وزارة التعليــم العالــي والبحــث العلمــي

Sétif 1 University-Ferhat ABBAS
Faculty of Sciences
Department of Mathematics

INTRODUCTION TO METRIC & TOPOLOGICAL SPACES

FOR THE SECOND YEAR LMD MATHEMATICS STUDENTS

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CONTENTS

INTRODUCTION

This course aims to provide a comprehensive and accessible introduction to the fundamental concepts of metric spaces, topological spaces, complete spaces, compact spaces, and connected spaces. These mathematical structures form the backbone of modern analysis and topology, and they have wide-ranging applications in fields such as geometry, functional analysis, and theoretical physics. Understanding these spaces is essential for anyone wishing to pursue advanced studies in mathematics or related disciplines.

We begin with metric spaces, one of the most intuitive and well-studied types of spaces. A metric space is a set equipped with a distance function, or metric, that assigns a non-negative real number to each pair of points, representing the "distance" between them. This simple yet powerful structure allows us to define and analyze concepts such as convergence, continuity, and compactness. Metric spaces also serve as a foundation for more advanced spaces, making them an ideal starting point for our study.

Building on the notion of metric spaces, we will then introduce topological spaces, a more abstract and general framework. Unlike metric spaces, topological spaces are defined by a collection of open sets that satisfy certain axioms. This abstraction allows mathematicians to study a wide range of spaces that may not have a natural notion of distance but still exhibit similar topological properties. Topological spaces provide a unifying language for various branches of mathematics, from analysis to algebraic geometry.

The concept of completeness is a natural extension in both metric and topological settings. A space is said to be complete if every Cauchy sequence converges to a limit within the space. Completeness is crucial in the study of functional spaces, as it guarantees the existence of solutions to various mathematical problems, such as differential equations. We will explore the importance of complete spaces and their role in the theory of Banach and Hilbert spaces, which are central to functional analysis.

Next, we will delve into the notion of compactness, a property that captures the idea of "smallness" or "boundedness" in a topological sense. Compact spaces are those in which every open cover has a finite subcover, and they exhibit many desirable properties that make them *indispensable in both pure and applied mathematics. For example, compactness ensures the existence of convergent subsequences and plays a critical role in optimization, integration, and the study of continuous functions.*

Finally, we will examine connectedness, a fundamental property that describes whether a space can be divided into disjoint open subsets. A connected space is one that cannot be split into two non-empty, disconnected parts. Connectedness is essential in understanding the behavior of continuous functions and the topological structure of spaces. It also provides a framework for analyzing geometric shapes and understanding how different parts of a space relate to each other.

Throughout this course, we will emphasize both the theoretical foundations and practical applications of these concepts. Each chapter will build on the previous ones, providing a logical progression from basic definitions to more advanced topics. By the end of this course, students will have a solid understanding of the core ideas in topology and analysis, enabling them to tackle more complex problems in mathematics and its applications.

In addition to theoretical discussions, we will include numerous examples and exercises to help students develop intuition and problem-solving skills. Historical notes will highlight the contributions of mathematicians such as Henri Poincaré, Karl Weierstrass, and Maurice Fréchet, whose work has shaped the development of these concepts.

By the end of this course, students will have a comprehensive understanding of these fundamental mathematical structures and their importance in various branches of mathematics. This knowledge will prepare them for more advanced topics and applications in areas such as functional analysis, differential equations, and mathematical physics. It consists of five chapters, outlined as follows:

- ***Chapter 1****: Explores metric spaces and their properties, introducing concepts like distance, open and closed balls, isometric spaces, and Lipschitz functions.*
- ***Chapter 2****: Develops the concept of complete spaces, focusing on Cauchy sequences and fixed points.*
- ***Chapter 3****: Introduces the structure and properties of topological spaces, convergent sequences, continuous functions, open and closed maps, and homeomorphisms.*
- ***Chapter 4****: Examines compactness in both topological and metric spaces.*
- ***Chapter 5****: Dedicates attention to connectedness in topological and metric spaces.*

This copy does not exempt you from attending the meetings or taking additional notes. It is there to avoid a copy work that sometimes prevents you from focusing on the explanations given orally.

CHAPTER 1

METRIC SPACES

1.1 Metric spaces

Metric spaces are a fertile field for examples that we will use to study topological spaces and their properties.

The notion of metric space is was introduced in 1906 by Maurice Fréchet and developed and named by Felix Hausdorff in 1914.

Definition 1.1. *Let $\mathbb{X}$ be a non-empty set and $d : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{R}_+$ a real valued function such that for all $x, y, z \in \mathbb{X}$ the following holds:*

C_1) $d(x,y) = 0 \Longleftrightarrow x = y$,

C_2) $d(x,y)=d(y,x)$; *(symmetry).*

C_3) $d(x,y) \leq d(x,z) + d(z,y)$; *(triangle inequality).*

Then d is said to be a metric(or distance) on $\mathbb{X}$, the pair $(\mathbb{X}, d)$ is called a metric space and $d(x,y)$ is referred to as the distance between x and y.

Remark 1.1. *A metric space $(\mathbb{X}, d)$ is a set $\mathbb{X}$ endowed with a metric d. When there is no possibility of confusion, we abbreviate by saying that $\mathbb{X}$ is a metric space.*

Example 1.1. *On $\mathbb{R}^n$ we have the following metrics :*

1) $d_1(x,y) = \sum\limits_{i=1}^{n} |x_i - y_i|$, ***2)*** $d_2(x,y) = \left(\sum\limits_{i=1}^{n} (x_i - y_i)^2\right)^{\frac{1}{2}}$,

3) $d_\infty(x,y) = \max\limits_{i=1,\dots,n} (|x_i - y_i|)$, ***4)*** $d_p(x,y) = \left(\sum\limits_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}}$, $p \geqslant 1$

The metric d_1 is called ℓ^1 metric, d_2 is called the the euclidean metric (or ℓ^2 metric), d_∞ is called the maximum metric(or ℓ^∞ metric) and d_p is called ℓ^p metric.

Example 1.2. *On $C([a,b],\mathbb{R})$(the set of continuous functions from [a,b] to $\mathbb{R}$) we have the following metrics :*

1) $d_1(f,g) = \int_a^b |f(t)-g(t)|dt$, ***2)*** $d_2(f,g) = [\int_a^b (f(t)-g(t))^2 dt]^{\frac{1}{2}}$,

3) $d_\infty(f,g) = \sup_{t\in[a,b]} (|f(t)-g(t)|)$, ***4)*** $d_p(f,g) = [\int_a^b |f(t)-g(t)|^p dt]^{\frac{1}{p}}$, $p \geq 1$.

Example 1.3. *The function $d_u : \mathbb{R} \longrightarrow \mathbb{R}_+$ given by $d(x,y) = |x-y|$ is a metric on $\mathbb{R}$ and is called usual metric(or euclidean metric) on $\mathbb{R}$.*

Example 1.4. *Let $\mathbb{X}$ be a non-empty set and $\delta : \mathbb{X}\times\mathbb{X} \longrightarrow \mathbb{R}$ the function defined by*

$$\delta(x,y) = \begin{cases} 1 \text{ if } x \neq y, \\ 0 \text{ if } x = y. \end{cases} \tag{1.1}$$

Then, δ is a metric on $\mathbb{X}$ and is called the discrete metric.

Proposition 1.1. *Let $(\mathbb{X},d)$ be a metric space. Then,*

$$|d(x,z)-d(y,z)| \leqslant d(x,y), \tag{1.2}$$

for all $x,y,z \in \mathbb{X}$

Proof. *Using the triangle inequality for metrics we obtain*

$$d(x,z)-d(y,z) \leqslant d(x,y). \tag{\textit{i}}$$

Again, by using the triangle inequality for metrics we can see that

$$-d(x,y) \leqslant d(x,z)-d(y,z). \tag{\textit{ii}}$$

*Then, the inequality (1.2) follows from **(i)** and **(ii)**.*

Definition 1.2. *Let $(\mathbb{X},d_{\mathbb{X}})$ be a metric space and let A be a subset of $\mathbb{X}$. We define a metric $d_A : A\times A \longrightarrow \mathbb{R}_+$ on A by $d_A(x,y) = d_{\mathbb{X}}(x,y)$ for all $x,y \in A$. Then, (A,d_A) is a metric space, which is said to be a subspace of $(\mathbb{X},d_{\mathbb{X}})$.*

Remark 1.2. *The metric d_A is just the function $d_{\mathbb{X}}$ restricted to the subset $A\times A$ of $\mathbb{X}\times\mathbb{X}$.*

1.2 Open balls, closed balls and spheres

Definition 1.3. *Let $(\mathbb{X},d)$ be a metric space. Let $a\in\mathbb{X}$ and r any positive real number. Then,*

1) *the open ball around a of radius r is defined as follows:*

$$B(a,r)=\{x\in\mathbb{X}\ /\ d(a,x)<r\}.$$

2) *the closed ball around a of radius r is defined as follows:*

$$B_f(a,r)=\{x\in\mathbb{X}\ /\ d(a,x)\leq r\}.$$

3) *the sphere centered at a of radius r is defined as follows:*

$$S(a,r)=\{x\in\mathbb{X}\ /\ d(a,x)=r\}.$$

Remark 1.3. $B_f(a,r)=B(a,r)\cup S(a,r)$ *for all $a\in\mathbb{X}$ and $r>0$.*

Example 1.5. *In $\mathbb{R}$ with the euclidean metric d_u we have:*

- $B(a,r)=\{x\in\mathbb{R}\ /\ |x-a|<r\}=(a-r,a+r)$.
- $B_f(a,r)=\{x\in\mathbb{R}\ /\ |x-a|\leq r\}=[a-r,a+r]$.
- $S(a,r)=\{x\in\mathbb{R}\ /\ |x-a|=r\}=\{a-r,a+r\}$.

Example 1.6. *In $\mathbb{R}^2$ with the euclidean metric d_2 we have:*

- $B(a,r)$ *is the open disc centered at $a=(a_1,a_2)\in\mathbb{R}^2$ of radius r.*
- $B_f(a,r)$ *is the closed disc centered at $a=(a_1,a_2)\in\mathbb{R}^2$ of radius r.*
- $S(a,r)$ *is the circle centered at $a=(a_1,a_2)\in\mathbb{R}^2$ of radius r.*

Example 1.7. *In $\mathbb{R}^2$, if we take $a=(0,0)\in\mathbb{R}^2$ and $r=1$ we obtain:*

- $B_{d_1}((0,0),1)=\Big\{(x_1,x_2)\in\mathbb{R}^2\ /\ d_1((x_1,x_2),(0,0))=|x_1|+|x_2|<1\Big\}$,
- $B_{d_2}((0,0),1)=\Big\{(x_1,x_2)\in\mathbb{R}^2\ /\ d_2((x_1,x_2),(0,0))=\sqrt{(x_1)^2+(x_2)^2}<1\Big\}$,
- $B_{d_\infty}((0,0),1)=\Big\{(x_1,x_2)\in\mathbb{R}^2\ /\ d_\infty((x_1,x_2),(0,0))=max(|x_1|,|x_2|)<1\Big\}$.

Hence, the unit ball (open ball) $B((0,0),1)$ takes the following forms:

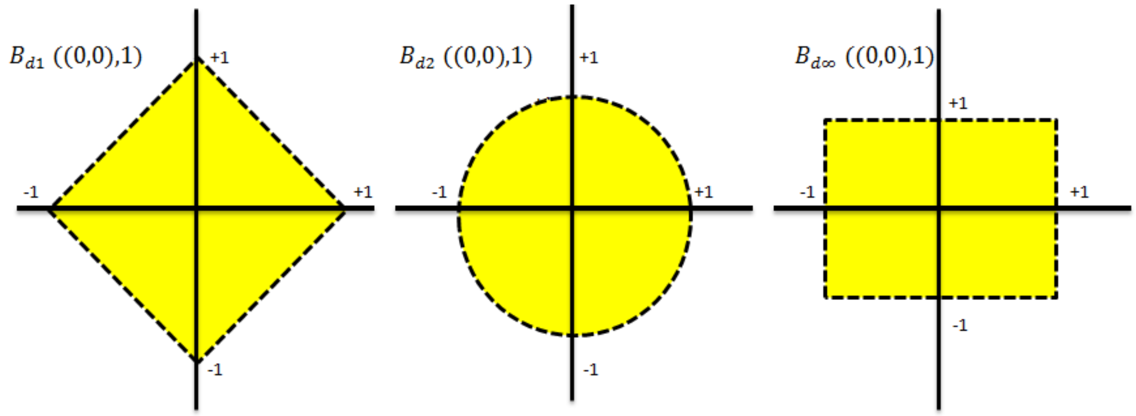


Figure 1.1: Open ball $B((0,0),1)$ in $\mathbb{R}^2$ with d_1, d_2 and d_∞

Example 1.8. *In $\mathbb{R}^3$ equipped with the euclidean metric d_2 we have:*

- *$S(a,r)$ is the sphere centered at a of radius r.*
- *$B(a,r)$ is the open ball (excluding the boundary points that constitute the sphere) centered at a of radius r.*
- *$B_f(a,r)$ is the closed ball (including the boundary points that constitute the sphere) centered at a of radius r.*

Example 1.9. *In the discrete metric $(\mathbb{X},\delta)$ (see Example (1.4)), we have:*

$$B(a,r) = \begin{cases} \{a\} & \text{if } r \leq 1, \\ \mathbb{X} & \text{if } r > 1, \end{cases}$$

for all $a \in \mathbb{X}$ and $r > 0$.

1.3 Open sets, closed sets and neighbourhood

Definition 1.4. *Let $(\mathbb{X},d)$ be a metric space. A set $O \subset \mathbb{X}$ is called open if every point $x \in O$ is the center of an open ball contained in O. That is,*

$$\forall x \in O,\ \exists r > 0 \text{ such that } B(x,r) \subseteq O.$$

Definition 1.5. *Let $(\mathbb{X},d)$ be a metric space. A subset F of $\mathbb{X}$ is said to be closed in $(\mathbb{X},d)$ if its complement, $C_{\mathbb{X}}F$ (or F^C), is open in $(\mathbb{X},d)$.*

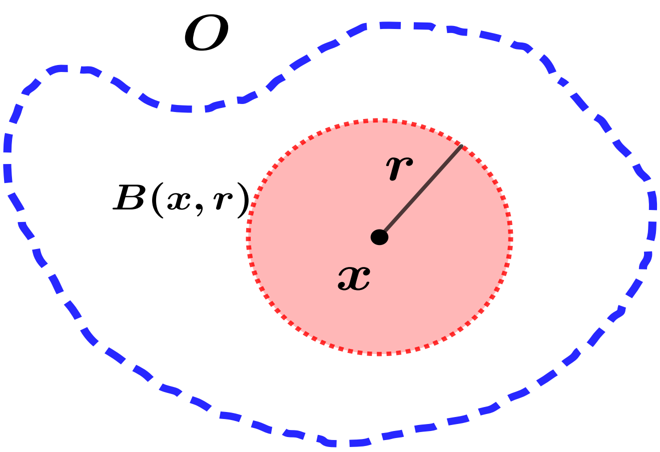
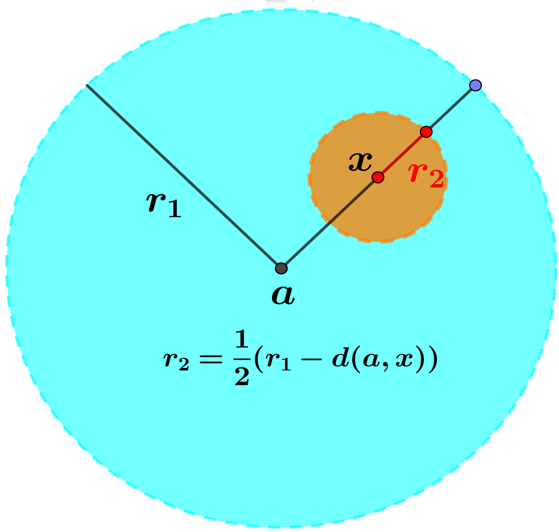


Figure 1.2: Open set **O**

Proposition 1.2. *Let $(\mathbb{X},d)$ be a metric space. Let $a \in \mathbb{X}$ and $r>0$. Then the open ball $B(a,r)$ is an open set.*

Proof. *Let $B(a,r_1)$ be an open ball in $(\mathbb{X},d)$. Then, for all $x \in B(a,r_1)$ we have $d(a,x)<r_1$. By taking $r_2 = \frac{1}{2}(r_1 - d(a,x))$ we obtain $B(x,r_2) \subset B(a,r_1)$.*



Remark **1.4.** *Using the previous preposition we conclude that the closed ball $B_f(a,r)$ is a closed set.*

Proposition 1.3. *Let $(\mathbb{X},d)$ be a metric space. Any open set in $\mathbb{X}$ is an union of open balls.*

Proof. *Let O be an open subset of $\mathbb{X}$. Then, for all $x \in O$ there exists $r > 0$ such that $B(x,r) \subseteq O$ which implies that $O = \bigcup\limits_{x\in O} \{x\} \subseteq \bigcup\limits_{x\in O} B(x,r) \subseteq O$. Hence, $O = \bigcup\limits_{x\in O} B(x,r)$.*

Proposition 1.4. *Let $(\mathbb{X},d)$ be a metric space. Then, the open sets in $\mathbb{X}$ satisfy the following properties.*

1) *$\mathbb{X}$ is open and $\emptyset$ is open.*

2) *Any union of open sets in $\mathbb{X}$ is an open set in $\mathbb{X}$.*

3) *Any finite intersection of open sets in $\mathbb{X}$ is an open set in $\mathbb{X}$.*

Proof.

1) *Note that it is vacuously true that the empty set contains an open ball about each of its points, since it contains no points. And the set $\mathbb{X}$ contains an open ball about each of its points because every open ball is a subset of $\mathbb{X}$.*

2) *Assume that $\{O_i : i \in I\}$ is a collection (finite or infinite) of open sets in $(\mathbb{X},d)$ and let $x \in \bigcup\limits_{i\in I} O_i$. So, there exists $i_0 \in I$ such that $x \in O_{i_0}$. Since, O_{i_0} is open, there is an $r > 0$ such that $B(x,r) \subseteq O_{i_0}$. Hence, $B(x,r) \subseteq O_{i_0} \subseteq \bigcup\limits_{i\in I} O_i$ which implies that $\bigcup\limits_{i\in I} O_i$ is an open set in $(\mathbb{X},d)$.*

3) *Assume that $\{O_1, O_2, ..., O_n\}$ is a finite collection of open sets in $(\mathbb{X},d)$ and let $x \in \bigcap\limits_{i=1}^{n} O_i$. Then $x \in O_i$ for each $i = 1,2,...,n$. So, for each $i = 1,2,...,n$, there is an $r_i > 0$ such that $B(x,r_i) \subseteq O_i$. Let $r = min(r_1, r_2, ..., r_n)$. Then $B(x,r) \subseteq O_i$ for all $i = 1,2,...,n$, which implies that $B(x,r) \subseteq \bigcap\limits_{i=1}^{n} O_i$. Hence, $\bigcap\limits_{i=1}^{n} O_i$ is an open set in $(\mathbb{X},d)$.*

Example 1.10. *In $\mathbb{R}$ with the Euclidean metric (usual metric), the set*

$$A = \bigcap_{i=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\},$$

is an intersection of open sets but is in fact not open.

By applying De Morgan's laws to the previous proposition, we can easily prove the following similar proposition for closed sets.

Proposition 1.5. *Let $(\mathbb{X}, d)$ be a metric space. Then, the closed sets in $\mathbb{X}$ satisfy the following properties.*

1. *$\mathbb{X}$ is closed and $\emptyset$ is closed.*
2. *Any intersection of closed sets in $\mathbb{X}$ is a closed set in $\mathbb{X}$.*
3. *Any finite union of closed sets in $\mathbb{X}$ is a closed set in $\mathbb{X}$.*

Proposition 1.6. *Let $(\mathbb{X}, d)$ be a metric space.Then, the sphere $S(x,r)$ is a closed set in $(\mathbb{X}, d)$.*

Proof. *The complement of the sphere $S(x,r)$ is the union $\complement_{\mathbb{X}} B_f(x,r) \bigcup B(x,r)$ which is an open set in $\mathbb{X}$ because it is the union of two open sets in $\mathbb{X}$. So the sphere $S(x,r)$ is a closed set in $(\mathbb{X}, d)$.*

Definition 1.6. *Let $(\mathbb{X}, d)$ be a metric space and let $x \in \mathbb{X}$. A subset $\mathcal{N}$ of $\mathbb{X}$ is said to be neighbourhood of x in $(\mathbb{X}, d)$ if there is an $r > 0$ such that $B(x,r) \subseteq \mathcal{N}$, that is, if $\mathcal{N}$ contains an open ball centered at x with radius r. We denote by $\mathcal{N}(x)$ the set of neighbourhoods of x.*

Example 1.11.

1) *In $\mathbb{R}$ with the Euclidean metric, the set $\mathbb{R}_+$ (the positive real numbers) is a neighbourhood of $x = 2$ because the open ball $B(2, 0.5)$ is completely contained in $\mathbb{R}_+$.*

2) *In $\mathbb{R}$ with the Euclidean metric, the set $\mathcal{Z}$ is not a neighbourhood of $x = 2$ because any open ball centered at $x = 2$ will contain some non-integers.*

Remark 1.5. *Using proposition (1.3) we can easily prove that a set is open if and only if it is a neighbourhood of each of its points (Do yourself!).*

Proposition 1.7. *Let $(\mathbb{X},d)$ be a metric space and let $x \in \mathbb{X}$.*

1. *Any union of neighborhoods of x is also its neighborhood.*
2. *Any finite intersection of neighborhoods of x is also its neighborhood.*

Proof.

1) *Suppose that $\{\mathcal{N}_i : i \in I\}$ is a collection (finite or infinite) of neighborhoods of x in $(\mathbb{X},d)$. Then for each $i \in I$, there is an $r_i > 0$ such that $B(x,r_i) \subseteq \mathcal{N}_i$. Hence $\bigcup\limits_{i\in I} B(x,r_i) \subseteq \bigcup\limits_{i\in I} \mathcal{N}_i$ which implies that there exists $i_0 \in I$ such that $B(x,r_{i_0}) \subseteq \bigcup\limits_{i\in I} \mathcal{N}_i$. So, $\bigcup\limits_{i\in I} \mathcal{N}_i$ is a neighborhood of x in $(\mathbb{X},d)$.*

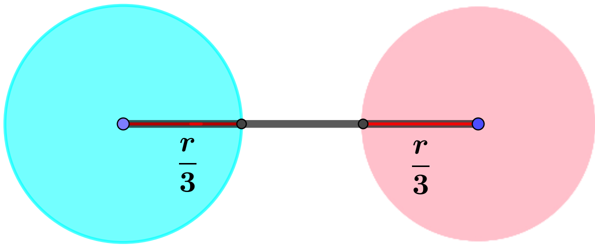
2) *Assume that $\{\mathcal{N}_1,\mathcal{N}_2,\ldots,\mathcal{N}_n\}$ is a finite collection of neighborhoods of x in $(\mathbb{X},d)$. Then, for each $i = 1,2,\ldots,n$, there is an $r_i > 0$ such that $B(x,r_i) \subseteq \mathcal{N}_i$. Let $r = min(r_1,r_2,\ldots,r_n)$. Then $B(x,r) \subseteq \mathcal{N}_i$ for all $i = 1,2,\ldots,n$, which implies that $B(x,r) \subseteq \bigcap\limits_{i=1}^{n} \mathcal{N}_i$. Hence, $\bigcap\limits_{i=1}^{n} \mathcal{N}_i$ is a neighborhood of x in $(\mathbb{X},d)$.*

Definition 1.7. *Let $(\mathbb{X},d)$ be a metric space. Let $x,y \in \mathbb{X}$. We say that x and y can be separated by neighborhoods if there exists a neighborhood $U \in \mathcal{N}(x)$ and a neighborhood $V \in \mathcal{N}(y)$ such that U and V are disjoint, i.e., $U \bigcap V = \emptyset$.*

Definition 1.8. *A metric space $(\mathbb{X},d)$ is said to be a Hausdorff space if every two distinct points of $\mathbb{X}$ have disjoint neighborhoods.*

Proposition 1.8. *Any metric space is a Hausdorff space.*

Proof. *Let $(\mathbb{X},d)$ be a metric space and let $x,y \in \mathbb{X}$ such that $x \neq y$. So there exists $r > 0$ such that $d(x,y) = r$. Hence, if we take $U = B(x,\frac{r}{3})$ and $V = B(y,\frac{r}{3})$ we obtain $U \bigcap V = \emptyset$, which shows that $(\mathbb{X},d)$ is a **Hausdorff** space.*



1.4 Interior, exterior, boundary and closure

Definition 1.9. *Let $(\mathbb{X}, d)$ be a metric space and let $A \subseteq \mathbb{X}$. A point $x \in A$ is called an interior point (or an inner point,) of A, if and only if there exists $r > 0$ such that $B(x,r) \subseteq A$. The set of all interior points of A is called the interior of A and is denoted by $Int(A)$(or $\mathring{A}$).*

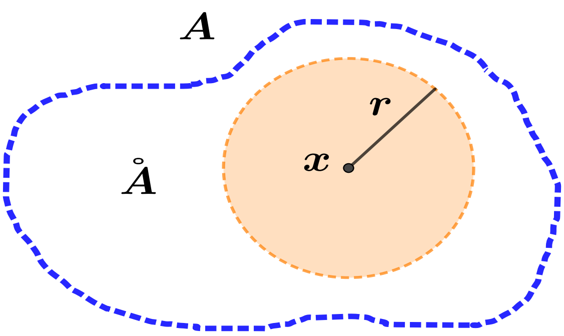


Figure 1.3: *Interior point*

Example 1.12. *In $\mathbb{R}$ with the Euclidean metric, we have $Int([0,1]) = (0,1)$ (because for any point other than 0 or 1, we can fit a ball inside $[0,1]$).*

Example 1.13. *In $\mathbb{R}$ with the Euclidean metric, we have $Int(\mathbb{Q}) = \emptyset$ (because for any $x \in \mathbb{Q}$, there is no ball $(x-r, x+r)$ that lies entirely within $\mathbb{Q}$).*

Example 1.14. *In $\mathbb{R}$ with the Euclidean metric, we have $Int((0,1)) = (0,1)$ (because for all $x \in (0,1)$, there is $r > 0$ such that $(x-r, x+r) \subseteq (0,1)$).*

Proposition 1.9. *Let $A \subset \mathbb{X}$ where $(\mathbb{X}, d)$ is a metric space. Then*

1. *$Int(A)$ is open.*

2. $Int(A)$ is the largest open subset contained in A.

Proof.

1) *If $x \in Int(A)$ then there exists $r > 0$ such that $B(x,r) \subset A$. Since $B(x,r)$ is open, it only contains interior points of A, thus $B(x,r) \subset Int(A)$. Hence $Int(A)$ is open per Definition(1.4).*

2) *Fix a set A and let $G \subset A$ be an open set. Let $x \in G$ arbitrary. Since G is open, x is an interior point of G and there exists some $r > 0$ such that $B(x,r) \subset G$. But since $G \subset A$, then $B(x,r) \subset A$ and thus $x \in Int(A)$, showing that $G \subset Int(A)$. Since $Int(A)$ is always open, $Int(A)$ is the largest open subset contained in A.*

Remark 1.6. *From the previous proposition we conclude that if A is open, then $Int(A) = A$ and if $Int(A) = A$ then A is open.*

Definition 1.10. *Let $(\mathbb{X}, d)$ be a metric space and let $A \subseteq \mathbb{X}$. A point $x \in A$ is called an exterior point of A, if and only if there exists $r > 0$ such that $B(x,r) \subseteq \complement_{\mathbb{X}} A$. The set of all exterior points of A is called the exterior of A and is denoted by $Ext(A)$.*

Example 1.15. *In $\mathbb{R}$ with the Euclidean metric (usual metric), we have*

$$Ext((0,1)) = (-\infty, 0) \cup (1, +\infty).$$

Definition 1.11. *Let $(\mathbb{X}, d)$ be a metric space and let $A \subseteq \mathbb{X}$. A point $x \in A$ is called an adherent point of A, if and only if for every real $r > 0$, we have $B(x,r) \cap A \neq \emptyset$. The set of all adherent points of A is called the closure of A and is denoted by $Cl(A)$ (or $\bar{A}$).*

Example 1.16. *In $\mathbb{R}$ with the Euclidean metric (usual metric), we have*

$$Cl((0,1)) = [0,1].$$

Proposition 1.10. *Let $A \subset \mathbb{X}$ where $(\mathbb{X}, d)$ is a metric space. Then*

$$Cl(\complement_{\mathbb{X}} A) = \complement_{\mathbb{X}} Int(A).$$

Proof. *Let $x \in \mathbb{X}$. Then we have*

$$\begin{aligned} x \in \complement_{\mathbb{X}} Int(A) &\iff x \notin Int(A) \\ &\iff \forall r > 0, B(x,r) \nsubseteq A \\ &\iff \forall r > 0, B(x,r) \subseteq \complement_{\mathbb{X}} A \\ &\iff \forall r > 0, B(x,r) \cap \complement_{\mathbb{X}} A \neq \emptyset \\ &\iff x \in Cl(\complement_{\mathbb{X}} A). \end{aligned}$$

Proposition 1.11. *Let $A \subset \mathbb{X}$ where $(\mathbb{X}, d)$ is a metric space. Then,*

1. *$Cl(A)$ is closed.*
2. *$Cl(A)$ is the smallest closed set containing A.*

Proof. *Do yourself!*

Remark 1.7. *From the previous proposition we conclude that if A is closed, then $Cl(A) = A$ and if $Cl(A) = A$ then A is closed.*

Definition 1.12. *Let $(\mathbb{X}, d)$ be a metric space and let $A \subseteq \mathbb{X}$. A point $x \in A$ is called a boundary point of A, if and only if x is neither an interior point nor an exterior point of A. The set of all boundary points of A is called the boundary of A and is denoted by $\partial(A)$.*

Example 1.17. *In $\mathbb{R}$ with the Euclidean metric (usual metric), we have*

$$\partial((0,1)) = \{0,1\}.$$

Remark 1.8. *We can define the boundary of A as follows:*

$$\begin{aligned} \partial(A) &= Cl(A) \cap Cl(\complement_{\mathbb{X}} A) (or \overline{A} \cap \overline{\complement_{\mathbb{X}} A}). \\ &= Cl(A) \setminus Int(A) (or \overline{A} \setminus \mathring{A}). \end{aligned}$$

Definition 1.13. *Let $(\mathbb{X}, d)$ be a metric space and let $A \subseteq \mathbb{X}$. A point $x \in A$ is called an accumulation point(or a limit point) of A, if and only if for every real $r > 0$, we have $(B(x,r) \setminus \{x\}) \cap A \neq \emptyset$. The set of all accumulation points of A is called the derived set of A and is denoted by A'.*

Example 1.18. *In $\mathbb{R}$ with the Euclidean metric (usual metric), we have*

$$\mathbb{Z}' = \emptyset, \qquad \mathbb{Q}' = \mathbb{R}.$$

Remark 1.9. *We have $Cl(A) = A \cup A'$ and a set A is closed if $A' \subset A$.*

Definition 1.14. *Let $(\mathbb{X}, d)$ be a metric space and let $A \subseteq \mathbb{X}$. A point $x \in A$ is called an isolate point in A, if and only if there exists a real $r > 0$ such that $(B(x,r) \setminus \{x\}) \cap A = \emptyset$ (or $B(x,r) \cap A = \{x\}$). The set of all isolate points in A is denoted by $Is(A)$.*

Remark 1.10. *If $x \in A$ is not an accumulation point, it is called isolated in A.*

Example 1.19. *In $\mathbb{R}$ with the Euclidean metric (usual metric), we have*

$$Is(\mathbb{Z}) = \mathbb{Z}, \qquad Is(\mathbb{Q}) = \emptyset.$$

1.5 Distance between two sets, Diameter

Definition 1.15. *Let $(\mathbb{X}, d)$ be a metric space with $A, B \subset \mathbb{X}$ and $a \in \mathbb{X}$. We define the distance between a point and a set, and the distance between two sets as follows:*

$$d(a, A) = \inf_{x \in A} d(a, x), \qquad d(A, B) = \inf_{x \in A, y \in B} d(x, y), \tag{1.3}$$

Remark 1.11.

