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Contribution of the Dynamic Programming Method to Solving the Dolichobrachistochrone Differential game

Presented by

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Thesis defended on ……………………, 2025, in front of the jury composed of:

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إهـــــــداء

بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

"وقل رب زدني علمًا"

الحمد لله الذي منحني القوة والعزيمة لأخوض غمار هذه الرحلة الطويلة، وألهمني الصبر والإصرار لتحقيق هذا الإنجاز. أقدم هذا العمل المتواضع كعربون وفاء وتقدير لكل من كان له أثر في حياتي، ولكل من كان دعمهم عونًا لي في هذا الطريق.

إلى من زرعت في قلبي بذور الأمل وسقتني حباً وحناناً، إلى أمي الحبيبة خثير فاطمة الزهراء، التي كانت لي المعلمة الأولى، واليد الحانية، والركن الدافئ في حياتي، رمز للتضحية والعطاء، رافقتني منذ خطواتي الأولى بحب لا يضاهيه حب.

إلى والدي العزيز عبد الوهاب السند الذي لا ينكسر، والذي أفنى لياليه ساهراً وأيامه عاملاً، متحدياً برد الشتاء القارس وحر الصيف اللاهب ليمنحنا العلم والحياة الكريمة.

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إلى أخي فارس، صاحب القلب الطيب الذي يحمل في نبضاته أصفى مشاعر المحبة والوفاء.

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أقول لكم جميعاً: هذا العمل هو ثمرة جهودكم، وعربون وفاء ومحبة لما قدمتموه لي من حب وتضحيات.

غانم عائشة

شــــكر وتــقـديـــــر

بســم الله الرحمن الرحيم

الحمد لله الذي وفقني وأعانني على إتمام هذا العمل، وأسبغ عليَّ من فضله وكرمه، فله الحمد والشكر كما ينبغي لجلال وجهه وعظيم سلطانه.

أود أن أتقدم بجزيل الشكر والتقدير إلى الأستاذ توفيق بورماني، المشرف على هذه الأطروحة، الذي كان له الأثر الكبير في توجيهي وإرشادي طوال مسيرتي العلمية. لم يبخل عليّ بتوجهاته السديدة ونصائحه القيمة التي كانت تُضيء لي الطريق في لحظات التحدي، فكان خير مشرف ومعلم. كما أود أن أعبر عن عميق امتناني للأستاذ جمال بن تـركي، الذي ساعدني بفضل علمه الواسع ورؤيته العميقة في تطوير هذا العمل. لقد كان دعمه لي مستمرًا وكان دائمًا مصدرًا للإلهام والتوجيه العلمي السليم. جزاهما الله خير الجزاء على ما بذلاه من جهد ووقت، وأسأل الله أن يبارك في علمهما ويزيدهما من فضله.

أتوجه بجزيل الشكر والتقدير إلى الأستاذ بشير مريخي رئيس لجنة المناقشة، الذي رافقني طيلة سنوات دراستي، حيث تعلمت منه العلم والتفاني في العمل، وكانت دعمه ورؤيته الثاقبة أكبر حافز لي.

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غانـم عائشة

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Scientific Production

List of publications

- A. Ghanem, T. Bouremani, D. Benterki, On the solution of Dolichobrachistochrone differential game via dynamic programming approach, Journal of Computational and Applied Mathematics. 461 (2025) 116460.

Paper submitted

- A. Ghanem, T. Bouremani, D. Benterki, A. Ziad, Dynamic programming solution of the Pursuit-Evasion Game, Applied Mathematics and Computation.

List of communications

International communications

- A. Ghanem, T. Bouremani and D. Benterki, Admissible Trajectories of Dolichobrachistochrone Differential Game Problem Based on Dynamic Programming Approach. The 6th International Workshop on Applied Mathematics and Modelling, 26-27/10/2022, Guelma University, Algeria.

- A. Ghanem, T. Bouremani, D. Benterki and A. Ghanem, The Brachistochrone parametric curves the solution in the non-singular case $k > 0$. International Conference of Young Mathematicians, 1-3/06/2023, Institute of Mathematics, Kyiv, Ukraine.

- A. Ghanem, T. Bouremani and D. Benterki, Characterization of Certain Admissible Trajectories of Isaacs' Wall Pursuit Differential Game. International Mathematics and Statistics Student Research, 13/04/2024, USA.

- A. Ghanem, T. Bouremani, D. Benterki and A. Ziad, Parametric Curves in Isaac's Pursuit-Evasion Game. International Conference on Mathematics and its Applications in Science and Technology, 15-16/12/2024, Setif, Algeria.

National communications

- A. Ghanem, T. Bouremani and D. Benterki, Characterization of a Certain Admissible Trajectories of Dolichobrachistochrone Differential Game Problem. The Second National Conference on Mathematics and its Applications, 16-17/11/2022, Bordj Bou Arréridj, Algeria.

- A. Ghanem, T. Bouremani and D. Benterki, On The Extended Hamiltonian Flow of Dolichobrachistochrone Differential Game Using Dynamic Programming Approach. New Trends in Theoretical and Computational Mathematics, 8-9/11/2022, University Tamanghasset, Algeria.

- A. Ghanem, T. Bouremani, D. Benterki and A. Ghanem, The Brachistochrone parametric curves without initial velocity the solution in the singular case k = 0. The Second Online National Conference on Pure and Applied Mathematics, 27/04/2023, Tebessa, Algeria.

List of abbreviations and notions

$\langle x, y\rangle = x^T y = x.y$	Scalar product of x and y;
$Int(S)$	Interior of S;
$Cl(S)$	Closure of S;
$co(S)$	Convex hull of S;
$\overline{co}(S)$	Closed convex hull of S;
$B_r(x)$	Ball of radius r centred at x;
$\overline{B}_r(x)$	Closed ball of radius r centred at x;
$L(\mathbb{R}^n, \mathbb{R}^p)$	Space of linear mappings (from $\mathbb{R}^n$ to $\mathbb{R}^p$);
$\mathcal{V}(x)$	the family of all neighborhoods of x;
$C^k(Y; X)$	the space of mappings of class C^k (from Y to X);
$T_x X$	the tangent space at $x \in X$;
$D_G f(x; v)$	Gateaux derivative of f at x along $v \in \mathbb{R}^n$;
$f_K^+(x; v)$	Contingent derivative of f at x along $v \in \mathbb{R}^n$;
$f_Q^+(x; v)$	Quasitangent differentiable of f at x along $v \in \mathbb{R}^n$;
$Q_x^+ X$	Cone quasitangent at $x \in X$;
$K_x^+ X$	Cone contingent at $x \in X$;
$C_x^+ X$	Clarke's tangent cone at $x \in X$;
$AC(\mathbb{R}^n, \mathbb{R}^p)$	Collection of all absolutely continuous functions defined from $\mathbb{R}^n$ to $\mathbb{R}^p$;
$Epi(f)$	The epigraph (the region on or above the graph of a function f);
$Sub(f)$	The subgraph (the region on or below the graph of f);
$\mathcal{P}(M)$	Family of all subsets of M;
$J_f(x)$	Jacobian matrix of $f(.)$ at x;
iff	if and only if;
resp	respectively;
$a.e.(I)$	almost everywhere in I;
i.e.	Latin phrase "id est" meaning "that is";

INTRODUCTION

The differential games theory provides a powerful framework for studying the dynamic interactions between multiple decision-makers or players in different real-world scenarios by finding optimal strategies for each player. This provides valuable insights into the behavior of complex systems and helps guide decision and competitive environments. The development of differential games in the literature can be traced through key milestones and contributions by the mathematician Rufus Isaacs who can be considered a pioneer in the field while working at the RAND Corporation in the early 1950s. After the publication of Isaacs's monograph [31] in 1965, which heuristically solved an astonishing number of conflict problems, there appeared very severe criticisms of his approach (e.g. Blaquière and Leitmann [8], Lidov [36], Berkovitz [7], Krasovskii and Subbotin [33], Chigir [18], Breakwell and John [14], Basar and Olsder [4], Bardi et al. [2], Kamneva et al. [32], Patsko and Turova [48],. . . ,etc.). That led to the appearance of many other theoretical concepts of strategies and value functions for a differential game. Unfortunately, these theories are less concrete than the real-life examples that need to be solved. Since 1982, viscosity theory has been initiated and significantly developed in [21], generating numerous descriptions of the value function as being the solution of a Hamilton-Jacobi-Isaac equation. Due to the great complexity of viscosity solution theory, it has unfortunately not provided a comprehensive solution to any specific problem presented in the literature. We would like to take this opportunity to highlight the significant contributions made to developing a new theory of dynamic programming [42], which provides a fundamental theoretical framework, laying the base for an important theoretical foundation consisting of a certain number of verification theorems with varying degrees of regularity for the value function from differentiability to semicontinuity. The initial verification theorem (Theorem 4.4.1) in [31], which is the only previously known (unjustifiably applied since the value function lacks differentiability). Unlike earlier approaches, the optimality is not defined in the class of saddle points [4, 24], but in class of relatively optimal feedback strategies. As mentioned by Krasovskii and Subbotin [33], Subbotin [51] and Mirică [42], the most practical way to proceed in a differential game is to use pre-calculated feedback strategies. Our approach has been successfully used to solve some famous problems in the literature, among which [11, 13] and more recently [6, 12] which deals with a model in military strategy. Recall that the essential techniques for studying a differential game problem are the Hamiltonian and the value function.

The objectif of our work is to apply the dynamic programming approach [42, 43] and to consolidate numerically these results to obtain a rigorous and complete solution of the Dolichobrachistochrone differential game [31]. This model considers a zero-sum game between two players with totally opposing interests. In fact, this problem has been the subject of much research throughout history with several critics. The first critic of Isaac's solution was Lidov [36], who showed that this solution contains "erroneous statements", which encouraged further research in this area, such as Chigir [18]; Basar and Olsder [4] who studied the problem in the same rather heuristically. In [32], an applicative approach on a modified form of the problem is numerically established without theoretical justification.

Beyond its theoretical significance, the methodology proposed in this study has practical applications across various fields. In robotics and autonomous systems, it can optimize trajectories in dynamic and competitive environments, such as self-driving vehicles navigating traffic or collaborative robots operating in confined spaces. In aerospace engineering, it offers tools for designing interception trajectories or evasion strategies in adversarial scenarios. Additionally, in economic and ecological modeling, the framework addresses competitive resource allocation problems, enabling agents to identify optimal strategies for balancing conflicting objectives. These real-world examples underscore the versatility and applicability of our approach, demonstrating its potential to tackle complex, real-world challenges.

Our thesis contains three chapters, organized as follows:

Chapter 1, serving as an introductory part, presents notations, definitions, and general results of broader interest that will be employed as tools for studying differential games. Initially, a concise overview is provided of basic concepts and results from non-smooth analysis, focusing on tangent cones and generalized derivatives of non-smooth functions. Emphasis is placed on contingent cones, which appear to be more effective than other approaches in the study of game problems. Furthermore, some general results on the monotonicity of real-valued functions are also introduced.

Chapter 2 is dedicated to the synthesis of the new theory of differential games initiated by Mirică (2002-2004). That in turn includes, considerations on the vague and rigorous formulations of a differential game problem, precise definitions of the concepts of admissible pairs and optimal feedback strategy pairs, the statements of verification theorems, and the general algorithm of dynamic programming for autonomous differential games.

Chapter 3 focuses on a detailed study of the Dolichobrachistochrone differential game problem, providing a rigorous, comprehensive and theoretically justifiable solution. This problem was previously incomplete as it had not been addressed in a thorough manner. The main directions explored in this chapter include:

- Developing a synthesis of recent results in non-smooth analysis and their applications in optimal control and differential games.
- Integrating highly complex differential systems with state constraints, obtaining new extended Hamiltonian flows and corresponding maximal intervals.
- Utilizing numerical procedures to determine the maximal intervals of admissible trajectories and the domains covered by these trajectories.
- Identifying a potential barrier (in the sense of Isaacs) that differs from the one previously proposed in Isaacs (1965), along with the possible domain of the associated value function.
- Identifying a feedback strategies providing a complete optimality solution using one of the verification theorems.

Chapter I

CONCEPTS AND AUXILIARY RESULTS FROM NON-SMOOTH ANALYSIS

1.1 Introduction

As is well known, the Hamiltonian and the value function serve as essential tools in the study of differential games. However, these functions are typically non-smooth and defined over arbitrary domains that lack the structure of open sets or differentiable manifolds. This inherent non-differentiability poses significant challenges, both analytically and computationally, in establishing a rigorous mathematical framework for differential games. To meet these challenges, the development of a comprehensive theory of dynamic programming has incorporated the principles of non-smooth analysis. This provides the necessary tools to handle functions that are not classically differentiable, extending the traditional calculus of variations into domains where standard techniques fall short. Using concepts such as subdifferentials, generalised gradients and tangent cones, this theory allows the formulation and solution of problems where the usual smoothness assumptions do not hold.

A cornerstone of non-smooth analysis is the introduction of the subdifferential, a generalised notion of derivative that captures the behaviour of non-smooth functions. This concept plays a central role in characterising optimality conditions and deriving necessary and sufficient criteria for solutions. In addition, the Clarke generalized gradient extends the classical gradient to Lipschitz continuous functions, providing a powerful framework for the analysis of non-smooth dynamics. Another important element is the use of viscosity solutions based on the Hamilton-Jacobi equations. These solutions circumvent the lack of differentiability in the value function and allow for a meaningful interpretation of the equations which govern optimal control problems. The viscosity approach ensures the stability and uniqueness of solutions, even in highly irregular domains. The interplay between non-smooth analysis and dynamic programming has also introduced new perspectives on set-valued mappings and variational inequalities, which are crucial for dealing with the constraints and discontinuities inherent in practical applications.

1.2 Analysis of differentiable mappings on smooth manifolds of $\mathbb{R}^n$

The study of differentiable mappings on submanifolds of $\mathbb{R}^n$ lies at the heart of modern differential geometry, offering profound insights into the interplay between smooth structures and analytical properties. These mappings not only preserve the inherent smoothness of submanifolds but also serve as critical tools for investigating their geometric behaviors and transformations.

In this section, we explore the foundational principles and nuanced conditions under which mappings between submanifolds are considered differentiable. Our investigation highlights their pivotal role in understanding the relationships between local and global geometrical structures. Differentiable mappings allow us to extend local smoothness properties to broader contexts, creating a bridge between abstract theoretical constructs and practical applications. Pioneering researchers like John M. Lee [35], Michael Spivak [41], and Serge Lang [34] have laid the groundwork for the modern understanding of differentiable mappings and their manifold applications. Their seminal works offer valuable perspectives on how these mappings influence the study of smooth structures and geometric frameworks.

Definition 1.2.1. *A function $f(.) : \mathbb{R}^n \to \mathbb{R}^p$ is* ***differentiable*** *at $x_0 \in \mathbb{R}^n$ if there exists a linear map $J_f(x_0) : \mathbb{R}^n \to \mathbb{R}^p$ satisfied:*

$$\lim_{h\to 0} \frac{\|f(x_0+h) - f(x_0) - < J_f(x_0), h > \|}{\|h\|} = 0, \tag{1.2.1}$$

where, $h \in \mathbb{R}^n$ and $J_f(x) = \left(\frac{\partial f_i}{\partial x_j}(x)\right)$, $i = \overline{1,p}$, $j = \overline{1,n}$.

In the case $p = 1$ the derivative defined in (1.2.1) takes the vectorial form:

$$J_f(x) = \nabla f(x), \tag{1.2.2}$$

and in the case where, $n = 1$ (1.2.1) becomes:

$$f'(x_0) = \lim_{t\to 0} \frac{f(x_0+t) - f(x_0)}{t}. \tag{1.2.3}$$

Definition 1.2.2. *The mapping $f(.) : X \subseteq \mathbb{R}^n \to \mathbb{R}^p$ is **Fréchet differentiable** at $x_0 \in Int(X)$ if there exists a mapping $Df(x_0) \in L(\mathbb{R}^n, \mathbb{R}^p)$ such that:*

$$\lim_{\|h\| \to 0} \frac{f(x_0 + h) - f(x_0) - D^T f(x_0)h}{\|h\|} = 0 \in \mathbb{R}^p, \tag{1.2.4}$$

the mapping $Df(x_0)$ is called the Fréchet derivative of $f(.)$ at x_0.

Definition 1.2.3. *Let $f(.) : X \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be a mapping. The **Gateaux directional derivative** of $f(.)$ at a point $x_0 \in X$ in the direction of a vector $v \in \mathbb{R}^n$ is defined as:*

$$D_G f(x_0; v) = \lim_{s \to 0} \frac{f(x_0 + sv) - f(x_0)}{s}. \tag{1.2.5}$$

Definition 1.2.4. *A subset $X \subseteq \mathbb{R}^n$ contains only **isolated points** if for every $x_0 \in X$, there exists $V_{x_0} \in \mathcal{V}(x_0) \subseteq \mathbb{R}^n$ then:*

$$V_{x_0} \cap X = \{x_0\},$$

in other words, there are no other points of X within any open neighborhood of x_0, except for x_0 itself.

Definition 1.2.5. *Let $X \subseteq \mathbb{R}^n$ be an open subset. $f(.) : X \to \mathbb{R}^n$ is called **diffeomorphism** if:*

1. *$f(.)$ is bijective.*
2. *$f(.)$ is smooth (infinitely differentiable) on X.*
3. *The inverse $f^{-1}(.)$ exists and is smooth.*

Definition 1.2.6. *Let $X \subseteq \mathbb{R}^n$ be a subset. $f(.) : X \to \mathbb{R}$ is **homeomorphism** if:*

1. *$f(.)$ is bijective.*
2. *$f(.)$ is continuous.*
3. *The inverse $f^{-1}(.)$ is continuous.*

1.2.1 Differentiable mappings on manifolds of $\mathbb{R}^n$

Definition 1.2.7. *A non-empty subset $X \subset \mathbb{R}^n$ is a **differentiable manifold** (submanifold) of dimension $m \in \{1, 2, ..., n-1\}$ of class C^k, $k \geq 1$, if for each point $x_0 \in X$ there exists $V_{x_0} \in \mathcal{V}(x_0)$, an neighborhood $U \times V$ of $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ and a C^k-diffeomorphism $\alpha(.,.) : U \times V \to V_{x_0}$ (i.e., $\exists \alpha^{-1}(.) : V_{x_0} \to U \times V$ of class C^k) such that:*

$$\alpha(0, 0) = x_0, \ \alpha(U \times \{0\}) = X \cap V_{x_0}.$$

Moreover, the tangent space to X at the point $x = \alpha(u, 0) \in X \cap V_{x_0}$ is the subspace defined by:

$$T_x X = D\alpha(u, 0), \ \text{if } x = \alpha(u, 0).$$

There exists a close relationship between differentiable manifolds, immersions and submersions in differential geometry. Differentiable manifolds provide the underlying smooth structure, while immersions and submersions describe how one manifold maps into another with specific properties regarding the differential. See [9, 14, 35].

Definition 1.2.8. *An **immersion** is a differentiable function $f(.) : X \to Y$ between differentiable manifolds X and Y such that its differential $Df(.) : T_x X \to T_{f(x)} Y$ is injective at every point $x \in X$. Moreover, if its differential $Df(.)$ is surjective at every point $x \in X$ then $f(.)$ is a **submersion**.*

Theorem 1.2.1. *[35] A function $f(.) : X \to Y$ is an immersion if $\dim(X) \leq \dim(Y)$ and $f(.)$ is a submersion if $\dim(X) \geq \dim(Y)$.*

1.2.2 Tangent space of a differentiable manifolds

The tangent space of a differentiable manifold is a critical concept in differential geometry, encapsulating the collection of all tangent vectors at a specific point. These vectors illustrate the possible local directions one can take within the manifold. This foundational idea has been shaped by influential scholars, including C. F. Gauss [26]; B. Riemann [49] and J. Lee [35], each contributing to our understanding of the geometry and structure of manifolds.

Definition 1.2.9. *Let $X \subset \mathbb{R}^n$ be a differentiable manifold and $x \in X$. Consider a differentiable function $\gamma(.) : (-\varepsilon, \varepsilon) \to X$ such that, $\gamma(0) = x$.*

*The **tangent space** of X at x, denoted by T_xX, is given by:*

$$T_xX = \left\{ v \in \mathbb{R}^n;\ v = \left.\frac{d\gamma(t)}{dt}\right|_{t=0} \right\}. \tag{1.2.6}$$

Remark 1.

- *If X is an open set, then it called n-dimensional manifold.*
- *If X contains only isolated points, it called 0-dimensional manifold ([42],[35]). Furthermore the tangent space becomes:*

$$T_xX = \begin{cases} \mathbb{R}^n, & \text{if } X \text{ is an open set,} \\ \{0\}, & \text{if } X \text{ contains only isolated points.} \end{cases} \tag{1.2.7}$$

Proposition 1.2.2. *(Proposition 2.1.4, [42]) If $k \in \mathbb{N}^*$, $m \in \{1, ..., n-1\}$ and $X \subset \mathbb{R}^n$ be a non-empty set then the following assertions are equivalent:*

1. *The set $X \subseteq \mathbb{R}^n$ is a differentiable manifold of class C^k and of dimension m.*
2. *For each $x_0 \in X$, there exists $V_{x_0} \in \mathcal{V}(x_0)$ and a submersion $F(x) \in C^k(V_{x_0}, \mathbb{R}^{n-m})$ (i.e. $DF(x) \in L(\mathbb{R}^n, \mathbb{R}^{n-m})$ is surjective for any $x \in V_{x_0}$), such that:*

$$X \cap V_{x_0} = \left\{ x \in V_{x_0};\ F(x) = 0 \in \mathbb{R}^{n-m} \right\}.$$

In this case, the tangent space at $x \in X \cap V_{x_0}$ is given by:

$$T_xX = \left\{ v \in \mathbb{R}^{n-m};\ DF(x)v = 0 \in \mathbb{R}^{n-m} \right\}. \tag{1.2.8}$$

3. *For each $x_0 \in X$ there exists $V_{x_0} \in \mathcal{V}(x_0)$, $U \in \mathcal{V}(0) \subset \mathbb{R}^m$ and a immersion $\gamma(x) \in C^k(U, X \cap V_{x_0})$ (i.e., $D\gamma(.) \in L(\mathbb{R}^m, \mathbb{R}^n)$ is injective for any $x \in V_{x_0}$) that is also a homeomorphism (i.e., $\gamma^{-1}(.) : X \cap V_{x_0} \to U$ is continuous) such that:*

$$X \cap V_{x_0} = \{\gamma(u);\ u = (u_1, \ldots, u_m) \in U \subseteq \mathbb{R}^m\}.$$

In this case the tangent space at $x \in X \cap V_{x_0}$ is given by:

$$T_xX = \{D\gamma(u)\overline{u} \in \mathbb{R}^n;\ \overline{u} \in \mathbb{R}^n\} \text{ if } x = \gamma(u) \in X \cap V_{x_0}, \tag{1.2.9}$$

where, X is described parametrically by $x = \gamma(u)$ with $u \in U$.

Definition 1.2.10. *A subset $X \subseteq \mathbb{R}^n$ is called **locally closed** if it can be expressed as the intersection of an open set U and a closed set F in $\mathbb{R}^n$, i.e.,*

$$X = U \cap F.$$

1.3 Stratified sets and mappings

The study of stratifications began with H. Whitney in the 1950s in [57], who introduced "Whitney conditions" to ensure the smooth compatibility of strata. R. Thom [52] expanded on these ideas, introducing stability and transversality concepts in singularity theory. H. Hironaka [29] contributed through his resolution of singularities. Next, J. Mather [40] formalized stratified spaces and their stability under mappings, while D. Trotman [53] and J. Brasselet [16] advanced the study of stratifications in relation to transversality and intersection homology. In this section, we will use the concept of stratification as discussed by Whitney [57] and Mirică [42, 47].

A stratified set S in Whitney concept [57] refers to a set of points in a smooth manifold X that is decomposed into strata, each of which is a smooth manifold itself. The stratification respects certain regularity conditions related to singularities and smoothness. In order to rigorously define the concept of stratification, we first introduce the notion of an at most countable partition, which serves as a fundamental component in its mathematical formulation.

Definition 1.3.1. *Let S be a non-empty set. An* ***at most countable*** *partition of S is a collection of subsets $\{S_i\}_{i\in I}$ satisfying the following conditions:*

1. *Disjointness:*
$$S_i \cap S_j = \emptyset, \ \forall i, j \in I \text{ with } i \neq j.$$
2. *Covering:*
$$S = \bigcup_{i\in I} S_i.$$

Definition 1.3.2. *A non-empty subset $X \subseteq \mathbb{R}^n$ is said to be* ***weakly $\mathcal{C}^1$-stratified*** *by S_X if S_X is at most countable and forms a partition of X into differentiable submanifolds of $\mathbb{R}^n$ called strata. The tangent space to X at $x \in X$ with respect to the stratification S_X is defined as:*

$$T_xX = T_xS \text{ if } x \in S \in S_X, \tag{1.3.1}$$

where T_xS denotes the tangent space in Definition 1.2.7. The stratification S_X is said to be of dimension $m \in \{0, 1, \dots, n\}$ if there exists $S_0 \in S_X$ such that $\dim(S_0) = m \geq \dim(S)$ $\forall S \in S_X$ *(for more details, see [42, 52, 57]).*

Lemma 1.3.1. *[57] Let X be a stratified set and let $\{X_i\}_{i\in I}$ be its strata. For any $i,j \in I$, if $X_i \cap Cl(X_j) \neq \emptyset$ then $X_i \subseteq Cl(X_j)$.*

This is the boundary condition of Whitney's stratification, which guarantees that the strata are arranged in a specific order based on their closures.

Next, stratified mappings are functions defined between stratified spaces, respecting their stratifications by preserving the structure of each stratum and ensuring smooth behavior across strata transitions.

Definition 1.3.3. *A mapping $f(.) : X \to \mathbb{R}^p$ is **differentiably-stratified** by S_X if S_X is a stratification of X in the sense of Definition 1.3.2 and for each $S \in S_X$ the restriction of $f(.)$ on S noted by $f_s(.)$ is differentiable. In this case, the derivative of $f(.)$ with respect to the stratification S_X is defined as:*

$$Df_s(x) = Df(x) \in L(T_xX, \mathbb{R}^p) \;\; \forall x \in S \in S_X. \tag{1.3.2}$$

Definition 1.3.4. *Let $w(.) : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a real-valued function. The **bilateral Dini derivatives** of $w(.)$ at $t \in I$ are defined as follows:*

$$\overline{D}w(t) = \lim_{s\to 0} \sup \frac{w(t+s)-w(t)}{s}, \; \underline{D}w(t) = \lim_{s\to 0} \inf \frac{w(t+s)-w(t)}{s},$$

where the function $w(.)$ is defined as the composite:

$$w(t) = W(x(t)), \;\; t \in I, \tag{1.3.3}$$

*and the **unilateral Dini derivatives** of $w(.)$ at $t \in I$ are described by:*

$$\overline{D}^{\pm}w(t) = \lim_{s\to 0^{\pm}} \sup \frac{w(t+s)-w(t)}{s}, \; \underline{D}^{\pm}w(t) = \lim_{s\to 0^{\pm}} \inf \frac{w(t+s)-w(t)}{s}. \tag{1.3.4}$$

Definition 1.3.5. *A function $x(.) \in AC([a,b], \mathbb{R})$, if it satisfies the following equivalent conditions:*

1. ***Condition based on partitions of the interval:*** *For every $\varepsilon > 0$, there exists a $\delta > 0$ such that for any finite collection of disjoint intervals $\{(a_i,b_i)\}$ of $[a,b]$, if:*

$$\sum_{i=1}^{n}(b_i - a_i) < \delta, \quad \text{then} \quad \sum_{i=1}^{n}|x(b_i) - x(a_i)| < \varepsilon.$$

2. ***Condition based on an integrable function:*** *There exists an integrable function $f(.) \in L^1([a,b])$ in such a way that:*

$$x(t) = x(a) + \int_a^t f(s)ds \; \forall t \in [a,b],$$

where, $L^1([a,b])$ denotes the space of Lebesgue integrable functions on the interval $[a,b]$, for which the absolute value of the function has a finite Lebesgue integral, i.e., $\int_a^b |f(x)|\, dx < +\infty$. We remark that, if $x(.) \in AC([a,b])$ then it is differentiable almost everywhere on $[a,b]$ and $x'(t)$ coincides with the integrable function $f(t)$ almost everywhere. Moreover, absolute continuity implies continuity, but not every continuous or uniformly continuous function is absolutely continuous.

Lemma 1.3.2. *(Lemma 2.2.2, [42]) Let $x(.) \in AC([a,b],X)$, if $X \subseteq \mathbb{R}^n$ is C^1-stratified by S_X and J_x^{is}, J_x^d, $J_x \subset [a,b]$ are the subsets defined by:*

$$\begin{aligned} &J_x^{is} = \left\{t \in [a,b];\; t \text{ is isolated in } x^{-1}(S),\; (i.e.,\; x(t) \in S \in S_X)\right\}, \\ &J_x^d = \{t \in [a,b];\; \nexists x'(t)\},\; J_x = J_x^d \cup J_x^{is}, \end{aligned} \tag{1.3.5}$$

where, $x^{-1}(S) = \{t \in [a,b];\; x(t) \in S\}$, then J_x^{is} is at most countable and $J_x \subset [a,b]$ is a null subset (i.e., of zero Lebesgue measure $\mu(J_x)$) and has the following property:

$$\exists x'(t) \in T_{x(t)}X,\;\; \forall t \in [a,b] \backslash J_x,$$

Definition 1.3.6. *Let $x(.) : I = [a,b] \subseteq \mathbb{R} \to \mathbb{R}^m$ is **regular** if:*

$$\exists x'(t^{\pm}) = x'_{\pm}(t) = \lim_{s \to 0^{\pm}} \frac{x(t+s) - x(t)}{s} \; \forall t \in I, \tag{1.3.6}$$

and satisfies the Leibnitz-Newton formula given by:

$$x(t) = x(a) + \int_a^t x'(s)ds,\;\; \forall t \in I. \tag{1.3.7}$$

Lemma 1.3.3. *(Chain Rules, Lemma 2.2.3, [42]) Let $x(.) : [a,b] \to X$ is absolutely continuous. If $W(.) : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable stratified by S_W and if J_x^{is}, J_x^d, $J_x \subset [a,b]$ with conditions in (1.3.6) and (1.3.7), then the bilateral Dini derivatives of the composite function $w\,(.)$ in (1.3.3) satisfy the inequalities:*

$$\overline{D}w(t) \;\geq\; DW(x(t))x'(t) \;\geq\; \underline{D}w(t) \; \forall t \in I \backslash J_x. \tag{1.3.8}$$

Specifically, if $J_w^d = \{t \in I;\; \nexists w'(t)\} \subseteq I$ is the subset of non-differentiability point, then the derivative of the function $w(.)$ is expressed as:

$$w'(t) = DW(x(t))x'(t) \; \forall t \in I \backslash (J_w^d \cup J_x). \tag{1.3.9}$$

The concepts discussed in this section are used in the following chapters to introduce "stratified Hamiltonian systems" and "stratified Hamiltonian flows" and to address and solve important problems in differential game theory.

1.4 Tangent cones and generalized derivatives

This section briefly introduces the notations and main results on tangent cones and related differentiability concepts. These concepts serve as alternatives to classical ones from the literature, particularly when mappings or their effective domains are not stratified or not differentiable on differentiable manifolds. Among the various concepts of tangent cones and tangential approximates discussed in the literature, we will primarily focus on the **Bouligand-Severi contingent** cones and **quasi-tangent** (intermediate) cones in the next chapters. For comparative purposes, we also mention the well-known Clarke's tangent cones and the related ideas of differentiability.

Historically, the contingent cone was simultaneously introduced by **Bouligand** (1930) and **Severi** (1930) and later developed under various names and definitions by researchers e.g., Abadie (1965); Flett (1980); Laurent (1972), Whitney (1965); and others. Similarly, the **quasi-tangent** cone traces its origins to **Federer** (1959), with equivalent formulations (particularly in finite-dimensional spaces) proposed by Girsanov (1972); Ursescu (1976); and others. **Clarke's tangent** cone, introduced by **Clarke** (1975), has been widely applied in numerous studies and publications, including Clarke (1983); Loewen (1993); Rockafellar (1979) and Mirica (2004) have significantly advanced the understanding of tangential structures.

