People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research

Setif 1 University Ferhat Abbas
Faculty of Sciences
Department of Mathematics

جامعـــة سطيف 1 فرحـــــات عباس
كـليـــة العلـــــــــوم
قســـــــم الرياضيـــــــــات

Doctoral Thesis

Submitted in Fulfillment of the Requirements
for the Degree of Doctor in Mathematics

Option : Mathematics and Applications

Theme :

CONTRIBUTIONS TO SOME EQUILIBRIUM PROBLEMS VIA PROXIMAL TYPE ALGORITHMS AND APPLICATIONS

Presented by

Miss. **ZEGHAD Bochra**

Supervisor : **Professor Nourreddine DAILI**

Thesis defended on 16/06/2025, in front of the jury composed of :

Mr.	Salim MESBAHI	Prof.	Setif 1 University Ferhat Abbas	President
Mr.	Nourreddine DAILI	Prof.	Setif 1 University Ferhat Abbas	Supervisor
Ms.	Chafia DAILI	MCA.	Setif 1 University Ferhat Abbas	Co- supervisor
Mr.	Nabil BEROUAL	MCA.	Setif 1 University Ferhat Abbas	Examiner
Mr.	Amar GUESMIA	Prof.	Skikda University August 20, 1955	Examiner
Mr.	Fateh ELLAGGOUNE	Prof.	Guelma University May 8, 1945	Examiner

2025

Liste of Publications

Published article

Bochra Zeghad and Nourreddine Daili. An improved inertial subgradient extragradient algorithm for pseudomonotone equilibrium problems and its applications. *J. Math. Model.* **13** (2025), 263–280.

Submitted articles

Bochra Zeghad and Nourreddine Daili. An Enhanced extragradient algorithm for pseudomonotone equilibrium problems in Hilbert Spaces.

Bochra Zeghad. Modified Bregman extragradient algorithm for equilibrium problems in Banach spaces.

Conference proceedings

Bochra Zeghad, Convergence of generalized proximal point algorithms, 1 st International Conference on Nonlinear Mathematical Analysis and its Applications (IC-NMAA'24), held on May, 14-15, 2024 at Bordj Bou Arréridj University.

Bochra Zeghad, The Bergman Popov's subgradient extragradient algorithm for strongly pseudomonotone equilibrium problems, International Conference of Young Mathematicians, June 1-3, 2023, National Academy of Sciences of Ukraine.

Bochra Zeghad, Bregman Proximal Method for Variational Inequality Problems, Second National Conference on Applied Mathematics and Didactics (2NCAMD 2023) held online on May 13, 2023, Ecole Normale Supérieure El Katiba Assia Djebar Constantine.

Bochra Zeghad and Nourreddine Daili, A modified Proximal Point algorithm and its Convergence, National Conference on Mathematics and Applications (NCMA 2023), held in Ferhat Abbas University Setif 1, on May 15-16, 2023.

Dedication

I dedicate this work

To our brothers in Gaza and all of Palestine.

To those who have renewed our purposes within us.

To those who taught us that every person has a position to defend. Thus, we have committed ourselves to the position of knowledge to strive through it.

To my beloved family, whose unwavering support and love have been my greatest strength and inspiration.

To my dear parents, who have taught me resilience, perseverance, and the value of hard work. Your sacrifices and unconditional love have shaped my journey.

To my brothers and sisters, for their endless love and encouragement. A special gratitude to my brother, Zouheyr, for his unwavering support and motivation throughout this journey.

To my dear friends, whose kindness, encouragement, and companionship have made this path brighter and more fulfilling.

Acknowledgments

Praise be to Allah, who guided me, directed my steps, and blessed me to complete my doctoral thesis.

*I extend my sincere thanks to my supervisor, **Prof. Nourreddine DAILI** and **Dr. Chafia DAILI**, for their patience, support, and invaluable guidance during the realization of this work.*

*I am also profoundly grateful to the jury president, **Prof. Salim MESBAHI**, for accepting the role of chair in this committee, and to the examiners **Dr. Nabil BEROUAL**, **Prof. Amar GUESMIA** and **Prof. Fateh ELLAGGOUNE** for their acceptance to be members of my thesis examination committee and for dedicating their time and effort to read and review my work.*

Table of contents

List of algorithms

List of tables

List of Figures

List of symbols and abbreviations

1. Symbols

- $\mathbb{R}$: The set of real numbers.
- $\mathbb{R}^n$: n-dimensional Euclidean vector space.
- $\mathcal{H}$: A real Hilbert space.
- E: A Banach space.
- E^*: The dual space of E.
- $\mathcal{T}_m$: The half-space.
- $\langle ., . \rangle$: Usual scalar product of $\mathbb{R}^n$.
- $\rightarrow$: The strong convergence.
- $\rightharpoonup$: The weak convergence.
- $\partial h(.)$: The subdifferential of h.
- $P_{\mathcal{S}}(.)$: The projection onto $\mathcal{S}$.
- J_φ: The proximal operator associated with φ.
- D_φ: The Bregman distance with respect to φ.
- dom (φ): The effective domain of φ.
- int (dom φ): The interior of dom φ.
- φ^*: The Fenchel conjugate of φ.
- φ°: The right-hand derivative of φ.
- D_m: Used to measure the error of the $m - th$ iteration step.

2. Abbreviations

- Iter: The number of iterations.
- CPU(s): The execution time in seconds.
- EP: Equilibrium problem.
- $EQ(B, \mathcal{S})$: The solution set of an equilibrium problem over the set $\mathcal{S}$.
- VIP: Variational inequality problem.
- $VI(\Psi; \mathcal{S})$: The solution set of variational inequality problem on $\mathcal{S}$.
- tol: Tolerance level, used as a stopping criterion in numerical algorithms.

Introduction

Equilibrium problems, variational inequalities, and optimization problems have emerged as fundamental pillars of modern applied mathematics. These areas are not only deeply rooted in theoretical advancements but also serve as essential tools for solving practical problems in diverse disciplines, including engineering, economics, machine learning, data science, and imaging. By providing a rigorous mathematical framework for analyzing and modeling complex systems, these problem classes enable researchers to address challenges related to decision-making, resource allocation, signal processing, and economic equilibrium analysis.

Among these, equilibrium problems hold a significant place among various problem classes, as they provide a unified and general framework capable of modeling a wide range of mathematical problems. This includes both optimization problems, where the goal is to find the best solution according to certain criteria, and variational inequalities, which involve finding a point that satisfies specific constraints. The equilibrium problem (EP) [3] is defined as follows:

$$\text{Find } v \in \mathcal{S} \text{ such that } B(v,t) \geq 0,\ \forall t \in \mathcal{S},$$

where B is a bifunction : $\mathcal{S} \times \mathcal{S} \to \mathbb{R}$, and $\mathcal{S}$ is a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{H}$. This formulation, often referred to as the Ky Fan inequality [11], has gained prominence due to its wide applicability in diverse fields, including game theory, optimization, saddle point problem, fixed point problem and Nash equilibrium problem [3, 22, 29].

The study of equilibrium problems is motivated both by theoretical and computational considerations. From a theoretical perspective, significant attention has been devoted to investigating the existence and uniqueness of solutions under various conditions (see, e.g., [18]). On the computational side, numerous iterative methods have been developed to efficiently solve equilibrium problems, particularly in Hilbert spaces. Classical approaches include the auxiliary problem principle [26], gap function-based techniques [27], the proximal point method ([12, 28]), subgradient extragradient methods [1, 34, 38, 45], inertial methods [14, 15], and other approaches discussed in [30, 40].

Among these methods, extragradient techniques and their variants play a crucial role in solving EP in Hilbert spaces, as highlighted by Hieu [17]. These methods have been widely studied due to their simplicity, convergence properties, and practical efficiency. The extragradient method involves solving two optimization problems over a closed convex set at each iteration while utilizing a monotonically decreasing step size sequence $\{\varrho_m\}$. The iterative scheme is formulated as follows:

$$\begin{cases} t_m = \underset{t\in\mathcal{S}}{\arg\min}\left(\varrho_m B(v_m,t) + \frac{1}{2}\|t-v_m\|^2\right), \\ v_{m+1} = \underset{t\in\mathcal{S}}{\arg\min}\left(\varrho_m B(t_m,t) + \frac{1}{2}\|t-v_m\|^2\right), \end{cases}$$

and

$$\varrho_{m+1} = \begin{cases} \min\left\{\frac{\varsigma\left(\|v_m-t_m\|^2+\|v_{m+1}-t_m\|^2\right)}{2M}, \varrho_m\right\}, & \text{if } M>0, \\ \varrho_m \quad \text{otherwise}. \end{cases}$$

Here, $M = B(v_m, v_{m+1}) - B(v_m, t_m) - B(t_m, v_{m+1})$. Under certain conditions, the algorithm has been shown to weakly converge to a solution of the equilibrium problem in Hilbert spaces.

Despite the flexibility of Hilbert spaces, many real-world problems involve complexities such as non-linearities, non-smoothness, and high-dimensionality, which exceed their scope. This has driven researchers to explore Banach spaces, which generalize Hilbert spaces by replacing the inner product with a norm. While Banach spaces offer greater flexibility, they introduce new challenges in algorithm design and convergence analysis. To address these challenges, the Bregman distance has been introduced. This distance, based on convex functions, adapts to the geometry of Banach spaces. Unlike Euclidean distance in Hilbert spaces, the Bregman distance is well-suited for solving equilibrium problems in Banach spaces [35–37].

Building on this, Eskandani et al. [10] further developed the extragradient algorithm by replacing the Euclidean distance with the Bregman distance and monotonically decreasing step size sequence $\{\varrho_m\}$ for solving equilibrium problems in Banach spaces. The iterative scheme is formulated as follows:

$$\begin{cases} t_m = \underset{t\in\mathcal{S}}{\arg\min}\left(\varrho_m B(v_m,t) + D_\varphi(t,v_m)\right), \\ v_{m+1} = \underset{t\in\mathcal{S}}{\arg\min}\left(\varrho_m B(t_m,t) + D_\varphi(t,v_m)\right), \end{cases}$$

and

$$\varrho_{m+1} = \begin{cases} \min\left\{\frac{\varsigma(D_\varphi(v_m,t_m)+D_\varphi(v_{m+1},t_m))}{M}, \varrho_m\right\}, & \text{if } M>0, \\ \varrho_m \quad \text{otherwise}, \end{cases}$$

where $M = B(v_m, v_{m+1}) - B(v_m, t_m) - B(t_m, v_{m+1})$.

Inspired by these studies, a natural question arises:

How can proximal algorithms be developed and applied to solve equilibrium problems in Hilbert and Banach spaces, and how can they be effectively utilized to address variational inequalities problems?

In light of these questions, this thesis aims to address the following objectives:

1. develop proximal algorithms for solving equilibrium problems in Hilbert and Banach spaces;

2. improve convergence speed and reduce computational costs by using adaptive step size criteria;

3. incorporate inertial term with proximal algorithms for strong convergence in pseudomonotone equilibrium problems;

4. apply Bregman distance and non-monotonic step size to enhance the effectiveness of algorithms in reflexive Banach spaces;

5. establish convergence results under suitable conditions and validate the performance of the proposed algorithms through numerical experiments.

The thesis is structured into four chapters:

➤ **Chapter 1** provides the fundamental concepts and mathematical tools that serve as the foundation of this thesis. The chapter is organized as follows: Section 1.1 introduces essential definitions and key lemmas. Sections 1.2, 1.3, and 1.4 provide an overview of optimization problems, equilibrium problems and variational inequalities, respectively, along with a discussion of classical proximal algorithms. Finally, Section 1.5 discusses the connection between optimization and equilibrium problems.

➤ **Chapter 2** introduces a proximal-based algorithm called an enhanced extragradient algorithm for solving EP in Hilbert spaces. The chapter is organized as follows: Section 2.1 introduces the algorithm and the underlying assumptions. In Section 2.2, we analyze its theoretical properties, including convergence analysis. Section 2.3 discusses the application of the proposed method to variational inequalities problems. Finally, in Section 2.4, we provide numerical experiments that demonstrate the algorithm's efficiency compared to classical methods.

➤ **Chapter 3** Proposes an algorithm that combines subgradient extragradient methods with inertial terms for pseudomonotone EP in Hilbert spaces. The chapter is organized as follows: Section 3.1 introduces the proposed algorithm, describing its structure, including the use of non-monotonic step sizes. In Section 3.2, we establish its strong convergence under appropriate conditions for the equilibrium bifunction B and the control parameters. Section 3.3 explores the application of the algorithm to variational inequalities problems. Finally, in Section 3.4, numerical experiments are presented to validate the proposed algorithm, comparing its performance with existing methods and demonstrating its computational advantages.[1]

➤ **Chapter 4** introduces a modified Bregman extragradient algorithm for solving pseudomonotone equilibrium problems in a real reflexive Banach space. The chapter is organized as follows: Section 4.1 discusses Bregman distance, Section 4.2 recall the known results. In Section 4.3 introduces the proposed algorithm. Section 4.4 covers weak and strong convergence analysis of the proposed algorithm. Finally, in Section 4.5, numerical experiments are conducted to validate the performance of the proposed algorithm, comparing its efficiency across different Bregman distances and analyzing its computational advantages compared to traditional methods.

[1]This part is published in Journal of Mathematical Modeling [54].

CHAPTER 1

PRELIMINARIES

In this chapter, we introduce the fundamental concepts and mathematical tools that form the foundation of this thesis. This includes essential definitions, key lemmas, an overview of optimization problems, equilibrium problems and variational inequalities. Additionally, we analyze some classical proximal algorithms, such as the extragradient algorithm [17] and the extragradient subgradient algorithm [8], which are designed to solve these problems.

1.1 Basic Definitions and Lemmas

In this section, we recall fundamental definitions and key lemmas that will be used in the following chapters. Let $\mathcal{S}$ be a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{H}$. The inner product is denoted by $\langle .,. \rangle$, and the Euclidean norm by $\|.\|$. The weak convergence and strong convergence of $\{v_m\}$ to v are represented by $\rightharpoonup$ and $\rightarrow$, respectively. For every $t \in \mathcal{H}$. The definitions are well-established in the literature (see, e.g., [25, 46]).

Definition 1.1. A sequence $\{v_m\}$ in a normed space is called a Cauchy sequence if for every $\epsilon > 0$, there exists an integer N such that for all $m, n \geq N$, we have

$$\|v_n - v_m\| < \epsilon. \tag{1.1}$$

Definition 1.2 (Banach Space)**.** A Banach space is a complete normed vector space, i.e., a vector space E equipped with a norm $\| \cdot \|$ such that every Cauchy sequence converges to a limit in E.

Definition 1.3 (Hilbert Space)**.** A Hilbert space is a complete inner product space, i.e., a vector space $\mathcal{H}$ equipped with an inner product $\langle \cdot, \cdot \rangle$ such that every Cauchy sequence converges to a limit within the space.

Definition 1.4. Let $\mathcal{S}$ be a subset of $\mathcal{H}$

- The subset $\mathcal{S}$ is said to be convex if, for any two points $v, t \in \mathcal{S}$, the line segment joining v and t lies entirely within $\mathcal{S}$. Mathematically, this is expressed as:

$$\lambda v + (1 - \lambda)t \in \mathcal{S}, \quad \forall v, t \in \mathcal{S}, \quad \forall \lambda \in [0, 1].$$

- A subset of $\mathcal{S}$ at $t \in \mathcal{S}$ defined by

$$N_{\mathcal{S}}(t) := \{t^* \in \mathcal{H} : < t^*, s - t > \leq 0, \quad \forall s \in \mathcal{S}\},$$

  is called the normal cone.

- The subdifferential of convex function $h : \mathcal{S} \to \mathbb{R}$ at $t \in \mathcal{S}$ is defined by

$$\partial h(t) := \{u \in \mathcal{H} : h(s) - h(t) \geq < u, s - t >, \quad \forall s \in \mathcal{S}\},$$

  an element $u \in \partial h(t)$ is called subgradient. In case that the function h is differentiable then $\partial h(t) = \{\nabla h(t)\}$, which is the gradient of h.

Definition 1.5. Let $\varphi : \mathcal{H} \to \mathbb{R}$. We say

- φ is proper if $dom(\varphi) \neq \emptyset$ with

$$dom(\varphi) := \{v \in E : \varphi(v) < +\infty\};$$

  which is denotes the domain of the function φ.

- φ is lower semicontinuous at $v_0 \in H$ if

$$\liminf_{v \to v_0} \varphi(v) \geq \varphi(v_0);$$

  Equivalently, for any sequence $\{v_n\} \subset H$ such that $v_n \to v_0$, we have

$$\liminf_{n \to \infty} \varphi(v_n) \geq \varphi(v_0);$$

- φ is convex if, for all $v, t \in \mathcal{H}$ and $\lambda \in [0, 1]$, the following inequality holds:

$$\varphi(\lambda v + (1 - \lambda)t) \leq \lambda\varphi(v) + (1 - \lambda)\varphi(t);$$

- φ is strongly convex with constant $\sigma > 0$, if and only if

$$\varphi(v) - \varphi(t) \geq \langle \nabla\varphi(t), t - v \rangle + \frac{\sigma}{2} \|v - t\|^2.$$

Lemma 1.1. *[46] Suppose $\mathcal{S}$ is a nonempty convex subset of $\mathcal{H}$. Consider $h : \mathcal{S} \to \mathbb{R} \cup \{+\infty\}$ as a convex function that is subdifferentiable and lower semicontinuous. Then, t^* is a solution to the following convex optimization problem:*

$$\min \{h(t) : t \in \mathcal{S}\},$$

if and only if

$$0 \in \partial h(t^*) + N_{\mathcal{S}}(t^*),$$

where $\partial h(t^*)$, $N_{\mathcal{S}}(t^*)$ are the subdifferential of h and the normal cone of $\mathcal{S}$ at t^*, respectively.

Lemma 1.2. *[6] Let $\{a_m\}, \{b_m\}$ be two nonnegative real sequences such that*

$$a_{m+1} \leq a_m - b_m.$$

Then, $\lim\limits_{m\to\infty} a_m \in \mathbb{R}$, and $\sum\limits_{m\geq 1} b_m < \infty$.

Lemma 1.3. *[31] Let $\{x_m\}$ be a sequence in $\mathcal{H}$ and $\mathcal{S} \subset \mathcal{H}$ such that*

(i) *for each $x \in \mathcal{S}$, $\lim\limits_{m\to\infty} \|x_m - x\|$ exists;*

(ii) *all sequentially weak cluster point of $\{x_m\}$ belongs to $\mathcal{S}$.*

Then, $\{x_m\}$ converges weakly to a point in $\mathcal{S}$.

Lemma 1.4. *[32] Let $\{a_m\}, \{b_m\}$ and $\{c_m\}$ be positive sequences such that*

$$a_{m+1} \leq a_m b_m + c_m, \ \forall m \in \mathbb{N}.$$

If $\{b_m\} \subset [1, \infty)$, $\sum\limits_{m=1}^{\infty} (b_m - 1) < \infty$ and $\sum\limits_{m=1}^{\infty} c_m < \infty$, then $\lim\limits_{m\to\infty} a_m$ exists.

Lemma 1.5. *[50] Let $\{J_m\} \subset [0, +\infty)$ and $\{L_m\} \subset \mathbb{R}$ be sequences satisfying*

$$J_{m+1} \leq (1 - \delta_m) J_m + \delta_m L_m, \ \forall m \in \mathbb{N},$$

where $\{\delta_m\} \subset (0,1)$, $\sum\limits_{m=1}^{+\infty} \delta_m = +\infty$. If $\limsup\limits_{m\to+\infty} L_m \leq 0$ for every subsequence $\{J_{m_k}\}$ of $\{J_m\}$ such that

$$\liminf_{k\to\infty} (J_{m_k+1} - J_{m_k}) \geq 0,$$

then $\lim\limits_{m\to\infty} J_m = 0$.

With these foundational concepts in consideration, we now turn our attention to optimization problems and proximal algorithms.

1.2 Optimization Problems

Optimization problems are mathematical frameworks that aim to find the best possible solution from a set of feasible solutions. Formally, an optimization problem can be expressed as:

$$\min_{v\in\mathcal{S}} \varphi(v), \tag{1.2}$$

where $\mathcal{S} \subseteq \mathcal{H}$ is the feasible set and $\varphi : \mathcal{H} \to \mathbb{R}$ is the objective function.

Optimization problems can be broadly classified into differentiable and non-differentiable categories:

Differentiable optimization is a well-established field where the objective function is assumed to be differentiable. Due to the regularity of these functions, a range of analytical and geometric tools can be employed to solve optimization problems efficiently.

The most common methods for solving such problems include gradient descent and its variants, such as the projected gradient method or Newton's method, which rely on the differentiability of h. These methods are based on the linear approximation of the function around a current point.

In contrast, for problems where the objective function is not necessarily differentiable, specialized techniques are required. Non-differentiable optimization problems are common in many practical applications, such as combinatorial optimization and model regularization.

The tools used for solving such problems differ from those in differentiable optimization. One common approach is to use subgradients instead of gradients. A subgradient generalizes the concept of a derivative in the case where the function is not differentiable at certain points.

1.2.1 Proximal algorithm

Let $\varphi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function and $\mathcal{S}$ is a nonempty, closed and convex set. The proximal operator J_φ of φ at $t \in \mathcal{H}$ is defined as

$$J_\varphi(t) := \arg\min_{s\in\mathcal{S}} \left(\varphi(s) + \frac{1}{2} \|t - s\|^2 \right), \quad t \in \mathcal{H}.$$

When φ is the indicator function $I_\mathcal{S}$

$$I_\mathcal{S}(v) = \begin{cases} 0 & v \in \mathcal{S} \\ +\infty & v \notin \mathcal{S}, \end{cases} \tag{1.3}$$

the proximal operator of φ reduces to the Euclidean projection onto $\mathcal{S}$, which defined by

$$P_\mathcal{S}(t) = \arg\min_{s\in\mathcal{S}} \|t - s\|.$$

For $\tau > 0$, the proximal operator corresponding to the scaled function $\tau\varphi$ can be written as

$$J_{\tau\varphi}(t) := \arg\min_{s\in\mathcal{S}} \left(\tau\varphi(s) + \frac{1}{2} \|s - t\|^2 \right), \quad t \in \mathcal{H}.$$

Lemma 1.6. *[2] For all $t \in \mathcal{H}$, $s \in \mathcal{S}$ and $\tau > 0$, the following inequality holds:*

$$\tau \{\varphi(s) - \varphi(J_{\tau h}(t))\} \geq \langle t - J_{\tau\varphi}(t), s - J_{\tau\varphi}(t) \rangle .$$

Remark 1.1. From Lemma 1.6, it is easy to show that if $t = J_{\tau\varphi}(t)$, then

$$t \in Arg\min \{\varphi(s) : s \in \mathcal{S}\} := \left\{ t \in \mathcal{S} : \varphi(t) = \min_{s\in\mathcal{S}} \varphi(s) \right\}.$$

A proximal algorithm is an algorithm for solving a convex optimization problem that uses the proximal operators of the objective terms. For example, the proximal point algorithm minimizes a convex function φ by repeatedly applying J_φ to some initial point t_0,i.e.,

$$t_{m+1} = J_{\tau\varphi}(t_m).$$

If φ has a minimum, then $\{t_m\}$ converges to the set of minimizes of φ and $\varphi(t_m)$ converges to its optimal value (see [2]). Now, we shift our focus to a related class of problems: equilibrium problems.