1. *From the previous definition we conclude that $d(A, B) = d(B, A)$.*
2. *$d(A, B)$ is not a distance on $\mathcal{P}(\mathbb{X})$ (the power set of $\mathbb{X}$). For example, in $(\mathbb{R}, |.|)$, if we take $A = [-2, 4]$ and $B = [4, 6]$ we obtain $d(A, B) = 0$ but $A \neq B$.*

3. $\forall A, B \subset \mathbb{X},\ A \cap B \neq \emptyset \Longrightarrow d(A,B) = 0$. *The reciprocal of the previous implication is not true. For example, in* $(\mathbb{R}, |.|)$, *if we take* $A = \left\{\frac{n+1}{n}, n \in \mathbb{N}^*\right\}$ *and* $B = \{1\}$ *we obtain* $d(A,B) = \underset{n\in\mathbb{N}^*}{inf} \left|1 - \frac{n+1}{n}\right| = 0$ *but* $A \cap B = \emptyset$.

Proposition 1.12. *Let* $(\mathbb{X}, d)$ *be a metric space, let* A *be a subset of* $\mathbb{X}$ *and let* $x \in \mathbb{X}$. *Then, we have*

$$\textbf{1)} \qquad x \in Cl(A) \iff d(x,A) = 0.$$

$$\textbf{2)} \qquad x \in Ext(A) \iff d(x,A) > 0.$$

Proof.

1. $\Longrightarrow)$ $x \in Cl(A) \Longrightarrow \forall \varepsilon > 0,\ B(x,\varepsilon) \cap A \neq \emptyset \Longrightarrow \forall \varepsilon > 0,\ d(x,A) < \varepsilon \Longrightarrow d(x,A) = 0$.
   $\Longleftarrow$) *Suppose that* $x \notin Cl(A)$, *then there exists* $r > 0$ *such that* $B(x,r) \cap A = \emptyset$ *(By negation). Hence*
   $$\forall y \in A,\ d(x,y) \geqslant r,$$
   which shows that
   $$d(x,A) = \underset{y\in A}{inf} f(x,y) \geqslant r > 0.$$
2. *By negation of* ***(1)****(Do yourself!).*

Definition 1.16. *Let* $(\mathbb{X}, d)$ *be a metric space and* A *a subset of* $\mathbb{X}$. *Then the set* A *is said to be bounded if there exists some* $x_0 \in \mathbb{X}$ *and* $r > 0$ *such that* $A \subseteq B(x_0, r)$.

Remark 1.12. *From the previous definition we conclude that the finite subsets of* $\mathbb{X}$ *are bounded.*

Definition 1.17. *Let* A *be a non-empty subset of a metric space* $(\mathbb{X}, d)$. *The diameter of* A *is defined by*

$$diam(A) = \sup_{x,y\in A} d(x,y). \tag{1.4}$$

Proposition 1.13. *A non-empty subset A of a metric space $(\mathbb{X}, d)$ is bounded if and only if $diam(A) < +\infty$.*

Proof.

$\Longrightarrow$) *Suppose A is bounded, then there exists $r > 0$ such that $A \subseteq B(x,r)$. Hence $diam(A) \leqslant 2r < +\infty$.*

$\Longleftarrow$) *Suppose $diam(A) < +\infty$, then for every $x \in A$ we have $A \subseteq B(x, diam(A))$ which implies that A is bounded.*

1.6 Equivalent metrics

Definition 1.18. *Let $(\mathbb{X}, d)$ be a metric space. Let $\mathcal{T}_d$ be the collection of subsets U of $\mathbb{X}$ such that for each $x \in U$ there exists $r > 0$ with $B(x,r) \subset U$. Then $(\mathbb{X}, \mathcal{T}_d)$ is called the topological space defined by the metric d and call $\mathcal{T}_d$ the topology on $\mathbb{X}$ defined by d.*

Sometimes different metrics on a set give rise to the same topology.

Definition 1.19. *Let d_1 and d_2 be metrics on a set $\mathbb{X}$. We say that d_1 and d_2 are equivalent if they define the same topology , i.e. if $\mathcal{T}_{d_1} = \mathcal{T}_{d_2}$.*

Proposition 1.14. *Suppose that metrics d_1 and d_2 on $\mathbb{X}$ are such that for some $\kappa > 0$ we have*

$$\frac{1}{\kappa} d_1(x,y) \leqslant d_2(x,y) \leqslant \kappa d_1(x,y),$$

for all $x, y \in \mathbb{X}$. Then d_1 and d_2 are equivalent (or Lipschitz-equivalent).

Proof. *Let $\mathcal{T}_1$ be the topology defined by d_1 (i.e. $\mathcal{T}_1 = \mathcal{T}_{d_1}$) and let $\mathcal{T}_2$ be the topology defined by d_2 (i.e. $\mathcal{T}_2 = \mathcal{T}_{d_2}$). We must show that a subset U of $\mathbb{X}$ belongs to $\mathcal{T}_1$ if and only if it belongs to $\mathcal{T}_2$.*
Suppose U belongs to $\mathcal{T}_1$. Let $x \in U$. Then there exists some $r > 0$ such that $B_{d_1}(x,r) \subset U$, i.e.

$$\{y \in \mathbb{X} \setminus d_1(x,y) < r\} \subset U.$$

Consider $B_{d_2}(x,r/\kappa)$. *If* $y \in B_{d_2}(x,r/\kappa)$ *then* $d_2(x,y) < r/\kappa$. *But* $\frac{1}{\kappa}d_1(x,y) \leqslant d_2(x,y)$ *and so, for* $y \in B_{d_2}(x,r/\kappa)$ *we have* $d_1(x,y) < \kappa d_2(x,y) < \kappa.\frac{r}{\kappa} = r$. *Hence* $y \in B_{d_1}(x,r)$ *whenever* $y \in B_{d_2}(x,r/\kappa)$. *But* $B_{d_1}(x,r) \subset U$ *and so* $B_{d_2}(x,r/\kappa) \subset B_{d_1}(x,r) \subset U$. *Thus, for* $x \in U$*, there exists some* $r' > 0$ *(namely* $r' = r/\kappa$*) such* $B_{d_2}(x,r') \subset U$. *Thus* U *is open in the topology determined by* d_2*, i.e.* $U \in \mathcal{T}_1$ *implies that* $U \in \mathcal{T}_2$

Now suppose that $U \in \mathcal{T}_2$. *For* $x \in U$ *there exists some* $r > 0$ *with* $B_{d_2}(x,r) \subset U$. *Now if* $d_1(y,x) < r/\kappa$ *we have*

$$d_2(x,y) \leqslant \kappa d_1(x,y) \leqslant \kappa.\frac{r}{\kappa} = r,$$

so $B_{d_1}(x,r/\kappa) \subset B_{d_2}(x,r) \subset U$*, and so* $U \in \mathcal{T}_1$. *Thus* $U \in \mathcal{T}_1$ *if and only if* $U \in \mathcal{T}_2$ *and hence* $\mathcal{T}_1 = \mathcal{T}_2$.

Proposition 1.15. *The three metrics,* d_1, d_2 *and* d_∞*, on* $\mathbb{R}^n$ *(see Example(1.1)) are equivalent.*

Proof. *Let* $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. *Note that*

$$\begin{aligned} d_2(x,y) &= \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} \\ &\leqslant \sqrt{\sum_{i=1}^{n}(d_\infty(x,y))^2} \qquad (\textit{since } |x_i - y_i| \leqslant d_\infty(x,y), \quad \forall 1 \leqslant i \leqslant n) \\ &= \sqrt{n}d_\infty(x,y) \end{aligned}$$

and so

$$d_2(x,y) \leqslant n d_\infty(x,y). \tag{1.5}$$

Also we have

$$\begin{aligned} d_\infty(x,y) &= \max_{i=1,\ldots,n}(|x_i - y_i|) \\ &= |x_j - y_j| \quad \textit{for some } j \\ &= \sqrt{(x_j - y_j)^2} \\ &\leqslant \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = d_2(x,y), \end{aligned}$$

and so $d_\infty(x,y) \leqslant d_2(x,y)$ *and certainly*

$$d_\infty(x,y) \leqslant n d_2(x,y). \tag{1.6}$$

Combining (1.5) and (1.6) we get

$$\frac{1}{n}d_2(x,y) \leqslant d_\infty(x,y) \leqslant n d_2(x,y),$$

and so d_2 and d_∞ are equivalent.

Clearly $d_\infty(x,y) \leqslant d_1(x,y)$ and so

$$\frac{1}{n}d_\infty(x,y) \leqslant d_1(x,y). \tag{1.7}$$

Also, $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$ and each $|x_i - y_i| \leqslant d_\infty(x,y)$ so that

$$d_1(x,y) \leqslant n d_\infty(x,y). \tag{1.8}$$

Combining (1.7) and (1.8) we get

$$\frac{1}{n}d_\infty(x,y) \leqslant d_1(x,y) \leqslant n d_\infty(x,y),$$

and so d_1 and d_∞ are equivalent.

We have now shown that d_2 and d_∞ define the same topology and that d_1 and d_∞ define the same topology and hence d_1, d_2 and d_∞ all define the same topology, i.e. d_1, d_2 and d_∞ are equivalent.

1.7 Finite metric products

Let $\{(\mathbb{X}_i, d_i) : i = 1, ..., n\}$ be a collection of metric spaces and let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ be arbitrary points in the product $\mathbb{X} = \prod_{i=1}^{n} \mathbb{X}_i$. Define

$$d(x,y) = max\{d_i(x_i, y_i) : 1 \leqslant i \leqslant n\}.$$

Proposition 1.16. *$(\mathbb{X}, d)$ is a metric space.*

Proof. *Clearly $d(x,y) \geqslant 0$ and $d(x,y) = 0$ if and only if $d_i(x_i, y_i) = 0$ for $1 \leqslant i \leqslant n$, which the case if and only if $x_i = y_i$ for $1 \leqslant i \leqslant n$, i.e., if and only if $x = y$. It is equally clear that $d(x,y) = d(y,x)$. It remains to verify the triangle inequality. Observe that*

$$d_i(x_i, z_i) \leqslant d_i(x_i, y_i) + d_i(y_i, z_i),$$

for all $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$, $z = (z_1, z_2, ..., z_n) \in \mathbb{X}$. *This implies*

$$d_k(x_k, z_k) \leqslant max\{d_i(x_i, y_i) : 1 \leqslant i \leqslant n\} + max\{d_i(y_i, z_i) : 1 \leqslant i \leqslant n\},$$

for $k = 1, 2, ..., n$. *So*

$$d(x, z) = max\{d_k(x_k, z_k) : 1 \leqslant k \leqslant n\} \leqslant d(x, y) + d(y, z).$$

Hence, $(\mathbb{X}, d)$ *is a metric space.*

Definition 1.20. *The metric space obtained by taking*

$$d(x, y) = max\{d_i(x_i, y_i) : 1 \leqslant i \leqslant n\}, \tag{1.9}$$

as the distance on $\mathbb{X}$, *is called the product of the metric spaces* $(\mathbb{X}_1, d_1)$, $(\mathbb{X}_2, d_2)$,..., $(\mathbb{X}_n, d_n)$.

Remark 1.13.

i) *The functions*

$$\textbf{\textit{1)}}\ d_1(x, y) = \sum_{i=1}^{n} d_i(x_i, y_i),$$

$$\textbf{\textit{2)}}\ d_2(x, y) = \left[\sum_{i=1}^{n} (d_i(x_i, y_i))^2\right]^{\frac{1}{2}},$$

where $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$, $z = (z_1, z_2, ..., z_n) \in \mathbb{X}$, *are also metrics on* $\mathbb{X}$. *The proof of the statement that* d_1 *and* d_2 *are metrics on* $\mathbb{X}$ *is almost trivial.*
ii) *The metrics* d, d_1 *and* d_2 *on* $\mathbb{X}$ *are equivalent. Indeed*

$$d(x, y) \leqslant d_2(x, y) \leqslant d_1(x, y) \leqslant nd(x, y).$$

Proposition 1.17. *The open ball* $B(x, r)$, $x = (x_1, x_2, ..., x_n)$ *and* $r > 0$, *in* $\mathbb{X}$ *is the product of the open balls* $B_1(x_1, r)$, $B_2(x_2, r)$,...$B_n(x_n, r)$. *That is*

$$B(x, r) = \prod_{i=1}^{n} B_i(x_i, r),$$

where $B_i(x_i, r)$ *is the open ball centered in* $x_i \in \mathbb{X}_i$ *with radius* $r > 0$.

Proof. *We have* $y \in B(x,r)$ *if and only if* $d(x,y) = max\{d_i(x_i,y_i) : 1 \leqslant i \leqslant n\} < r$, *i.e. if and only if* $d_i(x_i,y_i) < r$, $1 \leqslant i \leqslant n$. *So,* $y \in B(x,r)$ *if and only if* $y_i \in B(x_i,r)$, $1 \leqslant i \leqslant n$, *that is, if and only if* $y \in \prod_{i=1}^{n} B_i(x_i,r)$.

Proposition 1.18.

1. *If* $O_i \subseteq X$, $1 \leqslant i \leqslant n$ *are open subsets in* $\mathbb{X}_i$, *then* $\prod_{i=1}^{n} O_i$ *is open in* $\mathbb{X}$.
2. *If* $F_i \subseteq X$, $1 \leqslant i \leqslant n$ *are closed subsets in* $\mathbb{X}_i$, *then* $\prod_{i=1}^{n} F_i$ *is closed in* $\mathbb{X}$.

Proof.

1. *If* $x = (x_1,x_2,\ldots,x_n) \in \prod_{i=1}^{n} O_i$, *then there exist positive* $r_1,r_2,\ldots,r_n$ *such that* $B(x_i,r_i) \subseteq O_i$, $1 \leqslant i \leqslant n$. *Let* $r = min\{r_1,r_2,\ldots,r_n\}$. *Then,* $B(x,r) = \prod_{i=1}^{n} B_i(x_i,r) \subseteq \prod_{i=1}^{n} O_i$. *Hence,* $\prod_{i=1}^{n} O_i$ *is open.*
2. *Proof left to the reader.*

Example 1.20. *Since* $\mathbb{R}$, (x,y) *and* $(-\infty,z)$ *are open in* $\mathbb{R}$, *then* $(x,y) \times \mathbb{R}$ *and* $(x,y) \times (-\infty,z)$ *are open in* $\mathbb{R}^2$.

1.8 Continuity

1.8.1 Continuous Mappings

Definition 1.21. *Let* $(\mathbb{X},d_{\mathbb{X}})$ *and* $(\mathbb{Y},d_{\mathbb{Y}})$ *be metric spaces and* $A \subseteq \mathbb{X}$. *Then, a function* $f : A \longrightarrow \mathbb{Y}$ *is said to be continuous at* $x_0 \in A$, *if for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\forall x \in A,\ d_{\mathbb{X}}(x,x_0) < \delta \Longrightarrow d_{\mathbb{Y}}(f(x),f(x_0)) < \varepsilon. \tag{1.10}$$

Remark 1.14. *If* f *is continuous at every point of* A, *then it is said to be continuous on* A.

Example 1.21. *Let* $(\mathbb{X},d)$ *be a metric space. Every function* $f_a : \mathbb{X} \longrightarrow \mathbb{R}_+$ *defined by* $f_a(x) = d(a,x)$, *such that* $a \in \mathbb{X}$, *is continuous on* $\mathbb{X}$ *because* $|f_a(x) - f_a(y)| = |d(a,x) - d(a,y)| \leqslant d(x,y)$ *(it is enough to take* $\delta = \varepsilon$*).*

Example 1.22. *Let d_u and δ be the usual metric and the discrete metric on $\mathbb{R}$, respectively. Then, the function $f:(\mathbb{R},d_u)\longrightarrow(\mathbb{R},\delta)$ defined by $f(x)=x$ is not continuous on $\mathbb{R}$ because if $x\neq x_0$ and $\varepsilon<1$ we obtain $\delta(f(x),f(x_0))=\delta(x,x_0)=1$.*

Proposition 1.19. *Let $(\mathbb{X},d_{\mathbb{X}})$ and $(\mathbb{Y},d_{\mathbb{Y}})$ be two metric spaces. A function $f:(\mathbb{X},d_{\mathbb{X}})\longrightarrow(\mathbb{Y},d_{\mathbb{Y}})$ is continuous at a point $x_0\in\mathbb{X}$ if and only if for every $\varepsilon>0$, there exists $\delta>0$ such that*

$$B(x_0,\delta)\subseteq f^{-1}(B(f(x_0),\varepsilon)), \tag{1.11}$$

Proof. *The function $f:\mathbb{X}\longrightarrow\mathbb{Y}$ is continuous at $x_0\in\mathbb{X}$ only and only if for every $\varepsilon>0$, there exists $\delta>0$ such that*

$$\forall x\in\mathbb{X},\ d_{\mathbb{X}}(x,x_0)<\delta\Longrightarrow d_{\mathbb{Y}}(f(x),f(x_0))<\varepsilon,$$

i.e.,

$$x\in B(x_0,\delta)\Longrightarrow f(x)\in B(f(x_0),\varepsilon).$$

or

$$f(B(x_0,\delta))\subseteq B(f(x_0),\varepsilon).$$

This is equivalent to the condition (1.11).

1.8.2 Uniform Continuity

Let $(\mathbb{X},d_{\mathbb{X}})$ and $(\mathbb{Y},d_{\mathbb{Y}})$ be two metric spaces and let f be a function continuous at each point $x_0\in\mathbb{X}$. In the definition of continuity, when x_0 and ε are specified, we make a definite choice of δ so that

$$\forall x\in\mathbb{X},\ d_{\mathbb{X}}(x,x_0)<\delta\Longrightarrow d_{\mathbb{Y}}(f(x),f(x_0))<\varepsilon,$$

This describes δ as dependent upon x_0 and ε, say $\delta=\delta(x_0,\varepsilon)$. If $\delta(x_0,\varepsilon)$ can be chosen in such a way that its values have a lower positive bound when ε is kept fixed and x_0 is allowed to vary over $\mathbb{X}$, and if this happens for each positive ε, then we have the notion of uniform continuity. More precisely, we have the following definition:

Definition 1.22. *Let $(\mathbb{X},d_{\mathbb{X}})$, $(\mathbb{Y},d_{\mathbb{Y}})$ be tow metric spaces. A functionf $:\mathbb{X}\longrightarrow\mathbb{Y}$ is said to be uniformly continuous on $\mathbb{X}$ if, for every ε, there exists a δ (depending on ε alone)* *such that:*

$$\forall x, y \in \mathbb{X},\ d_{\mathbb{X}}(x,y) < \delta(\varepsilon) \Longrightarrow d_{\mathbb{Y}}(f(x), f(y)) < \varepsilon. \tag{1.12}$$

Example 1.23. *The function $f : (\mathbb{R}, |.|) \longrightarrow (\mathbb{R}, |.|)$ defined by $f(x) = x$ is uniformly continuous (it is enough to take $\delta = \varepsilon$).*

Using the previous definition we obtain the following result.

Proposition 1.20. *Every uniformly continuous function on $\mathbb{X}$ is necessarily continuous on $\mathbb{X}$. However, the converse may not be true.*

Example 1.24. *The function $f : (\mathbb{R}, |.|) \longrightarrow (\mathbb{R}, |.|)$ defined by $f(x) = x^2$ is continuous but not uniformly continuous. Take $\varepsilon = 1$ and let $\delta > 0$ be arbitrary. If we choose $x = \frac{\delta}{2} + \frac{1}{\delta}$ and $y = \frac{1}{\delta}$ we obtain*

$$|x - y| = \left| \frac{\delta}{2} + \frac{1}{\delta} - \frac{1}{\delta} \right| = \frac{\delta}{2} < \delta,$$

but

$$|f(x) - f(y)| = \left| \left(\frac{\delta}{2} + \frac{1}{\delta} \right)^2 - \left(\frac{1}{\delta} \right)^2 \right| = 1 + \frac{\delta^2}{4} > 1.$$

1.9 Homeomorphism

Definition 1.23. *Let $(\mathbb{X}, d_{\mathbb{X}})$ and $(\mathbb{Y}, d_{\mathbb{Y}})$ be two metric spaces and $f : \mathbb{X} \longrightarrow \mathbb{Y}$. We say that f is an homeomorphism from $\mathbb{X}$ to $\mathbb{Y}$ if:*

1. *f is a bijection (one-to-one and onto),*
2. *f is continuous,*
3. *the inverse function f^{-1} is continuous.*

If there exists an homeomorphism from $\mathbb{X}$ to $\mathbb{Y}$, we say that $\mathbb{X}$ and $\mathbb{Y}$ are homeomorphic.

Example 1.25.

1. *Let $\mathbb{X} = \mathbb{R}$ and $\mathbb{Y} = (-1, 1)$ endowed with the usual distance. The function $f : \mathbb{R} \longrightarrow (-1, 1)$ defined by $f(x) = \dfrac{x}{1 + |x|}$ is a homeomorphism. Consequently, $\mathbb{X}$ and $\mathbb{Y}$ are homeomorphic.*

2. *Let* $\mathbb{X} = (a,b)$ *and* $Y = \mathbb{R}$ *with the usual distance. The function* $f : (a,b) \longrightarrow \mathbb{R}$ *defined by* $f(x) = \dfrac{1}{x-a} + \dfrac{1}{x-b}$ *is a homeomorphism. Therefore,* $\mathbb{X}$ *and* $\mathbb{Y}$ *are homeomorphic.*

3. *Let* $\mathbb{X} = (0,1)$ *and* $\mathbb{Y} = (a,b)$ *endowed with the usual distance. The function* $f : (0,1) \longrightarrow (a,b)$ *defined by* $f(x) = (b-a)x + a$ *is a homeomorphism. Consequently,* $\mathbb{X}$ *and* $\mathbb{Y}$ *are homeomorphic.*

Remark 1.15. *In general, the bijectivity and continuity of* f *do not imply that* f *is a homeomorphism. For example, the map* $f : (\mathbb{R}, \delta) \longrightarrow (\mathbb{R}, d_u)$ *defined by* $f(x) = x$ *is a bijection and continuous, while* f^{-1} *is not continuous.*

1.9.1 Lipschitz and Contraction Mappings and Applications

Definition 1.24. *Let* $(\mathbb{X}, d_{\mathbb{X}})$ *and* $(\mathbb{Y}, d_{\mathbb{Y}})$ *two metric spaces. A mapping* $f : \mathbb{X} \longrightarrow \mathbb{Y}$ *is said to be k-Lipschitz if there exists a real number* $k > 0$ *such that*

$$\forall x, y \in \mathbb{X},\ d_{\mathbb{Y}}(f(x), f(y)) \leqslant k d_{\mathbb{X}}(x,y), \tag{1.13}$$

Definition 1.25. *Let* $(\mathbb{X}, d_{\mathbb{X}})$ *and* $(\mathbb{Y}, d_{\mathbb{Y}})$ *two metric spaces. A mapping* $f : \mathbb{X} \longrightarrow \mathbb{Y}$ *is said to be a contraction (or contraction mapping) if there exists a real number* $0 \leqslant k < 1$ *such that*

$$\forall x, y \in \mathbb{X},\ d_{\mathbb{Y}}(f(x), f(y)) \leqslant k d_{\mathbb{X}}(x,y). \tag{1.14}$$

Proposition 1.21. *Let* $f : \mathbb{R} \supseteq I \longrightarrow \mathbb{R}$ *be a differentiable mapping such that* $|f'(x)| \leqslant k$, *for all* $x \in I$. *Then,* f *is k-Lipschitz. Moreover, if* $|f'(x)| \leqslant k < 1$ *for all* $x \in I$, *then* f *is a contraction.*

Proof. *For all* $x, y \in I$, *we have* $|f(x) - f(y)| = |\int_x^y f'(t)dt| \leqslant k|x-y|$, *which implies that* f *is k-Lipschitz. Moreover, if* $|f'(x)| \leqslant k < 1$, *then* f *is a contraction.*

Using the previous definition we obtain the following result.

Proposition 1.22. *Every k-Lipschitz or contraction mapping is uniformly continuous.*

1.10 Isometry

Definition 1.26. *Let* $(\mathbb{X}, d_{\mathbb{X}})$ *and* $(\mathbb{Y}, d_{\mathbb{Y}})$ *be two metric spaces. A bijection* $f : \mathbb{X} \longrightarrow \mathbb{Y}$ *is called an isometry if*

$$d_{\mathbb{Y}}(f(x), f(y)) = d_{\mathbb{X}}(x, y), \ \forall x, y \in \mathbb{X}. \tag{1.15}$$

In this case, one says that $\mathbb{X}$ *and* $\mathbb{Y}$ *are isometric (or* $\mathbb{X}$ *is isometric to* $\mathbb{Y}$*).*

Remark 1.16. *In other words, an isometry between metric spaces is a bijection which preserves the distance between elements. Clearly,* $\mathbb{Y}$ *is isometric to* $\mathbb{X}$ *if and only if* $\mathbb{X}$ *is isometric to* $\mathbb{Y}$*.*

Example 1.26. *The mapping* $f : (\mathbb{R}, |.|) \longrightarrow (\mathbb{R}, |.|)$ *defined by* $f(x) = x \pm b$, $b \in \mathbb{R}$ *is an isometry.*

Example 1.27. *Let* $(\mathbb{X}, d_{\mathbb{X}})$ *and* $(\mathbb{Y}, d_{\mathbb{Y}})$ *be two metric spaces such that* $card(\mathbb{X}) = card(\mathbb{Y})$, $d_{\mathbb{X}} = \delta$ *(the discrete distance, see example (1.4)) and* $d_{\mathbb{Y}}(x, y) = \begin{cases} 2 \ si \ x \neq y \\ 0 \ si \ x = y \end{cases}$*. Then,* $(\mathbb{X}, d_{\mathbb{X}})$ *and* $(\mathbb{Y}, d_{\mathbb{Y}})$ *are not isometric because the distance between two different points of the first space is different to the distance between two different points of the second space.*

Remark 1.17.

1. *Every isometry is uniformly continuous (because it is 1-Lipschitz).*
2. *Every isometry is an homeomorphism (Exercise).*

1.11 Normed spaces

In functional analysis, a normed space is a vector space equipped with a norm, which is a function that assigns a non-negative length or size to each vector in the space.

Definition 1.27. *Let $\mathbb{X}$ be a vector space over the field $\mathbb{K}$ of real or complex numbers. A semi-norm on $\mathbb{X}$ is a function $\|\cdot\| : \mathbb{X} \to \mathbb{R}$ satisfying the following properties:*

1. *Non-negativity: $\|x\| \geqslant 0$ for all $x \in \mathbb{X}$.*
2. *Homogeneity: $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{R}$ (or $\mathbb{C}$) and $x \in \mathbb{X}$.*
3. *Triangle inequality: $\|x+y\| \leqslant \|x\| + \|y\|$ for all $x, y \in \mathbb{X}$.*

A semi-normed space $(\mathbb{X}, \|\cdot\|)$ is a vector space $\mathbb{X}$ equipped with a semi-norm.

Example 1.28.

1. *Both $\mathbb{R}$ and $\mathbb{C}$ are semi-normed space with $\|x\| = |x|$.*
2. *The function $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$\|(x,y)\| = |x-y|, \tag{1.16}$$

is a semi-norm.

Proposition 1.23. *Let $(\mathbb{X}, \|\cdot\|)$ be a semi-normed space, then*

1. $\|0\| = 0$.
2. $\forall x, y \in \mathbb{X}, \quad \|x-y\| = \|y-x\|$.
3. $\forall x, y \in \mathbb{X}, \quad |\|x\| - \|y\|| \leqslant \|x-y\|$.

Proof.

1. *Let $\lambda \in \mathbb{K}$ such that $\lambda \neq 1$, then*

$$\|0\| = \|\lambda.0\| = |\lambda|\,\|0\|,$$

which implies that

$$\|0\|\,(1-|\lambda|) = 0.$$

Hence, $\|0\| = 0$.

2. *For all $x, y \in \mathbb{X}$, we have*

$$\|x-y\| = \|-(y-x)\| = |-1|\,\|y-x\| = \|y-x\|.$$

3. *Let us write,*

$$y = y - x + x \Longrightarrow \|y\| = \|y - x + x\| \leqslant \|y - x\| + \|x\|,$$

from which it follows that

$$\|y\| - \|x\| \leqslant \|x - y\| \tag{i}$$

Similarly, we have

$$x = x - y + y \Longrightarrow \|x\| = \|x - y + y\| \leqslant \|x - y\| + \|y\|,$$

from which it follows that

$$\|x\| - \|y\| \leqslant \|x - y\| \tag{ii}$$

Finally, from inequalities ((i)) and ((ii)) we obtain

$$|\|x\| - \|y\|| \leqslant \|x - y\|.$$

Definition 1.28. *Let $\mathbb{X}$ be a vector space over the field of real or complex numbers. A norm on $\mathbb{X}$ is a semi-norm $\|\cdot\| : \mathbb{X} \to \mathbb{R}$ satisfying, furthermore, the following property:*

$$\|x\| = 0 \Longrightarrow x = 0.$$

A normed space $(\mathbb{X}, \|\cdot\|)$ is a vector space $\mathbb{X}$ equipped with a norm.

Example 1.29.

1. *Both $\mathbb{R}$ and $\mathbb{C}$ are normed space with $\|x\| = |x|$.*
2. *The function defined by (1.16) is not a norm because*

$$\|(1,1)\| = |1 - 1| = 0 \quad \textit{but} \quad (1,1) \neq (0,0).$$

3. *On $\mathbb{R}^n$, for all $x = (x_1, x_2, x_3, \dots, x_n) \in \mathbb{R}^n$ we have the following norms,*

$$\begin{array}{lrcl}
① & \|x\|_1 & = & \sum_{i=1}^{n} |x_i|, \\
② & \|x\|_2 & = & \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}, \\
③ & \|x\|_\infty & = & \max_{1 \leqslant i \leqslant n} |x_i|, \\
④ & \|x\|_p & = & \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}, \quad (1 \leqslant p < \infty).
\end{array}$$

The corresponding metric space of ③ is denoted by ℓ_∞^n and the corresponding metric space of ④ is denoted by ℓ_p^n.

4. *On the vector space $C([a,b],\mathbb{R})$, for all $f \in C([a,b],\mathbb{R})$ we have the following norms,*

$$\begin{aligned}
\|f\|_1 &= \int_a^b |f(x)|\,dx, \\
\|f(x)\|_2 &= \left(\int_a^b f(x)^2 dx\right)^{\frac{1}{2}}, \\
\|f(x)\|_\infty &= \max_{x\in[a,b]} |f(x)|, \\
\|f(x)\|_p &= \left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}}, \quad (1 \leqslant p < \infty)
\end{aligned}$$

Normed space are an important instance of metric spaces, as the following proposition asserts.

Proposition 1.24. *Let $(X, \|\cdot\|)$ be a normed space. Then,*

$$d(x,y) = \|x-y\|$$

defines a metric on $\mathbb{X}$. That is, every normed space is automatically a metric space with a canonical metric.

