وزارة التعليـم العالـي والبحـث العلمـي

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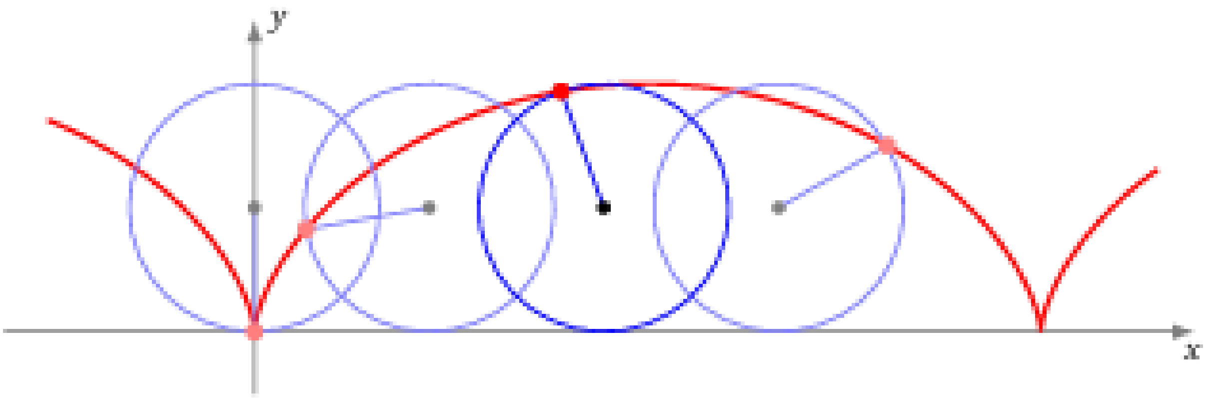
geometry
Course notes and exercises

Intended to the students of second year Bachelor in Mathematics



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Contents

General Introduction

Parametric curves are fascinating mathematical objects that allow us to precisely describe trajectories in the plane or in space. They play a vital role in many areas of mathematics, such as differential geometry, analysis, and physics.

This course, intended for second-year undergraduate students in Mathematics, aims to familiarize you with the fundamental concepts of planar and spatial parametric curves. We will explore their parameterization, their metric and local properties, as well as their curvature.

The first two parts of the course focuses on the parameterization of curves. We begin by studying the construction of plane curves defined by a parametric representation. You will learn how to describe these curves using functions that assign a point in the plane to each moment in time.

Next, we will examine the arc length of a curve, which allows us to measure the distance traveled along a trajectory.

Finally, in the third part of the course, we will address the curvature of planar and spatial curves. We will explore curvature in three-dimensional space as well as in the plane, focusing on key concepts related to curvature. You will also be introduced to the concept of the evolute of a curve, which describes the envelope of the family of normal lines to the curve.

This course will provide you with a solid foundation for understanding and working with planar and spatial parametric curves. Practical exercises will be offered throughout the course to help you consolidate your knowledge and develop your problem-solving skills. We also encourage you to consult the bibliography provided at the end of the course to deepen your understanding and explore further aspects of parametric curves.

We hope that this course will help you develop a deep and intuitive understanding of planar and spatial parametric curves, and recognize their importance in many areas of mathematics and beyond.

Chapter 1

Affine geometry

1.1 Definition of an affine space

An affine space is a geometric structure that generalizes the properties of Euclidean space but without a fixed origin.

Definition 1.1.1 *An affine space over a field K (often $\mathbb{R}$ or $\mathbb{C}$) is a set E together with a vector space E over K end a free and transitive action of V on E.*
That means:

1. *For every pair points $P, Q \in E$, there exists a unique vector $\vec{v} \in V$ such that moving P by $\vec{v}$ gives Q.*
   Wwe usually write this as:
   $$\vec{PQ} \in V$$
2. *Conversely, given a point $P \in E$ and a vector $\vec{v} \in V$; there exists unique point $Q \in E$ such that:*
   $$Q = P + \vec{v}$$

Example 1.1.1 *Every vector space $\vec{E}$ carries a natural structure of an affine space on itself.*
Meaning: if you take a vector space, you can also view it as an affine space by "forgetting the origin". The difference of two points is still a vector in E, so it naturally satisfies the definition of an affine space.
Every vector space E naturally carries the structure of an affine space. Indeed, in the definition

we simply take the underlying set of E as the set of points, with the associated vector space $\vec{E} = E$. The difference map is given by

$$\begin{aligned} E \times E &\to E \\ (u, v) &\to v - u \end{aligned}$$

Thus, any vector space can be regarded as an affine space once we "forget" the choice of origin. verification:

1. *Let* $M = \vec{u}, N = \vec{v}, P = \vec{w}$ *then*

$$\vec{MN} = \vec{v} - \vec{u},\ \vec{NP} = \vec{w} - \vec{v}$$

and

$$\begin{aligned} \vec{MN} + \vec{NP} &= \vec{v} - \vec{u} + \vec{w} - \vec{v} \\ &= \vec{w} - \vec{u} \\ &= \vec{MP} \end{aligned}$$

2. *For every* $\vec{u}$ *over* $\vec{E}$*, the application:*

$$\begin{aligned} M &\to \vec{OM} \\ \vec{v} &\to \vec{v} - \vec{u} = f_{\vec{u}}(\vec{v}) \end{aligned}$$

then

$$f_{\vec{u}}^{-1}(\vec{v} = \vec{v} + \vec{u} = \tau_{\vec{u}}(\vec{v})$$

the mapping from E to E is bijective.

proposition 1.1.1 *For every points $M, N, O \in E$, we have*

1. $\vec{MN} = \vec{O} \Leftrightarrow M = N$.
2. $\vec{NM} = -\vec{MN}$.
3. $\vec{MN} = \vec{ON} - \vec{OM}$

We have Chasles' relation

$$\vec{MM} + \vec{MM} = \vec{MM} = \vec{O}$$

$$\vec{MN} + \vec{NM} = 0 \implies \vec{MN} = -\vec{NM}$$

1.2 Examples of affine space structures on $\mathbb{R}^2$

$\mathbb{R}^2 \times \mathbb{R}^2 \implies \mathbb{R}^2$

$f(a, b) = b - a$

Chasles' relation

$f(a, b) = c - a + b - c = f(a, c) + f(c, b)$

Bijective application

$$\begin{aligned} \mathbb{R}^2 &\to \mathbb{R}^2 \\ y &\to f_x(y) = y - x \end{aligned}$$

where $f_x^{-1}(y) = y + x$

$$\begin{aligned} f_x(f_x^{-1}(y)) &= f_x(y + x) \\ &= y + x - x = y \end{aligned}$$

$$\begin{aligned} f_x^{-1}(f_x(y)) &= f_x^{-1}(y - x) \\ &= y - x + x = y \end{aligned}$$

1.3 Lines and hyperplanes in affine space

1.3.1 Cartesian equations of lines in an affine plane

Let (P) be an affine plane equipped with a Cartesian coordinate system $(o, \vec{i}, \vec{j})$. A line in the plane is a hyperplane of $\mathbb{R}^2$.

1.3.2 Equations of lines in $\mathbb{R}^2$

Let (Δ) be a line in the affine plane (P), given by the equation

$$ax + by + c = 0, \ (a;b) \neq 0$$

The direction of $\vec{\Delta}$ is determined by the associated homogeneous equation

$$ax + by = 0$$

A direction vector of Δ is $(-b, a)$.

The three points are collinearif only if

$$\Delta = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = x_1y_2 - x_2y_1 - x_1y_3 + x_3y_1 + x_2y_3 - x_3y_2 = 0$$

then

$$\vec{AB} = (x2 - x1, y_2 - y_1), \ \ \vec{AC} = (x_3 - x_1, y_3 - y_1)$$

The three points are collinear if

$$\vec{AB} \wedge \vec{Ac} = (0, 0, 0)$$

$$\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{pmatrix} = (x_1y_2 - x_2y_1 - x_1y_3 + x_3y_1 + x_2y_3 - x_3y_2)\vec{k} = 0$$

An equation of the line passing through $M_0(x_0, y_0)$ with direction vector $\vec{u} = (a, b)$ is given by:

$$\begin{vmatrix} x - x_0 & a \\ y - y_0 & b \end{vmatrix} = 0$$

An equation of the line passing through $M_0(x_0, y_0)$ and $M_1(x_1, y_1)$ is given by:

$$\begin{vmatrix} x - x_0 & x_1 - x_0 \\ y - y_0 & y_1 - y_0 \end{vmatrix} = 0$$

An alternative matrix representation

$$\begin{vmatrix} x_0 & x_1 & x \\ y_0 & y_1 & y \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Affine parametric equation of a line (D) **in** (P)

$$D = P_0 + \mathbb{R}\vec{u} = \{P_0 + \lambda\vec{u}, \lambda \in \mathbb{R}\}$$

where $P_0 \in D,\ \ \vec{u} \in \vec{D} - \{\vec{0}\}$

1.3.3 Cartesian equations of lines and plan in $\mathbb{R}^3$

Let X be a 3-dimensional affine space endowed with a Cartesian coordinate system $(o, \vec{i}, \vec{j}, \vec{k})$.
An affine plane of X is a hyperplane of $\mathbb{R}^3$.
Let (P) be such a plane with equation

$$ax + by + cz + d = 0.$$

The direction of $\vec{P}$ is given by the homogeneous equation

$$ax + by + cz = 0.$$

Let four points $A = (x_0, y_0, z_0),\ \ B = (x_1 y_1 z_1)$ and $C = (x_2, y_2, z_2)$ be give coplanarity if

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ z_0 & z_1 & z_2 & z_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

Affine parametric equation of a plan (P)

$$P = P_0 + \mathbb{R}\vec{u} + \mathbb{R}\vec{v}$$

where $(\vec{u}, \vec{v})$ is a linearly independent set of $\vec{P}$ and P_0 is a point of $\vec{P}$

The parametric equation of a line $D \subset \mathbb{R}^3$

$$D = P_0 + \mathbb{R}\vec{u} = \{P_0 + \lambda\vec{u}, \quad \lambda \in \mathbb{R}\{$$

where $P_0 \in D, \quad \vec{u} \in \vec{D} - \{\vec{0}\}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

where $\vec{u} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ of D then:

$$\begin{cases} x - x_0 = \lambda\alpha \\ y - y_0 = \lambda\beta \\ z - z_0 = \lambda\gamma \end{cases}$$

Thus we obtain the classical equation of a line in space.

$$\frac{x - x_0}{\alpha} = \frac{y - y_0}{\beta} = \frac{z - z_0}{\gamma}$$

1.3.4 Intersection of two planes (P) and (P')

If two planes are parallel or coincide, their intersection is either empty or equal to the entire plane. To determine this, one studies the normal vectors of the two planes. We assume that the intersection is a line, for which we aim to find a parametrization.

Example 1.3.1 *We have two planes P and P' implicitly defined as follows*

$$P: \quad x - 2y + z + 5 = 0$$

$$P': \quad 2x - y + 3z - 1 = 0$$

the respective normal vectors: $\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$

The two vectors are linearly independent. Therefore, the intersection of the two planes is non-empty and consists of a line. We now proceed to determine the parametric equations of this line.

$$\Delta = P \cap P'$$

then

$$x - 2y - z + 5 = 0 \tag{1.1}$$

and

$$x - 2y - z + 5 = 0 \tag{1.2}$$

In(1.1)

$$z = x - 2y + 5$$

and we replace z by this value in equation (1.2)

$$2x - y + 3(x - 2y + 5) - 1 = 0$$

so

$$\begin{cases} y = \dfrac{5x}{7} + 2 \\ z = -\dfrac{3x}{7} + 1 \end{cases}$$

Finally, we arrive at the parametric representation of the line D

$$D = D_0 + \lambda \vec{u} = \left\{ \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5/7 \\ -3/7 \end{pmatrix} \right\}$$

1.4 Notion of Barycenter

Definition 1.4.1 *A weighted point is defined as a pair (A, λ), where A is a point of X and $(\lambda \in \mathbb{R})$. The real number λ is called the weight of the weighted point.*

proposition 1.4.1 *Barycenter of a Weighted Family of Points Let (A_i, λ_i); $i = 1, 2, ..., n$ be a family of (n) weighted points of an affine space X, where each $(A_i \in X)$ and $(\alpha_i \in \mathbb{R})$,* *with*

$$\sum_{i=1}^{n} \lambda_i \neq 0$$

Then there exists a unique point $(G \in X)$, called the barycenter (or centroid) of the weighted family, such that for every origin O chosen in X:

$$G = O + \frac{\sum_{i=1}^{n} \lambda_i \vec{OA_i}}{\sum_{i=1}^{n} \lambda_i}$$

Proof. Let $O \in X$. Then

$$\sum_{i=1}^{n} \lambda_i \vec{GA_i} = 0$$

so

$$\sum_{i=1}^{n} \lambda_i \vec{GO} + \sum_{i=1}^{n} \lambda_i \vec{OA_i} = \vec{GO} \sum_{i=1}^{n} \lambda_i + \sum_{i=1}^{n} \lambda_i \vec{OA_i}$$

Hence the existence of G, independent of the choice of O. If G' is another point satisfying the same condition, then we must have $G = G'$. This proves the uniqueness of the barycenter

$$\sum_{i=1}^{n} \lambda_i \vec{G'A_i} = 0$$

therefore

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i \vec{G'G} + \sum_{i=1}^{n} \lambda_i \vec{GA_i} &= 0 \\ \Longrightarrow \vec{G'G} \sum_{i=1}^{n} \lambda_i &= 0 \end{aligned}$$

then $G = G'$. □

Definition 1.4.2 *The point G satisfying*

$$\sum_{i=1}^{n} \lambda_i \vec{GA_i} = 0$$

is called the barycenter of the weighted family $(A_i, \alpha_i)_{i=1}^{n}$.

Remarque 1.4.1 *f all the coefficients (α_i) are equal, then the barycenter G is called the centroid (or center of gravity), also referred to as the isobarycenter of the family $(A_i, \alpha_i)_{i=1}^{n}$.*

proposition 1.4.2 *Properties of the Barycenter of Weighted Points*

Let $((A_i, \alpha_i)_{i=1}^{n})$ *be a family of weighted points in an affine space X, with* $(\sum_{i=1}^{n} \alpha_i \neq 0)$. *Then:*

Scaling of weights

The barycenter is unchanged if all the weights α_i *are multiplied by the same non-zero scalar.*

2. ***Independence of ordering***

   The barycenter does not depend on the order in which the points are listed.

3. ***Replacement by barycenter***

   The barycenter of the family remains unchanged if a subset of weighted points is replaced by their own barycenter, endowed with the sum of the corresponding weights.

Proof.

1. Multiplying all weights by a non-zero scalar does not affect the ratio defining the barycenter, hence the barycenter remains the same.

2. The definition of the barycenter involves a sum over all weighted points, which is commutative; thus, the order of the points is irrelevant.

3. Let $(A_j, \alpha_j), \quad j \in J$ be a subset of the family, and let B be its barycenter with total weight $(\sum_{j \in J} \alpha_j)$.
   Substituting B and this total weight into the global sum yields the same vector equation, so the barycenter of the entire system is unchanged.

□

1.5 Affine subspace

Definition 1.5.1 *A subset* F *of an affine space* E *is called an affine subspace of* E *if there exists a point* $A \in F$ *such that*

$$\vec{F} = \{\vec{AM} / M \in F\}$$

where V *is a vector subspace of the direction space of* E.

proposition 1.5.1 *A non-empty subset F of an affine space E is an affine subspace of E if and only if every barycenter of points of F belongs to F.*

Proof. ($\Rightarrow$) Suppose F is an affine subspace of E.

Then $F = A + V$ for some point $A \in F$ and some vector subspace V.

For any weighted family of points (A_i, α_i) in F, each point can be written as $A_i = A + v_i$ with $v_i \in V$. The barycenter G of this family is given by

$$\overrightarrow{AG} = \frac{1}{\sum \alpha_i} \sum \alpha_i \overrightarrow{AA_i} = \frac{1}{\sum \alpha_i} \sum \alpha_i v_i \in V.$$

Hence $G = A + \overrightarrow{AG} \in A + V = F$. Therefore every barycenter of points of F belongs to F.

($\Leftarrow$) Conversely, suppose every barycenter of points of F belongs to F. Pick any point $A \in F$. Define

$$V = \overrightarrow{AB};|;B \in F.$$

One can check that V is a vector subspace of the direction space of E.

Then clearly,

$$F = A + V,$$

which shows that F is an affine subspace. □

1.6 Leibniz Vector Function

Definition 1.6.1 *Let $(A_i, \alpha_i)_{i=1}^{n}$ be a system of n weighted points in an affine space E. The Leibniz vector function associated with this system is defined as the mapping:*

$$f : E \longrightarrow \vec{E}, \quad M \mapsto \sum_{i=1}^{n} \alpha_i \overrightarrow{MA_i} \ ,$$

where $\vec{E}$ denotes the direction space of E.

proposition 1.6.1 *1. If the total weight $\sum_{i=1}^{n} \alpha_i = 0$, then the Leibniz vector function is constant.*

2. *If the total weight* $\sum_{i=1}^{n} \alpha_i \neq 0$*, then the Leibniz vector function is a bijection from* E *onto* E.

Proof. Let $(A_i, \alpha_i)_{i=1}^{n}$ be a system of weighted points in an affine space E. Define

$$f : E \longrightarrow \vec{E}, \qquad f(M) = \sum_{i=1}^{n} \alpha_i \overrightarrow{MA_i},$$

where $\vec{E}$ is the direction (vector) space of (E). Fix an arbitrary point $M_0 \in E$. For any $M \in E$ we have the decomposition

$$\overrightarrow{MA_i} = \overrightarrow{MM_0} + \overrightarrow{M_0A_i},$$

so

$$f(M) = \sum_{i=1}^{n} \alpha_i \big(\overrightarrow{MM_0} + \overrightarrow{M_0A_i}\big) = \sum_{i=1}^{n} \alpha_i \overrightarrow{MM_0} + \sum_{i=1}^{n} \alpha_i, \overrightarrow{M_0A_i}.$$

Write $S = \sum_{i=1}^{n} \alpha_i$ and $C = \sum_{i=1}^{n} \alpha_i \overrightarrow{M_0A_i}$ (a fixed vector depending on the system and the chosen M_0.

Then

$$f(M) = S\overrightarrow{MM_0} + C. \tag{1.3}$$

1. If $S = 0$ then by (1.3) we get $f(M) = C$ for every $M \in E$. Hence f is constant.

2. If $S \neq 0$ then the map $M \mapsto S\overrightarrow{MM_0}$ is a bijection from E onto $\vec{E}$ (its linear part is multiplication by the nonzero scalar S, which is invertible on the vector space). Adding the fixed vector C (a translation) preserves bijectivity. Concretely:

Injective: If $f(M) = f(N)$ then

$$S\overrightarrow{MM_0} = S\overrightarrow{NM_0},$$

hence

$$S\overrightarrow{MN} = 0.$$

As $S \neq 0$ this implies

$$\overrightarrow{MN} = 0,$$

so $M = N$.

Surjective: Given any vector $v \in \vec{E}$, set $\overrightarrow{MM_0} = \frac{1}{S}(v - C)$.

There exists a unique point $M \in E$ with that vector from M to M_0, and for that M we have $f(M) = v$.