1.3 Equilibrium Problem

In this section, we will present the notion of equilibrium problems, with some properties of the equilibrium bifunction B, and discuss two fundamental proximal-based methods for solving equilibrium problems.

An equilibrium problem (EP) in the sense of Blum and Oettli [3] consists of finding a point $v \in \mathcal{H}$ such that:

$$B(v,t) \geq 0, \; \forall t \in \mathcal{S}, \tag{1.4}$$

where B is the equilibrium bifunction : $\mathcal{S} \times \mathcal{S} \to \mathbb{R}$, and $\mathcal{S}$ is a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{H}$.

1.3.1 Equilibrium bifunction properties

First we recall some well-known definitions that we need in the sequel (see [3]) for more details.

Definition 1.6. A bifunction B: $E \times E \to \mathbb{R}$ is said

(A) $\gamma-$strongly monotone on $\mathcal{S}$ if there exists a constant γ such that if,

$$B(v,t) + B(t,v) \leq -\gamma \|v - t\|^2 , \qquad \forall v, t \in \mathcal{S};$$

(B) monotone on $\mathcal{S}$, if

$$B(v,t) + B(t,v) \leq 0, \qquad \forall v,t \in \mathcal{S};$$

(C) pseudomonotone on $\mathcal{S}$, if

$$B(v,t) \geq 0 \Rightarrow B(t,v) \leq 0, \qquad \forall v,t \in \mathcal{S};$$

(D) $\gamma-$strongly pseudomonotone on $\mathcal{S}$ if there exists a constant γ such that

$$B(v,t) \geq 0 \Rightarrow B(t,v) \leq -\gamma \left\| v-t \right\|^2, \qquad \forall v,t \in \mathcal{S};$$

(E) Lipschitz type continuous on $\mathcal{H}$ with two positive constants L_1 and L_2 if

$$B(v,t) + B(t,w) \geq B(v,w) - L_1 \left\| v-t \right\|^2 - L_2 \left\| t-w \right\|^2, \ \ \forall t,v,w \in \mathcal{S};$$

(F) subdifferentiable on $\mathcal{H}$ in the second argument if, for each fixed $s \in \mathcal{S}$, there exists at least one subgradient on $w \in \partial B(v,.)$ such that,

$$B(v,t) - B(v,s) \geq \left\langle w, t-s \right\rangle, \quad \forall \, t \in \mathcal{S};$$

(G) jointly weakly continuous on $\mathcal{S} \times \mathcal{S}$ if for any sequences $\{v_m\}, \{t_m\} \subset \mathcal{H}$ such that $v_m \rightharpoonup v$ and $t_m \rightharpoonup t$ (weak convergence), it holds that

$$B(v_m, t_m) \rightarrow B(v,t).$$

From the concepts described above, the following consequences holds

$$(A) \Longrightarrow (B) \Longrightarrow (D) \text{ and } (A) \Longrightarrow (C) \Longrightarrow (D)$$

1.3.2 Proximal type algorithms for EP

EP provide a unifying framework for various mathematical models, including optimization problems, variational inequalities, and Nash equilibrium problems. Due to their wide applicability in numerous fields, developing efficient algorithms to solve EP has become an active area of research. Among the proposed methods, proximal-like algorithms have attracted significant interest from researchers due to their convergence properties and adaptability to various problem structures, leading to numerous improvements and refinements.

In the following, we present two key proximal-like algorithms proposed in the literature for solving EP:

Extragradient algorithm for EP

The extragradient algorithm for solving (EP) in real Hilbert spaces was introduced and thoroughly studied by Hieu et al. in [17]. The extragradient algorithm is outlined below:

Algorithm 1 Extragradient Algorithm for EP in $\mathcal{H}$ [17]

Initialization: Given $x_0 \in \mathcal{S}$, $\varrho_0 > 0$, $\zeta \in (0, 1)$.

Step 1: Compute

$$t_m = \arg\min_{t \in \mathcal{S}} \left(\varrho_m B(v_m, t) + \frac{1}{2} \|t - v_m\|^2 \right) = J_{\varrho_m B(v_m, \cdot)}(v_m).$$

If $v_m = t_m$, then stop, and t_m is a solution. Otherwise, go to next step.

Step 2: Compute

$$v_{m+1} = \arg\min_{t \in \mathcal{S}} \left(\varrho_m B(t_m, t) + \frac{1}{2} \|t - v_m\|^2 \right) = J_{\varrho_m B(t_m, \cdot)}(v_m),$$

where

$$\varrho_{m+1} = \begin{cases} \min\left\{ \frac{\zeta\left(\|v_m - t_m\|^2 + \|v_{m+1} - t_m\|^2\right)}{2M}, \varrho_m \right\} & \text{if } M > 0, \\ \varrho_m & \text{otherwise}, \end{cases} \tag{1.5}$$

and $M = B(v_m, v_{m+1}) - B(v_m, t_m) - B(t_m, v_{m+1})$.

The authors proved that the sequence $\{v_k\}$ generated by Algorithm 1 converges weakly to a solution of the EP under the following conditions:

- B is pseudomonotone on $\mathcal{S}$ and $B(v, v) = 0$ for all $v \in \mathcal{S}$;
- B is Lipschitz type continuous on $\mathcal{H}$;
- $B(v, \cdot)$ is convex and subdifferentiable on $\mathcal{H}$ for each fixed $v \in \mathcal{S}$;
- $\limsup_{m \to \infty} B(v_m, t) \leq B(v, t)$, for every weakly convergent $\{v_m\} \subset \mathcal{S}$ to $v \in \mathcal{H}$ and $t \in \mathcal{S}$,

Extragradient subgradient algorithm for EP

Combining the strengths of the extragradient and subgradient methods, the extragradient subgradient algorithm is a hybrid approach designed to solve EP by Dadashi et al. in [8] . They replaced the second minimization problem onto a closed convex set in the extragradient method, with a subgradient projection onto a half-space $\mathcal{T}_m$. The extragradient subgradient algorithm proposed in [8] is outlined below:

Algorithm 2 Subgradient Extragradient Algorithm for EP in $\mathcal{H}$ [8]

Initialization: Given $v_0, v_1 \in \mathcal{S}$, $\varrho_0 > 0$, $\gamma > 0$, $\zeta \leq \min\left\{1, \frac{1}{2L_1}, \frac{1}{2L_2}\right\}$ and $\mu \in (0, \zeta)$.
Step 1: Compute

$$t_m = \arg\min_{t \in \mathcal{S}} \left(\varrho_m B(v_m, t) + \frac{1}{2} \|t - v_m\|^2 \right) = J_{\varrho_m B(v_m, \cdot)}(v_m).$$

If $v_m = t_m$, then stop, and t_m is a solution. Otherwise, go to next step.
Step 2:

$$v_{m+1} = \arg\min_{t \in \mathcal{T}_m} \left(\mu \varrho_m B(t_m, t) + \frac{1}{2} \|t - v_m\|^2 \right) = J_{\mu \varrho_m B(t_m, \cdot)}(v_m),$$

where the half-space $\mathcal{T}_m$ is given by

$$\mathcal{T}_m = \{ t \in \mathcal{H} : \langle v_m - \varrho_m u_m - t_m, t - t_m \rangle \leq 0 \}, u_m \in \partial B(v_m, t_m),$$

and

$$\varrho_{m+1} = \min \left\{ \zeta, \frac{\mu B(t_m, v_{m+1})}{B(v_m, v_{m+1}) - B(v_m, t_m) - L_1 \|v_m - t_m\|^2 - L_2 \|t_m - v_{m+1}\|^2 + 1} \right\}. \tag{1.6}$$

In [8], the authors proved the weak convergence of the extragradient subgradient algorithm under the conditions that B is pseudomonotone on $\mathcal{S}$, Lipschitz type continuous, convex, subdifferentiable in the second argument and jointly weakly continuous.

Remark 1.2. $\mathcal{T}_m$ is a half-space and so $\mathcal{T}_m$ is a closed and convex set in $\mathcal{H}$.

The extragradient algorithm [17] and the extragradient subgradient method [8] have been extensively studied and applied to solve EP. However, these algorithms exhibit certain limitations. Notably, they often require stringent assumptions, such as using monotonically decreasing step size $\{\varrho_m\}$ and the need to know Lipschitz constants (L_1, L_2), which can restrict their applicability. These limitations inspired us to develop and refine new algorithms that overcome these challenges and improve their convergence properties and broaden their applicability to variational inequalities.

1.4 Variational Inequalities Problems

In this section, we consider the classical variational inequality problem (VIP) formulated as follows [20,39]:

$$\begin{cases} \text{Find } v \in \mathcal{S} \text{ such that} \\ \langle \Psi(v), t - v \rangle \geq 0, \quad \forall t \in \mathcal{S}. \end{cases} \tag{1.7}$$

where $\Psi : \mathcal{S} \to \mathcal{S}$ is an operator. We assume that the solution set of (1.7), denoted by $VI(\Psi, \mathcal{S})$, is nonempty.

VIP are a fundamental tool for analyzing a wide range of problems in physics, engineering, economics, and optimization theory. Notably, they can be regarded as a special case of equilibrium problems by setting $B(v,t) = \langle \Psi(v), t-v \rangle$.

In the following, we explore key properties of the operator Ψ, followed by an analysis of proximal-like algorithms used to solve *VIP* efficiently.

1.4.1 Basic Properties of Ψ

Definition 1.7. A mapping Ψ: $E \times E \to \mathbb{R}$ is said to be

(A) $\gamma-$strongly monotone on $\mathcal{S}$, i.e., there exists a constant γ such that,

$$\langle \Psi(v), t-v \rangle + \langle \Psi(t), v-t \rangle \leq -\gamma \|v-t\|^2, \qquad \forall v, t \in \mathcal{S};$$

(B) monotone on $\mathcal{S}$, i.e.,

$$\langle \Psi(v), t-v \rangle + \langle \Psi(t), v-t \rangle \leq 0, \qquad \forall v, t \in \mathcal{S};$$

(C) pseudomonotone on $\mathcal{S}$, i.e.,

$$\langle \Psi(v), t-v \rangle \geq 0 \Rightarrow \langle \Psi(t), v-t \rangle \leq 0, \quad \forall v, t \in \mathcal{S},$$

(D) γ-strongly pseudomonotone on $\mathcal{S}$, i.e., there exists a constant γ such that

$$\langle \Psi(v), t-v \rangle \geq 0 \Rightarrow \langle \Psi(t), v-t \rangle \leq -\gamma \|v-t\|^2, \qquad \forall v, t \in \mathcal{S};$$

(E) Lipschitz continuous on $\mathcal{H}$ with $L > 0$, i.e.,

$$\|\Psi(v) - \Psi(t)\| \leq L \|v-t\|, \quad \forall v, t \in \mathcal{S},$$

1.4.2 Proximal type algorithms for *VIP*

The following two proximal type algorithms are designed to solve the *VIP* :

Extragradient algorithm for *VIP*

Yang et al. in [52] introduced a modified extragradient algorithm for solving *VIP* in $\mathcal{H}$. The algorithm is formulated as follows:

Algorithm 3 Extragradient Algorithm for VIP in $\mathcal{H}$ [52]

Initialization: Given $x_0 \in \mathcal{S}$, $\varrho_0 > 0$, $\zeta \in (0,1)$.
Step 1: Compute

$$t_m = P_{\mathcal{S}}\left(v_m - \varrho_m \Psi(v_m)\right).$$

If $v_m = t_m$, then stop, and t_m is a solution. Otherwise, go to next step.
Step 2: Compute

$$v_{m+1} = P_{\mathcal{S}}\left(v_m - \varrho_m \Psi(t_m)\right),$$

where

$$\varrho_{m+1} = \begin{cases} \min\left\{\frac{\zeta\left(\|v_m - t_m\|^2 + \|v_{m+1} - t_m\|^2\right)}{2\langle \Psi(v_m) - \Psi(t_m), v_{m+1} - v_m\rangle}, \varrho_m\right\} & \text{if } M > 0, \\ \varrho_m & \text{otherwise,} \end{cases} \tag{1.8}$$

Set $m := m+1$ and return to **Step 1**.

To establish the weak convergence of the sequence $\{x_k\}$ generated by Algorithm 3, the authors imposed the following assumptions:

- Ψ is monotone on $\mathcal{S}$;
- Ψ is Lipschitz type continuous on $\mathcal{H}$;

Remark 1.3. Algorithm 3 and the convergence results established in [52] for solving the VIP can be regarded as a special case of Algorithm 1 and the results in [17] for solving EP.

Extragradient subgradient algorithm for VIP

The extragradient subgradient algorithm has also been applied to VIP. In [8], the algorithm was initially adapted to solve EP, thereby extending its applicability to VIP.

Below is the description of the extragradient subgradient algorithm for VIP, as outlined in [8]

Algorithm 4 Subgradient Extragradient Algorithm for VIP in $\mathcal{H}$ [8]

Initialization: Given $v_0 \in \mathcal{S}$, $\varrho_0 > 0$, $\gamma > 0$, $\zeta \leq \min\left\{1, \frac{1}{2L}\right\}$ and $\mu \in (0, \zeta)$.
Step 1: Compute

$$t_m = P_{\mathcal{S}}\left(v_m - \varrho_m \Psi(v_m)\right).$$

If $v_m = t_m$, then stop, and t_m is a solution. Otherwise, go to next step.
Step 2: Compute

$$v_{m+1} = P_{\mathcal{T}_m}\left(v_m - \mu\varrho_m \Psi(t_m)\right).,$$

where the half-space $\mathcal{T}_m$ is given by

$$\mathcal{T}_m = \left\{t \in \mathcal{H} : \langle v_m - \varrho_m \Psi(v_m) - t_m, t - t_m \rangle \leq 0\right\},$$

and

$$\varrho_{m+1} = \min\left\{\zeta, \frac{\mu\left\langle \Psi(t_m), v_{m+1} - t_m\right\rangle}{\left\langle \Psi(v_m), v_{m+1} - v_m\right\rangle - \frac{1}{L}\left\|v_m - t_m\right\|^2 - \frac{1}{L}\left\|t_m - v_{m+1}\right\|^2 + 1}\right\}. \tag{1.9}$$

Set $m := m + 1$ and return to **Step 1**.

Dadashi et al. in [8] proved that the sequence $\{v_m\}$ generated by Algorithm 4 converges weakly to a solution of the VIP under the following conditions

- Ψ is pseudomonotone on $\mathcal{S}$;
- Ψ is Lipschitz type continuous on $\mathcal{H}$;
- Ψ is weak to strong continuous on $\mathcal{S}$ that is $\Psi(v_m) \to \Psi(v)$ for each sequence $\{v_m\} \subset \mathcal{H}$ converging weakly to v.

1.5 The Connection between Optimization Problems and Equilibrium Problems

Optimization problems and equilibrium problems are two fundamental mathematical frameworks that play a crucial role in various applications. Optimization focuses on minimizing or maximizing an objective function, while equilibrium problems determine a balanced state where certain conditions are satisfied. Interestingly, many optimization problems can be rewritten as equilibrium problems, offering new ways to analyze and solve them.

The connection between these two classes of problems can be observed through variational inequalities and fixed-point formulations. In particular, equilibrium problems can be expressed in terms of a bifunction B satisfying certain properties. One such formulation is given by:

$$B(v, t) = \varphi(t) - \varphi(v). \tag{1.10}$$

This formulation reveals a direct link between optimization and equilibrium problems. Specifically, consider the optimization problem (1.2). A necessary optimality condition for problem (1.2) states that at the optimal solution v^*, we must have

$$\varphi(t) - \varphi(v^*) \geq 0, \quad \forall t \in \mathcal{S}. \tag{1.11}$$

This condition is precisely the definition of an equilibrium problem, meaning that problem (1.2) can be equivalently rewritten as finding $v^* \in \mathcal{S}$ such that:

$$\langle \nabla\varphi(v^*), t - v^* \rangle \geq 0, \quad \forall t \in \mathcal{S}. \tag{1.12}$$

This last expression corresponds to a *VIP*, which generalizes optimization problems and provides a bridge between these two mathematical frameworks. Hence, *EP* not only encompass classical optimization formulations but also allow for broader problem modeling, including cases where traditional differentiability assumptions may not hold.

The interplay between optimization and equilibrium problems provides a rich ground for algorithmic development. Proximal point methods, gradient-based algorithms, and fixed-point techniques originally designed for optimization have been successfully adapted to solve equilibrium problems.

CHAPTER 2

AN ENHANCED EXTRAGRADIENT ALGORITHM FOR EP IN REAL HILBERT SPACES

In this chapter, we propose an enhanced extragradient algorithm to solve EP (1.4) in Hilbert spaces. The algorithm improves convergence speed and reduces computational costs by utilizing two parameters, μ and τ, along with a monotonically decreasing step size sequence $\{\varrho_m\}$ that is independent of the Lipschitz constants or line-search techniques. Furthermore application of the main result to variational inequalities problems is given. Additionally, numerical results confirm that the algorithm performs better than standard methods in terms of iterations and execution time. The proposed method extends and generalizes classical algorithms.
Let B be a bifunction: $\mathcal{H} \times \mathcal{H} \to \mathbb{R}$ and $EQ(B, \mathcal{S})$ denotes the solution set of an equilibrium problem over the set $\mathcal{S}$.

2.1 The Proposed Algorithm

In this section, we propose an enhanced extragradient algorithm for solving EPb (1.4) with the following conditions:

(H_1) the bifunction B is pseudomonotone on $\mathcal{S}$;

(H_2) B is Lipschitz type continuous on $\mathcal{H}$;

(H_3) $B(v, \cdot)$ is convex and subdifferentiable on $\mathcal{H}$ for each fixed $v \in \mathcal{S}$,

(H_4) $\limsup_{m\to\infty} B(v_m, t) \leq B(v, t)$, for every weakly convergent $\{v_m\} \subset \mathcal{S}$ to $v \in \mathcal{H}$ and $t \in \mathcal{S}$;

Remark 2.1.

- Note that, from (H_1) and (H_2), we have $B(v, v) = 0, \forall v \in \mathcal{S}$ (see [47]).
- The solution set $EQ(B, \mathcal{S})$ of the EP (1.4) is convex and closed under the conditions (H_1)-(H_4) ([45]).

Algorithm 5 Enhanced Extragradient Algorithm for EP in $\mathcal{H}$

Initialization: Given $v_0 \in \mathcal{S}$, $\varrho_0 > 0$, $\zeta \in (0,1)$, $\mu \in \left[\tau, \frac{1}{(2-\sqrt{2}-\theta)\zeta}\right)$,
$\tau \in \left(0, \frac{1}{2(2-\sqrt{2}-\theta)\zeta}\right)$ and $\theta \in (0, 2-\sqrt{2})$.
Step 1: Compute

$$t_m = \arg\min_{t\in\mathcal{S}} \left(\mu\varrho_m B(v_m,t) + \frac{1}{2}\|t-v_m\|^2\right) = J_{\mu\varrho_m B(v_m,\cdot)}(v_m).$$

If $v_m = t_m$, then stop, and t_m is a solution. Otherwise, go to next step.
Step 2: Compute

$$v_{m+1} = \arg\min_{t\in\mathcal{S}} \left(\tau\varrho_m B(t_m,t) + \frac{1}{2}\|t-v_m\|^2\right) = J_{\tau\varrho_m B(t_m,\cdot)}(v_m),$$

where

$$\varrho_{m+1} = \begin{cases} \min\left\{\frac{\zeta(2-\sqrt{2}-\theta)\left(\|s_m-t_m\|^2+\|v_{m+1}-t_m\|^2\right)}{2M}, \varrho_m\right\} & \text{if } M > 0, \\ \varrho_m & \text{otherwise,} \end{cases} \tag{2.1}$$

and $M = B(s_m, v_{m+1}) - B(s_m, t_m) - B(t_m, v_{m+1})$.

Remark 2.2. If ($\mu = 1$, $\tau = 1$), the algorithm becomes the extragradient method from [17]. Additionally, when $\theta = 1-\sqrt{2}$, Algorithm 5 extends Algorithm 2 in [48].

Lemma 2.1. *[48] The sequence $\{\varrho_m\}$ created by (2.1) is well defined and $\lim_{m\to+\infty} \varrho_m$ exists.*

2.2 Convergence Analysis

To establish the weak convergence of Algorithm 5, our initial steps involve proving the following fundamental results.

Lemma 2.2. *Let $\{t_m\}$ and $\{v_m\}$ be the sequences generated by Algorithm 5. Then*

(i) $\langle v_m - t_m, v_{m+1} - t_m\rangle \leq \mu\varrho_m \left(B(v_m, v_{m+1}) - B(v_m, t_m)\right)$,

(ii) $< v_m - v_{m+1}, t - v_{m+1} >\leq \tau\varrho_m \left(B(t_m, t) - B(t_m, v_{m+1})\right), \forall t \in \mathcal{S}$.

