}(a)}{k!}$.

Proposition 1.3.2. *If f is of class $\mathcal{C}^n$ in the vicinity of a point a, then f admits a limited development at point a of order n, which comes from the Taylor-Young formula:*

$$f(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + (x - a)^n \varepsilon(x),$$

where $\lim_{x \to a} \varepsilon(x) = 0$.

Remark 1.3.3. *1. If f is of class C^n in the vicinity of 0, then a LD at 0 of order n is given by:*

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + x^n\varepsilon(x), \quad \textit{with } \lim_{x\to 0}\varepsilon(x) = 0.$$

2. If f admits a LD at point a of order n, then f has a LD at a of order k, where $k < n$.

Indeed:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k +$$

$$+\frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1} + \cdots + \boxed{\frac{f^{(n)}(a)}{n!}(x-a)^n + (x-a)^n\varepsilon(x)} = (x-a)^k\eta(x),$$

where $\lim_{x\to a}\eta(x) = 0$.

1.3.2 Uniqueness of limited development

Proposition 1.3.4. *Let $f : I \to \mathbb{R}, x \in I$. If f admits a limited development (LD) at x of order n, then this LD is unique.*

Proof. Let us write two LDs of f at point a of order n.

$$f(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n + \varepsilon_1(x), \quad \lim_{x\to a}\varepsilon_1(x) = 0$$

and

$$f(x) = d_0 + d_1(x-a) + \cdots + d_n(x-a)^n + \varepsilon_2(x), \quad \lim_{x\to a}\varepsilon_2(x) = 0$$

By taking the difference we obtain

$$(*) \quad (c_0-d_0)+(c_1-d_1)(x-a)+(c_2-d_2)(x-a)^2+\cdots+(c_n-d_n)(x-a)^n+(\varepsilon_1(x)-\varepsilon_2(x)) = 0$$

When we make $x = a$ in this equality we find $c_0 - d_0 = 0 \Rightarrow [c_0 = d_0]$.
Now, $f(x) \neq x^2$, we divide the equality $(*)$ by $(x-a)$, we get

$$(c_1 - d_1) + (c_2 - d_2)(x-a) + \cdots + (c_n - d_n)(x-a)^{n-1} + (\varepsilon_1(x) - \varepsilon_2(x))$$

Evaluating at $x = a$, we get $c_1 = d_1 \Rightarrow c_1 = d_1$, etc.

Finally, we get $c_k = d_k,\ \forall k = 0, 1, \cdots, n.$ □

The polynomial parts are equal and therefore the rests too.

Corollairy 1.3.5. *If f is even (resp. odd), then the polynomial part of its limited development at 0 contains only even degrees (resp. odd degrees).*

Proof.

$$f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_nx^n + x^n\varepsilon(x), \quad \lim_{x\to a} \varepsilon(x) = 0$$

If f is even, then $f(-x) = f(x)$, i.e.

$$f(x) = f(-x) = c_0 + c_1(-x) + c_2(-x)^2 + \cdots + c_n(-x)^n + (-x)^n\varepsilon(-x)$$

$$= c_0 - c_1x + c_2x^2 - c_3x^3 + \cdots + (-1)^nc_nx^n + x^n\varepsilon(-x)$$

But the LD of f is unique at the point 0, then we get

$$c_1 = -c_1 \Rightarrow c_1 = 0, \quad c_3 = -c_3 \Rightarrow c_3 = 0, \quad \cdots \text{ (i.e. } c_n = 0 \text{ when n is odd)}$$

□

Example 1.3.6. ***1)*** *$f(x) = \cos x$ is an even function. The LD of f at 0 is given by:*

$$f(x) = \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

2) $f(x) = \sin x$ is an odd function. The LD of f at 0 is given by

$$f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Remark 1.3.7.

1. *The uniqueness of the LD and the Taylor-Young formula implies that if we know the LD of f and $f \in \mathcal{C}^n(I)$, then we can calculate the numbers c_i derived from the polynomial part by the formula: $c_i = \frac{f^{(k)}(a)}{k!}$. However, in the majority of cases we will do the reverse, i.e., we find the LD from the derivatives.*

2. *If f admits a LD at a point a of order n, then f is continuous at a and $c_0 = f(a)$.*

3. *If f admits a LD at a point a of order $n \geq 1$, then f is differentiable at a and we have $c_0 = f(a)$ and $c_1 = f'(a)$. Therefore $y = c_0 + c_1(x-a)$ is the equation of the tangent to the graph of f at the point of abscissa a.*

4. *f can admit a LD of order 2 at point a, without admitting a second derivative in a. For example: $f(x) = x^3 \sin\left(\frac{1}{x}\right)$. So, f is differentiable but f' is not differentiable. However, f admits a LD at 0 of order 2: $f(x) = 0 + x^2(x\sin(\frac{1}{x})) = 0 + x^2\varepsilon(x)$ with $\lim_{x\to 0}\varepsilon(x) = \lim_{x\to 0} x\sin(\frac{1}{x}) = 0$ (the polynomial part is zero).*

1.3.3 Limited developments of the usual functions at the origin

The following LDs in 0, come from the Taylor-Young formula (or Taylor-Maclaurin):

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + o(x^n)$,
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + o(x^n)$,
- $\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$,
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n+2})$,
- $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots + \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!}x^{2n-1} + o(x^{2n})$,
- $\arcsin x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots + \frac{(2n)!}{4^n(n!)^2(2n+1)}x^{2n+1} + o(x^{2n+2})$,
- $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n\frac{x^{2n+1}}{2n+1} + o(x^{2n+2})$,
- $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + o(x^n)$,
- $\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + o(x^n)$,
- $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \cdots + (-1)^{n-1}\frac{(2n-3)!!}{2^n n!}x^n + o(x^n)$.

Remark 1.3.8. *We can write the LD using formal sums for some functions, if it is possible. For example:*

$$e^x = \sum_{k=0}^{+\infty} \frac{x^k}{k!} + o(x^n), \quad \ln(1+x) = \sum_{k=1}^{n} (-1)^{k-1}\frac{x^k}{k} + o(x^n).$$

1.3.4 Limited development of function at any point

Definition 1.3.9. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ and $a \in I$. The function f admits a LD in the vicinity of a if and only if the function $g(h) = f(h + a)$ admits a LD in the vicinity of zero (0).*

Often, we bring the problem back to 0 by making the change of variable $h = x - a \Leftrightarrow x = h + a$ ($x \to a \Leftrightarrow h \to 0$).

Example 1.3.10. *1) Find the LD of $g(x) = e^x$ in the vicinity of $a = 1$.*

We set $(h = x - 1 \Leftrightarrow x = h + 1)$. $(x \to 1 \Leftrightarrow h \to 0)$.

We have:

$$f(x) = e^x = e^{1+(x-1)} = e \cdot e^{x-1} = e \cdot e^h \quad (h = x - 1, x \to 1 \Rightarrow h \to 0).$$

In the $V(0)$, e^h admits the following LD:

$$e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots + \frac{h^n}{n!} + \varepsilon(h), \quad \lim_{h\to 0} \varepsilon(h) = 0.$$

$$\Rightarrow f(x) = e^x = e \cdot e^h = e\left[1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots + \frac{h^n}{n!} + \varepsilon(h)\right] \text{ where } h = x - 1.$$

$$\Rightarrow f(x) = e + e(x-1) + \frac{e(x-1)^2}{2!} + \cdots + \frac{e(x-1)^n}{n!} + e \cdot \varepsilon(x-1).$$

$$\text{where } \lim_{x\to 1} \varepsilon(x-1) = 0.$$

2) Find the LD of $f(x) = \sin x$ in the vicinity of $a = \frac{\pi}{2}$.

We know that $\sin x = \sin\left(\frac{\pi}{2} + x - \frac{\pi}{2}\right) = \cos(x - \frac{\pi}{2})$.

Set $h = x - \frac{\pi}{2} \Leftrightarrow x = h + \frac{\pi}{2}$.

$$(x \to \frac{\pi}{2}) \Leftrightarrow (h \to 0).$$

We come back to the LD of $\cos(h)$ *in* $V(0)$. *We have:*

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \varepsilon(x), \quad \lim_{x \to 0} \varepsilon(x) = 0.$$

$$\Rightarrow f(x) = \sin x = \cos(x - \frac{\pi}{2}) = 1 - \frac{(x-\frac{\pi}{2})^2}{2!} + \frac{(x-\frac{\pi}{2})^4}{4!} + \cdots + (-1)^n \frac{(x-\frac{\pi}{2})^{2n}}{(2n)!} + \varepsilon(x - \frac{\pi}{2}).$$

$$\textit{with } \lim_{x \to \frac{\pi}{2}} \varepsilon(x - \frac{\pi}{2}) = 0.$$

3) Find the LD of $f(x) = \ln(1+3x)$ *in* $V(1)$ *of order* $3 \Rightarrow n = 3$.

Set $h = x - 1 \Leftrightarrow x = h + 1$.

$$x \to 1 \Leftrightarrow h \to 0.$$

We have:

$$f(x) = \ln(1+3x) = \ln(1+3(h+1)) = \ln(1+3+3h) = \ln(4+3h).$$

$$= \ln\left(4(1+\frac{3}{4}h)\right) = \ln(4) + \ln\left(1+\frac{3}{4}h\right).$$

We have $t = \frac{3}{4}h$, $t \in V(0)$.

$$\Rightarrow \ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} + \varepsilon(t), \quad \lim_{t \to 0} \varepsilon(t) = 0.$$

$$\Rightarrow f(x) = \ln(1+3x) = \ln(4) + \ln(1+t) \textit{ with } t = \frac{3}{4}(x-1).$$

$$= \ln(4) + \frac{3}{4}(x-1) - \frac{9}{32}(x-1)^2 + \frac{9}{64}(x-1)^3 + \varepsilon(x-1).$$

$$\textit{where } \lim_{x \to 1} \varepsilon(x-1) = 0.$$

1.4 Operations on limited developments

1.4.1 Sum and product

We suppose that f and g are two functions which admit limited developments at 0 to order n:

$$f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n + x^n\varepsilon_1(x), \quad \lim_{x\to 0}\varepsilon_1(x) = 0,$$

$$g(x) = d_0 + d_1x + d_2x^2 + \cdots + d_nx^n + x^n\varepsilon_2(x), \quad \lim_{x\to 0}\varepsilon_2(x) = 0.$$

Proposition 1.4.1.

- *The function $f + g$ admits a LD at 0 of order n, given by:*

$$(f + g)(x) = f(x) + g(x) = (c_0 + d_0) + (c_1 + d_1)x + \cdots + (c_n + d_n)x^n + x^n\varepsilon(x),$$

where $\varepsilon(x) = \varepsilon_1(x) + \varepsilon_2(x)$, and $\lim_{x\to 0}\varepsilon(x) = \lim_{x\to 0}(\varepsilon_1(x) + \varepsilon_2(x)) = 0$.

- *The function $f \times g$ (or $f \cdot g$) admits a LD at 0 of order n, given by:*

$$(f \times g)(x) = f(x) \times g(x) = T_n(x) + x^n\varepsilon(x),$$

where T_n is the polynomial

$$(c_0 + c_1x + \cdots + c_nx^n)(d_0 + d_1x + \cdots + d_nx^n),$$

truncated to order n, that means we only keep monomials of degree $\leq n$.

Example 1.4.2. *1) Calculate the LD of $h(x) = \cos x + \sin x$ at 0 of order 4.*

We take $h(x) = f(x) + g(x)$, if $f(x) = \cos x$ and $g(x) = \sin x$.

We have

$$f(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4) \quad \textit{LD order 4},$$

$$g(x) = \sin x = x - \frac{x^3}{3!} + o(x^4) \quad \textit{LD order 4}.$$

$$\Rightarrow h(x) = \cos x + \sin x = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + o(x^4).$$

2) Calculate the LD of $h(x) = x^2\ln(1 + x)$ at 0 of order 3.

We take $h(x) = f(x) \cdot g(x)$, if $f(x) = x^2$, $g(x) = \ln(1 + x)$.

We have

$$f(x) = x^2, \quad g(x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3),$$

$$\Rightarrow h(x) = \frac{1}{3}x^3 + o(x^3),$$

where we develop and we keep only degree $\leq n = 3$.

$$\Rightarrow h(x) = (x^2) \cdot \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) = x^3 - \frac{x^4}{2} + \frac{x^5}{3},$$

$$\Rightarrow h(x) = x^2 \ln(1+x) = x^3 - \frac{x^4}{2} + \frac{x^5}{3} + o(x^3).$$

1.4.2 Composition

Let f and g admit LD at 0 of order n.

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + o(x^n) = C(x) + o(x^n),$$

$$g(x) = d_0 + d_1 x + d_2 x^2 + \cdots + d_n x^n + o(x^n) = D(x) + o(x^n).$$

Proposition 1.4.3. *If* $g(0) = 0$ *(i.e.,* $d_0 = 0$*), then the function* $f \circ g$ *admits a LD at 0 of order* n *whose polynomial part is the polynomial truncated to order* n *of the composition* $C(D(x))$.

Example 1.4.4. *1) Calculate the LD of* $h(x) = \cdot e^x$ *at 0 of order* $n = 5$.

We have $e^{\cos(x)} = e^{\cos(x-1)+1} = ee^{\cos(x-1)}$.

We take $f(x) = e^x$ *and* $g(x) = \cos(x) - 1$.

We have $g(0) = \cos(0) - 1 = 0$ *and* f *and* g *admit LD of order 5 at 0.*

$$g(x) = u(x) = \cos x - 1 = -\frac{x^2}{2!} + \frac{x^4}{4!} + o(x^5)$$

and we have

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!} + o(x^5)$$

$$= 1 + (-\frac{x^2}{2!} + \frac{x^4}{4!}) + \frac{(-\frac{x^2}{2!} + \frac{x^4}{4!})^2}{2!} + o(x^5)$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} + \frac{1}{2}\left(\frac{x^4}{4}\right) + o(x^5)$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^4}{8} + o(x^5)$$

$$= 1 - \frac{x^2}{2} + \left(\frac{1}{24} + \frac{1}{8}\right)x^4 + o(x^5)$$

$$= 1 - \frac{x^2}{2} + \frac{1}{6}x^4 + o(x^5)$$

So,

$$h(x) = e^{\cos x - 1} = e^{u(x)} = 1 - \frac{x^2}{2} + \frac{1}{6}x^4 + o(x^5), \quad x \in \mathcal{V}(0)$$

2) Calculate the LD of $h(x) = \sin(\ln(1+x))$ at 0 of order 3.

We take $f(x) = \sin x$ and $g(x) = \ln(1+x)$.

We have $g(0) = 0$ and f and g admit LD at 0 of order 3.

Let $u = g(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$

$$\sin(u) = u - \frac{u^3}{6} + o(u^3) = (x - \frac{x^2}{2} + \frac{x^3}{3}) - \frac{(x - \frac{x^2}{2} + \frac{x^3}{3})^3}{6} + o(x^3)$$

We keep only degree ≤ 3.

So,

$$h(x) = \sin(\ln(1+x)) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3)$$

1.4.3 Division (Quotient)

Let f and g admit a LDs at 0 of order n,

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + o(x^n) = C(x) + o(x^n)$$

$$g(x) = d_0 + d_1 x + d_2 x^2 + \cdots + d_n x^n + o(x^n) = D(x) + o(x^n)$$

(C, D are polynomials of degree $\leq n$)

We will use the LD of $\frac{1}{1+u} = 1 - u + u^2 - u^3 + \cdots + (-1)^k u^k + o(u^n)$.

1. If $d_0 = 1$, we set $u = d_1x + d_2x^2 + \cdots + d_nx^n$, then we have

$$\frac{f(x)}{g(x)} = \frac{f(x)}{1+u} = f(x)(1 - u + u^2 - \cdots + (-1)^k u^k + o(u^n))$$

$$= (c_0 + c_1x + c_2x^2 - \cdots - c_nx^n)(1 - d_1x + d_2x^2 - \cdots + (-1)^n d_nx^n) + o(x^n)$$

$$= C(x)\left(1 - d_1x + d_2x^2 - \cdots + (-1)^n d_nx^n\right) + o(x^n)$$

We keep only monomials of degree $\leq n$,

$$= C(x) \cdot P_n(x) + o(x^n) = Q_n(x) + o(x^n)$$

we keep only degree $\leq n$.

2. If $d_0 \neq 0$ (arbitrary), then we return to the previous case by writing:

$$\tilde{g}(x) = \frac{g(x)}{d_0} = 1 + \frac{d_1}{d_0}x + \cdots + \frac{d_n}{d_0}x^n + o(x^n),$$

and

$$\frac{f(x)}{g(x)} = d_0\frac{f(x)}{1+u}.$$

3. If $d_0 = 0$, then we factor by x^k (for certain k) in order to reduce to the previous case.

Remark 1.4.5. *If $d_0 \neq 0$, the regular part (polynomial part) $Q_n(x)$ can be obtained by dividing $C(x)$ over $D(x)$ according to increasing powers until order n.*

Example 1.4.6. *1) Find the LD of $f(x) = \tan(x)$ at 0 ($\forall o(x^n)$) to order 5.*

We have $\tan(x) = \frac{\sin x}{\cos x} = \frac{f(x)}{g(x)}$,

$$f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)$$

$$g(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^5)$$

We have $g(0) = \cos(0) = 1 = d_0 \neq 0$.

Method 1: *We set $u = -\frac{x^2}{2} + \frac{x^4}{24}$,*

So,

$$\frac{1}{\cos x} = \frac{1}{1+u} = 1 + u + u^2 + u^3 + u^4 + u^5 + o(u^5)$$

$$= \left(1 + \frac{x^2}{2} + \frac{x^4}{24}\right) + \left(\frac{x^2}{2} + \frac{x^4}{24}\right)^2 + \left(\frac{x^2}{2} + \frac{x^4}{24}\right)^3 + o(x^5)$$

We do not take u^4 *and* u^5*, because in these two terms, the power of* x *is superior to 5* (> 5).

For the terms u^2 *and* u^3*, we develop and we take only the monomials of degree* ≤ 5.

Then,

$$\frac{1}{\cos x} = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + o(x^5)$$

$$\Rightarrow \tan x = \sin x \cdot \frac{1}{\cos x} = \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)\left(1 + \frac{x^2}{2} + \frac{5x^4}{24}\right) + o(x^5)$$

We develop and we keep only degrees ≤ 5.

Finally,

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + o(x^5), \quad \forall x \in \mathcal{V}(0)$$

Method 2: *We reduce* $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5) = d_0 + d_1 x + \cdots + o(x^5)$, $d_0 = 1 \neq 0$.

So,

$$\tan x = Q_n(x) + o(x^5)$$

Where $Q_n(x)$ *is obtained by dividing* $x - \frac{x^3}{6} + \frac{x^5}{120}$ *over* $(1 - \frac{x^2}{2} + \frac{x^4}{24})$*, according to increasing powers until order 5.*

2) Find the LD of $h(x) = \frac{1+x}{2+x}$ *at 0 of order 4.*

$$\frac{1+x}{2+x} = (1+x) \cdot \frac{1}{2+x} = (1+x) \cdot \frac{1}{2} \cdot \frac{1}{1 + \frac{x}{2}}$$

$$= \left(\frac{1+x}{2}\right)\left(1 - \frac{x}{2} + \left(\frac{x}{2}\right)^2 - \left(\frac{x}{2}\right)^3 + \left(\frac{x}{2}\right)^4\right) + o(x^4)$$

$$\Rightarrow \frac{1+x}{2}\left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \frac{x^4}{16}\right)$$

$$= \frac{1}{2} + \frac{x}{2} - \frac{x}{4} - \frac{x^2}{4} + \frac{x^2}{8} + \frac{x^3}{8} - \frac{x^3}{16} - \frac{x^4}{16} + \frac{x^4}{32} + o(x^4)$$

1.4.4 Integration

Theorem 1.4.7. *Let $f : I \to \mathbb{R}$ be a function of class C^n, whose LD at $c \in I$ of order n is given by*

$$f^{(n)} = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + (x-c)^n \cdot \varepsilon(x).$$

Let F be a primitive of f. Then, F admits a LD at the point c of order $n+1$, given by

$$F(x) = F(a) + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots + c_n\frac{(x-a)^{n+1}}{n+1} + (x-a)^{n+1}\varepsilon_1(x),$$

where $\lim_{x\to a}\varepsilon_1(x) = 0$.

This means that we integrate the polynomial part, term by term, to obtain the LD of $F(x)$ up to the constant $F(a)$.

Example 1.4.8. *1) Find the LD of* $\arctan(x) = F(x)$ *at* 0.
We know that $F(x)$ is a primitive of $f(x) = \dfrac{1}{1+x^2}$, i.e.,

$$F'(x) = \arctan(x) = \frac{1}{1+x^2} = f(x).$$

We have the LD of f as given by:

$$f(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + o(x^{2n}).$$

$\Rightarrow F(x) = \arctan(x) = F(0) + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n\frac{x^{2n+1}}{2n+1} + o(x^{2n+1})$.
Since $F(x) = \arctan(x)$, $F(0) = 0$, *then*

$$\arctan(x) = \sum_{k=0}^{n}\frac{(-1)^k}{2k+1}x^{2k+1} + o(x^{2n+1}).$$

2) Find the LD of $F(x) = \arcsin(x)$ *at* 0 *of order 5.*

We have:

$$(\arcsin(x))' = \frac{1}{\sqrt{1-x^2}}, \quad \arcsin(x) = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt.$$

We have:

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} = (1+u)^{-\frac{1}{2}}, \quad \text{with } u = -x^2.$$

$$\Rightarrow \frac{1}{\sqrt{1-x^2}} = 1 + \left(-\frac{1}{2}\right)(-x^2) + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{(-x^2)^2}{2!} + o(x^4),$$

(LD at 0 of order 4)

$$\Rightarrow = 1 + \frac{x^2}{2} + \frac{3}{8}x^4 + o(x^4).$$

Then,

$$F(x) = \arcsin(x) = F(0) + x + \frac{x^3}{6} + \frac{3x^5}{40} + o(x^5), \quad (F(0) = \arcsin(0) = 0)$$

$$\Rightarrow \arcsin(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + o(x^5).$$

1.4.5 Derivation

Theorem 1.4.9. *Let $f : I \to \mathbb{R}$ be a differentiable function over I and admits a LD of order n at a.*

If $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + o((x-a)^n)$, then this LD is obtained by derivation of the polynomial of part $g(x)$.

i.e., the LD of $f(x)' = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + nc_n(x-a)^{n-1} + o((x-a)^{n-1})$.

Example 1.4.10. *1) $f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$.*

So, by derivation we obtain the LD of $f'(x) = \frac{1}{(1-x)^2}$, so

$f'(x) = 1 + x + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + o(x^{n-1})$.

2) We have $f(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n+1}\frac{x^n}{n} + o(x^n)$, $\forall x \in V(0)$. Then, $f'(x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^{n-1}x^{n-1} + o(x^{n-1})$.

1.5 Limited development in $+\infty$

Let f be a function defined on an interval $I =]x_0, +\infty[$.

We say that f admits a LD at $+\infty$ of order n, if there exist real numbers $c_0, c_1, \ldots, c_n$, such that

$$f(x) = c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots + \frac{c_n}{x^n} + \frac{1}{x^n}\varepsilon(\frac{1}{x}),$$

where

$$\lim_{x \to +\infty} \varepsilon(\frac{1}{x}) = 0.$$

Remark 1.5.1. *1. The function $f(x)$ admits a LD at a of order $n \Leftrightarrow g(x) = f(\frac{1}{x})$ admits a LD at 0 of order n.*

2. A LD at a is also called an asymptotic development.

3. We can similarly define what is the LD at $-\infty$.

Example 1.5.2. $f(x) = \ln\left(1 + \frac{1}{x}\right), \quad x \in V(+\infty)$.

We set

$$u = \frac{1}{x} \Rightarrow x = \frac{1}{u}, \quad \textit{when } x \to +\infty, \textit{ then } u \to 0.$$

Let $g(u) = f\left(\frac{1}{u}\right) = \ln\left(1 + \frac{1}{u}\right) = \ln(1 + u^{-1})$.

- *f admits a LD at $+\infty$ of order $n \Leftrightarrow g$ admits a LD at 0 of order n.*

We have $u \in V(0)$, $\quad (u \to 0$ if $x \to +\infty)$.

$$g(u) = \ln(1 + u^{-1}) = u^{-1} - \frac{1}{2}u^{-2} + \frac{1}{3}u^{-3} + \cdots + (-1)^{n+1}\frac{1}{n}u^{-n} + o(u^{-n}),$$

$$\Rightarrow f(x) = \ln\left(1 + \frac{1}{x}\right) = \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} + \cdots + (-1)^{n+1}\frac{1}{nx^n} + o\left(\frac{1}{x^n}\right).$$

1.6 Generalized limited developments

Let $f : I \to \mathbb{R}$ be a function defined in the vicinity of 0. We suppose that f don't admit a LD at 0, but the function $x^\alpha f(x) \to \mathbb{R}$, $\quad$ (x positive real, $x \to 0$), admits a LD at 0 of order n.

So, we have:

$$x^\alpha f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + o(x^n),$$

which implies

$$f(x) = \frac{1}{x^\alpha}\left(c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + o(x^n)\right).$$

This expression is called a **generalized limited development** of f at 0.

Example 1.6.1. *The function* $f(x) = co\tan(x)$ *is not defined at* 0.
($\lim_{x\to 0} co\tan(x) = \infty$)*, so* f *doesn't admit a LD at* 0.

But $x \cdot co\tan(x) = \frac{x\cos x}{\sin x}$ *admits a LD at* 0 *of order* $n = 4$.

$$x \cdot co\tan(x) = 1 - \frac{x^2}{3} - \frac{x^4}{45} + o(x^4),$$

Then, cotan(x) *admits a generalized limited development at* 0*:*

$$cotan(x) = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} + o(x).$$

1.7 Applications of limited developments

Here are the most remarkable applications of limited developments.

