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MASTER THESIS

Field: Mathematics and Computer Science

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Speciality: Partial differential equation and nonlinear analysis

Theme

Studies on sequences and series

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Dedication

First and foremost, I would like to express my deepest gratitude to my advisor, **Pr. Abdelouahab Kadem**, whose expertise, understanding, and patience added considerably to my graduate experience. I appreciate his vast knowledge and skill in many areas, and his assistance in writing this dissertation. His invaluable guidance and persistent help are what made this work possible.

I also wish to extend my heartfelt thanks to the members of my thesis committee, **Pr. Salim Mesbahi and Dr. Salah Boutiah**, for their time, encouragement, insightful comments, and constructive feedback.

A special thanks to my fellow graduate students and colleagues in the Department Mathematics at Sétif **1** University, whose camaraderie and support have made my time here memorable. In particular.

I am deeply thankful to my family for their unwavering support and encouragement throughout my academic journey. To my parents, **Fatima , Rabeh**, for their constant love and encouragement; to my siblings, **Achouak , Abir and him husband (Sahraoui Khlifa)** for their support and understanding; and to my husband **Fahd Benkara** for his patience and unwavering belief in me.

Thank you all.

Amina Kaabeche
Sétif, 22-06-2024

Contents

Table of figures

Introduction

Sequences, defined as ordered collections of elements, are used to model sequential phenomena and discrete processes. Series, which are infinite sums of elements in a sequence, are used to explore concepts such as convergence, divergence and summation. These notions are particularly relevant to the study of functions, where Taylor and Fourier series are used to represent and analyze complex functions.

They are fundamental concepts in mathematics, playing a crucial role in various branches such as analysis, number theory and differential equations. Sequences and series play a crucial role in many scientific and technical disciplines. Here are a few notable applications in various fields

Their study goes back centuries, with significant contributions from famous mathematicians such as Isaac Newton, Gottfried Wilhelm Leibniz and Augustin-Louis Cauchy. Not only are sequences and series essential theoretical tools, they also have practical applications in fields such as physics Quantum mechanics: Fourier series are used to solve the Schrödinger equation in quantum systems. Eigen function series help to represent quantum states of particles in potentials. Electromagnetism: Laurent and Taylor series are used to solve Maxwell's equations in complex media. Fourier series are also used in the analysis of electrical circuits and signals. Engineering: Signal Processing: Fourier series and wavelet transforms are essential for signal processing, particularly in data compression, noise reduction and frequency analysis. Structural Analysis: In civil and mechanical engineering, series are used to analyze structural vibrations and model deformations under various loads. Economics and Finance: Time Series Models: Sequences and series are fundamental in time series models for the analysis of financial data. For example ARIMA (Auto Regressive Integrated Moving Average) models use sequence concepts to predict market movements. Compound interest calculation: Geometric series are used to calculate compound interest and investment returns over extended periods. Computer science: Algorithms : Recurrent sequences, such as Fibonacci sequences, are used in the design of efficient, optimized algorithms for a variety of problems, including sorting and searching. Data Compression: Fourier series and wavelets play a key role in image and video compression techniques, such as JPEG and MPEG.

Number Theory: Zeta functions: Infinite series are essential for the study of Riemann zeta functions, which have profound implications in number theory, particularly in relation to the distribution of prime numbers. Dirichlet series: used in the proofs of many number theorems, these series help to analyze the properties of arithmetic functions.

Numerical Analysis: Approximation Methods: Taylor polynomials and Fourier series are used to approximate complex functions, facilitating the numerical calculation of solutions for differential and integral equations. Solving Differential Equations: Series methods are commonly used to find analytical and numerical solutions to ordinary differential equations and partial differential equations.

Many authors can be found in the literature among others Bartle gives a classic text covering the fundamental concepts of real analysis, including sequences and series, Knopp has given a thorough reference on infinite series, exploring both theory and practical applications. Apostol in his book, covers many aspects of analysis, including Fourier series and Taylor series, In his reference in mathematical analysis, Rudin, offers a solid grounding in the theory of sequences and series. Zygmund, has studied trigonometric series in depth, with applications to Fourier analysis, and as research articles in his work Hardy has given an important background dealing with the convergence of multiple series, Dym in his article explores the properties of Fourier series and their applications. A reference on function theory, including discussions of Laurent and Taylor series is due to Titchmarsh. We can add to this bibliography, online resources Khan Academy - Sequences and Series. A series of educational videos covering the basic concepts of sequences and series, available on Khan Academy and MIT Open Course Ware - Single Variable Calculus, a free online course including material on Taylor and Fourier series, available on MIT Open Course Ware. In his thesis, Smith explored the applications of series in various engineering problems. A technical report discussing the convergence properties of special series used in mathematical physics was prepared by Brown. Finally, we end with review articles by Stein, featuring interesting book and related articles on complex analysis, including in-depth discussions of Laurent series. Gasquet, provides a comprehensive text combining Fourier analysis with practical applications, including the use of series for filtering and numerical computation.

This work, entitled "Studies in Sequences and Series", aims to offer an in-depth exploration of these topics. We'll start with a review of basic concepts and fundamental theorems, before delving into more advanced developments and their modern applications.

The aim is to provide readers not only with a solid understanding of the underlying principles, but also to familiarize them with the latest advances in this dynamic field. Whether you're a student, a researcher, or a practitioner, this document will serve as a comprehensive guide to navigating the rich landscape of sequences and series, and to applying these concepts to real and theoretical problems.

At the end we give a bibliography which offers a diverse set of resources for deepening understanding of sequences and series, covering both theoretical and practical aspects. The books, articles and online resources listed here provide a solid foundation for any advanced study of the subject.

Chapter 01

Sequences and Series

This chapter introduces **sequences** and **series,** important mathematical constructions that are useful when solving a large variety of mathematical problems, the content of this chapter is considerably different from the content of the chapters before it, while the material we learn here definitely, falls under the scope of "**calculus**", we will make very little use of derivatives or integrals. limits are extremely important , though , especially limits that involve infinity .

1-Sequences

We commonly refer to a set of events that occur one after the other as a sequence of events. In mathematics, we use word sequence to refer to an ordered set of numbers, **i.e., a** set of numbers that "**occur one after the other.**"

For instance, the numbers **2, 4, 6, 8, ... ,** from a sequence . The order is important; the first number is **2**, the second is **4**, etc . It seems natural to seek a formula that describes a given sequence, and often this can be done. for instance , the sequence above could be described by the function $a(n) = 2n$, for the values of $n = 1, 2, ...$ To find the 10^{th} term in the sequence we would compute $a(10)$. This leads us to the following formal definition of a sequence

Definition 01: Sequences , range and terms

* A sequence is a function $a(n)$ whose domain is $\mathbb{N}$
* The **range** of a sequence is the set of all distinct values of $a(n)$
* The **terms** of a sequence are the values $a(1)$, $a(2)$, **...** , which are usually denoted with subscripts as a_1, a_2 , ...

A sequence $a(n)$ is often denoted as $\{a_n\}$

Notation : We use $\mathbb{N}$ to describe the set of natural numbers, that is, the integers **1 , 2 , 3 , ...**

Definition 02: Factorial

the expression **3** ! refers to the number **3. 2. 1** = **6** In general,

$$n! = n.(n-1).(n-2)\dots 2.1$$

where n is a natural number. We define $0! = 1$. while this does not immediately make sense; it makes many mathematical formulas work properly.

Example 01: Listing terms of a sequence

List the first four terms of the following sequences.

1. $\{a_n\} = \left\{\frac{3^n}{n!}\right\}$

2. $\{a_n\} = \{4 + (-1)^n\}$

3. $\{a_n\} = \left\{\frac{(-1)^{\frac{n(n+1)}{2}}}{n^2}\right\}$

Solution

1. $\{a_n\} = \left\{\frac{3^n}{n!}\right\}$

n = 1 $\Longrightarrow$ $a_1 = \frac{3^1}{1!} = 3$

n = 2 ⟹ $a_2 = \frac{3^2}{2\,!} = \frac{9}{2\times1} = \frac{9}{2} = 4{,}5$

n = 3 ⟹ $a_3 = \frac{3^3}{3\,!} = \frac{27}{3\times2\times1} = \frac{9}{2} = 4{,}5$

n = 4 ⟹ $a_4 = \frac{3^4}{4\,!} = \frac{3^4}{4\times3\times2\times1} = \frac{3^3\times\cancel{3}}{4\times\cancel{3}\times2\times1} = \frac{3^3}{8} = \frac{27}{8} = 3,\ 38$

n = 5 ⟹ $a_5 = \frac{3^5}{5\,!} = \frac{243}{5\times4\times3\times2\times1} = \frac{81}{40} = 2,025$

n = 6 ⟹ $a_6 = \frac{3^6}{6\,!} = \frac{729}{6\times5\times4\times3\times2\times1} = \frac{81}{80} = 1,012$

n = 7 ⟹ $a_7 = \frac{3^7}{7\,!} = \frac{2187}{7\times6\times5\times4\times3\times2\times1} = \frac{243}{560} = 0,43$

Analytical

Using d'Alempart's rule: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$

1. $\{a_n\} = \left\{\frac{3^n}{n!}\right\}$

- $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}}$

$$= \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)!} \times \frac{n!}{3^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{3^n}\cdot 3^1}{(n+1)\cdot\cancel{n!}} \times \frac{\cancel{n!}}{\cancel{3^n}}$$

$$= \lim_{n\to\infty} \frac{3^1}{(n+1)}$$

$$= \lim_{n\to\infty} \frac{3}{(n+1)}$$

$$= \frac{3}{\infty}$$

$$= 0 < 1$$

$\{a_n\} = \left\{\frac{3^n}{n!}\right\}$ **(so sequence is convergent)**

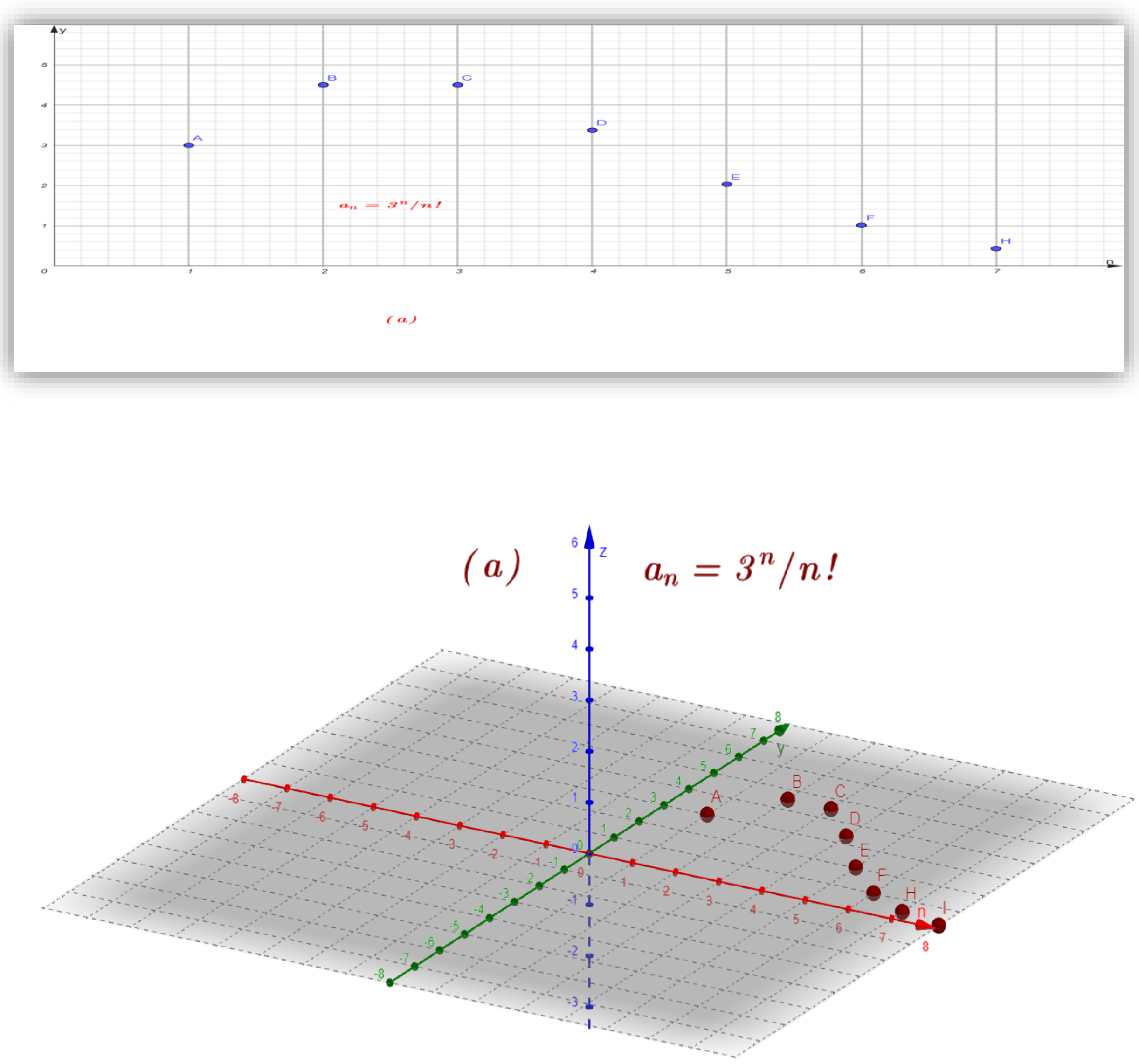


Figure A

We can plot the terms of a sequence with a scatter plot. The " x "-***axis*** is used for the values of, and the values of the terms are plotted on the " **y** "-**axis**. To visualize this sequence, see **Figure A**

2. $\{a_n\} = \{4 + (-1)^n\}$

n = 1 ⟹ $a_1 = 4 + (-1)^1 = 3$
n = 2 ⟹ $a_2 = 4 + (-1)^2 = 5$
n = 3 ⟹ $a_3 = 4 + (-1)^3 = 3$

n = 4 ⟹ $a_4 = 4 + (-1)^4 = 5$
n = 5 ⟹ $a_5 = 4 + (-1)^5 = 3$
n = 6 ⟹ $a_6 = 4 + (-1)^6 = 5$
n = 7 ⟹ $a_7 = 4 + (-1)^7 = 3$

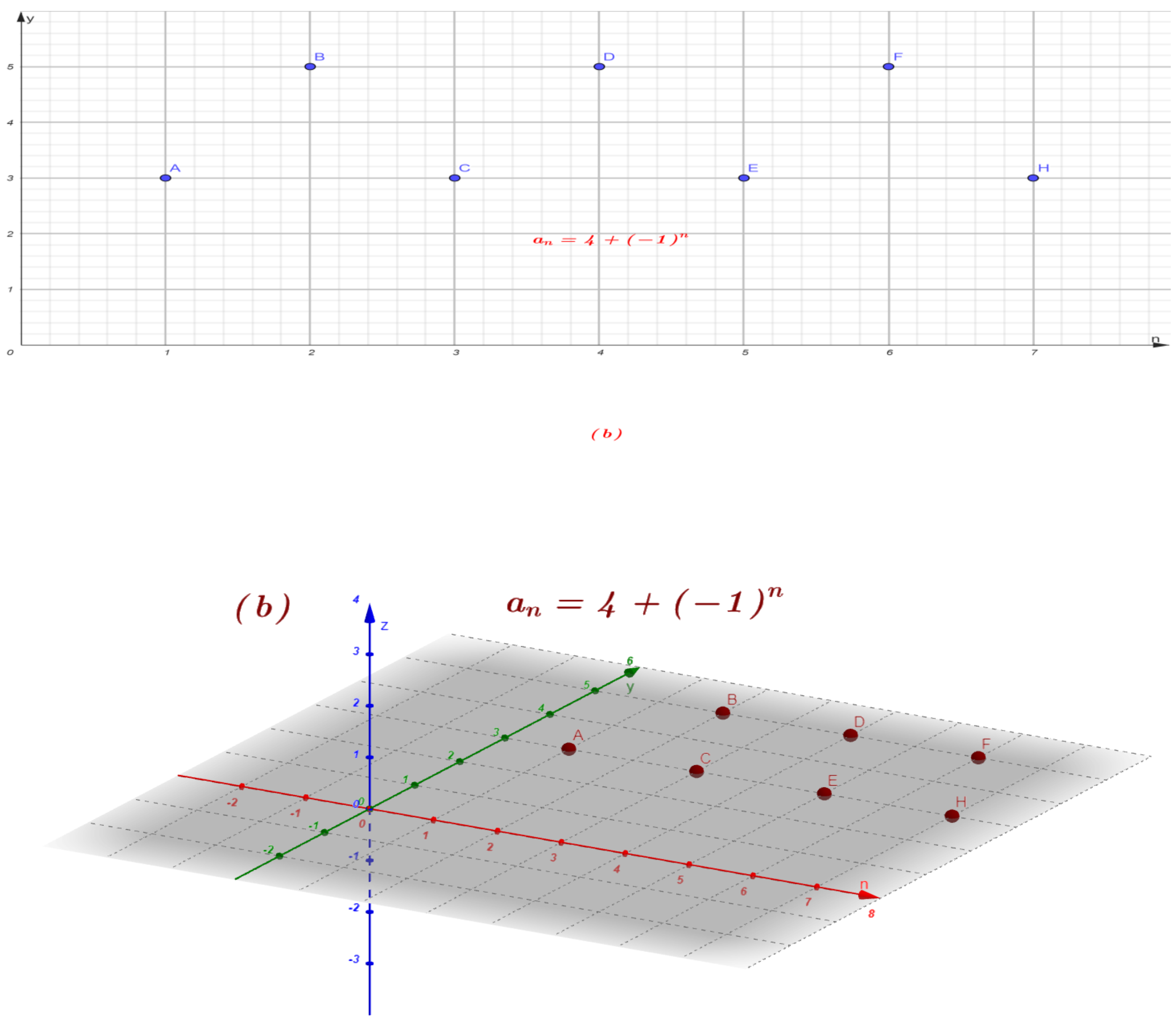


Figure B

Note that the range of this sequence is finite, consisting of only the values **3** ans **5**. This sequence is plotted in **figure B**

3. $\{a_n\} = \left\{\frac{(-1)^{\frac{n(n+1)}{2}}}{n^2}\right\}$

n = 1 $\Longrightarrow$ $a_1 = \frac{(-1)^{\frac{1(2)}{2}}}{1^2} = \frac{(-1)^1}{1} = -1$

n = 2 $\Longrightarrow$ $a_2 = \frac{(-1)^{\frac{2(3)}{2}}}{2^2} = \frac{(-1)^3}{4} = \frac{-1}{4}$

n = 3 $\Longrightarrow$ $a_3 = \frac{(-1)^{\frac{3(4)}{2}}}{3^2} = \frac{(-1)^6}{9} = \frac{1}{9} = 0,11$

n = 4 $\Longrightarrow$ $a_4 = \frac{(-1)^{\frac{4(5)}{2}}}{4^2} = \frac{(-1)^{10}}{16} = \frac{1}{16} = 0,06$

n = 5 $\Longrightarrow$ $a_5 = \frac{(-1)^{\frac{5(6)}{2}}}{5^2} = \frac{(-1)^{15}}{25} = -\frac{1}{16} = -0,06$

n = 6 $\Longrightarrow$ $a_6 = \frac{(-1)^{\frac{6(7)}{2}}}{6^2} = \frac{(-1)^{21}}{36} = -\frac{1}{36} = -0,03$

n = 7 $\Longrightarrow$ $a_7 = \frac{(-1)^{\frac{7(8)}{2}}}{7^2} = \frac{(-1)^{28}}{49} = \frac{1}{49} = 0,02$

Analytical

Using d'Alempart's rule: $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = L$

3. $\{a_n\} = \left\{\frac{(-1)^{\frac{n(n+1)}{2}}}{n^2}\right\}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(-1)^{(n+1)\cdot((n+1)+1)/2}}{(n+1)^2}}{\frac{(-1)^{n\cdot(n+1)/2}}{n^2}}$$

$$= \lim_{n\to\infty} \frac{\frac{(-1)^{(n+1)\cdot(n+2)/2}}{(n+1)^2}}{\frac{(-1)^{n\cdot(n+1)/2}}{n^2}}$$

$$= \lim_{n\to\infty} \frac{(-1)^{(n+1)\cdot\frac{n+2}{2}}}{(n+1)^2} \times \frac{n^2}{(-1)^{n\cdot\frac{n+1}{2}}}$$

$$= \lim_{n\to\infty} \frac{(-1)^{(n+2)/2}}{(n+1)^2} \times \frac{n^2}{(-1)^{n/2}}$$

$$= \lim_{n\to\infty} \frac{(-1)^{n/2}\cdot(-1)^1}{(n+1)^2} \times \frac{n^2}{(-1)^{n/2}}$$

$$= \lim_{n\to\infty} \frac{(-1)^1 \cdot n^2}{(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{-n^2}{n^2+2n+1}$$

$$= \lim_{n\to\infty} \frac{-\cancel{n^2}}{\cancel{n^2}}$$

$= -1 < 1.$ *(so sequence is convergent)*

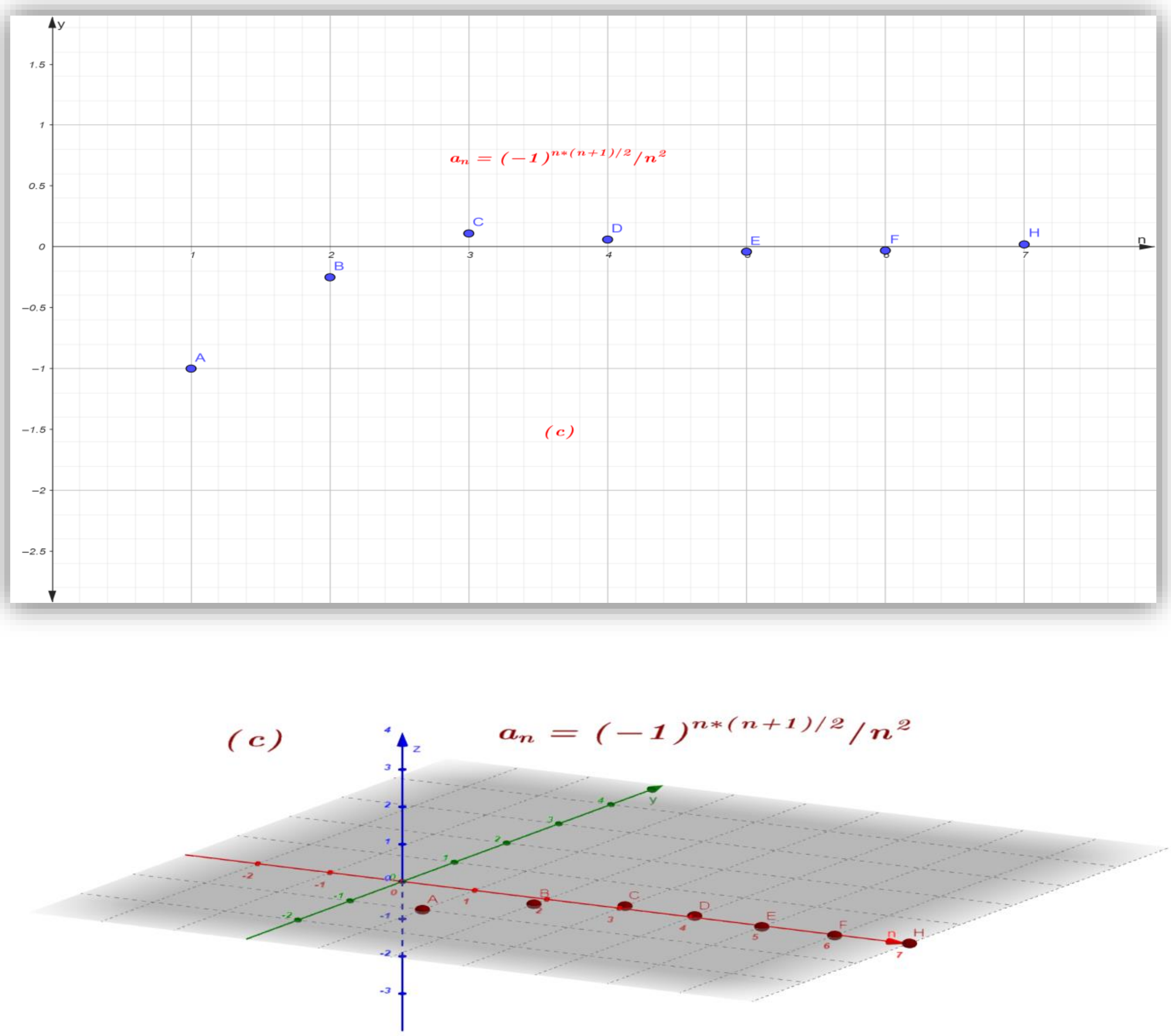


Figure C

We gave one extra term to begin to show the pattern of signs is $-,-,+,+,-,-,\ \ldots$, due to the fact that the exponent of $-$ **1** is a special quadratic. This sequence is plotted in **Figure C**

Example 02: Determining a formula for a sequence

Find the n^{th} term of the following sequences, i.e., find a function that describes each of the given sequences..

1. $2, 5, 8, 11, 14, \ldots$

2. $2, -5, 10, -17, 26, -37, \ldots$

3. $1, 1, 2, 6, 24, 120, 720, \ldots$

4. $\frac{5}{2}, \frac{5}{2}, \frac{15}{8}, \frac{5}{4}, \frac{25}{32}, \ldots$

Solution

we should first note that there is never exactly one function that describes a finite set of numbers as a sequence. There are many sequences that start with **2**, then **5**, as our first example does. We are looking for a simple formula that describes the terms given, knowing there is possibly more than one answer.

1. Note how each term is **3** more than the previous one. This implies a linear function would be appropriate: $a(n) = a_n = 3n + b$ for some appropriate value of **b**. As we want $a_1 = 2$ we set $b = -1$ thus $a_n = 3n - 1$, see Figure **a** (1)

Analytical

using d'Alempart's rule: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$

$$\{a_n\} = \{3n - 1\}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3(n+1) - 1}{3n - 1}$$

$$= \lim_{n\to\infty} \frac{3n+3-1}{3n-1}$$

$$= \lim_{n\to\infty} \frac{3n + 2}{3n - 1}$$

$$= \lim_{n\to\infty} \frac{\cancel{3n}}{\cancel{3n}}$$

$$= 1$$

The sequence is inconclusive

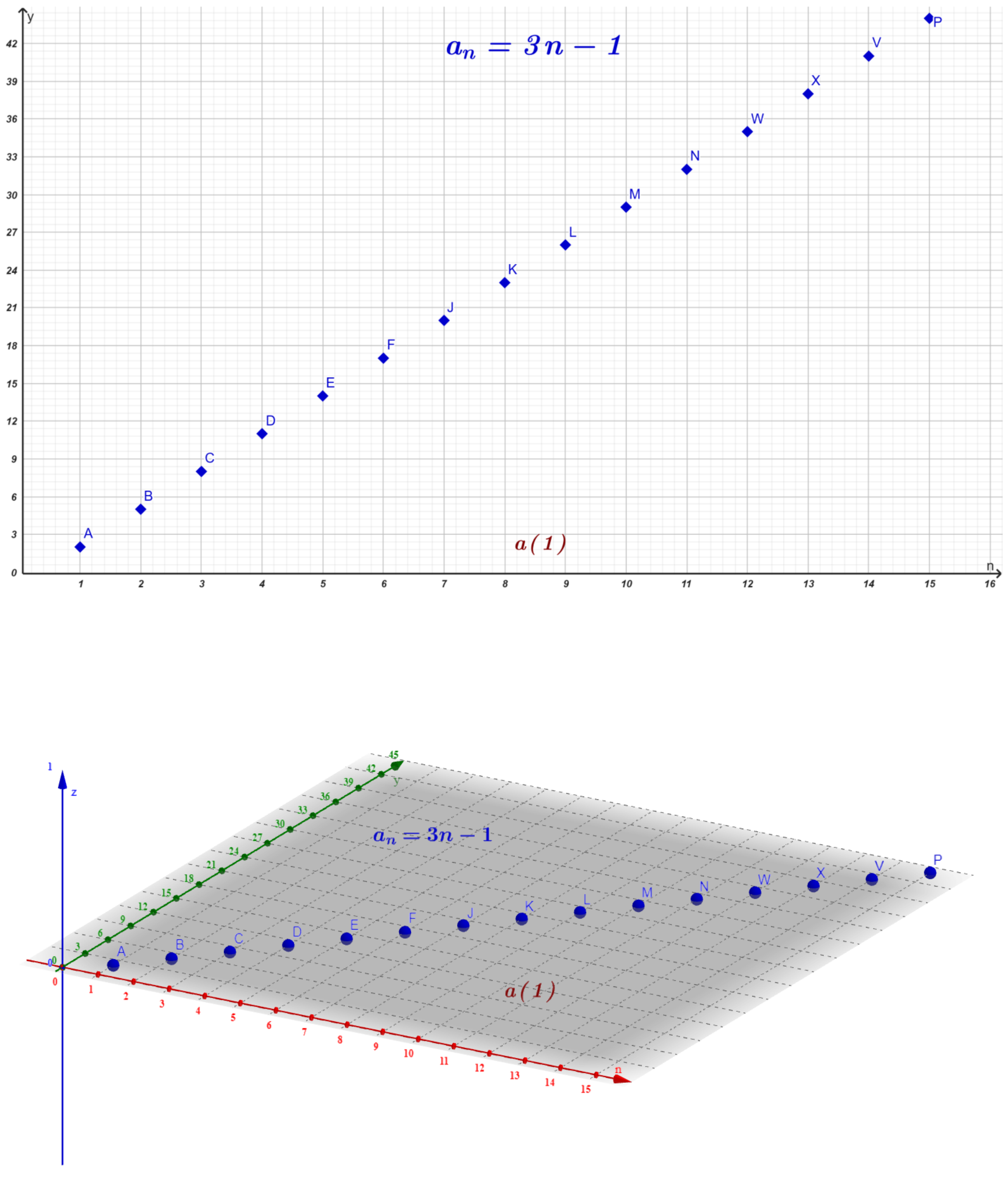


Figure a (1)

2. First notice how the sign changes from term to term. This is most commonly accomplished by multiplying the terms by either $(-\mathbf{1})^{n}$ or $(-\mathbf{1})^{n+1}$. Using $(-\mathbf{1})^{n}$ multiplies the odd terms by $(-\mathbf{1})$; using $(-\mathbf{1})^{n+1}$ multiplies the even terms by $(-\mathbf{1})$. As this sequence has negative even terms, we will multiply by $(-\mathbf{1})^{n+1}$.