We first recall that the unilateral contingent cone $K_x^{\pm}X$, the quasitangent cone $Q_x^{\pm}X$ and the Clarke cone $C_x^{\pm}X$ to a non-empty subset $X \subseteq \mathbb{R}^n$ at x are defined as subsets respectively, as follows:

$$
\begin{aligned}
K_x^{\pm}X = \;& \{v \in \mathbb{R}^n;\ \exists v_k \to v,\ \exists s_k \to 0^{\pm}:\ x + s_k v_k \in X\ \forall k \in \mathbb{N}\},\\
Q_x^{\pm}X = \;& \{v \in \mathbb{R}^n;\ \exists v_k \to v,\ \forall s_k \to 0^{\pm}:\ x + s_k v_k \in X\ \forall k \in \mathbb{N}\},\\
C_x^{\pm}X = \;& \{v \in \mathbb{R}^n;\ \forall x_k \in X,\ x_k \to x,\ \forall s_k \to 0^{\pm},\ \exists v_k \to v:\\
& x + s_k v_k \in X\ \forall k \in \mathbb{N}\},
\end{aligned}
\tag{1.4.1}
$$

it has a relation as follows:

$$
\text{if } \tau_x \in \{K_x,\ Q_x,\ C_x\},\ \tau_x^{-}X = -\tau_x^{+}X \text{ and } C_x^{\pm}X \subseteq Q_x^{\pm}X \subseteq K_x^{\pm}X, \tag{1.4.2}
$$

For each type of tangent cone $\tau_x \in \{K_x, Q_x, C_x\}$, the corresponding unilateral τ-set-valued directional derivative of a set-valued mapping $F(.) : X \subseteq \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^m)$ at a given point $(x, y) \in Graph(F(.)) = \{(x, y);\ x \in X,\ y \in F(x)\}$ is described as:

$$\tau_y F(x; u) = \left\{ v \in \mathbb{R}^m;\ (u, v) \in \tau_{(x,y)} Graph(F),\ \forall u \in \tau_x X \right\}. \tag{1.4.3}$$

From (1.4.1) it follows that the τ-set-valued directional derivative of $F(.)$ at the point $(x, y) \in Graph(F(.))$, as given in (1.4.3) in the direction $\tau_x X$ is expressed as:

$$\begin{aligned}
K_y^{\pm} F(x; u) = \quad & \{v \in \mathbb{R}^m;\ \exists (s_k, u_k, v_k) \to (0^{\pm}, u, v) : \\
& x + s_k u_k \in X,\ y + s_k v_k \in F(x + s_k u_k)\ \forall k \in \mathbb{N}\}, \\
Q_y^{\pm} F(x; u) = \quad & \{v \in \mathbb{R}^m;\ \forall s_k \to 0^{\pm}\ \exists (u_k, v_k) \to (u, v) : \\
& x + s_k u_k \in X,\ y + s_k v_k \in F(x + s_k u_k)\ \forall k \in \mathbb{N}\}, \\
C_y^{\pm} F(x; u) = \quad & \{v \in \mathbb{R}^m;\ \forall (x_k, y_k, s_k) \to (x, y, 0^{\pm}),\ \exists (u_k, v_k) \to (u, v) \\
& y_k \in F(x_k) : x_k + s_k u_k \in X,\ y_k + s_k v_k \in F(x_k + s_k u_k)\ \forall k \in \mathbb{N}\}.
\end{aligned} \tag{1.4.4}$$

Based on the inclusions in (1.4.2) and the derivatives in (1.4.4) are related as follows:

$$C_y^{\pm} F(x; u) \subseteq Q_y^{\pm} F(x; u) \subseteq K_y^{\pm} F(x; u). \tag{1.4.5}$$

While, the set valued contingent directional derivatives (respectively, quasitangent and Clarke) of $f(.) : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ at a point $x \in X$ in the direction $u \in K_x^{\pm} X$ (respectively, $u \in Q_x^{\pm} X$ and $u \in C_x^{\pm} X$), which are expressed as follows:

$$\begin{aligned}
K^{\pm} f(x; u) = \quad & \{v \in \mathbb{R}^m;\ \exists (s_k, u_k) \to (0^{\pm}, u) : x + s_k u_k \in X, \\
& \tfrac{f(x + s_k u_k) - f(x)}{s_k} \to v\ \forall k \in \mathbb{N}\},\ K^- f(x; u) = -K^+ f(x; -u). \\
Q^{\pm} f(x; u) = \quad & \{v \in \mathbb{R}^m;\ \forall s_k \to 0^{\pm}\ \exists u_k \to u : x + s_k u_k \in X \\
& \tfrac{f(x + s_k u_k) - f(x)}{s_k} \to v\ \forall k \in \mathbb{N}\}, \\
C^{\pm} f(x; u) = \quad & \{v \in \mathbb{R}^m;\ \forall x_k \in X\ \forall (x_k, s_k) \to (x, 0^{\pm})\ \exists u_k \to u : \\
& x_k + s_k u_k \in X,\ \tfrac{f(x_k + s_k u_k) - f(x_k)}{s_k} \to v\ \forall k \in \mathbb{N}\}.
\end{aligned} \tag{1.4.6}$$

Definition 1.4.1. *The mapping $f(.) : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is said to be* ***contingent differentiable*** *at $x \in X$ in direction $u \in K_x^+ X$ if:*

$$\exists f_K^+(x; u) = \lim_{(s,v) \to (0^+, u)} \frac{f(x + sv) - f(x)}{s},\ x + sv \in X, \tag{1.4.7}$$

or, equivalently if:

$$\exists f_K^-(x; -u) = \lim_{(s,v) \to (0^-, -u)} \frac{f(x + sv) - f(x)}{s} = -f_K^+(x; u), \tag{1.4.8}$$

and is said to be bilaterally contingent differentiable in direction $u \in K_x X$ *if:*

$$\exists f'_K(x;u) = \lim_{(s,v)\to(0,u)} \frac{f(x+sv)-f(x)}{s} = f_K^{\pm}(x;u); \tag{1.4.9}$$

or, equivalently if:

$$K_x^{cd}(f) = \left\{u \in K_x X;\ \exists f'_K(x;u) = f_K^{+}(x;u) = f_K^{-}(x;u)\right\} = K_x X. \tag{1.4.10}$$

A key aspect of our analysis involves the contingent set-valued directional derivatives of an absolutely continuous mapping $x(.) : [a,b] \subset \mathbb{R} \to \mathbb{R}$ at a specific point $t \in I = [a,b]$ in direction $r \in K_t^{\pm} I$ defined by:

$$\begin{aligned} K^{\pm}x(t;r) = \quad & \{v \in \mathbb{R}^n;\ \exists (s_m, r_m) \to (0^{\pm}, r) : t + s_m r_m \in I, \\ & \tfrac{x(t+s_m r_m)-x(t)}{s_m} \to v\}, \end{aligned} \tag{1.4.11}$$

which can have the following properties:

$$\begin{aligned} K^{\pm}x(t;1) = \quad & \{v \in \mathbb{R}^n; \exists s_m \to 0^{\pm},\ t + s_m \in I : \tfrac{x(t+s_m)-x(t)}{s_m} \to v\}, \\ K^{\pm}x(t;r) = \quad & \begin{cases} rK^{\pm}x(t;1) & \text{if } r > 0, \\ -rK^{\pm}x(t;-1) & \text{if } r < 0. \end{cases} \end{aligned} \tag{1.4.12}$$

1.5 Contingent derivatives of marginal functions

In the subsequent results, we will rely extensively on the extreme contingent derivatives in the sense of Mirică [42], let $g(.) : X \subseteq \mathbb{R}^n \to \mathbb{R}$, the associated extreme generalized derivatives at $x \in X$ defined by:

$$\begin{aligned} \overline{D}_\tau g(x;u) = \quad & \sup\left\{v \in \mathbb{R};\ (u,v) \in \tau_{(x,g(x))} Sub(g)\right\}, \\ \underline{D}_\tau g(x;u) = \quad & \inf\left\{v \in \mathbb{R};\ (u,v) \in \tau_{(x,g(x))} Epi(g)\right\},\ u \in \tau_x X, \end{aligned} \tag{1.5.1}$$

where, *epigraph subgraph* respectively, described by:

$$\begin{aligned} Epi(g) = \left\{(x,y); x \in X, y \geq g(x)\right\}, \\ Sub(g) = \left\{(x,y); x \in X, y \leq g(x)\right\}. \end{aligned} \tag{1.5.2}$$

Therefore:

$$\begin{aligned} \overline{D}_K^{\pm} g(x;u) \quad &= \lim_{(s,v)\to(0^{\pm},v)} \sup_{x+sv\in X} \tfrac{g(x+sv)-g(x)}{s},\ u \in K_x^{\pm} X. \\ \underline{D}_K^{\pm} g(x;u) \quad &= \lim_{(s,v)\to(0^{\pm},v)} \inf_{x+sv\in X} \tfrac{g(x+sv)-g(x)}{s}. \end{aligned} \tag{1.5.3}$$

Based on the properties in (1.4.8), these are connected through the following relationships:

$$\begin{aligned} \overline{D}_\tau^+ g(x;-v) &= -\underline{D}_\tau^- g(x;v) \\ \underline{D}_\tau^+ g(x;-v) &= -\overline{D}_\tau^- g(x;v),\ \forall v \in \tau_x^- X,\ x \in X. \end{aligned} \tag{1.5.4}$$

Additionally, certain results can be improved by applying the extreme quasitangent directional derivatives, similarly to how the quasitangent cones in (1.4.1) is used. However, these cones have more complex descriptions (see, for example Rockafellar [50]):

$$\begin{aligned} \overline{D}_Q^\pm g(x;u) &= \inf_{s_m \to 0^\pm} \sup_{u_m \to u} \lim_{m\to\infty} \inf_{x+s_m u_m \in X} \tfrac{g(x+s_m u_m)-g(x)}{s_m},\ \forall u \in Q_x^\pm X, \\ \underline{D}_Q^\pm g(x;u) &= \sup_{s_m \to 0^\pm} \inf_{u_m \to u} \lim_{m\to\infty} \sup_{x+s_m u_m \in X} \tfrac{g(x+s_m u_m)-g(x)}{s_m}, \end{aligned} \tag{1.5.5}$$

where, the sequences $s_m \to 0^\pm$, $u_m \to u$ are taken such that, $x + s_m u_m \in X\ \forall m \in \mathbb{N}$. On the other hand, Clarke's extreme directional derivatives defined as in (1.5.1) by the Clarke's tangent cones in (1.4.6) particularly, if $g(.)$ is locally-Lipschitz at $x \in Int(X)$ one has:

$$\underline{D}_C^- g(x;u) = \underline{D}_C^+ g(x;u) = \lim_{(y,s)\to(x,0^+)} \sup \frac{g(y+su)-g(y)}{s} = -\underline{D}_C^+ g(x;-u).$$

It is interesting to note that $\inf \emptyset = +\infty$, $\sup \emptyset = -\infty$. Certain results from nonsmooth analysis can be stated in weaker forms using the corresponding $\tau-$semidifferentials, which are defined by:

$$\begin{aligned} \underline{\partial}_\tau g(x) &= \left\{p \in \mathbb{R}^n;\ <p,v> \le \underline{D}_\tau^+ g(x;v)\ \forall v \in \tau_x^+ X\right\}, \\ \overline{\partial}_\tau g(x) &= \left\{p \in \mathbb{R}^n;\ <p,v> \ge \overline{D}_\tau^+ g(x;v)\ \forall v \in \tau_x^+ X\right\}. \end{aligned} \tag{1.5.6}$$

From (1.4.2) the generalized derivatives in (1.5.1) – (1.5.6) for $\tau_x \in \{K_x, Q_x, C_x\}$, are related in the following manner:

$$\begin{aligned} &\underline{D}_K^\pm g(x;u) \le \underline{D}_Q^\pm g(x;u) \le \underline{D}_C^\pm g(x;u), \\ &\overline{D}_K^\pm g(x;u) \ge \overline{D}_Q^\pm g(x;u) \ge \overline{D}_C^\pm g(x;u), \\ &\underline{\partial}_K g(x) \subseteq \underline{\partial}_Q g(x) \subseteq \underline{\partial}_C g(x), \\ &\overline{\partial}_K g(x) \subseteq \overline{\partial}_Q g(x) \subseteq \overline{\partial}_C g(x). \end{aligned} \tag{1.5.7}$$

According to the Proposition 2.3.8 in [42], for any $x \in X$ and $u \in K_x^\pm X$; the extreme contingent derivatives in (1.5.3) have the following properties:

$$\begin{aligned} &D^\pm g(x;\lambda u) = \lambda D^\pm g(x;u)\ \forall \lambda > 0,\ D^\pm \in \left\{\overline{D}_K^\pm, \underline{D}_K^\pm\right\}, \\ &\underline{D}_K^- g(x;-u) = -\overline{D}_K^+ g(x;u),\ \overline{D}_K^- g(x,-u) = -\underline{D}_K^+ g(x;u), \\ &\underline{D}_K^\pm g(x;0) \in \{-\infty, 0\},\ \overline{D}_K^\pm g(x;0) \in \{0, +\infty\}. \end{aligned} \tag{1.5.8}$$

According to some results (e.g., Bruckner (1978); Saks (1937); Thomson (1985); Mirică (1995),..., etc.), these properties apply to the real function $w(.) : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$, which like the mapping of a single real variable in (1.4.11), have the following additional properties:

$$\begin{aligned}
&D_K^{\pm} w(t;r) = rD_K^{\pm} w(t;1)\ \forall r>0,\ D_K^{\pm} \in \left\{\overline{D}_K^{\pm}, \underline{D}_K^{\pm}\right\},\\
&\overline{D}_K^{\pm} w(t;1) = \lim_{s\to 0^{\pm}} \sup \tfrac{w(t+s)-w(t)}{s} = -\underline{D}^{\pm} w(t;-1) = \overline{D}^{\pm} w(t),\\
&\underline{D}_K^{\pm} w(t;1) = \lim_{s\to 0^{\pm}} \inf \tfrac{w(t+s)-w(t)}{s} = -\overline{D}^{\pm} w(t;-1) = \underline{D}^{\pm} w(t),\\
&\overline{D}_K^{\pm} w(t;0^+) = \lim_{(s,r)\to(0^{\pm},0^{\pm})} \sup \tfrac{w(t+sr)-w(t)}{s},\\
&\underline{D}_K^{\pm} w(t;0^+) = \lim_{(s,r)\to(0^{\pm},0^{\pm})} \inf \tfrac{w(t+sr)-w(t)}{s},
\end{aligned} \tag{1.5.9}$$

where, $D^{\pm}w(x)$ denote the well known Dini derivatives in Definition 1.3.4.

Finally, we note that certain results can be expressed using the contingent semidifferentials of a real function $g(.) : X \subseteq \mathbb{R}^n \to \mathbb{R}$ at $x \in X$ as defined in (1.5.6), through the extreme contingent derivatives in (1.5.9):

$$\begin{aligned}
\overline{\partial}_K g(x) &= \left\{p \in \mathbb{R}^n; < p, u > \geq \overline{D}_K^{+} g(x;u)\ \forall u \in K_x^+ X\right\},\\
\underline{\partial}_K g(x) &= \left\{p \in \mathbb{R}^n; < p, u > \leq \underline{D}_K^{+} g(x;u)\ \forall u \in K_x^+ X\right\},
\end{aligned} \tag{1.5.10}$$

this can be equivalently defined by the left-contingent directional derivatives, given by:

$$\begin{aligned}
\overline{\partial}_K g(x) &= \left\{p \in \mathbb{R}^n; < p, u > \leq \underline{D}_K^{-} g(x;u)\ \forall u \in K_x^- X\right\},\\
\underline{\partial}_K g(x) &= \left\{p \in \mathbb{R}^n; < p, u > \geq \overline{D}_K^{-} g(x;u)\ \forall u \in K_x^- X\right\}.
\end{aligned} \tag{1.5.11}$$

In fact, there are other equivalent definitions, such as those proposed in [42]:

$$\begin{aligned}
\overline{\partial}_K g(x) &= \left\{p \in \mathbb{R}^n;\ \lim_{h\to 0} \sup_{x+h\in X} \tfrac{g(x+h)-g(x)-<p,h>}{\|h\|} \leq 0\right\},\\
\underline{\partial}_K g(x) &= \left\{p \in \mathbb{R}^n;\ \lim_{h\to 0} \inf_{x+h\in X} \tfrac{g(x+h)-g(x)-<p,h>}{\|h\|} \geq 0\right\}.
\end{aligned} \tag{1.5.12}$$

Using a term apparently introduced by Hiriart-Urruty (1978), we define a marginal function as an extended real-valued function of the following form:

$$f(x) = \inf_{h(y)=x} g(y), \tag{1.5.13}$$

where, $g(.) : Y \subseteq \mathbb{R}^m \to \mathbb{R}$ and $h(.) : Y \to X \subseteq \mathbb{R}^n$. The multifunction of minimal points and its effective domain is associated with:

$$\widehat{Y}(x) = \underset{h(y)=x}{\arg\min} g(y),\ x \in X. \tag{1.5.14}$$

We note that the apparently more general marginal functions of the form:

$$f(x) = \inf_{z \in F(x)} \widetilde{g}(x,z),\ \widehat{F}(x) = \{z \in F(x);\ \widetilde{g}(x,z) = f(x),\ x \in X\}\,. \tag{1.5.15}$$

It can clearly be expressed in the form, with suitable choices of $g(.)$ and $h(.)$, such that:

$$x = h(y),\ g(y) = \widetilde{g}(x,z) \text{ if } y = (x,z) \in Y = \{(x,z);\ x \in X, z \in F(x)\}\,. \tag{1.5.16}$$

In the following, we will primarily rely on the notations and definitions provided previously. We note that:

$$\begin{aligned} &g(y) \geq f(x)\ \forall y \in h^{-1}(x) = \{y \in Y;\ h(y) = x\}\,,\ x \in h(Y), \\ &g(y) = f(h(y)) = f(x)\ \forall y \in \widehat{Y}(x) \text{ if } x \in \widetilde{X}. \end{aligned} \tag{1.5.17}$$

Our main result concerning the extreme contingent derivatives of the marginal function in (1.5.13) is the following:

Theorem 1.5.1. *(Extreme contingent derivatives of marginal functions, [42]) If* $f(.)$ *is the marginal function in* (1.5.13)*,* $\widehat{Y}(.)$ *is the multifunction of minimal points and* $x \in \widetilde{X} = dom\widehat{Y}(.)$ *then, the following properties hold:*

1. The extreme contingent derivatives satisfy the inequalities:

$$\begin{aligned} &\overline{D}_K^- f(x;u) \geq \sup\{\underline{D}_K^- g(y;v);\ y \in \widehat{Y}(x),\ v \in K_y^- Y,\ u \in K^- h(y;v)\}, \\ &\underline{D}_K^+ f(x;u) \leq \inf\{\overline{D}_K^+ g(y;v);\ y \in \widehat{Y}(x),\ v \in K_y^+ Y,\ u \in K^+ h(y;v)\}. \end{aligned} \tag{1.5.18}$$

2. If $g(.)$ is **locally-Lipschitz** then:

$$\begin{aligned} &\overline{D}_K^- f(x;u) \geq \sup\{\overline{D}_K^- g(y;v);\ y \in \widehat{Y}(x),\ v \in Q_y^- Y,\ u \in Q^- h(y;v)\}, \\ &\underline{D}_K^+ f(x;u) \leq \inf\{\underline{D}_K^+ g(y;v);\ y \in \widehat{Y}(x),\ v \in Q_y^+ Y,\ u \in Q^+ h(y;v)\}. \end{aligned} \tag{1.5.19}$$

3. If $h(.)$ is **contingent differentiable** then one has:

$$\begin{aligned} &\overline{D}_K^- f(x;u) \geq \sup\{\overline{D}_K^- g(y;v);\ y \in \widehat{Y}(x),\ v \in Q_y^- Y,\ u = h_K^-(x;v)\}, \\ &\underline{D}_K^+ f(x;u) \leq \inf\{\underline{D}_K^+ g(y;v);\ y \in \widehat{Y}(x),\ v \in Q_y^+ Y,\ u = h_K^+(x;v)\}; \end{aligned} \tag{1.5.20}$$

4. If the marginal function itself, $f(.)$ is **locally-Lipschitz** then one has:

$$\begin{aligned} &\underline{D}_K^- f(x;u) \geq \sup\{\underline{D}_K^- g(y,v);\ y \in \widehat{Y}(x),\ v \in K_y^- Y,\ u \in K^- h(y;v), \\ &\overline{D}_K^+ f(x;u) \leq \inf\{\overline{D}_K^+ g(y,v);\ y \in \widehat{Y}(x),\ v \in K_y^+ Y,\ u \in K^+ h(y;v). \end{aligned} \tag{1.5.21}$$

1.6 Necessary conditions for monotonicity

In this section, we review classical monotonicity results and present recent advancements applicable to very general classes of functions, including those that may lack semi-continuity. Drawing from the extensive literature on real functions, we focus on monotonicity results that are particularly effective for composite function in (1.3.3) where, $x(.) : I \to Y \subseteq \mathbb{R}^n$ is absolutely continuous while, $W(.) : Y \to \mathbb{R}$ and respectively, $w(.)$ are often exhibit limited regularity. These monotonicity conditions are expressed in terms of "extreme contingent derivatives" computed using the chain rules outlined in Proposition 1.6.1. To apply the necessary conditions for monotonicity, we require derivative rules for composite functions in the previous part, that estimate the extreme contingent derivative of the function $w(.)$ in (1.3.3), in terms of the generalized derivatives of the mappings $W(.)$ and $x(.)$.

Definition 1.6.1. *A function $f(.) : \mathbb{R}^n \to \mathbb{R}$ is said to be **radially locally-Lipschitz** at $x \in \mathbb{R}^n$, if for every $v \in \mathbb{R}^n$ (with $v \neq 0$), there exists $\delta > 0$ and a constant $L > 0$ such that for all t_1, $t_2 \in (-\delta, \delta)$ the following inequality holds:*

$$|f(x + t_1 v) - f(x + t_2 v)| \leq L\,|t_1 - t_2|\,.$$

Proposition 1.6.1. *([42]) For any $t \in I = [a, b]$ and $r \in \{1, 0^+\}$ then, the following statements hold:*

1. The contingent derivatives in (1.5.9) satisfy:

$$\begin{aligned} \overline{D}_K^{\pm} w(t; r) &\geq \sup_{v \in K^{\pm} x(t; r)} \underline{D}_K^{\pm} W(x(t); v), \\ \underline{D}_K^{\pm} w(t; r) &\leq \inf_{v \in K^{\pm} x(t; r)} \overline{D}_K^{\pm} W(x(t); v). \end{aligned} \tag{1.6.1}$$

2. If $x(.)$ is in one-sided differentiable at t then one has:

$$\underline{D}_K^{\pm} W(x(t); x'_{\pm}(t)) \leq \underline{D}_K^{\pm} w(t; 1) \leq \overline{D}_K^{\pm} w(t; 1) \leq \overline{D}_K^{\pm} W(x(t); x'_{\pm}(t)). \tag{1.6.2}$$

3. If $x(.)$ is differentiable at t and $W(.)$ is locally-Lipschitz at $x(t) \in Y$ then one has:

$$\begin{aligned} \overline{D}_K^{\pm} w(t; 1) &= \overline{D}_K^{\pm} W(x(t); x'(t)), \\ \underline{D}_K^{\pm} w(t; 1) &= \underline{D}_K^{\pm} W(x(t); x'(t)), \\ \overline{D}_K^{\pm} w(t; 0^+) &= \overline{D}_K^{\pm} W(x(t); 0) = 0, \\ \underline{D}_K^{\pm} w(t; 0^+) &= \underline{D}_K^{\pm} W(x(t); 0) = 0. \end{aligned} \tag{1.6.3}$$

4. If $x(.)$ and $w(.)$ are differentiable at t and $W(.)$ is locally-Lipschitz at $x(t) \in Y$ then $W(.)$ is contingent differentiable at $x(t)$ in direction $x'(t) \in Q_{x(t)}Y$ and one has:

$$w'(t) = W_K^{\pm}(x(t); x'(t)) = W_Q^{\pm}(x(t); x'(t)). \quad (1.6.4)$$

Theorem 1.6.2. *(Necessary properties of increasing functions, [42]) If $w(.)$ is increasing then it has the following properties:*

1. $w(.)$ is regulated in the sense that: $\exists w(t^{\pm}) = \lim_{s \to 0^{\pm}} w(t+s)\ \forall t \in I$.
2. $w(.)$ is a.e differentiable and satisfies:

$$w'(t) = \overline{D}_K^{\pm} w(t; 1) \geq 0\ a.e.(I)\ \text{(i.e., } \forall t \in I \backslash N_w, \mu(N_w) = 0). \quad (1.6.5)$$

3. $w(.)$ is upper semicontinuous to the left in the sense that:

$$S_w^-(t) = \lim_{s \to 0^-} \sup w(t+s) \leq w(t)\ \forall t \in I, \quad (1.6.6)$$

and also lower semicontinuous to the right in the sense that:

$$I_w^+(t) = \lim_{s \to 0^+} \inf w(t+s) \geq w(t)\ \forall t \in I. \quad (1.6.7)$$

4. The extreme contingent and quasitangent derivatives in (1.5.3) at $t \in I$ in any direction $r \in \{1, 0^+\}$ satisfy:

$$\overline{D}_K^{\pm} w(t; r) \geq \max\left\{\overline{D}_Q^{\pm} w(t; r), \underline{D}_Q^{\pm} w(t; r)\right\} \geq \underline{D}_K^{\pm} w(t; r). \quad (1.6.8)$$

In what follows we consider a constrained differential inclusion:

$$\begin{gathered} x'(t) \in F(x(t)) \text{ a.e. } (I_Y(x(.)),\ x(0) = y \in Y, \\ x(t) \in Y\ \forall t \in I_Y(x(.)) = dom(x(.)) \subseteq \mathbb{R}, \end{gathered} \quad (1.6.9)$$

where, $F(.) : X \subseteq \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ and $Y \subseteq X$; for any $y \in Y$ denotes the set of absolutely continuous solutions of (1.5.18) by $S_{F,Y}(y)$. Let a real function $g_0(.,.) : Z \subseteq Y \times \mathbb{R}^n \to \mathbb{R}$, $W(.)$ is said to be $(F, g_0)-$increasing (resp, decreasing) if for any $y \in Y$, $x(.) \in S_{F,Y}(y)$ the associated real function:

$$w_x(t) = W(x(t)) + \int g_0(x(s), x'(s))ds,\ t \in I_Y(x(.)), \quad (1.6.10)$$

is increasing (resp., decreasing).

Theorem 1.6.3. *(For locally-Lipschitz functions, [42]) If the function $W(.)$ is* ***locally-Lipschitz****, $Q^{cd}_x(W)$, $w \in Y$ are the subsets of directions $v \in Q_xY$ at which $W(.)$ is bilaterally contingent differentiable at $x \in Y$ such that:*

$$Q^{cd}_x(W) = \left\{v \in Q_xY;\ \exists (W)'_K(x;v) = (W)^{\pm}_K(x;v)\right\}, \tag{1.6.11}$$

if $W(.)$ satisfies the following differential inequality:

$$(W)'_K(x;v) + g_0(x,v) \geq 0\ \forall v \in F_Q(x) \cap Q^{cd}_x(W),\ x \in Y, \tag{1.6.12}$$

then $W(.)$ is $(F, g_0)-$increasing.

To introduce the next chapter, we first outline the fundamental properties of "smooth Hamiltonian and characteristic flows" on differentiable manifolds. These properties are then used to construct both classical and generalized "characteristic solutions" of autonomous Hamilton-Jacobi equations, which are useful for applications in differential games and optimal control.

1.7 Smooth Hamiltonian and characteristic flows

This section explores how the value functions of autonomous problems in the calculus of variations and optimal control serve as generalized or classical (i.e., differentiable) solutions to autonomous Hamilton-Jacobi equations of the form:

$$H(x, DW(x)) = 0,\ x \in Y_0 \subset \mathbb{R}^n,\ W(x) = g(x),\ x \in Y_1 \subset Cl(Y_0), \tag{1.7.1}$$

defined by the Hamiltonians $H(.,.) : Z \subseteq \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $g(.) : Y_1 \to \mathbb{R}$; we significantly extend the classical results to the case in which the domain $Z \subseteq \mathbb{R}^n \times \mathbb{R}^n$ of the Hamiltonian $H(.,.)$ is differentiable submanifold. To provide fundamental tools for the upcoming sections, we will now introduce the key properties of a "smooth Hamiltonian" and its associated "characteristic flows". In this context, the main concept is defined as follows:

Definition 1.7.1. *If $Z \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is a differentiable manifold and the Hamiltonian $H(.) : Z \to \mathbb{R}$ is of class C^1, then* ***the geometric Hamiltonian orientor field*** *associated to $H(.)$ on all $z = (x,p) \in Z$ is the multifunction defined by:*

$$\begin{aligned} d^\sharp H(z) = \ & \{(x',p') \in T_zZ;\ < x',v > - < p',u >= \\ & DH(z).(u,v)\ \forall (u,v) \in T_zZ,\ z \in Z\}. \end{aligned} \tag{1.7.2}$$

To characterize the above Hamiltonian orientor field , we first consider the following hypothesis:

Hypothesis 1.7.1. *(Hypothesis 4.1.2, [42]) The subset $Z \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is a class C^2 submanifold, $H(.) : Z \to \mathbb{R}$ is of class C^2 and there exists $h(.) : Z \to \mathbb{R}^n \times \mathbb{R}^n$ of class C^1, a selection of the multifunction $d^\sharp H(.)$ in* (1.7.2)*, satisfying:*

$$h(z) = (h_1(z), h_2(z)) \in d^\sharp H(z) \; \forall z = (x, p) \in Z, \tag{1.7.3}$$

alternatively, a smooth function that meets the following condition:

$$\begin{aligned} &< h_1(z), v > - < h_2(z), u >= DH(z).(u, v), \\ &\forall (u, v) \in T_z Z, \;\; h(z) \in T_z Z \; \forall z \in Z. \end{aligned} \tag{1.7.4}$$

Remark 2. *If the domain $Z \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is a manifold of maximum dimension $2n$ (i.e., it is open subset) then the geometric orientor field in* (1.7.2) *is a single-valued and coincides with the classical Hamiltonian vector field given by:*

$$d^\sharp H(z) = \left\{ (\frac{\partial H}{\partial p}(z), -\frac{\partial H}{\partial x}(z)) \right\}, \;\; z \in Z = int(Z), \tag{1.7.5}$$

in this case, one has:

$$< DH(z), (u, v) >=< \frac{\partial H}{\partial x}(z), u > + < \frac{\partial H}{\partial p}(z), v > \;\; \forall u \in \mathbb{R}^n, \;\; v \in \mathbb{R}^n. \tag{1.7.6}$$

If the set $Z \subset \mathbb{R}^n \times \mathbb{R}^n$ is not of the maximum dimension $2n$, then at every point $z \in Z$, the value $d^\sharp H(z) \subset T_z Z$ is either empty or an affine manifold that is parallel to the linear subspace $d^\sharp H_0(z) \subseteq \mathbb{R}^n \times \mathbb{R}^n$, which is defined similarly by the null-Hamiltonian, given by:

$$\begin{aligned} &H_0(z) = 0 \; \forall z \in Z, \; \forall (u, v) \in T_z Z, \\ &d^\sharp H_0(z) = \{(x', p') \in T_z Z; \;\; < x', v > - < p', u >= 0\}, \end{aligned} \tag{1.7.7}$$

in the sense that:

$$d^\sharp H(z) = \{(x'_0, p'_0)\} + d^\sharp H_0(z) \; \forall (x'_0, p'_0) \in T_z Z. \tag{1.7.8}$$

As it is well known, in the classical case i.e., $Z = int(Z) \subseteq \mathbb{R}^n \times \mathbb{R}^n$ the essential tool for Cauchy's method of characteristics is the associated system:

$$\begin{cases} x' = \frac{\partial H}{\partial p}(z), & x(0) = \xi, \\ p' = -\frac{\partial H}{\partial x}(z), & p(0) = q \in Z(\xi), \\ v' =< p, \frac{\partial H}{\partial p}(z) >, & v(0) = v_0 \in \mathbb{R}, \end{cases} \tag{1.7.9}$$

where, $Z(\xi) = \{q \in \mathbb{R}^n;\ (\xi, q) \in Z\}$. Due to the particular form of the characteristic vector field:

$$c(z,v) = (x', p', v') = \left(\frac{\partial H}{\partial p}(z), -\frac{\partial H}{\partial x}(z), < p, \frac{\partial H}{\partial p}(z) > \right),\ (z,v) \in Z \times \mathbb{R}, \quad (1.7.10)$$

it is clearly simplified to the smooth (classical) Hamiltonian system:

$$\begin{cases} x' = \frac{\partial H}{\partial p}(z),\ x(0) = \xi, \\ p' = -\frac{\partial H}{\partial x}(z),\ p(0) = q \in Z(\xi), \end{cases} \quad (1.7.11)$$

which, in the case Hypothesis 1.7.1 is satisfied, is replaced by the generalized Hamiltonian system:

$$(x', p') = h(z),\ (x(0), p(0)) = (\xi, q) \in Z, \quad (1.7.12)$$

where, $h(.,.)$ defined in (1.7.3).

1.8 Cauchy's method of characteristics

Building on the results from the previous part, we derive both classical and generalized characteristic solutions for autonomous Hamilton-Jacobi equations of the form as in (1.7.1). These solutions are designed for effective use in optimal control. Our work significantly extends the classical Cauchy method of characteristics (e.g. Benton (1977); Hartman (1964); Mirică (1987); Subbotin (1994); Van et al (2000), etc.) in two important aspects:

1. The domain of the Hamiltonian in (1.7.1) is not necessarily an open set.

2. The first component of the Hamiltonian flow does not need to be invertible.

Definition 1.8.1. *A mapping $f(.,.) : D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ defines the ordinary differential equation:*

$$\frac{dx}{dt} = f(t,x),\ x(s) = y,\ (s,y) \in D, \quad (1.8.1)$$

*$f(.,.)$ is called **vector field**. If $I \subseteq \{(t \in \mathbb{R};\ \exists x \in \mathbb{R}^n,\ (t,x) \in D\}$ is an interval, then a mapping $\varphi(.) : I \to \mathbb{R}^n$ is called **a classical solution** of the Cauchy problem in* (1.8.1) *if $\varphi(.)$ is differentiable with*

$$\frac{d\varphi}{dt}(t) = f(t, \varphi(t)) \text{ for all } t \in I = dom(\varphi(.)),\ \varphi(s) = y.$$

The mapping $\varphi(.)$ *is called* ***Carthéodory solution*** *of the Cauchy problem in* (1.8.1) *if* $\varphi(.)$ *is locally absolutely continuous (i.e., it is absolutely continuous on every compact subinterval* $I_0 \subseteq I$*) and satisfies:*

$$\frac{d\varphi}{dt}(t) = f(t, \varphi(t)) \ a.e.(I), \ \varphi(s) = y.$$

We begin by addressing the case where the data $H(.,.)$ and $g(.)$ from problem (1.7.1) satisfy the following assumptions:

Hypothesis 1.8.1. *(Hypothesis 4.2.1, [42]) The Hamiltonian* $H(.,.) : Z \subseteq \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *and the Hamiltonian vector field* $h(.,.) : Z \to \mathbb{R}^n \times \mathbb{R}^n$ *satisfy Hyoithesis* (1.7.1) *(i.e., Z is a differentiable manifold,* $H(.,.)$ *and* $h(.,.)$ *are of class* C^1 *and satisfy* (1.7.3)*) the boundary set* $Y_1 \subset Cl(Y_0)$ *is a differentiable manifold, and* $g(.) : Y_1 \to \mathbb{R}$ *is a differentiable function the* ***initial characteristic strip*** Z_1^S *is described by:*

$$Z_1^S = \{(\xi, q) = (x(0), p(0)) \in Z; \ \xi \in Y_1, \ H(z) = 0, < q, v >= Dg(\xi).v \ \forall v \in T_\xi Y_1 \neq \emptyset\}. \tag{1.8.2}$$

In this case, the characteristic flow associated to the problem in (1.7.1) is described by:

Definition 1.8.2. *If the data* $H(.,.)$ *and* $g(.)$ *satisfy Hypothesis 1.8.1, and the characteristic flow* $C^*(.,.) = (X^*(.,.), V(.,.))$ *defined by:*

$$\begin{aligned} &B = \left\{(t, z) \in D_h \subseteq \mathbb{R} \times Z; \ z \in Z_1^S, \ X(t) \in Y_0 \ if \ t \neq 0\right\}, \\ &X^*(t, z) = (X(t, z), P(t, z)), \ (t, z) \in B, \\ &V(t, z) = g(\xi) + \int < P(t, z), X'(t, z)dt \ if \ z \in Z_1^S, \ (t, s) \in B, \end{aligned} \tag{1.8.3}$$

is called the ***classical characteristic flow*** *associated to the problem in* (1.7.1).

