People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research

Setif 1 University Ferhat Abbas
Faculty of Sciences
Department of Mathematics

جامعـــة سطيف 1 فرحـــــات عباس
كـليـــة العلـــــــــوم
قســـــم الرياضيــــــــــات

Doctoral Thesis

Submitted in Fulfillment of the Requirements
for the Degree of Doctor in Mathematics

Option : Mathematics and Applications

Theme :

DIFFERENTIAL-NONDIFFERENTIABLE GAMES AND APPLICATIONS IN ECONOMICS

Presented by

Miss. **BENMENNI Rania**

Supervisor : **Professor Nourreddine DAILI**

Thesis defended on 30/06/ 2025, in front of the jury composed of :

Mr.	Salim MESBAHI	Prof.	Setif 1 University Ferhat Abbas	President
Mr.	Nourreddine DAILI	Prof.	Setif 1 University Ferhat Abbas	Supervisor
Mr.	Abderrahmane ZIAD	Prof.	University of Caen Normandy, France	Co- supervisor
Mr.	Moussa AHMIA	Prof.	Jijel University Mohamed Seddik Ben Yahia	Examiner
Mr.	Mohamed ZERGUINE	Prof.	Batna 2 University Mustapha Ben Boulaid	Examiner
Mr.	Hamza GUEBBAI	Prof.	Guelma University 8 Mai 1945	Examiner
Mr.	Abdelhamid BENHOCINE	Prof.	Setif 1 University Ferhat Abbas	Guest

2025

Acknowledgements

*First and foremost, I would like to thank **Allah**, who supported me, and gave me the strength to complete this work. Without His guidance and blessing, this project would not have been possible.*

I would like to sincerely thank my supervisor, Prof. Nourreddine DAILI and Prof. Abderrahmane ZIAD, for their invaluable guidance and support throughout this research.

I would like to express my sincere thanks to the president of the committee, Prof. Salim MESBAHI, and the examiners, Prof. Moussa AHMIA, Prof. Mohamed ZERGUINE and Prof. Hamza GUEBBAI as well as the invited member, Prof. Abdelhamid BENHOUCINE, for their time and effort to read and review this thesis.

I would also like to thank my professors, Mr. Alia Ishak and Ms. Zeghdane Rebiha, for their valuable advice and help.

A special thank you to my parents, for their constant support, and trust. Their presence at every step of my journey has provided me with the strength and motivation to overcome challenges.

I also thank my sisters for their constant support and encouragement throughout this journey.

Finally, my gratitude to my friends who have accompanied and challenged me along my academic journey.

Dedication

To my parents, for their endless support and constant encouragement.

Publications

Published article

- Rania Benmenni and Nourreddine Daili, A Connection between the Adjoint Variables and Value Function for Differential Games, Stat, Optim. Inf. Comput , Vol. 13, January 2025, pp. 173-188. DOI : 10.19139/soic-2310-5070-2115.

Submitted article

- R.Benmenni, R.Zaghedane, N.Daili, Jacobi Spectral Method for Solving Differential Game Problems.

Conference proceedings

- Rania Benmenni, Open Loop Nash and Stackelberg Equilibrium in Nonzero-Sum Differential Games, 7 th International Workshop on Applied Mathematics and Modelling WIMAM 2023 ; held on 13 and 14 December 2023 at University 8 Mai 1945 Guelma.
- Rania Benmenni and Nourreddine Daili, A Nash Strategy For Differential Games, International Conference on Mathematics and its Applications in Science and Technology. ICMAST2024-Setif, December 15-16, 2024.
- Rania Benmenni, Feedback saddle point in linear quadratic zero-sum differential games, Second National Conference on Applied Mathematics and Didactics 2NCAMD2023 held online on May 13, 2023, Ecole Normale Supérieure El Katiba Assia Djebar Constantine.
- Rania Benmenni and Nourreddine Daili, On the Relation between the Maximum Principle and Dynamic Programming Principle for Differential Games, National Conference on Mathematics and Applications (NCMA 2023), Ferhat Abbas Universiy Setif 1, May 15-16, 2023.

Table of contents

List of tables

List of figures

Abbreviations and Notations

- **NZSDGs** : Nonzero-sum Differential Games.
- **MP** : Maximum Principle.
- **DPP** : Dynamic Programming Principle.
- **ODE** : Ordinary Differential Equation.
- **PDE** : Partial Differential Equation.
- **LQDGs** : Linear Quadratic Differential Games.
- **LSDGs** : Linear State Differential Games.
- **NE** : Nash Equilibrium.
- **SPE** : Saddle Point Equilibrium.
- **OLNE** : Open-Loop Nash Equilibrium.
- **FNE** : Feedback Nash Equilibrium.
- **HJB** : Hamilton-Jacobi-Bellman.
- **RDE** : Riccati Differential Equations.
- **TPBVPs** : Two-point Boundary Value Problems.
- **JPs** : Jacobi Polynomials.
- **JSM** : Jacobi Spectral Method.
- **CPM** : Chebyshev Pseudospectral Method.
- **BTM** : Bernoulli Tau method.
- **a.e** : Almost everywhere.
- $\mathbb{R}$: The set of real numbers.
- $\mathbb{R}^n$: Euclidean n -dimensional space.
- $\mathbb{R}^{m_i}$: Euclidean m_i-dimensional space.
- $\mathbb{R}^{n\times m}$: Space of $(n \times m)$ real matrices.
- $\mathbb{N}$: Set of natural numbers.
- $\langle .,. \rangle$: The inner product in some Euclidean space.

- $\|.\|$: The Euclidean norm.
- $z^{\top}$: The transpose of the vector (or matrix) z.
- ψ_y^i : The gradient or derivative of ψ^i with respect to y.
- $\mathcal{B}_i\,[0,T]$: The set of admissible controls of player i.
- $C\left([0,T]\,;\mathbb{R}^n\right)$: The set of all continuous functions $\psi : [0,T] \rightarrow \mathbb{R}^n$.
- $C^1\left([0,T]\,;\mathbb{R}^n\right)$: The set of continuous differentiable functions.
- $\mathbb{L}^2([0,T]\,;\mathbb{R}^n)$: Set of all measurable functions that are square integrable over all finite intervals $[0,T]$.
- $\mu^N(x)$: Spectral Approximation.
- $\mathcal{J}_r^{k,\ell}\,(x)$: Jacobi Polynomials of degree r with $k,\ell > -1$.
- $\omega^{k,\ell}(x)$: Jacobi weight function in $I =: (-1,1)$.
- $\mathbb{L}^2_{\omega^{k,\ell}}\,(I)$: Space of all measurable functions, with $\|\mu\|_{\omega^{k,\ell}} = (\int\limits_I |\mu(x)|^2\omega^{k,\ell}dx)^{\frac{1}{2}}$.
- $\mathbb{H}^m_{\omega^{k,\ell}}\,(I)$: The weighted Sobolev spaces.
- P_N : The space of all polynomials of degree at most N.
- $\mathcal{J}_{T,r}^{k,\ell}\,(s)$: Shifted Jacobi Polynomials of degree r.
- $Res_i\,(s)$: Residual functions.

Introduction

Differential games are a kind of dynamic game that evolve over time. The state of the game is represented by a system of differential equations involving multiple decision-makers, known as players. Each player aims to minimize or maximize his individual criteria ([35], [9]). Applications of differential games have been used in many fields, such as economics and management science [23], military defense [35] and biology.

On the other hand, differential games are an extension of optimal control problems (OCPs). Due to their connection, some of the concepts and techniques used in the solution of OCPs can also be applied in the solution of differential game problems such as Pontryagin maximum principle (MP) and Bellman's dynamic programming principle (DPP) serve as the main significant approaches for differential games (see e.g., [9]). The MP approach characterizes the open-loop Nash equilibrium (OLNE) solution of the differential games using Hamiltonian function and adjoint variables. An OLNE refers to a situation in nonzero-sum differential games where the strategies of the players depend on the initial state of the system and time. These strategies can be determined by solving two-point boundary value problems (TPBVPs), which are derived from MP. This principle provides the necessary conditions for the existence of an OLNE by describing how players must adjust their strategies over time to maximize or minimize their individual objectives, taking into account interactions with other players (see e.g., [9]). Whereas, the DPP characterizes the feedback Nash equilibrium (FNE) using the value function solution to the Hamilton-Jacobi-Bellman (HJB) equations (see e.g., [9]) and there is a close relationship between them. The relation between MP and DPP can be regarded as the connection between adjoint variables and the value function, or the Hamiltonian systems and the HJB equations ([57]). There is a lot of research on the study of relationship between them in deterministic and stochastic optimal control problems (Single player differential games) (see [19, 34, 37, 42, 45, 53, 57, 59]).

The connection between MP and DPP for optimal control problems with a smooth value function was established by Fleming and Rishel [29], Yong and Zhou [57] and further investigated by Shi in [49] for (zero-sum) stochastic differential games with jump diffusions. However, even in very simple cases, the value function is not smooth and the HJB equations may not have a smooth solution at all, so this equations must be studied in viscosity solution (VS). This new notion is a kind of nonsmooth solutions was first proposed by Crandall and Lions [21] (see also Crandall et al. [22] and [6]) to overcome the difficulty that the value function of differential games or single player differential games (OCPs) is not smooth. The VS provides researchers to explore relationships between adjoint variables and value functions of deterministic and stochastics OCPs (see [7, 19, 20, 34, 37, 41, 42, 45, 53, 57, 59]).

The connection between the adjoint variables in MP and the value function in DPP for optimal control problems has important applications in mathematical economics and finance. Yong and Zhou [57] discusses the economic interpretations of the adjoint variable, also known as the shadow price, in both smooth and nonsmooth of the value function. For zero-sum stochastic differential games with jump diffusions, Shi [49] discusses a portfolio optimization problem under model uncertainty in an incomplete financial market in the smooth case.

Differential game problems are often solved numerically, as analytical solutions are not always available. Several numerical methods have been proposed, including algorithms based on dynamic programming [28], direct and indirect methods [36], an iterative adaptive dynamic programming method for solving nonlinear zero-sum differential games [58], and the complementarity theory, which is specifically used for solving zero-sum pursuit-evasion differential games [54]. The most efficient and accurate numerical method for solving various types of differential equations are spectral methods (see, [48], [15], [16]), which are based on truncated series of orthogonal polynomials. These methods are commonly classified into three main categories : Galerkin, Tau, and collocation methods. Among the orthogonal polynomials, Jacobi polynomials (see eg., [24, 52]), including special cases such as Legendre and Chebyshev polynomials (see eg., [13, 15, 32]), are widely used in mathematical analysis and practical applications due to their strong convergence properties. A number of studies have investigated the application of spectral methods to solve open-loop Nash equilibrium in nonlinear differential games. Specifically, pseudospectral methods have been used to solve nonlinear two-point boundary value problems (TPBVPs) in nonzero-sum differential games and min-max optimal control problems with uncertainty ([43], [44]). In [3], the Legendre Tau method to find the OLNE of noncooperative nonzero-sum differential games. In addition, the Bernoulli Tau method has been used to compute these equilibria in nonlinear differential games [5].

The main objective of this thesis, which focuses on the study of the deterministic nonzero-sum differential games (NZSDGs) on a finite horizon, consist of two parts. In the first part, we present the connection between the adjoint variables in MP and the value function in the DPP for differential games. We analyze both the smooth and nonsmooth cases of the value function in terms of the derivatives and super- and subdifferentials, with their economic interpretations of the adjoint variables. This result represents a generalization of the results in [57] related to deterministic OCPs.

In the second part, we apply an appropriate numerical method based on the Jacobi spectral method (JSM) to approximately solve the nonlinear TPBVPs derived from MP are transferred to a system of algebraic equations in order to obtain the OLNE of NZSDGs.
The thesis is structured as follows :

- The first chapter provides a general overview of differential game theory by progressively introducing the basic concepts of optimal control theory, game theory, and differential games.

- The second chapter, we present a deterministic two-player NZSDGs on a finite horizon. We explore the connection between the adjoint variables in the maximum principle (MP) and the value function in the DPP for differential games, both in the smooth and nonsmooth cases of the value function. The connection is established in terms of derivatives as well as sub and super-differentials of the value function. This chapter represent a generalization of the results in [57] related to deterministic OCPs.
- In the third chapter, we provide an example of a producer-consumer game with sticky prices taken from [14] to illustrate the theoretical results from the second chapter. This application illustrates how this connection can be applied to real-world situations, focusing on the economic interpretations of adjoint variables in differential games.
- In the fourth chapter, we present a numerical method based on the MP, using the Jacobi spectral method (JSM), which allows us to approximately solve the nonlinear TPBVPs in order to determine the OLNE of NZSDGs in a finite horizon. We then discuss the application of the JSM to solve these differential games. Finally, we provide some examples to demonstrate the accuracy and usefulness of the proposed method.

The results presented in this thesis have been published or are scheduled for submission to international journals. Chapters 2 and 3 are included in [12]. The final Chapter is a preprint and will be submitted for publication.

CHAPTER 1

AN OVERVIEW OF DIFFERENTIAL GAMES THEORY

In this chapter, we provide an overview of the theoretical foundations of differential games, a branch of applied mathematics and game theory that focuses on dynamic systems in interaction. These systems are commonly used in contexts where agents' decisions are influenced by temporal and strategic factors. We begin by introducing optimal control theory, which offers the necessary tools to formulate and solve dynamic decision making problems. This theory focuses on optimizing performance functions subject to dynamic constraints, with methods such as the maximum principle and dynamic programming principle to finding optimal strategies over time. We then delve into game theory, which provides a framework for modelling and analyzing strategic interactions between rational agents (players). Key concepts such as Nash equilibrium, where no player has an incentive to unilaterally change their strategy, are explored. Finally, we examine differential games, which combine the principles of optimal control and game theory to address situations where agents' decisions evolve over time in dynamic systems. Special attention is given to LQDGs, where linear differential equations and quadratic cost functions. A more comprehensive background on differential game theory can be found in the references (see [9,10,35,50]).

1.1 Optimal Control Theory

This section deals with the theory of optimal control (OC). It can be seen as a theory for single-player differential games. First, we present the formulation of OC problems. Then we introduce the two main approaches to solving deterministic OC problems, namely, Maximum Principle and Dynamic Programming Principle. For more details on the concepts presented in this section, see reference [57].

1.1.1 Problem Formulation

Consider the following control system

$$\begin{cases} \dot{y}(s) = F(s, y(s), b(s)), \ s \in [0, T] \\ y(0) = y_0, \end{cases} \tag{1.1}$$

where $F : [0, T] \times \mathbb{R}^n \times B \to \mathbb{R}^n$ is a measurable map that represents the dynamical system. $(B; d)$ is a separable metric space representing the action space of the controller ; $y(.)$ is the state variable, and $y_0 \in \mathbb{R}^n$ is the initial state ; $T \in \bar{\mathbb{R}}_+$ is called the horizon of the system. For any instant $s \in [0, T]$, the controller (decision-makers) has to choose an action $b(s) \in B$ in order to influence the trajectory of the state of his system. Any measurable function $b : [0, T] \to B$ is called a control or a feasible strategy of the controller and $y(.)$ solution of (1.1) is called the corresponding state trajectory of $b(.)$.
The set of feasible controls is defined

$$\mathcal{B}_f [0, T] = \{b : [0, T] \to B, \ b(.) \text{ measurable}\}.$$

The cost function is presented as follows

$$J(b(.)) = \int_0^T G(s, y(s), b(s))\, ds + h(y(T)), \tag{1.2}$$

The functions $G : [0, T] \times \mathbb{R}^n \times B \to \mathbb{R}$ is called the running cost and $h : \mathbb{R}^n \to \mathbb{R}$ is terminal cost.

Definition 1.1. (Admissible Control). A feasible control $b(.) \in \mathcal{B}_f [0, T]$ is called an admissible control, and $(y(.), b(.))$ called an admissible pair, if :

1. the equation (1.1) has an unique solution $y(s)$;
2. $J(b(.)) \langle\infty$.

Denote $\mathcal{B}[0, T]$ the set of all admissible controls. The OC problem is stated as follows :
Problem (OC). Find a control $\bar{b}(\cdot) \in \mathcal{B}[0, T]$, such that

$$J(\bar{b}(\cdot)) = \inf_{b(\cdot) \in \mathcal{B}[0,T]} J(b(\cdot)). \tag{1.3}$$

Any control $\bar{b}(\cdot) \in \mathcal{B}[0, T]$ that satisfies (1.3) is considered as an optimal control to Problem(OC), the associated state trajectory $\bar{y}(s)$ is known as the optimal state trajectory, and $\left(\bar{y}(\cdot), \bar{b}(\cdot)\right)$ is called an optimal pair.

1.1.2 Approaches to Solving Optimal Control Problem

This section presents two of the most widely used approaches to solving optimal control problems specifically Maximum Principle, which involves the use of adjoint equations along with both necessary and sufficient optimality conditions, and the dynamic programming principle, which is coupled with the Hamilton-Jacobi-Bellman equations. These methods provide powerful tools for determining optimal solutions.

The Maximum Principle (MP)

Consider the OC problem (1.1)-(1.2) and we give the tools needed to state the necessary conditions for the MP.
We introduce the following assumptions :

(OC1) B is a separable metric space;

(OC2) F is continuous, linear growth and continuously differentiable in (y,b), and F_y, F_b are bounded and uniformly Lipchitz in (y,b). There existe $M > 0$, such that for all $y, \hat{y} \in \mathbb{R}^n, b, \hat{b} \in B$,

$$|F(s,y,b)| \leq M(1+|y|+|b|)$$

and

$$|F_y(s,y,b)| + |F_b(s,y,b)| \leq M$$
$$\left|F_y(s,y,b) - F_y\left(s,\hat{y},\hat{b}\right)\right| + \left|F_b(t,y,b) - F_b\left(s,\hat{y},\hat{b}\right)\right| \leq M\left(|y-\hat{y}| + \left|b-\hat{b}\right|\right);$$

(OC3) the functions G and h are C^1 in (y,b) and its derivatives are uniformly Lipchitz and linear growth in (y,b). There exist a constant $M > 0$, such that

$$|G_y(s,y,b)| + |G_b(s,x,b)| \leq M(1+|y|+|b|),$$
$$|h_y(y)| \leq M(1+|y|).$$
$$\left|G_y(s,y,b) - G_y\left(s,\hat{y},\hat{b}\right)\right| + \left|G_b(s,y,b) - G_b\left(s,\hat{y},\hat{b}\right)\right| \leq M\left(|y-\hat{y}| + \left|b-\hat{b}\right|\right)$$
$$|h_y(y) - h_y(\hat{y})| \leq M|y-\hat{y}|, \ \forall y, \hat{y} \in \mathbb{R}^n, b, \hat{b} \in B.$$

Under assumption (OC2) for any $(s,y) \in [0,T] \times \mathbb{R}^n$ and the controls $b(.) \in \mathbb{L}^2([0,T];\mathbb{R}^m)$, equation (1.1) admits a unique solution $y(\cdot)$ and under (OC3) the functional (1.2) is well-defined (see [57]).
We can define the following set of all admissible control as

$$\mathcal{B}([0,T]) = \{b(.) : [0,T] \to B | b(.) \in \mathbb{L}^2([0,T];\mathbb{R}^m)\}.$$

The Hamiltonian function, $H : [0,T] \times \mathbb{R}^n \times B \times \mathbb{R} \to \mathbb{R}$ is defined by

$$H(s,y,b,p) = \langle F(s,y,b), p\rangle + G(s,y,b),$$

where p is called the adjoint or co-state variable.
The following adjoint equation

$$\begin{cases} \dot{p}(s) = -H_y(s, y(s), b(s), p(s)), \ s \in [0,T] \\ p(T) = h_y(y(T)), \end{cases} \tag{1.4}$$

Theorem 1.1. *Let (OC1)-(OC3) hold. Suppose that $\left(\bar{y}(.), \bar{b}(.)\right)$ is an optimal pair of Problem (OC), then there exist a unique $(\bar{p}(.)) \in (C([0,T]; \mathbb{R}^n))$ solution of the adjoint equations*

$$\begin{cases} \dot{\bar{p}}(s) = -H_y\left(s, \bar{y}(s), \bar{b}(s), \bar{p}(s)\right), \ s \in [0,T] \\ \bar{p}(T) = h_y(\bar{y}(T)) \end{cases}, \tag{1.5}$$

and,

$$H\left(s, \bar{y}(s), \bar{b}(s), \bar{p}(s)\right) = \inf_{b \in B} H(s, \bar{y}(s), b, \bar{p}(s)),$$

Remark 1.1. The following system is called an Hamiltonian system.

$$\begin{cases} \dot{\bar{y}}(s) = F\left(s, \bar{y}(s), \bar{b}(s)\right), \\ \dot{\bar{p}}(s) = -H_y\left(s, \bar{y}(s), \bar{b}(s), \bar{p}(s)\right), \ s \in [0,T], \\ \bar{p}(T) = h_y(\bar{y}(T)), \quad \bar{y}(0) = y_0, \\ H\left(s, \bar{y}(s), \bar{b}(s), \bar{p}(s)\right) = \inf_{b \in B} H(s, \bar{y}(s), b, \bar{p}(s)). \end{cases}$$

Now, under some appropriate convexity conditions, we recall the following sufficiency conditions for optimal control problem.

(OC4) H is convex in (y,b) and h is convex in y, $\forall s \in [0,T]$.

Theorem 1.2. *Let (OC1)-(OC4) hold. Suppose that $\left(\bar{y}(.), \bar{b}(.)\right)$ is an admissible pair. Suppose there exist a solution $(\bar{p}(.)) \in C([0,T]; \mathbb{R}^n)$ of the adjoint equations (1.5). Then, $\left(\bar{y}(.), \bar{b}(.)\right)$ is an optimal pair if*

$$H\left(s, \bar{y}(s), \bar{b}(s), \bar{p}(s)\right) = \inf_{b \in B} H(s, \bar{y}(s), b, \bar{p}(s)),$$

The Dynamic Programming Principle (DPP)

Dynamic programming, introduced by Richard Bellman [11] in the early 1950s, involves solving a family of optimal control problems by the Hamilton-Jacobi-Bellman (HJB) equation, as a nonlinear first-order partial differential equation (PDE). When the HJB equation is solvable, it allows for the determination of an optimal feedback control by maximizing or minimizing the

Hamiltonian involved in the HJB equation; see for example [30] and [57] for a more detailed discussion.