Proof. **(i)** According to Lemma 1.1 and $t_m = J_{\mu\varrho_m B(v_m,.)}(v_m)$, we have

$$0 \in \partial\left(\mu\varrho_m B(v_m,t) + \frac{1}{2}\|t-v_m\|^2\right)(t_m) + N_{\mathcal{S}}(t_m).$$

Then, there exists $u_m \in \partial B(v_m, t_m)$ and $\vartheta \in N_{\mathcal{S}}(t_m)$, such that

$$\mu \varrho_m u_m + t_m - v_m + \vartheta = 0.$$

By the definition of $N_{\mathcal{S}}$, we get

$$< v_m - t_m, t - t_m >= \mu \varrho_m < u_m, t - t_m > + < \vartheta, t - t_m >\leq \mu \varrho_m < u_m, t - t_m >, \ \forall t \in \mathcal{S}.$$

Since $u_m \in \partial B(v_m, t_m)$, we have

$$< u_m, t - t_m >\leq B(v_m, t) - B(v_m, t_m), \ \forall t \in \mathcal{S}.$$

From the last two inequalities, we obtain

$$< v_m - t_m, t - t_m >\leq \mu \varrho_m \left(B(t_m, t) - B(v_m, t_m)\right), \ \forall t \in \mathcal{S}. \tag{2.2}$$

In particular, substituting $t = v_{m+1}$ in (2.2), we get

$$\langle v_m - t_m, v_{m+1} - t_m \rangle \leq \mu \varrho_m \left(B(v_m, v_{m+1}) - B(v_m, t_m)\right).$$

(ii) We have $v_{m+1} = J_{\tau \varrho_m B(t_m,.)}(v_m)$, as similar arguments to the proof of (**i**), we obtain

$$< v_m - v_{m+1}, t - v_{m+1} >\leq \tau \varrho_m \left(B(t_m, t) - B(t_m, v_{m+1})\right), \forall t \in \mathcal{S}.$$

□

Lemma 2.3. *For all $r \in EQ(B,\ \mathcal{S})$, the following inequality holds:*

$$\|v_{m+1} - r\|^2 \leq \|v_m - r\|^2 - \tau \left(\frac{1}{\mu} - \frac{\zeta\left(2 - \sqrt{2} - \theta\right)\varrho_m}{\varrho_{m+1}}\right)\left(\|v_{m+1} - t_m\|^2 + \|v_m - t_m\|^2\right). \tag{2.3}$$

Proof. By substituting $t = r$ in Lemma 2.2 (ii), we get

$$< v_m - v_{m+1}, r - v_{m+1} >\leq \tau \varrho_m \left(B(t_m, r) - B(t_m, v_{m+1})\right)$$

So, from the pseudo monotonicity of B, we have $B(r, t_m) \geq 0$. Thus $B(t_m, r) \leq 0$. Then

$$< v_m - v_{m+1}, r - v_{m+1} >\leq -\tau \varrho_m B(t_m, v_{m+1})$$

$$
\begin{aligned}
2\mu\tau\varrho_m\left(B(v_m,v_{m+1})-B(v_m,t_m)-B(t_m,v_{m+1})\right) &\geq 2\tau < v_m-t_m, v_{m+1}-t_m > \\
&+ 2\mu < v_m-v_{m+1}, r-v_{m+1} > \\
&\geq \tau\left(\|v_m-t_m\|^2+\|v_{m+1}-t_m\|^2-\|v_{m+1}-v_m\|^2\right) \\
&+ \mu\left(\|v_{m+1}-v_m\|^2+\|v_{m+1}-r\|^2-\|v_m-r\|^2\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|v_{m+1}-r\|^2 \leq \|v_m-r\|^2-\|v_{m+1}-v_m\|^2-\frac{\tau}{\mu}\left(\|v_m-t_m\|^2+\|v_{m+1}-t_m\|^2-\|v_{m+1}-v_m\|^2\right) \\
+ 2\tau\varrho_m\left(B(s_m,v_{m+1})-B(s_m,t_m)-B(t_m,v_{m+1})\right).
\end{aligned}
\tag{2.4}
$$

From the definition of ϱ_m, we have

$$
2\tau\varrho_m\left(B(s_m,v_{m+1})-B(s_m,t_m)-B(t_m,v_{m+1})\right) \leq \frac{\tau\left(2-\sqrt{2}-\theta\right)\zeta\varrho_m}{\varrho_{m+1}}\left(\|s_m-t_m\|^2+\|v_{m+1}-t_m\|^2\right). \tag{2.5}
$$

Substituting (2.5) into (2.4), we obtain

$$
\begin{aligned}
\|v_{m+1}-r\|^2 &\leq \|v_m-r\|^2-\|v_{m+1}-v_m\|^2-\frac{\tau}{\mu}\left(\|v_m-t_m\|^2+\|v_{m+1}-t_m\|^2-\|v_{m+1}-v_m\|^2\right) \\
&+ \frac{\tau\left(2-\sqrt{2}-\theta\right)\zeta\varrho_m}{\varrho_{m+1}}\left(\|v_m-t_m\|^2+\|v_{m+1}-t_m\|^2\right) \\
&\leq \|v_m-r\|^2-(1-\frac{\tau}{\mu})\|v_{m+1}-v_m\|^2-\frac{\tau}{\mu}\left(\|v_m-t_m\|^2+\|v_{m+1}-t_m\|^2\right) \\
&+ \left(\frac{\tau\left(2-\sqrt{2}-\theta\right)\zeta\varrho_m}{\varrho_{m+1}}\right)\left(\|v_m-t_m\|^2+\|v_{m+1}-t_m\|^2\right).
\end{aligned}
\tag{2.6}
$$

Then, by using the chosen values for the parameters τ and μ (noting that $\frac{\tau}{\mu}\in(0,1]$), we obtain

$$
\|v_{m+1}-r\|^2 \leq \|v_m-r\|^2-\tau\left(\frac{1}{\mu}-\frac{\left(2-\sqrt{2}-\theta\right)\zeta\varrho_m}{\varrho_{m+1}}\right)\left(\|v_{m+1}-t_m\|^2+\|v_m-t_m\|^2\right). \tag{2.7}
$$

This completes the proof. □

Theorem 2.1. *Suppose that the conditions (H_1)-(H_4) hold and $EQ(B,\ \mathcal{S})\neq\emptyset$. Then the sequence $\{v_m\}$ generated by Algorithm 5 converges weakly to a point $r\in EQ(B,\ \mathcal{S})$.*

Proof. To prove the result, we will show that the sequence $\{v_m\}$ satisfies the two conditions of the Lemma 1.3. Let $\epsilon\in\left(0,\frac{\tau}{\mu}-\left(2-\sqrt{2}-\theta\right)\tau\zeta\right)$ be some fixed number. Since $\lim\limits_{n\to\infty}\frac{\varrho_m}{\varrho_{m+1}}=1$ and using the assumptions on the parameters $\mu\in\left[\tau,\frac{1}{\left(2-\sqrt{2}-\theta\right)\zeta}\right)$, $\tau\in\left(0,\frac{1}{2\left(2-\sqrt{2}-\theta\right)\zeta}\right)$ and $\zeta\in(0,1)$, yields that

$$\lim_{m\to\infty}\tau\left(\frac{1}{\mu}-\frac{\left(2-\sqrt{2}-\theta\right)\zeta\varrho_m}{\varrho_{m+1}}\right)=\tau\left(\frac{1}{\mu}-\left(2-\sqrt{2}-\theta\right)\zeta\right)>\epsilon>0.$$

Thus, there exists $m_1\in\mathbb{N}$ such that

$$\tau\left(\frac{1}{\mu}-\frac{\left(2-\sqrt{2}-\theta\right)\zeta\varrho_m}{\varrho_{m+1}}\right)>\epsilon,\forall m\geq m_1.$$

From the relations (2.7) and the above fact, we have

$$a_{m+1}\leq a_m-b_m,$$

where

$$\begin{cases} a_m=\|v_m-r\|^2,\\ b_m=\epsilon\left(\|v_{m+1}-t_m\|^2+\|v_m-t_m\|^2\right). \end{cases}$$

Applying Lemma 1.2, we deduce that the limit of $\{a_m\}$ exists and $\lim_{n\to\infty}b_m=0$. This implies that the sequence $\{v_m\}$ is bounded and

$$\lim_{m\to\infty}\|v_{m+1}-t_m\|^2=\|v_m-t_m\|^2=0. \tag{2.8}$$

Based on the above facts, it follows that

$$\lim_{m\to\infty}\|v_{m+1}-v_m\|=0.$$

Now, we prove that each weak cluster point of $\{v_m\}$ is in $EQ(B,\ \mathcal{S})$. Let q denote a weak limit point of $\{v_m\}$ and $\{v_{m_k}\}$ be a subsequence of $\{v_m\}$ such that $v_{m_k}\rightharpoonup q$ as $k\to\infty$. By using (2.8) we also have $\{t_{m_k}\}\rightharpoonup q$ as $k\to\infty$. Since $\mathcal{S}$ is closed and convex set, so $\mathcal{S}$ is weakly closed, therefore we can confirm that $q\in\mathcal{S}$. By Lemma 2.2 with expression (2.1) yields the following

$$\begin{aligned}\tau\varrho_{m_k}B(t_{m_k},t)&\geq\tau\varrho_{m_k}B(t_{m_k},v_{m_k+1})+<v_{m_k}-v_{m_k+1},t-v_{m_k+1}>\\ &\geq\tau\varrho_{m_k}\left(B(v_{m_k},v_{m_k+1})-B(v_{m_k},t_{m_k})\right)\\ &\quad-\frac{\tau\left(2-\sqrt{2}-\theta\right)\zeta\varrho_{m_k}}{2\varrho_{m_k+1}}\left(\|v_{m_k}-t_{m_k}\|^2+\|v_{m_k+1}-t_{m_k}\|^2\right)\\ &\quad+<v_{m_k}-v_{m_k+1},t-v_{m_k+1}>\\ &\geq\frac{\tau}{\mu}\langle v_{m_k}-t_m,v_{m+1}-t_m\rangle-\frac{\tau\left(2-\sqrt{2}-\theta\right)\zeta\varrho_{m_k}}{2\varrho_{m_k+1}}\left(\|v_{m_k}-t_{m_k}\|^2+\|v_{m_k+1}-t_{m_k}\|^2\right)\\ &\quad+<v_{m_k}-v_{m_k+1},t-v_{m_k+1}>,\quad\text{for } t\in\mathcal{S}.\end{aligned}$$

Given τ, μ, $\varrho_{m_k} > 0$, condition (H$_4$), $v_{m_k} \rightharpoonup q$, $t_{m_k} \rightharpoonup q$ and (2.8), it follows that

$$B(q,t) \geq \limsup_{k\to\infty} B(t_{m_k}, t) \geq 0, \forall t \in \mathcal{S}.$$

Since $B(q,t) \geq 0$, $\forall t \in \mathcal{S}$, then $q \in EQ(B,\ \mathcal{S})$. Consequently, Lemma 1.5 confirms that $\{v_m\}$ and $\{t_m\}$ converges weakly to an element $r \in EQ(B,\ \mathcal{S})$. □

Remark 2.3. We can establish the same convergence results for the following iterative process

$$\begin{cases} t_m = \arg\min\limits_{t\in\mathcal{S}} \left(\mu\varrho_m B(v_m,t) + \frac{1}{2}\|t - v_m\|^2\right), \\ v_{m+1} = \arg\min\limits_{t\in\mathcal{T}_m} \left(\tau\varrho_m B(t_m,t) + \frac{1}{2}\|t - v_m\|^2\right), \end{cases}$$

where $\mathcal{T}_m = \{y \in \mathcal{H} : \langle v_m - \varrho_m u_m - t_m, y - t_m\rangle \leq 0\}$, $u_m \in \partial B(v_m, t_m)$ and ϱ_{m+1} is defined in (2.1).

2.3 Application to Variational Inequality Problem

In this section, we apply the main results (Theorem 2.1) to solve variational inequality problem (1.7) in Hilbert spaces.

Assume that the solution set of (1.7) (denoted by $VI(\Psi, \mathcal{S})$) is nonempty and the operator Ψ satisfies the following:

(H$_1'$) Ψ is pseudomonotone on $\mathcal{S}$;

(H$_2'$) Ψ is Lipschitz continuous on $\mathcal{H}$ with $L > 0$;

(H$_3'$) $\limsup\limits_{m\to\infty} \langle \Psi(v_m), t - v_m\rangle \leq \langle \Psi(v), t - v\rangle$ for every weakly convergent $\{v_m\} \subset \mathcal{S}$ to $v \in \mathcal{H}$ and $t \in \mathcal{S}$.

The sequence t_m rewritten as

$$\begin{aligned} t_m &= \arg\min_{t\in\mathcal{S}} \left(\mu\varrho_m B(v_m,t) + \frac{1}{2}\|t - v_m\|^2\right) \\ &= \arg\min_{t\in\mathcal{S}} \left(\mu\varrho_m \langle \Psi(v_m), t - v_m\rangle + \frac{1}{2}\|t - v_m\|^2\right) \\ &= \arg\min_{t\in\mathcal{S}} \left(\frac{1}{2}\|t - (v_m - \mu\varrho_m\Psi(v_m))\|^2\right) - \frac{1}{2}\|\mu\varrho_m\Psi(v_m)\|^2 \\ &= P_{\mathcal{S}}\left(v_m - \mu\varrho_m\Psi(v_m)\right). \end{aligned}$$

Similarly, $v_{m+1} = P_{\mathcal{T}_m}\left(v_m - \tau\varrho_m\Psi(t_m)\right).$

Corollary 2.1. *Assume that the conditions*$(H_1') - (H_3')$ *hold. Let* $\{v_m\}$ *and* $\{t_m\}$ *be two sequences created in the following way:*

(i) *Given* $v_0, v_1 \in \mathcal{S}$, $\varrho_1 > 0, \gamma > 0$, $\mu \in \left[\tau, \frac{1}{(2-\sqrt{2}-\theta)\zeta}\right)$, $\tau \in \left(0, \frac{1}{2(2-\sqrt{2}-\theta)\zeta}\right)$ *and* $\zeta \in (0,1)$.

(ii) *Compute*

$$\begin{cases} t_m = P_{\mathcal{S}}\left(v_m - \mu\varrho_m\Psi(v_m)\right), \\ v_{m+1} = P_{\mathcal{S}}\left(v_m - \tau\varrho_m\Psi(t_m)\right), \end{cases}$$

where

$$\varrho_{m+1} = \begin{cases} \min\left\{\frac{\zeta(2-\sqrt{2}-\theta)\left(\|v_m-t_m\|^2+\|v_{m+1}-t_m\|^2\right)}{2\langle\Psi(v_m)-\Psi(t_m),v_{m+1}-t_m\rangle}, \varrho_m\right\}, & \textit{if } \langle\Psi(v_m)-\Psi(t_m), v_{m+1}-t_m\rangle > 0, \\ \varrho_m, & \textit{otherwise}. \end{cases}$$

Then, the sequence $\{v_m\}$ *converges weakly to* r*, for each* $r \in VI(\Psi, \mathcal{S}) \neq \emptyset$.

2.4 Numerical Illustrations

In this section, we present numerical results to prove the efficiency of our proposed algorithm. All the programs were implemented in MATLAB (R2023a) on a Intel(R) Core(TM) i5-8265U CPU @ 1.60 GHz 1.80 GHz with RAM 8.00 GB.

Consider the Nash-Cournot oligopolistic equilibrium model in [45]:

$$\text{Find } v \in \mathcal{S} \text{ such that } \langle Pv + Qt + q, t - v\rangle \geq 0, \, \forall t \in \mathcal{S}, \tag{2.9}$$

where $q \in \mathbb{R}^n$ and $P, Q \in \mathbb{R}^{n\times n}$ are two matrices of order n such that Q is symmetric positive semidefinite and $Q - P$ is is symmetric negative semidefinite with the Lipschitz type constants $L_1 = L_2 = \frac{1}{2}\|Q - P\|$. It can be checked that all the conditions (H_1)-(H_4) are satisfied (for more details see [45]). Let $\mathcal{S}$ is a polyhedral convex set given by

$$\mathcal{S} := \{v \in \mathbb{R}^n : Av \leq b\},$$

where A is a random matrix of size $l \times n$ ($l = 10$ and $n = 200$ or $m = 300$) and b$\in \mathbb{R}^n$ such that v_0 =$(1, 1, ..., 1) \in \mathcal{S}$. The two matrices P, Q are generated randomly. We use the same stopping rule $D_m = \|t_m - v_m\| \leq tol$. In the numerical results presented in the following tables, 'Iter.' represents the number of iterations, while 'CPU(s)' denotes the execution time in seconds.
We will apply Algorithm 5 to solve the EP (2.9). The performance of Algorithm 5 was initially

evaluated for different values of μ, τ, with the parameters $\rho_0 = 2, \zeta = 0.3$ and $\theta = 0.05$ fixed. Combinations that do not satisfy the assumption were excluded and marked with - , as shown in Table 2.1. In view of this Table , we see that the proposed algorithm work better when $\mu = 0.5$ and $\tau = 4.5$.

Table 2.1: Comparison of iterations and CPU time for different μ and τ with ($n = 200$, $tol = 10^{-3}$).

Algorithm 5	$\tau = 1.5$		$\tau = 2.5$		$\tau = 3.5$		$\tau = 4.5$	
μ	Iter.	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)
0.5	31	0.29	18	0.17	13	0.14	11	0.08
1.5	62	0.48	29	0.24	17	0.16	14	0.13
2	-	-	46	0.37	33	0.31	26	0.22
2.5	-	-	53	0.42	38	0.32	29	0.25

Finally, the Algorithm 5 was compared with the explicit extragradient Algorithm suggested by Hieu et al. [17] (shortly, EEA) and the improved extragradient Algorithm introduced by Wairojjana et al. [48] (shortly, IEA) to assess its efficiency and effectiveness. The control parameters of all algorithms are choose as follows:

- Algorithm 5: $\varrho_0 = 2$, $\zeta = 0.3$, $\theta = 0.05$, $\mu = 0.5$ and $\tau = 4.5$.
- IEA : $\varrho_0 = 2$, $\zeta = 0.3$, $\theta = 0.05$.
- EEA : $\varrho_0 = 2$, $\zeta = 0.3$.

The numerical results for all algorithms are presented in Figs. 2.1-2.4 and Table 2.2. It can be observed that our algorithm (Algorithm 5) outperforms both IEA and EEA in terms of the number of iterations (Iter.) and execution time in seconds (CPU(s)), while achieving the same tolerance.

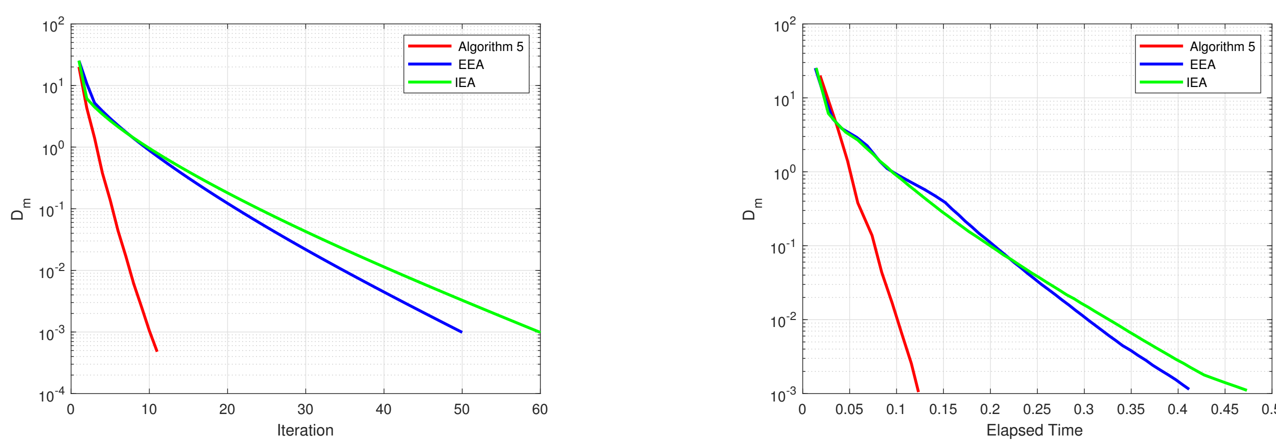

Figure 2.1: Numerical behavior of all algorithms with ($n = 200$, $tol = 10^{-3}$)

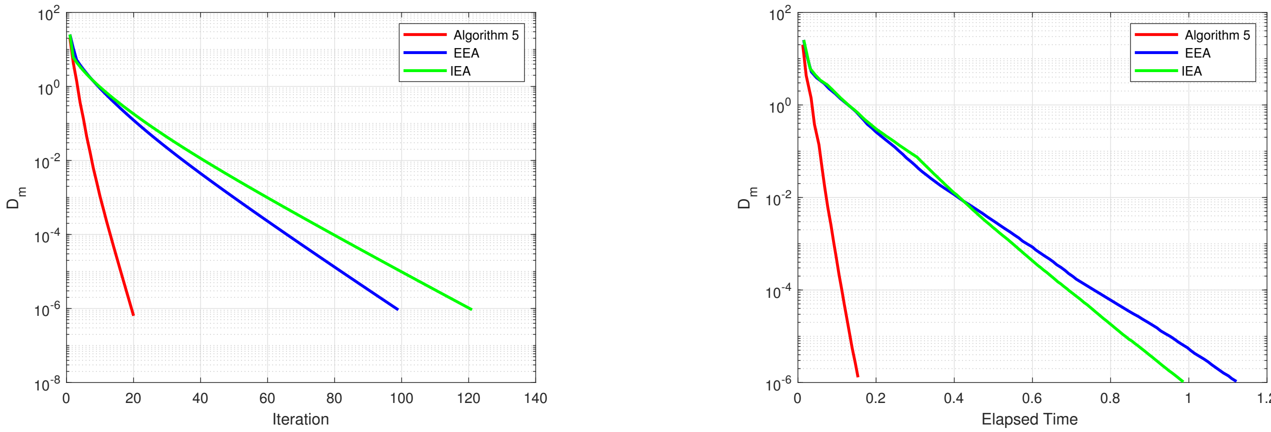

Figure 2.2: Numerical behavior of all algorithms with ($n = 300$, $tol = 10^{-3}$)