1.7.1 Limit calculations

Limited developments are very efficient to calculate limits with indeterminable forms. Just notice that if

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + o((x-a)^n),$$

then

$$\lim_{x\to a} f(x) = c_0.$$

Example 1.7.1. *1) We have*

$$\lim_{x\to 0} \frac{\sin x}{x} = \frac{0}{0} \quad \textit{(F.I). We take the L.D. at 0 of order 3:}$$

$$\Rightarrow \frac{\sin x}{x} = \frac{x - \frac{x^3}{6} + o(x^3)}{x} = 1 - \frac{x^2}{6} + o(x^2),$$

$$\Rightarrow \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

2) We have

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{\ln(1+x) - \tan(x) + x\sin x}{3x^2(\sin x)^2} = \frac{0}{0} \quad \textit{(F.I).}$$

We take LD at 0 of order 4:

$$\begin{aligned} f(x) &= \ln(1+x) - \tan(x) + \frac{1}{2}\sin^2 x \\ &= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\right) - \left(x + \frac{x^3}{3}\right) + x\left(x - \frac{x^3}{6}\right) + o(x^4), \end{aligned}$$

$$\Rightarrow f(x) = -\frac{5}{12}x^4 + o(x^4).$$

$$g(x) = 3x^2\sin(x)^2 = 3x^2\left(x - \frac{x^3}{6}\right)^2 = 3x^2\left(x^2 - \frac{x^4}{3}\right) = 3x^4 + o(x^4).$$

So,

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{-\frac{5}{12}x^4 + o(x^4)}{3x^4 + o(x^4)} = \lim_{x\to 0}\frac{-\frac{5}{12} + o(1)}{3 + o(1)} = \frac{-5}{36}.$$

We note that $o(1)$ *is a function tending towards* 0.

1.7.2 Position of a curve relative to its tangent

Proposition 1.7.2. *Let* $f : I \to \mathbb{R}$ *be a function admitting a LD at* $a \in I$,

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_k(x-a)^k + o((x-a)^k),$$

where k *is the smallest integer* ≥ 2, *such that* $c_k \neq 0$.

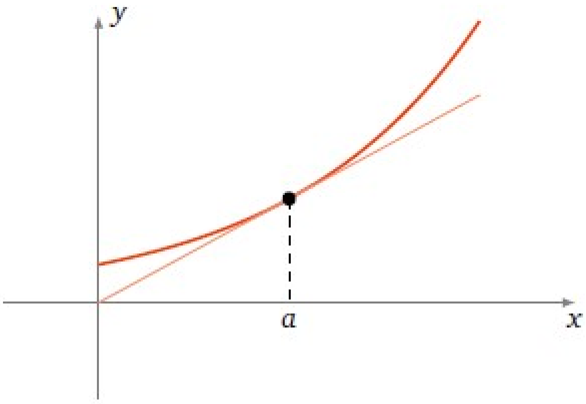
Then, the equation of the tangent to the curve of f *at* a *is:*

$$y = c_0 + c_1(x-a),$$

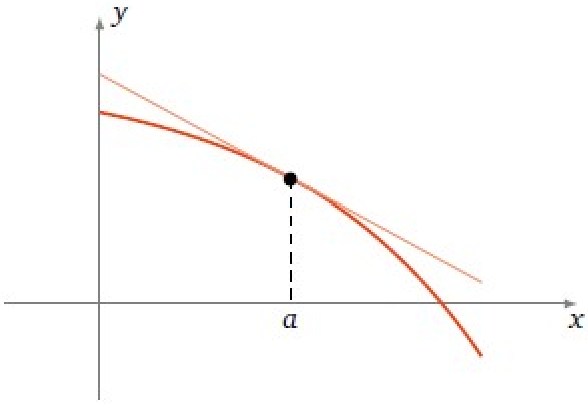
and the position of the curve with respect to the tangent for x *close to* a *is given by the sign of* $f(x) - y$, *i.e., the sign of* $c_k(x-a)^k$.

There are 3 possible cases:

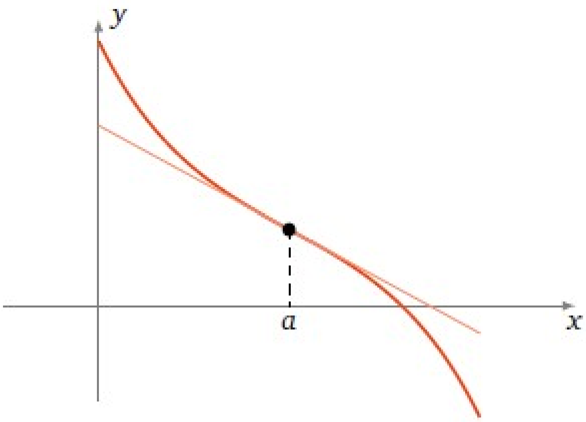
- **Case 1:** The sign is positive, the curve is above the tangent.



- **Case 2:** The sign is negative, the curve is below the tangent.



- **Case 3:** The sign changes (when $x > a$, $x < a$), the curve crosses the tangent $\Rightarrow$ is an inflection point.



Proposition 1.7.3. *We suppose that the function $x \mapsto f(x)$ admits a limit at $\pm\infty$ (or as $x \to a$),*

$$f(x) = c_0 x + c_1 + \frac{c_2}{x} + \frac{c_3}{x^2} + \cdots + \frac{c_k}{x^k} + o\left(\frac{1}{x^k}\right),$$

where k is the smallest integer ≥ 2 such that $c_k \neq 0$.

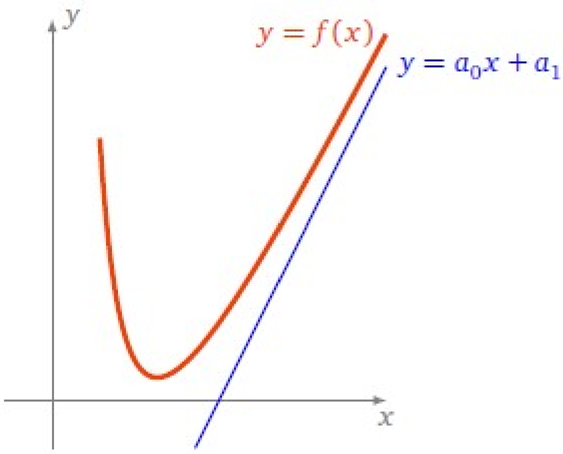
Then,

$$\lim_{x\to\infty} (f(x) - (c_0 x + c_1)) = 0 \quad (\textit{resp. } x \to -\infty),$$

and the line

$$y = c_0 x + c_1$$

*is an **asymptote** to the curve of f at ∞ (or $-\infty$), and the position of the curve relative to the asymptote is given by the sign of $f(x) - y$, i.e., the sign of $\frac{c_k}{x^{k-1}}$.*



Example 1.7.4. *(To do)*

1) Let $f(x) = x^4 - 2x^3 + 1$.

- *Find the tangent of the curve* (g) *at* $x = \frac{1}{2}$.
- *Determine the inflection points.*

2) Find the asymptotes of $f(x) = e^{-\frac{x}{\sqrt{x^2+1}}}$ *at* $+\infty$ *and* $-\infty$.

1.8 Exercises

Exercise 1:

Determine the limited developments (LD) at $x = 0$ up to order 3 of the following functions:

1) $f_1(x) = \sqrt{1+x}$

2) $f_2(x) = \dfrac{e^x - 1 - x}{x^2}$

3) $f_3(x) = \ln(2+x)$

Solution:

1) We have:

$$f_1(x) = \sqrt{1+x} = (1+x)^{1/2}$$

This is of the form $(1+x)^\alpha$, with $\alpha = \frac{1}{2}$. Then:

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + x^3\varepsilon(x)$$

2) We have:

$$e^x - 1 - x = \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + x^5\varepsilon(x)$$

Thus:

$$f_2(x) = \frac{e^x - 1 - x}{x^2} = \frac{1}{2} + \frac{1}{6}x + \frac{1}{24}x^2 + \frac{1}{120}x^3 + x^3\varepsilon(x)$$

3) We write:

$$f_4(x) = \ln(2+x) = \ln\left(2\left(1 + \frac{x}{2}\right)\right) = \ln(2) + \ln\left(1 + \frac{x}{2}\right)$$

Using the expansion $\ln(1+u) = u - \frac{1}{2}u^2 + \frac{1}{3}u^3 + u^3\varepsilon(x)$, with $u = \frac{x}{2}$, we get:

$$f_4(x) = \ln(2) + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{24}x^3 + x^3\varepsilon(x)$$

Exercise 2:

Determine the LD of the following functions:

1) $f_1(x) = (\cos x - 1)(\sinh x - x)$, to order 5 at $x = 0$

2) $f_2(x) = \ln\left(\frac{\sin x}{x}\right)$, to order 2 at $x = 0$

3) $f_3(x) = \frac{x^2+1}{x^2-2x+1}$, to order 4 at $x = 0$

4) $f_4(x) = x\sqrt{1+\frac{1}{x}}$, to order 2 at $x = \infty$

Solution:

1) Using the Taylor expansions of cos(x) and sinh(x), we develop and we keep only degree $\leq n = 5$, we obtain:

$$f_1(x) = (\cos x - 1)(\sinh x - x) = \left(-\frac{1}{2}x^2 + \frac{1}{24}x^4\right)\left(\frac{1}{6}x^3 + \frac{1}{120}x^5\right) + x^5\varepsilon(x)$$

$$f_1(x) = -\frac{1}{12}x^5 + x^5\varepsilon(x)$$

2) Using:

$$\sin x = x - \frac{x^3}{6} + x^3\varepsilon(x) \Rightarrow \frac{\sin x}{x} = 1 - \frac{x^2}{6} + x^2\varepsilon(x)$$

$$\Rightarrow \; f_2(x) = ln\left(\frac{\sin x}{x}\right) = -\frac{1}{6}x^2 + x^2\varepsilon(x)$$

3) Since $x^2 - 2x + 1 = (x-1)^2$, this function becomes:

$$f_3(x) = \frac{x^2+1}{(x-1)^2}$$

Then perform a Taylor expansion around $x = 0$ using known series.

4) Let:

$$f_4(x) = x\sqrt{1+\frac{1}{x}} = x\left(1 + \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{16x^3} + \frac{1}{x^3}\varepsilon\left(\frac{1}{x}\right)\right)$$

$$f_4(x) = x + \frac{1}{2} - \frac{1}{8x} + \frac{1}{16x^2} + \frac{1}{x^2}\varepsilon\left(\frac{1}{x}\right)$$

Exercise 3:

1) Compute the limited development at order 2 around $x_0 = 2$ for:

$$f(x) = \ln x, \quad g(x) = x^3 - x^2 - x - 2$$

2) Deduce the limit:

$$\lim_{x \to 2} \frac{\ln x - \ln 2}{x^3 - x^2 - x - 2}$$

Solution:

1) Let $t = x - 2$, then:

$$f(x) = \ln x = \ln(2 + t) = \ln 2 + \ln\left(1 + \frac{t}{2}\right)$$

$$\Rightarrow f(x) = \ln 2 + \frac{t}{2} - \frac{t^2}{8} + t^3\varepsilon(t)$$

Substituting $t = x - 2$, we get:

$$f(x) = \ln 2 + \frac{x-2}{2} - \frac{(x-2)^2}{8} + (x-2)^3\varepsilon(x-2)$$

Also, by Taylor expansion of $g(x)$ at 2:

$$g(x) = 7(x-2) + (x-2)^2 + (x-2)^3\varepsilon(x-2)$$

2) We have

$$\frac{\ln x - \ln 2}{x^3 - x^2 - x - 2} = \frac{\frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + (x-2)^3\varepsilon(x-2)}{7(x-2) + (x-2)^2 + (x-2)^3\varepsilon(x-2)}$$

Taking the limit:

$$\lim_{x \to 2} \frac{\ln x - \ln 2}{x^3 - x^2 - x - 2} = \frac{1}{14}$$

Exercise 4:

Let the function $f(x) = \dfrac{\ln(\cosh x)}{\sinh x}$, defined on $\mathbb{R}^*$.

1) Determine the limited development of f at $x = 0$ up to order 3.

2) Show that f can be extended by continuity at 0 and that the extension is differentiable.

3) Find the value of $f'(0)$.

Solution:

1) We have:

$$\cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{24} + x^4\varepsilon(x)$$

So,

$$\ln(\cosh x) = \ln(1+u), \quad u = \frac{x^2}{2} + \frac{x^4}{24} + x^4\varepsilon(x)$$

$$\Rightarrow \ln(\cosh x) = \frac{x^2}{2} - \frac{x^4}{12} + x^4\varepsilon(x)$$

and we have

$$\sinh x = x + \frac{x^3}{6} + x^3\varepsilon(x)$$

Therefore,

$$f(x) = \frac{\ln(\cosh x)}{\sinh x} = \frac{\frac{x^2}{2} - \frac{x^4}{12} + x^4\varepsilon(x)}{x + \frac{x^3}{6} + x^3\varepsilon(x)}$$

Divide numerator over denominator according to increasing powers until order 3 to get:

$$f(x) = \frac{x}{2} - \frac{x^3}{6} + x^3\varepsilon(x)$$

2) As $\lim_{x\to 0} f(x) = 0$, the function can be extended by continuity at 0.

Define:

$$g(x) = \begin{cases} \dfrac{\ln(\cosh x)}{\sinh x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Then,

$$g'(0) = \lim_{x\to 0} \frac{g(x) - g(0)}{x} = \lim_{x\to 0} \frac{f(x)}{x} = \frac{1}{2}$$

So, g is differentiable at $x = 0$ and $g'(0) = \frac{1}{2}$.

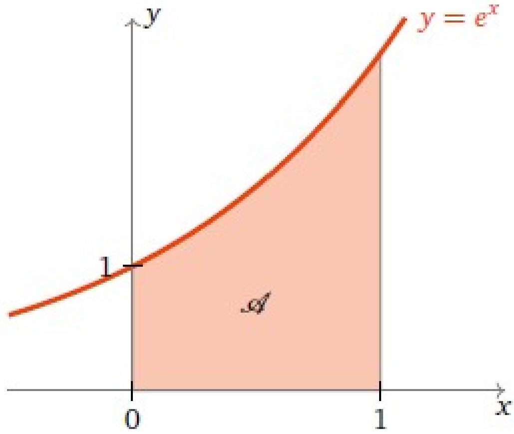
Chapter 2

Riemann Integral

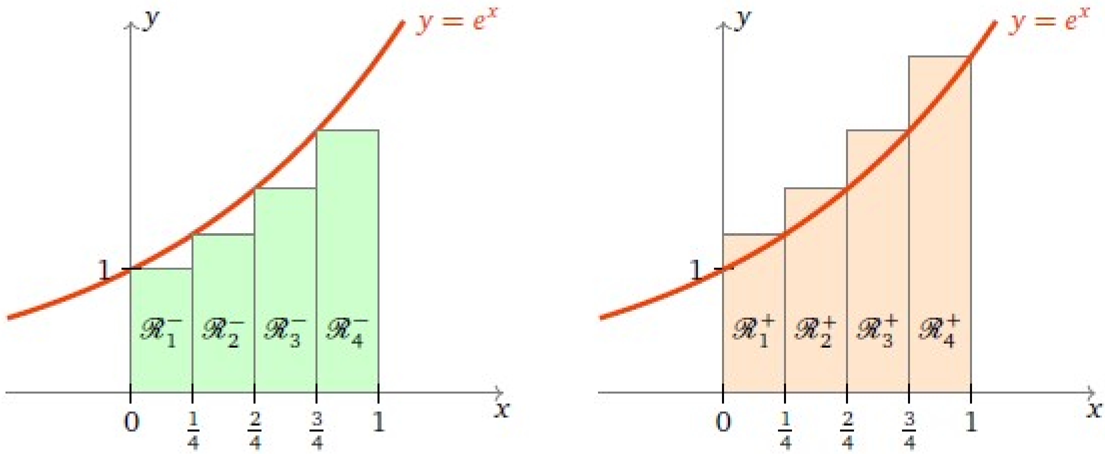
2.1 Introduction

We take the function $f(x) = e^x$, as an example to introduce the integral.

We want to calculate the area below the graph of f and between the lines of equation $x = a$ and $x = b$, and the (ox) axis, noted $\mathcal{A}$. We approximate this area $\mathcal{A}$ by sums of



areas of the rectangles located under the curve or by the sums of areas of the rectangles located above the curve.



Where $\mathcal{R}^- = \sum \mathcal{R}_i^-$, and $\mathcal{R}^+ = \sum \mathcal{R}_i^+$.

It's clear that the area $\mathcal{A}$ verify

$$\mathcal{R}^- < \mathcal{A} < \mathcal{R}^+,$$

i.e., the area $\mathcal{A}$ is greater than the sum of the areas of lower rectangles and it's less than the sum of the areas of the upper rectangles.

When we consider smaller and smaller subdivisions of the interval $[a, b]$ $(x_0 < x_1 < x_2 < \ldots < x_n = b)$, that is, when we make n tends to $+\infty$ $(n \to +\infty)$, then we obtain in the limit that the area of our region of it is framed by two areas $\mathcal{R}^-$ and $\mathcal{R}^+$, which tends towards themselves (i.e., $\mathcal{R}^+ - \mathcal{R}^- \to 0$), and in this case,

$$\mathcal{A} = \mathcal{R}^+ = \mathcal{R}^- \quad (\text{when } n \to +\infty).$$

2.2 Riemann integral

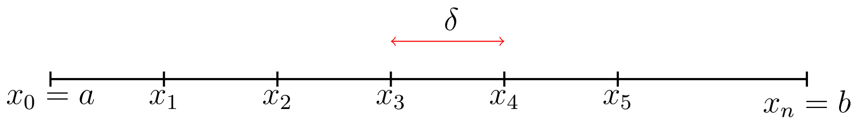
2.2.1 Riemann sums

Definition 2.2.1. *(Subdivision)*

- *Let $[a, b]$ be a closed bounded interval $(-\infty < a < b < \infty)$.*

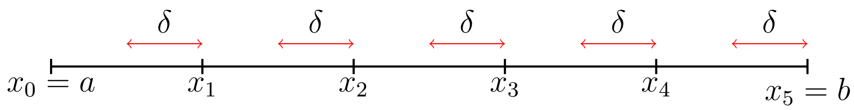
  We call a subdivision of the interval $[a, b]$ the finite family $\sigma = \{x_0, x_1, x_2, \ldots, x_n\}$ such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

- *The step of such a subdivision* n *is the number* $\delta = \max|x_k - x_{k-1}|$, $1 \leq k \leq n$ *is the length of the largest interval in the division of* $[a, b]$.



- *We say that* $\sigma = \{x_0, x_1, \ldots, x_n\}$ *is an equidistant subdivision if we have an equidistant division of* $[a, b]$ *into* n *intervals of identical length* $\delta = \dfrac{b-a}{n}$.
  The subdivision points are given by:

$$x_k = a + k \cdot \delta, \quad \textit{for } k \in \{0, 1, \ldots, n\}.$$



Definition 2.2.2. *(Riemann sum)*
Let f *be a function defined on* $[a, b]$, $\sigma = \{x_0, x_1, \ldots, x_n\}$ *a subdivision of* $[a, b]$ *and* $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *a family of reals such that:* $\forall\, k \in \{1, 2, \ldots, n\}$, $\lambda_k \in [x_{k-1}, x_k]$.
We define the ***Riemann sum*** *of the function* f *associated with* σ *and* Λ*, the number*

$$S(f, \sigma, \Lambda) = \sum_{k=1}^{n} (x_k - x_{k-1}) f(\lambda_k).$$

$S(f, \sigma, \Lambda)$ *represents the area of the union of the areas of the rectangles of base* $[x_{k-1}, x_k]$ *and of height* $f(\lambda_k)$.
If $\sigma = \{x_0, x_1, \ldots, x_n\}$ *is equidistant subdivision of* $[a, b]$*, we choose* λ_k *one of the limits of each sub-interval* $[x_{k-1}, x_k]$*, i.e.,*

$$x_k = a + k\frac{b-a}{n}, \quad 0 \leq k \leq n,$$

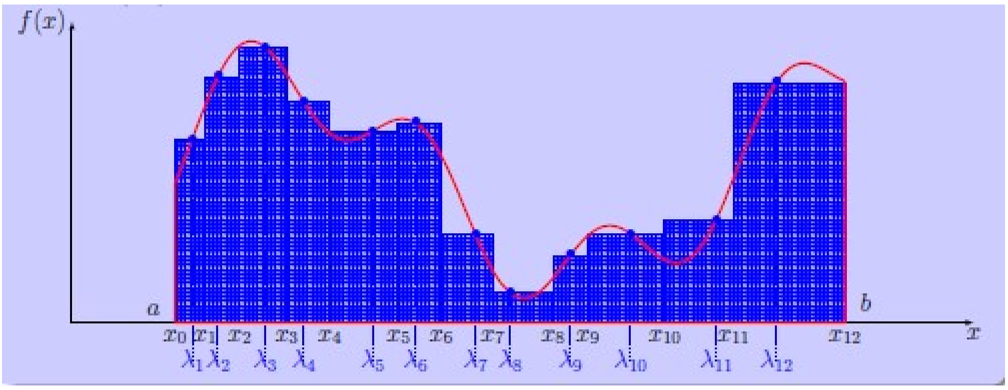
$$\lambda_k = x_k \quad \textit{or} \quad \lambda_k = x_{k-1}, \quad 1 \leq k \leq n.$$

Then, the corresponding Riemann sums are given by:

$$S(f, \sigma, \Lambda) = \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k \cdot \frac{b-a}{n}\right),$$

or

$$S(f,\sigma,\lambda) = \frac{b-a}{n}\sum_{k=0}^{n-1} f\left(a + k\cdot\frac{b-a}{n}\right).$$



Remark 2.2.3. *(Darboux sums)*

Let f be a continuous function on $[a,b]$, $\sigma = \{x_0, x_1, \ldots, x_n\}$ a subdivision of $[a,b]$.

Let $m_k = \min f(x)$ and $M_k = \max f(x)$ on $[x_{k-1}, x_k]$, $k = 1, 2, \ldots, n$.

By continuity, it reaches its limits, i.e.,

$$\exists \lambda_k^1 \in [x_{k-1}, x_k] \mid m_k = f(\lambda_k^1),$$

and

$$\exists \lambda_k^2 \in [x_{k-1}, x_k] \mid M_k = f(\lambda_k^2), \quad k = 1, 2, \ldots, n.$$

*The Riemann sums corresponding to the updated families $\Lambda_1 = \{\lambda_1^1, \lambda_2^1, \ldots, \lambda_n^1\}$ and $\Lambda_2 = \{\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2\}$ are called **Darboux sums**, given by:*

$$S_{D_1} = S(f,\sigma,\Lambda_1) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}),$$

$$S_{D_2} = S(f,\sigma,\Lambda_2) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}).$$

*S_{D_1} is called the **Lower Darboux sum**.*

*S_{D_2} is called the **Upper Darboux sum**.*

It is clear that the Riemann sums are between the two sums of Darboux:

$$S_{D_1} = S(f,\sigma,\Lambda_1) \leq S(f,\sigma,\Lambda) \leq S(f,\sigma,\Lambda_2) = S_{D_2}.$$

Definition 2.2.4. *(Integrability)*

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. If there exists a real number I such that $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall \sigma$ a subdivision of $[a, b]$, $\forall \mathcal{N} = (t_1, \dots, t_n)$ adapted to σ if $|\mathcal{S}(f, \sigma, \mathcal{N}) - I| < \varepsilon$, then we say that the function f is integrable (in the Riemann sense) on $[a, b]$, and the number I is the integral of f on $[a, b]$, denoted by

$$I = \int_a^b f(x)\, dx.$$

In other words, a function f is integrable if and only if all its sequences of Riemann sums, whose step of the associated subdivisions tends to zero, are convergent with the same finite limit I.

- *If the subdivision $\sigma = \{x_0, x_1, \dots, x_n\}$ is equidistant $\left(x_k = a + k \cdot \frac{b-a}{n},\ k = 0, 1, \dots, n\right)$, then,*

$$f \textit{is integrable on} [a, b] \Leftrightarrow \exists I \in \mathbb{R} / \lim_{n \to +\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k \cdot \frac{b-a}{n}\right) = I.$$

$$\Rightarrow \int_a^b f(x)\, dx = \lim_{n \to +\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k \cdot \frac{b-a}{n}\right).$$

Remark 2.2.5. *1. The variable used in the notation of the integral is called mute,*

$$\text{i.e., } I = \int_a^b f(x)\, dx = \int_a^b f(t)\, dt = \int_a^b f(z)\, dz = \dots$$

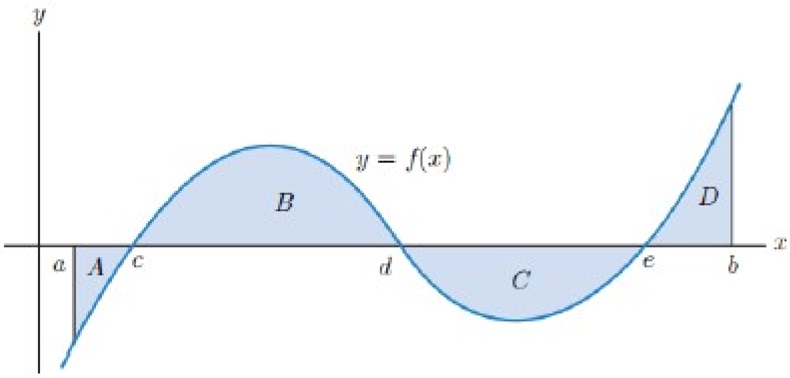
2. The number $I = \int_a^b f(x)\, dx$ represents the algebraic area between the curve of f, the abscissa axis and the lines $(x = a)$, $(x = b)$, counting negatively the parts below the axis and positively the parts above the axis.