Proof. *Leave to the reader (Immediate).*

Remark 1.18. *Note that metric spaces need not be vector spaces.*

Proposition 1.25. *The norm is a uniformly continuous function.*

Proof. *Using Proposition (1.23)$_3$ we deduce that the norm is 1-Lipschitz.*

Definition 1.29. *Two norms on a $\mathbb{K}$-vector space $\mathbb{X}$ are called equivalent if they define the same open subsets of $\mathbb{X}$.*

Proposition 1.26. *Let $\mathbb{X}$ be a $\mathbb{K}$-vector space. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{X}$ are equivalent if and only if there exist constants $\alpha > 0$ and $\beta > 0$ such that*

$$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1 \quad \textit{for all} \quad x \in \mathbb{X}.$$

Proof. *The proof is left to the readers.*

Example 1.30.

1. *The norms ①, ② and ③ defined in Example(1.29) are equivalent because for all $x \in \mathbb{R}^n$ we have*

$$\|x\|_2 \leqslant \|x\|_1 \leqslant \sqrt{n}\,\|x\|_2\,,$$

$$\|x\|_\infty \leqslant \|x\|_1 \leqslant n\,\|x\|_\infty\,.$$

2. *Let $\mathbb{X}$ the vector space defined by*

$$\mathbb{X} = \left\{ f \in C^1([0,1],) \quad / \quad f(0) = 1 \right\},$$

and equipped with the following norms:

$$\|f\|_1 = \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} \left|f'(x)\right|, \qquad \|f\|_2 = \sup_{x\in[0,1]} \left|f'(x)\right|.$$

Let us prove that the two previous norms are equivalent. On the one hand, by definition we have

(i) $$\|f\|_2 \leqslant \|f\|_1\,.$$

On the other hand, by application of the finite-increments formula we obtain

$$f(1) - f(x) = f'(c)(1-x), \quad 0 \leqslant x < c < 1.$$

Keeping in mind that $f(1) = 0$ we find

$$f(x) = f'(c)(x-1), \quad 0 \leqslant x < c < 1,$$

from which it follows that

$$\sup_{x\in[0,1]} |f(x)| \leqslant \sup_{x\in[0,1]} \left|f'(x)\right|.$$

Hence,

(ii) $$\|f\|_1 \leqslant 2\,\|f\|_2\,.$$

Finally, from the two previous inequality (i) and (ii) we deduce that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms on $\mathbb{X}$.

We have just seen that norms induce metrics. Next we look at a useful way to induce a norm.

Definition 1.30. *Let $\mathbb{X}$ be a vector space over either $\mathbb{R}$ or $\mathbb{C}$. An inner product (Sesquilinear form) on $\mathbb{X}$ is a function $\langle\cdot,\cdot\rangle : \mathbb{X}\times\mathbb{X} \to \mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$, such that for all vectors $x,y,z,w \in \mathbb{X}$ and all scalars $\alpha,\beta \in \mathbb{K}$, the following properties hold:*

1. ***Linearity in the first argument**: $\langle \alpha x+\beta y, z\rangle = \alpha\langle x,z\rangle + \beta\langle y,z\rangle$,*
2. ***Conjugate linearity in the second argument:** $\langle x, \alpha z+\beta w\rangle = \overline{\alpha}\langle x,z\rangle + \overline{\beta}\langle x,w\rangle$,*
3. ***Conjugate symmetry**: $\langle x,y\rangle = \overline{\langle y,x\rangle}$ (for complex vector spaces),*
4. ***Positive-definiteness**: $\langle x,x\rangle \geq 0$ and $\langle x,x\rangle = 0$ if and only if $x = 0$.*

We present two examples of inner products next; the reader is asked to verify that they satisfy the inner product axioms in definition(1.30).

Example 1.31. *On $\mathbb{R}^n$, the function $\langle\cdot,\cdot\rangle$ defined by*

$$\langle \mathbf{x},\mathbf{y}\rangle = \sum_{i=1}^{n} x_i y_i,$$

is an inner product. When we consider $\mathbb{R}^2$ or $\mathbb{R}^3$, this is often called the dot product.

Example 1.32. *On the vector space $C([0,1])$, the function $\langle\cdot,\cdot\rangle$ defined by*

$$\langle f,g\rangle = \int_0^1 f(t)g(t)\,dt,$$

is an inner product.

Our next result is incredibly useful.

Proposition 1.27. ***(Cauchy-Schwarz inequality)** Suppose that $\langle\cdot,\cdot\rangle$ is an inner product on a real vector space $\mathbb{X}$. Then for all $x,y \in \mathbb{X}$, we have*

$$|\langle x,y\rangle|^2 \leq \langle x,x\rangle\langle y,y\rangle. \tag{1.17}$$

Proof. *Let $x,y\in\mathbb{X}$. For all $\lambda\in\mathbb{K}$ we have,*

$$\langle\lambda x+y,\lambda x+y\rangle=\lambda\overline{\lambda}\langle x,x\rangle+\langle y,y\rangle+\lambda\langle x,y\rangle+\overline{\lambda}\langle y,x\rangle\geqslant 0. \tag{1.18}$$

By taking $a=\langle x,x\rangle$, $b=\langle x,y\rangle$ and $c=\langle y,y\rangle$ into (1.18), we obtain

$$a\lambda\overline{\lambda}+b\lambda+\overline{b\lambda}+c\geqslant 0. \tag{1.19}$$

If $a=c=0$, we set $\lambda=-\overline{b}$ and by substitution into (1.19) we find,

$$-b\overline{b}-b\overline{b}=-2\left|b\right|^{2}\geqslant 0,$$

which implies that $b=0$. Hence, the inequality (1.17) is verified.
If $a\neq 0$, we set $\lambda=-\dfrac{\overline{b}}{a}$ and by substitution into (1.19) we find,

$$a\left(-\frac{\overline{b}}{a}\right)\left(-\frac{b}{a}\right)-\frac{\overline{b}}{a}b-\frac{\overline{b\overline{b}}}{a}+c\geqslant 0,$$

i.e.,

$$-\frac{\left|b\right|^{2}}{a}+c\geqslant 0,$$

which implies that

$$\left|b\right|^{2}\leqslant ac,$$

Hence, the inequality (1.17) is verified.

Proposition 1.28. ***(Minkowski inequality)*** *If $\langle\cdot,\cdot\rangle$ is an inner product on the vector space $\mathbb{X}$, then we have*

$$\forall x,y\in\mathbb{X},\quad \sqrt{\langle x+y,x+y\rangle}\leqslant\sqrt{\langle x,x\rangle}+\sqrt{\langle y,y\rangle}. \tag{1.20}$$

Proof. *We know that*

$$\langle x+y,x+y\rangle=\langle x,x\rangle+2Re\langle x,y\rangle+\langle y,y\rangle,$$

Furthermore, we have

$$Re\langle x,y\rangle\leqslant\left|\langle x,y\rangle\right|\leqslant\sqrt{\langle x,x\rangle}.\sqrt{\langle y,y\rangle},$$

which implies that

$$\langle x+y,x+y\rangle\leqslant\left(\sqrt{\langle x,x\rangle}+\sqrt{\langle y,y\rangle}\right)^{2}.$$

From which the inequality (1.20) follows by taking square roots.

With the *Cauchy-Schwarz* inequality in hand, our final result of the section shows how inner products induce norms (which then induce metrics).

Proposition 1.29. ***(Inner Products Induce Norms)*** *If $\langle\cdot,\cdot\rangle$ is an inner product on the vector space $\mathbb{X}$, then the function $\|\cdot\|:\mathbb{X}\to\mathbb{R}$ defined by*

$$\|x\| = \sqrt{\langle x,x\rangle}, \tag{1.21}$$

is a norm on $\mathbb{X}$.

Proof. *It suffices to use the Minkowski inequality to obtain the triangle inequality. The other properties are left to the reader.*

Definition 1.31. *A pre-Hilbert space (or an inner product space) is a vector space with a norm induced by an inner product.*

Example 1.33.

1. *$\mathbb{X}=\mathbb{C}^n$ is a pre-Hilbert space (or an inner product space) with the following inner product*
$$\langle x,y\rangle = \sum_{i=1}^{n} x_i\overline{y}_i.$$
2. *$\mathbb{X}=C([0,1],\mathbb{C})$ is a pre-Hilbert space (or an inner product space) with the following inner product*
$$\langle f,g\rangle = \int_0^1 f(t)\overline{g(t)}\,dt.$$

CHAPTER 2

COMPLETE METRIC SPACES

You are familiar with the notion of a convergent sequence of real numbers. It is defined as follows: the sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is said to converge to the real number x if given any $\varepsilon > 0$ there exists n_0 such that for all $n \geqslant n_0$, $|x_n - x| < \varepsilon$.

It is obvious how this definition can be extended from $\mathbb{R}$ with the Euclidean metric to any metric space.

2.1 Convergence in a metric space

2.1.1 Convergence and limits

Definition 2.1. *A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space $(\mathbb{X}, d)$ is called convergent to $x_0 \in \mathbb{X}$ if,*

$$\forall \varepsilon > 0,\ \exists N_0(\varepsilon) \in \mathbb{N}\ /\ \forall n \in \mathbb{N},\ n \geqslant N_0 \Longrightarrow d(x_n, x_0) < \varepsilon. \tag{2.1}$$

The point x is called the limit of the sequence $(x_n)_{n\in\mathbb{N}}$, and we write

$$\lim_{n \longrightarrow +\infty} (x_n) = x_0 \ \text{or}\ \ x_n \longrightarrow x_0.$$

If the sequence does not converge, then it is said to diverge.

Remark 2.1. *The condition (2.1) means that from a certain rank N_0 the elements of the sequence (x_n) are in the open ball $B(x_0, \varepsilon)$. Thus, this ball contains an infinite elements of this sequence.*

Example 2.1. *In the metric space $(\mathbb{R}, |.|)$, the sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}^*}$ converge to 0 and we write $\lim\limits_{n\longrightarrow+\infty}\left(\frac{1}{n}\right) = 0$.*

Proof. *Let $\varepsilon > 0$ be given. By the Archimedean property, there is an integer $N_0 \in \mathbb{N}$ such that $N_0 > \frac{1}{\varepsilon}$, and thus for all $n \geqslant N_0$, we have $d_u\left(\frac{1}{n}, 0\right) = \left|\frac{1}{n}\right| < \varepsilon$.*

However, this sequence can be made to diverge by changing the metric on $\mathbb{R}$.

Example 2.2. *In the metric space $(\mathbb{R}, d_{disc})$, the sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}^*}$ diverges.*

Proof. *Using the definition of the discrete distance (1.1) we obtain $\delta\left(\frac{1}{n}, 0\right) = 1$ because $\frac{1}{n} \neq 0$. Hence, if we take $\varepsilon < 1$, we obtain $\delta(\frac{1}{n}, 0) > \varepsilon$.*

Proposition 2.1. *Let $(\mathbb{X}, d)$ be a metric space. The sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x_0 \in \mathbb{X}$ if and only if the sequence $(d(x_n, x_0))$ converges to 0 in $(\mathbb{R}, d_u)$.*

Proof. *Suppose first that $x_n \longrightarrow x_0$ in $(\mathbb{X}, d)$. Then for any $\varepsilon > 0$, there is an $N_0 \in \mathbb{N}$ such that $d(x_n, x_0) = |d(x_n, x_0)| < \varepsilon$ for all $n \geqslant N_0$, but this precisely the same as $d(x_n, x_0) \longrightarrow 0$ in $(\mathbb{R}, d_u)$. Conversely, if $d(x_n, x_0) \longrightarrow 0$ in $(\mathbb{R}, d_u)$, then for every $\varepsilon > 0$ there is an $N_0 \in \mathbb{N}$ such that $d(x_n, x_0) = |d(x_n, x_0)| < \varepsilon$ for all $n \geqslant N_0$, and this precisely what it means to have $x_n \longrightarrow x_0$ in $(\mathbb{X}, d)$.*

Example 2.3. *The sequence $(x_n)_{n\in\mathbb{N}^*}$ defined by $x_n = \left(\frac{1}{n}, \frac{1}{n}\right)$ converges to $(0,0)$ in the three metric spaces $(\mathbb{R}^2, d_1)$, $(\mathbb{R}^2, d_2)$ and $(\mathbb{R}^2, d_\infty)$.*

Proof. *We have*

$$\begin{aligned}
d_1\left(\left(\frac{1}{n}, \frac{1}{n}\right), (0,0)\right) &= \frac{2}{n}, \\
d_2\left(\left(\frac{1}{n}, \frac{1}{n}\right), (0,0)\right) &= \frac{\sqrt{2}}{n}, \\
d_\infty\left(\left(\frac{1}{n}, \frac{1}{n}\right), (0,0)\right) &= \frac{1}{n}.
\end{aligned}$$

The result follows immediately from Proposition(2.1).

Proposition 2.2. *If a sequence converges then its limit is unique.*

Proof. *Suppose that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to both x and y. Then, for every $\varepsilon > 0$ there exists a (sufficiently large) $N_0 \in \mathbb{N}$ such that*

$$\forall n \in \mathbb{N},\ n \geqslant N_0 \Longrightarrow d(x_n,x) < \frac{\varepsilon}{2} \text{ and } d(x_n,y) < \frac{\varepsilon}{2}.$$

from which we conclude that for every $\varepsilon > 0$,

$$\forall n \in \mathbb{N},\ n \geqslant N_0 \Longrightarrow d(x,y) < d(x,x_n) + d(x_n,y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

i.e., $d(x,y) = 0$, namely $x = y$.

Proposition 2.3. *Let F be a subset of a metric space $(\mathbb{X},d)$ and $\mathcal{L}$ is the set of all $x \in \mathbb{X}$ such that x is the limit of some sequence of elements of F. Then, F is closed if and only if $F = \mathcal{L}$.*

Proof.

$\Longrightarrow$) Suppose that F is closed.

• *$\mathcal{L} \subseteq F$?) Take $x \in \mathcal{L}$, meaning that x is a limit of a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of F. If $x \notin F$, then $x \in \complement_{\mathbb{X}} F$ (which is an open), implying that there exists $r > 0$ such that $B(x,r) \subseteq \complement_{\mathbb{X}} F$. Since $x_n \longrightarrow x$, there exists $N_0 \in \mathbb{N}$ such that for all, $n \geqslant N_0$ we have $d(x_n,x) < r$. This means $x_n \in B(x,r) \subseteq \complement_{\mathbb{X}} F$, which contradicts the fact that $x_n \in F$.*

• *$F \subseteq \mathcal{L}$?) Let $x \in F$, then we can consider x as a limit of the constant sequence $x_n = x$ which implies that $x \in \mathcal{L}$.*

$\Longleftarrow$) Suppose that the limit of all convergent sequence of F belongs to F. Let $x \in Cl(F)$. Then $B(x,r) \cap F \neq \emptyset$ for all $r > 0$. Thus, for all $n \in \mathbb{N}^*$, there exists x_n such that $x_n \in B(x,\frac{1}{n}) \cap F$. Then, (x_n) is a sequence of elements of F that satisfies $d(x_n,x) < \frac{1}{n}$ for all $n \in \mathbb{N}^*$, which implies, $x_n \longrightarrow x$. Therefore, $x \in F$ (by hypotheses), which implies that F is closed.

Using the previous proposition we obtain the following result.

Proposition 2.4. *Let $(\mathbb{X}, d)$ be a metric space and F be a subset of $\mathbb{X}$, then :*

$$Cl(F) = \left\{ x \in \mathbb{X} : \exists\ (x_n) \subset F \ /\ \lim_{n \longrightarrow +\infty} x_n = x \right\} \tag{2.2}$$

Proposition 2.5 (Sequential continuity)**.** *Let $(\mathbb{X}, d_{\mathbb{X}})$, $(\mathbb{Y}, d_{\mathbb{Y}})$ be two metric spaces and $f : (\mathbb{X}, d_{\mathbb{X}}) \longrightarrow (\mathbb{Y}, d_{\mathbb{Y}})$. Then, f is continuous at x_0 if and only if for every sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{X}$, we have*

$$\lim_{n \longrightarrow +\infty} x_n = x_0 \Longrightarrow \lim_{n \longrightarrow +\infty} f(x_n) = f(x_0) \quad (\ f \text{ is sequentially continuous}).$$

Proof.

$\Longrightarrow$) *Suppose that f is continuous at x_0 and let (x_n) be a sequence in $\mathbb{X}$ that converges to x_0. Since $x_n \longrightarrow x_0$,*

$$\forall \varepsilon > 0,\ \exists n_0(\varepsilon) \in \mathbb{N}\ /\ \forall n \in \mathbb{N},\ n \geqslant n_0 \Longrightarrow d_{\mathbb{X}}(x_n, x_0) < \varepsilon.$$

Since f is continuous at x_0,

$$\forall \varepsilon > 0,\ \exists \delta(\varepsilon, x_0)\ /\ \forall x \in \mathbb{X},\ d_{\mathbb{X}}(x, x_0) < \delta \Longrightarrow d_{\mathbb{Y}}(f(x), f(x_0)) < \varepsilon.$$

Then, it is enough to take $\varepsilon = \delta$ to obtain,

$$\forall\ n \geqslant n_0,\ d_{\mathbb{X}}(x_n, x_0) < \delta = \varepsilon \Longrightarrow d_{\mathbb{Y}}(f(x_n), f(x_0)) < \varepsilon,$$

which shows that f is sequentially continuous.

$\Longleftarrow$) *Suppose that f is not continuous at x_0. Then, there exists $\varepsilon > 0$ such that for all $\delta > 0$, there exists x_δ that satisfying,*

$$d_{\mathbb{X}}(x_\delta, x_0) < \delta \ \textit{et}\ d_{\mathbb{Y}}(f(x_\delta), f(x_0)) \geqslant \varepsilon.$$

Thus, for $\delta = \dfrac{1}{n}$ there exists a sequence (x_n) such that:

$$d_{\mathbb{X}}(x_n, x_0) < \frac{1}{n} \ \textit{et}\ d_{\mathbb{Y}}(f(x_n), f(x_0)) \geqslant \varepsilon.$$

This shows that (x_n) converges to x_0, but $(f(x_n))$ does not converge to $f(x_0)$. Therefore f does not sequentially continuous at x_0.

Remark 2.2. *In the third chapter (Topological Spaces), we will demonstrate that in the more general context of topological spaces, continuity always implies sequential continuity; however, the converse is not true.*

2.2 Cauchy sequences and completeness

The definition of *Cauchy sequences* in general metric spaces is a straightforward generalization of their definition in the real line.

Definition 2.2. *Let $(\mathbb{X},d)$ be a metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{X}$ is called a Cauchy sequence if for every $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that $d(x_n,x_m)<\varepsilon$ for all $n,m\geqslant n_0$. In other words,*

$$\forall\varepsilon>0,\ \exists\ n_0(\varepsilon)\in\mathbb{N}\ /\ \forall(n,m)\in\mathbb{N}^2,\ n,m\geqslant n_0\Longrightarrow d(x_n,x_m)<\varepsilon. \tag{2.3}$$

Example 2.4.

1. *The sequence $(x_n)_{n\geqslant1}$, where $x_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$, does not satisfy Cauchy's criterion for convergence. Indeed, we have*

$$|x_{2n}-x_n|=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\geqslant\frac{1}{2n}+\frac{1}{2n}+\cdots+\frac{1}{2n}=\frac{1}{2}.$$

Thus, it is not the case that $|x_n-x_m|\to0$ as n and m become large.

2. *In $(C[0,1],\mathbb{R})$, the sequence $(f_n)_{n\geqslant1}$ given by*

$$f_n(x)=\frac{nx}{n+x},\qquad x\in[0,1],$$

is Cauchy in the uniform metric. For $m\geqslant n$, the difference between the functions is given by

$$f_m(x)-f_n(x)=\frac{mx}{m+x}-\frac{nx}{n+x}=\frac{(m-n)x^2}{(m+x)(n+x)}.$$

Since this function is continuous on $[0,1]$, it attains its maximum at some point $x_0\in[0,1]$. Thus, we have

$$d(f_m,f_n)=\sup_{x\in[0,1]}|f_m(x)-f_n(x)|=\frac{(m-n)x_0^2}{(m+x_0)(n+x_0)}\leq\frac{x_0^2}{n+x_0}\leqslant\frac{1}{n}\to0,$$

for large m and n.

3. *If (x_n) is a Cauchy sequence in the discrete metric space $(\mathbb{X},\delta)$, then $\delta(x_n,x_m)<\varepsilon\Longrightarrow x_n=x_m$ for any $\varepsilon\leqslant1$. Thus, the sequence (x_n) is convergent.*

Proposition 2.6. *In a metric space $(\mathbb{X}, d)$, we have:*

1. *Every convergent sequence is a Cauchy sequence.*
2. *Every Cauchy sequence is bounded.*
3. *Every subsequence of a Cauchy sequence is also a Cauchy sequence.*
4. *Every Cauchy sequence that has a convergent subsequence is convergent.*

Proof.

1. *If $x_n \longrightarrow x_0$, then for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, we have $d(x_n, x_0) < \frac{\varepsilon}{2}$. Therefore, if $n \geqslant n_0$ and $m \geqslant n_0$, we obtain*
$$d(x_n, x_m) \leqslant d(x_n, x_0) + d(x_m, x_0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
2. *If (x_n) is a Cauchy sequence, then for $\varepsilon = 1$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n_0}) < 1$ for all $n \geq n_0$. Let $r = \max\left(d(x_{n_0}, x_1), \ldots, d(x_{n_0}, x_{n_0-1}), 1\right)$. Then, for all $n \in \mathbb{N}$, we have $d(x_n, x_{n_0}) < r$, which implies that $(x_n) \subset B(x_{n_0}, r)$.*
3. *Obvious.*
4. *Suppose $x_{n_k} \longrightarrow x_0$. Then, for every $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that for all $n_k \geqslant n_1$, we have $d(x_{n_k}, x_0) < \frac{\varepsilon}{2}$. Since (x_n) is a Cauchy sequence, there exists $n_2 \in \mathbb{N}$ such that for all $n, m \geqslant n_2$, we have $d(x_n, x_m) < \frac{\varepsilon}{2}$. Let $n_0 = \max(n_1, n_2)$ and choose k such that $n_k \geqslant n_0$ to obtain:*
$$\forall n \in \mathbb{N},\ n \geq n_0 \Longrightarrow d(x_n, x_0) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x_0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
Thus, $x_n \longrightarrow x_0$.

Definition 2.3. *A metric space $(\mathbb{X}, d)$ is said to be complete if every Cauchy sequence in $(\mathbb{X}, d)$ converges to a limit that is also in $\mathbb{X}$.*

Example 2.5.

1. *The space $(\mathbb{R}, |.|)$ is complete because every Cauchy sequence is convergent in $\mathbb{R}$.*
2. *The Cauchy sequence $\left(\frac{1}{n}\right)$ does not converge in $(0, 1)$, and therefore $(0, 1)$ is not complete.*

3. *In a discrete metric space, every Cauchy sequence is convergent (see Example 2.4(3)), and therefore every discrete metric space is complete.*

Proposition 2.7. *Let $(\mathbb{X}, d_{\mathbb{X}})$ and $(\mathbb{Y}, d_{\mathbb{Y}})$ be two metric spaces. If $f : (\mathbb{X}, d_{\mathbb{X}}) \longrightarrow (\mathbb{Y}, d_{\mathbb{Y}})$ is uniformly continuous and (x_n) is a Cauchy sequence in $(\mathbb{X}, d_{\mathbb{X}})$, then $(f(x_n))$ is a Cauchy sequence in $(\mathbb{Y}, d_{\mathbb{Y}})$.*

Proof. *If (x_n) is a Cauchy sequence in $(\mathbb{X}, d_{\mathbb{X}})$, then we have:*

$$\forall \varepsilon > 0,\ \exists\ n_0(\varepsilon) \in \mathbb{N} \text{ such that } \forall (n,m) \in \mathbb{N}^2,\ n,m \geqslant n_0 \Longrightarrow d_{\mathbb{X}}(x_n, x_m) < \varepsilon.$$

Since f is uniformly continuous, for $\varepsilon = \delta$, we obtain :

$$\forall (n,m) \in \mathbb{N}^2,\ n,m \geq n_0 \Longrightarrow d_{\mathbb{Y}}(f(x_n), f(x_m)) < \varepsilon.$$

Therefore $(f(x_n))$ is a Cauchy sequence in $\mathbb{Y}$.

Remark 2.3. *The previous proposition is false if f is only continuous. For example, consider the function $f : ((-1,1), |.|) \longrightarrow (\mathbb{R}, |.|)$ defined by $f(x) = \dfrac{x}{1-|x|}$. Let $x_n = 1 - \dfrac{1}{n}$. The sequence $(f(x_n))$ is not a Cauchy sequence.*

Proposition 2.8. *Let $(\mathbb{X}, d_{\mathbb{X}})$ and $(\mathbb{Y}, d_{\mathbb{Y}})$ be two metric spaces, and let $f : (\mathbb{X}, d_{\mathbb{X}}) \longrightarrow (\mathbb{Y}, d_{\mathbb{Y}})$ be a uniformly continuous homeomorphism. If $(\mathbb{Y}, d_{\mathbb{Y}})$ is complete, then $(\mathbb{X}, d_{\mathbb{X}})$ is also complete.*

Proof. *Let (x_n) be a Cauchy sequence in $\mathbb{X}$. Then, by Proposition (2.7), $(f(x_n))$ is a Cauchy sequence in $\mathbb{Y}$. Since $\mathbb{Y}$ is a complete space, we conclude that $(f(x_n))$ is convergent. This implies the convergence of (x_n) in $\mathbb{X}$ because f^{-1} is continuous.*

Remark 2.4. *The converse of the previous proposition is not true.*

Example 2.6. *Let $(\mathbb{X}, d_{\mathbb{X}})$ and $(\mathbb{Y}, d_{\mathbb{Y}})$ be two metric spaces defined as follows:*

- *$\mathbb{X} = [0,1]$ is the closed interval in $\mathbb{R}$, with the standard metric $d_{\mathbb{X}}(x,y) = |x-y|$. This space is complete because every Cauchy sequence in $[0,1]$ converges to a point in $[0,1]$.*
- *$\mathbb{Y} = (0,1)$ is the open interval in $\mathbb{R}$, with the standard metric $d_{\mathbb{Y}}(x,y) = |x-y|$. This space is not complete. For example, the sequence $x_n = \frac{1}{n}$ is a Cauchy sequence in $(0,1)$, but it converges*

to 0*, which is not in* $(0,1)$*.*
Now define a homeomorphism $f:[0,1]\to(0,1)$ *by:*

$$f(x)=\frac{x}{2}+\frac{1}{4}.$$

This function is uniformly continuous and is a homeomorphism because it is continuous, bijective, and its inverse is also continuous. Thus, the converse of the proposition is false: even though $(\mathbb{X},d_{\mathbb{X}})$ *is complete and* f *is a uniformly continuous homeomorphism,* $(\mathbb{Y},d_{\mathbb{Y}})$ *is not complete.*

Using the previous proposition, we obtain the following result.

Proposition 2.9. *If* $(\mathbb{X},d_{\mathbb{X}})$ *and* $(\mathbb{Y},d_{\mathbb{Y}})$ *are two isometric spaces, then* $(\mathbb{X},d_{\mathbb{X}})$ *is complete if and only if* $(\mathbb{Y},d_{\mathbb{Y}})$ *is complete.*

Proof. *Evident, because every isometry and its inverse are uniformly continuous homeomorphisms.*

Proposition 2.10. *.*

1. *Every complete subset in a metric space* $(\mathbb{X},d_{\mathbb{X}})$ *is closed.*
2. *Every closed subset in a complete metric space* $(\mathbb{X},d_{\mathbb{X}})$ *is complete.*

Proof.

1. *Let* A *be a complete subset of* $\mathbb{X}$*, and let* $x\in Cl(A)$*. Then there exists a sequence* (x_n) *of elements in* A *such that* $x_n\longrightarrow x$ *(see Proposition (2.4)). Since* (x_n) *is a Cauchy sequence in* A*, and* A *is complete, it follows that* $x\in A$*. This shows that* A *is closed.*
2. *Let* $(\mathbb{X},d_{\mathbb{X}})$ *be a complete metric space, and let* (x_n) *be a Cauchy sequence in a closed subset* $A\subset\mathbb{X}$*. Then* (x_n) *is a Cauchy sequence in* $\mathbb{X}$*, which is complete, so* $x_n\longrightarrow x\in\mathbb{X}$*. Given that* A *is closed, we deduce that* $x\in A$*, which shows that* A *is complete.*

Using the previous proposition, we obtain the following result.

Proposition 2.11. *Let $\mathbb{X}$ be a complete metric space, and let A be a subset of $\mathbb{X}$. Then, the metric subspace (A, d_A) is complete if and only if A is closed in $\mathbb{X}$.*

Example 2.7.

1. *The intervals (a,b), $(a,+\infty)$, and $(-\infty,b)$ are not complete because they are not closed.*
2. *The intervals $[a,b]$, $[a,+\infty)$, and $(-\infty,b]$ are complete because they are closed in $\mathbb{R}$.*

Proposition 2.12. *The product of a finite number of metric spaces is complete if and only if all its factors are complete.*

Proof. *Exercise.*

2.3 Contractive mapping theorem

Definition 2.4. *Let $\mathbb{X}$ be a set and let $f : \mathbb{X} \to \mathbb{X}$. A point $x \in \mathbb{X}$ is called a fixed point of f if $f(x) = x$.*

Theorem 2.1 (Banach fixed point theorem). *Let $(\mathbb{X}, d_{\mathbb{X}})$ be a complete metric space. If $f : \mathbb{X} \longrightarrow \mathbb{X}$ is a contraction (see Definition (1.25)), then it has a unique fixed point $x \in \mathbb{X}$.*

Proof. *Consider a recursive sequence given by $x_{n+1} = f(x_n)$ with $x_0 \in \mathbb{X}$. For all $n, m \in \mathbb{N}$, if we assume that $n > m$, we obtain:*

$$d(x_n, x_m) \leqslant \sum_{\ell=m}^{n-1} d(x_{\ell+1}, x_\ell) = \sum_{\ell=m}^{n-1} d\left(f^\ell(x_1), f^\ell(x_0)\right). \tag{1}$$

On the other hand, using the contraction property, we obtain:

$$d\left(f^\ell(x_1), f^\ell(x_0)\right) \leqslant k d\left(f^{\ell-1}(x_1), f^{\ell-1}(x_0)\right) \leqslant k^\ell d(x_1, x_0). \tag{2}$$

Now, keeping in mind that $0 \leqslant k < 1$ *and using **(1)** and **(2)**, we obtain:*

$$\textbf{(3)} \qquad d(x_n, x_m) \leqslant \sum_{\ell=m}^{n-1} k^{\ell} d(x_1, x_0) = k^m \frac{1-k^{n-m-1}}{1-k} d(x_1, x_0) \leqslant \frac{k^m}{1-k} d(x_1, x_0).$$

Since $\lim\limits_{m\to\infty} \dfrac{k^m}{1-k} = 0$*, we conclude from inequality **(3)** that* (x_n) *is a Cauchy sequence and therefore convergent to* $x \in \mathbb{X}$ *because* $(\mathbb{X}, d_{\mathbb{X}})$ *is complete. Since* f *is continuous, we obtain:*

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x.$$

Thus, x *is a fixed point of* f*.*
For uniqueness, suppose that x_1 *and* x_2 *are two fixed points of* f*. Then we have:*

$$d(x_1, x_2) = d(f(x_1), f(x_2)) \leqslant k d(x_1, x_2) \Longrightarrow (1-k) d(x_1, x_2) \leqslant 0.$$

Since $0 \leqslant k < 1$*, we conclude from the last inequality that* $x_1 = x_2$*.*

Remark 2.5.