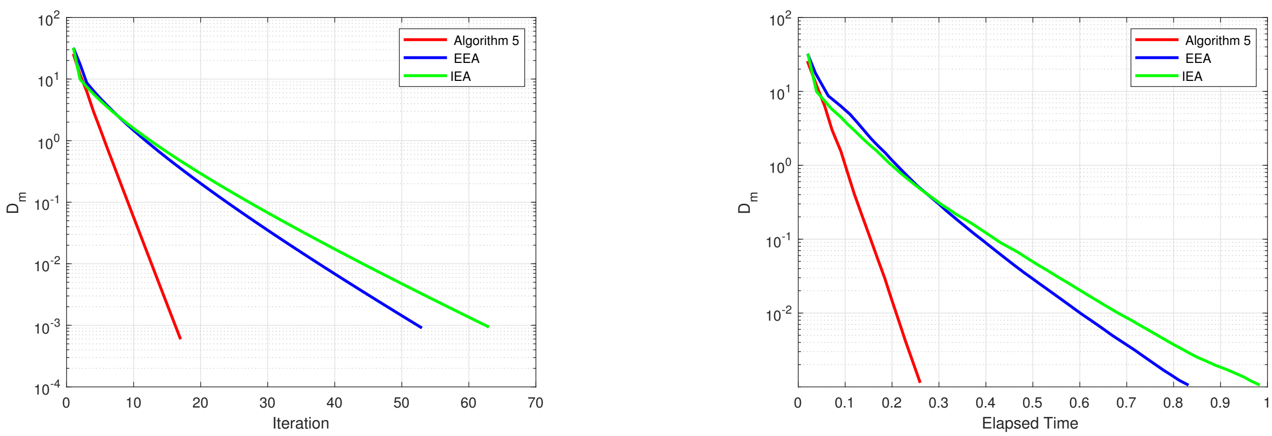

Figure 2.3: Numerical behavior of all algorithms with ($n = 200$, $tol = 10^{-6}$)

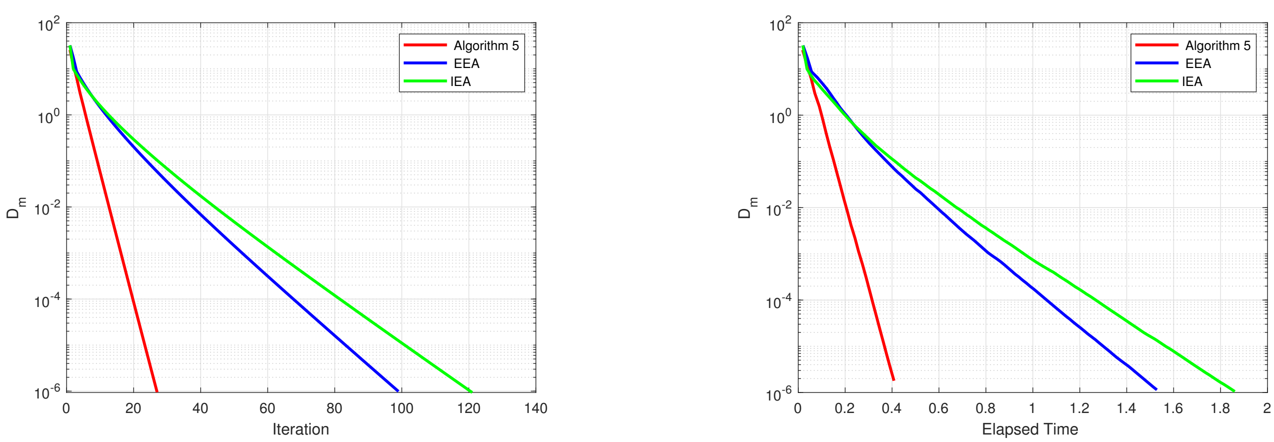

Figure 2.4: Numerical behavior of all algorithms with ($n = 300$, $tol = 10^{-6}$)

Table 2.2: Comparison of iterations and CPU time of all algorithms.

n	tol	Algorithm 5		**IEA**		**EEA**	
		Iter.	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)
200	10^{-3}	11	0.13	60	0.48	50	0.41
300	10^{-3}	17	0.27	63	0.99	53	0.84
200	10^{-6}	20	0.16	121	0.99	99	1.13
300	10^{-6}	27	0.41	121	1.87	99	1.54

Remark 2.4. From our numerical results, we observe the following:

- $\checkmark$ Algorithm 5 is the most efficient in both iteration count and CPU time.
- $\checkmark$ IEA requires the most iterations and has the longest CPU time.
- $\checkmark$ EEA performs better than IEA but remains slower than Algorithm 5.
- $\checkmark$ A smaller tol increases iterations for all algorithms, but Algorithm 5 is least affected.
- $\checkmark$ Increasing n raises CPU time and iterations, impacting IEA and EEA more than Algorithm 5.

CHAPTER 3

AN IMPROVED INERTIAL SUBGRADIENT EXTRAGRADIENT ALGORITHM FOR EP IN REAL HILBERT SPACES

In this chapter, an algorithm with a simple and straightforward structure is presented, combining the subgradient extragradient method with inertial terms to solve pseudomonotone equilibrium problems in a real Hilbert space. The main advantage of this algorithm is the use of two non-monotonic step size criteria that can adaptively operate without requiring Lipschitz constants or a line search technique. Strong convergence is established under appropriate conditions on the equilibrium bifunction B and the control parameters. Furthermore, application of the main result to variational inequalities problems is provided. Additionally, numerical examples are presented to demonstrate the efficiency of the proposed algorithm.

3.1 The Proposed Algorithm

In this section, we propose a modified subgradient extragradient algorithm with two non-monotonic step size criteria for solving EPb (1.4). Assume that the equilibrium bifunction B satisfies conditions (H1)-(H4) (as in Section 2.1). Moreover, assume that the following condition (H_5) holds

(H_5) Let $\{\varepsilon_m\}$ be a positive sequence such that $\lim\limits_{m\to\infty} \frac{\varepsilon_m}{\delta_m} = 0$, where $\{\delta_m\} \subset (0,1)$ satisfies $\lim\limits_{m\to\infty} \delta_m = 0$ and $\sum\limits_{m=1}^{\infty} \delta_m = \infty$. Also let $\{\sigma_m\} \subset [0,\infty)$ and $\{\omega_m\} \subset [1,\infty)$ such that $\sum\limits_{m=1}^{\infty} \sigma_m < \infty$ and $\sum\limits_{m=1}^{\infty} (\omega_m - 1) < \infty$.

Algorithm 6 Improved inertial subgradient extragradient algorithm for EP in $\mathcal{H}$

Initialization: Given $v_0, v_1 \in \mathcal{S}$, $\varrho_0 > 0$, $\gamma > 0$, $\zeta \in (0,1)$ and $\mu \in \left(0, \frac{2}{(1+\zeta)}\right)$. Select the sequences $\{\ \delta_m\}$, $\{\sigma_m\}$ and $\{\omega_m\}$ to satisfy (H_5).

Step 1: Compute

$$s_m = (1-\delta_m)(v_m + \gamma_m(v_m - v_{m-1})),$$

where $0 \le \gamma_m \le \widetilde{\gamma}_m$ such that

$$\widetilde{\gamma}_m = \begin{cases} \min\left\{\gamma, \frac{\varepsilon_m}{\|v_m - v_{m-1}\|}\right\} & \text{if } v_m \neq v_{m-1}, \\ \gamma & \text{else,} \end{cases} \tag{3.1}$$

Step 2: Compute

$$t_m = \arg\min_{t\in\mathcal{S}} \left(\varrho_m B(s_m, t) + \frac{1}{2}\|t - s_m\|^2\right) = J_{\varrho_m B(s_m,\cdot)}(s_m).$$

If $s_m = t_m$, then stop, and t_m is a solution. Otherwise, go to next step.

Step 3:

$$v_{m+1} = \arg\min_{t\in\mathcal{T}_m} \left(\mu\varrho_m B(t_m, t) + \frac{1}{2}\|t - s_m\|^2\right) = J_{\mu\varrho_m B(t_m,\cdot)}(s_m),$$

where the half-space $\mathcal{T}_m$ is given by

$$\mathcal{T}_m = \{t \in \mathcal{H} : \langle s_m - \varrho_m u_m - t_m, t - t_m\rangle \le 0\}, u_m \in \partial B(s_m, t_m),$$

and

$$\varrho_{m+1} = \begin{cases} \min\left\{\frac{\zeta\left(\|s_m - t_m\|^2 + \|v_{m+1} - t_m\|^2\right)}{2M}, \omega_m\varrho_m + \sigma_m\right\} & \text{if } M > 0, \\ \omega_m\varrho_m + \sigma_m & \text{otherwise,} \end{cases} \tag{3.2}$$

where $M = B(s_m, v_{m+1}) - B(s_m, t_m) - B(t_m, v_{m+1})$.

Remark 3.1.
- If we choose ($\delta_m = 0$, $\gamma = 2\theta$, $\sigma_m = 0$) or ($\delta_m = 0$, $\gamma_m = 0$, $\varrho_m = \varrho$, $\mu = 1$), the algorithm reduces to the Algorithm 3.4 in [41] and the standard extragradient algorithm in [45], respectively.
- Notice that, the sequence $\{\varrho_m\}$ defined by (3.2) is non-monotonic step size, independent to the Lipschitz constants and does not need any Armijo line-search technique.

Remark 3.2. From the expression (3.1), it is apparent that $\lim_{m\to+\infty} \frac{\gamma_m}{\delta_m}\|v_m - v_{m-1}\| = 0$. Indeed, $\gamma_m \le \frac{\varepsilon_m}{\|v_m - v_{m-1}\|}$ and $\lim_{m\to\infty} \frac{\varepsilon_m}{\delta_m} = 0$, implies

$$\lim_{m\to\infty} \frac{\gamma_m}{\delta_m}\|v_m - v_{m-1}\| \le \lim_{m\to\infty} \frac{\varepsilon_m}{\delta_m}\|v_m - v_{m-1}\| = 0.$$

3.2 Convergence Analysis

In order to establish the strong convergence of Algorithm 6, our initial steps involves proving the following basic results.

Lemma 3.1. *The sequence $\{\varrho_m\}$ created by (3.2) is well defined and $\lim_{m\to+\infty}\varrho_m$ exists.*

Proof. Since B fulfills (H_2), it follows that

$$\begin{aligned}\frac{\zeta\left(\|s_m-t_m\|^2+\|v_{m+1}-t_m\|^2\right)}{2\left(B(s_m,v_{m+1})-B(s_m,t_m)-B(t_m,v_{m+1})\right)} &\geq \frac{\zeta\left(\|s_m-t_m\|^2+\|v_{m+1}-t_m\|^2\right)}{2\left(L_1\|s_m-t_m\|^2+L_2\|v_{m+1}-t_m\|^2\right)}\\ &\geq \frac{\zeta}{2\max\{L_1,L_2\}}.\end{aligned}$$

This, in addition to the expression (3.2), gives $\varrho_{m+1}\geq\min\left\{\frac{\zeta}{2\max\{L_1,L_2\}},\varrho_m\right\}$. Moreover $\varrho_m\geq\min\left\{\frac{\zeta}{2\max\{L_1,L_2\}},\varrho_1\right\}$. In contrast, it becomes clear from expression (3.2) that

$$\varrho_{m+1}\leq\omega_m\varrho_m+\sigma_m,\ \forall m\geq 1.$$

It follows from condition (H_5) and Lemma 1.4 that $\lim_{m\to+\infty}\varrho_m$ exists. Since $\min\left\{\frac{\zeta}{2\max\{L_1,L_2\}},\varrho_1\right\}$ is the lower boundary of $\{\varrho_m\}$, then $\lim_{m\to+\infty}\varrho_m:=\varrho>0$. □

Lemma 3.2. *Let $\{s_m\}$, $\{t_m\}$ and $\{v_m\}$ be the sequences generated by Algorithm 6. Then*

(i) $< s_m-t_m, t-t_m >\leq\varrho_m\left(B(s_m,t)-B(s_m,t_m)\right),\ \forall t\in\mathcal{S}$,

(ii) *if $t_m=s_m$, then $t_m\in EQ(B,\mathcal{S})$,*

(iii) *for all $r\in EQ(B,\ \mathcal{S})$, the following inequality holds:*

$$\|v_{m+1}-r\|^2\leq\|s_m-r\|^2-\mu_m^*\left(\|v_{m+1}-t_m\|^2+\|s_m-t_m\|^2\right), \tag{3.3}$$

where

$$\mu_m^*=\begin{cases}\mu\left(1-\frac{\zeta\varrho_m}{\varrho_{m+1}}\right) & \text{if}\quad \mu\in(0,1),\\ 2-\mu-\frac{\zeta\mu\varrho_m}{\varrho_{m+1}} & \text{if}\quad \mu\in[1,\dfrac{2}{(1+\zeta)}).\end{cases}$$

Proof. **(i)** According to Lemma 1.1 and $t_m=J_{\varrho_m B(s_m,.)}(s_m)$, we have

$$0\in\partial\left(\varrho_m B(s_m,t)+\frac{1}{2}\|t-s_m\|^2\right)(t_m)+N_{\mathcal{S}}(t_m).$$

Then, there exists $u_m \in \partial B(s_m, t_m)$ and $\vartheta \in N_{\mathcal{S}}(t_m)$, such that

$$\varrho_m u_m + t_m - s_m + \vartheta = 0.$$

By the definition of $N_{\mathcal{S}}$, we get

$$< s_m - t_m, t - t_m >= \varrho_m < u_m, t - t_m > + < \vartheta, t - t_m >\leq \varrho_m < u_m, t - t_m >, \ \forall t \in \mathcal{S}.$$

Since $u_m \in \partial B(s_m, t_m)$, we have

$$< u_m, t - t_m >\leq B(s_m, t) - B(s_m, t_m), \ \forall t \in \mathcal{S}.$$

From the last two inequalities, we obtain

$$< s_m - t_m, t - t_m >\leq \varrho_m \left(B(t_m, t) - B(s_m, t_m)\right), \ \forall t \in \mathcal{S}. \tag{3.4}$$

(ii) If $t_m = s_m$, then from inequalities (3.4) and $\varrho_m > 0$, we find $B(t_m, t) \geq 0$, for all $t \in \mathcal{S}$. Thus $t_m \in EQ(B,\ \mathcal{S})$.

(iii) We have $v_{m+1} = J_{\mu\varrho_m B(t_m,.)}(s_m)$, as similar arguments to the proof of $(\mathbf{i})$, we obtain

$$< s_m - v_{m+1}, t - v_{m+1} >\leq \mu\varrho_m \left(B(t_m, t) - B(t_m, v_{m+1})\right), \ \forall t \in \mathcal{T}_m. \tag{3.5}$$

In particular, substituting $t = v_{m+1}$ in (3.4) and $t = r$ in (3.5), we get

$$\begin{cases} \langle s_m - t_m, v_{m+1} - t_m\rangle \leq \varrho_m \left(B(s_m, v_{m+1}) - B(s_m, t_m)\right), \\ \langle s_m - v_{m+1}, r - v_{m+1}\rangle \leq \mu\varrho_m \left(B(t_m, r) - B(t_m, v_{m+1})\right). \end{cases}$$

So, from the pseudo monotonicity of B, we have $B(r, t_m) \geq 0$. Thus $B(t_m, r) \leq 0$. Then

$$\begin{aligned} 2\mu\varrho_m \left(B(s_m, v_{m+1}) - B(s_m, t_m) - B(t_m, v_{m+1})\right) &\geq 2\mu < s_m - t_m, v_{m+1} - t_m > \\ &\quad + 2 < s_m - v_{m+1}, r - v_{m+1} > \\ &\geq \mu \left\|s_m - t_m\right\|^2 + \mu \left\|v_{m+1} - t_m\right\|^2 \\ &\quad - \mu \left\|v_{m+1} - s_m\right\|^2 + \left\|v_{m+1} - s_m\right\|^2 \\ &\quad + \left\|v_{m+1} - r\right\|^2 - \left\|s_m - r\right\|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \left\|v_{m+1} - r\right\|^2 \leq \left\|s_m - r\right\|^2 - \mu \left\|s_m - t_m\right\|^2 - \mu \left\|v_{m+1} - t_m\right\|^2 - (1 - \mu) \left\|v_{m+1} - s_m\right\|^2 \\ + 2\mu\varrho_m \left(B(s_m, v_{m+1}) - B(s_m, t_m) - B(t_m, v_{m+1})\right). \end{aligned} \tag{3.6}$$

From the definition of ϱ_m, we have

$$2\mu\varrho_m \left(B(s_m, v_{m+1}) - B(s_m, t_m) - B(t_m, v_{m+1})\right) \leq \frac{\mu\zeta\varrho_m}{\varrho_{m+1}} \left(\|s_m - t_m\|^2 + \|v_{m+1} - t_m\|^2\right). \tag{3.7}$$

Substituting (3.7) into (3.6), we obtain

$$\begin{aligned} \|v_{m+1} - r\|^2 &\leq \|s_m - r\|^2 - \mu \|s_m - t_m\|^2 - \mu \|v_{m+1} - t_m\|^2 - (1-\mu) \|v_{m+1} - s_m\|^2 \\ &\quad + \frac{\mu\zeta\varrho_m}{\varrho_{m+1}} \left(\|s_m - t_m\|^2 + \|v_{m+1} - t_m\|^2\right) \\ &\leq \|s_m - r\|^2 - (1-\mu) \|v_{m+1} - s_m\|^2 \\ &\quad - \mu \left(1 - \frac{\zeta\varrho_m}{\varrho_{m+1}}\right) \left(\|s_m - t_m\|^2 + \|v_{m+1} - t_m\|^2\right). \end{aligned} \tag{3.8}$$

If $\mu \in (0, 1)$, then

$$\|v_{m+1} - r\|^2 \leq \|s_m - r\|^2 - \mu \left(1 - \frac{\zeta\varrho_m}{\varrho_{m+1}}\right) \left(\|v_{m+1} - t_m\|^2 + \|s_m - t_m\|^2\right).$$

Note that

$$\|v_{m+1} - s_m\|^2 \leq (\|v_{m+1} - t_m\| + \|s_m - t_m\|)^2 \leq 2\left(\|v_{m+1} - t_m\|^2 + \|s_m - t_m\|^2\right),$$

which yields that

$$-(1-\mu) \|v_{m+1} - s_m\|^2 \leq -2(1-\mu) \left(\|v_{m+1} - t_m\|^2 + \|s_m - t_m\|^2\right), \ \ \forall \mu \geq 1.$$

From expression (3.8), we get

$$\|v_{m+1} - r\|^2 \leq \|s_m - r\|^2 - \left(2 - \mu - \frac{\zeta\mu\varrho_m}{\varrho_{m+1}}\right) \left(\|v_{m+1} - t_m\|^2 + \|s_m - t_m\|^2\right), \ \ \forall \mu \geq 1.$$

This completes the proof. □

Remark 3.3. It easy to cheek that $\lim_{m\to\infty} \mu_m^* > 0$. Using Lemma 3.1, we have

$$\lim_{m\to\infty} \mu_m^* = \begin{cases} \mu(1-\zeta) & \text{if} \quad \mu \in (0,1), \\ 2 - \mu(1+\zeta) & \text{if} \quad \mu \in [1, \frac{2}{(1+\zeta)}). \end{cases}$$

Moreover, there exists $m_0 \geq 0$ such that $\mu_m^* > 0$ for all $m \geq m_0$.

Lemma 3.3. *The sequence $\{v_m\}$ generated by Algorithm 6 is bounded. Consequently, $\{s_m\}$ and $\{t_m\}$ are bounded.*

Proof. From the definition of s_m, we have

$$\begin{aligned}\|s_m - r\| &= \|(1-\delta_m)(v_m + \gamma_m(v_m - v_{m-1})) - r\| \\ &= \|(1-\delta_m)(v_m - r) + (1-\delta_m)\gamma_m(v_m - v_{m-1}) - \delta_m r\| \\ &\leq (1-\delta_m)\|v_m - r\| + (1-\delta_m)\gamma_m \|v_m - v_{m-1}\| + \delta_m \|r\| \\ &\leq (1-\delta_m)\|v_m - r\| + \delta_m I_1, \qquad (3.9)\end{aligned}$$

where

$$(1-\delta_m)\frac{\gamma_m}{\delta_m}\|v_m - v_{m-1}\| + \|r\| \leq I_1.$$

From (3.3), we obtain

$$\|v_{m+1} - r\|^2 \leq \|s_m - r\|^2, \forall m \in \mathbb{N} \qquad (3.10)$$

Using (3.9), then

$$\begin{aligned}\|v_{m+1} - r\| &\leq (1-\delta_m)\|v_m - r\| + \delta_m I_1 \\ &\leq \max\{\|v_m - r\|, I_1\}, \quad \forall m \geq m_0 \\ &\leq ... \leq \max\{\|v_0 - r\|, I_1\}.\end{aligned}$$

Thus, we conclude that $\{\|v_m - r\|\}$ is bounded sequence which implies that $\{v_m\}$ is bounded. Therefor $\{s_m\}$ and $\{t_m\}$ are also bounded. □

Theorem 3.1. *Suppose that the conditions (H_1)-(H_5) hold and $EQ(B, \mathcal{S}) \neq \emptyset$. Then the sequence $\{v_m\}$ generated by Algorithm 6 converges in norm to r, where $\|r\| = \min\{\|q\| : q \in EQ(B, \mathcal{S})\}$, i.e., $r = P_{EQ(B, \mathcal{S})}(0)$.*

Proof. First, we show that the sequence $\{v_m\}$ and $\{s_m\}$ generated by Algorithm 6 achieves the following:

$$\begin{cases} J_{m+1} \leq (1-\delta_m)J_m + \delta_m L_m, \forall m \geq m_0, \\ \limsup\limits_{m\to\infty} L_m \leq 0, \end{cases}$$

where

$$\begin{cases} J_m = \|v_m - r\|^2, \\ L_m = \gamma_m \|v_m - v_{m-1}\| \frac{\gamma_m}{\delta_m}\|v_m - v_{m-1}\| + 2(1-\delta_m)\|v_m - r\| \frac{\gamma_m}{\delta_m}\|v_m - v_{m-1}\| \\ \qquad + 2\|r\|\|s_m - v_{m+1}\| + 2\langle r, r - v_{m+1}\rangle. \end{cases}$$

Indeed, according to the inequality (3.10) and the definition of $\{s_m\}$, we have

$$\begin{aligned}\|v_{m+1}-r\|^2 &\leq \|(1-\delta_m)(v_m-r)+(1-\delta_m)\gamma_m(v_m-v_{m-1})-\ \delta_m r\|^2\\ &= \|(1-\delta_m)(v_m-r)+(1-\delta_m)\gamma_m(v_m-v_{m-1})\|^2+\|\ \delta_m r\|^2\\ &\quad +2\ \delta_m\left\langle -r, s_m+r\right\rangle+2\delta_m\|r\|^2\\ &\leq (1-\delta_m)^2\|v_m-r\|^2+(1-\delta_m)^2\gamma_m^2\|v_m-v_{m-1}\|^2\\ &\quad +2\gamma_m(1-\delta_m)^2\|v_m-r\|\|v_m-v_{m-1}\|\\ &\quad +2\ \delta_m\left\langle -r, s_m-v_{m+1}\right\rangle+2\ \delta_m\left\langle -r, v_{m+1}+r\right\rangle.\end{aligned}$$

Since $\delta_m\subset(0,1)$, for all $m\geq m_0$, the above expression yields that

$$\begin{aligned}\|v_{m+1}-r\|^2 \leq&(1-\ \delta_m)\|v_m-r\|^2+\ \delta_m[\gamma_m\|v_m-v_{m-1}\|\frac{\gamma_m}{\delta_m}\|v_m-v_{m-1}\|\\ &+2(1-\ \delta_m)\|v_m-r\|\frac{\gamma_m}{\delta_m}\|v_m-v_{m-1}\|+2\|r\|\|s_m-v_{m+1}\|\\ &+2\left\langle r, r-v_{m+1}\right\rangle].\end{aligned}\tag{3.11}$$

The last inequality can be written as

$$J_{m+1}\leq(1-\ \delta_m)J_m+\ \delta_m L_m, \forall m\geq m_0.$$

Due to Lemma 1.5, suppose that $\{J_{m_k}\}$ is a subsequence of $\{J_m\}$ satisfies

$$\liminf_{k\to\infty}\left(J_{m_{k+1}}-J_{m_k}\right)>0.$$

Now, we prove that $\limsup_{m\to\infty} L_m\leq 0$. From (3.9), we have

$$\begin{aligned}\|s_m-r\|^2 &\leq \|(1-\ \delta_m)\|v_m-r\|+\ \delta_m I_1\|^2\\ &\leq (1-\ \delta_m)^2\|v_m-r\|^2+\ \delta_m^2 I_1^2+2I_1(1-\ \delta_m)\ \delta_m\|v_m-r\|\\ &\leq \|v_m-r\|^2+\ \delta_m\left(\ \delta_m I_1^2+2I_1(1-\ \delta_m)\|v_m-r\|\right)\\ &\leq \|v_m-r\|^2+\delta_m I_2,\ \text{for all } m\geq 1\end{aligned}\tag{3.12}$$

where $I_2=\sup_{m\in}\{(\delta_m I_1^2+2I_1(1-\ \delta_m)\|v_m-r\|)\}$.