Example 2.2.6. *1) Let f be a constant function: $f(x) = \alpha$, $\alpha \in \mathbb{R}$, $\forall x$.*

We have:

$$I = \int_a^b f(x)\, dx = \int_a^b \alpha\, dx = \lim_{n \to \infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)$$

$$= \lim_{n \to \infty} \frac{b-a}{n} \sum_{k=1}^{n} \alpha = \lim_{n \to \infty} \frac{b-a}{n} \cdot n \cdot \alpha = \alpha(b-a)$$



2) Let $f(x) = x$, calculate $\int_0^1 f(x)\,dx$.

We have:

$$I = \int_0^1 f(x)\,dx = \int_0^1 x\,dx$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(f\left(a + k\frac{k}{n}\right)\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k}{n} = \lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{n} + \frac{2}{n} + \cdots + \frac{n}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \lim_{n\to\infty} \frac{n(n+1)}{2n^2} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}$$

3) Let $f(x) = x^2$, calculate $\int_0^2 f(x)\,dx$.

We have:

$$I = \int_0^2 x^2\,dx = \lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{n} \left(\cdot f\left(\frac{2k}{n}\right)\right)$$

$$= \lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{n} \left(\frac{2k}{n}\right)^2 = \lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{n} \frac{4k^2}{n^2} = \lim_{n\to\infty} \frac{8}{n^3} \sum_{k=1}^{n} k^2$$

$$= \lim_{n\to\infty} \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} \frac{8}{6n^3} \cdot (n(n+1)(2n+1)) = \lim_{n\to\infty} \frac{8}{6} \cdot \frac{(n+1)(2n+1)}{n^2}$$

$$= \lim_{n\to\infty} \frac{8}{6} \cdot \left(\frac{2n+1}{n} \cdot \frac{n+1}{n}\right) = \frac{8}{6} \cdot 2 = \frac{8}{3}$$

4) Calculate $I = \int_0^1 e^x\,dx = \int_0^1 f(x)\,dx$

We have:

$$I = \int_0^1 e^x\,dx = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} e^{\left(\frac{k}{n}\right)^2})$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} e^{\frac{k}{n}} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} e^{\frac{k}{n}} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (e^{\frac{1}{n}})^k$$

this is sum of geometric sequence with $r = e^{\frac{1}{n}}$, $u_1 = e^{\frac{1}{n}}$,

$$= \lim_{n\to\infty} \frac{1}{n} \cdot \frac{e^{\frac{1}{n}}(e^{\frac{1}{n}n} - 1)}{e^{\frac{1}{n}} - 1} = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{e^{\frac{1}{n}}(e - 1)}{e^{\frac{1}{n}} - 1}$$

$$= (e-1) \lim_{n\to\infty} \frac{1}{n} \cdot \frac{e^{\frac{1}{n}}}{e^{\frac{1}{n}} - 1} \qquad = (e-1) \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{\frac{e^{\frac{1}{n}}-1}{e^{\frac{1}{n}}}}$$

$$= (e-1) \lim_{x\to 0} \frac{e^x}{\frac{(e^x-1)}{x}} = (e-1)$$

where $x = \frac{1}{n}$, $n \to \infty$ *implies* $x \to 0$.

So,

$$I = \int_0^1 e^x \, dx = e - 1.$$

5) Let be the ***Dirichlet function (indicator function of*** $\mathbb{Q}$***)****, given by*

$$1_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}, \quad 1_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}, \end{cases} \quad \text{(the set of rational numbers).}$$

Let be an interval $[a, b] \subset \mathbb{R}$. *Let* $\mathcal{T} = \{x_1, x_2, \dots, x_n\}$ *be a subdivision of* $[a, b]$ *of arbitrarily small step. We take* $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ *and* $\Lambda' = \{\lambda'_1, \dots, \lambda'_n\}$ *two adapted families to the subdivision* $\mathcal{T}$ *such that:*

$$\forall i \in \{1, 2, \dots, n\},\ \lambda_i \in \mathbb{Q} \text{ and } \lambda'_i \in \mathbb{R} \setminus \mathbb{Q} \quad (\lambda'_i \notin \mathbb{Q})$$

The corresponding Riemann sums are given by:

$$S(1_{\mathbb{Q}}, \mathcal{T}, \Lambda) = \sum_{k=1}^{n} (x_k - x_{k-1}) \cdot 1_{\mathbb{Q}}(\lambda_i) = \sum_{k=1}^{n} (x_k - x_{k-1}) \cdot 1 = b - a,$$

and

$$S(1_{\mathbb{Q}}, \mathcal{T}, \Lambda') = \sum_{k=1}^{n} (x_k - x_{k-1}) \cdot 1_{\mathbb{Q}}(\lambda'_k) = \sum_{k=1}^{n} (x_k - x_{k-1}) \cdot 0 = 0.$$

It is clear that these two Riemann sums cannot tend to a common limit. Thus, the indicator function of $\mathbb{Q}$ *is* ***not*** *integrable on any interval* $[a, b]$*, despite being bounded on this interval.*

2.3 Properties of the Riemann integral

Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

We will pose by definition or convention:

$$\int_a^a f(x)\,dx = 0, \quad \forall f,$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Theorem 2.3.1.

- *If $f : [a, b] \to \mathbb{R}$ is continuous, then f is integrable.*
- *If $f : [a, b] \to \mathbb{R}$ is piece wise continuous on $[a, b]$, then f is integrable on $[a, b]$.*
- *If $f : [a, b] \to \mathbb{R}$ is monotone, then f is integrable.*

Theorem 2.3.2. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function.*

1) *If f is integrable on $[a, b]$, then f is integrable on all intervals $[\alpha, \beta] \subseteq [a, b]$.*

2) *Let $c \in [a, b]$. If f is integrable on $[a, c]$ and on $[c, b]$, then f is integrable on $[a, b]$, and we have*
$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$$

3) *If f is integrable on $[a, b]$ and $c \in [a, b]$, then f is integrable on $[a, c]$ and $[c, b]$, and we have*
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

This relationship is called ***Chasles relation****.*

Corollairy 2.3.3. *Let $a = c_0 < c_1 < c_2 < \ldots < c_{k-1} < c_k = b$ be a subdivision of $[a, b]$. If f is integrable on each interval $[c_i, c_{i+1}]$ $(i = 1, \ldots, k)$, then f is integrable on $[a, b]$, and we have*
$$\int_a^b f(x)\,dx = \int_{c_0}^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \ldots + \int_{c_{k-1}}^{c_k} f(x)\,dx.$$

Proposition 2.3.4. *Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions on $[a, b]$. So, we have:*

1. *The function $f+g$ is integrable on $[a,b]$, and we have:*

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

2. *For all $\lambda \in \mathbb{R}$, the function λf is integrable on $[a,b]$, and we have:*

$$\int_a^b (\lambda f)(x)\,dx = \lambda \int_a^b f(x)\,dx.$$

3. *If f and g are different only at a finite number of points of $[a,b]$, then they have the same integral.*

4. *If $f \geq g$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.*

5. *If $f \geq 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq 0$.*
   In particular, if $f > 0$ on $[a,b]$, then $\int_a^b f(x)\,dx > 0$.

6. *If $f = 0$ (i.e., $f(x) = 0$, $\forall x \in [a,b]$), then $\int_a^b f(x)\,dx = 0$.*
   But $\int_a^b f(x)\,dx = 0 \nRightarrow f = 0$, for example $\int_{-a}^a f(x)\,dx = 0$ if f is an odd function.

7. *The function $|f|$ is integrable on $[a,b]$, and we have:*

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx.$$

8. *If there exists $m, M \in \mathbb{R}$ such that $m \leq f(x) \leq M$, $\forall x \in [a,b]$. Then:*

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Proposition 2.3.5. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then, $\exists c \in [a,b]$ such that*

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

The value $f(c)$ is called the **average value** of f on $[a,b]$.

Theorem 2.3.6. *(Inequality of Cauchy-Schwarz) Let $f, g : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. Then:*

$$\left(\int_a^b f(x)g(x)\,dx \right)^2 \leq \int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx.$$

2.4 Primitive of a function

Proposition 2.4.1. *Let $f : [a, b] \to \mathbb{R}$ be an integrable function on $[a, b]$. Let $F : [a, b] \to \mathbb{R}$ be defined by:*

$$F(x) = \int_a^x f(t)\,dt.$$

Then:

1. *F is continuous on $[a, b]$.*
2. *If f is continuous at $x_0 \in [a, b]$, then F is differentiable at x_0, and we have:*

$$F'(x_0) = f(x_0).$$

Definition 2.4.2. *Let $f : [a, b] \to \mathbb{R}$ and $F : [a, b] \to \mathbb{R}$, where F is differentiable on $[a, b]$. Then, F is a primitive of f if:*

$$\forall x \in [a, b], \quad F'(x) = f(x).$$

Proposition 2.4.3. *If F and G are two primitives of a function f, then $F - G$ is constant on $[a, b]$.*
i.e., if F is a primitive of f, then $F(x) + c$ ($c \in \mathbb{R}$) is also a primitive of f.

The theorem below is called **Fundamental Theorem of Analysis**:

Theorem 2.4.4. *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then*

$$F(x) = \int_a^x f(t)\,dt \quad (x \in [a, b])$$

is a primitive of f.

Theorem 2.4.5. *Let F be a primitive of a continuous function f on $[a, b]$. Then,*

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).$$

Definition 2.4.6. *(Indefinite Integral) The set of all the primitives of the function $f : [a, b] \to \mathbb{R}$ is called the **indefinite integral** of f.*

It is noted

$$\int f(x)\,dx.$$

So, if F is a primitive of f, then:

$$\int f(x)\,dx = F(x) + c, \quad c \in \mathbb{R}.$$

Example 2.4.7. *We have:*

$$\int \sin x\,dx = -\cos x + c, \quad c \in \mathbb{R}.$$

$$\int \cos x\,dx = \sin x + c, \quad c \in \mathbb{R}.$$

2.5 Primitives of usual functions

- $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c, \quad x \in \mathbb{R},\ n \in \mathbb{N},\ c \in \mathbb{R}$ (constant)
- $\int x^\alpha\,dx = \dfrac{x^{\alpha+1}}{\alpha+1} + c, \quad \alpha \in \mathbb{R} \setminus \{-1\}$ and $x > 0$
- $\int \dfrac{1}{x}\,dx = \ln|x| + c, \quad x \neq 0$
- $\int e^x\,dx = e^x + c, \quad x \in \mathbb{R}, \quad \int e^{\alpha x}\,dx = \dfrac{1}{\alpha}e^{\alpha x} + c, \quad \alpha \in \mathbb{R}^*,\ x \in \mathbb{R}$
- $\int a^x\,dx = \dfrac{a^x}{\ln(a)}, \quad x \in \mathbb{R}$ and $a > 0$
- $\int \cos x\,dx = \sin x + c, \quad x \in \mathbb{R}$
- $\int \sin x\,dx = -\cos x + c, \quad x \in \mathbb{R}$
- $\int \cosh x\,dx = \sinh x + c, \quad x \in \mathbb{R}$
- $\int \sinh x\,dx = \cosh x + c, \quad x \in \mathbb{R}$
- $\int \dfrac{1}{\cos^2 x}\,dx = \tan x + c, \quad x \in \mathbb{R} \setminus \left\{\dfrac{\pi}{2} + k\pi\right\},\ k \in \mathbb{Z}$
- $\int \dfrac{1}{\sin^2 x}\,dx = -\cot x + c, \quad x \in \mathbb{R} \setminus \{k\pi\},\ k \in \mathbb{Z}$
- $\int \tan x\,dx = -\ln|\cos x| + c, \quad x \notin \left\{\dfrac{\pi}{2} + k\pi\right\},\ k \in \mathbb{Z}$

- $\int \frac{1}{1+x^2}\,dx = \arctan x + c, \quad x \in \mathbb{R}$
- $\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + c, \quad x \in \mathbb{R}\setminus\{1\}$
- $\int \frac{1}{\sqrt{1+x^2}}\,dx = \operatorname{argsh} x + c, \quad x \in \mathbb{R}$ or $\int \frac{1}{\sqrt{1+x^2}}\,dx = \ln(x+\sqrt{x^2+1}) + c$
- $\int \frac{1}{\sqrt{x^2-1}}\,dx = \operatorname{argch} x + c, \quad x > 1$ or $\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln(x+\sqrt{x^2-1}) + c$
- $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + c, \quad |x| < 1$

2.6 General integration processes

2.6.1 Change of variables

Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$, and let $\varphi : [\alpha,\beta] \to [a,b]$ be a differentiable function of class $\mathcal{C}^1$ on $[\alpha,\beta]$, such that $\varphi(\alpha) = a$ and $\varphi(\beta) = b$, then the function $t \to f(\varphi(t)) \cdot \varphi'(t)$ is integrable on $[\alpha,\beta]$, and we have:

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t)) \cdot \varphi'(t)\,dt.$$

Just we make the change of variables

$$x = \varphi(t) \Rightarrow dx = \varphi'(t)\,dt \quad \text{with } \varphi(\alpha) = a \text{ and } \varphi(\beta) = b.$$

Example 2.6.1. *1) Calculate* $I = \int \sin^2 x \cos x\,dx$

We take: $t = \sin x \Rightarrow dt = \cos x\,dx$

$$\Rightarrow I = \int \underbrace{\sin^2 x}_{t^2}\underbrace{\cos x\,dx}_{dt} = \int t^2\,dt = \frac{t^3}{3} + C = \frac{\sin^3 x}{3} + C, \quad C \in \mathbb{R}.$$

2) Calculate $I = \int_0^{\frac{\pi}{2}} \sin x \cdot e^{\cos x}\,dx$

We take: $t = \cos x \Rightarrow dt = -\sin x\,dx \Rightarrow \sin x\,dx = -dt$

So,

$$I = \int_0^{\frac{\pi}{2}} \sin x \cdot e^{\cos x}\,dx = -\int_1^0 e^t\,dt = \int_0^1 e^t\,dt = [e^t]_0^1 = e - 1.$$

3) Calculate $I = \int_0^1 \sqrt{1-x^2}\,dx$

We take $x = \sin t \Rightarrow dx = \cos t\, dt$

$$\begin{aligned} &when\ x = 0 \Rightarrow 0 = \sin(t) \Rightarrow t = \arcsin(0) = 0,\\ &when\ x = 1 \Rightarrow 1 = \sin(t) \Rightarrow t = \arcsin(1) = \frac{\pi}{2}. \end{aligned}$$

So,

$$\begin{aligned} I &= \int_0^{\pi/2} \sqrt{1-\sin^2 t} \cdot \cos t\, dt \\ &= \int_0^{\pi/2} \sqrt{\cos^2 t} \cdot \cos t\, dt = \int_0^{\pi/2} |\cos t| \cdot \cos t\, dt. \end{aligned}$$

Since $\cos t \geq 0$ *on* $[0, \frac{\pi}{2}]$, $|\cos t| = \cos t$, *so:*

$$I = \int_0^{\pi/2} \cos^2 t\, dt = \int_0^{\pi/2} \frac{1+\cos(2t)}{2}\, dt.$$

$$= \left[\frac{t}{2} + \frac{1}{2} \cdot \frac{\sin(2t)}{2}\right]_0^{\pi/2} = \frac{\pi}{4}.$$

2.6.2 Integration by parts

Theorem 2.6.2. *Let* u *and* v *be two differentiable functions of class* $\mathcal{C}^1$ *on* $[a, b]$*, then:*

$$\int_a^b u(x)\, v'(x)\, dx = u(x)\, v(x) - \int_a^b u'(x)\, v(x)\, dx$$

and:

$$\int_a^b u(x)\, v'(x)\, dx = [u(x)\, v(x)]_a^b - \int_a^b u'(x)\, v(x)\, dx.$$

Proof. Indeed,

$$u\,v = \int (u\,v)' = \int u'\,v + u\,v' \Rightarrow \int u\,v' = u\,v - \int u'\,v.$$

□

Example 2.6.3. *1)* $I = \int x\, e^{-x}\, dx$, *we take*

$$\begin{cases} u(x) = x \Rightarrow u'(x) = 1 \\ v'(x) = e^{-x} \Rightarrow v(x) = -e^{-x} \end{cases}$$

So,

$$I=\int x\,e^{-x}\,dx = u\,v-\int u'\,v = x\,(-e^{-x})-\int 1\cdot(-e^{-x})\,dx,$$

$$=-x\,e^{-x}+\int e^{-x}\,dx=-x\,e^{-x}-e^{-x}+C=-e^{-x}(x+1)+C,\quad C\in\mathbb{R}.$$

2) $I=\int(\ln x)^2\,dx$, *we take*

$$\begin{cases} u=(\ln x)^2 \Rightarrow u'=2\,\frac{\ln x}{x} \\ v'=1\Rightarrow v=x \end{cases}$$

So,

$$I=\int(\ln x)^2\,dx = x\,(\ln x)^2-2\int 2x\,\frac{\ln x}{x}\,dx = x\,(\ln x)^2-2\int\ln x\,dx.$$

To calculate $\int\ln x\,dx$, *we use the integration by parts. We take:*

$$\begin{cases} u=\ln x \Rightarrow u'=\frac{1}{x} \\ v'=1\Rightarrow v=x \end{cases} \Rightarrow \int\ln x\,dx = x\ln x-\int x\cdot\frac{1}{x}\,dx = x\ln x-\int 1\,dx = x\ln x-x+C.$$

Finally,

$$I=\int(\ln x)^2\,dx = x(\ln x)^2-2x(\ln x-1)+C,\quad C\in\mathbb{R}.$$

3) $I=\int_0^1\arctan x\,dx$, *we take*

$$\begin{cases} u=\arctan x \Rightarrow u'=\frac{1}{1+x^2} \\ v'=1\Rightarrow v=x \end{cases}$$

So,

$$I=\int_0^1\arctan x\,dx=[x\arctan x]_0^1-\int_0^1\frac{x}{1+x^2}\,dx.$$

We have

$$\int\frac{x}{1+x^2}\,dx=\frac{1}{2}\int\frac{2x}{1+x^2}\,dx=\frac{1}{2}\ln(1+x^2)+C\quad\left(\int\frac{f'}{f}=\ln f\right).$$

So,

$$I=\left[x\arctan x-\frac{1}{2}\ln(1+x^2)\right]_0^1=\frac{\pi}{4}-\frac{1}{2}\ln(2).$$

2.7 Primitive of a rational function

Let f be a rational function, i.e., $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are two polynomials and $Q(x) \neq 0$.

For the integration of rational functions, we have two cases:

Case 1: If $d^\circ P \geq d^\circ Q$, we do an Euclidean division according to the decreasing powers of x, then

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},$$

where $d^\circ R < d^\circ Q$ and $S(x)$ is a polynomial.

Next, we decompose $\frac{R(x)}{Q(x)}$ into the sum of **simple elements**.

Case 2: If $d^\circ P < d^\circ Q$, we decompose $\frac{P(x)}{Q(x)}$ into the sum of **simple elements** after determining the **roots of** $Q(x)$.

Now, let see how to do the decomposition of a rational function $f(x) = \frac{P(x)}{Q(x)}$, $d^\circ P < d^\circ Q$, into the sum of simple elements.

We have two forms of simple elements for the rational functions:

- **Simple elements of the first kind of the form:**

$$\frac{A}{(x-a)^k}, \quad \text{where } A, a \in \mathbb{R},\ k \in \mathbb{N}^*.$$

- **Simple elements of the second kind of the form:**

$$\frac{Mx+N}{(x^2+ax+b)^k}, \quad \frac{Mx+N}{((x-\alpha)^2+\beta^2)^k},$$

where $M, N, a, b, \alpha, \beta \in \mathbb{R},\ k \in \mathbb{N}^*$ and $\Delta = a^2 - 4b < 0$.

Finally, all rational functions can be written as a sum of a polynomial and simple elements.

So, to integrate $f(x) = \frac{P(x)}{Q(x)}$, we must know how to integrate the simple elements.

2.7.1 Integration of simple elements of the first kind

Let $I = \int \frac{A}{(x-a)^k}\, dx, \quad k \in \mathbb{N}^*,\ A, a \in \mathbb{R}.$

- If $k = 1$, then
$$I = \int \frac{A}{x-a}\,dx = A\ln|x-a| + c, \quad c \in \mathbb{R}.$$

- If $k > 1$, then
$$I = \int \frac{A}{(x-a)^k}\,dx = \frac{A}{(1-k)(x-a)^{k-1}} + c, \quad c \in \mathbb{R}.$$

2.7.2 Integration of simple elements of the second kind

Let
$$I = \int \frac{Mx+N}{(x^2+ax+b)^k}\,dx, \quad a, b, M, N \in \mathbb{R},\ k \in \mathbb{N}^* \text{ and } \Delta = a^2 - 4b < 0.$$

We decompose the polynomial $x^2 + ax + b$ in the form of the sum of two squares:
$$x^2 + ax + b = \left(x + \frac{a}{2}\right)^2 + b - \left(\frac{a^2}{4}\right) = \left(x + \frac{a}{2}\right)^2 + \frac{4b - a^2}{4}.$$

We set $\alpha = -\frac{a}{2}$ and $\beta^2 = \frac{4b-a^2}{4}$, we obtain
$$x^2 + ax + b = (x-\alpha)^2 + \beta^2.$$

So,
$$\frac{Mx+N}{(x^2+ax+b)^k} = \frac{Mx+N}{[(x-\alpha)^2+\beta^2]^k} = \frac{Mx+N}{[\beta^{2k}((\frac{x-\alpha}{\beta})^2+1)^k}.$$

We take the following change of variables:
$$t = \frac{x-\alpha}{\beta} \Leftrightarrow x = \beta t + \alpha \Rightarrow dx = \beta dt.$$

$$I = \frac{Mx+N}{(x^2+ax+b)^k} = \frac{1}{\beta^{2k-1}} \int \frac{M(\beta t + \alpha) + N}{(t^2+1)^k}\,dt.$$

$$= \frac{M}{\beta^{2k-1}} \int \frac{t}{(t^2+1)^k}\,dt + \frac{M\alpha + N}{\beta^{2k-1}} \int \frac{1}{(t^2+1)^k}\,dt.$$

Let $I_k = \int \frac{t}{(t^2+1)^k}\,dt, \quad J_k = \int \frac{1}{(t^2+1)^k}\,dt.$

Calculation of $I_k = \int \frac{t}{(t^2+1)^k}\,dt$

- If $k = 1$, then
$$I_1 = \int \frac{t}{t^2+1}\, dt = \frac{1}{2}\ln(t^2+1) + c, \quad c \in \mathbb{R}.$$

- If $k > 1$, we do a change of variables:
$$u = 1 + t^2 \Rightarrow du = 2t\, dt.$$

So,
$$I_k = \int \frac{t}{(t^2+1)^k}\, dt = \frac{1}{2}\int \frac{1}{u^k}\, du = \frac{1}{2(1-k)}(1+t^2)^{1-k} + c, \quad c \in \mathbb{R}.$$

$$\Rightarrow I_k = \int \frac{t}{(t^2+1)^k}\, dt = \frac{1}{2(1-k)(t^2+1)^{k-1}} + c, \quad c \in \mathbb{R}.$$

Calculation of $J_k = \int \frac{1}{(t^2+1)^k}\, dt$

$$J_k = \int \frac{1 \cdot t}{(t^2+1)^k}\, dt = uv - \int vu'\, dt \Rightarrow \begin{cases} v' = 1 \Rightarrow v = t, \\ u = \frac{1}{(t^2+1)^k} \Rightarrow u' = \frac{-2kt}{(t^2+1)^{k+1}} dt \end{cases}$$

$$J_k = \frac{t}{(t^2+1)^k} + 2k\int \frac{t^2}{(t^2+1)^{k+1}}\, dt$$

$$= \frac{t}{(t^2+1)^k} + 2k\int \left(\frac{(t^2+1)-1}{(t^2+1)^{k+1}}\right) dt$$

$$= \frac{t}{(t^2+1)^k} + 2k\left[\int \frac{1}{(t^2+1)^k}\, dt - \int \frac{1}{(t^2+1)^{k+1}}\, dt\right]$$

$$= \frac{t}{(t^2+1)^k} + 2kJ_k - 2kJ_{k+1}$$

$$\Rightarrow J_k = \frac{t}{(t^2+1)^k} + 2kJ_k - 2kJ_{k+1}$$

$$\Rightarrow J_{k+1} = \frac{1}{2k} \cdot \frac{t}{(t^2+1)^k} + \frac{2k-1}{2k}J_k, \quad \text{for } k \geq 1.$$

$$\text{with } J_1 = \int \frac{dt}{(t^2+1)^1} = \int \frac{dt}{t^2+1} = \arctan(t) + C, \quad /C \in \mathbb{R}.$$

Application: for $k = 1$, we obtain:

$$J_2 = \int \frac{dt}{(t^2+1)} = \frac{1}{2} \cdot \frac{1}{(t^2+1)} + \frac{1}{2} \arctan(t) + C.$$

N.B. Once the integral calculations are finished, we don't forget to return to the variable x. Remind:

$$t = \frac{x-\alpha}{\beta} \Leftrightarrow \ x = \alpha + \beta t$$

Summary

Let $f(x) = \frac{P(x)}{Q(x)}$, $x \in I$, $Q(x) \neq 0$ for all $x \in I$. To determine a primitive of f over I, we proceed as follows:

1. Determine the integer part $S(x)$ of f, using the Euclidean division of $P(x)$ over $Q(x)$. If $d^\circ(P) \geq d^\circ(Q)$, then $f(x) = S(x) + \frac{R(x)}{Q(x)}$ where $d^\circ(R) < d^\circ(Q)$. If $d^\circ(P) < d^\circ(Q)$, then $S(x) = 0$, $f(x) = \frac{P(x)}{Q(x)}$.
2. Determine all the roots of $Q(x)$, then factor the polynomial $Q(x)$ into a product of irreducible polynomials.
3. Write the decomposition of f into the sum of simple elements, including integral part and simple elements of the 1^{st} and 2^{nd} kinds.
4. Integrate each term of the decomposition.