After this, we might feel a bit stuck as to how to proceed. At this point, we are just looking for a pattern of some sort: what do the numbers $\mathbf{2}$, $\mathbf{5}$, $\mathbf{10}$, $\mathbf{17}$, etc, have in common? There are many correct answers, but the one that we'll use here is that each is one more than a perfect square. That is, $\mathbf{2} = \mathbf{1}^{1} + \mathbf{1}$, $\mathbf{5} = \mathbf{2}^{2} + \mathbf{1}$, $\mathbf{10} = \mathbf{3}^{2} + \mathbf{1}$, etc . Thus our formula is
$a_n = (-1)^{n+1} \cdot (n^2 + 1)$, see Figure **b (1).**

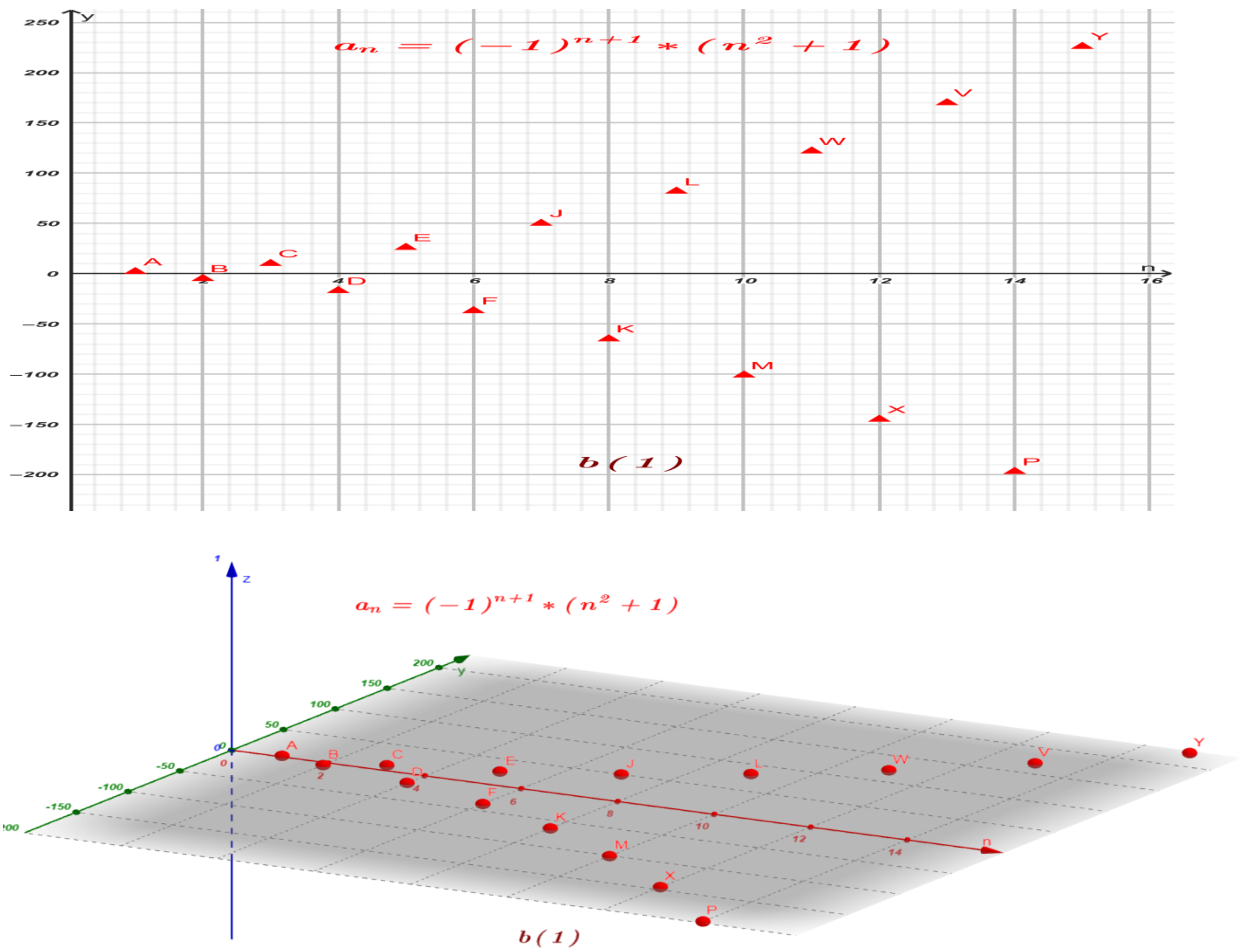


Figure b (1)

3. One who is familiar with the factorial function will readily recognize these numbers. They are $\mathbf{0}!$, $\mathbf{1}!$, $\mathbf{2}!$, $\mathbf{3}!$, **etc**. Since our sequences start with $\boldsymbol{n = 1}$, we cannot write $\boldsymbol{a_n = n!}$, for this misses the $\mathbf{0}!$ term. Instead, we shift by $\mathbf{1}$, and write $\boldsymbol{a_n = (n-1)!}$, *see* Figure **c** (1)

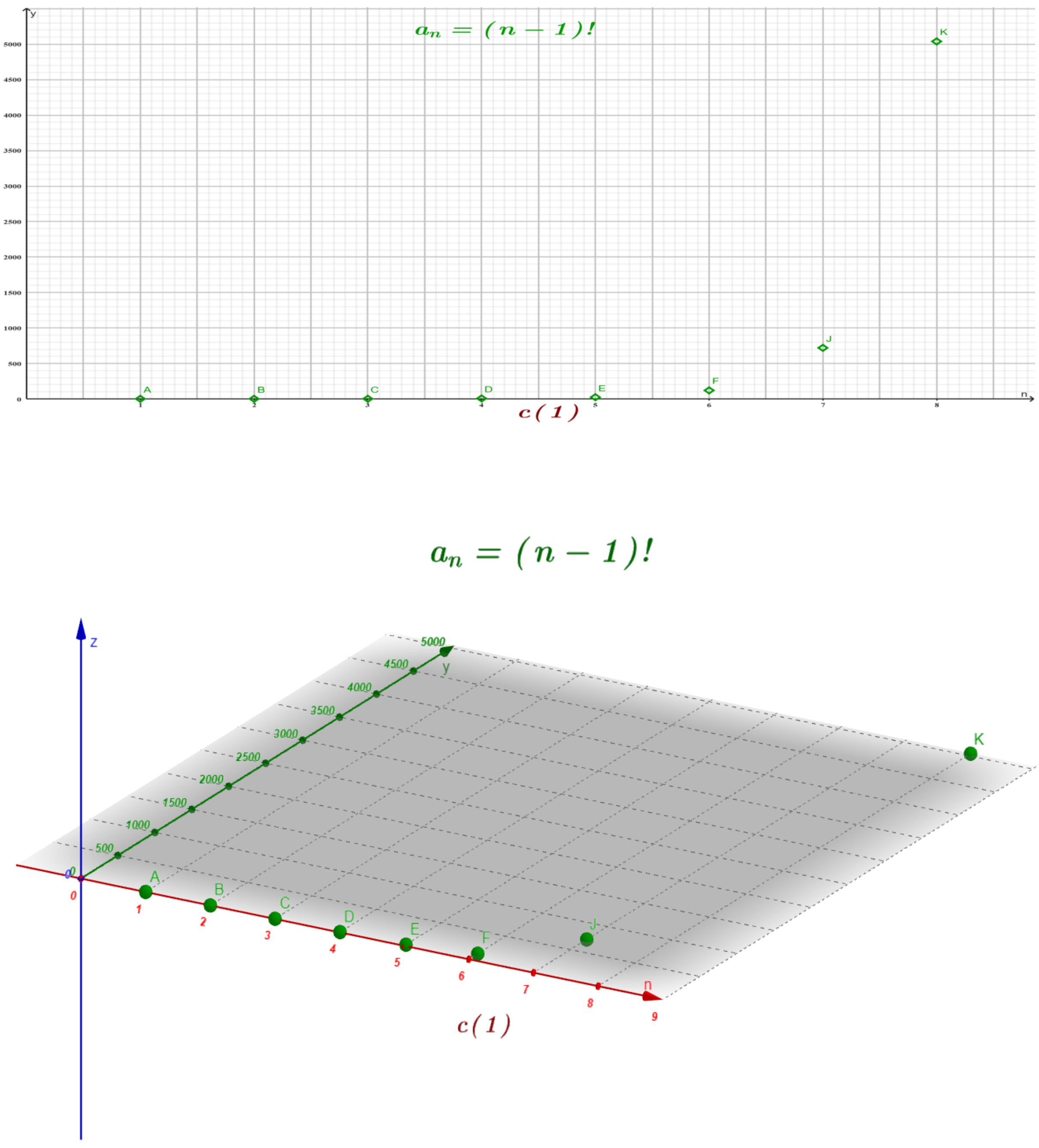


Figure c (1)

4. This one may appear difficult, especially as the first two terms are the same, but a little ``**sleuthing**" will help. Notice how the terms in the numerator are always multiples of **5** , and the terms in the denominator are always powers of **2** . Does something as simple as
$a_n = \frac{5n}{2^n}$ work?

When $n = 1$, we see that we indeed get $a_1 = \frac{5}{2}$ as desired. When $n = 2$ we get $a_2 = \frac{10}{4} = \frac{5}{2}$ Further checking shows that this formula indeed matches the other terms of the sequence, see
Figure d (1)

Analytical

using d'Alempart's rule: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$

- $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{5(n+1)}{2^{(n+1)}}}{\frac{5n}{2^n}}$

$$= \lim_{n\to\infty} \frac{5(n+1)}{2^{(n+1)}} \times \frac{2^n}{5n}$$

$$= \lim_{n\to\infty} \frac{5(n+1)}{2^n \cdot 2^1} \times \frac{2^n}{5n}$$

$$= \lim_{n\to\infty} \frac{5n+1}{2^1} \times \frac{1}{5n}$$

$$= \lim_{n\to\infty} \frac{5n+1}{2^1 \cdot 5n}$$

$$= \lim_{n\to\infty} \frac{5n+1}{10n}$$

$$= \lim_{n\to\infty} \frac{5n}{10n}$$

$$= \frac{1}{2} < 1 .$$

(So sequence is convergent)

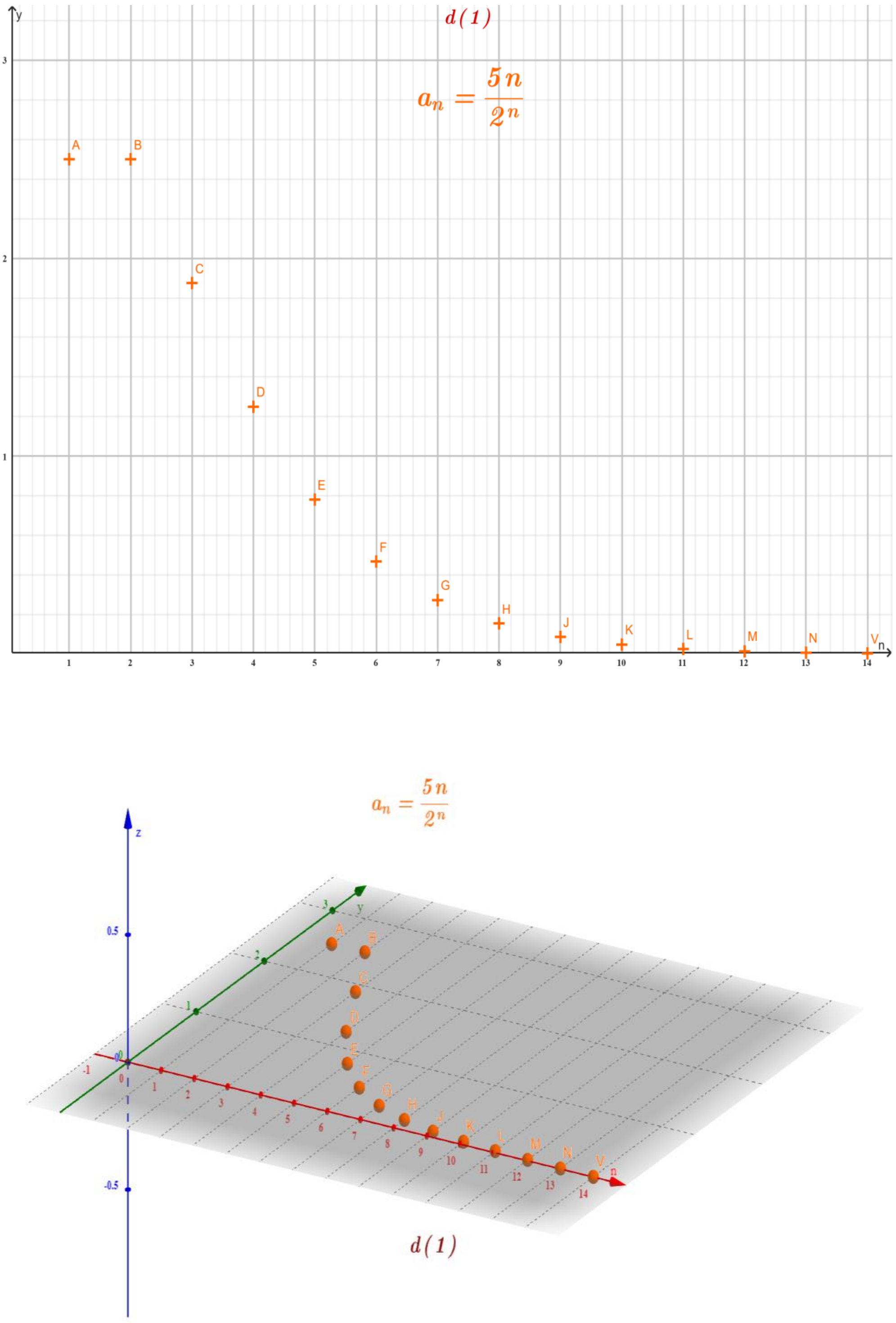


Figure d (1)

Definition 03: Limita of a sequence, convergent, divergent

Let $\{a_n\}$ be a sequence and let L be a real number. given any $\varepsilon > 0$, if an m can be found such that $|a_n - L| < \varepsilon$ for all $n > m$, then we say the **limit** of $\{a_n\}$, as **n approaches infinity , is L ,** denoted

$$\lim_{n\to\infty} a_n = L$$

If $\lim_{n\to\infty} a_n$ *exists , we say the sequence* ***converges*** *, otherwise , the sequence* ***diverges***

this definition states , informally , that if the limit of a sequence is L , then if you go far enough out along the sequence , all subsequent terms will be really close to L . of course , the terms "**far enough**" and "**really close**" are subjective terms , but hopefully the intent is clear

Theorem 01: Limit of a sequence

Let $\{a_n\}$ be a sequence and let $f(x)$ be a function whose domain contains the positive real numbers where $f(n) = a_n$ for all n in $\mathbb{N}$.

1. If $\lim_{x\to\infty} f(x)$ does not exist, we can not conclude that $\lim_{n\to\infty} a_n$ does not exist. It may, or may not, exist. For instance, we can define a sequence $\{a_n\} = \{\cos(2\pi n)\}$.
Let $f(x) = \cos(2\pi n)$. Since the cosine function oscillates over the real numbers, the limit $\lim_{x\to\infty} f(x)$ does not exist.

However, for every positive integer n , $\cos(2\pi n) = 1$, *so* $\lim_{n\to\infty} a_n = 1$

2. If we can not find a function $f(x)$ whose domain contains the positive real numbers where $f(n) = a_n$ for all **n in** $\mathbb{N}$, we can not conclude $\lim_{n\to\infty} a_n$ does not exist. It may, or may not, exist.

Example 03: Determining convergence / divergence of a sequence

determine the convergence or divergence of the following sequences.

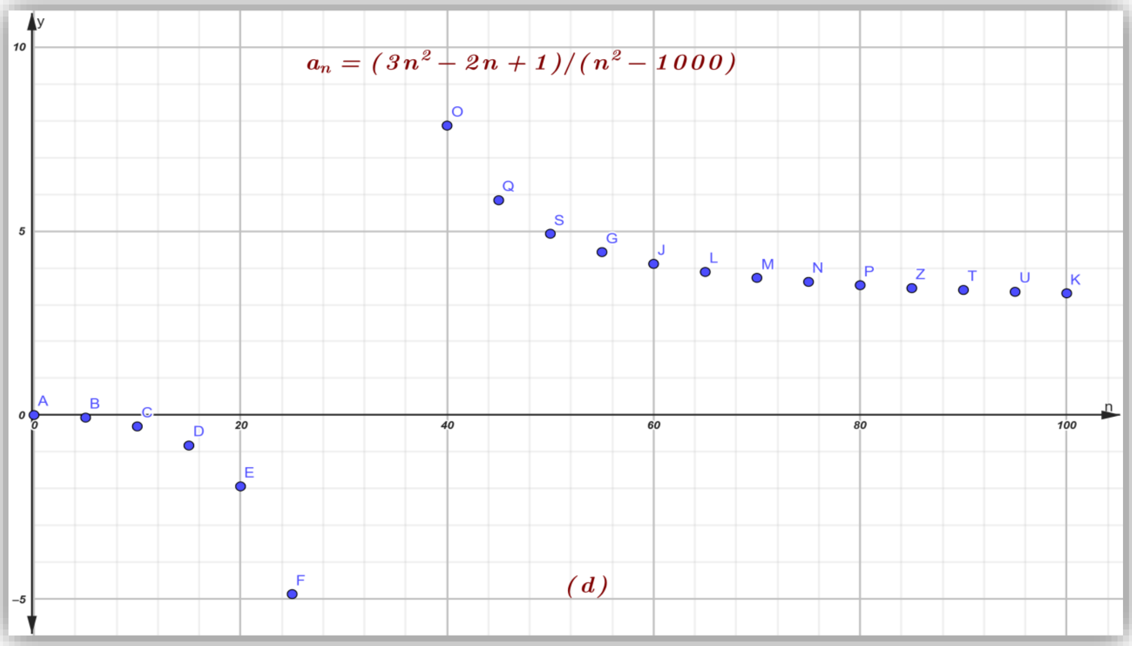
1. $\{a_n\} = \left\{\frac{3n^2-2n+1}{n^2-1000}\right\}$

2. $\{a_n\} = \{\cos n\}$

3. $\{a_n\} = \left\{\frac{(-1)^n}{n}\right\}$

Solution

1.We can state that $\lim_{x\to\infty} \frac{3x^2-2x+1}{x^2-1000} = 3$ (We could have also directly applied **l'H\^opital's Rule**. thus the sequence $\{a_n\}$ converges, and its limit is **3**. A scatter plot of every **5** values of a_n is given in **Figure D**



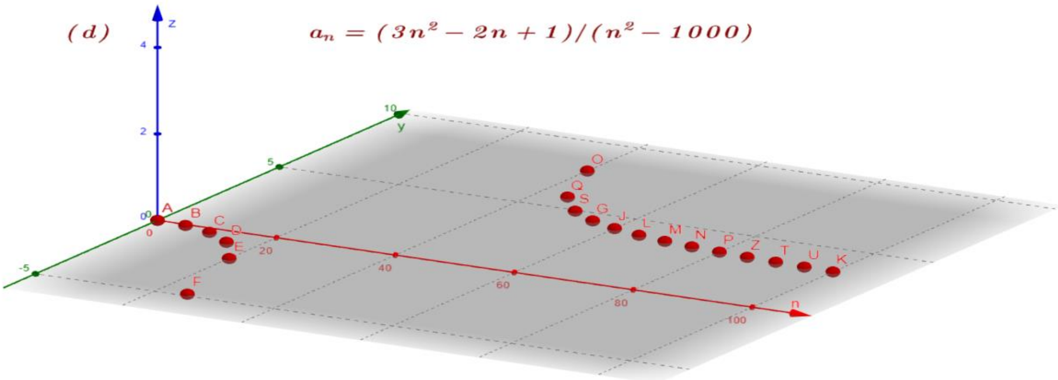


Figure D

2. The limit $\lim_{n\to\infty} \cos x$ does not exist, as $\cos x$ oscillates (and takes on every value in $[-1, 1]$ infinitely many times). Thus we cannot apply **Theorem 01**.
The fact that the cosine function oscillates strongly hints that $\cos n$, when n is restricted to $\mathbb{N}$, will also oscillate. **Figure E**, where the sequence is plotted, shows that this is true. Because only discrete values of cosine are plotted, it does not bear strong resemblance to the familiar cosine wave.

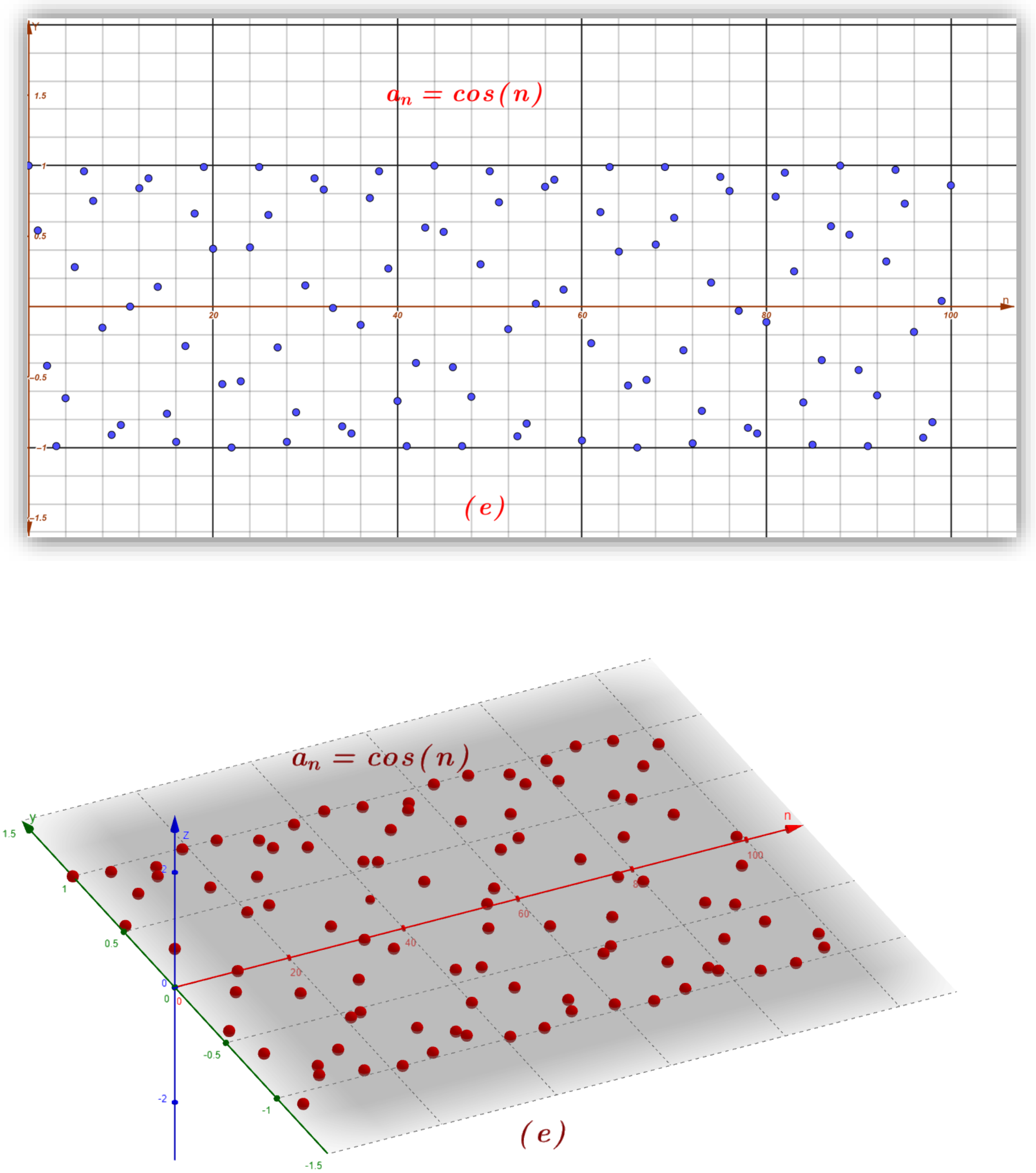


Figure E

We conclude that $\lim_{n\to\infty} a_n$ does not exist.

3. We cannot actually apply Theorem 01 here, as the function $f(x) = \frac{(-1)^x}{x}$ is not well defined. (What does $(-1)^{\sqrt{2}}$ mean? In actuality, there is an answer, but it involves *complex analysis*, beyond the scope of this text.) So for now we say that we cannot determine the limit. (But we will be able to very soon.) By looking at the plot in **Figure F,** we would like to conclude that the sequence converges to o. That is true, but at this point we are unable to decisively say so.

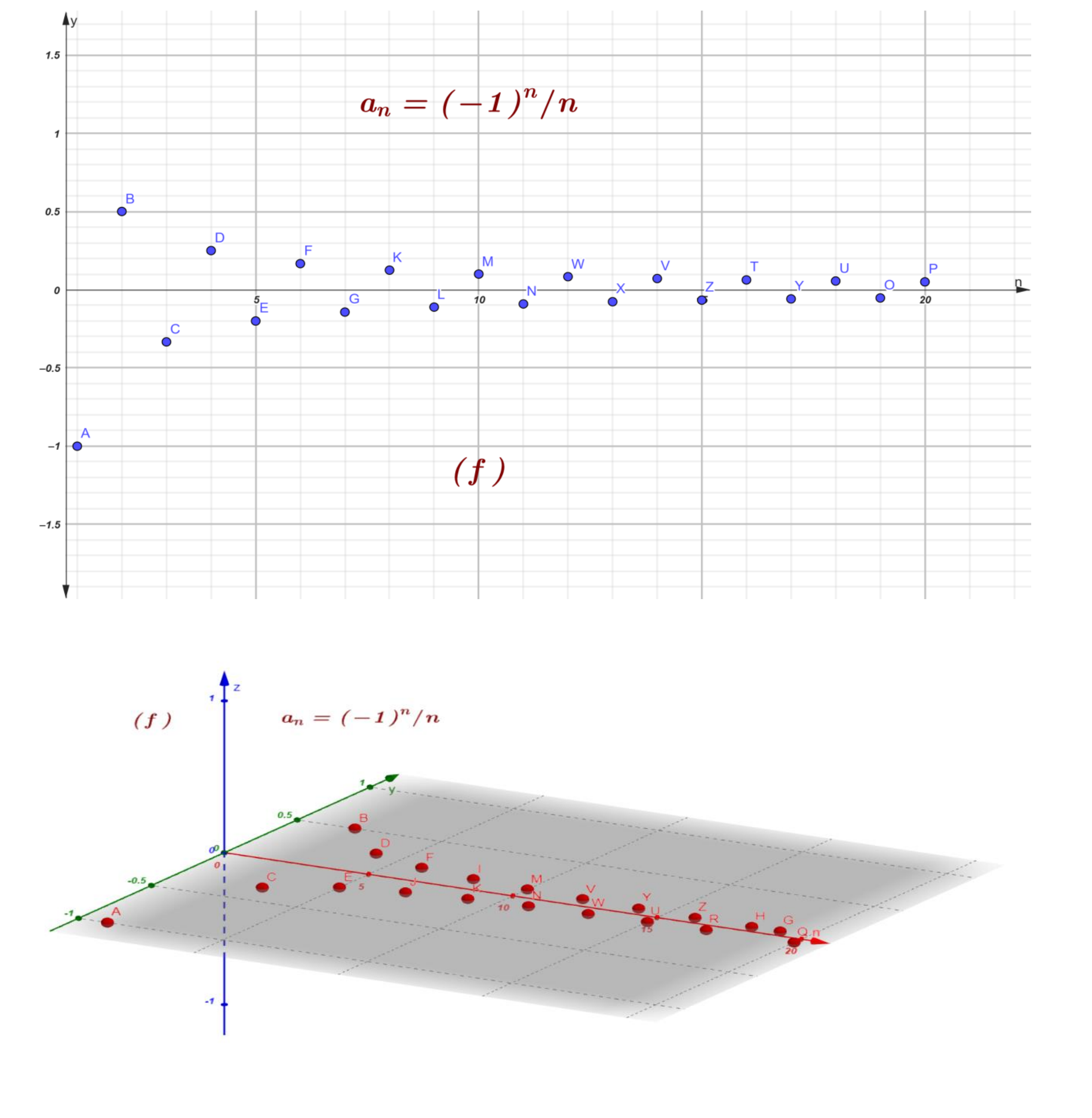


Figure F

It seems that $\frac{(-1)^n}{n}$ converges to o but we lack the formal tool to prove it. The following theorem gives us that tool.