For $t \in [0,T]$ and $x \in \mathbb{R}^n$, consider a control system given by the following ordinary differential equations (ODE)

$$\begin{cases} \dot{y}(s) = F(s, y(s), b(s)), \ s \in [t,T], \\ y(t) = x. \end{cases} \tag{1.6}$$

The cost functional defined by

$$J(t,x;b(\cdot)) = \int_t^T G(s, y(s), b(s))\, ds + h(y(T)). \tag{1.7}$$

We need to the following assumptions :

(OC5) F is uniformly Lipchitz and linear growth in (y,b). There existe $M>0$, such that for all $y, \hat{y} \in \mathbb{R}^n, b, \hat{b} \in B$,

$$|F(s,y,b)| \leq M(1 + |y| + |b|)$$
$$\left|F(s,y,b) - F\left(s, \hat{y}, \hat{b}\right)\right| \leq M(|y - \hat{y}|).$$

(OC6) The functions G and h are continuous and quadratic growth in (y,b).

Under assumption (OC5) for any $(s,y) \in [0,T] \times \mathbb{R}^n$ and the controls $b(.) \in \mathcal{B}[0,T]$, equation (1.6) admits a unique solution $y(\cdot)$ and under (OC6) the functional (1.7) is well-defined (see [57]). Then, we define the value function

$$\begin{cases} W(t,x) = \inf\limits_{b(\cdot)\in\mathcal{B}[t,T]} J(t,x,b(\cdot)) \\ W(T,x) = h(x), \end{cases} \tag{1.8}$$

which satisfies the Problem (OC).

Theorem 1.3. *Suppose (OC1), (OC5) and (OC6) hold. Then for any $(s,y) \in [0,T] \times \mathbb{R}^n$,*

$$W(t,x) = \inf_{b(\cdot)\in\mathcal{B}[t,T]} \left\{ \int_t^{\hat{t}} G(s, y(s), b(s))\, ds + W\left(\hat{t}, y\left(\hat{t}\right)\right) \right\}, \forall \hat{t} \in [t,T].$$

The HJB equations defined by,

$$\begin{cases} \dfrac{\partial W}{\partial t}(t,x) + H^*\left(t, x, \dfrac{\partial W}{\partial x}(t,x)\right) = 0, \forall (t,x) \in [0,T] \times \mathbb{R}^n \\ W(T,x) = h(x), \end{cases} \tag{1.9}$$

where $W(\cdot,\cdot) \in (C^1([0,T]\times\mathbb{R}^n);\mathbb{R})$ and

$$H^*\left(t,x,\frac{\partial W}{\partial x}(t,x)\right) = H\left(t,x,\bar{b},\frac{\partial W}{\partial x}(t,x)\right) = \inf_{b\in\mathcal{B}[t,T]} H\left(t,x,b,\frac{\partial W}{\partial x}(t,x)\right).$$

Theorem 1.4. *(Verification Theorem). Let assumptions (OC1), (OC5) and (OC6) hold. Assume that $W(\cdot,\cdot)\in C^1([0,T]\times\mathbb{R}^n)$ is a solution to equations (1.9). Then we have the following :*
(i) $W(t,x)\leq J(t,x;b(\cdot)), \forall (t,x)\in[0,T]\times\mathbb{R}^n, b(\cdot)\in\mathcal{B}[t,T];$
(ii)suppose

$$\frac{\partial W}{\partial t}(t,x) + H^*\left(t,x,\frac{\partial W}{\partial x}(t,x)\right) = 0, \forall (t,x)\in[0,T]\times\mathbb{R}^n,$$

and there exist an $(\bar{b}(\cdot))\in\mathcal{B}[t,T]$ admissible strategy with the corresponding state trajectory $\bar{y}(\cdot)$ for Problem (OC)

$$H^*\left(\hat{t},\bar{y}(\hat{t}),\frac{\partial W}{\partial x}(\hat{t},\bar{y}(\hat{t}))\right) = H\left(\hat{t},\bar{y}(\hat{t}),\bar{b}(\hat{t}),\frac{\partial W}{\partial x}(\hat{t},\bar{y}(\hat{t}))\right), \ \forall\hat{t}\in[t,T].$$

Then $\left(\bar{b}(\cdot),\bar{y}(\cdot)\right)$, is an optimal pair for Problem (OC) in (t,x).

1.2 Basic Notions of Game Theory

Game theory provides a framework for analyzing strategic interactions among rational multiple decision makers, called players, where each player seeks to achieve their own goals and each decision is influenced by the others. This section introduces game classification and strategic form games. It then presents the central concept of Nash equilibrium.

1.2.1 Game Classification

Games can be classified into various categories based on factors such as players interaction, available information, and the objective functions (see. eg [8]).

- **Simultaneous and Sequential** :
  In simultaneous or static games, players make their decisions at the same time, the game is usually represented in normal form (stategic form). For sequential (or dynamic) games are played in stages, with each player's decision depending on previous actions.
- **Complete information and incomplete** :
  In games with complete information, all factors of the game, such as the players, their strategies, and their objective functions (payoff or cost functions), are known to all players. Otherwise, the game is said to be incomplete.
- **Zero-sum and nonzero-sum game** :

A zero-sum game is a game in which the sum of the objective functions of two players is zero. A nonzero-sum game, on the other hand, is a game in which the sum of the players objective functions cannot be made to equal zero [8].

- **Cooperative and non-cooperative games** :
  Cooperative games are those in which players can form coalitions and agreements to achieve mutual benefits, while in non-cooperative games, players make independent decisions, without cooperating, each aiming to maximize (minimize) their own payoff (cost) based on the strategies of others.

In cooperative game theory, the Shapley value is one of the most fundamental solution concepts [47]. Aumann and Shapley [1] define the Shapley value specifically for nonatomic (market) games. Edhan [26] introduces new diagonal formulas for the Mertens value [38] and the Neyman value [40] for a large space of non-differentiable games. In this thesis, we focus on non-cooperative games, especially in the context of differential games.

1.2.2 Strategic-Form Games

Definition 1.2. (see. eg. [31]). A game in normal form (strategic form) consists of :

(i) a finite number of players $N = \{1, 2, ..., n\}$;

(ii) for each player i, a set X_i of available strategies;

(iii) for each player i, a payoff function (cost functions) $J_i : X \to \mathbb{R}$, which assigns a specific payoff to each player based on their strategy and the strategies of others,
where, $X = \prod_{i=1}^{n} X_i$ denotes the set of all possible strategy profiles.

The following notations can be presented as follows :

- for a strategy profile $x = (x_1, ..., ..., x_n)\,, \forall i \in \{1, 2, ..., n\}\,, x_i \in X_i$;
- for the strategy profile of all players except player i,

$$x_{-i} = (x_1, ..., x_{i-1}, x_{i+1} ..., x_n)\,;$$

- X_i represents the strategy set of player i, containing all possible strategies available to that player;
- X_{-i} is the set of strategies of all players except player i,

$$X_{-i} = X_1 \times ... \times X_{i-1} \times X_{i+1} \times ...X_n.$$

1.2.3 Nash Equilibrium

The Nash equilibrium (NE) is a key concept in non-cooperative game theory, where cooperation among players is often difficult. It describes a situation in which each player makes the best decision possible, taking into account the decisions of the others. In this state, no player has an incentive to change their strategy unilaterally. The concept was developed by the mathematician John Nash [39].
The best-response function is defined as follows :

Definition 1.3. The strategy x_i^* for a player i is considered a best response to a strategy profile x_{-i} if

$$J_i\left(x_i^*, x_{-i}\right) = \inf_{x_i \in X_i} J_i\left(x_i, x_{-i}\right).$$

The best response function can also be expressed as :

$$\begin{aligned} \mathcal{BR}_i : \quad & X_{-i} \rightarrow \mathcal{P}\left(X_i\right) \\ & x_{-i} \rightarrow \mathcal{BR}_i\left(x_{-i}\right) = \left\{x_i^* \in X_i,\ J_i\left(x_i^*, x_{-i}\right) \leq J_i\left(x_i, x_{-i}\right)\right\}, \forall x_i \in X_i. \end{aligned}$$

We note $\mathcal{BR}_i\left(x_{-i}\right)$ the set of rational responses of player i against strategies x_{-i}.

Definition 1.4. (Nash Equilibrium, see e.g., [8, 9, 31]). A strategy profile $x^* \in X$ is a Nash equilibrium if

$$x_i^* \in \mathcal{BR}_i\left(x_{-i}^*\right), \forall i \in \{1, 2, ..., n\}.$$

In other terms,

$$J_i\left(x_i^*, x_{-i}^*\right) \leq J_i\left(x_i, x_{-i}^*\right), \forall x_i \in X_i.$$

1.3 Differential Games

This section focuses mainly on differential games, an extension of game theory to dynamic contexts. We start with the problem formulation of differential games. Information structures and strategies are then introduced, and finally a special case of differential games, linear-quadratic differential games, is discussed.

1.3.1 Problem Formulation

The concepts introduced for the case where $n = 1$ for the optimal control problem, as discussed in Section 1.1 can be extended to the general case of n players. Consider the n-player nonzero-sum differential games on finite horizon and the dynamical system [9] :

$$\begin{cases} \dot{y}\left(s\right) = F\left(s, y\left(s\right), b_1\left(s\right), b_2\left(s\right), ..., , b_n\left(s\right)\right),\ s \in [0, T] \\ y(0) = y_0, \end{cases} \tag{1.10}$$

where $y(s) \in \mathbb{R}^n$ is the state variables of the game and $F : [0,T] \times \mathbb{R}^n \times \prod_{i=1}^{N} B_i \to \mathbb{R}^n$ is a function determining the evolution of the system, and T is a time horizon. The control (strategy) for the i-th player $b_i : [0,T] \to B_i$, for some given sets $B_i \subset \mathbb{R}^{m_i}$, $(B_i = B_1 \times B_2 \times ..... \times B_n, i = 1,2,...,n)$, and $\mathcal{B}_i$ is called admissible set of the $b_i(\cdot) = (b_1(\cdot), b_2(\cdot), ...b_n(\cdot))$ defined by the following :

$$\mathcal{B}_i\,[0,T] = \{b_i\,(.) : [0,T] \to B_i | b_i\,(.) \in \mathbb{L}^2([0,T]\,;\mathbb{R}^{m_i})\},\ i = 1,2,....,n;$$

and the cost functionals as follows

$$J_i(b_1(\cdot),....,b_n(\cdot)) = \int_0^T G_i\,(s, y\,(s)\,, b_1\,(s)\,, b_2\,(s)\,, ..., b_n\,(s))\,ds + h_i\,(y\,(T))\,,\ i = 1,2,\cdots,n. \quad (1.11)$$

Here G_i is a running cost and h_i is a terminal cost.
The aim of the i-th player is to minimize his own cost functionals (1.11). This problem is known as the n-player nonzero-sum differential game problem given as follows.
Problem (NZSDG). Find a $\bar{b}_i(\cdot) = \left(\bar{b}_1(\cdot), \bar{b}_2(\cdot), ..., \bar{b}_n(\cdot)\right) \in \mathcal{B}_i\,[0,T]\,, i = 1,2,...,n$ such that

$$J_i(\bar{b}_i(\cdot), \bar{b}_{-i}(\cdot)) = \inf_{b_i(\cdot)\in\mathcal{B}_i[0,T]} J_i(b_i(\cdot), b_{-i}(\cdot)). \quad (1.12)$$

where $b_i(\cdot)$ is the control (strategy) for the i-th player and $b_{-i}(\cdot)$ are the controls for the rest of the players $b_{-i} = b_j, (j \neq i)$.
An n-player NZSDGs (1.10)-(1.11) requires assumptions on the functions F, G_i and h_i to ensure the existence of a unique solution of (1.10) and the well-defined of the functionals (1.11). These assumptions commonly used in differential games (see e.g., [9, 10]) and are presented in Chapter 2 for two-player nonzero-sum differential games, which can be extended to the general case with n players. The following theorem provides conditions that ensure the existence and uniqueness of the state trajectory $y(.)$.

Theorem 1.5. *Let the differential games defined by (1.10)-(1.11). Then, if the function F satisfy,*

$$|F\,(s, y, b_1, ..., b_n) - F\,(s, \hat{y}, b_1, ..., b_n)| \leq M\,|y - \hat{y}|\,,$$
$$|F\,(s, y, b_1, ..., b_n)| \leq M\,(1 + |y|)\,,$$

then for any measurable $b_i\,(\cdot)\,, i = 1,...,n$, the equation (1.10) admits a unique state trajectory $y(.)$

The NE solution concept for the n-player differential game defined by (1.10) and (1.11) given as follows :

Definition 1.5. The control strategies $\bar{b}_i(\cdot) = (\bar{b}_1(\cdot), \bar{b}_2(\cdot), ..., \bar{b}_n(\cdot)) \in \mathcal{B}_i\,[0,T]$ is a NE of Problem

(NZSDG) if the following holds :

$$J_i(\bar{b}_i(\cdot), \bar{b}_{-i}(\cdot)) \leq J_i(b_i(\cdot), \bar{b}_{-i}(\cdot)),\ \forall b_i(\cdot) \in \mathcal{B}_i\left[0, T\right],\ i = 1, ..., n. \tag{1.13}$$

This means that the NE is a situation where, for each player i, the strategy $\bar{b}_i(\cdot)$ of that player is a best response to the strategies of the other players $\bar{b}_{-i}(\cdot)$.

1.3.2 Information Structures and Strategies

The information structure in a differential game represents the available information for each player at any given time and that significantly influences their decision making process. The NE depends on the information structure employed (see e.g., [9, 10]). In this context, we focus on open-loop Nash equilibrium (OLNE) and feedback Nash equilibrium (FNE) strategies. In an open-loop information structure, players' decisions depend on time and the initial state, while in feedback the decisions depend on both time and the current state (see [10]).

- **Open-loop strategy** : The control action is selected according to a decision rule ν_i, which is a function of the initial state y_0

$$b_i(s) = \nu_i(s, y_0), \forall y_0, \forall s \in \left[0, T\right],\ i = 1, ..., n.$$

- **Feedback strategy** : The control action is selected according to a feedback rule ν_i, which is a function of the current state

$$b_i(s) = \nu_i(s, y(s)), \forall s \in \left[0, T\right],\ i = 1, ..., n.$$

The MP and DPP approaches characterize the OLNE and FNE solutions of differential games. These approaches are used in solving differential games, respectively. In Chapter 2, we discuss them in detail, particularly in the context of two-player differential game.

Zero-sum differential games (ZSDGs)

A common special case of differential games when $n = 2$ are known as a two-player zero-sum differential games, where the sum of the cost functionals (1.11) for both players is zero, meaning that the players are adversaries. A gain for Player 1 implies an equal loss for Player 2 (see [9, 35, 56]).

$$\begin{cases} G_1\left(s, y, b_1, b_2\right) = -G_2\left(s, y, b_1, b_2\right), \\ h_1\left(y\right) = -h_2\left(y\right). \end{cases}$$

Then,

$$J(b_1(\cdot), b_2(\cdot)) = J_1(b_1(\cdot), b_2(\cdot)) = -J_2(b_1(\cdot), b_2(\cdot)).$$

In the ZSDGs, the NE $\left(\bar{b}_1(\cdot), \bar{b}_2(\cdot)\right)$ is called the Saddle point Equilibrium (SPE), defined as follows

$$J(\bar{b}_1(\cdot), b_2(\cdot)) \leq J(\bar{b}_1(\cdot), \bar{b}_2(\cdot)) \leq J(b_1(\cdot), \bar{b}_2(\cdot)),$$

$$\forall\, (b_1(\cdot), b_2(\cdot)) \in \mathcal{B}_1\, [0, T] \times \mathcal{B}_2\, [0, T]\, .$$

1.3.3 Linear Quadratic Differential Games (LQDGs)

Consider a special case of two-player nonzero-sum differential games on a finite time horizon in which the dynamical system of (1.10) is linear, and cost functionals (1.11) are quadratic and given respectively by

$$\begin{cases} \dot{y}(s) = A(s)y(s) + E_1(s)b_1(s) + E_2(s)b_2(s),\ s \in [0,\ T] \\ \\ y(0) = y_0 \in \mathbb{R}^n, \end{cases} \tag{1.14}$$

$$\begin{aligned} J_i(b_1(\cdot), b_2(\cdot)) = \textstyle\int_0^T \frac{1}{2}(y(t)^\top Z_i(s)\, y(s) + b_1(s)^\top R_{i1}(t) b_1(s) + b_2(s)^\top R_{i2}(s) b_2(s)) ds \\ + \frac{1}{2} y(T)^\top S_{iT}\, y(T), \quad i = 1, 2, \end{aligned} \tag{1.15}$$

where $y, b_1,$ and b_2 are the state and the strategies of players as defined in Section 1.3.1. The matrices $A(\cdot) \in \mathbb{R}^{n\times n}$, $E_i(\cdot) : [0, T] \to \mathbb{R}^{n\times m_i}, i = 1, 2.$
The weighting matrices $Z_i(\cdot)$, $S_{iT} \in \mathbb{R}^{n\times n}$, $R_{ii}(\cdot) \in \mathbb{R}^{m_i\times m_i}$, $R_{ij}(\cdot) \in \mathbb{R}^{m_j\times m_j}$ are symmetric and verify $Z_i(t) \geq 0,$ and $S_{iT} \geq 0, R_{ii} > 0,\ i = 1, 2,$
For the two player LQDGs, the characterization of the OLNE and FNE often involves using the MP and DPP. These problems necessitate the solution of Riccati differential equations (RDE), coupled to (1.14) equations. The Riccati equations correspond to the open-loop and the feedback Nash strategies for the players, which are defined respectively as follows (see e.g., [9,23,27,50]).

$$\begin{cases} \dot{K}_1(s) = -K_1(s)A - A^\top K_1(s) - Z_1(s) + K_1(s)Y_1(s)K_1(s) + K_1(s)Y_2(s)K_2(s) \\ \\ K_1(T) = S_{1T}, \\ \\ \dot{K}_2(s) = -K_2(s)A - A^\top K_2(s) - Z_2(s) + K_2(s)Y_2(s)K_2(t) + K_2(s)Y_1(s)K_1(s) \\ \\ K_2(T) = S_{2T};\ s \in [0, T]\, . \end{cases}$$

$$
\begin{cases}
\dot{K}_1(s) = -K_1(s)A - A^\top K_1(s) - Q_1(s) + K_1(s)Y_1(s)K_1(s) \\
\qquad +K_1(s)Y_2(s)K_2(s) + K_2(s)Y_2(s)K_1(s) - K_2(s)Y_{12}(s)K_2(s) \\
K_1(T) = S_{1T}, \\
\dot{K}_2(s) = -K_2(s)A - A^\top K_2(s) - Z_2(s) + K_2(s)Y_2(s)K_2(s) \\
\qquad +K_2(s)Y_1(s)K_1(s) + K_1(s)Y_1(s)K_2(s) - K_1(s)Y_{21}(s)K_1(s) \\
K_2(T) = S_{2T},\ ;\ s \in [0, T]\,,
\end{cases}
$$

where,

$$Y_i(s) = E_i(s)R_{ii}(s)^{-1}E_i^\top(s); \qquad i = 1, 2.$$

$$Y_{ij}(s) = E_j(s)R_{jj}(s)^{-1}R_{ij}(s)R_{jj}(s)^{-1}E_j^\top(s), \quad 1 \leq\ i, j \leq 2.$$

Linear State Differential Games (LSDGs)

Consider a deterministic LSDGs with linear scalar dynamics system (1.14) and the cost functions (1.15) which are quadratic scalar functions for the two player

$$
\begin{cases}
\dot{y}(s) = ay(s) + e_1 b_1(s) + e_2 b_2(s),\ s \in [0,\ T] \\
y(0) = y_0,
\end{cases}
$$

where $y\,(s)$ is the state of the system at time s, which is scalar in this case. $y_0 \in \mathbb{R}$ is the initial state of the system. The strategies of two players $b_i(s) \neq 0, i = 1, 2,$. The constants a, e_1 and e_2 define the dynamic system.
Each player aims to minimize their own cost function over $[0, T]$. The typical cost functions are of the form :

$$J_i(b_1(\cdot), b_2(\cdot)) = \textstyle\int_0^T \left(z_i(s)y(t)^2 + r_i(t)b_i^2(s)\right) ds$$

$$+s_{iT}\ y^2(T), \quad i = 1, 2.$$

The coefficients verify $z_i, z_{iT} \in \mathbb{R}, i = 1, 2,$.

CHAPTER 2

A CONNECTION BETWEEN THE ADJOINT VARIABLES AND VALUE FUNCTION FOR DIFFERENTIAL GAMES

In this chapter, we give the problem formulation of a deterministic two-player NZSDGs in a finite horizon and we recall the preliminaries results of the two main approaches, both the Maximum Principle (MP) and dynamic programming principle (DPP) (see e.g., [9]) in section 1. The last two subsequent sections contain the main results of the connection between the adjoint variables in the maximum principle and the value function in the dynamic programming principle where the value function is smooth and nonsmooth. For the smooth case, the connection between the solutions of the adjoint equations of the Maximum Principle and the derivatives of the value function are equal to each other along optimal trajectories. Furthermore, for the nonsmooth case, this relationship is given in terms of viscosity solutions (VS), which provides a more general framework to handle cases where the value function is not smooth and this relation is represented in terms of the adjoint variables and the first order super- and subdifferentials of the value function.