Therefore, if the total weight S is nonzero, f is a bijection from E onto $\vec{E}$. $\square$

1.7 exercices

Exercise 1.7.1 *Let $\mathbb{R}^3$ be equipped with an orthonormal coordinate system $(O, \vec{i}, \vec{j}, \vec{k})$. We are given three points $A(2,1,3), B(-3,-1,7), C(3,2,4)$ and a point H.*

1. *Show that the three points (A, B, C) are not collinear.*
2. *Let D be the affine line with the given parametric representation:*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7 \\ 0 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

   Show that the affine line D is orthogonal to the plane (ABC).
3. *Determine the Cartesian equation of the plane (ABC).*
4. *Show that (H) is the barycenter of the weighted points (A, α), (B, β), (C, γ) for appropriate weights (α, β, γ).*

Solution

In the orthonormal coordinate system $(O, \vec{i}, \vec{j}, \vec{k})$ of $\mathbb{R}^3$, we have:

$$A(2,1,3), \quad B(-3,-1,7), \quad C(3,2,4),$$

and the line D given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7 \\ 0 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \qquad \lambda \in \mathbb{R}.$$

1. Non-collinearity

   To show that A, B, C are not collinear, we verify that the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not parallel.

$$\overrightarrow{AB} = B - A = (-5, -2, 4),$$

$$\overrightarrow{AC} = C - A = (1, 1, 1).$$

Their cross product is

$$\overrightarrow{AB} \times \overrightarrow{AC} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & -2 & 4 \\ 1 & 1 & 1 \end{pmatrix} = (-6, 9, -3) \neq (0, 0, 0).$$

Since the cross product is non-zero, A, B, C are not collinear.

2. Orthogonality

   The direction vector of line D is

$$\mathbf{v} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix},$$

   and it passes through $P_0(-7, 0, 4)$.

   A normal vector to the plane (ABC) is given by

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = (-6, 9, -3).$$

   We notice that

$$\mathbf{n} = -3 \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = -3\mathbf{v}.$$

   Hence, the direction of D is parallel to the normal vector of the plane (ABC), which means that D is orthogonal to the plane (ABC).

3. Cartesian equation of plane (ABC)

   Since $\mathbf{n} = (-6, 9, -3)$ is a normal vector and $A(2, 1, 3)$ lies on the plane, the equation is

$$-6(x - 2) + 9(y - 1) - 3(z - 3) = 0.$$

Expanding and simplifying:

$$-6x + 12 + 9y - 9 - 3z + 9 = 0 \quad \Rightarrow \quad -6x + 9y - 3z + 12 = 0.$$

Dividing by -3, we get a simpler form:

$$\boxed{2x - 3y + z - 4 = 0.}$$

This is the Cartesian equation of the plane (ABC).

4. Intersection point $H = D \cap (ABC)$ and barycenter

   A general point on D is

$$M(\lambda) = (-7 + 2\lambda, , -3\lambda, , 4 + \lambda).$$

   Substitute into the plane equation:

$$2(-7 + 2\lambda) - 3(-3\lambda) + (4 + \lambda) - 4 = 0.$$

   Simplify:

$$-14 + 4\lambda + 9\lambda + 4 + \lambda - 4 = 14\lambda - 14 = 0 \quad \Rightarrow \quad \lambda = 1.$$

   Therefore, the intersection point is

$$H = M(1) = (-7 + 2, , -3, , 4 + 1) = (-5, -3, 5).$$

   We seek scalars (α, β, γ) (not all zero) such that

$$H = \frac{\alpha A + \beta B + \gamma C}{\alpha + \beta + \gamma}.$$

   This is equivalent to

$$\alpha(A - H) + \beta(B - H) + \gamma(C - H) = \mathbf{0}.$$

   Compute the vectors:

$$A - H = (7, 4, -2), \quad B - H = (2, 2, 2), \quad C - H = (8, 5, -1).$$

Hence, we must solve

$$\alpha(7, 4, -2) + \beta(2, 2, 2) + \gamma(8, 5, -1) = (0, 0, 0).$$

Solving this system yields a non-trivial solution, for example:

$$(\alpha, \beta, \gamma) = (-2, -1, 2).$$

Verification:

$$-2(7, 4, -2) - 1(2, 2, 2) + 2(8, 5, -1) = (0, 0, 0).$$

Since $\alpha + \beta + \gamma = -1 \neq 0$, the barycenter is well-defined and equals

$$H = \frac{-2A - 1B + 2C}{-1} = (-5, -3, 5).$$

Thus, H is the barycenter of points A, B, C with weights $(\alpha, \beta, \gamma) = (-2, -1, 2)$ (up to a non-zero scalar multiple).

Exercise 1.7.2 *in $\mathbb{R}^3$ with an Orthonormal Basis $(O, \vec{i}, \vec{j}, \vec{k})$.*

We are given the points:

$$A(3, -2, 1), \quad B(5, 2, -3), \quad C(6, -2, -2), \quad D(4, 3, 2).$$

1. *Right isosceles triangle.*

   Show that the triangle (ABC) is right isosceles.

2. *Normal vector.*

   Show that the vector $N = (2, 1, 2)$ is a normal vector to the plane (ABC).

3. *Equation of the plane.*

   Deduce the Cartesian equation of the plane (ABC).

4. *Distance.*

   Compute the distance from point D to the plane (ABC).

5. *Isobarycenter.*

   Calculate the coordinates of the isobarycenter of the tetrahedron $(ABCD)$.

Solution

Points in an orthonormal frame of $\mathbb{R}^3$:

$$A(3,-2,1),\quad B(5,2,-3),\quad C(6,-2,-2),\quad D(4,3,2).$$

1. Show that triangle (ABC) is a right isosceles triangle Compute the edge vectors (from the first point of each pair to the second):

$$\overrightarrow{AB} = B - A = (5-3, 2-(-2), -3-1) = (2,4,-4),$$
$$\overrightarrow{AC} = C - A = (6-3, -2-(-2), -2-1) = (3,0,-3),$$
$$\overrightarrow{BC} = C - B = (6-5, -2-2, -2-(-3)) = (1,-4,1).$$

Compute dot-products to detect perpendicularity:

$$\overrightarrow{AC} \cdot \overrightarrow{BC} = 3 \cdot 1 + 0 \cdot (-4) + (-3) \cdot 1 = 0.$$

Since $\overrightarrow{AC} \cdot \overrightarrow{BC} = 0$, vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$ are perpendicular. This shows the angle at point (C) is a right angle.
Compute squared lengths of these two legs:

$$|\overrightarrow{AC}|^2 = 3^2 + 0^2 + (-3)^2 = 18,$$
$$|\overrightarrow{BC}|^2 = 1^2 + (-4)^2 + 1^2 = 18.$$

Thus $|\overrightarrow{AC}| = |\overrightarrow{BC}| = \sqrt{18}$. The two legs meeting at (C) have equal length, so triangle (ABC) is isosceles with the equal legs (AC) and (BC).

Combining perpendicularity and equal legs, triangle (ABC) is right isosceles (right angle at (C)).

2. Show that $N = (2,1,2)$ is a normal vector to the plane (ABC)

A normal vector to the plane can be obtained by the cross product of two non-collinear vectors lying in the plane, for example $\overrightarrow{AB} \times \overrightarrow{AC}$.

Compute the cross product:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & -4 \\ 3 & 0 & -3 \end{vmatrix}$$
$$= \big(4 \cdot (-3) - (-4) \cdot 0, -\big(2 \cdot (-3) - (-4) \cdot 3\big), 2 \cdot 0 - 4 \cdot 3\big)$$
$$= (-12, -6, -12)$$

So

$$\overrightarrow{AB} \times \overrightarrow{AC} = (-12, -6, -12).$$

This vector is a scalar multiple of $N = (2, 1, 2)$:

$$(-12, -6, -12) = -6(2, 1, 2).$$

Hence $N = (2, 1, 2)$ is collinear with the normal

$$\overrightarrow{AB} \times \overrightarrow{AC},$$

so (N) is indeed a normal vector to the plane (ABC).

3. Cartesian equation of the plane (ABC)

Using normal $N = (2, 1, 2)$ and point $A(3, -2, 1)$, the plane equation is

$$2(x - 3) + 1(y - (-2)) + 2(z - 1) = 0.$$

Expand:

$$2x - 6 + y + 2 + 2z - 2 = 0 \quad \Rightarrow \quad 2x + y + 2z - 6 = 0.$$

So the Cartesian equation of (ABC) is

$$\boxed{, 2x + y + 2z - 6 = 0,}.$$

Equivalently, $2x + y + 2z = 6$.

4. Distance from $D(4, 3, 2)$ to the plane (ABC) Distance formula from point $P(x_0, y_0, z_0)$ to

plane $ax + by + cz + d = 0$ is

$$\operatorname{dist}(P, \Pi) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Here $a = 2,\ \underline{=}1,\ \ c = 2$. From the plane equation $2x + y + 2z - 6 = 0$ we have $d = -6$.

Compute numerator at $D(4, 3, 2)$:

$$|2 \cdot 4 + 1 \cdot 3 + 2 \cdot 2 - 6| = |8 + 3 + 4 - 6| = 9.$$

Denominator:

$$\sqrt{2^2 + 1^2 + 2^2} = \sqrt{4 + 1 + 4} = 3.$$

Thus the distance is

$$\operatorname{dist}(D, , (ABC)) = \frac{9}{3} = 3.$$

So $\boxed{3}$ is the distance.

5. Coordinates of the isobarycenter (centroid) of tetrahedron $(ABCD)$.

The isobarycenter (centroid) of four vertices with equal weights is the average of their coordinates:

$$G = \frac{A + B + C + D}{4}.$$

Compute coordinate-wise:

- x-coordinate: $(3 + 5 + 6 + 4)/4 = 18/4 = 4.5$.
- y-coordinate: $(-2 + 2 + (-2) + 3)/4 = 1/4 = 0.25$.
- z-coordinate: $(1 + (-3) + (-2) + 2)/4 = (-2)/4 = -0.5$.

So the centroid is

$$\boxed{G\,(4.5, 0.25, -0.5)}.$$

Exercise 1.7.3 *Let (ABC) be an equilateral triangle of side (3). Put a convenient coordinate system: place (BC) on the $x - axis$ with*

$$B\big(-\tfrac{3}{2}, 0\big), \qquad C\big(\tfrac{3}{2}, 0\big),$$

and then

$$A\big(0, h\big), \qquad h = \frac{3\sqrt{3}}{2}$$

since the height of an equilateral triangle of side (3) *is* $(3\sqrt{3}/2)$.

The centroid (G) *of triangle* (ABC) *is the average of the vertices:*

$$G\big(0, \tfrac{h}{3}\big).$$

Let (H) *be the reflection of* (G) *across the side* (BC) *(the* x*-axis). Thus*

$$H\big(0, -\tfrac{h}{3}\big).$$

1. (H) is the barycenter of $\{(A,1),(B,-2),(C,-2)\}$.

We check the barycenter condition: (H) is the barycenter of the weighted family $\{(A,1),(B,-2),(C,-2)\}$ *iff*

$$1\overrightarrow{HA} + (-2)\overrightarrow{HB} + (-2)\overrightarrow{HC} = \vec{0}.$$

Compute the vectors from (H) to the vertices:

$$\overrightarrow{HA} = A - H = \big(0, ; h - (-\tfrac{h}{3})\big) = \big(0, \tfrac{4h}{3}\big),$$

$$\overrightarrow{HB} = B - H = \big(-\tfrac{3}{2}, ; 0 - (-\tfrac{h}{3})\big) = \big(-\tfrac{3}{2}, \tfrac{h}{3}\big),$$

$$\overrightarrow{HC} = C - H = \big(\tfrac{3}{2}, \tfrac{h}{3}\big).$$

Now form the weighted sum:

$$\overrightarrow{S} = \overrightarrow{HA} - 2\overrightarrow{HB} - 2\overrightarrow{HC}.$$

For the x*-component:*

$$0 - 2\big(-\tfrac{3}{2}\big) - 2\big(\tfrac{3}{2}\big) = 0 + 3 - 3 = 0.$$

For the y*-component:*

$$\tfrac{4h}{3} - 2\cdot\tfrac{h}{3} - 2\cdot\tfrac{h}{3} = \tfrac{4h}{3} - \tfrac{2h}{3} - \tfrac{2h}{3} = 0.$$

So

$$\overrightarrow{S} = \vec{0}$$

. *Hence* (H) *satisfies the barycenter equation and therefore is the barycenter of* $\{(A,1),(B,-2),(C,-2)\}$.

2. Compute the scalar product $\overrightarrow{HA}\cdot\overrightarrow{HC}$.

We have

$$\overrightarrow{HA} = \left(0, \tfrac{4h}{3}\right), \qquad \overrightarrow{HC} = \left(\tfrac{3}{2}, \tfrac{h}{3}\right).$$

Thus

$$\overrightarrow{HA} \cdot \overrightarrow{HC} = 0 \cdot \tfrac{3}{2} + \tfrac{4h}{3} \cdot \tfrac{h}{3} = \frac{4h^2}{9}.$$

With $(h = \dfrac{3\sqrt{3}}{2})$ *we get*

$$h^2 = \frac{9 \cdot 3}{4} = \frac{27}{4}, \quad \textit{so} \quad \frac{4h^2}{9} = \frac{4 \cdot (27/4)}{9} = \frac{27}{9} = 3.$$

Therefore

$$\boxed{\overrightarrow{HA} \cdot \overrightarrow{HC} = 3}.$$

Chapter 2

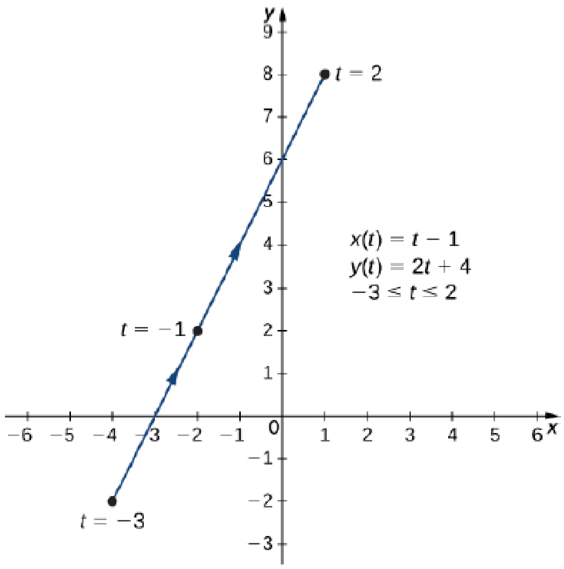
Parametric equations and plane curves in affine space

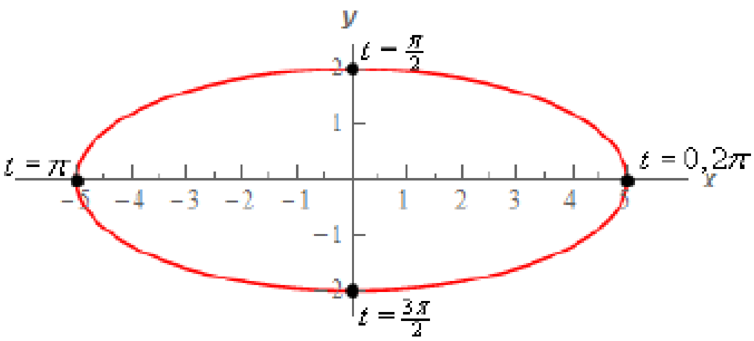
2.1 Parameterized curve

Definition 2.1.1 *Let $I = [a; b]$ be an interval. If $\gamma(t) = (f(t), g(t)) : I \to \mathbb{R}^2$ is a function defined on I, then the set of points $x = f(t),\ \ y = g(t)$ is called a* ***parametric curve.***
The relations $x = f(t),\ \ y = g(t)$ are called parametric equations.
The graph of parametric equations is called a parametric curve or plane curve, and is denoted by $C = \gamma(I)$. Sometimes the function $\gamma(t)$ itself is called a parametric curve. The variable t is called a parameter.

Example 2.1.1 *Sketch the curves described by the following parametric equations:*

- $x(t) = t - 1,\ \ y(t) = 2t + 4,\quad t \in [-3, 2].$
- $x(t) = 5\cos(t),\ \ y(t) = 2\sin(t),\quad t \in [0, 2\pi].$





Example 2.1.2 .

Find two different pairs of parametric equations to represent the graph of

$$y = 2x^2 - 3$$

1. First, it is always possible to parameterize a curve by defining $x(t) = t$, then replacing x with t in the equation for $y(t)$. This gives the parameterization $y(t) = 2t^2 - 3$.
2. We have complete freedom in the choice for the second parameterization. For example, we can choose $x(t) = 3t - 2$.

The only thing we need to check is that there are no restrictions imposed on$x$; that is, the range of $x(t)$ is all real numbers. This is the case for $x(t) = 3t - 2$.

Now since $y = 2x^2 - 3$, we can substitute $x(t) = 3t - 2$ for x. This gives

$$y(t) = 18t^2 - 24t + 6$$

Therefore, a second parameterization of the curve can be written as

$$x(t) = 2t - 3$$
$$y(t) = 18t^2 - 24t + 6$$

2.2 Eliminating the Parameter

To better understand the graph of a curve represented parametrically, it is useful to rewrite the two equations as a single equation relating the variables x and y.

Then we can apply any previous knowledge of equations of curves in the plane to identify the curve. For example, the equations describing the plane curve γ defined by $x(t) = t^2-3, \;\; y(t) = 2t+1, \;\; t \in [-2, 3]$

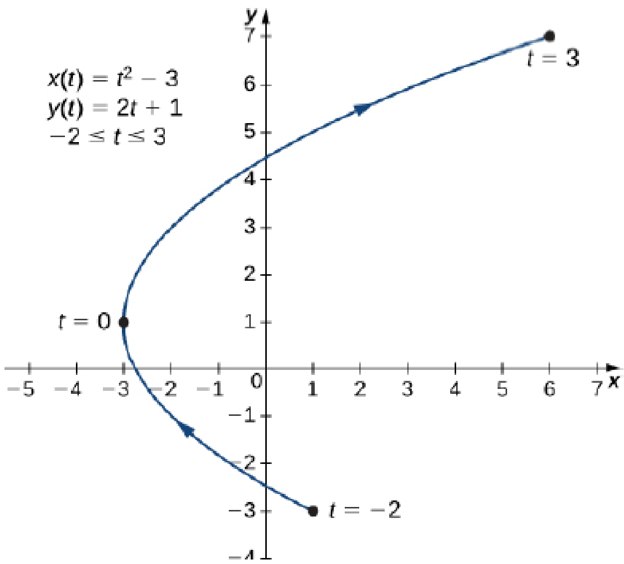
Solving the second equation for t gives

$$t = \frac{y-1}{2}$$

This can be substituted into the first equation:

$$x = \left(\frac{y-1}{2}\right)^2 - 3 = \frac{1}{4}y^2 - \frac{1}{2}y - \frac{11}{4}.$$

This equation describes x as a function of y. These steps give an example of eliminating the parameter. The graph of this function is a parabola opening to the right.



2.3 Tangents with Parametric Equations

Let's explore how to find tangent lines to curves given by parametric equations.