Theorem 02: Absolute value theorem

let a_n be a sequence. $\lim_{n\to\infty} |a_n| = 0$, *then* $\lim_{n\to\infty} a_n = 0$

Example 04: Determining the convergence / divergence of a sequence

determine the convergence or divergence of the following sequences.

1. $\{a_n\}=\left\{\frac{(-1)^n}{n}\right\}$

2. $\{a_n\}=\left\{\frac{(-1)^n(n+1)}{n}\right\}$

Solution

1. This appeared in ***Example 03*** we want to apply **Theorem 02**, so consider the limit of $\{a_n\}$

$$\lim_{n\to\infty}|a_n| = \lim_{x\to\infty}\left|\frac{(-1)^n}{n}\right|$$

$$= \lim_{n\to\infty}\frac{1}{n}$$

$$= 0$$

since this limit is **0**, we can apply ***Theorem 02*** and state that $\lim_{n\to\infty}\{a_n\} = 0$ *, see* **Figure F**

Analytical

$\{a_n\}=\left\{\frac{(-1)^n}{n}\right\}$

- $$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{(-1)^{n+1}}{(n+1)}}{\frac{(-1)^n}{n}}$$

$$= \lim_{n\to\infty}\frac{(-1)^{n+1}}{(n+1)} \times \frac{n}{(-1)^n}$$

$$= \lim_{n\to\infty}\frac{(-1)^n\cdot(-1)^1}{(n+1)} \times \frac{n}{(-1)^n}$$

$$= \lim_{n\to\infty}\frac{-n}{n+1}$$

$$= \lim_{n\to\infty}\frac{-n}{n}$$

$$= -1 < 1 .$$

(So sequence is convergent)

2. Because of the alternating nature of this sequence (i.e., every other term is multiplied by), we cannot simply look at the limit $\lim_{x\to\infty}\frac{(-1)^x(x+1)}{x}$, and we can try to apply the techniques of **Theorem *02***

$$\lim_{n\to\infty}|a_n| = \lim_{x\to\infty}\left|\frac{(-1)^n(n+1)}{n}\right| = \lim_{n\to\infty}\frac{n+1}{n} = 1$$

We have concluded that when we ignore the alternating sign, the sequence approaches **1**. This means we cannot apply ***Theorem 02***, it states the the limit must be 0 in order to conclude anything.

Since we know that the signs of the terms alternate *and* we know that the limit of $|a_n|$ is **1** , we know that as n approaches infinity, the terms will alternate between values close to **1** and **– 1** , meaning the sequence diverges. A plot of this sequence is given in **Figure G.**

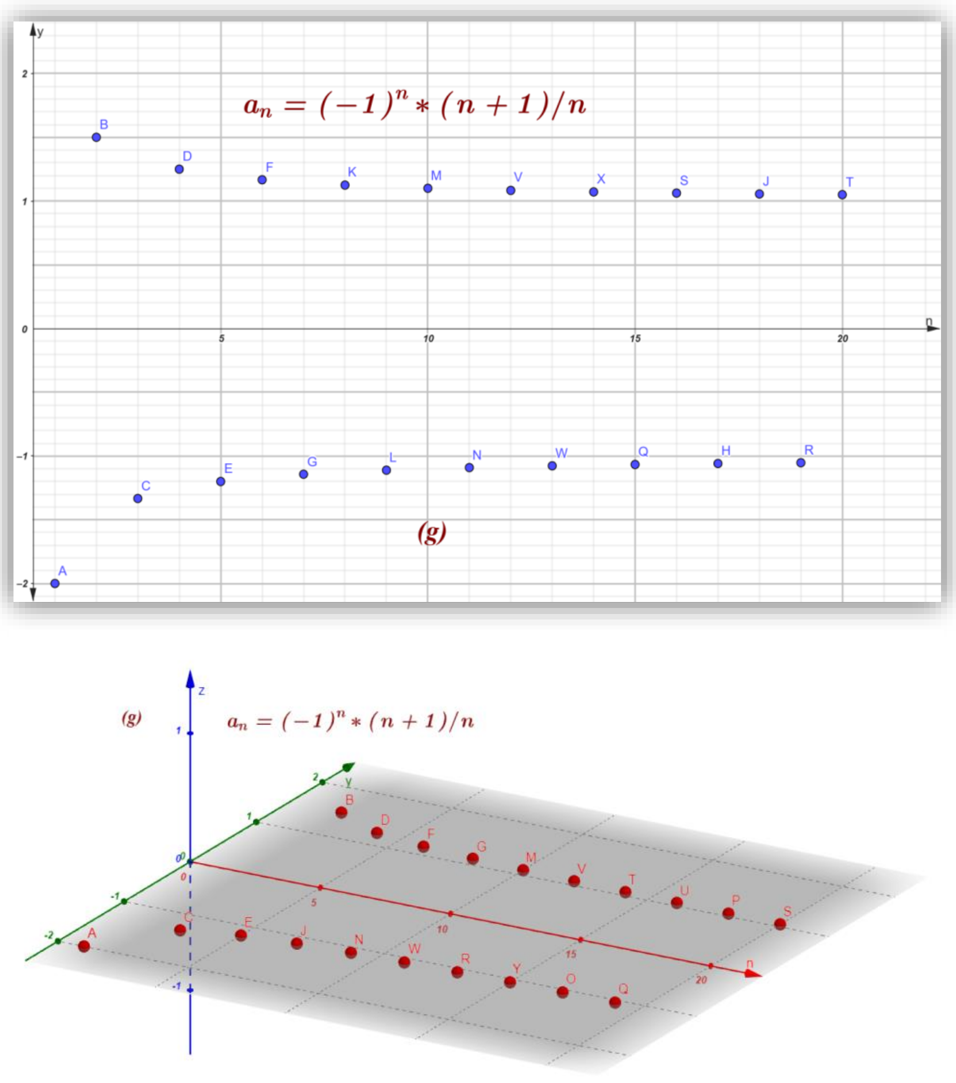


Figure G

We continue our study of the limits of sequences by considering some of the properties of these limits.

Theorem 03: Properties of the limits of sequences

Let $\{a_n\}$ *and* $\{b_n\}$*be sequences such that* $\lim\limits_{n\to\infty} a_n = L$, $\lim\limits_{n\to\infty} b_n = K$ *, and let* c *be a real number.*

1. $\lim\limits_{n\to\infty}(a_n \pm b_n)$ =*L±K*
2. $\lim\limits_{n\to\infty}(a_n \cdot b_n)$ =L·K
3. $\lim\limits_{n\to\infty}(a_n/b_n)$ =*L/K, K≠o*
4. $\lim\limits_{n\to\infty}(c \cdot a_n) = c \cdot L$

Example 05: Applying properties of limits of sequences

Let the following sequence, and their limits, be given:

- $\{a_n\}=\left\{\frac{n+1}{n}\right\}$ *,and* $\lim\limits_{n\to\infty} a_n = 0$
- $\{b_n\} = \left\{\left(1+\frac{1}{n}\right)^n\right\}$ *,and* $\lim\limits_{n\to\infty} b_n = e$
- $\{c_n\} =\left\{n \sin\left(\frac{5}{n}\right)\right\}$ *, and* $\lim\limits_{n\to\infty} c_n = 5$

Solution

We will use Theorem to answer each of these.

1. *Since* $\lim\limits_{n\to\infty} a_n$*=0 and* $\lim\limits_{n\to\infty} b_n = e$ *, we conclude that* $\lim\limits_{n\to\infty}(a_n + b_n)$*=0+ e= e.So even though we are adding* something to each term of the sequence b_n , we are adding something so small that the final limit is the same as before.

2. Since $\lim\limits_{n\to\infty} b_n = e$ and $\lim\limits_{n\to\infty} c_n = 5$, we conclude that $\lim\limits_{n\to\infty}(b_n \cdot c_n)$= $e \cdot 5=5\,e$

3. *Since* $\lim\limits_{n\to\infty} a_n$*=0, we have* $\lim\limits_{n\to\infty} 1000 \cdot a_n = 1000 \cdot 0 = 0$. It does not matter that we multiply each term by 1000; the sequence still approaches 0. (It just takes longer to get close to **0**.)

There is more to learn about sequences than just their limits. We will also study their range and the relationships terms have with the terms that follow. We start with some definitions describing properties of the range.

Definition 04: Bounded and unbounded sequences

1. A sequence $\{a_n\}$ is said to be **bounded** if there exists real numbers m and M such that $m < a_n < M$ for all n in ℕ .
2. A sequence $\{a_n\}$ is said to be **unbounded** if it is not bounded.
3. A sequence $\{a_n\}$ is said to be **bounded above** if there exists an **M** such that $a_n < M$ for all n in ℕ it is **bounded below** if there exists an m such that $m < a_n$ for all n in ℕ .

Example 06: Determining boundedness of sequences

Determine the boundedness of the following sequences.

1. $\{a_n\} = \left\{\frac{1}{n}\right\}$
2. $\{a_n\} = \{2^n\}$

Solution

1. The terms of this sequence are always positive but are decreasing, so we have $\boldsymbol{0} < \boldsymbol{a_n} < 2$ for all $\boldsymbol{n}$. Thus this sequence is bounded, **Figure H** illustrates this.

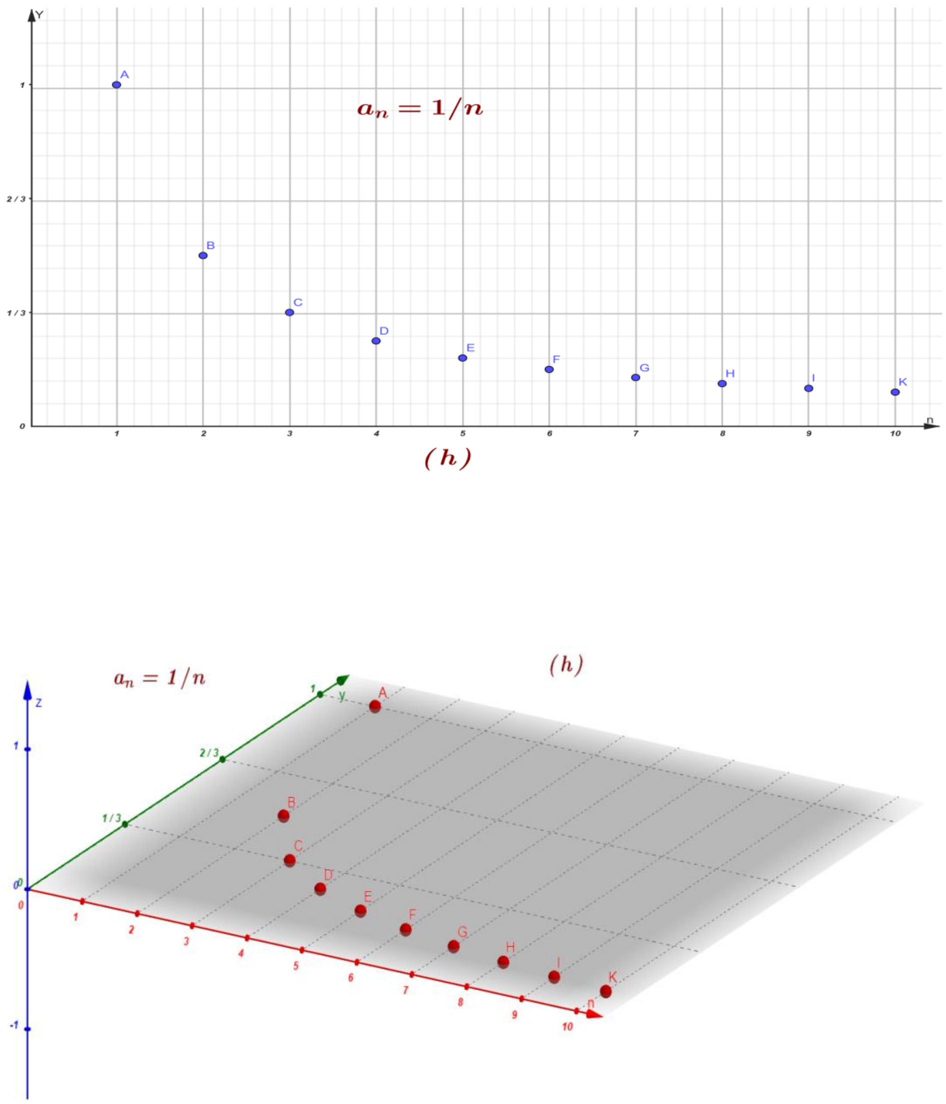


Figure H

2. The terms of this sequence obviously grow without bound. However, it is also true that these terms are all positive, meaning $0 < a_n$. Thus we can say the sequence is unbounded, but also bounded below. **Figure I** illustrates this.

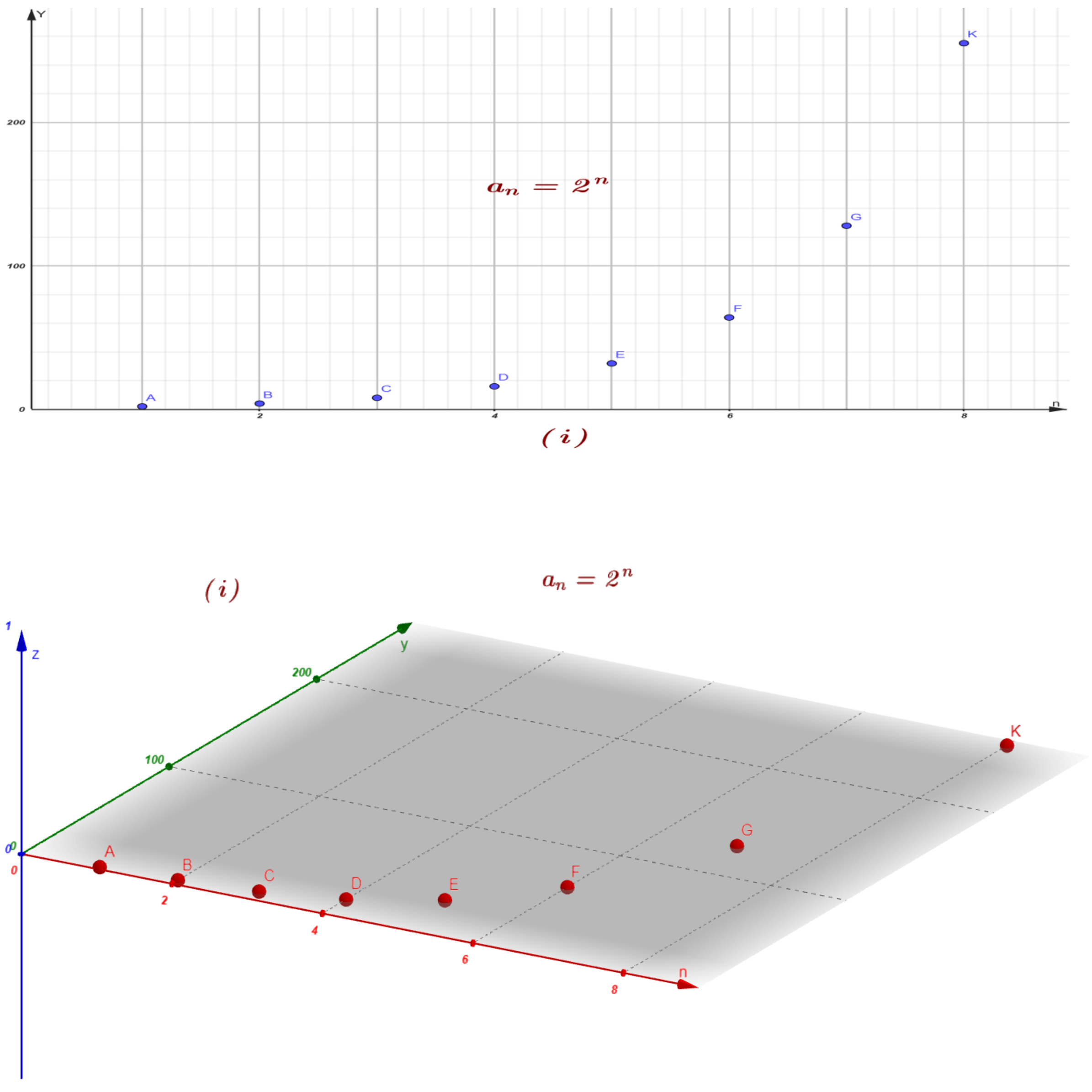


Figure I

Theorem 04: Convergent sequences are bounded

let $\{a_n\}$ be a convergent sequence. Then $\{a_n\}$ is bounded.

In **Example 05** we saw the sequence $\{b_n\} = \left\{(1+\frac{1}{n})^n\right\}$, where it was stated that $\lim_{n\to\infty} b_n = e$. (Note that this is simply restating part of **Theorem 05.**) Even though it may be difficult to intuitively grasp the behaviour of this sequence, we know immediately that it is bounded.

Another interesting concept to come out of **Example 06** again involves the sequence $\frac{1}{n}$. We stated, without proof that the terms of the sequence were decreasing. That is, that $a_{n+1} < a_n$ for all n. (This is easy to show. Clearly $n < n+1$. Taking reciprocals flips the inequality: $\frac{1}{n} > \frac{1}{(n+1)}$. This is the same as $a_n > a_{n+1}$.) Sequences that either steadily increase or decrease are important, so we give this property a name.

Definition 05: monotonic sequences

1. A sequence $\{a_n\}$ is **monotonically increasing** if $a_n \leq a_{n+1}$ for all **n** ,
$$a_1 \leq a_2 \leq a_3 \leq \cdots a_n \leq a_{n+1} \;\ldots.$$
2. A sequence $\{a_n\}$ is **monotonically decreasing** if $a_n \geq a_{n+1}$ for all **n** ,
$$a_1 \geq a_2 \geq a_3 \geq \ldots a_n \geq a_{n+1} \;\ldots.$$
3. A sequence is **monotonic** if it is monotonically increasing or monotonically decreasing.

NOTE: It is sometimes useful to call a monotonically increasing sequence *strictly increasing* if $a_n < a_{n+1}$ for all ; i.e , we remove the possibility that subsequent terms are equal . A similar statement holds for *strictly* decreasing

Example 07: Determining monotonicity

Determine the monotonicity of the following sequences.

1. $\{a_n\} = \left\{\frac{n+1}{n}\right\}$

2. $\{a_n\} = \left\{\frac{n^2+1}{n+1}\right\}$

3. $\{a_n\} = \left\{\frac{n^2-9}{n^2-10n+26}\right\}$

4. $\{a_n\} = \left\{\frac{n^2}{n!}\right\}$

Solution

In each of the following, we will examine $a_{n+1} - a_n$. If $a_{n+1} - a_n > 0$, we conclude that $a_n < a_{n+1}$ and hence the sequence is increasing. If $a_{n+1} - a_n < 0$, we conclude that $a_n > a_{n+1}$ and the sequence is decreasing. Of course, a sequence need not be monotonic and perhaps neither of the above will apply.

We also give a scatter plot of each sequence. These are useful as they suggest a pattern of monotonicity, but analytic work should be done to confirm a graphical trend.

1. $\{\boldsymbol{a_n}\} = \left\{\frac{\boldsymbol{n+1}}{\boldsymbol{n}}\right\}$

Analytical

$$\boldsymbol{a_{n+1} - a_n} = \frac{\boldsymbol{n+2}}{\boldsymbol{n+1}} - \frac{\boldsymbol{n+1}}{\boldsymbol{n}}$$

$$= \frac{(\boldsymbol{n+2})(\boldsymbol{n}) - (\boldsymbol{n+1})^2}{(\boldsymbol{n+1})\boldsymbol{n}}$$

$$= \frac{\boldsymbol{n^2+2n-(n^2+2n+1)}}{\boldsymbol{n(n+1)}}$$

$$= \frac{\cancel{\boldsymbol{n^2}} + \cancel{\boldsymbol{2n}} - \cancel{\boldsymbol{n^2}} - \cancel{\boldsymbol{2n}} - \boldsymbol{1}}{\boldsymbol{n(n+1)}}$$

$$= \frac{\boldsymbol{-1}}{\boldsymbol{n(n+1)}} < 0$$

Since $\boldsymbol{a_{n+1} - a_n} < 0$ *for all* $\boldsymbol{n}$ *, we conclude that the sequence is decreasing .*

Graphical

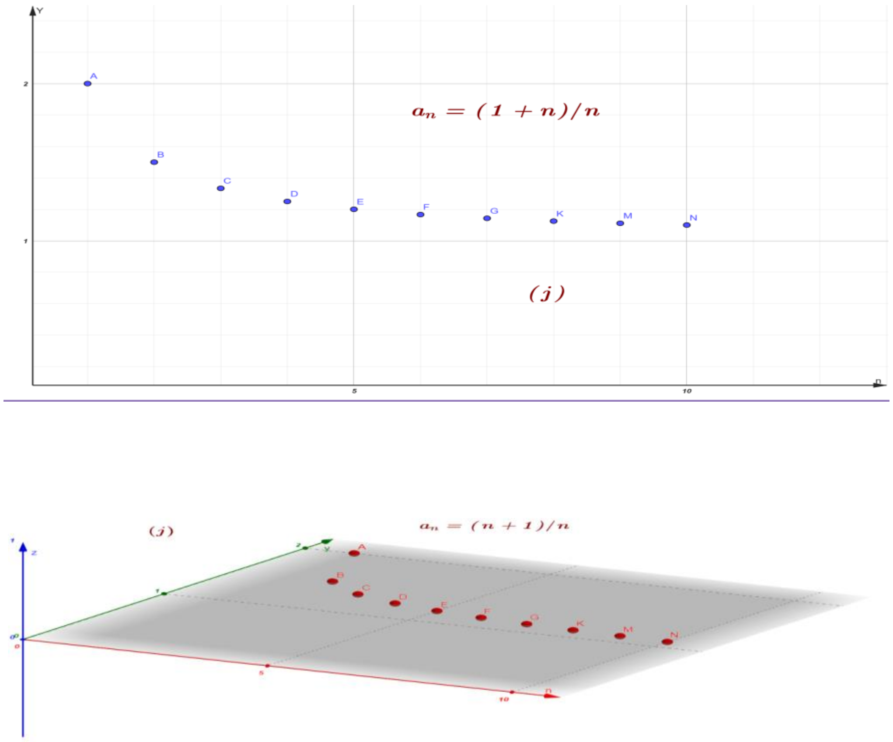


Figure J

2. $\{a_n\} = \left\{\frac{n^2+1}{n+1}\right\}$

Analytical

$$a_{n+1} - a_n = \frac{(n+1)^2+1}{n+2} - \frac{n^2+1}{n+1}$$

$$= \frac{\left((n+1)^2+1\right)(n+1) - (n^2+1)(n+2)}{(n+2)(n+1)}$$

$$= \frac{(n^2+1+2n+1)(n+1) - (n^2+1)(n+2)}{(n+2)(n+1)}$$

$$= \frac{(n^2+2+2n)(n+1) - (n^3+2n^2+n+2)}{(n+2)(n+1)}$$

$$= \frac{(n^3+2n^2+2n+n^2+2+2n) - (n^3+2n^2+n+2)}{(n+1)(n+2)}$$

$$= \frac{n^3+3n^2+4n+2-n^3-2n^2-n-2}{(n+1)(n+2)}$$

$$= \frac{n^2+3n}{(n+1)(n+2)} > 0$$

Since $\boldsymbol{a_{n+1} - a_n > 0}$ *for all* $\boldsymbol{n}$ *, we conclude that the sequence is increasing .*

Graphical

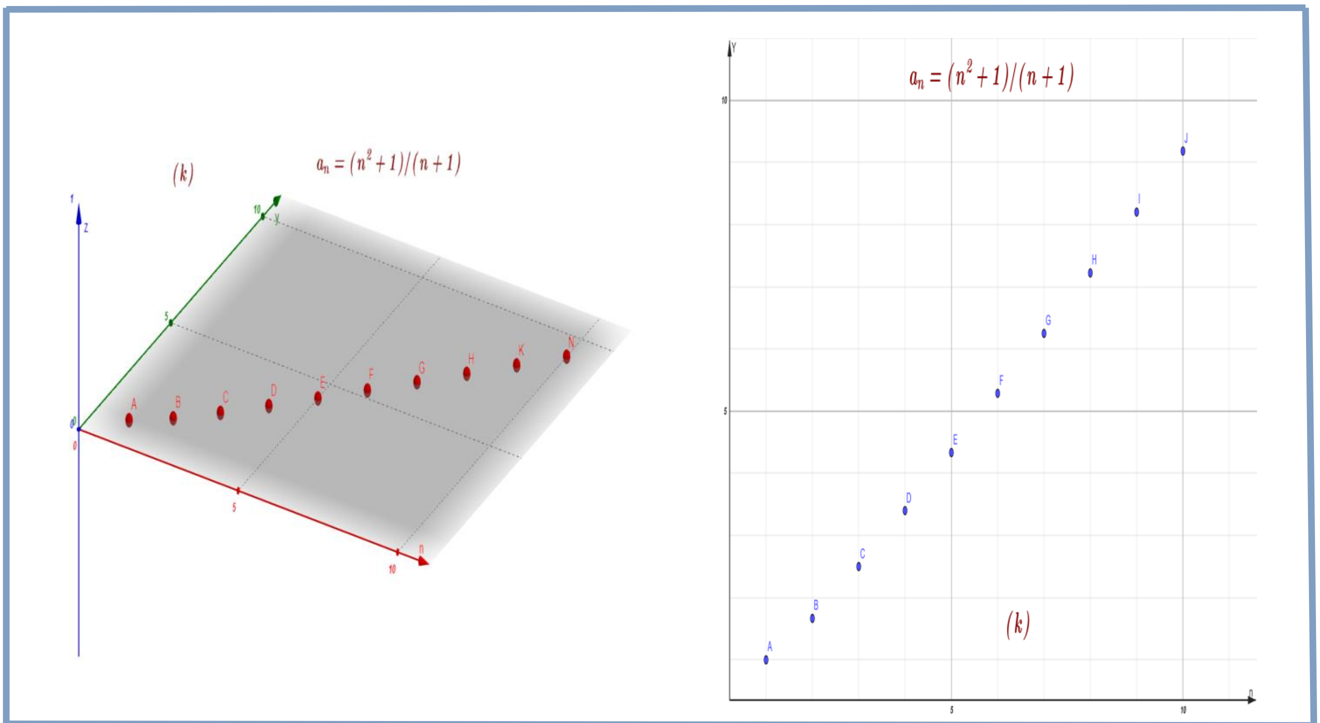


Figure K

3. $\{a_n\} = \left\{\frac{n^2-9}{n^2-10n+26}\right\}$

Analytical

we can clearly see in **Figure L** where the sequence is plotted, that it is not monotonic. However, it does seem that after the first 4 terms it is decreasing. To understand why, perform the same analysis as done before:

$$a_{n+1} - a_n = \frac{(n+1)^2 - 9}{(n+1)^2 - 10(n+1) + 26} - \frac{n^2-9}{n^2-10n+26}$$

$$= \frac{(n^2+1+2n)-9}{(n^2+1+2n)-10n-10+26} - \frac{n^2-9}{n^2-10n+26}$$

$$= \frac{n^2+2n-8}{n^2-8n+17} - \frac{n^2-9}{n^2-10n+26}$$

$$= \frac{(n^2+2n-8)(n^2-10n+26)-(n^2-9)(n^2-8n+17)}{(n^2-8n+17)(n^2-10n+26)}$$

$$= \frac{n^4-10n^3+26n^2+2n^3-20n^2+52n-8n^2+80n-208-n^4+8n^3-17n^2+9n^2-72n-153}{(n^2-8n+17)(n^2-10n+26)}$$

$$= \frac{-8n^3-2n^2+132n-208+8n^3-8n^2-72n+158}{(n^2-8n+17)(n^2-10n+26)}$$

$$= \frac{-10n^2+60n-55}{(n^2-8n+17)(n^2-10n+26)}$$

$-10n^2 + 60n - 55 = 0$ ⟶ $\Delta = (+60)^2 - 4 \times (-55) \times (-10)$

$\Delta = 1400$

$\sqrt{\Delta} = \sqrt{1400} = 10\sqrt{14}$ *has two solution* ; $\begin{cases} n_1 = \frac{-60+10\sqrt{14}}{-2\times10} = \frac{+6-\sqrt{14}}{2} = 1,13 \\ n_2 = \frac{-60-10\sqrt{14}}{-2\times10} = \frac{+6+\sqrt{14}}{2} = 4,87 \end{cases}$

We want to know when this is greater than, or less than, **0**. The denominator is always positive; therefore we are only concerned with the numerator. Using the quadratic formula, we can determine that $-10n^2 + 60n - 55 = 0$ when n_1, n_2 . So for, $n < 1,13$ the sequence is decreasing. Since we are only dealing with the natural numbers, this means that $a_1 > a_2$.

