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Faculty of Sciences
Department of Mathematics

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Meriem Chabekh

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President:	Pr. Toufik Serrar	Setif 1 University - Ferhat Abbas
Supervisor:	Dr. Nadhir Chougui	Setif 1 University - Ferhat Abbas
Examiner:	Pr. Salim A. Messaoudi	Sharjah University, UAE
Examiner:	Pr. Azedine Rahmoune	Bordj Bou Arreridj University
Examiner:	Pr. Salah Zitouni	Souk-Ahras University
Examiner:	Dr. Mohammed Kara	Setif 1 University - Ferhat Abbas
Guest:	Pr. Salah Drabla	Setif 1 University - Ferhat Abbas

Theoretical and Numerical Study of Some Models in Mechanics

A thesis submitted for the degree of

Doctor of Philosophy
in
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by

Meriem Chabekh
meriem.chabekh@univ-setif.dz

Advisor: Dr. Nadhir Chougui

Setif 1 University - Ferhat Abbas
Setif 19000, Algeria

Jun 2025

Ferhat Abbas Sétif 1 University, Algeria

PLEDGE

This is to certify that work presented in this thesis proposal titled ***theoretical and numerical study of some models in mechanics*** by *Chabekh Meriem* has been carried out under my supervision and is not submitted elsewhere for a degree.

______________	__________________________________
Date	Advisor: Dr. Chougui Nadhir

To the memory of my brother

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Abstract

In this dissertation, we study the existence and the long-term behavior of certain systems influenced by various dissipation mechanisms, damping effects, and delayed term. By imposing appropriate assumptions, we establish the well-posedness by the application of semigroup theory or the Faedo-Galerkin approach. To achieve the desired stability results for the systems, we employ the multiplier method.

To support the theoretical findings, a numerical analysis is conducted for each problem. Fully discrete approximations are formulated using the finite-element method combined with the implicit Euler scheme. Numerical simulations, implemented in MATLAB, are provided to illustrate the accuracy of the approximations and the behavior of the solutions.

ملخص

في هذه الرسالة، ندرس وجود وسلوك طويل الأمد لأنظمة معينة تتأثر بآليات التبديد المختلفة، تأثيرات التخميد، وحد تأخير. بوضع فرضيات مناسبة، نبرهن وجود و وحدانية الحل بواسطة تطبيق نظرية شبه الزمر أو تقريب Faedo-Galerkin . لتحقيق نتائج الإستقرار المطلوبة للأنظمة، نستخدم طريقة المضروبات.

لدعم النتائج النظرية، يتم إجراء تحليل عددي لكل مسألة. تتم صياغة التقريبات المتقطعة بالكامل باستخدام طريقة العناصر المحدودة جنبا إلى جنب مع مخطط Euler الضمني. كما يتم تقديم المحاكاة العددية، المنفذة باستخدام برنامج MATLAB ، لإظهار دقة التقريبات وسلوك الحلول.

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Chapter 1

Introduction

The study of global existence and stability in time of partial-differential-equations has been, a long time the focus of numerous works. Stabilization aims to attenuate vibrations through feedback mechanisms, ensuring that the energy of the solutions diminishes to zero at a controlled rate, facilitated by a dissipation process. In this regard, we study various problems and establish exponential, polynomial, or general decay results for certain thermoelastic evolution problems in one dimension. These include models such as Shear systems, laminated beams, and elastic solids with voids. Various dissipation mechanisms are considered, and their influence on the stability of these systems is analyzed.

1.1 Thermoelasticity

The theory of thermoelasticity merges the principles of elasticity and heat conduction. It addresses the effect of heat on the deformation of an elastic medium and the reciprocal effect of this deformation on the thermal state of the medium. Thermal stress arises when the time rate of variation of a heat source in the medium or the time rate of variation of thermal boundary conditions on the medium is compared with the structural oscillation characteristics. In this scenario, solutions for the temperature and stress fields must be obtained using the coupled equations of thermoelasticity.

The foundations of thermoelasticity theory were laid in the 19th century by scientists such as Duhamel [49] and Neumann [115]. The classical theory of thermoelasticity was established in the 1950s by Biot [27], who formulated the governing equations and constitutive relations. In this theory, the heat flux q and the temperature gradient θ_x are considered to happen simultaneously. Fourier's law of heat conduction, in its linear form, is given by

$$q(x,t) = -\kappa\theta_x(x,t). \tag{1.1}$$

This implies an instantaneous response and there is no difference between the cause and the effect of heat flow. As a result, heat propagation has an infinite speed (meaning that any thermal change at some point has an instantaneous effect elsewhere in the body regardless to its distance). However, experiments have shown that the speed of thermal wave propagation in some dielectric crystals at low temperatures is finite. This phenomenon in dielectric crystals is called second sound.

The classical theory has some limitations, particularly in dealing with high-frequency or short-time phenomena. This led to the development of generalized theories of thermoelasticity, which aim to overcome these limitations. In the modern theory of thermal propagation, there are several ways to overcome this physical paradox. The most known is the one proposed replacing Fourier's law of heat flux with Cattaneo's law ([33],1958) to obtain a heat conduction equation of hyperbolic type that describes the wave nature of heat propagation at low temperatures. At the turn of the century, Green and Naghdi introduced three other theories (known as thermoelasticity Type I, Type II, and Type III), based on the equality of entropy rather than the usual entropy inequality ([65]- [67],1991-1993). In each theory, the heat flux is determined by different appropriate assumptions. These three theories give a comprehensive and logical explanation that embodies the transmission of a thermal pulse and modifies the occurrence of the infinite un-physical speed of heat propagation induced by the classical theory of heat conduction. When the theory of type I is linearized, it aligns with the classical system of thermoelasticity. The systems arising in thermoelasticity of type III exhibit dissipative characteristics, while those in Type II do not sustain energy dissipation. It is a limiting case of thermoelasticity type III. For more details in this regard, we refer the reader to [38, 39, 110, 128, 154].

The wave theory of heat conduction was formulated by assuming that the heat flux vector and the temperature gradient occur at different moments in time. In this framework, a natural generalization of (1.1) can be written as

$$q(x, t+\tau) = -\kappa\theta_x(x, t), \tag{1.2}$$

where τ represents the time delay, known as the "relaxation time" in the wave theory of heat conduction [149]. The first-order Taylor expansion, which includes the linear effect of τ, results in

$$q + \tau q_t = -\kappa\theta_x, \tag{1.3}$$

which refers to the CV wave model (second sound) created by Cattaneo and Vernotte to resolve the paradox of infinite heat propagation speed resulted in Fourier's law [40]. The finite speed of heat propagation relates to the relaxation time by

$$\tau = \frac{\alpha_0}{C_0}, \tag{1.4}$$

where α_0 is the thermal diffusivity and C_0 represents the thermal wave speed [147]. As C_0 approaches infinity, the relaxation time τ decreases to zero, and the CV wave model (equations (1.2) or (1.3)) reduces to Fourier's law, as given by equation (1.1). Later on, two types of delay time were introduced. The first delay time, τ_θ, is attributed to microstructural interactions, such as small-scale heat transport mechanisms at the microscale, or small-scale effects of heat transport in space. This includes phenomena such as phonon-electron interactions or phonon scattering and is referred to as the phase lag of the temperature gradient. The second delay time, τ_q, is related to the relaxation time resulting from the fasttransient effects of thermal inertia (or small-scale effects of heat transport over time), and is called the phase-lag of the heat flux. Both delays are regarded as inherent thermal or structural features of the material. The dual-phase-lag model is intended to avoid the precedence assumption in the thermal wave model. For materials with $\tau_\theta < \tau_q$, the heat flux vector (effect) results from a temperature gradient (cause). The relationship is reversed when $\tau_\theta > \tau_q$ (see [40, 145, 147]). Mathematically, this can be represented by

$$q(x, t + \tau_q) = -\kappa\theta_x(x, t + \tau_\theta),\ \tau_q, \tau_\theta > 0, \tag{1.5}$$

when $\tau_q = \tau_\theta$, this law reduces to the classical Fourier's law (1.1), and the relation (1.2) corresponds to the particular case when $\tau_\theta = 0$ and $\tau_q = \tau > 0$. In [148], Tzou expanded both sides of equation (1.5) using Taylor's expansions and retaining terms up to the second order in τ_q and only the first order term in τ_θ. This resulted in the following generalization of the heat conduction law

$$q + \tau_q q_t + \frac{\tau_q^2}{2} q_{tt} = -\kappa(\tau_\theta \theta_{xt} + \theta_x). \tag{1.6}$$

This theory imposes certain constraints on the delay parameters τ_θ and τ_q to ensure the exponential stability (or at least stability) of the solutions. In [123, 124], exponential stability was proved whenever

$$\tau_\theta > \frac{\tau_q}{2}. \tag{1.7}$$

Tzou's experimental findings in [146] validate the physical relevance and the practical use of the dual-phase-lag model. This model plays a crucial role in several fields, particularly in thermal management for electronics [4], heat exchanger design [99], and biomedical applications [153].

1.2 Shear beam model

Beams are fundamental structural components in engineering and construction, crucial for supporting and distributing loads within various structures. These primarily horizontal members are designed to withstand vertical loads, shear forces, and bending moments, ensuring the stability and safety of buildings, bridges, and other infrastructures.

The beam theory proposed by Euler and Bernoulli in the 18th century [18, 57] has remarkably withstood the test of time and continues to be effectively utilized by contemporary scientists and engineers. The differential equation that describes the vibrations of a uniform beam in the classical Euler-Bernoulli theory can be expressed as follows

$$\rho A \varphi_{tt}(x, t) + \boldsymbol{EI} \varphi_{xxxx}(x, t) = 0, \tag{1.8}$$

where $x \in (0, L)$ and $t > 0$, A is the cross section area, ρ is the density of the material, $\boldsymbol{E}$ is Young's modulus, $\boldsymbol{I}$ is the geometric moment of inertia and φ denotes the deflection of the beam. Later advancements were made by Bresse and Rayleigh [28, 131], who incorporated the effect of rotary inertia into beam theory. This work was significantly furthered by Timoshenko [142, 143] in the early 20th century, who introduced the additional effect of shear deformation. This development led to what is now known as the classical Timoshenko beam,

$$\begin{cases} \rho_1 \varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) = 0, \\ \rho_2 \psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x + \psi)(x,t) = 0, \end{cases} \tag{1.9}$$

where φ represents the transverse displacement and ψ denotes the rotation of the neutral axis. $\rho_1 = \rho A$, $K = K'GA$, $\rho_2 = \rho \boldsymbol{I}$, $b = \boldsymbol{EI}$ where K' is the shear correction factor and G is the shear modulus. The Timoshenko beam formulation is typically utilized to analyze the dynamic behavior of beams at higher frequencies (higher wave numbers). In this context, two distinct frequency values emerge, leading to the presence of two wave speeds: $\sqrt{K/\rho_1}$ and $\sqrt{b/\rho_2}$. Notably, one of these wave speeds exhibits an infinite speed (blow-up) for lower wave numbers (see [89]). In order to overcome this physics drawback, simplified beam models have been proposed, such as the Truncated version introduced by Elishakoff [51], based on the system of Timoshenko given by

$$\begin{cases} \rho_1 \varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) = 0, \\ -\rho_2 \varphi_{ttx}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x + \psi)(x,t) = 0. \end{cases} \tag{1.10}$$

He demonstrated in the presence of a linear dissipation, that energy decays exponentially, regardless of the wave speed. This contrasts the Timoshenko beam, which is only exponentially stable when the wave speeds are equal. Another beam theory considered is the Shear beam model, which adds shear distortion to the Euler-Bernoulli model but omits rotary inertia [74],

$$\begin{cases} \rho_1 \varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) = 0, \\ -b\psi_{xx}(x,t) + K(\varphi_x + \psi)(x,t) = 0, \end{cases}$$

has a single finite wave speed, $\sqrt{K/\rho_1}$, for all wave numbers. The Timoshenko beam can be effectively analyzed using a single parameter because the rotary inertia and shear deformation parameters are related [15].