2.3.1 Derivative for Parametric Equations $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \text{ provided } \frac{dx}{dt} \neq 0.$$

2.3.2 Derivative for Parametric Equations $\frac{dx}{dy}$

$$\frac{dx}{dy} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}} \text{ provided } \frac{dy}{dt} \neq 0.$$

2.3.3 Second derivative for a parametric equation

If the relations $x = f(t)$, $y = g(t)$ define y as a twice differentiable function of x at the point where $\frac{dx}{dt} \neq 0$, then

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}}.$$

then In this section we want to find the tangent lines to the parametric equations given by,

$$x = f(t), \quad y = g(t)$$

To do this let's first recall how to find the tangent line to $y = F(x)$ at $(x(t_0), y(t_0))$. Here the tangent line is given by

$$\begin{vmatrix} x - x(t_0) & x'(t_0) \\ y - y(t_0) & y'(t_0) \end{vmatrix} = 0$$

where $x'(t) = \frac{dx}{dt}$ and $y'(t) = \frac{dy}{dt}$.
Then

$$y = \frac{y'(t_0)}{x'(t_0)}(x - x(t_0)) + y(t_0)$$

or

$$y = \frac{dy}{dx}(t_0)(x - x(t_0)) + y(t_0)$$

Example 2.3.1 *Find the tangent line(s) to the parametric curve given by*

$$\gamma(t) = (3t^2, 2t^3)$$

The tangent line to the curve γ at a point $M(t)$ is given by

$$\begin{vmatrix} x - 3t^2 & 6t \\ y - 2t^3 & 6t^2 \end{vmatrix} = 0 \implies y = tx - t^3$$

2.3.4 Horizontal Tangent for Parametric Equations

Horizontal tangents will occur where the derivative is zero and that means that we'll get horizontal tangent at values of t for which we have

$$\frac{dy}{dt} = 0, \text{provided} \frac{dx}{dt} \neq 0$$

2.3.5 Vertical Tangent for Parametric Equations

Vertical tangents will occur where the derivative is not defined and so we'll get vertical tangents at values of t for which we have,

$$\frac{dx}{dt} = 0, \text{provided} \frac{dy}{dt} \neq 0$$

Example 2.3.2 *Determine the $(x; y)$ coordinates of the points where the following parametric equations will have horizontal or vertical tangents.*

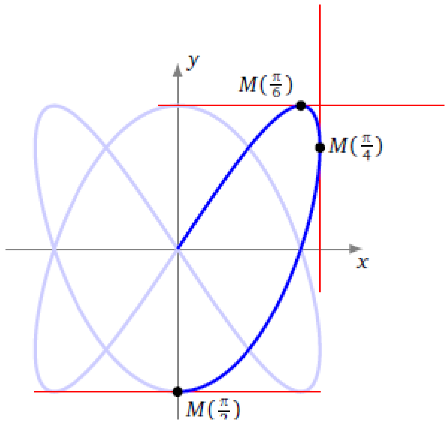
$$\begin{cases} x(t) = \sin(2t) \\ y(t) = \sin(3t) \end{cases} \quad t \in [-\pi, \pi]$$

Horizontal Tangents
$x'(t) = 0 \Leftrightarrow t = \frac{\pi}{4}$
then $(x(\frac{\pi}{4}), y(\frac{\pi}{4})) = (1, \frac{\sqrt{2}}{2})$.

Vertical Tangents

$y'(t) = 0 \Leftrightarrow t = \dfrac{\pi}{6}$ *or* $t = \dfrac{\pi}{2}$

then $(x(\dfrac{\pi}{2}), y(\dfrac{\pi}{2})) = (0, -1)$, *and* $(x(\dfrac{\pi}{6}), y(\dfrac{\pi}{6})) = (1/2, 1)$



2.4 Arc Length of a Parametric Curve

Let's go over the arc length of a parametric curve, a fundamental concept in calculus and differential geometry.

proposition 2.4.1 *If* (γ) *is given by* $(x, y) = (f(t), g(t))$, $a \leq t \leq b$, *and* $f' = \dfrac{df}{dt}$, $g' = \dfrac{dg}{dt}$ *are continuous and not simultaneously zero and* (γ) *is one-to-one. Then the length of* (γ) *is given by*

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2} dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Proof. We have

$$\|\gamma'(t)\| = \sqrt{(f'(t))^2 + (g'(t))^2}$$

and for any partition P of $[a, b]$; we have

$$\begin{aligned} l(\gamma, P) &= \sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\| \\ &= \sum_{i=1}^{n} \| \int_{t_{i-1}}^{t_i} \gamma'(t)dt\| \\ &\leq \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|\gamma'(t)\|dt \\ &= \int_a^b \|\gamma'(t)\|dt. \end{aligned}$$

So $L \leq \int_a^b \|\gamma'(t)\|dt$.

The corresponding inequality holds on any interval.

Now, for $a \leq t \leq b$, define $s(t)$ to be the arclength of the curve γ on the interval $[a, t]$. Then for $h > 0$ we have

$$\begin{aligned}\frac{\|\gamma(t+h) - \gamma(t)\|}{h} &\leq \frac{s(t+h) - s(t)}{h} \\ &\leq \frac{1}{h}\int_t^{t+h} \|\gamma'(u)\|du,\end{aligned}$$

since $s(t+h) - s(t)$ is the arclength of the curve γ on the interval $[t, t+h]$. Now

$$\begin{aligned}\lim_{h\to 0^+} \frac{\|\gamma(t+h) - \gamma(t)\|}{h} &= \|\gamma'(t)\| \\ &= \lim_{h\to 0^+} \frac{1}{h}\int_t^{t+h} \|\gamma'(u)\|du\end{aligned}$$

Therefore, by the squeeze principle,

$$\lim_{h\to 0^+} \frac{s(t+h) - s(t)}{h} = \|\gamma'(t)\|.$$

A similar argument works for $h < 0$, and we conclude that $s'(t) = \|\gamma'(t)\|$.

Therfore,

$$s(t) = \int_a^t \|\gamma'(u)\|du, \quad a \leq t \leq b.$$

and, in particular,$s(b) = L = \int_a^b \|\gamma'(t)\|dt$, as desired.

$\triangleright$ At each instant t, the velocity vector $\gamma'(t)$ gives the direction and speed of movement.

$\triangleright$ Taking the norm $\|\gamma'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}$ gives the instantaneous speed, and integrating that over the interval gives the total distance traveled. $\square$

Example 2.4.1 *Find the arc length of the semicircle defined by the equations*

$$\begin{cases} x(t) = 3\cos(t) \\ y(t) = 3\sin(t) \end{cases}, \quad 0 \leq t \leq \pi$$

solution.

We have

$$\frac{dx}{dt} = -3\sin(t),$$

and

$$\frac{dy}{dt} = 3\cos(t)$$

then

$$L = \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^{\pi} \sqrt{9\cos^2(t) + 9\sin^2(t)} dt = 3\pi.$$

Note that the formula for the arc length of a semicircle is πR and the radius of this circle is 3. This is a great example of using calculus to derive a known formula of a geometric quantity.

2.4.1 Length of a curve $y = f(x)$

When x = t in the parameterization, we obtain

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

2.5 Surface Area Generated by a Parametric Curve

2.5.1 Area under a Parametric Curve

theorem 2.1 *Consider the non-self-intersecting plane curve defined by the parametric equations*

$$x = f(t), \quad y = g(t) \ \text{ and } \ a \leq t \leq b$$

and assume that x is differentiable. The area under this curve is given by

$$A = \int_a^b g(t) f'(t) dt = \int_a^b y(t) x'(t) dt$$

2.5.2 Area of Surface of Revolution

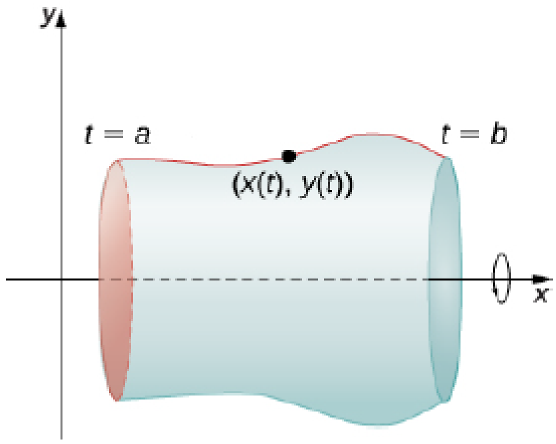
The area of the surface generated by revolving the parametric curve $(x, y) = (f(t), g(t)), a \leq t \leq b$ about (either x or y axis) is given as follows

$$A = 2\pi \int_a^b y ds \quad \text{if revolved about x-axis}$$

$$A = 2\pi \int_a^b x ds \quad \text{if revolved about y-axis}$$

where

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$



Example 2.5.1 *Find the surface area of revolution of the parameterized circle*

$$x = \cos(t), \quad y = 1 + \sin(t) \;\; \textit{and} \;\; 0 \leq t \leq 2\pi$$

about the x-axis.

$$\begin{aligned} A =& 2\pi \int_0^{2\pi} y\sqrt{(x'(t))^2 + (y'(t))^2} dt \\ =& 2\pi \int_0^{2\pi} (1 + \sin(t)) dt \\ =& 4\pi^2 \end{aligned}$$

Example 2.5.2 *Determine the surface area of the solid obtained by rotating $y = (x)^{\frac{1}{3}}$, $1 \leq y \leq 2$ about the $y - axis$.*

We have

$$y'(t) = \frac{1}{3} x^{-\frac{2}{3}}$$

and

$$\sqrt{1 + (y')^2} = \frac{\sqrt{9x^{4/3} + 1}}{3x^{2/3}}$$

We'll also need to get new limits. That isn't too bad however. All we need to do is plug in the given y's into our equation and solve to get that the range of x's is $1 \leq x \leq 8$. The integral for *the surface area is then,*

$$
\begin{aligned}
S &= 2\pi \int_1^8 x\sqrt{1+(y')^2}dx \\
&= 2\pi \int_1^8 x\frac{\sqrt{9x^{4/3}+1}}{3x^{2/3}}dx \\
&= \frac{2\pi}{3} \int_1^8 x^{1/3}\sqrt{1+9x^{4/3}}dx
\end{aligned}
$$

Using the substitution

$$u = 9x^{4/3}+1 \implies du = 12x^{1/3}dx$$

the integral becomes,

$$
\begin{aligned}
S &= \frac{\pi}{18}\int_{10}^{145} \sqrt{u}du \\
&= \frac{\pi}{27}\left[u^{3/2}\right]_{10}^{145} \\
&= \frac{\pi}{27}\left[(145)^{3/2}-(10)^{3/2}\right].
\end{aligned}
$$

2.6 Exercices

Exercise 2.6.1 *Find two different sets of parametric equations to represent the graph of* $y = x^2+2x$.

Solution We need two different sets of parametric equations:

1. **Simple substitution:** Let $x = t$. Then

$$y = t^2 + 2t.$$

   So a parametric representation is

$$\boxed{x = t, \quad y = t^2+2t, \quad t \in \mathbb{R}}.$$

2. **Completing the square:**

$$y = x^2+2x = (x+1)^2-1.$$

Let $x+1=t \implies x=t-1$. Then $y=t^2-1$. Therefore, another parametric representation is

$$\boxed{x=t-1, \quad y=t^2-1, \quad t\in\mathbb{R}}.$$

Exercise 2.6.2 *Find a parameterizaation for the ellips* $\dfrac{x^2}{4}+\dfrac{y^2}{25}=1$.

Solution.
Parametric Equations for the Ellipse $\dfrac{x^2}{4}+\dfrac{y^2}{25}=1$ is

$$x=a\cos t, \quad y=b\sin t, \quad t\in[0,2\pi].$$

Here $a=2$ and $b=5$, so we get

$$\boxed{x=2\cos t, \quad y=5\sin t, \quad t\in[0,2\pi]}.$$

Exercise 2.6.3 *Find two different sets of parametric equations to represent each of the following graph:*

1. $x^2+y^2=9, \ \ y+z=2$
2. x^2+2x

Eliminate the parameter for each of the plane curves described by the following parametric equations

1. $x(t)=4\cos(t)+3, \ \ y(t)=2\sin(t)+1$
2. $x(t)=t^2-3, \ \ y(t)=2t+1$

find all points on the curve that have the given slope.

1. $x(t)=4\cos(t), \ \ y(t)=4\sin(t) \ \ slope=\dfrac{dy}{dx}=0.5$
2. $x(t)=2\cos(t), \ \ y(t)=8\sin(t) \ \ slope=\dfrac{dy}{dx}=-1$
3. $x(t)=t+\dfrac{1}{t}, \ \ y(t)=t-\dfrac{1}{t}, \ \ slope=1$

Solution.

1. Two Sets of Parametric Equations

   1. $x^2 + y^2 = 9,\ y + z = 2$:

   $$x = 3\cos(t),\ y = 3\sin(t),\ z = 2 - 3\sin(t),\ t \in [0, 2\pi]$$

   Alternative:

   $$x = 3\cos(s),\ y = 3\sin(s),\ z = 2 - y$$

   2. $y = x^2 + 2x$ as above:

   $$x = t,\ y = t^2 + 2t \quad \text{or} \quad x = t - 1,\ y = t^2 - 1$$

2. Eliminate the Parameter

   1. $x(t) = 4\cos(t) + 3,\ y(t) = 2\sin(t) + 1$:

   $$\frac{(x-3)^2}{16} + \frac{(y-1)^2}{4} = 1$$

   2. $x(t) = t^2 - 3,\ y(t) = 2t + 1$:

   $$t = \frac{y-1}{2} \implies x = \left(\frac{y-1}{2}\right)^2 - 3 = \frac{(y-1)^2}{4} - 3$$

3. Points on Curves with Given Slope

   1. $x(t) = 4\cos(t),\ y(t) = 4\sin(t),\ dy/dx = 0.5$:

   $$\frac{dy}{dx} = \frac{-\sin(t)}{\cos(t)} = -\tan(t) = 0.5 \implies t = -\arctan(0.5) + k\pi$$

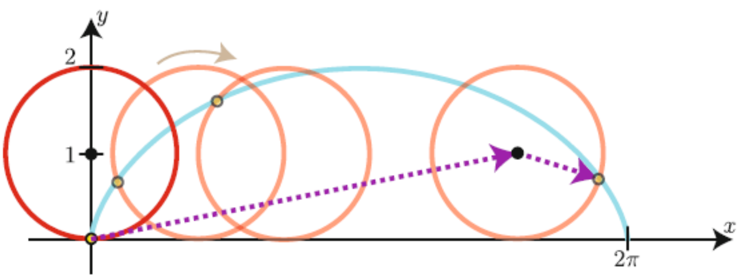
   2. $x(t) = 2\cos(t),\ y(t) = 8\sin(t),\ dy/dx = -1$:

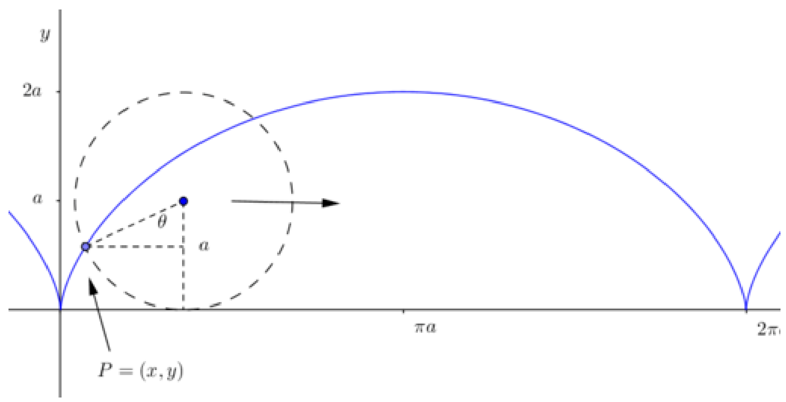
   $$\frac{dy}{dx} = \frac{8\cos(t)}{-2\sin(t)} = -4\cot(t) = -1 \implies t = \arctan(4) + k\pi$$

3. $x(t) = t + \frac{1}{t}$, $y(t) = t - \frac{1}{t}$, $dy/dx = 1$:

$$\frac{dy}{dx} = \frac{1+\frac{1}{t^2}}{1-\frac{1}{t^2}} = 1 \implies t^2 = 0 \text{ (no finite solution)}$$

Exercise 2.6.4 *Consider a circle of radius a rolling without slipping along the x-axis of a cartesian plane. Consider the point P on the circumference of this circle which is at the origin when its center is on the y-axis.*
Consider the cycloid traced out by the point P. Let (x, y) be the coordinates of P as it travels over the plane.





1. *What is the position of the center of the wheel after the tire has rotated through an angle of θ?*

2. *Show that, the point $P = (x, y)$ is described by the equations:*

$$\begin{cases} x(\theta) = a(\theta - \sin(\theta)) \\ y(\theta) = a(1 - \cos(\theta)) \end{cases}$$ *. (Called: parametric equation of Cycloid)*

3. *Find all points on the Cycloid curve at which horizontal and vertical tangents exist.*

4. *Eliminate the parameter t of the parametric equations (Cycloid).*

5. *Determine the length the parametric curve given by the last parametric equations where* $\theta \in [0; 2\pi]$

6. *Find the area under the curve of the cycloid where* $a = 1$ *and* $\theta \in [0; 2\pi]$.

Solution.

1. Center of circle after rotation θ: $(x_C, y_C) = (a\theta, a)$

2. Parametric equations of cycloid:

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta)$$

3. Horizontal tangents: $dy/d\theta = a\sin\theta = 0 \implies \theta = k\pi$

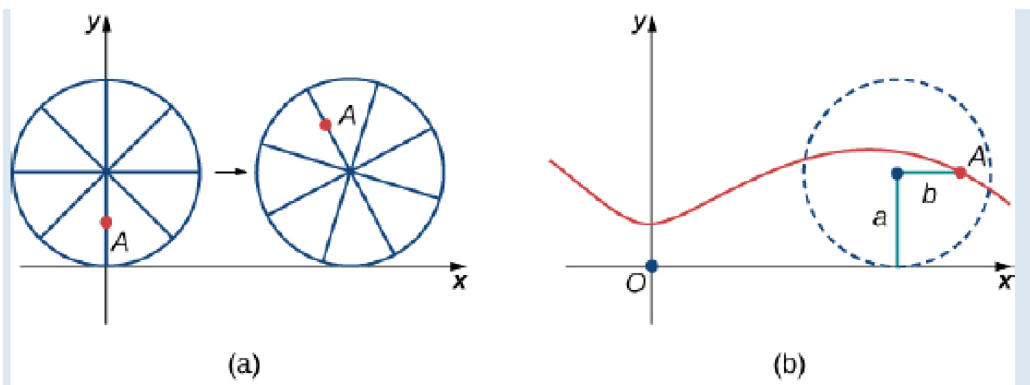
   Vertical tangents: $dx/d\theta = a(1 - \cos\theta) = 0 \implies \theta = 2k\pi$

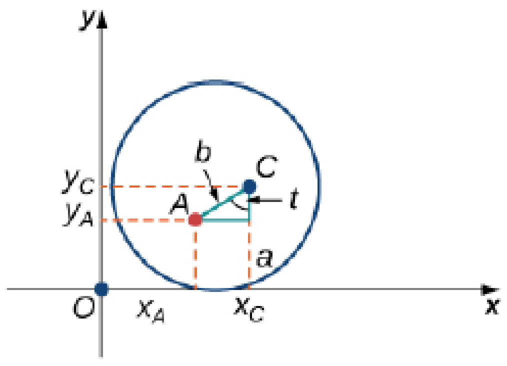
4. Eliminating θ: $y = a(1 - \cos((x + a\sin\theta)/a))$ (implicit)

5. Arc length for $\theta \in [0, 2\pi]$: $L = 8a$

6. Area under one arch of cycloid: $A = 3\pi$

Exercise 2.6.5 .





(c)

(a) The ant climbs up one of the spokes toward the center of the wheel.

(b)The ant's path of motion after he climbs closer to the center of the wheel. This is called a curtate cycloid.

(c) The new setup, now that the ant has moved closer to the center of the wheel.

1. *What is the position of the center of the wheel after the tire has rotated through an angle of t?*

2. *Use geometry to find expressions for $x_C - x_A$ and for $y_C - y_A$.*

3. *On the basis of your answers to parts 1 and 2, what are the parametric equations representing the curtate cycloid?*

Exercise 2.6.6 1. *Show that the total length of the ellipse $x(t) = a\sin(t), \;\; y(t) = b\cos(t)$ is*

$$L = 16\int_0^{\frac{\pi}{2}} \sqrt{1 - e^2\sin^2(t)}dt$$

where $e = \dfrac{c}{a}$ and $c = \sqrt{a^2 - b^2}$

2. *Find the length of the curve $x(t) = e^t - t, \;\; y(t) = 4e^{\frac{t}{2}}$ and $-8 \le t \le 3$*

3. *Find the surface area generated by revolving $x(t) = 3t^2, \;\; y(t) = 2t^3$, and $0 \le t \le 5$ about the y-axis.*

4. *Find the surface area generated by revolving $x = t^2, \;\; y = 2t, \;\; 0 \le t \le 4$ about the x-axis.*

Solution.