Example 2.7.1. *1) Calculate*

$$I = \int \frac{x^3+4x^2+9x+8}{x^2+2x+3}\, dx = \int g(x)\, dx$$

We have:

$$f(x) = \frac{P(x)}{Q(x)}, \quad d^\circ P = 3 > d^\circ Q = 2.$$

So, first we do an Euclidean division of P over Q.

$$\frac{x^3+4x^2+9x+8}{x^2+2x+3} = x+2+\frac{2x+2}{x^2+2x+3} \Rightarrow f(x) = S(x) + \frac{R(x)}{Q(x)}.$$

$$f(x) = x+2+\frac{2x+2}{x^2+2x+3}.$$

So:

$$I = \int f(x)dx = \int (x+2)dx + \int \frac{2x+2}{x^2+2x+3}dx,$$

$$\Rightarrow I = \frac{x^2}{2} + 2x + \ln|x^2+2x+3| + c.$$

2) Calculate

$$I = \int \frac{5x-1}{(x^2+4x+4)(x^2-1)}\,dx.$$

We have:

$$f(x) = \frac{P(x)}{Q(x)}, \quad d^\circ P < d^\circ Q, \Rightarrow S(x) = 0.$$

We have:

$$Q(x) = (x^2+4x+4)(x^2-1) = (x+2)^2(x-1)(x+1).$$

We will have only second kind of decomposition.

$$f(x) = \frac{5x-1}{(x+2)^2(x-1)(x+1)} = \frac{A}{x+2} + \frac{A_2}{(x+2)^2} + \frac{A_3}{x-1} + \frac{A_4}{x+1}$$

$$= \frac{29/9}{x+2} - \frac{11/3}{(x+2)^2} + \frac{2/9}{x-1} + \frac{3}{x+1}.$$

Then, we integrate each element, we obtain:

$$I = \int f(x)dx = -\frac{29}{9}\ln|x+2| + \frac{11}{3}\frac{1}{x+2} + \frac{2}{9}\ln|x-1| + 3\ln|x+1| + c.$$

3) Calculate

$$I = \int f(x)dx = \int \frac{1}{1+x^3}dx = \int \frac{P(x)}{Q(x)}dx.$$

We have:

$$d^\circ P = 0 < d^\circ Q = 3 \Rightarrow S(x) = 0.$$

$$Q(x) = 1 + x^3 = (x+1)(x^2 - x + 1).$$

For $x^2 - x + 1$, *we have* $\Delta = b^2 - 4ac = 1 - 4 = -3 < 0$.

So, $x^2 - x + 1$ *is irreducible polynomial.*

We obtain the decomposition of $f(x) = \frac{P(x)}{Q(x)}$ *as follows:*

$$f(x) = \frac{1}{1+x^3} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Mx+N}{x^2-x+1}.$$

$$= \frac{1/3}{x+1} + \frac{-1/3x + 2/3}{x^2-x+1}$$

$$= \frac{1}{3(x+1)} + \frac{1}{3} \cdot \frac{2-x}{x^2-x+1}.$$

So,

$$I = \int f(x)dx = \frac{1}{3}\int \frac{1}{x+1}dz + \frac{1}{3}\int \frac{2-x}{x^2-x+1}dx$$
$$= \frac{1}{3}\ln|x+1| + \frac{1}{3}I'$$

To calculate I', *we write* $x^2 - x + 1$ *as follows:*

$$x^2 - x + b = \left(x + \frac{\alpha}{2}\right)^2 + \frac{4b - \alpha^2}{4} = (x-\alpha)^2 + \beta^2 \quad \left(\sqrt{(x-\alpha)^2 + \beta^2}\right)$$

$$\Rightarrow x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} = (x - \frac{1}{2})^2 + \frac{3}{4} = \frac{3}{4}\left[\left(\frac{x - 1/2}{\sqrt{3}/2}\right)^2 + 1\right] = \frac{3}{4}(t^2+1)$$

We take the change of variables:

$$t = \frac{x-\alpha}{\beta} = \frac{x - \frac{1}{2}}{\sqrt{3}/2} = \frac{2x-1}{-\sqrt{3}} \quad (i.e.,\ x = \beta t + \alpha = \frac{\sqrt{3}}{2}t + \frac{1}{2}),\ and\ dx = \frac{\sqrt{3}}{2}dt$$

So,

$$I' = \int \frac{2-x}{x^2-x+1}dz = \int \frac{2-\left(\frac{\sqrt{3}}{2}t+\frac{1}{2}\right)}{\frac{3}{4}(t^2+1)} \cdot \frac{\sqrt{3}}{2}dt$$
$$= \int \frac{\sqrt{3}-t}{t^2+1}dt = \int \frac{\sqrt{3}}{1+t^2}dt - \int \frac{t}{1+t^2}dt$$

Then, we have

$$I' = \sqrt{3}\arctan(t) - \frac{1}{2}\ln(1+t^2) + c, \quad c \in \mathbb{R}$$
$$= \sqrt{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) - \frac{1}{2}\ln(x^2-x+1) + c$$

Finally,

$$I = \int \frac{f(x)}{1+x^3}dz = \frac{1}{3}\ln|x+1| + \frac{\sqrt{3}}{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) - \frac{1}{6}\ln(x^2-x+1) + c$$

$$= \frac{1}{3}\ln|x+1| + \frac{\sqrt{3}}{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) - \frac{1}{6}\ln(x^2-x+1) + c, \quad c \in \mathbb{R}.$$

4) This is an example to do:

Let $f(x) = \frac{P(x)}{Q(x)} = \frac{x^2-3x-2}{(x+1)(x^2+2x+1)}$ $\quad (d^\circ P = 2 < d^\circ Q = 6)$

- *Decompose* $f(x) = \frac{A_1}{x+1} + \frac{A_2}{(x+1)^2} + \frac{M_1x+N_1}{x^2+x+1} + \frac{M_2x+N_2}{(x^2+x+1)^2}$. *You will find:* $A_1 = -1, A_2 = 2, M_1 = -1, N_1 = -2, M_2 = 3, N_2 = -1$

- *Calculate* $I = \int f(x)dx$. *You will find:*

$$I = \frac{-2}{x+1} - \ln|x+1| + \frac{1}{2}\ln(x^2+x+1) - \frac{25}{3\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) - \frac{5x+7}{3(x^2+x+1)} + c$$

2.8 Primitive of a rational function of sine and cosine

Let be $I = \int f(\sin x, \cos x)\, dx$, where f is a rational function of sine and cosine.

General case

It is always possible to reduce this integration to that of a rational function of a variable t, using the following change variables:

$$t = \tan\left(\frac{x}{2}\right) \Rightarrow dx = \frac{2}{1+t^2}\,dt,$$

with:

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad \text{and} \quad \tan x = \frac{2t}{1-t^2}.$$

Example 2.8.1. *Let* $I = \int \frac{1}{\sin x}\,dx$.
We take $t = \tan\left(\frac{x}{2}\right) \Rightarrow dx = \frac{2}{1+t^2}\,dt$ *and* $\sin x = \frac{2t}{1+t^2}$.
So,

$$I = \int \frac{1}{\sin x}\,dx = \int \frac{1+t^2}{2t} \cdot \frac{2}{1+t^2}\,dt = \int \frac{1}{t}\,dt = \ln|t| + c = \ln\left|\tan\left(\frac{x}{2}\right)\right| + c, \quad c \in \mathbb{R}.$$

Special cases

The change variable $t = \tan\left(\frac{x}{2}\right)$ always succeeds, but can lead to quite long calculations. We can try other more suitable changes of variables.
In the following cases, the change of variables are evident:

- If $I = \int f(\sin x) \cdot \cos x\,dx$, we put $t = \sin x \Rightarrow dt = \cos x\,dx$.
  $\Rightarrow I = \int f(t)\,dt$.
- If $I = \int f(\cos x) \cdot \sin x\,dx$, we put $t = \cos x \Rightarrow dt = -\sin x\,dx$.
  $\Rightarrow I = -\int f(t)\,dt$.
- If $I = \int f(\tan x) \cdot \frac{1}{\cos^2 x}\,dx$, we put $t = \tan x \Rightarrow dt = \frac{1}{\cos^2 x}\,dx$.
  $\Rightarrow I = \int f(t)\,dt$.

Example 2.8.2. *Let* $I = \int \sin^5 x \cdot \cos x\,dx$, *we put* $t = \sin x \Rightarrow dt = \cos x\,dx$.

$$\Rightarrow I = \int t^5\,dt = \frac{t^6}{6} + c = \frac{\sin^6 x}{6} + c, \quad c \in \mathbb{R}.$$

Case of polynomial functions of sine and cosine

If $f(\sin(x), \cos(x))$ reduces to a polynomial (e.g., we will have $\sin(x)^p$ and $\cos(x)^q$), we can always express it linearly in terms of sine and cosine of multiples of x, using trigonometric transformation formulas. The integration becomes straightforward after such a transformation.

Example 2.8.3.

$$I = \int \cos^3 x \, dx$$

We have:

$$\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos(3x)$$

$$\Rightarrow I = \int \cos^3 x \, dx = \frac{3}{4}\int \cos x \, dx + \frac{1}{4}\int \cos(3x) \, dx$$

$$= \frac{3}{4}\sin x + \frac{1}{12}\sin(3x) + c \quad \textit{with } c \in \mathbb{R}.$$

Remark 2.8.4. ***Linearization of powers of*** $\cos^n(\theta)$ ***and*** $\sin^n(\theta)$*:*

For all $n \in \mathbb{N}^*$ and $\theta \in \mathbb{R}$, we can write $\cos^n(\theta)$ (resp. $\sin^n(\theta)$) as sums of terms of the form $\cos(m\theta)$ (resp. $\sin(m\theta)$), with $m \in \mathbb{N}$ and $m \leq n$.

The method consists in using Euler's formula and the Newton's binomial expansion:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}, \quad \textit{where } \binom{n}{k} = C_n^k = \frac{n!}{k!(n-k)!}.$$

- *For example,*

$$\cos^n(\theta) = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^n = \frac{1}{2^n}(e^{i\theta} + e^{-i\theta})^n$$

$$= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} e^{i(n-k)\theta} e^{-ik\theta} = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} e^{i(n-2k)\theta}$$

Thus, we have:

$$\cos^n(\theta) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} e^{i(n-2k)\theta} \quad (*)$$

Then, we group the terms in (*) into elements of the form $e^{im\theta} + e^{-im\theta}$. Using

Euler's formula, we obtain:

$$e^{im\theta} + e^{-im\theta} = 2\cos(m\theta)$$

- *Similarly for* $\sin(\theta)$*:*

$$\sin^n(\theta) = \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^n$$

Example 2.8.5.

$$\cos^4(\theta) = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^4 = \frac{1}{16}(e^{i\theta} + e^{-i\theta})^4 = \frac{1}{16}\sum_{k=0}^{4}\binom{4}{k}e^{i(4-2k)\theta}$$

$$\Rightarrow \cos^4(\theta) = \frac{1}{8}\left(\cos(4\theta) + 4\cos(2\theta) + 3\right).$$

2.9 Primitive of a rational function of e^x

Let $I = \int f(e^x)\,dx$, where f is a rational function of e^x.

We take the following change of variable:

$$t = e^x \Rightarrow dt = e^x dx = t dx \Rightarrow dx = \frac{dt}{t}$$

This leads to the integration of a rational function of t:

$$I = \int f(e^x)\,dx = \int \frac{f(t)}{t}\,dt$$

Example 2.9.1. *Calculate* $I = \int \frac{1}{1+e^x}dx$*.*

We take $t = e^x \Rightarrow dx = \frac{dt}{t}$*.*

$$\Rightarrow I = \int \frac{1}{1+e^x}dx = \int \frac{1}{1+t}\cdot\frac{dt}{t} = \int \frac{1}{t(1+t)}dt$$

$$= \int \left(\frac{1}{t} - \frac{1}{1+t}\right)dt$$

$$= \ln|t| - \ln|1+t| + c = \ln\left|\frac{t}{1+t}\right| + c$$

$$= \ln\left|\frac{e^x}{1+e^x}\right| + c, c \in \mathbb{R}.$$

2.10 Primitive of a rational function of $\sinh(x)$ and $\cosh(x)$

We have $\sinh x = \frac{e^x - e^{-x}}{2}$ and $dx = \frac{e^x + e^{-x}}{2} dt$, so the integration of a rational function of $\sinh(x)$ and $\cosh(x)$, $f(\sinh(x), \cosh(x))$, comes back to the integration of a rational function of e^x, and in this case we can make the change of variable:

$$t = e^x \Rightarrow dx = \frac{dt}{t} \text{ with } \sinh x = \frac{t^2-1}{2t}, \quad \cosh x = \frac{t^2+1}{2t}, \quad \tanh x = \frac{t^2-1}{t^2+1}.$$

We will obtain a rational function of t.

We can also take the change of variable of the form:

$$t = \tanh\left(\frac{x}{2}\right) \Rightarrow dt = \frac{1}{2}(1 - \tanh^2(\frac{x}{2}))dx = \frac{1}{2}(1-t^2)dx \Rightarrow dx = \frac{2}{1-t^2}dt$$

So,

$$t = \tanh\left(\frac{x}{2}\right) \Rightarrow dx = \frac{2}{1-t^2}dt$$

$$\text{with } \sinh x = \frac{2t}{1-t^2}, \quad \cosh x = \frac{1+t^2}{1-t^2}, \quad \tanh x = \frac{2t}{1+t^2}$$

Example 2.10.1. *Calculate* $I = \int \frac{1}{\sinh x} dx$

1st method: *We take* $t = e^x \Rightarrow dx = \frac{dt}{t}$, *and* $\sinh x = \frac{t^2-1}{2t}$

$$\Rightarrow I = \int \frac{1}{\sinh x} dx = \int \frac{2t}{t^2-1} \cdot \frac{dt}{t} = \int \frac{2}{t^2-1} dt = \int \frac{2}{(t-1)(t+1)} dt$$

$$= \int \left(\frac{1}{t-1} - \frac{1}{t+1}\right) dt = \ln|t-1| - \ln|t+1| + c = \ln\left|\frac{t-1}{t+1}\right| + c, \quad t = e^x$$

$$\Rightarrow I = \ln\left|\frac{e^x - 1}{e^x + 1}\right| + c = \ln\left|\tanh\left(\frac{x}{2}\right)\right| + c, \quad c \in \mathbb{R}.$$

2nd method: *We take* $u = \tanh\left(\frac{x}{2}\right) \Rightarrow dx = \frac{2}{1-u^2}du$*, and* $\sinh x = \frac{2u}{1-u^2}$

$$\Rightarrow I = \int \frac{dx}{\sinh x} = \int \frac{\frac{2}{1-u^2}du}{\frac{2u}{1-u^2}} = \int \frac{1}{u}du$$

$$= \ln|u| + c = \ln\left|\tanh\left(\frac{x}{2}\right)\right| + c = \ln\left|\frac{e^x - 1}{e^x + 1}\right| + c, c \in \mathbb{R}.$$

Example 2.10.2. *Calculate* $I = \int \frac{e^x}{\cosh x \cdot \sinh x}\, dx$

We take $t = e^x \Rightarrow dx = \frac{dt}{t}$*, and* $\cosh x = \frac{t^2+1}{2t}, \quad \sinh x = \frac{t^2-1}{2t}$

$$\Rightarrow I = \int \frac{e^x}{\cosh x \cdot \sinh x}\, dx = \int \frac{t \cdot \frac{dt}{t}}{\left(\frac{t^2+1}{2t}\right)\left(\frac{t^2-1}{2t}\right)} = \int \frac{4t^2}{(t^2+1)(t^2-1)} \cdot dt$$
$$= \int \frac{4t^2}{(t^2+1)(t-1)(t+1)}\, dt$$

We have:

$$\frac{4t^2}{(t^2+1)(t-1)(t+1)} = \frac{A_1}{t-1} + \frac{A_2}{t+1} + \frac{Mt+N}{t^2+1} \quad \textit{or} \quad = \frac{2}{t^2+1} + \frac{1}{t-1} - \frac{1}{t+1}$$

$$\Rightarrow I = \int \frac{2}{t^2+1}\, dt + \int \frac{1}{t-1}\, dt - \int \frac{1}{t+1}\, dt$$

$$= 2\arctan(t) + \ln|t-1| - \ln|t+1| + c, \quad c \in \mathbb{R}$$
$$= 2\arctan(t) + \ln\left|\frac{t-1}{t+1}\right| + c$$

Finally,

$$I = \int \frac{e^x}{\cosh x \cdot \sinh x}\, dx = 2\arctan(e^x) + \ln\left|\frac{e^x - 1}{e^x + 1}\right| + c, \quad c \in \mathbb{R}.$$

2.11 Exercises

Exercise 1:

1) Using Riemann sums, calculate the integrals:

$$a)\int_1^2 x^2dx, \qquad b)\int_0^t e^x dx.$$

2) Determine the functions associated with the following Riemann sums then calculate their limits

$$a)\sum_{k=0}^{n}\frac{k^2}{n^3+8k^3}, \qquad b)\sum_{k=0}^{n}\frac{1}{\sqrt{4n^2-k^2}} \qquad c)\sum_{k=0}^{n}\frac{1}{\sqrt{n^2+2kn}}.$$

Solution:

1) Riemann Sums

Riemann sum associated for a regular subdivision of $[a,b]$:

We pose: $x_k = a + \frac{k}{n}(b-a)$, then:

$$\begin{aligned} x_0 &= a, \\ x_1 &= a + \frac{1}{n}(b-a), \\ x_2 &= a + \frac{2}{n}(b-a), \\ &\ldots\ldots. \\ x_n &= b. \end{aligned}$$

Let be $f : [a,b] \to \mathbb{R}$, then:

$$\int_a^b f(x)dx = \lim_{n\to+\infty}\frac{b-a}{n}\sum_{k=0}^{n-1} f(a+\frac{k}{n}(b-a)).$$

a) We have

$$\begin{aligned}\int_1^2 x^2dx &= \lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1} f(1+\frac{k}{n}) = \lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1} f(1+\frac{k}{n})^2\\ &= \lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n}(1+2\frac{k}{n}+\frac{k^2}{n^2}) = \lim_{n\to+\infty}\frac{1}{n}(n+\frac{2}{n}\sum_{k=0}^{n-1}k+\frac{1}{n^2}\sum_{k=0}^{n-1}k^2)\\ &= \lim_{n\to+\infty}\frac{1}{n}(n+\frac{2}{n}\frac{n(n-1)}{2}+\frac{1}{n^2}\frac{n(n-1)(2n-1)}{6}) = \frac{7}{3}\end{aligned}$$

b) We have

$$\begin{aligned}\int_0^t e^x dx &= \lim_{n\to+\infty}\frac{t}{n}\sum_{k=0}^{n-1}\exp(\frac{kt}{n}) = \lim_{n\to+\infty}\frac{t}{n}\sum_{k=0}^{n-1}\exp(\frac{t}{n})^k\\ &= \lim_{n\to+\infty} -\frac{t}{n}.\frac{1-\exp(\frac{t}{n})^n}{1-\exp(\frac{t}{n})} = \lim_{n\to+\infty}\frac{1}{n}.\frac{1-\exp t}{1-\exp(\frac{t}{n})}\\ &= \lim_{n\to+\infty}(1-\exp t)\frac{\frac{t}{n}}{1-\exp(\frac{t}{n})}\end{aligned}$$

We pose $u=\frac{t}{n}$, then $\frac{\frac{t}{n}}{1-\exp(\frac{t}{n})}=\frac{u}{1-\exp u}\to -1$, when $u\to 0$. So, we obtain

$$\int_0^t e^x dx = (\exp t - 1).$$

2)

a) We have

$$\sum_{k=0}^{n}\frac{k^2}{n^3+8k^3} = \frac{1}{n}\sum_{k=0}^{n}\frac{(\frac{k}{n})^2}{1+8(\frac{k}{n})^3}$$

is the Riemann sum of the function $f(x)=\frac{x^2}{1+8x^3}$, defined on $[0,1]$, then:

$$\begin{aligned}\lim_{n\to+\infty}\sum_{k=0}^{n}\frac{k^2}{n^3+8k^3} &= \int_0^1\frac{x^2}{1+8x^3}dx = \frac{1}{24}\left[\ln(1+8x^3)\right]\\ &= \frac{1}{12}\ln 3.\end{aligned}$$

b) We have

$$\sum_{k=0}^{n}\frac{1}{\sqrt{4n^2-k^2}} = \frac{1}{n}\sum_{k=0}^{n}\frac{1}{\sqrt{4-(\frac{k}{n})^2}}$$

is the Riemann sum of the function $f(x) = \frac{1}{\sqrt{4-x^2}}$, defined on $[0, 1]$, then:

$$\begin{aligned}\lim_{n\to+\infty}\sum_{k=0}^{n}\frac{1}{\sqrt{4n^2-k^2}} &= \int_0^1 \frac{1}{\sqrt{4-x^2}}dx = \frac{1}{2}\int_0^1 \frac{1}{\sqrt{1-(\frac{x}{2})^2}}dx \\ &= \left[\frac{1}{2}\arcsin(\frac{x}{2})\right]_0^1 = \frac{\pi}{6}.\end{aligned}$$

c) We have

$$\sum_{k=0}^{n}\frac{1}{\sqrt{n^2+2kn}} = \frac{1}{n}\sum_{k=0}^{n}\frac{1}{\sqrt{1+2(\frac{k}{n})}}$$

is the Riemann sum of the function $f(x) = \frac{1}{\sqrt{1+2x}}$ defined on $[0, 1]$, then:

$$\begin{aligned}\lim_{n}\sum_{k=0}^{n}\frac{1}{\sqrt{n^2+2kn}} &= \int_0^1 \frac{1}{\sqrt{1+2x}}dx = \left[\sqrt{1+2x}\right]_0^1 \\ &= \sqrt{3}-1.\end{aligned}$$

Exercise 2:

Calculate the following integrals using the appropriate change of variables:

$$\begin{array}{ll} I_1 = \int_0^{\frac{\pi}{4}} \frac{\cos x(1+\tan^2 x)}{\sin x+\cos x}dx, & I_2 = \int_0^{\pi}\cos^2 x\sin^5 x dx, \\ I_3 = \int_0^{\sqrt{\frac{\pi}{4}}} \frac{x\sin x^2}{1+\cos x^2}dx, & I_4 = \int_0^1 (\frac{1}{(1+x)^2}+\frac{1}{(1+2x)})dx \\ I_5 = \int_0^1 (x-1)\exp(x-1)^2 dx, & I_6 = \int_{\ln 2}^{\ln 3}\frac{e^x+e^{-x}}{e^x-e^{-x}}dx. \end{array}$$

Solution:

1) We calculate I_1 :

$$I_1 = \int_0^{\frac{\pi}{4}} \frac{\cos x(1+\tan^2 x)}{\sin x+\cos x}dx = \int_0^{\frac{\pi}{4}} \frac{1+\tan^2 x}{1+\tan x}dx,$$

we put: $u = \tan x \Longrightarrow x = arc\tan u \Longrightarrow dx = \frac{du}{1+u^2}$,

$$\text{We have : }\begin{cases} x = 0 \Longrightarrow u = 0 \\ x = \frac{\pi}{4} \Longrightarrow u = 1 \end{cases}$$

$$I_1 = \int_0^1 \frac{1}{1+u}du = [\ln|1+u|]_0^1 = \ln(2).$$

2) We calculate I_2 :

$$I_2 = \int_0^{\pi} \cos^2 x \sin^5 x dx,$$

we put: $u = \cos x \Longrightarrow du = -\sin x dx,$

$$\text{We have : } \begin{cases} x = 0 \Longrightarrow u = 1 \\ x = \pi \Longrightarrow u = -1 \end{cases}$$

$$\begin{aligned} I_2 &= \int_0^{\pi} \cos^2 x \sin^5 x dx = \int_0^{\pi} \cos^2 x \sin^4 x \sin x dx \\ &= \int_{-1}^{1} u^2(1-u^2)(-du) = -\int_{-1}^{1}(u^6 - 2u^4 + u^2)du = 2(\frac{1}{7} - \frac{2}{5} + \frac{11}{3}) \end{aligned}$$

3) We calculate I_3 :

$$I_3 = \int_0^{\frac{\pi}{4}} \frac{x \sin x^2}{1 + \cos x^2} dx,$$

we put: $u = \cos(x^2) \Longrightarrow du = -2x \sin(x^2) dx,$

$$\text{We have : } \begin{cases} x = 0 \Longrightarrow u = 1 \\ x = \sqrt{\frac{\pi}{4}} \Longrightarrow u = 1/2 \end{cases}$$

$$I_3 = \int_0^{\frac{\pi}{4}} \frac{x \sin x^2}{1 + \cos x^2} dx = \int_{-1}^{1/2} \frac{-du}{2(1+u)} = 1/2 \ln(4/3).$$

4) We calculate I_4 :

$$\begin{aligned} I_4 &= \int_0^1 (\frac{1}{(1+x)^2} + \frac{1}{(1+2x)}) dx = \left[\frac{-1}{(x+1)} + 1/2 \ln(2x+1)\right]_0^1 \\ &= 1/2(1 + \ln 3). \end{aligned}$$

5) We calculate I_5 :

$$I_5 = \int_0^1 (x-1)\exp(x-1)^2 dx,$$

we put: $u = (x-1)^2 \Longrightarrow du = -2(x-1)dx,$

$$\text{We have : } \begin{cases} x = 0 \Longrightarrow u = 1 \\ x = 1 \Longrightarrow u = 0 \end{cases}$$

$$I_5 = \int_0^1 (x-1)\exp(x-1)^2 dx = \int_1^0 \frac{\exp u}{2} du = \left[\frac{1}{2}\exp u\right]_1^0 = \frac{1}{2}(1-e)$$

6) We calculate I_6 :

$$I_6 = \int_{\ln 2}^{\ln 3} \frac{e^x + e^{-x}}{e^x - e^{-x}} dx,$$

we put: $u = \exp x - \exp(-x) \Longrightarrow du = -\exp x + \exp(-x)dx,$

$$\begin{aligned} I_6 &= \int_{\ln 2}^{\ln 3} \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = \int_{\ln 2}^{\ln 3} \frac{du}{u} = [\ln u]_{\ln 2}^{\ln 3} \\ &= \ln\left[\exp(\ln 3) - \exp-(\ln 3)\right] - \ln\left[\exp(\ln 2) - \exp-(\ln 2)\right] \\ &= \ln|3 - 1/3| - \ln|2 - 1/2| = \ln\left|\frac{8}{3}\right| - \ln\left|\frac{3}{2}\right| = \ln\left|\frac{16}{8}\right|. \end{aligned}$$

Exercise 3:

Calculate the following primitives

1) $F(x) = \int \frac{4x-5}{x^2-1} dx,$ **2)** $G(x) = \int \cos^2 x \sin^2 x dx,$

3)$H(x) = \int \sin^4 x dx,$ **4)**$L(x) = \int \tan^2 x dx.$

Solution:

1) We have

$$F(x) = \int \frac{4x-5}{x^2-1} dx = 2\int \frac{2x}{x^2-1} dx - \frac{5}{x^2-1} dx,$$

We have: $\frac{5}{x^2-1} = \frac{5}{(x-1)(x+1)} = \frac{a}{x-1} + \frac{b}{x+1} \Longrightarrow a = 2/5, b = -2/5.$

$$\begin{aligned} F(x) = \int \frac{4x-5}{x^2-1}dx &= 2\int \frac{2x}{x^2-1}dx + \frac{2}{5}\int \frac{dx}{(x-1)} - \frac{2}{5}\int \frac{dx}{(x+1)} \\ &= 2\ln\left|x^2-1\right| + \frac{2}{5}\ln\left|\frac{x-1}{x+1}\right| + C. \end{aligned}$$

2) We have

$$G(x) = \int \cos^2 x \sin^2 x dx = \int \cos^p x \sin^q x dx,$$

p and q are even, so we linearize. We have: $\sin^2 x = (1-\cos^2 x)$.