1.3 Laminated beams

Layered composite structures are extensively utilized in structural engineering because of their high stiffness, strength, and low weight. The mechanical performance of these structures depends not only on the material properties of the individual beam layers but also on the quality of interfacial bonding. Perfect interfacial bonding can be achieved if the beam layers are connected with rigid connectors. However, typical connectors such as studs, nails, and viscoelastic adhesives lack rigidity, which can lead to some interfacial slip (see [151] and the numerous references therein).

Hansen [75] developed a model for a two-layered plate where a slip can occur along the interface. The model assumes that an adhesive layer, with negligible thickness and mass, bonds the two adjacent surfaces, and the restoring force generated by the adhesive is proportional to the amount of slip. Hansen and Spies [76] focused on the beam analog, with strain-rate damping included, of the described plate model ([75], equation (3.16)). In the absence of external forces, this model consists of three coupled hyperbolic equations. Precisely, the first two equations are based on the assumptions of Timoshenko beam theory and are coupled with the third equation, which describes the dynamics of slip. This coupled system is given as follows:

$$\begin{cases} \rho\omega_{tt} + G(\psi - \omega_x)_x = 0, \\ I_\rho(3s - \psi)_{tt} - D(3s - \psi)_{xx} - G(\psi - \omega_x) = 0, \\ I_\rho s_{tt} - D s_{xx} + G(\psi - \omega_x) + \dfrac{4}{3}\gamma s + \dfrac{4}{3}\beta s_t = 0, \end{cases} \tag{1.11}$$

where ω denotes the transverse displacement, ψ represents the rotation angle, s is proportional to the amount of slip along the interface at time t and longitudinal spatial variable x and $3s - \psi$ denotes the effective rotation angle. The positive parameters ρ, I_ρ, G, D, γ and β, represent the density, mass moment of inertia, shear stiffness, flexural rigidity, adhesive stiffness, and adhesive damping parameter, respectively. These structures have become widely popular and are known as laminated beams. It is clear that if s is zero at all points, then the standard Timoshenko system is retrieved.

Up to date, there have been several studies on laminated beam equations, primarily focusing on the global existence and stability of the associated systems. By incorporating appropriate damping mechanisms such as internal damping, boundary frictional damping, and viscoelastic damping, it has been demonstrated that adding linear damping terms to two of the three equations results in exponential stability of system (1.11) under the "equal wave speeds" condition ($\rho/I_\rho = G/D$). However, if linear damping terms are applied to all three equations, the system exhibits exponential decay even without the equal-wave-speed condition, as illustrated in references [7, 8, 12, 30, 70, 73, 114, 130, 141, 150].

1.4 The porous thermoelastic systems

Continuum mechanics is an established field that has received comprehensive treatment in numerous treatises. The theory of continuous media builds upon fundamental concepts like stress, motion, and deformation, as well as principles governing the conservation of mass, linear momentum, moment of momentum, and energy, in addition to constitutive relations. These constitutive relations define how a material reacts mechanically and thermally, while the fundamental conservation laws abstract the shared characteristics of all mechanical phenomena, regardless of the specific constitutive relations involved.

Goodman and Cowin [43,64] introduced a granular materials theory based on formal arguments from continuum mechanics. The fundamental idea of this theory is to expand the concept of mass distribution to accommodate granular materials. Specifically, the distribution of mass needs to be related to the volume distribution of granules. In their setting, the bulk density is written as the product of two fields, the matrix material density field and the volume fraction field. It is deeply discussed in the book of Ieșan [83]. This representation of the bulk density of the material introduces an additional degree of kinematic freedom in the theory that was developed by Nunziato and Cowin [117] for the behavior of porous solids in which the skeletal or matrix material is elastic and the interstices are void of material. The theory admits both finite deformations and nonlinear constitutive relations. The linearization of this theory has been established by Cowin and Nunziato [44]. Some applications and results of the linear and nonlinear theories for elastic materials with voids were presented in [45, 82, 84, 107, 125, 127]. The nonlinear theory of thermoelastic materials with voids has been established by Jarić and Golubović [85] and Jarić and Ranković [86]. Later, Ieșan [81] added the temperature to the linear theory.

Grot [69] advanced a thermodynamics theory concerning elastic materials exhibiting microstructure. In this theory, the microelements possess not only microdeformations but also microtemperatures. The foundation of this theory lies in continuum theories incorporating microstructure, wherein the microelements undergo uniform deformations known as microdeformations. It was widely developed by Eringen [52–54] and Eringen and Kafadar [55]. The importance of elastic materials with microstructure has been extensively evidenced by the vast number (thousands) of articles published over the past thirty years. These articles cover applications across various fields of physics and engineering, including the petroleum industry, materials science, and biology. Poroelasticity theory should be utilized for solids with small, distributed pores, such as rocks, soils, wood, ceramics, compressed powders, and biological materials like bones.

In one space dimension, The basic evolution equations governing porous materials theory with temperature and micro-temperature are given by

$$\rho u_{tt} = T_x,\ J\varphi_{tt} = H_x + G,\ \rho\eta_t = q_x,\ \rho E_t = P_x - Q,$$

where T is the stress tensor, H is the equilibrated stress vector, G is the equilibrated body force, η is the entropy, q is the heat flux vector, P is the first heat flux moment, Q is the mean heat flux and E is the first moment of energy. The constitutive equations $T,\ H,\ G,\ \eta,\ q,\ E,\ P$, and Q take the following form

$$\begin{cases} T = \mu u_x + b\varphi - \gamma\theta, \\ H = \delta\varphi_x - dw,\ G = -bu_x - \xi\varphi + m\theta, \\ \rho\eta = \gamma u_x + c\theta + m\varphi,\ q = k_0\theta_x - k_1 w, \\ \rho E = -\alpha w - d\varphi_x,\ P = -k_2 w_x,\ Q = -k_3 w - k_1\theta_x. \end{cases}$$

The functions u, φ, θ, w represent the displacement of the solid elastic material, the volume fraction, the temperature difference and the microtemperature vector, respectively. The constitutive parameters ρ, J, c, μ, b, δ, γ, ξ, m, d, k_1, k_2, k_3, α define the coupling among the different components of the materials.

1.5 Delay differential equations

Considering delays is essential for the accurate analysis of systems in science and engineering. A time delay occurs because it takes a finite amount of time to sense information and respond to it. Time delay has been extensively studied in various fields, such as biology [104], population dynamics [93], neural networks [19], feedback-controlled mechanical systems [80], and lasers [122]. Richard [133] also highlighted several other intriguing and challenging areas where delays play a significant role. Based on the causes of delays, they can be roughly classified into the following categories: physically inherent delays (found in physical or biological systems), technological delays, transmission delays, and information delays.

Mathematically, a simple delay differential equation for $x(t) \in \mathbb{R}$ is expressed as

$$\frac{d}{dt}x(t) = f(t, x_t),$$

where $x(t) = \{x(\tau) : \tau \leqslant t\}$ represents the past trajectory of the solution. The functional operator f takes a time input and a continuous function x_t and outputs a real number $\frac{d}{dt}x(t)$. An example of such an equation with a discrete/constant delay is

$$\frac{d}{dt}x(t) = f(t, x(t-\tau)).$$

where τ denotes the time delay.

In recent years, partial differential equations with time delay effects have emerged as a significant research focus in science and engineering, finding applications in various practical problems. Stabilizing a hyperbolic system that includes delay terms often requires the addition of control terms (refer to [116, 119, 134]). Delay terms can potentially lead to instability [46, 116, 152]. For instance, Nicaise and Pignotti [116] considered the following wave equation with a time-delay in the presence of a delay parameter ($\mu_2 > 0$),

$$u_{tt}(x,t) - \Delta u(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau) = 0, \tag{1.12}$$

they showed that this equation exhibits either exponential stability when the condition $\mu_2 < \mu_1$ is met or instability when $\mu_1 \leq \mu_2$. Said-Houari and Laskri [134] examined the Timoshenko system with a constant time delay in the internal feedback,

$$\begin{cases} \rho_1 \varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) = 0, \\ \rho_2 \psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x + \psi)(x,t) + \mu_1 \psi_t(x,t) + \mu_2 \psi_t(x, t-\tau) = 0, \end{cases} \tag{1.13}$$

and they proved that, in the case of equal-wave-speed, the energy decays exponentially if $\mu_2 < \mu_1$. To extend that result to a nonlinear framework, we refer to [60, 61].

1.6 About the finite element method

A significant number of natural laws and phenomena are represented through partial differential equations (PDEs). However, when the computational domains have irregular geometries, or when the initial conditions, boundary values, or source terms are complicated, it becomes impossible to derive analytical solutions. In such cases, we rely on numerical methods to obtain approximate solutions.

The origins of the finite-element method (FEM) date back to the early 20th century. During the 1940s and 1950s, engineers, particularly in the aerospace industry, began developing techniques to break down continuous structures into smaller, discrete elements to address challenges in structural analysis. This approach was driven by the need to handle the complexity of real-world engineering problems [144]. Over time, FEM has become a powerful tool for solving partial differential equations, including those of elliptic, parabolic, hyperbolic types, as well as complex hydrodynamic equations. Since its appearence, numerous reference books have been dedicated to this method (see [41, 62, 63, 88, 140]).

The theoretical proof of the existence and uniqueness of the finite-element (FE) solution for the FE equation is essential. In terms of numerical resolution, the finite-element method has been used in several research studies concerning control systems (see [14, 16, 50]).