1. *A function having one or multiple fixed points does not imply that it is a contracting function.*

2. *The assumption that "* f *is contracting " cannot generally be replaced by the weaker assumption* $d(f(x), f(y)) \leqslant d(x, y)$ *for all* $x \neq y$*, as demonstrated by the following example:*

$$f : (\mathbb{R}, |.|) \longrightarrow (\mathbb{R}, |.|) \text{ such that } f(x) = \sqrt{1+x^2}.$$

3. *The assumption that "*$\mathbb{X}$ *is complete" is fundamental. For example, if* $\mathbb{X} = \left(0, \dfrac{1}{4}\right)$ *(which is not complete) and* $f : \mathbb{X} \longrightarrow \mathbb{X}$ *is defined by* $f(x) = x^2$*, then* f *is a contraction on* $\mathbb{X}$ *that has no fixed point in* $\mathbb{X}$*.*

This theorem can be easily generalized in the following way.

Theorem 2.2. *Let* $(\mathbb{X}, d_{\mathbb{X}})$ *be a complete metric space and let* $f : \mathbb{X} \longrightarrow \mathbb{X}$*. If there exists* $n \in \mathbb{N}^*$ *such that* $f^{(n)}$ *is a contraction, then* f *has a unique fixed point.*

Proof. *Since* $(\mathbb{X}, d_{\mathbb{X}})$ *is complete and* $f^{(n)}$ *is a contraction,* $f^{(n)}$ *has a unique fixed point* $x_0 \in \mathbb{X}$*. Since* $f^{(n)}(f(x_0)) = f\left(f^{(n)}(x_0)\right) = f(x_0)$*, it follows, by the uniqueness of the fixed point of* $f^{(n)}$*, that* $f(x_0) = x_0$*, and thus* x_0 *is the unique fixed point of* f*.*

Proposition 2.13. *If* $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ *with* $|f'(x)| \leqslant k < 1$*, then* f *has a unique fixed point.*

Proof. *Using Proposition (1.21), we conclude that* f *is a contraction on* $\mathbb{R}$*, and since* $\mathbb{R}$ *is complete, we conclude by Theorem (2.1) that* f *has a unique fixed point.*

Example 2.8. *Let* $f\colon [0,1] \longrightarrow [0,1]$ *be defined by* $f(x) = \dfrac{e^x}{5}$*. We have* $|f'(x)| = \dfrac{e^x}{5} \leqslant \dfrac{e}{5} < 1$*, and* $[0,1]$ *is complete because it is a closed subset of* $\mathbb{R}$*, which is complete. Therefore, using Theorem (2.1), we conclude that* f *has a unique fixed point.*

CHAPTER 3

TOPOLOGICAL SPACES

3.1 Topology, Open sets and Closed sets

Let $\mathbb{X}$ be a non-empty set and $\mathcal{P}(\mathbb{X})$ be the power set of $\mathbb{X}$.

Definition 3.1. *A topology on $\mathbb{X}$ is a collection of sets $\mathcal{T} \subseteq \mathcal{P}(\mathbb{X})$ that satisfies :*

A_1) *$\emptyset$ and $\mathbb{X}$ are elements of $\mathcal{T}$,*

A_2) *any union (finite or infinite) of elements of $\mathcal{T}$ is an element of $\mathcal{T}$, that is, for any collection $\{O_i \in \mathcal{T} : \quad i \in I\}$ we have $\bigcup\limits_{i\in I} O_i \in \mathcal{T}$,*

A_3) *any finite intersection of elements of $\mathcal{T}$ is an element of $\mathcal{T}$, that is, for any collection $\{O_i \in \mathcal{T} : \quad 1 \leqslant i \leqslant n\}$ we have $\bigcap\limits_{i=1}^{n} O_i \in \mathcal{T}$.*

The pair $(\mathbb{X}, \mathcal{T})$ is called a topological space, and the elements of $\mathcal{T}$ are called open sets of the topology.

Example 3.1. *Let $\mathbb{X} = \{1,2\}$. The topologies defined on $\mathbb{X}$ are:*

$$
\begin{aligned}
\mathcal{T}_1 &= \{\emptyset, \mathbb{X}\}. \\
\mathcal{T}_2 &= \{\emptyset, \mathbb{X}, \{1\}\}. \\
\mathcal{T}_3 &= \{\emptyset, \mathbb{X}, \{2\}\}. \\
\mathcal{T}_4 &= \{\emptyset, \mathbb{X}, \{1\}, \{2\}\}.
\end{aligned}
$$

Example 3.2. *Let $\mathbb{X} = \{x,y,z,t,s,w\}$ and $\mathcal{T} = \{\mathbb{X}, \emptyset, \{x\}, \{z,t\}, \{x,z,t\}, \{y,z,t,s,w\}\}$. Then $\mathcal{T}$ is a topology on $\mathbb{X}$ as it satisfies conditions $(A_1), (A_2)$ and (A_3) of Definition(3.1).*

Example 3.3. *Let $\mathbb{X} = \{x,y,z,t,s\}$ and $\mathcal{T} = \{\mathbb{X}, \emptyset, \{x\}, \{z,t\}, \{x,z,s\}, \{y,z,t\}\}$. Then $\mathcal{T}$ is not a topology on $\mathbb{X}$ as the union $\{z,t\} \bigcup \{x,z,s\} = \{x,z,t,s\}$ of two members of $\mathcal{T}$ does not belong to $\mathcal{T}$; that is, $\mathcal{T}$ does not satisfy condition (A_2) of Definition(3.1).*

Example 3.4. *Let $\mathbb{N}$ the set of all natural numbers and let $\mathcal{T}$ the collection consisting of $\mathbb{N}$, $\emptyset$ and all finite subsets of $\mathbb{N}$. Then $\mathcal{T}$ is not a topology on $\mathbb{N}$, since the infinite union $\{3\} \cup \{4\} \cup \{5\} \cup \cdots \cup \{n\} \cup \cdots = \{3,4,5,\ldots,n,\ldots\}$ of members of $\mathcal{T}$ does not belong to $\mathcal{T}$; that is, $\mathcal{T}$ does not have property (A_2) of Definition(3.1).*

Definition 3.2. *Let $\mathbb{X}$ be any non-empty set and $\mathcal{T}$ the collection of all sets of $\mathbb{X}$ (the power set of $\mathbb{X}$). Then $\mathcal{T}$ is called the discrete topology on the set $\mathbb{X}$ and is denoted by $\mathcal{T}_{Disc}$. The topological space $(\mathbb{X}, \mathcal{T}_{Disc})$ is called a discrete space.*

Definition 3.3. *Let $\mathbb{X}$ be any non-empty set and $\mathcal{T} = \{\mathbb{X}, \emptyset\}$. Then $\mathcal{T}$ is called the indiscrete topology or trivial topology and is denoted by $\mathcal{T}_{Ind}$. The topological space $(\mathbb{X}, \mathcal{T}_{Ind})$ is called an indiscrete space.*

Remark 3.1. *Every set indeed admits at least two topologies.*

Definition 3.4. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. A subset F of $\mathbb{X}$ is said to be a closed set in $(\mathbb{X}, \mathcal{T})$ if its complement, namely $\complement_{\mathbb{X}} F$ or $\mathbb{X} \setminus F$, is open in $(\mathbb{X}, \mathcal{T})$. We denote by $\mathcal{F}$ the set of all closed subsets in $(\mathbb{X}, \mathcal{T})$.*

Example 3.5. *In Example (3.1), if we consider the topology $\mathcal{T}_2$, then the set $\{2\}$ is closed.*

Example 3.6. *In Example(3.2), the closed sets are*

$$\mathcal{F} = \{\emptyset, \mathbb{X}, \{y,z,t,s,w\}, \{x,y,s,w\}, \{y,s,w\}, \{x\}\}.$$

.

Example 3.7. *Let $\mathbb{X} = (\mathbb{R}, |\cdot|)$. Then $\mathbb{N}$ and $\mathbb{Z}$ are closed.*

Proposition 3.1. *Let $(\mathbb{X},\mathcal{T})$ be a topological space. Then, the collection $\mathcal{F}$ of closed sets in $\mathbb{X}$ satisfies the following properties:*

P_1) *$\mathbb{X}$ and $\emptyset$ are closed sets,*

P_2) *any finite union of closed sets is closed,*

P_3) *any arbitrary intersection of closed sets is closed.*

Proof. *These properties of closed sets directly follow from the properties verified by open sets in a topology. Indeed:*

- *We have seen that $\mathbb{X}$ and $\emptyset$ are open, and since $\complement_{\mathbb{X}}\emptyset = \mathbb{X}$ and $\complement_{\mathbb{X}}\mathbb{X} = \emptyset$, we conclude that $\mathbb{X}$ and $\emptyset$ are closed. Thus, (P_1) is verified.*
- *Let $\{F_i : i = 1,2,\dots,n\}$ be a finite family of closed sets in $\mathbb{X}$. Then, for all $i = 1,2,\dots,n$, their complements $\complement_{\mathbb{X}}F_i$ are open sets. But $\complement_{\mathbb{X}}\left(\bigcup_{i=1}^{n} F_i\right) = \bigcap_{i=1}^{n} \complement_{\mathbb{X}}F_i$ is an open set (because it is a finite intersection of open sets). Hence $\bigcup_{i=1}^{n} F_i$ is a closed set. Thus, (P_2) is verified.*
- *Let $\{F_i : i \in I\}$ be any family of closed sets of $\mathbb{X}$. Then, for all $i \in I$, their complements $\complement_{\mathbb{X}}F_i$ are open sets. But $\complement_{\mathbb{X}}\left(\bigcap_{i\in I} F_i\right) = \bigcup_{i\in I} \complement_{\mathbb{X}}F_i$ is an open set (because it is an union of any open sets). Hence, $\bigcap_{i\in I} F_i$ is a closed set. Thus, (P_3) is verified.*

Remark 3.2. *A topology can be defined either by the collection of its open sets or by the collection of its closed sets.*

Remark 3.3. *A subset of a topological space can be both open and closed. Moreover, a subset of a topological space can be neither open nor closed.*

Example 3.8. *If we consider Example(3.2), we see that*

1. *the set $\{x\}$ is both open and closed;*
2. *the set $\{y,z\}$ is neither open nor closed.*

Definition 3.5. *A subset $\mathcal{A}$ of a topological space $(\mathbb{X},\mathcal{T})$ is said to be clopen if it is both open and closed set in $(\mathbb{X},\mathcal{T})$.*

Example 3.9.

1. *In a discrete space all subsets in* $(\mathbb{X}, \mathcal{T}_{Dis})$ *are clopen.*
2. *In a indiscrete space the only clopen subsets in* $(\mathbb{X}, \mathcal{T}_{Ind})$ *are* $\mathbb{X}$ *and* $\emptyset$.
3. *In every topological space* $(\mathbb{X}, \mathcal{T})$ *both* $\mathbb{X}$ *and* $\emptyset$ *are clopen.*

Definition 3.6. *Let* $\mathbb{X}$ *be a non-empty set, and*

$$\mathcal{T}_{Cof} = \{O \subseteq \mathbb{X} : \complement_{\mathbb{X}} O \text{ is finite}\} \cup \{\emptyset\}.$$

Then, $(\mathbb{X}, \mathcal{T}_{Cof})$ *is a topology, and is called the cofinite topology on* $\mathbb{X}$.

Once again is necessary to check that $\mathcal{T}_{Cof}$ in the previous definition is indeed a topology; that is, that it satisfies each of the conditions of Definition(3.1).

3.2 Neighborhoods

Definition 3.7. *Let* $(\mathbb{X}, \mathcal{T})$ *be a topological space. A subset* N_x *of* $\mathbb{X}$ *is called a neighborhood of* x *in* $\mathbb{X}$ *if there exists an open set* O_x *of* $\mathbb{X}$ *such that* $x \in O_x \subseteq N_x$. *The collection of neighborhoods of* x *is denoted by* $\mathcal{N}(x)$ *and is called the neighborhood system at* x.

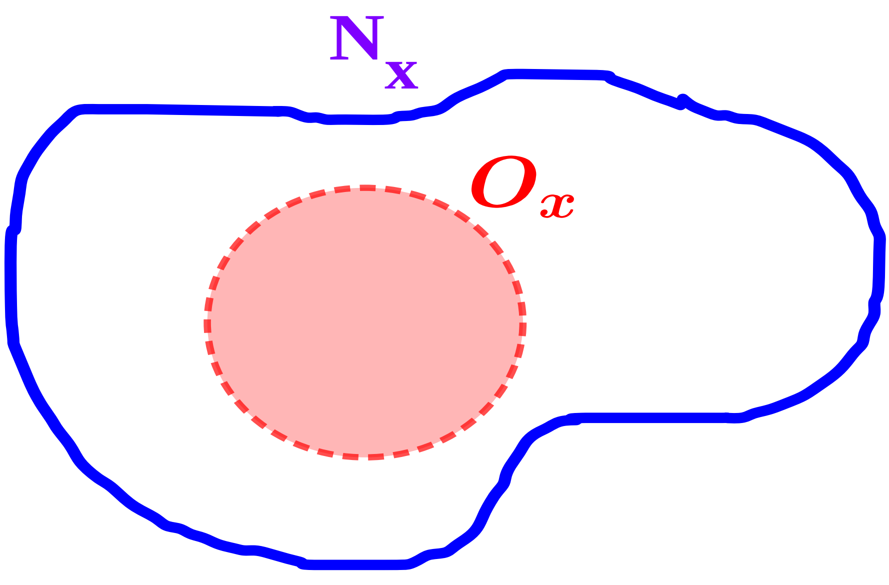


Figure 3.1: Neighborhood $\mathbf{N}_x$

The previous definition can be written in the following form:

$$(N_x \text{ is a neighborhood of } x) \Leftrightarrow (\exists\, O_x \in \mathcal{T} \ / \ x \in O_x \subseteq N_x). \tag{3.1}$$

Definition 3.8. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. We say that a subset N of $\mathbb{X}$ is a neighborhood of a non-empty subset A of $\mathbb{X}$ if there exists an open set O in $\mathcal{T}$ such that $A \subseteq O \subseteq N$. In other words:*

$$(N \text{ is a neighborhood of } A) \Leftrightarrow (\exists O \in \mathcal{T} \text{ such that } A \subseteq O \subseteq N). \tag{3.2}$$

Example 3.10.

1. *Let $(\mathbb{X}, \mathcal{T}_{Ind})$. Then, for all $x \in \mathbb{X}$ we have $\mathcal{N}(x) = \{\mathbb{X}\}$.*

2. *Let $(\mathbb{X}, \mathcal{T}_{Disc})$ and $x \in \mathbb{X}$. Then, every subset of $\mathbb{X}$ that contains x is an element of $\mathcal{N}(x)$.*

3. *Let $(\mathbb{X}, \mathcal{T}) = (\mathbb{R}, |.|)$ and $x \in \mathbb{R}$. Then, every subset of $\mathbb{R}$ that contains an interval centered at x is a neighborhood of x.*

4. *Let $\mathbb{X} = \{1,2,3,4\}$ and $\mathcal{T} = \{\emptyset, \mathbb{X}, \{1\}, \{4\}, \{1,4\}\}$. Then we have:*
   - $\mathcal{N}(1) = \{\{1\},\{1,2\},\{1,3\},\{1,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\mathbb{X}\}$;
   - $\mathcal{N}(2) = \{\mathbb{X}\}$;
   - $\mathcal{N}(\{1,4\}) = \{\{1,4\},\{1,2,4\},\{1,3,4\},\mathbb{X}\}$.

Remark 3.4. *It follows from the previous definition that if $B \subset A$, then every neighborhood of A is a neighborhood of B.*

Proposition 3.2. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space and A a subset of $\mathbb{X}$. Then, we have*

$$(N \text{ is a neighborhood of } A) \Longleftrightarrow (\forall x \in A : N \in \mathcal{N}(x)). \tag{3.3}$$

Proof.

$\Longrightarrow$) *Obvious.*

$\Longleftarrow$) *Suppose that N is a neighborhood of every point in A. Then, we have*

$$\forall x \in A, \exists O_x \in \mathcal{T} \ / \ x \in O_x \subseteq N, \tag{3.4}$$

from which we conclude that $A \subseteq \bigcup\limits_{x\in A} O_x \subseteq N$, and since $\bigcup\limits_{x\in A} O_x \in \mathcal{T}$, it follows that N is a neighborhood of A.

Proposition 3.3. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. A non-empty set A is an open set in $\mathbb{X}$ if and only if A is a neighborhood of each of its points.*

Proof.

$\Longrightarrow$) *Suppose A is an open set in $\mathbb{X}$. Then, using Definition (3.7), we conclude that A is a neighborhood of each of its points.*

$\Longleftarrow$) *Suppose A is a neighborhood of each of its points. Then, for every $x \in A$, there exists $O_x \in \mathcal{T}$ such that $x \in O_x \subseteq A$, hence $A = \bigcup\limits_{x\in A} O_x$. Therefore, A is open as a union of open sets.*

Proposition 3.4. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. The neighborhoods of a point satisfy the following properties:*

1. *For every $N \in \mathcal{N}(x)$, we have $x \in N$.*
2. *For every $N \in \mathcal{N}(x)$ and every $\mathbf{U} \subset \mathbb{X}$, if $N \subset \mathbf{U}$ then $\mathbf{U} \in \mathcal{N}(x)$.*
3. *Any finite intersection of neighborhoods of x is a neighborhood of x.*
4. *For every $N \in \mathcal{N}(x)$, there exists $W \in \mathcal{N}(x)$ such that for every $a \in W$, we have $N \in \mathcal{N}(a)$.*

Proof.

- *The two properties 1 and 2 are evident.*
- *For the third property, if* $\{N_i : i = 1,\dots,n\}$ *is a family of neighborhoods of* $x \in \mathbb{X}$*, then for all* $i = 1,\dots,n$*, there exists* $O_i \in \mathcal{T}$ *such that* $x \in O_i \subseteq N_i$*, from which we conclude that* $x \in \bigcap_{i=1}^{n} O_i \subseteq \bigcap_{i=1}^{n} N_i$*. We deduce that* $\bigcap_{i=1}^{n} N_i \in \mathcal{N}(x)$ *because* $\bigcap_{i=1}^{n} O_i \in \mathcal{T}$*.*
- *For the fourth property, if* $N \in \mathcal{N}(x)$*, then there exists* $O \in \mathcal{T}$ *such that* $x \in O \subseteq N$*. This implies that* N *is a neighborhood of every point* $a \in O$*. Then, it suffices to take* $W = O$ *to verify that property (4) is holds.*

3.3 Comparison of topologies

Definition 3.9. *Let* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *be two topologies on a set* $\mathbb{X}$*. We say that* $\mathcal{T}_1$ *is finer than* $\mathcal{T}_2$ *(or that* $\mathcal{T}_2$ *is coarser than* $\mathcal{T}_1$*) if* $\mathcal{T}_2 \subseteq \mathcal{T}_1$*. In other words,* $\mathcal{T}_1$ *is finer than* $\mathcal{T}_2$ *if one of the following three statements holds:*

1. *Every open set in* $(\mathbb{X}, \mathcal{T}_2)$ *is also an open set in* $(\mathbb{X}, \mathcal{T}_1)$*.*
2. *Every closed set in* $(\mathbb{X}, \mathcal{T}_2)$ *is also a closed set in* $(\mathbb{X}, \mathcal{T}_1)$*.*
3. *If* $x \in \mathbb{X}$*, then every neighborhood of* x *in* $(\mathbb{X}, \mathcal{T}_2)$ *is also a neighborhood of* x *in* $(\mathbb{X}, \mathcal{T}_1)$*.*

Remark 3.5. *If* $\mathcal{T}_1$ *is finer than* $\mathcal{T}_2$ *and* $\mathcal{T}_2$ *is finer than* $\mathcal{T}_1$*, we say that* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *are equivalent.*

Example 3.11. *For any topological space* $(\mathbb{X}, \mathcal{T})$*, the indiscrete topology on* $\mathbb{X}$ *is coarser than* $\mathcal{T}$ *which in turn is coarser than the discrete topology on* $\mathbb{X}$*.*

Example 3.12. *The Sierpinski space* $\mathbb{S}$ *consists of two points* $\{0,1\}$ *with the topology* $\{\emptyset, \{1\}, \{0,1\}\}$*. The topology of Sierpinski space is finer than the indiscrete topology* $\mathcal{T}_{Ind} = \{\emptyset, \{0,1\}\}$ *on* $\{0,1\}$ *but coarser than the discrete topology* $\mathcal{T}_{Disc} = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$ *on* $\{0,1\}$*.*

Example 3.13. *If* $\mathbb{X} = \{x,y,z\}$*, then* $\mathcal{T}_1 = \{\emptyset, \{x\}, \mathbb{X}\}$*,* $\mathcal{T}_2 = \{\emptyset, \{x,y\}, \mathbb{X}\}$*, and* $\mathcal{T}_3 = \{\emptyset, \{x\}, \{x,y\}, \mathbb{X}\}$ *are three distinct topologies on* $\mathbb{X}$*. The topologies* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *are coarser than* $\mathcal{T}_3$*; however,* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *are not comparable.*

Proposition 3.5. *Let $\{\mathcal{T}_i : i \in I\}$ be a collection of topologies on $\mathbb{X}$. Then, $\bigcap_{i\in I} \mathcal{T}_i$ is a topology on $\mathbb{X}$ that is the coarsest of each of the topologies $\mathcal{T}_i$.*

Proof. *Obvious.*

Proposition 3.6. *Let β be a family of subsets of $\mathbb{X}$. There exists a smallest topology that contains β. This topology is called the topology generated by β.*

Proof. *The set of topologies that contain β is not empty because it contains the discrete topology. Therefore, it is enough to take the intersection of these topologies.*

3.4 Base and Neighborhood base

Definition 3.10. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. A* ***basis*** *for the topology $\mathcal{T}$ is a family $\mathfrak{B} \subseteq \mathcal{T}$ such that every set in $\mathcal{T}$ is a union of sets from $\mathfrak{B}$.*

Example 3.14.

1. *Let the topological space $(\mathbb{R}, |.|)$ and $x \in \mathbb{R}$. The collection:*

$$\mathfrak{B} = \{]x, y[\ : x, y \in \mathbb{R}\},$$

is a basis for the usual topology.

2. *In the topological space $(\mathbb{X}, \mathcal{T}_{Disc})$, the collection:*

$$\mathfrak{B} = \{\{x\}\ : x \in \mathbb{X}\},$$

is a basis for the discrete topology.

3. *Let $\mathbb{X} = \{x, y, z\}$ and $\mathcal{T} = \{\emptyset, \{x\}, \{y\}, \{x, y\}, \mathbb{X}\}$. The collection:*

$$\mathfrak{B} = \{\{x\}, \{y\}, \mathbb{X}\},$$

is a basis for this topology.

4. *If $(\mathbb{X}, \mathcal{T})$ is a topological space, then $\mathcal{T}$ is a basis for itself.*

5. *In the topological space $(\mathbb{X}, \mathcal{T}_{Ind})$, the collection:*

$$\mathfrak{B} = \{\mathbb{X}\},$$

is a basis for the indiscrete topology.

Remark 3.6. *If $\mathfrak{B}$ is a basis for a topological space $(\mathbb{X}, \mathcal{T})$ and $\mathfrak{B}'$ is a family that contains $\mathfrak{B}$, then by using the previous definition, we conclude that $\mathfrak{B}'$ is another basis for $\mathcal{T}$. Therefore, a topological space can have multiple bases.*

Proposition 3.7. *Any basis $\mathfrak{B}$ of a topology $\mathcal{T}$ on $\mathbb{X}$ has the following two properties:*

1. *For every $x \in \mathbb{X}$, there exists $B \in \mathfrak{B}$ such that $x \in B$.*

2. *If $B_1, B_2 \in \mathfrak{B}$ and $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathfrak{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*

Proof. *Suppose that $\mathfrak{B}$ is a basis of the topology $\mathcal{T}$.*

1. *Since $\mathbb{X}$ is an open set, we have $\mathbb{X} = \bigcup_{B \in \mathfrak{B}} B$ (see definition (3.10)), from which it follows that for every $x \in \mathbb{X}$, there exists $B \in \mathfrak{B}$ such that $x \in B$.*

2. *If $B_1, B_2 \in \mathfrak{B}$, then $B_1, B_2 \in \mathcal{T}$ (since $\mathfrak{B} \subseteq \mathcal{T}$), which implies that $B_1 \cap B_2 \in \mathcal{T}$. Thus, $B_1 \cap B_2 = \bigcup_{B \in \mathfrak{B}} B$ (see definition (3.10)). Therefore, for every $x \in B_1 \cap B_2$, there exists $B_3 \in \mathfrak{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*

Proposition 3.8. *If $\mathfrak{B}$ is a family of subsets of a set $\mathbb{X}$ that satisfies the two properties of Proposition (3.7), then $\mathcal{T} = \{\bigcup B : B \in \mathfrak{B}\}$ is a topology on $\mathbb{X}$.*

Proof. *We leave it as an exercise.*

Now, using the two previous propositions, we obtain the following result:

Proposition 3.9. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. Then, a family of subsets $\mathfrak{B}$ of $\mathbb{X}$ is a basis for $\mathcal{T}$ if and only if $\mathfrak{B}$ satisfies the two properties of Proposition (3.7).*

Proposition 3.10. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space and $\mathfrak{B}$ a subset of $\mathcal{T}$. Then, $\mathfrak{B}$ is a basis for $\mathcal{T}$ if and only if for every $O \in \mathcal{T}$ and for every $x \in O$, there exists $U_x \in \mathfrak{B}$ such that: $x \in U_x \subseteq O$.*

Proof.

$\Longleftarrow$) *It is clear that $O = \bigcup\limits_{x \in O} U_x$, hence $\mathfrak{B}$ is a basis for $\mathcal{T}$.*

$\Longrightarrow$) *If $\mathfrak{B}$ is a basis for $\mathcal{T}$, then every subset O of $\mathcal{T}$ is a union of elements of $\mathfrak{B}$, which means that for each element $x \in O$, there exists $U_x \in \mathfrak{B}$ such that $x \in U_x \subset O$.*

Proposition 3.11. *Let $\mathfrak{B}_1$ be a basis of a topology $\mathcal{T}$ and $\mathfrak{B}_2$ a family of subsets of $\mathcal{T}$. If every element of $\mathfrak{B}_1$ is a union of elements of $\mathfrak{B}_2$, then $\mathfrak{B}_2$ is a basis for $\mathcal{T}$.*

Proof. *Let $O \in \mathcal{T}$. Then, there exists $\{O_i : i \in I \text{ and } O_i \in \mathfrak{B}_1\}$ such that $O = \bigcup\limits_{i \in I} O_i$ (because $\mathfrak{B}_1$ is a base for $\mathcal{T}$) and since every element of $\mathfrak{B}_1$ is a union of elements of $\mathfrak{B}_2$, there exists $\{U_{i,j} : j \in J \text{ and } U_{i,j} \in \mathfrak{B}_2\}$ such that $O_i = \bigcup\limits_{j \in J} U_{i,j}$ for all $i \in I$. Thus, we obtain $O = \bigcup\limits_{(i,j) \in I \times J} U_{i,j}$. Therefore, $\mathfrak{B}_2$ is a base for $\mathcal{T}$.*

Definition 3.11. *A collection $\mathcal{S}(x) \subseteq \mathcal{N}(x)$ is called a neighborhood base at x if for every neighborhood N_x, there is a neighborhood $W_x \in S(x)$ such that $W_x \subseteq N_x$. We refer to the sets in $\mathcal{S}(x)$ as basic neighborhoods of x.*

Example 3.15.

1. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. Then, we have:*

$$\mathcal{S}(x) = \{O \in \mathcal{T} : x \in O\}$$

   is a neighbourhoods base of x.

2. *In the topological space $(\mathbb{X}, \mathcal{T}_{Disc})$, we have:*

$$\mathcal{S}(x) = \{\{x\}\}$$

   is a neighborhoods base of x.

3. *Let the topological space* $(\mathbb{R}, |.|)$ *and* $x \in \mathbb{R}$. *Then, we have:*

$$\mathcal{S}(x) = \{(x-\varepsilon, x+\varepsilon) \ : \varepsilon > 0\}$$

is a neighborhoods base of x. *For example:*

$$\mathcal{S}(x) = \left\{\left(x - \frac{1}{n}, x + \frac{1}{n}\right) \ : n \in \mathbb{N}^*\right\}$$

is a countable neighborhoods base of x.

3.5 Interior points, Adherent points, Accumulation points, Isolated points, Boundary points, Exterior points and Dense sets.

3.5.1 Interior points

Definition 3.12. *Let* A *be a subset of a topological space* $(\mathbb{X}, \mathcal{T})$. *We say that* x *is an interior point of* A *if* A *is a neighborhood of* x, *in other words,*

$$x \text{ is an interior point of } A \Longleftrightarrow A \in \mathcal{N}(x). \tag{3.5}$$

The set of all interior points of A *is called the interior or the interior set of* A *and is denoted by* $Int(A)$.

Example 3.16.

1. *Consider the topological space* $(\mathbb{X}, \mathcal{T}_{Ind})$ *and let* $A \subseteq \mathbb{X}$. *Then, we have the following two cases:*
   - $\mathbb{X} = A \Longrightarrow Int(A) = \mathbb{X}$.
   - $\mathbb{X} \neq A \Longrightarrow Int(A) = \emptyset$.
2. *Consider the topological space* $(\mathbb{X}, \mathcal{T}_{Disc})$ *and let* $A \subseteq \mathbb{X}$. *Then,* $Int(A) = A$.
3. *For the topological space* $(\mathbb{R}, |.|)$, *we have:*
   - $\forall x \in \mathbb{R}, \ Int\{x\} = \emptyset$.
   - $\forall x, y \in \mathbb{R}, \ Int([x,y]) = Int([x,y)) = Int((x,y]) = Int((x,y)) = (x,y)$.
   - $Int(\mathbb{N}) = Int(\mathbb{Z}) = Int(\mathbb{Q}) = Int\left(\complement_{\mathbb{R}}\mathbb{Q}\right) = \emptyset$.

4. *If* $\mathbb{X}=\{x,y,z,t\}$ *and* $\mathcal{T}=\{\mathbb{X},\emptyset,\{x\},\{y\},\{x,y\}\}$, *then we have:*
   - $Int\{z\}=Int\{t\}=\emptyset$.
   - $Int\{x,z,t\}=\{x\}$.

Proposition 3.12. *$Int(A)$ is the largest open set contained in A.*

Proof. *We will show that $Int(A)$ is the union of all open subsets of A.*
For every $x\in Int(A)$, we have $A\in\mathcal{N}(x)$ (see definition (3.12)). Using definition (3.1), we conclude that: for every $x\in Int(A)$, there exists $O_x\in\mathcal{T}$ such that $x\in O_x\subset A$, which leads to:

(i) $$Int(A)\subset\bigcup_{x\in Int(A)}O_x\subset\bigcup_{x\in A}O_x.$$

Conversely, if $x\in\bigcup_{x\in A}O_x$ then $x\in O_x\subset A$, which implies $A\in\mathcal{N}(x)$, so $x\in Int(A)$. This means that:

(ii) $$\bigcup_{x\in A}O_x\subset Int(A).$$

From (i) and (ii) we conclude that:

$$Int(A)=\bigcup_{x\in A}O_x.$$

Finally, $Int(A)$ is the largest open set contained in A because it is the union of all open subsets of A.

Remark 3.7. *The previous proposition allows us to write the following result:*

(3.6) $$A \text{ is open in } \mathbb{X}\Longleftrightarrow A=Int(A).$$

Proposition 3.13. *Let $(\mathbb{X},\mathcal{T})$ be a topological space and A,B two subsets of $\mathbb{X}$. Then, we have:*

1. *If $A\subset B$ and A is open, then $A\subset Int(B)$.*
2. *If $A\subset B$, then $Int(A)\subset Int(B)$.*
3. $Int(A)=Int(Int(A))$.
4. $Int(A\cap B)=Int(A)\cap Int(B)$.