Then:

$$\int \cos^2 x \sin^2 x dx = \int (\cos^2 x - \cos^4 x)dx = \int \cos^2 x dx - \int (\cos^2 x)^2 dx.$$

We have:

$$\cos^2 x = \frac{1}{2}(1+\cos 2x),$$

Then:

$$\begin{aligned} (\cos^2 x)^2 &= \frac{1}{4}(1+\cos 2x)^2 = \frac{1}{4}(1+2\cos 2x + (\cos 2x)^2) \\ &= \frac{1}{4}(1+2\cos 2x) + \frac{1}{2}(1+\cos 4x). \end{aligned}$$

So we obtain:

$$\begin{aligned} G(x) &= \frac{1}{2}\int (1+2\cos 2x)dx - \frac{1}{4}\int (1+2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x)dx \\ &= \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x dx - \frac{1}{4}\int \frac{3}{2}dx - \frac{1}{2}\int \cos 2x dx - \frac{1}{8}\int \cos 4x dx \\ &= \frac{1}{2}\int dx - \frac{3}{8}\int dx - \frac{1}{32}\sin 4x + c - \frac{1}{8}\int \cos 4x dx \\ &= \frac{1}{8}\int dx - \frac{1}{8}\int \cos 4x dx \\ &= \frac{1}{8}x - \frac{1}{32}\sin 4x + c. \end{aligned}$$

3) We have

$$H(x) = \int \sin^4 x dx$$

We use the same idea like in 2), we linearize $\sin^4 x$, we get:

$$\begin{aligned} H(x) &= \int \sin^4 x dx = \frac{1}{8}\int \cos 4x dx - \frac{1}{2}\int \cos 2x dx + \frac{3}{8}\int dx \\ &= \frac{1}{32}\sin 4x - \frac{1}{4}\sin 2x + \frac{3}{8}x + c. \end{aligned}$$

4) We have

$$\begin{aligned} L(x) &= \int \tan^2 x dx = \int (\tan^2 x + 1 - 1)dx = \int (\tan^2 x + 1)dx - \int dx \\ &= \tan x - x + c. \end{aligned}$$

Exercise 4:

Calculate the following integrals using integration by part (IBP)

$$J_1 = \int_0^1 xe^x dx, \quad J_2 = \int_0^1 \frac{\ln x}{\sqrt[3]{x^2}}dx, \quad J_3 = \int_0^x \frac{t}{\sqrt{(1+t}}dt.$$

Solution:

1) We calculate J_1 :

$$J_1 = \int_0^1 xe^x dx,$$

$$\textbf{(IBP):} \int uv'dx = uv - \int u'vdx$$

We put:

$$\begin{cases} u = x \Longrightarrow u' = 1 \\ v' = \exp x \Longrightarrow v = \exp x \end{cases}$$

Then:

$$\begin{aligned} J_1 &= \int_0^1 xe^x dx = [x\exp x]_0^1 - \int_0^1 \exp x dx \\ &= [x\exp x]_0^1 - [\exp x]_0^1 \\ &= e - (e-1) = 1. \end{aligned}$$

2) We calculate $_2$:

$$J_2 = \int \frac{\ln x}{\sqrt[3]{x^2}} dx = \int x^{-2/3} \ln x dx$$

We put:

$$\begin{cases} u = \ln x \Longrightarrow u' = 1/x \\ v' = x^{-2/3} \Longrightarrow v = x^{1/3} \end{cases}$$

Then:

$$\begin{aligned} J_2 &= \int \frac{\ln x}{\sqrt[3]{x^2}} dx = \left[3x^{1/3} \ln x\right] - 3\int x^{-2/3} dx \\ &= 3x^{1/3} \ln x - 9x^{1/3} + c. \end{aligned}$$

$$J_3 = \int_0^x \frac{t}{\sqrt{(1+t}} dt = \int_0^x t(1+t)^{-1/2} dt$$

We put:

$$\begin{cases} u' = (1+t)^{-1/2} \Longrightarrow u = 2(1+t)^{1/2} \\ v = t \Longrightarrow v' = 1 \end{cases}$$

Then:

$$\begin{aligned} J_3 = \int_0^x t(1+t)^{-1/2} dt &= \left[2t(1+t)^{1/2}\right]_0^x - 2\int_0^x (1+t)^{1/2} dt \\ &= 2x(1+x)^{1/2} - 3\left[(1+t)^{3/2}\right]_0^x \\ &= 2x(1+x)^{1/2} - 3(1+x)^{3/2} - 3. \end{aligned}$$

Exercise 5:

We pose:

$$I = \int_0^{\ln 16} \frac{e^x + 3}{e^x + 4} dx, \qquad J = \int_0^{\ln 16} \frac{1}{e^x + 4} dx.$$

1) Calculate :$I + J$, and $I - 3J$.

2) Deduce the values of I and J.

Solution:

1) We have

$$\begin{aligned} I+J &= \int_0^{\ln 16} (\frac{e^x+3}{e^x+4} + \frac{1}{e^x+4})dx \\ &= \int_0^{\ln 16} (\frac{e^x+4}{e^x+4})dx = [x]_0^{\ln 16} = 4\ln 2. \end{aligned}$$

$$\begin{aligned} I-3J &= \int_0^{\ln 16} \frac{e^x+3}{e^x+4}dx - 3\int_0^{\ln 16} \frac{1}{e^x+4}dx \\ &= \int_0^{\ln 16} \frac{e^x}{e^x+4}dx = [\ln(e^x+4)]_0^{\ln 16} \\ &= \ln\left[e^{16}+4\right] - \ln 5 = \ln 20 - \ln 5 = 2\ln 2. \end{aligned}$$

2) Deduce the values of I and J. We have:

$$I+J = 4\ln 2 \quad (1)$$

$$I-3J = 2\ln 2 \quad (2)$$

So, (1)-(2)$\Longrightarrow 4J = 2\ln 2 \Longrightarrow J = \frac{1}{2}\ln 2$

Then: $I = \frac{7}{2}\ln 2$.

Chapter 3

Ordinary Differential Equations

3.1 Introduction

In the study of many physical problems, we are led to search for an unknown function that is the solution to an equation relating this function to its successive derivatives, this relation is called a "**Differential Equation**". Moreover, differential equations model a wide range of real world phenomena, from population growth, chemical reactions to the motion of objects, the behavior of physical systems, heat transfer, fluid flow, electrical circuits and much more.

In this chapter we will learn how to solve different types of differential equations.

3.2 Ordinary differential equations of order n

Definition 3.2.1. *Let be the application $F : \mathbb{R}^{n+2} \to \mathbb{R}(n \in \mathbb{N}^*)$.*

*1- An ordinary differential equation **(ODE)** of order n is an equation of the form:*

$$F(x, y(x), y'(x), ..., y^{(n)}(x) = 0, \tag{3.1}$$

where y is a function of variable x $(y = y(x))$ and $(y', y", ..., y^{(n)})$ are the successive derivatives of y until the order n $(y' = \frac{dy}{dx}, y" = \frac{d^2y}{dx^2}, ...)$.

2- The highest order of derivation in equation 3.1 determines the order of the differential equation.

3- An application $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$, is a solution of the equation (3.1) if φ admits derivatives up to order n on I and whose successive derivatives $(\varphi', \varphi", ..., \varphi^{(n)})$ verify, for *all $x \in I$, the equation (3.1) is verified, i.e.,*

$$F(x, \varphi(x), \varphi'(x), ..., \varphi^{(n)}(x)) = 0.$$

Examples:

1) The differential equation $y' = e^x(y' - e^x) = 0$, is an ODE of order 1.

2) The differential equation $y" + y' = \ln(x)$, is an ODE of order 2.

3) The differential equation $x^2y^{(4)} - \sin(x)y" + 3y' - \cos x = 0$, is an ODE of order 4.

3.3 First order ordinary differential equations

Definition 3.3.1. *Let be the application $F : \mathbb{R}^3 \to \mathbb{R}$.*

1- An ordinary differential equation of order 1 is all equation of the form:

$$F(x, y, y') = 0 \Longleftrightarrow y' = f(x, y), \text{ where } f : \mathbb{R}^2 \longrightarrow \mathbb{R},\ y = y(x) \text{ and } y' = \frac{dy}{dx}.$$

2- Let be the function $\varphi. : I \subseteq \mathbb{R} \to \mathbb{R}$

We say that φ is a solution of the equation $y' = f(x, y)$, if φ is differentiable and verifies:

$$\varphi'(x) = f(x, \varphi(x), \forall x \in I.$$

*3- We call a **Cauchy Problem,** all problem of the form:*

$$(P) \begin{cases} y' = f(x, y), \forall x \in I \\ y(x_0) = y_0 \end{cases}.$$

*$y(x_0) = y_0$, is called **initial condition**, which leads to a particular solution.*

Examples:

1) Let be the first order ODE

$$y' = 1 + e^x = f(x, y). \tag{3.2}$$

We have $y(x) = x + e^x + c\ /c \in \mathbb{R}$, is a solution of 3.2 (general solution of 3.2).

2) Let be the Cauchy problem:

$$(P)\begin{cases} y' = 1 + e^x \\ y(0) = 3 \end{cases}. \tag{3.3}$$

From the example (1), we have $y(x) = x + e^x + c$, is a general solution of 3.3.

We have $y(0) = 3 \Longrightarrow 0 + e^0 + c = 3 \Longrightarrow c = 2$.

So, the particular solution of (P) is

$$y(x) = x + e^x + 2.$$

In the following, we will see the different types of the first order ODE.

3.4 Ordinary differential equation with separate variables

Definition 3.4.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ and $g : I \subseteq \mathbb{R} \to \mathbb{R}$, be two continuous functions. A differential equation with* ***separate variables****, is an equation of the type:*

$$y' = \frac{f(x)}{g(y)} \Longleftrightarrow y'g(y) = f(x) \Longleftrightarrow g(y)dy = f(x)dx, \tag{3.4}$$

where $y' = \frac{dy}{dx}$.

To solve this equation, you just have to integrate both sides of the equation 3.4, we obtain:

$$\int g(y)dy = \int f(x)dx \Longrightarrow G(y) = F(x) + c \ /c \in \mathbb{R}.$$

Where G is a primitive of g and F is a primitive of f .

Remark 3.4.2. *Finding a primitive is already solving the differential equation $y' = f(x)$. This is why we often say "Integrate the differential equation" instead of "Find the solutions of the differential equation".*

Examples:

1) Integrate the differential equation:

$$y' = y. \tag{3.5}$$

We have:

$$\begin{aligned} y' &= y \Longleftrightarrow \frac{dy}{dx} = y \Longleftrightarrow \frac{dy}{y} = dx \ \ (y' = \frac{f(x)}{g(y)}/f(x) = 1, g(y) = \frac{1}{y}) \\ &\Longrightarrow \int \frac{dy}{y} = \int dx \Longrightarrow Ln\,|y| = x + c_1 \\ &\Longrightarrow |y| = e^{x+c_1} = e^{c_1}e^x = c_2e^x \ (c_2 = e^{c_1}) \\ &\Longrightarrow y = \pm c_2 e^x \ \ (c_2 > 0) \\ &\Longrightarrow y = ce^x/c \in \mathbb{R}. \end{aligned}$$

So, $y(x) = ce^x/c \in \mathbb{R}$, is a solution of equation 3.5.

2) Find the solution of the following differential equation:

$$y' = (1+x^2)y^2 \ \ (f(x) = 1 + x^2/g(y) = \frac{1}{y^2}).$$

We have:

$$\begin{aligned} y' &= (1+x^2)y^2 \Longleftrightarrow \frac{dy}{dx} = (1+x^2)y^2 \Longleftrightarrow \frac{dy}{y^2} = (1+x^2)dx \\ &\Longrightarrow \int \frac{dy}{y^2} = \int (1+x^2)dx \Longrightarrow -\frac{1}{y} = \frac{x^3}{3} + x + c \ \ / \ \ c \in \mathbb{R} \\ &\Longrightarrow y(x) = \frac{-3}{x^3+3x+k} \ \ / \ \ k \in \mathbb{R}. \end{aligned}$$

3) Solve the following Cauchy problem:

$$(P) \begin{cases} y' = \frac{e^x+1}{y} \\ y(0) = 1 \end{cases}.$$

First, we integrate the differential equation $y' = \frac{e^x+1}{y}$.

We have

$$\begin{aligned} y' &= \frac{e^x+1}{y} \Longleftrightarrow \frac{dy}{dx} = \frac{e^x+1}{y} \Longleftrightarrow ydy = (e^x+1)dx \\ &\Longrightarrow \int ydy = \int (e^x+1)dx \Longrightarrow -\frac{y^2}{2} = e^x + x + c \ \ / \ \ c \in \mathbb{R} \\ &\Longrightarrow y(x) = \sqrt{2(e^x+x+c)} \ \ / \ \ c \in \mathbb{R}. \end{aligned}$$

Now, we look for the particular solution, i.e., we determine the value of c.

We have:

$$\begin{aligned} y(0) &= 1 \Longrightarrow \sqrt{2(e^0+0+c)} = \sqrt{2(1+C)} = 1 \\ &\Longrightarrow 2+2c = 1 \Longrightarrow 2c = -1 \Longrightarrow c = \frac{-1}{2} \end{aligned}$$

So, the particular solution of (P) is:

$$y(x) = \sqrt{2(e^x + x - \frac{1}{2})} = \sqrt{2e^x + 2x - 1}$$

4) Integrate the following ODE:

$$y'\sqrt{1-x^2} = \sqrt{1-y^2}.$$

We have

$$\begin{aligned} y' &= \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} \Longleftrightarrow \frac{dy}{\sqrt{1-y^2}} = \frac{dx}{\sqrt{1-x^2}} \\ &\Longrightarrow \int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{\sqrt{1-x^2}} \Longrightarrow \arcsin y = \arcsin x + c \quad / \quad c \in \mathbb{R}. \\ &\Longrightarrow y = \sin(\arcsin x + c) \quad /(\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)) \\ &= \sin(\arcsin x)\cos(c) + \sin(c)\cos(\arcsin x). \\ &\quad / \quad \left[\sin(\arcsin x) = x \quad / \quad \cos(\arcsin x) = \sqrt{1-x^2}\right]. \\ &\Longrightarrow y(x) = \cos(x)x + \sin(c)\sqrt{1-x^2} \qquad / \quad c \in \mathbb{R}. \end{aligned}$$

N.B: To show that

$$\cos(\arcsin x) = \sqrt{1-x^2} \quad /x \in [-1,1].$$

We take

$$\theta = \arcsin x \Longrightarrow x = \sin\theta \quad /x \in [-1,1], \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right].$$

In other hand, we have

$$\begin{aligned}\sin^2(\theta)+\cos^2(\theta) &= 1\\ &\Longrightarrow \cos^2(\theta)=1-\sin^2(\theta)=1-x^2 \qquad (\sin\theta = x)\\ &\Longrightarrow \cos(\theta)=\sqrt{1-x^2} \qquad (\theta=\arcsin x)\\ &\Longrightarrow \cos(\arcsin x)=\sqrt{1-x^2} \quad / \quad x\in[-1,1].\end{aligned}$$

5) Integrate the ODE:

$$y'(x^2-1)-2xy=0.$$

We have

$$\begin{aligned}y'(x^2-1)-2xy &= 0 \Longleftrightarrow y'=\frac{2xy}{x^2-1}\Longleftrightarrow \frac{dy}{y}=\frac{2x}{x^2-1}dx\\ &\Longrightarrow \int\frac{dy}{y}=\int\frac{2x}{x^2-1}dx\\ &\Longrightarrow \ln|y|=\ln\left|x^2-1\right|+c \qquad / \quad c\in\mathbb{R}\\ &= \ln\left|x^2-1\right|+\ln k \qquad / \quad c=\ln k\\ &= \ln(k\left|x^2-1\right|)\\ &\Longrightarrow y=k(x^2-1) \quad / \; k\in\mathbb{R}.\end{aligned}$$

Or

$$\begin{aligned}\ln|y| &= \ln\left|x^2-1\right|+c\Longrightarrow |y|=e^{\ln\left|x^2-1\right|+c}\\ &\Longrightarrow y=k(x^2-1) \qquad /k=\pm e^c\end{aligned}$$

6) Find a particular solution of the following Cauchy problem:

$$(P)\begin{cases} yy'(1+e^x)=e^x\\ y(0)=1\end{cases}.$$

First, we solve the equation

$$\begin{aligned} yy'(1+e^x) &= e^x \\ &\Longrightarrow y\frac{dy}{dx} = \frac{e^x}{1+e^x} \Longrightarrow ydy = \frac{e^x}{1+e^x}dx \\ &\Longrightarrow \int ydy = \int \frac{e^x}{1+e^x}dx \Longrightarrow \frac{y^2}{2} = \ln(1+e^x)+c \quad / \; c \in \mathbb{R} \\ &\Longrightarrow y(x) = \sqrt{2\ln(1+e^x)+k} \quad / \; k = 2c \in \mathbb{R}. \end{aligned}$$

Now, we determine the value of the constant k using the initial condition $y(0) = 1$. We have

$$\begin{aligned} y(0) &= 1 \Longrightarrow \sqrt{2\ln(1+e^0)+k} = \sqrt{2\ln(2)+k} = 1 \\ &\Longrightarrow 2\ln(2)+k = 1 \Longrightarrow k = 1-\ln(4) \quad / \; (2\ln 2 = \ln 2^2 = \ln 4). \end{aligned}$$

So, the solution of (P) is

$$y(x) = \sqrt{2\ln(1+e^x)+1-\ln 4}.$$

3.5 Homogeneous ordinary differential equations

Definition 3.5.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$, be a continuous function. We call homogeneous differential equation in x and y all equation of the form:*

$$y' = f\left(\frac{y}{x}\right) \Longleftrightarrow dy = f\left(\frac{y}{x}\right) dx \tag{3.6}$$

To solve this equation we use the change variables

$$t = \frac{y}{x} \Longrightarrow y = tx \Longrightarrow dy = tdx + xdt$$

Then, by replacing this in equation 3.6, we obtain a differential equation with separate variables of x and t.

Indeed,we have:

$$\begin{aligned} dy &= f(\frac{y}{x})dx \Longleftrightarrow tdx + xdt = f(t)dx \\ &\Longleftrightarrow xdt = (f(t) - t)dx \\ &\Longleftrightarrow \frac{dt}{f(t) - t} = \frac{dx}{x} \Longrightarrow \int \frac{dt}{f(t) - t} = \int \frac{dx}{x} \end{aligned}$$

So, to solve homogeneous ODE, we are led to solve an ODE with separate variables, as following:

$$y' = f(\frac{y}{x}) \Longleftrightarrow \frac{dt}{f(t) - t} = \frac{dx}{x} \text{ where } t = \frac{y}{x}$$

Examples:

1) Integrate the ODE:

$$(2x + y)dx - (4x - y)dy = 0 \tag{3.7}$$

$$3.7 \Longleftrightarrow dy = \frac{2x + y}{4x - y}dx = \frac{x(2 + y/x)}{x(4 - y/x)}dx$$

$$\Longleftrightarrow dy = \frac{2 + y/x}{4 - y/x}dx = f(y/x)dx \tag{3.8}$$

We take:

$$\begin{aligned} t &= y/x \Longrightarrow y = tx \Longrightarrow dy = tdx + xdt \\ 3.8 &\Longleftrightarrow tdx + xdt = f(t)dx \quad /f(t) = \frac{2 + t}{4 - t} \\ &\Longrightarrow xdt = (f(t) - t)dx = (\frac{2 + t}{4 - t} - t)dx = (\frac{t^2 - 3t + 2}{4 - t})dx \\ &\Longrightarrow \frac{dt}{\frac{t^2 - 3t + 2}{4 - t}} = \frac{dx}{x} \Longrightarrow \frac{4 - t}{t^2 - 3t + 2}dt = \frac{1}{x}dx \\ &\Longrightarrow \int \frac{4 - t}{t^2 - 3t + 2}dt = \int \frac{1}{x}dx \end{aligned}$$

We decompose the rational function:

$$\frac{4 - t}{t^2 - 3t + 2} = \frac{4 - t}{(t - 1)(t - 2)} = \frac{a}{t - 1} + \frac{b}{t - 2} = \frac{3}{t - 1} + \frac{-2}{t - 2}$$

So,

$$\begin{aligned}
\int \frac{4-t}{t^2-3t+2}dt &= \int \frac{1}{x}dx \Longrightarrow 3\int \frac{dt}{t-1} - 2\int \frac{dt}{t-2} = \int \frac{dx}{x} \\
&\Longrightarrow 3\ln|t-1| - 2\ln|t-2| = \ln|x| + c \quad /c \in \mathbb{R} \\
&\Longrightarrow \ln\left(\frac{|t-1|^3}{|t-2|^2}\right) = \ln|x| + c \quad /(t = y/x) \\
&\Longrightarrow x = k\left(\frac{|y/x-1|^3}{|y/x-2|^2}\right) \qquad /k \in \mathbb{R}.
\end{aligned}$$

2) Solve the ODE:

$$xy' = xe^{y/x} + y$$

$$\Longrightarrow y' = \frac{x}{x}e^{x/y} + \frac{y}{x} = e^{x/y} + \frac{y}{x} = f(\frac{y}{x}) \tag{3.9}$$

Put

$$t = \frac{y}{x} \Longleftrightarrow y = tx \Longrightarrow dy = tdx + xdt \text{ and } f(t) = e^t + t.$$

So,

$$\begin{aligned}
3.9 \quad &\Longleftrightarrow \quad \frac{dt}{f(t)-t} = \frac{dx}{x} \Longrightarrow \frac{dt}{e^t+t-t} = \frac{dx}{x} \Longrightarrow \int \frac{dt}{e^t} = \int \frac{dx}{x} \\
&\Longrightarrow \quad -e^{-t} = \ln|x| + c \Longrightarrow e^{-t} = -\ln|x| - c = \ln(\frac{k}{|x|}) \quad /k \in \mathbb{R}_+^* \\
&\Longrightarrow \quad -t = \ln(\ln(\frac{k}{|x|})), t = \frac{y}{x} \\
&\Longrightarrow \quad t = \frac{y}{x} = -\ln(\ln(\frac{k}{|x|})) \\
&\Longrightarrow \quad y = -x\ln(\ln(\frac{k}{|x|}))
\end{aligned}$$

3) Find the solutions of the ODE:

$$xy' - y + x = 0$$

$$\Longrightarrow y' = \frac{y}{x} - 1 = f(\frac{y}{x}). \tag{3.10}$$

We take
$$t = \frac{y}{x} \Longleftrightarrow y = tx \Longrightarrow dy = tdx + xdt \text{ and } f(t) = t - 1.$$

So,
$$\begin{aligned} 3.10 \Longleftrightarrow & \frac{dt}{f(t)-t} = \frac{dx}{x} \Longleftrightarrow \frac{dt}{t-1\_t} = \frac{dx}{x} \Longleftrightarrow -dt = \frac{dx}{x} \\ \Longrightarrow & \int dt = -\int \frac{dx}{x} \Longrightarrow t = -\ln|x| + c \Longrightarrow t = \frac{y}{x} = -\ln|x| + c \\ \Longrightarrow & y = -x(\ln|x| + c) \quad / \, c \in \mathbb{R}. \end{aligned}$$

3.6 First order linear ordinary differential equations

Definition 3.6.1. *We call a first order linear differential equation,all differential equation of the form:*
$$a(x)y' + b(x)y = c(x) \tag{3.11}$$

Where, $a(x), b(x)$ and $c(x)$ are continuous function on $I \subseteq \mathbb{R}$,with $a(x) \neq 0$.

$c(x)$ is called the second member of the ODE 3.11.

The differential equation
$$a(x)y' + b(x)y = 0 \tag{3.12}$$

(without second member) is called the homogeneous equation associated to the equation 3.11.