If Hypothesis 1.8.1 is satisfied and $C^*(.,.) = (X^*(.,.), V(.,.))$ is the associated characteristic flow in (1.8.3) then:

$$H(X^*(t, z)) = 0 \ \forall (t, z) \in dom(C^*(.,.)). \tag{1.8.4}$$

According to Theorem 4.2.4 in [42]. Let Hypothesis 1.8.1 be satisfied, let $C^*(.,.) = (X^*(.,.), V(.,.))$ be the associated characteristic flow and let $\widetilde{B}_0 \subseteq B_0$ be a submanifold of dimension n of $\mathbb{R}^n$ such that $\widetilde{Y}_0 = X(\widetilde{B}_0) \subseteq \mathbb{R}^n$ is open and the restriction:

$$\widetilde{X}_0(.,.) = X(.,.) \mid \widetilde{B}_0 : \widetilde{B}_0 \to \widetilde{Y}_0 \subseteq Y_0, \tag{1.8.5}$$

is invertible where its differentiable inverse is $\widehat{B}_0(.) : \widetilde{Y}_0 \to \widetilde{B}_0$, i.e., satisfying:

$$\begin{aligned} &X(\widehat{B}_0(y)) = y \ \forall y \in \widetilde{Y}_0, \\ &\widehat{B}_0(X(t,z)) = (t,z) \ \forall (t,z) \in \widetilde{B}_0, \end{aligned} \tag{1.8.6}$$

then, the real function $\widetilde{W}(.)$ defined as:

$$\widetilde{W}(y) = \begin{cases} g(y) & \text{if } y \in Y_1, \\ V(\widehat{B}_0(y)) & \text{if } y \in \widetilde{Y}_0 \subseteq Y_0, \end{cases} \tag{1.8.7}$$

is a solution of the problem in (1.7.1) on the subset $\widetilde{Y} = \widetilde{Y}_0 \cup Y_1$, $Y_1 \subset Cl(Y_0)$. While in the case the first component, $X(.,.)$ is not invertible, the natural candidate for the value function should be the marginal function:

$$W_m(y) = \begin{cases} g(y) & \text{if } y \in Y_1, \\ \inf\limits_{y=X(t,a)} V(t,a) & \text{if } y \in Y_0, \end{cases} \tag{1.8.8}$$

moreover, the corresponding multifunction of minimal points and its domain:

$$\begin{aligned} &\widehat{B}_0(y) = \{(t,a) \in B_0;\ X(t,a) = y,\ V(t,a) = W(y)\}, \\ &\widetilde{Y}_0 = \{y \in X(B_0);\ \widehat{B}_0(y) \neq \emptyset\} \subseteq X(B_0) \subseteq Y_0, \end{aligned} \tag{1.8.9}$$

define in a natural way the corresponding "generalized field of extremals":

$$\mathcal{A}(y) = \{x_{t,a}(s) = X(s+t,a);\ \text{with } (t,a) \in \widehat{B}_0(y),\ (s+t) \in [t,0]\}, \tag{1.8.10}$$

whose "value function" is the function $W(.)$ in (1.8.8).

On the other hand, one may choose solutions defined by a "maximum-type characteristic":

$$W_M(y) = \begin{cases} g(y) & \text{if } y \in Y_1, \\ \sup\limits_{y=X(t,a)} V(t,a) & \text{if } y \in Y_0, \end{cases} \tag{1.8.11}$$

which can be used in other types of problems such as in differential game theory. We note that the smooth maximal flow $X_h^*(.,.) = (X(.,.), P(.,.)) : D_h \subseteq \mathbb{R} \times Z \to Z$, form (1.7.2), (1.7.3) and using Lemma 1.3.3, we obtain:

$$\frac{\partial H}{\partial t}(X_h^*(t,z)) = 0 \ \forall (t,z) \in D_h,$$

hence, $H\left(X_h^*(.,z)\right) \forall z \in Z$ is constant.

Chapter II

AUTONOMOUS DIFFERENTIAL GAMES

2.1 Introduction

Differential game theory represents a vast and significant field in mathematics, providing a foundation for analyzing dynamic interactions between decision-makers. In this chapter, we introduce a robust theoretical algorithm of Dynamic Programming developed by Mirică in [42, 43]. Dynamic Programming algorithm not only generalizes existing solutions with greater rigor compared to currently known approaches but also demonstrates the optimality of an admissible control pair by utilizing one of the most recent verification theorems in [44]. This work seeks to connect theoretical advancements with practical applications in differential games.

2.2 Formulation of a Differential Game

In the following, we deal with the problem of the autonomous differential game (DG) formulated as follows:

Problem 1. (DG) *Find:*

$$\inf_{u(.)\in\mathcal{U}_\alpha} \sup_{v(.)\in\mathcal{V}_\alpha} \mathcal{C}(y;u(.),v(.)),\ \ \forall y\in Y_0, \tag{2.2.1}$$

where

$$\mathcal{C}(y;u(.),v(.)) = g(x(T)) + \int_0^T f_0(x(t),u(t),v(t))dt, \tag{2.2.2}$$

subject to the following constraints:

$$x'(t) = f\left(x(t),u(t),v(t)\right)\ \ a.e.(0,T),\ \ x(0)=y, \tag{2.2.3}$$

$$u(t)\in U,\ \ v(t)\in V\ \ a.e.(0,T),\ \ (u(.),v(.))\in\mathcal{P}_1=\mathcal{U}_1\times\mathcal{V}_1, \tag{2.2.4}$$

$$x\left(.\right)\in\Omega_1 = AC\left([0,T];\mathbb{R}^n\right),\ \ f_0\left(x\left(.\right),u\left(.\right),v\left(.\right)\right)\in L^1\left([0,T];\mathbb{R}\right), \tag{2.2.5}$$

$$x(t)\in Y_0\ \forall t\in[0,T),\ \ x(T)\in Y_1, \tag{2.2.6}$$

the problem is defined by the following data:

- U, V *are two sets of control parameters;*
- $Y_0 \subset \mathbb{R}^n$ *and* $Y_1 \subset Cl(Y_0)$ *represent the set of initial states and the set of terminal states respectively;*
- $T > 0$ *is the terminal time;*
- $L^1([0,T];\mathbb{R})$ *represents the space of Lebesgue integrable functions;*
- $\mathcal{C}(y; u(.), v(.))$ *is the cost functionals;*
- $g(.) : Y_1 \to \mathbb{R}$ *is the terminal function;*
- $f_0(.,.,.) : D \times U \times V \to \mathbb{R}$ *is the running function;*
- $f(.,.,.) : D \times U \times V \to \mathbb{R}^n$ *is the parameterized vector field;*
- $\mathcal{P}_\alpha \in \{\mathcal{P}_1, \mathcal{P}_\infty, \mathcal{P}_r\}$ *is the class of admissible control-pairs associated to the class of admissible trajectories* $\Omega_\alpha \in \{\Omega_1, \Omega_\infty, \Omega_r\}$.

We will discuss the possible choices for the class of admissible trajectories Ω_α and their corresponding controls $\mathcal{P}_\alpha$. In many optimal control books, the class Ω_α is not always explicitly stated. As Mirică notes in (Section 1.1.3, [42]), it is typically assumed that the most general class of "absolutely continuous trajectories" is considered in such cases. However, in certain situations, particularly when dealing with mathematical models, specific classes of trajectories are required. These may include locally Lipschitzian trajectories or regular trajectories, depending on the nature of the problem. The class Ω_α, which is one of the following sets:

$$\begin{aligned}
\Omega_1 &= \left\{x(.) \in AC / \ \exists x'(.) \in L^1\right\}, \\
\Omega_\infty &= \{x(.) \in AC / \ \exists x'(.) \text{ piecewise continuous}\}, \\
\Omega_r &= \{x(.) \in AC / \ \exists x'(.) \text{ regulated}\}.
\end{aligned}$$

The specified class $\mathcal{U}_\alpha$ (resp, $\mathcal{V}_\alpha$) of admissible controls $u(.)$ (resp, $v(.)$) is usually one of the following subsets of the measurable mappings $M(I, U)$ (resp, $M(I, V)$) from I to U (resp, V):

$$\mathcal{U}_1 = \{u(.) : I \subset \mathbb{R} \to U/\ u(.) \text{ measurable}\},$$
$$\mathcal{U}_\infty = \{u(.) : I \subset \mathbb{R} \to U/\ u(.) \text{ piecewise continuous}\},$$
$$\mathcal{U}_r = \{u(.) : I \subset \mathbb{R} \to U/\ u(.) \text{ regulated}\}.$$

Remark 3. *As in [42, 43], we are dealing here with the largest class of absolutely continuous trajectories* $\Omega_1 = AC([0, T]; \mathbb{R}^n)$. *The corresponding* $\mathcal{P}_1 = \mathcal{U}_1 \times \mathcal{V}_1$ *represents the measurable admissible controls* $(u(.), v(.))$.

2.3 Admissible feedback strategies and their relative optimality

As noted by Krasovskii and Subbotin [33], Subbotin [51], and Mirică [42], the practical execution of differential games relies heavily on the use of pre-calculated feedback strategies. These strategies are essential as they provide a structured and actionable framework, enabling players to make decisions dynamically in response to the evolving state of the game.

Hypothesis 2.3.1. *Problem* 1 *satisfies the following assumptions:*

1. *The set* U, V *are compact and* $f(., ., .)$, $f_0(., ., .)$ *are continuous on the open domain* $D \subseteq \mathbb{R}^n$.
2. *The mapping* $f(., ., .)$ *is locally-Lipschitz with respect to the first variable.*

Remark 4. *[42] If Hypothesis 2.3.1 is satisfied, then for any pair of measurable control mappings* $(u(.), v(.)) : [0, \tau) \to U \times V$ *and any initial point* $y \in Y_0 \subset D$*, the differential system in* (2.2.3) *has a unique Carathéodory (AC) solution:*

$$x(.) = x_y^{u,v}(.) : [0, \tau) \to \mathbb{R}^n, \tag{2.3.1}$$

that automatically satisfies conditions (2.2.4) *and* (2.2.5). *However, for the trajectory* $x(.)$ *to be admissible, it must also satisfy the state and terminal constraints in* (2.2.6). *Specifically, there exists* $T \in (0, \tau)$ *such that* $x(.)$ *satisfies the state constraint* $x(t) \in Y_0$ $\forall t \in [0, T)$ *and the terminal constraint* $x(T) \in Y_1$*, where* $T = T(u(.), v(.)) \in (0, \tau)$ *(depending on the controls* $u(.)$ *and* $v(.)$*) at which* (2.2.6) *is satisfied and defines the terminal rules of the game.*

As can be seen, from Problem 1, there are two players $\mathbb{U}$ and $\mathbb{V}$ that at each initial state $y_0 \in Y_0$, may choose an admissible control $u(.) \in \mathcal{U}$ and $v(.) \in \mathcal{V}$ respectively. The sets $\mathcal{U}$ and $\mathcal{V}$ are defined by:

$$\mathcal{U} = \mathcal{M}([0,\tau);U),\ \mathcal{V} = \mathcal{M}([0,\tau);V), \tag{2.3.2}$$

where, $\mathcal{M}(I;U)$ (resp, $\mathcal{M}(I;V)$) represents the set of Lebesgue measurable functions from $I \subseteq \mathbb{R}$ to U (resp, V). The player $\mathbb{U}$ tries to minimize the functional $C(.,v(.))$, while player $\mathbb{V}$ tries to maximize the functional $C(u(.),.)$. In order to simplify some of the formulations in the sequel, we denote by:

$$\begin{aligned} &\Omega_\alpha(y) = \{x(.) \in \Omega_\alpha;\ \exists\,(u(.),v(.)) \in P_\alpha :\ x(.) = x_y^{u,v}(.)\},\ \forall y \in Y_0,\\ &P_\alpha(y) = \{(u(.),v(.)) \in P_\alpha;\ \exists x(.) := x_y^{u,v}(.) \in \Omega_\alpha(y)\}, \end{aligned} \tag{2.3.3}$$

where, $x_y^{u,v}(.)$ is a solution as defined in (2.3.1). Considering the formulation of Problem 1, we will introduce the following notations for the sets of relatively admissible controls-pairs:

$$\begin{aligned} \mathcal{U}_v(y) &= \{u(.) \in \mathcal{U}_\alpha;\ x_y^{u,v}(.) \in \Omega_\alpha(y)\},\ v(.) \in \mathcal{V}_\alpha,\ y \in Y_0,\\ \mathcal{V}_u(y) &= \{v(.) \in \mathcal{V}_\alpha;\ x_y^{u,v}(.) \in \Omega_\alpha(y)\},\ u(.) \in \mathcal{U}_\alpha. \end{aligned} \tag{2.3.4}$$

Definition 2.3.1. *A feedback strategy of player $\mathbb{U}$ is described as a set-valued mapping $\widetilde{U}(.) : Y_0 \to \mathcal{P}(U)$ which satisfies the condition $\widetilde{U}(x) \subset U(x)$ for all $x \in Y_0$. This mapping determines the following parameter-dependent differential inclusion:*

$$x' \in f\left(x, \widetilde{U}(x), v(t)\right),\ x(0) = y \in Y_0,\ v(.) \in \mathcal{V}_\alpha, \tag{2.3.5}$$

where, its admissible trajectories satisfying (2.2.3) *and* (2.2.6)*, are defined as follows:*

$$\begin{aligned} \overline{\Omega}_v(y) = \quad &\{\overline{x}_{y,v}(.);\ \exists \overline{u}_v(.) \in \mathcal{U}_v(y) : \overline{x}'_{y,v}(t) = f(\overline{x}_{y,v}(t), \overline{u}_v(t), v(t)),\\ &\overline{u}_v(t) \in \widetilde{U}\left(\overline{x}_{y,v}(t)\right) a.e.(0,\bar{t}_1),\ y \in Y_0,\ v(.) \in \mathcal{V}_\alpha\ \}, \end{aligned} \tag{2.3.6}$$

such that, the corresponding sets $\overline{\mathcal{U}}_v(y)$, $y \in Y_0$ of admissible controls that satisfy (2.3.6) *are denoted by:*

$$\overline{\mathcal{U}}_v(y) = \{\overline{u}_v(.) \in \mathcal{U}_v(y);\ \exists \overline{x}_{y,v} \in \overline{\Omega}_v(y)\},\ y \in Y_0, v(.) \in \mathcal{V}_\alpha, \tag{2.3.7}$$

the corresponding upper value function $\overline{W}_{\widetilde{U}}(.)$ (i.e., the best for player $\mathbb{V}$) is defined by:

$$\overline{W}_{\widetilde{U}}(y) = \sup_{v(.)\in\mathcal{V}_\alpha,\ \overline{u}_v(.)\in\overline{\mathcal{U}}_v(y)} C(\overline{u}_v(.), v(.)),\ y \in Y_0. \tag{2.3.8}$$

Similarly, a feedback strategy for the player $\mathbb{V}$ is described as a set-valued mapping $\widetilde{V}(.) : Y_0 \to \mathcal{P}(V)$ which satisfies the condition $\widetilde{V}(x) \subset V(x)$ for all $x \in Y_0$. This mapping determines the following parameter-dependent differential inclusion:

$$x' \in f\left(x, u(t), \widetilde{V}(x)\right), \ x(0) = y \in Y_0, \ u(.) \in \mathcal{U}_\alpha, \tag{2.3.9}$$

which, its admissible trajectories satisfying (2.2.3) and (2.2.6), are defined as follows:

$$\begin{aligned} \overline{\Omega}_u(y) = \ & \{\overline{x}_{y,u}(.); \ \exists \overline{v}_u(.) \in \mathcal{V}_u(y) : \overline{x}'_{y,u}(t) = f(\overline{x}_{y,u}(t), u(t), \overline{v}_u(t)), \\ & \overline{v}_u(t) \in \widetilde{V}\left(\overline{x}_{y,u}(t)\right) \ a.e.(0, \bar{t}_1), \ y \in Y_0, \ u(.) \in \mathcal{U}_\alpha\}, \end{aligned} \tag{2.3.10}$$

while the sets $\overline{\mathcal{V}}_u(y)$, $y \in Y_0$ of admissible controls that satisfy (2.3.10) are denoted by:

$$\overline{\mathcal{V}}_u(y) = \left\{\overline{v}_u(.) \in \mathcal{V}_u(y); \ \exists \overline{x}_{y,u} \in \overline{\Omega}_u\left(y\right)\right\}, \ y \in Y_0, \ u(.) \in \mathcal{U}_\alpha, \tag{2.3.11}$$

and the corresponding lower value function $\underline{W}_{\widetilde{V}}(.)$ (i.e., the best for player $\mathbb{U}$) is defined as:

$$\underline{W}_{\widetilde{V}}(y) = \inf_{u(.) \in \mathcal{U}_\alpha, \ \overline{v}_u(.) \in \overline{\mathcal{V}}_u(y)} C\left(u(.), \overline{v}_u(.)\right), \ y \in Y_0. \tag{2.3.12}$$

We note that $(\widetilde{U}(.), \widetilde{V}(.)) : Y_0 \to \mathcal{P}(U) \times \mathcal{P}(V)$ represents a pair of feedback strategies that define the differential inclusion:

$$x' \in f\left(x, \widetilde{U}(x), \widetilde{V}(x)\right), \ x(0) = y \in Y_0. \tag{2.3.13}$$

Its admissible trajectories, which satisfy (2.2.3) and (2.2.6), are defined as follows:

$$\begin{aligned} \overline{\Omega}(y) = \ & \{x_y(.); \ \exists (u_y(.), v_y(.)) : x'_y(t) = f(x_y(t), u_y(t), v_y(t)), \\ & u_y(t) \in \widetilde{U}(x_y(t)), \ v_y(t) \in \widetilde{V}(x_y(t)) \ a.e. \left(0, T\left(x_y(t)\right)\right) \ \}, \ y \in Y_0, \end{aligned} \tag{2.3.14}$$

while the sets of pairs $P_{\widetilde{U},\widetilde{V}}(y)$, $y \in Y_0$ of corresponding admissible controls that satisfy conditions (2.3.14) are defined as:

$$P_{\widetilde{U},\widetilde{V}}(y) = \{(u_y(.), v_y(.)) : \exists x_y(.) \in \overline{\Omega}(y) \text{ satisfying } (2.3.14)\}, \tag{2.3.15}$$

and $\left(\overline{W}_{\widetilde{U}}(.), \underline{W}_{\widetilde{V}}(.)\right)$ are the pairs of the upper (respectively, lower) value functions associated, which are defined as follows:

$$\begin{aligned} \underline{W}_{\widetilde{U},\widetilde{V}}(y) &= \inf_{(u_y,v_y) \in P_{\widetilde{U},\widetilde{V}}(y)} C(y; u_y(.), v_y(.)), \\ \overline{W}_{\widetilde{U},\widetilde{V}}(y) &= \sup_{(u_y,v_y) \in P_{\widetilde{U},\widetilde{V}}(y)} C(y; u_y(.), v_y(.)). \end{aligned} \tag{2.3.16}$$

Definition 2.3.2. *We say that the multifunction* $\left(\widetilde{U}(.),\widetilde{V}(.)\right) : \widetilde{Y}_0 \subseteq Y_0 \to \mathcal{P}(U) \times \mathcal{P}(V)$ *represents* ***an admissible pair of feedback strategies*** *for the restriction of Problem 1 on the subset* $\widetilde{Y}_0 \subseteq Y_0$*, if the following conditions are satisfied:*

- *For each* $y \in \widetilde{Y}_0$*, the set* $\overline{\Omega}(y)$ *of (AC) solutions* $x_y(.) : [0,\widetilde{t}_1] \to \widetilde{Y}_0$ *of the parametrized differential inclusion in* (2.3.13) *that satisfy the constraints:*

$$x_y(t) \in \widetilde{Y}_0 \ \forall t \in \left[0,\widetilde{t}_1\right), \ x_y\left(\widetilde{t}_1\right) \in \widetilde{Y}_1, \tag{2.3.17}$$

is not empty.

- *If* $\widetilde{P}_\alpha(y) = \{(u_y(.), v_y(.))\}$*,* $y \in \widetilde{Y}_0$ *is the set of corresponding control-pairs in* (2.3.15)*, which satisfy:*

$$\begin{aligned} &x_y'(t) = f(x_y(t), u_y(t), v_y(t)), \\ &u_y(t) \in \widetilde{U}(x_y(t)), \ v_y(t) \in \widetilde{V}(x_y(t)) \ \text{a.e.} \left(0,\widetilde{t}_1\right), \end{aligned} \tag{2.3.18}$$

thus, the corresponding value function is well-defined as follows:

$$\widetilde{W}(y) = \begin{cases} g(y) \ \text{ if } y \in \widetilde{Y}_1, \\ \widetilde{W}_0(y) = C(u_y(.), v_y(.)) \ \text{ if } (u_y(.), \ v_y(.)) \in \widetilde{P}_\alpha(y), \ y \in \widetilde{Y}_0, \end{cases} \tag{2.3.19}$$

where the subset $\widetilde{Y}_1 \subseteq Y_1$ *of endpoints of all admissible trajectories in* $\overline{\Omega}(y)$ *for* $y \in \widetilde{Y}_0$ *is defined as:*

$$\widetilde{Y}_1 = \{x_y(\widetilde{t}_1); \ x_y(.) \in \overline{\Omega}(y), \ y \in \widetilde{Y}_0\}. \tag{2.3.20}$$

The second formula in (2.3.19) implies that if the sets $\widetilde{P}_\alpha(y)$ contain more than one element they are considered admissible in the sense that:

$$\begin{aligned} &C(u_y^1(.), v_y^1(.)) = C(u_y^2(.), v_y^2(.)) \ y \in \widetilde{Y}_0. \\ &\forall (u_y^i(.), v_y^i(.)) \in \widetilde{P}_\alpha(y), \ i = 1, 2, \end{aligned} \tag{2.3.21}$$

Definition 2.3.3. *The multifunction* $\left(\widetilde{U}(.),\widetilde{V}(.)\right) : \widetilde{Y}_0 \subseteq Y_0 \to \mathcal{P}(U) \times \mathcal{P}(V)$ *is considered to define* ***a relatively optimal pair*** *of feedback strategies for the restriction of Problem 1 on the subset* $\widetilde{Y}_0 \subseteq Y_0$*, if it satisfies the admissibility condition in the sense of Definition 2.3.2 and its associated value function* $\widetilde{W}_0(.)$ *in* (2.3.19)*, satisfies the following conditions:*

1. *Whene player $\mathbb{V}$ selects the feedback strategy $\widetilde{V}(.)$, the best choice for player $\mathbb{U}$ is given by $\widetilde{U}(.)$, in the sense that:*

$$\widetilde{W}_0(y) = \underline{W}_{\widetilde{V}}(y) = \inf_{u(.)\in\mathcal{U}_\alpha,\ \overline{v}_u(.)\in\overline{\mathcal{V}}_u(y)} C\left(y; u\left(.\right), \overline{v}_u\left(.\right)\right),\ y \in \widetilde{Y}_0. \tag{2.3.22}$$

2. *Symetrically, when player $\mathbb{U}$ chooses the feedback strategy $\widetilde{U}(.)$, the optimal response for player $\mathbb{V}$ is $\widetilde{V}(.)$, in the sense that:*

$$\widetilde{W}_0(y) = \overline{W}_{\widetilde{U}}(y) = \sup_{v(.)\in\mathcal{V}_\alpha,\ \overline{u}_v(.)\in\overline{\mathcal{U}}_v(y)} C(y; \overline{u}_v(.), v(.)),\ y \in \widetilde{Y}_0. \tag{2.3.23}$$

Remark 5. *From Definitions 2.3.2-2.3.3 the optimal pair of feedback strategies $\left(\widetilde{U}(.), \widetilde{V}(.)\right)$ for the restriction of Problem 1 on the subset $\widetilde{Y}_0 \subseteq Y_0$ is **a siddle point** such that:*

$$\overline{W}_{\widetilde{U},V_1}(y) \le \widetilde{W}_0(y) = \overline{W}_{\widetilde{U}}(y) = \underline{W}_{\widetilde{V}}(y) \le \underline{W}_{U_1,\widetilde{V}}(y),\ \forall y \in \widetilde{Y}_0, \tag{2.3.24}$$

for any other choices of feedback strategies $U_1(.)$, $V_1(.)$.

2.4 Verification theorems for admissible feedback strategies

The term verification Theorem was apparently first used by R. Isaacs (1965) and more explicitly by W.H. Fleming and R.W. Rishel (1975). Which in turn can be regarded as certain sufficient optimality conditions of the dynamic programming type. The starting point for the possible contributions of the dynamic programming method to differential game theory is the following result, which is a natural extension of the dynamic programming principle in optimal control theory. Using the fact that, the value function $\widetilde{W}(.)$ in (2.3.19) coincides with each of the value function of two symmetric Bolza optimal control problems $\mathcal{B}_1$ and $\mathcal{B}_2$, the first one denoted by $\mathcal{B}_1$, focuses on minimizing the following functional:

$$\begin{aligned} &\min_{u\in U} C_1(y; x(.)) \text{ where} \\ &C_1(y; x(.)) = g(x(T)) + \textstyle\int_0^T f_0^1(x(t), x'(t))\, dt,\ y \in Y_0,\ \text{with} \\ &f_0^1(x, x') = f_0(x, u, v),\ x' = f(x, u, v) \text{ if } u \in U,\ v \in \widetilde{V}(x), \end{aligned} \tag{2.4.1}$$

subject to the differential inclusion:

$$x' \in F_{\widetilde{V}}(x) = f(x, U, \widetilde{V}(x)),\ x(0) = y, \tag{2.4.2}$$

with the initial and terminal constraints as in (2.2.6). The second one, $\mathcal{B}_2$ consists to maximize the following functional:

$$\begin{aligned}&\max_{v\in V} C_2(y;x(.)) \text{ where}\\ &C_2(y;x(.)) = g(x(T)) + \textstyle\int_0^T f_0^2(x(t),x'(t))\,dt,\ y\in Y_0,\ \text{with}\\ &f_0^2(x,x') = f_0(x,u,v) \text{ if } x' = f(x,u,v),\ u\in \widetilde{U}(x),\ v\in V,\end{aligned} \tag{2.4.3}$$

subject to the differential inclusion:

$$x' \in F_{\widetilde{U}}(x) = f(x,\widetilde{U}(x),V),\ x(0) = y, \tag{2.4.4}$$

with the initial and terminal constraints as in (2.2.6).

In fact, the development of sufficient optimality conditions in a realistic form is credited by Mirică [42], who introduced new theory in dynamic programming. This work is based on a foundational theoretical framework comprising seven verification theorems [44], each addressing different levels of regularity for the value function $\widetilde{W}(.)$ in (2.3.19), ranging from differentiability to semicontinuity. These results substantially extend the applicability of the elementary verification Theorem (Theorem 4.4.1) in [31, 51] which was previously the only known result and was improperly applied to problems where the value function is not differentiable. In the following, we will present here among them the three verification theorems that will be useful in our work.

Theorem 2.4.1. *(Abstract verification Theorem, [44, 51]) A pair* $\left(\widetilde{U}(.),\widetilde{V}(.)\right)$ *of admissible feedback strategies as defined in Definition 2.3.2, is optimal according to Definition 2.3.3 if and only if the corresponding value function* $\widetilde{W}(.)$ *satisfies the following monotonicity properties:*

1. *For any* $y \in \widetilde{Y}_0$,*with* $u(.) \in \mathcal{U}$ *and* $\overline{v}_u(.) \in \overline{\mathcal{V}}_u(y)$, *the real-valued function* $\overline{w}_u(.)$ *defined as:*

$$\overline{w}_u(t) = \widetilde{W}(\overline{x}_u(t)) + \int_0^t f_0(\overline{x}_u(s),u(s),\overline{v}_u(s))ds,\ t\in[0,T], \tag{2.4.5}$$

increases over the interval $[0,T]$.

2. *For any* $y \in \widetilde{Y}_0$, *with* $v(.) \in \mathcal{V}$ *and* $\overline{u}_v \in \overline{\mathcal{U}}_v(y)$, *the real-valued function* $\overline{w}_v(.)$ *defined as:*

$$\overline{w}_v(t) = \widetilde{W}(\overline{x}_v(t)) + \int_0^t f_0(\overline{x}_v(s),\overline{u}_v(s),v(s))ds,\ t\in[0,T], \tag{2.4.6}$$

decreases over the interval $[0, T]$.

In the context of optimal control, more practical verification theorems can be derived by applying different assumptions on the value function $\widetilde{W}(.)$, using the corresponding monotonicity properties of real-valued functions and the appropriate chain rules in Lemma 1.3.3 for the composite functions $\widetilde{W}(\overline{x}_u(.))$, $\widetilde{W}(\overline{x}_v(.))$ as presented in (2.4.5) and (2.4.6).

Because of the difficulty in verifying the generalized differential inequalities in (2.4.5) and (2.4.6), we assume that the restriction of $\widetilde{W}(.)$ on $\widetilde{Y}_0(.)$, denoted by $\widetilde{W}_0(.)$ in (2.3.19), is differentiable on its open domain i.e., $\widetilde{Y}_0 = Int(\widetilde{Y}_0)$.

Theorem 2.4.2. *(Elementary verification Theorem, [31, 42]) Let* $\left(\widetilde{U}(.), \widetilde{V}(.)\right)$ *be a pair of admissible feedback strategies, and assume that the associated value function* $\widetilde{W}(.)$*, as given in* (2.3.19)*, satisfies the following conditions:*

1. $\widetilde{W}(.)$ *is continuous at the points in* $\widetilde{Y}_1$ *as defined in* (2.3.20)*;*
2. *The subset* $\widetilde{Y}_0 \subseteq Y_0$ *is open and the restriction of* $\widetilde{W}(.)$ *in* $\widetilde{Y}_0$ *denoted by* $\widetilde{W}_0(.)$ *is differentiable and satisfies the fundamental differential inequalities:*

$$\begin{aligned} \sup_{v \in V, \overline{u} \in \widetilde{U}(x)} &\left[D\widetilde{W}_0(x).f(x, \overline{u}, v) + f_0(x, \overline{u}, v)\right] \leq 0. \\ \inf_{u \in U, \overline{v} \in \widetilde{V}(x)} &\left[D\widetilde{W}_0(x).f(x, u, \overline{v}) + f_0(x, u, \overline{v})\right] \geq 0; \end{aligned} \tag{2.4.7}$$

3. *Alternatively, the admissible controls are regulated i.e.,* $\mathcal{P}_\alpha = \mathcal{P}_r$ *or* $\widetilde{W}_0$ *is locally Lipschitz specifically, of class* C^1*.*

Therefore, the pair $\left(\widetilde{U}(.), \widetilde{V}(.)\right)$ *is optimal in the sense of Definition 2.3.3.*

Remark 6. *It can be noted that the differential inequalities in* (2.4.7) *are equivalent to the fundamental equation of Isaacs in [31]:*

$$\begin{aligned} &\min_{u \in U(x)} \max_{v \in V(x)} \left[D\widetilde{W}_0(x).f(x, u, v) + f_0(x, u, v)\right], \\ &= \max_{v \in V(x)} \min_{u \in U(x)} \left[D\widetilde{W}_0(x).f(x, u, v) + f_0(x, u, v)\right], \\ &= D\widetilde{W}_0(x).f(x, \overline{u}, \overline{v}) + f_0(x, \overline{u}, \overline{v}) = 0 \ \forall (\overline{u}, \overline{v}) \in \widetilde{U}(x) \times \widetilde{V}(x). \end{aligned} \tag{2.4.8}$$

Theorem 2.4.2 extends Isaacs' verification Theorem (Theorem 4.4.1 in [31]) and deals with problems such that $\widetilde{W}(.)$ does not satisfy the required conditions, when the domain $\widetilde{Y}_0$ of $\widetilde{W}_0(.)$ is not open and/or the restriction of $\widetilde{W}(.)$ on $\widetilde{Y}_0$ is not differentiable. To overcome this difficulty, we use various refinements and extensions of the Theorem 2.4.2 based on the concepts of non-smooth analysis presented in the previous chapter.

Theorem 2.4.3. *(Verification Theorem for locally-Lipchitz value functions, [44]) Let $\left(\widetilde{U}(.), \widetilde{V}(.)\right)$ represent a pair of admissible feedback strategies as defined in Definition 2.3.2, and assume that $\widetilde{W}(.)$ in* (2.3.19) *has the following properties:*

1. *$\widetilde{W}(.)$ is continuous at the terminal points in $\widetilde{Y}_1$.*

2. *The restriction of $\widetilde{W}(.)$ on $\widetilde{Y}_0$ is locally-Lipschitz. Furthermore, if the multifunctions $U'_K(.;.)$ and $V'_K(.;.)$ are defined as follows:*

$$\begin{aligned} U'_K(x,v) &= \left\{u \in U_K(x;v);\ f(x,u,v) \in D_K(\widetilde{W}_0;x)\right\},\ v \in V, \\ V'_K(x,u) &= \left\{v \in V_K(x;u);\ f(x,u,v) \in D_K(\widetilde{W}_0;x)\right\},\ u \in U, \end{aligned} \tag{2.4.9}$$

then, its bilaterally contingent differentiable in (1.4.9) *satisfy the following differential inequalities:*

$$\begin{aligned} \inf_{u \in U'_K(x,\overline{v}),\ \overline{v} \in \widetilde{V}(x)} & \left[(\widetilde{W}_0)'_K(x; f(x,u,\overline{v})) + f_0(x,u,\overline{v})\right] \geq 0, \\ \sup_{v \in V'_K(x,\overline{v}),\ \overline{u} \in \widetilde{U}(x)} & \left[(\widetilde{W}_0)'_K(x; f(x,\overline{u},v)) + f_0(x,\overline{u},v)\right] \leq 0. \end{aligned} \tag{2.4.10}$$

Consequently, the pair $\left(\widetilde{U}(.), \widetilde{V}(.)\right)$ is optimal in the sense of Definition 2.3.3.