5. $Int(A) \cup Int(B) \subset Int(A \cup B)$.

6. $A \in \mathcal{N}(B) \Longleftrightarrow B \subset Int(A)$.

Proof. *(Exercise).*

Remark 3.8. *We have* $Int\left(\bigcap_{i\in I} A_i\right) \subset \bigcap_{i\in I} Int(A_i)$ *if* I *is infinite.*

3.5.2 Adherent points

Definition 3.13. *Let* $(\mathbb{X}, \mathcal{T})$ *be a topological space,* $A \subset \mathbb{X}$*, and* $x \in \mathbb{X}$*. We say that* x *is an adherent point to* A *if and only if every* $N \in \mathcal{N}(x)$ *contains at least one point of* A*. In other words:*

$$x \text{ is an adherent point of } A \Longleftrightarrow \forall N \in \mathcal{N}(x), N \cap A \neq \emptyset. \tag{3.7}$$

The set of all adherent points to A *is called the closure of* A*, and it is denoted by Cl(A).*

Remark 3.9. *It follows from this definition that* $A \subseteq Cl(A)$.

Example 3.17.

1. *If* A *is a subset of* $\mathbb{X}$ *with the indiscrete topology* $\mathcal{T}_{Ind}$*, then* $Cl(A) = \mathbb{X}$.

2. *If* A *is a subset of* $\mathbb{X}$ *with the discrete topology* $\mathcal{T}_{Disc}$*, then* $Cl(A) = A$.

3. *Consider the topological space* $(\mathbb{R}, |.|)$*, then:*
   - $\forall x \in \mathbb{R},\ Cl(\{x\}) = \{x\}$.
   - $\forall x, y \in \mathbb{R},\ Cl([x,y]) = Cl([x,y)) = Cl((x,y]) = Cl((x,y)) = [x,y]$.
   - $Cl(\mathbb{Q}) = Cl(\complement_{\mathbb{R}}\mathbb{Q}) = \mathbb{R}, \quad Cl(\mathbb{N}) = \mathbb{N}, \quad Cl(\mathbb{Z}) = \mathbb{Z}$.

4. *If* $\mathbb{X} = \{x,y,z,t\}$ *and* $\mathcal{T} = \{\mathbb{X}, \emptyset, \{x\}, \{y\}, \{x,y\}\}$*, then for example:*
   - $Cl(\{x\}) = \{x,z,t\}, \quad Cl(\{y\}) = \{y,z,t\}$.
   - $Cl(\{z\}) = \{z,t\}, \quad Cl(\{t\}) = \{z,t\}$.

Proposition 3.14. $Cl(A)$ *is the smallest closed set that contains* A.

Proof. *We will show that $Cl(A)$ is the intersection of all closed sets that contain A. Let $F = \bigcap\limits_{i\in I} F_i$, where F_i is a closed set that contains A for all $i \in I$.*

- $\left(F \overset{?}{\subseteq} Cl(A)\right)$ *Let $x \notin Cl(A)$. Then there exists an open set $O \in \mathcal{N}(x)$ such that $O \cap A = \emptyset$, which implies $A \subset \complement_{\mathbb{X}} O$. Therefore, $\complement_{\mathbb{X}} O$ is a closed set that contains A and $x \notin \complement_{\mathbb{X}} O$ which leads to $x \notin F$. Thus, we have:*

$$(i) \qquad F \subseteq Cl(A).$$

- $\left(Cl(A) \overset{?}{\subseteq} F\right)$ *Now, let $x \notin F$. Then $x \in \complement_{\mathbb{X}} F$ (which is open), but $\complement_{\mathbb{X}} F \cap A = \emptyset$, leading to $x \notin Cl(A)$. Thus, we have:*

$$(ii) \qquad Cl(A) \subseteq F.$$

From (i) and (ii), we conclude that $Cl(A) = F$. Therefore, $Cl(A)$ is the smallest closed set that contains A.

Remark 3.10. *The previous proposition allows us to write the following result:*

$$A \text{ is closed in } \mathbb{X} \Longleftrightarrow A = Cl(A). \tag{3.8}$$

Proposition 3.15. *Let A and B be two subsets of the topological space $(\mathbb{X}, \mathcal{T})$. Then, we have:*

1. $A \subseteq B \Longrightarrow Cl(A) \subseteq Cl(B)$.
2. $Cl(A \cup B) = Cl(A) \cup Cl(B)$.
3. $Cl(A \cap B) \subset Cl(A) \cap Cl(B)$.
4. $\complement_{\mathbb{X}} Cl(A) = Int(\complement_{\mathbb{X}} A)$.
5. $Cl(\complement_{\mathbb{X}} A) = \complement_{\mathbb{X}} Int(A)$.
6. $Cl(Cl(A)) = Cl(A)$.

Proof. *(Exercise).*

Example 3.18. *If $A = (1,2)$ and $B = (2,3)$, then $Cl(A \cap B) = \emptyset$. However, $Cl(A) \cap Cl(B) = [1,2] \cap [2,3] = \{2\}$. This example shows that, in general, the inclusion in (3) is not an equality.*

3.5.3 Accumulation points

Definition 3.14. *Let* $(\mathbb{X},\mathcal{T})$ *be a topological space,* $A\subset\mathbb{X}$*, and* $x\in\mathbb{X}$*. We say that* x *is an accumulation point of* A *if and only if every* $N\in\mathcal{N}(x)$ *contains at least one point of* A *other than* x*. In other words:*

$$x \text{ is an accumulation point of } A \Longleftrightarrow \forall N\in\mathcal{N}(x),\ (N\setminus\{x\})\cap A\neq\emptyset. \tag{3.9}$$

The set of all accumulation points of A *is called the derived set of* A *and is denoted by* A'*.*

It follows from this definition that any point adherent to A but not belonging to A is an accumulation point. Therefore, we have the following result:

$$A'\cup A = Cl(A). \tag{3.10}$$

Example 3.19.

1. *Let* $\mathbb{X}=\{x,y,z,t,s\}$*,* $\mathcal{T}=\{\emptyset,\mathbb{X},\{x,y\},\{z,t,s\}\}$*, and* $A=\{x,y,z\}$*. Then, we have* $A'=\{x,y,t,s\}$*.*
2. *If* A *is a subset of a topological space* $(\mathbb{X},\mathcal{T}_{Disc})$*, then* $A'=\emptyset$*.*
3. *In* $(\mathbb{R},|\cdot|)$*, we have* $\mathbb{N}'=\mathbb{Z}'=\emptyset$*.*

Proposition 3.16. *A subset* A *of a topological space* $(\mathbb{X},\mathcal{T})$ *is closed if and only if it contains all of its accumulation points.*

Proof. *Evident (from relation (3.10)).*

3.5.4 Isolated Points

Definition 3.15. *Let* $(\mathbb{X},\mathcal{T})$ *be a topological space and* $A\subset\mathbb{X}$*. We say that a point* $x\in A$ *is an isolated point if and only if there exists* $N\in\mathcal{N}(x)$ *such that* N *contains no other points of* A *except* x*. That is:*

$$x \text{ is an isolated point in } A \Longleftrightarrow \exists N\in\mathcal{N}(x),\quad N\cap A=\{x\}.$$

The set of all isolated points of A *is denoted by* $Is(A)$*.*

Example **3.20.**

1. *In the topological space* $(\mathbb{R}, |\cdot|)$, *we have* $Is(\mathbb{N}) = \mathbb{N}$ *and* $Is(\mathbb{Z}) = \mathbb{Z}$.

2. *Every point in a topological space* $(\mathbb{X}, \mathcal{T}_{Disc})$ *is isolated.*

3. *Let* $\mathbb{X} = \{x, y, z, t, s\}$, $\mathcal{T} = \{\emptyset, \mathbb{X}, \{x\}, \{y\}, \{x, y\}\}$, *and* $A = \{y, z, t\}$. *Then,* $Is(A) = \{y\}$.

3.5.5 Boundary points

Definition 3.16. *Let* $(\mathbb{X}, \mathcal{T})$ *be a topological space,* $A \subset \mathbb{X}$, *and* $x \in \mathbb{X}$. *We say that* x *is a boundary point of* A *if it adheres to both* A *and* $\complement_{\mathbb{X}} A$. *In other words:*

$$x \text{ is a boundary point of } A \iff x \in Cl(A) \cap Cl(\complement_{\mathbb{X}} A).$$

The set of all boundary points of A *is called the boundary of* A *and is denoted by* $\partial(A)$.

Remark **3.11.** *Using property (5) of Proposition (3.15), we obtain:*

$$\begin{aligned} \partial(A) &= Cl(A) \cap Cl(\complement_{\mathbb{X}} A) \\ &= Cl(A) \cap \complement_{\mathbb{X}} Int(A) \\ &= Cl(A) - Int(A). \end{aligned} \tag{3.11}$$

Proposition 3.17. *Let* A *be a subset of a topological space* $(\mathbb{X}, \mathcal{T})$. *Then,*

1. $\partial(A)$ *is a closed set.*

2. A *is both open and closed* $\iff \partial(A) = \emptyset$.

3. A *is open* $\iff \partial(A) \cap A = \emptyset$.

4. A *is closed* $\iff \partial(A) \subseteq A$.

Proof. *(Exercise).*

Example **3.21.**

1. *If* A *is a subset of a topological space* $(\mathbb{X}, \mathcal{T}_{Disc})$, *then* $\partial(A) = \emptyset$.

2. *In the space* $(\mathbb{R}, |\cdot|)$:

- *If* $A = (a,b)$*, then* $\partial(A) = Cl(A) - Int(A) = [a,b] - (a,b) = \{a,b\}$.
- *If* $A = \mathbb{Z}$*, then* $\partial(A) = Cl(A) - Int(A) = \mathbb{Z} - \emptyset = \mathbb{Z}$.

3.5.6 Exterior points

Definition 3.17. *Let* $(\mathbb{X}, \mathcal{T})$ *be a topological space,* $A \subset \mathbb{X}$*, and* $x \in \mathbb{X}$*. We say that* x *is an exterior point of* A *if it belongs to the interior of* $\complement_{\mathbb{X}} A$*. In other words:*

$$x \text{ is an exterior point of } A \Longleftrightarrow x \in Int(\complement_{\mathbb{X}} A).$$

The set of all exterior points of A *is called the exterior of* A*, and it is denoted by* $Ext(A)$.

Remark 3.12. *Using property (4) from Proposition (3.15), we obtain the following result:*

$$Ext(A) = Int(\complement_{\mathbb{X}} A) = \complement_{\mathbb{X}} Cl(A).$$

Proposition 3.18. *Let* A *and* B *be two subsets of a topological space* $(\mathbb{X}, \mathcal{T})$*. Then,*

1. $Ext(A)$ *is an open set.*
2. $Ext(A) \subseteq \complement_{\mathbb{X}} A$.
3. $Ext(A) = Ext(\complement_{\mathbb{X}} Ext(A))$.
4. $Ext(A \cup B) = Ext(A) \cap Ext(B)$.
5. $Cl(A) = \mathbb{X} \Longleftrightarrow Ext(A) = \emptyset$.

Proof. *(Exercise).*

3.5.7 Dense sets

Definition 3.18. *Let* $(\mathbb{X}, \mathcal{T})$ *be a topological space, and let* A *and* B *be two subsets of* $\mathbb{X}$*. We say that* A *is dense in* B *if and only if every point of* B *is an adherent point of* A*, in other words:*

$$A \text{ is dense in } B \Longleftrightarrow B \subseteq Cl(A), \tag{3.12}$$

and we say that A is dense in $\mathbb{X}$ if and only if $Cl(A) = \mathbb{X}$ or $Int(\complement_{\mathbb{X}} A) = \emptyset$.

Example 3.22.

1. *If $\mathbb{X}$ is equipped with the indiscrete topology, then every non-empty subset of $\mathbb{X}$ is dense in $\mathbb{X}$.*

2. *If $\mathbb{X}$ is equipped with the discrete topology, and A and B are subsets of $\mathbb{X}$ such that $B \subset A$, then A is dense in B. Moreover, no subset $A \neq \mathbb{X}$ is dense in $\mathbb{X}$.*

3. *In $(\mathbb{R}, |\cdot|)$, let $A = [a,b)$ and $B = (a,b)$. It is clear that A is dense in B because $B \subseteq Cl(A) = [a,b]$.*

4. *We have seen that $\mathbb{Q}$ is dense in $\mathbb{R}$ since $Cl(\mathbb{Q}) = \mathbb{R}$.*

5. *Let $\mathbb{X} = \{x,y,z,t\}$ and $\mathcal{T} = \{\emptyset, \mathbb{X}, \{x\}, \{x,y\}\}$. Define $A = \{t\}$ and $B = \{x,z\}$; we find that B is dense in A because $A \subseteq Cl(B) = \mathbb{X}$, but A is not dense in B since $B \not\subseteq Cl(A) = \{z,t\}$.*

Proposition 3.19. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space, and consider three subsets A, B, and C of $\mathbb{X}$. If A is dense in B and B is dense in C; then, A is dense in C.*

Proof. *On the one hand, since A is dense in B, we have $B \subseteq Cl(A))$, which implies that*

$$Cl(B) \subseteq Cl(A). \tag{i}$$

On the other hand, since B is dense in C, we have

$$C \subseteq Cl(B). \tag{ii}$$

From (i) and (ii), we conclude that $C \subseteq Cl(A)$, so A is dense in C.

Remark 3.13. *The previous proposition shows that density is a transitive property.*

The following property is a very practical characterization of dense subsets.

Proposition 3.20. *Let $(\mathbb{X}, \mathcal{T})$ be a metric space, and let $A \subseteq \mathbb{X}$. Then, A is dense in $\mathbb{X}$ if and only if every non-empty open set in $\mathbb{X}$ contains at least one element of A.*

Proof.

$\Longrightarrow$) *Suppose that A is a dense subset of $\mathbb{X}$ and O is a non-empty open set in $\mathbb{X}$. Since $Cl(A) = \mathbb{X}$, it follows that $O \subseteq Cl(A)$. Thus, $A \cap O \neq \emptyset$ because O is a neighborhood of each of its points.*

$\Longleftarrow$) *Assume that $A \cap O \neq \emptyset$ for every open set O in $\mathbb{X}$. This implies that for any neighborhood N of any point $x \in \mathbb{X}$, we also have $N \cap A \neq \emptyset$, since N contains a non-empty open set. Therefore, $x \in Cl(A)$, and consequently, $Cl(A) = \mathbb{X}$.*

3.6 Separated Spaces (Hausdorff Spaces))

Definition 3.19. *A topological space $(\mathbb{X}, \mathcal{T})$ is said to be separated or Hausdorff if and only if, for any two distinct points x and y in $\mathbb{X}$, there exist neighborhoods $N \in \mathcal{N}(x)$ and $W \in \mathcal{N}(y)$ such that $N \cap W = \emptyset$.*

Example 3.23.

1. *The space $(\mathbb{X}, \mathcal{T}_{Disc})$ is separated.*
2. *If $\mathrm{card}(\mathbb{X}) \geq 2$, the space $(\mathbb{X}, \mathcal{T}_{Ind})$ is not separated.*
3. *The metric space $(\mathbb{R}, |.|)$ is Hausdorff.*
4. *The space $(\mathbb{X}, \mathcal{T}_{Cof})$ is not separated.*

Proposition 3.21. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. Then, $\mathbb{X}$ is separated if and only if for every $x \in \mathbb{X}$, we have $\{x\} = \bigcap_{N_x \in \mathcal{N}(x)} N_x$, where N_x is a closed neighborhood of x.*

Proof.

$\Longrightarrow$) *Let $\mathbb{X}$ be a separated topological space and $x \in \mathbb{X}$. We want to show that*

$$\{x\} = \bigcap_{N_x \in \mathcal{N}(x)} N_x, \tag{i}$$

where N_x is a closed neighborhood of x. Suppose there exists $y \in \bigcap_{N_x \in \mathcal{N}(x)} N_x$ such that $y \neq x$. Then, there exist two open neighborhoods U and W of x and y, respectively, such that $U \cap W = \emptyset$. This means that $C_{\mathbb{X}} W$ is a closed neighborhood of x (since it contains U), which contradicts the fact that y belongs to all closed neighborhoods of x.

$\Longleftarrow$) *Conversely, let $x, y \in \mathbb{X}$ such that $x \neq y$. From the equality (i), it follows that there exists a closed neighborhood N_x of x that does not contain y. Therefore, there exists an open set O such that $x \in O \subset Cl(O) \subset N_x$, which implies that $y \notin Cl(O)$. Finally, we conclude that O and $C_{\mathbb{X}}Cl(O)$ are two disjoint open sets containing x and y, respectively, which shows that $\mathbb{X}$ is a separated space.*

Using the previous proposition, we obtain the following result:

Proposition 3.22. *Every singleton in a separated space is closed, and in general, every finite set in a separated space is closed.*

Proposition 3.23. *Let $(\mathbb{X}, \mathcal{T})$ be a separated topological space and $x \in \mathbb{X}$. Then, x is an accumulation point of a subset A of $\mathbb{X}$ if and only if every neighborhood N_x of x contains infinitely many elements of A.*

Proof.

$\Longleftarrow$) *Obvious.*

$\Longrightarrow$) *Suppose there exists a neighborhood $N_x \in \mathcal{N}(x)$ that contains a finite number of elements $\{x_1, x_2, \ldots, x_n\}$ of A. Then, $W = N_x \setminus \{x_1, x_2, \ldots, x_n\}$ is a neighborhood of x and $(W \setminus \{x\}) \cap A = \emptyset$. Therefore, x is not an accumulation point of A.*

Remark 3.14. *It follows from the previous proposition that any finite subset of a separated topological space has no accumulation points.*

3.7 Induced topology, Product topology

3.7.1 Induced topology

Definition 3.20. *Let $(\mathbb{X}, \mathcal{T})$ a topological space and A a subset of $\mathbb{X}$. Then,*

$$\mathcal{T}_A = \{O_A = A \cap O : O \in \mathcal{T}\}, \tag{3.13}$$

is a topology in A. The open sets in A are the intersections of open sets in $\mathbb{X}$ with A. This topology is called the induced topology or relative topology of A in $\mathbb{X}$, and $(A, \mathcal{T})$ is *called a topological subspace of* $(\mathbb{X}, \mathcal{T})$.

Exercise. *Show that* $\mathcal{T}_A$ *is a topology on* A.

Example 3.24. *Consider the following topology*

$$\mathcal{T} = \{\mathbb{X}, \emptyset, \{x\}, \{z,t\}, \{x,z,t\}, \{y,z,t,s\}\}$$

on $\mathbb{X} = \{x,y,z,t,e\}$ *and the subset* $A = \{x,t,s\}$ *of* $\mathbb{X}$. *Then we have:* $\mathbb{X} \cap A = A$, $\emptyset \cap A = \emptyset$, $\{x\} \cap A = \{x\}$, $\{z,t\} \cap A = \{t\}$, $\{x,z,t\} \cap A = \{x,t\}$, *and* $\{y,z,t,s\} \cap A = \{t,s\}$. *Thus, the topology induced by* $\mathcal{T}$ *on* A *is*

$$\mathcal{T}_A = \{A, \emptyset, \{x\}, \{t\}, \{x,t\}, \{t,s\}\}.$$

Example 3.25. *Consider the usual topology on* $\mathbb{R}$ *and the induced topology* $\mathcal{T}_A$ *on the closed interval* $A = [4,9]$. *Note that the half-open interval* $[4,6[$ *is an open set in the topology* $\mathcal{T}_A$ *because* $[4,6[=]3,6[\cap A$, *where* $]3,6[$ *is an open set in* $\mathbb{R}$. *Thus, we see that a set can be open relative to a subspace but neither open nor closed in the entire space.*

Example 3.26. *Consider the usual topology on* $\mathbb{R}$ *and the induced topology* $\mathcal{T}_A$ *on* $A = \mathbb{N}$. *Then, for any* $n \in \mathbb{N}$, *we have* $\mathbb{N} \cap (n-1, n+1) = \{n\} \in \mathcal{T}_A$. *We conclude that* $(\mathbb{N}, \mathcal{T}_{\mathbb{N}} = \mathcal{P}(\mathbb{N}))$ *is a discrete space.*

Proposition 3.24. *Let* $(A, \mathcal{T}_A)$ *be a subspace of a topological space* $(\mathbb{X}, \mathcal{T})$, *and let* F' *be a subset of* A. *Then,* F' *is closed in* A *with respect to the induced topology* $\mathcal{T}_A$ *if and only if there exists* $F \in \mathcal{F}$ *(where* $\mathcal{F}$ *is the set of closed sets in* $\mathbb{X}$*) such that* $F' = A \cap F$.

Proof. *We have that* F' *is closed in* A *if and only if* $\complement_A F'$ *is open in* A, *i.e., if and only if there exists* $O \in \mathcal{T}$ *such that* $\complement_A F' = A \cap O$. *Therefore,* F' *is closed in* A *if and only if there exists* $O \in \mathcal{T}$ *such that*

$$F' = \complement_A(\complement_A F') = \complement_A(A \cap O) = A \cap (\complement_{\mathbb{X}} O),$$

i.e., if and only if there exists $F = \complement_{\mathbb{X}} O \in \mathcal{F}$ *such that* $F' = A \cap F$.

We can easily show the following result.

Proposition 3.25. *Let $(A, \mathcal{T}_A)$ be a subspace of a topological space $(\mathbb{X}, \mathcal{T})$, and let B be a subset of A. If B is open (resp. closed) in $\mathbb{X}$, then B is open (resp. closed) in A.*

Proof. *It suffices to see that $B = B \cap A$.*

Remark 3.15. *The two examples (3.25) and (3.26) show that the converse of the previous result is not necessarily true.*

Proposition 3.26. *Let $(A, \mathcal{T}_A)$ be a subspace of a topological space $(\mathbb{X}, \mathcal{T})$. Then every open (resp. closed) set in A is an open (resp. closed) set in $\mathbb{X}$ if and only if A is an open (resp. closed) set in $\mathbb{X}$.*

Proof.

$\Longrightarrow$) *Suppose that every open set in A is an open set in $\mathbb{X}$, then A is an open set in $\mathbb{X}$.*

$\Longleftarrow$) *Suppose that A is an open set in $\mathbb{X}$ and let O_A be an open set in A. Then there exists $O \in \mathcal{T}$ such that $O_A = A \cap O$, which is an open set in $\mathbb{X}$ since $A \in \mathcal{T}$.*
By similar arguments, this result can be shown for closed sets.

Proposition 3.27.

1. *If $x \in A$, then N' is a neighborhood of x in A if and only if there exists $N \in \mathcal{N}(x)$ such that $N' = N \cap A$.*
2. *If $\mathcal{S}(x)$ is a neighborhood base of x in $\mathbb{X}$, then $\{N \cap A : N \in \mathcal{S}(x)\}$ is neighborhood base of x in A for the induced topology $\mathcal{T}_A$.*
3. *If B is a subset of A, then we have:*
   - ***a**) $Cl(B)_A = A \cap Cl(B)$ (where $Cl(B)_A$ and $Cl(B)$ are the closures of B for $\mathcal{T}_A$ and $\mathcal{T}$, respectively).*
   - ***b**) $Cl(B)_A = Cl(B) \Longleftrightarrow A$ is closed in $\mathbb{X}$.*
   - ***c**) $A \cap Int(B) \subset Int(B)_A$*
4. *If $\mathfrak{B}$ is a base for $(\mathbb{X}, \mathcal{T})$, then $\mathfrak{B}_A = \{\beta \cap A : \beta \in \mathfrak{B}\}$ is a base for $(A, \mathcal{T}_A)$.*

Proof.

1. *If N' is a neighborhood of x in A, then there exists an open set $A\cap O\in\mathcal{T}_A$ (i.e., there exists $O\in\mathcal{T}$) such that $x\in A\cap O\subset N'$. Thus, if we define $N=O\cup N'$, we obtain $x\in O\subset N$, so N is a neighborhood of x in $\mathbb{X}$, and we have:*

$$N\cap A=(O\cup N')\cap A=(O\cap A)\cup(N'\cap A)=(O\cap A)\cup N'=N'.$$

Conversely, if $N\in\mathcal{N}(x)$, then there exists $O\in\mathcal{T}$ such that $x\in O\subset N$. Thus, $x\in A\cap O\subset A\cap N$, and therefore $N'=A\cap N$ is a neighborhood of x in A because $A\cap O$ is open in A.

2. *Let $N'=N\cap A$ be a neighborhood of x in A for the induced topology $\mathcal{T}_A$, with N being a neighborhood of x in $\mathbb{X}$. If $\mathcal{S}(x)$ is a neighborhood base of x in $\mathbb{X}$, then there exists $W\in\mathcal{S}(x)$ such that $W\subset N$, so $W\cap A\subset N'$. This leads to the conclusion that $\{N\cap A : N\in\mathcal{S}(x)\}$ is neighborhood base of x in A.*

3. ***a**) If $x\in Cl(B)_A$, then for every $N\in\mathcal{N}(x)$ (for the topology $\mathcal{T}$), we have $(N\cap A)\cap B\neq\emptyset$, and therefore $x\in A$ and $x\in Cl(B)$, from which we obtain*

   (i) $$x\in A\cap Cl(B).$$

   On the other hand, if $x\in A\cap Cl(B)$, then every neighborhood $N\cap A$ of x in A intersects B because N intersects B and $B\subset A$, from which we obtain

   (ii) $$x\in Cl(B)_A.$$

   Finally, from (i) and (ii), we conclude that $Cl(B)_A=A\cap Cl(B)$.

   ***b**) Suppose that for every subset B of A, we have $Cl(B)_A=Cl(B)$, then $A=Cl(A)_A=Cl(A)$ because A is closed in A, hence A is closed in $\mathbb{X}$.*

   Conversely, if A is closed in $\mathbb{X}$, then $Cl(B)\subset Cl(A)=A$, and thus $Cl(B)_A=A\cap Cl(B)=Cl(B)$.

   ***c**) We have $A\cap Int(B)$ is an open set in A contained in B, so $A\cap Int(B)\subset Int(B)_A$.*

4. *Let U be an open set of A, then there exists $O\in\mathcal{T}$ such that $U=A\cap O$, but $O=\bigcup_{i\in I}\beta_i$, where $\beta_i\in\mathfrak{B}$ for all $i\in I$, from which we obtain*

$$U=A\cap\left(\bigcup_{i\in I}\beta_i\right)=\bigcup_{i\in I}(A\cap\beta_i),$$

which completes the proof.

Definition 3.21. *A topological property is hereditary if whenever a topological space possesses this property, it also holds for each of its sub-spaces.*

Proposition 3.28. *Every subspace of a separated space is separated.*

Proof. *Let $(A, \mathcal{T}_A)$ be a topological subspace of a separated topological space $(\mathbb{X}, \mathcal{T})$, and let $x, y \in A$ such that $x \neq y$. Since $\mathbb{X}$ is separated, there exist neighborhoods $N \in \mathcal{N}(x)$ and $W \in \mathcal{N}(y)$ such that $N \cap W = \emptyset$, hence $(A \cap N) \cap (A \cap W) = \emptyset$. Therefore, $(A \cap N)$ and $(A \cap W)$ are disjoint neighborhoods of x and y, respectively, within A, which shows that A is separated.*

The following result shows the transitivity of the induced topology.

Proposition 3.29. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space and $B \subset A \subset \mathbb{X}$ two subsets of $\mathbb{X}$. We denote by $\mathcal{T}'_B$ the topology induced on B by $\mathcal{T}_A$. Then, we have*

$$\mathcal{T}_B = \mathcal{T}'_B.$$

Proof. *If $U \in \mathcal{T}_B$, then there exists $O \in \mathcal{T}$ such that $U = B \cap O$, and since $A \cap O \in \mathcal{T}_A$, we obtain $U = B \cap O = B \cap (A \cap O) \in \mathcal{T}'_B$.*

Conversely, if $U \in \mathcal{T}'_B$, then there exists $O_A \in \mathcal{T}_A$ such that $U = B \cap O_A$, and since $O_A \in \mathcal{T}_A$, there exists $O \in \mathcal{T}$ such that $O_A = A \cap O$. Thus, $U = B \cap (A \cap O) = B \cap O$, and therefore $U \in \mathcal{T}_B$.

3.7.2 Product topology

Definition 3.22. *Let $\{(\mathbb{X}_i, \mathcal{T}_i) : i = 1, \ldots, n\}$ be a collection of topological spaces. The box topology or product topology on the product $\mathbb{X} = \prod_{i=1}^{n} \mathbb{X}_i$ is the topology with basis*

$$\mathcal{B} = \left\{ \prod_{i=1}^{n} O_i \ : \ O_i \in \mathcal{T}_i \text{ for each } 1 \leqslant i \leqslant n \right\}.$$

So we can always make the product of topological space into a topological space using the box topology.

Proof.

1. *We have* $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2 \times \cdots \times \mathbb{X}_n \in \mathcal{T}$ *and* $\underbrace{\emptyset \times \emptyset \times \cdots \times \emptyset}_{n \text{ times}} \in \mathcal{T}$ *because they are elements of* $\mathcal{B}$.

2. *If* $\{O_i : i \in I\}$ *is a family of open subsets of* $\mathbb{X}$*, then we have:*

$$\bigcup_{i\in I} O_i = \bigcup_{i\in I}\left(\bigcup_{j\in J}\left(O^1_{i,j} \times O^2_{i,j} \times \cdots \times O^n_{i,j}\right)\right) = \bigcup_{(i,j)\in I\times J}\left(O^1_{i,j} \times O^2_{i,j} \times \cdots \times O^n_{i,j}\right) \in \mathcal{T},$$

because $O^1_{i,j} \times O^2_{i,j} \times \cdots \times O^n_{i,j} \in \mathcal{B}$ *for all* $i \in I$ *and* $j \in J$.

3. *It suffices to show that if* $O_1, O_2 \in \mathcal{T}$*, then* $O_1 \cap O_2 \in \mathcal{T}$*. Since* $O_1, O_2 \in \mathcal{T}$*, we have* $O_1 = \bigcup\limits_{i\in I} N_i$ *and* $O_2 = \bigcup\limits_{j\in J} W_j$ *where* N_i*,* $W_j \in \mathcal{B}$ *for all* $i \in I$ *and* $j \in J$*. Therefore, we obtain:*

$$O_1 \cap O_2 = \left(\bigcup_{i\in I} N_i\right) \cap \left(\bigcup_{j\in J} W_j\right) = \bigcup_{(i,j)\in I\times J} (N_i \cap W_j).$$

It remains to show that $N_i \cap W_j \in \mathcal{B}$ *for all* $i \in I$ *and* $j \in J$*. By definition, we have* $N_i = R^i_1 \times \cdots \times R^i_n$ *and* $W_j = K^j_1 \times \cdots \times K^j_n$ *where* $R^i_\alpha \in \mathcal{T}_\alpha$ *and* $K^j_\alpha \in \mathcal{T}_\alpha$ *for all* $\alpha = 1, \ldots, n$*. This allows us to write:*

$$N_i \cap W_j = (R^i_1 \cap K^j_1) \times (R^i_2 \cap K^j_2) \times \cdots \times (R^i_n \cap K^j_n).$$

Since $R^i_\alpha \cap K^j_\alpha$ *are open sets in* $\mathcal{T}_\alpha$ *for all* $\alpha = 1, \ldots, n$*, we deduce that* $N_i \cap W_j \in \mathcal{B}$ *for all* $i \in I$ *and* $j \in J$*, which implies that* $O_1 \cap O_2 \in \mathcal{T}$*. Finally, we conclude that* $\mathcal{T}$ *is a topology on* $\mathbb{X}$.

Example 3.27.