1.7 Methodology

In this thesis, the well-posedness of the problems is addressed using the theory of semigroups and the Faedo–Galerkin method. Specifically, the Faedo–Galerkin approach involves selecting a set of basis functions in an appropriate Sobolev space and solving approximate problems within a finite-dimensional subspaces spanned by these basis functions. The local existence of solutions to the approximate problem is established through the well-known local existence theorem for ordinary differential equations. Compactness estimates are then utilized to extract a convergent subsequence of the approximate solutions, leading to a solution of the original problem. The uniqueness of the solution for the original problem requires a separate proof. In the context of semigroup theory, the Lumer–Phillips theorem serves as a crucial tool. This theorem establishes a relationship between the energy dissipation properties of an unbounded operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ with the existence, uniqueness, and regularity of solutions to an evolution equation (Cauchy problem).

$$\begin{cases} U'(t) = \mathcal{A}U(t),\ t > 0, \\ U(0) = U_0. \end{cases}$$

We apply the multiplier method to derive the desired stability results for the systems. This approach primarily involves constructing an appropriate Lyapunov functional L that is equivalent to the energy E of the solution. The equivalence $L \sim E$ is expressed as:

$$c_1 E(t) \leqslant L(t) \leqslant c_2 E(t),\ \forall t \geqslant 0,$$

To establish exponential stability, we demonstrate that L satisfies the inequality:

$$L'(t) \leqslant -cL(t),\ \forall t \geqslant 0,$$

for some $c > 0$. By integrating this inequality over the interval $(0, t)$ and using the equivalence relationship above, the desired result of exponential stability is obtained (refer to Chapters 3, 4, and 5). More precisely, the stabilization problem involves determining the asymptotic behavior of the system by estimating the decay rate of the energy to zero. Various types of stabilization exist, including polynomial decay (see Chapter 4). For general decay, the achieved decay rate depends on the relaxation function g (see Chapter 6).

To compute numerical solutions for boundary value problems, we employ the finite-element method, a powerful and versatile approach. This method begins by replacing the original system with an equivalent idealized system composed of discrete elements. These elements are connected at specific points, referred to as nodes. After defining the elements of the system, direct physical reasoning is used to derive the equations for each individual element based on relevant variables. In the final step, the equations for the individual elements are assembled to construct the system equations for the entire model, which are then solved to determine the unknown nodal variables.

1.8 Structure of the thesis

The thesis consists of five chapters in addition to the introduction.

Chapter 2. This chapter summarizes some concepts, definitions and theorems needed in the proof of our results in the next four chapters.

Chapter 3. In this chapter, we conduct an analysis of a one-dimensional linear problem that describes the vibrations of a connected suspension bridge. In this model, the single-span roadbed is represented as a thermoelastic Shear beam without rotary inertia. We incorporate thermal dissipation into the transverse displacement equation, following Green and Naghdi's theory. Our work demonstrates the existence of a global solution by employing classical Faedo–Galerkin approximations and three a priori estimates. Furthermore, we establish the exponential stability through the application of the energy method. For numerical study, we propose a spatial discretization using finite elements and a temporal discretization through an implicit Euler scheme. In doing so, we prove discrete stability properties and a priori error estimates for the discrete problem. To provide a practical aspect to our theoretical findings, we present a set of numerical simulations. The results presented in this chapter have been published in [35].

Chapter 4. We focus on a thermoelastic laminated beam from both mathematical and numerical perspectives, where the dual-phase-lag heat conduction theory is used to model the heat transfer. In this theory, two delay parameters τ_θ and τ_q are considered. Under the condition $2\tau_\theta > \tau_q$, we establish the well-posedness of the system and prove both exponential and polynomial stability depending on a stability parameter χ. For the numerical study, we propose fully discrete approximations using the finite-element method combined with the implicit Euler scheme. Through this approach, we demonstrate the discrete stability property and provide a priori error estimates. To showcase the accuracy of the approximation and illustrate the practical application of our theoretical findings, we present numerical simulations. The results of this chapter have been published in [37].

Chapter 5. This chapter is devoted to the study of a one-dimensional system known as Shear beam model (no rotary inertia) where the transverse displacement equation is subjected to a delay. Under suitable assumptions on the weight of the delay, we first achieve the global well-posedness of the system by using the classical Faedo-Galerkin approximations along with three a priori estimates. Next, we study the asymptotic behavior of solution using the energy method. Later we propose a discretization based on P1-finite element for space and implicit Euler scheme for time, where a discrete stability property and a priori error estimates of the discrete problem are proved. Finally, numerical simulations are provided. This work has been published in [34].

Chapter 6. In this chapter, we consider a one-dimensional porous thermoelastic system with microtemperatures effects, past history term acting only on the porous equation and a delay term in the internal feedback. Under an appropriate assumptions on the kernel and between the weight of the delay and the weight of the damping, we prove the well-posedness of the system. Furthermore, we establish a general decay rate result for the energy, which allows a wider class of relaxation functions, and thus generalize some results in the literature. Finally, some numerical experiments are presented. The findings of this chapter have been published in [36].

Notation. Throughout this thesis, $(\cdot,\cdot)$ denotes the scalar product of $L^2(0,L)$, and $\|\cdot\|$ refers to the usual norm $\|\cdot\|_{L^2(0,L)}$.

Chapter 2

Preliminaries and materials needed

In this chapter, we review essential concepts in functional analysis that are necessary for proving our results in the upcoming chapters.

2.1 Functional spaces

2.1.1 Banach spaces, Inner product spaces, Hilbert spaces

For any normed vector space E, we denote its topological dual as E', that is, the space of continuous linear functionals on E. For $f \in E'$ and $x \in E$ we introduce the duality bracket

$$(f, x)_{E',E} = f(x).$$

A norm on E defines a metric d on E which is given by $d(x, y) = \|x - y\|$ for all $x, y \in E$ and is called the metric induced by the norm.

Definition 2.1.1 ([56]). *A Banach space is a normed vector space that is complete, that is, every Cauchy sequence in the metric induced by the norm of E converges to a limit in E.*

Definition 2.1.2 ([56]). *An inner product space (or pre-Hilbert space) is a vector space E with an inner product defined on E , which is a mapping of $E \times E$ into the scalar field K of E, such that for all vectors x, y, z and scalars α we have*

1. $(x + y, z) = (x, z) + (y, z)$,
2. $(\alpha x, y) = \alpha(x, y)$,
3. $(x, y) = \overline{(y, x)}$,
4. $(x, x) \geqslant 0$ *and* $(x, x) = 0 \Leftrightarrow x = 0$.

An inner product on E defines a norm on E given by $\|x\| = \sqrt{(x, x)}$ and a metric on E given by $d(x, y) = \|x - y\| = \sqrt{(x - y, x - y)}$, for all x, $y \in E$. A Hilbert space is a complete inner product space (complete in the metric defined by the inner product). Hence inner product spaces are normed spaces, and Hilbert spaces are Banach spaces.

Theorem 2.1.3 (Lax–Milgram, [26]). *Let V be a Hilbert space, $a : V \times V \to \mathbb{R}$ a bilinear form, and $l : V \to \mathbb{R}$ a linear form.*

Assume that a and l are continuous and that a is coercive, that is,

$$\exists \alpha > 0,\ a(u,u) \geqslant \alpha \|u\|_V^2, \forall u \in V,$$

then there exists a unique solution $u \in V$ to the problem

$$a(u,v) = l(v),\ \forall v \in V. \tag{2.1}$$

Moreover, this solution satisfies

$$\|u\|_V \leqslant \frac{\|l\|_{V'}}{\alpha}.$$

2.1.2 Weak topology

Let E be a Banach space and denote by E' its dual space with norm

$$\|f\|_{E'} = \sup_{x \in E,\ \|x\|_E \leqslant 1} |(f,x)|.$$

Definition 2.1.4 ([29]). *The weak topology $\sigma(E,E')$ on E is defined to be the coarsest topology under which each element of E' remains continuous on E.*

If $x_n \to x$ in $\sigma(E,E')$, we shall write $x_n \rightharpoonup x$ and say that the sequence (x_n) converges weakly to x in E.

Proposition 2.1.1 ([29]). *Let (x_n) be a sequence in E. Then*

$$x_n \rightharpoonup x \Leftrightarrow (f,x_n) \to (f,x), \forall f \in E'.$$

2.1.3 Weak* topology

Let E'' be the bidual space (the dual of E') with norm

$$\|\xi\|_{E''} = \sup_{f \in E',\ \|f\|_{E'} \leqslant 1} |(\xi,f)|.$$

There is a canonical injection $J : E \longrightarrow E''$. Indeed any element $x \in E$ defines an element $J_x \in E''$ by

$$(J_x, f) = (f,x), \forall f \in E'.$$

It is clear that J is linear and that J is an isometry, that is, $\|J_x\|_{E''} = \|x\|_E$ (by Corollary 1.4, page 4, [29]). This will allow us to define a new topology on E'.

Definition 2.1.5 ([29]). *The weak* topology $\sigma(E', E)$ on E' is the coarsest topology under which every element $x \in E$ corresponds to a continuous map on E'.*

If $f_n \to f$ in $\sigma(E', E)$, we shall write $f_n \rightharpoonup^* f$ and say that the sequence (f_n) converges weakly * to f in E'.

Proposition 2.1.2 ([29]). *Let (f_n) be a sequence in E'. Then*

$$f_n \rightharpoonup^* f \Leftrightarrow (f_n, x) \to (f, x), \forall x \in E.$$

2.1.4 Reflexive, separable spaces

Definition 2.1.6 ([29]). *E is reflexive, if $J(E) = E''$.*

Theorem 2.1.7 ([29]). *Let E be a separable Banach space. Then for any bounded sequence (x_n) in E, there exists a subsequence (x_{n_k}) that converges in $\sigma(E, E')$.*

Definition 2.1.8 ([29]). *A metric space is separable if it contains a dense and countable subset.*

Theorem 2.1.9 ([29]). *Let E be a separable Banach space. Then for any bounded sequence (f_n) in E', there exists a subsequence (f_{n_k}) that converges in $\sigma(E', E)$.*

2.1.5 Lebesgue spaces

Definition 2.1.10 ([29]). *Let $(\Omega, \mathcal{M}, \mu)$ be a measure space and $1 \leqslant p < \infty$. The space $L^p(\Omega)$ consists of equivalence classes of measurable functions $f : \Omega \to \mathbb{R}$ such that*

$$\int |f|^p d\mu < \infty.$$

Theorem 2.1.11 (Dominated convergence theorem, Lebesgue, [29]). *Let (f_n) be a sequence of functions in L^1 that satisfy*

1. *$f_n(x) \longrightarrow f(x)$ a.e. on Ω,*

2. *there is a function $g \in L^1$ such that for all n, $|f_n(x)| \leqslant |g(x)|$ a.e. on Ω.*

Then

$$f \in L^1(\Omega) \text{ and } \|f_n - f\|_{L^1} \longrightarrow 0.$$

Theorem 2.1.12 (Fubini, [29])**.** *Assume that $F \in L^1(\Omega_1 \times \Omega_2)$. Then for a.e. $x \in \Omega_1$, $F(x,y) \in L^1_y(\Omega_2)$ and $\int_{\Omega_2} F(x,y)d\mu_2 \in L^1_x(\Omega_1)$. Similarly, for a.e. $y \in \Omega_2$, $F(x,y) \in L^1_x(\Omega_1)$ and $\int_{\Omega_1} F(x,y)d\mu_1 \in L^1_y(\Omega_2)$. Moreover, one has*

$$\int_{\Omega_1} d\mu_1 \int_{\Omega_2} F(x,y)d\mu_2 = \int_{\Omega_2} d\mu_2 \int_{\Omega_1} F(x,y)d\mu_1 = \int\int_{\Omega_1\times\Omega_2} F(x,y)d\mu_1 d\mu_2.$$