1. Ellipse $x = a \sin t, y = b \cos t$:

$$L = 16 \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 t} \, dt, \quad e = \frac{c}{a}, \ c = \sqrt{a^2 - b^2}$$

2. $x = e^t - t, y = 4e^{t/2}, -8 \leq t \leq 3$:

$$L = \int_{-8}^{3} \sqrt{(e^t - 1)^2 + (2e^{t/2})^2} \, dt = \int_{-8}^{3} \sqrt{e^{2t} - 2e^t + 1 + 4e^t} \, dt = \int_{-8}^{3} \sqrt{e^{2t} + 2e^t + 1} \, dt$$

3. Surface area about y-axis: $x = 3t^2, y = 2t^3, 0 \leq t \leq 5$:

$$S = 2\pi \int_0^5 x \sqrt{(dx/dt)^2 + (dy/dt)^2} \, dt = 2\pi \int_0^5 3t^2 \sqrt{(6t)^2 + (6t^2)^2} \, dt$$

4. Surface area about x-axis: $x = t^2, y = 2t, 0 \leq t \leq 4$:

$$S = 2\pi \int_0^4 y \sqrt{(dx/dt)^2 + (dy/dt)^2} \, dt = 2\pi \int_0^4 2t \sqrt{(2t)^2 + 2^2} \, dt$$

Chapter 3

Plotting a Curve in cartesian and Polar Coordinates

3.1 Plotting a Curve in cartesian

3.1.1 Singular and regular points

At a point $M_0 = M(x(t_0), y(t_0))$ where $M'(x(t_0), y(t_0)) \neq \vec{0}$, we say that the curve has a regular point, and a point $M(x(t_0), y(t_0))$ where $M'(x(t_0), y(t_0)) = \vec{0}$ is called a singular point.

Example 3.1.1 *For* $\gamma'(t) = (t; t^2)$

$$\gamma'(t) = (1, 2t) \neq (0, 0)$$

This derivative is never zero, so every point is regular.

Example 3.1.2 *For* $\gamma(t^2 + 1, t^3 - 2)$.
Then $\gamma'(t) = (2t, 3t^2)$*,*
At $t = 0 : \gamma'(0) = (0, 0)$*, then, this is a singular point*

3.1.2 Symmetry in cartesian Coordinates

1-X**-Axis Symmetry.**

A graph is said to be symmetric about the X-axis if

$$\begin{cases} f(a-t) = f(t) \\ g(a-t) = -g(t) \end{cases} \quad a \in \mathbb{R}$$

2-Y**-Axis Symmetry.**

A graph is said to be symmetric about the Y-axis if

$$\begin{cases} f(a-t) = -f(t) \\ g(a-t) = g(t) \end{cases} \quad a \in \mathbb{R}$$

3- Origin symmetry.

A graph is said to be symmetric about the origin if

$$\begin{cases} f(a-t) = -f(t) \\ g(a-t) = -g(t) \end{cases} \quad a \in \mathbb{R}$$

4- symmetric about the line $x = y$.

A graph is said to be symmetric about the line $x = y$ if

$$\begin{cases} f(a-t) = g(t) \\ g(a-t) = h(t) \end{cases} \quad a \in \mathbb{R}$$

Periodicity

If the applications f and g admit a common period T, that is to say for all t of I, we have

$$\begin{cases} f(t+T) = f(t) \\ g(t+T) = g(t) \end{cases}$$

The period T can be obtained by looking for the $L.C.M$ (least common multiple) of the period of f and g.

3.1.3 Table of variation.

t	
f'	
f	
g	
g'	

Example 3.1.3 *Determine the symmetry of the graph determined by the equation*

$$\begin{cases} x = f(t) = a\cos^3(t) \\ y = g(t) = a\sin^3(t) \end{cases} \quad a \in \mathbb{R}^+$$

Sketch the curve.

The domain of definition is $D_f \cap D_g = \mathbb{R}$.

- ***Peridicity.*** *The function f is periodic with period 2π and the function g is periodic with period 2π then $T = L.C.M(2\pi, 2\pi) = 2\pi$.*
  Therfore $D = [-\frac{T}{2}, \frac{T}{2}]$.

- ***Symmetry.***

  1. $\forall t \in D$ we have

  $$\begin{cases} f(-t) = a\cos^3(-t) = a\cos^3(t) \\ g(-t) = a\sin^3(-t) = -a\sin^3(t) \end{cases} \quad a \in \mathbb{R}^+$$

  the graph is symmetric with respect to the x-axis. Then $D = [0, \frac{T}{2}] = [0, \pi]$

  2. For $t \in D$ there exists a reel π such that $\pi - t \in D$ and

  $$\begin{cases} f(\pi - t) = a\cos^3(\pi - t) = -a\cos^3(t) \\ g(\pi - t) = a\sin^3(\pi - t) = a\sin^3(t) \end{cases} \quad a \in \mathbb{R}^+$$

the graph is symmetric with respect to the y-axis. Then $D = [0, \frac{\pi}{2}]$

3. As $\frac{\pi}{2} - t \in D$ *we have*

$$\begin{cases} f(\frac{\pi}{2} - t) = a\cos^3(\frac{\pi}{2} - t) = a\sin^3(t) \\ g(\frac{\pi}{2} - t) = a\sin^3(\frac{\pi}{2} - t) = a\cos^3(t) \end{cases} \quad a \in \mathbb{R}^+$$

the graph is symmetric with respect to the line $y = x$. Then $D = [0, \frac{\pi}{4}]$

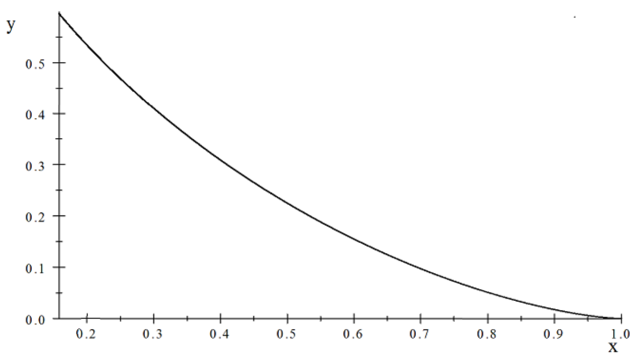
Variations of f and g on $D = [0, \frac{\pi}{4}]$*. We have:*

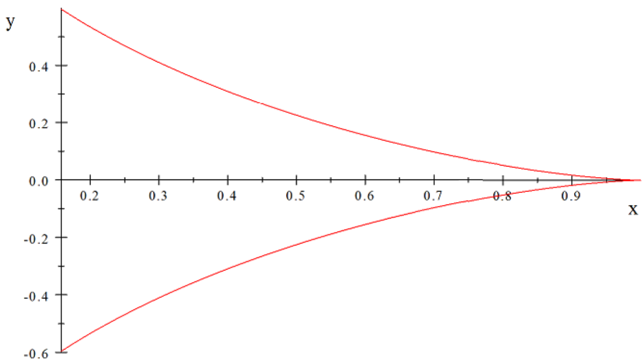
$$\begin{cases} f'(t) = -3a\sin(t)\cos^2(t) = 0 \\ g'(t) = 3a\cos(t)\sin^2(t) = 0 \end{cases} \Longrightarrow \begin{cases} t = 0 \\ t = 0 \end{cases}$$

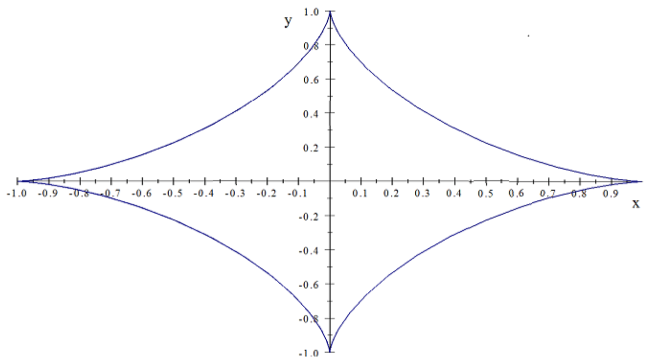
Table of variation

t	0		$\frac{\pi}{4}$
f'	0	$-$	
f	a	↘	$a\frac{\sqrt{2}}{4}$
g	0	↗	$a\frac{\sqrt{2}}{4}$
g'	0	$+$	

The graph







Exercise 3.1.1 *Let the curve (γ) be defined by the parametric equations*

$$\begin{cases} f(t) = \sin^2(t)\cos(t) \\ g(t) = \sin(t) \end{cases} \qquad t \in \mathbb{R}$$

1. *Determine the symmetry of the graph (γ).*

2. *Determine the Table of variation on $[0, \frac{\pi}{2}]$.*

3. *Sketch the graph (γ).*

Solution.

1. The periodicidy

$x(t + 2\pi) = x(t)$.

$y(t + 2\pi) = y(t)$

Therefore, the functions x and y are periodic with period 2π.

Let's study the parity of the functions x an y

$x(-t) = x(t)$ *and* $y(-t) = -y(t)$. *The function x is even and the function y odd, so the curve (γ) is symmetric with respect to the x-axis.*

$x(\pi - t) = -x(t)$.

$y(\pi - t) = y(t)$.

The the curve (γ) is symmetric with respect to the y-axis.

2. The variation

$$\begin{cases} x'(t) = \sin(t)(2\cos^2(t) - \sin^2(t)) \\ y'(t) = \cos(t) \end{cases}$$

then

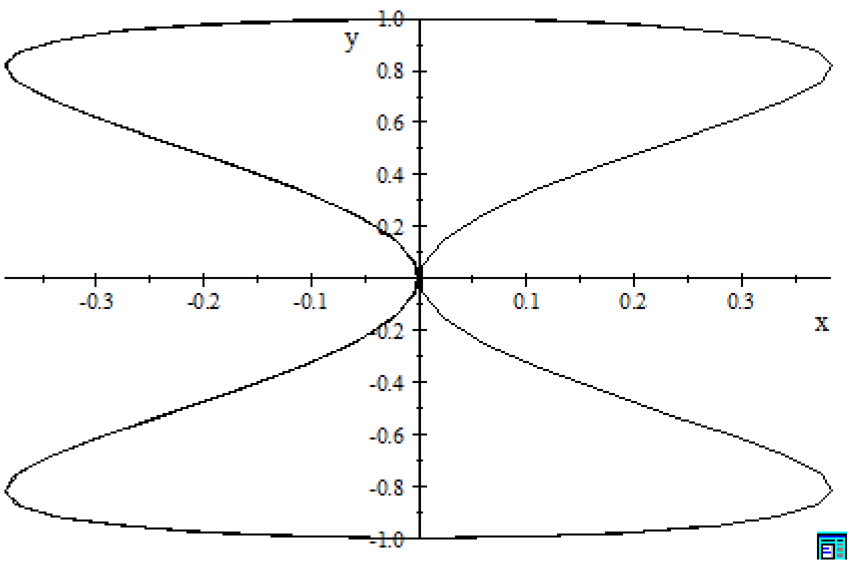
$$\begin{cases} x'(t) = \sin(t)(3\cos^2(t) - 1)) \\ y'(t) = \cos(t) \end{cases}$$

$$\begin{cases} x'(t) = 0 \\ y' = 0 \end{cases} \Rightarrow \begin{cases} t = 0 \textit{ ou } t = \arccos(\frac{1}{\sqrt{3}}) \\ t = \frac{\pi}{2} \end{cases}$$

Table of variation

t	0		$\arccos(\frac{1}{\sqrt{3}})$		$\frac{\pi}{2}$
x'	0	$+$	0	$-$	
x	0	↗	$\frac{2}{3\sqrt{3}}$	↘	0
y	0	↗	$\sqrt{\frac{2}{3}}$	↗	1
y'		$+$		$+$	0

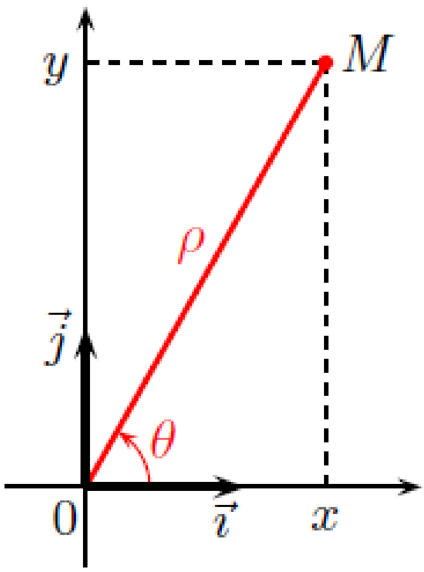
3. The graph



3.2 Polar Curves

3.2.1 Polar to Cartesian Conversion Formulas

To define the polar coordinate, we fix the origin O (also called a pole) and an initial ray from O. The point M has Cartesian coordinates (x, y). The line segment connecting the origin to the point M measures the distance from the origin to M and has length ρ.
The angle between the positive x -axis and the line segment has measure θ. This observation suggests a natural correspondence between the coordinate pair (x, y) and the values ρ and θ. This correspondence is the basis of the polar coordinate system. Note that every point in the Cartesian plane has two values (hence the term ordered pair) associated with it. In the polar coordinate system, each point also two values associated with it: ρ and θ.



Using right-triangle trigonometry, the following equations are true for the point M:

$$x = \rho\cos(\theta), \quad y = \rho\sin(\theta)$$

Each point (x, y) in the Cartesian coordinate system can therefore be represented as an ordered pair (ρ, θ) in the polar coordinate system.
Converting from Cartesian is almost as easy. Let's first notice the following.

$$\rho^2 = x^2 + y^2,$$

Furthermore,

$$\tan(\theta) = \frac{y}{x}$$

Summarizing then gives the following formulas for converting from Cartesian coordinates to polar coordinates.,

$$\rho = \sqrt{x^2 + y^2},\ \theta = \arctan(\frac{y}{x})$$

Example 3.2.1 *Convert each of the following into an equation in the given coordinate system.*

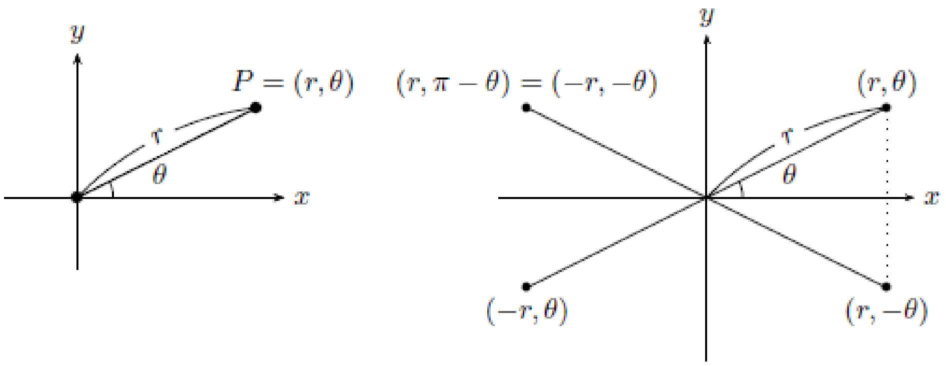
1. *Convert $2x - 5x^3 = 1 + xy$ into polar coordinates.*

2. *Convert $\rho = -8\cos(\theta)$ into Cartesian coordinates.*

3.2.2 Periodicity

The smallest period (if exists) of the function $\rho = f(\theta)$ is the smallest positive number T that satisfies the equality $f(\theta + T) = f(\theta)$

3.2.3 Symmetry



Let $P = (r, \theta)$ be a given point. The point $(r, -\theta)$ is symmetric to the point (r, θ) with respect to the x-axis, while the point $(r, \pi - \theta)$ is symmetric to the point (r, θ) with respect to the y-axis. Finally the point $(-r, \theta)$ (or $(r, \pi + \theta)$) is symmetric to the point (r, θ) about the origin. Hence we have the following result.

proposition 3.2.1 *The graph of $\rho = f(\theta)$ is symmetric with respect to*

1. *x-axis if $f(-\theta) = f(\theta)$ or $f(\pi - \theta) = -f(\theta)$,*

2. *y-axis if $f(-\theta) = -f(\theta)$ or $f(\pi - \theta) = f(\theta)$,*

3. *the origin if* $f(\pi+\theta)=f(\theta)$ *or* $f(\pi-\theta)=f(-\theta)$,

4. *the line* $y=x$ *if* $f(\frac{\pi}{2}-\theta)=f(\theta)$.

5. *the line* $\theta=\frac{\theta_0}{2}$ *if* $f(\theta_0-\theta)=f(\theta)$,

6. *the line* $\theta=\frac{\theta_0}{2}+\frac{\pi}{2}$ *if* $f(\theta_0-\theta)=-f(\theta)$.

Example 3.2.2 *Find the symmetry of the rose defined by the equation* $\rho=2\cos(2\theta)$.

$$f(-\theta)=2\cos(-2\theta)=2\cos(2\theta)=f(\theta).$$

Hence it is symmetric about the x*-axis. On the other hand,*

$$f(\pi-\theta)=2\cos(2\pi-2\theta)=2\cos(-2\theta)=f(\theta).$$

Hence it is symmetric about the y*-axis. Also, we see that*

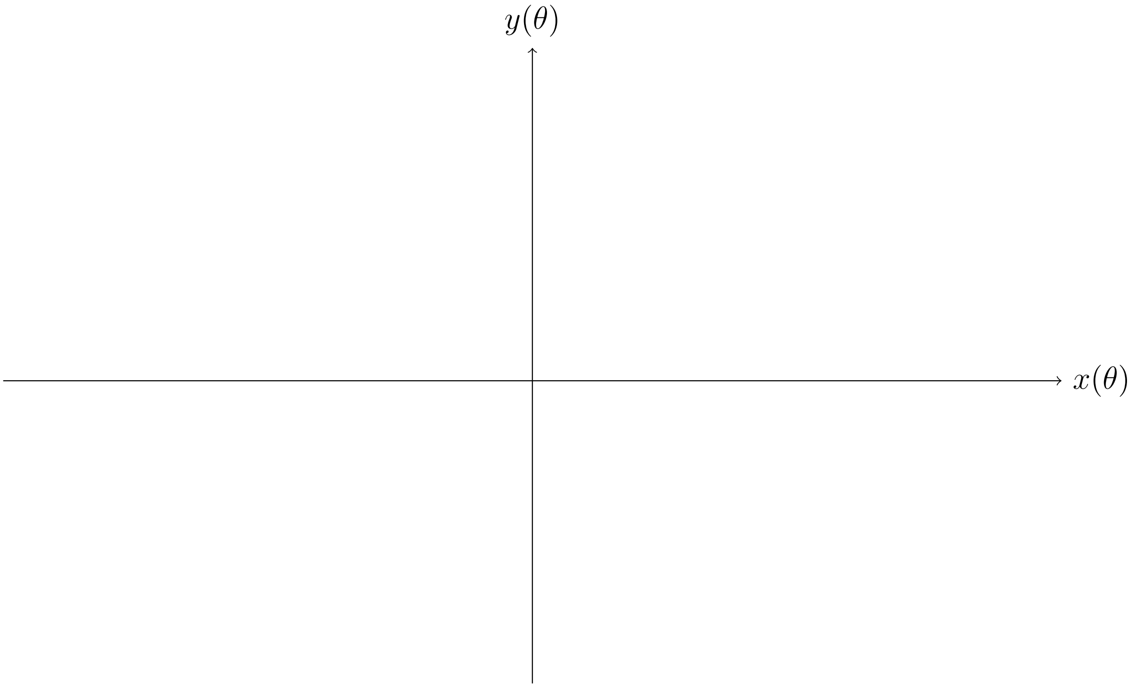
$$f(\frac{\pi}{2}-\theta)=2\cos(\pi-2\theta)=-2\cos(2\theta)=-f(\theta).$$

Hence it is symmetric about the line $\theta=\frac{\pi}{4}+\frac{\pi}{2}=\frac{3\pi}{4}$.