Between n_1 and n_2 , for $n = 2, 3$ *and* 4 , we have that $a_{n+1} > a_n$ and the sequence is increasing. (That is, when $= 2, 3$ *and* 4 , the numerator $-10n^2 + 60n - 55$ from the fraction above is.)

When, $n > n_2$ for ≥ 5 , we have that $-10n^2 + 60n - 55 < 0$, hence $a_{n+1} - a_n < 0$, so the sequence is decreasing.

In short, the sequence is simply not monotonic. However, it is useful to note that for ≥ 5 , the sequence is monotonically decreasing.

Graphical

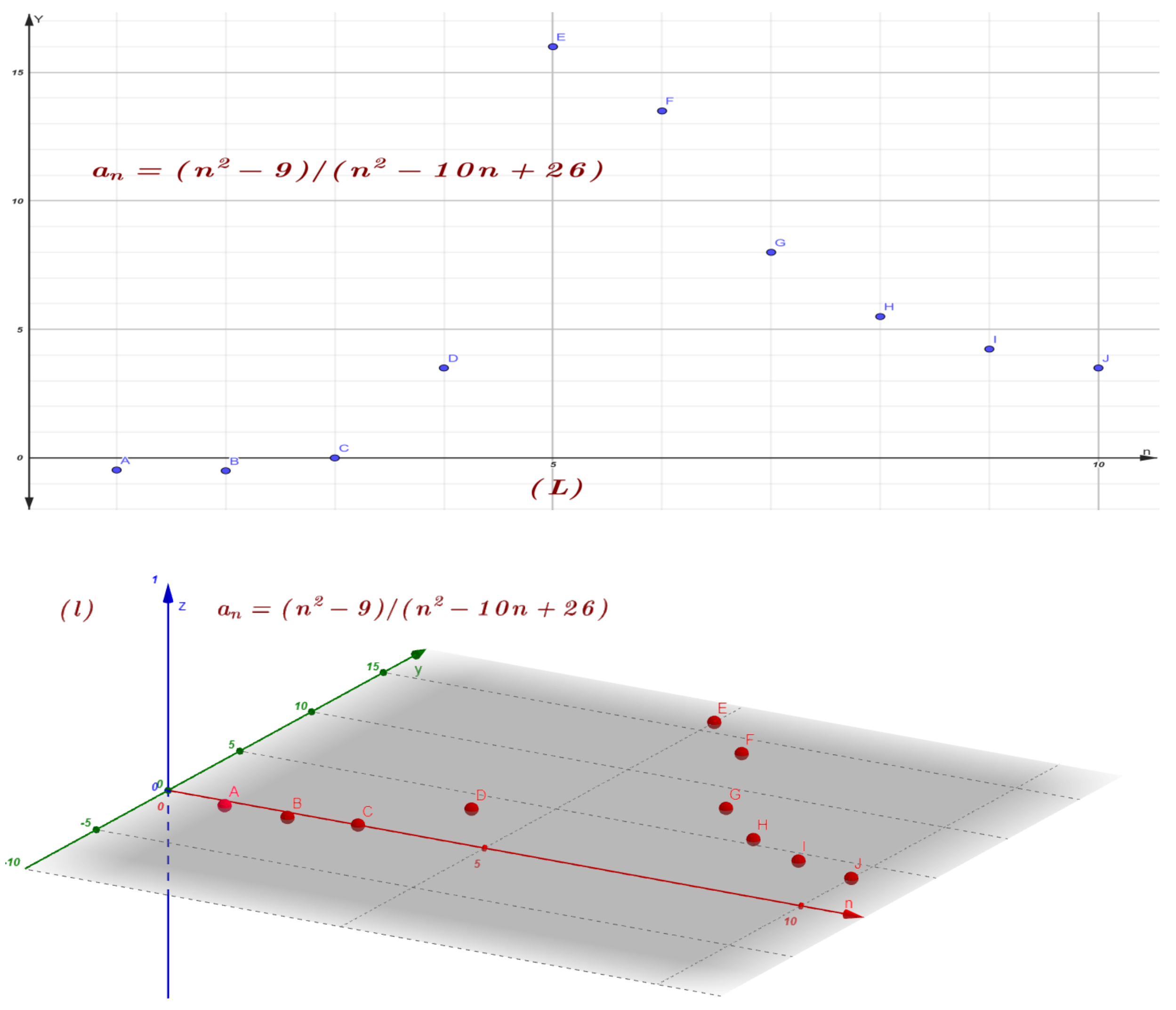


Figure L

4. $\{a_n\} = \left\{\frac{n^2}{n!}\right\}$

Analytical

Again, the plot in **Figure M** shows that the sequence is not monotonic, but it suggests that it is monotonically decreasing after the first term. We perform the usual analysis to confirm this.

$$a_{n+1} - a_n = \frac{(n+1)^2}{(n+1)!} - \frac{n^2}{n!}$$

$$= \frac{(n+1)^2 n! - n^2 (n+1)!}{n!(n+1)!}$$

$$= \frac{(n+1)^2 \cancel{n!} - n^2 (n+1)\cancel{n!}}{(n+1)!\cancel{n!}}$$

$$=\frac{\cancel{n^2}+2n+1-n^3-\cancel{n^2}}{(n+1)!}$$

$$=\frac{-n^3+2n+1}{(n+1)!}$$

When $n = 1$, the above expression is< 0 , for ≥ 2 , the above expression is < 0 . Thus this sequence is not monotonic, but it is monotonically decreasing after the first term.

Graphical

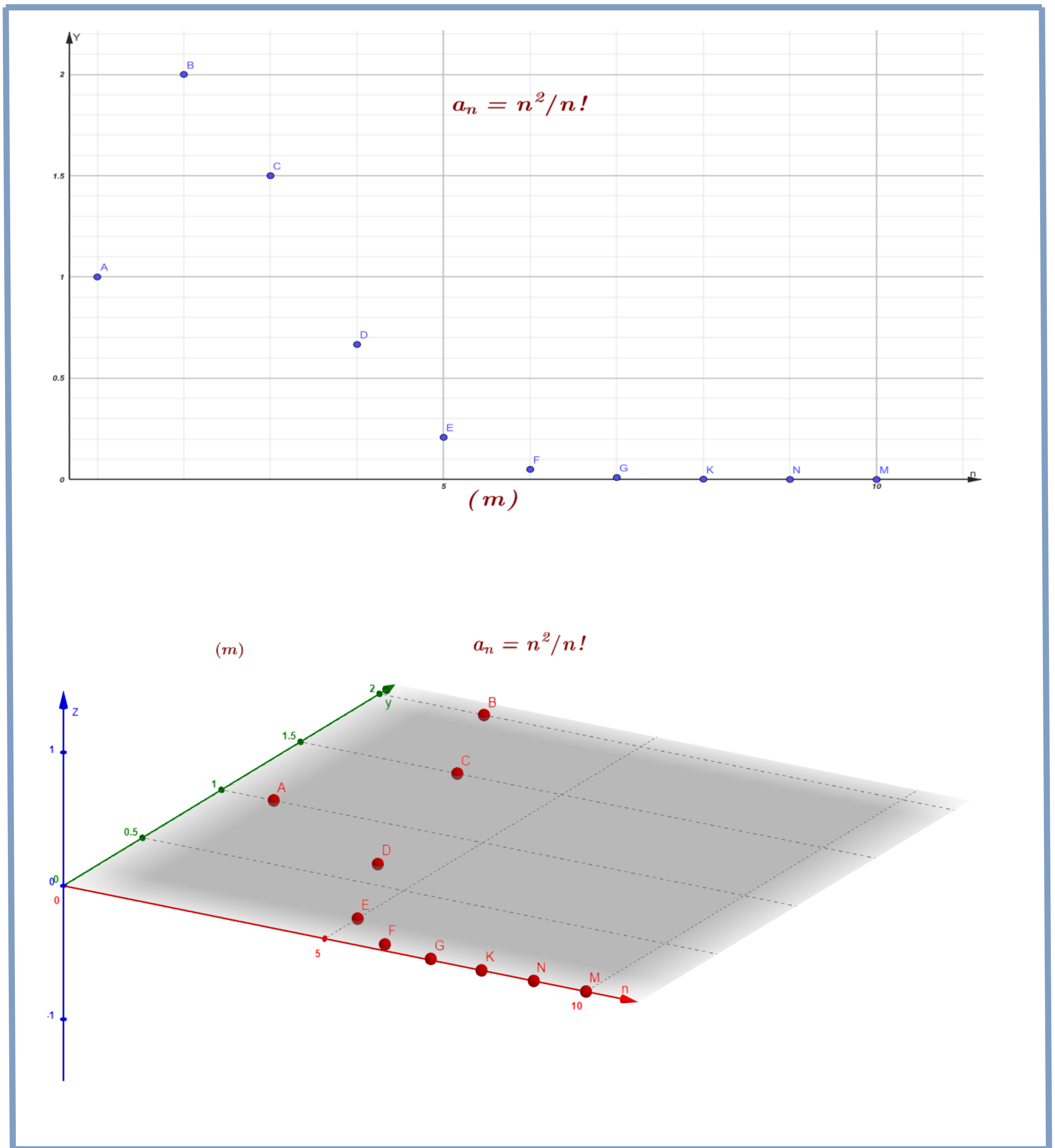


Figure M

Theorem 05: Bounded monotonic sequence are convergent

1. Let $\{a_n\}$ be a bounded, monotonic sequence. Then $\{a_n\}$ converges; $\lim_{n\to\infty} a_n$ exists.

2. Let $\{a_n\}$ be a monotonically increasing sequence that is bounded abouve. Then $\{a_n\}$ converges.

3. Let $\{a_n\}$ be a monotonically decreasing sequence that is bounded below. Then $\{a_n\}$ converges.

2-Infinite Series

Given the sequence $\{a_n\} = \{1/2^n\}$

$n = 1 \Longrightarrow a_1 = 1/2$

$n = 2 \Longrightarrow a_2 = 1/4$

$n = 3 \Longrightarrow a_3 = 1/8$

$n = 4 \Longrightarrow a_4 = 1/16$

consider the following sums:

$$a_1 = 1/2 = 1/2$$

$$a_1 + a_2 = 1/2 + 1/4 = 3/4$$

$$a_1 + a_2 + a_3 = 1/2 + 1/4 + 1/8 = 7/8$$

$$a_1 + a_2 + a_3 + a_4 = 1/2 + 1/4 + 1/8 + 1/16 = 15/16$$

In general, we can show that:

$$a_1 + a_2 + a_3 + \ldots\ldots + a_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$$

Let S_n be the sum of the first n terms of the sequence $\{1/2^n\}$. From the above, we see that $S_1 = 1/2$, $S_2 = 3/4$, etc. Our formula at the end shows that $S_n = 1 - 1/2^n$.

Now consider the following limit:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}(1 - 1/2^n) = 1$$

This limit can be interpreted as saying something amazing: *the sum of all* the terms of the sequence $\{1/2^n\}$ **is 1.**} This example illustrates some interesting concepts that we explore in this section. We begin this exploration with some definitions.

Definition 06: Infinite **series,** n^{th} partial sums, convergence, divergence

let $\{a_n\}$ be a sequence.

1. The sum $\sum_{n=1}^{\infty} a_n$ is an **infinite series** (or, simply **series**)

2. Let $S_n = \sum_{i=1}^{n} a_i$, the sequence $\{S_n\}$ is the sequence of n^{th} **partial sums** of $\{a_n\}$

3. If the sequence $\{S_n\}$ converges to L, we say the series $\sum_{n=1}^{\infty} a_n$ **converges to L,** and we write $\sum_{n=1}^{\infty} a_n = L$

*4. If the sequence $\{S_n\}$ diverges, the series $\sum_{n=1}^{\infty} a_n$ **diverges***

Using our new terminology, we can state that the series $\sum_{n=1}^{\infty} 1/2^n$ converges, and $\sum_{n=1}^{\infty} 1/2^n = 1$

We will explore a variety of series in this section. We start with two series that diverge, showing how we might discern divergence.

Theorem 06: Properties of summations

1. $\sum_{i=1}^{n} c = c \cdot n$ **,where c is a constant.**
2. $\sum_{i=m}^{n} (a_i \pm b_i) = \sum_{i=m}^{n} a_i \pm \sum_{i=m}^{n} b_i$
3. $\sum_{i=m}^{n} c \cdot a_i = c \cdot \sum_{i=m}^{n} a_i$
4. $\sum_{i=m}^{j} a_i + \sum_{i=j+1}^{n} a_i = \sum_{i=m}^{n} a_i$
5. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
6. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
7. $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

Example 08: Showing series diverge

1. Let $\{a_n\} = \{n^2\}$. Show $\sum_{n=1}^{\infty} a_n$ diverges.

2. Let $\{b_n\} = \{(-1)^{n+1}\}$. Show $\sum_{n=1}^{\infty} b_n$ diverges.

Solution

1. $\{a_n\} = \{n^2\}$

Consider S_n, the n^{th} partial sum.

$$S_n = a_1 + a_2 + a_3 + \dots + a_n$$

$$= 1^2 + 2^2 + 3^2 + \dots + n^2$$

By **Theorem 06**, this is

$$=\frac{n(n+1)(2n+1)}{6}$$

since $\lim_{n\to\infty} S_n = \infty$, we conclude that the series $\sum_{n=1}^{\infty} n^2$ diverges. It is instructive to write $\sum_{n=1}^{\infty} n^2 = \infty$ for this tells us *how* the series diverges: it grows without bound.

A scatter plot of the sequences $\{a_n\}$ and $\{S_n\}$ is given in **Figure N.** The terms of $\{a_n\}$ are growing, so the terms of the partial sums $\{S_n\}$ are growing even faster, illustrating that the series diverges.

Graphical

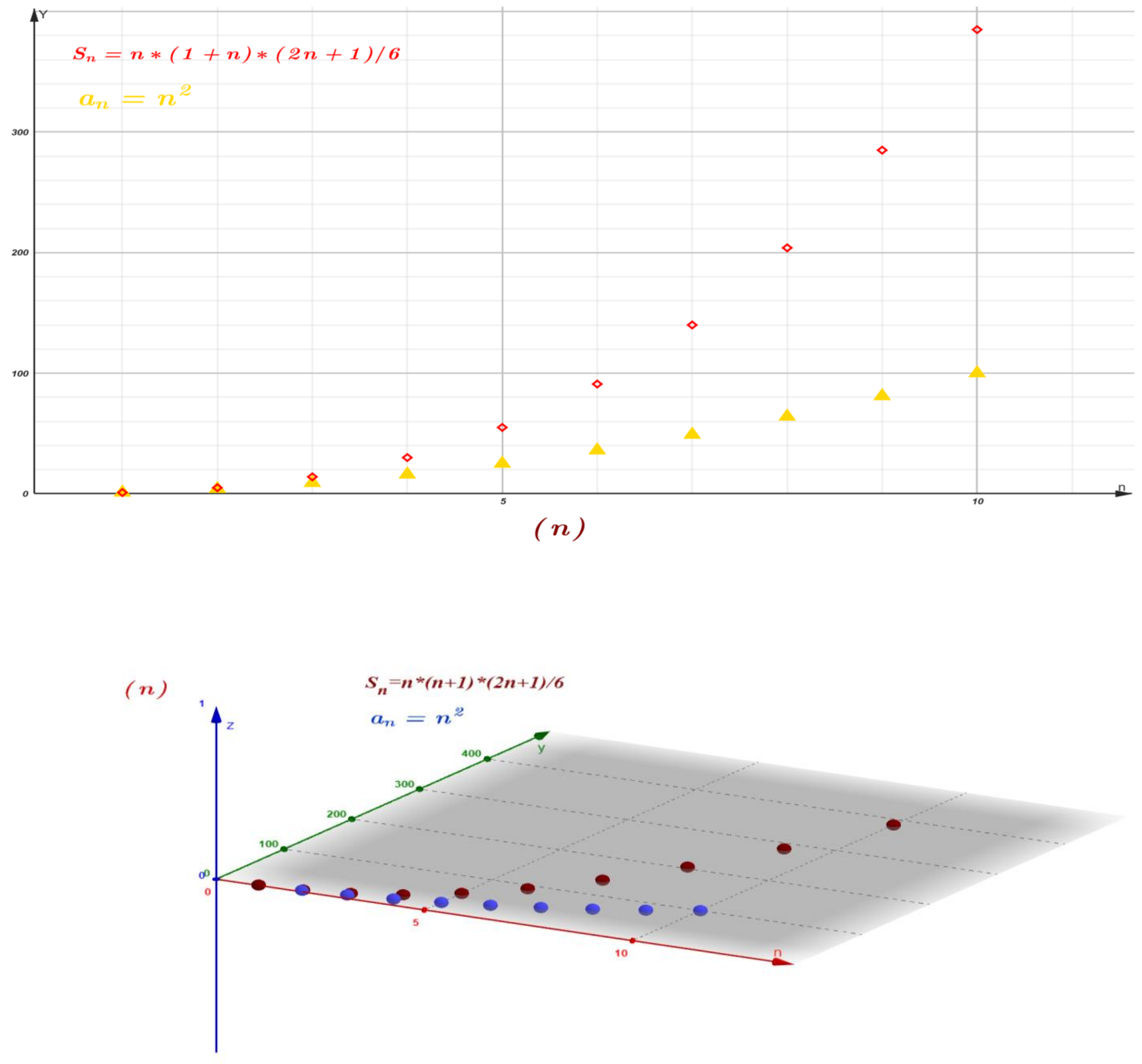


Figure N

2. $\{b_n\} = \{(-1)^{n+1}\}$

The sequence $\{b_n\}$ starts with $1, -1, 1, -1$ Consider some of the partial sums $\{S_n\}$ of $\{b_n\}$:

$$S_1 = 1$$
$$S_2 = 0$$
$$S_3 = 1$$
$$S_4 = 0$$

This pattern repeats; we find that $S_n = \begin{cases} 1 & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$

As $\{S_n\}$ oscillates , repeating $1, 0, 1, 0, \ldots,$ we conclude that $\lim_{n\to\infty} S_n$ does not exist , hence $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges .

A scatter plot of the sequence $\{b_n\}$ and the partial sums $\{S_n\}$ is given in **Figure O.** When n is odd, $b_n = S_n$ so the marks for b_n are drawn oversized to show they coincide.

Graphical

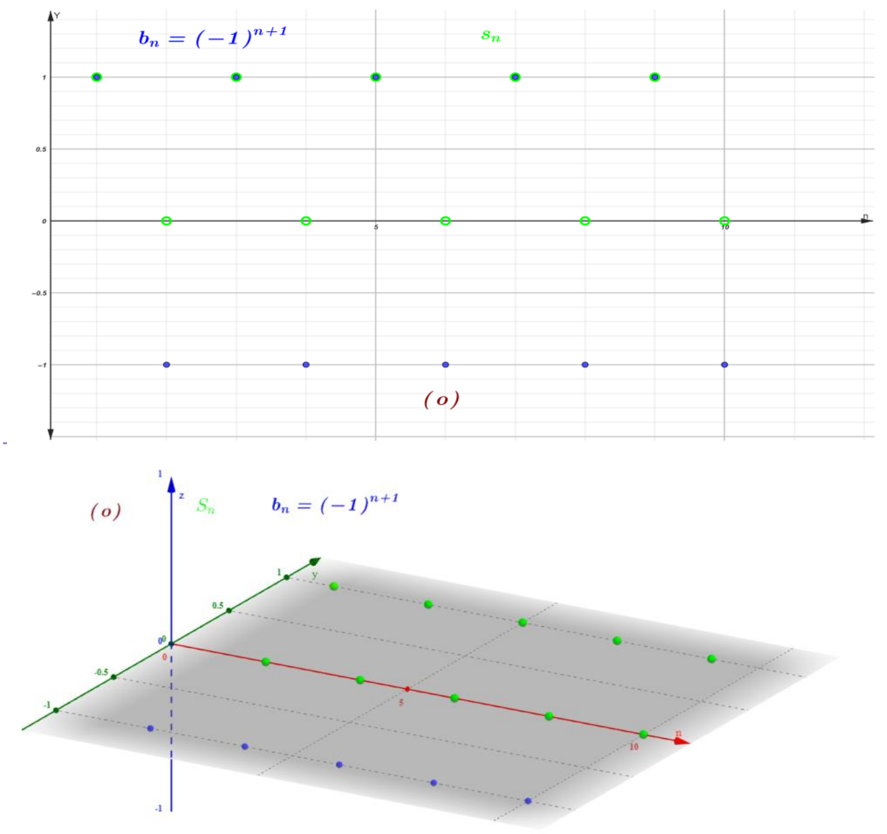


Figure O

2-1 Geometric Series

One important type of series is a *geometric series.*

Definition 07: Geometric series

A *geometric series* is a series of the form

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots + r^n + \cdots$$

Note that the index starts at $n=0$, not $n=1$.

We started this section with a geometric series, although we dropped the first term of *1*. One reason geometric series are important is that they have nice convergence properties.

Theorem 07: Convergence of geometric series

consider the geometric series $\sum_{n=0}^{\infty} r^n$.

1. The n^{th} partial sum is: $S_n = \frac{1-r^{n+1}}{1-r}$

2. The series converges if, and only if **1** . When $|r| < 1$,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

Example 09: Exploring geometric series

check the convergence of the following series. If the series converges, find its sum.

1. $\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n$ **2.** $\sum_{n=0}^{\infty} 3^n$

Solution

1. $\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n$

Since $r = 3/4 < 1$, this series converges. By **Theorem 07**, we have that

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = \frac{1}{1-3/4} = \frac{1}{\frac{4-3}{4}} = \frac{1}{\frac{1}{4}} = 1 \times 4 = 4$$

However, note the subscript of the summation in the given series: we are to start with $\boldsymbol{n} = \mathbf{2}$ Therefore we subtract off the first two terms, giving:

$$\sum_{n=2}^{\infty}\left(\frac{3}{4}\right)^{n} = 4 - 1 - \frac{3}{4} = \frac{9}{4}$$

Graphical

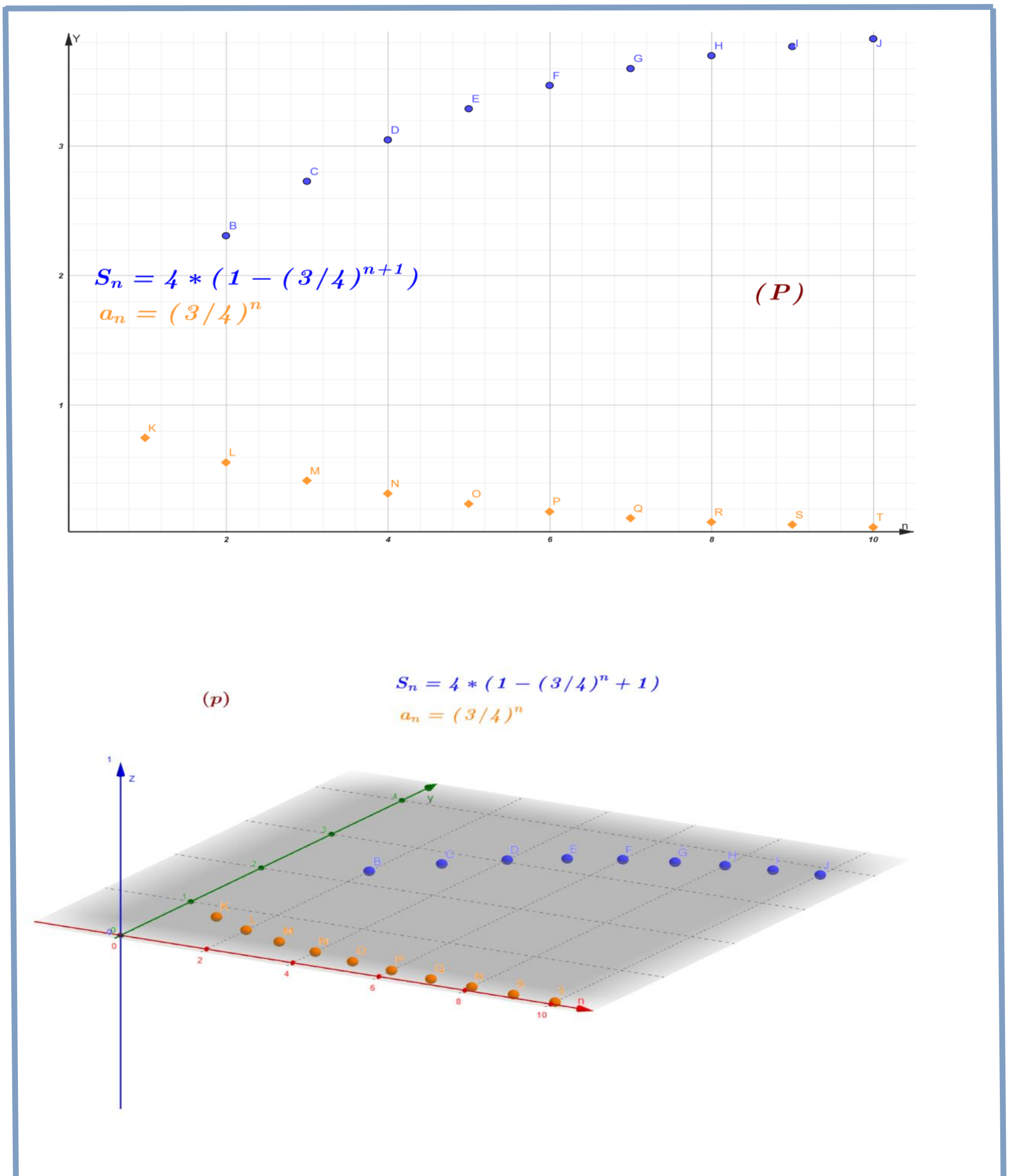


Figure P

2. $\sum_{n=0}^{\infty} 3^n$

Since $r > 1$, the series diverges. (This makes "common sense"; we expect the sum

$$1 + 3 + 9 + 27 + 81 + 243 + \cdots$$

to diverge.) This is illustrated in **Figure R.**

Graphical

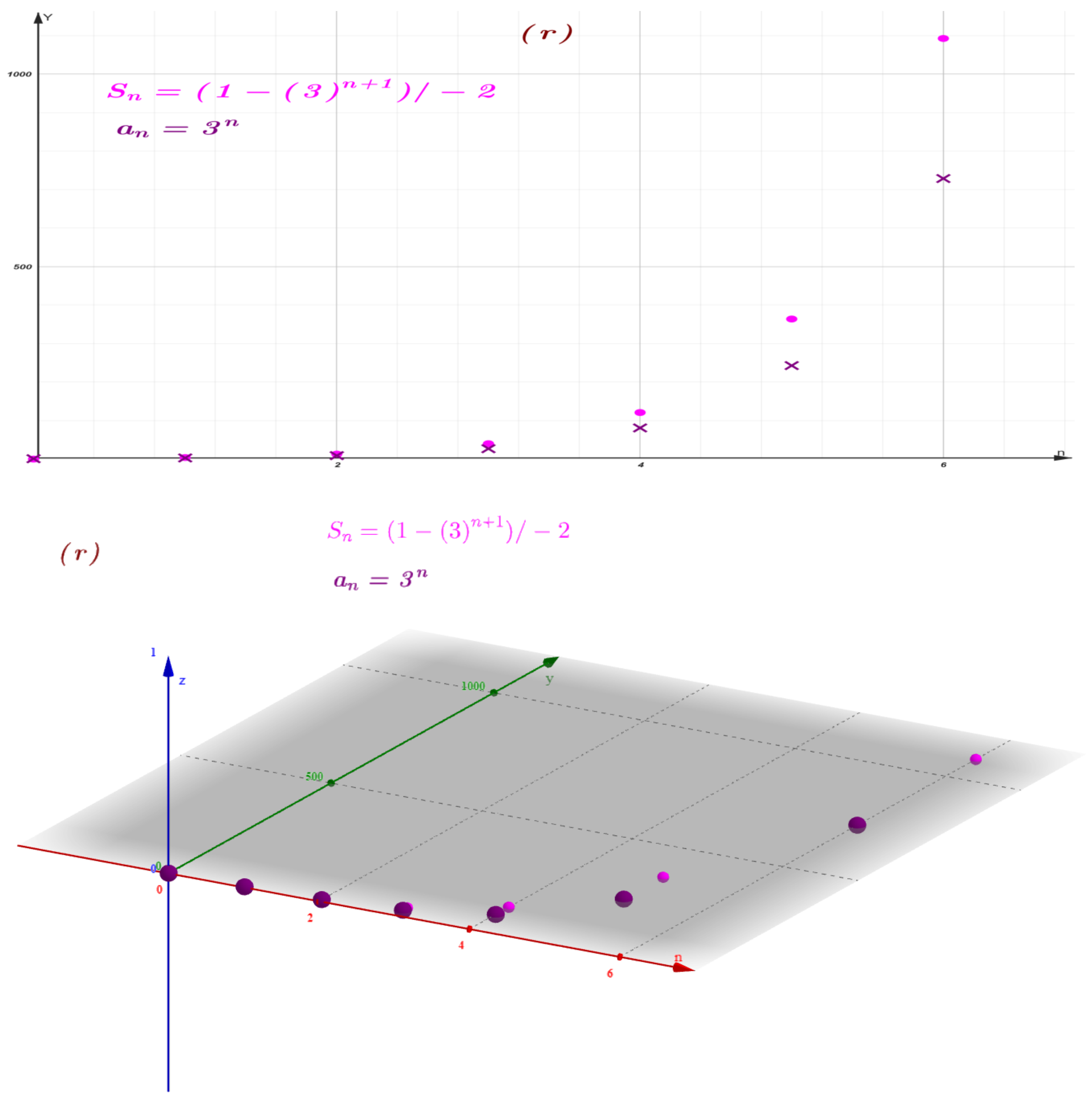


Figure R

2-2 p-Series

Another important type of series is the ***p-series***.