Remark 7. *Alternatively, if the value function $\widetilde{W}_0(.)$ in* 2.3.19 *is locally-Lipschitz, the differential inequalities in* (2.4.10) *can be substituted by what may appear to be "weaker" inequalities:*

$$\begin{aligned} \inf_{u \in U_K(x,\overline{v}),\ \overline{v} \in \widetilde{V}(x)} & \left[\max\left\{\overline{D}^+_K, \overline{D}^-_K\right\} \widetilde{W}_0\left(x; f(x,u,\overline{v})\right) + f_0(x,u,\overline{v})\right] \geq 0, x \in \widetilde{Y}_0, \\ \sup_{v \in V_K(x,\overline{u}),\ \overline{u} \in \widetilde{U}(x)} & \left[\min\left\{\underline{D}^+_K, \underline{D}^-_K\right\} \widetilde{W}_0\left(x; f(x,\overline{u},v)\right) + f_0(x,\overline{u},v)\right] \leq 0. \end{aligned} \tag{2.4.11}$$

Remark 8. *We recall that, in the case of zero-sum differential games studied by Elliott [24], a pair of strategies $\left(\widetilde{U}(.), \widetilde{V}(.)\right) \in \mathcal{P}$ is called optimal with respect to the initial point $x \in \widetilde{Y}_0$ if the saddle-point condition in* (2.3.24) *is satisfied. In contrast, our approach* *defines the optimality of a pair of admissible feedback strategies based on the verification of weaker conditions, expressed by the inequalities in* (2.4.7) , (2.4.8) *and* (2.4.10), *which are easier to check. Additionally, as discussed in Chapter 3, the application of saddle-point optimality conditions appears impractical because the value function is* ***implicitly*** *defined.*

2.5 The general algorithm of Dynamic Programming

To facilitate the application of the extensive results presented in the previous sections to solve the concrete *autonomous differential games problem* (DG) described in Section 2.2, we summarize these findings in the form of a theoretical algorithm:

Step 1: Preliminary operations

1.1. Statement of the problem and identification of data:

Firstly, we start with identifying the problem data:

$$f(.,.,.),\ f_0(.,.,.),\ g(.),\ U,\ V,\ Y_0 \text{ and } Y_1.$$

1.2. Compute the auxiliary Hamiltonians and the sets of extremal points:

$$\begin{aligned}
&\mathcal{H}(z,u,v) =< p, f(x,u,v) > + f_0(x,u,v), \\
&z = (x,p) \in Y \times \mathbb{R}^n,\ Y = Y_0 \cup Y_1, \\
&\mathcal{H}^+(z,u) = \sup_{v\in V} \mathcal{H}(z,u,v),\ \mathcal{H}^-(z,v) = \inf_{u\in U} \mathcal{H}(z,u,v), \\
&H^+(z) = \inf_{u\in U} \mathcal{H}^+(z,u),\ H^-(z) = \sup_{v\in V} \mathcal{H}^-(z,v), \\
&\widehat{U}^+(z) = \{\overline{u} \in U;\ \mathcal{H}^+(z,u) = H^+(z)\}, \\
&\widehat{V}^-(z) = \{\overline{v} \in V;\ \mathcal{H}^-(z,v) = H^-(z)\}, \\
&\widehat{U}(z) = \widehat{U}^+(z),\ \widehat{V}(z) = \widehat{V}^-(z),
\end{aligned} \tag{2.5.1}$$

the Isaacs' condition described by:

$$H(z) = H^+(z) = H^-(z),\ z \in Z \text{ where, } Z \subseteq Y \times \mathbb{R}^n,\ Y = Y_0 \cup Y_1, \tag{2.5.2}$$

where the domain Z of Isaacs' Hamiltonian is given by:

$$Z = \left\{ z \in Y_0 \times \mathbb{R}^n;\ H^+(z) = H^-(z),\ \widehat{U}(z) \neq \emptyset,\ \widehat{V}(z) \neq \emptyset \right\}. \tag{2.5.3}$$

The Hamiltonian at the terminal points:

$$Z_1 = \left\{ z_1 = (\xi, q) \in Y_1 \times \mathbb{R}^n;\ \exists H(z_1) = \lim_{z\to z_1} H(z) \right\}\ z \in Z. \tag{2.5.4}$$

Remark 9. *If Y_0 is not open, then the sets of control parameters $U(.)$ and $V(.)$, should be replaced by the following sets:*

$$\begin{aligned} &U_T(x,v) = \{u \in U(x);\ f(x,u,v) \in T_x Y_0\},\ v \in V(x),\\ &V_T(x,u) = \{v \in V(x);\ f(x,u,v) \in T_x Y_0\},\ u \in U(x),\ x \in Y_0. \end{aligned} \tag{2.5.5}$$

In the general case one should use the contingent cones in (1.4.1) *to define the restricted sets of control parameters given by:*

$$\begin{aligned} &U_K(x,v) = \{u \in U(x);\ f(x,u,v) \in K_x Y_0\},\ v \in V(x),\\ &V_K(x,u) = \{v \in V(x);\ f(x,u,v) \in K_x Y_0\},\ v \in U(x),\ x \in Y_0, \end{aligned} \tag{2.5.6}$$

which produce a contingent Isaacs' Hamiltonian noted by $H_K(.,.)$. To simplify the notation, we will consider only $Y_0 = Int(Y_0)$.

Step 2. Construction of generalized Hamiltonian and characteristic flows

2.1. Set of terminal transversality points:

If the terminal function $g(.) : Y_1 \to \mathbb{R}$ is stratified differentiable (in particular, if Y_1 is a differentiable manifold and $g(.)$ is differentiable), then the set of terminal transversality points is defined as:

$$Z_1^S = \{z_1 = (\xi, q);\ \xi \in Y_1,\ H(z_1) = 0,\ < q, \overline{\xi} >= Dg(\xi)\overline{\xi}\ \forall \overline{\xi} \in T_\xi Y_1\}. \tag{2.5.7}$$

In other cases, use the contingent semidifferentials in (1.5.12) defined by the expression:

$$Z_1^K = \{z_1 = (\xi, q) \in Z_1;\ H(z_1) = 0,\ q \in \partial_K g(\xi)\}.$$

2.2. The Hamiltonian orientor field:

If Isaacs' Hamiltonian $H(.) : Z \to \mathbb{R}$ is stratified differentiable, then the stratified Hamiltonian orientor field $d_s H(.)$ is given by:

$$\begin{aligned} &d_s^{\#} H(z) = \{(x', p') \in T_z Z;\ x' \in f\left(x, \widehat{U}(z), \widehat{V}(z)\right),\\ &< x', \overline{p} > - < p', \overline{x} >= DH(z).(\overline{x}, \overline{p})\ \forall (\overline{x}, \overline{p}) \in T_z Z\}\ z \in Z. \end{aligned} \tag{2.5.8}$$

Noting that on open strata $S \in S_H$ i.e., $\dim(S) = 2n$, the Hamiltonian oriented field described in (2.5.8) coincides with the classical Hamiltonian vector field given in (1.7.5).

On the other hand, if $H(.)$ is not differentiably stratified, then use the contingent semidifferentials in (1.5.10) to compute the contingent Hamiltonian orientor field, described by:

$$\begin{aligned} d_k^\sharp H(z) = \;& \{(x',p') \in K_z Z;\; x' \in f\left(x, \widehat{U}(z), \widehat{V}(z)\right), \\ & (-p',x') \in \partial_K H(z)\},\; z \in Z, \end{aligned}$$

when the Hamiltonian $H(z)$ is differentiable for all $z \in Int(Z)$, then $d_k^\sharp H(z)$ coincides with the classical Hamiltonian orientor field as described in (1.7.5).

2.3. Construction of a generalized Hamiltonian flow:

Choose a Hamiltonian field $d^\sharp H(.) \in \left\{d_s^\sharp H(.), d_k^\sharp H(.)\right\}$ with the corresponding terminal points $Z_1^* = \left\{Z_1^s, Z_1^k\right\}$. Through integrate backwardly for $t \leq 0$, the Hamiltonian differential inclusion:

$$(x',p') \in d^\sharp H(z),\;\; (x(0),p(0)) = (\xi,q) \in Z_1^*, \tag{2.5.9}$$

for each terminal point $z^* = (\xi,q) \in Z_1^*$, we determine the set of maximal solutions $X^*(.)$ such that:

$$X^*(.) = (X(.), P(.)) : I(z^*) = \left(t^-(z^*), 0\right] \to \mathbb{R}^n \times \mathbb{R}^n, \tag{2.5.10}$$

where, $X(.)$ is of type Ω_a, which satisfies the following conditions:

$$\begin{aligned} & X^*(0) = (X(0), P(0)) = z^* \in Z_1^*, \\ & X(t) \in Y_0 \;\forall t \in I_0(z^*) = (t^-(z^*), 0), \\ & H(X(t), P(t)) = 0,\;\; \forall t \in I(z^*), \\ & X'(t) = f(X(t), u_a(t), v_a(t))\;\; a.e.\,(I_0(z^*)), \\ & u_a(t) \in \widehat{U}(X^*(t)),\;\; v_a(t) \in \widehat{V}(X^*(t)),\;\; \forall t \in I_0(z^*). \end{aligned} \tag{2.5.11}$$

If for the same terminal point $z^* \in Z_1^*$ there exists more than one solution (i.e., $card(X^*(z)) > 1$), the set of solutions $X^*(.)$ is parametrized in the following manner:

$$\begin{aligned} & X^*(.,a) = (X(.,a), P(.,a)),\; \text{where} \\ & a = (z^*,\lambda),\;\; z^* = (\xi,q) \in Z_1^*,\;\; \lambda \in \Lambda(z) \subset \mathbb{R}^m, \end{aligned} \tag{2.5.12}$$

then, we determine $t^-(a)$, $I(a)$, $I_0(a)$ as well as B and B_0 given by:

$$\begin{aligned} t^-(a) &= \inf_{\tau} \{\tau < 0;\ X(t,a) \in Y_0\ \forall t \in (\tau, 0]\}, \\ I(a) &= (t^-(a), 0],\ a = (z^*, \lambda) \in A = Z_1^* \times \Lambda(z), \\ I_0(a) &= (t^-(a), 0), \\ B &= \{(t,a);\ a \in A,\ t \in I(a)\}, \\ B_0 &= \{(t,a);\ a \in A,\ t \in I_0(a)\}. \end{aligned} \tag{2.5.13}$$

2.4. The generalized characteristic flow:

Characterize the function $V(.,a)$ given by:

$$V(t,a) = g(\xi) + \int_0^t < P(s,a), X'(s,a) > ds,\ a = (z^*, \lambda) \in A, \tag{2.5.14}$$

where, $X'(.,a)$ is the derivative of the mapping $X(.,a)$. Note also that if the subset $B_0 \subseteq (-\infty, 0) \times A$ is defined as:

$$\begin{aligned} B_0 &= \{(t,a);\ a \in A,\ t \in I_0(a)\}, \\ A &= \{a = (z^*, \lambda);\ z^* \in Z_1^*,\ \lambda \in \Lambda(z^*)\}. \end{aligned} \tag{2.5.15}$$

Then, from the definition above, it follows that for any $(t,a) \in B_0$, the functions:

$$\begin{aligned} u_{t,a}(s) &= u_a(t+s),\ v_{t,a}(s) = v_a(t+s), \\ x_{t,a}(s) &= X(t+s, a),\ s \in [0, -t], \end{aligned} \tag{2.5.16}$$

are the admissible controls (respectively, the corresponding trajectories) with respect to the initial point $y = X(t,a) \in Y_0$. Furthermore, the cost function in (2.2.2) is given by the formula:

$$\mathcal{C}(y; u_{t,a}(.), v_{t,a}(.)) = V(t,a) \text{ if } y = X(t,a) \in X(B_0). \tag{2.5.17}$$

Step 3. Admissible value functions and feedback strategies

3.1. Admissible extreme "proper" value functions:

Compute the following marginal characteristic value functions:

$$\begin{aligned} & W_0^m(y) = \inf_{X(t,a)=y} V(t,a),\ W_0^M(y) = \sup_{X(t,a)=y} V(t,a), \\ & \widehat{B}_m(y) = \{(t,a) \in B_0;\ X(t,a) = y,\ V(t,a) = W_0^m(y)\}, \\ & \widehat{B}_M(y) = \left\{(t,a) \in B_0;\ X(t,a) = y,\ V(t,a) = W_0^M(y)\right\}, \\ & Y_0^m = \left\{y \in X(B_0);\ \widehat{B}_m(y) \neq \emptyset\right\}, \\ & Y_0^M = \left\{y \in X(B_0);\ \widehat{B}_M(y) \neq \emptyset\right\}. \end{aligned} \tag{2.5.18}$$

Let, $\widetilde{W}_0(.) \in \{W_0^m(.), W_0^M(.)\}$ and the corresponding elements $\widehat{B}_0(y) \in \left\{\widehat{B}_m(y), \widehat{B}_M(y)\right\}$ where $y \in \widetilde{Y}_0 \in \{Y_0^m, Y_0^M\}$. We verify that the function $\widetilde{W}_0(.)$ is admissible in the sense that the set $\widetilde{B}_0(.)$ is defined as:

$$\widetilde{B}_0(y) = \left\{(t,a) \in \widehat{B}_0(y);\ x_{t,a}(s) \in \widetilde{Y}_0, (t+s,a) \in \widehat{B}_0(x_{t,a}(s))\ \forall s \in [0,-t]\right\} \quad (2.5.19)$$

and satisfies:

$$\widetilde{B}_0(y) \neq \emptyset\ \forall y \in \widetilde{Y}_0 \in \{Y_0^m, Y_0^M\}. \quad (2.5.20)$$

3.2. Admissible "intermediate" proper value functions:

If the proper value functions $W_0^m(.)$ and $W_0^M(.)$ as defined in (2.5.18) are not admissible in the sense of (2.5.20) then, an "intermediate" proper value function can be introduced. This function satisfies:

$$\widetilde{W}_0(y) \in \left[W_0^m(y), W_0^M(y)\right],\ y \in \widetilde{Y}_0 \subseteq Y_0^M \cup Y_0^m.$$

Furthermore, the multifunction $\widehat{B}_0(.)$ as defined in (2.5.18) is expressed as:

$$\widehat{B}_0(y) = \left\{(t,a) \in B;\ X(t,a) = y,\ V(t,a) = \widetilde{W}_0(y)\right\} \neq \emptyset, \quad (2.5.21)$$

additionally, $\widetilde{B}_0(.)$ as defined in (2.5.19) corresponds to an admissible $\widetilde{W}_0(.)$ in the sense of (2.5.20).

Remark 10. *If $X(.,.)$ is invertible at (t,a), with its inverse $\widehat{B}_0(y) = (X(.,.))^{-1}(y) = \widehat{B}_m(y) = \widehat{B}_M(y)$, then:*

$$W_0^m(y) = W_0^M(y) = V(\widehat{B}_0(y)). \quad (2.5.22)$$

Moreover, if the function $\widetilde{W}_0(.) = W_0^m(.) = W_0^M(.)$ is differentiable at the point $y \in Int(\widetilde{Y}_0)$, then its derivative is given by:

$$D\widetilde{W}_0(y) = \widetilde{P}(y) = P\left(\widehat{B}_0(y)\right), \quad (2.5.23)$$

and satisfies the following relations:

$$\begin{aligned} &D\widetilde{W}_0(y).f(y,\overline{u},\overline{v}) + f_0(y,\overline{u},\overline{v}) = 0,\ \forall \overline{u} \in \widetilde{U}(y),\ \overline{v} \in \widetilde{V}(y), \\ &\widetilde{U}(y) = \widehat{U}(y,\widetilde{P}(y)) = \{\widetilde{u}(y)\},\ \widetilde{V}(y) = \widehat{V}(y,\widetilde{P}(y)) = \{\widetilde{v}(y)\}, \end{aligned} \quad (2.5.24)$$

where, $\widetilde{U}(.)$ and $\widetilde{V}(.)$ are the "corresponding candidates" for the optimal feedback strategies.

Moreover, it follows from (2.5.1) *that the value function* $\widetilde{W}_0(.)$ *verifies Isaacs' basic equation:*

$$\begin{aligned} \min_{u\in U(y)} \max_{v\in V(y)} \left[D\widetilde{W}_0(y).f(y,u,v)+f_0(y,u,v)\right] &= \\ \max_{v\in V(y)} \min_{u\in U(y)} \left[D\widetilde{W}_0(y).f(y,u,v)+f_0(y,u,v)\right] &= 0. \end{aligned} \tag{2.5.25}$$

3.3. Selections of admissible controls and trajectories:

For each proper value function $\widetilde{W}_0(.)$ identified in **step 3.1-3.2**, which is admissible in the sense of (2.5.20), determine the set of terminal points:

$$\widetilde{Y}_1 = \left\{x_{t,a}\left(-t\right) = \xi \in Y_1;\ (t,a) \in \widetilde{B}_0\left(y\right),\ a = (\xi,q,\lambda) \in A\right\}. \tag{2.5.26}$$

The corresponding selections of "admissible" controls and trajectories:

$$\begin{aligned} \widetilde{P}_\alpha(y) &= \left\{\left(u_{t,a}\left(.\right), v_{t,a}\left(.\right)\right);\ (t,a) \in \widetilde{B}_0\left(y\right)\right\}, \\ \widetilde{\Omega}_\alpha\left(y\right) &:= \left\{x_{t,a}\left(.\right);\ (t,a) \in \widetilde{B}_0\left(y\right)\right\}, \end{aligned} \tag{2.5.27}$$

where, $u_{t,a}\left(.\right), v_{t,a}\left(.\right)$ and $x_{t,a}\left(.\right)$ are the mappings in (2.5.16). In addition, the corresponding value function is defined as:

$$\widetilde{W}(y) = \begin{cases} g(y) & \text{if } y \in \widetilde{Y}_1, \\ \widetilde{W}_0(y) & \text{if } y \in \widetilde{Y}_0, \end{cases} \tag{2.5.28}$$

and, together with a suitable pair of feedback strategies, it must satisfy the admissibility conditions described in (2.3.17) – (2.3.19).

3.4. Admissible feedback strategies:

For each $a = (\xi, q, \lambda) \in A$, identify the two sets $\mathcal{U}\left(a\right), \mathcal{V}\left(a\right)$ of controls $u_a\left(.\right), v_a\left(.\right)$, respectively, which satisfy the two last conditions in (2.5.11). Additionally, compute the corresponding multifunctions:

$$\begin{aligned} \overline{U}(t,a) &= \left\{u_a(t);\ u_a(.) \in \mathcal{U}(a)\right\}, \\ \overline{V}(t,a) &= \left\{v_a(t);\ v_a(.) \in \mathcal{V}(a)\right\}. \end{aligned} \tag{2.5.29}$$

Therefore, the corresponding feedback strategies pair $\left(\widetilde{U}(.), \widetilde{V}(.)\right)$ in the sense of Definition 2.3.17 is given by:

$$\begin{aligned} \widetilde{U}(y) &= \overline{U}(\widetilde{B}_0(y)), \\ \widetilde{V}(y) &= \overline{V}(\widetilde{B}_0(y)),\ y \in \widetilde{Y}_0. \end{aligned} \tag{2.5.30}$$

Additionally, we can consider $\widetilde{\Omega}_\alpha(y)$, $y \in \widetilde{Y}_0$ in (2.5.27), as the (multi)-selection of admissible trajectories (which may also be optimal) satisfying the conditions in (2.5.12) , (2.5.17) and (2.5.18).

For certain points $y \in \widetilde{Y}_0$, these feedback strategies can be expressed in a simpler form:

$$\begin{aligned} \widetilde{U}\,(y) &= \widehat{U}(y, \widetilde{P}\,(y)), \\ \widetilde{V}\,(y) &= \widehat{V}(y, \widetilde{P}\,(y)), \\ \widetilde{P}\,(y) &= P(\widetilde{B}_0\,(y)), \end{aligned} \tag{2.5.31}$$

where $\widehat{U}\,(.,.)$ and $\widehat{V}\,(.,.)$ are the multifunctions defined in (2.5.1) and $P\,(.,.)$ is the component of the Hamiltonian flow described in (2.5.11).

Step 4. Proof of the relative optimality:

In order to prove optimality in the sense of Definition 2.3.3 for the restriction of the problem (DG) on $\widetilde{Y}_0$, we select one of a admissible value functions $\widetilde{W}\,(.)$ in (2.5.18) (naturally, choose the one with the best regularity properties), a corresponding admissible feedback strategies $\left(\widetilde{U}\,(.)\,,\widetilde{V}\,(.)\right)$ in (2.5.20) , (2.5.21) and use one of the verification theorems in Section 2.4

Chapter III

ON THE SOLUTION OF DOLICHOBRACHISTOCHRONE DIFFERENTIAL GAME VIA DYNAMIC PROGRAMMING APPROACH

3.1 Introduction

In this chapter, we explore the application of Mirică's dynamic programming algorithm, described in section 2.5, to solve the well-known Dolichobrachistochrone differential game, originally proposed by Isaacs in (1965). In addition, it aims to identify, for the first time, feedback strategies, a novel contribution that offers significant advantages in game theory over other types of strategies. Among them, they promote efficiency through dynamic performance optimization, leading to improved resource utilization and goal attainment. In addition, the simplicity of these strategies makes them easier to analyse and implement, while reducing the complexity of the computations. The essential tool in our approach, involves the use of a certain refinement of Cauchy's method of characteristics for stratified Hamilton–Jacobi equations, to describe a large class of admissible trajectories and to identify a domain in which the value function exists. As a rigorous criterion for proving the optimality of these admissible feedback strategies, we use the well-known verification Theorem for locally Lipschitz value functions as a sufficient optimality condition.

The set of results obtained in this chapter has been the subject of an article published in "Journal of Computational and Applied Mathematics" [28].

3.2 Position of the problem

In [31], Isaacs has been considered the Dolichobrachistochrone differential game that consists of optimizing the cost functional:

$$\mathcal{C}(\phi, \psi) = T, \tag{3.2.1}$$

and defined by the dynamical system with restrictions:

$$\begin{cases} x_1' = \sqrt{x_2}\cos\phi + \frac{w}{2}(\psi+1),\ x_1(0) > 0, \\ x_2' = \sqrt{x_2}\sin\phi + \frac{w}{2}(\psi-1),\ x_2(0) > 0, \\ \phi \in [0, 2\pi],\ \psi \in [-1, 1], \\ x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^+ \times \mathbb{R}^+,\ t \in [0, T]. \end{cases} \tag{3.2.2}$$

The involved functions have the following significance:

ϕ, ψ: are the directions of the first and second player respectively, which are actually the control functions;

$x_1(t)$, $x_2(t)$: represent the positions of players;

w: is the speed of the first player.

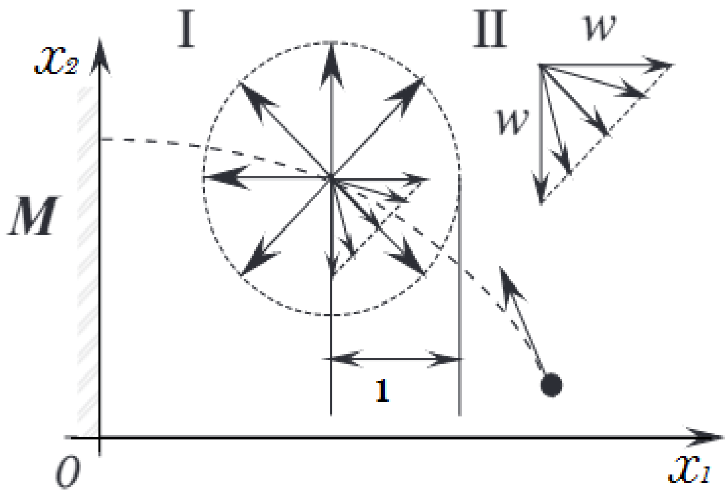

Figure 1: The vectograms of the players

Isaacs considered the first quadrant as the state space. The terminal set was the positive semiaxis x_2. The vectograms of the players were:

- The circle of radius 1 for the first player.
- The diagonal of a square with the side w for the second player.

The first player minimizes the time of attaining the terminal set M, the second player has the opposite objective. We recall that, the Dolichobrachistochrone problem described above is an extension of the well-known Brachistochrone problem in the calculus of variations (see for instance, Mirică [42]).

3.2.1 Dynamic programming formulation

In order to use the Dynamic Programming Approach described in [42, 43], we reformulate the previous problem using standard notations in differential games theory and embedding this problem in a set of problems associated to each initial point in the phase-space as follows:

Problem 2. *Find:*

$$\inf_{u(.)\in\mathcal{U}_1(.)} \sup_{v(.)\in\mathcal{V}_1(.)} C\left(y; u\left(.\right), v\left(.\right)\right) \ \forall y \in Y_0, \tag{3.2.3}$$

subject to:

$$\begin{gathered} C(y,u(.),v(.)) = g(x(T)) + \int_0^T f_0(x(t),u(t),v(t))dt, \\ x'(t) = f(x(t),u(t),v(t)) \ a.e.(0,T),\ x(0) = y, \\ u(t) \in U(x(t)),\ v(t) \in V(x(t)) \ a.e.(0,T), \\ x\left(.\right) \in \Omega,\ \left(u\left(.\right), v\left(.\right)\right) \in \mathcal{P}, \\ x(t) \in Y_0,\ \forall t \in [0,T),\ x(T) \in Y_1, \end{gathered} \tag{3.2.4}$$

defined, in our case, by the following data:

$$\begin{aligned} & f(x,u,v) = (\sqrt{x_2}u_1 + \tfrac{w}{2}(v+1), \sqrt{x_2}u_2 + \tfrac{w}{2}(v-1)), \\ & f_0(x,u,v) = 1, \\ & U(x) = U = S_1\left(0\right) = \left\{u = (u_1,u_2) \in \mathbb{R}^2;\ \ \|u\| = 1\right\},\ V(x) = V = [-1,1], \\ & g(\xi) = 0,\ \forall \xi \in Y_1,\ Y_0 = (0,+\infty)^2,\ Y_1 = \{0\} \times (0,+\infty), \end{aligned} \tag{3.2.5}$$

where $\mathcal{P} = \mathcal{U} \times \mathcal{V}$ is the class of measurable admissible control functions $\left(u\left(\cdot\right), v\left(\cdot\right)\right)$ and Ω is the corresponding class of absolutely continuous admissible trajectories.

From the intuitive formulation of the problem in (3.2.3) and (3.2.4), it is understood that, there are two players, $\mathbb{U}$ and $\mathbb{V}$ respectively, having an opposite objective. They can choose, an optimal strategy $\widetilde{u}\left(.\right), \widetilde{v}\left(.\right)$ respectively, for which the dynamic system in (3.2.4) generate a trajectory $\widetilde{x}\left(.\right) = \widetilde{x}_{\widetilde{u}\widetilde{v}}\left(.\right)$ and such that, the player $\mathbb{U}$ tries to minimize the cost functional $\mathcal{C}(., \widetilde{v}\left(.\right))$, while the second one $\mathbb{V}$, tries to maximize $\mathcal{C}(\widetilde{u}\left(.\right), .)$.

3.2.2 The Hamiltonian and set of transversality terminal points

The pseudo-Hamiltonian, $\mathcal{H}\left(x,p,u,v\right) =< p, f\left(x,u,v\right) > +f_0\left(x,u,v\right)$ is given in our case by:

$$\mathcal{H}\left(x,p,u,v\right) = \sqrt{x_2}\left\langle p,u\right\rangle + \frac{w}{2}\left[\left(p_1+p_2\right)v + p_1 - p_2\right] + 1.$$

The Isaacs' Hamiltonian for $(x,p)\in Z=dom(H(.,.))$, is defined by:

$$H\left(x,p\right)=\min_{u\in U}\ \max_{v\in V}\ \mathcal{H}\left(x,p,u,v\right)=\max_{v\in V}\ \min_{u\in U}\ \mathcal{H}\left(x,p,u,v\right).$$

Using the well-known fact that:

$$\inf_{u\in S_1(0)} < p,u > = < p,-\frac{p}{\|p\|} > = -\|p\|\,,\quad \sup_{v\in[-1,1]}\left[\left(p_1+p_2\right)v\right]=\left|p_1+p_2\right|, \tag{3.2.6}$$

then, the corresponding extremal value of the control parameters turn out to be given by:

$$\begin{array}{ll} H(x,p) & = -\sqrt{x_2}\,\|p\|+\frac{w}{2}\left[\left|p_1+p_2\right|+p_1-p_2\right]+1. \\ \widehat{U}\times\widehat{V} & = \begin{cases} \left(-\frac{p}{\|p\|},1\right) & \text{if } h(x,p)=p_1+p_2>0, \\ \left(-\frac{p}{\|p\|},-1\right) & \text{if } h(x,p)<0, \\ \left\{-\frac{p}{\|p\|}\right\}\times V & \text{if } h(x,p)=0, (p_1,p_2)\neq(0,0). \end{cases} \end{array} \tag{3.2.7}$$

Next, we need to compute the set of terminal transversality values defined in the general case by:

$$Z^*=\left\{(\xi,q)\in Y_1\times\mathbb{R}^2;\ H\left(\xi,q\right)=0,\ \left\langle q,\bar{\xi}\right\rangle=Dg(\xi)\bar{\xi},\ \forall\bar{\xi}\in T_\xi Y_1\right\}. \tag{3.2.8}$$

Since, the tangent space at the point $\xi\in Y_1$ is $T_\xi Y_1=\{0\}\times\mathbb{R}$ then, it follows from (3.2.8) that:

$$q_1\bar{\xi}_1+q_2\bar{\xi}_2=q_2\bar{\xi}_2=0,\ \bar{\xi}_1=0,\ \forall\bar{\xi}_2\in\ \mathbb{R},$$

and we obtain $q_2=0$. Starting from the fact that $H\left(\xi,q\right)=0$, from here, we can extract three cases:

Case 1. *If $q_1+q_2=0$, then, $q_1=0$ and $H\left(\xi,(0,0)\right)=1\neq 0$. Therefore, this case is impossible.*

Case 2. *If $q_1+q_2>0$, then, $q_1>0$ and $H\left(\xi,q\right)=(w-\sqrt{\xi_2})q_1+1=0$. Consequently, we have:*

$$q_1=\frac{1}{\sqrt{\xi_2}-w}.$$

From this, we can derive two sub-cases:

Case 2.1. *If $\sqrt{\xi_2}-w>0$, then $q_1>0$ for all $\sqrt{\xi_2}>w$.*

Case 2.2. *If $\sqrt{\xi_2}-w<0$, then $q_1<0$, which contradicts the condition that $q_1>0$.*

Case 3. *If $q_1+q_2<0$, then $q_1<0$ and $H\left(\xi,q\right)=\sqrt{\xi_2}q_1+1=0$. Therefore, we find that:*

$$q_1=-\frac{1}{\sqrt{\xi_2}}<0,\ \forall\xi_2>0.$$

Without in any way limiting the generality of the considered problem and for simplicity, in what follows, we parameterize the terminal set Y_1 in (3.2.5) by:

$$Y_1 = \left\{(0, s^2); s > 0\right\},$$

from here, we get that for $\xi = (\xi_1, \xi_2) = \left(0, s^2\right) \in Y_1$ it follows that, $\xi_1 = 0$ and $\sqrt{\xi_2} = s$. Therefore, the set of terminal transversality values Z^* is stratifed by:

$$\begin{aligned} Z^*_+ &= \{\left(\left(0, s^2\right), (q_1, 0)\right), \, q_1 = \tfrac{1}{s-w}, \, s > w\}, \\ Z^*_- &= \left\{\left(\left(0, s^2\right), (q_1, 0)\right), \; q_1 = -\tfrac{1}{s}, \; s > 0\right\}, \\ Z^* &= Z^*_+ \cup Z^*_-. \end{aligned} \tag{3.2.9}$$

3.3 The generalized Hamiltonian and characteristic flows

The first main computational operation consists in the backward integration for $t \leq 0$ of the Hamiltonian inclusion:

$$\left(x', p'\right) \in d_s^{\#} H\left(x, p\right), \;\; \left(x\left(0\right), p\left(0\right)\right) = z = \left(\xi, q\right) \in Z^*, \tag{3.3.1}$$

defined by the following generalized Hamiltonian orientor field $d_s^{\#} H\left(.,.\right)$:

$$\begin{aligned} d_s^{\#} H\left(x, p\right) \quad &= \Big\{(x', p') \in T_{(x,p)} Z, \; x' \in f(x, \widehat{U}\left(x, p\right), \widehat{V}\left(x, p\right)), \\ &< x', \overline{p} > - < p', \overline{x} >= DH\left(x, p\right).\left(\overline{x}, \overline{p}\right), \; \forall \left(\overline{x}, \overline{p}\right) \in T_{(x,p)} Z\Big\}. \end{aligned} \tag{3.3.2}$$

As specified in [42, 43], for each terminal point $z = (\xi, q) \in Z^*$, we shall identify the maximal solutions $X^*\left(.\right) = \left(X\left(.\right), P\left(.\right)\right) : I\left(z\right) = \left(t^-\left(z\right), 0\right] \rightarrow Z$, of the Hamiltonian inclusion in (3.3.1) that satisfy the conditions:

$$\begin{aligned} &X\left(t\right) = \left(X_1\left(t\right), X_2\left(t\right)\right) \in Y_0, \, \forall t \in I_0\left(z\right) = \left(t^-\left(z\right), 0\right), \\ &H\left(X\left(t\right), P\left(t\right)\right) = 0, \, \forall t \in I\left(z\right) \\ &X'\left(t\right) = f\left(X\left(t\right), u\left(t\right), v\left(t\right)\right) \text{ a.e. } I_0\left(z\right), \, \left(u\left(.\right), v\left(.\right)\right) \in \mathcal{P}_1, \\ &u\left(t\right) \in \widehat{U}\left(X^*\left(t\right)\right), \, v\left(t\right) \in \widehat{V}\left(X^*\left(t\right)\right) \text{ a.e. } I_0\left(z\right). \end{aligned} \tag{3.3.3}$$

In the case in which there exist more such solutions for the same terminal point $z = (\xi, q) \in Z^*$, we shall parameterize by $\lambda \in \Lambda\left(z\right)$ the set of these solutions obtaining a generalized Hamiltonian flow: $X^*\left(.,.\right) = \left(X\left(.,.\right), P\left(.,.\right)\right) : B = \{(t, a)\, ; \; t \in I\left(z\right), \; a \in A\} \rightarrow Z, \; A = graph\left(\Lambda\left(.\right)\right), \; a = \left(z, \lambda\right)$. We also recall the fact that, for each $(t, a) \in B_0 = \{(t, a) \in B; \; t \neq 0\}$, the Hamiltonian flow $X^*\left(.,.\right)$ defines the controls and, respectively, trajectories:

$$u_t(s) = u(t+s),\ v_t(s) = v(t+s), s \in [0,-t],\ x_{t,a}(s) = X(t+s,a),$$

which are admissible with respect to the initial point $y = X(t,a) \in Y_0$, for which the value of the cost functional in (3.2.4) is given by the function $V(.,.)$ defined by:

$$V(t,a) = g(\xi) + \int_0^t < P(\sigma,a), X'(\sigma,a) > d\sigma, \quad \text{if } a = (z,\lambda), \tag{3.3.4}$$

and which together with the Hamiltonian flow $X^*(.,.)$ defines the generalized characteristic flow $C^*(.,.) = (X^*(.,.), V(.,.))$, using the definition of the Hamiltonian $H(.,.)$ and the second condition in (3.3.3), we obtain:

$$< P(\sigma,s), X'(\sigma,s) >= -f_0(X(\sigma,s), \widehat{u}(X^*(\sigma,s)), \widehat{v}(X^*(\sigma,s))) = -1.$$

It follows from (3.3.4) that, in our case the function $V(.,.)$ is given by:

$$V(t,s) = -t,\ \forall (t,s) \in B.$$

First, we remark that the Hamiltonian $H(.,.)$ in (3.2.7) as well as the domain $Z \subset \mathbb{R}^2 \times \mathbb{R}^2$ are $\mathcal{C}^1$-stratified by the stratification $S_H = \{Z_+, Z_-, Z_0\}$ defined by:

$$Z_+ = \{(x,p) \in Z;\ h(x,p) > 0\}, \text{ where } h(x,p) = p_1 + p_2,$$
$$Z_- = \{(x,p) \in Z;\ h(x,p) < 0\},$$
$$Z_0 = \{(x,p) \in Z;\ h(x,p) = 0\}.$$

If we denote by, $H_\pm(.,.) = H(.,.)|Z_\pm$, $H_0(.,.) = H(.,.)|Z_0$, then:

$$H_+(x,p) = -\sqrt{x_2}\,\|p\| + wp_1 + 1,\ (x,p) \in Z_+,$$
$$H_-(x,p) = -\sqrt{x_2}\,\|p\| - wp_2 + 1,\ (x,p) \in Z_-,$$
$$H_0(x,p) = -\sqrt{2x_2}\,|p_1| + wp_1 + 1,\ (x,p) \in Z_0,$$

$$d_s^{\#}H(x,p) = \begin{cases} d_s^{\#}H_\pm(x,p),\ (x,p) \in Z_\pm, \\ d_s^{\#}H_0(x,p),\ (x,p) \in Z_0. \end{cases}$$

Since the manifolds Z_+, $Z_- \subset Z$ are open subsets, the Hamiltonian orientor fields $d_s^{\#}H_\pm(.,.)$ in (3.3.2) coincides with the classical Hamiltonian vector fields:

$$d_s^{\#}H_\pm(x,p) = \left\{\left(\frac{\partial H_\pm}{\partial p}(x,p), -\frac{\partial H_\pm}{\partial x}(x,p)\right)\right\},\ (x,p) \in Z_\pm,$$

which are easy to compute and will be described and studied later, while on the singular stratum $Z_0 \subset Z$ the corresponding Hamiltonian fields are more difficult to compute.