1. *The* box topology *or* product topology *on* $\mathbb{R}^n$*, such that* $\mathbb{R}$ *is equipped with the usual topology, is the topology with basis*

$$\mathcal{B} = \left\{\prod_{i=1}^{n}]a_i, b_i[\ : \ a_i, b_i \in \mathbb{R} \text{ for each } 1 \leqslant i \leqslant n\right\}.$$

2. *Let* $\{(\mathbb{X}_i, \mathcal{T}_i) : i = 1, \ldots, n\}$ *be a family of indiscrete spaces. Then, the product* $\mathbb{X} = \prod\limits_{i=1}^{n} \mathbb{X}_i$ *is an indiscrete space. Indeed, if* $O = \prod\limits_{i=1}^{n} O_i \neq \mathbb{X}$*, then there exists an index* i_0 *such that* $O_{i_0} \neq \mathbb{X}_{i_0}$*. Since* $\mathcal{T}_{i_0} = \{\mathbb{X}_{i_0}, \emptyset\}$*, we obtain* $O_{i_0} = \emptyset$*, and hence* $O = \emptyset$*. Therefore, the family* $\{\mathbb{X}, \emptyset\}$ *forms a basis for the product topology on* $\mathbb{X}$*, which shows that* $\mathbb{X}$ *is an indiscrete space.*

Proposition 3.30. *Let $\mathbb{X} = \prod_{i=1}^{n} \mathbb{X}_i$ be a product of topological spaces, and let $x = (x_1, \dots, x_n) \in \mathbb{X}$. Let $\mathcal{S}$ denote the family of sets of the form $N_1 \times \cdots \times N_n$, where $N_i \in \mathcal{N}(x_i)$ in $\mathbb{X}_i$ for $i = 1, \dots, n$. Then, $\mathcal{S}$ is a basic neighborhoods of x in $\mathbb{X}$.*

Proof.

If $N_i \in \mathcal{N}(x_i)$, then there exists $O_i \in \mathcal{T}_i$, for all $i = 1, \dots, n$, such that $x_i \in O_i \subset V_i$. Therefore, we obtain $x \in O_1 \times \cdots \times O_n \subset N_1 \times \cdots \times N_n$, and since $O_1 \times \cdots \times O_n$ is an open set in $\mathbb{X}$, we conclude that $N_1 \times \cdots \times N_n$ is a neighborhood of x in $\mathbb{X}$.

Now, let $N \in \mathcal{N}(x)$ in $\mathbb{X}$. Then, there exists an open set $O \subset \mathbb{X}$ such that $x \in O \subset N$. Thus, there exists $W = O_1 \times \cdots \times O_n$ an open set containing x (since $\mathcal{B}$ is a basis for the product topology on $\mathbb{X}$ (see Definition 3.22)). Hence, $W \in \mathcal{S}$ because $O_i \in \mathcal{N}(x_i)$ for all $i = 1, \dots, n$, which implies that $W \subset N$.

Example 3.28. *Let $\mathbb{R}^n$ be equipped with the usual topology, and let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. The family*

$$\left\{ \prod_{i=1}^{n} (x_i - \varepsilon_i, x_i + \varepsilon_i) \ : (\varepsilon_1, \dots, \varepsilon_n) \in (\mathbb{R}_+^*)^n \right\},$$

is a basic neighborhoods of x. Similarly, the family

$$\left\{ \prod_{i=1}^{n} (x_i - \varepsilon, x_i + \varepsilon) \ : \varepsilon \in \mathbb{R}_+^* \right\},$$

is also a basic neighborhoods of x.

Proposition 3.31. *Consider $A = \prod_{i=1}^{n} A_i$, a subset of the product space $\mathbb{X} = \prod_{i=1}^{n} \mathbb{X}_i$. The closure of A, denoted by $Cl(A)$, is given by:*

$$Cl(A) = \prod_{i=1}^{n} Cl(A_i).$$

Proof. *Let $x = (x_1, \dots, x_n) \in \mathrm{Cl}(A)$. Then, for every $N_i \in \mathcal{N}(x_i)$, we have:*

$$(N_1 \cap A_1) \times \cdots \times (N_n \cap A_n) = (N_1 \times \cdots \times N_n) \cap A \neq \emptyset,$$

which implies $N_i \cap A_i \neq \emptyset$ for all $i = 1, \dots, n$. Thus, $x_i \in Cl(A_i)$ for all $i = 1, \dots, n$, showing that $x \in \prod_{i=1}^{n} Cl(A_i)$.

Conversely, if $x \in \prod\limits_{i=1}^{n} Cl(A_i)$*, then for every* $N_i \in \mathcal{N}(x_i)$*,* $i = 1, \ldots, n$*, we have* $N_i \cap A_i \neq \emptyset$*. Therefore,*

$$(N_1 \cap A_1) \times \cdots \times (N_n \cap A_n) = (V_1 \times \cdots \times N_n) \cap A \neq \emptyset,$$

which shows that $x \in Cl(A)$*.*

Using the previous proposition, we obtain the following result.

Proposition 3.32. *Let* $A = \prod\limits_{i=1}^{n} A_i$ *be a subset of a product space* $\mathbb{X} = \prod\limits_{i=1}^{n} \mathbb{X}_i$*. Then* A *is closed in* $\mathbb{X}$ *if and only if* A_i *is closed in* $\mathbb{X}_i$ *for every* $i = 1, \ldots, n$*.*

Proposition 3.33. *A product space* $\mathbb{X} = \prod\limits_{i=1}^{n} \mathbb{X}_i$ *is Hausdorff if and only if each* $\mathbb{X}_i$ *is Hausdorff for every* $i = 1, \ldots, n$*.*

Proof.

$\Longrightarrow$) *Suppose that* $\mathbb{X} = \prod\limits_{i=1}^{n} \mathbb{X}_i$ *is Hausdorff, and let* $x_{i_0}, y_{i_0} \in \mathbb{X}_{i_0}$ *such that* $x_{i_0} \neq y_{i_0}$*. For any* $x' = (x_1, ..., x_{i_0-1}, x_{i_0+1}, ..., x_n) \in \prod\limits_{\substack{i=1 \\ i \neq i_0}}^{n} \mathbb{X}_i$*, there exists a neighborhood* O *of* $(x_1, \ldots, x_{i_0}, \ldots, x_n)$ *and a neighborhood* O' *of* $(x_1, \ldots, y_{i_0}, \ldots, x_n)$ *such that* $O \cap O' = \emptyset$*. Let* $O = N_1 \times N_2$ *and* $O' = N'_1 \times N'_2$*, where* $N_1 \in \mathcal{N}(x_{i_0})$*,* $N_2 \in \mathcal{N}(x')$*,* $N'_1 \in \mathcal{N}(y_{i_0})$*, and* $N'_2 \in \mathcal{N}(x')$*. Thus, we obtain:*

$$O \cap O' = (N_1 \cap N'_1) \times (N_2 \cap N'_2) = \emptyset \Longrightarrow N_1 \cap N'_1 = \emptyset,$$

and therefore $\mathbb{X}_{i_0}$ *is Hausdorff.*

$\Longleftarrow$) *Let* $x = (x_1, \ldots, x_n)$*,* $y = (y_1, \ldots, y_n) \in \mathbb{X} = \prod\limits_{i=1}^{n} \mathbb{X}_i$ *such that* $x \neq y$*. Then there exists at least one* $i_0 \in \{1, \ldots, n\}$ *such that* $x_{i_0} \neq y_{i_0}$*. Since* $\mathbb{X}_{i_0}$ *is Hausdorff, there exist a neighborhood* N *of* x_{i_0} *and a neighborhood* W *of* y_{i_0} *such that* $V \cap W = \emptyset$*. By setting* $O_x = \mathbb{X}_1 \times \cdots \times \mathbb{X}_{i_0-1} \times N \times \mathbb{X}_{i_0+1} \times \cdots \times \mathbb{X}_n$ *and* $O_y = \mathbb{X}_1 \times \cdots \times \mathbb{X}_{i_0-1} \times W \times \mathbb{X}_{i_0+1} \times \cdots \times \mathbb{X}_n$*, we obtain* $O_x \in \mathcal{N}(x)$*,* $O_y \in \mathcal{N}(y)$*, and* $O_x \cap O_y = \emptyset$*, which shows that* $\mathbb{X}$ *is Hausdorff.*

3.8 Convergent sequences

Definition 3.23. *A "sequence of elements" of a set $\mathbb{X}$ is defined as any function from $\mathbb{N}$ (or a subset of $\mathbb{N}$) into $\mathbb{X}$, which associates with each integer n in $\mathbb{N}$ an element of $\mathbb{X}$ denoted by x_n. The sequence with general term x_n is denoted by $(x_n)_{n\in\mathbb{N}}$.*

Definition 3.24. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. Consider a sequence $(x_n)_{n\in\mathbb{N}}$ of elements in $\mathbb{X}$ and a point $l \in \mathbb{X}$. We say that l is the limit of the sequence $(x_n)_{n\in\mathbb{N}}$ (or that $(x_n)_{n\in\mathbb{N}}$ converges to l) as n tends to infinity, if for every neighborhood N of l in $\mathbb{X}$, there exists an integer n_0 such that $x_n \in N$ for all $n \geq n_0$. In other words,*

$$\forall N \in \mathcal{N}(l),\ \exists n_0 \in \mathbb{N} \ :\ \forall n \in \mathbb{N},\ n \geqslant n_0 \Rightarrow x_n \in N.$$

In this case, we write:

$$\lim_{n\to\infty} x_n = l.$$

A sequence that does not converge is called divergent.

Example 3.29.

1. *Every constant sequence is convergent in all topological spaces.*
2. *A sequence in an indiscrete space is convergent to every point of that space.*
3. *If $(\mathbb{X}, \mathcal{T})$ is a discrete space, then a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{X}$ converges to l if and only if there exists n_0 such that $x_n = l$ for all $n \geqslant n_0$.*
4. *The sequence (x_n) of the general term $x_n = \dfrac{1}{n}$ is convergent to 0 in $(\mathbb{R}, |\cdot|)$, and it is divergent in $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$.*

Proposition 3.34. *If $(\mathbb{X}, \mathcal{T})$ is a Hausdorff topological space, then every convergent sequence in $\mathbb{X}$ has a unique limit.*

Proof. *Let us reason by contradiction. Let (x_n) be a convergent sequence in $\mathbb{X}$. Suppose it has two distinct limits $l_1 \neq l_2$. Since $(\mathbb{X}, \mathcal{T})$ is a Hausdorff space, there exist neighborhoods $N_1 \in \mathcal{N}(l_1)$ and $N_2 \in \mathcal{N}(l_2)$ such that $N_1 \cap N_2 = \emptyset$. According to the definition (3.24), there exist integers n_1 and n_2 such that:*

$$\forall n \geqslant n_1, x_n \in N_1 \quad \text{and} \quad \forall n \geqslant n_2, x_n \in N_2.$$

Let $n_0 = \max(n_1, n_2)$. *Then, for all* $n \geqslant n_0$, *we have*

$$x_n \in N_1 \cap N_2,$$

which contradicts the fact that $N_1 \cap N_2 = \emptyset$. *Therefore,* $l_1 = l_2$.

Example 3.30. *The trivial topology (indiscrete topology) on a set* $\mathbb{X}$ *is a non-Hausdorff topology because every element* $x \in \mathbb{X}$ *has only one neighborhood, namely* $\mathbb{X}$ *itself. Therefore, if* $(x_n)_{n\in\mathbb{N}}$ *is a sequence in* $\mathbb{X}$, *every point* $x \in \mathbb{X}$ *is a limit for this sequence. Hence, the limit is not unique.*

Definition 3.25. *A cluster point or accumulation point of a sequence* $(x_n)_{n\in\mathbb{N}}$ *in a topological space* $(\mathbb{X}, \mathcal{T})$ *is a point* x *such that, for every neighborhood* N *of* x, *there are infinitely many natural numbers* n *such that* $x_n \in N$.

Remark 3.16. *According to the previous definition, we conclude that the limit of a sequence is an accumulation (cluster point) point of this sequence.*

Example 3.31.

1. *In* $(\mathbb{R}, |.|)$, $x = 1$ *is the unique accumulation point (cluster point) of the sequence* $(x_n)_{n\in\mathbb{N}} = (1 + e^{-n})_{n\in\mathbb{N}}$, *and this value is the limit of the sequence. Moreover,* $x_n = 1 + e^{-n}$ *is an adherent point for every* $n \in \mathbb{N}$, *but it is not an accumulation point (cluster point).*

2. *In* $(\mathbb{R}, |.|)$, *the sequence* $(x_n)_{n\in\mathbb{N}} = ((-1)^n)_{n\in\mathbb{N}}$ *has two accumulation points (cluster points),* -1 *and* 1, *but it is a divergent sequence.*

According to the previous example and the definition (3.25), we conclude that every accumulation point is an adherent point, but the converse is not true.

Proposition 3.35. *If* $(\mathbb{X}, \mathcal{T})$ *is a Hausdorff (separated) topological space, then every convergent sequence in* $\mathbb{X}$ *has a unique accumulation point (cluster points), which is its limit.*

Proof. *By arguments similar to those used in the proof of the previous proposition.*

Remark 3.17.

1. *A sequence that has at least two accumulation points diverges.*

2. *The converse of the previous proposition is false. For example, the sequence defined by* $x_n = (1-(-1)^n) \times n$ *has only* 0 *as an accumulation point but diverges.*

Definition 3.26. *Let* (x_n) *be a sequence in a topological space* $(\mathbb{X}, \mathcal{T})$. *We call a subsequence or extracted sequence of* (x_n) *any sequence of the form* $(x_{\phi(n)})$, *where* $\phi(n)$ *is a strictly increasing function from* $\mathbb{N}$ *to* $\mathbb{N}$.

Example 3.32. *If* (x_n) *is a sequence in a topological space* $(\mathbb{X}, \mathcal{T})$ *and* $\phi(n) = 2n+1$, *then* $(x_{2n+1}) = \{x_1, x_3, x_5, x_7, \ldots, x_{2n+1}, \ldots\}$ *is a subsequence of* (x_n).

Using the definitions (3.24) and (3.25), we obtain the following two results.

Proposition 3.36.

1. *Every subsequence of a convergent sequence is convergent (towards the same limit).*

2. *The limit of a subsequence extracted from a sequence* (x_n) *is a cluster point of this sequence.*

Proposition 3.37. *Let* $(z_n) = \{z_n^1, z_n^2, \ldots, z_n^k\}$ *be a sequence in a space* $\mathbb{X} = \prod_{i=1}^{k} \mathbb{X}_i$. *Then,* (z_n) *converges to* $z = (z^1, z^2, \ldots, z^k)$ *if and only if for all* $i = 1, \ldots, k$, *the sequence* (z_n^i) *converges in* $\mathbb{X}$ *to* z^i.

Proof.

$\Longrightarrow$) *Suppose that* $(z_n) = \{z_n^1, z_n^2, \ldots, z_n^k\}$ *converges in* $\mathbb{X}$ *to* $z = (z^1, z^2, \ldots, z^k)$. *Let* N_i *be a neighborhood of* z_i *in* $\mathbb{X}_i$, *for* $i = 1, \ldots, k$. *Then,* $W = \mathbb{X}_1 \times \cdots \times \mathbb{X}_{i-1} \times N_i \times \mathbb{X}_{i+1} \times \cdots \times \mathbb{X}_k$ *is a neighborhood of* z, *so there exists* $n_0 \in \mathbb{N}$ *such that*

$$n \geqslant n_0 \Longrightarrow z_n \in W.$$

Consequently, we obtain:

$$n \geqslant n_0 \Longrightarrow z_n^i \in N_i,$$

which shows that for all $i = 1, \ldots, k$, the sequence (z_n^i) converges to z^i in $\mathbb{X}_i$.

$\Longleftarrow$) *Suppose that for all $i = 1, \ldots, k$, the sequence (z_n^i) converges to z^i in $\mathbb{X}_i$. Let W be a neighborhood of $z = (z^1, z^2, \ldots, z^k)$ in $\mathbb{X} = \prod_{i=1}^{k} \mathbb{X}_i$. According to proposition (3.30), W contains a neighborhood of the form $N_1 \times \cdots \times N_k$, where N_i is a neighborhood of z^i in $\mathbb{X}_i$ for all $i = 1, \ldots, k$. Thus, for all $i = 1, \ldots, k$, and for all $N_i \in \mathcal{N}(z^i)$, there exists n_0^i such that:*

$$n \geqslant n_0^i \Longrightarrow z_n^i \in N_i.$$

If we set $n_0 = \max(n_0^1, \ldots, n_0^k)$, we obtain:

$$n \geqslant n_0 \Longrightarrow z_n \in N_1 \times \cdots \times N_k,$$

which leads to:

$$n \geqslant n_0 \Longrightarrow z_n \in W.$$

Therefore, z_n is a sequence converging to z in $\mathbb{X}$.

Proposition 3.38. *If $x = (x^1, \ldots, x^k)$ is a cluster point of (z_n) in $\mathbb{X} = \prod_{i=1}^{k} \mathbb{X}_i$, then x^i is a cluster point of (z_n^i) for all $i = 1, \ldots, k$.*

Proof. *Let $N_i \in \mathcal{N}(x^i)$ for all $i = 1, \ldots, k$, then $W = \mathbb{X}_1 \times \cdots \times \mathbb{X}_{i-1} \times N_i \times \mathbb{X}_{i+1} \times \cdots \times \mathbb{X}_k$ is a neighborhood of x in $\mathbb{X}$. Consequently, we obtain:*

$$card\{n \in \mathbb{N} : z_n \in W\} = +\infty,$$

which leads to:

$$card\left\{n \in \mathbb{N} : z_n^i \in N_i\right\} = +\infty,$$

from which it follows that x^i is a cluster point of (z_n^i) for all $i = 1, \ldots, k$.

The previous result is generally false. For example, in $\mathbb{R}^2$, if we take the sequence $z_n = (x_n, y_n)$ defined by the following relations:

$$\begin{cases} x_{2n} & = n \\ x_{2n+1} & = \frac{1}{n} \end{cases}, \quad \begin{cases} y_{2n} & = \frac{1}{n} \\ y_{2n+1} & = n \end{cases}$$

It is clear that 0 is a cluster point of (x_n) and (y_n), but $(0,0)$ is not a cluster point of (z_n).

3.9 Continuous applications

Definition 3.27 (Pointwise continuity). *Let $(\mathbb{X},\mathcal{T}_{\mathbb{X}})$ and $(\mathbb{Y},\mathcal{T}_{\mathbb{Y}})$ be two topological spaces. We say that a function $f:\mathbb{X}\longrightarrow\mathbb{Y}$ is continuous at $x_0\in\mathbb{X}$ if and only if for every neighborhood $N\in\mathcal{N}_{\mathbb{Y}}(f(x_0))$, there exists $U\in\mathcal{N}_{\mathbb{X}}(x_0)$ such that $f(U)\subseteq N$. In other words,*

$$\forall N\in\mathcal{N}_{\mathbb{Y}}(f(x_0)),\exists U\in\mathcal{N}_{\mathbb{X}}(x_0),\ f(U)\subseteq N\Longleftrightarrow f \text{ is continuous at } x_0. \tag{3.14}$$

Using the preimage, we obtain $U\subseteq f^{-1}(N)$, hence $f^{-1}(N)$ is a neighborhood of x_0. Therefore, we can write the previous definition in the following form.

Definition 3.28. *Let $(\mathbb{X},\mathcal{T}_{\mathbb{X}})$ and $(\mathbb{Y},\mathcal{T}_{\mathbb{Y}})$ be two topological spaces. We say that a function $f:\mathbb{X}\longrightarrow\mathbb{Y}$ is continuous at $x_0\in\mathbb{X}$ if and only if the preimage of any neighborhood of $f(x_0)$ in $\mathbb{Y}$ is a neighborhood of x_0 in $\mathbb{X}$. In other words,*

$$\forall N\in\mathcal{N}_{\mathbb{Y}}(f(x_0)),\ f^{-1}(N)\in\mathcal{N}_{\mathbb{X}}(x_0) \tag{3.15}$$

Remark 3.18. *In both previous definitions, we can replace $\mathcal{N}_{\mathbb{X}}(x_0)$ and $\mathcal{N}_{\mathbb{Y}}(f(x_0))$ with the basic neighborhoods of x_0 and $f(x_0)$.*

Example 3.33.

1. *The function $f:(\mathbb{R},|.|)\longrightarrow(\mathbb{R},\mathcal{P}(\mathbb{R}))$ such that for all $x\in\mathbb{R}$, $f(x)=x$, is not continuous on $\mathbb{R}$, because $N=\{x\}$ is a neighborhood of x in $(\mathbb{R},\mathcal{P}(\mathbb{R}))$, but $f^{-1}(N)=\{x\}$ is not a neighborhood of x in $(\mathbb{R},|.|)$.*

2. *Let $\mathbb{X}=\{x_1,x_2,x_3,x_4\}$ and $\mathcal{T}_{\mathbb{X}}=\{\emptyset,\mathbb{X},\{x_1\},\{x_2\},\{x_1,x_2\},\{x_2,x_3,x_4\}\}$, and let $\mathbb{Y}=\{y_1,y_2,y_3,y_4\}$ and $\mathcal{T}_{\mathbb{Y}}=\{\emptyset,\mathbb{Y},\{y_1\},\{y_1,y_2\},\{y_1,y_2,y_3\}\}$. We define the function $f:\mathbb{X}\longrightarrow\mathbb{Y}$ by $f(x_4)=y_4$, $f(x_3)=y_2$, and $f(x_1)=f(x_2)=y_1$.*
   - *For example, we have $\mathcal{N}_{\mathbb{Y}}(f(x_4))=\mathcal{N}_{\mathbb{Y}}(y_4)=\{\mathbb{Y}\}$, and $f^{-1}(\mathbb{Y})=\mathbb{X}\in\mathcal{N}_{\mathbb{X}}(x_4)$, so f is continuous at x_4.*
   - *We also have $\mathcal{N}_{\mathbb{Y}}(y_2)=\{\{y_1,y_2\},\{y_1,y_2,y_3\},\{y_1,y_2,y_4\},\mathbb{Y}\}$. If we take $N=\{y_1,y_2\}$, we obtain $f^{-1}(N)=\{x_1,x_2,x_3\}\notin\mathcal{N}_{\mathbb{X}}(x_3)$, so f is not continuous at x_3.*

Proposition 3.39 (Transitivity of continuity). *Let $\mathbb{X}, \mathbb{Y}$ and $\mathbb{T}$ be three topological spaces. Consider the two functions $f : \mathbb{X} \longrightarrow \mathbb{Y}$ and $g : \mathbb{Y} \longrightarrow \mathbb{T}$. If f is continuous at a point $x_0 \in \mathbb{X}$ and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.*

Proof. *Let $W \in \mathcal{N}_{\mathbb{T}}(g \circ f(x_0))$. Since g is continuous at $f(x_0)$, there exists $N \in \mathcal{N}_{\mathbb{Y}}(f(x_0))$ such that $g(N) \subseteq W$, and since f is continuous at x_0, there exists $U \in \mathcal{N}_{\mathbb{X}}(x_0)$ such that $f(U) \subseteq N$. From this, we deduce that $g \circ f(U) \subseteq W$, which implies that $g \circ f$ is continuous at x_0.*

Remark 3.19. *The converse in the previous proposition is not always true.*
Consider the function f as shown in example (3.33(2)) and let $g : (\mathbb{Y}, \mathcal{T}_{\mathbb{Y}}) \longrightarrow (\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ be a function defined as follows: $g(y_4) = x_4$, $g(y_3) = x_1$, $g(y_2) = x_3$, $g(y_1) = x_2$. On one hand, we have $\mathcal{N}_{\mathbb{X}}(g(f(x_4))) = \mathcal{N}_{\mathbb{X}}(g(y_4)) = \mathcal{N}_{\mathbb{X}}(x_4) = \{\{x_2, x_3, x_4\}, \mathbb{X}\}$. But, $g^{-1}(\{x_2, x_3, x_4\}) = \{y_1, y_2, y_4\} \notin \mathcal{N}_{\mathbb{Y}}(y_4)$, which means that g is not continuous at $f(x_4) = y_4$. On the other hand, we have $(g \circ f)(x_4) = g(f(x_4)) = g(y_4) = x_4$ and $\mathcal{N}_{\mathbb{X}}(x_4) = \{\{x_2, x_3, x_4\}, \mathbb{X}\}$, and $(g \circ f)^{-1}(\{x_2, x_3, x_4\}) = f^{-1}(g^{-1}(\{x_2, x_3, x_4\})) = f^{-1}(\{y_1, y_2, y_4\}) = \mathbb{X} \in \mathcal{N}_{\mathbb{X}}(x_4)$. Since $(g \circ f)^{-1}(\mathbb{X}) = \mathbb{X}$, we conclude that $g \circ f$ is continuous at x_4.

Proposition 3.40. *Let $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ and $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ be two topological spaces and $f : \mathbb{X} \longrightarrow \mathbb{Y}$. The following statements are equivalent.*

1. *f is continuous.*
2. *$f(Cl(A)) \subseteq Cl(f(A))$ for every subset A of $\mathbb{X}$.*
3. *$f^{-1}(F)$ is closed in $\mathbb{X}$ for every closed set F in $\mathbb{Y}$.*
4. *$f^{-1}(O)$ is open in $\mathbb{X}$ for every open set O in $\mathbb{Y}$.*
5. *$f^{-1}(\beta)$ is open in $\mathbb{X}$ for every element β of a basis $\mathfrak{B}$ of $\mathcal{T}_{\mathbb{Y}}$.*
6. *$f^{-1}(IntB) \subseteq Intf^{-1}(B)$ for every subset B of $\mathbb{Y}$.*
7. *$Cl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$ for every subset B of $\mathbb{Y}$.*

Proof.

- ***(1)$\Longrightarrow$ (2)*** *Let $a \in Cl(A)$ and $N \in \mathcal{N}_{\mathbb{Y}}(f(a))$. Then $f^{-1}(N) \in \mathcal{N}_{\mathbb{X}}(a)$ because f is continuous. Consequently, $f^{-1}(N) \cap A \neq \emptyset$. Thus, if $x \in f^{-1}(N) \cap A$, we obtain $f(x) \in N \cap f(A)$, i.e., $N \cap f(A) \neq \emptyset$. Therefore, $f(a) \in Cl(f(A))$, which shows that $f(Cl(A)) \subset Cl(f(A))$.*
- ***(2)$\Longrightarrow$ (3)*** *Let F be a closed subset of $\mathbb{Y}$. Define $A = f^{-1}(F)$, so it is sufficient to show*

that $A = Cl(A)$. *By definition, we have* $A \subseteq Cl(A)$, *and according to* ***(2)***, *we have* $f(Cl(A)) \subseteq Cl(f(A)) \subseteq Cl(F) = F$ *(since* F *is closed), hence* $Cl(A) \subseteq f^{-1}(F) = A$. *Consequently,* $A = Cl(A)$, *which shows that* $f^{-1}(F)$ *is closed in* $\mathbb{X}$.

- ***(3)*** $\Longrightarrow$ ***(4)*** *Let* O *be an open subset of* $\mathbb{Y}$, *then* $\complement_{\mathbb{Y}} O$ *is a closed set in* $\mathbb{Y}$. *Therefore, by* ***(3)***, *the set* $f^{-1}(\complement_{\mathbb{Y}} O)$ *is closed in* $\mathbb{X}$. *Since* $f^{-1}(\complement_{\mathbb{Y}} O) = \complement_{\mathbb{X}} f^{-1}(O)$, *we deduce that* $f^{-1}(O)$ *is open in* $\mathbb{X}$.
- ***(4)*** $\Longrightarrow$ ***(5)*** *Obvious.*
- ***(5)*** $\Longrightarrow$ ***(6)*** *Let* B *be a subset of* $\mathbb{Y}$. *Then,* $Int(B) = \bigcup_{i \in I} \beta_i$ *such that* $\{\beta_i : i \in I\}$ *is a family of elements from a basis* $\mathfrak{B}$ *of* $\mathcal{T}_{\mathbb{Y}}$. *Using the inverse image, we obtain*

$$f^{-1}(Int(B)) = f^{-1}\left(\bigcup_{i \in I} \beta_i\right) = \bigcup_{i \in I} f^{-1}(\beta_i).$$

Thus, $f^{-1}(Int(B))$ *is an open set in* $\mathbb{X}$ *(according to* ***(5)****), and since* $f^{-1}(Int(B)) \subseteq f^{-1}(B)$, *we conclude that* $f^{-1}(Int(B)) \subseteq Int(f^{-1}(B))$ *(see Proposition (3.12)).*

- ***(6)*** $\Longrightarrow$ ***(7)***. *Let* B *be a subset of* $\mathbb{Y}$. *Using Proposition (3.15(4)) and* (6), *we obtain:*

$$\complement_{\mathbb{X}} f^{-1}(Cl(B)) = f^{-1}(\complement_{\mathbb{Y}} Cl(B)) = f^{-1}(\mathrm{Int}\, \complement_{\mathbb{Y}} B)$$

$$\subseteq Int f^{-1}(\complement_{\mathbb{Y}} B) = Int \complement_{\mathbb{X}} f^{-1}(B) = \complement_{\mathbb{X}} Cl f^{-1}(B),$$

which shows that

$$Cl f^{-1}(B) \subseteq f^{-1}(Cl(B)).$$

- ***(7)*** $\Longrightarrow$ ***(1)*** *Let* $x_0 \in \mathbb{X}$ *and* O *be an open neighborhood of* $f(x_0)$. *Then,* $\complement_{\mathbb{Y}} O$ *is closed in* $\mathbb{Y}$. *Using* ***(7)***, *we obtain*

$$Cl f^{-1}(\complement_{\mathbb{Y}} O) \subseteq f^{-1}(Cl(\complement_{\mathbb{Y}} O)) = f^{-1}(\complement_{\mathbb{Y}} O)$$

(since $\complement_{\mathbb{Y}} O$ *is closed), and thus* $f^{-1}(\complement_{\mathbb{Y}} O) = \complement_{\mathbb{X}} f^{-1}(O)$ *is closed. Consequently,* $f^{-1}(O)$ *is open in* $\mathbb{X}$. *Finally, since* $x_0 \in f^{-1}(O)$, *we conclude that* $f^{-1}(O) \in \mathcal{N}_{\mathbb{X}}(x_0)$, *which shows that* f *is continuous.*

Proposition 3.41. *Let* $(A, \mathcal{T}_A)$ *be a subspace of a topological space* $(\mathbb{X}, \mathcal{T})$. *Then the canonical injection* $i : A \longrightarrow \mathbb{X}$ *defined by* $i(a) = a$, *for all* $a \in A$ *is continuous.*

Proof. *Let* O *be an open set in* $\mathbb{X}$. *Then* $i^{-1}(O) = O \cap A$, *which is open in* $(A, \mathcal{T}_A)$, *so* i *is continuous.*

Proposition 3.42. *Let $f : (\mathbb{X}, \mathcal{T}_{\mathbb{X}}) \longrightarrow (\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ be a continuous mapping and $A \subset \mathbb{X}$. Then the restriction $f_{|_A} : (A, \mathcal{T}_A) \longrightarrow (\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ is continuous.*

Proof. *Given that $f_{|_A} = f \circ i$, it follows that $f_{|_A}$ is continuous because it is the composition of two continuous functions.*

Proposition 3.43. *Let $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ and $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ be two topological spaces. If $f : (\mathbb{X}, \mathcal{T}_{\mathbb{X}}) \longrightarrow (\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ is continuous and injective, and $\mathbb{Y}$ is separated, then $\mathbb{X}$ is separated.*

Proof. *Let $x, y \in \mathbb{X}$ such that $x \neq y$, then $f(x) \neq f(y)$ (since f is injective), and since $\mathbb{Y}$ is separated, there exist two disjoint open sets O_1 and O_2 such that $f(x) \in O_1$ and $f(y) \in O_2$. Therefore, $f^{-1}(O_1)$ and $f^{-1}(O_2)$ are two disjoint open sets such that $x \in f^{-1}(O_1)$ and $y \in f^{-1}(O_2)$, which shows that $\mathbb{X}$ is separated.*

Definition 3.29 (Sequential Continuity). *Let $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ and $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ be two topological spaces. We say that f is sequentially continuous at x_0 if for every sequence (x_n) that converges to x_0, the sequence $f(x_n)$ converges to $f(x_0)$.*

Remark 3.20. *We say that f is continuous (resp. sequentially continuous) on $\mathbb{X}$ if it is continuous (resp. sequentially continuous) at every point of $\mathbb{X}$.*

Proposition 3.44. *Every continuous function is sequentially continuous.*

Proof. *Let f be a function continuous at x_0 and let (x_n) be a sequence converging to x_0. Then, if N is a neighborhood of $f(x_0)$, $f^{-1}(N)$ is a neighborhood of x_0, so there exists $n_0 \in \mathbb{N}$ such that:*

$$n \geqslant n_0 \Rightarrow x_n \in f^{-1}(N),$$

or, equivalently,

$$n \geqslant n_0 \Rightarrow f(x_n) \in N,$$

which demonstrates that $\lim\limits_{n \to \infty} f(x_n) = f(x_0)$.

