THE THEORY OF DYNAMICAL SYSTEMS, AND ITS APPLICATIONS TO ECONOMICS

Dr. BENDJILALI Boualem
Prof. de Maths et expert économique
Banque Islamique De Développement, Arabie Saoudite

Abstract

This paper is devoted to the theory of dynamical systems and its applications to Economics. First, the paper presents some adjustment economic processes, and studies the question of stability of equilibrium. Second, some economic models will be presented that exhibit bifurcation and chaotic phenomena.

Key words: Adjustment economic processes, Stability of Equilibrium, Lyapunov function, Hopf bifurcation; Chaos.

ملخص:

يهدف هذا البحث إلى ربط نظرية النماذج الحركية بالنماذج الاقتصادية، يقدم البث في الجزء الأول بعض النماذج الديناميكية الاقتصادية، ثم تدرس الورقة مشكلة التوازن والاستقرار. ويطرح البحث في الجزء الثالث بعض النماذج الاقتصادية التي تظهر التفرع(bifurcation) والفوضى أو الانفلاق (chaos).

الكلمات الأساسية: استقرار التوازن، دالة ليبابونوف، تفرع هوبف، الفوضى أو الانفلاق.

I. Introduction.

The objective of this paper is to study the stability of equilibrium because of

1. The stability of competitive equilibrium is an essential part of General Equilibrium, for unless the stability of Competitive equilibrium can be established equilibrium prices and quantities are uninteresting as a model of the state of actual economic affairs.
2. Without an argument establishing the existence of an economic adjustment process that converges to equilibrium prices, equilibrium states, even if they exist and are optimal, loose both descriptive and normative appeal.
3. Saari (1995) states " The powerful moderating force of the market which ,if just left alone would steadily drive prices towards an equilibrium with the desired balance between demand and supply …No mathematical theory exists to justify it " (page 284)
4. Saari supports his contention that no mathematical theory exists to justify a belief in the stability of equilibrium by showing that even very simple economic models can exhibit complex dynamics and that such dynamics generally do not converge to equilibrium.

II. Preliminaries and Definitions

We consider a competitive exchange economy with n+1 commodities and m consumers ($m \geq 2$) .Each consumer is characterized by his consumption set, initial endowments, and preference relation. For each consumer i we assume the following :

a. The consumption set C^i is a compact ,convex subset of R^{n+1},containing the set

$\{ x \in X^{n+1} \text{ s.t } 0 < x_j < \sum w_j^h +1 \text{ for all } j \varepsilon I^{n+1}\}$

where $w^i = (w_1^i ,\dots W^i_{n+1})$ is the vector of endowments of the n+1 commodities of consumer i .

b. $w^i_j > 0$ for all $j \varepsilon I^{n+1}$

c. The preference relation $\geq_i$ is continuous ,monotonic ,and strictly convex.

Let $B^i (p) = \{ x \in X^i \text{ s.t } p^T x < p^T w^i \}$ denote the budget set of consumer i given the price vector p. We assume that each consumer maximizes his utility over his budget set i.e he chooses a maximal element with respect to his preferences in the budget set $B^i (p)$.

Let

$X^i (p) \in \{ y^* \in B^i(p) \text{ s.t } y^* \geq_i y \text{ for all y in } B^i(p) \}$

be a vector of commodities of consumer i given the price p. Under assumptions (a)- (c) ,for every price p this utility maximizing element is unique and it satisfies the budget constraint with equality. Therefore,

$p^T x^i (p) = p^T w^i$, for each consumer i in I^m and for every $p \in R_+^{n+1} \backslash\{ 0^{n+1}\}$

Furthermore, the *demand function* of consumer i denoted by

$x^i : R_+^{n+1} \backslash\{ 0^{n+1}\} \rightarrow R_+^{n+1}$ and point wise defined by the above is continuous .