It follows from (3.3) and (3.12) that

$$\mu_m^*\left(\|v_{m+1}-t_m\|^2+\|s_m-t_m\|^2\right)\leq\|v_m-r\|^2-\|v_{m+1}-r\|^2+\ \delta_m I_2, \forall m\geq m_0.\tag{3.13}$$

From (3.13), $\lim_{m\to\infty} \delta_m = 0$ and Remark 3.3, we get

$$\mu^*_{m_k}\left(\|v_{m_k+1}-t_{m_k}\|^2+\|s_{m_k}-t_{m_k}\|^2\right) \le \limsup_{k\to\infty}\left(J_{m_k}-J_{m_{k+1}}\right)+\limsup_{k\to\infty}\delta_{m_k} I_2,$$
$$\le -\liminf_{k\to\infty}\left(J_{m_{k+1}}-J_{m_k}\right)$$
$$\le 0.$$

Then

$$\lim_{k\to\infty}\|v_{m_k+1}-t_{m_k}\|=0 \text{ and } \|s_{m_k}-t_{m_k}\|=0. \tag{3.14}$$

Consequently,

$$\lim_{k\to\infty}\|v_{m_k+1}-s_{m_k}\|=0 \text{ and } \lim_{k\to\infty}\|v_{m_k+1}-s_{m_k}\|\,\|r\|=0. \tag{3.15}$$

Furthermore,

$$\begin{aligned}\|v_{m_k}-s_{m_k}\| &= \|(1-\ \delta_{m_k})\gamma_{m_k}(v_{m_k}-v_{m_k-1})-\ \delta_m v_{m_k}\|\\ &\le \|(1-\ \delta_{m_k})\gamma_{m_k}(v_{m_k}-v_{m_k-1})\|+\|\ \delta_m v_{m_k}\|\\ &= \ \delta_{m_k}\left[(1-\ \delta_{m_k})\frac{\gamma_{m_k}}{\delta_{m_k}}\|v_{m_k}-v_{m_k-1}\|+\|v_{m_k}\|\right],\end{aligned}$$

then, we deduce that

$$\lim_{k\to\infty}\|v_{m_k}-s_{m_k}\|=0. \tag{3.16}$$

Consequently

$$\lim_{k\to\infty}\|v_{m_k+1}-v_{m_k}\|=0. \tag{3.17}$$

Next we show that $\limsup_{k\to\infty}\langle r, r-v_{m_k+1}\rangle=0$. Due to the reflexive property of the Hilbert space $\mathcal{H}$, the boundedness of $\{v_{m_k}\}$ guarantee the existence of a subsequence $\left\{v_{m_{k_j}}\right\}$ of $\{v_{m_k}\}$ converges weakly to q as $j\to\infty$. Moreover

$$\limsup_{k\to\infty}\langle r, r-v_{m_k}\rangle=\lim_{j\to\infty}\left\langle r, r-v_{m_{k_j}}\right\rangle=\langle r, r-q\rangle. \tag{3.18}$$

It follows from (3.14) and (3.16) that $t_{m_k}\rightharpoonup q$ and $s_{m_k}\rightharpoonup q$. By means of $t_{m_k}=J_{\varrho_{m_k}B(s_{m_k},\cdot)}(s_{m_k})$, we have

$$\mu\varrho_{m_k}B(t_{m_k},v_{m_k+1})+<s_{m_k}-v_{m_k+1}, t-v_{m_k+1}>\le \mu\varrho_{m_k}B(t_{m_k},t), \text{ for } t\in\mathcal{T}_m. \tag{3.19}$$

From (3.2), we get

$$2\mu\varrho_{m_k}B(t_{m_k},v_{m_k+1})\geq 2\mu\varrho_{m_k}\left(B(s_{m_k},v_{m_k+1})-B(s_{m_k},t_{m_k})\right)$$
$$-\frac{\mu\zeta\varrho_{m_k}}{\varrho_{m_k+1}}\left(\|s_{m_k}-t_{m_k}\|^2+\|v_{m_k+1}-t_{m_k}\|^2\right), \tag{3.20}$$

Substituting (3.20) into (3.19), we obtain

$$\mu\varrho_{m_k}B(t_{m_k},t)\geq \mu\varrho_{m_k}\left(B(s_{m_k},v_{m_k+1})-B(s_{m_k},t_{m_k})\right)-\frac{\mu\zeta\varrho_{m_k}}{2\varrho_{m_k+1}}\left(\|s_{m_k}-t_{m_k}\|^2+\|v_{m_k+1}-t_{m_k}\|^2\right)$$
$$+<s_{m_k}-v_{m_k+1},t-v_{m_k+1}>,\quad \text{for } t\in\mathcal{T}_m.$$

Due to μ, $\varrho_{m_k}>0$, condition (H$_4$) $v_{m_k}\rightharpoonup q$ and (3.14), we have

$$0\leq\limsup_{k\to\infty}B(t_{m_k},t)\leq B(q,t),\,\forall t\in\mathcal{T}_m.$$

Since $\mathcal{S}\subset\mathcal{T}_m$, $B(q,t)\geq 0, \forall t\in\mathcal{S}$ and hence $q\in EQ(B,\,\mathcal{S})$. By using (3.18) and the definition of r, we obtain

$$\lim_{k\to\infty}\langle r,r-v_{m_k}\rangle=\langle r,r-q\rangle\leq 0.$$

This with (3.17) gives,

$$\limsup_{k\to\infty}\langle r,r-v_{m_k+1}\rangle\leq\limsup_{k\to\infty}\langle r,r-v_{m_k}\rangle+\limsup_{k\to\infty}\langle r,v_{m_k+1}-v_{m_k}\rangle\leq 0. \tag{3.21}$$

Moreover, from (3.15), (3.21) and $\lim_{m\to\infty}\frac{\gamma_m}{\delta_m}\|v_m-v_{m-1}\|=0$, we concludes that

$$\limsup_{k\to\infty}L_{m_k}\leq 0.$$

Applying Lemma 1.5, we obtain $\lim_{m\to\infty}\|v_m-r\|=0$, as desired. □

3.3 Application to variational inequality problem

In this section, we apply the main results (Theorem 3.1) to solve VIP (1.7) in real Hilbert spaces.

Let $B(v,t)=\langle\Psi(v),t-v\rangle$, with $\Psi:\ \mathcal{S}\to\mathcal{S}$ be an operator then the EPb (1.4) turn to VIP (1.7). Assume that the solution set of (1.7) (denoted by $VI(\Psi,\mathcal{S})$) is nonempty and the operator Ψ satisfies conditions (H$_1'$)-(H$_3'$) (as in Section 2.3).

It easy to cheek that the sequence $\{t_m\}$ can be written as

$$t_m=P_{\mathcal{S}}\left(s_m-\varrho_m\Psi(s_m)\right).$$

Similarly, $v_{m+1} = P_{\mathcal{T}_m}\left(s_m - \mu\varrho_m\Psi(t_m)\right).$

Corollary 3.1. *Assume that the conditions$(H_1') - (H_3')$ hold. Let $\{v_m\}$ and $\{t_m\}$ be two sequences created in the following way:*

(i) *Given $v_0, v_1 \in \mathcal{S}$, $\varrho_1 > 0$, $\gamma > 0$, $\zeta \in (0,1)$ and $\mu \in \left(0, \dfrac{2}{(1+\zeta)}\right)$. Select the sequences $\{\delta_m\}$, $\{\omega_m\}$ and $\{\sigma_m\}$ to satisfy (H_5) .*

(ii) *Compute $s_m = (1-\delta_m)(v_m + \gamma_m(v_m - v_{m-1}))$, where γ_m is defined in (3.1).*

(iii) *Compute*

$$\begin{cases} t_m = P_{\mathcal{S}}\left(s_m - \varrho_m\Psi(s_m)\right), \\ v_{m+1} = P_{\mathcal{T}_m}\left(s_m - \mu\varrho_m\Psi(s_m)\right), \end{cases}$$

where $\mathcal{T}_m = \{t \in \mathcal{H} : \langle s_m - \varrho_m\Psi(s_m) - t_m, t - t_m\rangle \le 0\}$, and

$$\varrho_{m+1} = \begin{cases} \min\left\{\frac{\zeta\left(\|s_m - t_m\|^2 + \|v_{m+1} - t_m\|^2\right)}{2\langle\Psi(s_m) - \Psi(t_m), v_{m+1} - t_m\rangle}, \omega_m\varrho_m + \sigma_m\right\}, & \text{if } \langle\Psi(s_m) - \Psi(t_m), v_{m+1} - t_m\rangle > 0, \\ \omega_m\varrho_m + \sigma_m, & \text{otherwise.} \end{cases}$$

Then, the sequence $\{v_m\}$ converges in norm to r, for each $r \in VI(\Psi, \mathcal{S}) \neq \emptyset$.

3.4 Numerical Illustrations

In this section, we give numerical example to prove the computational efficiency of the proposed algorithm compared to some related results. All the programs are implemented in MATLAB.

Consider the equilibrium bifunction $B : \mathbb{R}^5 \times \mathbb{R}^5 \to \mathbb{R}$ in [45]:

$$B(v,t) = \langle Mv + Ft + g, t - v\rangle,$$

where $g \in \mathbb{R}^5$ and $M, Q \in \mathbb{R}^{5\times 5}$ are two matrices of order 5 given by

$$M = \begin{pmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, F = \begin{pmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix},$$

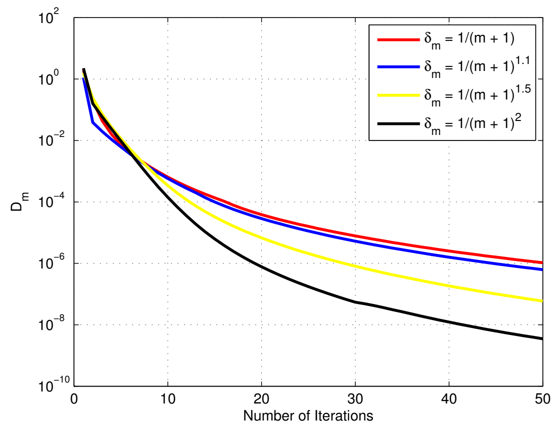

Figure 3.1: Numerical behavior of the proposed algorithm with different δ_m

with $g = (1, -2, -1, 2, -1)^t$, and $\mathcal{S}$ given by

$$\mathcal{S} := \{v \in \mathbb{R}^n : -5 \leq v_i \leq 5, i = 1, ..., 5\},$$

It can be checked that all the conditions (H_1)-(H_4) are satisfied (for more details see [45]). The initial values v_0 and v_1 are randomly generated by MATLAB function $rand(5,1)$. We use the maximum number of iterations 50 as a common stopping criterion for all algorithms and $D_m = \|v_{m+1} - v_m\|^2$ is used to measure the error of the $m - th$ iteration step. We compare the proposed Algorithm (Algorithm 6) with the Algorithm 3.1 suggested by Tan et al. [41] (shortly, BSJ Alg. 3.1) and the Algorithm 1 introduced by Panyanak et al. [33] (shortly, BCNN Alg. 1). The control parameters of all algorithms are choose as follows:

1. (Algorithm 6): $\varepsilon_m = \frac{100}{(m+1)^2}$, $\delta_m = \frac{1}{20(m+1)^2}$, $\sigma_m = \frac{1}{(m+100)^3}$, $\omega_m = 1 + \frac{1}{20(m+1)^{1.1}}$.
2. (BCNN Alg. 1) in [33]: $\varepsilon_m = \frac{100}{(m+1)^2}$, $\omega_m = 1 + \frac{1}{20(m+1)^{1.1}}$, $\alpha_m = \frac{(1-\alpha)}{10}$, $\beta_m = \frac{1}{5m+2}$, $T(x) = \frac{x}{5}$.
3. (BSJ Alg. 3.1) in [41]: $\varepsilon_m = \frac{100}{(m+1)^2}$, $\alpha_m = 1/(4m+1)$, $\beta_m = 0.5$, $S(x) = x$, $\varphi(x) = 0.5x$.

We first test the numerical behavior of the proposed algorithm with different parameter δ_m, as shown in Fig 3.1. and Table 3.1. Finally, the numerical results of all algorithms with different parameters are shown in Fig. 3.2-3.5 and Table 3.2.

Table 3.1: Numerical results of the proposed algorithm with different δ_m

Algorithm 6	D_n	CPU
$\delta_m = \dfrac{1}{(m+1)^2}$	$3.5E-9$	1.78
$\delta_m = \dfrac{1}{(m+1)^{1.5}}$	$5.91E-8$	1.80
$\delta_m = \dfrac{1}{(m+1)^{1.1}}$	$6.24E-7$	1.59
$\delta_m = \dfrac{1}{m+1}$	$1.04E-6$	1.62

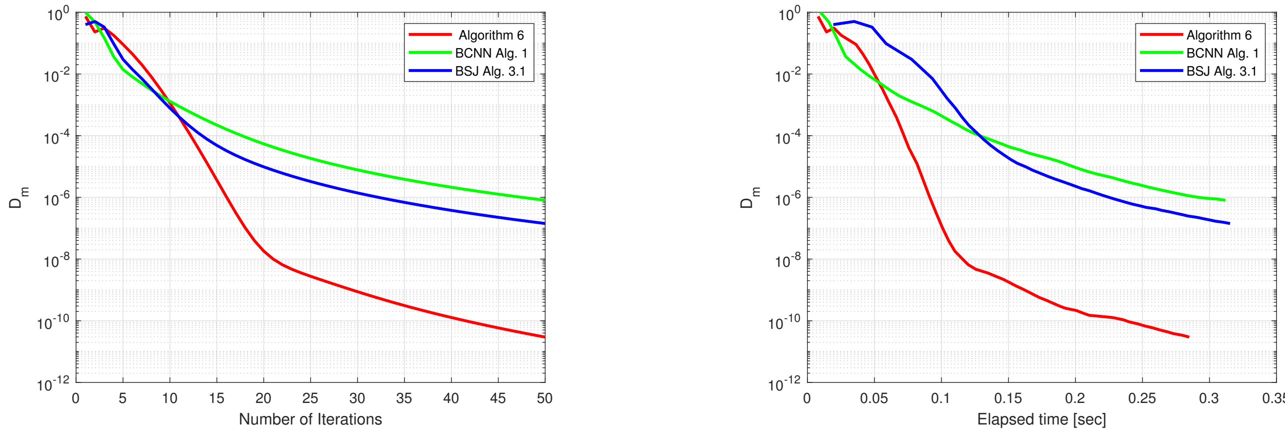

Figure 3.2: Numerical behavior of all algorithms with ($\gamma = 0.4$, $\zeta = 0.5$, $\mu = 1.25$, $\rho_0 = 0.1$)

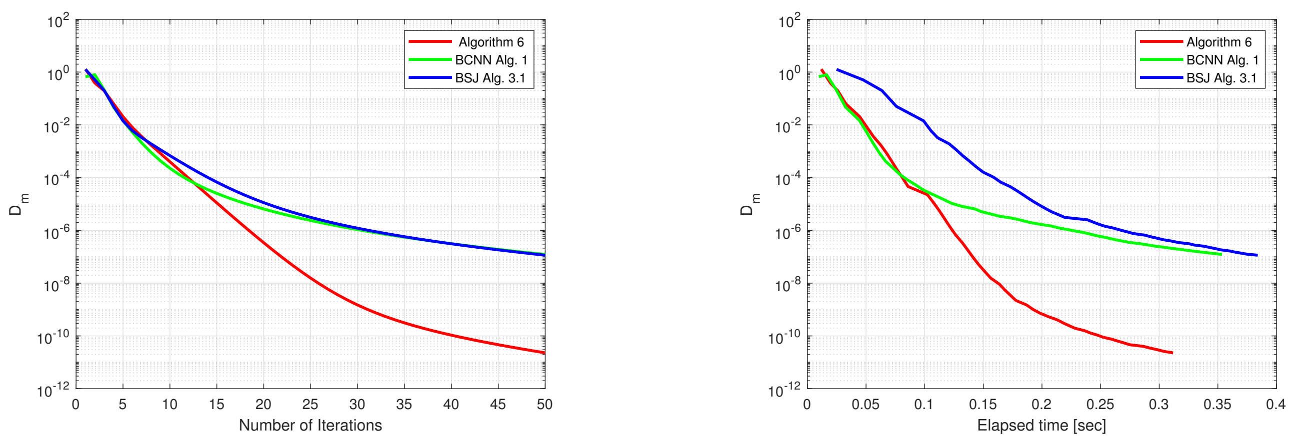

Figure 3.3: Numerical behavior of all algorithms with ($\gamma = 0.2$, $\zeta = 0.264$, $\mu = 0.5$, $\rho_0 = 0.36$)

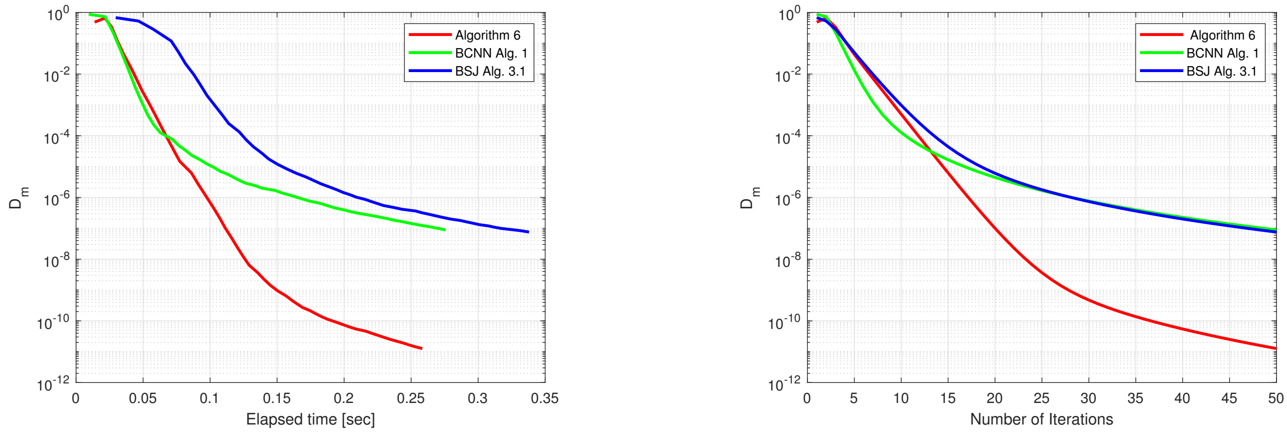

Figure 3.4: Numerical behavior of all algorithms with ($\gamma = 0.2$, $\zeta = 0.5$, $\mu = 0.5$, $\rho_0 = 0.5$)

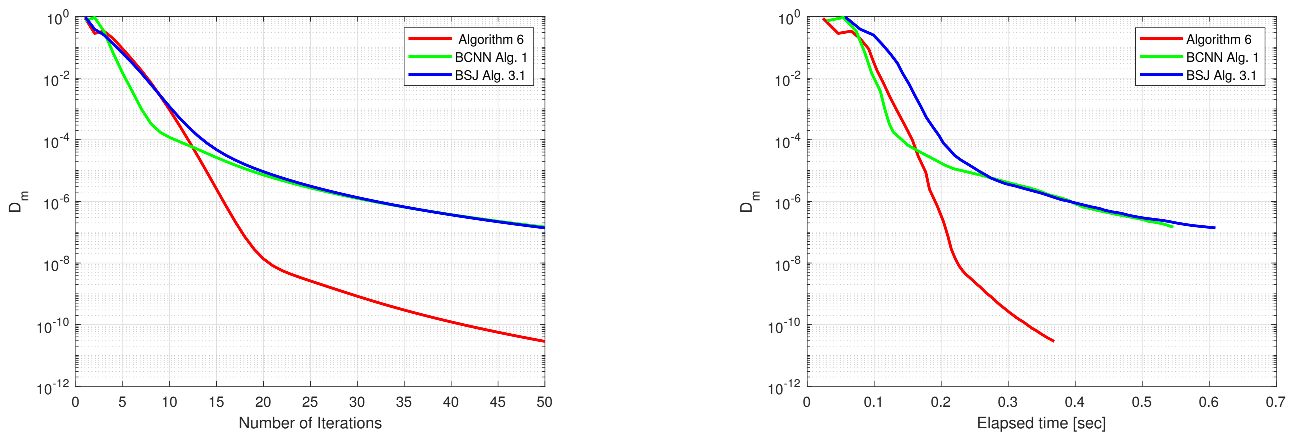

Figure 3.5: Numerical behavior of all algorithms with ($\gamma = 0.4$, $\zeta = 0.5$, $\mu = 0.5$, $\rho_0 = 0.3$)

Table 3.2: Numerical results of all algorithms with different parameters

Algorithms	Algorithm 6		BCNN Algo. 1		BSJ Algo 3.1	
	D_n	CPU	D_n	CPU	D_n	CPU
$\gamma = 0.4$, $\zeta = 0.5$, $\mu = 1.25$, $\rho_0 = 0.1$	$2.95E-11$	0.28	$8.05E-7$	0.31	$1.42E-7$	0.31
$\gamma = 0.2$, $\zeta = 0.264$, $\mu = 0.5$, $\rho_0 = 0.36$	$2.28E-11$	0.31	$1.23E-7$	0.35	$1.15E-7$	0.38
$\gamma = 0.2$, $\zeta = 0.5$, $\mu = 0.5$, $\rho_0 = 0.5$	$1.27E-11$	0.26	$8.90E-8$	0.27	$7.57E-8$	0.34
$\gamma = 0.4, \zeta = 0.5, \mu = 0.5$, $\rho_0 = 0.3$	$2.85E-11$	0.37	$1.47E-7$	0.55	$1.38E-7$	0.61

Remark 3.4. Based on our numerical experiments, we have the following observation:

√ Algorithm 6 shows clear superiority on the other algorithms in terms of convergence speed and accuracy.