Remark 3.6.2. ***1-*** *All solution of 3.11.will be written as: $y = y_1 + y_2$, where y_1 is the general solution of the homogeneous equation (4) $(a(x)y' + b(x)y = 0)$, and y_2 is a particular solution of the equation (3), i.e., solution of $a(x)y' + b(x)y = c(x)$).*

2-*All first order linear ODE can be written as $y' = \alpha(x)y + \beta(x)$, where $\alpha(x) = \frac{-b(x)}{a(x)}$, $\beta(x) = \frac{c(x)}{a(x)}$ with $a(x) \neq 0$.*

Resolution method

1- Resolution of the associated homogeneous equation

$$a(x)y' + b(x)y = 0,$$

to find y_1.

We notice that $y = 0$ is a solution of 3.12,that we call a "**trivial solution".** So, we look for a solution $y \neq 0$.

We have:

$$\begin{aligned} a(x)y' + b(x)y &= 0 \Longrightarrow a(x)\frac{dy}{dx} = -b(x)y \\ &\Longrightarrow \frac{dy}{y} = \frac{-b(x)}{a(x)} \\ &\Longrightarrow \int \frac{dy}{y} = \int \frac{-b(x)}{a(x)} \\ &\Longrightarrow \ln|y| = F(x) + k_1 \quad / F(x) = \int \frac{-b(x)}{a(x)} dx, k_1 \in \mathbb{R} \\ &\Longrightarrow y = e^{F(x)+k_1} = e^{F(x)}e^{k_1} = Ke^{F(x)} \end{aligned}$$

So,

$$y_1 = Ke^{F(x)} \quad / \ K \in \mathbb{R}$$

y_1 is a general solution of the homogeneous equation 3.12.

2- Find a particular solution y_2 of the equation 3.11

$$a(x)y' + b(x)y = c(x)$$

If the particular solution y_2 of 3.11 is clear or trivial, we can give directly (without calculation) the expression of the general solution: $y = y_1 + y_2$ (general solution of 3.11).

If the particular solution y_2 of 3.11 is not clear (trivial), then to calculate it, we use the **constant variation method**, i.e., we consider the constant c as a variable by taking $K = K(x)$.

We have:

$$\begin{aligned} y &= Ke^{F(x)} = K(x)e^{F(x)} \\ &\Longrightarrow y' = K'(x)e^{F(x)} + F'(x)K(x)e^{F(x)} \quad / \ F'(x) = \frac{-b(x)}{a(x)}. \end{aligned}$$

We substitute in equation 3.11,we obtain

$$a(x)(K'(x)e^{F(x)} + F'(x)K(x)e^{F(x)}) + b(x)(K(x)e^{F(x)}) = c(x)$$

$$\begin{aligned}
&\Longrightarrow \quad a(x)K'(x)e^{F(x)} - a(x)\frac{b(x)}{a(x)}K(x)e^{F(x)} + b(x)K(x)e^{F(x)} = c(x)\\
&\Longrightarrow \quad a(x)K'(x)e^{F(x)} = c(x)\\
&\Longrightarrow \quad K'(x) = \frac{c(x)}{a(x)}e^{-F(x)} \Longrightarrow K(x) = \int \frac{c(x)}{a(x)}e^{-F(x)}dx.
\end{aligned}$$

Finally,we get the general solution y of 3.11 as follows:

$$y = K(x)e^{F(x)} = y_1 + y_2$$

Where, $F(x) = \int \frac{-b(x)}{a(x)}dx$, $K(x) = \int \frac{c(x)}{a(x)}e^{-F(x)}dx$.

Examples:

1) Solve the ODE given by:

$$(1 - x^2)y' - xy = 1 \Longleftrightarrow a(x)y' + b(x)y = c(x) \quad /a(x) = 1 + x^2, b(x) = -x, c(x) = 1. \tag{3.13}$$

We have equation 3.13 is a first order linear ODE.

First, we solve the homogeneous equation :

$$(1 + x^2)y' - xy = 0. \tag{3.14}$$

$$\begin{aligned}
(6) \quad &\Longleftrightarrow \quad (1 - x^2)\frac{dy}{dx} = xy \Longrightarrow \frac{dy}{y} = \frac{x}{1 + x^2}dx\\
&\Longrightarrow \quad \int \frac{dy}{y} = \frac{1}{2}\int \frac{2x}{1 + x^2}dx\\
&\Longrightarrow \quad \ln|y| = \frac{1}{2}\ln(1 + x^2) + k_1 = \ln\left(\sqrt{1 + x^2}\right) + k_1 \quad / \; k_1 \in \mathbb{R}\\
&\Longrightarrow \quad y_1 = K\sqrt{1 + x^2} \quad / \; K = e^{k_1} \in \mathbb{R} \text{ (solution of 3.14).}
\end{aligned}$$

We notice that $y_2 = x$ is a particular solution of the equation 3.13.

$$(1 + x^2)y_2' - xy_2 = (1 + x^2).1 - xx = 1 + x^2 - x^2 = 1$$

So, the general solution of equation 3.13 is

$$y = y_1 + y_2 = K\sqrt{1+x^2} + x \quad /K \in \mathbb{R}.$$

2) Integrate the following ODE

$$y' + 2xy = 2xe^{-x^{2}.} \tag{3.15}$$

$$(7) \Longleftrightarrow a(x)y' + b(x)y = c(x) \quad / \; a(x) = 1, b(x) = 2x, c(x) = 2xe^{-x^2}.$$

The equation 3.15 is a first order linear ODE.

First, we solve the homogeneous equation associated to 3.15, given by

$$y' + 2xy = 0. \tag{3.16}$$

We have:

$$\begin{aligned} y' + 2xy &= 0 \Longrightarrow \frac{dy}{dx} = -2xy \Longrightarrow \frac{dy}{y} = -2xdx \\ &\Longrightarrow \ln|y| = -x^2 + k \Longrightarrow |y| = e^{-x^2+k} = e^k e^{-x^2}, \text{ put } c = \pm e^k. \end{aligned}$$

So, the general solution of 3.16 is

$$y_1 = ce^{-x^2} \quad / \; c \in \mathbb{R}.$$

In the second time, we will find a particular solution y_2 of equation 3.15,using the constant variation method, i.e., we put $c = c(x)$.

We have:

$$\begin{aligned} y &= ce^{-x^2} = c(x)e^{-x^2} \\ &\Longrightarrow y' = c'(x)e^{-x^2} - 2xc(x)e^{-x^2} \end{aligned}$$

We substitute in equation 3.15, to obtain:

$$\begin{aligned} y' + 2xy &= 2xe^{-x^2} \\ &\Longrightarrow c'(x)e^{-x^2} - 2xc(x)e^{-x^2} + 2xc(x)e^{-x^2} = 2xe^{-x^2} \\ &\Longrightarrow c'(x)e^{-x^2} = 2xe^{-x^2} \\ &\Longrightarrow c'(x) = 2x \Longrightarrow c(x) = x^2 + K \quad /K \in \mathbb{R}. \end{aligned}$$

So,

$$\begin{aligned} y &= c(x)e^{-x^2} = (x^2 + k)e^{-x^2} \\ &\Longrightarrow y = Ke^{-x^2} + x^2e^{-x^2} = y_1 + y_2, \end{aligned}$$

where $y_1 = Ke^{-x^2}$, is a general solution of 3.16 (homogeneous equation) and $y_2 = x^2e^{-x^2}$is a particular solution of equation 3.15.

3) Solve the differential equation:

$$2y' - 5y = 0. \tag{3.17}$$

3.17 is a linear ODE of order 1, with $a(x) = 2$, $b(x) = 5$, $c(x) = 0$.

We notice that the second member of 3.17 is zero, so the general solution of equation 3.17 is $y = y_1$ (y_1 solution of homogeneous equation).

We have:

$$\begin{aligned} 2y' - 5y &= 0 \Leftrightarrow y' = \frac{5}{2}y \\ &\Longleftrightarrow \frac{dy}{dx} = \frac{5}{2}y \Longleftrightarrow \frac{dy}{y} = \frac{5}{2}dx \\ &\Longrightarrow \int \frac{dy}{y} = \frac{5}{2}\int dx \Longrightarrow \ln|y| = \frac{5}{2}x + c \\ &\Longrightarrow |y| = e^{\frac{5}{2}x+c} = e^c e^{\frac{5}{2}x}, \text{ put } K = \pm e^c \\ &\Longrightarrow y = Ke^{\frac{5}{2}x} \quad / K \in \mathbb{R} \end{aligned}$$

Proposition 3.6.3. ***(Superposition principle)***

Let be $a(x), b(x), c_1(x), c_2(x), ..., c_n(x)$ continuous applications on $I \subseteq \mathbb{R}$. If for all $k \in \{1, 2, 3, ..., n\}$ the function $\varphi_k(x)$ is a particular solution on I of the differential *equation:*

$$a(x)y' + b(x)y = c_k(x), \tag{3.18}$$

then, $\varphi(x) = \sum_{k=1}^{n} \varphi_k(x)$ *is a particular solution on* I *of the differential equation:*

$$a(x)y' + b(x)y = \sum_{k=1}^{n} c_k(x). \tag{3.19}$$

Example:

Solve the differential equation

$$2y' - 5y = 4 - 10x + e^{3x}. \tag{3.20}$$

The general solution $y = y_1 + y_2$, where y_1 is a solution of the homogeneous equation

$$2y' - 5y = 0. \tag{3.21}$$

$$\Longrightarrow y_1 = Ke^{\frac{5}{2}x} \quad / \; K \in \mathbb{R}.$$

The particular solution of 3.20 is $y_2 = y_{p1} + y_{p2}$,where:

y_{p1} is the particular solution of :

$$2y' - 5y = 4 - 10x. \tag{3.22}$$

It is clear that $y_{p1} = 2x$ is particular solution of 3.22.

To calculate y_{p2}, the particular solution of:

$$2y' - 5y = e^{3x}. \tag{3.23}$$

We use the variation constant method.

We substitute in 3.23, we get:

$$2K'(x)e^{\frac{5}{2}x} + 5K(x)e^{\frac{5}{2}x} - 5K(x)e^{\frac{5}{2}x} = e^{3x}$$

$$\Longrightarrow K'(x) = \frac{1}{2}e^{3x-\frac{5}{2}x} = \frac{1}{2}e^{\frac{1}{2}x} \Longrightarrow K(x) = e^{\frac{1}{2}x} + c \quad / \; c \in \mathbb{R}.$$

Then, the general solution of 3.23 is

$$y = K(x)e^{\frac{5}{2}x} = ce^{\frac{5}{2}x} + e^{3x} \quad / \ c \in \mathbb{R}.$$

So,

$$y = y_{p1} + y_{p2} = 2x + e^{3x} \text{ (is a particular solution of 3.20)}$$

Finally, the general solution of 3.20 is

$$y = y_{p1} + y_{p2} = ce^{\frac{5}{2}} + 2x + e^{3x} \quad / \ c \in \mathbb{R}.$$

3.7 Bernoulli differential equation

Definition 3.7.1. *The* ***Bernoulli*** *differential equation is an equation of the form:*

$$(BE) \qquad y' + a(x)y = b(x)y^{\alpha} \quad /\alpha \in \mathbb{R}\backslash\{0, 1\}$$

$*$ If $\alpha = 0$ or $\alpha = 1$, we obtain a linear differential equation.

- $\alpha \ = \ 0, \ (BE) \Longleftrightarrow y' + a(x)y = b(x) \quad (LE)$
- $\alpha \ = \ 0, \ (BE) \Longleftrightarrow y' + a(x)y = b(x)y \Longleftrightarrow y' + [a(x) - b(x)]y = 0 \quad (LE)$

$*$ If $\alpha \neq 0$ or $\alpha \neq 1$, then to solve (BE) we transform it to a linear differential equation as follows:

We have $y' + a(x)y = b(x)y^{\alpha} \quad /\alpha \neq 0$ and $\alpha \neq 1$.

We multiply both sides of the equation (BE) by $y^{-\alpha}$, so

$$\begin{aligned} (BE) \quad &\Longleftrightarrow \quad y^{-\alpha}(y' + a(x)y) = b(x)y^{\alpha}y^{-\alpha} \\ &\Longleftrightarrow \quad y^{-\alpha}y' + a(x)y^{1-\alpha} = b(x)..........(*) \end{aligned}$$

We make a change of variables by posing:

$z = y^{1-\alpha} \Longrightarrow z' = (1 - \alpha)y'y^{-\alpha} \Longrightarrow y^{-\alpha}y' = \frac{z'}{1-\alpha}$

We substitute in equation $(*)$, we obtain:

$$\frac{z^{'}}{1-\alpha}+a(x)z=b(x)\Longleftrightarrow\frac{1}{1-\alpha}y^{'}+a(x)z=b(x)..........(**)$$

The equation $(**)$ is a linear ODE of order 1.

Finally,

$$\begin{aligned}(BE)\quad &: \quad y^{'}+a(x)y=b(x)y^{\alpha}\ /\ \alpha\neq 0 \text{ and } \alpha\neq 1\\ &\Longleftrightarrow \frac{1}{1-\alpha}z^{'}+a(x)z=b(x)\qquad \text{(linear equation)}.\\ \text{Where } z &= y^{1-\alpha}.\end{aligned}$$

Example: Solve the differential equation:

$$(E)\quad y^{'}-\frac{4}{x}y=x\sqrt{y}\Longleftrightarrow y^{'}+a(x)y=b(x)y^{\alpha}.$$

So, (E) is a Bernoulli equation, where $\alpha=\frac{1}{2}$, $a(x)=-\frac{4}{x}$ and $b(x)=x$.

We multiply both sides of the equation (E) by $y^{-\alpha}=y^{-1/2}$

We have:

$$\begin{aligned}y^{'}-\frac{4}{x}y &= x\sqrt{y}\Longleftrightarrow y^{-1/2}(y^{'}-\frac{4}{x}y)=xy^{1/2}y^{-1/2}\\ &\Longleftrightarrow y^{-1/2}y^{'}-\frac{4}{x}y^{1/2}=x\qquad (*)\end{aligned}$$

We take the change of variables:

$$z=y^{1-\alpha}=y^{1-\frac{1}{2}}=y^{1/2}=\sqrt{y}\Longrightarrow z=\sqrt{y}.$$

We substitute in equation $(*)$, we obtain a linear ODE

$$\begin{aligned}\frac{1}{1-\alpha}z^{'}+a(x)z &= b(x)\Longleftrightarrow\frac{1}{1-\frac{1}{2}}z^{'}-\frac{4}{x}z=x\\ &\Longleftrightarrow 2z^{'}-\frac{4}{x}z=x\qquad (**)\end{aligned}$$

We solve the linear ODE $(**)$

1) First, we solve the homogeneous equation associated to $(**)$

$$\begin{aligned} 2z^{'} - \frac{4}{x}z &= 0 \Longrightarrow \frac{2dz}{dx} = \frac{4}{x}z \Longrightarrow \frac{dz}{z} = \frac{2dx}{x} \\ &\Longrightarrow \ln|z| = 2\ln|x| + k = \ln(x^2) + k \\ &\Longrightarrow |z| = e^{\ln x^2}e^k \Longrightarrow z_1 = cx^2 \ / \ c = \pm e^k \in \mathbb{R}. \end{aligned}$$

z_1 is a general solution of the homogeneous equation.

2) To calculate z_2, the particular solution of $(**)$, we use the constant variation method, i.e., we put $c = c(x)$.

So, $z = cx^2 = c(x)x^2 \Longrightarrow z^{'} = c^{'}(x)x^2 + 2xc(x)$.

We substitute in $(**)$ as follows

$$\begin{aligned} 2z^{'} - \frac{4}{x}z &= x \Longrightarrow 2c^{'}(x)x^2 + 4xc(x) - \frac{4}{x}c(x)x^2 = x \\ &\Longrightarrow 2c^{'}(x)x^2 = x \Longrightarrow c^{'}(x) = \frac{1}{2x} \\ &\Longrightarrow c(x) = \frac{1}{2}\ln|x| + k, \quad k \in \mathbb{R}. \end{aligned}$$

Then:

$$\begin{aligned} z &= c(x)x^2 = (\frac{1}{2}\ln|x| + k)x^2 \\ &\Longrightarrow z = z_1 + z_2 = kx^2 + x^2\ln\sqrt{|x|}. \end{aligned}$$

Finally, we have $z = \sqrt{y} \Longleftrightarrow y = z^2$.

So, the solution of (E) is

$$y = x^4(\ln\sqrt{|x|} + k)^2 \ / \ k \in \mathbb{R}$$

3.8 Riccati differential equation

Definition 3.8.1. *We call* ***Riccati*** *differential equation, all differential equation of the form:*

$$y^{'} + a(x)y + b(x)y^2 = f(x) \quad (RE)$$

* If $f(x) = 0$, we obtain the Bernoulli equation with $\alpha = 2$.

* If $f(x) \neq 0$, to solve (RE) we proceed as follows:

Resolution method

To solve the equation (RE), it is necessary to have a particular solution of (RE) **in advance**.

We suppose that there exists a particular solution y_0 of the equation (RE). So, y_0 verifies

$$y_0^{'} + a(x)y_0 + b(x)y_0^2 = f(x) \qquad (*)$$

By subtraction $(RE) - (*)$, we obtain

$$y^{'} + a(x)y + b(x)y^2 - y_0^{'} - a(x)y_0 - b(x)y_0^2 = f(x) - f(x)$$

$$\Longrightarrow \quad (y^{'} - y_0^{'}) + a(x)(y - y_0) + b(x)(y^2 - y_0^2) = 0. \qquad (**)$$

We make the change of variables: $z = y - y_0 \Longleftrightarrow y = z + y_0$

We substitute z in $(**)$, we get

$$z^{'} + a(x)z + b(x)(y - y_0)(y + y_0) = 0 \ / \ y = z + y_0 \Longrightarrow y + y_0 = z + 2y_0$$

$$\begin{aligned} &\Longleftrightarrow \quad z^{'} + a(x)z + b(x)z(z + 2y_0) = 0 \\ &\Longleftrightarrow \quad z^{'} + a(x)z + b(x)(z^2 + 2y_0 z) = 0 \\ &\Longleftrightarrow \quad z^{'} + (a(x) + 2y_0 b(x))z = -b(x)z^2 \qquad (BE) \end{aligned}$$

It is a **Bernoulli** equation for z with $\alpha = 2$.

Finally,

$$y^{'} + a(x)y + b(x)y^2 = f(x) \qquad (RE)$$

$$\Longleftrightarrow \begin{cases} z^{'} + (a(x) + 2y_0 b(x))z = -b(x)z^2 \qquad (BE)\ (\alpha = 2) \\ \text{where } z = y - y_0 \text{ and } y_0 \text{ particular solution of } (RE). \end{cases}$$

Example: Solve the following Riccati equation:

$$y^{'} - x^2 y + y^2 = (x + 1)^2 \qquad (RE)$$

We give $y_0 = x^2 + 1$ as a particular solution of (RE).

Indeed,

$$y_0^{'} - x^2 y_0 + y_0^2 = 2x - x^2(x^2+1) + (x^2+1)^2 = x^2 + 2x + 1 = (x+1)^2.$$

We have, $a(x) = -x^2$, $b(x) = 1$ and $f(x) = (x+1)^2$.

We put $z = y - y_0 = y - (x^2+1) \Longleftrightarrow y = z + x^2 + 1$.

So,

$$y^{'} - x^2 y + y^2 = (x+1)^2 \Longleftrightarrow z^{'} + (a(x) + 2y_0 b(x))z = -b(x)z^2$$

$$\Longleftrightarrow z^{'} + (-x^2 + 2(x^2+1))z = -z^2$$

$$\Longleftrightarrow z^{'} + (x^2+2)z = -z^2 \qquad (BE) \ \text{(Bernoulli equation with } \alpha = 2\text{)}$$

$$\Longleftrightarrow z^{-2}(z^{'} + (x^2+2)z) = -z^2 z^{-2} \ \ \text{(we multiply by } z^{-\alpha} = z^{-2}\text{)}$$

$$\Longleftrightarrow z^{-2}z^{'} + (x^2+2)z^{-1} = -1 \qquad (*)$$

We put $t = z^{1-\alpha} = z^{1-2} = z^{-1} \Longrightarrow t^{'} = \frac{-1}{z^2}z^{'} = -z^{-2}z^{'}$

So,

$$\begin{aligned} (*) \quad &\Longleftrightarrow \quad -t^{'} + (x^2+2)t = -1 \\ &\Longleftrightarrow \quad t^{'} - (x^2+2)t = 1 \qquad (LE) \ \text{(Linear ODE of order 1)} \end{aligned}$$

We solve the linear differential equation (LE), we determine the solution $t = t(x)$, then we get $z = \frac{1}{t}$ and finally we obtain $y = z + y_0$.

3.9 Second order linear differential equation with constant coefficients

Definition 3.9.1. *A second order linear differential equation with constant coefficients, is all equation of the form:*

$$ay^{''} + by^{'} + cy = f(x) \qquad (E)$$

Where $a, b, c \in \mathbb{R}$ and $a \neq 0$ and f is continuous on an interval $I \subseteq \mathbb{R}$.

∗ *The associated homogeneous equation is:*

$$ay'' + by' + cy = 0 \qquad (HE)$$

∗ *The general solution of (E) is of the form:*

$$y = y_H + y_P$$

where, y_H is the solution of the homogeneous equation (HE) and y_P is a particular solution of (E).

Resolution method

1) **Solving the associated homogeneous equation**

We have the associated homogeneous equation to (E) given by:

$$ay'' + by' + cy = 0 \qquad (HE)$$

We look for solutions of equation (HE), if they exist, in the form: $y = e^{rx}, r \in \mathbb{R}$.

Then, we have:

$$y = e^{rx} \Longrightarrow y' = re^{rx} \Longrightarrow y'' = r^2 e^{rx}$$

So, the equation (HE) becomes:

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0 \Longleftrightarrow e^{rx}(ar^2 + br + c) = 0$$
$$\Longrightarrow ar^2 + br + c = 0$$

Definition 3.9.2. *The equation: $ar^2 + br + c = 0$, is called the characteristic equation (CE) of the homogeneous equation (HE).*

Proposition 3.9.3. *Depending on the sign of the discriminant $\triangle = b^2 - 4ac$, we have the following results:*

Case 1: *If $\triangle > 0$, then the equation (CE) admits 2 distinct real roots $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$ $\left(r_1 = \frac{-b-\sqrt{\triangle}}{2a}, r_2 = \frac{-b+\sqrt{\triangle}}{2a}\right)$.*

In this case, the functions $y_1 = e^{r_1x}$ and $y_2 = e^{r_2x}$ are particular solutions of the homogeneous equation (HE).

So, the general solution of equation (HE) is:

$$y_H = \lambda_1 y_1 + \lambda_2 y_2 = \lambda_1 e^{r_1x} + \lambda_2 e^{r_2x} \ / \ \lambda_1, \lambda_2 \in \mathbb{R}.$$

Case 2: *If $\triangle = 0$, then the equation (CE) admits a double root $r = \frac{-b}{2a}$.*

The two functions $y_1 = xe^{rx}$ and $y_2 = e^{rx}$ are particular solutions of the equation (HE).

So, the general solution of equation (HE) is:

$$y_H = \lambda_1 y_1 + \lambda_2 y_2 = \lambda_1 x e^{rx} + \lambda_2 e^{rx} \ / \ \lambda_1, \lambda_2 \in \mathbb{R}.$$

Case 3: *If $\triangle < 0$, then the equation (CE) admits two conjugate complex roots:*

$$\begin{cases} r_1 = \delta + i\omega \\ r_2 = \delta - i\omega \end{cases}, \ where \ \begin{cases} \delta = \frac{-b}{2a}. \\ \omega = \frac{\sqrt{|\triangle|}}{2a}. \end{cases}.$$

The two functions $y_1 = e^{\delta x}\cos(\omega x)$ and $y_2 = e^{\delta x}\sin(\omega x)$ are particular solutions of the homogeneous equation (HE).

So, the general solution of equation (HE) is:

$$y_H = \lambda_1 y_1 + \lambda_2 y_2 = e^{\delta x}(\lambda_1 \cos(\omega x) + \lambda_2 \sin(\omega x)) \ / \ \lambda_1, \lambda_2 \in \mathbb{R}.$$

2) **Calculation of the particular solution of equation** (E)

Let be the equation

$$ay'' + by' + cy = f(x) \qquad (E)$$

We have the general solution of (E) is in the form $y = y_H + y_P$, where y_H is the solution of homogeneous equation (HE) calculated as seen above.

Now, we are looking for y_P the particular solution of (E).

We distinguish several cases according to the expression of $f(x)$.

$*$ 1^{st}**case:** If $f(x) = e^{\alpha x}P(x) \ /\alpha \in \mathbb{R}, P \in \mathbb{R}[X]$ (Polynomial) then, we look for a particular solution y_P in the form:

$$y_P = e^{\alpha x}x^m Q(x), \text{ where } Q \text{ is polynomial with the same degree of } P\text{, i.e., } d^\circ Q = d^\circ P.$$

More precisely, we have:

- If α is not a root of (CE) ($i.e$ $a\alpha^2 + b\alpha + c \neq 0$), then $y_P = e^{\alpha x}Q(x)$ $(m = 0)$.
- If α is a simple root of (CE), then $y_P = e^{\alpha x}xQ(x)$ $(m = 1)$.
- If α is a dooble root of (CE), then $y_P = e^{\alpha x}x^2Q(x)$ $(m = 2)$.

Remark 3.9.4. *In this case, where $f(x) = e^{\alpha x}P(x)$, we can observe 2 special cases:*

1) *If $\alpha = 0$, then $f(x) = P(x)$ (polynomial), $d^{\circ}P = k$.*

So, the particular solution y_P is given in the form:

- $y_P = Q(x)$, *if 0 is not a root of (CE) (i.e., $a.0 + b.0 + c = c \neq 0$).*
- $y_P = xQ(x)$, *if 0 is a simple root of (CE) (i.e., $c = 0 \wedge a \neq 0 \wedge b \neq 0$).*
- $y_P = x^2Q(x)$, *if 0 is a double root of (CE) (i.e $a \neq 0 \wedge b = 0 \wedge c = 0$).*

where Q is polynomial with $d^{\circ}Q = d^{\circ}P = k$.