Definition 2.1.13 ([29])**.** *Let $p \in \mathbb{R}$ with $1 < p < \infty$, we set*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R};\ f \text{ is measurable and } |f|^p \in L^1(\Omega) \right\}$$

with the norm

$$\|f\|_{L^p} = \|f\|_p = \left(\int_\Omega |f(x)|^p d\mu \right)^{\frac{1}{p}}.$$

Definition 2.1.14 ([29])**.** *We set*

$$L^\infty(\Omega) = \{ f : \Omega \to \mathbb{R};\ f \text{ is measurable and } \exists C : |f(x)| \leqslant C \text{ a.e. on } \Omega \}$$

with the norm

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf \{ C;\ |f(x)| \leqslant C \text{ a.e. on } \Omega \}.$$

Notation 1. *Let $1 \leqslant p \leqslant \infty$, we denote by q the conjugate exponent,*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Theorem 2.1.15 ([29])**.** *L^p is reflexive (separable) for any p, $1 < p < \infty$ $(1 \leqslant p < \infty)$. L^q is the dual space of L^p.*

Theorem 2.1.16 (Hölder's inequality, [29])**.** *Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $1 \leqslant p \leqslant \infty$. Then $fg \in L^1(\Omega)$ and*

$$\|fg\|_{L^1(\Omega)} \leqslant \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

In the particular case where $p = q = 2$, we obtain Cauchy-Schwartz inequality :

$$\|fg\|_{L^1(\Omega)} \leqslant \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.$$

2.1.6 Sobolev spaces

Definition 2.1.17 ([106])**.** *Let Ω be an open subset of $\mathbb{R}^n$ and $m \in \mathbb{N}$. We call Sobolev space of order m and we denote it by $H^m(\Omega)$, the set*

$$H^m(\Omega) = \left\{ u \in L^2(\Omega) / D^\alpha u \in L^2(\Omega),\ \forall \alpha \in \mathbb{N}^n,\ |\alpha| \leqslant m \right\},$$

where $D^\alpha u = \dfrac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$ *is the derivative of order* α *in the sense of distributions with* $\alpha = (\alpha_1, ..., \alpha_n)$ $\in \mathbb{N}^n$ *and* $|\alpha| = \sum_{i=1}^n \alpha_i$*.*

Some properties of space $H^m(\Omega)$:

1. We equip the space $H^m(\Omega)$ with the dot product

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha| \leqslant m} (D^\alpha u, D^\alpha v)_{L^2}, \forall\, u, v \in H^m(\Omega)$$

   and the associated norm

$$\|u\|_{H^m(\Omega)} = \sqrt{(u,u)_{H^m(\Omega)}} = \left[\sum_{|\alpha| \leqslant m} \int_\Omega (D^\alpha u)^2 dx \right]^{\frac{1}{2}}, \forall u \in H^m(\Omega).$$

   It is well known that this space is a Hilbert space.

2. For $m = 0$, $H^0(\Omega) = L^2(\Omega)$ and for any $m_1 > m_2$, we have

$$H^{m_1}(\Omega) \subset H^{m_2}(\Omega) \text{ with continuous embedding.}$$

3. For any $m \geqslant 0$, $H^m(\Omega)$ is a separable space and for $0 < m < +\infty$, $H^m(\Omega)$ is reflexive.

4. For any $m \geqslant 0$, we denote by $H_0^m(\Omega)$ the closure of $D(\Omega)$ in $H^m(\Omega)$:

$$H_0^m(\Omega) = \overline{D(\Omega)} \text{ in } H^m(\Omega),$$

   and by $H^{-m}(\Omega)$ the topological dual space of $H_0^m(\Omega)$.

5. Thanks to the applications of trace formula, the spaces $H_0^m(\Omega)$ can be defined as follows

$$H_0^m(\Omega) = \left\{ u \in H^m(\Omega) / \ \frac{\partial^j u}{\partial \eta^j} = 0, \forall j = 0, ..., m-1 \right\}.$$

   where $\dfrac{\partial}{\partial \eta}$ is the outer normal derivative of u at $\Gamma = \partial\Omega$,

$$\frac{\partial u}{\partial \eta}(x) = \frac{\partial u}{\partial x_i}(x) \eta_i, \forall x \in \Gamma.$$

Theorem 2.1.18 (Poincaré-Friedrich's inequality, [132])**.** *If* Ω *is bounded, there exists a constant* $C = C(\Omega) > 0$*, such that*

$$\|u\|_{L^2(\Omega)} \leqslant C \|\nabla u\|_{L^2(\Omega)}, \forall u \in H_0^1(\Omega).$$

2.1.7 Bochner spaces

We introduce some spaces that involve time, consisting of functions that map time into Banach spaces. These spaces will be essential in constructing weak (strong) solutions in chapters 3 and 5.

Definition 2.1.19 ([26, 58]). *Let X be a Banach space and T be a strictly positive real number. For $p \in [1, +\infty[$, we denote by $L^p(0,T;X)$, the set of (classes of) Lebesgue measurable functions defined on $]0,T[$ and with values in X, such that $t \mapsto \|f(t)\|_X^p$ is integrable on $]0,T[$. This is a Banach space for the norm*

$$\|f\|_p = \|f\|_{L^p(0,T;X)} = \left(\int_0^T \|f(t)\|_X^p dt\right)^{\frac{1}{p}}.$$

Similarly, we define, for $p = +\infty$, the Banach space $L^\infty(0,T;X)$ equipped by the norm

$$\|f\|_\infty = \|f\|_{L^\infty(0,T;X)} = ess \sup_{t\in[0,T]} \|f(t)\|_X < +\infty.$$

We define

$$C([0,T], X) = \{f \mid f : [0,T] \to X \text{ continuous}\},$$

and this is a Banach space for the norm

$$\|f\| = \sup_{t\in[0,T]} \|f(t)\|_X.$$

If f is m-times continuously differentiable, we define

$$C^m([0,T], X) = \left\{f \mid f^{(i)} \in C([0,T], X) \text{ for all } 0 \leqslant i \leqslant m\right\},$$

is a Banach space equipped with the norm

$$\|f\| = \sum_{i=1}^{m} \sup_{t\in[0,T]} \|f^{(i)}(t)\|_X.$$

Proposition 2.1.3 ([26, 87]). *Let $p \in [1, +\infty[$.*

1. *If X is reflexive and $1 < p < \infty$, then $L^p(0,T;X)$ is reflexive.*

2. *If X is separable, then $L^p(0,T;X)$ is separable as well.*

3. *The dual of $L^p(0,T;X)$ is given by $L^q(0,T;X')$.*

4. *$C^m([0,T], X)$ is dense in $L^p(0,T;X)$.*

We generalise the concept of time derivatives for functions defined on an interval $]0,T[$ of $\mathbb{R}$ and with values in a Banach space X

Definition 2.1.20 ([106]). *Let $]0,T[\subset \mathbb{R}$ and X be a Banach space, we say that $D'(0,T;X)$ is the distributions space on $]0,T[$ with values in X (i.e. the space of continuous linear maps of $D(0,T)$ in X).*

Definition 2.1.21 ([106]). *Let $f \in D'(0,T;X)$ and m a positive integer, we define the derivative of f of order m, in the distribution sense, by*

$$\left(\frac{d^m f}{dt^m}, \varphi\right) = (-1)^m \left(f, \frac{d^m \varphi}{dt^m}\right).$$

Definition 2.1.22 ([58]). *The sobolev space $W^{1,p}(0,T;X)$ consists of all functions $u \in L^p(0,T;X)$ such that u' exists in the distribution sense and belongs to $L^p(0,T;X)$. Furthermore, we set*

$$\|u\|_{W^{1,p}(0,T;X)} = \begin{cases} (\int_0^T \|u(t)\|^p + \|u'(t)\|^p \, dt)^{1/p}, & (1 \leqslant p \leqslant \infty), \\ ess\sup_{0\leqslant t\leqslant T}(\|u(t)\| + \|u'(t)\|), & (p=\infty). \end{cases}$$

We write $H^1(0,T;X) = W^{1,2}(0,T;X)$.

2.1.8 Compactness result

The result of compactness in Banach-valued functions spaces is given by the Aubin–Lions–Simon theorem.

Definition 2.1.23 (Continuous embedding, embedding compact, [58]). *Let X and Y be two Banach spaces. We say that X is compactly embedded in Y, and write $X \hookrightarrow Y$, if*

1. *X is continuously embedded in Y, i.e. there is a constant C such that*

$$\|x\|_Y \leqslant C\|x\|_X, \forall x \in X.$$

2. *each bounded sequence in X is precompact in Y, i.e. every bounded sequence has a subsequence that is Cauchy in the norm $\|.\|_Y$.*

Theorem 2.1.24 (Aubin–Lions–Simon [26]). *Let $B_0 \subset B_1 \subset B_2$ be three Banach spaces. We assume that the embedding of B_1 in B_2 is continuous and that the embedding of B_0 in B_1 is compact. Let p, r such that $1 \leqslant p, r \leqslant \infty$. For $T > 0$, we define*

$$W_{p,r} = \left\{\chi \in L^p(]0,T[,B_0), \frac{d\chi}{dt} \in L^r(]0,T[,B_2)\right\}.$$

1. *If $p < \infty$, the embedding of $W_{p,r}$ in $L^p(]0,T[,B_1)$ is compact.*
2. *If $p = \infty$ and if $r > 1$, the embedding of $W_{p,r}$ in $C([0,T],B_1)$ is compact.*

2.2 Green's formula and some inequalities

Definition 2.2.1 ([132]). *An open set Ω in $\mathbb{R}^n$ is said to be m-regular if Ω is bounded and its boundary Γ is a C^m manifold of dimension $n-1$.*

Theorem 2.2.2 (Green's formula, [132]). *Let Ω be a regular open set in $\mathbb{R}^n$ (for example, Ω of class C^1 with Γ bounded). Then, we have Green's formula*

$$\int_\Omega u(x)\frac{\partial v_i}{x_i}(x)dx = -\int_\Omega v(x)\frac{\partial u_i}{x_i}(x)dx + \int_\Gamma u(x)v(x)\eta_i(x)d\sigma,\ \forall u, v \in H^1(\Omega),$$

where $\eta_i = \eta.e_i$ is the i-th coordonnate of η, the outward unit normal vector to Γ.

Based on this theorem, the following result follows:

Corollary 2.2.3 ([132]). *For any $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, the following formula holds:*

$$\int_\Omega \Delta u(x)v(x)dx = -\int_\Omega \nabla u(x).\nabla v(x)dx + \int_\Gamma \frac{\partial u}{\partial \eta}v(x)d\sigma,$$

where $\nabla u = \left(\frac{\partial u}{\partial x_i}\right)_{0\leqslant i\leqslant n}$ is the gradient vector of u, and $\frac{\partial u}{\partial \eta} = \nabla u.\eta$.

In the sequel, we will review some lemmas and concepts that are commonly used in the discussion of the existence, uniqueness, and asymptotic behavior of solutions.