3.2.4 Table of variation.

θ	D
$f'(\theta)$	The sign of the function f'
$\rho=f(\theta)$	the variation of the function f

Passages through the origin. We solve the equation $\rho = f(\theta) = 0$.



3.2.5 Plotting the polar equation curv

Plotting the curve with the polar equation $\rho = f(\theta)$

1. If ρ is positive and increasing, the point moves away from the origin.

2. If ρ is positive and decreasing, the point moves toward the origin.

3. If ρ is negative and decreasing, the point moves away from the origin.

4. If ρ is negative and increasing, the point moves toward the origin.

Example 3.2.3 *Consider the curve (γ) defined by:*

$$\rho(\theta) = 2\cos(2\theta), \qquad \theta \in \mathbb{R}$$

1. *Determine the solutions of the equation $\rho(\theta) = 0$.*

2. *Create the table of variation of the polar curve (γ) on $[0, \frac{\pi}{2}]$. Draw its graph on $[0, \frac{\pi}{2}]$ and complete by symmetry.*

Solution.

1- We solve the equation $\rho = 0$

$$\rho = 0 \Leftrightarrow \theta = (2k+1)\frac{\pi}{4}$$

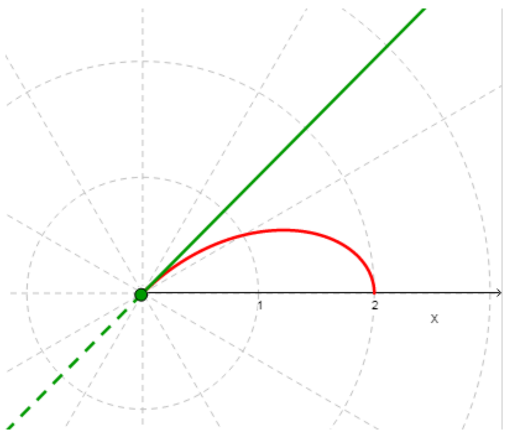
if $k = 0$ then $\theta = \frac{\pi}{4}$
2- Table of variation

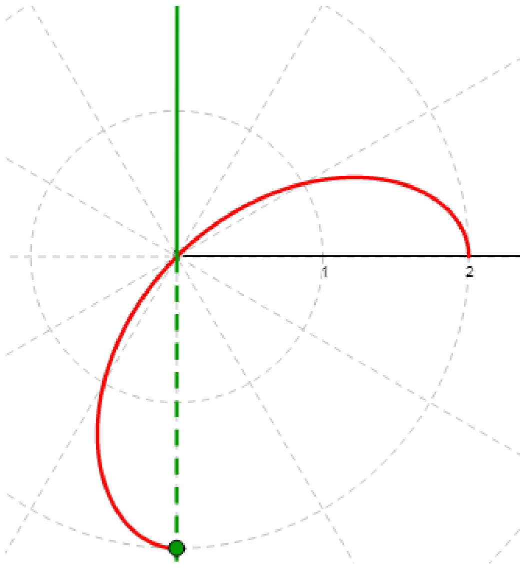
$$\rho' = f'(\theta) = -4\sin(2\theta)$$

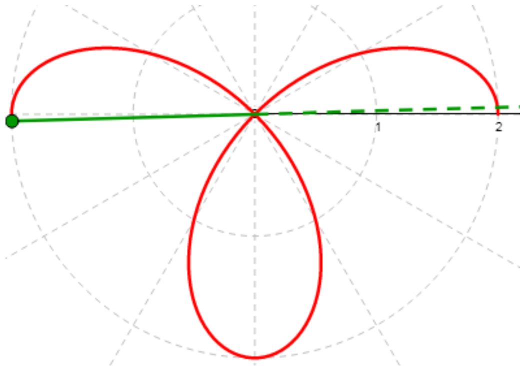
so $f'(\theta) = 0 \implies \theta = \frac{k}{2}\pi$ where $k \in \mathbb{Z}$

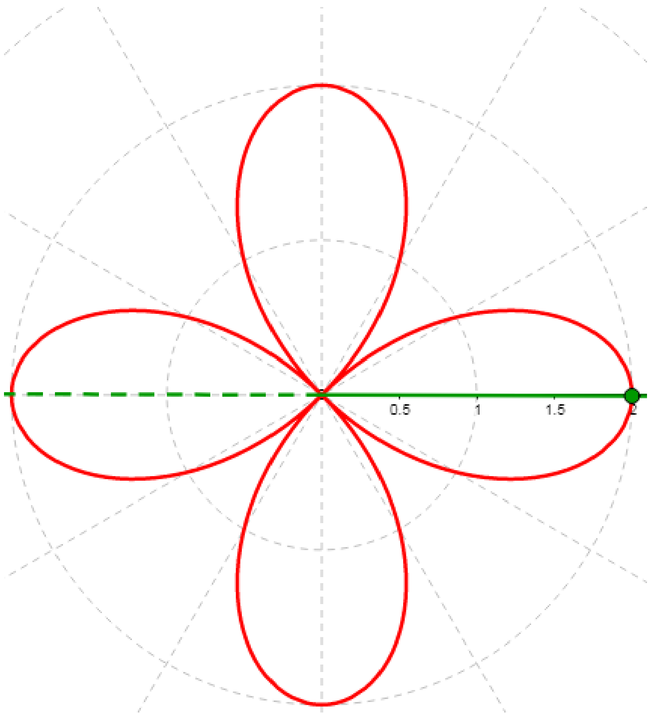
θ	0		$\frac{\pi}{2}$
ρ'	0	$-$	
ρ	2	$\searrow$	-2

• The graph









Exercise 3.2.1 *Consider the curve* (γ) *defined by:*

$$\forall \theta \in \mathbb{R}, \quad \rho(\theta) = \cos(2\theta) + \cos^2(\theta).$$

1. *Find the symmetry of the polar curve* (γ).

2. *Solve the equation* $\rho(\theta) = 0$.

3. *Create the table of variation of the polar curve* (γ) *on* $[0, \frac{\pi}{2}]$*. Draw its graph on* $[0, \frac{\pi}{2}]$ *and complete by symmetry.*

Solution.

1. The symmetry

the function ρ is even, then the graph of an even function is symmetric with respect to the x-axis.

On the other hand,

$$\rho(\pi - t) = \cos(2\pi - 2t) + \cos^2(\pi - t) = \cos(2t) + \cos^2(t) = \rho(t)$$

Hence it is symmetric about the y-axis.

2. We solve the equation $\rho(\theta) = 0$

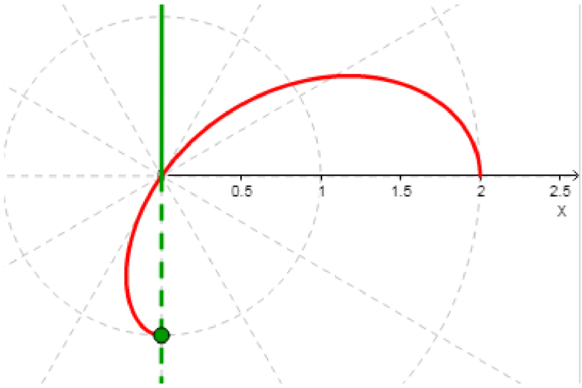
$$3\cos^2(\theta) - 1 = 0 \Longrightarrow \cos(\theta) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

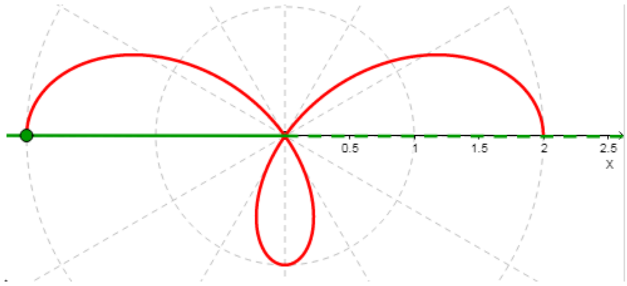
then $\theta_0 = \arccos(\frac{\sqrt{3}}{3})$.

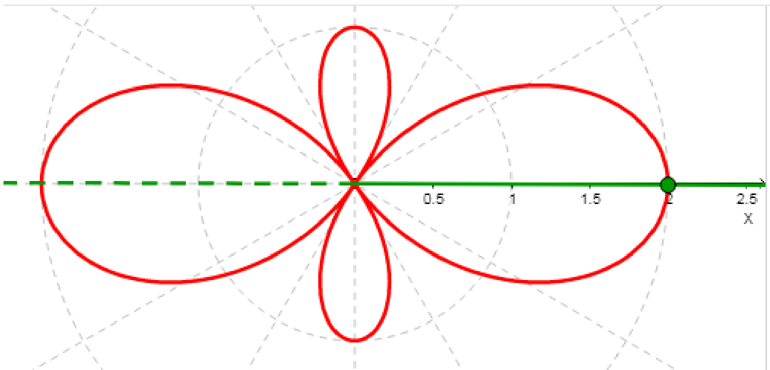
3.Variation table and the graph

$$\rho'(t) = -6\sin(t)\cos(t) = 0 \Longrightarrow t = 0 \text{ ou } t = \frac{\pi}{2}.$$

t	0		$\frac{\pi}{2}$
ρ'	0	$-$	0
ρ	2	↘	-1







Exercise 3.2.2 *Consider the curve* (γ) *defined by:*

$$\rho = f(\theta) = a(2\cos(\theta) - \cos(2\theta))$$

1. *Study the periodicity and the symmetry of the parametric curve* (γ).
2. *Determine the solutions of the equation* $f(\theta) = 0$.
3. *Find the variation table of the parampetric curve* (γ) *on* $[0, \pi]$ *and draw its graph on* $[-\pi, \pi]$.

1- As the function $\cos(\theta)$ is periodic of period 2π and function $\cos(2\theta)$ is periodic of period π, then $T = LCM(\pi, 2\pi) = 2\pi$.

Find the symmetry of the curve defined by the equation $\rho = a(2\cos(\theta) - \cos(2\theta))$.

$$f(-\theta) = a(2\cos(-\theta) - \cos(-2\theta)) = a(2\cos(\theta) - \cos(2\theta)) = f(\theta).$$

Hence it is symmetric about the x-axis. On the other hand,

$$f(\pi-\theta)=a(2\cos(\pi-\theta)-\cos(2\pi-2\theta))=a(-2\cos(\theta)-\cos(2\theta)).$$

Dont's a symmetry.

2- Determine the solutions of the equation $f(\theta)=0$.

$$\begin{aligned} a(2\cos(\theta)-\cos(2\theta))=0 &\implies 2\cos(\theta)-\cos^2(\theta)+\sin^2(\theta)=0 \\ &\implies -2\cos^2(\theta)+2\cos(\theta)+1=0 \end{aligned}$$

suppose $\cos(\theta)=X$, so

$$-2X^2+2X+1=0 \implies X_1=\frac{1-\sqrt{3}}{2} \;\; X_2=\frac{1+\sqrt{3}}{2}$$

then $\cos(\theta)=\dfrac{1-\sqrt{3}}{2} \implies \theta=111.47.$

3- Find the variation table of the parampetric curve

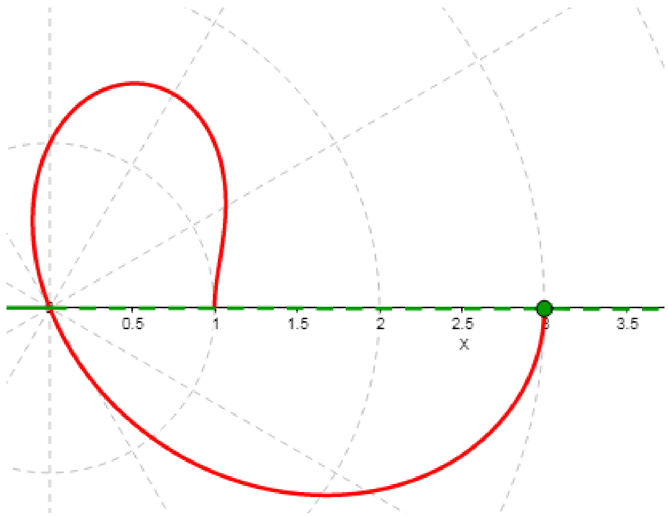
$$f'(\theta)=a(-2\sin(\theta)+2\sin(2\theta))=2a\sin(\theta)(-1+2\cos(\theta))$$

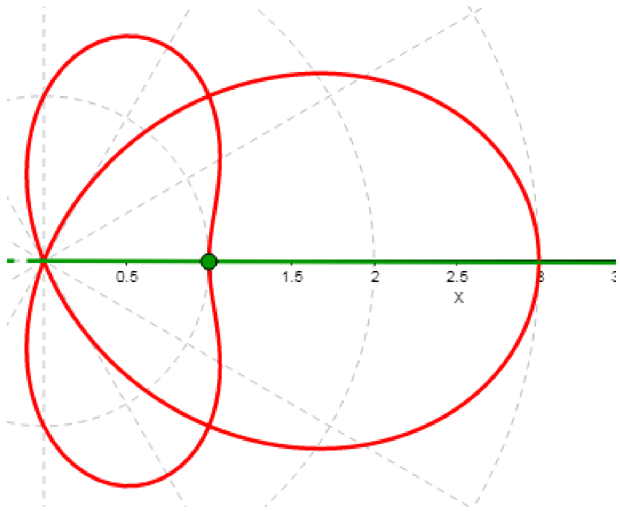
So

$$f'(\theta)=0 \implies \theta=0 \;\; or \;\; \theta=\frac{\pi}{3}$$

θ	0		$\frac{\pi}{3}$		π
ρ'	0	$+$	0	$-$	
ρ	a	↗	$\frac{3}{2}a$	↘	$-3a$

Th graph (if $a=1$)





Applications of Curves

▷ Physics: Curves are used to describe the paths of particles, planets, and other objects in motion, such as trajectories in classical mechanics and orbits in astrophysics.

▷ Engineering: Curves are used in design, such as in the creation of roads, rails, and other paths that need to follow a smooth or optimal trajectory.

▷ Computer Graphics: Curves are essential in graphics for modeling smooth surfaces, designing paths for animation, and generating shapes in CAD software.

▷ Robotics: Curves are used in motion planning to determine smooth paths for robots and vehicles.

3.3 Exercises

Exercise 3.3.1 *Consider the numerical functions f and g of the real variable θ defined by*

$$f(\theta) = 4\sqrt{2}\sin(\theta), \quad g(\theta) = \sin(2\theta)$$

Let a curve (γ) be defined by the parametric equations

$$\begin{cases} x = f(\theta) \\ y = g(\theta) \end{cases}$$

1. *Study the periodicity of functions f and g. Calculator $f(\pi - \theta)$, $g(\pi - \theta)$.*
2. *Study the variations of the functions f and g for $\theta \in [0, \frac{\pi}{2}]$.*
3. *Find the points on (γ) where the tangent is horizontal.*
4. *Find the points on (γ) where the tangent is vertical.*
5. *Sketch the parametric curve (γ) on $[0, \frac{\pi}{2}]$ and complete by symmetry.*
6. *Find the arc lenght of the curve (γ).*

Exercise 3.3.2 *Let (γ) be the curve defined by*

$$\begin{cases} x(t) = 2\cos(t) - \cos(2t) \\ y(t) = 2\sin(t) - \sin(2t) \end{cases} \quad t \in \mathbb{R}$$

1. *Study the periodicity and the symmetry of the parametric curve (γ).*
2. *Study the variation of f and g over $D = [0, \pi]$. Create a variation table.*
3. *Sketch the graph (γ) on $\mathbb{R}$.*

Exercise 3.3.3 *Let (γ) be the curve defined by:*

$$\begin{cases} x(t) = \sin(t)\cos^2(t) \\ y(t) = \cos(t) \end{cases} \quad t \in \mathbb{R}$$

1. *Study the periodicity and the symmetry of the parametric curve (γ).*

2. *Study the variations of x and y as a function of t on $D = [0; \frac{\pi}{2}]$.*

3. *Find the vertical and horizontal tangent lines of the curve (γ).*

4. *Sketch the graph (γ) on $\mathbb{R}$.*

Exercise 3.3.4 *Consider the curve (γ) defined by:*

$$\rho(\theta) = \cos(\frac{\theta}{2}), \qquad \theta \in \mathbb{R}$$

1. *Find the period T and the symmetry of the curve (γ).*

2. *Determine the solutions of the equation $\rho(\theta) = 0$.*

3. *Create the table of variation of the parampetric curve (γ) on $[0, \pi]$ and draw its graph on $[-2\pi, 2\pi]$.*

Exercise 3.3.5 .

1. *Find the period T and the symmetry of the rose defined by the equation $\rho = 3\sin(2\theta)$.*
2. *Study the variation of the function ρ on $[0, \frac{\pi}{4}]$. Create the table of variation.*
3.*Determine the solutions of the equation $\rho(\theta) = 0$. Sketch the graph.*

Exercise 3.3.6 .

1. *Show that the arc length of a polar curve $\rho = \rho(\theta)$ between $\theta = a$ and $\theta = b$ is given by the integral: $L_a^b = \displaystyle\int_a^b \sqrt{\rho^2(\theta) + \rho'^2(\theta)}d\theta$.*

2. *Find the arc lenght of the cardioid defined by the equation $\rho = 1+\cos(t)$ whith $0 \leq t \leq 2\pi$.*

Chapter 4

Curvature of Plane and Space Curves

4.1 Arc length Parametrization

4.1.1 Arc Length Function

Definition 4.1.1 *If $\alpha : [a, b] \to \mathbb{R}^i$ where $i = 2; 3$ is a parametrized curve, then for any $a < t < b$, we define its arc length function from a to t to be*

$$s(t) = \int_a^t \|\alpha'(u)\| du.$$

Where

in two dimensions $(i = 2)$, $\alpha(t) = (x(t), y(t))$ and

$$\|\alpha'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}$$

in tree dimensions $(i = 3)$, $\alpha(t) = (x(t), y(t), z(t))$ and

$$\|\alpha'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}.$$

The idea is to replace t with s, where s represents the distance traveled along the curve. To reparametrize in terms of arc length, we need to express the curve in terms of s, rather than t.

4.1.2 Steps to Perform Arc Length Parametrization

Let $\alpha : [a, b] \to \mathbb{R}^i$ where $i = 2; 3$ is a parametrized curve.

1. Compute the arc length function $s(t)$.
2. Solve for $s(t)$: After obtaining $s(t)$, solve for t as a function of s. This may be done explicitly or implicitly.
3. Substitute s for t in the original curve: Once you have $t(s)$, substitute it into the original vector function $\alpha(t)$ to obtain the new parametrization in terms of s.

Example 4.1.1 *Arc Length Parametrization of a Curve in two dimensions.*
The standard parametrization of the circle of radius a is $\alpha(t) = (a\cos(t), a\sin(t))$, $t \in [0; 2\pi]$.

1. *Compute the arc length function $s(t)$*

$$\alpha'(t) = (-a\sin(t), a\cos(t))$$

and

$$\|\alpha'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2} = a$$

The arc length $s(t)$ from $t = 0$ to t is:

$$s(t) = \int_0^t \|\alpha'(u)\| du = at.$$

2. *Solve for $t(s)$: Once you have the arc length functions(t), you would solve for t as a function of s.*

$$t(s) = \frac{s}{a}.$$

In general, if $s(t)$ is invertible, you can write $t = t(s)$.

3. *Reparametrize the curve:*

$$\alpha(s) = (a\cos(\frac{s}{a}), a\sin(\frac{s}{a})), s \in [0, 2\pi a]$$

Example 4.1.2 *Arc Length Parametrization of a Curve in tree dimensions.*

Where on the curve $\alpha(t) = (2t, 3\sin(2t), 3\cos(2t))$ *are we after traveling for a distance of* $\frac{\pi\sqrt{10}}{3}$.