Let $z^i : R_+^{n+1} \backslash\{ 0^{n+1}\} \rightarrow R_+^{n+1}$ be the *excess demand function* of consumer i defined by

- $z^i(p) = x^i(p) - w^i$ for every $p \in R_+^{n+1} \backslash\{ 0^{n+1}\}$

Then , $p^T z^i(p) = 0$ For every $p \in R_+^{n+1} \backslash\{ 0^{n+1}\}$. *The aggregate excess demand* function $z : R_+^{n+1} \backslash\{ 0^{n+1}\} \rightarrow R_+^{n+1}$ defined by

- $Z(p) = \sum_{i \varepsilon I^m} z^i(p)$ for every $p \in R_+^{n+1} \backslash\{ 0^{n+1}\}$

is continuous function satisfying the following properties

1. Walras' Law : $p^T z (p) = 0$ For every $p \in R_+^{n+1} \backslash\{ 0^{n+1}\}$

The Walras law says that any prices where excess demand is well defined ,the value of the excess demand evaluated at prevailing prices is zero

2. Desirability : $z_j (p) > 0$ whenever $p_j =0$

3. Homogeneity of Degree zero in prices :

$z(\mu .p) = z(p)$ for all scalars $\mu > 0$ and all $p \in R_+^{n+1} \setminus \{0^{n+1}\}$

Note that the aggregate excess demand of each commodity depends on its own price as well as on the prices of other commodities . The property of homogeneity zero in prices allows the normalization of the price space to the

n-dimensional unit simplex defined as:

$$\Delta = \left\{ p \in R_+^{n+1} \mid \sum_{j=1}^{n+1} p_j = 1 \right\}$$

Definition 1 (Price Adjustment Process**):** A price *adjustment process*, F is a map from the space of prices P into the space of prices P . If prices are normalized to lie in the unit simplex Δ , then $F : \Delta \rightarrow \Delta$.

Definition 2 (Local Stability) **:** An adjustment process is *locally stable* if for any starting point in the neighborhood of equilibrium the solution path generated by F approaches an equilibrium as $t \rightarrow \infty$

Definition 3 (Global Stability) : A price adjustment process is *globally stable* if for any starting point $p^0 \in \Delta$ the solution path $p(t; p^0)$, generated by F , approaches an equilibrium price vector as $t \rightarrow \infty$

Definition 4 (Universal adjustment process) : F is universal if it generates a solution path that converges to equilibrium for any type of excess demand map.

Given these definitions and preliminaries we now study some possible price adjustment mechanisms and their stability properties .

Definition 5 (Excess demand map) Let $w^i = (w_1^i ,\dots W_l^i)$ denote agent's i initial resources and $w = \sum w^i$ be the total initial resources , assumed to lie in the unit simplex of the prices space P. The ith agent's demand set at the price system p is defined as the maximal elements w.r.t the preference –indifference relation $\geq_i$ (reflexive, transitive ,complete) in the budget set $B^i(p) = \{ x \in X^i \text{ s.t } p^T x < p^T w^i \}$ of consumer i given the price vector p .

III. Adjustment Economic processes and their Stability

3.1 Walras' Tatonnement Process

The walras'Tatonnement process is based on the following assumptions;

- If p(t) is a price vector at time t then the price is changed in some manner if and only if p(t) is not an equilibrium .

- Agents are permitted to trade if and only if p(t) is an equilibrium
- One market at a time is considered and a price is sought to clear that market before the next market is worked on.

Markets reach their equilibrium with the help of a fictitious Auctioneer who announced prices then collected orders (or demands) from consumers as how much of each good they would wish to purchase at the announced prices .If the demand in a market exceeds the supply the price is adjusted upward. If the supply exceeds the demand, the price is adjusted downward. Walras reasoned that this procedure would cause the economy to eventually settle into equilibrium.

Mathematically, this procedure can be modeled (once we assume sufficient conditions to generate a differentiable excess demand map) as a simple differential equation in the prices of the form

$$\dot{p} = \frac{dp}{dt} = Z(p) \qquad (1)$$

This defines a process of adjusting prices, which increase the prices of any good whose excess demand is positive and decreases the price of any good whose excess demand is negative, the adjustment stops when

$Z(p) = 0$.

3.2 Samuelson's tatonnement Process

The Samuelson's tatonnement process is based on the above two first conditions but the third one is replaced by

- Prices in all markets move at the same time in response to excess demand experience in them.

If $Z_i(p)$ denotes the total excess demand for good I at price p then the process can be described mathematically by:

$$\begin{aligned} \dot{p} &= \lambda_i Z_i(p) \quad \text{for all } i = 1 \dots l \text{ where } \lambda_i \succ 0 \\ &= 0 \quad \text{if } p_i = 0 \text{ and } Z_i(p) \prec 0 \end{aligned} \qquad (2)$$

The process in (2) is economically attractive because it demands relatively little information and also because it seems mirror an intuitively reasonable state of affairs in which the price of a good is driven up by excess demand and down by excess supply.