2.1 Formulation of the Game Problem

In this section, we give the problem formulation of (NZSDGs) and we recall some preliminary of the MP and DPP without proofs (see e.g., [9]) necessary for the main results. We first give a brief exposition of the MP, introduce the Hamiltonian, the adjoint equations, and the necessary and sufficient maximum principle for differential games to characterize open-loop Nash equilibrium (OLNE). This section also includes the DPP. This principle leads to Hamilton Jacobi Bellman (HJB) equation, a nonlinear first-order PDE, and if the HJB equation is solvable, then the feedback Nash equilibrium (FNE) is obtained by minimizing the Hamiltonian involved in the HJB equation. Furthermore, the HJB equation is satisfied by the verification theorem. However, even in very simple cases, the value function is not smooth and the HJB equations may not have a smooth solution at all, so these equations must be studied in viscosity solution (VS). This new notion is a kind of nonsmooth solutions. It was first proposed by Crandall

and Lions [21] (see also Crandall et al. [22] and [6]) to overcome the difficulty that the value function of differential games or single player differential games (OCPs) is not smooth; see for example [30] and [57] for a more detailed discussion.
Let us consider a non-cooperative two-player nonzero-sum differential games equivalent to (1.10)-(1.11) on the finite horizon and the dynamical system are described by (ODE)

$$\begin{cases} \dot{y}(s) = F(s, y(s), b_1(s), b_2(s)), \ s \in [0, T] \\ y(0) = y_0, \end{cases} \tag{2.1}$$

where $y(s) \in \mathbb{R}^n$ is the state variables of the game at time $s \in [0, T]$ that is influenced by both players and the control (strategy) for the i-th player $b_i : [0, T] \to B_i$, where B_i is closed subset of $\mathbb{R}^{m_i}$, $(B_i = B_1 \times B_2, i = 1, 2)$. $T > 0$ is a fixed time horizon, and $\mathcal{B}_i$ is called admissible set of the control $b_i(\cdot) = (b_1(\cdot), b_2(\cdot))$ defined by the following :

$$\mathcal{B}_i([0, T]) = \{b_i(.) : [0, T] \to B_i | b_i(.) \in \mathbb{L}^2([0, T]; \mathbb{R}^{m_i})\}, \ i = 1, 2.$$

The cost functional for the two players is :

$$J_i(s, y_0; b_1(\cdot), b_2(\cdot)) = \int_0^T G_i(s, y(s), b_1(s), b_2(s))\, ds + h_i(y(T)), \ i = 1, 2. \tag{2.2}$$

We give the following assumptions for the coefficients of (2.1) and (2.2).

(DG1) The function $F : [0, T] \times \mathbb{R}^n \times B_1 \times B_2 \to \mathbb{R}^n$ is continuous and there exists a constant $M > 0$ such that for every $s \in [0, T], y, \hat{y} \in \mathbb{R}^n, b, \hat{b} \in B_i$ with $b = (b_1, b_2)$, we have

$$\left| F(s, y, b) - F\left(s, \hat{y}, \hat{b}\right) \right| \leq M \left(|y - \hat{y}| + \left| b - \hat{b} \right| \right),$$
$$|F(s, y, b)| \leq M(1 + |y| + |b|),$$

(DG2) The functions $G_i : [0, T] \times \mathbb{R}^n \times B_1 \times B_2 \to \mathbb{R}$ and $h_i : \mathbb{R}^n \to \mathbb{R}$ are continuous, and there exists a constant $M > 0$ such that

$$\left| G_i(s, y, b) - G_i\left(s, \hat{y}, \hat{b}\right) \right| \leq M \left(|y - \hat{y}| + \left| b - \hat{b} \right| \right),$$
$$|h_i(y) - h_i(\hat{y})| \leq M |y - \hat{y}|,$$
$$|G_i(s, y, b_1, b_2)| + |h_i(y)| \leq M(1 + |y|), \forall s \in [0, T], y, \hat{y} \in \mathbb{R}^n, b, \hat{b} \in B_i, \ i = 1, 2.$$

Under assumption (DG1) for any $(s, y) \in [0, T] \times \mathbb{R}^n$ and the controls $b_i(.) \in \mathcal{B}_i[0, T]$, equation (2.1) admits a unique solution $y(\cdot) = y^{s, y_0, b_i(\cdot)}(\cdot)$ and under (DG2) the functional (2.2) is well-defined. (see ([57]).
Consider the following nonzero-sum differential game problem.

Problem (NZSDG). For given $(s, y) \in [0, T] \times \mathbb{R}^n$, find a $\bar{b}_i(\cdot) \in \mathcal{B}_i [0, T]$, $i = 1, 2$, such that

$$J_i(s, y_0; \bar{b}_i(\cdot)) = \inf_{b_i(\cdot) \in \mathcal{B}_i[0,T]} J_i(s, y_0; b_i(\cdot)). \tag{2.3}$$

The NE concept was defined for n-player differential games in Definition 1.5. It is now given again for the two-player case.

Definition 2.1. For $i = 1, 2$, here $\bar{b}_i(\cdot) \in \mathcal{B}_i [0, T]$ satisfying (2.3) is called a NE of Problem (NZSDG) if for any other admissible control actions $b_i(\cdot)$ the following inequalities hold :

$$J_1(s, y_0; \bar{b}_1(\cdot), \bar{b}_2(\cdot)) \leq J_1(s, y_0; b_1(\cdot), \bar{b}_2(\cdot)),\ \forall b_1(\cdot) \in \mathcal{B}_1 [0, T],$$

$$J_2(s, y_0; \bar{b}_1(\cdot), \bar{b}_2(\cdot)) \leq J_2(s, y_0; \bar{b}_1(\cdot), b_2(\cdot)),\ \forall b_2(\cdot) \in\ \mathcal{B}_2 [0, T].$$

This implies that the controls $(\bar{b}_1(\cdot), \bar{b}_2(\cdot))$ represent a NE, indicating that neither player can benefit by changing their own control, making it the optimal choice for both [39].

2.1.1 Maximum Principle (MP)

Here, we present the approach to find the equilibrium, based on the Pontryagin's maximum principle (MP) for differential games (e.g., [9]). First, consider the maximum principle for Problem (NZSDG), as published in multiple articles (see e.g., [46, 57] and [9]), by using the necessary conditions (2.4), (2.5), (2.6) and (2.7) for an OLNE $\bar{b}_i(\cdot) = (\bar{b}_1(\cdot), \bar{b}_2(\cdot)) \in \mathcal{B}_1 [0, T] \times \mathcal{B}_2 [0, T]$ and the assumption is as follows :

(DG3) F is C^1 in (y, b) and its derivatives are bounded and uniformly Lipchitz in (y, b). In addition, G_i and h_i are C^1 in (y, b), and the partial derivatives G^i_y, G^i_b, h^i_y are uniformly Lipchitz and linear growth.

The Hamiltonian functions associated with this game $H_i : [0, T] \times \mathbb{R}^n \times B_i \times \mathbb{R} \to \mathbb{R}$ is defined by

$$H_i (s, y, b_1, b_2, p_i) = \langle F (s, y, b_1, b_2), p_i\rangle + G_i (s, y, b_1, b_2)\ ,\ i = 1, 2. \tag{2.4}$$

The determination of Nash equilibrium is related to the minimization of the Hamiltonian. Under the assumptions (DG1)-(DG3), let $(\bar{b}_1(\cdot), \bar{b}_2(\cdot))$ be an OLNE of Problem (NZSDG) and $\bar{y}(s)$ is the corresponding state trajectory, there exist a unique adjoint variables $(\bar{p}_i (.)) \in (C ([0, T] ; \mathbb{R}^n))$ for $i = 1, 2$ solution of the adjoint equations

$$\begin{cases} \dot{\bar{p}}_i (s) = -H^i_y \left(s, \bar{y} (s), \bar{b}_i (s), \bar{p}_i (s)\right),\ s \in [0, T] \\ \bar{p}_i (T) = h^i_y (\bar{y} (T)) \end{cases},\ i = 1, 2, \tag{2.5}$$

and the infimum condition

$$H_i^*\left(s, \bar{y}(s), \bar{p}_i(s)\right) = H_i\left(s, \bar{y}(s), \bar{b}_i(s), \bar{p}_i(s)\right) = \inf_{b_i(\cdot)\in\mathcal{B}_i[0,T]} H_i\left(s, \bar{y}(s), b_i(s), \bar{p}_i(s)\right). \tag{2.6}$$

Such that,

$$H_{b_i}^i\left(s, \bar{y}(s), \bar{b}_i(s), \bar{p}_i(s)\right) = 0,\ s \in [0,T]. \tag{2.7}$$

(DG4) $H_i, i = 1,2$, is convex in (y, b_1, b_2) and $h_i, i = 1,2$, is convex in y, $\forall s \in [0,T]$.

Under some appropriate convexity conditions (DG4) we can recall the sufficient maximum principle for an OLNE can be regarded as an extension of the MP for single player differential games in (see e.g., [57], we introduce the following theorem (see e.g., [57]).

Theorem 2.1. *Let (DG1)-(DG4) hold. Suppose that $(\bar{b}_1(\cdot), \bar{b}_2(\cdot))$ admissible strategy with the corresponding state trajectory $\bar{y}(\cdot)$. Suppose there exist a solution $(\bar{p}_i(.)) \in (C([0,T];\mathbb{R}^n), i = 1,2)$ of the adjoint equations (2.5) such that the infimum conditions hold*

$$H_i^*\left(s, \bar{y}(s), \bar{p}_i(s)\right) = \inf_{b_i(\cdot)\in\mathcal{B}_i[0,T]} H_i\left(s, \bar{y}(s), b_i(s), \bar{p}_i(s)\right),\ i = 1,2.$$

Then, $(\bar{b}_1(\cdot), \bar{b}_2(\cdot))$ is an open-loop Nash equilibrium.

2.1.2 Dynamic Programming Principle (DPP)

Now, we present the DPP (see e.g., [9]), HJB Equations, verification theorem and viscosity solution for the (NZSDG) problem when the controls $\bar{b}_i(\cdot) = (\bar{b}_1(\cdot), \bar{b}_2(\cdot)) \in \mathcal{B}_1[s,T] \times \mathcal{B}_2[s,T]$ is feedback Nash equilibrium (FNE). However, before presenting this approach, we must first adopt a dynamic formulation of the NZSDG problem.
For $t \in [0,T]$ and $x \in \mathbb{R}^n$, we rewrite (2.1) and (2.2) as the following :

$$\begin{cases} \dot{y}(s) = F\left(s, y(s), b_1(s), b_2(s)\right),\ s \in [t,T] \\ y(t) = x, \end{cases} \tag{2.8}$$

The objective of the players is to minimize

$$J_i(t,x;b_1(\cdot),b_2(\cdot)) = \int_t^T G_i\left(s, y(s), b_1(s), b_2(s)\right) ds + h_i\left(y(T)\right),\ i = 1,2. \tag{2.9}$$

Then, we define the value function as

$$\begin{cases} W_i(t,x) = \inf_{b_i(\cdot)\in\mathcal{B}_i[t,T]} J_i(t,x;b_i(\cdot)) \\ W_i(T,x) = h_i(x),\ i = 1,2. \end{cases} \tag{2.10}$$

It represents the minimum cost that can be achieved starting at time t with state x under the optimal decision strategy $\bar{b}_i$.
We present the following Bellman's Principle of optimality [11] for the Problem (NZSDG) :

$$W_i\left(t,x\right)=\inf_{b_i(\cdot)\in\mathcal{B}_i[t,T]}\left\{\int_t^{\hat{t}} g_i\left(s,y\left(s\right),b_i\left(s\right)\right)ds+W_i\left(\hat{t},y\left(\hat{t}\right)\right)\right\},\forall\hat{t}\in\left[t,T\right],\ i=1,2. \tag{2.11}$$

Similarly to the Pontryagin's MP approach, the search for the FNE is related to the minimization of the Hamiltonian (2.6). The development of the principle of optimality to equation (2.11), leads immediately to HJB equations

$$\begin{cases} \dfrac{\partial W_i}{\partial t}\left(t,x\right)+H_i^*\left(t,x,\dfrac{\partial W_i}{\partial x}\left(t,x\right)\right)=0,\forall\left(t,x\right)\in\left[0,T\right]\times\mathbb{R}^n \\ W_i\left(T,x\right)=h_i\left(x\right), \end{cases} \quad i=1,2, \tag{2.12}$$

where $W_i(\cdot,\cdot)\in\left(C^{1,1}\left(\left[0,T\right]\times\mathbb{R}^n\right);\mathbb{R}\right)$ and

$$H_i^*\left(t,x,\frac{\partial W_i}{\partial x}\left(t,x\right)\right)=H_i\left(t,x,\bar{b}_i,\frac{\partial W_i}{\partial x}\left(t,x\right)\right)=\inf_{b_i\in\mathcal{B}_i[t,T]}H_i\left(t,x,b_i,\frac{\partial W_i}{\partial x}\left(t,x\right)\right).\ i=1,2.$$

Now, we are going to state the following verification theorem that is a generalization of similar results from (e.g., [57]) for a single player differential game that gives a sufficient condition for a FNE. It allows us to verify that an admissible strategy is optimal.

Theorem 2.2. *(Verification Theorem). Let assumptions (DG1)-(DG2) hold. Assume that $W_i(\cdot,\cdot)\in C^{1,1}\left(\left[0,T\right]\times\mathbb{R}^n\right)$ is a solution to equations (2.12). Then we have the following :*
(i) $W_i\left(t,x\right)\leq J_i\left(t,x;b_i(\cdot)\right),\forall\left(t,x\right)\in\left[0,T\right]\times\mathbb{R}^n,b_i(\cdot)\in\mathcal{B}_i\left[t,T\right];$
(ii) suppose

$$\frac{\partial W_i}{\partial t}\left(t,x\right)+H_i^*\left(t,x,\frac{\partial W_i}{\partial x}\left(t,x\right)\right)=0,\forall\left(t,x\right)\in\left[0,T\right]\times\mathbb{R}^n,\ i=1,2,$$

and there exist an $(\bar{b}_1(\cdot),\bar{b}_2(\cdot))\in\mathcal{B}_1\left[t,T\right]\times\mathcal{B}_2\left[t,T\right]$ admissible strategy with the corresponding state trajectory $\bar{y}(\cdot)$ for Problem (NZSDG)

$$H_i^*\left(\hat{t},\bar{y}\left(\hat{t}\right),\frac{\partial W_i}{\partial x}\left(\hat{t},\bar{y}\left(\hat{t}\right)\right)\right)=H_i\left(\hat{t},\bar{y}\left(\hat{t}\right),\bar{b}_1\left(\hat{t}\right),\bar{b}_2\left(\hat{t}\right),\frac{\partial W_i}{\partial x}\left(\hat{t},\bar{y}\left(\hat{t}\right)\right)\right),\ \forall\hat{t}\in\left[t,T\right].$$

Then $\left(\bar{b}_1\left(\cdot\right),\bar{b}_2\left(\cdot\right)\right)$ is a feedback Nash equilibrium (FNE) with the optimal state $\bar{y}\left(\cdot\right)$ for Problem (NZSDG) in the point $\left(t,x\right)$.

As the value function $W_i\left(\cdot,\cdot\right)$ is nonsmooth, it is crucial to recall the definition of viscosity solution (VS) (see [22] and [57]).

Definition 2.2. (Viscosity Solution) A continuous function w_i on $[0,T] \times \mathbb{R}^n$ is a viscosity subsolution (respectively, supersolution) of (2.12), if $w_i(T,y) \leq (\geq) h_i(y)$ for all $y \in \mathbb{R}^n$ and

$$\phi_s^i(s,y) + H_i^*\left(s,y,\phi_y^i(s,y)\right) \geq (\leq) 0,\ i=1,2,$$

whenever $w_i - \phi_i$ attains a local maximum (respectively, minimum) at $(s,y) \in [0,T] \times \mathbb{R}^n$ for $\phi_i \in C^{1,1}([0,T] \times \mathbb{R}^n)$. A function w_i is called a viscosity Solution to (2.12) if it is both a viscosity subsolution and viscosity supersolution to (2.12).

Thus, the following result is the uniqueness of viscosity solution of the HJB equations (2.12) (see, [57]).

Proposition 2.1. *Suppose (DG1)-(DG2) hold. Then, (2.10) satisfies*

$$|W_i(t,x) - W_i(t,x^*)| \leq M(|x-x^*| + |t-t^*|),\ \forall t,t^* \in [0,T],x,x^* \in \mathbb{R}^n,$$

and

$$|W_i(t,x)| \leq M(1+|x|),\ \forall (t,x) \in [0,T] \times \mathbb{R}^n,\ i=1,2.$$

Furthermore, $W_i(\cdot,\cdot)$ is the viscosity solution to (2.12).

2.2 The Connection Between MP and DPP : Smooth Case

The following theorem states that the connection between the MP and the DPP is same to the connection between the adjoint variables and the derivatives of the value function along optimal trajectories.

Theorem 2.3. *Assume (DG1)-(DG3) hold and $(t,x) \in [0,T) \times \mathbb{R}^n$ be fixed. Let $(\bar{b}_1(\cdot),\bar{b}_2(\cdot))$ is a Nash equilibrium with the optimal state $\bar{y}(\cdot)$ for Problem (NZSDG) and $\bar{p}_i$ be the corresponding solution of the adjoint equations (2.5). Assume that $W_i(\cdot,\cdot) \in (C^{1,1}([0,T] \times \mathbb{R}^n);\mathbb{R})$, then*

$$\begin{aligned} -\frac{\partial W_i}{\partial s}(s,\bar{y}(s)) =& H_i\left(s,\bar{y}(s),\bar{b}_1(s),\bar{b}_2(s),\frac{\partial W_i}{\partial y}(s,\bar{y}(s))\right) \\ =& \inf_{b_i \in \mathcal{B}_i[s,T]} H_i\left(s,\bar{y}(s),b_1(s),b_2(s),\frac{\partial W_i}{\partial y}(s,\bar{y}(s))\right),\ i=1,2, \end{aligned} \tag{2.13}$$

$\forall s \in [t,T]$. *Further, if $W_i(\cdot,\cdot) \in (C^{1,2}([0,T] \times \mathbb{R}^n);\mathbb{R})$ and W_{sx}^i is continuous, then*

$$\bar{p}_i(s) = \frac{\partial W_i}{\partial y}(s,\bar{y}(s)),\forall s \in [t,T],\ i=1,2. \tag{2.14}$$

Proof. By the optimality of $(\bar{y}(\cdot), \bar{b}_1(\cdot), \bar{b}_2(\cdot))$ for Problem (NZSDG)

$$\begin{cases} \dot{\bar{y}}(s) = F\left(s, \bar{y}(s), \bar{b_1}(s), \bar{b_2}(s)\right), \ s \in [t, T] \\ \bar{y}(t) = x, \end{cases} \tag{2.15}$$

and the cost functional :

$$W_i(t,x) = J_i(t,x;\bar{b_1}(\cdot),\bar{b_2}(\cdot)) = \int_t^T G_i\left(s, \bar{y}(s), \bar{b_1}(s), \bar{b_2}(s)\right) ds + h_i(\bar{y}(T)), \ i = 1,2, \forall t \in [s,T]. \tag{2.16}$$

Differentiating both sides of the (2.16) with respect to s :

$$\frac{\partial W_i}{\partial s}(s,\bar{y}(s)) + \frac{\partial W_i}{\partial y}(s,\bar{y}(s))\,\dot{\bar{y}}(s) = -G_i\left(s, \bar{y}(s), \bar{b_1}(s), \bar{b}_2(s)\right), \ i = 1,2$$

According to (2.15), we can deduce that

$$\frac{\partial W_i}{\partial s}(s,\bar{y}(s)) + \left\langle F\left(s, \bar{y}, \bar{b_1}(s), \bar{b}_2(s)\right), \frac{\partial W_i}{\partial y}(s,\bar{y}(t))\right\rangle = -G_i\left(s, \bar{y}(s), \bar{b_1}(s), \bar{b}_2(s)\right)$$

By (2.4), we get the first equality in (2.13)

$$-\frac{\partial W_i}{\partial s}(s,\bar{y}(s)) = H_i\left(s, \bar{y}(s), \bar{b}_1(s), \bar{b}_2(s), \frac{\partial W_i}{\partial y}(s,\bar{y}(s))\right), \ i = 1,2,$$

Since $W_i \in C^{1,1}([0,T] \times \mathbb{R}^n)$ be a solution of the equations (2.12), we obtain that, for each $y \in \mathbb{R}^n$,

$$\begin{aligned} &\frac{\partial W_i}{\partial s}(s,\bar{y}(s)) + H_i\left(s, \bar{y}(s), \bar{b}_1(s), \bar{b}_2(s), \frac{\partial W_i}{\partial y}(s,\bar{y}(s))\right) \\ &= 0 \leq \frac{\partial W_i}{\partial s}(s,y) + H_i\left(s, y, \bar{b}_1(s), \bar{b}_2(s), \frac{\partial W_i}{\partial y}(s,y)\right) \end{aligned}$$

Thus we have the second equality in (2.13).
Therefore, if $W_i(\cdot,\cdot) \in (C^{1,2}([0,T] \times \mathbb{R}^n); \mathbb{R})$ and W^i_{sy} is continuous, thus

$$\frac{\partial}{\partial y}\left\{\frac{\partial W_i}{\partial s}(s,y) + H_i\left(s, y, \bar{b}_1(s), \bar{b}_2(s), \frac{\partial W_i}{\partial y}(s,y)\right)\right\}|_{y=\bar{y}(s)} = 0$$

This implies that

$$\begin{aligned} &\frac{\partial}{\partial s}\left\{\frac{\partial W_i}{\partial y}(s,\bar{y}(s))\right\} + \frac{\partial^2 W_i}{\partial y^2}(s,\bar{y}(s))\,F\left(s, \bar{z}(s), \bar{b_1}(s), \bar{b}_2(s)\right) \\ &+ \frac{\partial W_i}{\partial y}(t,\bar{y}(s))\,F_y\left(s, \bar{y}(s), \bar{b}_1(s), \bar{b}_2(s)\right) + G^i_y\left(s, \bar{y}(s), \bar{b_1}(s), \bar{b}_2(s)\right) = 0. \ i = 1,2. \end{aligned}$$

We have
$$\frac{\partial}{\partial s}\frac{\partial W_i}{\partial y}\left(s,\bar{y}(s)\right)=-H_y^i\left(s,\bar{y}\left(s\right),\bar{b}_1\left(s\right),\bar{b}_2\left(s\right),\frac{\partial W_i}{\partial y}\left(s,\bar{y}(s)\right)\right),$$
where,
$$H_y^i\left(s,\bar{y}\left(s\right),\bar{b}_1\left(s\right),\bar{b}_2\left(s\right),\frac{\partial W_i}{\partial y}\left(s,\bar{y}(s)\right)\right)=\frac{\partial^2 W_i}{\partial y^2}\left(s,\bar{y}(s)\right)F\left(s,\bar{y}\left(s\right),\bar{b}_1\left(s\right),\bar{b}_2\left(s\right)\right)$$
$$+\frac{\partial W_i}{\partial y}\left(s,\bar{y}(s)\right)F_y\left(s,\bar{y}\left(s\right),\bar{b}_1\left(s\right),\bar{b}_2\left(s\right)\right)+G_y^i\left(s,\bar{y}\left(s\right),\bar{b}_1\left(s\right),\bar{b}_2\left(s\right)\right).\ i=1,2.$$

Noting that $\frac{\partial W_i}{\partial y}\left(T,\bar{y}(T)\right)=h_y^i\left(\bar{y}\left(T\right)\right)$, and $\frac{\partial W_i}{\partial y}\left(s,\bar{y}(s)\right)$ satisfies the equation (2.5). Then by the uniqueness of the solutions to the adjoint equation (2.5), we get (2.14). □

Remark 2.1. The Theorem 2.3 is proved by Shi [49] in particular case of differential games (zero sum stochastic differential games) with jump diffusions.