Lemma 2.2.4 (Gronwall's inequality, [106]). *Let $y \in L^\infty(0,T)$ and $g \in L^1(0,T)$ be nonnegative functions and C a positive constant, such that:*

$$y(t) \leqslant C + \int_0^t g(s)y(s)ds \quad a.\ e.\ in\ \ [0,T],$$

then, we have for almost all $t \in [0,T]$,

$$y(t) \leqslant C\exp(\int_0^t g(s)ds).$$

Lemma 2.2.5 (Discrete Gronwall inequality, [79]). *Let (y_n) and (g_n) be nonnegative sequences and C a nonnegative constant. If*

$$y_n \leqslant C + \sum_{0\leqslant k<n} g_k y_k, \quad for\ \ n \geqslant 0,$$

then,

$$y_n \leqslant C\exp\left(\sum_{0\leqslant k<n} g_k\right), \quad for\ \ n \geqslant 0.$$

Lemma 2.2.6 (Young's inequality, [6]). *Let p et q be strictly positive real numbers such that $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\forall (a,b)\in \mathbb{R}_+^2, \forall \varepsilon>0,\ ab \leqslant \frac{\varepsilon}{p}a^p+\frac{1}{q\varepsilon^{\frac{q}{p}}}b^q.$$

In particular, we find the Cauchy inequality if $p=q=2$

$$\forall (a,b)\in \mathbb{R}_+^2,\ ab \leqslant \frac{a^2}{2}+\frac{b^2}{2}.$$

In practice, we use

$$\forall (a,b)\in \mathbb{R}_+^2, \forall \varepsilon>0,\ ab \leqslant \varepsilon a^2+\frac{1}{4\varepsilon}b^2.$$

Theorem 2.2.7 (Continuous dependence on initial conditions, [78]). *Let $F : U \subset \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ be continuous and Lipschitz continuous with respect to the $\mathbb{R}^n$-variable, with Lipschitz constant L. Let $(x_0,t_0),(x_0^*,t_0)\in U$ and*

$$\varphi_{x_0},\ \varphi_{x_0^*} : [t_0-\varepsilon, t_0+\varepsilon] \longrightarrow \mathbb{R}^n,$$

be solutions of the differential equation $x' = F(x,t)$ with initial values $\varphi_{x_0}(t_0)=x_0$ and $\varphi_{x_0^*}(t_0)=x_0^*$. Then:

$$\|\varphi_{x_0}(t_0)-\varphi_{x_0^*}(t_0) \leqslant \|x_0-x_0^*\|.e^{L|t-t_0|}, \forall t\in[t_0-\varepsilon,t_0+\varepsilon].$$

Theorem 2.2.8 (Leibnitz's rule (differentiation under the integral sign), [68]). *Let $f(x,t)$ be a function defined on a domain where both f and $\frac{\partial f}{\partial t}$ are continuous. If $a(t)$ and $b(t)$ are differentiable functions defining the limits of integration, then the derivative of the integral with respect to t is given by:*

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}dx + f(b(t),t)\frac{db(t)}{dt} - f(a(t),t)\frac{da(t)}{dt}.$$

2.3 Basic theory of semigroups associated with dissipative systems

We will cover concepts and results pertaining to strongly continuous semigroups of bounded linear operators on a Banach space.

2.3.1 Strongly continuous semigroups

Definition 2.3.1 ([121]). *Let X be a Banach space. A family $S(t)$, $(0<t<\infty)$ of bounded linear operators from X into X is called to form a strongly continuous semigroup (in short, a C_0-semigroup) if*

1. $S(0)=I$, *(I is the identity operator on X).*

2. $S(t_1+t_2)=S(t_1)S(t_2),\ \forall\, t_1,t_2\geqslant 0.$

3. $\lim_{t\to 0}\|S(t)x-x\|=0,\ \forall\, t\geqslant 0.$

Theorem 2.3.2 ([121]). *Let $S(t)$ be a C_0-semigroup. There exist constants $w \geqslant 0$ and $M \geqslant 1$ such that*

$$\|S(t)\| \leqslant Me^{wt}, \text{ for } 0 \leqslant t < \infty.$$

Definition 2.3.3 ([121]). *Let $S(t)$ be a strongly continuous semigroup of bounded linear operators. The linear operator $\mathcal{A}$ defined by*

$$\mathcal{D}(\mathcal{A}) = \left\{x \in X : \lim_{t \to 0} \frac{S(t)x - x}{t} \text{ exists}\right\}$$

and

$$\mathcal{A}x = \lim_{t \to 0} \frac{S(t)x - x}{t}, \text{ for } x \in \mathcal{D}(\mathcal{A})$$

is called the infinitesimal generator of the semigroup $S(t)$.

Notation 2. *We also denote $S(t)$ by $e^{t\mathcal{A}}$.*

Theorem 2.3.4 ([121]). *Let $S(t)$ be a C_0-semigroup and let $\mathcal{A}$ be its infinitesimal generator. Then*

1. *For $x \in X$, $\int_0^t S(\tau)x d\tau \in \mathcal{D}(\mathcal{A})$ and*

$$\mathcal{A}\left(\int_0^t S(\tau)x d\tau\right) = S(t)x - x.$$

2. *For $x \in \mathcal{D}(\mathcal{A})$, $S(t)x \in \mathcal{D}(\mathcal{A})$ and*

$$\frac{d}{dt}S(t)x = \mathcal{A}S(t)x = S(t)\mathcal{A}x.$$

2.3.2 The Lumer Phillips Theorem

Definition 2.3.5 ([121]). *$S(t)$ is called a C_0-semigroup of contractions if*

$$S(t) \leqslant 1, \ \forall t \geqslant 0.$$

Definition 2.3.6 ([102]). *Let $\mathcal{H}$ be a real or complex Hilbert space equipped with the inner product $(,)$ and the induced norm $\|.\|$. Let $\mathcal{A}$ be a densely defined linear operator on $\mathcal{H}$, i.e. $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subseteq \mathcal{H} \to \mathcal{H}$. We say that $\mathcal{A}$ is dissipative if for any $x \in \mathcal{D}(\mathcal{A})$,*

$$Re(\mathcal{A}x, x) \leqslant 0.$$

Theorem 2.3.7 (Lumer-Phillips, [102]). *Let $\mathcal{A}$ be a linear operator with dense domain $\mathcal{D}(\mathcal{A})$ in a Hilbert space $\mathcal{H}$. If $\mathcal{A}$ is dissipative and there is a $\lambda_0 > 0$ such that the range $R(\lambda_0 I - \mathcal{A})$ is $\mathcal{H}$, then $\mathcal{A}$ is the infinitesimal generator of a C_0- semigroup of contractions on $\mathcal{H}$.*

2.3.3 Wellposedness and regularity

Definition 2.3.8 ([121]). *Let X be a Banach space and let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subseteq X \to X$ be a linear operator. Given $u_0 \in X$, the abstract Cauchy problem for $\mathcal{A}$ with initial data u_0 consists of finding a solution $u(t)$ to the initial value problem*

$$\begin{cases} u'(t) = \mathcal{A}u(t),\ t > 0, \\ u(0) = u_0, \end{cases} \tag{2.2}$$

where by a solution we mean that $u(t)$ continuous for $t \geqslant 0$, continuously differentiable and $u(t) \in \mathcal{D}(\mathcal{A})$ for $t > 0$ and (2.2) is satisfied.

Note that since $u(t) \in \mathcal{D}(\mathcal{A})$ for $t > 0$ and u is continuous at $t = 0$, (2.2) cannot have a solution for $u_0 \notin \overline{\mathcal{D}(\mathcal{A})}$.

Theorem 2.3.9 ([118, 121]). *Let $\mathcal{A}$ generates a C_0-semigroup on X. Then the function $u(t) = S(t)u_0$ solves (2.2) for any given $u_0 \in \mathcal{D}(\mathcal{A})$.*

2.4 The Standard finite-element method in $\mathbb{R}$

Although the existence, uniqueness, and stability of generalized solutions for partial differential equations (PDEs) can be demonstrated theoretically, analytical solutions are often unattainable for cases involving complex data and domains. As a result, numerical solutions become essential. These solutions are derived by substituting the generalized solution spaces with finite-element (FE) spaces constructed by piecewise interpolating polynomials. Consequently, it is crucial to first address the subdivision of the computational domain, the properties of interpolating polynomials, and their associated error estimates.

2.4.1 (Ritz-)Galerkin method

The weak form of the problem (2.1) is defined within a vector space V comprising admissible functions. The finite element method fundamentally relies on incorporating this variational structure (the weak form) into the discretization process. To achieve this, the infinite-dimensional vector space V is approximated by a finite-dimensional space V_h. The objective is to construct V_h such that the solution u_h, computed by the computer, is sufficiently close to the actual continuous solution u.

At this point, assume that $(u_j)_{1 \leqslant j \leqslant N(h)}$ represents a basis for V_h. Then, any $u_h \in V_h$ can be decomposed as

$$u_h = \sum_{j=1}^{N(h)} u_j \varphi_j \tag{2.3}$$

and we can rewrite the problem as follows:

$$\text{find} \quad u_h \in V_h \quad \text{such that} \quad \sum_{j=1}^{N(h)} u_j a(\varphi_j, u_i) = l(u_i),\ i = 1, ..., N(h).$$

Setting $A_h = a_{ij}$ with $a_{ij} = a(\varphi_j, \varphi_i)$ and $F_h = (f_i)$ with $f_i = l(v_i)$, for all $i = 1, ..., N(h)$, we solve the linear system

$$A_h U = F_h.$$

2.4.2 Lagrange P_1 elements

In order to establish the FE space, it is necessary to divide the computational domain $\bar{I} = [0, L] \subset \mathbb{R}$ into some FEs. For the spatial approximation, we introduce a uniform partition $(\Gamma_h)_h$ of the interval $[0, L]$ into M subintervals, such that $0 = x_0 < x_1 < \cdots < x_M = L$, with a uniform length $h = x_{i+1} - x_i = 1/M$. Then, to approximate the variational space $H^1(0, L)$, we construct the finite-dimensional space S^h, defined as

$$S_h = \left\{ v_h \in H^1(I) : \forall \, [x_i, x_{i+1}] \in \Gamma_h, \ v_h|_{[x_i, x_{i+1}]} \in \mathcal{P}_1([x_i, x_{i+1}]) \right\}$$

and its subspaces

$$S_h^* = S_h \cap H_*^1(I), \ S_h^0 = S_h \cap H_0^1(I).$$

Here, h is the spatial discretization parameter, and $\mathcal{P}([x_i, x_{i+1}])$ represents the space of polynomials of degree $\leqslant 1$ in the subinterval $[x_i, x_{i+1}]$. In other words, the finite element space consists of continuous and piecewise affine functions (the vector space of the lowest order Lagrange finite elements). In this special case, the decomposition (2.3) for $u_h \in S_h$ can be rewritten as

$$u_h(x) = \sum_{j=0}^{M} u_h(x_j) \varphi_j(x), \ \forall x \in [0, L],$$

where $x_0, ..., x_M$ represent the mesh nodes, which correspond to the vertices of the simplices. The coefficients to be determined, also known as the degrees of freedom, are directly associated with the values of the function u_h at each node. The basis functions $\varphi_0, ..., \varphi_M$ are defined as hat functions, which take a value of one at their corresponding node and zero at all other nodes in the mesh.