1. *Compute the arc length function* $s(t)$

   $$\alpha'(t) = (2, 6\cos(2t), -6\sin(2t))$$

   and

   $$\|\alpha'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} = 2\sqrt{10}$$

   The arc length $s(t)$ *from* $t = 0$ *to* t *is:*

   $$s(t) = \int_0^t \|\alpha'(u)\| du = 2\sqrt{10}t.$$

2. *Solve for* $t(s)$*: Once you have the arc length functions*(t)*, you would solve for* t *as a function of* s.

   $$t(s) = \frac{s}{2\sqrt{10}}.$$

   In general, if $s(t)$ *is invertible, you can write* $t = t(s)$.

3. *Reparametrize the curve:*

   $$\alpha(s) = \left(\frac{s}{\sqrt{10}}, 3\sin(\frac{s}{\sqrt{10}}), 3\cos(\frac{s}{\sqrt{10}})\right)$$

   Then, to determine where we are all that we need to do is plug in $s = \frac{\pi\sqrt{10}}{3}$ *into this and we'll get our location.*

   $$\alpha(\frac{\pi\sqrt{10}}{3}) = \left(\frac{\pi}{3}, \frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$$

So, after traveling a distance of $\frac{\pi\sqrt{10}}{3}$ *along the curve we are at the point* $\left(\frac{\pi}{3}, \frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$

4.2 Curvature and center of curvarture of Space Curves (3D Curves)

4.2.1 Unit Tangent Vector

The unit tangent vector at any point on the curve gives the direction of the curve at that point, and it has a magnitude of 1.

For a curve defined by a vector function $\alpha(t) = (x(t), y(t), z(t))$, the unit tangent vector $T(t)$ is given by:

$$T(t) = \frac{d\alpha(t)}{ds} = \frac{d\alpha(t)}{dt}\frac{dt}{ds} = \frac{\alpha'(t)}{\|\alpha'(t)\|}$$

where

▷ $\alpha'(t) = (x'(t), y'(t), z'(t))$ is the derivative of the position vector with respect to t.

▷ $\|\alpha'(t)\| = \sqrt{(x'(t)^2) + (y'(t)^2) + (z'(t)^2)}$ is the magnitude of the derivative of the position vector.

4.2.2 Principal normal vector

The principal normal vector (denoted as $N(t)$) of a curve provides the direction of the curve's acceleration or the direction in which the curve is bending. It is perpendicular to the unit tangent vector $T(t)$, and it points in the direction of the curve's instantaneous change in direction.

To find the principal normal vector, you need to follow these steps:

1. Find the unit tangent vector $T(t)$.
2. Find the derivative of the unit tangent vector $T'(t) = \dfrac{dT}{ds}$:
   You need to compute the derivative of$T(t)$ with respect to t.
   The result will give you the change in the direction of the tangent vector, which is related to how the curve bends.

$$T'(t) = \frac{dT}{ds} = \frac{dT}{dt}\frac{dt}{ds}$$

3. Find the normal vector (principal normal vector):

$$N(t) = \frac{\frac{dT}{ds}}{\|\frac{dT}{ds}\|} = \frac{\frac{dT}{dt}}{\|\frac{dT}{dt}\|}.$$

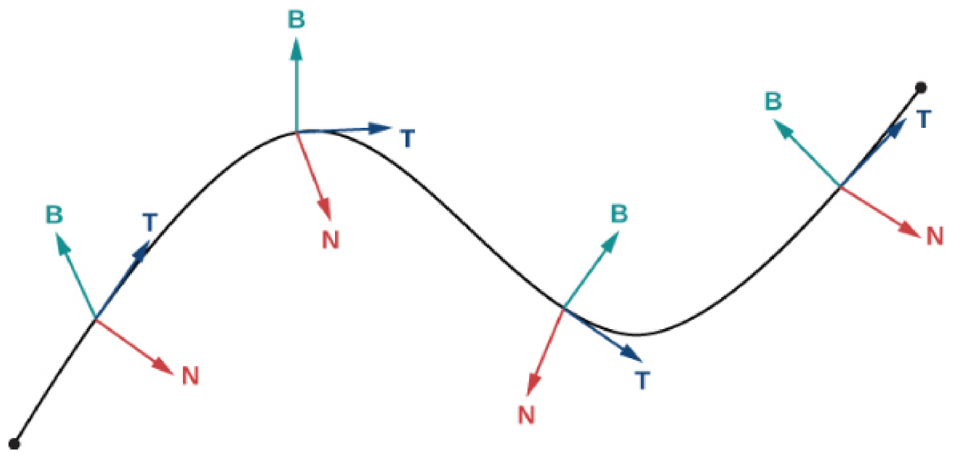
This vector points in the direction of the curve's instantaneous bending (acceleration), and it is always perpendicular to $T(t)$

4.2.3 Binormal Vector

The binormal vector is one of the three vectors in the Frenet–Serret frame (also known as the TNB frame), which describes the local geometry of a space curve (i.e., a curve in 3D space). Let $\alpha(t) = (x(t), y(t), z(t))$ be a regular smooth space curve. Then:

$$\vec{B(t)} = \vec{T} \times \vec{N}$$

Because the binormal vector is defined to be the cross product of the unit tangent and unit normal vector we then know that the binormal vector is orthogonal to both the tangent vector and the normal vector.



Example 4.2.1 *Find the normal and binormal vectors for:* $\alpha(t) = (t, 3\sin(t), 3\cos(t))$
Solution

1. *Tangent Vector:*

$$\alpha'(t) = (1, 3\cos(t), -3\sin(t)), \;\; \textit{and} \;\; \|\alpha'(t)\| = \sqrt{10}$$

$$T = \frac{d\alpha}{ds} = (1, 3\cos(t), -3\sin(t))\frac{1}{\sqrt{10}}$$

2. *Normal Vector:*

the unit normal vector will now require the derivative of the unit tangent and its magni-

tude.

$$\frac{dT}{dt} = (0, -3\sin(t), -3\cos(t))\,\frac{1}{\sqrt{10}}$$

and

$$\|\frac{dT}{dt}\| = \frac{3}{\sqrt{10}}.$$

The unit normal vector is then,

$$\begin{aligned} N(t) &= \frac{\frac{dT}{dt}}{\|\frac{dT}{dt}\|} \\ &= (0, -3\sin(t), -3\cos(t))\,\frac{1}{\sqrt{10}}\frac{\sqrt{10}}{3} \\ &= (0, -\sin(t), -\cos(t)) \end{aligned}$$

3. *Finally, the binormal vector is,*

$$\vec{B(t)} = \vec{T} \times \vec{N} = (-3, \cos(t), -\sin(t))\frac{1}{\sqrt{10}}$$

4.2.4 Curvature of space curves (3D)

The curvature $\kappa(s)$ (or $\kappa(t)$) of the curve measures how quickly the curve is changing direction at any given point. Curvature is defined as the magnitude of the derivative of the unit tangent vector with respect to the arc length parameter.

The formula for curvature is:

$$\kappa(s) = \|\frac{dT}{ds}\|$$

so

$$\kappa(t) = \|\frac{dT}{ds}\| = \|\frac{dT}{dt}\frac{dt}{ds}\| = \frac{\|\frac{dT}{dt}\|}{\|\frac{d\alpha}{dt}\|}$$

So far, we have

$$\frac{dT}{ds} = \kappa(s)N(s)$$

proposition 4.2.1 *Let $\alpha(t) = (x(t), y(t), z(t))$ Then, the curvature is given by:*

$$\kappa(t) = \frac{\|\alpha'(t) \times \alpha''(t)\|}{\|\alpha'(t)\|^3} = \frac{\sqrt{\begin{vmatrix} y' & z' \\ y'' & z'' \end{vmatrix}^2 + \begin{vmatrix} x' & z' \\ x'' & z'' \end{vmatrix}^2 + \begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}^2}}{(x'^2 + y'^2 + z'^2)^{3/2}}$$

where

▷ $\alpha'(t)$ *is the first derivative of the position vector,*

▷ $\alpha''(t)$ *is the second derivative of the position vector,*

▷ $\times$ *represents the cross product.*

Example 4.2.2 *Determine the curvature for*

$$\begin{cases} x(t) = r.\cos(t) \\ y(t) = r.\sin(t) & r \text{ et } h \text{ costantes} > 0 \\ z(t) = h.r.t \end{cases}$$

1- Find the unit tangent vector.

We have:

$$\frac{ds}{dt} = \sqrt{r^2 + h^2 r^2}$$

and

$$\frac{d\alpha(t)}{ds} = \frac{d\alpha(t)}{dt}\frac{dt}{ds}$$

so

$$\frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{-\sin(t)}{\sqrt{1+h^2}},$$
$$\frac{dy}{ds} = \frac{\cos(t)}{\sqrt{1+h^2}},$$
$$\frac{dz}{ds} = \frac{h}{\sqrt{1+h^2}}$$

then

$$T = (-\sin(t), \cos(t), h)\frac{1}{\sqrt{1+h^2}}$$

The derivative of the unit tangent is,

$$\frac{dT}{ds} = \left(\frac{-\cos(t)}{\sqrt{1+h^2}}, \frac{-\sin(t)}{\sqrt{1+h^2}}, 0\right)\frac{1}{r\sqrt{1+h^2}}$$

The curvature is then,

$$\kappa = \|\frac{d\vec{T}}{ds}\| = \sqrt{(\frac{da}{ds})^2 + (\frac{db}{ds})^2 + (\frac{dc}{ds})^2} = \frac{1}{r(1+h^2)}.$$

Second method.

$$\kappa(t) = \frac{\sqrt{\begin{vmatrix} y' & z' \\ y'' & z'' \end{vmatrix}^2 + \begin{vmatrix} x' & z' \\ x'' & z'' \end{vmatrix}^2 + \begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}^2}}{(x'^2 + y'^2 + z'^2)^{3/2}} = \frac{r^2(1+h^2)^{1/2}}{r^3(1+h^2)^{3/2}} = \frac{1}{r(1+h^2)}$$

4.2.5 Radius of cuevature

The radius of curvature R is a measure of how sharply a curve bends at a given point. Specifically, it is the radius of the osculating circle (in 2D) or osculating sphere (in 3D) that best approximates the curve near that point. A small radius indicates a sharply bent curve, while a large radius indicates a gently curved or almost straight curve.
The radius of curvature is related to curvature κ as:

$$R = \frac{1}{\kappa}$$

As $\frac{d\vec{T}}{ds} = \|\frac{d\vec{T}}{ds}\|\vec{N}$, we obtain

$$\vec{N} = R\frac{d\vec{T}}{ds}$$

4.2.6 Center of curvature

The center of curvature of a curve is the point at which the osculating circle (the best-fit circle) at a point on the curve is centered. It is the center of the osculating sphere in the case of space curves (3D curves), as the curvature in space can bend in multiple directions.
The center of curvature by the vector $\vec{M\Omega}$ (called the curvature vector)

$$\vec{M\Omega} = R.\vec{N} = R^2\frac{d\vec{T}}{ds}$$

Vector equality $\vec{O\Omega} = \vec{OM} + \vec{M\Omega}$ Gives:

$$\begin{cases} x_\Omega = x(t) + R^2\dfrac{da}{ds} \\ y_\Omega = y(t) + R^2\dfrac{db}{ds} \\ z_\Omega = z(t) + R^2\dfrac{dc}{ds} \end{cases}$$

where

$$\frac{d\vec{T}}{ds} = (\frac{da}{ds}, \frac{db}{ds}, \frac{dc}{ds}),$$

and

$$a = \frac{dx}{ds}, b = \frac{dy}{ds}, c = \frac{dz}{ds}$$

Example 4.2.3 *let (γ) be the curve of the circular helix with a parametric equation.*

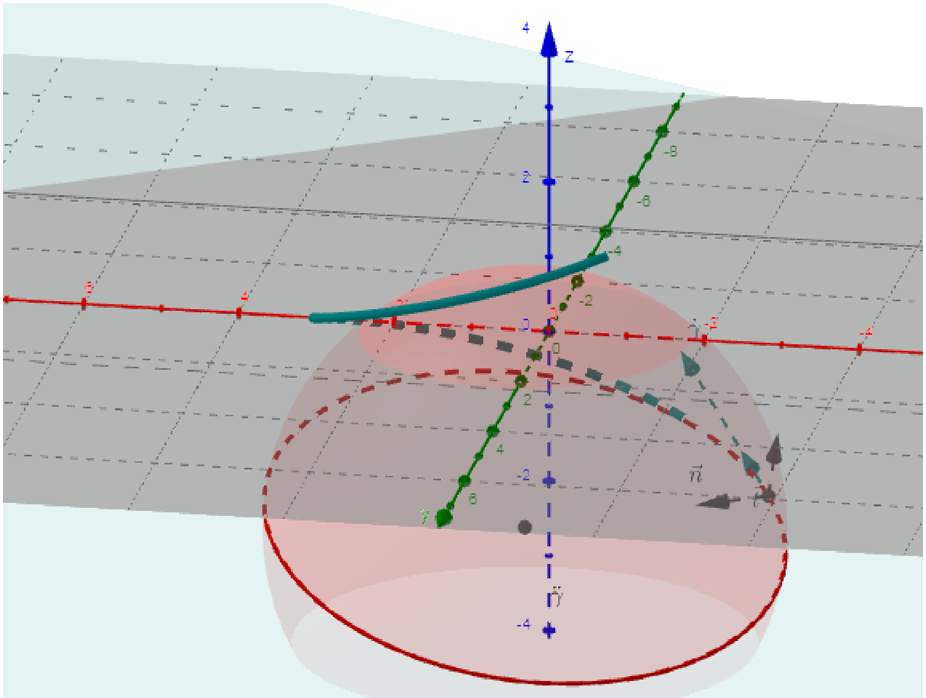
$$\begin{cases} x(t) & = r.\cos(t) \\ y(t) & = r.\sin(t) \quad r \text{ and } h \text{ constants} > 0 \\ z(t) & = h.r.t \end{cases}$$

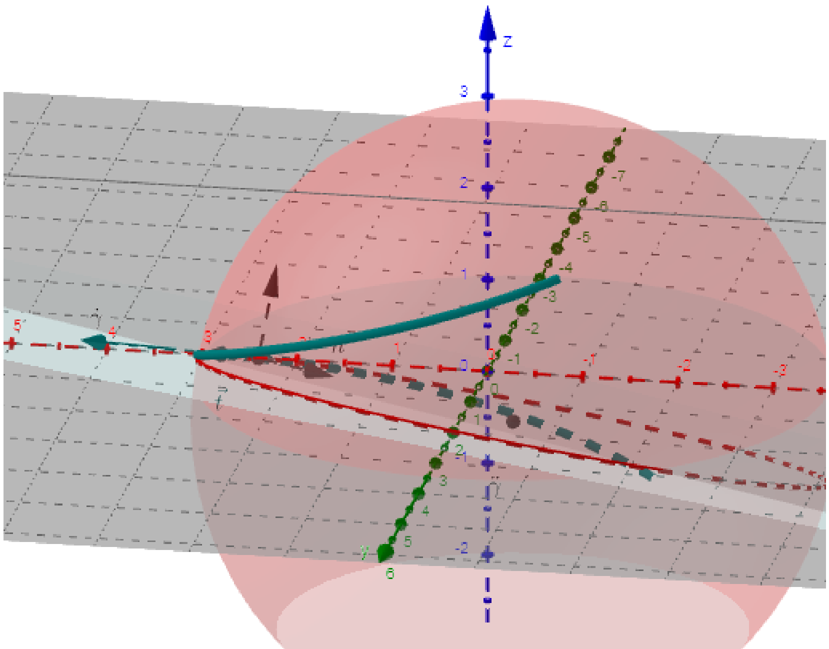
We have

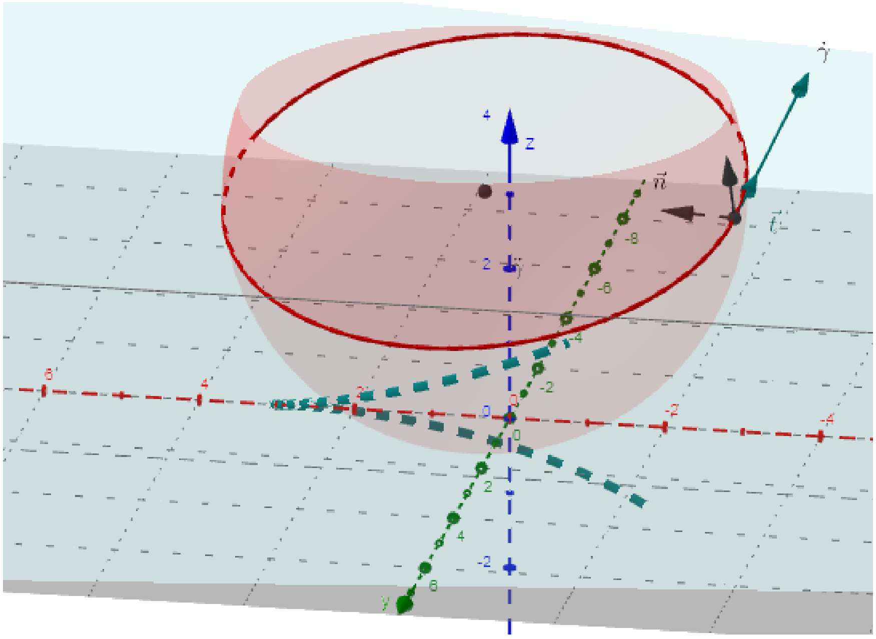
$$\begin{cases} x_\Omega = x(t) + R^2\dfrac{da}{ds} \\ y_\Omega = y(t) + R^2\dfrac{db}{ds} \\ z_\Omega = z(t) + R^2\dfrac{dc}{ds} \end{cases}$$

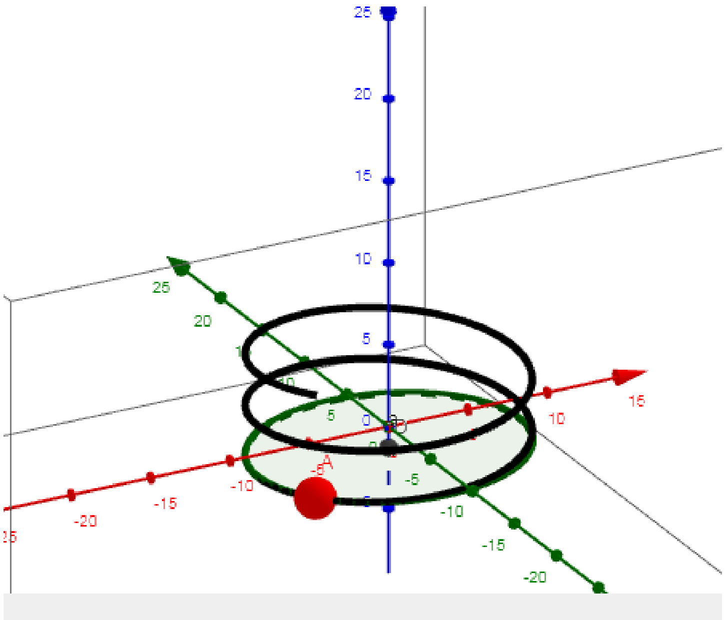
then

$$\begin{cases} x_\Omega & = -h^2.r.\cos(t) \\ y_\Omega & = -h^2.r.\sin(t) \\ z_\Omega & = h.r.t \end{cases}$$









4.3 Frenet–Serret Formulas

4.3.1 Torsion

The torsion of a space curve measures how much the curve twists out of the plane of curvature. If you think of curvature as how sharply a curve bends, torsion tells you how much it twists in 3D space.