2) *If $P(x) = A \in \mathbb{R}$ (constant) (i.e., $d^{\circ}P(x) = 0$) $\Longrightarrow f(x) = Ae^{\alpha x}$*

So, the particular solution y_P is given in the form:

- $y_P = Be^{\alpha x} = \frac{A}{a\alpha^2+b\alpha+c}e^{\alpha x}$, *if α is not a root of (CE), i.e $a\alpha^2 + b\alpha + c \neq 0$.*
- $y_P = Bxe^{\alpha x} = \frac{A}{2a\alpha+b}xe^{\alpha x}$, *if α is a simple root of (CE).*
- $y_P = Bx^2e^{\alpha x} = \frac{A}{2a}x^2e^{\alpha x}$, *if α is a double root of (CE).*

$*$ 2^{nd}**case:** $f(x) = e^{\alpha x}(P_1(x)\cos(\beta x) + P_2\sin(\beta x))$ where $\alpha, \beta \in \mathbb{R}$ and $P_1, P_2 \in \mathbb{R}[X]$ (polynomials), then we look for a particular solution y_P of the form:

- $y_P = e^{\alpha x}(Q_1(x)\cos(\beta x) + Q_2\sin(\beta x))$, if $\alpha + i\beta$ is not a root of the equation (CE).
- $y_P = xe^{\alpha x}(Q_1(x)\cos(\beta x) + Q_2\sin(\beta x))$, if $\alpha + i\beta$ is a root of the equation (CE).

In both cases, Q_1 and Q_2 are polynomials of degree n where, $n = \max\{d^{\circ}P_1, d^{\circ}P_2\}$.

$*$ **General case: we use the constant variation method**

Let be $y = \lambda_1 y_1 + \lambda_2 y_2$ a solution of the homogeneous equation:

$$ay" + by' + cy = 0 \qquad (HE) \qquad / \ \lambda_1, \lambda_2 \in \mathbb{R}.$$

We look for a particular solution of the equation (E)

$$ay" + by' + cy = f(x) \qquad (E)$$

in the form

$$y = \lambda_1 y_1 + \lambda_2 y_2 = \lambda_1(x)y_1 + \lambda_2(x)y_2$$

i.e., we consider λ_1 and λ_2 as two functions that's verify the system:

$$(S)\begin{cases} \lambda_1' y_1 + \lambda_2' y_2 = 0 \\ \lambda_1' y_1' + \lambda_2' y_2' = \frac{f(x)}{a} \end{cases}$$

Indeed, if $y_P = \lambda_1 y_1 + \lambda_2 y_2$ and verifies the system (S), then

$$\begin{aligned} y_P' &= \lambda_1' y_1 + \lambda_1 y_1' + \lambda_2' y_2 + \lambda_2 y_2' \ \ / \ (\lambda_1' y_1 + \lambda_2' y_2 = 0) \\ &\Longrightarrow \ y_P' = \lambda_1 y_1' + \lambda_2 y_2' \\ &\Longrightarrow \ y_P'' = \lambda_1' y_1' + \lambda_1 y_1'' + \lambda_2' y_2' + \lambda_2 y_2'' \ \ / \ \left(\lambda_1' y_1' + \lambda_2' y_2' = \frac{f(x)}{a}\right) \\ &\Longrightarrow \ y_P'' = \frac{f(x)}{a} + \lambda_1 y_1'' + \lambda_2 y_2'' \ . \end{aligned}$$

We substitute y_P, y_P' and y_P'' in equation (E), we obtain:

$$\begin{aligned} a y_P'' + b y_P' + c y_P &= a\left(\frac{f(x)}{a} + \lambda_1 y_1'' + \lambda_2 y_2''\right) + b(\lambda_1 y_1' + \lambda_2 y_2') + c(\lambda_1 y_1 + \lambda_2 y_2) \\ &= f(x) + \lambda_1(a y_1'' + b y_1' + c y_1) + \lambda_2(a y_2'' + b y_2' + c y_2) \\ &= f(x) \ , \ \text{ because } y_1 \text{ and } y_2 \text{ are solutions of } (HE)) \end{aligned}$$

So, $y_P = \lambda_1 y_1 + \lambda_2 y_2$ is a solution of equation (E).

We have used the fact that $y_1$and $y_2$are solutions of the homogeneous equation (HE).

The system (S) is easily solved giving $\lambda_1'$and λ_2', then by integration we obtain $\lambda_1(x)$ and $\lambda_2(x)$ and finally

$$y_P = \lambda_1(x) y_1 + \lambda_2(x) y_2.$$

Proposition 3.9.5. ***(Superposition principle)***

Let $a, b, c \in \mathbb{R}$ and $f_1, f_2, ..., f_n$ continuous functions on $I \subseteq \mathbb{R}$. If for all $k \in \{1, 2, ..., n\}$ the function $\varphi_k(x)$ is a particular solution of the differential equation (E_k) on I,

$$a y'' + b y' + c y = f_k(x) \qquad (E_k)$$

then,the function $\varphi(x) = \sum_k^n \varphi_k(x)$ is a particular solution on I of the differential equation :

$$a y'' + b y' + c y = \sum_{k=1}^{n} f_k(x) = f(x) \qquad (E)$$

Examples:

1) Solve the differential equation:

$$y'' + y = x + \cos(3x) \Longleftrightarrow a y'' + b y' + c y = f(x) \qquad (E)$$

• We start with the homogeneous equation:

$$y^{''}+y=0 \qquad (HE)/(a=1,b=0,c=1)$$

The characteristic equation is:

$$ar^2+br+c=0 \Longleftrightarrow r^2+1=0$$

we have:

$$\begin{aligned}\triangle &= b^2-4ac=-1<0 \Longrightarrow \begin{cases} r_1=\delta+iw=+i \\ r_2=\delta-iw=-i \end{cases} /\delta=0, w=1 \\ &\Longrightarrow y_H=\lambda_1 y_1+\lambda_2 y_2=e^{\delta x}(\lambda_1\cos(wx)+\lambda_2\sin(wx)) \\ &\Longrightarrow y_H=\lambda_1\cos(x)+\lambda_2\sin(x) \quad /\lambda_1,\lambda_2\in\mathbb{R}.\end{aligned}$$

• We have the general solution of (E) is:

$$y=y_H+y_P.$$

y_H is the general solution of (HE) and y_P is a particular solution of (E).

We observe that

$$f(x)=x+\cos(3x)=f_1(x)+f_2(x).$$

So, $y_P=y_{P_1}+y_{P_2}$ where:

$y_{P_1}$is a particular solution of:

$$y^{''}+y=f_1(x)=x \qquad (E1)$$

$y_{P_2}$is a particular solution of:

$$y^{''}+y=f_2(x)=\cos(3x) \qquad (E2)$$

• For the equation (E_1), it is clear that $y_{P_1}=x$ is a particular solution $(y^{''}_{P_1}+y_{P_1}=0+x=x)$

- For the equation (E_2)

$$(E_2) \quad y''+y=\cos(3x)=e^{\alpha x}(P_1(x)\cos(\beta x)+P_2(x)\sin(\beta x)\ /\ \alpha=0, \beta=3, P_1(x)=1, P_2(x)=0$$

We have $\alpha+i\beta=3i$ is not a root of $(CE)(r^2+1=0)$, because

$$(3i)^2+1=9i^2+1=-9+1=8\neq 0$$

So, the particular solution y_{P_2} of (E_2) is in the form:

$$y_{P_2}=e^{\alpha x}(Q_1(x)\cos(\beta x)+Q_2(x)\sin(\beta x))\ /\ \alpha=0, \beta=3$$

Where $d^\circ Q_1=d^\circ Q_2=\max\left\{d^\circ P_1, d^\circ P_2\right\}=\max\{0,0\}=0.$

$$\begin{aligned}
&\Longrightarrow \quad Q_1(x)=A, Q_2(x)=B\ /\ A,B\in\mathbb{R}\\
&\Longrightarrow \quad y_{P_2}=A\cos(3x)+B\sin(3x) \qquad (*)\\
&\Longrightarrow \quad y'_{P_2}=-3A\sin(3x)+3B\cos(3x)\\
&\Longrightarrow \quad y''_{P_2}=-9A\cos(3x)-9B\sin(3x) \qquad (**)
\end{aligned}$$

By substituting $(*)$ and $(**)$ in equation (E_2), we obtain:

$$-9A\cos(3x)-9B\sin(3x)+A\cos(3x)+B\sin(3x)=\cos(3x)$$

$$\Longrightarrow -8A\cos(3x)-8B\sin(3x)=\cos(3x).$$

By identification we get $B=0$ and $A=\frac{-1}{8}$.

So,

$$y_{P_2}=\frac{-1}{8}\cos(3x).$$

Then, the particular solution y_P of equation (E) (with second member) is:

$$y_P=y_{P_1}+y_{P_2}=x-\frac{1}{8}\cos(3x).$$

Finally, the general solution of the equation $(E): y^{''} + y = x + \cos(3x)$, is given by:

$$y = y_H + y_P = \lambda_1 \cos x + \lambda_2 \sin(x) + x - \frac{1}{8}\cos(3x) \ / \ \lambda_1, \lambda_2 \in \mathbb{R}$$

2) Solve the following equation:

$$y^{''} + y = \frac{1}{\sin^3(x)} \qquad (E) \Longleftrightarrow ay^{''} + by' + cy = f(x).$$

The general solution of (E) is:$y = y_H + y_P$, where y_H is the solution of homogeneous equation :

$$y^{''} + y = 0 \qquad (HE).$$

From example **(1)** we have

$$\begin{aligned} y_H &= \lambda_1 \cos(x) + \lambda_2 \sin(x) \ /\lambda_1, \lambda_2 \in \mathbb{R} \\ &= \lambda_1 y_1 + \lambda_2 y_2. \end{aligned}$$

Now, we search a particular solution y_P of (E), using the constant variation method. We put:

$$\begin{aligned} y &= \lambda_1(x)\cos(x) + \lambda_2(x)\sin(x) \\ &= \lambda_1(x) y_1 + \lambda_2(x) y_2. \end{aligned}$$

We obtain the system:

$$(S)\begin{cases} \lambda_1' y_1 + \lambda_2' y_2 = 0 \\ \lambda_1' y_1^{'} + \lambda_2' y_2^{'} = \frac{f(x)}{a} \end{cases} \Longleftrightarrow (S)\begin{cases} \lambda_1' \cos(x) + \lambda_2' \sin(x) = 0 \\ -\lambda_1' \sin(x) + \lambda_2' \cos(x) = \frac{1}{\sin^3(x)} \end{cases}$$

We calculate λ_1' and λ_2' using Cramer method.

We have:

$$\triangle = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0$$

$$\begin{aligned}
\lambda_1' &= \frac{\begin{vmatrix} 0 & \sin x \\ \frac{1}{\sin^3(x)} & \cos x \end{vmatrix}}{\triangle} = \frac{-\sin x}{\sin^3(x)} = -\frac{1}{\sin^2(x)} \\
&\Longrightarrow \lambda_1(x) = \int \frac{-1}{\sin^2(x)} dx = \frac{\cos x}{\sin x}. \\
\lambda_2' &= \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \frac{1}{\sin^3(x)} \end{vmatrix}}{\triangle} = \frac{\cos x}{\sin^3 x} \\
&\Longrightarrow \lambda_2(x) = \int \frac{\cos x}{\sin^3 x} dx = \frac{-1}{2\sin^2 x}.
\end{aligned}$$

So, the particular solution is:

$$\begin{aligned}
y_P &= \lambda_1(x)\cos(x) + \lambda_2(x)\sin(x) \\
&= \frac{\cos x}{\sin x}\cos x - \frac{1}{2\sin^2 x}\sin x = \frac{\cos^2 x - 1}{2\sin x} = \frac{\cos 2x}{2\sin x} \\
&\Longrightarrow y_P = \frac{\cos 2x}{2\sin x}.
\end{aligned}$$

Finally, the general solutions of $y'' + y = \frac{1}{\sin^3 x}$ are in the form:

$$y = y_H + y_P = \lambda_1 \cos x + \lambda_2 \sin x + \frac{\cos(2x)}{2\sin x} \;/\; \lambda_1, \lambda_2 \in \mathbb{R}.$$

3.10 Exercises

Exercise 1:

Solve the following differential equations with separate variables:

$$\begin{cases}
1)\ xy' = y, & 2)\ (x^2+1)y' = y^2+1, & 3)\ (x^2+1)xy' = 2xy \\
4)\ y' = y\cos x, & 5)\ 2yy'(1+\exp(x)) = \exp(x), & 6)\ y' = y\cos x, \\
7)\ 2yy'(1+e^x) = e^x, & 8)\ y' + y\sin x = \sin x. &
\end{cases}$$

Solution:

We solve the following equations:

1) We have

$$\begin{aligned} xy' &= y \Longrightarrow \frac{dy}{y} = \frac{dx}{x} \\ &\Longrightarrow \ln|y| = \ln|x| + c \Longrightarrow y = kx, \qquad k = e^c \in \mathbb{R}. \end{aligned}$$

2) We have

$$\begin{aligned} (x^2+1)y' &= y^2+1 \\ &\Longrightarrow \frac{dy}{y^2+1} = \frac{dx}{x^2+1} \Longrightarrow Arc\tan y = Arc\tan x + c \\ &\Longrightarrow y = \tan(Arc\tan x + c), c \in \mathbb{R}. \end{aligned}$$

3) We have

$$\begin{aligned} (x^2+1)xy' &= 2xy \\ &\Longrightarrow \frac{dy}{y} = \frac{2dx}{x^2+1} \Longrightarrow \ln|y| = 2Arc\tan x + c \\ &\Longrightarrow y = k\exp(2Arc\tan x), k \in \mathbb{R}. \end{aligned}$$

6) We have

$$\begin{aligned} y' &= y\cos x \quad \forall y \neq 0 \\ \int \frac{dy}{y} &= \int \cos x dx = \sin x + c \\ &\Longrightarrow \ln|y| = ke^{\sin x}, \quad k > 0. \end{aligned}$$

7) We have

$$\begin{aligned} 2yy'(1+e^x) &= e^x \\ &\Longrightarrow \int 2ydy = \int \frac{e^x}{1+e^x}dx \Longrightarrow y^2 = (1+e^x) + c \\ &\Longrightarrow y = [\ln(1+e^x) + c]^{1/2}, c \in \mathbb{R}. \end{aligned}$$

Exercise 2:

Solve the following homogeneous differential equations:

1) $x^2y^{'} = xy - y^2$, 2) $2x^2y^{'} = x^2 + y^2$, 3) $xy^{'} = y + x\cos^2(\frac{y}{x})$,
4) $xy^{'} = y + \sqrt{x^2 - y^2}$.

Solution:

1) We have

$$\begin{aligned} x^2y^{'} &= xy - y^2, \text{ for } x \neq 0, \text{ we have:} \\ y^{'} &= \frac{y}{x} - \left(\frac{y}{x}\right)^2 \\ \text{We put: } z = \frac{y}{x} \Longrightarrow y^{'} &= z - z^2, \text{ then } xz^{'} = -z^2 \end{aligned}$$

The constant function is a trivial solution. Then for $z \neq 0$, we have:

$$\begin{aligned} xz^{'} &= -z^2 \Longrightarrow \int -\frac{dz}{z^2} = \int \frac{dx}{x} \Longrightarrow \frac{1}{z} = \ln|x| + c \\ &\Longrightarrow \frac{x}{y} = \ln|x| + c \Longrightarrow y = \frac{x}{\ln cx}, \ c \in \mathbb{R}. \end{aligned}$$

2) We have

$$\begin{aligned} 2x^2y^{'} &= x^2 + y^2, \text{ for } x \neq 0, \text{ we have:} \\ 2y^{'} &= 1 + \left(\frac{y}{x}\right)^2. \text{ We put: } z = \frac{y}{x}, \text{then the diff eq is: } 2z^{'} = 1 + z^2 \\ \text{where } y^{'} &= z + xz^{'}. \text{ Then: } (1 - z^2) = 2xz^{'} \Longleftrightarrow \int \frac{2dz}{(1 - z^2)} = \int \frac{dx}{x} \\ &\Longleftrightarrow \frac{2}{(1 - z)} = \ln|x| + c \Longleftrightarrow (1 - z) = \frac{2}{\ln cx} \Longleftrightarrow z = 1 - \frac{2}{\ln cx} \\ \text{and } y &= x + \frac{2}{\ln kx}, \ k \in \mathbb{R}. \end{aligned}$$

3) We have

$$\begin{aligned}
xy' &= y + x\cos^2(\frac{y}{x}),\ \text{We put: } z = \frac{y}{x}\text{,then: } y' = z + xz' \\
\text{where, } y' &= \frac{y}{x} + x\cos^2(\frac{y}{x}),\ \text{then: } y' = z + \cos^2 z \\
&\Longrightarrow z + xz' = z + \cos^2 z \Longrightarrow xz' = \cos^2 z \\
&\Longrightarrow \int \frac{dz}{\cos^2 z} = \int \frac{dx}{x} \Longrightarrow \tan z = \ln kx \Longrightarrow \frac{y}{x} = Arc\tan(\ln kx) \\
\text{Then, } y &= xArc\tan(\ln kx),\ k \in \mathbb{R}.
\end{aligned}$$

4) We have

$$\begin{aligned}
xy' &= y + \sqrt{x^2 - y^2} \Longrightarrow y' = \frac{y}{x} + \sqrt{1 - \left(\frac{y}{x}\right)^2} \\
\text{We put } z &= \frac{y}{x},\ \text{and } z' = xz + z \Longrightarrow xz' + z = z + \sqrt{1 - z^2} \\
&\Longrightarrow \int \frac{dz}{\sqrt{1 - z^2}} = \int \frac{dx}{x} \Longleftrightarrow \frac{y}{x} = \sin(\ln kx) \\
&\Longrightarrow y = x\sin(\ln kx),\ k \in \mathbb{R}.
\end{aligned}$$

Exercise 3:

Solve the following first order linear differential equations:

1) $(1 + y) - 2y'\sqrt{x} = 0,$

2) $x^2y' + y - a = 0,$

3) $y' + \frac{1}{x}y = \ln x,$

4) $(1 + x)y' + y = \frac{Arc\tan x}{(1+x)^2}.$

Solution:

1) We have

$$\begin{aligned}
(1 + y) - 2y'\sqrt{x} &= 0 \Longrightarrow 2y'\sqrt{x} = (1 + y),\ \text{for } y \neq 1,\ \text{we have:} \\
\int \frac{dy}{(1 + y)} &= \int \frac{dx}{2\sqrt{x}} \Longrightarrow \ln(1 + y) = \sqrt{x} + c,\ c \in \mathbb{R}. \\
\text{Then, } &: \ y = Ke^{\sqrt{x}} - 1.
\end{aligned}$$

2) We have

$$\begin{aligned}
x^2y^{'}+y-a &= 0, \text{ we put } z=y-a,\\
\text{We oblian} &: \quad x^2z^{'}+z=0, \text{ is a lin hom eq}\\
&\Longrightarrow \int\frac{dz}{z}=-\int\frac{dx}{x^2}\Longrightarrow \ln|z|=\frac{1}{x}+c, \quad c\in\mathbb{R}.\\
&\Longleftrightarrow z=ke^{\frac{1}{x}}, \text{ then: } y=ke^{\frac{1}{x}}+a.
\end{aligned}$$

3) We have

$$\begin{aligned}
(1+x)y^{'}+y &= \frac{Arc\tan x}{(1+x)^2}, \qquad (E)\\
(1+x)y^{'}+y &= 0, \qquad (E_0)\\
&\Longrightarrow \int\frac{dy}{y}=-\int\frac{dx}{1+x}\Longrightarrow \ln|y_0|=-\ln|1+x|+c, \quad c\in\mathbb{R}\\
&\Longrightarrow y_0=\frac{k}{1+x}, \quad k\in\mathbb{R}
\end{aligned}$$

We use the variation constant method (VCM). We suppose that:

$$\begin{aligned}
y &= \frac{k(x)}{1+x}\\
&\Longrightarrow y^{'}=\frac{k^{'}}{1+x}-\frac{k}{(1+x)^2},\\
\text{We report in eq (E), we obtain} &:\\
k^{'} &= \frac{Arc\tan x}{1+x^2}\Longrightarrow k(x)=\int\frac{Arc\tan x}{1+x^2}dx\\
&\Longrightarrow k(x)=\int\left(\frac{1}{1+x^2}\right)Arc\tan x dx\\
&\Longrightarrow k(x)=\frac{(Arc\tan x)^2}{2}+c\\
\text{Where} &: \quad y=\frac{k(x)}{1+x}\\
\text{Then} &: \quad y=\frac{(Arc\tan x)^2}{2(1+x)}+\frac{c}{1+x}, \quad c\in\mathbb{R}..
\end{aligned}$$

4) We have

$$\begin{aligned} y' + \frac{1}{x}y &= \ln x, \qquad (E) \\ y' + \frac{1}{x}y &= 0, \qquad (E_0) \quad (H\ Eq) \\ &\Longrightarrow \frac{dy}{y} = -\frac{dx}{x}, \quad \forall y \neq 0, \\ &\Longrightarrow \ln|y| = -\ln|x| + c \Longrightarrow |y_0| = \exp(-\ln|x| + c) \\ \text{Then} \quad &: \quad |y_0| = \frac{k}{x}. \end{aligned}$$

We vary the constant, we suppose that:

$$y = \frac{k(x)}{x}, \text{ then: } y' = \frac{k'x - k}{x^2}$$

We report in Eq (E), we obtain:

$$\begin{aligned} k'(x) &= \ln x \Longrightarrow \int dk = \int \ln x dx \\ &\Longrightarrow k(x) = x\ln x - x + c \\ \text{Where} \quad &: \quad y = \frac{k(x)}{x}, \text{ then: } y = \ln x - 1 + \frac{c}{x},. \end{aligned}$$

Exercise 4:

Solve the following Bernoulli equations:

1) $y' + \frac{y}{x} = -xy^2$,

2) $xy' - y = y^2 \ln x$,

3) $y' = \frac{y}{x} + x\sqrt{y}$,

4) $y' + \frac{y}{x} - \frac{1}{x^4}y^{-\frac{3}{4}} = 0$.