Definition 08: Monotonic sequences

1. A p-series is a series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ , where } p > 0 .$$

2. A **general** p-series is a series of the form

$$\sum_{n=1}^{\infty} \frac{1}{(an+b)^p} \text{ , where } p > 0 \text{ and } p > 0 \text{ and } a , b \text{ are real numbers.}$$

Theorem 08: Convergence of general p-series

a general p-series $\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$ will converge if, and only if. $p > 1$.

Example 10: Determining convergence of series

Determine the convergence of the following series.

1. $\sum_{n=1}^{\infty} \frac{1}{n}$
2. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$
3. $\sum_{n=11}^{\infty} \frac{1}{(\frac{1}{2}n-5)^3}$
4. $\sum_{n=1}^{\infty} \frac{1}{2^n}$

Solution

1. this is a p-series with $p = 1$. By **Theorem 08** this series diverges.
This series is a famous series, called the *Harmonic Series*, so named because of its relationship to *harmonics* in the study of music and sound.

2. This is a p-series with $p = \frac{1}{2}$ the theorem states that it diverges.

3. This is a general p-series with $p = 3$, therefore it converges.

4. This is not a p-series, but a geometric series with $r = \frac{1}{2}$. It converges.

Example 11: Telescoping series

Evaluate the sum $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

Solution

It will help to write down some of the first few partial sums of this series.

$n = 1 \longrightarrow S_1 = \frac{1}{1} - \frac{1}{2}$

$= 1 - \frac{1}{2}$

$n = 2 \longrightarrow S_2 = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right)$

$= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \frac{1}{3}$

$= 1 - \frac{1}{3}$

$n = 3 \longrightarrow S_3 = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right)$

$= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \frac{1}{4}$

$= 1 - \frac{1}{4}$

$n = 4 \longrightarrow S_4 = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right)$

$= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \frac{1}{5}$

$= 1 - \frac{1}{5}$

$n = 5 \longrightarrow S_5 = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right)$

$= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{5}} - \frac{1}{6}$

$= 1 - \frac{1}{6}$

$n = 6 \longrightarrow S_6 = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{7}\right)$

$= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{5}} - \cancel{\frac{1}{6}} + \cancel{\frac{1}{6}} - \frac{1}{7}$

$= 1 - \frac{1}{7}$

Note how most of the terms in each partial sum are cancelled out! In general, we see that $S_n = 1 - \frac{1}{n+1}$. The sequence $\{S_n\}$ converges, as $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$ and so we conclude that $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 1$. Partial sums of the series are plotted in **Figure S**

Graphical

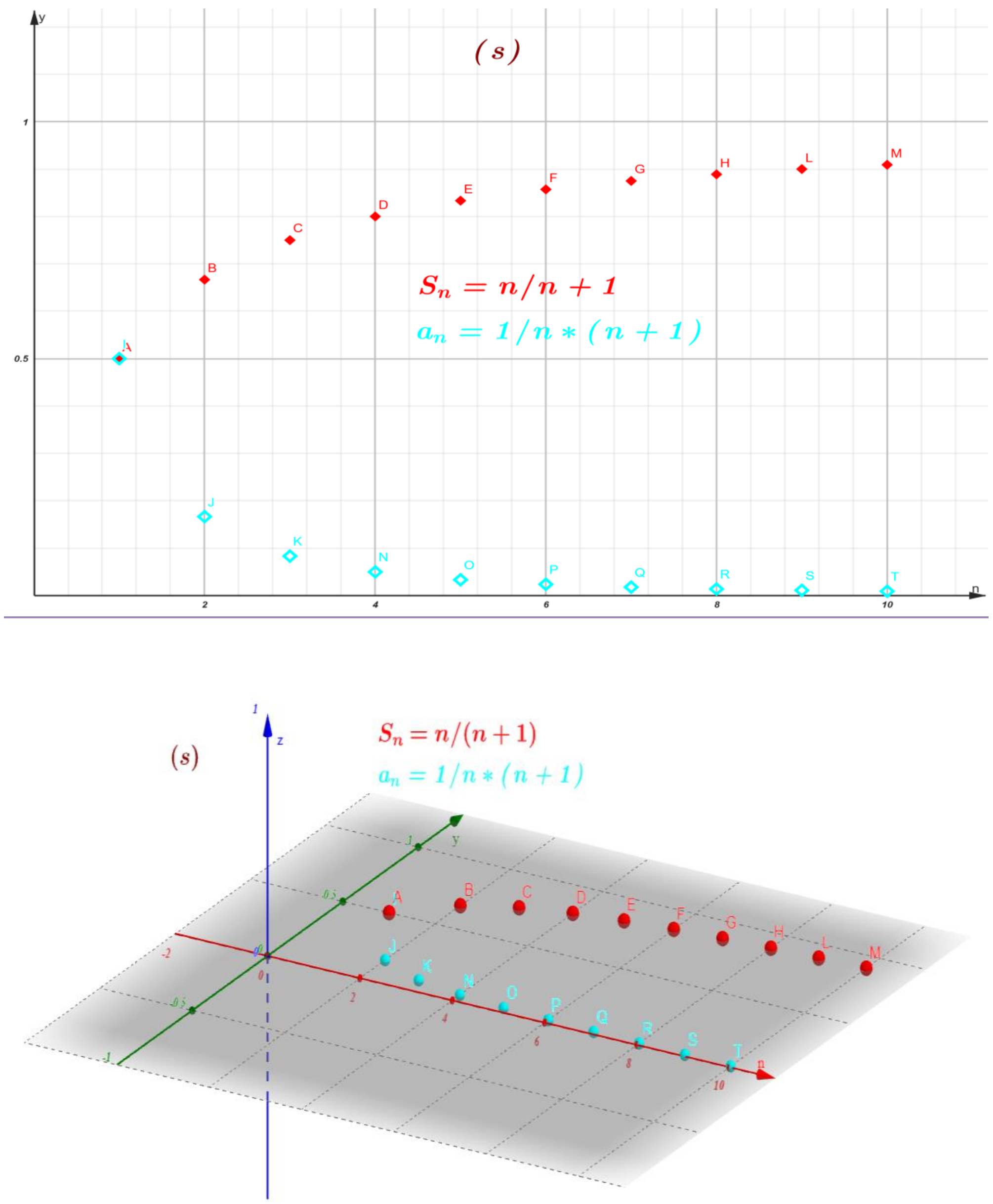


Figure S

Example 12: Evaluating series

evaluate each of the following infinite series.

1. $\sum_{n=1}^{\infty} \frac{2}{n^2+2n}$ **2.** $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

Solution

1. $\sum_{n=1}^{\infty} \frac{2}{n^2+2n}$

1. We can decompose the fraction $2/(n^2+2n)$ as

$$\frac{2}{n^2+2n} = \frac{2}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

$$= \frac{A(n+2)+Bn}{n(n+2)}$$

$$= \frac{An+2A+Bn}{n(n+2)}$$

$$= \frac{n(A+B)+2A}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$$

$$\begin{cases} 2A = 2 \\ (A+B) = 0 \end{cases} \longrightarrow \begin{cases} A = 1 \\ 1 + B = 0\,, B = -1 \end{cases}$$

Expressing the terms of $\{S_n\}$ is now more instructive:

$$n = 1 \longrightarrow S_1 = 1 - \frac{1}{3}$$

$$= 1 - \frac{1}{3}$$

$$n = 2 \longrightarrow S_2 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= 1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4}$$

$$n = 3 \longrightarrow S_3 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right)$$

$$= 1 + \frac{1}{2} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \frac{1}{4} - \frac{1}{5}$$

$$= 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5}$$

$$n = 4 \longrightarrow S_4 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right)$$

$$= 1 + \frac{1}{2} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \frac{1}{5} - \frac{1}{6}$$

$$= 1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6}$$

$\mathbf{n = 5}$ ⟶ $S_5 = \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)$

$$= 1+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\frac{1}{4}-\frac{1}{5}+\frac{1}{5}-\frac{1}{6}-\frac{1}{7}$$

$$= 1+\frac{1}{2}-\frac{1}{6}-\frac{1}{7}$$

$\mathbf{n = 6}$ ⟶ $S_6 = \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{6}-\frac{1}{8}\right)$

$$= 1+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\frac{1}{4}-\frac{1}{5}+\frac{1}{5}-\frac{1}{6}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}$$

$$= 1+\frac{1}{2}-\frac{1}{7}-\frac{1}{8}$$

We again have a telescoping series. In each partial sum, most of the terms cancel and we obtain the formula $\{S_n\}=\left\{1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right\}$ taking limits allows us to determine the convergence of the series:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right) = \lim_{n\to\infty}\left(1+\frac{1}{2}-\frac{1}{\infty}-\frac{1}{\infty}\right)$$

$$= 1+\frac{1}{2}-0-0$$

$$= \frac{1\times 2}{1\times 2}+\frac{1}{2}$$

$$= \frac{2}{2}+\frac{1}{2}$$

$$= \frac{3}{2}$$

So, $\sum_{n=1}^{\infty}\frac{2}{n^2+2n} = \frac{3}{2}$, This is illustrated in **Figure T**

2. $\sum_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right)$

2. We begin by writing the first few partial sums of the series:

$n = 1$ ⟶ $S_1 = \ln(2)$

$n = 2$ ⟶ $S_2 = \ln(2) + \ln\left(\frac{3}{2}\right)$

$$= \ln\left(2\cdot\frac{3}{2}\right)$$

$$= \ln(3)$$

$n = 3$ ⟶ $S_3 = \ln(2) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right)$

$$= \ln\left(2\cdot\frac{3}{2}\cdot\frac{4}{3}\right)$$

$$= \ln(4)$$

$n = 4$ $\longrightarrow$ $S_4 = \ln(2) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right) + \ln\left(\frac{5}{4}\right)$

$$= \ln\left(2\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\right)$$

$$= \ln(5)$$

$n = 5$ $\longrightarrow$ $S_5 = \ln(2) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right) + \ln\left(\frac{5}{4}\right) + \ln\left(\frac{6}{5}\right)$

$$= \ln\left(2\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdot\frac{6}{5}\right)$$

$$= \ln(6)$$

$n = 6$ $\longrightarrow$ $S_6 = \ln(2) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right) + \ln\left(\frac{5}{4}\right) + \ln\left(\frac{6}{5}\right) + \ln\left(\frac{7}{6}\right)$

$$= \ln\left(2\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdot\frac{6}{5}\cdot\frac{7}{6}\right)$$

$$= \ln(7)$$

At first, this does not seem helpful, but recall the logarithmic identity: $\ln x + \ln y = \ln(xy)$. Applying this to S_6 gives:

$$S_6 = \ln(2) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right) + \ln\left(\frac{5}{4}\right) + \ln\left(\frac{6}{5}\right) + \ln\left(\frac{7}{6}\right) = \ln(7)$$

We can conclude that $\{S_n\}=\{\ln(n+1)\}$. This sequence does not converge, as $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln(n+1) = \infty$. Therefore $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \infty$

The series diverges. Note in **Figure U**

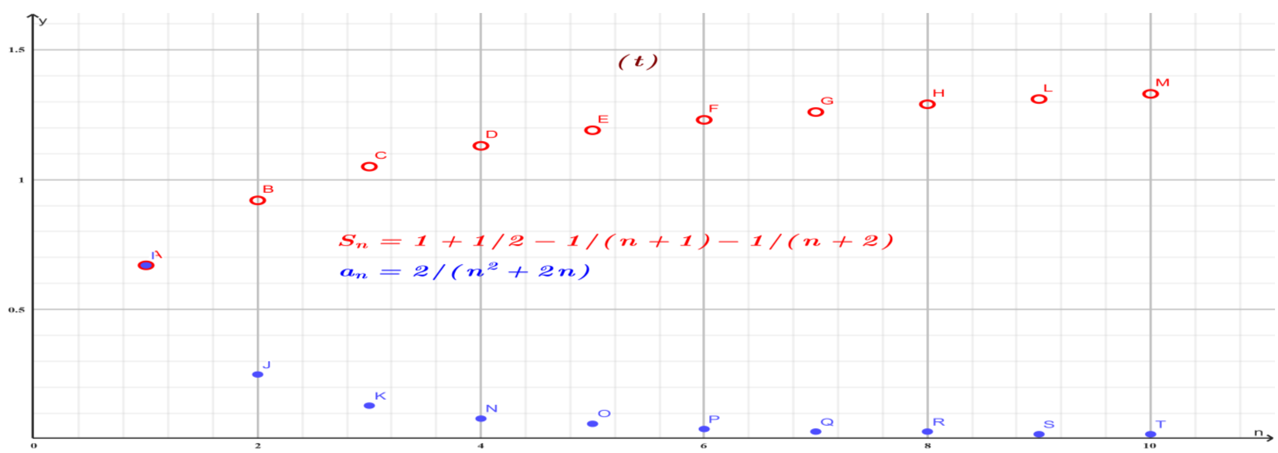


Figure T

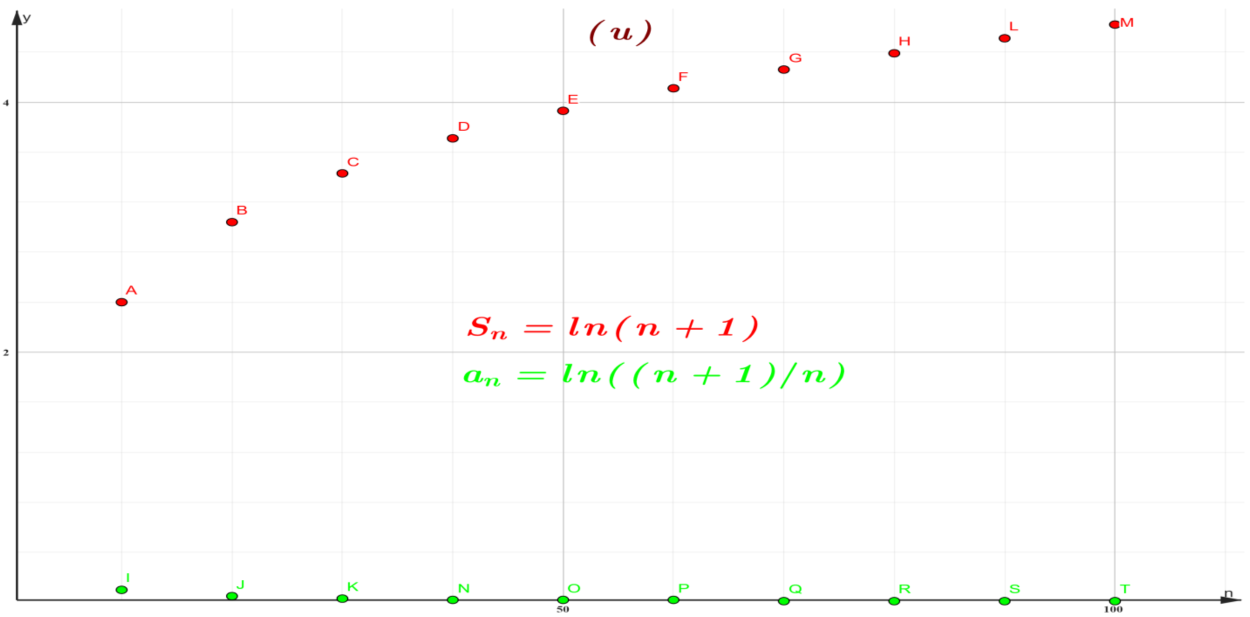


Figure U

Theorem 09: Properties of infinite series

let $\sum_{n=1}^{\infty} a_n = L$, $\sum_{n=1}^{\infty} b_n = k$ *, and let* c *be* a *constant.*

1. Constant Multiple Rule : $\sum_{n=1}^{\infty} c \cdot a_n = c \cdot \sum_{n=1}^{\infty} a_n = c \cdot L$.

2. Sum/Difference Rule : $\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$= L±$K$.

Key Idea 01: Important series

1. $\sum_{n=0}^{\infty} \frac{1}{n!} = e$ (Note that the index starts with $n = 0$)

2. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

3. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

4. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$

5. $\sum_{n=1}^{\infty} \frac{1}{n}$ Diverges (This is called the Harmonic series)

6. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2)$ (This is called the Alternating Harmonic Series)

Example 13: Evaluating series

Evaluate the given series.

1. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n^2-n)}{n^3}$ **2.** $\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\frac{1}{49}+\cdots$

Solution

1. We start by using algebra to break the series apart:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n^2-n)}{n^3}=\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}n^2-(-1)^{n+1}n}{n^3}\right)$$

$$=\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}n^2}{n^3}-\frac{(-1)^{n+1}n}{n^3}\right)$$

$$=\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}\cancel{n^2}}{\cancel{n^3}}-\frac{(-1)^{n+1}\cancel{n}}{\cancel{n^3}}\right)$$

$$=\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n}-\frac{(-1)^{n+1}}{n^2}\right)$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$$

$$=\ln(2)-\frac{\pi^2}{12}$$

$$\approx -0.1293$$

2. The denominators in each term are perfect squares; we are adding $\sum_{n=4}^{\infty}\frac{1}{n^2}$ (note we start with $n=4$, not $=1$). This series will converge. Using the formula from **Key Idea 01**, we have the following:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\sum_{n=1}^{3}\frac{1}{n^2}+\sum_{n=4}^{\infty}\frac{1}{n^2}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=1}^{3}\frac{1}{n^2}=\sum_{n=4}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{6}-\left(\frac{1}{1}+\frac{1}{4}+\frac{1}{9}\right)=\sum_{n=4}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{6}-\left(\frac{1\times4\times9}{1\times4\times9}+\frac{1\times9}{4\times9}+\frac{1\times4}{9\times4}\right)=\sum_{n=4}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{6}-\left(\frac{36}{36}+\frac{9}{36}+\frac{4}{36}\right)=\sum_{n=4}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{6}-\frac{49}{36}=\sum_{n=4}^{\infty}\frac{1}{n^2}$$

$$0,2838\approx\sum_{n=4}^{\infty}\frac{1}{n^2}$$

Theorem 10: n^{th} term test for convergence/divergence

Consider the series $\sum_{n=1}^{\infty} a_n$.

1. If $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

2. If $\lim_{n\to\infty} a_n \neq 0$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges*

Important! This theorem *does not state* that if $\lim_{n\to\infty} a_n = 0$ *then* $\sum_{n=1}^{\infty} a_n$ *converges*. The standard example of this is the Harmonic Series, as given in **Key Idea 01**. The Harmonic Sequence $\left\{\frac{1}{n}\right\}$ converges to **0** the Harmonic Series, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Theorem 11: Infinite nature of series

The convergence or divergence remains unchanged by the addition or subtraction of any finite number of terms. That is:

1. A divergent series will remain divergent with the addition or subtraction of any finite number of terms.

2. A convergent series will remain convergent with the addition or subtraction of any finite number of terms. (Of course, the *sum* will likely change.)

Consider once more the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges; that is, the sequence of partial sums $\{S_n\}$ grows (very, very slowly) without bound. One might think that by removing the "**large**" terms of the sequence that perhaps the series will converge.

This is simply not the case. For instance, the sum of the first **10 million** terms of the Harmonic Series is about **16,7**. Removing the first **10 million** terms from the Harmonic Series changes the n^{th} partial sums, effectively subtracting **16,7** from the sum. However, a sequence that is growing without bound will still grow without bound when **16,7** is subtracted from it.

The equations below illustrate this. The first line shows the infinite sum of the Harmonic Series split into the sum of the first **10 million** terms plus the sum of "**everything else.**" The next equation shows us subtracting these first **10 million** terms from both sides. The final equation employs a bit of "**psuedo--math**": subtracting **16,7** from "**infinity**" still leaves one with "**infinity.**"

$$\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{10.000.000} \frac{1}{n} + \sum_{n=10.000.001}^{\infty} \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{10.000.000} \frac{1}{n} = \sum_{n=10.000.001}^{\infty} \frac{1}{n}$$

$$\infty - 16.7 = \infty$$

This section introduced us to series and defined a few special types of series whose convergence properties are well known: we know when a **p-series** or a geometric series converges or diverges. Most series that we encounter are not one of these types, but we are still interested in knowing whether or not they converge. The next three sections introduce tests that help us determine whether or not a given series converges.

3-Integral and comparison tests

knowing whether or not a series converges is very important, especially when we discusses Power Series. **Theorems 07** and **08** give criteria for when Geometric and -series converge, and **Theorem 10** gives a quick test to determine if a series diverges. There are many important series whose convergence cannot be determined by these theorems, though, so we introduce a set of tests that allow us to handle a broad range of series. We start with the Integral Test.

3-1 Integral test

We stated in Section **(sequences)** that a sequence $\{a_n\}$ is a function $a(n)$ whose domain is $\mathbb{N}$, the set of natural numbers. If we can extend $a(n)$ to $\mathbb{R}$, the real numbers, and it is both positive and decreasing on $[1, \infty)$, then the convergence of $\sum_{n=1}^{\infty} a_n$ is the same as $\int_1^{\infty} a(x)dx$.

Theorem 12: Integral test

let a sequence $\{a_n\}$ be defined by $a_n = a(n)$, where $a(n)$ is continuous, positive and decreasing on $[1, \infty)$. Then $\sum_{n=1}^{\infty} a_n$
converges, if, and only if , $\int_1^{\infty} a(x)dx$ converges.

We can demonstrate the truth of the Integral Test with two simple graphs. In **Figure x** , the height of each rectangle is $a_n = a(n)$ for $n = 1, 2, \dots$, , and clearly the rectangles enclose more area than the area under $y = a(x)$. Therefore we can conclude that

$$\int_1^{\infty} a(x)dx < \sum_{n=1}^{\infty} a_n \quad \text{.............................. (1)}$$

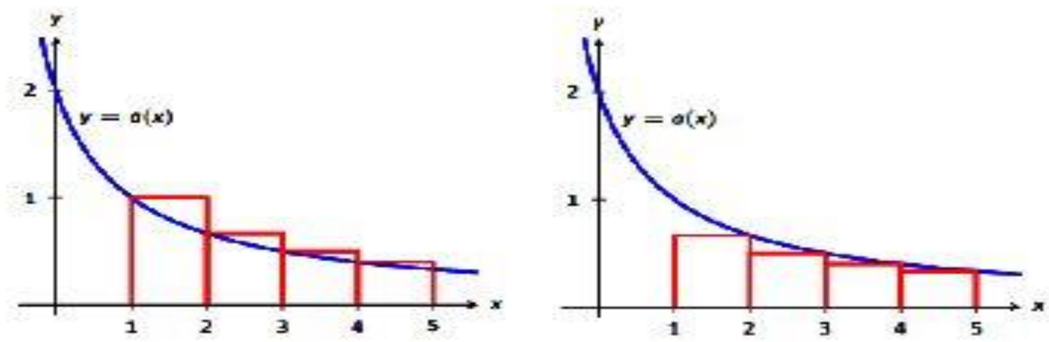


Figure x

In **Figure x**, we draw rectangles under $\boldsymbol{y} = \boldsymbol{a}(\boldsymbol{x})$ with the Right-Hand rule, starting with $\boldsymbol{n} = \mathbf{2}$
This time, the area of the rectangles is less than the area under $= \boldsymbol{a}(\boldsymbol{x})$,
so $\sum_{n=2}^{\infty} a_n < \int_1^{\infty} a(x)dx$. Note how this summation starts with $= \mathbf{2}$; adding $\boldsymbol{a}_1$ to
Both sides lets us rewrite the summation starting with $= \mathbf{1}$:

$$\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} a(x)dx \quad \text{.................................. (2)}$$

Combining Equations **(1)** And **(2)**, we have

$$\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} a(x)dx < a_1 + \sum_{n=1}^{\infty} a_n \quad \text{.................................. (3)}$$

Theorem 13:

From Equation **03** we can make the following two statements:

1. If $\sum_{n=1}^{\infty} a_n$ *diverges , so does* $\int_1^{\infty} a(x)dx$
(because $\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} a(x)dx$ *)*

2. If $\sum_{n=1}^{\infty} a_n$ *converges, so does* $\int_1^{\infty} a(x)dx$
(because $\int_1^{\infty} a(x)dx < \sum_{n=1}^{\infty} a_n$ *.)*

Therefore the series and integral either both converge or both diverge.

Example 14: Using the integral test

Determine the convergence of $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ (The terms of the sequence $\{a_n\}$=$\{\ln n/n^2\}$ and the n^{th} partial sums are given in **Figure Z.**

Solution

Figure Z implies that $a(n) = (\ln n)/n^2$ is positive and decreasing on $[2, \infty)$. We can determine this analytically, too. We know $a(n)$ is positive as both $\ln n$ and n^2 are positive on $2, \infty)$. To determine that $a(n)$ is decreasing, consider $a'(n) = (1 - 2\ln n)/n^3$, which is negative for ≥ 2 . Since $a'(n)$ is negative, $a(n)$is decreasing.

$$a(n) = (\ln n)/n^2 \longrightarrow a'(n) = \frac{\frac{1}{n} \times n^2 - 2n \times \ln n}{(n^2)^2}$$

$$= \frac{n \times (1 - 2\ln n)}{n^4}$$

$$= \frac{(1 - 2\ln n)}{n^3}$$

$$= (1 - 2\ln n)/n^3$$

Graphical

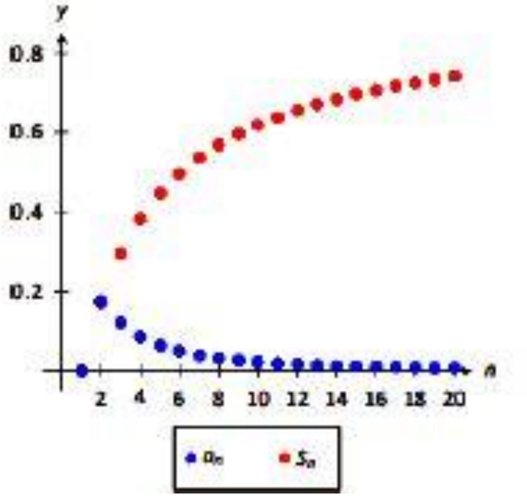


Figure Z

Applying the Integral Test, we test the convergence of $\int_1^{\infty} \frac{\ln x}{x^2}\, dx$. Integrating this improper integral requires the use of Integration by Parts.

$$u = \ln x \quad \longrightarrow \quad v' = \frac{1}{x^2}$$

$$u' = \frac{1}{x} \quad \longrightarrow \quad v = \int_a^b \frac{1}{x^2}\, dx$$

$$= \int_a^b x^{-2}\, dx$$

$$= \left[\frac{x^{-2+1}}{-2+1}\right]_a^b$$

$$= \left[\frac{x^{-1}}{-1}\right]_a^b$$

$$= \left[\frac{-1}{x}\right]_a^b$$

Note : $\int_a^b u \cdot v'\, dx = u \cdot v - \int_a^b v \cdot u'\, dx$

$$\int_1^{\infty} \frac{\ln x}{x^2}\, dx = \lim_{b\to\infty} \left[\ln x \cdot \left(\frac{-1}{x}\right) - \int_1^b \left(\frac{-1}{x}\right) \cdot \left(\frac{1}{x}\right) dx\right]$$

$$= \lim_{b\to\infty} \left[\left(\frac{-\ln x}{x}\right)_1^b + \int_1^b \left(\frac{1}{x^2}\right) dx\right]$$

$$= \lim_{b\to\infty} \left[\left(\frac{-\ln x}{x}\right)_1^b + \left(-\frac{1}{x}\right)_a^b\right]$$

$$= \lim_{b\to\infty}\left[\ \left(\frac{-\ln x}{x}\right)_1^b - \left(\frac{1}{x}\right)_a^b\right]$$

$$= \lim_{b\to\infty}\left[\ \left(\frac{-\ln b}{b} + \frac{\ln 1}{1}\right) - \left(\frac{1}{b} - \frac{1}{1}\right)\right]$$

$$= \lim_{b\to\infty}[\ (0 + \ 0) - (0 - \ 1)] = 1$$

Since $\int_1^\infty \frac{\ln x}{x^2}\, dx$ converges, so does $\sum_{n=1}^\infty \frac{\ln n}{n^2}$

Key Idea 02: Convergence of improper integrals $\int_1^\infty \frac{1}{x^p}\, dx$ and $\int_0^1 \frac{1}{x^p}\, dx$.

1. The improper integral $\int_1^\infty \frac{1}{x^p}\, dx$ converges when $p > 1$ and diverges when $p \le 1$.

2. The improper integral $\int_0^1 \frac{1}{x^p}\, dx$ converges when $p < 1$ and diverges when $p \ge 1$.

Example 15: Using the integral test to establish theorem 08

Use the Integral Test to prove that $\sum_{n=1}^\infty \frac{1}{(an+b)^p}$ converges if, and only if, > 1 .