The research at that time was focused on finding conditions on preferences /demands that ensure the global stability of (2) . The sufficient conditions that ensure global stability are the Weak Axiom of Revealed Preferences (WARP), the Diagonal Dominance (DD) and the Gross Substitutability (GS).

Definition . (WARP). An excess demand function Z(p) satisfies the weak axiom of revealed preference if for any price vectors p,p' for which $p.Z(p') < 0$ and $Z(p) \neq Z(p')$ it follows that $p'.Z(P) > 0$

Definition (GS). An excess demand function has the gross substitute property if whenever two price vectors p and p' are such that $p_k' > p_k$ for some commodity k while $p_h = p_h$ for all commodities $h \neq k$,it is true that $Z_h(p') > Z_h(p)$ for all $h \neq k$. In the case where excess demands are differentiable this means $\frac{\partial Z_i}{\partial p_i} \succ 0$, for all $h \neq k$.

Definition (DD). A set of excess demand functions satifies diagonal dominance at $p > 0$ if

$$\frac{\partial Z_i}{\partial p_i} \prec 0 \qquad \text{for all i and}$$

(b) There is a vector $h(p) = h_1(p), ...h_1(p) \succ\succ 0$ such that

(a)

$$h_i(p).\frac{\partial Z_i}{\partial p_i} \prec \sum_{n \succ j \neq i} \left| .\frac{\partial Z_i}{\partial p_j} \right| .h_j(p)$$

These conditions are found to be too restrictive. How restrictive these conditions are , are illustrated by Scarf's Example (1960) .The conclusion from Scarf's example is that there is nothing in the microeconomics of an economy that generates the stability of the process like (2).

Since there is nothing in the microeconomics of the economy to ensure the excess demand map, or its Jacobian, to have any particular structure the bulk of the stability literature turned to the issue of finding adjustment processes that converge for arbitrary excess demand maps.

The intuitively mechanism for attaining equilibrium was widely accepted (even though there were no formal proof for its validity) until 1960's when Scarf example demonstrated the existence of an open set of economies having a unique equilibrium which was unstable under the Walrasian tatonnement .

3.3 Smale's Global Newton Process.

Smale (1976) constructed a tatonnement process of the form

$$D_p Z(p)\frac{dp}{dt} = -\lambda(p) Z(p) \qquad (3)$$

Where $D_pZ(p)$ denotes the Jacobian Matrix of the excess demand function ,with the last row and column removed and

$$Sign[\lambda(p)] = (-1)^{l-1} \, Sign\,[\det D_p Z(p)] \text{ assuming that } \det D_p Z(p) \neq 0$$

The Smale process may then be written as

$$\dot{p} = -\lambda(p) D_p Z(p)^{-1} . Z(p) \qquad (4)$$

Smale showed that the process (4) converges to equilibrium for any arbitrary excess demand map Z(p) provided the process starts on the boundary of the price space . However, the equilibrium is not globally stable . In addition ,this process requires information not only about the prices but also at each step the adjustment process needs the information on all excess demand Z(p) along with all the gradients of almost all the excess demands $D_pZ(p)$. In other words , the construction of this mechanism requires information not only about prices ,but also about higher-order derivatives of all agents' utility functions.

3.4 Financial Market Equilibrium.

The formulation of the Capital Asset Pricing Model (CAPM) may be described as follows.

There are two dates t=0,1 with uncertainty about the state of nature at t=1 . A single good is available for competition at t=1,there is no consumption at t=0. Let J+1 assets that yield claims at date t=1 are traded at t=0. We assume that asset payoffs are non negative. The first asset is risk less , yielding a unit of consumption in every state ,all other assets are risky.

There are N investors in the market. Investor n is endowed with h_n^0 in R_+ units of the risk less security and a vector z_n^0 in R^J_+ of units of the risky securities.

We assume that investors agree on the mean and variance of any portfolio .

Let D_j (s) = return of the jth risky asset in state s of any portfolio

μ = vector of expected payoffs of the risky assets

Δ = [cov (D_j , D_k)] be the covariance matrix .

We assume that investors trade off mean against variance. Thus an investor who holds *h* units of the risk less asset and the vector *z* of risk assets enjoys utility

$$U_n\ (h,z) = h + [\, z.u\,] - b_n/2\ [\, z.\ \Delta\,] \qquad 3.4.1$$

Let p denotes the vector of prices of risky assets ,while the price of the riskless asset is normalized to unity.