2.3 The Connection Between MP and DPP : Nonsmooth Case

In this section, we present the connection between the adjoint variables in the MP and the value function in the DPP within the framework of the Viscosity Solution (VS), which represents this relationship in terms of the adjoint variables and the first-order super- and subdifferentials of the value function.

Now, we recall the notion of the first-order super- and subdifferentials (see, e.g; [57]). For $w_i\in C\left([0,T]\times\mathbb{R}^n\right)$ and $(s,y)\in[0,T]\times\mathbb{R}^n$, we have :

$$D_{s,y}^{1,+}w_i\left(s,y\right)=\left\{\left(q_i,p_i\right)\in\mathbb{R}\times\mathbb{R}^n|\limsup_{t\to s,t\in[0,T),x\to y}\frac{w_i\left(t,x\right)-w_i\left(s,y\right)-q_i\left(t-s\right)-\langle p_i,x-y\rangle}{|t-s|+|x-y|}\leq 0\right\}$$
$$D_{s,y}^{1,-}w_i\left(s,y\right)=\left\{\left(q_i,p_i\right)\in\mathbb{R}\times\mathbb{R}^n|\liminf_{t\to s,t\in[0,T),x\to y}\frac{w_i\left(t,x\right)-w_i\left(s,y\right)-q_i\left(t-s\right)-\langle p_i,x-y\rangle}{|t-s|+|x-y|}\geq 0\right\}$$

Next, the viscosity solution to HJB equation (2.12) can be expressed equivalently in terms of super- and subdifferentials (see, [57]). $w_i\in C\left([0,T]\times\mathbb{R}^n\right)$ is a VS of the equations (2.12) and for all $(s,y)\in[0,T]\times\mathbb{R}^n$,

$$\begin{cases} q_i+H_i^*\left(s,\bar{y},\bar{p}_i\right)\geq 0,\ \forall\left(q_i,\bar{p}_i\right)\in D_{s,y}^{1,+}w_i\left(s,y\right) \\ \\ q_i+H_i^*\left(s,\bar{y},\bar{p}_i\right)\leq 0,\ \forall\left(q_i,\bar{p}_i\right)\in D_{s,y}^{1,-}w_i\left(s,y\right),\quad i=1,2 \\ \\ w_i(T,y)=h_i\left(y\right). \end{cases} \tag{2.17}$$

The following theorem establishes the connection between MP and DPP in terms of the connection between the adjoint variables and the first-order super- and subdifferentials of the value function.

Theorem 2.4. *Assume (DG1)-(DG3) hold. Let $(t,x) \in [0,T] \times \mathbb{R}^n$ be fixed and $\left(\bar{b}_1(\cdot), \bar{b}_2(\cdot)\right)$ is a Nash equilibrium with the optimal state trajectory $\bar{y}(\cdot)$ for Problem (NZSDG). Let $\bar{p}_i(\cdot)$ be the solution to equation (2.5). Suppose that the value function $W_i(\cdot,\cdot) \in \left(C\left([0,T] \times \mathbb{R}^n\right); \mathbb{R}\right)$. Then*

$$D_{s,y}^{1,-} W_i\left(s, \bar{y}(s)\right) \subseteq \left\{\left(\mathcal{H}_i\left(s, \bar{y}(s), \bar{b}_1(s), \bar{b}_2(s), \bar{p}_i(s)\right), \bar{p}_i(s)\right)\right\} \subseteq D_{s,y}^{1,+} W_i\left(s, \bar{y}(s)\right) \tag{2.18}$$

where

$$\mathcal{H}_i\left(s, \bar{y}(s), \bar{b}_1(s), \bar{b}_2(s), \bar{p}_i(s)\right) = -H_i\left(s, \bar{y}(s), \bar{b}_1(s), \bar{b}_2(s), \bar{p}_i(s)\right), \ i = 1,2,$$

$$D_y^{1,-} W_i\left(s, \bar{y}(s)\right) \subseteq \left\{\bar{p}_i(s)\right\} \subseteq D_y^{1,+} W_i\left(s, \bar{y}(s)\right), \ i = 1,2, \ \forall s \in [s,T], \tag{2.19}$$

and

$$\bar{q}_i = \mathcal{H}_i\left(s, \bar{y}(s), \bar{b}_1(s), \bar{b}_2(s), \bar{p}_i(s)\right) = \inf_{b_i(\cdot) \in \mathcal{B}_i[0,T]} \mathcal{H}_i\left(s, \bar{y}(s), b_1(s), b_2(s), \bar{p}_i(s)\right), \ i = 1,2, \tag{2.20}$$

$$\forall \left(\bar{q}_i, \bar{p}_i\right) \in D_{s,y}^{1,+} W_i\left(s, \bar{y}(s)\right) \cup D_{s,y}^{1,-} W_i\left(s, \bar{y}(s)\right), \ \forall s \in [s,T],$$

Proof. Note that

$$\lim_{h \to 0} \frac{1}{h} \int_s^{s+h} \psi(\vartheta) d\vartheta = \psi(s), \ a.e. \ s \in (t,T), \tag{2.21}$$

and $\psi(\vartheta) = F\left(\vartheta, \bar{y}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right), G_i\left(\vartheta, \bar{y}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right), \ i = 1,2$. Fix $s \in (t,T)$ such that (2.21) holds.

For any $\eta \in \mathbb{R}^n$ and $\tau \in [t,T]$, consider the following ODE :

$$\begin{cases} \dot{y}^{\tau,\eta}(\vartheta) = F\left(\vartheta, y^{\tau,\eta}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right), \ \vartheta \in [\tau,T] \\ y^{\tau,\eta}(\tau) = \eta. \end{cases} \tag{2.22}$$

Denote by $y^{\tau,\eta}(\cdot)$ the solution of (2.22) starting from (τ,y) under the controls $\bar{b}_i(\cdot) = \left(\bar{b}_1(\cdot), \bar{b}_2(\cdot)\right)$, for $i = 1,2$,

$$y^{\tau,\eta}(\vartheta) = \eta + \int_\tau^\vartheta F\left(\alpha, y^{\tau,\eta}(\alpha), \bar{b}_1(\alpha), \bar{b}_2(\alpha)\right) d\alpha, \ \vartheta \in [\tau,T],$$

and $\bar{y}(\cdot)$ the solution of ODE

$$\bar{y}(\vartheta)=\bar{y}(s)+\int_{s}^{\vartheta}F\left(\alpha,\bar{y}(\alpha),\bar{b}_1(\alpha),\bar{b}_2(\alpha)\right)d\alpha,\ \vartheta\in[\tau,T],$$

Then $\tau<s$ and for any $\vartheta\in[\tau,T]$, we have

$$\begin{aligned} y^{\tau,\eta}(\vartheta)-\bar{y}(\vartheta) &= \eta-\bar{y}(s)-\int_{s}^{\tau}F\left(\alpha,\bar{y}(\alpha),\bar{b}_1(\alpha),\bar{b}_2(\alpha)\right)d\alpha \\ &\quad+\int_{\tau}^{\vartheta}\left[F\left(\alpha,y^{\tau,\eta}(\alpha),\bar{b}_1(\alpha),\bar{b}_2(\alpha)\right)-F\left(\alpha,\bar{y}(\alpha),\bar{b}_1(\alpha),\bar{b}_2(\alpha)\right)\right]d\alpha \\ &= \eta-\bar{y}(s)-\int_{s}^{\tau}F\left(\alpha,\bar{y}(\alpha),\bar{b}_1(\alpha),\bar{b}_2(\alpha)\right)d\alpha \\ &\quad+\int_{\tau}^{\vartheta}F_y\left(\alpha,\bar{y}(\alpha),\bar{b}_1(\alpha),\bar{b}_2(\alpha)\right)\left(y^{\tau,\eta}(\alpha)-\bar{y}(\alpha)\right)d\alpha \\ &\quad+\int_{\tau}^{\vartheta}\epsilon_{\tau,\eta}(\alpha)\left(y^{\tau,\eta}(\alpha)-\bar{y}(\alpha)\right)d\alpha. \end{aligned}\tag{2.23}$$

We obtain the second equality of (2.23) by using the variational equation for $\xi(\vartheta)=y^{\tau,\eta}(\vartheta)-\bar{y}(\vartheta)$ given by

$$\begin{cases} \dot{\xi}(\vartheta)=F_y\left(\vartheta,\bar{y}(\vartheta),\bar{b}_1(\vartheta),\bar{b}_2(\vartheta)\right)\xi(\vartheta)+\epsilon_{\tau,\eta}(\vartheta)\xi(\vartheta),\ \vartheta\in[\tau,T] \\ \xi(\vartheta)=\eta-\bar{y}(s)-\int_{s}^{\tau}F\left(\vartheta,\bar{y}(\vartheta),\bar{b}_1(\vartheta),\bar{b}_2(\vartheta)\right)d\vartheta. \end{cases}\tag{2.24}$$

where,

$$\begin{cases} \epsilon_{\tau,\eta}(\alpha)=\int_{0}^{1}\left\{F_y\left(\alpha,\bar{y}(\alpha)+\beta\left(y^{\tau,\eta}(\alpha)-\bar{y}(\alpha)\right),\bar{b}_1(\alpha),\bar{b}_2(\alpha)\right)\right. \\ \qquad\qquad \left.-F_y\left(\alpha,\bar{y}(\alpha),\bar{b}_1(\alpha),\bar{b}_2(\alpha)\right)\right\}d\beta \\ \lim\limits_{\tau\to s,\eta\to\bar{y}(s)}\epsilon_{\tau,\eta}(\alpha)=0,\quad\forall\alpha\in[0,T], \\ \sup\limits_{\alpha,\tau,\eta}|\epsilon_{\tau,\eta}(\alpha)|\leq K. \end{cases}\tag{2.25}$$

In this case, the assumption (DG3) was employed.
By the definition of $W_i(\tau,\eta)$

$$W_i(\tau,\eta)\leq\int_{\tau}^{T}G_i\left(\vartheta,y^{\tau,\eta}(\vartheta),\bar{b}_1(\vartheta),\bar{b}_2(\vartheta)\right)d\vartheta+h_i\left(y^{\tau,\eta}(T)\right),\ i=1,2,$$

and the optimality of $(\bar{y}(\cdot), \bar{b}_1(\cdot), \bar{b}_2(\cdot))$, we get

$$W_i(s, \bar{y}(t)) = \int_s^T G_i\left(\vartheta, \bar{y}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right) d\vartheta + h_i(\bar{y}(T)), \ i = 1, 2.$$

Then, compute $W_i(\tau, \eta) - W_i(s, \bar{y}(s))$ we obtain

$$\begin{aligned}
&W_i(\tau, \eta) - W_i(s, \bar{y}(s)) \\
&\leq \int_\tau^T \left\{G_i\left(\vartheta, y^{\tau,\eta}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right) - G_i\left(\vartheta, \bar{y}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right)\right\} d\vartheta \\
&\qquad + \left\{h_i(y^{\tau,\eta}(T)) - h_i(\bar{y}(T))\right\} - \int_t^\tau G_i\left(\vartheta, \bar{y}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right) d\vartheta \\
&= \int_\tau^T \left\langle G_y^i\left(\vartheta, \bar{y}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right), y^{\tau,\eta}(\vartheta) - \bar{y}(\vartheta)\right\rangle d\vartheta \\
&\qquad + \left\langle h_y^i(\bar{y}(T)), y^{\tau,\eta}(T) - \bar{y}(T)\right\rangle - \int_s^\tau G_i\left(\vartheta, \bar{y}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right) d\vartheta \\
&\qquad + \int_\tau^T \tilde{\epsilon}_{\tau,\eta}(\vartheta)\left(y^{\tau,\eta}(\vartheta) - \bar{y}(\vartheta)\right) d\vartheta + o\left(\left|y^{\tau,\eta}(T) - \bar{y}(T)\right|\right), \ i = 1, 2.
\end{aligned} \tag{2.26}$$

where $\tilde{\epsilon}_{\tau,\eta}(.)$ is defined similar to $\epsilon_{\tau,\eta}(.)$, with the substitution of F_y for G_y^i and has the same properties are present in (2.25)(see, [57]). Then, by the duality relation between the adjoint equation (2.5) $\bar{p}_i(.)$ and the variational equation (2.24) $y^{\tau,\eta}(.) - \bar{y}(.)$, we have

$$\begin{aligned}
&\left\langle h_y^i(\bar{y}(T)), \xi(T)\right\rangle \\
&\quad = \langle \bar{p}_i(T), \xi(T)\rangle \\
&\quad = \langle \bar{p}_i(T), \xi(T)\rangle - \langle \bar{p}_i(\tau), \xi(\tau)\rangle + \langle \bar{p}_i(\tau), \xi(\tau)\rangle \\
&\quad = \int_\tau^T \langle \dot{\bar{p}}_i(\vartheta), \xi(\vartheta)\rangle d\vartheta + \int_\tau^T \left\langle \bar{p}_i(\vartheta), \dot{\xi}(\vartheta)\right\rangle d\vartheta + \langle \bar{p}_i(\tau), \xi(\tau)\rangle \\
&\quad = \langle \bar{p}_i(\tau), \xi(\tau)\rangle + \int_\tau^T \langle \bar{p}_i(\vartheta), \epsilon_{\tau,\eta}(\vartheta)\xi(\vartheta)\rangle d\vartheta \\
&\quad - \int_\tau^T \left\langle G_y^i\left(\vartheta, \bar{y}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right), \xi(\vartheta)\right\rangle d\vartheta
\end{aligned} \tag{2.27}$$

After that, with respect to the term on the right side of (2.27)

$$\begin{aligned}
&\langle \bar{p}_i(\tau), \xi(\tau)\rangle \\
&\quad = \left\langle \bar{p}_i(s), \eta - \bar{y}(s) - \int_s^\tau F\left(\vartheta, \bar{y}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right) d\vartheta \right\rangle \\
&\quad + \left\langle \bar{p}_i(\tau) - \bar{p}_i(s), \eta - \bar{y}(s) - \int_s^\tau F\left(\vartheta, \bar{y}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right) d\vartheta \right\rangle \\
&\quad = \left\langle \bar{p}_i(s), \eta - \bar{y}(s) - \int_s^\tau F\left(\vartheta, \bar{y}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right) d\vartheta \right\rangle \\
&\quad + \left\langle \int_s^\tau \left[-F_y\left(\vartheta, \bar{y}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right)\bar{p}_i(\vartheta) - G_y^i\left(\vartheta, \bar{y}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right)\right] d\vartheta \right. \\
&\qquad \left. , \eta - \bar{y}(s) - \int_s^\tau F\left(\vartheta, \bar{y}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right) d\vartheta \right\rangle \\
&\quad = \langle \bar{p}_i(s), \eta - \bar{y}(s)\rangle - \left\langle \bar{p}_i(s), \int_s^\tau F\left(r, \bar{y}(r), \bar{b}_1(r), \bar{b}_2(r)\right) dr \right\rangle + o\left(|\tau - s| + |\eta - \bar{y}(s)|\right)
\end{aligned} \tag{2.28}$$

Here the properties presented in (2.25) was empolyed (see, [57]), for $\xi(\vartheta) = y^{\tau,\eta}(\vartheta) - \bar{y}(\vartheta)$, we have

$$\sup_{\tau \le \vartheta \le T} |\xi(\vartheta)| \le M\left[|\eta - \bar{y}(s)| + |\tau - s|\right],$$

and,

$$\int_\tau^T |\epsilon_{\tau,\eta}(\vartheta)\, \xi(\vartheta)|\, d\vartheta \le C\left[|\eta - \bar{y}(s)| + |\tau - s|\right]$$

Thus, by (2.26)-(2.28), we obtain

$$\begin{aligned}
&W_i(\tau, \eta) - W_i(s, \bar{y}(s)) \\
&\le \langle \bar{p}_i(s), \eta - \bar{y}(s)\rangle - \left\langle \bar{p}_i(s), \int_s^\tau F\left(\vartheta, \bar{y}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right) d\vartheta \right\rangle \\
&\qquad - \int_s^\tau G_i\left(\vartheta, \bar{y}(\vartheta), \bar{b}_1(\vartheta), \bar{b}_2(\vartheta)\right) d\vartheta + o\left(|\tau - s| + |\eta - \bar{y}(s)|\right) \\
&= \langle \bar{p}_i(s), \eta - \bar{y}(s)\rangle + (\tau - s)\,\mathcal{H}_i\left(s, \bar{y}(s), \bar{b}_1(s), \bar{b}_2(s), \bar{p}_i(s)\right) \\
&\qquad + o\left(|\tau - s| + |\eta - \bar{y}(s)|\right),\ i = 1, 2,
\end{aligned} \tag{2.29}$$

which implies

$$\left(\mathcal{H}_i\left(s, \bar{y}(s), \bar{b}_1(s), \bar{b}_2(s), \bar{p}_i(s)\right), \bar{p}_i(s)\right) \subseteq D_{s,y}^{1,+} W_i(s, \bar{y}(s)), i = 1, 2,\ \forall s \in [t, T],$$

by the definition of superdifferential and for such a s, $D_{s,y}^{1,+} W_i(s, \bar{y}(s))$ is nonempty.