Solution

Consider the integral $\int_1^\infty \frac{1}{(an+b)^p}\, dx$; assuming $p \ne 1$,

$$\int_1^\infty \frac{1}{(an+b)^p}\, dx \ = \lim_{c\to\infty} \int_1^c \frac{1}{(an+b)^p}\, dx$$

$$= \lim_{c\to\infty} \int_1^c (an+b)^{-p}\ dx$$

$$= \lim_{c\to\infty}\left[\frac{1}{a(-p+1)} \times (an+b)^{-p+1}\ \right]_1^c$$

$$= \lim_{c\to\infty}\left[\frac{1}{a(1-p)} \times (an+b)^{1-p}\ \right]_1^c$$

$$= \lim_{c\to\infty} \frac{1}{a(1-p)} \times \left[(ac+b)^{1-p} - (a+b)^{1-p}\right]$$

This limit converges if, and only if, > 1 . It is easy to show that the integral also diverges in the case of $= \mathbf{1}$. (This result is similar to the work preceding **Key Idea 02**.)

Therefore $\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$ converges if, and only if, $\mathbf{p > 1}$.

3-2 Direct comparison test

Theorem 14: Direct comparison test

Let $\{a_n\}$ and $\{b_n\}$ be positive sequences where $a_n \leq b_n$ for all $\geq \mathbf{N}$, for some $\mathbf{N} \geq \mathbf{1}$.
1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Example 16: Applying the direct comparison test
Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{3^n+n^2}$

Solution

This series is neither a geometric or **p-series**, but seems related. We predict it will converge, so we look for a series with larger terms that converges. (Note too that the Integral Test seems difficult to apply here.)

Since $\mathbf{3^n} < \mathbf{3^2} + \mathbf{n^2}$, $\frac{1}{3^n} > \frac{1}{3^n+n^2}$ for all $\mathbf{n} \geq \mathbf{1}$. The series $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is a convergent geometric series; by **Theorem 13**, $\sum_{n=1}^{\infty} \frac{1}{3^n+n^2}$ converges.

Example 17: Applying the direct comparison test

Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{n-\ln n}$.

Solution

we know the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and it seems that the given series is closely related to it, hence we predict it will diverge.

Since $n \geq n - \ln n$ for all $\geq \mathbf{1}$, $\frac{1}{n} \leq \frac{1}{n-\ln n}$ for all $n \geq \mathbf{1}$.
The Harmonic Series diverges, so we conclude that $\sum_{n=1}^{\infty} \frac{1}{n-\ln n}$ diverges as well.

3-3 Large limit comparison test

Theorem 15: Limit comparison test

Let $\{a_n\}$ and $\{b_n\}$ be positive sequences.

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, where L is a positive real number, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then if $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$

3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then if $\sum_{n=1}^{\infty} b_n$ diverges, then so does $\sum_{n=1}^{\infty} a_n$

Key Idea 03: L'Hopital's Rule

1. List the different indeterminate forms described in this section.

2. T/F: l'Hopital's Rule provides a faster method of computing derivatives.

3. T/F: l'Hopitals Rule states that $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)}{g'(x)}$.

4. Explain what the indeterminate form "1^{∞}" means.

5. Fill in the blanks" The Quotient Rule is applied to $\frac{f(x)}{g(x)}$ when taking l'Hopital's Rule is applied when taking
certain.

6. Create (but do not evaluate) a limit that returns "∞^0".

7. Create a function $f(x)$ such that $\lim_{x\to 1} f(x)$ returns "0^{∞}".

Example 18: Applying the limit comparison test

Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{n+\ln n}$ using the Limit Comparison Test.

Solution

we compare the terms of $\sum_{n=1}^{\infty} \frac{1}{n+\ln n}$ to the terms of the Harmonic Sequence $\sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n+\ln n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{n+\ln n} \times n$$

$$= \lim_{n\to\infty} \frac{n}{n+\ln n}$$

$$= \lim_{n\to\infty} \frac{(n)'}{(n+\ln n)'}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{n+1}{n}}$$

$$= \lim_{n\to\infty} 1 \times \frac{n}{n+1}$$

$$= \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \lim_{n\to\infty} \frac{n}{n}$$

$$= 1$$

(after applying L'Hopital's Rule) .

Since the Harmonic Series diverges, we conclude that $\sum_{n=1}^{\infty} \frac{1}{n+\ln n}$ diverges as well.

Example 19: Applying the limit comparison test

Determine the convergence of $\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{n^2-n+1}$

Solution

We naively attempt to apply the rule of thumb given above and note that the dominant term in the expression of the series is $\frac{1}{n^2}$. Knowing that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we attempt to apply the Limit Comparison Test:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n}+3}{n^2-n+1}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}+3}{n^2-n+1} \times n^2$$

$$= \lim_{n\to\infty} \frac{(\sqrt{n}+3)\cdot n^2}{n^2-n+1}$$

$$= \lim_{n\to\infty} \frac{(n^2\sqrt{n}+3\cdot n^2)'}{(n^2-n+1)'}$$

$$= \lim_{n\to\infty} \frac{\frac{4n^2+n^2+12n\sqrt{n}}{2\sqrt{n}}}{2n-1}$$

$$= \lim_{n\to\infty} \frac{5n^2+12n\sqrt{n}}{2\sqrt{n}} \times \frac{1}{2n-1}$$

$$= \lim_{n\to\infty} \frac{5n^2+12n\sqrt{n}}{2\sqrt{n}\cdot(2n-1)}$$

$$= \lim_{n\to\infty} \frac{(5n^2+12n\sqrt{n})'}{(4n\sqrt{n}-2\sqrt{n})'}$$

$$= \lim_{n\to\infty} \frac{(5n^2+12n\sqrt{n})'}{\left(4n\cdot(n)^{\frac{1}{2}}-2(n)^{\frac{1}{2}}\right)'}$$

$$= \lim_{n\to\infty} \frac{\left(10n+12\sqrt{n}+\frac{6n}{\sqrt{n}}\right)}{\left(6\cdot(n)^{\frac{1}{2}}-(n)^{\frac{-1}{2}}\right)}$$

$$= \lim_{n\to\infty} \frac{\left(10n+12\sqrt{n}+\frac{6n}{\sqrt{n}}\right)'}{\left(6\cdot(n)^{\frac{1}{2}}-(n)^{\frac{-1}{2}}\right)'}$$

$$= \lim_{n\to\infty} \frac{\left(10+\frac{6}{\sqrt{n}}+\frac{8n}{4n\cdot\sqrt{n}}\right)}{\left(3\cdot(n)^{\frac{-1}{2}}+(n)^{\frac{-3}{2}}\right)}$$

$$= \lim_{n\to\infty} \frac{\left(10+\frac{8}{\sqrt{n}}\right)}{\left(3\cdot(n)^{\frac{-1}{2}}+\frac{1}{2}\cdot(n)^{\frac{-3}{2}}\right)}$$

$$= \lim_{n\to\infty} \frac{\left(10+\frac{8}{\sqrt{n}}\right)}{\left(\frac{3}{\sqrt{n}}+\frac{1}{2\cdot\sqrt[3]{n^2}}\right)}$$

$$= \lim_{n\to\infty} \frac{10}{0}$$

$$= \infty$$

(Apply L'Hopital's Rule).

We conclude that $\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{n^2-n+1}$ diverges as well.

4-Ratio and Root Tests

The n^{th}-Term Test of **Theorem 15** states that in order for a series $\sum_{n=1}^{\infty} a_n$ to converge, $\lim_{n\to\infty} a_n = 0$. That is, the terms of $\{a_n\}$ must get very small. Not only must the terms approach **0**, they must approach **0** "**fast enough**": while $\lim_{n\to\infty} \frac{1}{n} = 0$, the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n} = 0$ diverges as the terms of $\left\{\frac{1}{n}\right\}$ do not approach **0** "**fast enough.**"

The comparison tests of the previous section determine convergence by comparing terms of a series to terms of another series whose convergence is known. This section introduces the Ratio and Root Tests, which determine convergence by analyzing the terms of a series to see if they approach **0** "**fast enough.**"

4-1 Ratio test

Theorem 16: Ratio test

Les $\{a_n\}$ be a positive sequence where $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$.

1. If < 1 , then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $L > 1$ or $L = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $= 1$, the Ratio Test is inconclusive.

Example 20: Applying the ratio test

Use the Ratio Test to determine the convergence of the following series:

1 . $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

2. $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

3. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

4. $\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!}$

Solution

1 . $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}$$

$$= \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \times \frac{n!}{2^n}$$

$$= \lim_{n\to\infty} \frac{2^n \cdot 2^1}{(n+1)!} \times \frac{n!}{2^n}$$

$$= \lim_{n\to\infty} \frac{2^1 \cdot n!}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{2 \cdot n!}{(n+1) \cdot n!}$$

$$= \lim_{n\to\infty} \frac{2}{(n+1)}$$

$$= \lim_{n\to\infty} \frac{2}{n}$$

$$= 0 < 1$$

Since the limit is $0 < 1$, by the Ratio Test $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges.

2. $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{3^{n+1}}{(n+1)^3}}{\frac{3^n}{n^3}}$$

$$= \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)^3} \times \frac{n^3}{3^n}$$

$$= \lim_{n\to\infty} \frac{3^n \cdot 3^1}{(n+1)^3} \times \frac{n^3}{3^n}$$

$$= \lim_{n\to\infty} \frac{3n^3}{(n+1)^3}$$

$$= \lim_{n\to\infty} \frac{3n^3}{n^3 + 3n^2 + 3n + 1}$$

$$= \lim_{n\to\infty} \frac{3n^3}{n^3}$$

$$= 3 > 1$$

Since the limit is > 1 , by the Ratio Test $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$ diverges.

3. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{((n+1)^2+1)}}{\frac{1}{n^2+1}}$$

$$= \lim_{n\to\infty} \frac{1}{((n+1)^2+1)} \times \frac{(n^2+1)}{1}$$
$$= \lim_{n\to\infty} \frac{n^2+1}{((n+1)^2+1)}$$
$$= \lim_{n\to\infty} \frac{n^2+1}{n^2+2n+2}$$
$$= \lim_{n\to\infty} \frac{n^2}{n^2}$$
$$= 1$$

Since the limit is **1**, the **Ratio Test** is inconclusive. We can easily show this series converges using the Direct or Limit Comparison Tests, with each comparing to the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

4. $\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!}$

Before we begin, be sure to note the difference between $(2n)!$ and ! . When, $n = 4$ the former is ! = $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40320$, whereas the latter is $2(4 \cdot 3 \cdot 2 \cdot 1) = 48$.
Applying the Ratio Test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!\,(n+1)!}{(2(n+1))!}}{\frac{n!\,n!}{(2n)!}}$$

$$= \lim_{n\to\infty} \frac{(n+1)!(n+1)!}{(2(n+1))!} \times \frac{(2n)!}{n!n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)\cdot n!(n+1)\cdot n!}{(2n+2)!} \times \frac{(2n)!}{n!n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)\cdot(n+1)\cdot(2n)!}{(2n+2)!}$$

$$= \lim_{n\to\infty} \frac{(n+1)\cdot(n+1)\cdot(2n)!}{(2n+2)\cdot(2n+1)\cdot(2n)!}$$

$$= \lim_{n\to\infty} \frac{(n+1)\cdot(n+1)}{2(n+1)\cdot(2n+1)}$$

$$= \lim_{n\to\infty} \frac{(n+1)}{2\cdot(2n+1)}$$

$$= \lim_{n\to\infty} \frac{n+1}{4n+2}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}}{4\cancel{n}}$$

$$= \frac{1}{4} < 1$$

Since the limit is $\frac{1}{4} < 1$, by the Ratio Test we conclude $\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!}$ converges.

4-2 Root Test

The final test we introduce is the Root Test, which works particularly well on series where each term is raised to a power, and does not work well with terms containing factorials.

Theorem 17: Root test

let $\{a_n\}$ be a positive sequence. And let $\lim_{n\to\infty} (a_n)^{1/n} = L$

1. If < 1 , then $\sum_{n=1}^{\infty} a_n$ converges.
2. If > 1 or $L = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $= 1$, the Root Test is inconclusive.

Example 21: Applying the root test

Determine the convergence of the following series using the Root Test:

1. $\sum_{n=1}^{\infty} \left(\frac{3n+1}{5n-2}\right)^n$ **2.** $\sum_{n=1}^{\infty} \frac{n^4}{(\ln n)^n}$

Solution

1. $\sum_{n=1}^{\infty} \left(\frac{3n+1}{5n-2}\right)^n$

$$\lim_{n\to\infty} \left(\left(\frac{3n+1}{5n-2}\right)^n\right)^{1/n} = \lim_{n\to\infty} \frac{3n+1}{5n-2} = \lim_{n\to\infty} \frac{3n}{5n} = \frac{3}{5}$$

Since the limit is less than **1**, we conclude the series converges. Note: it is difficult to apply the Ratio Test to this series.

2. $\sum_{n=1}^{\infty} \frac{n^4}{(\ln n)^n}$

$$\lim_{n\to\infty}\left(\frac{n^4}{(\ln n)^n}\right)^{1/n} = \lim_{n\to\infty}\frac{(n^{1/n})^4}{\ln n} = \lim_{n\to\infty}\frac{e^{(\ln n)^{4/n}}}{\ln n} = \lim_{n\to\infty}\frac{e^{\frac{4\cdot\ln n}{n}}}{\ln n} = \lim_{n\to\infty}\frac{e^{4\cdot\frac{\ln n}{n}}}{\ln n}$$

$$= \lim_{n\to\infty}\frac{e^{4\times 0}}{\ln n} = \lim_{n\to\infty}\frac{e^{0}}{\ln n} = \lim_{n\to\infty}\frac{1}{\ln n} = \frac{1}{\infty} = 0$$

*As grows, the numerator approaches 1 **(apply L'Hopital's Rule)** and the denominator grows to infinity.*

Because: $\left(\lim_{n\to\infty}\frac{\ln n}{n} = 0\right)$; $(a^{b/a})^n = a^{\frac{b\cdot n}{a}} = e^{(\ln a)^{b\cdot n/a}} = e^{\frac{b\cdot n}{a}\cdot\ln a}$

5-Alternating series and absolute convergence

All of the series convergence tests we have used require that the underlying sequence $\{a_n\}$ be a positive sequence. (We can relax this with **Theorem 11** and state that there must be an $N > 0$ such that $a_n > 0$ for all $> N$; that is, $\{a_n\}$ is positive for all but a finite number of values of n.)

In this section we explore series whose summation includes negative terms. We start with a very specific form of series, where the terms of the summation alternate between being positive and negative

Definition 09: Alternating series

let $\{a_n\}$ be a positive sequence. An **alternating series** is a series of either the form

$\sum_{n=1}^{\infty}(-1)^n a_n$ **or** $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$

Recall the terms of Harmonic Series come from the Harmonic Sequence $\{a_n\} = \left\{\frac{1}{n}\right\}$. An important alternating series is the **Alternating Harmonic Series**:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Geometric Series can also be alternating series when $r < 0$. For instance, if $r = -1/2$, the geometric series is

$$\sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots$$

Theorem 07 states that geometric series converge when $|r| < 1$ and gives the sum: $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$. When $r = -1/2$ as above, we find

$$\sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^{n} = \frac{1}{1-(-1/2)} = \frac{1}{3/2} = \frac{2}{3}$$

A powerful convergence theorem exists for other alternating series that meet a few conditions.

Theorem 18: Alternating series test

Let $\{a_n\}$ be a positive, decreasing sequence where $\lim_{n\to\infty} a_n = 0$. Then

$$\sum_{n=1}^{\infty}(-1)^n a_n \ \ and \ \sum_{n=1}^{\infty}(-1)^{n+1} a_n \ \ converge\,.$$

Example 22: Applying the alternating series test

Determine if the Alternating Series Test applies to each of the following series.

1. $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1}{n}$

2. $\sum_{n=1}^{\infty}(-1)^{n}\cdot\frac{\ln n}{n}$

Solution

1. This is the Alternating Harmonic Series as seen previously. The underlying sequence is $\{a_n\} = \{1/n\}$, which is positive, decreasing, and approaches 0 as $n \to \infty$. Therefore we can apply the Alternating Series Test and conclude this series converges.

While the test does not state what the series converges to, we will see later that $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1}{n} = \ln 2$.

2. The underlying sequence is $\{a_n\} = \{\ln n/n\}$. This is positive and approaches 0 as $n \to \infty$ **(use L'Hopital's Rule).**

However, the sequence is not decreasing for all . It is straightforward to compute $a_1 = 0$, $a_2 \approx 0.347$, $a_3 \approx 0.366$, $and\ a_4 \approx 0.347$: the sequence is increasing for at least the first 3 terms.

We do not immediately conclude that we cannot apply the Alternating Series Test. Rather, consider the long--term behaviour Of $\{a_n\}$. Treating $a_n = a(n)$ as a continuous function of n defined on $[1,\infty)$, we can take its derivative:

$$a'(n) = \frac{1-\ln n}{n^2}$$

The derivative is negative for all $n \geq 3$ (actually, for all $> e$), meaning $a_n = a(n)$ is decreasing on $[3,\infty)$. We can apply the Alternating Series Test to the series when we start with $n = 3$ and conclude that $\sum_{n=3}^{\infty}(-1)^{n}\cdot\frac{\ln n}{n}$ converges; adding the terms with $n = 1\ and\ n = 2$ do not change the convergence **(we apply Theorem 11).**

The important lesson here is that as before, if a series fails to meet the criteria of the Alternating Series Test on only a finite number of terms, we can still apply the test.

Theorem 19: The alternating series approximation theorem

let $\{a_n\}$ be a sequence that satisfies the hypotheses of the Alternating Series Test, and let S_n and L be the n^{th} partial sums and sum, respectively, of either $\sum_{n=1}^{\infty}(-1)^n a_n$ *or* $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ then

1. $|S_n - L| < a_{n+1}$, and
2. L is between S_n and S_{n+1}.

Part 1 of **Theorem 19** states that the partial sum of a convergent alternating series will be within a_{n+1} of its total sum. Consider the alternating series we looked at before the statement of the theorem, $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$. Since, $a_{14} = \frac{1}{14^2} \approx 0.0051$,we know that $S_{13} \approx 0.8252$ is within 0.0051 of the total sum.

Moreover, **Part 2** of the theorem states that since $S_{13} \approx 0.8252$ and $S_{14} \approx 0.8252$, we know the sum L lies between 0.8252 and $.8252$. One use of this is the knowledge that S_{14} is accurate to two places after the decimal.

Some alternating series converge slowly. In **Example 22** we determined the series $\sum_{n=1}^{\infty}(-1)^{n+1} \cdot \frac{\ln n}{n}$ converged. With $= 1001$, we find $\frac{\ln n}{n} \approx 0.0069$, meaning that $S_{1000} \approx 0.1633$ is accurate to one, maybe two, places after the decimal. Since $S_{1001} \approx 0.1564$, we know the sum L is $0.1564 \le L \le 0.1633$.

Example 23: Approximating the sum of convergent alternating series

Approximate the sum of the following series, accurate to within **0,001**.

$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^3}$

Solution

Using **Theorem 19**, we want to find n where $/n^3 < 0.001$:

$$\frac{1}{n^3} \le 0.001 = \frac{1}{1000}$$

$$n^3 \ge 1000$$

$$n \ge \sqrt[3]{1000}$$

$$n \ge 10.$$

Let L be the sum of this series. By **Part 01** of the theorem, $|S_9 - L| < a_{10} = 1/1000$. We can compute $S_9 = 0.902116$, which our theorem states is within 0.001 of the total sum.

We can use **Part 02** of the theorem to obtain an even more accurate result. As we know the $\mathbf{10^{th}}$ term of the series is $-\mathbf{1/1000}$, we can easily compute $\boldsymbol{S_{10}} = \mathbf{0.901116}$. **Part 02** of the theorem states that $\boldsymbol{L}$ is between $\boldsymbol{S_9}$ and $\boldsymbol{S_{10}}$, so $\mathbf{0.901116} < L < 0.902116$

Definition 10: Absolute and conditional convergence

1. A series $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if $\sum_{n=1}^{\infty} |a_n|$ converges.

2. A series $\sum_{n=1}^{\infty} a_n$ **converges conditionally** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Example 24: Determining absolute and conditional convergence

Determine if the following series converge absolutely, conditionally, or diverge.

1. $\sum_{n=1}^{\infty} (-1)^n \frac{n+3}{n^3+2n+5}$ *2.* $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+2n+5}{2^n}$

Solution

1. $\sum_{n=1}^{\infty} (-1)^n \frac{n+3}{n^3+2n+5}$

1. We can show the series

$$\sum_{n=1}^{\infty} \left|(-1)^n \frac{n+3}{n^3+2n+5}\right| = \sum_{n=1}^{\infty} \frac{n+3}{n^3+2n+5}$$

Diverges using the Limit Comparison Test, comparing with $\frac{1}{n}$.

The series $\sum_{n=1}^{\infty} (-1)^n \frac{n+3}{n^3+2n+5}$ converges using the Alternating Series Test; we conclude it converges conditionally.

2. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+2n+5}{2^n}$

2. We can show the series

$$\sum_{n=1}^{\infty} \left|(-1)^n \frac{n^2+2n+5}{2^n}\right| = \sum_{n=1}^{\infty} \frac{n^2+2n+5}{2^n}$$

converges using the Ratio Test.

Therefore we conclude $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+2n+5}{2^n}$ converges absolutely.

6-Power series

So far, our study of series has examined the question of "Is the sum of these infinite terms finite?," , "Does the series converge?" We now approach series from a different perspective: as a function. Given a value of $\boldsymbol{x}$, we evaluate $\boldsymbol{f(x)}$ by finding the sum of a particular series that depends on $\boldsymbol{x}$ (assuming the series converges). We start this new approach to series with a definition.

Definition 11: Power series

let $\{a_n\}$ be a sequence, let be a variable, and let be a real number.

1. The **power series in x** is the series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \cdots$$

2. The **power series in x** centered at c is the series

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c)^1 + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

Example 25: Examples of power series

Write out the first five terms of the following power series:

1. $\sum_{n=0}^{\infty} x^n$ **2.** $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x+1)^n}{n}$

Solution

1. One of the conventions we adopt is that $x^0 = 1$ regardless of the value of x.Therefore

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

This is a geometric series in .

2. This series is centered at $= -1$. Note how this series starts with $= 1$. We could rewrite this series starting at $n = 0$ with the understanding that $a_0 = 0$, and hence the first term is 0 .

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x+1)^n}{n} = (x+1) - \frac{(x+1)^2}{2} + \frac{(x+1)^3}{3} - \frac{(x+1)^4}{4} + \frac{(x+1)^5}{5} - \cdots$$

Theorem 20: Convergence of power series

Let a power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ be given, then one of the following is true.

1. The series converges only at $x = c$.

2. There is an $R > 0$ such that the series converges for all x in $(c - R, c + R)$ and diverges for all $x < c - R$ and $x > c + R$.

3. The series converges for all x.

The value of R is important when understanding a power series, hence it is given a name in the following definition. Also, note that **part 2** of **Theorem 20** makes a statement about the interval $(c - R, c + R)$, but the not the endpoints of that interval. A series may/may not converge at these endpoints.

Definition 12: Radius and interval of convergence

1. The number R given in **Theorem 20** is the **radius of convergence** of a given series. When a series converges for only $= c$, we say the radius of convergence is **0,** $R = 0$. When a series converges for all , we say the series has an infinite radius of convergence, $R = \infty$

2. The **interval of convergence** is the set of all values of for which the series converges.

To find the values of x for which a given series converges, we will use the convergence tests we studied previously (especially the Ratio Test). However, the tests all required that the terms of a series be positive. The following theorem gives us a work--around to this problem.

Theorem 21: The radius of convergence of a series and absolute convergence

The series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} |a_n x^n|$, have the same radius of convergence .

Theorem 21 allows us to find the radius of convergence R of a series by applying the Ratio Test **(or any applicable test)** to the absolute value of the terms of the series. We practice this in the following example.

Example 26: Determining the radius and interval of convergence

Find the radius and interval of convergence for each of the following series:

1. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ **2.** $\sum_{n=0}^{\infty} 2^n (x-3)^n$

Solution

1. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

1.We apply the Ratio Test to the series $\sum_{n=0}^{\infty} \left|\frac{x^n}{n!}\right|$

$$\lim_{n\to n} \frac{|x^{n+1}/(n+1)!|}{|x^n/n!|} = \lim_{n\to n} \left|\frac{x^{n+1}}{x^n} \cdot \frac{n!}{(n+1)!}\right|$$

$$= \lim_{n\to n} \left|\frac{x^n \cdot x}{x^n} \cdot \frac{n!}{(n+1)n!}\right|$$

$$= \lim_{n\to n}\left|\frac{x}{n+1}\right|$$

$$= 0$$

The Ratio Test shows us that regardless of the choice of , the series converges. Therefore the radius of convergence is = ∞ , and the interval of convergence is (−∞, +∞) .

2. $\sum_{n=0}^{\infty} 2^n (x-3)^n$

2. We apply the Ratio Test to the series $\sum_{n=0}^{\infty}|2^n(x-3)^n|$

$$\lim_{n\to\infty}\frac{|2^{n+1}(x-3)^{n+1}|}{|2^n(x-3)^n|} = \lim_{n\to\infty}\left|\frac{2^{n+1}(x-3)^{n+1}}{2^n(x-3)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2^{n+1}}{2^n}\cdot\frac{(x-3)^{n+1}}{(x-3)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{2^n}\cdot 2}{\cancel{2^n}}\cdot\frac{\cancel{(x-3)^n}\cdot(x-3)}{\cancel{(x-3)^n}}\right|$$

$$= \lim_{n\to\infty}|2(x-3)|$$

According to the Ratio Test, the series converges when $|2(x-3)| < 1 \Longrightarrow |x-3| < \frac{1}{2}$. the series is centered at 3, and x must be within $\frac{1}{2}$ of 3 in order for the series to converge. Therefore the radius of convergence is $R = 1/2$, and we know that the series converges absolutely for all x in $(3 - 1/2\,, 3 + 1/2) = (2.5\,, 3.5)$.

We check for convergence at the endpoints to find the interval of convergence. When $x = 2.5$ we have:

$$\sum_{n=0}^{\infty} 2^n (2.5-3)^n = \sum_{n=0}^{\infty} 2^n (-1/2)^n = \sum_{n=0}^{\infty}(-1)^n$$

which diverges. A similar process shows that the series also diverges at $= 3.5$. Therefore the interval of convergence is $(2.5\,, 3.5)$.

Theorem 22: Derivatives and indefinite integrals of power series functions

let $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ be a function defined by a power series, with radius of convergence R.

1. $f(x)$ is continuous and differentiable on $(c - R\,, c - R\,)$.

2. $f'(x) = \sum_{n=1}^{\infty} a_n \cdot n \cdot (x-c)^{n-1}$, with radius of convergence R.

3. $\int f(x)dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}$, with radius of convergence R.

Example 27: Derivatives and indefinite integrals of power series

Let $(x) = \sum_{n=0}^{\infty} x^n$. Find $f'(x)$ and $(x) = \int f(x)dx$, along with their respective intervals of convergence.

Solution

We find the derivative and indefinite integral of (x) , following **Theorem 22**.

1. $f'(x) = \sum_{n=1}^{\infty} n \cdot x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

In **Example 25**, we recognized that $\sum_{n=0}^{\infty} x^n$ is a geometric series in . We know that such a geometric series converges when $|x| < 1$; that is, the interval of convergence is $(-1, 1)$.

To determine the interval of convergence of $f'(x)$, we consider the endpoints of $(-1, 1)$:

$$f'(-1) = 1 - 2 + 3 - 4 + \cdots, \textit{which diverges}$$

$$f'(1) = 1 + 2 + 3 + 4 + \cdots, \textit{which diverges}$$

Therefore, the interval of convergence of $f'(x)$ is $(-1, 1)$.

2. $F(x) = \int f(x)dx = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

To find the interval of convergence of, we again consider the endpoints of $(-1, 1)$:

$$F(-1) = C - 1 + 1/2 - 1/3 + 1/4 + \cdots$$

The value of C is irrelevant; notice that the rest of the series is an Alternating Series that whose terms converge to 0. By the Alternating Series Test, this series converges. (In fact, we can recognize that the terms of the series after C are the opposite of the Alternating Harmonic Series. We can thus say that $F(-1) = C - \ln 2$.)

$$F(1) = C + 1 + 1/2 + 1/3 + 1/4 + \cdots$$

Notice that this summation is $C +$, the Harmonic Series, which diverges. Since F converges for $x = -1$ and diverges for $= 1$, the interval of convergence of $F(x)$ is $[-1, 1)$.

the previous example showed how to take the derivative and indefinite integral of a power series without motivation for why we care about such operations. We may care for the sheer mathematical enjoyment "that we can", which is motivation enough for many. However, we would be remiss to not recognize that we can learn a great deal from taking derivatives and indefinite integrals.