3.3.1 Hamiltonian field on the singular stratum Z_0

As can be seen that, the singular stratum Z_0 is naturally split as:

$$Z_0^+ = \{(x,p);\ h(x,p) = p_1 + p_2 = 0,\ p_1 > 0\},$$

$$Z_0^- = \{(x,p);\ h(x,p) = p_1 + p_2 = 0,\ p_1 < 0\}.$$

Lemma 3.3.1. *For any $(x,p) \in Z_0^\pm$, one has $d_s^\# H_0^\pm(x,p) = \emptyset$.*

Proof. In order to compute the generalized Hamiltonian field $d_s^\# H_0^+(.,.)$, we note first that, according to certain classical results as in [42], the tangent space to the 3-dimensional manifolds $Z_0^\pm$ are given by:

$$T_{(x,p)}Z_0^\pm = \left\{(\overline{x},\overline{p}) \in \mathbb{R}^2 \times \mathbb{R}^2;\ \overline{p}_1 + \overline{p}_2 = 0\right\},$$

and $DH_0^+(x,p).(\overline{x},\overline{p}) = -\frac{p_1}{\sqrt{2x_2}}\overline{x}_2 + \left(w - \sqrt{2x_2}\right)\overline{p}_1$, therefore a vector $(x',p') \in d_s^\# H_0^+(x,p)$ is fully characterized by the properties:

$$\left[x_1' - x_2' + \sqrt{2x_2} - w\right]\overline{p}_1 - p_1'\overline{x}_1 + \left[-p_2' + \frac{p_1}{\sqrt{2x_2}}\right]\overline{x}_2 = 0,\ \forall \overline{p}_1, \overline{x}_1, \overline{x}_2 \in \mathbb{R},$$
$$p_1' + p_2' = 0,\ x' \in f\left(x, \widehat{U}(x,p), \widehat{V}(x,p)\right),$$
$$\widehat{U}(x,p) = \left\{-\frac{p}{\|p\|}\right\},\ \widehat{V}(x,p) = \{+1,-1\}.$$

It follows that, at each point $(x,p) \in Z_0^+$ we obtain:

$$x_1' - x_2' + \sqrt{2x_2} - w = 0,\ p_1' = 0,\ p_2' = \frac{p_1}{\sqrt{2x_2}}. \tag{3.3.5}$$

Using the fact that, $(x',p') \in T_{(x,p)}Z_0^+$ then, $p_1' + p_2' = 0$ and it follows from (3.3.5) that, $p_2' = 0$ hence, $p_1 = 0$ this contradicts the fact that $(x,p) \in Z_0^+$.

Symmetrically, using the same type of computations and arguments as in previous case and we deduce that, a vector $(x',p') \in d_s^\# H_0^-(x,p)$ is fully characterized by the properties:

$$\left[x_1' - x_2' - \sqrt{2x_2} - w\right]\overline{p}_1 - p_1'\overline{x}_1 - \left[p_2' + \frac{p_1}{\sqrt{2x_2}}\right]\overline{x}_2 = 0,\ \forall \overline{p}_1, \overline{x}_1, \overline{x}_2 \in \mathbb{R},$$
$$p_1' + p_2' = 0,\ x' \in f(x, \widehat{U}(x,p), \widehat{V}(x,p)),$$
$$\widehat{U}(x,p) = \left\{-\frac{p}{\|p\|}\right\},\ \widehat{V}(x,p) = \{+1,-1\}.$$

It follows that, at each point $(x,p) \in Z_0^-$ is characterized by the formulas:

$$x_1' - x_2' - \sqrt{2x_2} - w = 0,\ p_1' = 0,\ p_2' = -\frac{p_1}{\sqrt{2x_2}}. \tag{3.3.6}$$

Since that, $(x',p') \in T_{(x,p)}Z_0^-$ then, $p_1' + p_2' = 0$. Also, from (3.3.6) it results that, $p_2' = 0$ hence, $p_1 = 0$ which leads to the same contradiction. This completes the proof. ∎

3.4 Construction of the Hamiltonian flow

3.4.1 The Hamiltonian flow ending on the stratum Z_+

On the open stratum Z_+ for which $h(x,p) = p_1 + p_2 > 0$, the differential inclusion in (3.3.1) coincides with the smooth Hamiltonian system described by:

$$\begin{cases} x' = \left(-\sqrt{x_2}\dfrac{p_1}{\|p\|} + w, -\sqrt{x_2}\dfrac{p_2}{\|p\|}\right), \\ p' = \left(0, \dfrac{\|p\|}{2\sqrt{x_2}}\right). \end{cases} \tag{3.4.1}$$

Let's describe the partial Hamiltonian flow whose trajectories have terminal points on the stratum Z_+. An admissible trajectory $X_+^*(.,z) = (X^+(.,z), P^+(.,z))$, $z \in Z_+^*$, of system (3.4.1) must satisfy the terminal conditions from the set of transversality terminal points Z_+^* in the sense that, $(x(0), p(0)) = \left((0, s^2), (q_1(s), 0)\right)$, $q_1(s) = \frac{1}{s-w}$, $\forall s > w$.

Lemma 3.4.1. *For any $s > w$, system* (3.4.1) *admits a unique solution in the form of a maximal Hamiltonian flow $X_+^*(.,s) = (X^+(.,s), P^+(.,s))$ whose components are given by the formulas:*

$$\begin{cases} X_1^+(t,s) = -\frac{s^2}{2}\sin\frac{t}{s} + (w - \frac{s}{2})t, \; t < 0 \\ X_2^+(t,s) = s^2\cos^2\frac{t}{2s}, \\ P_1^+(t,s) = q_1(s) = \frac{1}{s-w}, \\ P_2^+(t,s) = \frac{1}{s-w}\tan\frac{t}{2s}. \end{cases} \tag{3.4.2}$$

Proof. As can be seen, the third component follows immediately, hence, $P_1^+(t,s) = q_1(s)$. Moreover, due to the fact that, $H\left(X_+^*(t,s)\right) = 0$ then:

$$x_2(t) = \frac{s^2 q_1^2(s)}{q_1^2(s) + p_2^2(t)}, \tag{3.4.3}$$

and from here as well as from least expression in (3.4.1), we obtain a separable equation of the form:

$$\frac{dp_2}{q_1^2(s) + p_2^2} = \frac{s-w}{2s}dt, \; s > w,$$

which has as a general solution:

$$p_2(t) = q_1(s)\tan\left[q_1(s)\left(\frac{s-w}{2s}\right)t + c\right], \; c \in \mathbb{R},$$

using the terminal conditions $p_2(0) = 0$, we get $c = 0$. To obtain the expression of $x_2(.)$, we replace $P_2(.)$ in (3.4.3). Also in the same context, to describe $x_1(.)$, we replace $x_2(.)$ and $p_2(.)$ in the first equation of (3.4.1). This completes the proof. ∎

From the dynamic programming algorithm described in [42, 43], it follows that, we must retain only the trajectories $X_+^*(t,s)$ $s > w$ that satisfy the following admissible conditions:

$$X^+(t,s) \in Y_0,\ t < 0,\ s > w,$$
$$X_+^*(t,s) = (X^+(t,s), P^+(t,s)) \in Z_+,$$

which can be expressed as follows:

$$\begin{aligned} &X_1^+(t,s) > 0,\ X_2^+(t,s) > 0,\ t < 0,\ s > w \\ &h_+(t,s) = h\left(X_+^*(t,s)\right) = P_1^+(t,s) + P_2^+(t,s) > 0, \end{aligned} \tag{3.4.4}$$

on the maximal intervals $I^+(s) = (\tau^+(s), 0)$, hence the extremity $\tau^+(.)$ is defined by:

$$\begin{aligned} &\tau^+(s) = \max\left\{\tau_1^+(s), \tau_2^+(s)\right\}, \\ &\tau_1^+(s) = \inf\left\{\tau < 0;\ X_+^*(t,s) \in Z_+,\ \forall t \in (\tau, 0)\right\}, \\ &\tau_2^+(s) = \inf\left\{\tau < 0;\ X^+(t,s) \in Y_0,\ \forall t \in (\tau, 0)\right\}. \end{aligned} \tag{3.4.5}$$

Trying to describe an explicit formula for extremity $\tau^+(.)$. Based on this claim, we prove the following results.

Lemma 3.4.2. *There exists an extremity $\tau_0(s) = -\frac{\pi s}{2}$, $s \geq s_0 = \frac{2\pi w}{\pi+2}$ such that, the conditions in* (3.4.4) *are verified for any $t \in (\tau_0(s), 0)$. It is also verified:*

$$X_1^+(\tau_0(s_0), s_0) = 0,\ X_1^+(\tau_0(s), s) > 0,\ \forall s > s_0.$$

Proof. It follows from (3.4.2) that, $X_2^+(t,s) > 0$ for all $t < 0$. While from (3.4.4) one has:

$$h_+(t,s) = q_1(s)\left(1 + \tan\frac{t}{2s}\right),$$

and therefore, $h_+(t,s) > 0$ this implies that, $\tan\frac{t}{2s} > \tan\left(-\frac{\pi}{4}\right)$ hence, $t > \tau_0(s)$ and $h_+(t,s) < 0$, for $t < \tau_0(s)$. Further, from (3.4.2) we obtain:

$$\frac{\partial X_1^+}{\partial t}(t,s) = -\frac{s}{2}\left(1 + \cos\frac{t}{s}\right) + w,$$

hence, $\frac{\partial X_1^+}{\partial t}(t,s) = 0$ for, $\cos\frac{t}{s} = \frac{2w}{s} - 1$. Since $\frac{t}{s} \in (-\frac{\pi}{2}, 0)$, this gives $(\frac{2w}{s} - 1) \in (0,1)$, then there exists a unique extremity $t_1(s) \in (\tau_0(s), 0)$ given by:

$$t_1(s) = -s\arccos(\frac{2w}{s} - 1),\ \frac{\partial X_1^+}{\partial t}(t_1(s), s) = 0, \tag{3.4.6}$$

that checks the properties:

$$\begin{aligned} &\frac{\partial X_1^+}{\partial t}(t,s) > 0, \text{ if } t \in (\tau_0(s), t_1(s)) \\ &\frac{\partial X_1^+}{\partial t}(t,s) < 0, \text{ if } t \in (t_1(s), 0). \end{aligned} \tag{3.4.7}$$

Next, let $\alpha(.)$ be an extremity defined as, $\alpha(s) = X_1^+(\tau_0(s), s) = \frac{s}{2}\left[(\frac{\pi+2}{2})s - \pi w\right]$. Note that if $s = s_0 = \frac{2\pi w}{\pi+2} > w$, then $\alpha(s) = 0$ and $\alpha(s) > 0$, $\forall s > s_0$. Therefore, conditions (3.4.4) are verified for any $t \in (\tau_0(s), 0)$, $s \geq s_0$. This completes the proof.

■

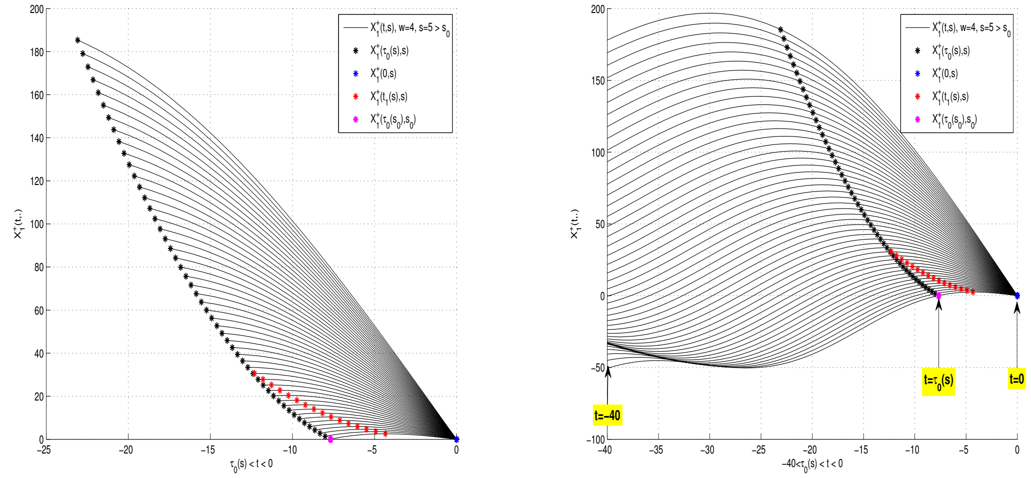

Figure 2: Variations of $X_1^+(.,.)$, $s \in [s_0, +\infty)$

In the following, we denote by:

$$G = \left\{ X^+(\tau_0(s), s) = \left(\alpha(s), \frac{s^2}{2}\right), \ s \in [s_0, +\infty) \right\},$$

which may be expressed as:

$$\begin{aligned} G &= \{(\alpha(\sqrt{2x_2}), x_2), \ x_2 \in [\tfrac{s_0^2}{2}, +\infty)\}, \\ &= \{(\beta(x_2), x_2), \ x_2 \in [\tfrac{s_0^2}{2}, +\infty)\} \\ \beta(x_2) &= \alpha(\sqrt{2x_2}) = \left(\tfrac{\pi}{2} + 1\right) x_2 - \tfrac{\pi w}{\sqrt{2}} \sqrt{x_2}. \end{aligned} \tag{3.4.8}$$

Lemma 3.4.3. *G is a parabolic curve.*

Proof. As can be seen, the function $\beta(.)$ is continuous on $[\frac{s_0^2}{2}, +\infty)$ as well as from the fact that, $\beta'(x_2) = \left(\frac{\pi}{2} + 1\right) - \frac{\pi w}{2\sqrt{2x_2}} > 0$, $\forall x_2 \in [\frac{s_0^2}{2}, +\infty)$ hence, $\beta(.)$ is strictly increasing and if $\beta'(x_2) = 0$, then, $x_2 = \frac{s_0^2}{8} \notin [\frac{s_0^2}{2}, +\infty)$. Moreover, from (3.4.8) we obtain, $\beta''(x_2) = \frac{\pi w}{4\sqrt{2x_2}\sqrt{x_2}} > 0$, $\forall x_2 \in [\frac{s_0^2}{2}, +\infty)$. This completes the proof. ■

Lemma 3.4.4. *If $s \in (w, s_0)$ then there exists a unique $t_2(s) \in (\tau_0(s), t_1(s))$ such that:*

$$X_1^+(t_2(s), s) = 0, \ X_1^+(t, s) > 0, \ \forall t \in (t_2(s), 0),$$

where, $t_1(.)$ *denotes the extremity given in* (3.4.6).

Proof. If $s \in (w, s_0)$ then, $X_1^+(\tau_0(s), s) < 0$, and from (3.4.7) according to *Intermediate value Theorem*, there exists a unique $t_2(s) \in (\tau_0(s), t_1(s))$ such that, $X_1^+(t_2(s), s) = 0$. Also, from (3.4.7) we deduce that, the component $X_1(., s)$, $s \in (w, s_0)$ is strictly increasing on $(\tau_0(s), t_1(s))$ and therefore:

$$X_1^+(t, s) < X_1^+(t_2(s), s) = 0, \ \forall t \in (\tau_0(s), t_2(s))$$
$$X_1^+(t, s) > X_1^+(t_2(s), s) = 0, \ \forall t \in (t_2(s), t_1(s)).$$

Finally, as $X_1^+(0, s) = 0$ and from (3.4.7), we can easily see that, $X_1^+(t_1(s), s) > 0$ and hence, $X_1^+(t, s) > 0, \forall t \in (t_2(s), 0)$. This completes the proof. ■

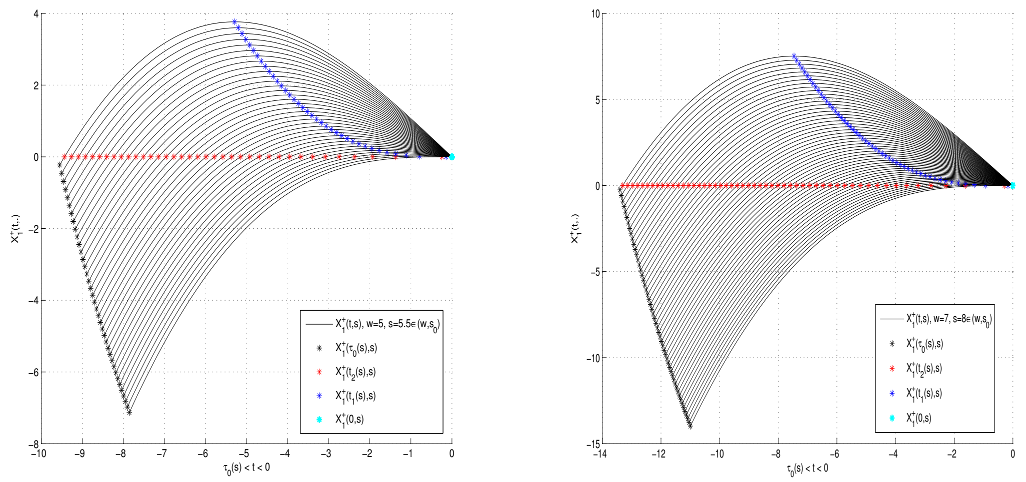

Figure 3: Variations of $X_1^+(., .)$, $s \in (w, s_0)$

Therefore, due to *Lemma* 3.4.2 and *Lemma* 3.4.4, the extremity $\tau^+(.)$ defined in (3.4.5) is given by:

$$\tau^+(s) = \begin{cases} \tau_0(s) = -\frac{\pi s}{2}, & \text{if } s \in [s_0, +\infty), \\ t_2(s), & \text{if } s \in (w, s_0). \end{cases}$$

One may note here that, geometrically the trajectories $X^+(., .) : B^+ \to \widetilde{Y}_+$ are the curves in Figure 4 and cover the domain $\widetilde{Y}_+$ given by:

$$B^+ = \{(t, s); \ t \in (\tau^+(s), 0), \ s > w\},$$
$$\widetilde{Y}_+ = X^+(B^+) = \left\{(x_1, x_2); \ x_1 \in (0, \beta(x_2)), \ x_2 \in [\tfrac{s_0^2}{2}, +\infty)\right\}.$$

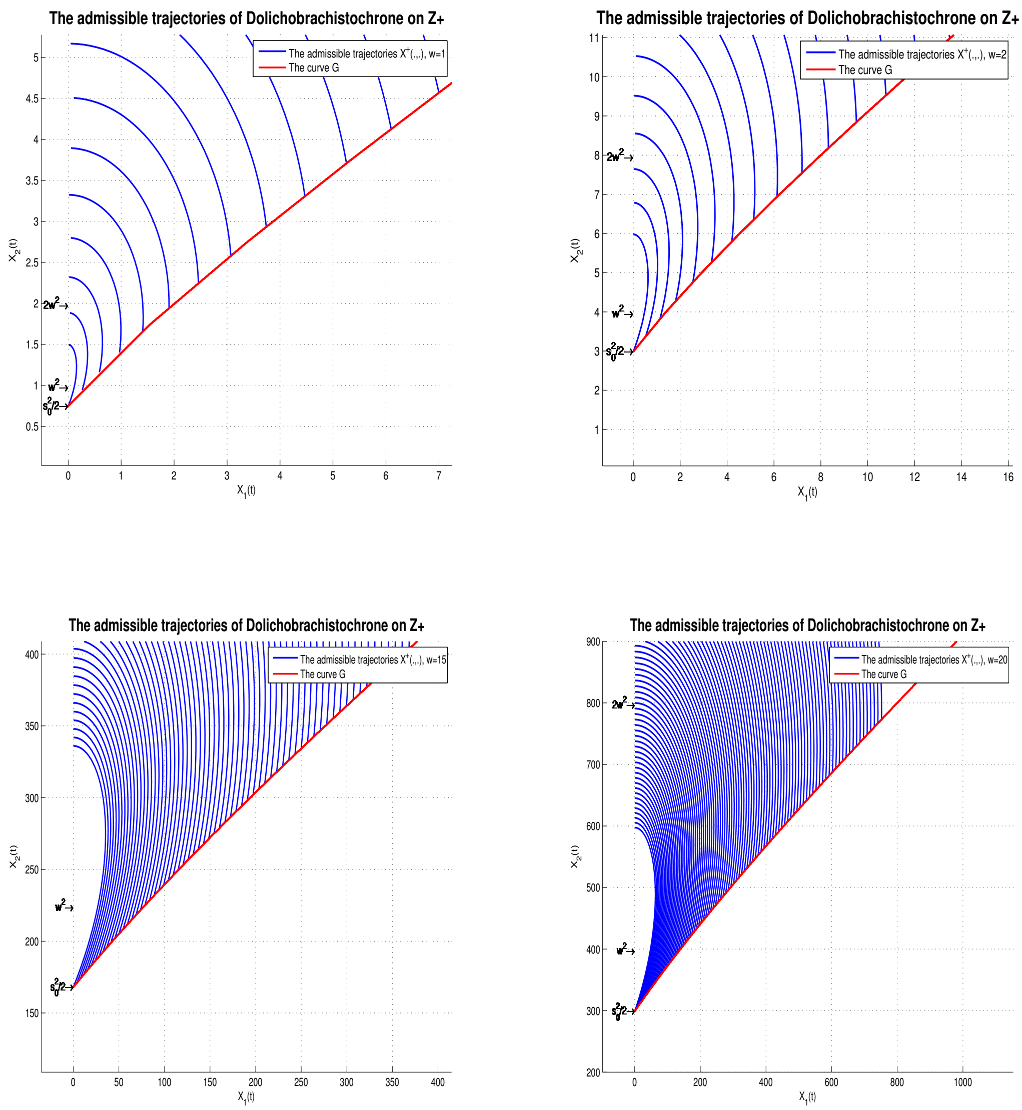

Figure 4: Admissible trajectories $X^{+}(.,.)$

3.4.2 The Hamiltonian system on the stratum Z_{-}

On the open stratum Z_{-} for which $h(x,p) = p_1 + p_2 < 0$, the differential inclusion in (3.3.1) coincides with the smooth Hamiltonian system described by:

$$\begin{cases} x' = \left(-\sqrt{x_2}\dfrac{p_1}{\|p\|}, -\sqrt{x_2}\dfrac{p_2}{\|p\|} - w\right), \\ p' = \left(0, \dfrac{\|p\|}{2\sqrt{x_2}}\right). \end{cases} \tag{3.4.9}$$

Symmetrically, whose Hamiltonian flow $X_{-}^{*}(.,.) = (X^{-}(.,.), P^{-}(.,.))$, must satisfy the terminal conditions from the set of transversality terminal points Z_{-}^{*} in (3.2.9) such that, $(x(0), p(0)) = \left((0, s^2), (q_0(s), 0)\right)$, $q_0(s) = -\frac{1}{s}$, $\forall s > 0$.

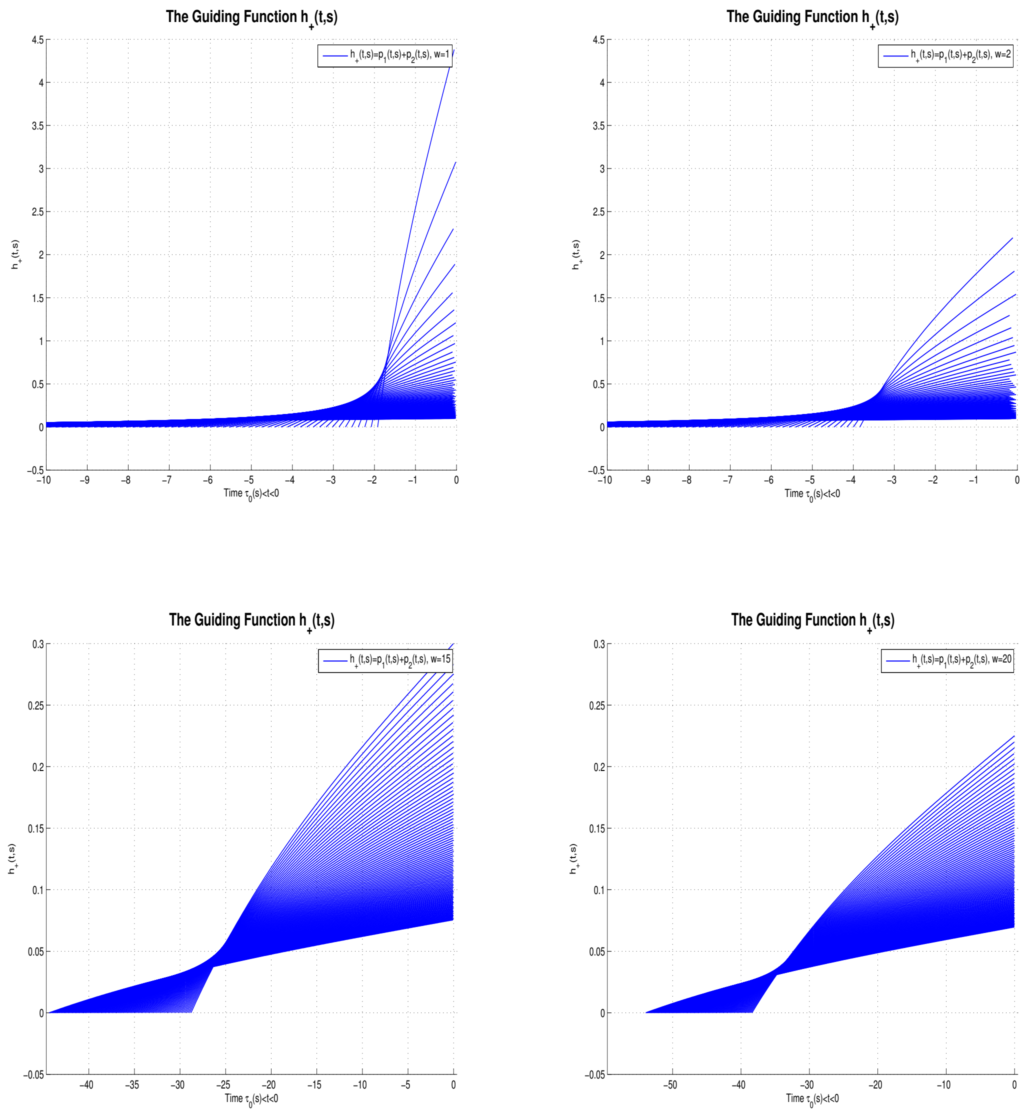

Figure 5: Guiding function $h_+(.,.)$

Lemma 3.4.5. *If $z = (\xi, q) \in Z_-^*$ then, the Hamiltonian system has a unique solution $X_{-,0}^*(.,z) = \left(X^{-,0}(.,z), P^{-,0}(.,z)\right)$ such that:*

$$X^{-,0}(t,z) \notin Y_0, \ \forall t < 0.$$

Proof. If $X_{-,0}^*(.,z)$, $z \in Z_-^*$ is a solution of system (3.4.9) then, from (3.3.3) it follows that, $H_-\left(X_{-,0}^*(t,z)\right) = -\sqrt{x_2}\,\|p\| - wp_2 + 1 = 0$, hence:

$$\sqrt{x_2} = \frac{1 - wp_2}{\sqrt{q_0^2(s) + p_2^2}} > 0, \ \forall p_2 < \frac{1}{w},$$

using the first equation in (3.4.9), we deduce that:

$$x_1'(t) = \frac{1}{s}\left[\frac{1 - wp_2(t)}{q_0^2(s) + p_2^2(t)}\right] > 0, \, \forall p_2(t) < \frac{1}{w},$$

and therefore, $x_1(.)$ is strictly increasing so that, $\forall t<0$, $x_1(t)<x_1(0)=0$ which means that, the trajectories $X^{-,0}(.,s)$ $s>0$ is outside its domain Y_0. This completes the proof. ■

Remark 11. *The results of the Lemma* 3.4.5 *show that, the trajectories* $X^*_{-,0}(.,s)$, $s>0$ *are not admissible and therefore must be eliminated from the study of Problem* 2.

3.4.3 Continuation of trajectories on the stratum Z_-

As can be seen, the extremity:

$$Z^*_{\oplus}=X^*_+(\tau_0(s),s)=((\alpha(s),\frac{s^2}{2}),(q_1(s),-q_1(s))),\ s>s_0, \tag{3.4.10}$$

belongs to the stratum Z_+ and boundary of the open stratum Z_-; examining the possibility of continuation for $t<\tau_0(s)$ of the trajectories $X^*_+(.,s)$, $s>s_0$, we note that, this is possible only on the open stratum Z_-.

Considering the fact that, $H_-(X^+(\tau_0(s),s),P^+(\tau_0(s),s))=0$, the possibility of continuation of the trajectories $X^*_+(.,s)$, $s>s_0$, for $t<\tau_0(s)$ on Z_- (for which $h(x,p)<0$) is first guaranteed by the condition $\frac{d}{dt}h(X^+(\tau_0(s),s),P^+(\tau_0(s),s))=\frac{q_1(s)}{\sqrt{2s}}>0$ since, in this case the function $h^*_-(t,s)=h\left(X^*_+(t,s)\right)$ is increasing on an interval of the form $(\tau^+(s)-\delta,\tau^+(s))$, $\delta>0$.

In this case, the trajectories in (3.4.2) may be continued by the trajectories: $X^*_-(.,s)=(X^-(.,s),P^-(.,s))$, $s>s_0$, which are solutions of the Hamiltonian system in (3.4.9) that satisfy $X^*_-(\tau_0(s),s)=((\alpha(s),\frac{s^2}{2}),\ (q_1(s),-q_1(s)))$, $s>s_0$ and for which there exists an extremity $\tau^-(s)<\tau^+(s)$, $s>s_0$ such that:

$$X^-(t,s)\in Y_0,\ X^*_-(t,s)\in Z_-,\ \forall t\in\left(\tau^-(s),\tau^+(s)\right),\ s>s_0. \tag{3.4.11}$$

Therefore, considering the system (3.4.9) and terminal conditions taken in (3.4.10). The characterization of trajectories $X^*_-(.,s)$, $s>s_0$ is proved in the following result.

Lemma 3.4.6. *The system in* (3.4.9) *together with terminal conditions* (3.4.10) *has a unique solution* $X_{-}^{*}(.,s) = (X^{-}(.,s), P^{-}(.,s))$*,* $s > s_0$*, that checks* (3.4.11)*. Moreover, its components are defined parametrically by the functions:*

$$\begin{cases} T_1(\xi,s) = & \frac{2}{q_1(s)}\arctan(\frac{\xi}{q_1(s)}) - w\ln\left(q_1^2(s)+\xi^2\right) + \frac{\pi}{2q_1(s)} + \\ & w\ln\left(2q_1^2(s)\right) + \tau_0(s), \\ \widetilde{X}_1(\xi,s) = & \frac{\xi}{q_1^2(s)+\xi^2}\left[q_1(s)(w^2 - \frac{1}{q_1^2(s)}) + \frac{2w\xi}{q_1(s)}\right] - \\ & (w^2 + \frac{1}{q_1^2(s)})\arctan(\frac{\xi}{q_1(s)}) + \frac{(2-\pi)w^2}{2}, \\ \widetilde{X}_2(\xi,s) = & \frac{(1-w\xi)^2}{q_1^2(s)+\xi^2},\ \xi \in (-\infty, -q_1(s)),\ s > s_0, \end{cases} \tag{3.4.12}$$

in the sense that, the function $T_1(.,s)$*,* $s > s_0$ *is invertible with inverses:*

$$\widetilde{\xi}_1(t,s) = (T_1(.,s))^{-1}(t),\ t \in (-\infty, \tau_0(s)), s > s_0, \tag{3.4.13}$$

and the components of $X_{-}^{*}(.,s)$*,* $s > s_0$ *are given by the formulas:*

$$\begin{aligned} &X_1^{-}(t,s) = \widetilde{X}_1(\widetilde{\xi}_1(t,s),s),\ X_2^{-}(t,s) = \widetilde{X}_2(\widetilde{\xi}_1(t,s),s), \\ &P_1^{-}(t,s) = q_1(s),\ P_2^{-}(t,s) = \widetilde{\xi}_1(t,s),\ t \in (-\infty, \tau_0(s)). \end{aligned} \tag{3.4.14}$$

Proof. First, from system (3.4.9), it is obviously that, $P_1^{-}(t,s) = q_1(s)$. It follows from the second condition in (3.3.3) that, $H_{-}(X^{-}(\tau_0(s),s), P^{-}(\tau_0(s),s)) = 0$ and we obtain:

$$\sqrt{x_2} = \frac{1 - wp_2}{\sqrt{q_1^2 + p_2^2}},\ \forall p_2 < \frac{1}{w}, \tag{3.4.15}$$

together with the last equation of system (3.4.9) results a separable differential equation of the form:

$$\frac{2(1 - wp_2)}{q_1^2(s) + p_2^2}dp_2 = dt,$$

whose general solution is given in implicit form by the formula:

$$\frac{2}{q_1(s)}\arctan\frac{p_2}{q_1(s)} - w\ln\left(q_1^2(s) + p_2^2\right) = t + c_1,\ c_1 \in \mathbb{R}, \tag{3.4.16}$$

using the terminal condition in (3.4.10) one has, $c_1 = -\frac{\pi}{2q_1(s)} - w\ln\left(2q_1^2(s)\right) - \tau_0(s)$ and therefore:

$$t = T_1(\xi,s),\ \xi < \frac{1}{w},\ s > s_0,$$

where $T_1(.,.)$ denotes the function defined in (3.4.12). Starting from the fact that:

$$\frac{\partial T_1}{\partial \xi}(\xi,s) = \frac{2(1-w\xi)}{q_1^2(s)+\xi^2} > 0,\ \forall \xi < \frac{1}{w}, \tag{3.4.17}$$

we deduce that, $T_1(.,s)$, $s > s_0$ is strictly increasing, therefore, there exists a unique function $\widetilde{\xi}_1(.,.)$ of the form as in (3.4.14).