- √ Algorithm 6 converges faster to lower error levels even as the number of iterations increases, reaching an error tolerance of 10^{-11} which is reached after a few iterations, while the BCNN Alg. 1 and BSJ Alg. 3.1 require significantly more iterations to achieve comparable results.

- √ Compared to BCNN Alg. 1 and BSJ Alg. 3.1, Algorithm 6 performs consistently well across with different parameters. It achieves the highest precision with a reasonable computation time, making it more efficient for large-scale problems.

CHAPTER 4

BREGMAN EXTRAGRADIENT ALGORITHM FOR EP IN REAL BANACH SPACES

This chapter introduces a modified Bregman extragradient algorithm designed to solve pseudomonotone equilibrium problems in a real reflexive Banach space. The algorithm guarantees weak convergence under mild assumptions and establishes strong convergence under additional conditions. In the proposed algorithm, we utilize two parameters along with the Bregman distance and a non-monotonic step size, which is independent of the Bregman Lipschitz constant, to enhance its effectiveness. Furthermore, numerical experiments are conducted to validate the performance of the proposed algorithm, demonstrating significant improvements in efficiency compared to traditional algorithms in similar settings.

4.1 Bregman Distance

Definition 4.1. ([4]) Assume that $\varphi : E \to \mathbb{R} \cup \{+\infty\}$ is a differentiable convex function. The Bregman distance with respect to φ is the bifunction

$$D_\varphi : dom(\varphi) \times int(dom(\varphi)) \to [0, +\infty) \,,$$

defined by

$$D_\varphi(v,t) := \varphi(v) - \varphi(t) - < \nabla\varphi(t), v - t >, \; \forall\, v \in dom(B), \; t \in int(dom(B)).$$

Unlike standard metrics, the Bregman distance neither exhibits symmetry nor satisfies the triangle inequality. However, it generalizes certain well-known distances. It satisfies the three-point identity:

$$D_\varphi(v,t) + D_\varphi(t,s) - D_\varphi(v,s) =< \nabla\varphi(s) - \nabla\varphi(t), v - t >, \tag{4.1}$$

and the four-point identity

$$D_\varphi(v,t) + D_\varphi(w,s) - D_\varphi(v,s) - D_\varphi(w,t) =< \nabla\varphi(s) - \nabla\varphi(t), v - w >, \tag{4.2}$$

for any $v, w \in dom\ \varphi$ and $t, s \in int(dom\ \varphi)$.
The Bregman projection ([4]) with respect to φ of $v \in int(dom\ \varphi)$ onto $\mathcal{S}$ is characterized as the unique vector $\pi_{\mathcal{S}}^{\varphi}$ fulfilling

$$\pi_{\mathcal{S}}^{\varphi}(v) := \inf_{t \in \mathcal{S}}(D_{\varphi}(t, v)).$$

Now, we present several examples of convex functions φ and the corresponding Bregman distances D_{φ}.

Example 4.1. The function $\varphi(v) := \sum_{i=1}^{n} v_i \log(v_i)$, which is called Shannon entropy. The corresponding Bregman distance is the Kullback-Leibler Divergence (KLD) defined as:

$$D_{\varphi}(v, t) := \sum_{i=1}^{n} \left(v_i \log\left(\frac{v_i}{t_i}\right) + t_i - v_i \right).$$

Example 4.2. The function $\varphi(v) := \frac{1}{2}\|v\|^2$, which is the quadratic function. This is a typical choice in convex optimization problems where the goal is to minimize a quadratic cost function. The corresponding Bregman distance is the squared Euclidean distance (SED), defined as:

$$D_{\varphi}(v, t) := \frac{1}{2}\|v - t\|^2$$

The squared Euclidean distance is often used in optimization algorithms.

Example 4.3. The function $\varphi(v) := -\sum_{i=1}^{n} \log(v_i)$, which is another convex function called Burg entropy. The corresponding Bregman distance is the Itakura-Saito Distance (ISD) defined as:

$$D_{\varphi}(v, t) := \sum_{i=1}^{n} \left(\log\left(\frac{v_i}{t_i}\right) + \frac{v_i}{t_i} - 1 \right).$$

The examples presented above show the flexibility of Bregman distances and their various applications in fields such as information theory, signal processing, and optimization. Each Bregman distance is associated with a specific convex function, and its properties make it a useful tool for a wide range of mathematical and applied problems.

4.2 Known Results

This section presents key definitions and some important results in Banach spaces, which provide the mathematical foundation for analyzing algorithms used in solving equilibrium problems. Let $\mathcal{S}$ be a nonempty, closed, and convex subset of a reflexive real Banach space E, with its dual space denoted by E^*. The duality pairing between E and E^* is represented by $\langle ., . \rangle$, while the norm is denoted by $\|.\|$ (not necessarily Euclidean). Denote by $\rightharpoonup$ and $\rightarrow$ the weak convergence and strong convergence, respectively. Let B be a bifunction : $E \times E \to \mathbb{R}$ and $\varphi : E \to \mathbb{R}$ at $v \in \mathcal{S}$ is defined by

$$\partial\varphi(v) := \{v^* \in E^* : h(t) - h(v) \geq < v^*, t - v >, \quad \forall t \in E\}.$$

The function φ is called a Legendre function if it fulfills the following two conditions

- int (dom φ)$\neq \emptyset$ and $\partial\varphi$ is single-valued on its domain;
- int(dom φ^*) $\neq \emptyset$ and $\partial\varphi^*$ is single-valued on its domain.

Where $\varphi^* : E^* \to \mathbb{R} \cup \{+\infty\}$ is the Fenchel conjugate function of φ given by

$$\varphi^*(v^*) = \sup\{\langle v, v^*\rangle - \varphi(v) : v \in E\}.$$

Definition 4.2. Let $\varphi : E \to \mathbb{R} \cup \{+\infty\}$ be a function.

1. The function φ is called Gâteaux differentiable at a point $v \in$ int(dom φ) if the limit
$$\varphi^\circ(v, t) := \lim_{h \to 0^+} \frac{\varphi(v + ht) - \varphi(v)}{h}, \tag{4.3}$$
exists for any $t \in E$.
2. φ is Gâteaux differentiable if it is Gâteaux differentiable for every $v \in int$(dom φ).
3. We say that φ is Fréchet differentiable at $v \in int$(dom φ) if the limit in (4.3) is attained uniformly in $\|t\| = 1$.
4. φ is Fréchet differentiable on a subset $\mathcal{S}$ of E if the limit in (4.3) is attained uniformly for $v \in \mathcal{S}$ and $\|t\| = 1$.
5. The function φ is supercoercive if $\lim\limits_{\|v\| \to \infty} \dfrac{\varphi(v)}{\|v\|}$;
6. φ is weakly sequentially continuous if $v_m \rightharpoonup v$ implies $\varphi(v_m) \rightharpoonup \varphi(v)$ as $m \to \infty$.

Definition 4.3. ([7]) The modulus of total convexity at a point $v \in$ int(dom φ) is defined as a function $\nu_\varphi(v, .) : [0, +\infty) \to [0, \infty]$, given by:

$$\nu_\varphi(v, d) = \inf\{D_\varphi(t, v) : y \in \text{dom } g, \parallel t - v \parallel = d\}.$$

If $\nu_\varphi(v, d)$ is strictly positive for all $d > 0$, the function φ is said to be totally convex at v. For a non-empty set $\mathcal{S} \subseteq E$, the modulus of total convexity of φ on $\mathcal{S}$ is expressed as:

$$\nu_\varphi(\mathcal{S}, d) = \inf\{\nu_\varphi(v, d) : v \in \mathcal{S} \cap \text{int(dom } \varphi)\}.$$

The function φ is referred to as totally convex on bounded subsets if $\nu_g(\mathcal{S}, d)$ remains positive for all $d > 0$ and for any bounded, non-empty subset $\mathcal{S}$.

Lemma 4.1. *([35]) A uniformly Fréchet differentiable function $\varphi : E \to \mathbb{R}$ that is bounded on bounded subsets of E ensures that $\nabla\varphi$ is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Lemma 4.2. *[6] The function $\varphi : E \to \mathbb{R} \cup \{+\infty\}$ is totally convex on bounded subsets of E iff for any two sequences $\{v_m\}$ and $\{t_m\}$ in int(dom φ) and dom φ, respectively, such that the first one is bounded,*

$$\lim_{m\to\infty} D_\varphi(t_m, v_m) = 0 \Rightarrow \lim_{m\to\infty} \|t_m - v_m\| = 0.$$

Lemma 4.3. *[37] Let the function $\varphi : E \to \mathbb{R}$ be Gâteaux differentiable such that $\nabla\varphi^*$ is bounded on bounded subsets of dom φ^*. Let $v_0 \in E$ and $\{v_m\} \subset$ dom φ. If $D_\varphi(v_0, v_m)$ is bounded, then the sequence $\{v_m\}$ is also bounded.*

Theorem 4.1. *[53] Let $\varphi : E \to \mathbb{R}$ be a convex function which is bounded on bounded subsets of E. Then, the following are equivalent:*

(i) *φ is supercoercive and uniformly convex on bounded subsets of X;*

(ii) $\operatorname{dom}\varphi^* = E^*$, *$E^*$ is bounded on bounded subsets and uniformly smooth on bounded subsets of E^*;*

(iii) $\operatorname{dom}, \varphi^* = E^*$, *$\varphi^*$ is Fréchet differentiable and $\nabla\varphi^*$ is uniformly norm-to-norm continuous on bounded subsets of E^*.*

Theorem 4.2. *[7] Suppose that $\varphi : E \to \mathbb{R}\cup\{+\infty\}$ is a Legendre function. The function φ is uniformly convex on bounded subsets of E if and only if φ is totally convex on bounded subsets of E.*

4.3 The proposed Algorithm

In this section, by using tow parameters, the Bregman distance and non-monotonic adaptive step size criterion, we propose modified extragradient algorithm (Algorithm 7) for solving the EP (1.4) in E. Assume that the solution set of EP (1.4), represented by $EQ(B, \mathcal{S})$, is nonempty.

Remark 4.1. From Algorithm 7, we have

- If $\tau = \mu = \omega_m = 1$ and $\sigma_m = 0$, Algorithm 1 reduces to the Bregman extragradient algorithm introduced in [10]. The inclusion of the new parameters $(\tau,\ \mu,\ \omega_m,\ \sigma_m)$ significantly improves the numerical performance, yielding better results than the original formulation.

Algorithm 7 Bregman extragradient algorithm for EP in E

Initialization: Given $v_0 \in \mathcal{S}$, $\varrho_0 > 0$, $\zeta \in (0,1)$, $\mu \in \left(0, \dfrac{1}{2\zeta}\right)$ and $\tau \in \left[\mu, \frac{1}{\zeta}\right)$. Select the sequences $\{\sigma_m\} \subset [0,\infty)$ and $\{\omega_m\} \subset [1,\infty)$ such that $\sum_{m=1}^{\infty} \sigma_m < \infty$ and $\sum_{m=1}^{\infty} (\omega_m - 1) < \infty$.

Step 1: Compute

$$t_m = \arg\min_{t\in\mathcal{S}} \left(\tau \varrho_m B(v_m, t) + D_\varphi(t, v_m)\right) = J^{\varphi}_{\tau\varrho_m B(v_m,.)}\left(v_m\right).$$

If $t_m = v_m$, then stop, and t_m is a solution. Otherwise, go to next step.

Step 2: Compute

$$v_{m+1} = \arg\min_{t\in\mathcal{S}} \left(\mu \varrho_m B(t_m, t) + D_\varphi(t, v_m)\right) = J^{\varphi}_{\mu\varrho_m B(t_m,.)}\left(v_m\right),$$

where

$$\varrho_{m+1} = \begin{cases} \min\left\{\frac{\zeta(D_\varphi(t_m,v_m)+D_\varphi(v_{m+1},t_m))}{M}, \omega_m\varrho_m + \sigma_m\right\} & \text{if } M > 0, \\ \omega_m\varrho_m + \sigma_m & \text{otherwise,} \end{cases} \tag{4.4}$$

and $M = B(v_m, v_{m+1}) - B(v_m, t_m) - B(t_m, v_{m+1})$. Set $m := m+1$ and go to **Step 1**.

- Furthermore, by setting $\varphi(.) = \frac{1}{2}\|.\|_2^2$, where $\|.\|_2$ denotes the Euclidean norm, and assuming E is a real Hilbert space, Algorithm 7 can be viewed as an extension and enhancement of the method in [17].

- Although similar extragradient algorithms have been studied in Hadamard spaces, such as in [43], our work focuses on Banach spaces and employs Bregman distance with two parameters τ and μ, leading to different approaches in terms of convergence analysis and numerical behavior.

4.4 Convergence Analysis

In this section, we focus on the convergence analysis of the proposed algorithm, presenting both weak and strong convergence results.

4.4.1 Weak convergence

For the weak convergence theorem, consider the following assumptions.

Assumption 4.1. (C_1) φ is a supercoercive and Legendre function which is bounded;

(C_2) φ is uniformly Frechet differentiable;

(C_3) φ is totally convex on bounded subsets of E;

(C_4) $\nabla\varphi$ is weakly sequentially continuous.

Assumption 4.2. (H_1) The bifunction B is pseudomonotone on $\mathcal{S}$;

(H_2) B is Bregman Lipschitz type continuous on $\mathcal{H}$ with two positive constants L_1 and L_2, i.e.,

$$B(v,t) + B(t,w) \geq B(v,w) - L_1 D_\varphi(t,v) - L_2 D_\varphi(w,t), \ \ \forall t,v,w \in \mathcal{S};$$

(H_3) $B(v,.)$ is convex and subdifferentiable on $\mathcal{H}$ for each fixed $v \in \mathcal{S}$;

(H_4) for every sequence $\{v_m\} \subset \mathcal{S}$ and $v \in \mathcal{H}$ such that $v_m \rightharpoonup v$ and $\limsup_{m\to\infty} B(v_m,t) \geq 0$, for all $t \in \mathcal{S}$, then $B(v,t) \geq 0$.

Example 4.4. [19] Let $E = \ell^p := \{z = (z_1, z_2, \dots) : \left(\sum_{i=1}^{\infty} |z_i|^p < \infty\}\right)^{\frac{1}{p}}$ for $1 < p < \infty$ and $\varphi : E \to \mathbb{R}$ defined by $\varphi(x) = \|x\|^p$. Let

$$\mathcal{S} = \{z = (z_1, z_2, \dots) \in \ell^p : z_i \geq 0 \text{ and } \|z\| \leq p, \quad \forall i \in \mathbb{N}\}$$

and $B : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ defined by

$$B(v,t) = (p - \|v\|) \langle v, t - v\rangle \quad \text{for all } v,t \in \mathcal{S}.$$

Clearly, $EQ(B,\mathcal{S}) \neq \emptyset$, and B satisfies Lipschitz condition (H_2), pseudomonotonicity. Indeed, let $v,t \in \mathcal{S}$ be such that

$$B(v,t) = (p - \|v\|) \langle v, t - v\rangle \geq 0,$$

implies that $\langle v, t-v\rangle \geq 0$. Then

$$\begin{aligned} B(t,v) &= (p - \|t\|) \langle t, v - t\rangle \\ &\leq (p - \|t\|) \langle t, v - t\rangle + (p - \|t\|) \langle v, t - v\rangle \\ &\leq (p - \|t\|) \langle t, v - t\rangle - (p - \|t\|) \langle v, v - t\rangle \\ &\leq (\|t\| - p)\|v - t\|^2 \leq 0. \end{aligned}$$

Thus, B is pseudomonotone on $\mathcal{S}$. Also, by the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
B(v,s)-g(v,t)-g(t,s) =&(p-\|v\|)\langle v,s-v\rangle-(p-\|v\|)\langle v,t-v\rangle-(p-\|t\|)\langle t,s-t\rangle\\
=&(p-\|v\|)\langle v,s-t\rangle-(p-\|t\|)\langle t,s-t\rangle\\
=&\langle (p-\|v\|)v-(p-\|t\|)t,s-t\rangle\\
\leq&\|(p-\|v\|)v-(p-\|t\|)t\|\,\|s-t\|\\
=&\|p(v-t)-\|v\|(v-t)-(\|v\|-\|t\|)t\|\,\|s-t\|\\
\leq&[p\|v-t\|+\|v\|\,\|v-t\|+\|\|v\|-\|t\|\|\,\|t\|]\|\,\|s-t\|\\
\leq&[p\|v-t\|+p\|v-t\|+p\|v-t\|]\,\|s-t\|\\
=&3p\|v-t\|\,\|s-t\|\\
\leq&\frac{3p}{2}\|v-t\|+\frac{3p}{2}\|s-t\|
\end{aligned}$$

Thus, B satisfies (H_2) with $L_1=L_2=\frac{3p}{2}$.

We begin by proving the following necessary results:

Lemma 4.4. *The sequence $\{\varrho_m\}$ created by (4.4) is well defined and $\lim\limits_{m\to+\infty}\varrho_m$ exists.*

Proof. Since B fulfills (H$_2$), it follows that

$$\begin{aligned}
\frac{\zeta\left(D_\varphi(t_m,v_m)+D_\varphi(v_{m+1},t_m)\right)}{\left(B(v_m,v_{m+1})-B(v_m,t_m)-B(t_m,v_{m+1})\right)} &\geq \frac{\zeta\left(D_\varphi(t_m,v_m)+D_\varphi(v_{m+1},t_m)\right)}{\left(L_1D_\varphi(t_m,v_m)+L_2D_\varphi(v_{m+1},t_m)\right)}\\
&\geq \frac{\zeta}{\max\{L_1,L_2\}}.
\end{aligned}$$

Thus, in addition to the expression (4.4), gives $\varrho_{m+1}\geq\min\left\{\frac{\zeta}{\max\{L_1,L_2\}},\varrho_m\right\}$. Moreover $\varrho_m\geq\min\left\{\frac{\zeta}{\max\{L_1,L_2\}},\varrho_1\right\}$. In contrast, it becomes clear from expression (4.4) that

$$\varrho_{m+1}\leq\omega_m\varrho_m+\sigma_m,\forall m\geq 1.$$

It follows from conditions on $\{\omega_m\}$, $\{\sigma_m\}$ and Lemma 1.4 that $\lim\limits_{m\to+\infty}\varrho_m$ exists. Since $\min\left\{\frac{\zeta}{\max\{L_1,L_2\}},\varrho_1\right\}$ is the lower boundary of $\{\varrho_m\}$, then $\lim\limits_{m\to+\infty}\varrho_m:=\varrho>0$.

□

Remark 4.2. From the definition of $\{t_m\}$ and Lemma 1.1, we have

$$0\in\partial\left(\tau\varrho_mB(v_m,t)+D_\varphi(t,v_m)\right)(t_m)+N_{\mathcal{S}}(t_m).$$

This implies that there exists $w\in\partial B(v_m,t_m)$ and $\eta\in N_{\mathcal{S}}(t_m)$, such that

$$\tau\varrho_mw+\nabla\varphi(t_m)-\nabla\varphi(v_m)+\eta=0.$$

Thus, we have from the definition of $N_{\mathcal{S}}$

$$\begin{aligned}
&< \nabla\varphi(v_m) - \nabla\varphi(t_m), t - t_m >= \tau\varrho_m < w, t - t_m > + < \eta, t - t_m >, \\
&\leq \tau\varrho_m < w, t - t_m >, \ \forall t \in \mathcal{S}.
\end{aligned}$$

Since, $w \in \partial B(v_m, t_m)$, we have

$$< w, t - t_m > \leq B(v_m, t) - B(v_m, t_m), \ \forall t \in \mathcal{S}.$$

From the last two inequalities, we get

$$< \nabla\varphi(v_m) - \nabla\varphi(t_m), t - t_m > \leq \tau\varrho_m \left(B(v_m, t) - B(v_m, t_m)\right), \ \forall t \in \mathcal{S}. \tag{4.5}$$

If $v_m = t_m$, then from (4.5) and $\tau,\ \varrho_m > 0$, we obtain $B(t_m, t) \geq 0$, for all $t \in \mathcal{S}$. Thus $t_m \in EQ(B, \mathcal{S})$.