Given prices p ,consumer n's choice set consists of portfolio (h, z) that yield non negative consumption in each state and satisfy the budget constraint

$$h + p.z < h_n^0 + p.z_n^0 \qquad 3.4.2$$

From the first order condition that characterize optima ,investor n's demand for risky assets given prices p is

$$z_n(p) = \frac{1}{b_n} \Delta^{-1} (\mu - p) \qquad 3.4.3$$

An equilibrium consists of prices p for assets and portfolio choices (h_n, z_n) for each investor so that investors optimize in their budget sets and markets clear. Solving for equilibrium prices yields

$$p = \mu - \left(\sum \frac{1}{b_n}\right)^{-1} (\Delta z^0) \qquad 3.4.4$$

Let $B = \frac{1}{N} \sum \frac{1}{b_n}$ the mean inverse risk aversion and $\tilde{z} = \frac{1}{N} \sum z_n^0$ the mean endowment.

The above equation may then be written as

$$p = \mu - B^{-1} \Delta \tilde{Z} \qquad 3.4.5$$

3.5 Basic Walrasian Model of equilibrium Discovery.

In this process ,prices are called out and agents react by submitting revised demands .Prices change subsequently as a reaction to the new excess demand.

The walrasian model supposes that an auctioneer calls out prices and that there is no intermediate trade before excess demand reaches zero. In this new context, however, the latter feature is irrelevant ,as demand for risky securities does not depend on endowment ,the Walrasian model is competitive ,as agents do not lie about their demands .Moreover, it is short sighted it assumes that agents do not speculate about subsequent price changes. In the Walrasian model, price changes are related to excess demand .Later, we will allow agents to have foresight . Agents speculate about the ability to fill demand given the state of the market and scale back demand in anticipation of demand frustration.

Reaction of demand to perceived excess demand will then become the driving force of the adjustment process and not reaction of demand to price changes .

In the prototype Walrasian adjustment process the price changes are proportional to excess demand.

$$dp = \lambda \, Z^{e}(p)\,\mathrm{dt} \qquad 3.5.1$$

where z^e denotes the vector of per capita excess demand

and λ a positive constant denoting the speed of the adjustment process. In this case there are no cross security effects. The price of a security reacts only to its own excess demand . Let $\lambda = 1$ then,

$$dp = [\,Z(p) - \tilde{Z}\,]\mathrm{dt} \qquad 3.5.2$$

where

$$Z(p) = \frac{1}{N}\sum z_n(p) \qquad \text{Morespecifically}$$

$$\dot{p} + B\Delta^{-1} p = K \qquad \text{where} \quad K = B\Delta^{-1}\mu - \tilde{Z} \qquad 3.5.3$$

This is a set of differential equations with simple matrix exponential solution.

$$p_t = p^* + e^{-B\Delta^{-1}t}(p_0 - p^*) \qquad 3.5.4$$

where

$p^* = \mu - B^{-1}\Delta\tilde{Z}$ is the equilibrium price vector and where p_0 indicates the initial price vector.

From the above we can observe the following:

1. The convergence process is exponential .It does not exhibit the damped cycling characteristic.
2. The price trajectories are determined by the covariance matrix of the payoff , Δ , which means that they inherit the correlation structure from the payoffs .This can be shown explicitly by taking small t so that the exponential matrix can be approximated in a linear fashion

$$p_t = p^* + (I - B\Delta^{-1}t)(p_0 - p^*) \qquad 3.5.4$$

Setting $\operatorname{cov}(p_0) = I_J$ the J- dimensional identity matrix it follows that

$$Cov(p_t - p_0) = \mathrm{Cov}[\,p^* - p_0 + (I - B\Delta^{-1}t)(p_0 - p\ \ )]$$

$$= \mathrm{Cov}(-B\Delta^{-1}t\,p_0) = B\Delta^{-1}t\,\mathrm{Cov}(p_0)\Delta^{-1}t^2$$

$$= B\Delta^{-1}t^2$$

3. because Δ is a covariance matrix and hence symmetric positive definite ,the solution in 3.5.2 always converges.