On the other hand, the second condition in (3.4.11) may be expressed as follows:

$$h_-(t,s) = h\left(X_-^*(t,s)\right) = P_1^-(t,s) + P_2^-(t,s) < 0,\ t < \tau_0(s),$$

because, from the last equation in system (3.4.9), we have, $P_2^-(.,s)$ strictly increasing for $s > s_0$. Hence, $\forall t < \tau_0(s) : P_2^-(t,s) < P_2^-(\tau_0(s),s) = -q_1(s) = -P_1^-(t,s)$ which means that, $h_-(t,s) < 0$.

Taking into account:

$$P_2^-(t,s) < -q_1(s) < \frac{1}{w},\ \forall t < \tau_0(s),\ s > s_0.$$

Further, from (3.4.12) we get:

$$\lim_{\xi\to-\infty} T_1(\xi,s) = -\infty,\ \lim_{\xi\to-q_1(s)} T_1(\xi,s) = \tau_0(s),\ s > s_0. \tag{3.4.18}$$

Moreover, since the function $T_1(.,s)$, $s > s_0$ is of class $\mathcal{C}^1$ and strictly increasing then, the inverse function $P_2^-(.,s)$, $s > s_0$ has the same properties hence, it is a diffeomorphism of class $\mathcal{C}^1$. Therefore:

$$\lim_{t\to-\infty} P_2^-(t,s) = -\infty,\ \lim_{t\to\tau_0(s)} P_2^-(t,s) = -q_1(s),\ s > s_0, \tag{3.4.19}$$

and it follows from (3.4.15) that:

$$X_2^-(t,s) = \frac{\left(1 - wP_2^-(t,s)\right)^2}{q_1^2(s) + \left(P_2^-(t,s)\right)^2} > 0,\ \forall t \in (-\infty, \tau_0(s)),\ s > s_0, \tag{3.4.20}$$

as well as from (3.4.9) it results that, the component $X_1^-(.,s)$, $s > s_0$ is strictly decreasing and due to Lemma 3.4.2 one has:

$$\forall t \in (-\infty, \tau_0(s)),\ X_1^-(t,s) > \alpha(s) = X_1^-(\tau_0(s),s) > 0,\ \forall s > s_0,$$

therefore, the conditions (3.4.11) are verified. Now, consider the following parametrization:

$$\widetilde{X}_1(\xi,s) = X_1^-(T_1(\xi,s),s),\ \xi \in (-\infty, -q_1(s)),\ s > s_0,$$
$$\widetilde{X}_2(\xi,s) = X_2^-(T_1(\xi,s),s),$$

and one has:

$$\frac{\partial \widetilde{X}_1}{\partial \xi}(\xi,s) = \frac{\partial X_1^-}{\partial t}(T_1(\xi,s),s)\,\frac{\partial T_1}{\partial \xi}(\xi,s).$$

Moreover, the first equation in (3.4.9) and (3.4.17) lead to the following expression:

$$\frac{\partial \widetilde{X}_1}{\partial \xi}(\xi, s) = -2q_1(s)\frac{(1-w\xi)^2}{\left(q_1^2(s)+\xi^2\right)^2} < 0,\ \forall \xi \in (-\infty, -q_1(s)),\ s > s_0. \tag{3.4.21}$$

Elementary computations and arguments show that:

$$\frac{(1-w\xi)^2}{(q_1^2(s)+\xi^2)^2} = \frac{w^2}{q_1^2(s)+\xi^2} + \frac{(1-w^2q_1^2(s))}{(q_1^2(s)+\xi^2)^2} - 2w\frac{\xi}{(q_1^2(s)+\xi^2)^2}, \tag{3.4.22}$$

next, we replace (3.4.22) in (3.4.21) and by integration of both members with terminal conditions (3.4.10), we get, $\widetilde{X}_1(-q_1(s), s) = X_1^-(T_1(-q_1(s), s) = X_1^-(\tau_0(s), s) = \alpha(s)$, which gives the second expression in (3.4.12). This completes the proof. ■

Corollary 3.4.1. *The following statements are satisfied:*

1. *The component $P_2^-(., s) : (-\infty, \tau_0(s)) \to (-\infty, -q_1(s))$ is a diffeomorphism of class $\mathcal{C}^1$.*

2. *There exists:*

$$\begin{aligned} &\lim_{t\to-\infty} X^-(t,s) = \left(k_2(s), w^2\right) \in Y_0, \\ &k_2(s) = \tfrac{2w}{q_1(s)} + \tfrac{\pi}{2q_1^2(s)} + w^2. \end{aligned} \tag{3.4.23}$$

Proof. The first statement follows immediately using the fact that, $T_1(., s),\ s > s_0$ in (3.4.12) is of class $\mathcal{C}^1$ and strictly increasing, hence, its inverse function $P_2^-(., s),\ s > s_0$ given in (3.4.14) has the same properties, so it's a diffeomorphism of class $\mathcal{C}^1$.

Next, we show that the fact that, the first component of $X^-(., s),\ s > s_0$, checks expression in (3.4.23). To this end, it follows from (3.4.19) and (3.4.20) that $\lim_{t\to-\infty} X_2^-(t, s) = w^2$.

While, from (3.4.12) one has:

$$\lim_{\xi\to-\infty} \widetilde{X}_1(\xi, s) = k_2(s) = \frac{1}{2}\left[\pi s^2 - 2w(\pi-2)s + (\pi-2)w^2\right]. \tag{3.4.24}$$

The discriminant $\Delta = -2w^2(\pi - 2)$ of the second-order equation $k_2(s) = 0$ is negative, hence, $k_2(s) > 0$, $\forall s > s_0$ therefore:

$$\begin{aligned} &\lim_{t\to-\infty} X_1^-(t,s) = \lim_{\xi\to-\infty} X_1^-(T_1(\xi,s), s) = \lim_{\xi\to-\infty} \widetilde{X}_1(\xi, s) = k_2(s) > 0, \\ &k_2(s_0) = \tfrac{1}{2}\left(\frac{\pi^3 + 2\pi^2 + 12\pi - 8}{(\pi+2)^2}\right) w^2 > 0. \end{aligned}$$

Finally, we get $\lim_{t\to-\infty} X^-(t, s) = \left(k_2(s), w^2\right) \in Y_0$. This completes the proof. ■

To give a clear image of the trajectories, it's necessary to study the variations of parametrized trajectories $\widetilde{X}_1(.,s)$, $\widetilde{X}_2(.,s)$, $s > s_0$. To do that, it follows from (3.4.12) that:

$$\frac{\partial \widetilde{X}_2}{\partial \xi}(\xi, s) = -\frac{2(1-w\xi)}{\left(q_1^2(s)+\xi^2\right)^2}\left(wq_1^2(s)+\xi\right),\ \xi \in (-\infty, -q_1(s)),\ s > s_0, \tag{3.4.25}$$

therefore, $\frac{\partial \widetilde{X}_2}{\partial \xi}(\xi_M, s) = 0$ for $\xi_M = -wq_1^2(s)$, also $\frac{\partial \widetilde{X}_2}{\partial \xi}(\xi, s) > 0$, $\forall \xi < \xi_M$ and $\frac{\partial \widetilde{X}_2}{\partial \xi}(\xi, s) < 0$, $\forall \xi > \xi_M$.

From here and together with the fact that, $\xi_M + q_1(s) = \frac{s-2w}{(s-w)^2}$, $s > s_0$, we extract two cases:

Case 1. *If $s \geq 2w$, then $\xi_M > -q_1(s)$. In this case, the point $\xi_M \notin (-\infty, -q_1(s))$ and hence, $\widetilde{X}_2(.,s)$ with $s \geq 2w$ is strictly increasing.*

Case 2. *If $s \in (s_0, 2w)$, then $\xi_M \in (-\infty, -q_1(s))$. Therefore, the point $\widetilde{X}_{2M} = \widetilde{X}_2(\xi_M, s) = 2w^2 + s^2 - 2sw > 0$ is a maximum point of the component $\widetilde{X}_2(.,s)$, with $s \in (s_0, 2w)$.*

On the other hand, from (3.4.21), we deduce that, the parametrized component $\widetilde{X}_1(.,s)$, $s > s_0$ is strictly decreasing which prove that, there exists a unique function:

$$\widehat{\xi}_1(x_1, s) = (\widetilde{X}_1(.,s))^{-1}(x_1),\ x_1 \in (\alpha(s), k_2(s)). \tag{3.4.26}$$

Next, let $\hat{X}_2(.,.)$ be a function defined by:

$$x_2 = \hat{X}_2(x_1, s) = \widetilde{X}_2(\widehat{\xi}_1(x_1, s), s),\ x_1 \in (\alpha(s), k_2(s)), s > s_0, \tag{3.4.27}$$

since that:

$$\frac{\partial \widehat{\xi}_1}{\partial x_1}(x_1, s) = \frac{1}{\frac{\partial \widetilde{X}_1}{\partial \xi}(\widehat{\xi}_1(x_1, s), s)} = -\frac{(q_1^2(s) + (\widehat{\xi}_1(x_1, s))^2)^2}{2q_1(s)(1 - w\widehat{\xi}_1(x_1, s))^2}, \tag{3.4.28}$$

therefore:

$$\begin{aligned} \frac{\partial \widehat{X}_2}{\partial x_1}(x_1, s) &= \frac{\partial \widetilde{X}_2}{\partial \xi}(\widehat{\xi}_1(x_1, s), s)\frac{\partial \widehat{\xi}_1}{\partial x_1}(x_1, s), \\ &= \frac{\widehat{\xi}_1(x_1, s) + wq_1^2(s)}{q_1(s)(1 - w\widehat{\xi}_1(x_1, s))},\ x_1 \in (\alpha(s), k_2(s)),\ s > s_0, \end{aligned} \tag{3.4.29}$$

from here, we deduce that, $\frac{\partial \widehat{X}_2}{\partial x_1}(x_1, s) < 0$, $\forall x_1 \in (\alpha(s), k_2(s)), s \geq 2w$ and also, $\frac{\partial^2 \widehat{X}_2}{\partial x_1^2}(x_1, s) = -\frac{1}{2}\widetilde{X}_{2M}(s)\frac{(q_1^2(s)+(\widehat{\xi}_1(x_1,s))^2)^2}{\left(1-w\widehat{\xi}_1(x_1,s)\right)^4} < 0$, $\forall x_1 \in (\alpha(s), k_2(s))$, $s > s_0$ which prove that, $\widehat{X}_2(.,s)$, $\forall s \geq 2w$ is a strictly concave function.

Geometrically, the trajectories $\widetilde{X}(.,.) : B^- \to \widetilde{Y}_-$ are the curves in Figure 6 and cover the domain $\widetilde{Y}_-$ given by:

$$B^- = \{(\xi, s);\ \xi \in (-\infty, -q_1(s)),\ s > s_0\},$$

$$\widetilde{Y}_- = \{\widetilde{X}(\xi, s) = (\widetilde{X}_1(\xi, s), \widetilde{X}_2(\xi, s)),\ (\xi, s) \in B^-\}.$$

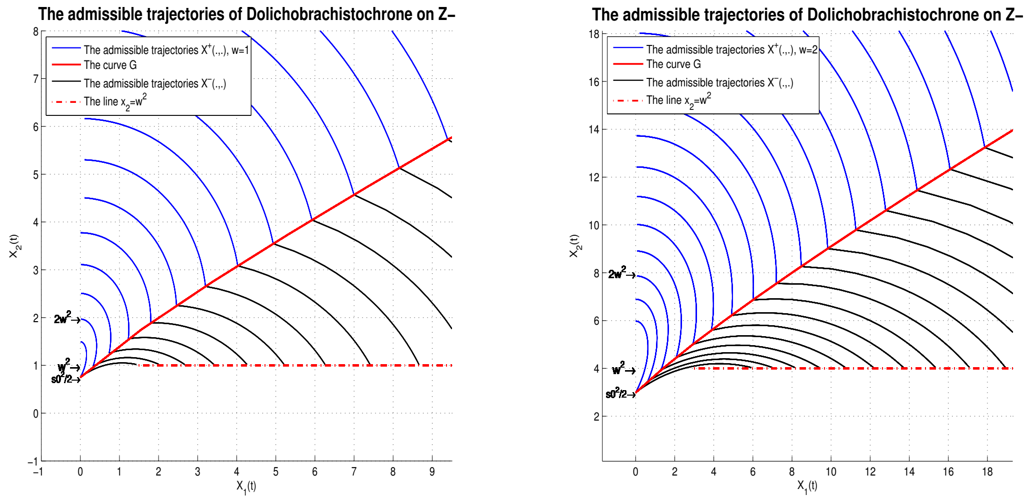

Figure 6: Admissible trajectories $X^-(.,.)$

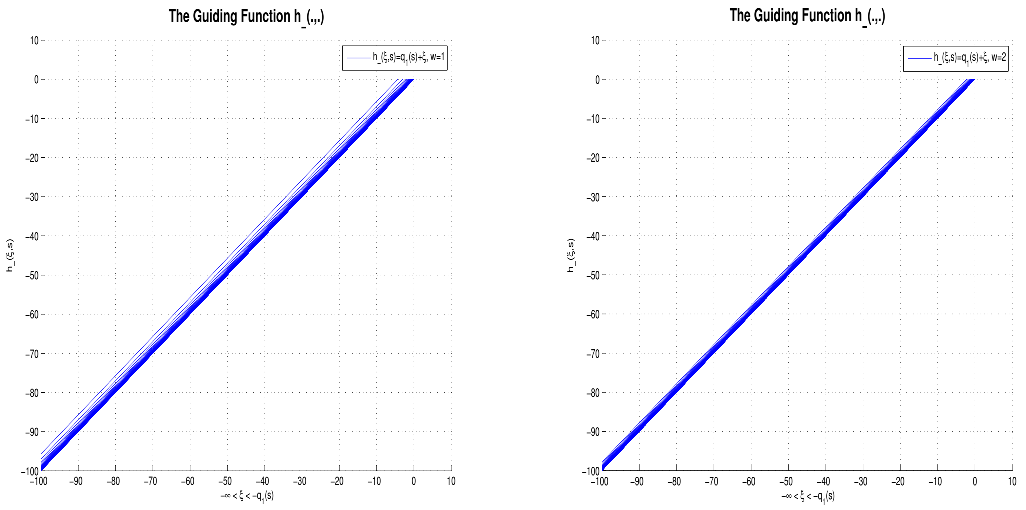

Figure 7: Guiding function $h_-(.,.)$

3.4.4 Other admissible trajectories

A special role is played by other admissible trajectories of the system (3.4.9), which don't start from the terminal set Z_{-}^{*}, but they start from the set given by:

$$Z_{1,0}^{*} = \left\{ ((0, \frac{s^2}{2}), (q_1(s), -q_1(s))),\ s \in (w, s_0) \right\}. \tag{3.4.30}$$

In which though $H(\xi, q) = 0$, the transversality conditions in (3.2.8) is not verified.

Remark 12. *The idea of constructing new admissible trajectories on the stratum Z_{-} starting from $Z_{1,0}^{*}$ arises from the fact that, $h_{-}(0, s) = h\left((0, \frac{s^2}{2}), (q_1(s), -q_1(s))\right) = q_1(s) - q_1(s) = 0$. Additionally, it follows from* (3.4.2) *that, the guiding function $h_{-}(., s)$ is strictly increasing for $s \in (w, s_0)$. Thus, $\forall t \in (-\infty, 0) : h_{-}(t, s) < h_{-}(0, s) = 0$, which guarantees the existence of admissible trajectories on the stratum Z_{-}.*

Lemma 3.4.7. *For any $s \in (w, s_0)$, the Hamiltonian system in* (3.4.9) *together with terminal conditions in* (3.4.30) *admits a solution $\overline{X}_{*}(., s) = \left(\overline{X}(., s), \overline{P}(., s)\right)$, such that:*

$$\overline{X}_{*}(t, s) \in Z_{-}\ \forall t \in (-\infty, 0), \tag{3.4.31}$$

moreover, its components are defined parametrically by the functions:

$$\begin{cases} T_2(\xi, s) = T_1(\xi, s) - \tau_0(s),\ \xi \in (-\infty, -q_1(s)),\ s \in (w, s_0), \\ \widetilde{\widetilde{X}}_1(\xi, s) = \widetilde{X}_1(\xi, s) - \alpha(s), \\ \widetilde{\widetilde{X}}_2(\xi, s) = \widetilde{X}_2(\xi, s), \end{cases} \tag{3.4.32}$$

in sense that, the function $T_2(., s)$, $s \in (w, s_0)$ is invertible with inverses given by:

$$\widetilde{\xi}_2(t, s) = (T_2(., s))^{-1}(t),\ t \in (-\infty, 0), \tag{3.4.33}$$

and the components of the solution $\overline{X}_{*}(., s)$, $s \in (w, s_0)$ are given by the formulas:

$$\begin{aligned} &\overline{X}_1(t, s) = \widetilde{\widetilde{X}}_1(\widetilde{\xi}_2(t, s), s),\ \overline{X}_2(t, s) = \widetilde{\widetilde{X}}_2(\widetilde{\xi}_2(t, s), s), \\ &\overline{P}_1(t, s) = P_1^{-}(t, s),\ \overline{P}_2(t, s) = \widetilde{\xi}_2(t, s),\ t \in (-\infty, 0). \end{aligned} \tag{3.4.34}$$

Proof. The proof is done in the same way as in Lemma 3.4.6. First, it is obviously, $\overline{P}_1(t,s) = P_1^-(t,s) = q_1(s)$ and $H_-\left(\overline{X}_*(0,s)\right) = 0$ such that, $\overline{X}(0,s) = (0, \frac{s^2}{2})$, $\overline{P}(0,s) = (q_1(s), -q_1(s))$ then the conditions in (3.4.30) are verified. Further, it follows from (3.4.16) and (3.4.30) that, $c_1 = \overline{k}_1(s) = -\frac{\pi}{2q_1(s)} - w\ln\left(2q_1^2(s)\right)$, $s \in (w, s_0)$, and we get:

$$t = T_2(\xi, s),\ \xi \in (-\infty, \frac{1}{w}),\ s \in (w, s_0),$$

since $\frac{\partial T_2}{\partial \xi}(\xi,s) = \frac{\partial T_1}{\partial \xi}(\xi,s)$ then, from (3.4.17) it results the existence and uniqueness of the function $\widetilde{\xi}_2(.,.)$ defined as in (3.4.33) and (3.4.34).

Furthermore, since the set in which the point $\overline{X}_*(t,s)$ $t \in (-\infty, 0)$ $s \in (w, s_0)$ was located is determined by the sign of the function:

$$\overline{h}(t,s) = \overline{P}_1(t,s) + \overline{P}_2(t,s),\ t \in (-\infty, 0),\ s \in (w, s_0),$$

then, it follows from the last equation in (3.4.9) that, the component $\overline{P}_2(.,s)$, $s \in (w, s_0)$ in (3.4.34) is strictly increasing:

$$\overline{P}_2(t,s) < \overline{P}_2(0,s) = -q_1(s) = -\overline{P}_1(t,s),\ \forall t \in (-\infty, 0),$$

therefore, $\overline{h}(t,s) < 0$ and hence, condition in (3.4.31) is verified. We also take into account the relation:

$$\overline{P}_2(t,s) < -q_1(s) < \frac{1}{w},\ \forall t \in (-\infty, 0),\ s \in (w, s_0),$$

and from (3.4.18) and (3.4.32), we deduce that:

$$\lim_{t \to -\infty} T_2(\xi, s) = -\infty,\ \lim_{t \to -q_1(s)} T_2(\xi, s) = 0,\ s \in (w, s_0).$$

For the same reasons as in Corollary 3.4.1, it can be verified that, the function $\overline{P}_2(.,.)$ is a diffeomorphism of class $\mathcal{C}^1$ and therefore:

$$\lim_{t \to -\infty} \overline{P}_2(t,s) = -\infty,\ \lim_{t \to 0} \overline{P}_2(t,s) = -q_1(s),\ s \in (w, s_0), \tag{3.4.35}$$

while, from (3.4.20), we get:

$$\overline{X}_2(t,s) = \frac{\left(1 - w\overline{P}_2(t,s)\right)^2}{q_1^2(s) + \overline{P}_2^2(t,s)} > 0,\ \forall t \in (-\infty, 0),\ s \in (w, s_0). \tag{3.4.36}$$

Next, taking into account the first equation of system (3.4.9) as well as the expressions in (3.4.30) and (3.4.34) we deduce that, the mapping $\overline{X}_1(., s)$, $s \in (w, s_0)$ is strictly decreasing, which means that:

$$\overline{X}_1(t, s) > \overline{X}_1(0, s) = 0, \ \forall t \in (-\infty, 0).$$

In what follows, we consider the parameterization:

$$\begin{aligned} &\widetilde{\widetilde{X}}_1(\xi, s) = \overline{X}_1(T_2(\xi, s), s), \ \xi \in (-\infty, -q_1(s)), \ s \in (w, s_0), \\ &\widetilde{\widetilde{X}}_2(\xi, s) = \overline{X}_2(T_2(\xi, s), s), \end{aligned}$$

since, $\frac{\partial \widetilde{\widetilde{X}}_1}{\partial \xi}(\xi, s) = \frac{\partial \overline{X}_1}{\partial t}(T_2(\xi, s), s) \frac{\partial T_2}{\partial \xi}(\xi, s) = \frac{\partial \widetilde{X}_1}{\partial \xi}(\xi, s), \ \xi \in (-\infty, -q_1(s))$ then, $\widetilde{\widetilde{X}}_1(\xi, s) = \widetilde{X}_1(\xi, s) + c_2(s)$ together with conditions (3.4.30), we get $\widetilde{\widetilde{X}}_1(-q_1(s), s) = \overline{X}_1(T_2(-q_1(s), s), s) = \overline{X}_1(0, s) = 0$ and therefore, $c_2(s) = -\alpha(s)$. This completes the proof. ■

Corollary 3.4.2. *The following statements are satisfied:*

1. *The component $\overline{P}_2(., s) : (-\infty, 0) \to (-\infty, -q_1(s))$ is a diffeomorphism of class $\mathcal{C}^1$.*

2. *There exists:*

$$\begin{aligned} &\lim_{t \to -\infty} \overline{X}(t, s) = \left(\overline{k}_2(s), w^2\right) \in Y_0, \\ &\text{where } \overline{k}_2(s) = k_2(s) - \alpha(s). \end{aligned} \tag{3.4.37}$$

Proof. The first statement is shown in the same way as in Corollary 3.4.1. Further, it follows from (3.4.35) and (3.4.36) that, $\lim_{t \to -\infty} \overline{X}_2(t, s) = w^2$ also, from (3.4.24) and the second expression in (3.4.32) one has, $\lim_{\xi \to -\infty} \widetilde{\widetilde{X}}_1(\xi, s) = \overline{k}_2(s)$ hence, $\lim_{t \to -\infty} \overline{X}_1(t, s) = \overline{k}_2(s)$. According to Corollary 3.4.2, we have, $\overline{k}_2(s) - k_2(s) = -\alpha(s) > 0$, $\forall s \in (w, s_0)$ with $\overline{k}_2(s_0) = k_2(s_0)$ and therefore, $\overline{k}_2(s) > k_2(s)$, $\forall s \in (w, s_0)$. This gives $\lim_{t \to -\infty} \overline{X}(t, s) = \left(\overline{k}_2(s), w^2\right) \in Y_0$. This completes the proof. ■

As in the previous case, in order to give a clear image of the trajectories, it is necessary to study the variation of $\widetilde{\widetilde{X}}_1(., s)$, $\widetilde{\widetilde{X}}_2(., s)$, $s \in (w, s_0)$. Using the fact that, $\widetilde{\widetilde{X}}_2(\xi, s) = \widetilde{X}_2(\xi, s)$, $\xi \in (-\infty, -q_1(s))$, $s \in (w, s_0)$, then, the point $\xi_M = -wq_1^2(s) \in (-\infty, -q_1(s))$. Hence, the study of the problem is done in the same way as in the proof of Corollary 3.4.2.

On the other hand, the second expression in (3.4.32) leads to the fact that, the component $\widetilde{\widetilde{X}}_1(.,s)$ $s \in (w, s_0)$ is strictly decreasing, hence, there exists a unique function $\widehat{\xi}_2(.,.)$ given by:

$$\widehat{\xi}_2(x_1, s) = (\widetilde{\widetilde{X}}_1(.,s))^{-1}(x_1), \; x_1 \in \left(0, \overline{k}_2(s)\right), \; s \in (w, s_0). \tag{3.4.38}$$

Next, let $\widehat{\widehat{X}}_2(.,.)$ be a mapping defined as:

$$\widehat{\widehat{X}}_2(x_1, s) = x_2 = \widetilde{X}_2(\widehat{\xi}_2(x_1, s), s), \; x_1 \in \left(0, \overline{k}_2(s)\right), \; s \in (w, s_0). \tag{3.4.39}$$

Since, $\frac{\partial \widehat{\widehat{X}}_2}{\partial x_1}(x_1, s) = \frac{\widehat{\xi}_2(x_1,s)+wq_1^2(s)}{q_1(s)\left(1-w\widehat{\xi}_2(x_1,s)\right)}$, $x_1 \in \left(0, \overline{k}_2(s)\right)$, $s \in (w, s_0)$ then, $\frac{\partial^2 \widehat{\widehat{X}}_2}{\partial x_1^2}(x_1, s) = -\frac{1}{2}\widetilde{X}_{2M}(s)\frac{\left(q_1^2(s)+\left(\widehat{\xi}_2(x_1,s)\right)^2\right)^2}{\left(1-w\widehat{\xi}_2(x_1,s)\right)^4} < 0$, $\forall x_1 \in \left(0, \overline{k}_2(s)\right)$, $s \in (w, s_0)$, hence $\widehat{\widehat{X}}_2(.,s)$, $s \in (w, s_0)$ is a strictly concave function. As in other case, the trajectories $\overline{X}(.,.)$ are the curves in Figure 8 which cover the domain $\widetilde{Y}_{\ominus}$ taken as:

$$B^{\ominus} = \{(\xi, s); \; \xi \in (-\infty, -q_1(s)), \; s \in (w, s_0)\},$$
$$\widetilde{Y}_{\ominus} = \{\widetilde{\widetilde{X}}(\xi, s) = (\widetilde{\widetilde{X}}_1(\xi, s), \widetilde{\widetilde{X}}_2(\xi, s)), \; (\xi, s) \in B^{\ominus}\}.$$

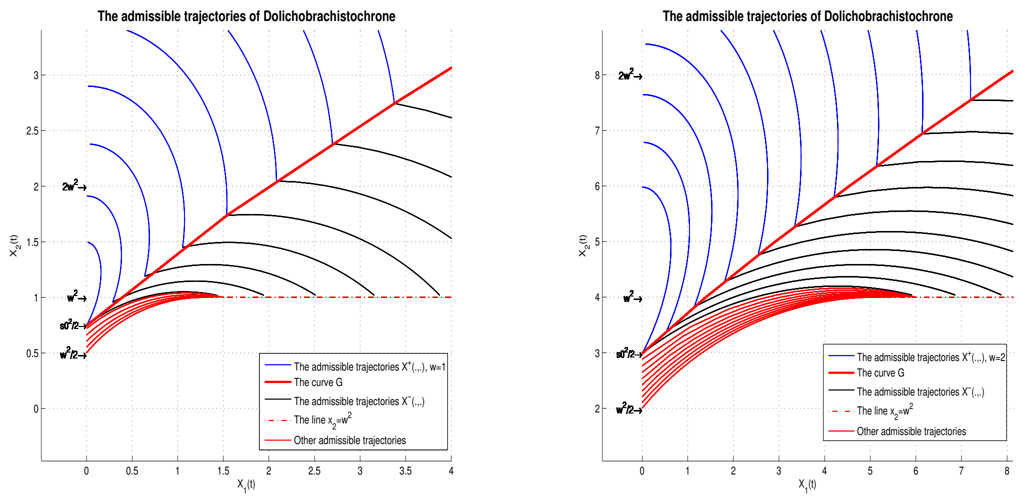

Figure 8: Admissible trajectories $\overline{X}(.,.)$

3.5 Value function and optimal feedback strategies

As mentioned in [42, 43], the natural candidates for value functions and optimal strategies of the Problem 2 are the extreme ones, defined by the following optimization process:

$$\begin{aligned}
&W^m(x) = \begin{cases} g(x), & \text{if } x \in Y_1, \\ W_0^m(x) \quad = \inf\limits_{X(t,a)=x,\ (t,a)\in B} V(t,a), & \text{if } x \in Y_0, \end{cases} \\
&W^M(x) = \begin{cases} g(x), & \text{if } x \in Y_1, \\ W_0^M(x) \quad = \sup\limits_{X(t,a)=x,\ (t,a)\in B} V(t,a), & \text{if } x \in Y_0, \end{cases} \\
&\widehat{B}_m(x) = \{(t,a) \in B;\ X(t,a) = x,\ V(t,a) = W_0^m(x)\}, \\
&\widehat{B}_M(x) = \big\{(t,a) \in B;\ X(t,a) = x,\ V(t,a) = W_0^M(x)\big\}, \\
&\widetilde{U}_m(x) = \overline{U}\left(\widehat{B}_m(x)\right),\ \widetilde{V}_m(x) = \overline{V}\left(\widehat{B}_m(x)\right), \\
&\widetilde{U}_M(x) = \overline{U}\left(\widehat{B}_M(x)\right),\ \widetilde{V}_M(x) = \overline{V}\left(\widehat{B}_M(x)\right), \\
&\overline{U}(t,a) = \big\{u_a(t);\ u_a(t) \in \overline{\mathcal{U}}(a)\big\}, \\
&\overline{V}(t,a) = \big\{v_a(t);\ v_a(t) \in \overline{\mathcal{V}}(a)\big\},
\end{aligned} \tag{3.5.1}$$

where $\overline{\mathcal{U}}(a) = \{u_a(.)\}$, $\overline{\mathcal{V}}(a) = \{v_a(.)\}$ denote the sets of control mappings that satisfy (3.3.3), it can be noted that:

$$\overline{U}(t,a) \subseteq \widehat{U}(X^*(t,a)),\ \overline{V}(t,a) \subseteq \widehat{V}(X^*(t,a)),\ \forall (t,a) \in B,$$

as well as, the fact that if $X(.,.)$ is invertible in (t,a) with the inverse $\widehat{B}(x) = (X(.,.))^{-1}(x)$, then we have:

$$W_0^m(x) = W_0^M(x) = V(\widehat{B}(x)),\ \widehat{B}_m(x) = \widehat{B}_M(x) = \widehat{B}_0(x), \tag{3.5.2}$$

it follows that if the function $W_0(.) = W_0^m(.) = W_0^M(.)$ is differentiable at the point $x \in Int(\widetilde{Y}_0)$, then its derivative is given by:

$$DW_0(x) = \widetilde{P}(x) = P(\widehat{B}_0(x)), \tag{3.5.3}$$

and checks the relations:

$$\begin{aligned}
&DW_0(x)\, f(x,\overline{u},\overline{v}) + f_0(x,\overline{u},\overline{v}) = 0\ \forall \overline{u} \in \widetilde{U}(x),\ \overline{v} \in \widetilde{V}(x), \\
&\widetilde{U}(x) = \widehat{U}(x,\widetilde{P}(x)),\ \widetilde{V}(x) = \widehat{V}(x,\widetilde{P}(x)),
\end{aligned} \tag{3.5.4}$$

where $\widetilde{U}(x), \widetilde{V}(x)$ are the corresponding candidates for the optimal feedback strategies. Moreover, it follows from (3.3.3) that, $W_0(.)$ satisfies Isaacs' basic equation:

$$\begin{aligned}&\min_{u\in U(x)}\max_{v\in V(x)}[DW_0(x)f(x,u,v)+f_0(x,u,v)]\\&=\max_{u\in V(x)}\min_{v\in U(x)}[DW_0(x)f(x,u,v)+f_0(x,u,v)]=0.\end{aligned} \tag{3.5.5}$$

3.5.1 Invertibility of the trajectories $X^+(.,.)$

In this section, we show that, the mapping $X^+(.,.)$ given in (3.4.2) is invertible in sense of (3.5.2) with inverse $\widehat{B}_+(.)=(\widehat{t}_+(.),\widehat{s}_+(.)) : G_+\subset\widetilde{Y}_+\to\widetilde{B}^+\subset B^+$. First taking into account the second expression in (3.4.2), we deduce that, $\cos(\frac{t}{2s})=\frac{\sqrt{x_2}}{s}\in(0,1)$, $\forall x_2\in(0,s^2)$, therefore, there exists a unique function $\widehat{t}_1(.,.)$ defined by:

$$\widehat{t}_1(x_2,s)=-2s\arccos\frac{\sqrt{x_2}}{s}. \tag{3.5.6}$$

Denote $\gamma_1(.,.)$ by:

$$\gamma_1(x_2,s)=X_1^+(\widehat{t}_1(x_2,s),s)=\sqrt{x_2}\sqrt{s^2-x_2}-s(2w-s)\arccos\frac{\sqrt{x_2}}{s}. \tag{3.5.7}$$

In the following lemmas, we prove that the Hamiltonian flow $X_+^*(.,.)$ described in (3.4.2) and the corresponding value function defined as in (3.5.1), may characterize a partial solution of the problem on its domain $\widetilde{Y}_+\subset Y_0$.

Lemma 3.5.1. *If $s>s_0$ then, the function $\widehat{t}_1(.,.)$ in (3.5.6) verifies the following condition:*

$$\widehat{t}_1(x_2,s)\in(\tau_0(s),0),\ \forall x_2\in(\frac{s^2}{2},s^2), \tag{3.5.8}$$

moreover, there exists a unique function:

$$\widehat{s}_+(x)=(\gamma_1(x_2,.))^{-1}(x_1)\ \forall x_1\in(0,\beta(x_2)) \tag{3.5.9}$$

and

$$\widehat{t}_+(x)=\widehat{t}_1(x_2,\widehat{s}_+(x)). \tag{3.5.10}$$

Proof. If $s>s_0$ then, from the second equation in (3.4.2) one has, $\frac{s^2}{2}<x_2<s^2$ this implies that, $\frac{1}{\sqrt{2}}<\frac{\sqrt{x_2}}{s}<1$, hence, $\arccos(\frac{\sqrt{x_2}}{s})\in(0,\frac{\pi}{4})$ and therefore, expression (3.5.8) is verified.