Remark 3.21. *The converse in the previous proposition is not true in general.*

3.10 Open and closed maps

Let $f : \mathbb{X} \to \mathbb{Y}$ be a continuous function.
• If O is an open set in $\mathbb{X}$, then $f(O)$ is not necessarily open in $\mathbb{Y}$.
• If F is a closed set in $\mathbb{X}$, then $f(F)$ is not necessarily closed in $\mathbb{Y}$.
In other words, the continuous image of an open set (resp. closed set) is not necessarily an open set (resp. closed set).

Example 3.34.

1. *The function $f : (\mathbb{R}, |.|) \to (\mathbb{R}, |.|)$ defined by $f(x) = \sin(x)$ is continuous on $\mathbb{R}$, but $f(\mathbb{R}) = [-1,1]$ is not an open set in $\mathbb{R}$.*

2. *The function $f : (\mathbb{R}, |.|) \to (\mathbb{R}, |.|)$ defined by $f(x) = e^x$ is continuous on $\mathbb{R}$, but $f(\mathbb{R}) = (0,+\infty)$ is not a closed set in $\mathbb{R}$.*

Definition 3.30. *Let $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ and $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ be two topological spaces, and let $f : \mathbb{X} \to \mathbb{Y}$. We say that f is an open map (resp. closed map) if the image of every open set (resp. closed set) in $\mathbb{X}$ is an open set (resp. closed set) in $\mathbb{Y}$.*

Example 3.35.

1. *Let $\mathbb{X}$ be a topological space and $A \subseteq \mathbb{X}$. The canonical map $i : (A, \mathcal{T}_A) \to \mathbb{X}$ defined by $i(x) = x$ is open (resp. closed) if A is an open (resp. closed) subset of $\mathbb{X}$.*

2. *Let $f : (\mathbb{R}, |.|) \to (\mathbb{R}, |.|)$ be the function defined by $f(x) = c \in \mathbb{R}$. If F is closed in $\mathbb{R}$, then $f(F) = \{c\}$ is also closed in $\mathbb{R}$. However, if O is open in $\mathbb{R}$, then $f(O) = \{c\}$ is not open in $\mathbb{R}$. Therefore, $f(x) = c$ is a closed map but is not an open map.*

Proposition 3.45. *Let $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ and $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ be two topological spaces, and let $f : \mathbb{X} \to \mathbb{Y}$. Then, for any $A \subseteq \mathbb{X}$, we have:*

1. *f is open $\Longleftrightarrow f(IntA) \subseteq Int(f(A))$.*

2. *f is closed $\Longleftrightarrow Cl(f(A)) \subseteq f(ClA)$.*

Proof.

1. $\Longrightarrow$) *Suppose that f is open; then $f(Int(A))$ is open in $\mathbb{Y}$. Consequently, $f(IntA) \subset Int(f(A))$ (since $Int(A) \subset A$).*
   $\Longleftarrow$) *Suppose that $f(IntA) \subset Int(f(A))$ and let A be an open set in $\mathbb{X}$. Then $f(A) = f(IntA) \subset Int(f(A))$, so $f(A) = Int(f(A))$, which shows that f is open.*

2. *Exercise: using arguments similar to those used in **(1)**.*

Proposition 3.46. *Let $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ and $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ be two topological spaces, and let $f : \mathbb{X} \to \mathbb{Y}$. Then, for any $A \subseteq \mathbb{X}$ and $B \subseteq \mathbb{Y}$, we have:*

1. *f is continuous and open $\Longleftrightarrow f^{-1}(IntB) = Int(f^{-1}(B))$.*

2. *f is continuous and closed $\Longleftrightarrow Cl(f(A)) = f(ClA)$.*

Proof.

1. $\Longrightarrow$) *Suppose f is open and continuous. Then we obtain*

   (i) $$f^{-1}(IntB) \subset Int(f^{-1}(B)),$$

   according to Proposition (3.40(6)). On the other hand, since $Int(f^{-1}(B))$ is open in $\mathbb{X}$, we have that $f(Int(f^{-1}(B)))$ is open in $\mathbb{Y}$ (since f is open). Consequently, $f(Int(f^{-1}(B))) = Intf(Int(f^{-1}(B))) \subseteq Int(f(f^{-1}(B))) \subseteq IntB$, so

   (ii) $$Int(f^{-1}(B)) \subseteq f^{-1}(IntB).$$

   Finally, the two inclusions (i) and (ii) show that $f^{-1}(IntB) = Int(f^{-1}(B))$.

   $\Longleftarrow$) *Suppose $f^{-1}(IntB) = Int(f^{-1}(B))$. Then f is continuous (see Proposition 3.40(6)). Moreover, if A is an open set in $\mathbb{X}$, we have*

   $$A = Int(A) \subset Int(f^{-1}(f(A))) = f^{-1}(Int(f(A)),$$

   and thus $f(A) \subset Int(f(A))$. Hence, $f(A)$ is open, so f is open.

2. *Clear (using Proposition (3.40(2)) and Proposition (3.45(**2**))).*

3.11 Homeomorphism

Definition 3.31. *Let $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ and $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ be two topological spaces and $f : \mathbb{X} \longrightarrow \mathbb{Y}$. We say that f is an homeomorphism from $\mathbb{X}$ to $\mathbb{Y}$ if:*

1. *f is a bijection (one-to-one and onto),*
2. *f is continuous,*
3. *the inverse function f^{-1} is continuous (f is an open mapping).*

If there exists an homeomorphism from $\mathbb{X}$ to $\mathbb{Y}$, we say that $\mathbb{X}$ and $\mathbb{Y}$ are homeomorphic or topologically equivalent, and we denote this by $\mathbb{X} \cong \mathbb{Y}$. Any property preserved by an homeomorphism is called a topological property.

Example 3.36.

1. *Let $\mathbb{X} = \mathbb{R}$ and $\mathbb{Y} = (-1, 1)$ endowed with the usual topology. The function $f : \mathbb{R} \longrightarrow (-1, 1)$ defined by $f(x) = \dfrac{x}{1 + |x|}$ is a homeomorphism. Consequently, $\mathbb{X}$ and $\mathbb{Y}$ are homeomorphic.*
2. *Let $\mathbb{X} = (a, b)$ and $Y = \mathbb{R}$ with the usual topology. The function $f : (a, b) \longrightarrow \mathbb{R}$ defined by $f(x) = \dfrac{1}{x - a} + \dfrac{1}{x - b}$ is a homeomorphism. Therefore, $\mathbb{X}$ and $\mathbb{Y}$ are homeomorphic.*
3. *Let $\mathbb{X} = (0, 1)$ and $\mathbb{Y} = (a, b)$ endowed with the usual topology. The function $f : (0, 1) \longrightarrow (a, b)$ defined by $f(x) = (b - a)x + a$ is a homeomorphism. Consequently, $\mathbb{X}$ and $\mathbb{Y}$ are homeomorphic.*

Remark 3.22.

1. *In general, the bijectivity and continuity of f do not imply that f is a homeomorphism. For example, the map $f : (\mathbb{R}, \mathcal{P}(\mathbb{R})) \longrightarrow (\mathbb{R}, |\cdot|)$ defined by $f(x) = x$ is a bijection and continuous, while f^{-1} is not continuous.*
2. *Homeomorphisms are, by definition, open and closed maps.*

CHAPTER 4

COMPACT SPACES

4.1 Compactness in Topological Spaces

4.1.1 Compact Spaces and Sets

Let $(\mathbb{X}, \mathcal{T})$ be a topological space and $\{O_i : i \in I\}$ a family of open sets in $\mathbb{X}$.

Definition 4.1. *We say that the family* $\{O_i : i \in I\}$ *is an open cover of* $\mathbb{X}$ *if* $\mathbb{X} = \bigcup_{i \in I} O_i$.

Definition 4.2. *We say that the family* $\{O_i : i \in I\}$ *is an open cover of a subset* A *of* $\mathbb{X}$ *if* $A \subseteq \bigcup_{i \in I} O_i$.

Definition 4.3 (**Borel-Lebesgue**)**.** *The topological space* $(\mathbb{X}, \mathcal{T})$ *is said to be compact if it is Hausdorff (separated) and for every open cover* $\{O_i : i \in I\}$ *of* $\mathbb{X}$*, one can extract a finite subcover. In other words:*

$$\left(\mathbb{X} = \bigcup_{i \in I} O_i\right) \Longrightarrow \left(\exists J \text{ (finite) } \subset I \text{ such that } \mathbb{X} = \bigcup_{i \in J} O_i\right). \tag{4.1}$$

The following definition characterize compactness in terms of closed subsets of the space.

Definition 4.4. *The topological space $(\mathbb{X}, \mathcal{T})$ is said to be compact if it is Hausdorff (separated), and for every family of closed sets $\{F_i : i \in I\}$ in $\mathbb{X}$ with an empty intersection, one can extract a finite subfamily whose intersection is also empty. In other words:*

$$\left(\bigcap_{i\in I} F_i = \emptyset\right) \Longrightarrow \left(\exists J \text{ (finite) } \subset I \text{ such that } \bigcap_{i\in J} F_i = \emptyset\right). \tag{4.2}$$

Example 4.1.

1. *The space $(\mathbb{R}, |.|)$ is Hausdorff, but it is not compact because the family $\{(-n, +n) : n \in \mathbb{N}\}$ is an open cover of $\mathbb{R}$ that does not have any finite subcover of $\mathbb{R}$.*

2. *The space $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ is Hausdorff, but it is not compact because the family $\{\{x\} : x \in \mathbb{R}\}$ is an open cover of $\mathbb{R}$ that does not have any finite subcover of $\mathbb{R}$.*

3. *Any finite Hausdorff space is compact.*

Definition 4.5. *A subset A of a Hausdorff topological space $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ is said to be compact if the subspace topology $(A, \mathcal{T}_A)$ is compact. In other words:*

$$\left(A \subset \bigcup_{i\in I} O_i\right) \Longrightarrow \left(\exists J \text{ (finite) } \subset I \text{ such that } A \subset \bigcup_{i\in J} O_i\right). \tag{4.3}$$

Remark 4.1. *The Borel-Lebesgue property in $(A, \mathcal{T}_A)$ is expressed using the open sets of $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ in the form (4.3).*

Proposition 4.1. *A subset A of a Hausdorff topological space $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ is compact if and only if for every family of closed sets $\{F_i : i \in I\}$ in $\mathbb{X}$, we have:*

$$\left(A \cap \left(\bigcap_{i\in I} F_i\right) = \emptyset\right) \Longrightarrow \left(\exists J \text{ (finite) } \subset I \text{ such that } A \cap \left(\bigcap_{i\in J} F_i\right) = \emptyset\right). \tag{4.4}$$

Proof.

$\Longrightarrow$) *On the one hand, if A is compact then we have:*

$$\left(A\cap\left(\bigcap_{i\in I}F_i\right)=\emptyset\right)\Longrightarrow\left(A\subset\complement_{\mathbb{X}}\left(\bigcap_{i\in I}F_i\right)=\bigcup_{i\in I}\complement_{\mathbb{X}}F_i\right)$$

Using definition (4.5), since $\left\{\complement_{\mathbb{X}}F_i : i\in I\right\}$ is a family of open sets in $\mathbb{X}$, we deduce that:

$$\exists J\ \text{(finite)}\ \subset I\ \text{such that}\ A\subset\complement_{\mathbb{X}}\left(\bigcap_{i\in J}F_i\right)=\bigcup_{i\in J}\complement_{\mathbb{X}}F_i.$$

This shows that:

$$\exists J\ \text{(finite)}\ \subset I\ \text{such that}\ A\cap\left(\bigcap_{i\in J}F_i\right)=\emptyset.$$

$\Longleftarrow$) *On the other hand, we have:*

$$\left(A\subset\complement_{\mathbb{X}}\left(\bigcap_{i\in J}F_i\right)=\bigcup_{i\in J}\complement_{\mathbb{X}}F_i\right)\Longleftrightarrow\left(A\cap\left(\bigcap_{i\in I}F_i\right)=\emptyset\right).$$

Now, using (4.4) we obtain:

$$\left(A\cap\left(\bigcap_{i\in I}F_i\right)=\emptyset\right)\Longrightarrow\left(\exists J\ \text{(finite)}\ \subset I\ \text{such that}\ A\cap\left(\bigcap_{i\in J}F_i\right)=\emptyset\right)$$

$$\Longrightarrow\left(\exists J\ \text{(finite)}\ \subset I\ \text{such that}\ A\subset\complement_{\mathbb{X}}\left(\bigcap_{i\in J}F_i\right)=\bigcup_{i\in J}\complement_{\mathbb{X}}F_i\right).$$

Since $\left\{\complement_{\mathbb{X}}F_i : i\in I\right\}$ is a family of open sets in $(\mathbb{X},\mathcal{T}_{\mathbb{X}})$, we conclude that A is compact.

Example **4.2.**

1. *$A=(0,1]$ is not compact because $I_n=(\frac{1}{n},1]$ is a sequence of open sets in A covering A, and no finite subcover can be extracted.*

2. *Any finite subset of a Hausdorff space is compact.*

4.1.2 Properties of Compact Topological Spaces

Proposition 4.2. *In a Hausdorff topological space, a compact subset is closed.*

Proof. *Let K be a compact subset in a Hausdorff topological space $(\mathbb{X},\mathcal{T}_{\mathbb{X}})$. It suffices to show that $\complement_{\mathbb{X}}K$ is open. Let $x \in \complement_{\mathbb{X}}K$. Since $(\mathbb{X},\mathcal{T}_{\mathbb{X}})$ is Hausdorff, for every $y \in K$, there exist two open sets $N_{x,y} \in \mathcal{N}(x)$ and $W_{x,y} \in \mathcal{N}(y)$ such that $N_{x,y} \cap W_{x,y} = \emptyset$. The family $\{W_{x,y} : y \in K\}$ is an open cover of K and is compact, so we can extract a finite subcover $\{W_{x,y_i} : i = 1,\ldots,n\}$ such that $K \subset \bigcup_{i=1}^{n} W_{x,y_i}$. If we take $N = \bigcap_{i=1}^{n} N_{x,y_i}$, we obtain $N \in \mathcal{N}(x)$ and $N \subset \complement_{\mathbb{X}}K$, which shows that $\complement_{\mathbb{X}}K$ is open (because it is a neighborhood of each of its points).*

Proposition 4.3. *If $(\mathbb{X},\mathcal{T}_{\mathbb{X}})$ is a compact topological space and $F \subset \mathbb{X}$, then F is compact if and only if F is a closed subset of $\mathbb{X}$.*

Proof.

$\Longrightarrow$ *) This is evident from the previous proposition.*
$\Longleftarrow$ *) Suppose that F is a closed subset of $\mathbb{X}$. Then, if $\{F_i : i \in I\}$ is a family of closed subsets of X such that $F \cap \left(\bigcap_{i\in I} F_i\right) = \emptyset$, we obtain $\bigcap_{i\in I}(F \cap F_i) = \emptyset$. Therefore, by definition (4.4), there exists a finite subset $J \subset I$ such that $\emptyset = \bigcap_{i\in J}(F \cap F_i) = F \cap \left(\bigcap_{i\in J} F_i\right)$. Thus, F is compact by proposition (4.1).*

Proposition 4.4. *In a Hausdorff topological space, a finite union of compact sets is compact.*

Proof. *Let $\{K_k : k = 1,\ldots,n\}$ be a finite family of compact sets in a topological space $\mathbb{X}$ and let $K = \bigcup_{k=1}^{n} K_k$. Then, any open cover $\{O_i : i \in I\}$ of K is also an open cover of each K_k, for each $k = 1,\ldots,n$. Therefore, there exists a finite subset $J_k \subset I$ such that $K_k \subset \bigcup_{i\in J_k} O_i$, for each $k = 1,\ldots,n$. Taking $J = J_1 \cup \cdots \cup J_n$, we see that $\bigcup_{i\in J} O_i$ is a finite subcover of K, which shows that K is compact.*

Proposition 4.5. *In a Hausdorff topological space, any intersection of compact sets is compact.*

Proof. *Let* $\{K_i : i \in I\}$ *be a family of compact sets in a Hausdorff topological space* $\mathbb{X}$*, and let* $K = \bigcap_{i \in I} K_i$*. Then* K *is closed (since it is an intersection of closed sets) within the compact set* K_{i_0} *for some* $i_0 \in I$*. Therefore, by proposition (4.3),* K *is compact.*

Lemma 4.1 (Bolzano-Weierstrass). *Let* $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ *be a compact topological space. Then, every infinite subset of* $\mathbb{X}$ *has at least one accumulation point.*

Proof. *If* A *is an infinite subset of* $\mathbb{X}$ *with no accumulation points, then for each* $x \in \mathbb{X}$*, there exists an open neighborhood* $N_x \in \mathcal{N}(x)$ *such that*

$$N_x \cap A = \begin{cases} \{x\} & \text{if } x \in A, \\ \emptyset & \text{if } x \notin A. \end{cases}$$

Thus, the family $\{N_x : x \in \mathbb{X}\}$ *forms an open cover of* $\mathbb{X}$*, which is compact. Therefore, we can extract a finite subcover* $\{N_{x_i} : i = 1, \dots, n\}$ *such that*

$$\mathbb{X} = \bigcup_{i=1}^{n} N_{x_i}.$$

However, we also have

$$A = A \cap \mathbb{X} = A \cap \left(\bigcup_{i=1}^{n} N_{x_i} \right) = \bigcup_{i=1}^{n} (A \cap N_{x_i}),$$

which implies that A *contains at most* n *elements, contradicting the assumption that* A *is infinite.*

Lemma 4.2 (Weierstrass). *Let* $(\mathbb{X}, \mathcal{T}_X)$ *and* $(\mathbb{Y}, \mathcal{T}_Y)$ *be two topological spaces such that* $(Y, \mathcal{T}_Y)$ *is Hausdorff, and let* $f : \mathbb{X} \to \mathbb{Y}$ *be a continuous map. If* A *is a compact subset of* $\mathbb{X}$*, then* $f(A)$ *is a compact subset of* $\mathbb{Y}$*.*

Proof. *Let* $\{O_i : i \in I\}$ *be an open cover of* $f(A)$*, i.e.,* $f(A) \subset \bigcup_{i \in I} O_i$*. Since* f *is continuous, the family* $\{f^{-1}(O_i) : i \in I\}$ *is an open cover of* A*. By the compactness of* A*, there exists a finite subset* $J \subset I$ *such that*

$$A \subset \bigcup_{i \in J} f^{-1}(O_i) = f^{-1}\left(\bigcup_{i \in J} O_i \right).$$

Since $f(A) \subset f\left(f^{-1}\left(\bigcup_{i\in J} O_i\right)\right) \subset \bigcup_{i\in J} O_i$, *we conclude that* $f(A)$ *is a compact subset of* $\mathbb{Y}$.

Remark 4.2. *According to the previous proposition, we conclude that compactness is a topological property.*

The following corollary is a version of the extreme value theorem.

Corollaire 4.1 (Heine). *If* $(\mathbb{X},\mathcal{T})$ *is a compact topological space and* $f:\mathbb{X}\to(\mathbb{R},|.|)$ *is a continuous function, then* f *is bounded on* $\mathbb{X}$, *and there exist points* $a,b\in\mathbb{X}$ *such that*

$$f(a)=\max_{x\in\mathbb{X}} f(x) \quad \text{and} \quad f(b)=\min_{x\in\mathbb{X}} f(x).$$

Proof. *Since* f *is continuous and* $\mathbb{X}$ *is compact,* $f(\mathbb{X})$ *is a compact subset of* $(\mathbb{R},|.|)$ *(see Lemma (4.2)). It follows that* $f(X)$ *is closed and bounded (see Proposition (4.2)). Let* $M=\sup f(\mathbb{X})$. *Since* $f(\mathbb{X})$ *is closed, we conclude that* $M\in f(\mathbb{X})$, *and therefore, there exists* $a\in\mathbb{X}$ *such that* $f(a)=M=\max_{x\in\mathbb{X}} f(x)$. *Similarly, we can show that the minimum is attained.*

Remark 4.3. *The previous corollary shows that continuous functions on a compact set and with values in* $\mathbb{R}$ *attain their bounds.*

Proposition 4.6. *Let* $(\mathbb{X},\mathcal{T}_{\mathbb{X}})$ *be a compact space,* $(\mathbb{Y},\mathcal{T}_{\mathbb{Y}})$ *be a Hausdorff space, and* $f:(\mathbb{X},\mathcal{T}_{\mathbb{X}})\longrightarrow(\mathbb{Y},\mathcal{T}_{\mathbb{Y}})$ *be a continuous function, then* f *is closed.*

Proof. *Let* F *be a closed subset of* $\mathbb{X}$, *then* F *is compact (see proposition (4.3)), and thus* $f(F)$ *is compact (see lemma (4.2)), from which it follows that* $f(F)$ *is closed (see proposition (4.2)).*

Proposition 4.7. *Let* $(\mathbb{X},\mathcal{T}_{\mathbb{X}})$ *be a compact space,* $(\mathbb{Y},\mathcal{T}_{\mathbb{Y}})$ *be a Hausdorff space, and* $f:(\mathbb{X},\mathcal{T}_{\mathbb{X}})\longrightarrow(\mathbb{Y},\mathcal{T}_{\mathbb{Y}})$ *be a continuous bijection, then* f *is a homeomorphism.*

Proof. *It is enough to show that* f^{-1} *is continuous. Let* $g=f^{-1}$. *If* F *is a closed set in* $\mathbb{X}$, *then it is compact, and thus* $f(F)$ *is compact. However, a compact subset of a separated space is closed, so* $g^{-1}(F)=f(F)$ *is closed. Therefore,* $g=f^{-1}$ *is continuous.*

Theorem 4.1 (**Tychonoff's Theorem**). *Let $\{(\mathbb{X}_i, \mathcal{T}_i) : i \in I\}$ be a family of topological spaces, then $\prod_{i\in I} \mathbb{X}_i$ is compact if and only if $\mathbb{X}_i$ is compact for all $i \in I$.*

Proof. *We will assume the result in the general case. Here, we simply prove it for the finite product of compact spaces. Therefore, it is sufficient to prove it for the product of two compact spaces.*

$\Longrightarrow$) *If $\mathbb{X} \times \mathbb{Y}$ is compact, then $\mathbb{X}$ and $\mathbb{Y}$ are compact because they are the images of the continuous projections $P_{\mathbb{X}}(\mathbb{X} \times \mathbb{Y}) = \mathbb{X}$ and $P_{\mathbb{Y}}(\mathbb{X} \times \mathbb{Y}) = \mathbb{Y}$.*

$\Longleftarrow$) *Suppose that $\mathbb{X}$ and $\mathbb{Y}$ are compact. Let $\{O_i : i \in I\}$ be an open cover of $\mathbb{X} \times \mathbb{Y}$. Then, for every $(x,y) \in \mathbb{X} \times \mathbb{Y}$, there exist $U_{(x,y)} \in \mathcal{T}_{\mathbb{X}}$ and $V_{(x,y)} \in \mathcal{T}_{\mathbb{Y}}$ such that $(x,y) \in U_{(x,y)} \times V_{(x,y)} \subseteq O_{(x,y)}$ with $O_{(x,y)} \in \{O_i : i \in I\}$.*

Notice that for each $x \in \mathbb{X}$, the family $\left\{V_{(x,y)} : y \in \mathbb{Y}\right\}$ is an open cover of the compact space $\mathbb{Y}$, and so we can extract a finite subcover $\left\{V_{(x,y_i)} : i = 1, \ldots, n\right\}$ for it. On the other hand, if we take $W_x = \bigcap_{i=1}^{n} U_{(x,y_i)}$, then the family $\{W_x : x \in \mathbb{X}\}$ is an open cover of the compact space $\mathbb{X}$, and so we can extract a finite subcover $\left\{W_{x_j} : j = 1, \ldots, m\right\}$.

We deduce that the family $\left\{W_{x_j} \times V_{(x_j,y_i)} : i = 1, \ldots, n, j = 1, \ldots, m\right\}$ is a finite cover of $\mathbb{X} \times \mathbb{Y}$. Moreover, we have:

$$W_{x_j} \times V_{(x_j,y_i)} \subseteq U_{(x_j,y_i)} \times V_{(x_j,y_i)} \subseteq O_{(x_j,y_i)}, \quad \forall 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m.$$

Thus, the family $\left\{O_{(x_j,y_i)} : i = 1, \ldots, n \text{ and } j = 1, \ldots, m\right\}$ is a finite open cover of $\mathbb{X} \times \mathbb{Y}$, which shows that $\mathbb{X} \times \mathbb{Y}$ is compact.

Definition 4.6 (**Relative compactness**). *A set A is said to be relatively compact in a topological space $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ if $Cl(A)$ is compact.*

Example 4.3.

1. *Every non-empty subset of a compact space is relatively compact.*
2. *Every compact set is relatively compact.*

Definition 4.7 (Local compactness). *A space $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ is said to be locally compact if it is Hausdorff and every point of $\mathbb{X}$ has at least one compact neighborhood.*

Example 4.4.

1. *$(\mathbb{R}, |\cdot|)$ is locally compact because it is Hausdorff and $[x-r, x+r]$ is a compact neighborhood of every $x \in \mathbb{R}$.*

2. *Every discrete space is locally compact because it is Hausdorff and $\{x\}$ is a compact neighborhood of every point x in this space.*

4.2 Compactness in metric spaces

The definitions of compactness in a metric space $(\mathbb{X}, d)$ are the same as those we saw in a topological space (see the previous section (4.1)).

Remark 4.4. *In an abstract topological space, there is no notion of distance, and therefore we do not talk about bounded sets.*

4.2.1 Precompact spaces and sequentially compact spaces

Definition 4.8. *Let A be a subset of a metric space $(\mathbb{X}, d)$. We say that A is bounded if there exists a ball $B(x, r)$ with center x and radius $r > 0$ such that $A \subseteq B(x, r)$.*

Definition 4.9. *Let $(\mathbb{X}, d)$ be a metric space and A a subset of $\mathbb{X}$. We say that A is sequentially compact if every sequence in A has a convergent sub-sequence.*

Definition 4.10. *Let $(\mathbb{X}, d)$ be a metric space and A a subset of $\mathbb{X}$. We say that A is precompact (or totally bounded) if for every $r > 0$, there exist points $x_1, \dots, x_n$ in A such that $A \subseteq \bigcup_{i=1}^{n} B(x_i, r)$.*

Remark 4.5. *According to the two definitions (4.8) and (4.10), every precompact set is bounded.*

Example 4.5.

1. *Any finite subset A of a metric space $(\mathbb{X}, d)$ is sequentially compact because if $(x_n)_{n\in\mathbb{N}}$ is a sequence in A, then at least one of the elements $x \in A$ must repeat infinitely many times in this sequence, and thus the sequence $(x_0, \ldots, x_i, x, x, \ldots)$ is convergent.*

2. *Every finite subset A of a metric space $(\mathbb{X}, d)$ is precompact, because for any $r > 0$, there exist points $x_1, \ldots, x_n$ in A such that:$A \subseteq \bigcup_{i=1} B(x_i, r)$.*

3. *If $(x_n)_{n\in\mathbb{N}}$ is a sequence converging to x_0 in a metric space $(\mathbb{X}, d)$, then the set $A = \{x_n : n \geqslant 0\} \cup \{x_0\}$ is precompact. This is because for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\forall n \in \mathbb{N}, n \geqslant n_0 \Longrightarrow d(x_n, x_0) < \varepsilon.$$

Therefore,

$$A \subseteq \bigcup_{k=1}^{n_0-1} B(x_k, \varepsilon) \cup B(x_0, \varepsilon).$$

Lemma 4.3. *Let $(\mathbb{X}, d)$ be sequentially compact metric space. If $\{O_i : i \in I\}$ is an open cover of $\mathbb{X}$, then there exists $r > 0$ such that for every $x \in \mathbb{X}$, the ball $B(x, r)$ is contained in one of the open sets O_i.*

Proof. *Suppose that for every $n \in \mathbb{N}^*$, there exists a point $x_n \in \mathbb{X}$ such that the ball $B(x_n, \frac{1}{n})$ is not contained in any of the open sets O_i, i.e., $B(x_n, \frac{1}{n}) \cap \complement_{\mathbb{X}} O_i \neq \emptyset$ for all $i \in I$. Since $(\mathbb{X}, d)$ is sequentially compact, the sequence (x_n) has a convergent subsequence (x_{n_k}). Let $x_{n_k} \to x \in \mathbb{X}$. Then, there exists at least one $i_0 \in I$ such that $x \in O_{i_0}$, and therefore there exists $r > 0$ such that $B(x, r) \subset O_{i_0}$.*

On the other hand, since $x_{n_k} \to x$, the ball $B(x, \frac{r}{2})$ contains infinitely many points of (x_{n_k}). Thus, there exists $n_p > \frac{2}{r}$ such that $x_{n_p} \in B(x, \frac{r}{2})$, which implies that $B(x_{n_p}, \frac{1}{n_p}) \subset B(x, r) \subset O_{i_0}$, leading to a contradiction.

4.2.2 Properties of Compact Metric Spaces

Proposition 4.8. *Let $(\mathbb{X}, d)$ be a metric space and A a closed subset of $\mathbb{X}$. Then, the following statements are equivalent.*

1. *A is compact.*
2. *If* $\mathcal{F} = \{F_i : i \in I\}$ *is a family of closed subsets of* A *such that for every finite family sets* $F_1, \ldots, F_n \in \mathcal{F}$ *we have* $\bigcap\limits_{i=1}^{n} F_i \neq \emptyset$, *then* $\bigcap\limits_{i \in I} F_i \neq \emptyset$.
3. *A is sequentially compact.*
4. *Every infinite subset of A has an accumulation point.*
5. *A is complete and precompact.*

Proof.

Let us assume that A *is compact and let* $\mathcal{F} = \{F_i : i \in I\}$ *be a family of closed subsets of* A *such that for every finite collection* $F_1, \ldots, F_n \in \mathcal{F}$, *we have* $\bigcap\limits_{i=1}^{n} F_i \neq \emptyset$. *Now, if* $\bigcap\limits_{i \in I} F_i = \emptyset$, *it follows that the family* $\{\complement_{\mathbb{X}} F_i : i \in I\}$ *is an open cover of* $\mathbb{X}$ *and, therefore, of* A. *Since* A *is compact, there exist* $F_1, \ldots, F_n \in \mathcal{F}$ *such that* $A \subseteq \bigcup\limits_{i=1}^{n} \complement_{\mathbb{X}} F_i = \complement_{\mathbb{X}} \bigcap\limits_{i=1}^{n} F_i$. *Given that* $F_i \subseteq A$ *for every* $i = 1, \ldots, n$, *we conclude that* $\bigcap\limits_{i=1}^{n} F_i = \emptyset$, *which contradicts the assumption that* $\bigcap\limits_{i=1}^{n} F_i \neq \emptyset$.