Recall that $f(x) = \sum_{n=0}^{\infty} x^n$ in **Example 27** is a geometric series. According to **Theorem 07**, this series converges to $1/(1 - x)$, when $|x| < 1$. Thus we can say

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad , on \ (-1, 1).$$

Integrating the power series, (as done in **Example 27**,) we find

$$F(x) = C_1 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \text{ .... } (01)$$

While integrating the function $f(x) = 1/(1-x)$ gives

$$F(x) = -\ln|1-x| + C_2 \text{ .... } (02)$$

Equating Equations **(1)** and **(2)**, we have

$$F(x) = C_1 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln|1-x| + C_2$$

Letting $x = 0$, we have $F(0) = C_1 = C_2$. This implies that we can drop the constants and conclude

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln|1-x|$$

We established in **Example 27** that the series on the left converges at $= -1$; substituting $x = -1$ on both sides of the above equality gives

$$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots = -\ln 2$$

On the left we have the opposite of the Alternating Harmonic Series; on the right, we have $-\ln 2$. We conclude that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2$$

Example 28: Analyzing power series functions

Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ .Find $f'(x)$ and $\int f(x)dx$, and use these to analyze the behavior of $f(x)$.

Solution

we start by making two notes: first, in **Example 26**, we found the interval of convergence of this power series is $(-\infty, \infty)$. Second, we will find it useful later to have a few terms of the series written out:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \text{ .... } (03)$$

We now find the derivative:

$$f'(x) = \sum_{n=1}^{\infty} n \cdot \frac{x^{n-1}}{n!}$$

$$= \sum_{n=1}^{\infty} n \cdot \frac{x^{n-1}}{n(n-1)!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = 1 + x + \frac{x^2}{2} + \cdots$$

since the series starts at $n = 1$ and each term refers to $(n-1)$, we can re-index the series starting with $n = 0$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$= f(x)$$

We found the derivative of $f(x)$ is (x) . The only functions for which this is true are of the form $y = ce^x$ for some constant .
As $f(0) = 1$ (**see Equation 03**), c must be 1. Therefore we conclude that

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

for all x .
We can also find $\int f(x)dx$:

$$\int f(x)dx = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!\,(n+1)}$$

$$= C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!}$$

We write out a few terms of this last series:

$$C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} = C + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

The integral of $f(x)$ differs from $f(x)$ only by a constant, again indicating that $f(x) = e^x$.

Example 29: Solving a differential equation with a power series

Give the first **4** terms of the power series solution to $y' = 2y$, where $y(0) = 1$.

Solution

The differential equation $y' = 2y$ describes a function $y = f(x)$ where the derivative of y is twice y and $y(0) = 1$. This is a rather simple differential equation; with a bit of thought one should realize that if $y = Ce^{2x}$, then $y' = 2 \cdot Ce^{2x}$, and hence $y' = 2y$. By letting $C = 1$ we satisfy the initial condition of $y(0) = 1$.

Let's ignore the fact that we already know the solution and find a power series function that satisfies the equation. The solution we seek will have the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

for unknown coefficients a_n . We can find $f'(x)$ using **Theorem 22**:

$$f'(x) = \sum_{n=0}^{\infty} a_n \cdot n \cdot x^n = a_1 x + 2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4 + \cdots$$

Since $f'(x) = 2f(x)$, we have

$$\begin{aligned} a_1 x + 2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4 + \cdots &= 2(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) \\ &= 2a_0 + 2a_1 x + 2a_2 x^2 + 2a_3 x^3 + \cdots \end{aligned}$$

The coefficients of like powers of x must be equal, so we find that

$$a_1 = 2a_0 ,\ 2a_2 = 2a_1 ,\ 3a_3 = 2a_2 , 4a_4 = 2a_3 , \qquad etc$$

The initial condition $y(0) = f(0) = 1$ indicates that $a_0 = 1$; with this, we can find the values of the other coefficients:

$$a_0 = 1 \ and \quad a_1 = 2a_0 \ \Rightarrow \ a_1 = 2 ;$$

$$a_1 = 2 \ and \ \ 2a_2 = 2a_1 \ \Rightarrow \ a_2 = 4/2 = 2;$$

$$a_2 = 2 \ and \ \ 3a_3 = \ 2a_2 \ \Rightarrow \ a_3 = 8/(2 \cdot 3) = 4/3;$$

$$a_3 = 4/3 \ and \ \ 4a_4 = 2a_3 \ \Rightarrow \ a_4 = 16/(2 \cdot 3 \cdot 4) = 2/3 .$$

Thus the first 5 terms of the power series solution to the differential equation $y' = 2y$ is

$$f(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots$$

$$y = e^{2x}$$

Chapter 02

Taylor Polynomials and Taylor Series

In mathematics, the **Taylor series** or **Taylor expansion** of a function is an infinite sum of terms that are expressed in terms of the function's derivatives at a single point. For most common functions, the function and the sum of its Taylor series are equal near this point. Taylor series are named after **Brook Taylor**, who introduced them in **1715**. A Taylor series is also called a **Maclaurin series** when **0** is the point where the derivatives are considered, after Colin Maclaurin, who made extensive use of this special case of Taylor series in the 18th century.

The partial sum formed by the first $n + 1$ terms of a Taylor series is a polynomial of degree n that is called the nth **Taylor polynomial** of the function. Taylor polynomials are approximations of a function, which become generally more accurate as n increases. Taylor's theorem gives quantitative estimates on the error introduced by the use of such approximations. If the Taylor series of a function is convergent, its sum is the limit of the infinite sequence of the Taylor polynomials. A function may differ from the sum of its Taylor series, even if its Taylor series is convergent. A function is analytic at a point x if it is equal to the sum of its Taylor series in some open interval containing x. This implies that the function is analytic at every point of the interval

1-Taylor Polynomials

Consider a function $y = f(x)$ and a point $(c, f(c))$. The derivative, $f'(c)$, gives the instantaneous rate of change of f at $x = c$. Of all lines that pass through the point $(c, f(c))$, the line that best approximates f at this point is the tangent line; that is, the line whose slope (rate of change) is $f'(c)$.
In **Figure Z**, we see a function $y = f(x)$ graphed. The table below the graph shows that $f(0) = 2$ and $f'(0) = 1$; therefore, the tangent line to f at $x = 0$ is $p_1(x) = 1(x - 0) + 2 = x + 2$. The tangent line is also given in the figure. Note that "near" $= 0$, $p_1(x) \approx f(x)$; that is, the tangent line approximates f well.

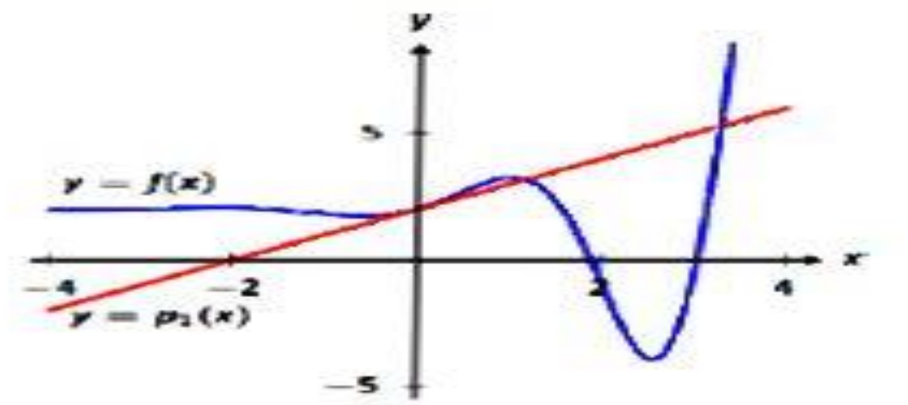


Figure Z' : plotting $y = f(x)$ (blue) and $y = p_1(x)$ (red).

$f(0) = 2$	$f'''(0) = -1$
$f'(0) = 1$	$f^4(0) = -12$
$f''(0) = 2$	$f^5(0) = -19$

One shortcoming of this approximation is that the tangent line only matches the slope of ; it does not, for instance, match the concavity of f. We can find a polynomial, $p_2(x)$, that does match the concavity without much difficulty, though. The table gives the following information:

$f(0) = 2$ $\quad$ $f'(0) = 1$ $\quad$ $f''(0) = 2$

Therefore, we want our polynomial $p_2(x)$ to have these same properties. That is, we need

$$p_2(0) = 2 \qquad p_2'(0) = 1 \qquad p_2''(0) = 2$$

This is simply an initial-value problem. We can solve this using the techniques first described in Section 5.1. To keep $p_2(x)$ as simple as possible, we'll assume that $p_2'(0) = 1$ not only $p_2''(0) = 2$, but that $p_2''(x) = 2$. That is, the second derivative p_2 of is constant.
If $p_2''(x) = 2$, then $p_2'(x) = 2x + C$ for some constant C. Since we have determined that

$p_2{}'(0) = 1$, we find that $C = 1$ and so $p_2{}'(x) = 2x + 1$. Finally, we can compute $p_2(x) = x^2 + x + C$. Using our initial values, we know $p_2(0) = 2$ $so\ C = 2$ so We conclude that $p_2(x) = x^2 + x + 2$. This function is plotted with in **Figure A'** .

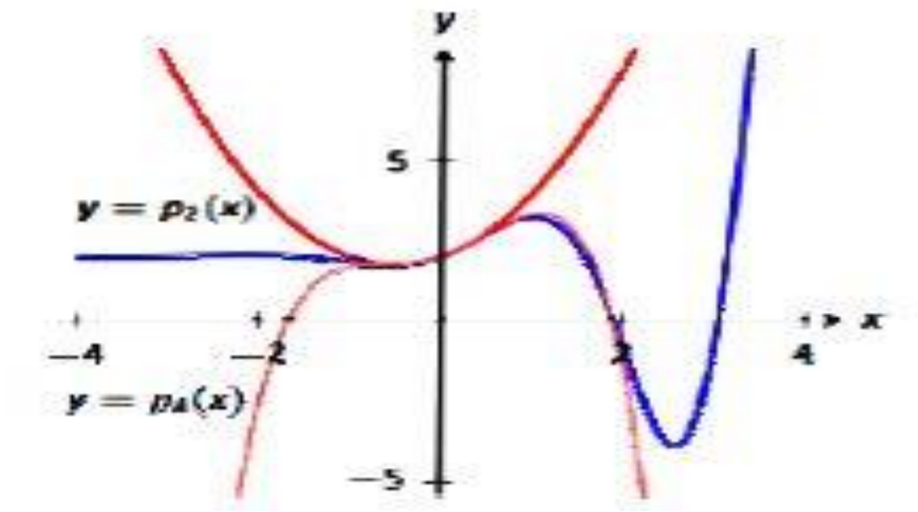


Figure A' : plotting f $(blue)$, $p_2(red)$ and p_4 (light red) .

We can repeat this approximation process by creating polynomials of higher degree that match more of the derivatives of f at $= 0$. In general, a polynomial of degree n can be created to match the first n derivatives of . **Figure A'** also shows $p_4(x) = -x^4/2 - x^3/6 + x^2 + x + 2$ whose first four derivatives at $\mathbf{0}$ match those of . (Using the table in **Figure Z** , start with $p_4{}^4(x) = -12$ and solve the related initial-value problem.)

As we use more and more derivatives, our polynomial approximation to f gets better and better. In this example, the interval on which the approximation is "good" gets bigger and bigger. **Figure B'** shows $p_{13}(x)$; we can visually affirm that this polynomial approximates f very well on $[-2, 3]$

The polynomial $p_{13}(x)$ is not particularly "**nice**". It is

$$p_{13}(x) = \frac{16901x^{13}}{6227020800} + \frac{13x^{12}}{1209600} - \frac{1321x^{11}}{39916800} - \frac{779x^{10}}{1814400} - \frac{359x^9}{362880} + \frac{x^8}{240} + \frac{139x^7}{5040} + \frac{11x^6}{360} - \frac{19x^5}{120} - \frac{x^4}{2} - \frac{x^3}{6} + x^2 + x + 2$$

The polynomials we have created are examples of ***Taylor polynomials***, named after the British mathematician Brook Taylor who made important discoveries about such functions. While we created the above Taylor polynomials by solving initial-value problems, it can be shown that Taylor polynomials follow a general pattern that make their formation much more direct. This is described in the following definition.

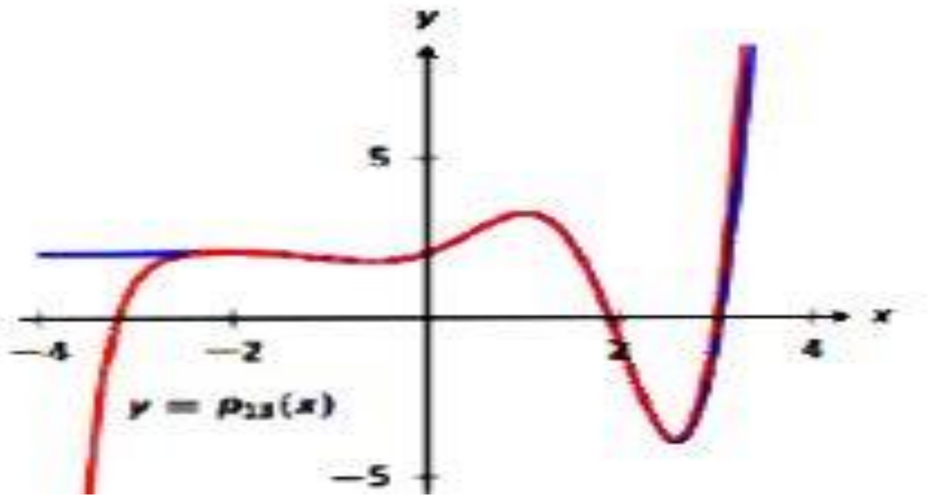


Figure C' : $plotting\ f\ (\ blue\ )\ and\ p_{13}\ (\ red\ )$

Definition 13: Taylor polynomials and maclaurin polynomials

Let f be a function whose first n derivatives exist at $x = c$.

1. The Taylor polynomial of degree of at $x = c$ is

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^n(c)}{n!}(x-c)^n$$

2. A special case of the Taylor polynomial is the Maclaurin polynomial, where $= 0$. That is, the **Maclaurin polynomial of degree n** of f is

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^n(0)}{n!}x^n$$

Example 30: Finding and using maclaurin polynomials

1. Find n^{th} the Maclaurin polynomial for $(x) = e^x$.
2. Use $p_5(x)$ to approximate the value of .

Solution the derivatives of $f(x) = e^x$ evaluated at $x = 0$

$f(x) = e^x \quad \Rightarrow \quad f(0) = 1$

$f'(x) = e^x \quad \Rightarrow \quad f'(0) = 1$

$f''(x) = e^x \quad \Rightarrow \quad f''(0) = 1$

. .
. .
. .

$f^n(x) = e^x$ $\qquad$ $f^n(0) = 1$

1. We start with creating a table of the derivatives of e^x evaluated at $= 0$. In this particular case, this is relatively simple.

By the definition of the Maclaurin series, we have

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^n(0)}{n!}x^n$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots + \frac{1}{n!}x^n$$

2.Using our answer from part **1**, we have

$$p_5 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$$

To approximate the value of , note that $e = e^1 = f(1) \approx p_5(1)$. It is very straightforward to evaluate $p_5(1)$:

$p_5(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = \frac{163}{60} \approx 2.71667$. ....**(1)**

A plot of $f(x) = e^x$ and $p_5(x)$ is given in **Figure D'**.

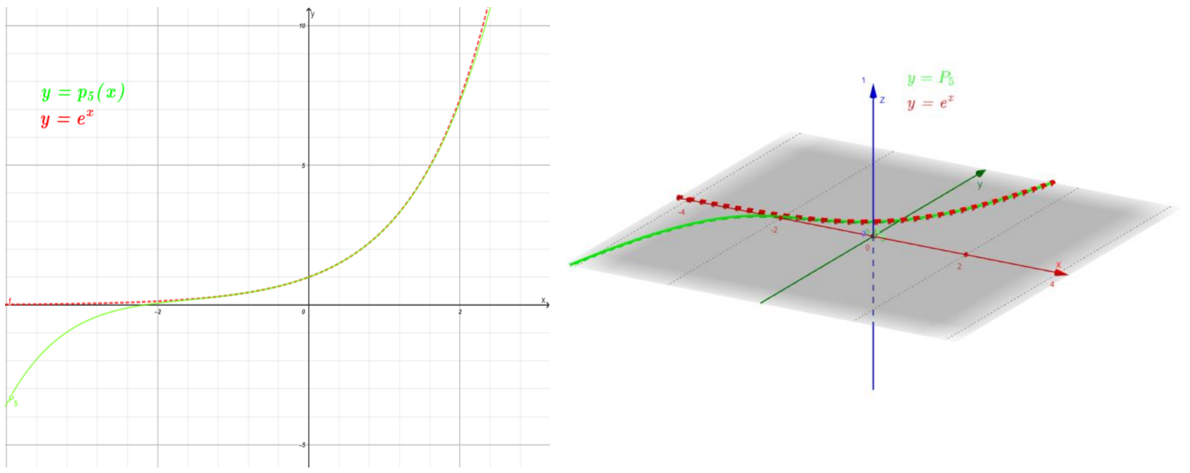


Figure D'

Example 31: Finding and using taylor polynomials

1. Find n^{th} the Taylor polynomial of $y = \ln x$ at $= 1$.
2. Use $p_6(x)$ to approximate the value of $\ln 1.5$.
3. Use $p_6(x)$ to approximate the value of $\ln 2$.

Solution

$$f(x) = \ln x \quad \Rightarrow \quad f(1) = 0$$

$$f'(x) = \frac{1}{x} \quad \Rightarrow \quad f'(1) = 1$$

$$f''(x) = \frac{-1}{x^2} \quad \Rightarrow \quad f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3} \quad \Rightarrow \quad f'''(1) = 2$$

$$f^4(x) = \frac{-6}{x^4} \quad \Rightarrow \quad f^4(1) = -6$$

$$\vdots \qquad\qquad \vdots$$

$$f^n(x) = \frac{(-1)^{n+1}(n-1)!}{x^n} \quad \Rightarrow \quad f^n(x) = (-1)^{n+1}(n-1)!$$

Derivatives of $\ln x$ evaluated at $x = 1$

We begin by creating a table of derivatives of $\ln x$ evaluated at $x = 1$. While this is not as straightforward as it was in the previous example, a pattern does emerge.
Using **Definition 13**, we have

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^n(c)}{n!}(x-c)^n$$

$$= 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 \ldots + \frac{(-1)^{n+1}}{n}(x-1)^n$$

Note how the coefficients of the $(x-1)$ terms turn out to be "nice."

2. We can compute $p_6(x)$ using our work above:

$$p_6(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6 \ldots\ldots\ldots\ldots\ldots (2)$$

Since $p_6(x)$ approximates $\ln x$ well near $= 1$, we approximate $\ln 1.5 \approx p_6(1.5)$:

$$p_6(1.5) = (1.5-1) - \frac{1}{2}(1.5-1)^2 + \frac{1}{3}(1.5-1)^3 - \frac{1}{4}(1.5-1)^4 + \frac{1}{5}(1.5-1)^5 - \frac{1}{6}(1.5-1)^6.$$

$$= \frac{259}{640}$$

$$\approx 0.404688$$

This is a good approximation as a calculator shows that $\ln 1,5 \approx 0,4055$ **Figure 01** plots $y = \ln x$ with $= p_6(x)$. We can see that $\ln 1,5 \approx p_6(1.5)$.

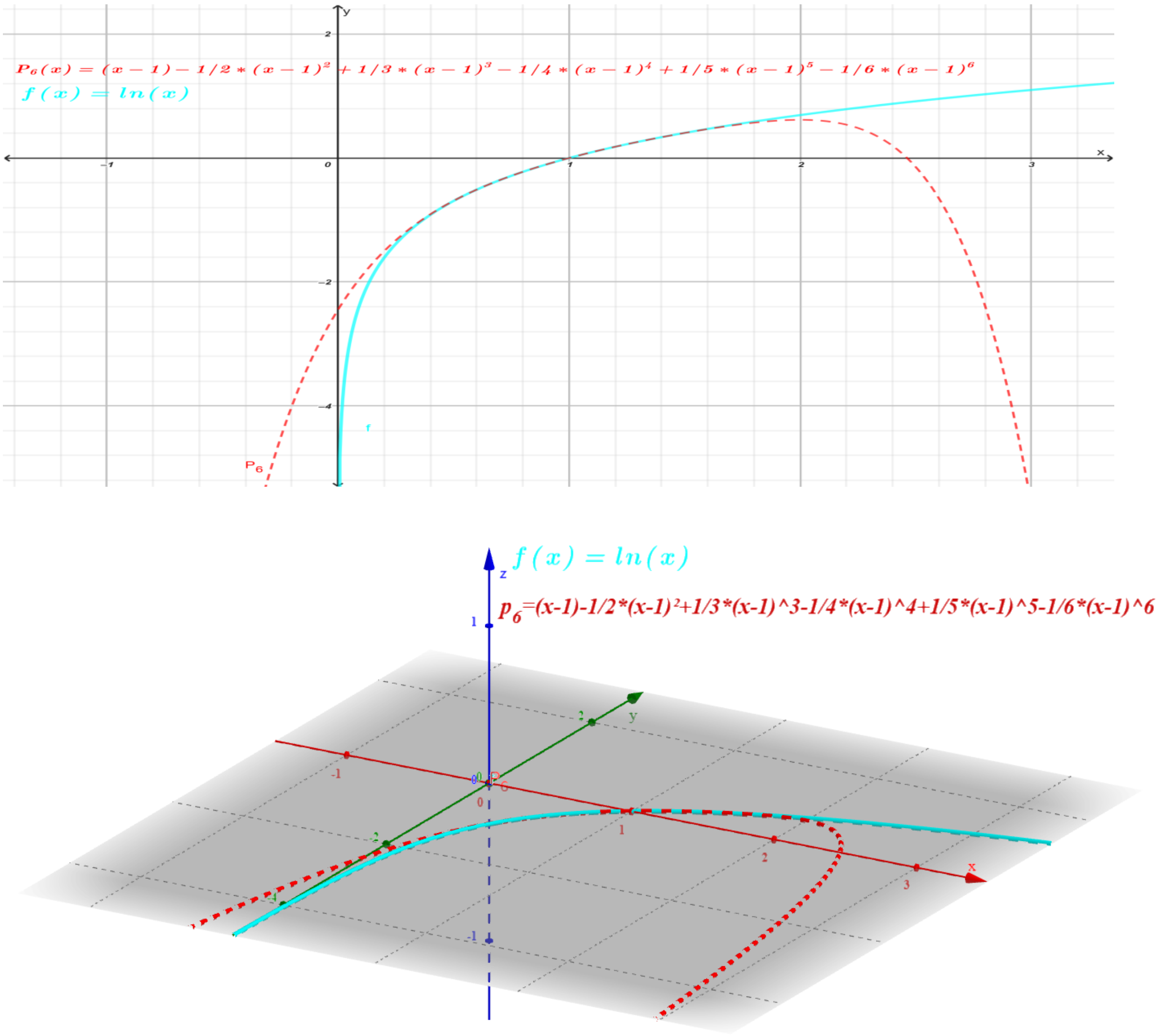


Figure E'

3. We approximate $\ln 2$ with $p_6(2)$:

$$p_6(2) = (2-1) - \frac{1}{2}(2-1)^2 + \frac{1}{3}(2-1)^3 - \frac{1}{4}(2-1)^4 + \frac{1}{5}(2-1)^5 - \frac{1}{6}(2-1)^6.$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}$$

$$= \frac{37}{60}$$

$$\approx 0.61666$$

This approximation is not terribly impressive: a hand held calculator shows that
$\ln 2 \approx \ 0.693147$

Surprisingly enough, even the 20^{th} degree Taylor polynomial fails to approximate $\ln x$ for > 2, as shown in **Figure F'**. We'll soon discuss why this is.

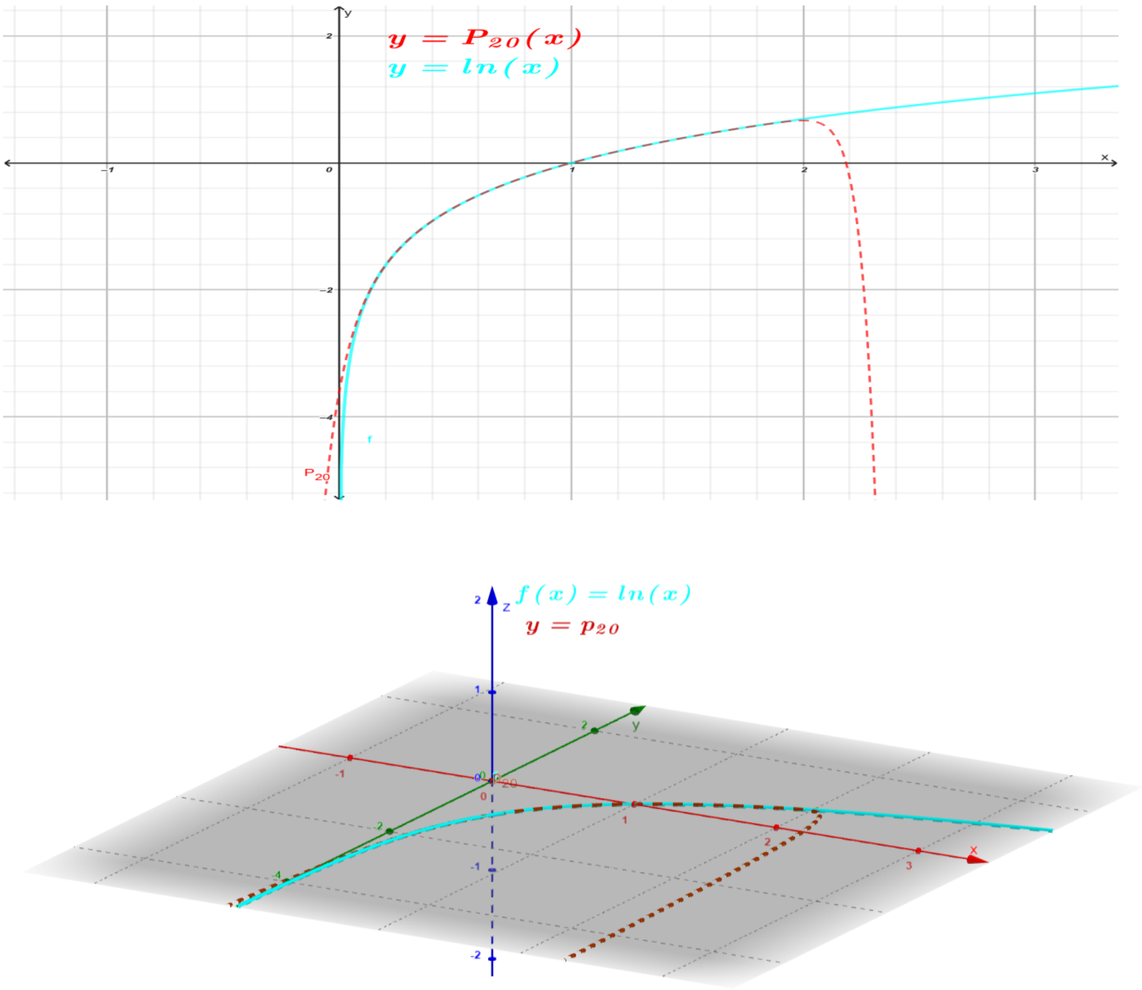


Figure F'

Theorem 23: Taylor theorem

1. Let f be a function whose $n+\ 1^{th}$ derivative exists on an interval I and let c be in I Then, for each x in I , there exists z_x between x and c such that

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^n(c)}{n!}(x-c)^n + R_n(x)$$

Where $R_n(x) = \frac{f^{(n+1)}(z_x)}{(n+1)!} \cdot (x-c)^{(n+1)}$

2. $|R_n(x)| \leq \frac{max|f^{(n+1)}(z_x)|}{(n+1)!} \cdot \left|(x-c)^{(n+1)}\right|$

The first part of Taylor's Theorem states that $f(x) = p_n(x) + R_n(x)$, where $p_n(x)$ is the n^{th} order Taylor polynomial and $R_n(x)$ is the remainder, or error, in the Taylor approximation. The second part gives bounds on how big that error can be. If the $(n+1)^{th}$ derivative is large, the error may be large; if x is far from c , the error may also be large. However, the $(n+1)!$ term in the denominator tends to ensure that the error gets smaller as n increases.

The following example computes error estimates for the approximations of and made in **Example 31**

Example 32: Finding error bounds of a taylor polynomial

Use **Theorem 23** to find error bounds when approximating $\ln 1,5$ and $\ln 2$ with $p_6(x)$, the Taylor polynomial of degree 6 of $f(x) = \ln x$ at $x = 1$, as calculated in **Example 31**

Solution

1. We start with the approximation of $\ln 1,5$ with $p_6(1,5)$. The theorem references an open interval I that contains both x and . The smaller the interval we use the better; it will give us a more accurate (and smaller!) approximation of the error. We let $= (0.9, 1.6)$, as this interval contains both $c = 1$ and $x = 1,5$.

The theorem references $\left|f^{(n+1)}(z)\right|$. In our situation, this is asking "How big can the 7^{th} derivative of $y = \ln x$ be on the interval $(0.9, 1.6)$?" The seventh derivative is $= \frac{-6!}{x^7}$. The largest value it attains on **I** is about **1506**. Thus we can bound the error as:

$$|R_6(1.5)| \leq \frac{max|f^{(7)}(z)|}{7!} \cdot \left|(1.5-1)^{(7)}\right|$$

$$\leq \frac{1506}{5040} \cdot \frac{1}{2^7}$$

$$\approx 0.0023.$$

We computed $p_6(1.5) = 0.404688$; using a calculator, we find $\ln 1.5 \approx 0.45465$, so the actual error is about 0.000778 , which is less than our bound of 0.0023 . This affirms Taylor's Theorem; the theorem states that our approximation would be within about 2 thousandths of the actual value, whereas the approximation was actually closer.