Lemma 4.5. *Let $\{v_m\}$ and $\{t_m\}$ be the two sequences generated by Algorithm 7. Fix $v^* \in EQ(B, \mathcal{S})$. Then*

$$D_\varphi(v^*, v_{m+1}) \leq D_\varphi(v^*, v_m) - \mu\left(\frac{1}{\tau} - \frac{\zeta\varrho_m}{\varrho_{m+1}}\right) D_\varphi(t_m, v_m) - \mu\left(\frac{1}{\tau} - \frac{\zeta\varrho_m}{\varrho_{m+1}}\right) D_\varphi(v_{m+1}, t_m).$$

Proof. According to the definition of $\{v_{m+1}\}$ and Remark 4.2, one has

$$< \nabla\varphi(v_m) - \nabla\varphi(v_{m+1}), t - v_{m+1} > \leq \mu\varrho_m \left(B(t_m, t) - B(t_m, v_{m+1})\right), \forall t \in \mathcal{S}. \tag{4.6}$$

In particular, substituting $t = v_{m+1}$ in (4.5), we get

$$< \nabla\varphi(v_m) - \nabla\varphi(t_m), v_{m+1} - t_m > \leq \tau\varrho_m \left(B(v_m, v_{m+1}) - B(v_m, t_m)\right), \tag{4.7}$$

Addining (4.6) with (4.7)

$$\begin{aligned}
\tau\mu\varrho_m \left(B(v_m, v_{m+1}) - B(v_m, t_m) - B(t_m, v_{m+1})\right) \ \geq \ & \mu\left(< \nabla\varphi(v_m) - \nabla\varphi(t_m), v_{m+1} - t_m >\right) \\
& + \tau\left(< \nabla\varphi(v_m) - \nabla\varphi(v_{m+1}), t - v_{m+1} >\right) \\
& + \tau\mu\varrho_m B(t_m, t) \quad (\forall t \in \mathcal{S})\,. \qquad (4.8)
\end{aligned}$$

From the definition of ϱ_m, we have

$$\left(B(v_m, v_{m+1}) - B(v_m, t_m) - B(t_m, v_{m+1})\right) \leq \frac{\zeta}{\varrho_{m+1}}\left(D_\varphi(t_m, v_m) + D_\varphi(v_{m+1}, t_m)\right). \tag{4.9}$$

By Bregman three point identity (4.1), it follows that

$$< \nabla\varphi(v_m) - \nabla\varphi(t_m), v_{m+1} - t_m >= D_\varphi(v_{m+1}, t_m) + D_\varphi(t_m, v_m) - D_\varphi(v_{m+1}, v_m), \quad (4.10)$$

and

$$< \nabla\varphi(v_m) - \nabla\varphi(v_{m+1}), t - v_{m+1} >= D_\varphi(t, v_{m+1}) + D_\varphi(v_{m+1}, v_m) - D_\varphi(t, v_m). \quad (4.11)$$

Appling (4.9), (4.10) and (4.11) into (4.8), we obtain

$$\begin{aligned}\frac{\tau\mu\zeta\varrho_m}{\varrho_{m+1}}\left(D_\varphi(t_m, v_m) + D_\varphi(v_{m+1}, t_m)\right) \geq\ & \mu\left(D_\varphi(v_{m+1}, t_m) + D_\varphi(t_m, v_m) - D_\varphi(v_{m+1}, v_m)\right) \\ & +\tau\left(D_\varphi(t, v_{m+1}) + D_\varphi(v_{m+1}, v_m) - D_\varphi(t, v_m)\right) \\ & +\tau\mu\varrho_m B(t_m, t) \quad (\forall t \in \mathcal{S}).\end{aligned}$$

Then

$$\begin{aligned}D_\varphi(t, v_{m+1}) \leq\ & D_\varphi(t, v_m) - D_\varphi(v_{m+1}, v_m) + \frac{\mu\zeta\varrho_m}{\varrho_{m+1}}\left(D_\varphi(t_m, v_m) + D_\varphi(v_{m+1}, t_m)\right) \\ & -\frac{\mu}{\tau}\left(D_\varphi(v_{m+1}, t_m) + D_\varphi(t_m, v_m) - D_\varphi(v_{m+1}, v_m)\right) \\ & +\mu\varrho_m B(t_m, t) \quad (\forall t \in \mathcal{S}). \quad (4.12)\end{aligned}$$

Therefore, it follows from relation (4.12) that

$$\begin{aligned}D_\varphi(t, v_{m+1}) \leq D_\varphi(t, v_m) - \left(\frac{\mu}{\tau} - \frac{\mu\zeta\varrho_m}{\varrho_{m+1}}\right) D_\varphi(t_m, v_m) - \left(\frac{\mu}{\tau} - \frac{\mu\zeta\varrho_m}{\varrho_{m+1}}\right) D_\varphi(v_{m+1}, t_m) \\ - \left(1 - \frac{\mu}{\tau}\right) D_\varphi(v_{m+1}, v_m) + \mu\varrho_m B(t_m, t) \quad (\forall t \in \mathcal{S}).\end{aligned}$$

Noting that $\frac{\mu}{\tau} \in (0, 1]$ then, we have

$$\begin{aligned}D_\varphi(t, v_{m+1}) \leq\ & D_\varphi(t, v_m) - \left(\frac{\mu}{\tau} - \frac{\mu\zeta\varrho_m}{\varrho_{m+1}}\right) D_\varphi(t_m, v_m) \\ & - \left(\frac{\mu}{\tau} - \frac{\mu\zeta\varrho_m}{\varrho_{m+1}}\right) D_\varphi(v_{m+1}, t_m) + \mu\varrho_m B(t_m, t) \quad (\forall t \in \mathcal{S}). \quad (4.13)\end{aligned}$$

Let $t = v^* \in EQ(B, \mathcal{S})$. Therefore, from the pseudo monotonicity of B, we have $B(v^*, t_m) \geq 0$. Thus, $B(t_m, v^*) \leq 0$. Hence from (4.13), we get

$$\begin{aligned}D_\varphi(v^*, v_{m+1}) \leq\ & D_\varphi(v^*, v_m) - \mu\left(\frac{1}{\tau} - \frac{\zeta\varrho_m}{\varrho_{m+1}}\right) D_\varphi(t_m, v_m) \\ & -\mu\left(\frac{1}{\tau} - \frac{\zeta\varrho_m}{\varrho_{m+1}}\right) D_\varphi(v_{m+1}, t_m). \quad (4.14)\end{aligned}$$

□

Lemma 4.6. *The sequences $\{v_m\}$ and $\{t_m\}$ generated by Algorithm 7 are bounded.*

Proof. From Lemma 4.4, one knows that $\lim_{m\to+\infty} \frac{\varrho_m}{\varrho_{m+1}} = 1$. This together with the assumptions on the parameters $\zeta \in (0,1)$, $\mu \in \left(0, \frac{1}{2\zeta}\right)$ and $\tau \in \left[\mu, \frac{1}{\zeta}\right)$ yields that

$$\lim_{m\to\infty} \left(\frac{\mu}{\tau} - \frac{\mu\zeta\varrho_m}{\varrho_{m+1}}\right) = \mu\left(\frac{1}{\tau} - \zeta\right) > 0.$$

Let $\epsilon \in \left(0, \mu\left(\frac{1}{\tau} - \zeta\right)\right)$. Consequently, there exists $m_0 \in \mathbb{N}$ satisfying

$$\left(\frac{\mu}{\tau} - \frac{\mu\zeta\varrho_m}{\varrho_{m+1}}\right) > \epsilon > 0, \ \forall m \geq m_0. \tag{4.15}$$

From Lemma 4.5, we have

$$D_\varphi(v^*, v_{m+1}) \leq D_\varphi(v^*, v_m) - \epsilon\left(D_\varphi(t_m, v_m) + D_\varphi(v_{m+1}, t_m)\right), \tag{4.16}$$

which take the form

$$a_{m+1} \leq a_m - b_m, \tag{4.17}$$

where

$$\begin{cases} a_m = D_\varphi(v^*, v_m), \\ b_m = \epsilon\left(D_\varphi(t_m, v_m) + D_\varphi(v_{m+1}, t_m)\right). \end{cases}$$

Thus, from Lemma 1.2, it follows that the limit of a_m and $\lim_{m\to\infty} b_m = 0$ for all $m \geq 0$. Hence, from the definition of b_m, we have

$$\lim_{m\to\infty} D_\varphi(t_m, v_m) = \lim_{m\to\infty} D_\varphi(v_{m+1}, t_m) = 0. \tag{4.18}$$

From Lemma 4.2 , we conclude that

$$\lim_{m\to+\infty} \|t_m - v_m\| = \lim_{m\to+\infty} \|v_{m+1} - t_m\| = 0. \tag{4.19}$$

Consequntly,

$$\lim_{m\to+\infty} \|v_{m+1} - v_m\| = 0.$$

Therefore, from Lemma 4.3 we have

$$\lim_{m\to+\infty} \|\nabla\varphi(v_{m+1}) - \nabla\varphi(v_m)\| = 0. \tag{4.20}$$

From Theorems 4.1 and 4.2, φ^* is bounded on bounded subsets of E^* and hence $\nabla\varphi^*$ is also bounded on bounded subsets of E^*. From this, (4.16) and Lemma 4.3, the sequence $\{v_n\}$ is bounded. As a result, $\{t_m\}$ is also bounded. □

Now, we prove that the sequences $\{v_m\}$ and $\{t_m\}$ generated by Algorithm 7 converge weakly to an element $v^* \in EQ(B, \mathcal{S})$.

Theorem 4.3. *Let Assumptions 4.1–4.2 be satisfied. Then for each $v^* \in EQ(B, \mathcal{S}) \neq \emptyset$, the sequences $\{v_m\}$ and $\{t_m\}$ generated by Algorithm 7, converge weakly to v^*.*

Proof. To show that $\{v_m\}$ converges to a solution of EP (1.4), it is left to prove that any cluster point of $\{v_n\}$ belongs to $EQ(B, \mathcal{S})$. Let $\bar{v}$ be a cluster point of $\{v_m\}$. Hence $\{v_m\}$ is bounded, there exists a subsequence $\{v_{m_k}\}$ of $\{v_m\}$ such that $v_{m_k} \rightharpoonup \bar{v}$ as $k \to \infty$. From (4.19), we also have $t_{m_k} \rightharpoonup \bar{v}$. Next, we show that $\bar{v} \in EQ(B, \mathcal{S})$. Letting $m = m_k$ in (4.13) and using $\left(\frac{\mu}{\tau} - \frac{\mu\zeta\varrho_{m_k}}{\varrho_{m_k+1}}\right) > 0$, we have

$$\tau\mu\varrho_{m_k} B(t_{m_k}, t) \geq D_\varphi(t, v_{m_k+1}) - D_\varphi(t, v_{m_k}). \tag{4.21}$$

Passing to the limit in (4.21), we obtain

$$\begin{aligned}
\limsup_{k\to\infty} \tau\mu\varrho_{m_k} B(t_{m_k}, t) &\geq \limsup_{k\to\infty} \left(D_\varphi(t, v_{m_k+1}) - D_\varphi(t, v_{m_k})\right), \\
&\geq \limsup_{k\to\infty} \left(D_\varphi(t, v_{m_k+1}) - D_\varphi(t, v_{m_k}) - D_\varphi(v_{m_k}, v_{m_k+1})\right), \\
&= \limsup_{k\to\infty} < \nabla\varphi(v_{m_k}) - \nabla\varphi(v_{m_k+1}), t - v_{m_k} > .
\end{aligned}$$

It follows from (4.20), boundedness of $\{v_m\}$, the parameters $\tau, \mu, \varrho_{m_k} \geq 0$ and Condition (H_4) that

$$0 \leq \limsup_{k\to\infty} B(t_{m_k}, t) \leq B(\bar{v}, t), \quad (\forall t \in \mathcal{S}). \tag{4.22}$$

Then $\bar{v} \in EQ(B, \mathcal{S})$. By utilizing equations (4.4) and (4.15), it follows that v_m is Bregman monotone with respect to $EQ(B, \mathcal{S})$. Consequently, the desired result is obtained by applying [10, Lemmas 10-12]. □

4.4.2 Strong convergence

Next, we examine the strong convergence of the Algorithm 7, which guarantees that the iterates converge in a stronger sense than weak convergence. The specific assumption required for strong convergence will be outlined in the following.

Assumption 4.3. Assume the following conditions

(H'$_1$) The bifunction B is $\gamma-$strongly pseudomonotone on $\mathcal{S}$;

(H$_2$) B is Bregman Lipschitz type continuous on $\mathcal{H}$;

(H$_3$) $B(v, .)$ is convex and subdifferentiable on $\mathcal{H}$ for each fixed $v \in \mathcal{S}$;

(H_4) for all bounded sequences v_m and t_m in $\mathcal{S}$,

$$\|v_m - t_m\| \to 0 \Rightarrow B(v_m, t_m) \to 0.$$

Theorem 4.4. *Let Assumption 4.3 and (C_1, C_2, C_3) in Assumptions 4.1 be satisfied. Then. for each $v^* \in EQ(B, \mathcal{S}) \neq \emptyset$, the sequences $\{v_m\}$ and $\{t_m\}$ generated by Algorithm 7, converges strongly to v^*.*

The proof of strong convergence for Algorithm 7 is based on the same reasoning as in [10].

Proof. As shown in Theorem 4.3, all cluster points of the sequence $\{v_m\}$ are elements of $EQ(B, \mathcal{S})$. Now, consider arbitrary subsequences $\{v_{m_k}\}$ and $\{v_{m_n}\}$ of $\{v_m\}$ that converge strongly to p and q, respectively. From (4.2), it follows that

$$< p - q, \nabla\varphi(v_{m_k}) - \nabla\varphi(v_{m_n}) >= D_\varphi(p, v_{m_k}) - D_\varphi(q, v_{m_n}) - D_\varphi(p, v_{m_n}) - D_\varphi(q, v_{m_k}).$$

According to (4.16), $\lim_{m\to\infty} D_\varphi(p, v_m)$ and $\lim_{m\to\infty} D_\varphi(q, v_m)$ exist. By utilizing this fact, Lemma 4.1, and letting $m \to \infty$, it follows that $p = q$. Hence, the sequence $\{v_m\}$ converges strongly to a point in $EQ(B, \mathcal{S})$. Next, we show that if $v_{m_k} \rightharpoonup \bar{v}$, then $v_{m_k} \to \bar{v}$. Assume that $v_{m_k} \rightharpoonup \bar{v}$. Therefore, by (4.19), $t_{m_k} \rightharpoonup \bar{v}$. Substituting $t = \bar{v}$ into (4.6), we get

$$\begin{aligned} 0 &\leq \mu\varrho_{m_k}\Big(B(t_{m_k}, \bar{v}) - B(t_{m_k}, v_{m_k})\Big) - \langle \nabla\varphi(v_{m_k}) - \nabla\varphi(v_{m_k+1}), v_{m_k+1} - \bar{v}\rangle, \\ &= \mu\varrho_{m_k}\Big(B(t_{m_k}, \bar{v}) - B(t_{m_k}, v_{m_k+1})\Big) + \langle \nabla\varphi(v_{m_k}) - \nabla\varphi(v_{m_k+1}), v_{m_k+1} - \bar{v}\rangle, \\ &\leq \mu\varrho_{m_k}\Big(B(t_{m_k}, \bar{v}) - B(t_{m_k}, v_{m_k+1})\Big) + \|\nabla\varphi(v_{m_k}) - \nabla\varphi(v_{m_k+1})\| \, \|v_{m_k+1} - \bar{v}\|. \end{aligned}$$

Using (4.19), (4.20), Lemma 4.4, condition H_4, and the boundedness of $\{v_m\}$, it follows that

$$\liminf_{k\to\infty} B(t_{m_k}, \bar{v}) \geq 0.$$

Given that $B(t_{m_k}, \bar{v}) \geq 0$, there exists a constant γ such that $B(t_{m_k}, \bar{v}) \leq -\gamma\|t_{m_k} - \bar{v}\|^2$. Combining this with (18), we conclude that

$$0 \leq \liminf_{k\to\infty} B(t_{m_k}, \bar{v}) \leq \liminf_{k\to\infty}\left(-\gamma\|t_{m_k} - \bar{v}\|^2\right) \leq -\gamma\left(\limsup_{k\to\infty}\|t_{m_k} - v\|^2\right) \leq 0.$$

Consequently, $t_{m_k} \to \bar{v}$, and therefore, $v_{m_k} \to \bar{v}$. □

4.5 Numerical Illustrations

The numerical results are presented in this section to prove the performance of our proposed algorithm. All the programs were implemented in MATLAB (R2023a) on a Intel(R) Core(TM) i5-8265U CPU @ 1.60 GHz 1.80 GHz with RAM 8.00 GB.

Example 4.5. Consider the bifunction $B : \mathcal{S}\times\mathcal{S} \to \mathbb{R}$ defined in the context of the Nash-Cournot equilibrium model as follows:

$$B(v,t) = \langle Pv + Qt + q, t - v\rangle, \quad \forall\, v, t \in \mathcal{S},$$

where $\mathcal{S} \subset \mathbb{R}^n$ is the feasible set, $q \in \mathbb{R}^n$ is a given vector, P and $Q \in \mathbb{R}^{n\times n}$ are matrices, where Q is symmetric positive semidefinite, and $Q - P$ is symmetric negative semidefinite, ensuring that f is monotone. The two matrices P, Q are generated randomly (Generate two random orthogonal matrices O_1 and O_2 using the `RandOrthMat` function. Create diagonal matrices A_1 and A_2 with values within $[0, 2]$ and $[-2, 0]$, respectively. Define $B_1 = O_1A_1O_1^T$ (positive semi-definite) and $B_2 = O_2A_2O_2^T$ (negative semi-definite). Set $Q = B_1 + B_1^T$, $T = B_2 + B_2^T$, and $P = Q-T$. Randomly generate the vector q with elements in the range $[-1, 1]$). We use the same stopping rule $D_m = \|t_m - v_m\|^2 \leq 10^{-6}$. In the numerical results presented in the following tables, 'Iter.' represents the number of iterations, while 'CPU(s)' denotes the execution time in seconds. The set $\mathcal{S}$ is given by:

$$\mathcal{S} = \{v \in \mathbb{R}^n : -5 \leq v_i \leq 5, \quad i = 1, 2, \ldots, n\}.$$

In all experiments, we selected the parameters for Algorithm 7 as follows: $\varrho_0 = 0.1, \zeta = 0.1$, $\mu = 4.75$, $\tau = 4.75$, $\omega_m = \dfrac{1}{20\,(m+1)^{1.1}}$ and $\sigma_m = \dfrac{1}{(m+1)^3}$. The performance of Algorithm 7 was initially evaluated for different Bregman distances and different values n $(n = 60, 120, 180, 240)$. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be defined by

i $\varphi(v) := \sum\limits_{i=1}^{n} v_i \log(v_i)\,.$

ii $\varphi(v) := \dfrac{1}{2}\|v\|^2\,.$

iii $\varphi(v) := -\sum\limits_{i=1}^{n} \log(v_i).$

Additionally, we establish that the corresponding Bregman distances can be expressed as

i $D_\varphi(v,t) := \sum\limits_{i=1}^{n} \left(v_i \log\left(\frac{v_i}{t_i}\right) + t_i - v_i\right)$, which is called the Kullback-Leibler distance(shortly denoted by KLD);

ii $D_\varphi(v,t) := \frac{1}{2}\|v - t\|^2$, which is called the squared Euclidean distance(denoted by SED);

iii $D_\varphi(v,t) := \sum\limits_{i=1}^{n} \left(\log\left(\frac{v_i}{t_i}\right) + \frac{v_i}{t_i} - 1\right)$, which is called Itakura-Saito distance (ISD).

The numerical results shown in Fig. 4.1 and Table 4.1 indicate that the proposed algorithm achieves superior performance when the Bregman distance is chosen as the Kullback-Leibler distance (KLD).

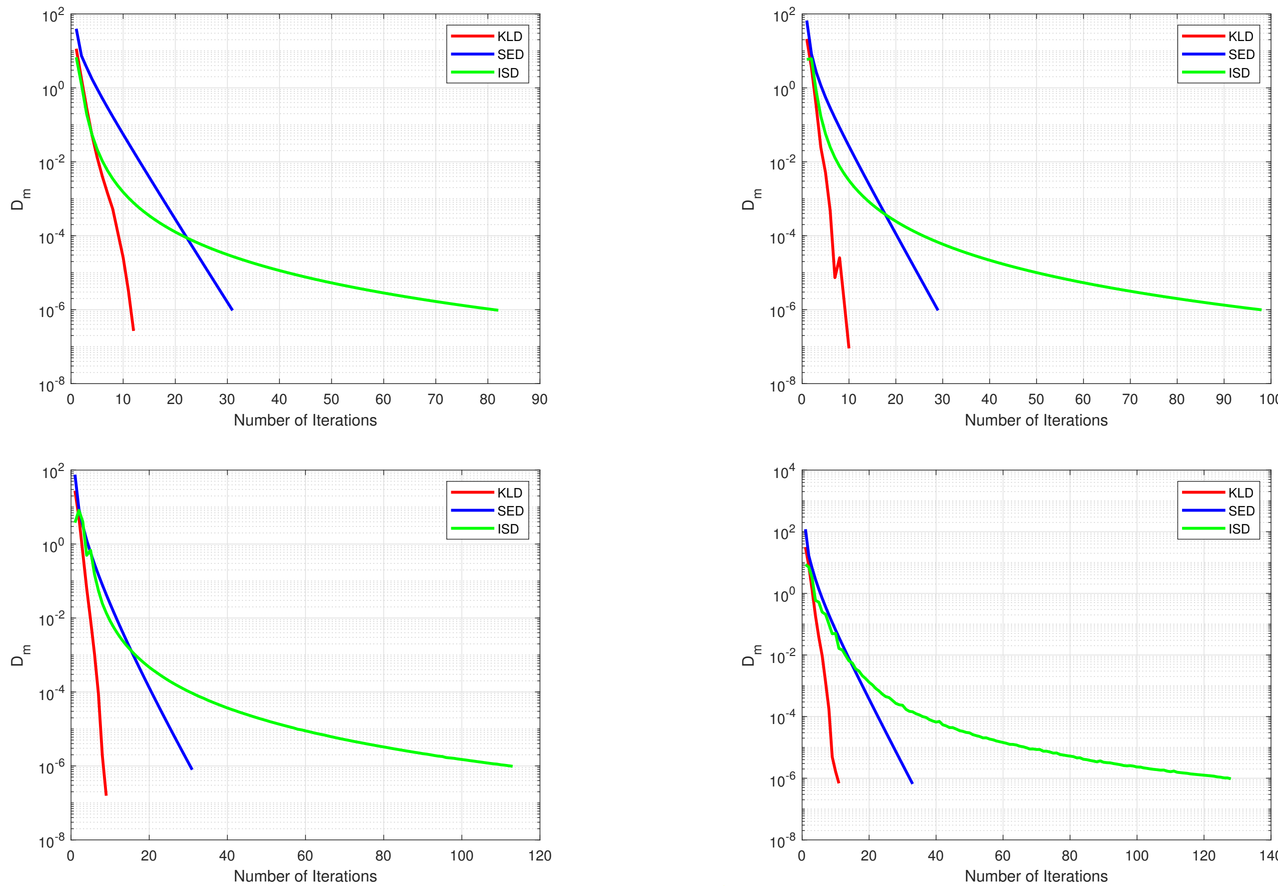

Figure 4.1: Example 4.5, Top Left: n = 60; Top Right: n= 120, Bottom Left: n = 180 ; Bottom Right: n = 240.