Let us now introduce random shocks to the convergence process. The simplest way to introduce Gaussian randomness is to add increments of a J-dimensional Brownian motion (dw) to equation 3.5.3.This will give the following system of stochastic differential equations which is a version of the Ornstein Uhlenbeck process.

$$dp = (K - B\Delta^{-1} p)\,dt + \sigma\,dw \qquad 3.5.5$$

where σ is defined to be the J-dimensional matrix of diffusion coefficient.

3.6 Walrasian Price Discovery with Cross Security Effects.

If there are cross-security effects , a non diagonal matrix multiplies Δ^{-1} in 3.5.4. If the product generates a matrix that does not have eigenvalues with negative real parts, convergence may fail.

We introduce cross security effects by putting a general adjustment matrix A in front of the excess demand in equation 3.5.1. Deleting λ,his means that equa. 3.5.1 becomes:

$$dp = A\,z^{e}(p)\,dt$$

or

$$\frac{dp}{dt} + BA\Delta^{-1} p = AK \qquad 3.6.1$$

The solution of equation 3.5.6 is of the form.

$$p_t = p^* + e^{-BA\Delta^{-1} t}(p_0 - p^*) \qquad 3.5.7$$

where

$$p^* = \mu - B^{-1}\Delta\tilde{z}$$

denotes the equilibrium price and p_0 denotes the initial price vector. We notice that the correlation of changes in prices along the adjustment path is now partly influence by the matrix A. More importantly, divergence or definite cycling as in Scarf (1960) may be obtained .That is stability is not a foregone conclusion any more.

Let us consider the case where we have two risky assets . Let a_{ij} be entry (i,j) of the matrix A . Stability can only be obtained if and only if the following conditions are satisfied.

(*i*) $\det(A) \succ 0$

(ii) $a_{1,1}\Delta_{2,2} - (a_{1,2} + a_{2,1})\Delta_{1,2} + a_{2,2}\Delta_{1,1} \succ 0$

Instability would be easily be obtained I, for instance if

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

This problem of possible outcomes when there are cross security effects forces us to think more carefully about the nature of the adjustment matrix A. Walrasian analysis does not provide an answer.

3.7 A Globally Convergent Process

In this process we imagine an autonomous market where investors adjust their demand to perceived excess demand instead of a Walrasian auctioneer who adjusts prices to excess demand. This means that the investors re-adjust the prices at which they are willing to trade (marginal utility at new demand). In this new approach, demand is an autonomous system and prices are but a derivative of the demand process.

The reaction of demand to perceived excess demand can be viewed as resulting from foresight (speculation): If investors sense excess demand , they rationally expect their demand to be frustrated in the future,and they shave back their demand ; if there is excess supply ,they rightly become optimistic that they can fill even more demand . This speculative activity is not part of the standard Walrasian system where investors are assumed to myopically adjust demand to announced prices. This new speculative demand adjustment process makes the equilibrium process globally convergent.

How does this work ? A clear sign of excess demand emerges if investors post bids which do not get filled, or when asks are quickly taken. Conversely, excess supply emerges if asks are not filled ,yet bids are immediately hit. That is ,activity in the book of a market can provide an indication of excess demand or supply.

Mathematically, we posit quantities to adjust in an autonomous fashion:

$$dz = -\lambda z^e dt \qquad (3.7.1)$$

Prices adjust because investors' marginal valuation of the securities changes as a result of the changes in demand that they hope will be fillable . In the aggregate, investors'marginal valuations can be summarized by that of an aggregate investor with risk aversion 1/B (existence of such an aggregate investor is specific to our CAPM framework) To put differently, per capita aggregate demand can be viewed as originating in the optimization of the following utility

$$u_a(h^a, z^a) = \mathrm{h}^a + \left[\mathrm{z}^a.\mu\right] - \frac{1}{2\mathrm{B}}\left[z^a.\Delta \mathrm{z}^a\right]$$

subject to

$$\mathrm{h}^a + \mathrm{p.z}^a \leq \bar{\mathrm{h}} + \mathrm{p}.\tilde{\mathrm{z}} \ ,$$