(2) $\Longrightarrow$ ***(1)*** *Let* $\{O_i : i \in I\}$ *be an open cover of* A, *i.e.,* $A \subseteq \bigcup\limits_{i \in I} O_i$, *and thus* $\bigcap\limits_{i \in I} \complement_A O_i = \emptyset$, *which implies* $\bigcap\limits_{i=1}^{n} \complement_A O_i = \emptyset$ *(according to* ***(2)****), showing that* $A \subseteq \bigcup\limits_{i=1}^{n} O_i$, *and hence* A *is a compact set.*

(3) $\Longrightarrow$ ***(4)*** *If* K *is an infinite subset of* A, *then* K *contains a sequence of distinct points* (x_n); *by* ***(3)***, *there exists a subsequence* (x_{n_k}) *converging to* x. *Therefore,* x *is an accumulation point of* K.

(4) $\Longrightarrow$ ***(3)*** *Let* (x_n) *be a sequence of distinct points in* A. *Using equation* (4), *we conclude that* (x_n) *has an accumulation point* $x \in A$ *because* A *is closed. The ball* $B(x, 1)$ *contains infinitely many elements of the sequence* (x_n), *so we choose* $x_{n_1} \in B(x, 1)$. *The ball* $B(x, \frac{1}{2})$ *contains infinitely many elements of the sequence* (x_n), *so we choose* $x_{n_2} \in B(x, \frac{1}{2})$ *with* $n_2 > n_1$. *We repeat this process, choosing* $x_{n_3} \in B(x, \frac{1}{3})$ *with* $n_3 > n_2$. *Therefore, we can select a subsequence* (x_{n_k}), *such that* $n_{k+1} > n_k$ *and* $x_{n_k} \in B(x, \frac{1}{k})$ *for all* $k = 1, 2, \ldots$. *It is clear that this subsequence converges to* x.

(1) $\Longrightarrow$ ***(4)*** *(See Lemma (4.1))*

(1) $\Longrightarrow$ ***(5)*** *Assume that* A *is compact.*

- *Let* (x_n) *be a Cauchy sequence in* A. *Since* ***(1)*** $\Longrightarrow$ ***(4)*** $\Longleftrightarrow$ ***(3)***, *there exists a subsequence* (x_{n_k}) *such that* $x_{n_k} \to x$, *so* $x_n \to x$, *and hence* A *is complete.*
- *For every* $r > 0$, *the family* $\{B(x, r) : x \in A\}$ *is an open cover of* A, *so there exists a finite open subcover* $\{B(x_i, r) : i = 1, \ldots, n\}$ *of* A, *that is, there exist points* $x_1, \ldots, x_n$ *in* A *such that* $A \subseteq \bigcup\limits_{i=1}^{n} B(x_i, r)$, *which shows that* A *is precompact.*

(5) $\Longrightarrow$ ***(3)*** *Let* (x_n) *be a sequence in* A *and* (r_n) *a decreasing sequence of positive numbers such that* $(r_n) \to 0$. *Using* ***(5)***, *we conclude that there exists a finite cover of* A *by the balls* $\{B(x_i, r_1) : i = 1, \ldots, n\}$. *Therefore, there exists a ball* $B(y_1, r_1)$ *that contains infinitely many elements of* (x_n). *Let* $\mathbb{N}_1 = \{n \in \mathbb{N} : d(y_1, x_n) < r_1\}$. *Now, consider the sequence*

$\{x_n : n \in \mathbb{N}_1\}$ and the balls of radius r_2. We repeat the process; there exists $y_2 \in A$ such that $\mathbb{N}_2 = \{n \in \mathbb{N}_1 : d(y_2, x_n) < r_2\}$ is an infinite set. By induction, we can show that for each $i \geq 1$, we choose a point $y_k \in A$ and an infinite set $\mathbb{N}_k$ such that $\mathbb{N}_{k+1} \subset \mathbb{N}_k$ and $\{x_n : n \in \mathbb{N}_k\} \subset B(y_k, r_k)$. If we define $F_k = Cl\{x_n : n \in \mathbb{N}_k\}$, then $F_{k+1} \subset F_k$ and $diam(F_k) \leqslant 2r_k$. Since A is complete, Cantor's theorem implies that $\bigcap\limits_k F_k = \{x\}$. If we choose $n_k \in \mathbb{N}_k$, then (x_{n_k}) is a subsequence of (x_n) converging to x, and hence A is sequentially compact.

(5)* $\Longrightarrow$ *(1) *Let $G = \{O_i : i \in I\}$ be an open cover of A. Since A is precompact, for every $r > 0$, there exist points $x_1, \ldots, x_n \in A$ such that $A \subseteq \bigcup\limits_{i=1}^{n} B(x_k, r)$. But for each $1 \leq k \leq n$, there exists $O_k \in G$ such that $x_k \in O_k$. Thus, it is sufficient to choose $r > 0$ such that $B(x_k, r) \subset O_k$ for all $1 \leq k \leq n$ (see Lemma 4.3). We then deduce that $A \subseteq \bigcup\limits_{i=1}^{n} B(x_k, r) \subset \bigcup\limits_{i=1}^{n} O_k$, which shows that the family $\{O_k : k = 1, \ldots, n\}$ is a finite open cover of A, and therefore A is compact.*

(3)* $\Longrightarrow$ *(5) *Let (x_n) be a Cauchy sequence in A, then **(3)** implies that there exists a subsequence (x_{n_k}) such that $x_{n_k} \to x$. Since $d(x_n, x) \leqslant d(x_n, x_{n_k}) + d(x_{n_k}, x)$, it follows that $x_n \to x$.*

Proposition 4.9.

1. *For all $a, b \in \mathbb{R}$, the closed interval $[a, b]$ is compact in $(\mathbb{R}, |\cdot|)$.*
2. *A subset A of $(\mathbb{R}, |\cdot|)$ is compact if and only if it is closed and bounded.*

Proof.

1. *The interval $[a, b]$ is closed in $\mathbb{R}$, which is complete, and therefore complete. Thus, according to the previous proposition, it is enough to show that it is precompact. Indeed, for every $r > 0$, we can find points $a = x_1, x_2, \ldots, x_n = b$ such that $x_i - x_{i-1} < r$, and $[a, b] \subseteq \bigcup\limits_{i=1}^{n} (x_i - r, x_i + r)$.*

2. $\Longrightarrow$ *) Let A be a compact subset of $(\mathbb{R}, |\cdot|)$, then it is complete and precompact, according to the previous theorem, and therefore it is closed and bounded (see Proposition 2.10 and Remark 4.5).*

   $\Longleftarrow$ *) Let A be a closed and bounded subset of $(\mathbb{R}, |\cdot|)$, then there exists a closed and bounded interval $[a, b]$ such that $A \subseteq [a, b]$, and since $[a, b]$ is compact, A is compact (see Proposition 4.3).*

Example 4.6.

1. *Every bounded and closed subset of $(\mathbb{R}^2, d_2)$ is compact.*
2. *Any subset of $(\mathbb{R}^2, d_2)$ that is either unbounded or not closed is not compact.*

3. *The closed disk* $\left\{x \in \mathbb{R}^2 : d_2(x,y) \leqslant r\right\}$ *is compact.*

Lemma 4.4 (Heine). *If* $f : (\mathbb{X}, d_{\mathbb{X}}) \to (\mathbb{Y}, d_{\mathbb{Y}})$ *is continuous and* $\mathbb{X}$ *is compact, then* f *is uniformly continuous.*

Proof. *Suppose that* f *is continuous but not uniformly continuous. Then, there exist* $\varepsilon > 0$ *and two sequences* (x_n) *and* (y_n) *in* $\mathbb{X}$ *such that:*

$$d_{\mathbb{X}}(x_n, y_n) < \frac{1}{n} \quad \text{and} \quad d_{\mathbb{Y}}(f(x_n), f(y_n)) \geqslant \varepsilon.$$

Since $\mathbb{X}$ *is compact, there exists* $x \in \mathbb{X}$ *and a subsequence* x_{n_k} *such that* $d(x_{n_k}, x) \longrightarrow 0$. *We deduce that*

$$d(y_{n_k}, x) \leqslant d(y_{n_k}, x_{n_k}) + d(x_{n_k}, x) \leqslant \frac{1}{n} + d(x_{n_k}, x),$$

hence $y_{n_k} \longrightarrow x$. *Since* f *is continuous, there exists* $\delta > 0$ *such that:*

$$d_{\mathbb{X}}(x_{n_k}, x) < \delta \Longrightarrow d_{\mathbb{Y}}(f(x_{n_k}), f(x)) < \frac{\varepsilon}{2},$$

$$d_{\mathbb{X}}(y_{n_k}, x) < \delta \Longrightarrow d_{\mathbb{Y}}(f(y_{n_k}), f(x)) < \frac{\varepsilon}{2}.$$

Consequently, we obtain:

$$d_{\mathbb{Y}}(f(x_{n_k}), f(y_{n_k})) \leqslant d_{\mathbb{Y}}(f(x_{n_k}), f(x)) + d_{\mathbb{Y}}(f(x), f(y_{n_k})) < \varepsilon,$$

which contradicts the hypotheses.

Using the previous lemma, we obtain the following result.

Corollaire 4.2. *If* $f : [a,b] \to \mathbb{R}$ *is a continuous function, then* f *is uniformly continuous on* $[a,b]$.

Proposition 4.10. *Let* $(\mathbb{X}, d)$ *be a metric space. Then, we have:*

1. *Every relatively compact subset is precompact.*
2. *If* $(\mathbb{X}, d)$ *is complete, then every precompact subset is relatively compact.*

Proof.

1. *Let A be a relatively compact subset of $\mathbb{X}$; then $Cl(A)$ is precompact, and hence A is precompact.*

2. *Suppose $(\mathbb{X}, d)$ is complete. If A is precompact, then $Cl(A)$ is also precompact. Furthermore, $Cl(A)$ is closed in $\mathbb{X}$, which is complete, and thus $Cl(A)$ is complete as well, showing that $Cl(A)$ is compact (see Proposition (4.8)).*

CHAPTER 5

CONNECTED SPACES

5.1 Connectivity in Topological Spaces

5.1.1 Connected Spaces and Subsets

Let the two spaces $(\mathbb{X}, |.|)$ and $(\mathbb{Y}, |.|)$ be such that $\mathbb{X} =]2,3[\cup]4,5[$ and $\mathbb{Y} = [2,3]\cup]3,4[$. The two subsets $O_1 =]2,3[$ and $O_2 =]4,5[$ are both open and closed in $\mathbb{X}$ because $O_1 = \mathbb{X}\cap]2,3[= \mathbb{X}\cap[2,3]$ and $O_2 = \mathbb{X}\cap]4,5[= \mathbb{X}\cap[4,5]$. Moreover, we have $\mathbb{X} = O_1 \cup O_2$, so the family $\{O_1, O_2\}$ is a partition of $\mathbb{X}$ into two disjoint open (and closed) sets. In this case, we say that $\mathbb{X}$ is not connected, whereas $\mathbb{Y}$ is connected because it can be written in the form $\mathbb{Y} = [2,4[$. The concept of connectivity, which we will define below, intuitively means that a space is "in one piece" or that it cannot be split into two "separated" parts.

Definition 5.1. *Let $(\mathbb{X}, \mathcal{T})$ a topological space. $\mathbb{X}$ is said to be disconnected if it is the union of two disjoint non-empty open sets. In other words, a space is connected if it does not have a partition consisting of two non-empty open sets. We write then,*

$$\mathbb{X} \text{ is connected} \iff \begin{cases} \text{There do not exist } O_1, O_2 \in \mathcal{T} \text{ such that:} \\ \bullet\ O_1 \cup O_2 = \mathbb{X}, \\ \bullet\ O_1 \cap O_2 = \emptyset, \\ \bullet\ O_1 \neq \emptyset \text{ and } O_2 \neq \emptyset. \end{cases}$$

An equivalent definition of the connectivity of $\mathbb{X}$ is as follows.

Definition 5.2. *$\mathbb{X}$ is connected if for any partition of $\mathbb{X}$ into two open sets O_1 and O_2, we have $O_1 = \emptyset$ or $O_2 = \emptyset$.*

Proposition 5.1. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. The following assertions are equivalent.*

1. *$\mathbb{X}$ is connected.*
2. *There does not exist a partition of $\mathbb{X}$ into two non-empty open sets.*
3. *There does not exist a partition of $\mathbb{X}$ into two non-empty closed sets.*
4. *$\emptyset$ and $\mathbb{X}$ are the only sets that are both open and closed (clopen sets) in $\mathbb{X}$.*
5. *Any subset $A \subset \mathbb{X}$ such that $A \neq \emptyset$ and $A \neq \mathbb{X}$ has a non-empty boundary.*
6. *There is no continuous and surjective map from $\mathbb{X}$ to a discrete space $\mathbb{Y}$ containing two elements.*
7. *Every continuous map $f : \mathbb{X} \longrightarrow \mathbb{Y} = \{a, b\}$ is constant.*

Proof.

1. $\Longrightarrow$ ***2.*** *By definition.*

2. $\Longrightarrow$ ***3.*** *Suppose there exists a partition of $\mathbb{X}$ into two non-empty closed sets F_1 and F_2, i.e., $F_1 \cup F_2 = \mathbb{X}$ and $F_1 \cap F_2 = \emptyset$. Then F_1 and F_2 are two non-empty open sets that form a partition of $\mathbb{X}$ because $\complement_{\mathbb{X}} F_1 = F_2$ and $\complement_{\mathbb{X}} F_2 = F_1$.*

3. $\Longrightarrow$ ***4.*** *Suppose there exists a set A that is both open and closed, and different from $\mathbb{X}$ and $\emptyset$. We deduce that A and $\complement_{\mathbb{X}} A$ form a partition of $\mathbb{X}$ into two non-empty closed sets.*

4. $\Longrightarrow$ ***5.*** *Suppose A is a subset of $\mathbb{X}$ such that $A \neq \emptyset$, $A \neq \mathbb{X}$, and $Cl(A) = \emptyset$. We deduce that A is both open and closed.*

5. $\Longrightarrow$ ***6.*** *Suppose there exists a continuous and surjective map $f : \mathbb{X} \longrightarrow \mathbb{Y} = \{a, b\}$. Then, the set $\{a\}$ is both open and closed. Thus, $f^{-1}(\{a\})$ is a set that is both open and closed, such that $f^{-1}(\{a\}) \neq \emptyset$ and $f^{-1}(\{a\}) \neq \mathbb{X}$. Moreover, $Cl(f^{-1}(\{a\})) = \emptyset$.*

6. $\Longrightarrow$ ***7.*** *Suppose there exists a continuous map $f : \mathbb{X} \longrightarrow \mathbb{Y} = \{a, b\}$ that is not constant. Then f is surjective.*

7. $\Longrightarrow$ ***1.*** *Suppose $\mathbb{X}$ is not connected. Then there exist two non-empty open sets $O_1, O_2 \subset \mathbb{X}$ such that $O_1 \cup O_2 = \mathbb{X}$ and $O_1 \cap O_2 = \emptyset$. Then, the map $f : \mathbb{X} \longrightarrow \mathbb{Y} = \{a, b\}$ defined by $f(x) = a$ if $x \in O_1$ and $f(x) = b$ if $x \in O_2$ is continuous but not constant.*

Example 5.1.

1. *$\mathbb{R}$ is connected.*

2. *Any discrete space $(\mathbb{X}, \delta)$ such that card$(\mathbb{X}) \geqslant 2$ is not connected. Indeed, if $x \in \mathbb{X}$, then we have $\{x\} \cup \complement_{\mathbb{X}}\{x\} = \mathbb{X}$ and $\{x\} \cap \complement_{\mathbb{X}}\{x\} = \emptyset$, with $\{x\}$ and $\complement_{\mathbb{X}}\{x\}$ being two open (two closed) sets.*

3. *It is evident that any space equipped with the trivial topology is connected.*

4. *Let $\mathbb{X} = \{a, b, c, d\}$ and $\mathcal{T} = \{\emptyset, \mathbb{X}, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. It is clear that $(\mathbb{X}, \mathcal{T})$ is connect.*

Definition 5.3. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space and A a non-empty subset of $\mathbb{X}$. We say that A is connected if the subspace $(A, \mathcal{T}_A)$ is connected. Classically, we consider the empty set as connected.*

Example 5.2.

1. *Every interval in $\mathbb{R}$ is connected.*

2. *Every open (closed) ball in $\mathbb{R}^n$ is connected.*

3. *The space $(\mathbb{R}^*, |\cdot|)$ is not connected (why ?).*

5.1.2 Properties of Connected Spaces

Proposition 5.2. *If a subset A of a topological space $(\mathbb{X}, \mathcal{T})$ is connected, then the existence of two open sets $O_1, O_2 \in \mathcal{T}$ such that $A \subset O_1 \cup O_2$ and $O_1 \cap O_2 = \emptyset$ implies that $A \subset O_1$ or $A \subset O_2$.*

Proof. *Suppose A is connected and let $O_1, O_2 \in \mathcal{T}$ such that $A \subset O_1 \cup O_2$ and $O_1 \cap O_2 = \emptyset$. Then, $A = (A \cap O_1) \cup (A \cap O_2)$ and $(A \cap O_1) \cap (A \cap O_2) = \emptyset$. Since A is connected, we obtain $(A \cap O_1 = \emptyset)$ or $(A \cap O_2 = \emptyset)$, from which it follows that $A \subset O_2$ or $A \subset O_1$.*

Proposition 5.3. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space and A, B two subsets of $\mathbb{X}$ such that A is connected and $A \subset B \subset Cl(A)$. Then, we have:*

1. *If A is connected, then B is connected.*

2. *If A is connected, then $Cl(A)$ is connected.*

3. *If A is a connected and dense subset of $\mathbb{X}$, then $\mathbb{X}$ is connected.*

Proof.

1. *Let $f : B \longrightarrow \{0,1\}$ be a continuous function. Since A is connected and f is continuous on A, we obtain that f is constant on A. Since f is continuous on B, the set $G = \{x \in B : f(x) \in f(A)\}$ is a closed set of B containing A, so $Cl(A)_B \subset G$. Thus, f is constant on the closure of A in B, which is $Cl(A)_B = B \cap Cl(A) = B$. We conclude that f is constant on B. Therefore, B is connected.*

2. *It is sufficient to take $B = Cl(A)$ in **(1)**).*

3. *We have $Cl(A) = \mathbb{X}$ because A is dense in $\mathbb{X}$, and $Cl(A)$ is connected because A is connected (see question **(2)**). We conclude that $\mathbb{X}$ is connected.*

Proposition 5.4. *Let $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ and $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ be two topological spaces, and let $f : \mathbb{X} \longrightarrow \mathbb{Y}$ be a continuous function. If $\mathbb{X}$ is connected, then $f(\mathbb{X})$ is a connected subset of $\mathbb{Y}$.*

Proof. *Let G be a subset of $f(\mathbb{X})$ that is both open and closed in the induced topology. Since f is continuous as a function with values in $f(\mathbb{X})$, we deduce that $f^{-1}(G)$ is both open and closed in $\mathbb{X}$. Since $\mathbb{X}$ is connected, we deduce that $f^{-1}(G) = \emptyset$ or $f^{-1}(G) = \mathbb{X}$. Since $f(f^{-1}(G)) = G$, we obtain that $G = \emptyset$ or $G = f(\mathbb{X})$, which shows that $f(\mathbb{X})$ is connected.*

Remark **5.1.** *According to the previous proposition, connectedness is a topological property.*

Proposition 5.5. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space.*

1. *If $\{A_i : i \in I\}$ is an arbitrary family of connected subsets of $\mathbb{X}$ such that $\bigcap\limits_{i \in I} A_i \neq \emptyset$, then $\bigcup\limits_{i \in I} A_i$ is connected.*

2. *If $\{A_i : i \in I\}$ is an arbitrary family of connected subsets of $\mathbb{X}$ such that $A_i \cap A_j \neq \emptyset$ for all $i, j \in I$, then $\bigcup\limits_{i \in I} A_i$ is connected.*

3. *If $\{A_i : i \in I\}$ is an arbitrary totally ordered family of connected subsets of $\mathbb{X}$, then $\bigcup\limits_{i \in I} A_i$ is connected.*

We will provide the proof for the first case only.

Proof. *Let $a \in \bigcap_{i\in I} A_i \neq \emptyset$. If $f : \bigcup_{i\in I} A_i \longrightarrow \{0,1\}$ is a continuous function, then $f_{|A_i}$ is continuous, and thus constant by the connectedness of A_i. Since $a \in A_i$ for all $i \in I$, we obtain $f(x) = f(a)$ for all $x \in A_i$. Therefore, $f(x) = f(a)$ for all $x \in \bigcup_{i\in I} A_i$, i.e., f is constant on $\bigcup_{i\in I} A_i$, which shows that $\bigcup_{i\in I} A_i$ is connected.*

Proposition 5.6. *A subset A of $\mathbb{R}$ is connected if and only if it is an interval.*

Proof.

$\Longrightarrow$) *Suppose that the set $A \subset \mathbb{R}$ is not an interval in $\mathbb{R}$. Then, there exist points $x, y \in A$ and $z \notin A$ such that $x < z < y$. Define $O_1 =]-\infty, z[\cap A$ and $O_2 =]z, +\infty[\cap A$, which are two non-empty open subsets of A. Furthermore, we have $O_1 \cap O_2 = \emptyset$ and $O_1 \cup O_2 = A$. Therefore, A is not connected.*

$\Longleftarrow$) *Let A be a non-empty interval in $\mathbb{R}$. Suppose $A = O_1 \cup O_2$, where O_1 and O_2 are two non-empty open subsets of A with $O_1 \cap O_2 = \emptyset$. Let $x \in O_1$ and $y \in O_2$ such that $x < y$, and let $z = \sup(O_1 \cap [x, y])$.*

On the one hand, if $z \in O_1$, then $z < y$, which implies the existence of a real number $r > 0$ such that $[z, z + r[\subset O_1 \cap [x, y]$, contradicting the definition of z.

On the other hand, if $z \in O_2$, then $z > x$, which implies the existence of a real number $r > 0$ such that $]z - r, z] \subset O_2 \cap [x, y]$, again contradicting the definition of z.

Thus, we conclude that $z \notin O_1$ and $z \notin O_2$, which is impossible because $[x, y] \subset A$. Therefore, A is connected.

Proposition 5.7. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space and $f : \mathbb{X} \longrightarrow \mathbb{R}$ a continuous function.*

1. *The image of any connected subset of $\mathbb{X}$ is an interval in $\mathbb{R}$.*
2. *Let $a, b \in f(\mathbb{X})$. If $\mathbb{X}$ is connected, then the equation $f(x) = c$ has a solution for every $c \in [a, b]$.*

Proof.

1. *Let A be a connected subset of X. Then, $f(A)$ is connected in $\mathbb{R}$ (see Proposition 5.4), which implies that $f(A)$ is an interval (see Proposition 5.6).*

2. *Using the two propositions (5.4) and (5.6), we conclude that $f(\mathbb{X})$ is an interval. Then, $[a,b] \subset f(\mathbb{X})$ which implies that*

$$\forall c \in [a,b], \quad c \in f(\mathbb{X}).$$

Therefore, there exists $x \in \mathbb{X}$ such that $f(x) = c$.

Proposition 5.8. *Let $((\mathbb{X}, \mathcal{T}_{\mathbb{X}}))$ and $((\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ be two topological spaces. Then $\mathbb{X} \times \mathbb{Y}$ is connected if and only if $\mathbb{X}$ and $\mathbb{Y}$ are connected.*

Proof.

$\Longrightarrow$) *Suppose that $\mathbb{X} \times \mathbb{Y}$ is connected. We have $p_{\mathbb{X}}(\mathbb{X} \times \mathbb{Y}) = \mathbb{X}$ and $p_{\mathbb{Y}}(\mathbb{X} \times \mathbb{Y}) = \mathbb{Y}$, where $p_{\mathbb{X}}$ and $p_{\mathbb{Y}}$ are the continuous canonical projections. It follows that $\mathbb{X}$ and $\mathbb{Y}$ are connected.*

$\Longleftarrow$) *Suppose that $\mathbb{X}$ and $\mathbb{Y}$ are connected, and let $f : \mathbb{X} \times \mathbb{Y} \longrightarrow \{0,1\}$ be a continuous function. Then, it suffices to show that f is constant. Since $\mathbb{Y}$ is connected, the function $f(x,\cdot) : \mathbb{Y} \longrightarrow \{0,1\}$ is constant, meaning $f(x,y_1) = f(x,y_2)$ for all $x \in \mathbb{X}$.*

Since $\mathbb{X}$ is connected, the function $f(\cdot,y) : \mathbb{X} \longrightarrow \{0,1\}$ is constant, meaning $f(x_1,y) = f(x_2,y)$ for all $y \in \mathbb{Y}$. Therefore, $f(x_1,y_1) = f(x_2,y_2)$ for all $(x_1,y_1),(x_2,y_2) \in \mathbb{X} \times \mathbb{Y}$, which shows that f is constant. Thus, $\mathbb{X} \times \mathbb{Y}$ is connected.

In the general case, we have the following result.

Proposition 5.9. *Let $\{(\mathbb{X}_i, \mathcal{T}_i) : i \in I\}$ be a family of topological spaces. Then $\prod_{i \in I} \mathbb{X}_i$ is connected if and only if $\mathbb{X}_i$ is connected for every $i \in I$.*

5.1.3 Connected components, locally connected spaces

Definition 5.4. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. For each $x \in \mathbb{X}$, we call the connected component of x, denoted by $\mathcal{C}(x)$, the equivalence class of x under the relation $\mathcal{R}$ defined*

. *by " $x\mathcal{R}y \Longleftrightarrow x$ and y belong to the same connected subset of $\mathbb{X}$."*

Remark 5.2. *According to the previous definition, we conclude that the connected component of a point x is the union of all connected subsets containing x. In other words, it is the largest connected subset containing x. Moreover, the connected components of $\mathbb{X}$ form a partition of $\mathbb{X}$.*

Definition 5.5. *A connected component of a space $\mathbb{X}$ is a maximal connected subset of $\mathbb{X}$, i.e., a connected subset that is not contained in any other (strictly) larger connected subset of X.*

Example 5.3.

1. *The only connected component in $(\mathbb{R}, |\cdot|)$ is $\mathbb{R}$ itself.*
2. *$(\mathbb{R}^*, |\cdot|)$ has two connected components: $\mathbb{R}^*_-$ and $\mathbb{R}^*_+$.*

Definition 5.6. *Let $(X, \mathcal{T})$ be a topological space, and let $A \subset X$. The connected components of A are defined as the connected components of $(A, \mathcal{T}_A)$.*

Proposition 5.10. *Every connected component is closed.*

Proof. *Let A be a connected component. Then A is connected, and thus $Cl(A)$ is a connected subset containing A, so $Cl(A) = A$, which shows that A is closed.*

Definition 5.7. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. We say that $\mathbb{X}$ is locally connected if every point $x \in \mathbb{X}$ admits a neighborhood basis consisting of open connected sets.*

Example 5.4.

1. *$\mathbb{R}$ is locally connected.*
2. *$\mathbb{Q}$ is not locally connected.*
3. *Every discrete space is locally connected. Indeed, $\mathcal{N}(x) = \{\{x\}\}$ forms a neighborhood basis consisting of open connected sets for each point $x \in \mathbb{X}$.*

Proposition 5.11. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. X is locally connected if and only if every connected component of every open set in $\mathbb{X}$ is open.*

Proof.

$\Longrightarrow$) *Suppose that $\mathbb{X}$ is locally connected. Let O be an open set in $\mathbb{X}$, and let $\mathcal{C}(O)$ be a connected component of O. Then, for every $x \in \mathcal{C}(O)$, there exists $N \in \mathcal{N}(x)$ such that N is connected and $N \subset O$. Thus, $N \subset \mathcal{C}(O)$, which shows that $\mathcal{C}(O)$ is open (a neighborhood of each of its points).*

$\Longleftarrow$) *Let $x \in X$ and N be an open neighborhood of x. Then, the connected component of x in N is open, which shows that X is locally connected.*

5.1.4 Path-connectedness

Definition 5.8. *Let $(X, \mathcal{T})$ be a topological space and $[x,y]$ an interval in $\mathbb{R}$. A path in a subset A of X is any continuous function $\gamma : [x,y] \longrightarrow A$. The image $\gamma([x,y])$ of the path is called an arc with starting point $\gamma(x)$ and endpoint $\gamma(y)$.*

Remark 5.3. *We can replace $[x,y]$ with $[0,1]$ because they are homeomorphic.*

Definition 5.9. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space and A a subset of X. We say that A is arc-connected if for every $a, b \in A$, there exists an arc contained in A with starting point a and endpoint b.*

Example 5.5.

1. *$\mathbb{R}$ is arc-connected. It is enough to take as a path in $\mathbb{R}$ the map $\gamma : [0,1] \longrightarrow \mathbb{R}$ defined by $\gamma(x) = a + x(b-a)$, for all $a, b \in \mathbb{R}$.*

2. *$\mathbb{Q}$ and $\complement_{\mathbb{R}}\mathbb{Q}$ are not arc-connected.*

Proposition 5.12. *An arc-connected space is connected.*

Proof. *Suppose that $\mathbb{X}$ is an arc-connected space and let $a \in \mathbb{X}$. Then, for every $b \in \mathbb{X}$, there exists a continuous function $\gamma_b : [0,1] \to \mathbb{X}$ such that $\gamma_b(0) = a$ and $\gamma_b(1) = b$. Therefore, the collection $\{\gamma_b([0,1]) : b \in \mathbb{X}\}$ forms a family of connected sets whose intersection is non-empty (since it contains a), and $\mathbb{X} = \bigcup_{b\in\mathbb{X}} \gamma_b([0,1])$ which implies that $\mathbb{X}$ is connected.*

5.2 Connectedness in Metric Spaces

The definitions and properties of connectedness in metric spaces are the same as those we have seen in topological spaces. Therefore, it is enough to give a brief reminder of these definitions and properties.

5.2.1 Definitions and properties of connectivity in metric spaces

- $\mathbb{X}$ is connected if and only if the only subsets of $\mathbb{X}$ that are both open and closed are the empty set $\emptyset$ and $\mathbb{X}$.
- $\mathbb{X}$ is connected if and only if there is no partition of $\mathbb{X}$ into two non-empty open sets.
- $\mathbb{X}$ is connected if and only if there is no partition of $\mathbb{X}$ into two non-empty closed sets.
- $\mathbb{X}$ is connected if and only if every continuous function $f : (\mathbb{X}, d) \longrightarrow (\{0,1\}, \delta)$ is constant.
- The continuous image of a connected set is connected.
- Connectivity is a topological property.
- $\mathbb{X} \times \mathbb{Y}$ is connected if and only if both $\mathbb{X}$ and $\mathbb{Y}$ are connected.
- If A is connected and $A \subset B \subset Cl(A)$, then B is connected.
- If A is connected, then $Cl(A)$ is also connected.
- $\mathbb{X}$ is arc-connected if for all $a, b \in \mathbb{X}$, there exists a continuous function $f : [0,1] \longrightarrow \mathbb{X}$ such that $f(0) = a$ and $f(1) = b$.
- Every arc-connected space is connected.
- A connected space is not necessarily arc-connected.
- A subset $A \subset \mathbb{R}$ is connected if and only if A is an interval.
- If $\mathbb{X}$ is connected and $f : \mathbb{X} \longrightarrow \mathbb{R}$ is continuous, then $f(\mathbb{X})$ is an interval.
- If $f : [a,b] \longrightarrow \mathbb{R}$ is continuous, then $f([a,b]) = [c,d]$.

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