Now we prove that

$$D_{s,y}^{1,-}W_i\left(s,\bar{y}(s)\right)\subseteq\left\{\left(\mathcal{H}_i\left(s,\bar{y}(s),\bar{b}_1\left(s\right),\bar{b}_2\left(s\right),\bar{p}_i(s)\right),\bar{p}_i(s)\right)\right\},$$

with $s\in(t,T)$ such that (2.21) holds. For any $(\bar{q}_i,\bar{p}_i)\in D_{s,y}^{1,-}W_i\left(s,\bar{y}(s)\right)$, by definition of subdifferential and (2.29), we have

$$\begin{aligned}0 &\leq \liminf_{\tau\uparrow s}\left\{\frac{W_i\left(\tau,\eta\right)-W_i\left(s,\bar{y}\left(s\right)\right)-\bar{q}_i\left(\tau-s\right)-\langle\bar{p}_i,\eta-\bar{y}\left(s\right)\rangle}{\left|\tau-s\right|+\left|\eta-\bar{y}\left(s\right)\right|}\right\}\\ &\leq \liminf_{\tau\uparrow s}\left\{\frac{\left(\mathcal{H}_i\left(s,\bar{y}(s),\bar{b}_1\left(s\right),\bar{b}_2\left(s\right),\bar{p}_i(s)\right)-\bar{q}_i\right)\left(\tau-s\right)+\langle\bar{p}_i\left(s\right)-\bar{p}_i,\eta-\bar{y}\left(s\right)\rangle}{\left|\tau-s\right|+\left|\eta-\bar{y}\left(s\right)\right|}\right\}\end{aligned}$$

Thus, the first inclusion of (2.18) holds.
Let us show (2.19) by taking $\tau=s$ from the above proof of the inclusion in (2.18). Then we do not need s to satisfy (2.21). As a consecontly, (2.19) holds for all $s\in[t,T]$. Finally, we prove (2.20). Taking $s\in(t,T)$ such that (2.21) holds. If $\forall\left(\bar{q}_i,\bar{p}_i\right)\in D_{s,y}^{1,+}W_i\left(s,\bar{y}(s)\right)$, then by the definition of superdifferential and Bellman's Principle of optimality (2.11) we have

$$\begin{aligned}0 &\geq \limsup_{\vartheta\downarrow s}\left\{\frac{W_i\left(\vartheta,\bar{y}(\vartheta)\right)-W_i\left(s,\bar{y}(s)\right)-\bar{q}_i(\vartheta-s)-\langle\bar{p}_i,\bar{y}(\vartheta)-\bar{y}(s)\rangle}{|\vartheta-s|+|\bar{y}(\vartheta)-\bar{y}(s)|}\right\}\\ &=\limsup_{\vartheta\downarrow s}\left\{\frac{1}{|\vartheta-s|+|\bar{y}(\vartheta)-\bar{y}(s)|}\left[-\int_s^\vartheta G_i\left(\alpha,\bar{y}(\alpha),\bar{b}_1(\alpha),\bar{b}_2(\alpha)\right)\,d\alpha\right.\right.\\ &\quad\left.\left.-\bar{q}_i(\vartheta-s)-\int_s^\vartheta\left\langle\bar{p}_i(\tau),F\left(\alpha,\bar{y}(\alpha),\bar{b}_1(\alpha),\bar{b}_2(\alpha)\right)\right\rangle\,d\alpha\right]\right\}\\ &=\limsup_{\vartheta\downarrow s}\left\{\frac{1}{|\vartheta-s|}\left[-\int_s^\vartheta G_i\left(\alpha,\bar{y}(\alpha),\bar{b}_1(\alpha),\bar{b}_2(\alpha)\right)\,d\alpha\right.\right.\\ &\quad\left.\left.-\bar{q}_i(\vartheta-s)-\int_s^\vartheta\left\langle\bar{p}_i(\tau),F\left(\alpha,\bar{y}(\alpha),\bar{b}_1(\alpha),\bar{b}_2(\alpha)\right)\right\rangle\,d\alpha\right]\cdot\frac{|\vartheta-s|}{|\vartheta-s|+|\bar{y}(\vartheta)-\bar{y}(s)|}\right\}\end{aligned}\tag{2.30}$$

By using (2.21) and the limit of the first term on the right-hand side exists (constant). Because $\left|\bar{y}\left(\vartheta\right)-\bar{y}\left(s\right)\right|\leq C\left|\vartheta-s\right|$ for some constant $C>0$, the inequality (2.30) yields

$$\begin{aligned}0&\geq -G_i\left(s,\bar{y}\left(s\right),\bar{b}_1\left(s\right),\bar{b}_2\left(s\right)\right)-\bar{q}_i-\left\langle\bar{p}_i,F\left(s,\bar{y}\left(s\right),\bar{b}_1\left(s\right),\bar{b}_2\left(s\right)\right)\right\rangle\\ &=\mathcal{H}_i\left(s,\bar{y}(s),\bar{b}_1\left(s\right),\bar{b}_2\left(s\right),\bar{p}_i\right)-\bar{q}_i,\end{aligned}$$

Then

$$\bar{q}_i\geq\mathcal{H}_i\left(s,\bar{y}(s),\bar{b}_1\left(s\right),\bar{b}_2\left(s\right),\bar{p}_i(s)\right).\tag{2.31}$$

Similarly, letting $\vartheta \uparrow s$, we can conclude

$$\bar{q}_i \leq \mathcal{H}_i \left(s, \bar{y}(s), \bar{b}_1\left(s\right), \bar{b}_2\left(s\right), \bar{p}_i(s)\right). \tag{2.32}$$

Then, from (2.31) and (2.32) the first equality in (2.20) holds.
Next, since W_i is the VS of the equations (2.12) by (2.17) we have,

$$\bar{q}_i - \inf_{b_i(\cdot)\in\mathcal{B}_i[0,T]} \mathcal{H}_i\left(s, \bar{y}(s), b_1(s), b_2\left(s\right), \bar{p}_i(s)\right) \geq 0,\ i = 1, 2,$$

which yields the second equality of (2.20). □

Remark 2.2. We note that :
(i) When W_i is differentiable, the inclusions (2.18)-(2.19) is reduced to (2.13) and (2.14) in Theorem 2.3;
(ii) the principal results of this study might be considered as an extension of similar results in [57] related to deterministic optimal control problem.

CHAPTER 3

APPLICATIONS TO ECONOMIC

In this chapter, we provide an example of a producer-consumer game with sticky prices, taken from [14] to illustrate the connection between adjoint variables in the MP and value function in the DPP, both in the smooth and nonsmooth cases. This application illustrates how this connection can be applied to real-world situations, focusing on the economic interpretations of the adjoint variables and showing their role in determining Nash equilibria in differential games. More specifically, we will apply the results seen in Chapter 2, about as, we start by presenting the formulation of the producer-consumer game problem, which we can see this problem as a deterministic two-player nonzero-sum differential games between producer and consumer. Then we move on to present the two of the most important approaches, both the MP ([14]) and the DPP of this game problem with their economic applications. In the last section, the connection between MP and DPP, both in the smooth and nonsmooth cases of the value function is established with their application to economic.

3.1 Producer-Consumer Game with Sticky Price

The producer-consumer model is a concept in economics that represents the interaction between producers and consumers in a market economy. This model is used to understand economic phenomena such as price setting, resource allocation and market efficiency. We present this example formulated as a two-player NZSDG, taken from ([14]) to illustrate the theoretical results (Theorem 2.3 and Theorem 2.4).

3.1.1 Formulation of the Problem

Here, we present the formulation of the producer-consumer game problem, which we can consider this problem as a deterministic two-player nonzero-sum differential games between producer and consumer.

Consider a company manufacturing a good, let $y(s)$ denote the sale price of a good at time s, and this good is produced at rate $b_1(s)$ by the company and consumed at rate $b_2(s)$ by the consumer.

The dynamical system (2.1) represent the variation of the price in time is given by the following

ODE

$$\begin{cases} \dot{y}(s) = y(s)\left(b_2(s) - b_1(s)\right), \ s \in [0,T] \\ y(0) = y_0, \end{cases} \tag{3.1}$$

The controls $b_i(s), i = 1, 2$ represent the rate of production and consumption of a good at time s, respectively. According to (3.1), the price increases when the consumption is larger than the production of goods, and decreases otherwise.
To simplify, let $c_i(r), i = 1, 2$ denote the cost function of company i. We assume that

$$c_1(r) = \frac{r^2}{2}, \qquad c_2(r) = 2\sqrt{r}.$$

The producer's payoff is given by the profit generated from sales minus the cost of production $c_1(b_1(s))$, depending on the rate of production $b_1(s)$. The consumer's payoff is measured by an utility function $c_2(b_2(s))$, which represents the benefit obtained from consuming the goods minus the price paid to purchase the goods. The payoff functional for the two player over the $[0,T]$ are given by

$$\begin{aligned} J_1(s, y_0; b_1(s), b_2(s)) &= \int_0^T \left[y(s) b_2(s) - c_1(b_1(s))\right] ds, \\ J_2(s, y_0; b_1(s), b_2(s)) &= \int_0^T \left[c_2(b_2(s)) - y(s) b_2(s)\right] ds, \end{aligned} \tag{3.2}$$

The problem is to maximize the payoffs for both the producer and the consumer (3.2), which can be rewrittten as the minimization of

$$\begin{aligned} J_1(s, y_0; b_1(s), b_2(s)) &= -\int_0^T \left[y(s) b_2(s) - c_1(b_1(s))\right] ds, \\ J_2(s, y_0; b_1(s), b_2(s)) &= -\int_0^T \left[c_2(b_2(s)) - y(s) b_2(s)\right] ds. \end{aligned} \tag{3.3}$$

Maximum Principle approach :

Now, we apply maximum principle approach as mentioned in the section 2.1.1 to a two-player NZSDG involving a producer and a consumer (see, e.g [14]).
To derive the OLNE of this producer-consumer game, the Hamiltonian (2.4) for producer (Player

1) and consumer (Player 2) is defined as follows :

$$H_1(s, y, b_1, b_2, p_1) = p_1(s)\left(y(s)\left(b_2(s) - b_1(s)\right)\right) - y(s)b_2(s) + \frac{b_1^2(s)}{2},$$

$$H_2(s, y, b_1, b_2, p_2) = p_2(s)\left(y(s)\left(b_2(s) - b_1(s)\right)\right) - 2\sqrt{b_2}(s) + y(s)b_2(s).$$

By minimizing $H_1(s, y, b_1, b_2, p_1)$ and $H_2(s, y, b_1, b_2, p_2)$ with respect to b_1 and b_2 using (2.7) we get the OLNE $\bar{b}_1$ and $\bar{b}_2$ for producer and consumer are given, respectively, by

$$\bar{b_1}(s) = \bar{p}_1(s)\,\bar{y}(s),$$

$$\bar{b_2}(s) = \frac{1}{\bar{y}^2(s)\left(\bar{p}_2(s) + 1\right)^2},$$

here $\bar{y} > 0, \bar{p}_1 \geq 0$ and $\bar{p}_2 > -1$.
The adjoint equations (2.5) for the two player are given by

$$\begin{cases} \dot{\bar{p}}_1(s) = \bar{p}_1^2(s)\,\bar{y}(s) - \frac{\bar{p}_1(s)-1}{\bar{y}^2(s)(\bar{p}_2(s)+1)^2} \\ \\ \bar{p}_1(T) = 0, \end{cases} \tag{3.4}$$

and

$$\begin{cases} \dot{\bar{p}}_2(s) = \bar{p}_1(s)\,\bar{p}_2(s)\,\bar{y}(s) - \frac{\bar{p}_2(s)+1}{\bar{y}^2(s)(\bar{p}_2(s)+1)^2} \\ \\ \bar{p}_2(T) = 0. \end{cases} \tag{3.5}$$

The state equation (2.1) is given by the following (ODE)

$$\begin{cases} \dot{\bar{y}}(s) = \frac{1}{\bar{y}(s)(\bar{p}_2(s)+1)} - \bar{p}_1(s)\,\bar{y}^2(s) \\ \\ \bar{y}(0) = y_0, \end{cases}$$

Dynamic Programming approach :

Here, we apply the dynamic programming approach as mentioned in the section 2.1.2 to a two-player NZSDG involving a producer and a consumer.

The value functions (2.10) are defined as the following :

$$W_1\left(s,y\right)=J_1(s,y;\bar{b}_1(\cdot),\bar{b}_2(\cdot)),$$

$$W_2\left(s,y\right)=J_2(s,y;\bar{b}_1(\cdot),\bar{b}_2(\cdot)).$$

Where $W_i\left(s,y\right), i=1,2$ is the value function for the problem of minimizig (3.3), which represents the minimum achievable cost or loss that either the producer or the consumer can incur, given the initial time s and the state y.

Thus, by introducing the Hamiltonian functions (2.4)

$$H_1\left(s,y,b_1,b_2,\frac{\partial W_1}{\partial y}\left(s,y(s)\right)\right)=\frac{\partial W_1}{\partial y}\left(s,y(s)\right)\left(y(s)\left(b_2(s)-b_1(s)\right)\right)-y(s)b_2(s)+\frac{b_1^2(s)}{2},$$

$$H_2\left(s,y,b_1,b_2,\frac{\partial W_2}{\partial y}\left(s,y(s)\right)\right)=\frac{\partial W_2}{\partial y}\left(s,y(s)\right)\left(y(s)\left(b_2(s)-b_1(s)\right)\right)-2\sqrt{b_2(s)}+y(s)b_2(s).$$

The value function W_i satisfies HJB equations (2.12), as established in Theorem 2.2, for the two player as follows

$$\begin{cases}\dfrac{\partial W_1}{\partial s}\left(s,y\right)+\inf_{b_1(s)\in\mathcal{B}_1[s,T]}\left\{\left(y\left(s\right)\left(b_2\left(s\right)-b_1\left(s\right)\right)\right)\dfrac{\partial W_1}{\partial y}\left(s,y\right)-y\left(s\right)b_2\left(s\right)+\frac{b_1^2(s)}{2}\right\}=0,\\ \\ W_1\left(T,x\right)=0,\end{cases}\tag{3.6}$$

and

$$\begin{cases}\dfrac{\partial W_2}{\partial s}\left(s,y\right)+\inf_{b_2(s)\in\mathcal{B}_2[s,T]}\left\{\left(y\left(s\right)\left(b_2\left(s\right)-b_1\left(s\right)\right)\right)\dfrac{\partial W_2}{\partial y}\left(s,y\right)-2\sqrt{b_2\left(s\right)}+y\left(s\right)b_2\left(s\right)\right\}=0\\ \\ W_2\left(T,x\right)=0,\end{cases}\tag{3.7}$$

Minimizing $H_1\left(s,y,b_1,b_2,\dfrac{\partial W_1}{\partial y}\left(s,y(s)\right)\right)$ and $H_2\left(s,y,b_1,b_2,\dfrac{\partial W_2}{\partial y}\left(s,y(s)\right)\right)$ with respect to b_1 and b_2 leads to FNE for our problem, which we can write as :

$$\bar{b_1}\left(s\right)=\bar{y}\left(s\right).\frac{\partial W_1}{\partial y}\left(s,\bar{y}(s)\right),$$

$$\bar{b_2}\left(s\right)=\frac{1}{\bar{y}^2\left(s\right)\left(\dfrac{\partial W_2}{\partial y}\left(s,\bar{y}\left(s\right)\right)+1\right)^2}.$$

When FNE is substituted into the HJB equations (3.6) and (3.7), the HJB equations take the forms :

$$\begin{cases} \dfrac{\partial W_1}{\partial s}(s,y) + \left\{ \bar{y}(s) \left(\dfrac{1}{\bar{y}^2(s)\left(\frac{\partial W_2}{\partial y}(s,\bar{y}(s))+1\right)^2} - \bar{y}(s) \, . \dfrac{\partial W_1}{\partial y}(s,\bar{y}(s)) \right) \dfrac{\partial W_1}{\partial y}(s,y) \right. \\ \qquad \left. -\bar{y}(s) \dfrac{1}{\bar{y}^2(s)\left(\frac{\partial W_2}{\partial y}(s,\bar{y}(s))+1\right)^2} + \dfrac{\left(\bar{y}(s).\frac{\partial W_1}{\partial y}(s,\bar{y}(s))\right)^2}{2} \right\} = 0, \\ W_1(T,x) = 0, \end{cases} \tag{3.8}$$

and

$$\begin{cases} \dfrac{\partial W_2}{\partial s}(s,y) + \left\{ \bar{y}(s) \left(\dfrac{1}{\bar{y}^2(s)\left(\frac{\partial W_2}{\partial y}(s,\bar{y}(s))+1\right)^2} - \bar{y}(s) \, . \dfrac{\partial W_1}{\partial y}(s,\bar{y}(s)) \right) \dfrac{\partial W_2}{\partial y}(s,y) \right. \\ \qquad \left. -2\sqrt{\dfrac{1}{\bar{y}^2(s)\left(\frac{\partial W_2}{\partial y}(s,\bar{y}(s))+1\right)^2}} + \bar{y}(s) \dfrac{1}{\bar{y}^2(s)\left(\frac{\partial W_2}{\partial y}(s,\bar{y}(s))+1\right)^2} \right\} = 0 \\ W_2(T,x) = 0, \end{cases} \tag{3.9}$$

3.2 The Connection Between MP and DPP

In this section, we present the economic interpretation of the connection between the MP and the DPP in both the smooth and nonsmooth cases.

3.2.1 Smooth Case

In order to explain the results of Theorem 2.3, we can derive the equality (2.13) directly from equations (3.6)-(3.9). About equality (2.14), the adjoint variables $\bar{p}_i(s), i = 1, 2$ represent the marginal value (also known as the shadow prices) of the sale price $\bar{y}(s)$. This provides an economic interpretation to the adjoint variables (see [25], [57], [2]). In addition, the change in the value of the sale price of the system from state $\bar{y}(s)$ to $\bar{y}(s) + \gamma y(s)$ is

$$W_i(s, \bar{y}(s) + \gamma y(s)) - W_i(s, \bar{y}(s)) \approx \bar{p}_i(s) \gamma y(s) \, . \, i = 1, 2. \tag{3.10}$$

This implies (Fréchet) differentiability of $W_i(s, \bar{y}(s))$ at $\bar{y}(s)$ (see e.g., [2]). Thus, $\bar{p}_i(s), i = 1, 2$ represents the marginal value of the rate of change in the profit W_i for slight adjustments in the sale price $\bar{y}(s)$. As sale prices increase due to increased consumption, $\bar{p}_1(s)$ decreases for the producer while $\bar{p}_2(s)$ increases for the consumer. Furthermore, the marginal value for producer $\bar{p}_1(s)$ can be interpreted as the incremental profit of producing and selling another product, and for consumers, $\bar{p}_2(s)$ represents the maximum price they are actually willing to pay for the last thing they consume.

3.2.2 Nonsmooth Case

Similar to the smooth case, we illustrate the result of the Theorem 2.4 when the value function $W_i(\cdot,\cdot)$ is nonsmooth, satisfying the VS; see [57]. As we have seen in Section 3.2.1, since the second inclusion in (2.19) and when the increment $\gamma y(s)$ is small, the increase in the value of the system from state $\bar{y}(s)$ to $\bar{y}(s)+\gamma y(s)$ is defined as

$$W_i(s,\bar{y}(s)+\gamma y(s)) - W_i(s,\bar{y}(s)) \le \bar{p}_i(s)\,\gamma y(s)\,. \tag{3.11}$$

Due to the positivity of both sides (3.11), we conclude that

$$|W_i(s,\bar{y}(s)+\gamma y(s)) - W_i(s,\bar{y}(s))| \le \bar{p}_i(s)\,|\gamma y(s)|.$$

This indicates that the effect of slight changes $\gamma y(s)$ in the sale price on the producer's and the customer's payoffs is dependent on their individual marginal values. Then, as the sale price increases, the producer's marginal value $\bar{p}_1(s)$ decreases, suggesting that the rate of increase in the producer's reward per unit sold slows down. Meanwhile, the consumer's marginal value $\bar{p}_2(s)$ increases, suggesting that consumer are prepared to pay more for each unit they consume. The other side, the decrease in the value of the sale price state from state $\bar{y}(s)$ to $\bar{y}(s)-\gamma y(s)$, then

$$W_i(s,\bar{y}(s)-\gamma y(s)) - W_i(s,\bar{y}(s)) \le -\bar{p}_i(s)\,\gamma y(s)\,. \tag{3.12}$$

Both sides of (3.12) are negative ($\gamma y(s)>0$). So,

$$|W_i(s,\bar{y}(s)-\gamma y(s)) - W_i(s,\bar{y}(s))| \ge \bar{p}_i(s)\,|\gamma y(s)|.$$

When the sale prices decrease, the producer's marginal value $\bar{p}_1(s)$ increases, proving that the additional profit made from producing and selling more units rises as well. Conversely, the consumer's marginal value $\bar{p}_2(s)$ decreases, which indicates that consumers are less able to pay for every product that they consume as prices decrease. Similarly, we can also interpret the Hamiltonian $H_i(s,y,b_1,b_2,p_i)\,, i=1,2$ as the rate of change for the maximum profit with respect to time using (2.18).

In a producer-consumer game, the adjoint variable $\bar{p}_i(s)$ illustrates how changes in the state $\bar{y}(s)$ (the sale prices) affect the optimal cost or payoff $W_i(s,\bar{y}(s))$ (value function). To be more precise, the rate at which changes in the state variable affect the value function is indicated by the adjoint variable, often known as the marginal value. Essentially, $\bar{p}_i\left(s\right)$ expresses the sensitivity of optimal costs or payoffs to small state adjustments in the state.

CHAPTER 4

JACOBI SPECTRAL METHOD FOR SOLVING DIFFERENTIAL GAMES

This chapter presents a numerical method based on the MP, as mentioned in section 2.1.1, using the Jacobi spectral method (JSM), which allows us to approximately solve the nonlinear two-point boundary value problems (TPBVPs) derived from MP is transferred to a system of algebraic equations in order to determine the OLNE of nonzero-sum differential games (NZSDGs) in a finite horizon. First, we present the formulation of the problem for deterministic n-player NZSDGs. We then discuss the application of the JSM to solve these differential games. Finally, we provide some examples to demonstrate the accuracy and usefulness of the proposed method.

4.1 Problem Statement

In this section, consider the n-player nonzero-sum differential game with the finite horizon which has the following nonlinear differential equation [9,55] :

$$\begin{cases} \dot{y}\left(s\right) = F\left(s, y\left(s\right), b_1\left(s\right), b_2\left(s\right), ..., , b_n\left(s\right)\right), \ s \in [0, T] \\ \\ y(0) = y_0, \end{cases} \tag{4.1}$$

and cost functionals given as

$$J_i(b_1(\cdot), ..., b_n(\cdot)) = \int_0^T G_i\left(s, y\left(s\right), b_1\left(s\right), b_2\left(s\right), ..., b_n\left(s\right)\right) ds + h_i\left(y\left(T\right)\right), \ i = 1, 2, \cdots, n. \tag{4.2}$$

Each players aim is to minimize the functional (4.2) by finding the control $\bar{b}_i(\cdot) = (b_1(\cdot), b_2(\cdot), ., b_n(\cdot)) \in \mathcal{B}_i\left[0, T\right], i = 1, 2, ., n$. In particular, $\bar{b}_i(\cdot)$ an OLNE solves the Problem (NZSDG) :

$$J_i(\bar{b}_i(\cdot), \bar{b}_{-i}(\cdot)) = \inf_{b_i(\cdot) \in \mathcal{B}_i[0,T]} J_i(b_i(\cdot), b_{-i}(\cdot)).$$

where b_i is the control strategy for the i-th player and b_{-i} are the controls for the rest of the

players $b_{-i} = b_j, (j \neq i)$.
Now, we apply the necessary conditions of the MP approach as presented in section 2.1.1 for two-player differential games, which can be directly extended to the general case with n players, as follows :

$$\dot{y}(s) = F\left(s, y(s), b_1(s), b_2(s), ..., b_n(s)\right), \quad y(0) = y_0, \tag{4.3}$$

$$\dot{p}_i(s) = -H_y^i\left(s, y(s), b_i(s), b_{-i}(s), p_i(s)\right), \quad p_i(T) = h_y^i\left(y(T)\right) \tag{4.4}$$

$$H_{b_i}^i\left(s, y(s), b_i(s), b_{-i}(s), p_i(s)\right) = 0, \quad i = 1, 2, ..., n. \tag{4.5}$$

According to the equation (4.5), we get the OLNE $\bar{b}_i(s) = \bar{b}_i\left(s, \bar{y}(s), \bar{p}_i(s)\right)$. Then, by substituting into equations (4.3) and (4.4), we obtain the following system of TPBVPs :

$$\dot{\bar{y}}(s) = F\left(s, \bar{y}(s), \bar{b}_1(s), \bar{b}_2(s), ..., \bar{b}_n(s)\right), \quad \bar{y}(0) = y_0, \tag{4.6}$$

$$\dot{\bar{p}}_i(s) = -H_y^i\left(s, \bar{y}(s), \bar{b}_i(s), \bar{b}_{-i}(s), \bar{p}_i(s)\right), \quad \bar{p}_i(T) = h_y^i\left(\bar{y}(T)\right), \ i = 1, 2, ..., n, \tag{4.7}$$

We can use an Algorithm 4.1 to summarize all we have discussed so far on the steps for determining OLNE in differential games, similar to the case of FNE in [4].