Where $\bar{\mathrm{h}} = \frac{1}{\mathrm{N}} \sum \mathrm{h}_n^0$,

per capita supply of riskfree securities, and $\tilde{\mathrm{z}}$ denotes the per capita supply of risky securities ,as before. Aggregate utility is optimized if the following first order conditions are satisfied :

$$z^a = \mathrm{B}\Delta^{-1}(\mu - \mathrm{p}) \qquad (3.7.2)$$

This shows that the optimal demand under this aggregate utility equals the actual per capita aggregate demand. Rearranging equation (3.7.2) gives

$$p = \mu - \mathrm{B}^{-1}\Delta z$$

using the implicit function theorem gives

$$\mathrm{dp} = -\mathrm{B}^{-1}\Delta dz \qquad (3.7.3)$$

Combining this with the autonomous demand evolution (3.7.1) we conclude

$$\mathrm{dp} = \mathrm{B}^{-1}\,\Delta \mathrm{z}^e\,\mathrm{dt}$$

In this formulation ,price changes derive from quantity changes. Still because the excess demand z^e is uniquely defined for any price vector p, we can write this system of differential equations only in terms of prices by making the relationship between z^e and p explicit .

$$dp = B^{-1}\Delta (B \Delta^{-1} (\mu - p) - \tilde{z}) dt$$

or

$$\frac{dp}{dt} + p = L \qquad (3.7.4)$$

Where

$$L = \mu - B\Delta^{-1} \tilde{z} = p^* ,$$

the equilibrium price vector. This new process is equivalent to the general Walrasian adjustment price system (3.6.1) provided we substitute $B^{-1} \Delta$ for the adjustment matrix A. Notice that $B^{-1} \Delta$ equals the Hessian of the utility function of the aggregate investor. In deriving the price discovery system in (3.7.4) we made use of the existence of an aggregate investor whose demand coincides with the per capita aggregate demand. In general, this is not possible , and the changes in the market-wide marginal valuation of the risky securities will be a complex function of the Hessian of all individual investors.

The solution to (3.7.4) is

$$P_t = p^* + e^{-t} (p_0 - p^*) \qquad (3.7.5)$$

Where

$$p^* = \mu - B\Delta^{-1} \tilde{z}$$

is the equilibrium price vector, and p_0 is the initial value for the price vector. This solution is globally convergent ,with exponential trajectories.

Unlike in the simple Walrasian case price change along the equilibration paths do not inherit the correlation structure from the payoffs.

3.8 The Kaldor Model .

Consider a Keynesian economy where the money price level is fixed and liquidity is supplied to the economy in such a way as to hold the interest rate constant. Let time flow continuously.

Suppose that the level of income ,Y adjusts gradually to differences between desired investment, I, and desired saving, S. Assume further that the desired level of net investment ,I , depends on the level of income and the capital stock K, and that the desired level of net saving depends on the same variables:

$$\dot{Y} = \alpha [I(Y,K) - S(Y,K)]$$

$$\dot{K} = I(Y,K) \qquad (3.8.1)$$

Here the state x = (Y,K) consists of the level of income Y ,and the capital stock, K. An equilibrium of the system of differential equations (3.8.1) consists of a level of income Y^* and a capital stock K^* such that

$$I(Y^*,K^*) = S(Y^*,K^*) = 0$$

This means that equilibrium occurs when desired net investment is zero ,so that capital stock is constant over time, and when desired net saving is also zero.

The dynamical system (3.8.1) is a non linear system

3.9 The Ricardian Model

If we look at the Ricardian model ,we see that the accumulation equation in population ,N_t takes the form

$$N_{t+1} = \frac{a}{w} N_t - \frac{b}{w} N_t^2 \qquad (3.9.1)$$

The equilibration of this system is the stationary state where employed labor is constant and profits are zero:

$$N^* = \frac{a}{w} N^* - \frac{b}{w} N^{*2} \qquad (3.9.2)$$

But we can rewrite the accumulation equation as

$$n_{t+1} = An_t (1-n_t) \qquad (3.9.3)$$

Here $n_t = \frac{bN_t}{a} \prec 1$, since the marginal product of land is a $-bN_t > 0$ and $A = a/w$ where a is the output per unit land (or worker) on the best land , and w is the subsistence wage. Since this is just the logistic equation , **values of A near 4 will produce chaotic trajectories of the model** . Rather than setting down at the stationary state, as Ricardo probably envisioned and surely many later economists working on this type of model assumed without much question, the population and capital stock being a complex chaos-like oscillation around the stationary state , with periods of over accumulation of capital alternative with periods of under accumulation.