For a curve $\alpha(t) = (x(t), y(t), z(t))$, the torsion $\tau(t)$ is given by:

$$\tau(t) = \frac{(\alpha' \times \alpha'').\alpha'''}{\|\alpha' \times \alpha''\|^2}$$

Example 4.3.1 *Let's compute the torsion of the space curve:*

$$\left(e^t \cos(t), e^t \sin(t), e^t\right)$$

- *The Derivatives $\alpha'(t)$, $\alpha''(t)$ and $\alpha'''(t)$:*

$$\begin{aligned}
\alpha'(t) &= \left(e^t(\cos(t) - \sin(t)), e^t(\cos(t) + \sin(t)), e^t\right) \\
\alpha''(t) &= \left(-2e^t \sin(t), 2e^t \cos(t), e^t\right) \\
\alpha'''(t) &= \left(-2e^t(\cos(t) + \sin(t)), 2e^t(\cos(t) - \sin(t)), e^t\right)
\end{aligned}$$

- *Compute $\alpha'(t) \times \alpha''(t)$ and $\|\alpha'(t) \times \alpha''(t)\|^2$*

$$\begin{aligned}
\alpha'(t) \times \alpha''(t) &= \left(\begin{vmatrix} y' & z' \\ y'' & z'' \end{vmatrix}, - \begin{vmatrix} x' & z' \\ x'' & z'' \end{vmatrix}, \begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix} \right) \\
&= \left(e^{2t}(\cos(t) - \sin(t), \cos(t) + \sin(t), -2)\right)
\end{aligned}$$

$$\|\alpha'(t) \times \alpha''(t)\|^2 = e^{4t}\left((\cos(t) - \sin(t))^2 + (\cos(t) + \sin(t))^2 + 4\right) = 6e^{4t}$$

- *Compute numerator of torsion*

$$\begin{aligned}
(\alpha'(t) \times \alpha''(t)).\alpha'''(t) &= \left[-2(\cos^2(t) - \sin^2(t)) + 2(\cos^2(t) - \sin^2(t)) - 2\right] \\
&= -2e^{3t}
\end{aligned}$$

- *Final Expression for Torsion*

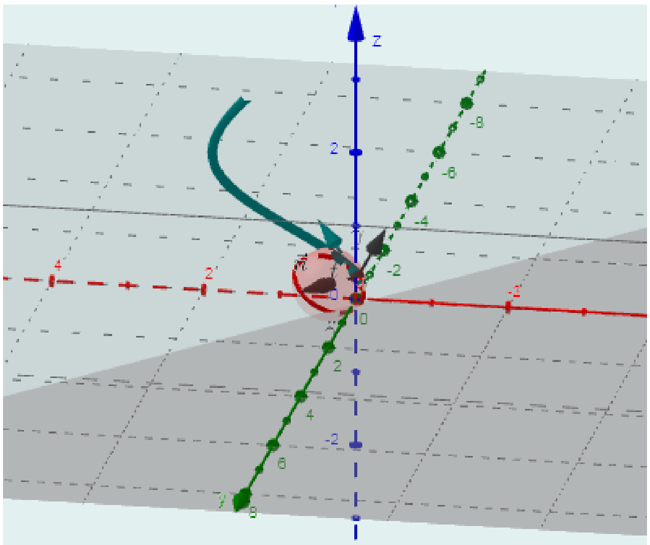
$$\tau(t) = \frac{(\alpha' \times \alpha'').\alpha'''}{\|\alpha' \times \alpha''\|^2} = \frac{-2e^{3t}}{6e^{4t}} = -\frac{1}{3}e^{-t}.$$

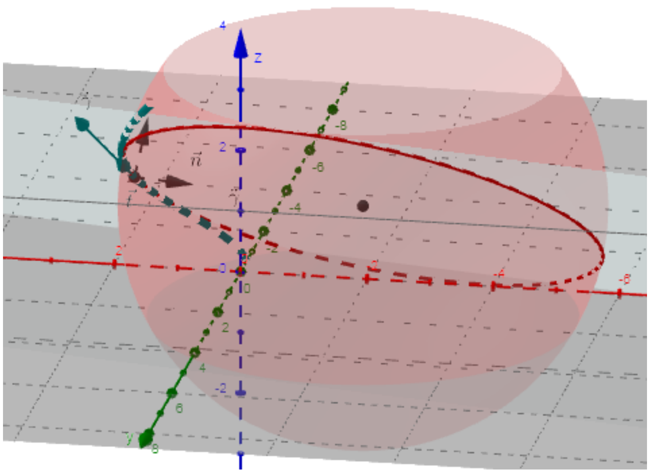
4.3.2 Frenet-Equations

Let $\alpha(s)$, $s \in [a,b]$ be a regular unit speed curve such that $\kappa(s) \neq 0$ for all $s \in [a,b]$.
Along α weare going to introduce the vector fiels,

$$T = T(s), \quad \textit{unit tangent vector field}$$
$$N = N(s), \quad \textit{principal normal vector field}$$
$$B = B(s), \quad \textit{binormal vector field}$$

$\{T, N, B\}$ is called a Frenet frame.
At each point α of $\{T, N, B\}$ forms an orthonormal basis, i.e. T, N, B are mutually perpendicular unit vectors





theorem 4.1 *The Frenet-Serret Equations*

We have

$$T' = \kappa N,$$
$$N' = -\kappa T + \tau B,$$
$$B' = -\tau N$$

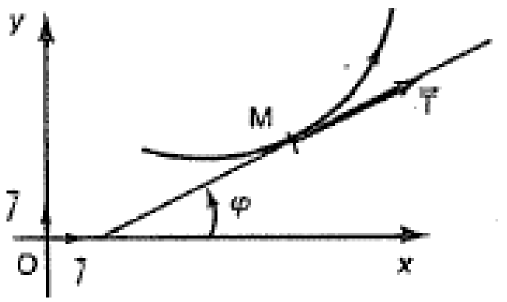
The Frenet-Serret equations (often just called the Frenet equations) describe the geometric properties of a smooth curve in 3D space using a moving coordinate system. They're foundational in differential geometry and are especially useful in physics, robotics, and computer graphics when analyzing the motion along a path.

4.4 Curvature and center of curvarture of of a Planar Curve (2D Curves)

Let a curve (α) defined by the parametric equations $(x(t), y(t))$.
Let us remember that $T = \dfrac{d\alpha(t)}{ds} = \dfrac{\alpha'(t)}{\|\alpha'(t)\|}$ is the unit tangent vector.
In parametric form, if you have $\alpha(t) = (x(t), y(t))$, then the unit normal vector N at point M is typically given by:

$$N = \frac{(-y'(t), x'(t))}{\|\alpha'(t)\|}$$

We denote by α the polar angle of the unit tangent vector $\vec{T}$ i.e $\varphi = (\vec{ox}, \vec{T})$



The components of $\vec{T}$ are, on the one hand, $\dfrac{dx}{ds}, \dfrac{dy}{ds}$ and on the other hand, $\cos(\varphi)$, $\sin(\varphi)$ (since $\vec{T}$ is a unit vector), hence

$$\frac{dx}{ds} = \cos(\varphi), \frac{dy}{ds} = \sin(\varphi)$$

$$\vec{T} = (\cos(\varphi), \sin(\varphi))$$

This confirms that

$$tg(\varphi) = \frac{\sin(\varphi)}{\cos(\varphi)} = \frac{\frac{dy}{ds}}{\frac{dx}{ds}} = \frac{\frac{dy}{dt}\frac{dt}{ds}}{\frac{dx}{dt}\frac{dt}{ds}} = \frac{y'(t)}{x'(t)}$$

4.4.1 Curvature

Definition 4.4.1 *Let a curve planar $\alpha(t) = (x(t), y(t))$. The curvature in terms of parametric equations is given by:*

$$\kappa(t) = \frac{\begin{vmatrix} x' & x'' \\ y' & y'' \end{vmatrix}}{\|\alpha'(t)\|^3} = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{((x'(t))^2 + (y'(t))^2)^{\frac{3}{2}}}.$$

Then the radius of curvature R is given by:

$$R(t) = \frac{1}{\kappa(t)}.$$

So, if you know the curvature at a point, you can calculate the radius of the osculating circle.

Remarque 4.4.1 *Algebraic radius of curvature $\Re$ is given by*

$$\Re = \frac{ds}{d\alpha},$$

and

$$R = |\Re|$$

Example 4.4.1 *Let us compute the curvature of the astroid.*

$$\begin{cases} x(t) & = \cos^3(t) \\ y(t) & = \sin^3(t) \end{cases}$$

$$\frac{ds}{dt} = 3/2\sin(2t).$$

On the other hand,

$$tg(\varphi) = \frac{y'}{x'} = \frac{3\cos(t)\sin^2(t)}{-3\sin(t)\cos^2(t)} = -tg(t).$$

One can take

$$\varphi(t) = -t,$$

then

$$\frac{d\varphi}{dt} = -1$$

We now conclude $\kappa(t)$

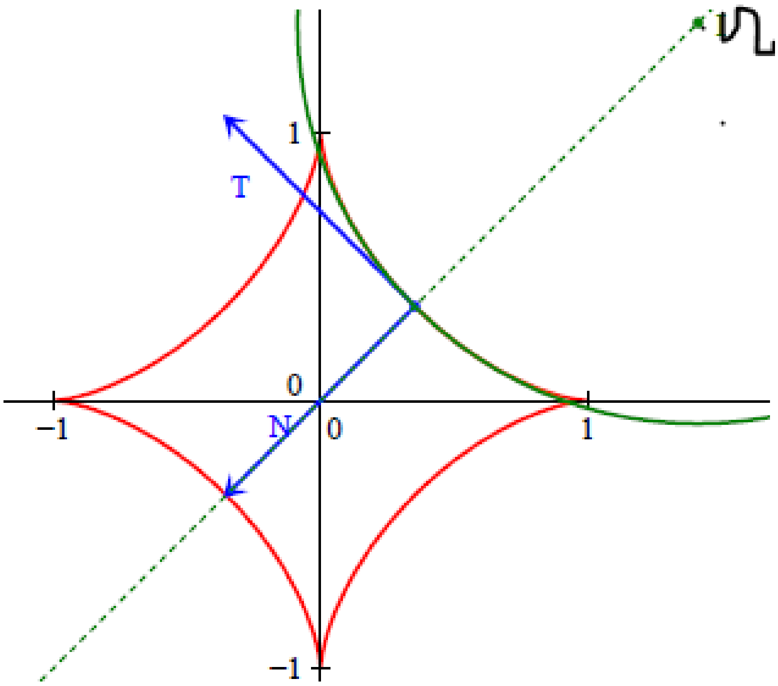
$$\kappa(t) = |\frac{d\varphi}{dt}\frac{dt}{ds}| = |\frac{2}{3\sin(t)}|$$

so

$$\Re(t) = \frac{ds}{d\alpha} = -\frac{3}{2}\sin(2t)$$

The negative sign is not surprising, on the segment of the curve under consideration, the normal will be directed inward toward the curve, while the center of curvature is clearly located on the outside.

A concrete example, in $\frac{\pi}{4}$, $R = -\frac{3}{2}$. *With the Frenet frame, the center of curvature and the osculating circle at the point with parameter* $\frac{\pi}{4}$*:*



Example 4.4.2 *Calculation of the radius of curvature at a point* M *of the cycloid defined by*

$$\begin{cases} x(t) = a(t - \cos(t)) \\ y(t) = a(1 + \sin(t)) \end{cases} \quad a > 0$$

For $t \in [0, 2\pi]$, we first compute

$$\begin{cases} x'(t) = a(1+\sin(t)) \\ y'(t) = a(\cos(t)) \end{cases} \quad a > 0$$

$$\begin{cases} x''(t) = a\cos(t) \\ y''(t) = -a\sin(t) \end{cases} \quad a > 0$$

then

$$\begin{aligned} R(t) = \frac{(x'^2+y'^2)^{3/2}}{|y''x' - y'x''|} &= \frac{(4a\sin(t/2))^{3/2}}{|a^2(\cos(t) - \cos^2(t)) - \sin^2(t))|} \\ &= \frac{(4a\sin(t/2))^{3/2}}{|a^2(\cos(t) - 1)|} \\ &= \frac{(4a\sin(t/2))^{3/2}}{|2a^2\sin^2(t/2)|} \\ &= 4a\sin(t/2) \end{aligned}$$

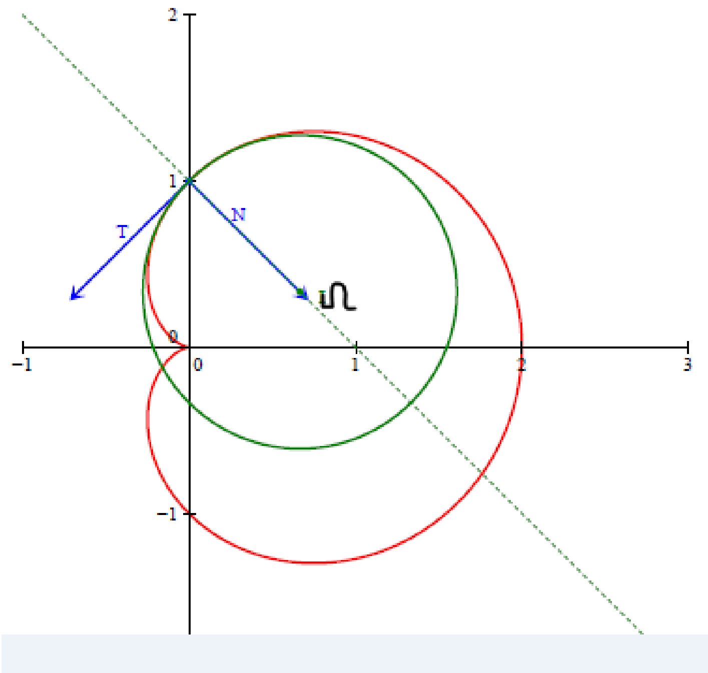
4.4.2 Center of curvature

The center of curvature Ω is the center of the osculating circle at a given point, and the radius of that circle is the inverse of the curvature.
The center of curvature Ω lies on the normal line to the curve at the point of interest, and its position can be described as:

$$\vec{M\Omega} = R\vec{N}$$

where: M is the point on the curve with coordinates $(x(t), y(t))$,
also, we have $\vec{M\Omega} = (x_\Omega - x_M, y_\Omega - y_M)$ then

$$\begin{cases} x_\Omega = x(t) - \dfrac{y'(x'^2+y'^2)}{|x'y''-x''y'|} \\ y_\Omega = y(t) + \dfrac{x'(x'^2+y'^2)}{|x'y''-x''y'|} \end{cases}$$



4.4.3 Evolute (or development) of a Curve

The evolute of a curveα is the locus (i.e., the path) of all the centers of curvature Ω of α. It tells us how the curvature of the curve changes along its length. For a curve $\alpha(t) = (x(t), y(t))$, the evolute is given by:

$$\begin{cases} X = x(t) - \dfrac{y'(x'^2 + y'^2)}{x'y'' - x''y'} \\ Y = y(t) + \dfrac{x'(x'^2 + y'^2)}{x'y'' - x''y'} \end{cases}$$

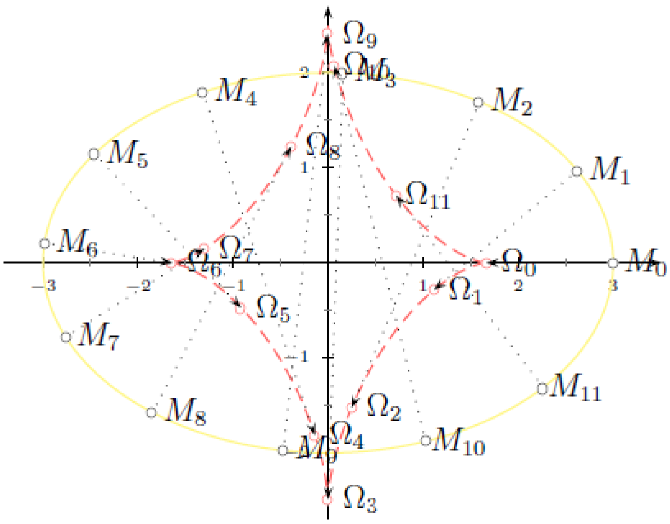
Example 4.4.3 *Let's walk through the development (evolute) of an ellipse, using its parametric representation* $(a\cos(t), b\sin(t))$

$$\begin{cases} X = x(t) - \dfrac{y'(x'^2 + y'^2)}{x'y'' - x''y'} = a\cos(t) - b\cos(t)\dfrac{a^2\sin^2(t) + b^2\cos^2(t)}{ab} \\ Y = y(t) + \dfrac{x'(x'^2 + y'^2)}{x'y'' - x''y'} = b\sin(t) - a\sin(t)\dfrac{a^2\sin^2(t) + b^2\cos^2(t)}{ab} \end{cases}$$

$$\begin{cases} X = \dfrac{1}{ab}(a^2 b\cos(t)(1 - \sin^2(t)) - b^3\cos(t)) \\ Y = \dfrac{1}{ab}(a^2 b\sin(t)(1 - \cos^2(t)) - b^3\sin(t)) \end{cases}$$

then

$$\begin{cases} X = \dfrac{a^2 - b^2}{a} \cos^3(t) \\ Y = \dfrac{b^2 - a^2}{ab} \sin^3(t) \end{cases}$$



The evolute of an ellipse is a hypocycloid-like closed curve with four cusps (sharp points), located at the extremities of the major and minor axes.

Exercise 4.4.1 *Let $a, b > 0$. We consider the parametrized curve (γ) defined by:*

$$M(t) = \begin{cases} x(t) = a\cos(t) \\ y(t) = b\sin(t) \end{cases} \quad t \in \mathbb{R}$$

1. *Compute the unit tangent vector T and the radius of curvature R at the point corresponding to parameter t on the curve (γ).*
2. *Define the mapping $\Omega : t \longrightarrow M(t) + R\dfrac{d\vec{T}}{d\varphi}$.*

   (a) *Let $\Omega(t) = (X(t), Y(t))$. Show that*

   $$X(t) = \frac{a^2 - b^2}{a}\cos^3(t), \qquad Y(t) = \frac{b^2 - a^2}{b}\sin^3(t)$$

   (b) *Study the periodicity and the symmetries of the curve Ω.*

   (c) *For $a = 2$ et $b = 1$, create a variation table on the interval $[0, \frac{\pi}{4}]$. Plot the curve Ω over $\mathbb{R}$.*

Solution.
1- Compute the unit tangent vector T and the radius of curvature R at the point corresponding to parameter t on the curve (γ).
Unit tangent vector

$$\begin{aligned} T &= \frac{dM}{ds} \\ &= \frac{dM}{dt}\frac{dt}{ds} \\ &= (-a\sin(t), b\cos(t))\frac{dM}{ds} \end{aligned}$$

on the other hand, we have

$$\frac{ds}{dt} = \sqrt{x'^2(t) + y'^2(t)} = \sqrt{a^2\sin^2(t) + b\cos^2(t)},$$

then

$$T = (-a\sin(t), b\cos(t))\frac{1}{\sqrt{a^2\sin^2(t) + b\cos^2(t)}}$$

Radius of curvature R

$$R(t) = \frac{(x'^2(t) + y'^2(t))^{3/2}}{x'(t)y''(t) - x''(t)y'(t)}$$

$$\begin{cases} x'(t) = -a\sin(t) \\ y'(t) = b\cos(t) \end{cases}$$

$$\begin{cases} x''(t) = -a\cos(t) \\ y''(t) = -b\sin(t) \end{cases}$$

This gives

$$x'(t)y''(t) - x''(t)y'(t) = ab$$
$$x'^2(t) + y'^2(t) = \sqrt{a^2\sin^2(t) + b\cos^2(t)}$$

Therefore

$$R = \frac{(a^2\sin^2(t) + b^2\cos^2(t))^{3/2}}{ab}$$

2.a)

$$\begin{cases} X = x(t) - \dfrac{y'(x'^2 + y'^2)}{x'y'' - x''y'} = a\cos(t) - b\cos(t)\dfrac{a^2\sin^2(t) + b^2\cos^2(t)}{ab} \\ Y = y(t) + \dfrac{x'(x'^2 + y'^2)}{x'y'' - x''y'} = b\sin(t) - a\sin(t)\dfrac{a^2\sin^2(t) + b^2\cos^2(t)}{ab} \end{cases}$$

$$\begin{cases} X = \dfrac{1}{ab}(a^2b\cos(t)(1 - \sin^2(t)) - b^3\cos^2(t)) \\ Y = \dfrac{1}{ab}(ab^2\sin(t)(1 - \cos^2(t)) - a^3\sin(t)) \end{cases}$$

then

$$\begin{cases} X = \dfrac{a^2 - b^2}{a}\cos^3(t) \\ Y = \dfrac{b^2 - a^2}{b}\sin^3(t) \end{cases}$$

2.b) - **The periodicidy**

Since the functions X andet Y are periodic with period 2π, the curve (Ω) is periodic with period 2π.