Algorithm 4.1. ***Input*** *:The nonzero-sum differential game (4.1)-(4.2).*
Step 1*. Write down the necessary condition for a MP (4.3)-(4.5).*
Step 2*. Minimize the Hamiltonian functions using (4.5) and find the optimal control strategies given by* $\bar{b}_i(s) = \bar{b}_i\left(s, \bar{y}(s), \bar{p}_i(s)\right), i = 1, 2, ..., n.$
Step 3*. Insert the obtained optimal control strategies as the function of* $\bar{y}$ *and* $\bar{p}_i, i = 1, 2, ..., n$ *from Step 2 in the system (4.3)-(4.5) in Step 1. This leads to the system of TPBVPs (4.6)-(4.7).*
Step 4*. Solve the obtained system of TPBVPs from Step 3 and find the state variable* $\bar{y}(s)$ *and adjoint variables* $\bar{p}_i(s), i = 1, 2, ..., n.$
Step 5*. According to Steps 2 and 4, write down the optimal control strategies* $\bar{b}_i(s)$.
Output *: OLNE* $\bar{b}_i(\cdot), i = 1, 2, ..., n.$

The difficulties in solving this system of TPBVPs (4.6)-(4.7) are mainly due to the combination of nonlinearity and split boundary conditions. Therefore, obtaining an exact analytical solution is highly complex, and the application of appropriate numerical methods is essential to solve these problems (see e.g., [5, 43]).

4.2 Jacobi Spectral Method for Nonzero-sum Differential Games

In this section, we present the application of the JSM using Jacobi polynomials (JPs), which play a crucial role in solving the system of TPBVPs (4.6)-(4.7) and finding the OLNE of a nonzero-sum differential games (4.1)-(4.2).
The main concept of this method is consists of approximating a function $\mu(x) \in \mathbb{L}^2(-1,1)$ by converting it into a finite series expansion of basis functions. This can be written as :

$$\mu(x) \simeq \mu^N(x) = \sum_{j=0}^{N} \mu_j \mathcal{J}_j^{k,\ell}(x),$$

where $\mathcal{J}_j^{k,\ell}(x), j = 0, 1, ..., N$ are Jacobi polynomials (basis functions) and μ_j, $j = 0, 1, ..., N$ are spectral coefficients (see e.g., [43,48]).
We recall the definition of classical JPs and their properties as follows.

Definition 4.1. The Jacobi polynomials $\mathcal{J}_r^{k,\ell}(x), (r \geq 0)$ for $k, \ell > -1$ are the eigenfunctions of the singular Sturm Liouville problem (see e.g., [52])

$$\left(1-x^2\right) Y'' + \left(\ell - k - (k+\ell+2)\, x\right) Y' + r\,(r+k+\ell+1)\, Y = 0, \quad Y = \mathcal{J}_r^{k,\ell}(x).$$

Hence, JPs are orthogonal in $\mathbb{L}^2_{\omega^{k,\ell}}(-1,1)$ with respect to the weight function $\omega^{k,\ell}(x) = (1-x)^k(1+x)^\ell$ (see e.g., [33]).

$$\int_{-1}^{1} \mathcal{J}_r^{k,\ell}(x)\, \mathcal{J}_m^{k,\ell}(x)\, \omega^{k,\ell}(x) dx = h_r^{k,\ell} \delta_{r,m},$$

where $\delta_{r,m}$ is the Kronecker function, and

$$h_r^{k,\ell} = \frac{2^{k+\ell+1}\Gamma(r+k+1)\,\Gamma(r+\ell+1)}{(2r+k+\ell+1)\,\Gamma(r+1)\Gamma(r+k+\ell+1)}.$$

The following recurrence formula for JPs as follows (see e.g., [52]).

$$\begin{cases} 2r\,(r+k+\ell)\,(2r+k+\ell-2)\,\mathcal{J}_r^{k,\ell}(x) = \\ (2r+k+\ell-1)\left[(k^2-\ell^2) + (2r+k+\ell)\,(2r+k+\ell-2)\,x\right] \mathcal{J}_{r-1}^{k,\ell}(x) \\ -2\,(r+k-1)\,(r+\ell-1)\,(2r+k+\ell)\,\mathcal{J}_{r-2}^{k,\ell}(x), r = 2, 3, ...; \\ \mathcal{J}_0^{k,\ell}(x) = 1, \\ \mathcal{J}_1^{k,\ell}(x) = \frac{1}{2}(k+\ell+2)\,x + \frac{1}{2}(k-\ell), \end{cases}$$

Remark 4.1. There are particular JPs that form a family of orthogonal polynomials, including several well-known special cases.

- Legendre polynomials are a particular case of Jacobi polynomials when $k = \ell = 0$.
- Chebyshev polynomials of the first kind correspond to $k = \ell = -\frac{1}{2}$, while Chebyshev polynomials of the second kind $k = \ell = \frac{1}{2}$.
- Chebyshev polynomials of the third and fourth kind correspond to $k = -\ell = \pm\frac{1}{2}$,
- JPs can be symmetric when $k = \ell$ and non-symmetric of JPs when $k \neq \ell$.

The following two theorems have been applied to Chebyshev and Legendre polynomials ([43], [3]). The application is extended to JPs, which generalize the above polynomials.

Theorem 4.1. *[15]. Let $\mu(x) \in \mathbb{H}^m_{\omega^{k,\ell}}(-1,1)$, $\mu^N(x) = \sum_{j=0}^{N} \mu_j \mathcal{J}^{k,\ell}_j(x)$ be the best approximation of $\mu(x)$ in $\mathbb{L}^2_{\omega^{k,\ell}}$- norm, then*

$$\left\|\mu(x) - \mu^N(x)\right\|_{\mathbb{L}^2_{\omega^{k,\ell}[-1,1]}} \leq M_0 N^{-m} \left\|\mu(x)\right\|_{\mathbb{H}^m_{\omega^{k,\ell}(-1,1)}}.$$

where M_0 is a positive constant, which depends on the selected norm, independent of $\mu(x)$ and N.

The main results of the presented approach, as well as the theoretical analysis of its convergence, are related to the well-known Weierstrass approximation theorem [48].

Theorem 4.2. *Suppose that $\mu \in \mathbb{L}^2_\omega[-1,1]$ and $N \in \mathbb{N}$. Then there exists a unique $\mu^{N*} \in P_N$, such that*

$$\left\|\mu - \mu^{N*}\right\|_\omega = \inf_{\mu^N \in P_N} \left\|\mu - \mu^N\right\|_\omega,$$

where

$$\mu^{N*}(s) = \sum_{j=0}^{N} \hat{\mu}_j \psi_j(s), \quad \hat{\mu}_j = \frac{\langle \mu, \psi_j \rangle_\omega}{\|\psi_j\|^2_\omega},$$

and $\{\psi_j\}_{j=0}^N$ form an $\mathbb{L}^2_\omega$-orthogonal basis of P_N.

To apply the JPs on the interval $[0,T]$, we defined shifted JPs through a change of variable $x = \frac{2s}{T} - 1$, which satisfies the same properties mentioned above and is defined by $\mathcal{J}^{k,\ell}_{T,r}(s)$.
In order, to present the application of the JSM for solving the system of TPBVPs (4.6)-(4.7) in nonzero-sum differential games in the finite horizon T, we can approximate the state of game $\bar{y}(s)$ and the adjoint variables $\bar{p}_i(s), i = 1, 2, ...n$ as finite expansions of shifted JPs that have the following form

$$\bar{y}(s) \simeq y^N(s) = \sum_{j=0}^{N} c_j \mathcal{J}^{k,\ell}_{T,j}(s) = C^\top \mathcal{J}^{k,\ell}_T(s), \tag{4.8}$$

$$\bar{p}_i(s) \simeq p^N_i(s) = \sum_{j=0}^{N} d_{ij} \mathcal{J}^{k,\ell}_{T,j}(s) = D_i^\top \mathcal{J}^{k,\ell}_T(s), i = 1, ..., n, \tag{4.9}$$

where $C^\top = [c_1, c_2, c_3, ..., c_N]$ and $D_i^\top = [d_{i0}, d_{i1}, d_{i2}...., d_{iN}]$ are unknown coefficients and $\mathcal{J}^{k,\ell}_T(s) = \left[\mathcal{J}^{k,\ell}_{T,0}(s), \mathcal{J}^{k,\ell}_{T,1}(s), ..., \mathcal{J}^{k,\ell}_{T,N}(s)\right]^\top$ is the shifted JPs on the interval $[0,T]$.

Furthermore, we can approximate the derivatives of $\bar{y}^N(s)$ and $\bar{p}_i^N(s)$ in terms of the derivatives of the shifted JPs.
Now, by substituting the approximations (4.8) and (4.9) into the differential equations of the system of TPBVPs (4.6)-(4.7), we can define the following residual functions

$$Res_y(s) = \dot{y}^N(s) - F\left(s, y^N(s), b_1^N(s), b_2^N(s), ..., b_n^N(s)\right), \tag{4.10}$$

$$Res_i(s) = \dot{p}_i^N(s) + H_{y^N}^i\left(s, y^N(s), b_i^N(s), b_{-i}^N(s), p_i^N(s)\right) \quad , \; i = 1, ..., n.$$

By multiplying these residuals by the shifted JPs $\mathcal{J}_{T,j}^{k,\ell}(s), j = 0, ...N-1$ and integrating over the interval $[0, T]$, and then setting the result equal to zero along with the boundary values, a system of $(n+1)(N+1)$ algebraic equations is obtained.

$$\begin{cases} \int\limits_0^T Res_y(s)\,\mathcal{J}_{T,j}^{k,\ell}(s)\,ds = 0, \\ \int\limits_0^T Res_i(s)\,\mathcal{J}_{T,j}^{k,\ell}(s)\,ds = 0, \\ y^N(0) = y_0, \\ p_i^N(T) = h_{y^N}^i\left(y^N(T)\right) \quad , \; i = 1, ..., n. \end{cases} \tag{4.11}$$

Subsequently, Newton's iteration method can be used to solve this system and determine the unknown coefficients $C^\top$ and $D_i^\top, i = 1, ...n$.

4.3 Numerical Examples

The purpose of this section is to apply the proposed method to three examples. The first two (Example 4.1 and Example 4.2) are LQDGs. The solutions obtained by our method are compared with exact solutions as well as with those obtained by existing methods : the Bernoulli Tau method (BTM) [5] for Example 4.1, and the Chebyshev pseudospectral method (CPM) [43] and BTM [5] for Example 4.2. Example 4.3 is a differential game arising from an economic model based on a nonlinear system of TPBVPs, for which no exact solution is available. This example is introduced in [5]. In this case, a residual function is defined to assess the performance of the proposed method. For these examples, we follow the steps of Algorithm 4.1, which represents the necessary conditions of the MP. In addition, different values of the Jacobi parameters are used.

Example 4.1. Consider the following differential game problem defined by the system [27]

$$\begin{cases} \dot{y}(s) = b_1(s) + b_2(s), \ s \in [0,1] \\ y(0) = 1. \end{cases}$$

The cost functional for the two players who want to minimize are as follows :

$$J_1(b_1(\cdot), b_2(\cdot)) = \int_0^1 \left(-y^2(s) + b_1^2(s)\right) ds,$$

$$J_2(b_1(\cdot), b_2(\cdot)) = \int_0^1 \left(2y^2(s) + b_2^2(s)\right) ds + y^2(1).$$

The exact solution for OLNE of this problem is [27]

$$\bar{b_1}(s) = -\frac{1}{e} + e^{-s},$$

$$\bar{b_2}(s) = \frac{1}{e} - 2e - s.$$

Thus, the exact values of the cost functionals for Player 1 and Player 2 are

$$\bar{J_1}\left(\bar{b_1}(s), \bar{b_2}(s)\right) = -0.32975303263305,$$

$$\bar{J_2}\left(\bar{b_1}(s), \bar{b_2}(s)\right) = 1.9344880850240.$$

The Hamiltonian for Player 1 and Player 2 are defined by

$$H_1(s, y, b_1, b_2, p_1) = p_1(s)\left(b_1(s) + b_2(s)\right) - y^2(s) + b_1^2(s),$$

$$H_2(s, y, b_1, b_2, p_2) = p_2(s)\left(b_1(s) + b_2(s)\right) + 2y^2(s) + b_2^2(s).$$

where the adjoint variables for two player are denoted by p_1, p_2.
By minimizing $H_1(s, y, b_1, b_2, p_1)$ and $H_2(s, y, b_1, b_2, p_2)$ with respect to b_1 and b_2 we get the OLNE $\bar{b}_1$ and $\bar{b}_2$ for two players are given, respectively, by

$$\bar{b_1}(s) = -\frac{\bar{p}_1(s)}{2},$$

$$\bar{b_2}(s) = -\frac{\bar{p}_{2(s)}}{2},$$

Then, the system of TPBVPs (4.6)-(4.7) of this example can be expressed as follows

$$\dot{\bar{y}}\left(s\right) = -\frac{\bar{p}_1\left(s\right)}{2} - \frac{\bar{p}_2\left(s\right)}{2}, \quad \bar{y}(0) = 1,$$

$$\dot{\bar{p}}_1\left(s\right) = 2\bar{y}\left(s\right), \quad \bar{p}_1(1) = 0,$$

$$\dot{\bar{p}}_2\left(s\right) = -4\bar{y}\left(s\right), \quad \bar{p}_2(1) = 2\bar{y}\left(1\right),$$

A comparison of optimal cost functionals J_1 and J_2 obtained using JSM with the exact solutions is presented in Table 4.1, Table 4.2, Table 4.3 and Table 4.4, showing the results with various choices of k, ℓ and different values of N for Example 4.1. In Figure 4.1, we show the approximate solutions of $y(s), b_1(s)$ and $b_2(s)$ along with the exact solutions and absolute errors for $k = \ell = 0$ and $N = 10$.

N	J_{1JSM}	J_{2JSM}	$\lvert\bar{J}_1 - J_{1JSM}\rvert$	$\lvert\bar{J}_2 - J_{2JSM}\rvert$
4	-0.32975302954236861650594603530789	1.934488137365525879500666069862	3.091e-009	5.234e-008
6	-0.32975303263303305145518126348297	1.934488085024296353144451807013	1.693e-014	2.963e-013
8	-0.32975303263304656749214145627909	1.9344880850240688000077478452531	3.435e-015	6.875e-014
10	-0.32975303263304656750904904722969	1.9344880850240687997237177743588	3.435e-015	6.875e-014

TABLE 4.1 – A comparison of the optimal cost functionals J_1 and J_2 for Example 4.1 using JSM with the exact solutions for $k = \ell = 0$.

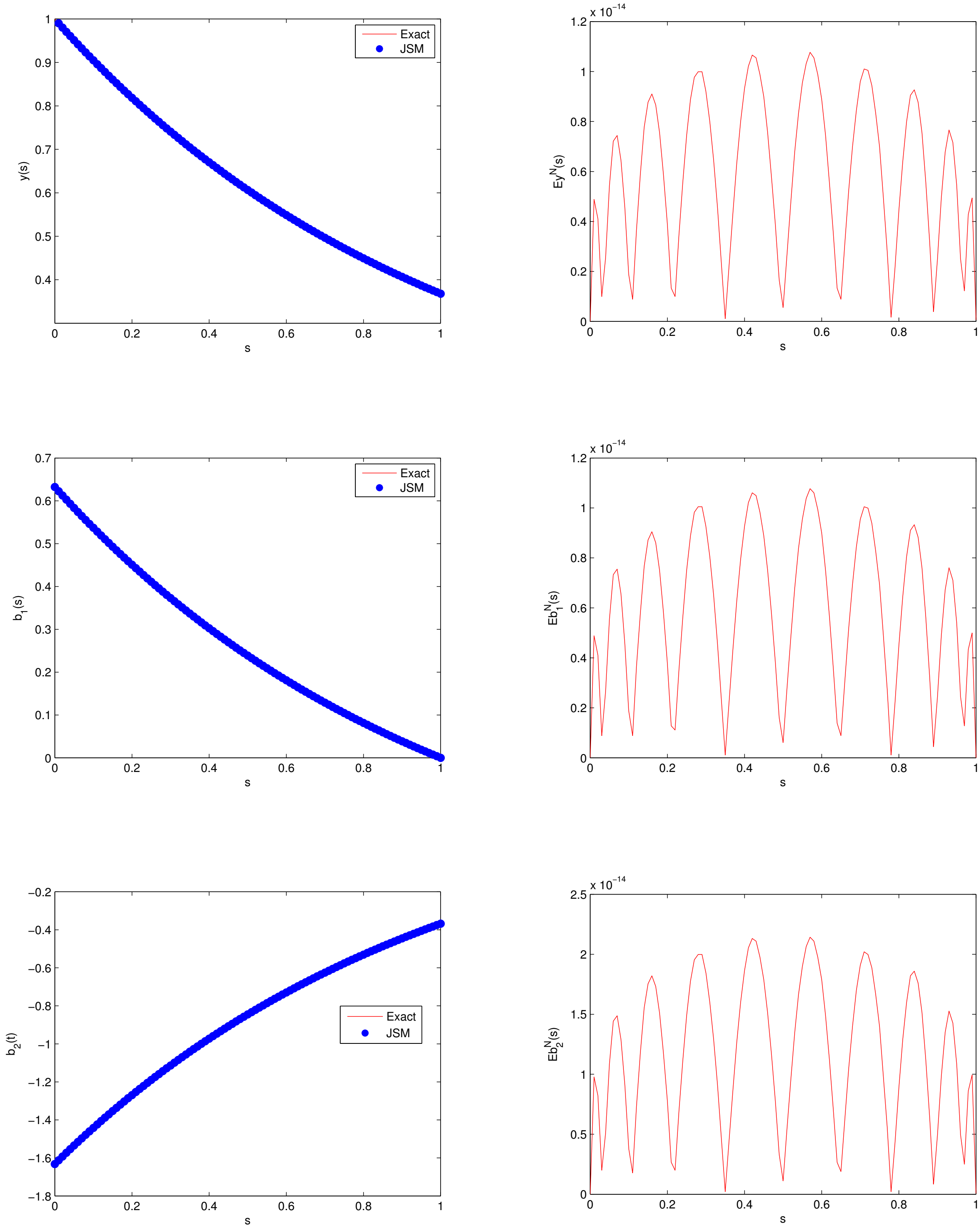

FIGURE 4.1 – The graphs of the numerical and the exact solutions with absolute errors for $k = \ell = 0$ and $N = 10$ for Example 4.1.

N	J_{1JSM}	J_{2JSM}	$\lvert \bar{J}_1 - J_{1JSM} \rvert$	$\lvert \bar{J}_2 - J_{2JSM} \rvert$
4	-0.32975302954236861650594603530789	1.934488137365525879500666069862	3.091e-009	5.234e-008
6	-0.32975303263303305145518126348297	1.934488085024296353144451807013	1.695e-014	2.963e-013
8	-0.32975303263304656749214145627909	1.9344880850240688000077478452531	3.435e-015	6.875e-014
10	-0.32975303263304656750904904692834	1.934488085024068799723717774528	3.435e-015	6.875e-014

TABLE 4.2 – A comparison of the optimal cost functionals J_1 and J_2 for Example 4.1 using JSM with the exact solutions for $k = \ell = \frac{-1}{2}$.

N	J_{1JSM}	J_{2JSM}	$\lvert \bar{J}_1 - J_{1JSM} \rvert$	$\lvert \bar{J}_2 - J_{2JSM} \rvert$
4	-0.32975302954236861650594603530789	1.934488137365525879500666069862	3.091e-009	5.234e-008
6	-0.32975303263303305145518126348297	1.934488085024296353144451807013	1.695e-014	2.963e-013
8	-0.32975303263304656749214145627909	1.9344880850240688000077478452531	3.435e-015	6.875e-014
10	-0.32975303263304656750904904722969	1.9344880850240687997237177743588	3.435e-015	6.875e-014

TABLE 4.3 – A comparison of the optimal cost functionals J_1 and J_2 for Example 4.1 using JSM with the exact solutions for $k = \ell = 1$.

N	J_{1JSM}	J_{2JSM}	$\lvert \bar{J}_1 - J_{1JSM} \rvert$	$\lvert \bar{J}_2 - J_{2JSM} \rvert$
4	-0.32975302954236861650594603530789	1.934488137365525879500666069862	3.091e-009	5.234e-008
6	-0.32975303263303305145518126348297	1.934488085024296353144451807013	1.695e-014	2.963e-013
8	-0.32975303263304656749214169210629	1.9344880850240688000077476123017	3.435e-015	6.875e-014
10	-0.32975303263304656750904904722969	1.9344880850240687997237177743588	3.435e-015	6.875e-014

TABLE 4.4 – A comparison of the optimal cost functionals J_1 and J_2 for Example 4.1 using JSM with the exact solutions for $k = -\frac{1}{2}, \ell = \frac{1}{2}$.

Furthermore, the results of the BTM [5] are presented in Table 4.5.

N	J_{1BTM}	J_{2BTM}
4	-0.32975302954236861650	1.93448814833633875533
6	-0.32975303263303305145	1.93448808502434431993
8	-0.32975303263304656749	1.93448808502406878964
10	-0.32975303263304656750	1.93448808502406878929

TABLE 4.5 – The optimal cost functionals J_1 and J_2 for Example 4.1 by BTM.