2.4.3 Convergence of the method

Definition 2.4.1 (Interpolation, [17, 63])**.** *The linear mapping $P_h : H^1(0, L) \longrightarrow S_h$ defined for every $u \in H^1(0, L)$ as*

$$(P_h u)(x) = \sum_{j=0}^{M} u(x_j) \varphi_j(x), \ \forall x \in [0, L]$$

and

$$\forall v_h \in S_h, \ ((P_h u - u)_x, v_{hx}) = 0,$$

is called $\mathcal{P}_1$ interpolation (projection) operator. The $\mathcal{P}_1$-interpolate of a function u is the unique piecewise affine function that coincides with u at the mesh vertices x_j . Furthermore, for every $u \in H^1(0, L)$, *the interpolation operator is such that*

$$\lim_{h\to 0}\|u - P_h u\|_{H^1(0,L)} = 0.$$

Lemma 2.4.2 (Interpolation error, [17, 63])**.** *The operator P_h preserves the values at all end points of the elements in Γ_h and satisfies the following estimate for all $u \in H^1(0,L)$:*

$$\|P_h u - u\| \leqslant Ch\|u_x\| \tag{2.4}$$

and for more regular functions u,

$$\|P_h u - u\| + h\|(P_h u - u)_x\| \leqslant Ch^2\|u_{xx}\|, \tag{2.5}$$

where C is a constant independent of h.

Now, we can establish the convergence of the finite-element method.

Theorem 2.4.3 (Convergence, [17, 63])**.** *Suppose that $u \in H_0^1(0,L)$ and $u_h \in V_h^0$. The Lagrange P_1 finite-element method converges, i.e. we have:*

$$\lim_{h\to 0}\|u - u_h\|_{H^1(0,L)} = 0.$$

The operators

$$P_h^0 : H_0^1(0,L) \longrightarrow S_h^0,\ P_h^* : H_*^1(0,L) \longrightarrow S_h^*$$

have similar properties.

2.4.4 Non-stationary PDEs

For non-stationary partial differential equations, time discretization involves more than just ensuring accuracy (method order). It also requires careful consideration of stability, dissipation, and dispersion (including spectral resolution and phase deviation, which are crucial for wave propagation) and all the computational aspects such as runing time, memory usage, scalability, and, increasingly, the energy cost of performing the simulation play a significant role.

In the next chapters, to discretize the time derivatives for a given final time $T > 0$ and a given positive integer N, we define the time step $\Delta t = T/N$ and the nodes $t_n = n\Delta t$, $n = 0, \ldots, N$.

2.4.5 Numerical errors

The finite-element discretization (the discrete problem) can be applied to practical problems and reformulated as a linear system (see [62]). At this stage, a computer can be used to solve it. The resulting vector of nodal unknowns, $\widetilde{U}$ may slightly deviate from the theoretical (exact) vector U mentioned above and that serves to reconstruct u_h. The difference between $\widetilde{U}$ and U is called the numerical error.

2.4.6 Numerical illustration

Numerical simulations can be performed using MATLAB. The MATLAB scripts outlined in [92] provide a comprehensive understanding of all implementation aspects while minimizing technical complexities and keeping close to the mathematical theory.

Chapter 3

Analysis of a Shear beam model with suspenders in thermoelasticity of type III

3.1 Introduction

A cable-suspended beam is a structural design comprising a beam that is upheld by one or more cables. These cables have the dual role of bearing the beam weight and preserving its shape to enhance stability and provide additional support. Cable-suspended beams find applications in a range of engineering contexts, including cable-stayed bridges and suspension bridges [113].

In this chapter, we address a thermomechanical problem associated with a cable-suspended beam structure, exemplified by the suspension bridge. The key characteristic of this structure is that the roadbed sectional dimensions are significantly smaller than its length (span of the bridge). Suspension bridges with large span lengths exhibit higher flexibility in comparison to alternative bridge structures. This increased flexibility renders them vulnerable to various dynamic loads, including wind, earthquakes, and the movement of vehicles. The distinctive structural features of suspension bridges elevate the significance of understanding their dynamic response to oscillations, presenting a crucial engineering challenge. Therefore, we model the roadbed as an extensible thermoelastic Shear-type beam. The primary suspension cable is represented as an elastic string, and it is linked to the roadbed through a distributed network of elastic springs. Our model aligns with the configuration of a Shear-suspended-beam system within thermoelasticity of type III, as described in the following problem:

$$\begin{cases} \rho u_{tt} - \alpha u_{xx} - \lambda(\varphi - u) + \mu u_t = 0, \text{ in } (0, L) \times (0, \infty), \\ \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x + \lambda(\varphi - u) + \gamma \varphi_t \\ \qquad\qquad + \beta \theta_x = 0, \text{ in } (0, L) \times (0, \infty), \\ -b\psi_{xx} + K(\varphi_x + \psi) = 0, \text{ in } (0, L) \times (0, \infty), \\ \rho_3 \theta_{tt} - \delta \theta_{xx} + \beta \varphi_{xtt} - \kappa \theta_{xxt} = 0, \text{ in } (0, L) \times (0, \infty), \\ u(0, t) = u(L, t) = \varphi(0, t) = \varphi(L, t) = 0, \ t \geqslant 0, \\ \psi(0, t) = \psi(L, t) = \theta(0, t) = \theta(L, t) = 0, \ t \geqslant 0, \\ u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x), \ x \in (0, L), \\ \varphi(x, 0) = \varphi_0(x), \ \varphi_t(x, 0) = \varphi_1(x), \ x \in (0, L), \\ \theta(x, 0) = \theta_0(x), \ \theta_t(x, 0) = \theta_1(x), \ x \in (0, L). \end{cases} \tag{3.1}$$

Here, the symbol t represents the time variable, while x denotes the distance along the centerline of the beam in its equilibrium configuration. Both L and the beam length coincide with the cable length. Within this framework, we use the following notations: u represents the vertical displacement of the vibrating spring in the primary cable; φ is the transverse displacement or vertical deflection of the beam cross-section; ψ denotes the angle of rotation of a cross-section; while θ is employed to represent the thermal moment of the beam. We consider the suspender cables (ties) as linear elastic springs with a shared stiffness parameter $\lambda > 0$, and the constant $\alpha > 0$ defines the elastic modulus of the string that connects the cable to the deck. The constants ρ, μ, ρ_1, ρ_3, K, γ, δ, κ, δ, and $\beta \neq 0$ are positive. The initial data, denoted as $(u_0, u_1, \varphi_0, \varphi_1, \theta_0, \theta_1)$, belong to an appropriate functional space.

For many years, there has been substantial research interest in the dynamic behavior and nonlinear vibrations of suspension bridges [2,3,5,48]. Suspension bridges are complex structures with distinctive dynamic characteristics and they can display nonlinear responses to various external forces and loads. This nonlinearity may stem from factors such as large-amplitude vibrations, material properties, and environmental conditions. The appearance of string-beam systems that model a nonlinear coupling of a beam (the roadbed) and main cable (the string) were came out of the pioneering works of Lazer and McKenna [95,96]. Elimination of nonlinear coupling terms in the equations of motion governing vertical and torsional vibrations of suspension bridges is achieved by linearization, as indicated in reference [1]. Previous approaches have often employed models where the roadbed was based on the Euler-Bernoulli beam theory [21–24]. The Timoshenko beam theory also has demonstrated superior performance in anticipating the vibrational response of a beam compared to a model grounded in the classical Euler-Bernoulli beam theory [77].

It's worth noting that the Euler-Bernoulli beam theory and the Timoshenko beam theory are two different approaches to modeling the behavior of beams. The Euler-Bernoulli beam theory assumes that the beam is slender and that the cross-section remains plane and perpendicular to the longitudinal axis of the beam during deformation. This theory is suitable for modeling long and slender beams that are subjected to bending loads. On the other hand, the Timoshenko beam theory takes into account the effects of shear deformation and rotational bending, which makes this theory more accurate for analyzing short and thick beams.

The Euler-Bernoulli beam theory, used to represent the single-span roadbed, neglects the effects of shear deformation and rotary inertia. These effects can be taken into account by using more accurate models, such as the Timoshenko beam theory to have a deeper range of applicability and to be physically more realistic [13, 94]. Bochicchio et al. examined a linear problem that characterizes the vibrations of a coupled suspension bridge [20]. In their analysis, the single-span roadbed is represented as an extensible thermoelastic beam following the Timoshenko model, with heat governed by Fourier's law. They demonstrated the existence and uniqueness of solutions by using the semigroup-theory and the exponential decay property by employing the energy method. Additionally, they conducted several numerical experiments to further support their findings. Mukiawa et al. utilized the same methods to establish the existence and uniqueness of a weak global solution and to demonstrate exponential stability in a thermal-Timoshenko-beam system [112]. This system incorporated suspenders and Kelvin–Voigt damping and was founded on the principles of thermoelasticity, as described by Cattaneo's law.

Almeida Júnior et al. [89] and Ramos et al. [129] were pioneers in investigating the well-posedness and stability characteristics of the Shear beam model. This model constitutes an improvement over the Euler-Bernoulli beam model by adding the shear distortion effect but without rotary inertia. Consequently, one can analyze long-span bridges, unlike the Timoshenko beam theory, which is better suited for modeling beams with relatively short spans. This is precisely our focus here – the examination of the Shear model (3.1) with suspenders and thermal dissipation, given by thermoelasticity of Type III.

The structure of this chapter is outlined as follows. In Section 3.2, we provide some preliminary information. To demonstrate the global existence and uniqueness of solutions, in Section 3.3, we use the Faedo–Galerkin method, along with three estimates, previously discussed in references such as [47, 97]. In Section 3.4, we employ the energy method to construct several Lyapunov functionals, establishing the property of exponential decay. In Section 3.5, we propose a finite-element-discretization approach to solve the problem at hand. We obtain discrete stability results and a priori error estimates. Finally, in Section 3.6, we present some numerical simulations using MATLAB.

Throughout the chapter, C is used to represent a generic positive constant.