Moreover, it follows from (3.5.7) that:

$$\frac{\partial \gamma_1}{\partial s}(x_2, s) = 2(s-w)\left[\frac{\sqrt{x_2}}{\sqrt{s^2 - x_2}} + \arccos\frac{\sqrt{x_2}}{s}\right], \tag{3.5.11}$$

it is easy to see that, $\frac{\partial \gamma_1}{\partial s}(x_2, s) > 0 \ \forall x_2 \in (\frac{s^2}{2}, s^2)$ hence, the function $\gamma_1(x_2, .) : (\sqrt{x_2}, \sqrt{2x_2}) \to (0, \beta(x_2))$ is strictly increasing hence, there exists a unique function $\widehat{s}_+(.)$ defined as in (3.5.9) and the function $\widehat{t}_+(.)$ is well defined. This completes the proof. ■

Also, we get the following extreme limits:

$$\begin{cases} \lim\limits_{s \overset{>}{\to} \sqrt{x_2}} \dfrac{\partial \gamma_1}{\partial s}(x_2, s) = +\infty, \\ \lim\limits_{s \overset{<}{\to} \sqrt{2x_2}} \dfrac{\partial \gamma_1}{\partial s}(x_2, s) = 2(\sqrt{2x_2} - w)(1 + \frac{\pi}{4}). \end{cases}$$

Lemma 3.5.2. *If $s \in (w, s_0]$ then, the function $\widehat{t}_1(.,.)$ in (3.5.6) verifies:*

$$\begin{aligned} &\widehat{t}_1(x_2, s) \in (t_2(s), 0), \ \forall x_2 \in (\alpha_1(s), s^2), \\ &\textit{where } \alpha_1(s) = X_2^+(t_2(s), s), \ \textit{such that } \alpha_1(w) = w^2, \ \alpha_1(s_0) = \tfrac{s_0^2}{2}, \end{aligned} \tag{3.5.12}$$

moreover, there is a unique function $\widehat{s}_+(.)$ defined as in (3.5.9) checks expression (3.5.10).

Proof. Assuming by the absurdity that statement (3.5.12) is not true then, there exists $x_2 \in (\alpha_1(s), s^2)$ such that, $\widehat{t}_1(x_2, s) \leq t_2(s)$. Using the monotonicity of the component $X_2^+(., s)$, $s \in (w, s_0]$ we deduce that, $X_2^+(\widehat{t}_1(x_2, s), s) = x_2 < X_2^+(t_2(s), s) = \alpha_1(s)$ which implies, $\alpha_1(s) < x_2 < \alpha_1(s)$, which is impossible, then $\widehat{t}_1(x_2, s) \in (t_2(s), 0)$, $\forall x_2 \in (\alpha_1(s), s^2)$.

Next, in order to prove the existence and uniqueness of the function $\widehat{s}_+(x)$, we proceed as follow. Firstly, it remains to show that, $\alpha_1(.)$ is strictly decreasing. Note that, the component $X_2^+(.,.)$ is of class $\mathcal{C}^1$ therefore, from (3.4.2) it follows that:

$$\alpha_1'(s) = \frac{2}{s}\alpha_1(s) + \frac{1}{2}(t_2(s) - 2st_2'(s))\sin\frac{t_2(s)}{s},$$

and due to the first condition in Lemma 3.5.2, we get:

$$\sin\frac{t_2(s)}{s} = (\frac{2w - s}{s^2})t_2(s). \tag{3.5.13}$$

Since the component $X_1^+(.,.)$ is of class $\mathcal{C}^1$ then also, according to Lemma 3.5.1 one has:

$$\begin{aligned} &\tfrac{\partial X_1^+}{\partial t}(t_2(s), s)t_2'(s) + \tfrac{\partial X_1^+}{\partial s}(t_2(s), s) = 0, \\ &\tfrac{\partial X_1^+}{\partial t}(t_2(s), s) = -\tfrac{1}{s}\alpha_1(s) + w, \\ &\tfrac{\partial X_1^+}{\partial s}(t_2(s), s) = \tfrac{t_2(s)}{s^2}\alpha_1(s) - t_2(s) - s\sin\tfrac{t_2(s)}{s}, \end{aligned} \tag{3.5.14}$$

and from here, we deduce that, $s(sw-\alpha_1(s))t_2'(s)+(\alpha_1(s)-2ws)t_2(s)=0$. We remark that $\alpha_1(s)-sw\neq 0$, because if $\alpha_1(s)=sw$ then we get, $(\alpha_1(s)-2sw)t_2(s)=0$, $\forall s\in(w,s_0]$ hence, $\alpha_1(s)=2sw\neq sw$ because $t_2(s)<0, \forall s\in(w,s_0]$ which gives:

$$t_2'(s)=\left[\frac{\alpha_1(s)-2ws}{s(\alpha_1(s)-sw)}\right]t_2(s), \tag{3.5.15}$$

moreover, it follows from (3.5.13), (3.5.14) and (3.5.15) that:

$$\alpha_1'(s)=\frac{2}{s}\alpha_1(s)+\frac{w}{2s(\alpha_1(s)-sw)(2w-s)}s^4(\sin\frac{t_2(s)}{s})^2, \tag{3.5.16}$$

since, $s^4(\sin\frac{t_2(s)}{s})^2=4\alpha_1(s)(s^2-\alpha_1(s))$ from here together with (3.5.16) it results that:

$$\alpha_1'(s)=-\frac{2(s-w)}{s(2w-s)}\alpha_1(s)\left[1+\frac{sw}{sw-\alpha_1(s)}\right]. \tag{3.5.17}$$

As $\alpha_1(s)\in[\frac{s_0^2}{2},w^2)$, $\forall s\in(w,s_0]$ then $\alpha_1(s)<w^2<ws$, $\forall s\in(w,s_0]$, consequently, it follows from (3.5.17) that, $\alpha_1'(s)<0$, $\forall s\in(w,s_0]$ so, the function $\alpha_1(.)$ is strictly decreasing on $(w,s_0]$. Therefore there is a unique function $\alpha_1^{-1}(x_2)\in(w,s_0]$, $\forall x_2\in(\frac{s_0^2}{2},w^2)$ from here, we can easily prove that, $s\in(\alpha_1^{-1}(x_2),\sqrt{2x_2})$. The existence and uniqueness of the function $\widehat{s}_+(.)$ (respectively, $\widehat{t}_+(.)$) defined as in (3.5.9) and (3.5.10) is done in the same way as in Lemma 3.5.1, we only change the domain by $s\in(\alpha_1^{-1}(x_2),\sqrt{2x_2})$. Obviously, the function $\gamma_1(x_2,.)$, $x_2\in(\frac{s_0^2}{2},+\infty)$ is derivable and strictly increasing inside the intervals, $s\in(\sqrt{x_2},\sqrt{2x_2})$ and $s\in(\alpha_1^{-1}(x_2),\sqrt{2x_2})$. Hence, the function $\widehat{s}_+(.,x_2)$, $x_2\in(\frac{s_0^2}{2},+\infty)$ has the same properties as $\gamma_1(x_2,.)$, $x_2\in(\frac{s_0^2}{2},+\infty)$. Therefore, there is a unique function $\widehat{s}_+(.)$ defined as in (3.5.9) and checks expression (3.5.10). This completes the proof. ■

Now, in order to precise the nature of the partial derivatives $\frac{\partial\widehat{t}_+}{\partial x_i}(.)$, $i=1,2$, we need to compute the following limits. In fact, according to Lemmas 3.5.1 and 3.5.2, we have:

$$\begin{cases} \lim\limits_{x_1\overset{>}{\to}0}\widehat{s}_+(x)=\begin{cases}\sqrt{x_2},\ x_2>w^2,\\ \alpha_1^{-1}(x_2),\ x_2\in(\frac{s_0^2}{2},w^2),\end{cases}\\ \lim\limits_{x_1\overset{<}{\to}\beta(x_2)}\widehat{s}_+(x)=\sqrt{2x_2},\ x_2>\frac{s_0^2}{2},\\ \lim\limits_{(x_1,x_2)\to(0,\frac{s_0^2}{2})}\widehat{s}_+(x)=\alpha_1^{-1}(\frac{s_0^2}{2}).\end{cases} \tag{3.5.18}$$

On the other hand, taking into consideration (3.5.6) and (3.5.18) together with the fact that, $\widehat{t}_1(x_2,.)$, $x_2 \in (\frac{s_0^2}{2}, +\infty)$ is continuous we get:

$$\begin{cases} \lim\limits_{x_1 \overset{>}{\to} 0} \quad \widehat{t}_+(x) = \begin{cases} 0, \text{ if } x_2 > w^2, \\ -2\alpha_1^{-1}(x_2) \arccos \dfrac{\sqrt{x_2}}{\alpha_1^{-1}(x_2)}, \text{ if } x_2 \in (\frac{s_0^2}{2}, w^2), \end{cases} \\ \lim\limits_{x_1 \overset{<}{\to} \beta(x_2)} \quad \widehat{t}_+(x) = -2\sqrt{2x_2} \arccos \frac{\sqrt{x_2}}{\sqrt{2x_2}} = -\frac{\pi}{\sqrt{2}}\sqrt{x_2}, \text{ if } x_2 > \frac{s_0^2}{2}, \\ \lim\limits_{(x_1,x_2) \to (0, \frac{s_0^2}{2})} \quad \widehat{t}_+(x) = -2\alpha_1^{-1}(\frac{s_0^2}{2}) \arccos \left[\dfrac{s_0}{\sqrt{2}\alpha_1^{-1}(\frac{s_0^2}{2})} \right]. \end{cases}$$

Further, it follows from (3.5.9) that:

$$\frac{\partial \widehat{s}_+}{\partial x_1}(x) = \frac{\sqrt{s^2 - x_2}}{2(s-w)(\sqrt{x_2} + (\sqrt{s^2 - x_2}) \arccos(\frac{\sqrt{x_2}}{s}))}, \quad s = \widehat{s}_+(x), \tag{3.5.19}$$

using the fact that, $\frac{\partial \gamma_1}{\partial s}(x_2,.)$, $x_2 \in (\frac{s_0^2}{2}, +\infty)$ is continuous then:

$$\lim_{\substack{x_1 \overset{>}{\to} 0 \\ x_1 \overset{<}{\to} \beta(x_2) \\ (x_1,x_2) \to (0, \frac{s_0^2}{2})}} \frac{\partial \gamma_1}{\partial s}(x_2, \widehat{s}_+(x)) = \frac{\partial \gamma_1}{\partial s}(x_2, \lim_{\substack{x_1 \overset{>}{\to} 0 \\ x_1 \overset{<}{\to} \beta(x_2) \\ (x_1,x_2) \to (0, \frac{s_0^2}{2})}} \widehat{s}_+(x)),$$

from here, we can extract the following limits:

$$\begin{cases} \lim\limits_{x_1 \overset{>}{\to} 0} \quad \dfrac{\partial \gamma_1}{\partial s}(x_2, \widehat{s}_+(x)) = \begin{cases} +\infty, \text{ if } x_2 > w^2, \\ \theta(x_2), \text{ if } x_2 \in (\frac{s_0^2}{2}, w^2), \end{cases} \\ \theta(x_2) = 2\left(\alpha_1^{-1}(x_2) - w\right) \left[\frac{\sqrt{x_2}}{\sqrt{\left(\alpha_1^{-1}(x_2)\right)^2 - x_2}} + \arccos \frac{\sqrt{x_2}}{\alpha_1^{-1}(x_2)} \right], \\ \lim\limits_{x_1 \overset{<}{\to} \beta(x_2)} \quad \dfrac{\partial \gamma_1}{\partial s}(x_2, \widehat{s}_+(x)) = 2(\sqrt{2x_2} - w)(1 + \frac{\pi}{4}) \text{ if } x_2 > \frac{s_0^2}{2}, \\ \lim\limits_{(x_1,x_2) \to (0, \frac{s_0^2}{2})} \quad \dfrac{\partial \gamma_1}{\partial s}(x_2, \widehat{s}_+(x)) = \theta(\frac{s_0^2}{2}), \end{cases}$$

and therefore:

$$\begin{cases} \lim\limits_{x_1 \overset{>}{\to} 0} \quad \dfrac{\partial \widehat{s}_+}{\partial x_1}(x) = \begin{cases} 0, \text{ if } x_2 > w^2, \\ \dfrac{1}{\theta(x_2)}, \text{if } x_2 \in (\frac{s_0^2}{2}, w^2), \end{cases} \\ \lim\limits_{x_1 \overset{<}{\to} \beta(x_2)} \quad \dfrac{\partial \widehat{s}_+}{\partial x_1}(x) = \dfrac{2}{(\sqrt{2x_2} - w)(4 + \pi)}, \text{ if } x_2 > \frac{s_0^2}{2}, \\ \lim\limits_{(x_1,x_2) \to (0, \frac{s_0^2}{2})} \quad \dfrac{\partial \widehat{s}_+}{\partial x_1}(x) = \dfrac{1}{\theta(\frac{s_0^2}{2})}, \end{cases} \tag{3.5.20}$$

also, it follows from (3.5.7) and (3.5.9) that:

$$\frac{\partial \widehat{s}_+}{\partial x_2}(x) = -\frac{\frac{\partial \gamma_1}{\partial x_2}(x_2, \widehat{s}_+(x))}{\frac{\partial \gamma_1}{\partial s}(x_2, \widehat{s}_+(x))}, \quad \frac{\partial \gamma_1}{\partial x_2}(x_2, s) = \frac{sw - x_2}{\sqrt{x_2}\sqrt{s^2 - x_2}}, \tag{3.5.21}$$

due to expression (3.5.11), the partial derivative in (3.5.21) becomes:

$$\begin{cases} \dfrac{\partial \widehat{s}_+}{\partial x_2}(x) = \dfrac{x_2 - sw}{2(s-w)\sqrt{x_2}\left[\sqrt{x_2} + (\sqrt{s^2 - x_2}) \arccos \frac{\sqrt{x_2}}{s}\right]}, \; s = \widehat{s}_+(x), \\ \lim\limits_{x_1 \overset{>}{\to} 0} \dfrac{\partial \widehat{s}_+}{\partial x_2}(x) = \begin{cases} \dfrac{1}{2\sqrt{x_2}}, \text{ if } x_2 > w^2, \\ \overline{\theta}(x_2), \text{ if } x_2 \in (\frac{s_0^2}{2}, w^2), \end{cases} \\ \overline{\theta}(x_2) = \dfrac{x_2 - \alpha_1^{-1}(x_2) w}{2(\alpha_1^{-1}(x_2) - w)\left[x_2 + (\sqrt{x_2}\sqrt{\left(\alpha_1^{-1}(x_2)\right)^2 - x_2}) \arccos \frac{\sqrt{x_2}}{\alpha_1^{-1}(x_2)}\right]}, \\ \lim\limits_{x_1 \overset{<}{\to} \beta(x_2)} \dfrac{\partial \widehat{s}_+}{\partial x_2}(x) = \dfrac{2(\sqrt{x_2} - \sqrt{2}w)}{\sqrt{x_2}(\sqrt{2x_2} - w)(\pi + 4)}, \text{ if } x_2 > \frac{s_0^2}{2}, \\ \lim\limits_{(x_1,x_2) \to (0, \frac{s_0^2}{2})} \dfrac{\partial \widehat{s}_+}{\partial x_2}(x) = \overline{\theta}(\frac{s_0^2}{2}), \end{cases}$$

further, it follows from (3.5.6) , (3.5.10) together with (3.5.19) that:

$$\begin{aligned} &\frac{\partial \widehat{t}_+}{\partial x_1}(x) = \frac{\partial \widehat{t}_1}{\partial s}(x_2, \widehat{s}_+(x)) \frac{\partial \widehat{s}_+}{\partial x_1}(x), \\ &\frac{\partial \widehat{t}_1}{\partial s}(x_2, s) = -2 \arccos \frac{\sqrt{x_2}}{s} + \frac{2\sqrt{x_2}}{\sqrt{x_2 - s^2}}, \\ &\frac{\partial \widehat{t}_+}{\partial x_1}(x) = \frac{1}{w - \widehat{s}_+(x)}, \end{aligned}$$

and we extract here the following limits:

$$\begin{cases} \lim\limits_{x_1 \overset{>}{\to} 0} \dfrac{\partial \widehat{t}_+}{\partial x_1}(x) = \begin{cases} \dfrac{1}{w - \sqrt{x_2}}, \text{ if } x_2 > w^2, \\ \dfrac{1}{w - \alpha_1^{-1}(x_2)}, \text{ if } x_2 \in (\frac{s_0^2}{2}, w^2), \end{cases} \\ \lim\limits_{x_1 \overset{<}{\to} \beta(x_2)} \dfrac{\partial \widehat{t}_+}{\partial x_1}(x) = \dfrac{1}{w - \sqrt{2x_2}}, \text{ if } x_2 > \frac{s_0^2}{2}, \\ \lim\limits_{(x_1,x_2) \to (0, \frac{s_0^2}{2})} \dfrac{\partial \widehat{t}_+}{\partial x_1}(x) = \dfrac{1}{w - \alpha_1^{-1}(\frac{s_0^2}{2})}. \end{cases}$$

Using the same type of computations as in the previous case, we obtain:

$$\begin{aligned} &\frac{\partial \widehat{t}_+}{\partial x_2}(x) = \frac{\partial \widehat{t}_1}{\partial x_2}(x_2, \widehat{s}_+(x)) + \frac{\partial \widehat{t}_1}{\partial s}(x_2, \widehat{s}_+(x)) \frac{\partial \widehat{s}_+}{\partial x_2}(x), \\ &\frac{\partial \widehat{t}_1}{\partial x_2}(x_2, s) = \frac{s}{\sqrt{x_2}\sqrt{s^2 - x_2}}, \\ &\frac{\partial \widehat{t}_+}{\partial x_2}(x) = \frac{s^2 + sw - 2x_2}{(s-w)\sqrt{x_2}\sqrt{s^2 - x_2}}, \; s = \widehat{s}_+(x), \; x = (x_1, x_2), \end{aligned}$$

therefore:

$$\begin{cases} \lim\limits_{x_1 \overset{>}{\to} 0} \dfrac{\partial \widehat{t}_+}{\partial x_2}(x) = \begin{cases} -\infty, \; x_2 > w^2, \\ \dfrac{s^2 + sw - 2x_2}{(s-w)\sqrt{x_2}\sqrt{s^2 - x_2}}, \; s = \alpha_1^{-1}(x_2), \; x_2 \in (\dfrac{s_0^2}{2}, w^2), \end{cases} \\ \lim\limits_{x_1 \overset{<}{\to} \beta(x_2)} \dfrac{\partial \widehat{t}_+}{\partial x_2}(x) = \dfrac{\sqrt{2}w}{\sqrt{x_2}(\sqrt{2x_2} - w)}, \; x_2 > \dfrac{s_0^2}{2}, \\ \lim\limits_{(x_1,x_2) \to (0, \frac{s_0^2}{2})} \dfrac{\partial \widehat{t}_+}{\partial x_2}(x) = \dfrac{s^2 + sw - 2x_2}{(s-w)\sqrt{x_2}\sqrt{s^2 - x_2}}, \; s = \alpha_1^{-1}(\dfrac{s_0^2}{2}). \end{cases}$$

Remark 13. *As can be seen, the invertibility domain* $G_+ \subset \widetilde{Y}_+$ *is given by:*

$$G_+ = \left\{(x_1, x_2);\ x_1 \in (0, \beta(x_2)),\ x_2 \in (\alpha_2(s), s^2),\ s > w\right\},$$
$$\alpha_2(s) = \begin{cases} \dfrac{s^2}{2},\ \text{if } s > s_0, \\ \alpha_1(s),\ \text{if } s \in (w, s_0]. \end{cases}$$

Moreover, the partial derivatives $\dfrac{\partial \widehat{t}_+}{\partial x_i}(.)$, $i = 1, 2$ *are bounded on* $G_+ \subset \widetilde{Y}_+$ *then the functions* $\widehat{t}_+(x_1, .)$, $x_1 \in (0, \beta(x_2))$ *and* $\widehat{t}_+(., x_2)$, $x_2 \in (\frac{s_0^2}{2}, w^2)$ *are lipschitzian. Moreover, these functions are continuous for* $x_2 > w^2$ *on the terminal set* Y_1.

3.5.2 Invertibility of the trajectories $X^-(.,.)$

Since the trajectories $X_-^*(., s)$, $s > s_0$ is obtained in implicit form for this reason, we will only deal with the study of the invertibility problem for the parameterized trajectories $\widetilde{X}(.,.)$ in (3.4.12) by showing the existence and uniqueness of the functions $\widehat{s}_-(.)$ and $\widehat{\xi}_-(.)$ such that $\widetilde{X}(.,.) : B^- \to \widetilde{Y}_-$ is invertible at $(\xi, s) \in B^-$ with the inverse, $\widehat{B}_-(.) = (\widehat{\xi}_-(.), \widehat{s}_-(.)) : G_- \subset \widetilde{Y}_- \to \widetilde{B}^- \subset B^-$ such that:

$$\begin{aligned} &\widetilde{X}(\widehat{B}_-(x)) = \widetilde{X}((\widehat{\xi}_-(x), \widehat{s}_-(x))) = x,\ \forall x = (x_1, x_2) \in G_-, \\ &\widehat{B}_-(\widetilde{X}(\xi, s)) = (\widehat{\xi}_-(\widetilde{X}(\xi, s)), \widehat{s}_-(\widetilde{X}(\xi, s))) = (\xi, s),\ \forall (\xi, s) \in \widetilde{B}^-. \end{aligned} \tag{3.5.22}$$

Lemma 3.5.3. *If* $s > s_0$ *and* $\widehat{\xi}_1(.,.)$ *is the function defined in* (3.4.26) *then, there exists a unique function* $\widehat{B}_-(.) : G_- \to B^-$ *checks* (3.5.22) *with:*

$$\begin{cases} \widehat{\xi}_-(x) = \widehat{\xi}_1(x_1, \widehat{s}(x)),\ x = (x_1, x_2) \in G_- \\ \widehat{t}_-(x) = T_1(\widehat{\xi}_-(x), \widehat{s}_-(x)), \end{cases} \tag{3.5.23}$$

where,

$$\begin{aligned} &G_- = \left\{(x_1, x_2);\ x_2 \in (\gamma(x_1), \beta^{-1}(x_1)),\ \beta(x_2) = \alpha(\sqrt{2x_2})\right\}, \\ &\text{with } \gamma(x_1) = \begin{cases} \widehat{X}_2(x_1, s_0),\ x_1 \in (0, k_2(s_0)), \\ w^2,\ x_1 \geq k_2(s_0), \end{cases} \end{aligned} \tag{3.5.24}$$

such that $k_2(.)$ *denotes the extremity in* (3.4.24).

Proof. It follows from (3.4.27), that:

$$\frac{\partial \widehat{X}_2}{\partial s}(x_1,s) = \frac{\partial \widetilde{X}_2}{\partial \xi}(\widehat{\xi}_1(x_1,s),s)\frac{\partial \widehat{\xi}_1}{\partial s}(x_1,s) + \frac{\partial \widetilde{X}_2}{\partial s}(\widehat{\xi}_1(x_1,s),s) \tag{3.5.25}$$

while, from (3.4.26), we deduce that:

$$\begin{aligned} \frac{\partial \widehat{\xi}_1}{\partial s}(x_1,s) &= -\frac{\frac{\partial \widetilde{X}_1}{\partial s}(\xi,s)}{\frac{\partial \widetilde{X}_1}{\partial \xi}(\xi,s)},\ \xi = \widehat{\xi}_1(x_1,s), \\ &= -\frac{\xi\left[\xi^2 - wq_1^2(s)\xi + 2q_1^2(s)\right]}{q_1(s)(1-w\xi)} - \frac{(q_1^2(s)+\xi^2)^2}{q_1^2(s)(1-w\xi)^2}\arctan\frac{\xi}{q_1(s)}, \end{aligned} \tag{3.5.26}$$

therefore, from (3.4.25), (3.5.25) and (3.5.26), we obtain:

$$\frac{\partial \widehat{X}_2}{\partial s}(x_1,s) = \frac{2}{q_1(s)}[1 + \frac{\partial \widehat{X}_2}{\partial x_1}(x_1,s)\arctan\frac{\xi}{q_1(s)}],\ \xi = \widehat{\xi}_1(x_1,s). \tag{3.5.27}$$

Case 1: If $s \geq 2w$, using (3.4.29) and the fact that, $\arctan\frac{\xi}{q_1(s)} \in (-\frac{\pi}{2}, -\frac{\pi}{4})$, we obtain:

$$\frac{\partial \widehat{X}_2}{\partial s}(x_1,s) > 0,\ \forall x_1(\alpha(s),k_2(s)). \tag{3.5.28}$$

Case 2: If $s \in (s_0, 2w)$ and $\xi \in (\xi_M, -q_1(s))$, we replace (3.4.29) in expression (3.5.27) for $\xi = \widehat{\xi}_1(x_1,s)$ we obtain:

$$\frac{\partial \widehat{X}_2}{\partial s}(x_1,s) = \frac{2}{q_1^2(s)(1-w\xi)}\left[q_1(s)(1-w\xi) + (\xi + wq_1^2(s))\arctan\frac{\xi}{q_1(s)}\right],$$

we denote $\phi(.,.)$ by:

$$\phi(\xi,q_1(s)) = q_1(s)(1-w\xi) + (\xi + wq_1^2(s))\arctan\frac{\xi}{q_1(s)},$$

then elementary computations and arguments show that:

$$\frac{\partial \phi}{\partial \xi}(\xi,q_1(s)) = \arctan\frac{\xi}{q_1(s)} - \frac{q_1(s)\xi(w\xi-1)}{q_1^2(s)+\xi^2},$$

and therefore, if $\frac{\partial \phi}{\partial \xi}(\xi,q_1(s)) = 0$, we obtain:

$$\arctan\frac{\xi}{q_1(s)} = \frac{q_1(s)\xi(w\xi-1)}{q_1^2(s)+\xi^2},$$

also, we put $\theta_\xi = \frac{\xi}{q_1(s)}$ we get:

$$\arctan\theta_\xi = \frac{\theta_\xi(wq_1(s)\theta_\xi - 1)}{\theta_\xi^2+1},\ \theta_\xi \in (-wq_1(s),-1),$$

let $g_1(.)$ to be a function defined by:

$$\begin{aligned} g_1(\theta_\xi) &= \tfrac{\theta_\xi(wq_1(s)\theta_\xi-1)}{\theta_\xi^2+1},\ \theta_\xi \in (-wq_1(s),-1) \\ g_1'(\theta_\xi) &= \tfrac{\theta_\xi^2+2wq_1(s)\theta_\xi-1}{(\theta_\xi^2+1)^2}, \end{aligned}$$

and we obtain:

$$\lim_{\theta_\xi \to -wq_1(s)} g_1(\theta_\xi) = wq_1(s), \lim_{\theta_\xi \to -1} g_1(\theta_\xi) = \frac{wq_1(s)+1}{2},$$
$$g_1(-wq_1(s)) - g_1(-1) = \frac{2w-s}{2(s-w)} > 0, \ \forall s \in (s_0, 2w).$$

From the below Figure 9, we remark that, $\arctan \theta_\xi < g_1(\theta_\xi)$, $\forall \theta_\xi \in (-wq_1(s), -1)$, hence, $\frac{\partial \phi}{\partial \xi}(\xi, q_1(s)) < 0 \ \forall \xi \in (\xi_M, -q_1(s))$ this implies that, $\phi(., q_1(s))$, $s \in (s_0, 2w)$ is strictly decreasing, so, $\forall \xi \in (\xi_M, -q_1(s))$ one has:

$$\phi(\xi, q_1(s)) > \phi(-q_1(s), q_1(s)) = \frac{1}{4} q_1^2(s)((\pi+4)s - 2\pi w) > 0, \forall s \in (s_0, 2w).$$

Therefore, the expression in (3.5.28) is also verified for $\xi \in (\xi_M, -q_1(s))$, $s \in (s_0, 2w)$.

Hence $\widehat{X}_2(x_1, .)$, $x_1 \in (\alpha(s), k_2(s))$, $s > s_0$ is strictly increasing which prove the existence and uniqueness of the function:

$$\widehat{s}_-(x) = (\widehat{X}_2(x_1, .))^{-1}(x_2), \ x \in G_-. \tag{3.5.29}$$

Besides the existence and uniqueness of the function $\widehat{\xi}_1(.,.)$ in (3.4.26), we can define the two functions $\widehat{\xi}_-(.)$ and $\widehat{t}_-(.)$ given in (3.5.23), which in turn are well-defined and unique. This completes the proof. ■

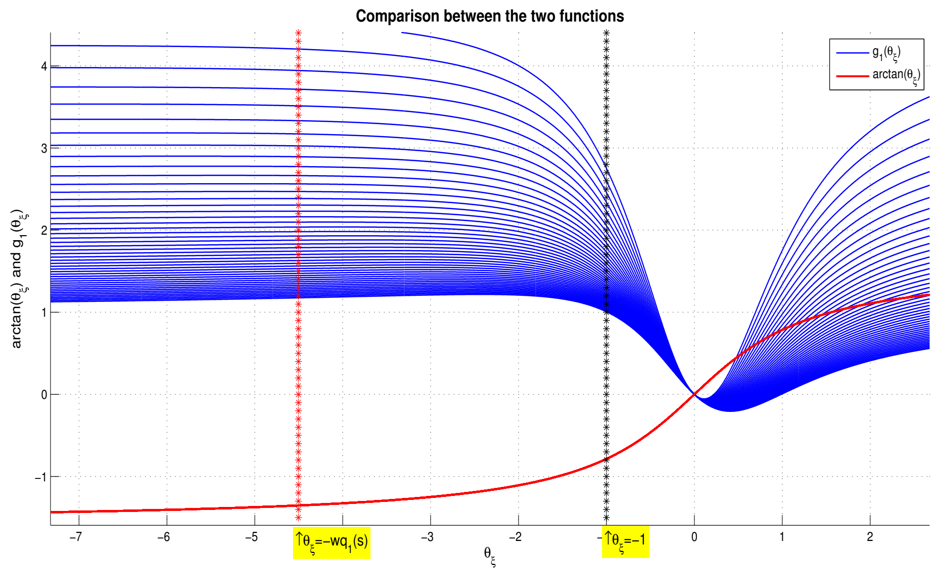

Figure 9: Curves of functions $g_1(.)$ and $arctan(.)$

Now, in order to precise the nature of the partial derivatives $\frac{\partial \widehat{t}_-}{\partial x_i}(.)$, $i = 1, 2$, we need to compute the following limits. In fact, according to the previous study in Lemma 3.5.3, we have:

$$\begin{cases} \lim\limits_{x_2 \overset{>}{\to} \gamma(x_1)} \widehat{s}_-(x) = \begin{cases} \sqrt{2\widehat{X}_2(x_1, s_0)}, \; x_1 \geq k_2(s_0), \\ k_2^{-1}(x_1), \; x_1 \in (0, k_2(s_0)), \end{cases} \\ \lim\limits_{x_2 \overset{<}{\to} \beta^{-1}(x_1)} \widehat{s}_-(x) = \alpha^{-1}(x_1), \; x_1 > 0. \end{cases} \tag{3.5.30}$$

$$\begin{cases} \lim\limits_{x_2 \overset{>}{\to} \gamma(x_1)} \widehat{\xi}_-(x_1, x_2) = \begin{cases} \dfrac{1}{w - \sqrt{2\widehat{X}_2(x_1, s_0)}}, \; x_1 \geq k_2(s_0), \\ \dfrac{1}{w - k_2^{-1}(x_1)}, \; x_1 \in (0, k_2(s_0)), \end{cases} \\ \lim\limits_{x_2 \overset{<}{\to} \beta^{-1}(x_1)} \widehat{\xi}_-(x_1, x_2) = -\infty, \; x_1 > 0. \end{cases} \tag{3.5.31}$$

On the other hand, it follows from (3.5.23) that:

$$\begin{cases} \frac{\partial \widehat{t}_-}{\partial x_1}(x) &= \frac{\partial T_1}{\partial \xi}(\widehat{\xi}_-(x), \widehat{s}_-(x)) \left[\frac{\partial \widehat{\xi}_1}{\partial x_1}(x_1, \widehat{s}_-(x)) + \frac{\partial \widehat{\xi}_1}{\partial s}(x_1, \widehat{s}_-(x))\right] \\ &+ \frac{\partial T_1}{\partial s}(\widehat{\xi}_-(x), \widehat{s}_-(x)) \frac{\partial \widehat{s}_-}{\partial x_1}(x), \\ \frac{\partial \widehat{t}_-}{\partial x_2}(x) &= \frac{\partial T_1}{\partial \xi}(\widehat{\xi}_-(x), \widehat{s}_-(x)) \frac{\partial \widehat{\xi}_1}{\partial s}(x_1, \widehat{s}_-(x)) \frac{\partial \widehat{s}_-}{\partial x_2}(x) \\ &+ \frac{\partial T_1}{\partial s}(\widehat{\xi}_-(x), \widehat{s}_-(x)) \frac{\partial \widehat{s}_-}{\partial x_2}(x), \; x = (x_1, x_2), \end{cases} \tag{3.5.32}$$

while from (3.4.12), elementary computations show that:

$$\frac{\partial T_1}{\partial s}(\xi, s) = 2 \arctan \frac{\xi}{q_1(s)} + \frac{2q_1(s)(\xi + wq_1^2(s))}{q_1^2(s) + \xi^2} - 2wq_1(s) + \pi, \tag{3.5.33}$$

also, from (3.5.29) one has:

$$\begin{cases} \dfrac{\partial \widehat{s}_-}{\partial x_1}(x) = -\dfrac{\frac{\partial \widehat{X}_2}{\partial x_1}(x_1, \widehat{s}_-(x))}{\frac{\partial \widehat{X}_2}{\partial s}(x_1, \widehat{s}_-(x))}, \; x = (x_1, x_2) \in G_-, \\ \dfrac{\partial \widehat{s}_-}{\partial x_2}(x) = -\dfrac{1}{\frac{\partial \widehat{X}_2}{\partial s}(x_1, \widehat{s}_-(x))}, \end{cases}$$

from here, together with (3.4.29) and (3.4.8) we get:

$$\begin{cases} \dfrac{\partial \widehat{s}_-}{\partial x_1}(x) = -\dfrac{q_1(s)}{2} \dfrac{\xi + wq_1^2(s)}{q_1(s)(1 - w\xi) + (\xi + wq_1^2(s)) \arctan \frac{\xi}{q_1(s)}}, \\ \dfrac{\partial \widehat{s}_-}{\partial x_2}(x) = \dfrac{q_1^2(s)(1 - w\xi)}{2\left[q_1(s)(1 - w\xi) + (\xi + wq_1^2(s)). \arctan \frac{\xi}{q_1(s)}\right]}, \\ \xi = \widehat{\xi}_-(x), \; s = \widehat{s}_-(x), \; x = (x_1, x_2) \in G_-, \end{cases} \tag{3.5.34}$$

from here, together with (3.5.30) and (3.5.31), we obtain:

$$\begin{cases} \lim\limits_{x_2 \overset{>}{\to} \gamma(x_1)} \dfrac{\partial \widehat{s}_-}{\partial x_1}(x) = \dfrac{2q_1(s)(1-wq_1(s))}{(\pi+4)+(4-\pi)wq_1(s)}, \\ \lim\limits_{x_2 \overset{<}{\to} \beta^{-1}(x_1)} \dfrac{\partial \widehat{s}_-}{\partial x_1}(x) = \dfrac{1}{\pi(\alpha^{-1}(x_1)-w)+2w}, \; x_1 > 0, \\ \lim\limits_{x_2 \overset{>}{\to} \gamma(x_1)} \dfrac{\partial \widehat{s}_-}{\partial x_2}(x) = \dfrac{2s}{(s-w)\left[(\pi+4)s - 2\pi w\right]}, \\ s = \begin{cases} \sqrt{2\widehat{X}_2(x_1, s_0)}, \; x_1 \geq k_2(s_0), \\ k_2^{-1}(x_1), \; x_1 \in (0, k_2(s_0)), \end{cases} \\ \lim\limits_{x_2 \overset{<}{\to} \beta^{-1}(x_1)} \dfrac{\partial \widehat{s}_-}{\partial x_2}(x) = \dfrac{wq_1^2(s)}{2wq_1(s)+\pi}, \; s = \alpha^{-1}(x_1), \; x_1 > 0. \end{cases} \tag{3.5.35}$$

On the other hand, it follows from (3.4.17), (3.4.28), (3.5.26), (3.5.33) and (3.5.34) that, the expressions in (3.5.32) becomes:

$$\begin{cases} \frac{\partial \widehat{t}_-}{\partial x_1}(x) & = -\left(\frac{2\xi(\xi+wq_1^2(s))}{q_1^2(s)(1-w\xi)} \arctan \frac{\xi}{q_1(s)} + \frac{2(\xi - wq_1^2(s))}{q_1(s)}\right) \frac{\partial \widehat{s}_-}{\partial x_1}(x) \\ & - (2wq_1(s)-\pi) \frac{\partial \widehat{s}_-}{\partial x_1}(x) - \frac{q_1^2(s)+\xi^2}{q_1(s)(1-w\xi)}, \\ \frac{\partial \widehat{t}_-}{\partial x_2}(x) & = -\left(\frac{2\xi(\xi+wq_1^2(s))}{q_1^2(s)(1-w\xi)} \arctan \frac{\xi}{q_1(s)} + \frac{2(\xi - wq_1^2(s))}{q_1(s)}\right) \frac{\partial \widehat{s}_-}{\partial x_2}(x) \\ & - (2wq_1(s)-\pi) \frac{\partial \widehat{s}_-}{\partial x_2}(x), \\ & \xi = \widehat{\xi}_-(x), \; s = \widehat{s}_-(x), \; x = (x_1, x_2) \in G_-, \end{cases}$$

and due to relations (3.5.35) we get:

$$\begin{cases} \lim\limits_{x_2 \overset{>}{\to} \gamma(x_1)} \dfrac{\partial \widehat{t}_-}{\partial x_1}(x) = \left[\frac{-2q_1(s)}{1+wq_1(s)} - \frac{2q_1(s)(1-wq_1(s))}{(\pi+4)+(4-\pi)wq_1(s)}\left[\frac{\pi}{2}\frac{wq_1(s)-1}{1+wq_1(s)} - wq_1(s)\right]\right], \\ \lim\limits_{x_2 \overset{<}{\to} \beta^{-1}(x_1)} \dfrac{\partial \widehat{t}_-}{\partial x_1}(x) = +\infty, \\ \lim\limits_{x_2 \overset{>}{\to} \gamma(x_1)} \dfrac{\partial \widehat{t}_-}{\partial x_2}(x) = -\frac{2q_1(s)(1+wq_1(s))}{\pi+4+(4-\pi)wq_1(s)} \left[\frac{\pi}{2}\frac{wq_1(s)-1}{1+wq_1(s)} - wq_1(s) - \pi - 2\right], \\ s = \begin{cases} \sqrt{2\widehat{X}_2(x_1, s_0)}, \; x_1 \geq k_2(s_0), \\ k_2^{-1}(x_1), \; x_1 \in (0, k_2(s_0)), \end{cases} \\ \lim\limits_{x_2 \overset{<}{\to} \beta^{-1}(x_1)} \dfrac{\partial \widehat{t}_-}{\partial x_2}(x) = +\infty. \end{cases}$$

Remark 14. *The partial derivatives* $\dfrac{\partial \widehat{t}_-}{\partial x_i}(.)$, $i = 1, 2$ *are bounded on* $G_- \subset \widetilde{Y}_-$ *then the functions* $\widehat{t}_-(x_1, .)$, *for* $x_1 > 0$ *and* $\widehat{t}_-(., x_2)$, *for* $x_2 \in (\gamma(x_1), \beta^{-1}(x_1))$ *are lipschitzian. Moreover these functions are continuous for* $x_2 > w^2$ *on the terminal set* Y_1.