3.10 Example of Bifurcation Analysis

We consider the Richard Goodwin's predator prey model of distribution in a capitalist economy with the 2-dimensional differential equation system:

$$\frac{\dot{u}}{u} = f(v) - \alpha$$

$$(3.10.1)$$

$$\frac{\dot{v}}{v} = \frac{1-u}{\sigma} - (\alpha + \beta)$$

Here σ is the capital output ratio in a fixed proportions production function, α is the exogenously given rate of growth of labor productivity, β is the rate of growth of the labor force ,μ is the labor share in output ,and v is the employment ratio, the proportion of the labor force employed. f (.) is a function relating the rate of growth of the real wage to the employment ratio. The above model is the Lotka- Volterra type model. It produces neutral cycles for any starting point.

For the purpose of clarifying the methods of **bifurcation analysis** ,let us modify Godwin's model to include an effect of the labor share on the rate of growth of real wages. If we represent this effect by a function g(u), the system becomes

$$\frac{\dot{u}}{u} = g(u) + f(v) - \alpha$$

(3.10.2)

$$\frac{\dot{v}}{v} = \frac{1-u}{\sigma} - (\alpha + \beta)$$

If we make the following transformation

X = lnu and y = lnv

The system becomes

$$\dot{x} = g(x) + f(y) - \alpha$$

(3.10.3)

$$\dot{y} = \frac{1-e^{x}}{\sigma} - (\alpha + \beta)$$

The linearization of the above system around the equilibrium point gives

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} g' & f' \\ -\frac{e^{X^*}}{\sigma} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{3.10.4}$$

The characteristic equation is :

$$\lambda^2 - g'\lambda + \frac{f'e^{X^*}}{\sigma} = 0$$

The eigevalues are

$$\lambda_{1,2} = \frac{g'}{2} \pm \frac{1}{2}\sqrt{g'^2 - 4\frac{f'e^{X^*}}{\sigma}} \tag{3.10.5}$$

Suppose that f' is positive, this will imply that there exists a pair of complex conjugate eigenvalues for $|g'| \prec 2\sqrt{\frac{f' e^{X^*}}{\sigma}}$. The real part Reλ = g'/2, so the linearized system is stable if g' < 0 and unstable if g' > 0 . Let us **take μ = g'/2 to be the bifurcation parameter**, and we see that this system undergoes a Hopf bifurcation at μ = 0 . Since we are in two dimensions, the center manifold is also the center subspace which is the whole space at the bifurcation point. At the bifurcation point the eigenvalues are

$\lambda^*_{1,2} = \pm i\,\omega$ where

$$\omega = \sqrt{\frac{f' e^{X^*}}{\sigma}}$$

Now let us look at the stability of the bifurcation point . This may be analyzed in two ways.

First let us try to analyse the stability of the non linear system in the center manifold at the bifurcation point directly. The exact nonlinear system at the bifurcation is:

$$\dot{x} = f' y + \frac{g'''}{6} x^3$$

$$\dot{y} = -\frac{e^{X^*}}{\sigma} x \qquad (3.10.6)$$

Define the Lyapunov function V (X,Y) by

$$V(X,Y) = \frac{1}{2}\left(X^2 + \frac{\sigma f'}{e^{X^*}} Y^2\right) \qquad (3.10.7)$$

V(0,0) = 0 and V(X,Y) ≥ 0 , so V is a possible Lyapunov function for the system . This has the derivative :

$$\dot{V} = X\dot{X} + \frac{\sigma f'}{e^{X^*}} Y\dot{Y} = f'XY + \frac{g'''}{6} X^4 - f'XY = \frac{g'''}{6} X^4 \qquad (3.10.8)$$

Thus if g ''' is negative then $\dot{V} < 0$ every where,

and this is sufficient to prove the asymptotic stability of the system. Thus g''' < 0 is a sufficient condition for the bifurcation to be stable.

An alternative approach is to apply the Hopf theorem directly. We see that $\frac{d \operatorname{Re} \lambda(\mu)}{d\mu} = 1 \neq 0$, and the system has a pair of pure imaginary eigenvalues for $\mu = 0$,so that the hypotheses of the Hopf theorem are satisfied.

Hopf bifurcations are probably the most studied type of bifurcations. Such bifurcations occur at points at which the system has a non hyperbolic equilibrium with a pair of purely imaginary eigenvalues ,but without zero eigenvalues. Also additional transversality conditions must be satisfied. (Hopf Theorem in Guckenheimer and Holmes (1983).

Hopf bifurcation requires the presence of a pair of purely imaginary eigenvalues. Hence the dimension of the system needs to be at least two. The transversality conditions which are rather lengthy are given in Glendinning (1994).

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