- The symmetries

$$\begin{cases} X(-t) = \dfrac{a^2-b^2}{a}\cos^3(-t) = \dfrac{a^2-b^2}{a}\cos^3(t) = X(t) \\ Y(-t) = \dfrac{b^2-a^2}{b}\sin^3(-t) = -\dfrac{b^2-a^2}{b}\sin^3(t) = -Y(t) \end{cases}$$

So the curve (Ω) is symmetric with respect to the x-axis and x-axis .

$$\begin{cases} X(\pi-t) = \dfrac{a^2-b^2}{a}\cos^3(\pi-t) = -\dfrac{a^2-b^2}{a}\cos^3(t) = -X(t) \\ Y(\pi-t) = \dfrac{b^2-a^2}{b}\sin^3(\pi-t) = \dfrac{b^2-a^2}{b}\sin^3(t) = Y(t) \end{cases}$$

So the curve (Ω) is symmetric with respect to the x-axis and y-axis

2.c) **For $a = 2$ et $b = 1$, create a variation table on the interval $[0, \dfrac{\pi}{4}]$. Plot the curve Ω over $\mathbb{R}$.**

$$\begin{cases} X(t) = 3/2\cos^3(t) \\ Y(t) = -3\sin^3(t) \end{cases}$$

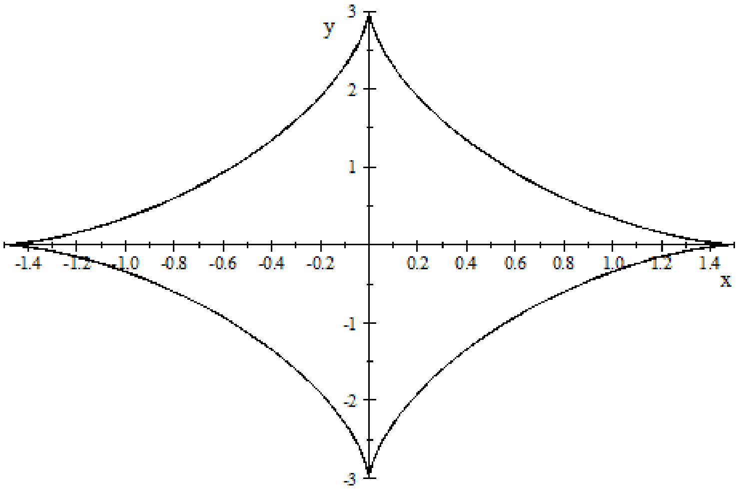
Variations of X and Y. We have:

$$\begin{cases} X'(t) = -9/2\sin(t)\cos^2(t) = 0 \\ Y'(t) = -9\cos(t)\sin^2(t) = 0 \end{cases} \Longrightarrow \begin{cases} t = 0 \; ou \; t = \pi/2 \\ t = 0 \; ou \; t = \pi/2 \end{cases}$$

Hence the table of variations:

t	0		$\frac{\pi}{2}$
f'	0	$-$	
f	$\frac{3}{2}$	↘	0
g	0	↘	-3
g'	0	$-$	

The graph of (Ω)



4.5 Exercises

Exercise 4.5.1 *In space, with respect to a coordinate system* $(o, \vec{i}, \vec{j}, \vec{(k)})$, *we consider the curve* (Γ) *given by the parametric equations:*

$$\begin{cases} x(t) = \displaystyle\int_0^t f(u)\sin(u)du \\ y(t) = \displaystyle\int_0^t f(u)\cos(u)du \\ z(t) = \displaystyle\int_0^t f(u)\tan(u)du \end{cases}$$

where $f(u) = \cos(u)\sqrt{1 + \cos^2(u)}$

Show that the curvature of the curve (Γ) *is constant.*

Solution.

1. Compute derivatives:

$$\mathbf{r}'(t) = \big(f(t)\sin t, f(t)\cos t, f(t)\tan t\big)$$

2. Compute the second derivative:

$$\mathbf{r}''(t) = \big(f'(t)\sin t + f(t)\cos t, f'(t)\cos t - f(t)\sin t, f'(t)\tan t + f(t)\sec^2 t\big)$$

3. Curvature formula in space:

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = 1$$

Exercise 4.5.2 *Consider the helix* $\alpha(t) = (a\cos(t), a\sin(t), bt)$

1. *Calculate* $\alpha'(t)$ *and* $\|\alpha'(t)\|$.
2. *Reparametrize* α *by arc length function* S.

Solution.

Curve: $\alpha(t) = (a\cos t, a\sin t, bt)$

1. Derivative:

$$\alpha'(t) = (-a\sin t, a\cos t, b)$$

$$\|\alpha'(t)\| = \sqrt{a^2 + b^2}$$

2. Arc-length parametrization s:

$$s = \int_0^t \|\alpha'(\tau)\| d\tau = \sqrt{a^2 + b^2}\, t \implies t = \frac{s}{\sqrt{a^2 + b^2}}$$

$$\alpha(s) = \left(a\cos\frac{s}{\sqrt{a^2 + b^2}}, a\sin\frac{s}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}} s \right)$$

Exercise 4.5.3 *Let* $a, b, c \in \mathbb{R}$ *with* $a^2 + b^2 = c^2$ *an* $a \neq 0$.

Consider the following parameterized curve

$$\gamma : \mathbb{R} \longrightarrow \mathbb{R}^3$$
$$t \longmapsto (a\cos(t), a\sin(t), bt)$$

1. *Reparametrize* γ *by arc length function* S.
2. *After traveling a distance of* $\frac{\pi c}{4}$, *where are we along the curve.*
3. *Find the normal* $N(t)$ *and binormal* $B(t)$ *vectors.*
4. *Calculate the curvature* κ *and the torsion* τ *of* γ.
5. *Find the center of curvature.*

Solution.

Curve: $\gamma(t) = (a\cos t, a\sin t, bt), a^2 + b^2 = c^2$

1. Arc-length parametrization:

$$s = \int_0^t \|\gamma'(\tau)\| d\tau = \int_0^t \sqrt{a^2 + b^2} d\tau = ct \implies t = \frac{s}{c}$$

2. After traveling $\pi c/4$:

$$s = \frac{\pi c}{4} \implies t = \frac{\pi}{4} \implies \gamma\left(\frac{\pi}{4}\right) = \left(a\frac{\sqrt{2}}{2}, a\frac{\sqrt{2}}{2}, b\frac{\pi}{4}\right)$$

3. Tangent, normal, binormal:

$$T(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|} = \frac{(-a\sin t, a\cos t, b)}{c}$$

$$\gamma''(t) = (-a\cos t, -a\sin t, 0), \quad N(t) = \frac{T'(t)}{\|T'(t)\|} = (-\cos t, -\sin t, 0)$$

$$B(t) = T(t) \times N(t) = \frac{1}{c}(b\sin t, -b\cos t, a)$$

4. Curvature and torsion:

$$\kappa = \frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3} = \frac{a}{c^2}, \quad \tau = \frac{(\gamma' \times \gamma'') \cdot \gamma'''}{\|\gamma' \times \gamma''\|^2} = \frac{b}{c^2}$$

5. Center of curvature:

$$C(t) = \gamma(t) + \frac{1}{\kappa}N(t) = (a\cos t - \frac{c^2}{a}\cos t, a\sin t - \frac{c^2}{a}\sin t, bt)$$

Exercise 4.5.4 *Find the curvature and the torsion of the following curves:*

1. $\alpha_1(t) = \left(e^t, e^{-t}, \sqrt{2}t\right)$*;*

2.$\alpha_1(t) = \left(\cos^3(t), \sin^3(t), \cos(2t)\right)$.

Solution.

Curvature and Torsion of Given Curves

1. Curve: $\alpha_1(t) = \left(e^t, e^{-t}, \sqrt{2}\,t\right)$

Derivatives

$$\alpha_1'(t) = (e^t, -e^{-t}, \sqrt{2}), \quad \alpha_1''(t) = (e^t, e^{-t}, 0), \quad \alpha_1'''(t) = (e^t, -e^{-t}, 0)$$

Curvature

$$\kappa(t) = \frac{\|\alpha_1'(t) \times \alpha_1''(t)\|}{\|\alpha_1'(t)\|^3}$$

$$\alpha_1' \times \alpha_1'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^t & -e^{-t} & \sqrt{2} \\ e^t & e^{-t} & 0 \end{vmatrix} = (-\sqrt{2}e^{-t}, \sqrt{2}e^t, 2)$$

$$\|\alpha_1' \times \alpha_1''\| = \sqrt{2e^{-2t} + 2e^{2t} + 4} = \sqrt{2}(e^t + e^{-t})$$

$$\|\alpha_1'\| = \sqrt{e^{2t} + e^{-2t} + 2} = e^t + e^{-t}$$

$$\boxed{\kappa(t) = \frac{\sqrt{2}}{(e^t + e^{-t})^2}}$$

Torsion

$$\tau(t) = \frac{(\alpha_1' \times \alpha_1'') \cdot \alpha_1'''}{\|\alpha_1' \times \alpha_1''\|^2}$$

$$(\alpha_1' \times \alpha_1'') \cdot \alpha_1''' = (-\sqrt{2}e^{-t})(e^t) + (\sqrt{2}e^t)(-e^{-t}) + 2 \cdot 0 = -2\sqrt{2}$$

$$\|\alpha_1' \times \alpha_1''\|^2 = 2(e^t + e^{-t})^2$$

$$\boxed{\tau(t) = -\frac{\sqrt{2}}{(e^t + e^{-t})^2}}$$

Exercise 4.5.5 *Let ρ be a C^2 function from an interval $I \subset \mathbb{R}$.*

A plane curve is defined in polar coordinates by $\rho = \rho(t)$.

1. *Show that the curvature at a regular point is given by*

$$\kappa(t) = \frac{\rho^2(t) + 2\rho'^2(t) - \rho(t)\rho''(t)}{(\rho^2(t) + \rho'^2(t))^{3/2}}$$

2. *compute the curvature $\kappa(t)$ for a polar curve given by $\rho(t) = ae^{-bt}$ where $a, b > 0$.*

3. *Give the coordinates of the center of curvature at the point $M(0)$.*

Solution.

Curvature and Center of Curvature in Polar Coordinates

1. Curvature formula for a plane curve in polar coordinates

Let a plane curve be defined in polar coordinates by $\rho = \rho(t)$, with t as the parameter.

In Cartesian coordinates:

$$x(t) = \rho(t)\cos t, \quad y(t) = \rho(t)\sin t$$

Derivatives:

$$x'(t) = \rho'(t)\cos t - \rho(t)\sin t, \quad y'(t) = \rho'(t)\sin t + \rho(t)\cos t$$

Second derivatives:

$$x''(t) = \rho''\cos t - 2\rho'\sin t - \rho\cos t, \quad y''(t) = \rho''\sin t + 2\rho'\cos t - \rho\sin t$$

The curvature formula in Cartesian coordinates is:

$$\kappa(t) = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}}$$

Compute the numerator:

$$\begin{aligned} x'y'' - y'x'' &= (\rho' \cos t - \rho \sin t)(\rho'' \sin t + 2\rho' \cos t - \rho \sin t) \\ &\quad - (\rho' \sin t + \rho \cos t)(\rho'' \cos t - 2\rho' \sin t - \rho \cos t) \\ &= \rho^2 + 2(\rho')^2 - \rho\rho'' \quad \text{(after simplification)} \end{aligned}$$

Compute the denominator:

$$x'^2 + y'^2 = \rho^2 + (\rho')^2$$

Thus, the curvature in polar coordinates is:

$$\boxed{\kappa(t) = \frac{\rho^2(t) + 2\rho'^2(t) - \rho(t)\rho''(t)}{(\rho^2(t) + \rho'^2(t))^{3/2}}}$$

—

2. Curvature for $\rho(t) = ae^{-bt}$, $a, b > 0$

Compute derivatives:

$$\rho'(t) = -abe^{-bt}, \quad \rho''(t) = ab^2e^{-bt}$$

Substitute into the curvature formula:

$$\kappa(t) = \frac{(ae^{-bt})^2 + 2(-abe^{-bt})^2 - (ae^{-bt})(ab^2e^{-bt})}{\left((ae^{-bt})^2 + (-abe^{-bt})^2\right)^{3/2}}$$

Simplify numerator:

$$a^2e^{-2bt} + 2a^2b^2e^{-2bt} - a^2b^2e^{-2bt} = a^2e^{-2bt}(1 + b^2)$$

Simplify denominator:

$$(x'^2 + y'^2)^{3/2} = \left(a^2e^{-2bt} + a^2b^2e^{-2bt}\right)^{3/2} = \left(a^2e^{-2bt}(1+b^2)\right)^{3/2} = a^3e^{-3bt}(1+b^2)^{3/2}$$

Hence:

$$\kappa(t) = \frac{a^2e^{-2bt}(1+b^2)}{a^3e^{-3bt}(1+b^2)^{3/2}} = \frac{e^{bt}}{a\sqrt{1+b^2}}$$

$$\boxed{\kappa(t) = \frac{e^{bt}}{a\sqrt{1+b^2}}}$$

3. Coordinates of the center of curvature at $M(0)$

The coordinates of the center of curvature in polar coordinates are:

$$X_c = x - \frac{y'(x'^2 + y'^2)}{x'y'' - y'x''}, \quad Y_c = y + \frac{x'(x'^2 + y'^2)}{x'y'' - y'x''}$$

At $t = 0$:

$$\rho(0) = a, \quad \rho'(0) = -ab, \quad \rho''(0) = ab^2$$

Cartesian coordinates of $M(0)$:

$$x = \rho(0)\cos 0 = a, \quad y = \rho(0)\sin 0 = 0$$

$$x'(0) = \rho'(0)\cos 0 - \rho(0)\sin 0 = -ab, \quad y'(0) = \rho'(0)\sin 0 + \rho(0)\cos 0 = a$$

$$x''(0) = \rho''(0)\cos 0 - 2\rho'(0)\sin 0 - \rho(0)\cos 0 = ab^2 - a = a(b^2 - 1)$$

$$y''(0) = \rho''(0)\sin 0 + 2\rho'(0)\cos 0 - \rho(0)\sin 0 = 2(-ab) = -2ab$$

Compute denominator:

$$x'y'' - y'x'' = (-ab)(-2ab) - (a)(a(b^2-1)) = 2a^2b^2 - a^2(b^2-1) = a^2(b^2+1)$$

Compute $x'^2 + y'^2$:

$$(-ab)^2 + (a)^2 = a^2(b^2+1)$$

Finally:

$$X_c = x - \frac{y'(x'^2+y'^2)}{x'y''-y'x''} = a - \frac{a\cdot a^2(b^2+1)}{a^2(b^2+1)} = a - a = 0$$

$$Y_c = y + \frac{x'(x'^2+y'^2)}{x'y''-y'x''} = 0 + \frac{-ab\cdot a^2(b^2+1)}{a^2(b^2+1)} = -ab$$

$$\boxed{C = (0, -ab)}$$

Exercise 4.5.6 *Determine the coordinates of the center of curvature at a point $M(t)$ for certain curves*

1/ $\alpha_1 = (3t - t^3, 2t^2)$*;*

2/ $\alpha_2 = (2\cos(t) + \cos(2t), 2\sin(t) - \sin(2t))$*;*

3/ $xy = 1$

Solution.

1. Curve : $\alpha_1(t) = (3t - t^3, 2t^2)$

$$x(t) = 3t - t^3, \quad y(t) = 2t^2$$

$$x'(t) = 3 - 3t^2, \quad x''(t) = -6t, \quad y'(t) = 4t, \quad y''(t) = 4$$

$$x'y'' - y'x'' = (3-3t^2)(4) - (4t)(-6t) = 12 - 12t^2 + 24t^2 = 12(1+t^2)$$

$$x'^2 + y'^2 = (3-3t^2)^2 + (4t)^2 = 9(1-t^2)^2 + 16t^2$$

the coordinates :

$$\boxed{X_c(t) = 3t - t^3 - \frac{4t\big(9(1-t^2)^2+16t^2\big)}{12(1+t^2)}, \quad Y_c(t) = 2t^2 + \frac{(3-3t^2)\big(9(1-t^2)^2+16t^2\big)}{12(1+t^2)}}$$

2. Curve : $\alpha_2(t) = (2\cos t + \cos 2t, 2\sin t - \sin 2t)$

$$x(t) = 2\cos t + \cos 2t, \quad y(t) = 2\sin t - \sin 2t$$

$$x'(t) = -2\sin t - 2\sin 2t, \quad x''(t) = -2\cos t - 4\cos 2t$$

$$y'(t) = 2\cos t - 2\cos 2t, \quad y''(t) = -2\sin t + 4\sin 2t$$

$$x'y'' - y'x'' = (-2\sin t - 2\sin 2t)(-2\sin t + 4\sin 2t) - (2\cos t - 2\cos 2t)(-2\cos t - 4\cos 2t)$$

$$x'^2 + y'^2 = (-2\sin t - 2\sin 2t)^2 + (2\cos t - 2\cos 2t)^2$$

the coordinates :

$$\boxed{X_c(t) = x(t) - \frac{y'(t)(x'^2+y'^2)}{x'y''-y'x''}, \quad Y_c(t) = y(t) + \frac{x'(t)(x'^2+y'^2)}{x'y''-y'x''}}$$

3. $xy = 1$

The curve $y = y(x)$:

$$y' = \frac{dy}{dx} = -\frac{1}{x^2}, \quad y'' = \frac{2}{x^3}$$

$$X_c = x - \frac{-\frac{1}{x^2}\left(1+\frac{1}{x^4}\right)}{\frac{2}{x^3}} = x + \frac{x(1+1/x^4)}{2} = x + \frac{x}{2} + \frac{1}{2x^3} = \frac{3x}{2} + \frac{1}{2x^3}$$

$$Y_c = \frac{1}{x} + \frac{1 + 1/x^4}{2/x^3} = \frac{1}{x} + \frac{x^3 + 1/x}{2} = \frac{1}{x} + \frac{x^3}{2} + \frac{1}{2x} = \frac{3}{2x} + \frac{x^3}{2}$$

$$\boxed{C(x) = \left(\frac{3}{2}x + \frac{1}{2x^3}, \frac{3}{2x} + \frac{x^3}{2}\right)}$$

Exercise 4.5.7 *We are given an astroid, which is a particular type of hypocycloid, and it's defined parametrically by:*

$$\begin{cases} x(t) = a\cos^3(t) \\ y(t) = a\sin^3(t) \end{cases}$$

We are asked to find the parametric equations of the evolute (i.e., the developed curve) of this astroid.

Solution.

Curve: $x = a\cos^3 t, y = a\sin^3 t$

-
- First derivatives: $x' = -3a\cos^2 t \sin t, y' = 3a\sin^2 t \cos t$
- Second derivatives: $x'' = -3a(\cos^3 t - 2\cos t \sin^2 t), y'' = 3a(2\sin t \cos^2 t - \sin^3 t)$
- Curvature:

$$\kappa = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}}$$

- Center of curvature (evolute):

$$X = x - \frac{y'(x'^2 + y'^2)}{x'y'' - y'x''}, \quad Y = y + \frac{x'(x'^2 + y'^2)}{x'y'' - y'x''}$$

After simplification, the evolute has parametric equations:

$$\boxed{X = \frac{a}{4}\cos 3t, \quad Y = \frac{a}{4}\sin 3t}$$

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