It can be observed that applying the JSM with variables k and ℓ for Example 4.1, gives accurate approximations. As shown in Figure 4.1, there is a strong agreement between the approximate and exact solutions. Tables 4.1, Table 4.2, Table 4.3 and Table 4.4 show that the results are sufficiently accurate. For different values of k and ℓ, the differences in the approximate solutions are minimal at $N = 6$, which is due to the effect of the Jacobi polynomial coefficients.

Example 4.2. Consider the linear quadratic nonzero-sum differential game defined by the state equation ([27]) as :

$$\begin{cases} \dot{y}(s) = 2y(s) + b_1(s) + b_2(s), \ s \in [0,3] \\ y(0) = 1. \end{cases}$$

The cost functional for the two players who want to minimize are as follows :

$$J_1(b_1(\cdot), b_2(\cdot)) = \int_0^3 \left(y^2(s) + b_1^2(s)\right) ds,$$

$$J_2(b_1(\cdot), b_2(\cdot)) = \int_0^3 \left(4y^2(s) + b_2^2(s)\right) ds + 5y^2(3).$$

The exact solution for OLNE of this problem is ([27])

$$\bar{b_1}(s) = -e^{-3s} + \frac{1}{e^{-3}} e^{-2s},$$

$$\bar{b_2}(s) = -4e^{-3s} - \frac{1}{e^{-3}} e^{-2s},$$

$$\bar{y}(s) = e^{-3s}.$$

Thus, the exact values of the cost functionals for Player 1 and Player 2 are given, respectively, by

$$\bar{J_1}\left(\bar{b_1}(s), \bar{b_2}(s)\right) = 0.3140381912,$$

$$\bar{J_2}\left(\bar{b_1}(s), \bar{b_2}(s)\right) = 3.4136123279.$$

The Hamiltonian for two player are defined by

$$H_1(s, y, b_1, b_2, p_1) = p_1(s)\left(2y(s) + b_1(s) + b_2(s)\right) + y^2(s) + b_1^2(s),$$

$$H_2(s, y, b_1, b_2, p_2) = p_2(s)\left(2y(s) + b_1(s) + b_2(s)\right) + 4y^2(s) + b_2^2(s).$$

By minimizing $H_1(s, y, b_1, b_2, p_1)$ and $H_2(s, y, b_1, b_2, p_2)$ with respect to b_1 and b_2 we get the OLNE

$\bar{b}_1$ and $\bar{b}_2$ for two players are given by

$$\bar{b_1}(s) = \frac{-\bar{p}_1(s)}{2},$$

$$\bar{b_2}(s) = \frac{-\bar{p}_2(s)}{2},$$

Then, the system of TPBVPs (4.6)-(4.7) of this example can be expressed as follows

$$\dot{\bar{y}}(s) = 2\bar{y}(s) - \frac{\bar{p}_1(s)}{2} - \frac{\bar{p}_2(s)}{2}, \quad \bar{y}(0) = 1,$$

$$\dot{\bar{p}}_1(s) = -2\bar{y}(s) - 2\bar{p}_1(s), \quad \bar{p}_1(3) = 0,$$

$$\dot{\bar{p}}_2(s) = -8\bar{y}(s) - 2\bar{p}_2(s), \quad \bar{p}_2(3) = 10\bar{y}(3),$$

The values of optimal cost functionals determined through the JSM and the comparison with the exact solutions are shown in Table 4.6. In Figure 4.2, we plot the approximate solutions of $y(s), b_1(s)$ and $b_2(s)$ with the exact solutions and absolute errors for $k = -\frac{1}{2}, \ell = \frac{1}{2}$ and $N = 20$.

N	J_{1JSM}	J_{2JSM}	$\lvert\bar{J}_1 - J_{1JSM}\rvert$	$\lvert\bar{J}_2 - J_{2JSM}\rvert$
10	0.31403763402282735215	3.4136147802083237797	5.572e-007	2.452e-006
15	0.31403819123820213086	3.413612327973873465	3.820e-011	7.387e-011
20	0.31403819124108284469	3.4136123279613940355	4.108e-011	6.139e-011

TABLE 4.6 – A comparison of the optimal cost functionals J_1 and J_2 for Example 4.2 using JSM with the exact solutions for $k = -\frac{1}{2}; \ell = \frac{1}{2}$.

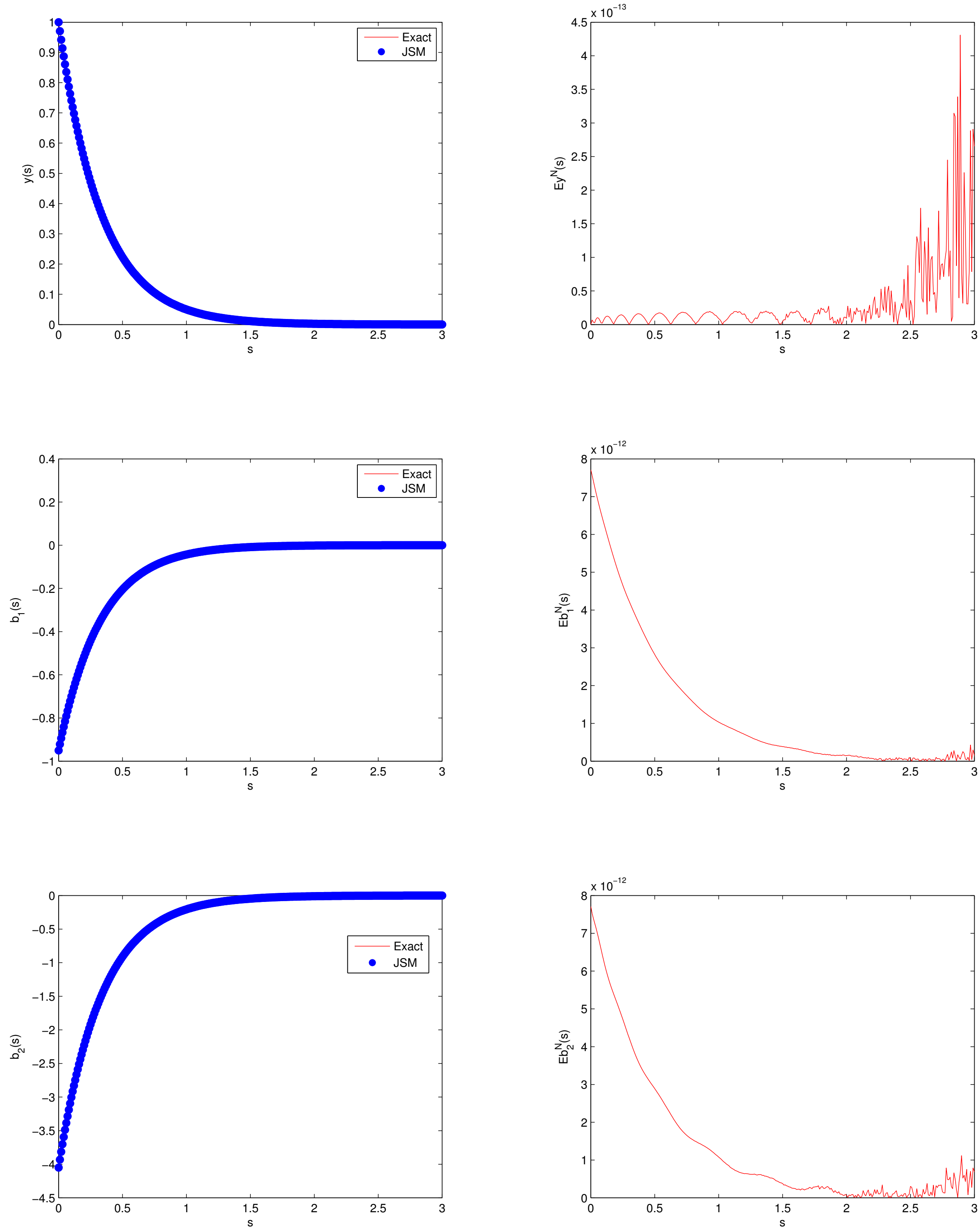

FIGURE 4.2 – The graphs of the numerical and the exact solutions with absolute errors for $k = -\frac{1}{2}, \ell = \frac{1}{2}$ and $N = 20$ for Example 4.2.

The results obtained by the JSM with different parameters for k, ℓ , and different values of N for Example 4.2 are given in Table 4.7, Table 4.8, Table 4.9, Table 4.10, Table 4.11 and Table 4.12, respectively. In Figure 4.3, the approximate solutions of $y(s), b_1(s)$ and $b_2(s)$ are shown alongside the exact solutions and absolute errors for $k = 0, \ell = \frac{1}{2}$ and $N = 20$.

N	J_{1JSM}	J_{2JSM}	$\lvert\bar{J}_1 - J_{1JSM}\rvert$	$\lvert\bar{J}_2 - J_{2JSM}\rvert$
10	0.31403763402282735215	3.4136147802083237797	5.572e-007	2.452e-006
15	0.31403819123820213086	3.413612327973873465	3.820e-011	7.387e-011
20	0.31403819124108284469	3.4136123279613940355	3.788e-011	7.526e-011

TABLE 4.7 – A comparison of the optimal cost functionals J_1 and J_2 for Example 4.2 using JSM with the exact solutions for $k = -\frac{1}{2}, \ell = -\frac{1}{2}$.

N	J_{1JSM}	J_{2JSM}	$\lvert\bar{J}_1 - J_{1JSM}\rvert$	$\lvert\bar{J}_2 - J_{2JSM}\rvert$
10	0.31403763402282735215	3.4136147802083237797	5.572e-007	2.452e-006
15	0.31403819123820213086	3.413612327973873465	3.820e-011	7.387e-011
20	0.31403819124108284469	3.4136123279613940355	3.949e-011	6.830e-011

TABLE 4.8 – A comparison of the optimal cost functionals J_1 and J_2 for Example 4.2 using JSM with the exact solutions for $k = \ell = 1$.

N	J_{1JSM}	J_{2JSM}	$\lvert\bar{J}_1 - J_{1JSM}\rvert$	$\lvert\bar{J}_2 - J_{2JSM}\rvert$
10	0.31403763402282735215	3.4136147802083237797	5.572e-007	2.452e-006
15	0.31403819123820213086	3.413612327973873465	3.820e-011	7.387e-011
20	0.31403819124108284469	3.4136123279613940355	3.658e-011	8.092e-011

TABLE 4.9 – A comparison of the optimal cost functionals J_1 and J_2 for Example 4.2 using JSM with the exact solutions for $k = \ell = 0$.

N	J_{1JSM}	J_{2JSM}	$\lvert\bar{J}_1 - J_{1JSM}\rvert$	$\lvert\bar{J}_2 - J_{2JSM}\rvert$
10	0.31403763402282735242	3.4136147802083237785	5.572e-007	2.452e-006
15	0.31403819124203047349	3.4136123279572897834	4.203e-011	5.729e-011
20	0.3140381912439577814	3.4136123279489403343	4.396e-011	4.894e-011

TABLE 4.10 – A comparison of the optimal cost functionals J_1 and J_2 for Example 4.2 using JSM with the exact solutions for $k = 0, \ell = \frac{1}{2}$.

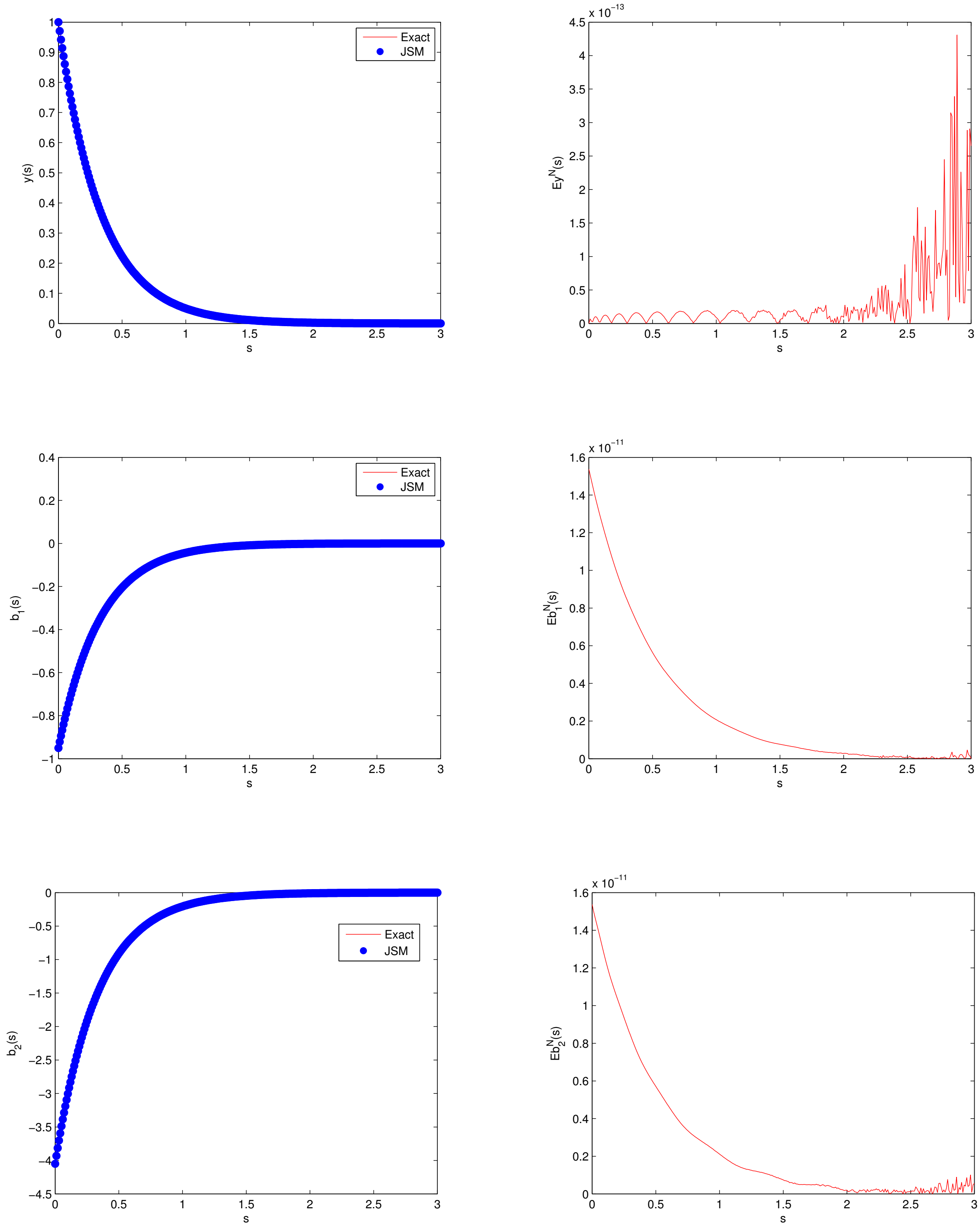

FIGURE 4.3 – The graphs of the numerical and the exact solutions with absolute errors for $k = 0, \ell = \frac{1}{2}$ and $N = 20$ for Example 4.2.

N	J_{1JSM}	J_{2JSM}	$\lvert\bar{J}_1 - J_{1JSM}\rvert$	$\lvert\bar{J}_2 - J_{2JSM}\rvert$
10	0.31403763402282735242	3.41361478020832377785	5.572e-007	2.452e-006
15	0.31403819124332684495	3.41361232795167413888	4.333e-011	5.167e-011
20	0.31403819124071578177	3.41361232796298408847	4.072e-011	6.298e-011

TABLE 4.11 – A comparison of the optimal cost functionals J_1 and J_2 for Example 4.2 using JSM with the exact solutions for $k = 0, \ell = \frac{3}{2}$.

N	J_{1JSM}	J_{2JSM}	$\lvert\bar{J}_1 - J_{1JSM}\rvert$	$\lvert\bar{J}_2 - J_{2JSM}\rvert$
10	0.31403763402282735242	3.41361478020832377785	5.572e-007	2.452e-006
15	0.31403819123820213086	3.413612327973873465	3.820e-011	7.387e-011
20	0.31403819123737936103	3.41361232797743685509	3.738e-011	7.744e-011

TABLE 4.12 – A comparison of the optimal cost functionals J_1 and J_2 for Example 4.2 using JSM with the exact solutions for $k = \ell = \frac{1}{2}$.

In addition, the CPM [43] and BTM [5] results are shown in Table 4.13 and Table 4.14, respectively.

N	J_{1CPM}	J_{2CPM}
10	0.3140689582	3.4134809955
15	0.3140381906	3.4136123306
20	0.3140381912	3.4136123279

TABLE 4.13 – The optimal cost functionals J_1 and J_2 for Example 4.2 by CPM.

N	J_{1BTM}	J_{2BTM}
10	0.31403763402282	3.41361478021289
15	0.31403819123820	3.41361232797387
20	0.31403819123819	3.41361232797391

TABLE 4.14 – The optimal cost functionals J_1 and J_2 for Example 4.2 by BTM.

The analysis show that the JSM offers highly accurate approximations across all cases studied for Example 4.2. There is good agreement between the approximate and exact solutions, especially as N increases, reducing errors, as shown in Figures 4.2 and 4.3. For instance, at $N = 15$ with different values of k and ℓ, the absolute errors were very small. Although slight differences in errors were observed with different values of k and ℓ, these differences remained within acceptable limits, indicating minimal impact on accuracy. Furthermore, the accuracy improves as N increases, with significantly smaller errors at $N = 20$, demonstrating the ability of the method to improve accuracy with more points.

Example 4.3. The differential game described below models the competition between two players striving to harvest a natural renewable resource.
Consider the dynamic state of this game defined as follows

$$\begin{cases} \dot{y}(s) = 0.1y(s) - 0.001y^2(s) - y(s)\,b_1(s) - y(s)\,b_2(s)\,, \ s \in [0,1] \\ y(0) = 1. \end{cases}$$

The payoff for each player over $[0, T]$ who want to maximize is as follows :

$$J_1(b_1(\cdot), b_2(\cdot)) = \int_0^1 \left(3y(s)\,b_1(s) - \frac{1}{2}b_1^2(s)\right) ds,$$

$$J_2(b_1(\cdot), b_2(\cdot)) = \int_0^1 \left(2y(s)\,b_2(s) - \frac{1}{2}b_2^2(s)\right) ds,$$

where the value $y(s) > 0$ is the resource level and the amounts $b_1(s) \geq 0$ and $b_2(s) \geq 0$ are the players' efforts for harvesting this resource, all at time s. In addition, $\frac{1}{2}b_1^2(s)$ and $\frac{1}{2}b_2^2(s)$ represent the costs of the harvest at each level of effort $b_1(s)$ and $b_2(s)$, respectively [17].
The Hamiltonian for two players are defined by

$$H_1(s, y, b_1, b_2, p_1) = p_1(s)\left(0.1y(s) - 0.001y^2(s) - y(s)b_1(s) - y(s)b_2(s)\right) + 3y(s)b_1(s) - \frac{1}{2}b_1^2(s),$$

$$H_2(s, y, b_1, b_2, p_2) = p_2(s)\left(0.1y(s) - 0.001y^2(s) - y(s)b_1(s) - y(s)b_2(s)\right) + 2y(s)b_2(s) - \frac{1}{2}b_2^2(s).$$

By maximizing $H_1(s, y, b_1, b_2, p_1)$ and $H_2(s, y, b_1, b_2, p_2)$ with respect to b_1 and b_2 we get the OLNE $\bar{b}_1$ and $\bar{b}_2$ for two players are given by,

$$\bar{b_1}(s) = 3\bar{y}(s) - \bar{p}_1(s)\bar{y}(s),$$

$$\bar{b_2}(s) = 2\bar{y}(s) - \bar{p}_2(s)\bar{y}(s).$$

Remark 4.2. (see, [51]). By the linearity of the dynamic state of this game with respect to the controls $b_i, i = 1, 2,$ and the concavity of performance $J_i, i = 1, 2,$ with respect to b_i, (since $\dfrac{\partial^2 J_i}{\partial b_i^2} = -1 < 0, i = 1, 2$), it leads to the open-loop strategy exists and is unique for this game concerning the Filippov-Cesari existence theorem [18].

Then, the system (4.6)-(4.7) of this game is obtained as

$$\begin{cases} \dot{\bar{y}} = 0.1\bar{y} - 5.001\bar{y}^2 + \bar{y}^2\bar{p}_1 + \bar{y}^2\bar{p}_2, \\ \dot{\bar{p}}_1 = -9\bar{y} - 0.1\bar{p}_1 + 8.002\bar{y}\bar{p}_1 - \bar{y}\bar{p}_1^2 - \bar{y}\bar{p}_1\bar{p}_2, \\ \dot{\bar{p}}_2 = -4\bar{y} - 0.1\bar{p}_2 + 7.002\bar{y}\bar{p}_2 - \bar{y}\bar{p}_2^2 - \bar{y}\bar{p}_1\bar{p}_2, \\ \bar{y}(0) = 1, \quad \bar{p}_1(1) = 0, \quad \bar{p}_2(1) = 0, \end{cases}$$

Table 4.15, Table 4.16, Table 4.17 and Table 4.18 presents the numerical results obtained by the JSM with various choices of k, ℓ and different values of N for Example 4.3. It is worth mentioning that since the exact solution to this differential game is not available, to verify the accuracy and the validity of the JSM for the differential game concerned, the residuals error is determined as follows :

$$\|\mathcal{R}\|^2 = \int_0^1 \left(Res_y^2(s) + Res_1^2(s) + Res_2^2(s)\right) ds,$$

where $Res_y, Res_i, i = 1, 2,$ are the residuals functions defined in (4.10).
Figure 4.4, presents the sum of squared residuals for the state of game and adjoint variables $\bar{p}_i(s), i = 1, 2$. Figure 4.5 displays the numerical solutions of $\bar{y}(s)$ and $\bar{b}_i(s), i = 1, 2$, while Figure 4.6 shows the approximate solutions for the adjoint variables $\bar{p}_i(s), i = 1, 2$ of both players. In general, these figures represent the approximate solutions for the various variables involved in the game with $k = 0, \ell = \frac{1}{2}$ and $N = 6$.