3.5.3 Invertibility of the trajectories $\overline{X}(.,.)$

As we can see, the trajectories in (3.4.34) are characterized parametrically as those in (3.4.14) hence, the invertibility problem is done in the same way as in previous section. The problem refers to showing that, the parametrically trajectories $\widetilde{\widetilde{X}}(.) : B^{\ominus} \to \widetilde{Y}_{\ominus}$ are invertible with inverses, $\widehat{B}_{\ominus}(.) = (\widehat{\xi}_{\ominus}(.), \widehat{s}_{\ominus}(.)) : G_{\ominus} \subseteq \widetilde{Y}_{\ominus} \to B^{\ominus}$ such that:

$$\widetilde{\widetilde{X}}(\widehat{B}_{\ominus}(x)) = x \ \forall x = (x_1, x_2) \in G_{\ominus}$$
$$\widehat{B}_{\ominus}(\widetilde{\widetilde{X}}(\xi, s)) = (\xi, s) \ \forall (\xi, s) \in B^{\ominus}, \tag{3.5.36}$$

the existence and uniqueness of the inverse $\widehat{B}_{\ominus}(.)$ is illustrated in the following result. The demonstration is similar to Lemma 3.5.3.

Lemma 3.5.4. *If $s \in (w, s_0)$ and $T_2(.)$, $\widehat{\xi}_2(.,.)$, $\widehat{\widehat{X}}_2(.,.)$ are the functions defined in* (3.4.32) , (3.4.38) *and* (3.4.39) *then, there exists a unique function $\widehat{B}_{\ominus}(.) : G_{\ominus} \subset \widetilde{Y}_{\ominus} \to B^{\ominus}$ checks* (3.5.36) *such that:*

$$\widehat{\xi}_{\ominus}(x) = \widehat{\xi}_2(x_1, \widehat{s}_{\ominus}(x)), \ x = (x_1, x_2) \in G_{\ominus}$$
$$\widehat{t}_{\ominus}(x) = T_2(\widehat{\xi}_{\ominus}(x), \widehat{s}_{\ominus}(x)),$$
$$G_{\ominus} = \{x;\ x_2 \in (\delta(x_1), \gamma(x_1)),\ x_1 \in (0, \overline{k}_2(w))\},$$

where, $\delta(x_1) = \widehat{\widehat{X}}_2(x_1, w)$, $\overline{k}_2(.)$ and $\gamma(.)$ denote the functions in (3.4.37) *and* (3.5.24) *respectively.*

Remark 15. *Similar as remark* 14*, the partial derivatives $\dfrac{\partial \widehat{t}_{\ominus}}{\partial x_i}(.)$, $i = 1, 2$ are bounded on $G_{\ominus} \subset \widetilde{Y}_{\ominus}$ then, the functions $\widehat{t}_{\ominus}(x_1, .)$, $x_1 > 0$ and $\widehat{t}_{-}(., x_2)$, $x_2 \in (\delta(x_1), \gamma(x_1))$ are lipschitzian and are also, continuous for $x_2 > w^2$ on the terminal set Y_1.*

The results in Lemmas 3.5.1 − 3.5.4 show that the characteristic flows $C^*_{\pm}(.,.) = (X^*_{\pm}(.,.), V(.,.))$ and $C^*_{\ominus}(.,.) = (X^*_{\ominus}(.,.), V(.,.))$ described above are invertible in the sense of (3.5.2) and define the smooth partial proper value function

$$W_0(x) = \begin{cases} W_0^{\pm}(x) = -\widehat{t}_{\pm}(x), \text{ if } x \in G_{\pm} \\ W_0^{\ominus}(x) = -\widehat{t}_{\ominus}(x), \text{ if } x \in G_{\ominus}, \end{cases} \tag{3.5.37}$$

which is lipschitzian, knowing that $\widehat{t}_{\pm}(.)$ and $\widehat{t}_{\ominus}(.)$ have the same property. Also, the function $W_0(.)$ can be extended by $W(\xi) = 0 \ \forall \xi \in Y_1$ to the corresponding terminal sets defined in (3.2.5).

Moreover, from (3.2.7) and (3.5.4) it follows that, the corresponding admissible feedback strategies $\widetilde{U}(.)$, $\widetilde{V}(.)$ are given by:

$$\begin{aligned} \widetilde{U}(x) &= \begin{cases} \widetilde{U}^{\pm}(x) = \{\widetilde{u}^{\pm}(x)\} \text{ if } x \in G_{\pm} \\ \widetilde{U}^{\ominus}(x) = \{\widetilde{u}^{\ominus}(x)\} \text{ if } x \in G_{\ominus} \end{cases} \\ \widetilde{V}(x) &= \begin{cases} \widetilde{V}^{\pm}(x) = \{\widetilde{v}^{\pm}(x)\} \text{ if } x \in G_{\pm} \\ \widetilde{V}^{\ominus}(x) = \{\widetilde{v}^{\ominus}(x)\} \text{ if } x \in G_{\ominus}, \end{cases} \end{aligned} \tag{3.5.38}$$

while, the control parameters $\widetilde{u}^{\pm}(.)$, $\widetilde{v}^{\pm}(.)$, $\widetilde{u}^{\ominus}(.)$ and $\widetilde{v}^{\ominus}(.)$ are given by the formulas:

$$\begin{cases} \widetilde{u}^{+}(x) = -\frac{\widetilde{P}_{+}(x)}{\|\widetilde{P}_{+}(x)\|} = (-\frac{\sqrt{x_2}}{\widehat{s}_{+}(x)}, \sqrt{1-\frac{x_2}{\widehat{s}_{+}^2(x)}}),\ x \in G_{+} \\ \widetilde{u}^{-}(x) = -\frac{\widetilde{P}_{-}(x)}{\|\widetilde{P}_{-}(x)\|} = -\frac{1}{\sqrt{q_1^2(s)+\widetilde{\xi}_1^2(t,s)}}(q_1(s), \widetilde{\xi}_1(t,s)),\ x \in G_{-} \\ \qquad \text{where } t = \widehat{t}_{-}(x),\ s = \widehat{s}_{-}(x), \\ \widetilde{u}^{\ominus}(x) = -\frac{\widetilde{P}_{\ominus}(x)}{\|\widetilde{P}_{\ominus}(x)\|} = -\frac{1}{\sqrt{q_1^2(s)+\widetilde{\xi}_2^2(t,s)}}(q_1(s)), \widetilde{\xi}_2(t,s)),\ x \in G_{\ominus} \\ \qquad \text{where } t = \widehat{t}_{\ominus}(x),\ s = \widehat{s}_{\ominus}(x), \\ \widetilde{v}^{\pm}(x) = \pm 1,\ x \in G_{\pm},\ \widetilde{v}^{\ominus}(x) = -1,\ x \in G_{\ominus} \\ \widetilde{P}_{\pm}(x) = P^{\pm}(\widehat{B}_{\pm}(x)),\ \widetilde{P}_{\ominus}(x) = P^{\ominus}(\widehat{B}_{\ominus}(x)), \end{cases} \tag{3.5.39}$$

where, $\widetilde{\xi}_1(.,.)$, $\widetilde{\xi}_2(.,.)$ denote the functions defined in (3.4.13) and (3.4.33) respectively.

Now, we are able to present the fundamental Theorem of optimality.

Theorem 3.5.1. *The following statements hold:*

1. *The function $W_0(.)$ of* (3.5.37) *is a solution of Isaacs' equation* (3.5.5) *on the corresponding domain $G = G_{+} \cup G_{-} \cup G_{\ominus}$. Moreover, it is the value function in the sense* (3.5.1) *of the corresponding admissible feedback strategies* (3.5.38).

2. *The corresponding admissible feedback strategies $\widetilde{U}(.)$, $\widetilde{V}(.)$ in* (3.5.38) *are optimal for the restriction on their domain G.*

Proof. (1) The function $W_0(.)$ is solution of Isaacs' equation (3.5.5) on its domain G follows from Lemma 3.5.1 − 3.5.4 and the classical theory of smooth Hamiltonian-Jacobi equations (e.g. [17, 42, 43]). In fact, from (3.2.5), (3.5.3) and (3.5.38), if $x \in G_{+}$ one has:

$$\begin{aligned} f(x, \overline{u}, \overline{v}) &= \left(w - \frac{x_2}{\widehat{s}_{+}(x)}, \sqrt{x_2}\sqrt{1-\frac{x_2}{\widehat{s}_{+}^2(x)}}\right),\ f_0(x, \overline{u}, \overline{v}) = 1, \\ DW_0^{+}(x) &= \widetilde{P}_{+}(x) = q_1(\widehat{s}_{+}(x))\left(1, -\frac{\widehat{s}_{+}(x)}{\sqrt{x_2}}\sqrt{1-\frac{x_2}{\widehat{s}_{+}^2(x)}}\right), \end{aligned}$$

and by direct inspection, the relation (3.5.4) becomes:

$$\begin{aligned} & DW_0^+(x).f(x,\overline{u},\overline{v}) + f_0(x,\overline{u},\overline{v}) \\ & = q_1(\widehat{s}_+(x))\left[w - \tfrac{x_2}{s} - \widehat{s}_+(x)(1 - \tfrac{x_2}{\widehat{s}_+^2(x)}) + \widehat{s}_+(x) - w\right] = 0, \end{aligned}$$

while, if $x \in G_-$, it follows from (3.2.5), (3.3.3), (3.5.3), (3.5.5) and (3.5.38) that:

$$\begin{aligned} & \min_{u\in\widetilde{U}(x)} \max_{v\in\widetilde{V}(x)} DW_0^-(x)f(x,u,v) + f_0(x,u,v) \\ & = \min_{u\in\widetilde{U}(x)} \max_{v\in\widetilde{V}(x)} \mathcal{H}(x,\widetilde{P}_-(x),u,v) \\ & = \mathcal{H}(x,\widetilde{P}_-(x),\widetilde{u}^-(x),\widetilde{v}^-(x)) = H_-(x,\widetilde{P}_-(x)) = 0, \end{aligned}$$

next, if $x \in G_\ominus$ the proof is done in a similar way as in the case $x \in G_-$.

(2) Due to the fact that, the function $W_0(.)$ of (3.5.37) is a lipschitzian function, the optimality of the admissible feedback strategies $\widetilde{U}(.), \widetilde{V}(.)$ follows from the so called *Verification Theorem for locally-Lipschitz functions (Theorem 5.4,* [44]), according to which a sufficient optimality condition for the admissible feedback strategies $\widetilde{U}(.), \widetilde{V}(.)$ is the verification of the differential inequalities:

$$\begin{aligned} & \inf_{u\in U,\overline{v}\in\widetilde{V}(x)} \left[\max\{\overline{D}_k^-,\overline{D}_k^+\}W_0(x;f(x,u,\overline{v})) + f_0(x,u,\overline{v})\right] \geq 0 \\ & \sup_{v\in V,\overline{u}\in\widetilde{U}(x),} \left[\min\left\{\underline{D}_k^-,\underline{D}_k^+\right\} W_0(x;f(x,\overline{u},v)) + f_0(x,\overline{u},v)\right] \leq 0, \end{aligned} \tag{3.5.40}$$

where, $\overline{D}_k^\pm W_0(.;.)$ and $\underline{D}_k^\pm W_0(.;.)$ denote the extreme contingent derivatives of $W_0(.)$ (e.g. [44, 42]). In order to prove the above inequalities (3.5.40), we use certain classical results as in ([11],[42],[46]):

$$\begin{aligned} & \max\{\overline{D}_k^-,\overline{D}_k^+\}W_0(x;f(x,u,\overline{v})) + f_0(x,u,\overline{v}) \geq \\ & DW_0^\pm(x)f(x,u,\overline{v}) + f_0(x,u,\overline{v}) \geq 0,\ u \in U,\ \overline{v} \in \widetilde{V}^\pm(x) \\ & \min\left\{\underline{D}_k^-,\underline{D}_k^+\right\} W_0(x;f(x,\overline{u},v)) + f_0(x,\overline{u},v) \leq \\ & DW_0^\pm(x)f(x,\overline{u},v) + f_0(x,\overline{u},v) \leq 0,\ \overline{u} \in \widetilde{U}^\pm(x),\ v \in V. \end{aligned} \tag{3.5.41}$$

For the first inequality of (3.5.41), if $x \in G_+$ and $u \in U = S_1(0)$, $v \in \widetilde{V}^+(x) = \{1\}$ then, $f(x,u,\widetilde{v}^+(x)) = (\sqrt{x_2}u_1 + w, \sqrt{x_2}u_2)$, $f_0(x,u,\widetilde{v}^+(x)) = 1$, therefore:

$$\begin{aligned} & DW_0^+(x)f(x,u,\widetilde{v}^+(x)) + f_0(x,u,\widetilde{v}^+(x)) \\ & = q_1(s)[\sqrt{x_2}u_1 - s\sqrt{1 - \tfrac{x_2}{s^2}}u_2] + sq_1(s),\ s = \widehat{s}_+(x), \end{aligned}$$

and from (3.2.6) it follows that:

$$\begin{aligned} \inf_{u\in U}[\sqrt{x_2}u_1 - s\sqrt{1 - \tfrac{x_2}{s^2}}u_2] & = \inf_{u\in U}[< (\sqrt{x_2}, -s\sqrt{1 - \tfrac{x_2}{s^2}}), u >] \\ & = -\sqrt{x_{2+}s^2(1 - \tfrac{x_2}{s^2})} = -s, \end{aligned}$$

hence:

$$\inf_{u\in U}\left[DW_0^+(x)f(x,u,\widetilde{v}^+(x))+f_0(x,u,\widetilde{v}^+(x))\right]$$
$$=-sq_1(\widehat{s}_+(x))+sq_1(\widehat{s}_+(x))=0.$$

Checking the second inequality in (3.5.41) then, if $u\in\widetilde{U}^+(x)$ and $v\in V=[-1,1]$ one has, $f(x,\widetilde{u}^+(x),v)=(-\frac{x_2}{s}+\frac{w}{2}(v+1),\sqrt{x_2}\sqrt{1-\frac{x_2}{s^2}}+\frac{w}{2}(v-1))$, $f_0(x,\widetilde{u}^+(x),v)=1$ and we obtain:

$$DW_0^+(x)f(x,\widetilde{u}^+(x),v)+f_0(x,\widetilde{u}^+(x),v)$$
$$=q_1(s)\left[\frac{w}{2}(\frac{s}{\sqrt{x_2}}\sqrt{1-\frac{x_2}{s^2}}-1)+\frac{w}{2}(1-\frac{s}{\sqrt{x_2}}\sqrt{1-\frac{x_2}{s^2}})v\right],\ s=\widehat{s}_+(x),$$

since, $(1-\frac{s}{\sqrt{x_2}}\sqrt{1-\frac{x_2}{s^2}})\in(0,1)$ then, $\sup_{v\in V}\left[(1-\frac{s}{\sqrt{x_2}}\sqrt{1-\frac{x_2}{s^2}})v\right]=1-\frac{s}{\sqrt{x_2}}\sqrt{1-\frac{x_2}{s^2}}$ hence,

$$\sup_{v\in V}[DW_0^+(x)f(x,\widetilde{u}^+(x),v)+f_0(x,\widetilde{u}^+(x),v)]=0.$$

To check inequalities (3.5.41) on G_-, we note here that, if $u\in U=S_1(0)$ and $v\in\widetilde{V}^-(x)=\{-1\}$ then:

$$\inf_{u\in U}\ DW_0^-(x)f(x,u,\widetilde{v}^-(x))+f_0(x,u,\widetilde{v}^-(x))=\inf_{u\in U}\ \mathcal{H}(x,\widetilde{P}_-(x),u,\widetilde{v}^-(x))$$
$$=\mathcal{H}(x,\widetilde{P}_-(x),\widetilde{u}^-(x),\widetilde{v}^-(x))=H_-(x,\widetilde{P}_-(x))=0,$$

if $u\in\widetilde{U}^-(x)$ and $v\in V=[-1,1]$ one has:

$$\sup_{v\in V}\ DW_0^-(x)f(x,\widetilde{u}^-(x),v)+f_0(x,\widetilde{u}^-(x),v)=\sup_{v\in V}\ \mathcal{H}(x,\widetilde{P}_-(x),\widetilde{u}^-(x),v)$$
$$=\mathcal{H}(x,\widetilde{P}_-(x),\widetilde{u}^-(x),\widetilde{v}^-(x))=H_-(x,\widetilde{P}_-(x))=0.$$

While, if $x\in G_\ominus$ the proof is done in the same way as in the previous case. Hence, the optimality of the admissible feedback strategies $\widetilde{U}(.)$, $\widetilde{V}(.)$. This completes the proof. ■

GENERAL CONCLUSION

This thesis has developed a systematic approach to addressing the well-known Dolicho-brachistochrone differential game by integrating theoretical dynamic programming method with advanced numerical techniques. Using the tools of non-smooth analysis, this work has provided significant insights into the problem and contributed to a deeper theoretical understanding. The results obtained not only improve the existing solutions, but also provide a robust and comprehensive framework for addressing similar challenges in the literature. Finally, we compared our results with those already available and drew some conclusions:

- The first solution of this famous differential game was given by R. Isaacs in his monograph [31]. However, this solution is not **correct** (this is, in fact, also the view of Lidov [36]). Namely, Isaacs assumed that, the value function is infinite below the horizontal line $x_2 = w^2$ which is marked as a barrier. This is far from the truth since, as shown in this study, there are other types of admissible trajectories $X^-(.,.)$ and $\overline{X}(.,.)$ that exceed this line and cover a part of the region below this line. Moreover, as can be seen from the above reasoning as well as from the images of the trajectories, a new barrier is that first curve starting from the bottom of the trajectories $\overline{X}(.,.)$ given in (3.4.34) and therefore, the curve $\overline{G} = \left\{\overline{X}(t,w)\,;\ t \in (-\infty, 0)\right\}$ is the only correct barrier.

- If $s \in (w, s_0)$, Isaacs [31] assumed that, there are no solutions since the value function of the game is infinite below the barrier line $x_2 = w^2$, which contradicts the fact that, the partial proper value functions $W_0^-(.)$ and $W_0^{\ominus}(.)$ in (3.5.37) are well defined. In the same rather heuristically, Chigir [18] and Isaacs [31] in their study did not explicitly characterize the trajectories under the curve G. Also, most of them don't even specify the significance in their sense of optimality. While Basar and Olsder [4] were given some strategies in which, the optimality is proved using the well-known saddle point inequalities.

However, Chigir [18] was used the well-known ks-trajectories (called, perfect trajectories in the authors' terminology) defined in [33, 51] as uniform limits of the $\triangle$-approximate trajectories of Isaacs [31]. Which in turn, do not necessarily have to be absolutely continuous (AC), not even differentiable. Therefore, such trajectories do not admit the concatenation property. Working as in [31], the authors of [4] tried to identify certain admissible strategies for which, their optimality is closely related to the verification of the *saddle point condition* (eg. [7, 24, 51]) for the cost $\mathcal{C}(.,.)$. Instead, in our approach, the optimality of a pair of admissible feedback strategies $\left(\widetilde{U}(.), \widetilde{V}(.)\right)$ is the verification of the weak conditions given by the differential inequalities in (3.5.40) which are easier to verify, and much more efficient because don't require the presence of all pairs of admissible strategies, $\left(\widetilde{U}(.), V(.)\right)$ and $\left(U(.), \widetilde{V}(.)\right)$ which in return, are required when checking the saddle point optimality condition. This apparently minor fact, paves the way for the use of dynamic programming-type optimality sufficient conditions known as verification theorems [44].

In summary, the present study provides our contributions in the following directions:

1. The use of some recent concepts and results from *Non-Smooth Analysis* and relevant applications in the differential games theory, as well as the use of the synthesis of the very recent theory in [44, 42, 43] regarding the rigorous approach and constructive of differential game problems.

2. The integration of very complex differential systems with state restrictions, obtaining the new extended Hamiltonian flows, and the corresponding maximal intervals.

3. The identification of a possible barrier (in the sense of Isaacs) different from the one previously proposed in [31] as well as the possible domain of the associated value function.

4. The identification for the first time of a pair of feedback strategies, as well as the corresponding complete solution and the rigorous demonstration of its optimality.

REFERENCES

[1] J.P. Aubin, A. Cellina, Differential inclusions, Springer, New York, Berlin, Heidelberg, 1984.

[2] M. Bardi, I. Capuzzo, Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations, Birkhauser, Berlin, 1997.

[3] M. Bardi and I. Capuzzo-Dolcetta, Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations. Birkhauser, Boston, 1997. https://doi.org/10.1007/978-0-8176-4755-1

[4] T. Basar, G.J. Olsder, Dynamic noncooperative game theory, Academic Press, London, 1982.

[5] E. Bednarczuk, W. Song, Contingent epiderivative and its applications to set-valued optimization, Control and Cybernetics, 27(3), 375-386, 1998.

[6] S. Benghebrid, T. Bouremani, D. Benterki, On Isaac's War Game of Attrition and Attack Using Dynamic Programming Approach, Games 15(6) (2024) 35. https://doi.org/10.3390/g15060035.

[7] L.D. Berkovitz, Optimal control theory, Springer, New York, 1974.

[8] A. Blaquière, G. Leitmann, Jeux quantitatifs, Gauthier-Villars, Paris, 1969.

[9] W. M. Boothby, An Introduction to Differentiable Manifolds and Riemannian Geometry. Academic Press, 1986.

[10] G. Bouligand, Sur l'existence des demi-tangentes à une courbe de Jordan. Fundamenta Mathematicae, 15(1), 215-218, 1930.

[11] T. Bouremani, Ş. Mirică, On the solution of a simple differential game with a singular focal line, Bull. Math. Soc. Sci. Math. Roumanie 49(97) (2006) 113-139.

[12] T. Bouremani, Y. Slimani, Study of a warfare differential game via dynamic programming approach, Dyn. Games Appl. (2023) 113-139.

[13] Ş. Mirică, T. Bouremani, On the solution of the homicidal chauffeur differential game, Math. Rep. 8(58)1 (2006) 53-81.

[14] J.V. Breakwell, Zero-sum differential games with terminal payoff, Differ. Games Appl. 3 (1977) 70-95.

[15] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer New York, 2011.

[16] M. Goresky, R. MacPherson, Intersection homology theory. Topology, 19(2), 135-162, (1980).

[17] L. Cesari, Optimization theory and applications, Springer, New York, 1983.

[18] A.S. Chigir, The game problem on the Dolichobrachistochrone, J. Appl. Math. Mech. 40 (1976) 1003-1013.

[19] F. H. Clarke, Nonsmooth analysis and optimization, In Proceedings of the international congress of mathematicians (Vol. 5, pp. 847-853), 1983.

[20] F. H. Clarke, Optimization and non-smooth analysis, Society for industrial and Applied Mathematics, 1990.

[21] M.G. Crandall, L.C. Evans, P.L. Lions, Some properties of viscosity solutions of Hamilton-Jacobi equations, Trans. Am. Math. Soc. 282(2) (1984) 487-502.

[22] M. G. Crandall and P. L. Lions, Viscosity Solutions of Hamilton-Jacobi Equations. Transactions of the American Mathematical Society, 277, 1-42, 1983. https://doi.org/10.1090/S0002-9947-1983-0690039-8

[23] D. L. Donoho, Higher-Dimensional Data Analysis: Contingent Differentiability. Journal of Computational and Graphical Statistics, 14(3), 1-21, 2005.

[24] R.J. Elliott, Viscosity solutions and optimal control, Pitman Res. Notes Math. Ser., Longman Sci. Tech., Harlow, 1987.

[25] M. Fréchet, Sur les espaces fonctionnels. Mathematica (Cluj), 1(1), 1-23, 1906.

[26] C. F. Gauss, Disquisitiones generales circa superficies curvas. Dieterich, 1828.

[27] J. Gâteaux, Sur les dérivées directionnelles, Comptes Rendus de l'Académie des Sciences, 137 577-580, 1903.

[28] A. Ghanem, T. Bouremani, D. Benterki, On the solution of Dolichobrachistochrone differential game via dynamic programming approach, J. Comput. Appl. Math. 461 (2025) 116460. https://doi.org/10.1016/j.cam.2024.116460.

[29] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero: II. Annals of Mathematics, 79(2), 205-326, 1964.

[30] J. Hiriart-Urruty and C. Lemaréchal, Convex Analysis and Minimization Algorithms I, Springer, 2001.

[31] R. Isaacs, Differential games, Wiley, New York, 1965.

[32] L.V. Kamneva, V.S. Patsko, V.L. Turova, Analytical and numerical study of the Dolichobrachistochrone problem, Anal. Control Deterministic Stoch. Evol. Equ. (2000).

[33] N.N. Krasovskii, A.I. Subbotin, Positional differential games, Nauka Publishing House, Moscow, 1974.

[34] S. Lang, Fundamentals of Differential Geometry. Springer, 1999.

[35] J. M. Lee, Introduction to Smooth Manifolds (2nd ed.). Springer, 2013.

[36] M.L. Lidov, On a differential game problem (in Russian), Avtomat. i Telemekhan. 1(4) (1971) 173-175.

[37] J. L. Lions, Contrôle optimal et calcul des variations. Dunod, 1971.

[38] F. Severi, Le curve intuitive. Rendiconti del Circolo Matematico di Palermo (1884-1940), 54(1), 51-66, 1930.

[39] L. Abadie, Applications des méthodes topologiques à la théorie des jeux différentiel. Bulletin of the Malaysian Mathematical Society, 2(2), 101-113. 1965.

[40] J. Mather, Stratifications and mappings. In : Dynamical systems. Academic Press. (1973) 195-232.

[41] M. Spivak, A Comprehensive Introduction to Differential Geometry (1). Publish or Perish, 1979.

[42] Ş. Mirică, Constructive dynamic programming in optimal control: Autonomous problems, Editura Academiei Române, Bucharest, 2004.

[43] Ş. Mirică, User's guide on dynamic programming for autonomous differential games and optimal control problems, Rev. Roum. Math. Pures Appl. 49(5-6) (2004) 501-529.

[44] Ş. Mirică, Verification theorems for optimal feedback strategies in differential games, Int. Game Theory Rev. 5(1) (2003) 167-189.

[45] Ş. Mirică, Generalized solutions by Cauchy's method of characteristics, Rend. Sem. Mat. Univ. Padova 77(1) (1987) 317-349.

[46] Ş. Mirică, Hamilton-Jacobi equations on possibly non-symplectic differentiable manifolds, Bull. Math. Soc. Sci. Math. Roumanie 41(87) (1998) 23-69.

[47] Ş. Mirică, Monotonicity and differential properties of the value functions in optimal control, International Journal of Mathematics and Mathematical Sciences. 65 3513-3540 2004.

[48] V.S. Patsko, V.L. Turova, Level sets of the value function in differential games with the homicidal chauffeur dynamics, Int. Game Theory Rev. 3(1) (2001) 67-112.

[49] B. Riemann, Über die Hypothesen, welche der Geometrie zu Grunde liegen. Physikalische Blätter, 10(7), 296-306, 1954.

[50] R. T. Rockafellar, Convex Analysis, Princeton University Press, 1970. `https://doi.org/10.1515/9781400873173`.

[51] A.I. Subbotin, Generalization of the main equation of differential games, J. Optim. Theory Appl. 43(1) (1984) 103-134.

[52] R. Thom, Singularities of Differentiable Maps. In Publications Mathématiques de l'IHÉS, 1970.

[53] D. J. Trotman, Stability of transversality to a stratification implies Whitney (a)-regularity. Inventiones mathematicae, 50(3), 273-277, 1978.

[54] K. Wang, C. Mu, Learning-based control with decentralized dynamic event-triggering for vehicle systems, IEEE Trans. Ind. Inform. 19(3) (2023) 2629-2639.

[55] K. Wang, C. Mu, Asynchronous learning for actor critic neural networks and synchronous triggering for multiplayer system, ISA Trans. 129 (2022) 295-308.

[56] F. W. Warner, Foundations of Differentiable Manifolds and Lie Groups. Springer, 1983.

[57] H. Whitney, Geometric Integration Theory. Princeton University Press, 1957.

ملخـــص :

في هذه الأطروحة، نهتم بتطبيق طريقة البرمجة الديناميكية لـ St. Mirică لحل لعبة دوليخوبراخستوكرون التفاضلية التي قدمها R. Isaacs. نقترح استراتيجيات التغذية الراجعة كإسهام جديد يوفر التكيف، الفعالية والبساطة مع تقليل تكلفة الخوارزمية. تعتمد هذه المقاربة على طريقة محسنة لخصائص كوشي لمعالجة معادلات هاميلتون-جاكوبي الطبقية مع ضمان وجود دالة القيمة. يتم التحقق بدقة من مثالية استراتيجيات التغذية الراجعة باستخدام مبرهنة التحقق لوظائف القيمة محليًا لليبشيتز، كما يتم تعزيزها باختبارات عددية مثبتة.

الكلمات المفتاحية: لعبة تفاضلية، الاحتواء التفاضلي، استراتيجيات التغذية الراجعة، البرمجة الديناميكية، التدفق الهاميلتوني، دالة القيمة، مبرهنة التحقق.

Abstract:

In this thesis, we focus on applying St. Mirică's Dynamic Programming method to solve the Dolichobrachistochrone differential game introduced by R. Isaacs. We propose feedback strategies as a new contribution that offers adaptability, efficiency, and simplicity while reducing algorithmic complexity. This approach employs a refined Cauchy characteristics method to handle stratified Hamilton-Jacobi equations while ensuring the existence of the value function. The optimality of the feedback strategies is rigorously validated using the Verification Theorem for locally Lipschitz value functions and further supported by established numerical tests.

Key words : Differential game, Differential inclusion, Feedback strategies, Dynamic programming, Hamiltonian flow, Value function, Verification theorem

Résumé :

Dans cette thèse, on s’intéresse à l'application de la méthode de Programmation Dynamique de St. Mirică pour résoudre le jeu différentiel Dolichobrachistochrone introduit par R. Isaacs. On propose les stratégies de rétroaction comme nouvelle contribution qui offre l’adaptabilité, l’efficacité et la simplicité tout en réduisant la complexité algorithmique. Cette approche utilise une méthode raffinée des caractéristiques de Cauchy pour traiter les équations de Hamilton-Jacobi stratifiées tout en garantissant l'existence de la fonction de valeur. L'optimalité des stratégies de rétroaction est rigoureusement validée à l'aide du Théorème de vérification pour les fonctions de valeur localement Lipschitziennes et consolidée par des tests numériques établis.

Mots-clés : Jeu différentiel, Inclusion différentielle, Stratégies de rétroaction, Programmation dynamique, Flux Hamiltonien, Fonction de valeur, Théorème de vérification.