Solution:

We solve the following Bernoulli equations:

1) We have

$$y' + \frac{y}{x} = -xy^2$$

is a Bernoulli equation with $k = 2$. We multiply this equation by (y^{-2}) we obtain:

$$y'y^{-2} + \frac{y^{-1}}{x} = -x, \qquad (1)$$

We put:

$$y^{-1} = z \Longrightarrow y = 1/z \Longrightarrow -z' = y'y^{-2}$$

We report in Eq (1), Then we obtain:

$$\begin{array}{rcl} -z' + \frac{z}{x} & = & -x \\ z' - \frac{z}{x} & = & x \quad (E) \text{ is a linear Equation} \\ \text{Then} \qquad z' - \frac{z}{x} & = & 0 \quad (E_0) \\ & \Longrightarrow & \int \frac{dz}{z} = \int \frac{dx}{x} \Longrightarrow \ln z = \ln x + c \\ & \Longrightarrow & z_0 = k.x, \qquad k \in \mathbb{R} \end{array}$$

We use the variable change method (VCM), we suppose that :

$$z = k(x).x \Longrightarrow z' = k'.x + k$$

We report in Eq (E), we obtain:

$$\begin{array}{rcl} \frac{dk}{dx} & = & 1 \Longrightarrow \int dk = \int dx \\ & \Longrightarrow & k(x) = x + \lambda, \qquad \lambda \in \mathbb{R} \\ & \Longrightarrow & z = (x + \lambda)x \\ & \Longrightarrow & y = \frac{1}{(x+\lambda)x} \end{array}$$

2) We have

$$xy' - y = y^2 \log x,$$

is a Bernoulli equation with $k = 2$. We multiply it by (y^{-2}), we obtain:

$$xy'y^{-2} - y^{-1} = \log x \qquad (1)$$

We put :

$$z = 1/y \Longrightarrow y = 1/z \Longrightarrow y'y^{-2} = -z'$$

Then the equation (1), will be

$$-xz^{'}-z=\log x \qquad (E), \text{ is a linear equation.}$$

Then :

$$\begin{aligned} -xz^{'}-z &= 0 \qquad (E_0), \text{ is the Homogeneous Eq} \\ &\Longrightarrow \int \frac{dz}{z}=\int -\frac{dx}{x} \Longrightarrow z_0=\frac{c}{x}, \quad c\in\mathbb{R} \end{aligned}$$

We use the V.C.M. We suppose that :

$$z=\frac{c(x)}{x}, \text{ then: } z^{'}=\frac{c^{'}.x-c}{x^2}$$

We report in (E), we obtain:

$$\begin{aligned} \frac{dc}{dx} &= \log x \Longrightarrow \int dc=\int \log x dx \\ &\Longrightarrow c(x)=x-x\log x+k, \quad k\in\mathbb{R} \\ \text{Then} &: \\ z &= 1-\log x+\frac{k}{x}. \end{aligned}$$

3) We have

$$y^{'}=\frac{y}{x}+x\sqrt{y}$$

The equation can be written as :

$$y^{'}-\frac{y}{x}=x\sqrt{y}$$

is a Bernoulli equation with $(k=1/2)$, we multiply it by $y^{-1/2}$, we obtain:

$$y^{'}y^{-1/2}-\frac{y^{1/2}}{x}=x$$

We put:

$$z = y^{1/2} \Longrightarrow z' = \frac{1}{2} y^{-1/2} y$$

Then the equation will be

$$\begin{aligned}
2z' - \frac{z}{x} &= x, \qquad (E), \text{ is a linear Equation} \\
2z' - \frac{z}{x} &= 0, \qquad (E_0) \\
&\Longrightarrow 2z' = \frac{z}{x} \Longrightarrow \int \frac{dz}{z} = \frac{1}{2} \int \frac{dx}{x} \\
&\Longrightarrow \ln|z| = \frac{1}{2} \ln|x| + c, \quad c \in \mathbb{R} \\
&\Longrightarrow z_0 = k\sqrt{x}, \qquad k \in \mathbb{R}
\end{aligned}$$

We use the V.C.M. We suppose that :

$$z = k(x)\sqrt{x} \Longrightarrow z' = k'.\sqrt{x} + \frac{k}{2\sqrt{x}}$$

We report in (E), we obtain

$$\begin{aligned}
\frac{dk}{sx} &= \frac{1}{2}\sqrt{x} \Longrightarrow k(x) = \frac{1}{2} \int \sqrt{x} dx = \frac{1}{3} x^{3/2} + c \\
\text{Where} &: \\
z &= k(x)\sqrt{x} \\
\text{Then} &: \\
z &= \frac{1}{3} x^2 + c\sqrt{x} \\
\text{And} &: \\
z &= y^{1/2} \Longrightarrow y = z^2 \\
\text{Then} &: \\
y &= \left(\frac{1}{3} x^2 + c\sqrt{x}\right)^2.
\end{aligned}$$

4) We have

$$\begin{aligned} y' + \frac{y}{x} - \frac{1}{x^4} y^{-\frac{3}{4}} &= 0 \\ y' + \frac{y}{x} &= \frac{1}{x^4} y^{-\frac{3}{4}} \qquad (1) \end{aligned}$$

(1) is a Bernoulli equation with $(k = -3/4)$, we multiply it by $y^{3/4}$, we obtain:

$$y' y^{3/4} + \frac{y^{7/4}}{x} = \frac{1}{x^2}, \qquad (2)$$

We put:

$$z = y^{7/4} \Longrightarrow z' = \frac{7}{4} y^{3/4} y'$$

we report in (2), we obtain:

$$\begin{aligned} \frac{4}{7} z' + \frac{z}{x} &= \frac{1}{x^4}, \qquad (E), \text{ is a linear Eq} \\ \frac{4}{7} z' + \frac{z}{x} &= 0, \qquad (E_0), \text{ is the Homeg Eq} \\ &\Longrightarrow \int \frac{dz}{z} = -\frac{7}{4} \int \frac{dx}{x}, \text{ for } z \neq 0, \text{ we have:} \\ \ln|z| &= -\frac{7}{4} \ln|x| + c, \quad c \in \mathbb{R} \\ &\Longrightarrow z_0 = k \exp(-\frac{7}{4} \ln|x|) \Longrightarrow z_0 = k x^{-7/4} \end{aligned}$$

We use the V.C.M. We suppose that:

$$z = k(x) x^{-7/4} \Longrightarrow z' = k' x^{-7/4} - \frac{7}{4} k x^{-11/4}$$

we report in (E), we obtain:

$$\begin{aligned}
\frac{4}{7}k' x^{-7/4} &= \frac{1}{x^4} \Longrightarrow \int dx = \frac{7}{4}\int x^{-9/4}dx \\
&\Longrightarrow k(x) = -\frac{7}{5}x^{-5/4} + c \quad c \in \mathbb{R} \\
\text{Then} &: \\
z(x) &= (-\frac{7}{5}x^{-5/4} + c)x^{-7/4} \\
\text{Where} &: \\
z(x) &= y^{7/4} \Longrightarrow y = z^{4/7} \\
\text{Then} &: \\
y(x) &= ((-\frac{7}{5}x^{-5/4} + c)x^{-7/4})^{4/7} \\
&\Longrightarrow y(x) = (-\frac{7}{5}x^{-3} + cx^{-7/4})^{4/7}.
\end{aligned}$$

Exercise 5:

Solve the following Riccati equations:

$$\begin{aligned}
&1)\ y' + \tfrac{y}{x} - y^2 = -\tfrac{1}{x^2}, && \text{such that: } s(x) = \tfrac{1}{x}. \\
&2)\ y' - y^2 + (2x+1)y = x^2 + 2x, && \text{such that: } s(x) = x.
\end{aligned}$$

Solution:

We solve the following Riccati equations: **1)**

$$\begin{aligned}
y' + \frac{y}{x} - y^2 &= -\frac{1}{x^2} \qquad (1) \\
s(x) &= \frac{1}{x} \text{ is a particular solution.}
\end{aligned}$$

We put:

$$y = z + \frac{1}{x} \Longrightarrow y' = z' - \frac{1}{x^2}$$

We report in (1), we obtain:

$$z' - \frac{z}{x} - z^2 = 0 \qquad (2), \text{ is a Bernouli Eq with } k = 2.$$

We multiply the Eq (2) by z^{-2} then:

$$z^{'}z^{-2} - \frac{1}{xz} = 1, \quad \forall z \neq 0$$

We put :

$$t = \frac{1}{z} \Longrightarrow t^{'} = -z^{'}z^{-2} \qquad (3)$$

Then the Eq (3) will be :

$$t^{'} + \frac{t}{x} = -1 \qquad (4), \text{ is a linear Eq.}$$

The homogeneous equation is :

$$\begin{aligned} t^{'} + \frac{t}{x} &= 0 \Longrightarrow \int \frac{dt}{t} = -\int \frac{dx}{x} \\ &\Longleftrightarrow \ln|t| = -\ln|x| + c \Longrightarrow t_0 = \frac{k}{x}, \quad k \in \mathbb{R}. \end{aligned}$$

We use the C.V.M. We suppose that:

$$t = \frac{k(x)}{x} \Longrightarrow t^{'} = \frac{k^{'}x - k}{x^2}$$

we report in (4), we obtain:

$$\begin{aligned} \frac{k^{'}x - k}{x^2} + \frac{k}{x^2} &= -1 \Longrightarrow \int dk = -\int x dx \\ &\Longrightarrow k(x) = -\frac{x^2}{2} + c \quad c \in \mathbb{R} \end{aligned}$$

Then :

$$t(x) = \frac{-x}{2} + \frac{c}{x}$$

where :

$$z(x) = \frac{1}{t} \Longrightarrow z = \frac{2x}{-x^2 + 2c}$$

The general solution of Riccati equation is : $y = z + \frac{1}{x}$

$$y = \frac{2x}{-x^2 + 2c} + \frac{1}{x} = \frac{x^2 + 2c}{x(2c - x^2)}.$$

Exercise 6:

Solve the following second order linear differential equations:

$$\begin{aligned} &a/.\ y'' + y' - 2y = 0, \quad b/.\ y'' + 4y' + 4y = 0, \\ &c/.\ y'' + y' + y = 0, \qquad d/.\ y'' + 9y = 0. \end{aligned}$$

Solution:

We solve the following equations:

1)

$$\begin{aligned} y'' + y' - 2y &= 0, \text{ is the homogenuous equation} \\ r^2 + r - 2 &= 0, \text{ is the Characteristic Equation} \\ &\Longrightarrow (r+2)(r-1) = 0 \Longrightarrow r_1 = -2 \text{ and } r_2 = 1 \end{aligned}$$

Then:

$$y_{\text{hom}} = \lambda_1 \exp(-2x) + \lambda_2 \exp(x), \quad \lambda_1, \lambda_2 \in \mathbb{R}$$

2)

$$\begin{aligned} y'' + 4y' + 4y &= 0, \text{ homogenuous equation} \\ r^2 + 4r + 4 &= 0, \text{ is the Characteristic Equation} \\ \triangle &= 0 \Longrightarrow r_1 = r_2 = \frac{-b}{a} = -2 \end{aligned}$$

Then:

$$y_{\text{hom}} = (\lambda_1 x + \lambda_2) \exp(-2x), \quad \lambda_1, \lambda_2 \in \mathbb{R}$$

3)

$$\begin{aligned} y^{''} + y^{'} + y &= 0, \text{ homogenuous equation} \\ r^2 + r + 1 &= 0, \text{ is the Characteristic Equation} \\ \triangle &= -3 = 3i^2 \Longrightarrow r_1 = \frac{-1}{2} - \frac{\sqrt{3}}{2}i, \text{ and } r_2 = \frac{-1}{2} + \frac{\sqrt{3}}{2}i \end{aligned}$$

Then:

$$y_{\text{hom}} = \exp(-\frac{-x}{2})\left(\lambda_1 \cos\frac{\sqrt{3}}{2}x + \lambda_2 \sin\frac{\sqrt{3}}{2}x\right), \quad \lambda_1, \lambda_2 \in \mathbb{R}$$

4)

$$\begin{aligned} y^{''} + 9y &= 0, \text{ homogeneous equation} \\ r^2 + 9 &= 0, \text{ is the Characteristic Equation} \\ r^2 &= -9 \Longrightarrow r_1 = 3i, \text{ and } r_2 = -3i \end{aligned}$$

Then:

$$y_{\text{hom}} = (\lambda_1 \cos 3x + \lambda_2 \sin 3x), \quad \lambda_1, \lambda_2 \in \mathbb{R}$$

Exercise 7:

Solve the following second order linear differential equations:

$$\begin{aligned} &1) \ y^{''} + 2y^{'} + 5y = 0, &&\text{with: } y(0) = 0 \text{ and } y^{'}(0) = 3, \\ &2) \ 9y^{''} + 6y^{'} + 5y = 0 &&\text{with: } y(0) = 3 \text{ and } y^{'}(0) = 0, \\ &3) \ y^{''} + 4y^{'} = 0, &&\text{with: } y(0) = 0 \text{ and } y^{'}(0) = 1. \end{aligned}$$

Solution:

We solve the following equations:

1)

$$\begin{aligned} y'' + 2y' + 5y &= 0, \text{ homogeneous equation} \\ r^2 + 2r + 5 &= 0, \text{ is the characteristic equation} \\ \triangle &= -16 = 16i^2 \Longrightarrow r_1 = -1 - 2i, \text{ and } r_2 = -1 + 2i \end{aligned}$$

Then:

$$\begin{aligned} y_{\text{hom}} &= \exp(-x)(\lambda_1 \cos 2x + \lambda_2 \sin 2x), \quad \lambda_1, \lambda_2 \in \mathbb{R} \\ y'_{\text{hom}} &= -\exp(-x)(\lambda_1 \cos 2x + \lambda_2 \sin 2x) + \exp(-x)(-2\lambda_1 \sin 2x + 2\lambda_2 \cos 2x) \end{aligned}$$

Or

$$\begin{aligned} y(0) &= 0 \Longrightarrow \lambda_1 = 0, \\ y'(0) &= 0 \Longrightarrow -\lambda_1 + 2\lambda_2 = 0 \Longrightarrow \lambda_2 = 1, \end{aligned}$$

Then:

$$y_{\text{hom}} = \exp(-x) \sin 2x.$$

3)

$$y'' + 4y' = 0, \text{ with: } y(0) = 0 \text{ and } y'(0) = 4.$$

The characteristic equation :

$$r^2 + 4 = 0 \Longrightarrow r_1 = 2i, \text{ and } r_2 = -2i$$

Then:

$$y_{\text{hom}} = (\lambda_1 \cos 2x + \lambda_2 \sin 2x), \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

Or

$$y(0) = 0 \Longrightarrow \lambda_1 = 0,$$
$$y^{'}(0) = 4 \Longrightarrow -2\lambda_2 = 4 \Longrightarrow \lambda_2 = 2.$$

Exercise 8:

Solve the following differential equations:

$$\begin{cases} 1)\ y^{''} + y^{'} - 2y = 2x^2 - 3x + 1, \\ 2)\ 2y^{''} + 2y^{'} + 3y = x^2 + 2 - 1. \end{cases}$$

Solution:

1) We have

$$y^{''} + y^{'} - 2y = 2x^2 - 3x + 1$$

The homogeneous equation: $y^{''} + y^{'} - 2y = 0$

The characteristic equation : $r^2 + r - 2 = 0 \Longrightarrow r_1 = -2$, and $r_2 = 1$.

Then:

$$y_{\text{hom}} = \lambda_1 \exp(-2x) + \lambda_2 \exp(x), \quad \lambda_1, \lambda_2 \in \mathbb{R}$$

The CVM which consists of looking for a solution in the form:

$$y = \lambda_1(x) \exp(-2x) + \lambda_2(x) \exp(x),$$

with $y_1(x) = \exp(-2x)$, $y_2(x) = \exp(x)$ and $\lambda_1(x)$, and $\lambda_2(x)$ two differential functions and verifies

$$\begin{cases} \lambda_1^{'}(x) y_1(x) + \lambda_2^{'}(x) y_2(x) = 0, \\ \lambda_1^{'}(x) y_1^{'}(x) + \lambda_2^{'}(x) y_2^{'}(x) = \frac{f(x)}{a} = 2x^2 - 3x + 1. \end{cases}$$

Then:

$$\begin{cases} \lambda_1^{'}(x) \exp(-2x) + \lambda_2^{'}(x) \exp(x) = 0, & (1) \\ -2\lambda_1^{'}(x) \exp(-2x) + \lambda_2^{'}(x) \exp(x) = 2x^2 - 3x + 1. & (2) \end{cases}$$

$$\begin{aligned} 2\times(1)+(2) &\Longrightarrow 3\lambda_2^{'}(x)\exp(x)=2x^2-3x+1 \\ &\Longrightarrow \lambda_2^{'}(x)=\frac{1}{3}(2x^2-3x+1)\exp(-x) \end{aligned}$$

Then:

$$\lambda_2(x)=\int \frac{1}{3}(2x^2-3x+1)\exp(-x)dx.$$

Two Integration by parts gives:

$$\lambda_2(x)=-\frac{1}{3}(2x^2+x+2)\exp(-x)+k_2, \quad k_2\in\mathbb{R}$$

In other way we have:

$$\begin{aligned} \lambda_1^{'}(x) &= -\lambda_2^{'}(x)\exp(3x) \\ &= -\frac{1}{3}(-2x^2-3x+1)\exp(2x) \\ \Longrightarrow & \lambda_1(x)=\int -\frac{1}{3}(-2x^2-3x+1)\exp(2x)dx \end{aligned}$$

Two Integration by parts gives:

$$\lambda_1(x)=-\frac{1}{6}(2x^2-5x+\frac{7}{2})\exp(2x)+k_1, \quad k_1\in\mathbb{R}$$

Then the general solution of (E) is:

$$\begin{aligned} y &= -\left[-\frac{1}{6}(2x^2-5x+\frac{7}{2})\exp(2x)+k_1\right]+\left[-\frac{1}{3}(2x^2+x+2)\exp(-x)+k_2\right] \\ y &= -k_1\exp(-2x)-k_2\exp(x)-x^2+\frac{1}{2}x-\frac{5}{4} \end{aligned}$$

Exercise 8:

Solve the following differential equations:

$$\begin{cases} 1)\ y'' - 5y' + 6y = 2\exp(3x) + \exp(4x), \\ 2)\ y'' - 2y' + 2y = (5x+3)\exp(-x) + (x^2 - 1), \\ 3)\ y'' + y' + y = 3\exp(x) + (13x - 4)\cos 2x + 6\sin 2x. \end{cases}$$

Solution:

1) We have

$$y'' - 5y' + 6y = 2\exp(3x) + \exp(4x). \qquad (E)$$

The homogeneous equation: $y'' - 5y' + 6y = 0$

The characteristic equation: $r^2 - 5r + 6 = 0 \Longrightarrow r_1 = 3$ and $r_2 = 2$.

Then:

$$y_{\text{hom}} = \lambda_1 \exp(3x) + \lambda_2 \exp(2x), \qquad \lambda_1, \lambda_2 \in \mathbb{R}$$

For the particular solution we use the superposition principle, we obtain:

$$y_p = y_{p1} + y_{p2},$$

y_{p1} is a particular solution of the equation: $y'' - 5y' + 6y = 2\exp(3x)$, (1)

y_{p2} is a particular solution of the equation: $y'' - 5y' + 6y = \exp(4x)$. (2)

* As $r = 3$, is a root of characteristic equation, then: y_{p1} is in the form:

$$\begin{aligned} y_{p1} &= \alpha x \exp(3x), \\ y_{p1}' &= \alpha(1+3x)\exp(3x), \\ y_{p1}'' &= \alpha(6+9x)\exp(3x). \end{aligned}$$

We report in Eq (1), we obtain : $\alpha \exp(3x) = 2\exp(3x) \Longrightarrow \alpha = 2$.

Then:

$$y_{p1} = 2x\exp(3x)$$

** As $r = 4$, is not a root of the Characteristic Equation.Then: y_{p2} is in the form:

$$\begin{aligned} y_{p2} &= \beta \exp(4x), \\ y'_{p2} &= 4\beta \exp(4x), \\ y''_{p2} &= 16\beta \exp(4x). \end{aligned}$$

We report in Eq (2), we obtain : $\beta = 1/2$

Then:

$$y_{p2} = \frac{1}{2} \exp(4x)$$

Finally:

$$y_p = y_{p1} + y_{p2} = 2x \exp(3x) + \frac{1}{2} \exp(4x)$$

Then the general solution of (E) is:

$$\begin{aligned} y_g &= y_{\text{hom}} + y_p \\ &= \lambda_1 \exp(3x) + \lambda_2 \exp(2x) + 2x \exp(3x) + \frac{1}{2} \exp(4x). \end{aligned}$$

2)

$$y'' - 2y' + 2y = (5x + 3) \exp(-x) + (x^2 - 1), \qquad (E)$$

The homogeneous equation: $y'' - 2y' + 2y = 0$,

The characteristic equation : $r^2 - 2r + 2 = 0 \Longrightarrow r_1 = 1 - i$, and $r_2 = 1 + i$

$$y_{\text{hom}} = \exp(x)(\lambda_1 \cos x + \lambda_2 \sin x), \quad \lambda_1, \lambda_2 \in \mathbb{R}$$

is in the form:

$$\begin{aligned} y_{p1} &= (\alpha x+\beta)\exp(-x), \\ y_{p1}^{'} &= (\alpha-\beta-\alpha x)\exp(-x), \\ y_{p1}^{''} &= (-2\alpha+\beta+\alpha x)\exp(-x). \end{aligned}$$

we report in the equation (1), we obtain:

$$\begin{cases} -4\alpha+5\beta=3 \\ 5\alpha=5 \end{cases}$$
$$\Longrightarrow \alpha=1, \text{ and } \beta=7/5$$

Then:

$$y_{p1}=(x+7/5)\exp(-x).$$

* As $r=0$, is not a root of Characteristic Equation.Then: y_{p2} is in the form:

$$\begin{aligned} y_{p2} &= ax^2+bx+c, \\ y_{p2}^{'} &= 2ax+b, \\ y_{p2}^{''} &= 2a. \end{aligned}$$

We report in Eq (1), we obtain :

$$\begin{cases} 2a=1 \\ -4a+2b=0 \\ 2a-2b+2c=-1 \end{cases} \Longrightarrow \begin{cases} a=1/2 \\ b=1 \\ c=0 \end{cases}$$

Then:

$$y_{p2}=\frac{1}{2}x^2+x$$

Finally:

$$y_p = y_{p1} + y_{p2} = (x + 7/5)\exp(-x) + \frac{1}{2}x^2 + x$$

Then the general solution of (E) is:

$$y_g = y_{\text{hom}} + y_p = \exp(x)(\lambda_1 \cos x + \lambda_2 \sin x) + (x + 7/5)\exp(-x) + \frac{1}{2}x^2 + x.$$

3)

$$y^{''} + y^{'} + y = 3\exp(x) + (13x - 4)\cos 2x + 6\sin 2x \qquad (E)$$

The homogeneous equation: $y^{''} + y^{'} + y = 0$

The characteristic equation : $r^2 + r + 1 = 0 \Longrightarrow r_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, and $r_1 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

$$y_{\text{hom}} = \exp(-x/2), \qquad \lambda_1, \lambda_2 \in \mathbb{R}.$$

* $r = 1$, is not a root of characteristic equation,Then: y_{p1} is in the the form:

$$\begin{aligned} y_{p1} &= \alpha\exp(x), \\ y_{p1}^{'} &= \alpha\exp(x), \\ y_{p1}^{''} &= \alpha\exp(x). \end{aligned}$$

We report in Eq (1), we obtain : $3\alpha = 3 \Longrightarrow \alpha = 1$.

Then:

$$y_{p1} = \exp(x).$$

y_{p2} is in the form:

$$\begin{aligned} y_{p2} &= (ax + b)\cos(2x) + (cx + d)\sin(2x) \\ y_{p2}^{'} &= (a + 2d + 2cx)\cos(2x) + (c - 2b - 2ax)\sin(2x) \\ y_{p2}^{''} &= (4c - 4ax - 4b)\cos(2x) + (-4a - 4cx - 4d)\sin(2x) \end{aligned}$$

$$\Longrightarrow \begin{cases} a = -3 \\ b = 43/13 \\ c = 2 \\ d = 6/19 \end{cases}$$

Then:

$$y_{p2} = (-3x + 43/13)\cos(2x) + (2x + 6/19)\sin(2x)$$

$$\begin{aligned} y_p &= y_{p1} + y_{p2} \\ &= \exp(x) + (-3x + 43/13)\cos(2x) + (2x + 6/19)\sin(2x) \end{aligned}$$

Then the general solution of (E) is:

$$\begin{aligned} y_g &= y_{\text{hom}} + y_p \\ &= \exp(-x/2)(\lambda_1 \cos\frac{\sqrt{3}}{2}x + \lambda_2 \sin\frac{\sqrt{3}}{2}x) + \exp(x) + \\ &\quad (-3x + 43/13)\cos(2x) + (2x + 6/19)\sin(2x). \end{aligned}$$

Exercise 9:

Solve the following differential equations:

1) $y'' + y = 2\cos x,$ with: $y(0) = 0$, and $y'(0) = -1.$
2) $y'' + 2y' + 5y = 3,$ with: $y(0) = 1$, and $y'(0) = 0.$
3) $-2y'' + 3y' + 2y = \sin x,$ with: $y(0) = 0$, and $y'(0) = 0.$

Solution:

1) We have

$$(P)\begin{cases} y'' + y = 2\cos x, \\ y(0) = 0,\ y'(0) = -1 \end{cases}$$

The homogeneous equation: $y^{''} + y = 0, \qquad (E)$

The characteristic equation : $r^2 + 1 = 0 \Longrightarrow r_1 = i$, and $r_2 = -i$

Then:

$$y_{\text{hom}} = \lambda_1 \cos x + \lambda_2 \sin x, \qquad \lambda_1, \lambda_2 \in \mathbb{R}$$

The particular solution?

The second member in the equation (E): $f(x) = 2\cos x$

* $r = i$, is a root of characteristic equation. Then: y_p is in the form:

$$\begin{aligned} y_p &= x(a\cos x + b\sin x), \\ y_p^{'} &= (a\cos x + b\sin x) + (-a\sin x + b\cos x) \\ y_p^{''} &= (-a\sin x + b\cos x) + (-a\sin x + b\cos x) + x(-a\cos x - b\sin x) \end{aligned}$$

We report in Eq (E). Then we obtain:

$$\begin{cases} -2a = 0 \Longrightarrow a = 0 \\ 2b = 2 \Longrightarrow b = 1 \end{cases}$$

Then :

$$y_p = x\sin x$$

Then the general solution of (E) is:

$$\begin{aligned} y_g &= y_{\text{hom}} + y_p \\ &= \lambda_1 \cos x + \lambda_2 \sin x + x\sin x. \end{aligned}$$

and

$$y_g^{'} = -\lambda_1 \sin x + \lambda_2 \cos x + \sin x + x\cos x$$

Or

$$y(0) = 0 \Longrightarrow \lambda_1 = 0$$
$$y(0) = 0 \Longrightarrow \lambda_2 = -1$$

Then the solution of the problem (P) is:

$$y = (x-1)\sin x$$

3)

$$(P)\begin{cases} -2y'' + 3y' + 2y = \sin x, & (E) \\ y(0) = 0, y'(0) = 0. \end{cases}$$

The homogeneous equation: $-2y'' + 3y' + 2y = 0, \quad (E)$

The characteristic equation : $-2r^2 + 3r + 2 = 0 \Longrightarrow (2r+1)(2-r) = 0 \Longrightarrow r_1 = -1/2,$ and $r_2 = 2$

Then:

$$y_{\text{hom}} = \lambda_1 \exp(-x/2) + \lambda_2 \exp(2x), \qquad \lambda_1, \lambda_2 \in \mathbb{R}$$

$y_p = ?$

The second member in Eq (E) is $f(x) = \sin x$

* $r = i$, is not a root of characteristic equation,Then: y_{p1} is in the form:

$$y_p = a\cos x + b\sin x,$$
$$y_p' = -a\sin x + b\cos x,$$
$$y_p'' = -a\cos x - b\sin x.$$

We report in (E), we obtain:

$$\begin{aligned}(4a+3b)\cos x+(4b-3a)\sin x &= \sin x\\ &\Longrightarrow \begin{cases} 4a+3b=0 \\ 4b-3a=1 \end{cases} \Longrightarrow \begin{cases} a=-3/25, \\ b=4/25. \end{cases}\\ y_p &= -3/25\cos x+4/25\sin x,\end{aligned}$$

Then the general solution of (E) is:

$$\begin{aligned} y_g &= y_{\text{hom}}+y_p \\ &= \lambda_1\exp(-x/2)+\lambda_2\exp(2x)-3/25\cos x+4/25\sin x \end{aligned}$$

And

$$y_g^{'}=-(\lambda_1/2)\exp(-x/2)+2\lambda_2\exp(2x)+3/25\sin x+4/25\sin x$$

Or

$$\begin{aligned} y(0) &= 0 \Longrightarrow \lambda_1+\lambda_2-3/25=0, \qquad (1)\\ y^{'}(0) &= 0 \Longrightarrow -(\lambda_1/2)+2\lambda_2+4/25=0, \qquad (2) \end{aligned}$$

$(1)+2\times(2)$, gives: $\lambda_2=-1/25, \quad \lambda_1=4/25$

Then the solution of the problem (P) is:

$$y=(4/25)\exp(-x/2)+(-1/25)\exp(2x)-3/25\cos x+4/25\sin x.$$

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