2. We again find an interval I that contains both $= 1$ *and* $x = 2$; we choose $I = (0.9, 2.1)$. The maximum value of the seventh derivative of f on this interval is again about **1506** (as the largest values come near $= 0.9$). Thus

$$|R_6(2)| \le \frac{max|f^{(7)}(z)|}{7!} \cdot |(2-1)^{(7)}|$$

$$\le \frac{1506}{5040} \cdot 1^7$$

$$\approx 0.30$$

This bound is not as nearly as good as before. Using the degree **6** Taylor polynomial at $x = 1$ will bring us within **0,3** of the correct answer. As $p_6(2) \approx 0.61667$, our error estimate guarantees that the actual value of **ln 2** is somewhere between 0.31667 and 0.91667. These bounds are not particularly useful.

In reality, our approximation was only off by about **0,07**. However, we are approximating ostensibly because we do not know the real answer. In order to be assured that we have a good approximation, we would have to resort to using a polynomial of higher degree.

Example 33: Finding sufficiently accurate taylor polynomials

Find n such that the n^{th}Taylor polynomial of $f(x) = \cos x$ *at* $x = 0$ approximates $\cos 2$ to within 0.001 of the actual answer. What is $p_n(2)$?

Solution

Following Taylor's theorem, we need bounds on the size of the derivatives of $(x) = \cos x$. In the case of this trigonometric function, this is easy. All derivatives of cosine are $\pm \sin x$ *or* $\pm \cos x$. In all cases, these functions are never greater than **1** in absolute value. We want the error to be less than 0.001 . To find the appropriate n , consider the following inequalities:

$$\frac{max|f^{(n+1)}(z)|}{(n+1)!} \cdot |(2-0)^{(n+1)}| \le 0.01$$

$$\frac{1}{(n+1)!} \cdot 2^{(n+1)} \le 0.001$$

We find an n that satisfies this last inequality with trial-and-error. When $n = 8$, we have $\frac{2^{8+1}}{(8+1)!} \approx 0.0014$; when $n = 9$, we have $\frac{2^{9+1}}{(9+1)!} \approx 0.000282 < 0.001$. Thus we want to approximate $\cos 2$, with $p_9(2)$.

We now set out to compute$p_9(x)$. We again need a table of the derivatives of $f(x) = \cos x$ evaluated at $x = 0$.

$$f(x) = \cos x \quad \Rightarrow \quad f(0) = 1$$

$$f'(x) = -\sin x \quad \Rightarrow \quad f'(0) = 0$$

$$f''(x) = -\cos x \quad \Rightarrow \quad f''(0) = -1$$

$$f'''(x) = \sin x \quad \Rightarrow \quad f'''(0) = 0$$

$$f^4(x) = \cos x \quad \Rightarrow \quad f^4(0) = 1$$

$$f^5(x) = -\sin x \quad \Rightarrow \quad f^5(0) = 0$$

$$f^6(x) = -\cos x \quad \Rightarrow \quad f^6(0) = -1$$

$$f^7(x) = \sin x \quad \Rightarrow \quad f^7(0) = 0$$

$$f^8(x) = \cos x \quad \Rightarrow \quad f^8(0) = 1$$

$$f^9(x) = -\sin x \quad \Rightarrow \quad f^9(0) = 0$$

Notice how the derivatives, evaluated at $x = 0$, follow a certain pattern. All the odd powers of x in the Taylor polynomial will disappear as their coefficient is 0. While our error bounds state that we need $p_9(x)$, our work shows that this will be the same as $p_8(x)$.

Since we are forming our polynomial at $x = 0$, we are creating a Maclaurin polynomial, and:

$$p_8(x) = \quad f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^8(0)}{8!}x^n$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8$$

We finally approximate $\cos 2$:

$$\cos 2 \approx p_8(2) = \frac{131}{315} \approx -0.41587$$

Our error bound guarantee that this approximation is within 0.001 of the correct answer. Technology shows us that our approximation is actually within about 0.0003 of the correct answer.

Figure G' shows a graph of $y = p_8(x)$ and $y = \cos x$. Note how well the two functions agree on about $(-\pi, +\pi)$.

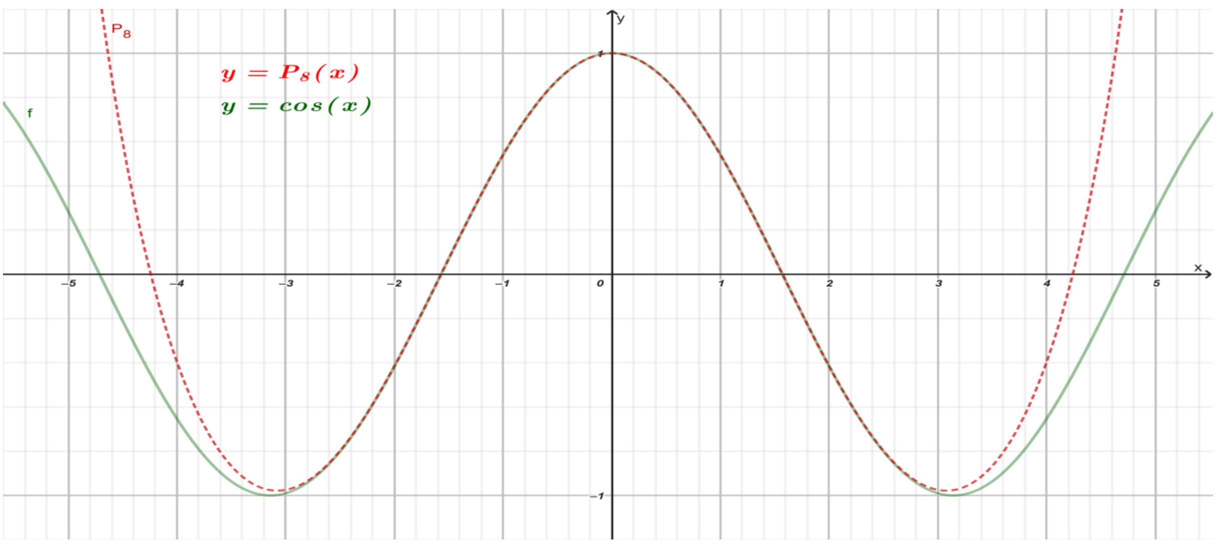


Figure G'

Example 34: Finding and using Taylor polynomials

1. Find the degree 4 Taylor polynomial, $p_4(x)$, for $f(x) = \sqrt{x}$, at $x = 4$
2. Use $p_4(x)$to approximate $\sqrt{3}$.
3. Find bounds on the error when approximating $\sqrt{3}$ with $p_4(3)$.

$$f(x) = \sqrt{x} \quad \Rightarrow \quad f(4) = 2$$

$$f'(x) = \frac{1}{2\sqrt{x}} \quad \Rightarrow \quad f'(4) = \frac{1}{4}$$

$$f''(x) = \frac{-1}{4x^{3/2}} \quad \Rightarrow \quad f''(4) = \frac{-1}{32}$$

$$f'''(x) = \frac{3}{8x^{5/2}} \quad \Rightarrow \quad f'''(4) = \frac{3}{256}$$

$$f^4(x) = \frac{-15}{16x^{7/2}} \quad \Rightarrow \quad f^4(4) = \frac{-15}{2048}$$

1. We begin by evaluating the derivatives of f at $x = 4$. These values allow us to form the Taylor polynomial $p_4(x)$:

$$p_4(x) = 2 + \frac{1}{4}(x-4) + \frac{-1/32}{2!}(x-4)^2 + \frac{3/256}{3!}(x-4)^3 + \frac{-15/2048}{4!}(x-4)^4.$$

2. As $p_4(x) \approx \sqrt{x}$, near $x = 4$, we approximate $\sqrt{3}$ with $p_4(3) = 1.73212$.
3. To find a bound on the error, we need an open interval that contains $x = 3$ and $x = 4$. We set $= (2.9 , 4.1)$. The largest value the fifth derivative of $f(x) = \sqrt{x}$ takes on this interval is near $x = 2.9$, at about $.0273$. Thus

$$|R_4(3)| \leq \frac{0.0273}{5!}\left|(3-4)^5\right| \approx 0.00023 .$$

This shows our approximation is accurate to at least the first **2** places after the decimal. (It turns out that our approximation is actually accurate to **4** places after the decimal.) A graph of $f(x) = \sqrt{x}$ ***and*** $p_4(x)$ is given in **Figure** H' . Note how the two functions are nearly indistinguishable on $(2, 7)$.

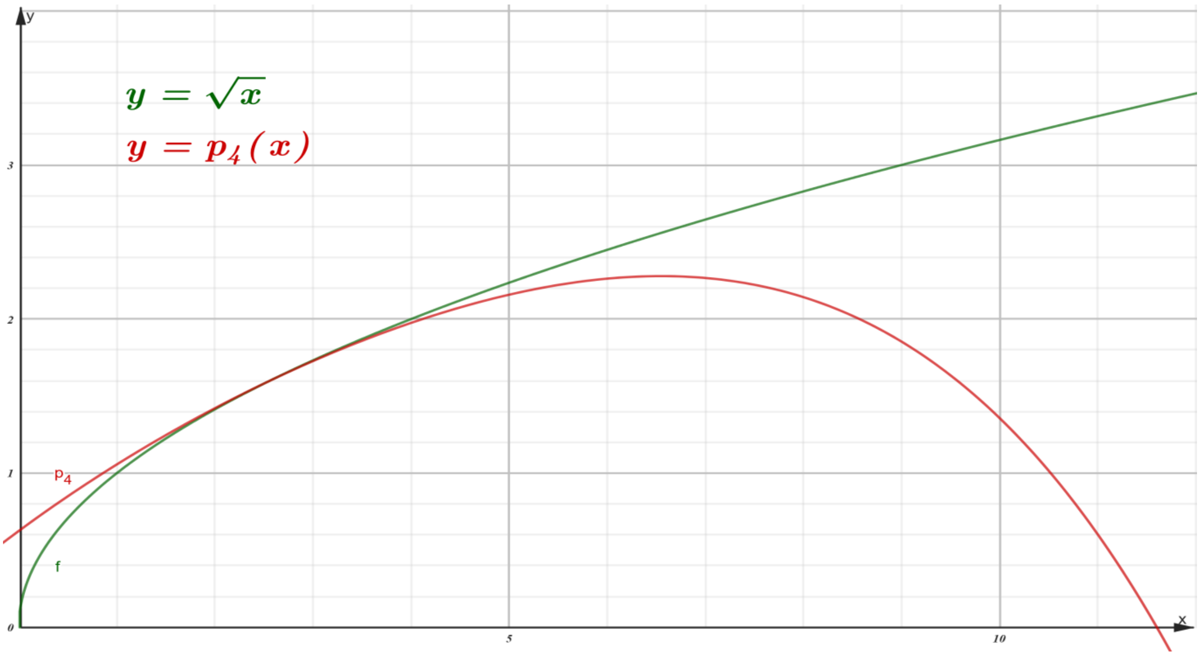


Figure H'

Example 35: Approximating an unknown function

A function $y = f(x)$ is unknown save for the following two facts.
1. $y(0) = f(0) = 1$, and
2. $y' = y^2$
(This second fact says that amazingly, the derivative of the function is actually the function squared!)

Find the degree **3** Maclaurin polynomial $p_3(x)$ of $= f(x)$.

Solution

One might initially think that not enough information is given to find $p_3(x)$. However, note how the second fact above actually lets us know what $y'(0)$ is:

$$y' = y^2 \;\Rightarrow\; y'(0) = y^2(0)$$

Since $y(0) = 1$, we conclude that$y'(0) = 1$.
Now we find information about y'' . Starting with $y' = y^2$, take derivatives of both sides, *with respect to* x . That means we must use implicit differentiation.

$$y' = y^2$$

$$\frac{d}{dx}(y') = \frac{d}{dx}(y^2)$$

$$y'' = 2y \cdot y'$$

Now evaluate both sides at $x = 0$:

$$y''(0) = 2y(0) \cdot y'(0)$$

$$y''(0) = 2$$

We repeat this once more to find $y'''(0)$. We again use implicit differentiation; this time the Product Rule is also required.

$$\frac{d}{dx}(y'') = \frac{d}{dx}(2yy')$$

$$y''' = 2y' \cdot y' + 2y \cdot y''$$

Now evaluate both sides at $x = 0$:

$$y'''(0) = 2y'(0)^2 + 2y(0) \cdot y''(0)$$

$$y'''(0) = 2 + 4 = 6$$

In summary, we have:

$$y(0) = 1 \quad y'(0) = 1 \quad y''(0) = 2 \quad y'''(0) = 6$$

We can now form $p_3(x)$:

$$p_3(x) = 1 + x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3$$

$$= 1 + x + x^2 + x^3$$

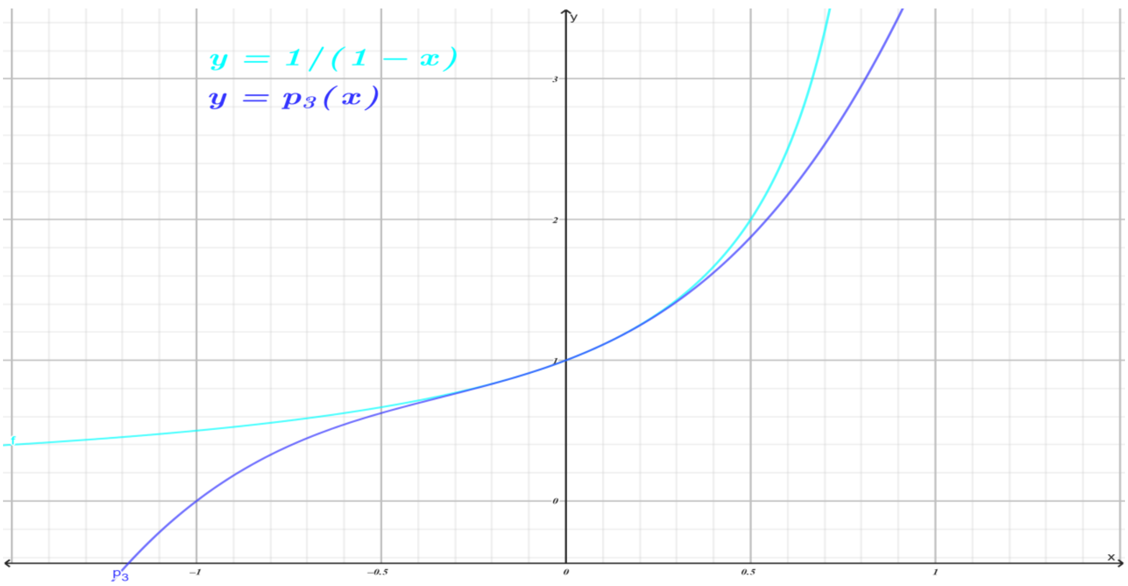


Figure I'

It turns out that the differential equation we started with, $y' = y^2$, where $y(0) = 1$, can be solved without too much difficulty: $y = \frac{1}{1-x}$. **Figure** I'shows this function plotted with $p_3(x)$.
Note how similar they are near $= 0$. .

2-Taylor Series

In **Section 6**, we showed how certain functions can be represented by a power series function. In **7**, we showed how we can approximate functions with polynomials, given that enough derivative information is available. In this section we combine these concepts: if a function $f(x)$ is infinitely differentiable, we show how to represent it with a power series function.

Definition 14: Taylor and maclaurin series

Let $f(x)$ have derivatives of all orders at $= c$.

1. The **Taylor Series of** (x) , **centered at** c is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

2. Setting $c = 0$ gives the **Maclaurin Series of** $f(x)$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

The difference between a *Taylor polynomial* and a *Taylor series* is the former is a polynomial, containing only a finite number of terms, whereas the latter is a series, a summation of an infinite set of terms. When creating the Taylor polynomial of degree n for a function $f(x)$ at $x = c$,we needed to evaluate f,and the first n derivatives of f ,at $x = c$ .When creating the Taylor series of f , it helps to find a pattern that describes the n^{th} derivative of f at $x = c$ .We demonstrate this in the next two examples.

Example 36: The maclaurin series of $f(x) = \cos x$

Find the Maclaurin series of $(x) = \cos x$.

Solution

In **Example 33** we found the degree Maclaurin polynomial of $\cos x$.

$$f(x) = \cos x \quad \Rightarrow \quad f(0) = 1$$

$$f'(x) = -\sin x \quad \Rightarrow \quad f'(0) = 0$$

$$f''(x) = -\cos x \quad \Rightarrow \quad f''(0) = -1$$

$$f'''(x) = \sin x \quad \Rightarrow \quad f'''(0) = 0$$

$$f^4(x) = \cos x \quad \Rightarrow \quad f^4(0) = 1$$

$$f^5(x) = -\sin x \quad \Rightarrow \quad f^5(0) = 0$$

$$f^6(x) = -\cos x \quad \Rightarrow \quad f^6(0) = -1$$

$$f^7(x) = \sin x \quad \Rightarrow \quad f^7(0) = 0$$

$$f^8(x) = \cos x \quad \Rightarrow \quad f^8(0) = 1$$

$$f^9(x) = -\sin x \quad \Rightarrow \quad f^9(0) = 0$$

Notice how $f^{(n)}(0) = 0$ when n is odd, $f^{(n)}(0) = 1$ when n is divisible by 4, and $f^{(n)}(0) = -1$ when n is even but not divisible by 4. Thus the Maclaurin series of $\cos x$ is

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots$$

We can go further and write this as a summation. Since we only need the terms where the power of is even, we write the power series in terms of x^{2n}:

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Example 37: The taylor series of $f(x) = \ln x$ at $x = 1$

Find the Taylor series of $f(x) = \ln x$ centered at $x = 1$.

Solution

Figure 1 shows the n^{th} derivative of $\ln x$ evaluated at $x = 1$ for $n = 0, \ldots \ldots, 5$,along with an expression for the n^{th} term:

$$f^{(n)}(1) = (-1)^{n+1}(n+1)!\,, n \geq 1$$

Remember that this is what distinguishes Taylor series from Taylor polynomials; we are very interested in finding a pattern for the n^{th} term, not just finding a finite set of coefficients for a polynomial.

$$f(x) = \ln x \quad \Rightarrow \quad f(1) = 0$$

$$f'(x) = \frac{1}{x} \quad \Rightarrow \quad f'(1) = 1$$

$$f''(x) = \frac{-1}{x^2} \quad \Rightarrow \quad f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3} \quad \Rightarrow \quad f'''(1) = 2$$

$$f^4(x) = \frac{-6}{x^4} \quad \Rightarrow \quad f^4(1) = -6$$

$$f^5(x) = \frac{24}{x^5} \quad \Rightarrow \quad f^5(1) = 2$$

$$f^n(x) = \frac{(-1)^{n+1}(n-1)!}{x^n} \quad \Rightarrow f^n(x) = (-1)^{n+1}(n-1)! \ldots \ldots \ldots \boldsymbol{Figure}\ (1)$$

Since $f(1) = \ln 1 = 0$,we skip the first term and start the summation with $n = 1$,giving the Taylor series for $\ln x$,centered at $x = 1$,as

$$\sum_{n=1}^{\infty}(-1)^{n+1}(n-1)! \cdot \frac{1}{n!} \cdot (x-1)^n = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}$$

It is important to note that **Definition 14** defines a Taylor series given a function (x) ; however, we *cannot* yet state that $f(x)$ *is equal* to its Taylor series. We will find that "most of the time" they are equal, but we need to consider the conditions that allow us to conclude this.

Theorem 23 states that the error between a function $f(x)$and its n^{th} --degree Taylor polynomial $p_n(x)$ is $R_n(x)$,where

$$|R_n(x)| \leq \frac{max|f^{(n+1)}(z)|}{(n+1)!}\left|(x-c)^{(n+1)}\right|$$

If $R_n(x)$ goes to 0 for each x in an interval I as n approaches infinity, we conclude that the function is equal to its Taylor series expansion.

Theorem 24: Function and taylor series equality

Let $f(x)$ have derivatives of all orders at $x = c$,let $R_n(x)$ be as stated in **Theorem 23**, and let I be an interval on which the Taylor series of $f(x)$ converges.

If $\lim_{n\to\infty} R_n(x) = 0$ for all x in I ,then

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!} \cdot (x-c)^n \text{ on } I$$

Example 38: Establishing equality of a function and its taylor series

Show that $f(x) = \cos x$ is equal to its Maclaurin series, as found in **Example 36**, for all .

Solution

Given a value x ,the magnitude of the error term $R_n(x)$ is bounded by

$$|R_n(x)| \leq \frac{max|f^{(n+1)}(z)|}{(n+1)!}\left|x^{(n+1)}\right|$$

Since all derivatives of $\cos x$ are $\pm\sin x$ or $\pm\cos x$,whose magnitudes are bounded by 1 ,we can state

$$|R_n(x)| \leq \frac{1}{(n+1)!}\left|x^{(n+1)}\right|$$

which implies

$\frac{-|x^{(n+1)}|}{(n+1)!} \le |R_n(x)| \le \frac{|x^{(n+1)}|}{(n+1)!}$ .... (*)

For any , $\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$. Applying the Squeeze Theorem to Equation (*) , we conclude that $\lim_{n\to\infty} R_n(x) = 0$ for all x ,and hence

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \text{for all } x.$$

Example 39: The binomial series

Find the Maclaurin series of $f(x) = (1+x)^k$, $k \neq 0$.

Solution

When $k$is a positive integer, the Maclaurin series is finite. For instance, when $k = 4$,we have

$$f(x) = (1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

The coefficients of x when k is a positive integer are known as the ***binomial coefficients***, giving the series we are developing its name. When
$k = 1/2$ $\quad$ $we\ have\ f(x) = \sqrt{1+x}$.Knowing a series representation of this function would give a useful way of approximating $\sqrt{1,3}$,for instance.

To develop the Maclaurin series for $f(x) = (1+x)^k$ for , $k \neq 0$ any value of f ,we consider the derivatives of evaluated at $x = 0$:

$f(x) = (1+x)^k \quad \Rightarrow \quad f(0) = 1$

$f'(x) = k(1+x)^{k-1} \quad \Rightarrow \quad f'(0) = k$

$f''(x) = k(k-1)\cdot(1+x)^{k-2} \quad \Rightarrow \quad f''0 = k(k-1)$

$f'''(x) = k(k-1)\cdot(k-2)\cdot(1+x)^{k-3} \quad \Rightarrow \quad f'''(0) = k(k-1)\cdot(k-2)$

$\vdots \qquad\qquad \vdots$

$f^n(x) = k(k-1)\cdots(k-(k-2))\cdot(1+x)^{k-n} \quad \Rightarrow \quad f^n(x) = k(k-1)\cdots(k-(k-1))$

Thus the Maclaurin series for $f(x) = (1+x)^k$ is

$$1 + k + \frac{k(k-1)}{2!} + \frac{k(k-1)\cdot(k-2)}{3!} + \cdots + \frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!} + \cdots$$

It is important to determine the interval of convergence of this series. With

$$a_n = \frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!} x^n$$

we apply the Ratio Test:

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\left|\frac{k(k-1)\cdots(k-n)}{(n+1)!}x^{n+1}\right| \Big/ \left|\frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!}x^{n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{k-n}{n}x\right|$$

$$= |x|$$

The series converges absolutely when the limit of the Ratio Test is less than $\mathbf{1}$; therefore, we have absolute convergence when $|x| < 1$.

While outside the scope of this text, the interval of convergence depends on the value of $\boldsymbol{k}$. When $\boldsymbol{k} > 0$,the interval of convergence is $[-\mathbf{1}, +\mathbf{1}]$ .When $-\mathbf{1} < k < 0$,the interval of convergence is $[-\mathbf{1}\,, +\mathbf{1})$ .If $\boldsymbol{k} \leq -\mathbf{1}$,the interval of convergence is $(-\mathbf{1}, +\mathbf{1})$.

Key Idea 04: Important taylor series expasions

Function and Series	First Few Terms	Interval of Convergence
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$(-\infty, \infty)$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$(-\infty, \infty)$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$(-\infty, \infty)$
$\ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$	$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$	$(0, 2]$
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	$1 + x + x^2 + x^3 + \cdots$	$(-1, 1)$
$(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-(n-1))}{n!} x^n$	$1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots$	$(-1, 1)^a$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$[-1, 1]$

Theorem 25: Algebra of power series

let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $g(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$ *and let* $h(x)$ *be continuous*

1. $f(x) \pm g(x) = \sum_{n=0}^{\infty}(a_n \pm b_n) x^n$ \quad for , $|x| < R$.

2. $f(x) \cdot g(x) = (\sum_{n=0}^{\infty} a_n x^n) \cdot (\sum_{n=0}^{\infty} b_n x^n) = \sum_{n=0}^{\infty}(a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) x^n$ for $|x| < R$.

3. $f(h(x)) = \sum_{n=0}^{\infty} a_n (h(x))^n$ for $|h(x)| < R$.

Example 40: Combining taylor series

Write out the first **3** terms of the Taylor Series for $f(x) = e^x \cos x$ using **Key Idea 04** and **Theorem 25**.

Solution

Key Idea 04 informs us that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{and } \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

Applying **Theorem 25**, we find that

$$e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)$$

Distribute the right hand expression across the left :

$$= 1\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \frac{x^2}{2!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \frac{x^3}{3!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \frac{x^4}{4!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \cdots$$

Distribute again and collect like terms

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + \frac{x^7}{630} + \cdots$$

While this process is a bit tedious, it is much faster than evaluating all the necessary derivatives of $e^x \cos x$ and computing the Taylor series directly.

Because the series for e^x and $\cos x$ both converge on (-∞, +∞),so does the series expansion for $e^x \cos x$.

Example 41: Creating new taylor series

Use **Theorem 25** to create series for $y = \sin(x^2)$ *and* $y = \ln(\sqrt{x})$.

Solution

Given that

$$\text{Sin } x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

we simply substitute x^2 for x in the series, giving

$$\text{Sin}(x^2) = \sum_{n=0}^{\infty}(-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$$

Since the Taylor series for $\sin x$ has an infinite radius of convergence, so does the Taylor series for $\text{Sin}(x^2)$.

The Taylor expansion for $\ln x$ given in **Key Idea 04** is centered at $x = 1$,so we will center the series for $\ln\sqrt{x}$ at $x = 1$ as well.
With

$$\ln x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$$

we substitute $\sqrt{x}$ for x to obtain

$$\ln(\sqrt{x}) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(\sqrt{x}-1)^n}{n} = (\sqrt{x}-1) - \frac{(\sqrt{x}-1)^2}{2} + \frac{(\sqrt{x}-1)^3}{3} - \cdots$$

While this is not strictly a power series, it is a series that allows us to study the function $\ln(\sqrt{x})$ Since the interval of convergence of $\ln x$ is $(0,2]$,and the range of $\sqrt{x}$ on $(0,4]$ is $(0,2]$,the interval of convergence of this series expansion of $\ln(\sqrt{x})$ is $(0,4]$.

Example 42: Using taylor series to evaluate definite integrals

Use the Taylor series of e^{-x^2} to evaluate $\int_0^1 e^{-x^2}dx$.

Solution

We learned, when studying Numerical Integration, that e^{-x^2} does not have an antiderivative expressible in terms of elementary functions. This means any definite integral of this function must have its value approximated, and not computed exactly.

We can quickly write out the Taylor series for e^{-x^2} using the Taylor series of e^x :

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

and so

$$e^{-x^2} = \sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{n!}$$

$$= \mathbf{1} - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

We use **Theorem 23** to integrate:

$$\int_0^1 e^{-x^2} dx = C + x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots$$

This *is* the antiderivative of e^{-x^2};while we can write it out as a series, we cannot write it out in terms of elementary functions. We can evaluate the definite integral $\int_0^1 e^{-x^2} dx$ using this antiderivative; substituting $\mathbf{1}$ and $\mathbf{0}$ for x and subtracting gives

$$\int_0^1 e^{-x^2} dx = 1 - \frac{1}{3} + \frac{1}{5 \cdot 2!} - \frac{1}{7 \cdot 3!} + \frac{1}{9 \cdot 4!} + \cdots$$

Summing the $\mathbf{5}$ terms shown above give the approximation of $\mathbf{0.74749}$ Since this is an alternating series, we can use the Alternating Series Approximation Theorem, **(Theorem19),** to determine how accurate this approximation is. The next term of the series is $1/(11 \cdot 5!) \approx 0.00075758$ .Thus we know our approximation is within $\mathbf{0.00075758}$ of the actual value of the integral. This is arguably much less work than using Simpson's Rule to approximate the value of the integral.

Conclusion

Sequences and series are powerful and versatile mathematical tools that play a central role in many scientific and technical fields. Their ability to model, analyze and solve complex problems makes them indispensable for theoretical research and practical applications. They play a crucial role in many scientific and technical disciplines. Here are a few notable applications in various fields:

This dissertation explores not only the theoretical underpinnings of these concepts, but also recent developments and their implications in a variety of fields, providing a comprehensive and integrated overview of the importance of sequences and series in today's scientific and technological landscape.

Here is a list of key references covering theoretical aspects, applications and recent developments in sequences and series.

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