Table 4.1: Comparison of iterations and CPU time for different dimensions.

n	**KLD**		**SED**		**ISD**	
	Iter.	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)
60	12	3.78	31	1.79	82	28.01
120	10	2.77	29	2.13	98	29.02
180	9	3.01	31	4.36	113	45.79
240	11	4.87	33	7.73	128	70.71

Finally, the Algorithm 7, was compared with Algorithm 6, the explicit extragradient Algorithm suggested by Hieu et al. [17] (shortly, EEG Alg) and the Bregman explicit extragradient Algorithm proposed by Eskandani et al. [10] (shortly, BEEG Alg) to assess its efficiency and effectiveness. The control parameters of all algorithms are choose as follows:

- Algorithm 7: Similar parameters as mentioned above with $\varphi(v) = -\sum_{i=1}^{n} v_i \log(v_i)$.
- Algorithm 6 : $\varrho_0 = 0.1$, $\zeta = 0.1$, $\mu = 0.99$, $\gamma = 0.2$, $\varepsilon_m = 2$, $\omega_m = \dfrac{1}{20(m+1)^{1.1}}$, $\sigma_m = \dfrac{1}{(m+1)^3}$.
- EEG Alg : $\varrho_0 = 0.1$, $\zeta = 0.1$.
- BEEG Alg : $\varrho_0 = 0.1$, $\zeta = 0.1$ and $\varphi(v) = -\sum_{i=1}^{n} v_i \log(v_i)$.

We test the algorithms for different values of n $(n = 50, 100, 150, 200)$. The numerical results for all algorithms are presented in Fig. 4.2 and Table 4.2. It can be observed that our algorithm (Algorithm 7) outperforms then (Algorithm 6), EEG Alg and BEEG Alg in terms of the number of iterations (Iter.) and execution time in seconds (CPU(s)), while achieving the same tolerance.

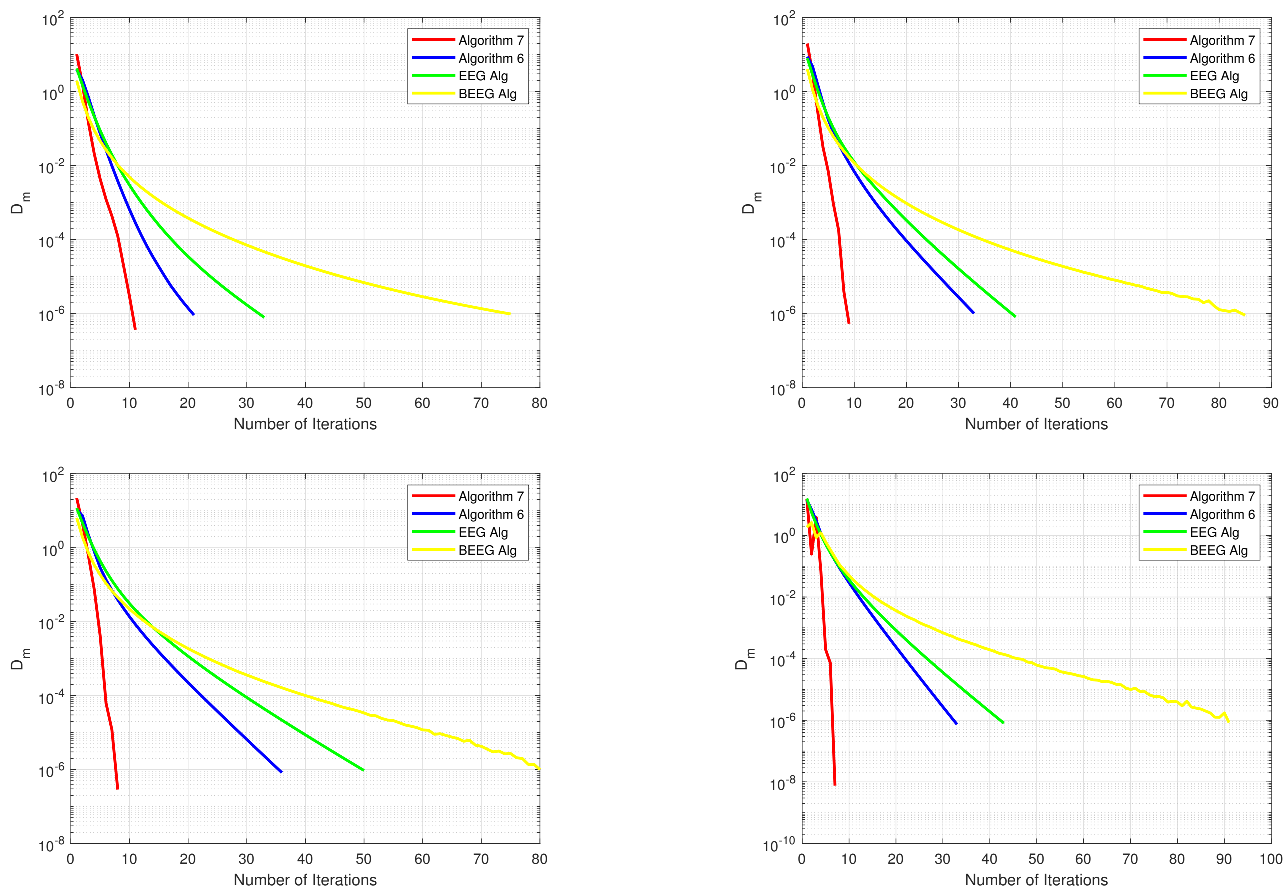

Figure 4.2: Example 4.5 , Top Left: n = 50; Top Right: n= 100, Bottom Left: n = 150; Bottom Right: n = 200.

Table 4.2: Comparison of iterations and CPU time for different dim.

n	Algorithm 7		Algorithm 6		EEG Alg		BEEG Alg	
	Iter.	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)	Iter.	CPU(s)
50	11	2.45	21	0.42	33	0.57	75	13 .92
100	9	2.74	33	1.66	41	1.80	85	26.09
150	8	3.13	36	3.60	50	4.57	80	31.83
200	7	3.19	33	6.77	43	7.03	91	43.72

Remark 4.3. Based on our numerical experiments, we have the following observation:

- √ Algorithm 7 shows the fastest convergence, requiring fewer iterations to reach the desired accuracy.
- √ Algorithm 6 performs well but is slightly slower than Algorithm 7.
- √ EEG Alg converges more slowly compared to Algorithm 6 and Algorithm 7.
- √ BEEG Alg proves the slowest convergence, requiring significantly more iterations.
- √ As n increases, all algorithms exhibit similar behavior, but the number of iterations required increases.
- √ The proposed Algorithm 7 proves to be the most efficient in terms of iteration count and execution time.

Conclusion

This thesis presents a comprehensive study on the development, analysis, and implementation of advanced proximal-type algorithms for solving equilibrium problems in both Hilbert and Banach spaces. The proposed algorithms integrate key techniques such as subgradient and extragradient methods, enhanced with inertial terms to accelerate convergence, and incorporate Bregman distances to improve adaptability in non-Euclidean geometries.

The primary contribution of this research lies in the formulation of new iterative schemes that significantly improve upon existing methods in terms of convergence speed and computational efficiency. By establishing both weak and strong convergence results under relatively mild conditions, the thesis provides rigorous theoretical guarantees that support the reliability and robustness of the proposed algorithms. These results not only strengthen the mathematical foundation of equilibrium problem but also extend the applicability of these algorithms to a broader class of variational inequality problem.

Furthermore, extensive numerical experiments were conducted to verify the practical effectiveness of the proposed methods. These experiments demonstrate that the new algorithms consistently outperform classical techniques in terms of iteration count, execution time, and solution accuracy. In particular, the incorporation of Bregman distance allows for better adaptation to the underlying geometry of the problem space, especially in Banach spaces, where traditional Euclidean-based methods are often less effective. In summary, the thesis offers several key contributions:

- The design of novel, efficient algorithms that generalize and improve upon existing methods in equilibrium problem solving.
- Theoretical analysis ensuring strong and weak convergence under mild assumptions.
- Demonstrated computational advantages through rigorous numerical simulations.
- A unified approach that enhances the understanding of the relationship between equilibrium problems and broader classes of optimization models.

These contributions collectively advance the current state of research in this field and open new directions for further investigation, such as extending these algorithms to large-scale equilibrium problems, as well as exploring applications in other fields

Bibliography

[1] Anh, P.N. and An, L.T.H. *The subgradient extragradient method extended to equilibrium problems*, Optimization, **64** (2012), 225–248. https://doi.org/10.1007/s11590-019-01474-1

[2] Bauschke, H. and Combettes, P. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer-Verlag, 2011.

[3] Blum, E. and Oettli, W. *From optimization and variational inequality to equilibrium problems*, Math. Stud., **63** (1994), 127–149.

[4] Bregman, L. M. *A relaxation method for finding the common point of convex sets and its application to the solution of problems in convex programming*, USSR Comput. Math. Math. Phys., **7** (1967), 200–217. https://doi.org/10.1016/0041-5553(67)90040-7

[5] Browder, F. and Petryshyn, W. *Construction of fixed points of nonlinear mappings in Hilbert space*, Journal of Mathematical Analysis and Applications, **20** (1967), 197–228. https://doi.org/10.1016/0022-247X(67)90085-6

[6] Butnariu, D. and Iusem, A.N. *Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization*, Kluwer Academic Publishers, Dordrecht, 2000.

[7] Butnariu, D. and Iusem, A.N.; Zălinescu, C. *On uniform convexity, total convexity and convergence of the proximal point and outer Bregman projection algorithms in Banach Spaces*, J. Convex Anal., **10** (2003), 35–61.

[8] Dadashi, V., Iyiola, O. S. and Shehu, Y. *The subgradient extragradient method for pseudomonotone equilibrium problems*, Optimization, (2020). https://doi.org/10.1080/02331934.2020.1778017

[9] Dadashi, V. and Khatibzadeh, H. *On the weak and strong convergence of the proximal point algorithm in reflexive Banach spaces*, Optimization, **66**(9) (2017), 1487–1494. https://doi.org/10.1080/02331934.2017.1337764

[10] Eskandani, G. Z.; Raeisi, M. An Iterative Explicit Algorithm for Solving Equilibrium Problems in Banach Spaces. *Bull. Malays. Math. Sci. Soc.* **44** (2021), 4299–4321. https://doi.org/10.1007/s40840-021-01168-x

[11] Fan, K. A minimax inequality and applications. In: O. Shisha, (ed.) *Inequalities III*, pp. 103–113. Academic Press, San Diego, 1972.

[12] Flåm, S. D. and Antipin, A. S. Equilibrium programming using proximal-like algorithms. *Math. Program.* **78** (1996), 29–41. https://doi.org/10.1007/BF02614504

[13] Hieu, D. V. Halpern subgradient extragradient method extended to equilibrium problems. *Revista de la Real Academia de Ciencias Exactas, FÃsicas y Naturales. Serie A. MatemÃ¡ticas.* **111** (2016), 823–840. https://doi.org/10.1007/s13398-016-0328-9.

[14] Hieu, D. V. An inertial-like proximal algorithm for equilibrium problems. *Math. Methods Oper. Res.* **88** (2018), 399–415. https://doi.org/10.1007/s00186-018-0640-6

[15] Hieu, D. V.; Cho, Y. J. and Xiao, Y. Modified extragradient algorithms for solving equilibrium problems. *Optimization* **67** (2018), 2003–2029. https://doi.org/10.1080/02331934.2018.1505886

[16] Hieu, D. V. and Gibali, A. Strong convergence of inertial algorithms for solving equilibrium problems. *Optim. Lett.* **14** (2020), 1817–1843. https://doi.org/10.1007/s11590-019-01479-w

[17] Hieu, D. V., Quy, P. K. and Vy, L. V. Explicit iterative algorithms for solving equilibrium problems. *Calcolo* (2019). https://doi.org/10.1007/s10092-019-0308-5

[18] Iusem, A. N., Kassay, G. and Sosa, W. On certain conditions for the existence of solutions of equilibrium problems. *Mathematical Programming.* **116** (2009), 259–273. https://doi.org/10.1007/s10107-007-0125-5

[19] Jolaoso, L.O., Okeke, C.C. and Shehu, Y. Extragradient Algorithm for Solving Pseudomonotone Equilibrium Problem with Bregman Distance in Reflexive Banach Spaces. *Netw Spat Econ* **21** (2021), 873–903. https://doi.org/10.1007/s11067-021-09554-5

[20] Konnov, I. V. On systems of variational inequalities. *Russ. Math. C/C Izv. Vyss. Uchebnye Zaved. Mat.* **41** (1997), 77–86.

[21] Konnov, I. Combined Relaxation Methods for Variational Inequalities. *Lecture Notes in Economics and Mathematical Systems*, Springer-Verlag, Berlin, **495** (2001).

[22] Konnov, I. Equilibrium Models and Variational Inequalities. *Mathematics in Science and Engineering*, Elsevier B. V., Amsterdam, **210** (2007).

[23] Lin, L. J. and Park, S. On some generalized quasi-equilibrium problems. *J. Math. Anal. Appl.* **224** (1998), 167–181. https://doi.org/10.1006/jmaa.1998.5964

[24] Lyashko, S. I. and Semenov, V. V. A new two-step proximal algorithm of solving the problem of equilibrium programming. *Optimization and Its Applications in Control and Data Sciences.* **115** (2016), 315–325. https://doi.org/10.1007/978-3-319-42056-1_10.

[25] Maddox, I. J. Elements of Functional Analysis. Cambridge University Press, 1970.

[26] Mastroeni, G. On auxiliary principle for equilibrium problems. *Equilibrium Problems and Variational Models*, Daniele, P., et al. (eds.), 289–298, Kluwer Academic Publishers, Dordrecht, 2003.

[27] Mastroeni, G. Gap functions for equilibrium problems. *J. Global Optim.* **27** (2003), 411–426. https://doi.org/10.1023/A:1026050425030

[28] Moudafi, A. Proximal point algorithm extended to equilibrium problems. *J. Nat. Geom.* **15** (1999), 91–100.

[29] Muu, L. D. and Oettli, W. Convergence of an adaptive penalty scheme for finding constrained equilibria. *Nonlinear Anal.* **18** (1992), 1159–1166. https://doi.org/10.1016/0362-546X(92)90159-C

[30] Muu, L. D. and Quoc, T. D. Regularization algorithms for solving monotone Ky Fan inequalities with application to a Nash-Cournot equilibrium model. *J. Optim. Theory Appl.* **142** (2009), 185–204.

[31] Opial, Z. Weak convergence of the sequence of successive approximations for nonexpansive mappings. *Bull. Amer. Math. Soc.* **73** (1967), 591–598.

[32] Osilike, M. O. and Aniagbosor, S. C. Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings. *Math. Comput. Model.* **32** (2000), 1181–1191. https://doi.org/10.1016/S0895-7177(00)00199-0

[33] Panyanak, B., Khunpanuk, C.; Pholasa, N. and Pakkaranang, N. Dynamical inertial extragradient techniques for solving equilibrium and fixed-point problems in real Hilbert spaces. *J. Inequal. Appl.* **2023**, 7 (2023). https://doi.org/10.1186/s13660-023-02912-6

[34] ur Rehman, H., Kumam, P., Cho, Y. J.; et al. Weak convergence of explicit extragradient algorithms for solving equilibrium problems. *J. Inequal. Appl.* **2019**, 282 (2019). https://doi.org/10.1186/s13660-019-2233-1

[35] Reich, S. and Sabach, S. A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces. *J. Nonlinear Convex Anal.* **10** (2009), 471–485.

[36] Reich, S. and Sabach, S. Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces. *Nonlinear Anal.* **73** (2010), 122–135. https://doi.org/10.1016/j.na.2010.03.005

[37] Reich, S. and Sabach, S. Two strong convergence theorems for a proximal method in reflexive Banach spaces. *Numer. Funct. Anal. Optim.* **31** (2010), 22–44. https://doi.org/10.1080/01630560903499852

[38] Santos, P. and Scheimberg, S. An inexact subgradient algorithm for equilibrium problems. *Comput. Appl. Math.* **30** (2011), 91–107. https://doi.org/10.1590/S1807-03022011000100005

[39] Stampacchia, G. Formes bilinéaires coercitives sur les ensembles convexes. *C. R. Acad. Sci. Paris* **258** (1964), 4413–4416.

[40] Strodiot, J. J., Van Nguyen, T. T. and Nguyen, V. A new class of hybrid extragradient algorithms for solving quasi-equilibrium problems. *J. Global Optim.* **56** (2013), 373–397. https://doi.org/10.1007/s10898-011-9814-y

[41] Tan, B., Cho, S. and Yao, J. C. Accelerated inertial subgradient extragradient algorithms with non-monotonic step sizes for equilibrium problems and fixed point problems. *J. Nonlinear Var. Anal.* **6** (2022), 89–122. 10.23952/jnva.6.2022.1.06

[42] Tan, B. and Li, S. Adaptive inertial subgradient extragradient methods for finding minimum-norm solutions of pseudomonotone variational inequalities. *J. Ind. Manag. Optim.* **19** (2023), 7640–7659. https://doi.org/10.3934/jimo.2023012

[43] Tan, B., Qin, X. and Yao, J. C. Extragradient algorithms for solving equilibrium problems on Hadamard manifolds. *Appl. Numer. Math.* **201** (2024), 187–216. https://doi.org/10.1016/j.apnum.2024.03.009

[44] Thong, D. V., Cholamjiak, P., Rassias, M. T. and Cho, Y. J. Strong convergence of inertial subgradient extragradient algorithm for solving pseudomonotone equilibrium problems. *Optim. Lett.* **16** (2021), 545–573. https://doi.org/10.1007/s11590-021-01734-z

[45] Tran, D. Q., Dung, M. L. and Nguyen, V. H. Extragradient algorithms extended to equilibrium problems. *Optimization* **57** (2008), 749–776. https://doi.org/10.1080/02331930601122876

[46] Van Tiel, J. *Convex Analysis: An Introductory Text*. John Wiley and Sons, Inc., New York, 1984.

[47] Vinh, N. T. and Muu, L. D. Inertial extragradient algorithms for solving equilibrium problems. *Acta Math. Vietnam.* **44** (2019), 639–663. https://doi.org/10.1007/s40306-019-00338-1

[48] Wairojjana, N., Pholasa, N., Pakkaranang, N. and Panyanak, B. Two new extragradient methods for solving pseudomonotone equilibrium problems in Hilbert spaces. *Carpathian J. Math.* **41** (2025), 205–222.

[49] Wang, S., Zhang, Y., Ping, P., Cho, Y. and Guo, H. New extragradient methods with non-convex combination for pseudomonotone equilibrium problems with applications in Hilbert spaces. *Filomat* **33** (2019), 1677–1693. https://doi.org/10.2298/FIL1906677W

[50] Xu, H. K. Iterative algorithms for nonlinear operators. *J. Lond. Math. Soc.* **66** (2002), 240–256. https://doi.org/10.1112/S0024610702003332

[51] Xu, H. K. Averaged Mappings and the Gradient-Projection Algorithm. *J Optim Theory Appl* **150** (2011), 360–378 . https://doi.org/10.1007/s10957-011-9837-z

[52] Yang, J., Liu, H. and Liu, Z. Modified subgradient extragradient algorithms for solving monotone variational inequalities. *Optimization* **2018**. https://doi.org/10.1080/02331934.2018.1523404

[53] Zălinescu, C. *Convex Analysis in General Vector Spaces*. World Scientific Publishing, Singapore, 2002. https://doi.org/10.1142/5021

[54] Zeghad, B. and Daili, N. An improved inertial subgradient extragradient algorithm for pseudomonotone equilibrium problems and its applications. *J. Math. Model.* **13** (2025), 263–280. https://doi.org/10.22124/jmm.2024.28107.2479

ملخّص: تركز هذه الأطروحة على تطوير وتحليل خوارزميات من النوع التقريبي لحل مسائل التوازن شبه الرتيبة في فضائي هيلبرت وباناخ، مع تطبيقات على مسائل المتباينات التغايرية، حيث تعتبر حالة خاصة من مشاكل التوازن. تم اقتراح ثلاث خوارزميات محسّنة، تعتمد على طرق التدرج الفرعي والتدرج الإضافي مع أحجام خطوات غير رتيبة، إلى جانب تحسينات تستند إلى مسافة بَرِغمان. يتم إثبات التقارب الضعيف والقوي لهذه الخوارزميات تحت فرضيات مناسبة. وتُظهر الطرق المقترحة تفوقًا ملحوظًا في الأداء، من حيث سرعة التقارب، ودقة الحل، وتقليل زمن الحساب، كما تؤكد ذلك التجارب العددية مقارنة بالخوارزميات التقليدية.

الكلمات المفتاحية: الخوارزميات القريبة ; التقارب ; خوارزميات التدرج الفرعي ; مشاكل توازن; متباينات المغايرة.

Abstract : This thesis focuses on the development and analysis of proximal-type algorithms for solving pseudomonotone equilibrium problems in Hilbert and Banach spaces, with application to variational inequalities problems, which is special case of equilibrium problems. Three enhanced algorithms are proposed, incorporating subgradient and extragradient methods with non-monotonic step sizes, along with improvements based on Bregman distance. Weak and strong convergence of these algorithms are established under suitable assumptions. The proposed methods demonstrate significant improvements in performance, including faster convergence, higher accuracy, and reduced computational time, as confirmed by numerical experiments compared to traditional algorithms.

Keywords: Proximal algorithms; Convergence; Extragradient Algorithm; Equilibrium Problem ; Variational Inequalities.

Résumé : Cette thèse porte sur le développement et l'analyse d'algorithmes de type proximal pour résoudre des problèmes d'équilibre pseudo-monotones dans les espaces de Hilbert et de Banach, avec des applications aux problèmes d'inégalités variationnelles, qui constituent un cas particulier des problèmes d’équilibre. Trois algorithmes améliorés sont proposés, intégrant les méthodes du sous-gradient et de l’extragradient avec des tailles de pas non monotones, ainsi que des améliorations basées sur la distance de Bregman. La convergence faible et forte de ces algorithmes est établie sous des hypothèses appropriées. Les méthodes proposées montrent des améliorations significatives en termes de performance, notamment une convergence plus rapide, une meilleure précision, et un temps de calcul réduit, comme le confirment les expériences numériques comparées aux algorithmes classiques

Mots-clés : Algorithmes proximaux ; Convergence ; extragradient Algorithmes ; Problème d'équilibre ; Inégalités variationnelles.