N	J_{1JSM}	J_{2JSM}	$\|\mathcal{R}\|^2$
3	0.94699053011063	0.45265191690172	1.70224879159856510e-002
4	0.94617331743847	0.45217675952914	2.29690776765495560e-003
5	0.94616311331662	0.45217517076851	2.9836687468064280e-004
6	0.94616143782921	0.45217455203462	3.74812018674081340e-005

TABLE 4.15 – Optimal payoff functionals J_1 and J_2 and residuals error using JSM for Example 4.3 for $k = -\frac{1}{2}, \ell = \frac{1}{2}$.

N	J_{1JSM}	J_{2JSM}	$\|\mathcal{R}\|^2$
3	0.94699053011063	0.45265191690172	1.70224879159855820e-002
4	0.94617331743847	0.45217675952914	2.29690776765471230e-003
5	0.94616311331662	0.45217517076851	2.9836687468064280e-004
6	0.94616143782921	0.45217455203462	3.74812018674076260e-005

TABLE 4.16 – Optimal payoff functionals J_1 and J_2 and residuals error using JSM for Example 4.3 for $k = 0, \ell = \frac{1}{2}$.

N	J_{1JSM}	J_{2JSM}	$\|\mathcal{R}\|^2$
3	0.94699053011063	0.45265191690172	1.70224879159856510e-002
4	0.94617331743847	0.45217675952914	2.29690776765471230e-003
5	0.94616311331662	0.45217517076851	2.9836687468064280e-004
6	0.94616143782921	0.45217455203462	3.74812018674267890e-005

TABLE 4.17 – Optimal payoff functionals J_1 and J_2 and residuals error using JSM for Example 4.3 for $k = -\frac{1}{2}, \ell = -\frac{1}{2}$.

N	J_{1JSM}	J_{2JSM}	$\|\mathcal{R}\|^2$
3	0.94699053011063	0.45265191690172	1.70224879159856370e-002
4	0.94617331743847	0.45217675952914	2.29690776765471230e-003
5	0.94616311331662	0.45217517076851	2.9836687468064280e-004
6	0.94616143782921	0.45217455203462	3.74812018674075650e-005

TABLE 4.18 – Optimal payoff functionals J_1 and J_2 and residuals error using JSM for Example 4.3 for $k = \ell = 0$.

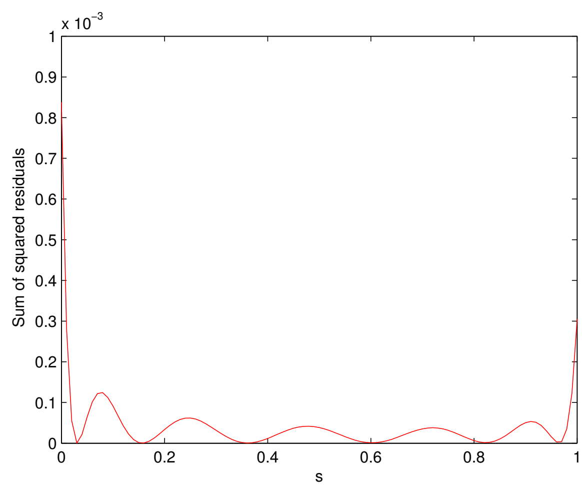

FIGURE 4.4 – The graphs of the sum of squared residuals for the state of game and adjoint variables with $k = 0, \ell = \frac{1}{2}$, for $N = 6$ for Example 4.3.

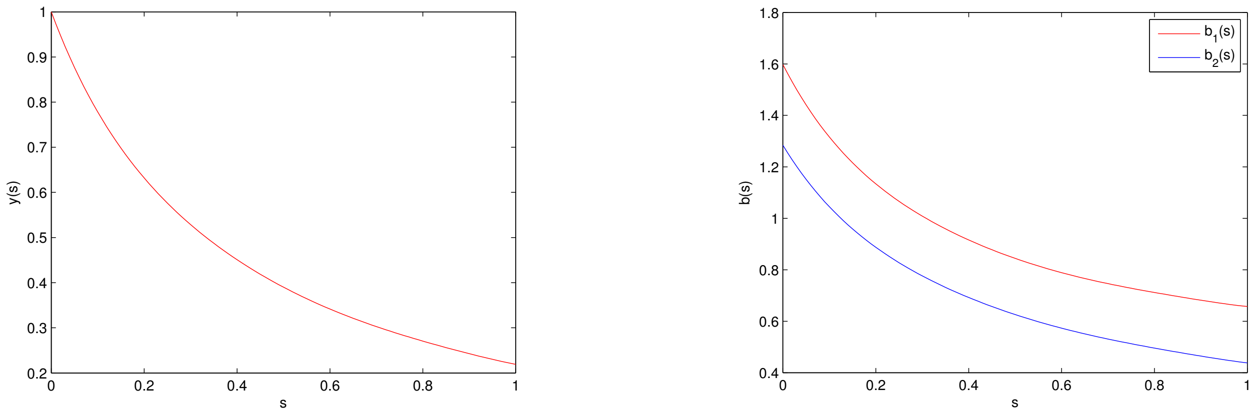

FIGURE 4.5 – The graphs of the numerical solutions for the state of game and the OLNE with $k = 0, \ell = \frac{1}{2}$, for $N = 6$ for Example 4.3.

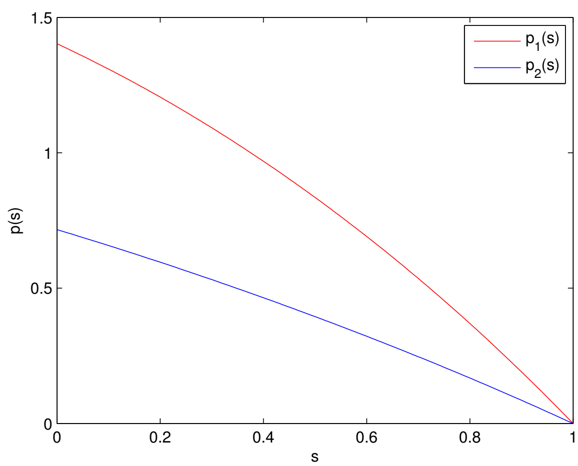

FIGURE 4.6 – The graphs of the numerical solutions for the adjoint variables with $k = 0, \ell = \frac{1}{2}$, for $N = 6$ for Example 4.3.

The results of Example 4.3 show that increasing N generally leads to a reduction in error, consistent with the theoretical properties of spectral methods. Higher N improves the accuracy of the numerical solution, leading to lower residual error. While the effect of k and ℓ on the error was less pronounced compared to N, variations in these parameters still contributed to a slight reduction in the error, highlighting their role in refining the solution.

Conclusion

The main contributions of the work presented in this thesis, focusing on the study of deterministic nonzero-sum differential games (NZSDGs) on a finite horizon, are divided into two parts.

In the first part, we established the connection between the adjoint variables in the maximum principle and the value function in the dynamic programming principle for two-player nonzero-sum differential games. We extended existing results to nonzero-sum differential games and addressed this relationship in both smooth and nonsmooth scenarios using viscosity solutions. This relationship is established in terms of derivatives, as well as sub- and super-differentials of the value function, with economic interpretations related to the adjoint variables.

In the second part, we presented a numerical method based on the Jacobi spectral method (JSM) to solve the nonlinear two-point boundary value problems (TPBVPs) derived from the maximum principle. These problems were converted into a system of algebraic equations to obtain the open-loop Nash equilibrium (OLNE) for nonzero-sum differential games. Some examples were provided to validate the accuracy and effectiveness of the proposed method.

For future research, this work can inspire further directions in the field, such as extending the results to stochastic differential games, in order to explore the connection between adjoint variables and the value function in both smooth and nonsmooth cases. Another research direction involves using the numerical method introduced to find feedback Nash equilibrium in both deterministic and stochastic differential games.

Bibliography

[1] R.J. Aumann, L.S. Shapley, *Values of non-atomic games*, Princeton University Press, Princeton, 1974.

[2] S.M. Aseev, V.M Veliov, *Another view of the maximum principle for infinite-horizon optimal control problems in economics*, Russian Mathematical Surveys, vol. 74, no. 6, pp. 963–1011, 2019.

[3] M.D. Banadaki, H. Navidi, *Numerical Solution of Open-Loop Nash Differential Games Based on the Legendre Tau Method*, Games, vol. 11, no. 3, 2020.

[4] M.D. Banadaki, H. Navidi, *A Bernoulli Tau method for numerical solution of feedback Nash differential games with an error estimation*, Computational Methods for Differential Equations, vol. 10, no. 4, pp. 849-904, 2022.

[5] M.D. Banadaki, H. Navidi, *A numerical treatment based on Bernoulli Tau method for computing the open-loop Nash equilibrium in nonlinear differential games*, Iranian Journal of Numerical Analysis and Optimization, vol. 12, no. 2, pp. 467-482, 2022.

[6] M. Bardi and I. Capuzzo-Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Birkhauser, Boston, 1997.

[7] E. N. Barron and R. Jensen, *The Pontryagin maximum principle from dynamic programming and viscosity solutions to first-order partial differential equations*, Transactions of the American Mathematical Society, vol. 298, no. 2, pp. 635–641, 1986.

[8] T. Başar, *Lecture notes on non-cooperative game theory*, Hamilton Institute, Dublin, Ireland, 2010.

[9] T. Başar and G. J. Olsder, *Dynamic Non-cooperative Game Theory*, second ed, SIAM, Philadelphia, 1999.

[10] T. Başar, A. Haurie, and G. Zaccour, *Nonzero-Sum Differential Garmes*, editors, Handbook of Dynamic Game Theory, pp. 61-110, Springer International Publishing, 2018.

[11] R. Bellman, *Dynamic programming*, Princeton University Press, Princeton, 1957.

[12] R. Benmenni and N. Daili, *A Connection between the Adjoint Variables and Value Function for Differential Games*, Statistics, Optimization & Information Computing, Vol. 13, no. 1, pp. 173-188, 2025, DOI : 10.19139/soic-2310-5070-2115.

[13] J.P. Boyd, *Chebyshev and Fourier Spectral Methods*, Dover, 2001.

[14] A. Bressan, *Noncooperative differential games*, Milan Journal of Mathematics, vol. 79, no. 2, pp. 357–427, 2011.

[15] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral Methods in Fluid Dynamics*, Springer Verlag, 1987.

[16] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A Zang, *Spectral Methods : Fundamentals in Single Domains*, Springer : New York, NY, USA, 2006.

[17] D. A. Carlson and G. Leitmann, *An extension of the coordinate transformation method for open-loop Nash equilibria*, Journal of optimization theory and applications, vol. 123, no. 1, pp. 27-47, 2004.

[18] L. Cesari, *Optimization-theory and applications : problems with ordinary differential equations*, Springer Science and Business Media, Berlin Heidelberg, Germany, 2012.

[19] L.Y. Chen, Q. Lu, *Relationships between the maximum principle and dynamic programming for infinite dimensional stochastic control systems*, Journal of Differential Equations, vol. 358, pp. 103–146, 2023.

[20] F. H. Clarke, R. B. Vinter, *The relationship between the maximum prinple and dynamic programming*, SIAM Journal on Control and Optimization, vol. 25, no. 5, pp. 1291–1311, 1987.

[21] M. G. Crandall, P. L. Lions, *Viscosity solutions of Hamilton-Jacobi equations*, Transactions of the American mathematical society, vol. 277, no. 1, pp. 1–42, 1983.

[22] M. G. Crandall, L. C. Evans, P. L. Lions, *properties of viscosity solutions of Hamilton-Jacobi equations*, Transactions of the American mathematical society, vol. 282, no. 2, pp. 487–502, 1984.

[23] E. J. Dockner ,S. Jorgensen, N. V. Long and G. Sorger, *Differential Games in Economies and Management Science*, Cambridge University Press, 2000.

[24] E. H. Doha, A. H. Bhrawy, and R. M. Hafez, *On shifted Jacobi spectral method for high-order multi-point boundary value problems*, Communications in Nonlinear Science and Numerical Simulation, vol. 17, pp. 38023810, 2012.

[25] R. Dorfman, *An economic interpretation of optimal control theory*, The American Economic Review, vol. 59, no. 5, pp. 817–831, 1969.

[26] O. Edhan, *Values of nondifferentiable vector measure games*, International Journal of Game Theory, vol. 42, no. 4, pp. 947-972, 2013.

[27] J.C. Engwerda, *LQ Dynamic Optimization and Differential Games*, John Wiley and Sons, 2005.

[28] M. Falcone, *Numerical methods for differential games based on partial differential equations*, International Game Theory Review, vol. 8, no. 2, pp. 231-272, 2006.

[29] W. H. Fleming and R. W. Rishel, *Deterministic and stochastic optimal control*, in Applications of Mathematics, Springer-Verlag, New York, 1975.

[30] W.H. Fleming et H.M. Soner, *Controlled Markov Processes and Viscosity Solutions*,Springer-Verlag, 1993.

[31] D. Fudenberg and J. Tirole, *Game Theory*, MIT Press, 1991.

[32] D. Gottlieb and S. A. Orszag, *Numerical Analysis of Spectral Methods : Theory and Applications*, SIAM CBMS, Philadelphia, 1977.

[33] B.Y, Guo, J. Shen and L.L. Wang, *Generalized Jacobi polynomials/functions and their applications*, Applied Numerical Mathematics, vol. 59, no. 5, pp. 1011-1028, 2009.

[34] M.S. Hu, S.L. Ji, and X.L. Xue, *Stochastic maximum principle, dynamic programming principle, and their relationship for fully coupled forward-backward stochastic controlled systems*, ESAIM Control, Optimisation and Calculus of Variations, Vol. 26, no. 81, 2020.

[35] R. Isaacs, *Differential games*, Wiley, New York, 1965.

[36] P.A. Johnson, *Numerical solution methods for differential game problems (MS thesis)*, Massachusetts Institute of Technology, 2009.

[37] X.J. Li, *Relationship between maximum principle and dynamic programming principle for stochastic recursive optimal control problem under volatility uncertainty*, Optimal Control Applications and Methods, vol. 44, no. 5, pp. 2457–2475, 2023.

[38] J.F Mertens, *The Shapley value in the non-differentiable case*. International Journal of Game Theory, vol. 17, no. 1, pp. 1-65, 1988.

[39] J.F. Nash, *Non-cooperative games*, Annals of Mathematics, vol. 54, pp. 286–295, 1951.

[40] A. Neyman, *Values of non-atomic vector measure games*, Israel Journal Of Mathematics, vol. 124, pp. 1-27, 2001.

[41] T. Nie, J. Shi and Z. Wu, *Connection between MP and DPP for stochastic recursive optimal control problems : Viscosity solution framework in local case*, in Proceedings of the 2016 American Control Conference, Boston, pp. 7225–7230, 2016.

[42] T. Nie, J.T. Shi and Z. Wu, Connection between MP and DPP for stochastic recursive optimal control problems : viscosity solution framework in the general case, *SIAM Journal on Control and Optimization*, vol. 55, no. 5, pp. 3258–3294, 2017.

[43] Z. Nikooeinejad, A. Delavarkhalafi, and M. Heydari, *A numerical solution of open-loop Nash equilibrium in nonlinear differential games based on Chebyshev pseudospectral method*, Journal of Computational and Applied Mathematics, vol. 300, pp. 369-384, 2016.

[44] Z. Nikooeinejad, A. Delavarkhalafi, and M. Heydari, *Application of shifted Jacobi pseudospectral method for solving (in) finite-horizon min-max optimal control problems with uncertainty*, International Journal of Control, vol. 91, no. 3, pp. 725-739, 2018.

[45] A. Pakniyat, P.E. Caines, *On the Relation between the Minimum Principle and Dynamic Programming for Classical and Hybrid Control Systems*, IEEE Transactions on Automatic Control, vol. 62, no. 9, pp. 4347-4362, 2017.

[46] L. Pontryagin, V. Boltyanskii, R. Gamkrelidze, and E.Mishchenko, *Mathematical theory of optimal processes*, Wiley, New York, 1962.

[47] L.S. Shapley, *A value for n-person games*, In : Contributions to the theory of Games, Princeton, Princeton University Press, vol. II, pp. 307-317, 1953.

[48] J. Shen, T. Tang, and L. L. Wang, *Spectral Methods :(Algorithms, Analysis and Applications)*, Springer, 2011.

[49] J.T. Shi, *Relationship between maximum principle and dynamic programming for stochastic differential games of jump diffusions*, International Journal of Control. vol. 87, no. 4, pp. 693–703, 2014.

[50] A. W. Starr and Y. C. Ho, *Nonzero-sum differential games*, Journal of Optimization Theory and Applications, vol. 3, no. 3, pp. 184-206, 1969.

[51] G. Sorger, *Competitive dynamic advertising : A modification of the case game*, Journal of Economic Dynamics and Control, vol. 13, no. 1 , pp. 55-80, 1989.

[52] G. Szegö, *Orthogonal Polynomials*, (fourth edition), vol. 23, American Mathematical Society Colloquium Publications, 1975.

[53] B. Wang, J.T. Shi, *Relationship between general MP and DPP for the stochastic recursive optimal control problem with jumps : Viscosity solution framework*, Preprint, arXiv :2403.09044v2 [math.OC] 2024.

[54] Z. Yazdaniyan, M. Shamsi, Z. Foroozandeh and MR. de Pinho, *A numerical method based on the complementarity and optimal control formulations for solving a family of zero-sum pursuit-evasion differential games*, Journal of Computational and Applied Mathematics, vol. 368, 2020.

[55] D.W.K. Yeung, L.A. Petrosyan, *Cooperative Stochastic Differential Games*, Springer Science & Business Media, 2006.

[56] J. Yong, *Differential games : a concise introduction*, World Scientific Publisher, Singapore, 2015.

[57] J. Yong and X.Y. Zhou, *Stochastic Controls : Hamiltonian Systems and HJB Equations*, Springer-Verlag, New York, 1999.

[58] H. Zhang, Q. Wei, D. Liu, *An iterative dynamic programming method for solving a class of nonlinear zero-sum differential games*, Automatica, Vol. 47, pp. 207-214, 2011.

[59] X.Y. Zhou, *Maximum principle, dynamic programming, and their connection in deterministic control*, Journal of Optimization Theory and Applications, vol. 65, no. 2, pp. 363–373, 1990.

ملخّص: الهدف الرئيسي من هذه الأطروحة هو عرض العلاقة بين المتغيرات المرافقة في مبدأ الحد الأقصى ودالـــــــة القيمة في مبدأ البرمجة الديناميكية للألعاب التفاضلية غير الصفرية للاعبين، وذلك في كل من الحالات الملساء وغير الملساء لدالة القيمة. يتم إثبات هذه العلاقة من حيث المشتقات في الحالة الملساء ومن خلال حلول اللزوجة عندما لا تكون دالة القيمة ملساء، مع تفسيرات اقتصادية تتعلق بالمتغيرات المرافقة.

في الجزء الثاني قمنا بتطبيق طريقة عددية تعتمد على طريقة جاكوبي الطيفية لحل مشاكل القيمة الحدية في نقطتين غير الخطية المشتقة من مبدأ الحد الأقصى. ثم يتم تحويل هده المشاكل إلى نظام من المعادلات الجبرية وذلك من أجل تحديد توازن ناش في الحلقة المفتوحة للألعاب التفاضلية غير الصفرية. يتم تقديم أمثلة توضيحية لإظهار فعالية وصحة الطريقة المقترحة.

الكلمات المفتاحية: الألعاب التفاضلية غير الصفرية. الحد الأقصى. البرمجة الديناميكية. حلول اللزوجة.

طريقة جاكوبي الطيفية. توازن ناش في الحلقة المفتوحة.

Abstract: The main objective of this thesis is to present the connection between the adjoint variables in the maximum principle (MP) and the value function in the dynamic programming principle (DPP) for two-player nonzero-sum differential games, both in the smooth and nonsmooth cases. This relationship is established in terms of derivatives in the smooth case and through viscosity solutions when the value function is not smooth, with economic interpretations related to the adjoint variables.

In the second part, we apply a numerical method based on the Jacobi spectral method (JSM) to solve the nonlinear two-point boundary value problems (TPBVPs) derived from the maximum principle. These problems are then transferred into a system of algebraic equations to determine the open-loop Nash equilibrium (OLNE) for nonzero-sum differential games. Illustrative examples are presented to demonstrate the effectiveness and validity of the proposed method.

Keywords: Nonzero-sum differential games, maximum principle, dynamic programming principle, viscosity solutions, Jacobi spectral method, open-loop Nash equilibrium.

Résumé : L'objectif principal de cette thèse est de présenter la relation entre les variables adjointes dans le principe du maximum et la fonction valeur dans le principe de la programmation dynamique pour les jeux différentiels à deux joueurs à somme non nulle, dans les cas lisses et non lisses. Cette relation est établie en termes de dérivées dans le cas lisse et par des solutions de viscosité lorsque la fonction valeur n'est pas lisse, avec des interprétations économiques liées aux variables adjointes.

Dans la deuxième partie, nous appliquons une méthode numérique basée sur la méthode spectrale de Jacobi pour résoudre les problèmes aux limites non linéaires à deux points dérivés par le principe du maximum. Ces problèmes sont ensuite convertis en un système d'équations algébriques pour déterminer l'équilibre de Nash à boucle ouverte pour les jeux différentiels à somme non nulle. Des exemples illustratifs sont présentés pour démontrer l'efficacité et la validité de la méthode proposée.

Mots-clés : Jeux différentiels à somme non nulle, principe du maximum, programmation dynamique, solution de viscosité, méthode spectrale de Jacobi, l'équilibre de Nash à boucle ouverte.