3.2 Preliminaries

In order to exhibit the dissipative nature of system (3.1), we introduce the new variable (heat displacement)

$$w(x,t) = \int_0^t \theta(x,s)ds + \eta(x), \tag{3.2}$$

where $\eta(x)$ solves

$$\begin{cases} \delta\Delta\eta = \rho_3\theta_1 - \kappa\Delta\theta_0 + \beta\nabla\cdot\varphi_1, \\ \eta(0) = \eta(L) = 0. \end{cases} \tag{3.3}$$

Performing an integration of $(3.1)_4$ with respect to t, we get

$$\rho_3\int_0^t \theta_{tt}ds - \delta\int_0^t \theta_{xx}ds + \beta\int_0^t \varphi_{xtt}ds - \kappa\int_0^t \theta_{xxt}ds = 0,$$

taking into account (3.2), we arrive at

$$\rho_3 w_{tt} - \rho_3\theta_1 - (\delta w_{xx} - \delta\Delta\eta) + \beta\varphi_{xt} - \beta\nabla\cdot\varphi_1 - \kappa w_{xxt} + \kappa\Delta\theta_0 = 0.$$

Then (3.3) gives

$$\rho_3 w_{tt} - (\delta w_{xx} - \delta\Delta\eta) + \beta\varphi_{xt} - \kappa w_{xxt} - \delta\Delta\eta = 0.$$

The last equation leads to

$$\rho_3 w_{tt} - \delta w_{xx} + \beta\varphi_{xt} - \kappa w_{xxt} = 0,$$

and problem (3.1) takes the form

$$\begin{cases} \rho u_{tt} - \alpha u_{xx} - \lambda(\varphi - u) + \mu u_t = 0, \text{ in } (0,L)\times(0,\infty), \\ \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x + \lambda(\varphi - u) + \gamma\varphi_t + \beta w_{xt} = 0, \text{ in } (0,L)\times(0,\infty), \\ -b\psi_{xx} + K(\varphi_x + \psi) = 0, \text{ in } (0,L)\times(0,\infty), \\ \rho_3 w_{tt} - \delta w_{xx} + \beta\varphi_{xt} - \kappa w_{xxt} = 0, \text{ in } (0,L)\times(0,\infty), \\ u(0,t) = u(L,t) = \varphi(0,t) = \varphi(L,t) = 0, \ t \geqslant 0, \\ \psi(0,t) = \psi(L,t) = w(0,t) = w(L,t) = 0, \ t \geqslant 0, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in (0,L), \\ \varphi(x,0) = \varphi_0(x), \ \varphi_t(x,0) = \varphi_1(x), \ x \in (0,L), \\ w(x,0) = w_0(x), \ w_t(x,0) = w_1(x), \ x \in (0,L). \end{cases} \tag{3.4}$$

Lemma 3.2.1. *Let (u,φ,ψ,w) be the solution of (3.4). Then the energy functional E, defined by*

$$E(t) = \frac{1}{2}\int_0^L \Big\{\rho u_t^2 + \alpha u_x^2 + \lambda(\varphi - u)^2 + \rho_1\varphi_t^2 + K(\varphi_x + \psi)^2 + b\psi_x^2 + \rho_3 w_t^2 + \delta w_x^2\Big\}dx \tag{3.5}$$

satisfies

$$\frac{dE(t)}{dt} = -\mu\int_0^L u_t^2dx - \gamma\int_0^L \varphi_t^2dx - \kappa\int_0^L w_{xt}^2dx \leqslant 0, \ \forall\, t \geqslant 0. \tag{3.6}$$

Proof. We multiply the equations in (3.4) by u_t, φ_t, ψ_t, w_t, respectively, and integrate by parts to get

$$\rho\int_0^L u_{tt}u_t dx + \alpha\int_0^L u_x u_{xt} dx - \lambda\int_0^L \varphi u_t dx + \lambda\int_0^L u u_t dx = -\mu\int_0^L u_t^2 dx,$$

$$\begin{aligned}\rho_1\int_0^L \varphi_{tt}\varphi_t dx + K\int_0^L \varphi_x\varphi_{xt} dx + K\int_0^L \psi\varphi_{xt} dx + \lambda\int_0^L \varphi\varphi_t dx \\ - \lambda\int_0^L u\varphi_t dx + \beta\int_0^L w_{xt}\varphi_t dx = -\gamma\int_0^L \varphi_t^2 dx,\end{aligned}$$

$$b\int_0^L \psi_x\psi_{xt} dx + K\int_0^L \varphi_x\psi_t dx + K\int_0^L \psi\psi_t dx = 0,$$

$$\rho_3\int_0^L w_{tt}w_t dx + \delta\int_0^L w_{xx}w_t dx + \beta\int_0^L \varphi_{xt}w_t dx = -\kappa\int_0^L w_{xt}^2 dx.$$

From the above four equations, we conclude

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\rho\int_0^L u_t^2 dx\right) + \frac{1}{2}\frac{d}{dt}\left(\alpha\int_0^L u_x^2 dx\right) \\ &+ \frac{1}{2}\frac{d}{dt}\left(\lambda\int_0^L u^2 dx\right) - \lambda\int_0^L \varphi u_t dx = -\mu\int_0^L u_t^2 dx,\end{aligned} \tag{3.7}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\rho_1\int_0^L \varphi_t^2 dx\right) + \frac{1}{2}\frac{d}{dt}\left(K\int_0^L \varphi_x^2 dx\right) + K\int_0^L \psi\varphi_{xt} dx \\ &+ \frac{1}{2}\frac{d}{dt}\left(\lambda\int_0^L \varphi^2 dx\right) - \lambda\int_0^L u\varphi_t dx + \beta\int_0^L w_{xt}\varphi_t dx = -\gamma\int_0^L \varphi_t^2 dx,\end{aligned} \tag{3.8}$$

$$\frac{1}{2}\frac{d}{dt}\left(b\int_0^L \psi_x^2 dx\right) + \frac{1}{2}\frac{d}{dt}\left(K\int_0^L \psi^2 dx\right) + K\int_0^L \varphi_x\psi_t dx = 0, \tag{3.9}$$

$$\frac{1}{2}\frac{d}{dt}\left(\rho_3\int_0^L w_t^2 dx\right) + \beta\int_0^L \varphi_{xt}w_t dx + \frac{1}{2}\frac{d}{dt}\left(\delta\int_0^L w_x^2 dx\right) = -\kappa\int_0^L w_{xt}^2 dx. \tag{3.10}$$

Adding (3.7)–(3.10), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^L \left\{\rho u_t^2 + \alpha u_x^2 + \lambda(\varphi - u)^2 + \rho_1\varphi_t^2 + K(\varphi_x + \psi)^2 + b\psi_x^2 + \rho_3 w_t^2 + \delta w_x^2\right\} dx \\ &= -\mu\int_0^L u_t^2 dx - \gamma\int_0^L \varphi_t^2 dx - \kappa\int_0^L w_{xt}^2 dx,\end{aligned}$$

then (3.6) holds. These calculations are done for any regular solution. Nonetheless, the same result holds valid for weak solutions by using density arguments. □

3.3 Global well-posedness

In this section we prove the existence and uniqueness of regular and weak solutions for system (3.4) by using the Faedo-Galerkin method.

Theorem 3.3.1. *1. If the initial data $u_0, \varphi_0, \psi_0, w_0 \in H_0^1(0,L)$ and $u_1, \varphi_1, w_1 \in L^2(0,L)$, then problem (3.4) has a unique weak solution satisfying*

$$u, \varphi, w \in C([0,T], H_0^1(0,L)) \cap C^1([0,T], L^2(0,L)),\ \psi \in C([0,T], H_0^1(0,L)). \tag{3.11}$$

2. If the initial data $u_0, \varphi_0, \psi_0, w_0 \in H^2(0,L) \cap H_0^1(0,L)$ and $u_1, \varphi_1, w_1 \in H_0^1(0,L)$, then the weak solution (3.11) *has higher regularity:*

$$u, \varphi, w \in C([0,T], H^2(0,L) \cap H_0^1(0,L)) \cap C^1([0,T], H_0^1(0,L)) \cap C^2([0,T], L^2(0,L)),$$
$$\psi \in C([0,T], H^2(0,L) \cap H_0^1(0,L)).$$

Proof. We prove the result in six steps.

Step 1: approximated problem. Suppose that u_0, φ_0, ψ_0, $w_0 \in H^2(0,L) \cap H_0^1(0,L)$, u_1, φ_1, w_1 $\in H_0^1(0,L)$. Let $\{v_i\}_{i\in\mathbb{N}}$ be a smooth orthonormal basis of $L^2(0,L)$, which is also orthogonal in $H^2(0,L) \cap H_0^1(0,L)$, given by the eigenfunctions of $-v_{xx} = \varsigma v$ with boundary condition $v(0) = v(1) = 0$, such that

$$-v_{ixx} = \varsigma_i v_i. \tag{3.12}$$

Here ς_i is the eigenvalue corresponding to v_i. For any $m \in \mathbb{N}$, denote by V_m the finite-dimensional subspace

$$V_m = span\,\{v_1, v_2, \ldots, v_m\} \subset H^2(0,L) \cap H_0^1(0,L).$$

To construct the Galerkin approximation $(u^m, \varphi^m, \psi^m, w^m)$ of the solution, let us denote

$$u^m(t) = \sum_{i=1}^{m} g_{im}(t) v_i,\ \varphi^m(t) = \sum_{i=1}^{m} \hat{g}_{im}(t) v_i,$$

$$\psi^m(t) = \sum_{i=1}^{m} f_{im}(t) v_i,\ w^m(t) = \sum_{i=1}^{m} \hat{f}_{im}(t) v_i,$$

where functions g_{im}, $\hat{g}_{im}$, f_{im}, $\hat{f}_{im}$ are given by the solution of the approximated system

$$\begin{cases} \rho(u_{tt}^m(t), v_i) + \alpha(u_x^m(t), v_{ix}) - \lambda(\varphi^m(t) - u^m(t), v_i) + \mu(u_t^m(t), v_i) = 0, \\ \rho_1(\varphi_{tt}^m(t), v_i) + K(\varphi_x^m(t) + \psi^m(t), v_{ix}) + \lambda(\varphi^m(t) - u^m(t), v_i) \\ \qquad\qquad + \gamma(\varphi_t^m(t), v_i) + \beta(w_{xt}^m(t), v_i) = 0, \\ b(\psi_x^m(t), v_{ix}) + K(\varphi_x^m(t) + \psi^m(t), v_i) = 0, \\ \rho_3(w_{tt}^m(t), v_i) + \delta(w_x^m(t), v_{ix}) + \beta(\varphi_{xt}^m(t), v_i) + \kappa(w_{xt}^m(t), v_{ix}) = 0, \end{cases} \tag{3.13}$$

with the initial conditions

$$u^m(0) = u_0^m = \sum_{i=1}^m (u_0, v_i)v_i \xrightarrow{m\to\infty} u_0 \text{ in } H^2(0,L) \cap H_0^1(0,L), \tag{3.14}$$

$$\varphi^m(0) = \varphi_0^m = \sum_{i=1}^m (\varphi_0, v_i)v_i \xrightarrow{m\to\infty} \varphi_0 \text{ in } H^2(0,L) \cap H_0^1(0,L), \tag{3.15}$$

$$\psi^m(0) = \psi_0^m = \sum_{i=1}^m (\psi_0, v_i)v_i \xrightarrow{m\to\infty} \psi_0 \text{ in } H^2(0,L) \cap H_0^1(0,L), \tag{3.16}$$

$$w^m(0) = w_0^m = \sum_{i=1}^m (w_0, v_i)v_i \xrightarrow{m\to\infty} w_0 \text{ in } H^2(0,L) \cap H_0^1(0,L), \tag{3.17}$$

$$u_t^m(0) = u_1^m = \sum_{i=1}^m (u_1, v_i)v_i \xrightarrow{m\to\infty} u_1 \text{ in } H_0^1(0,L), \tag{3.18}$$

$$\varphi_t^m(0) = \varphi_1^m = \sum_{i=1}^m (\varphi_1, v_i)v_i \xrightarrow{m\to\infty} \varphi_1 \text{ in } H_0^1(0,L), \tag{3.19}$$

$$w_t^m(0) = w_1^m = \sum_{i=1}^m (w_1, v_i)v_i \xrightarrow{m\to\infty} w_1 \text{ in } H_0^1(0,L). \tag{3.20}$$

The system (3.13) can be rewritten as

$$\begin{cases} g'_{im} + (\alpha\varsigma_i + \lambda)g_{im} - \lambda\hat{g}_{im} = 0, \\ \rho g''_{im} + \mu\rho_1\hat{g}''_{im} + (K\varsigma_i + \lambda)\hat{g}_{im} - \lambda g_{im} \\ \qquad + \beta\sum_{j=1}^m (v_{jx}, v_i)\hat{f}_{jm} = K\sum_{j=1}^m (v_{jx}, v_i)f_{jm}, \\ (b\varsigma_i + K)f_{im} + K\sum_{j=1}^m (v_{jx}, v_i)\hat{g}_{jm} = 0, \\ \rho_3\hat{f}''_{im} + \kappa\varsigma_i\hat{f}'_{im} + \delta\varsigma_i\hat{f}_{im} + \beta\sum_{j=1}^m (v_{jx}, v_i)\hat{g}'_{jm} = 0. \end{cases} \tag{3.21}$$

The third equation of (3.21) gives

$$f_{im} = -\frac{K}{b\varsigma_i + K}\sum_{j=1}^{m}(v_{jx}, v_i)\hat{g}_{jm}, \tag{3.22}$$

then, we have

$$\begin{cases} \rho g''_{im} + \mu g'_{im} + (\alpha\varsigma_i + \lambda)g_{im} - \lambda\hat{g}_{im} = 0 \\ \rho_1\hat{g}''_{im} + (K\varsigma_i + \lambda)\hat{g}_{im} - \lambda g_{im} + \beta\sum_{j=1}^{m}(v_{jx}, , v_i)\hat{f}_{jm}, \\ \qquad = -\sum_{j,k=1}^{m}\frac{K^2}{b\varsigma_j + K}(v_{jx}, v_i)(v_{kx}, v_j)\hat{g}_{km}, \\ \rho_3\hat{f}''_{im} + \kappa\varsigma_i\hat{f}'_{im} + \delta\varsigma_i\hat{f}_{im} + \beta\sum_{j=1}^{m}(v_{jx}, v_i)\hat{g}'_{jm} = 0. \end{cases} \tag{3.23}$$

Using the theory of ODEs, the problem (3.23) has a unique solution $(g_{im}, \hat{g}_{im}, \hat{f}_{im}) \in (C^2[0, T_m))^3$. Then, from (3.22) we deduce that $f_{im} \in C^2[0, T_m)$. As a result, system (3.13)–(3.20) has a unique local solution $(u^m(t), \varphi^m(t), \psi^m(t), w^m(t))$ defined on $[0, T_m)$ with $0 < T_m < T$. Our next step is to show that the local solution is extended to $[0, T]$ for any given $T > 0$.

Step 2: first a priori estimate. Multiplying $(3.13)_1$, $(3.13)_2$, $(3.13)_3$ and $(3.13)_4$ by g'_{im}, $\hat{g}'_{im}$, f'_{im} and $\hat{f}'_{im}$, respectively, and taking the L^2 inner product to get

$$\rho(u^m_{tt}(t), g'_{im}(t)v_i) + \alpha(u^m_x(t), g'_{im}(t)v_{ix}) - \lambda(\varphi^m(t) - u^m(t), g'_{im}(t)v_i) \\ + \mu(u^m_t(t), g'_{im}(t)v_i) = 0,$$

$$\rho_1(\varphi^m_{tt}(t), \hat{g}'_{im}(t)v_i) + K(\varphi^m_x(t) + \psi^m(t), \hat{g}'_{im}(t)v_{ix}) + \lambda(\varphi^m(t) - u^m(t), \hat{g}'_{im}(t)v_i) \\ + \gamma(\varphi^m_t(t), \hat{g}'_{im}(t)v_i) + \beta(w^m_{xt}(t), \hat{g}'_{im}(t)v_i) = 0,$$

$$b(\psi^m_x(t), f'_{im}(t)v_{ix}) + K\left(\varphi^m_x(t) + \psi^m(t), f'_{im}(t)v_i\right) = 0,$$

$$\rho_3(w^m_{tt}(t), \hat{f}'_{im}(t)v_i) + \delta(w^m_x(t), \hat{f}'_{im}(t)v_{ix}) + \beta(\varphi^m_{xt}(t), \hat{f}'_{im}(t)v_i) \\ + \kappa(w^m_{xt}(t), \hat{f}'_{im}(t)v_{ix}) = 0.$$

Next, summing up over i from 1 to m, to obtain

$$\rho(u^m_{tt}(t), \sum_{i=1}^{m} g'_{im}(t)v_i) + \alpha(u^m_x(t), \sum_{i=1}^{m} g'_{im}(t)v_{ix}) - \lambda(\varphi^m(t) - u^m(t), \sum_{i=1}^{m} g'_{im}(t)v_i) \\ + \mu(u^m_t(t), \sum_{i=1}^{m} g'_{im}(t)v_i) = 0,$$

$$\rho_1(\varphi_{tt}^m(t), \sum_{i=1}^m \hat{g}'_{im}(t)v_i) + K(\varphi_x^m(t) + \psi^m(t), \sum_{i=1}^m \hat{g}'_{im}(t)v_{ix})$$
$$+ \lambda(\varphi^m(t) - u^m(t), \sum_{i=1}^m \hat{g}'_{im}(t)v_i) + \gamma(\varphi_t^m(t), \sum_{i=1}^m \hat{g}'_{im}(t)v_i) + \beta(w_{xt}^m(t), \sum_{i=1}^m \hat{g}'_{im}(t)v_i) = 0,$$
$$b(\psi_x^m(t), \sum_{i=1}^m f'_{im}(t)v_{ix}) + K\left(\varphi_x^m(t) + \psi^m(t), \sum_{i=1}^m f'_{im}(t)v_i\right) = 0,$$
$$\rho_3(w_{tt}^m(t), \sum_{i=1}^m \hat{f}'_{im}(t)v_i) + \delta(w_x^m(t), \sum_{i=1}^m \hat{f}'_{im}(t)v_{ix}) + \beta(\varphi_{xt}^m(t), \sum_{i=1}^m \hat{f}'_{im}(t)v_i)$$
$$+ \kappa(w_{xt}^m(t), \sum_{i=1}^m \hat{f}'_{im}(t)v_{ix}) = 0.$$

Then, we have

$$\begin{cases} \rho(u_{tt}^m(t), u_t^m(t)) + \alpha(u_x^m(t), u_{xt}^m(t)) - \lambda(\varphi^m(t) - u^m(t), u_t^m(t)) \\ \qquad + \mu(u_t^m(t), u_t^m(t)) = 0, \\ \rho_1(\varphi_{tt}^m(t), \varphi_t^m(t)) + K(\varphi_x^m(t) + \psi^m(t), \varphi_{xt}^m(t)) + \lambda(\varphi^m(t) - u^m(t), \varphi_t^m(t)) \\ \qquad + \gamma(\varphi_t^m(t), \varphi_t^m(t)) + \beta(w_{xt}^m(t), \varphi_t^m(t)) = 0, \\ b(\psi_x^m(t), \psi_{xt}^m) + K(\varphi_x^m(t) + \psi^m(t), \psi_t^m) = 0, \\ \rho_3(w_{tt}^m(t), w_t^m(t)) + \delta(w_x^m(t), w_{xt}^m(t)) + \beta(\varphi_{xt}^m(t), w_t^m(t)) + \kappa(w_{xt}^m(t), w_{xt}^m(t)) = 0. \end{cases} \tag{3.24}$$

Taking the sum of the resulting equations in (3.24) and using integration by parts, with a similar way to the proof of lemma 3.2.1, we obtain

$$\begin{aligned} &\frac{d}{dt}\Big(\frac{\rho}{2}\|u_t^m(t)\|^2 + \frac{\alpha}{2}\|u_x^m(t)\|^2 + \frac{\lambda}{2}\|\varphi^m(t) - u^m(t)\|^2 + \frac{\rho_1}{2}\|\varphi_t^m(t)\|^2 \\ &\quad + \frac{K}{2}\|\varphi_x^m(t) + \psi^m(t)\|^2 + \frac{b}{2}\|\psi_x^m(t)\|^2 + \frac{\rho_3}{2}\|w_t^m(t)\|^2 + \frac{\delta}{2}\|w_x^m(t)\|^2\Big) \\ &\quad + \mu\|u_t^m(t)\|^2 + \gamma\|\varphi_t^m(t)\|^2 + \kappa\|w_{xt}^m(t)\|^2 = 0. \end{aligned} \tag{3.25}$$

Integrating (3.25) over $(0, t)$, we find

$$\mathcal{E}_1^m(t) + \mu\int_0^t \|u_t^m(s)\|^2 ds + \gamma\int_0^t \|\varphi_t^m(s)\|^2 ds + \kappa\int_0^t \|w_{xt}^m(s)\|^2 ds = \mathcal{E}_1^m(0), \tag{3.26}$$

where

$$\begin{aligned} \mathcal{E}_1^m(t) = \frac{1}{2}\Big(&\rho\|u_t^m(t)\|^2 + \alpha\|u_x^m(t)\|^2 + \lambda\|\varphi^m(t) - u^m(t)\|^2 + \rho_1\|\varphi_t^m(t)\|^2 \\ &+ K\|\varphi_x^m(t) + \psi^m(t)\|^2 + b\|\psi_x^m(t)\|^2 + \rho_3\|w_t^m(t)\|^2 + \delta\|w_x^m(t)\|^2\Big). \end{aligned}$$

From (3.26), we deduce that there exists C, independent of m, such that

$$\mathcal{E}_1^m(t) \leqslant \mathcal{E}_1^m(0) \leqslant C, \ t \geqslant 0. \tag{3.27}$$

Consequently, we have

$$\begin{aligned} &\|u_t^m(t)\|^2 + \|u_x^m(t)\|^2 + \|\varphi^m(t) - u^m(t)\|^2 + \|\varphi_t^m(t)\|^2 \\ &+ \|\varphi_x^m(t) + \psi^m(t)\|^2 + \|\psi_x^m(t)\|^2 + \|w_t^m(t)\|^2 + \|w_x^m(t)\|^2 \leqslant C, \ t \geqslant 0. \end{aligned} \tag{3.28}$$

The estimate (3.28) implies that the solution $(u^m, \varphi^m, \psi^m, w^m)$ exists globally in $[0, \infty)$ and, for any $m \in \mathbb{N}$,

$$u^m, \ \varphi^m, \ \psi^m, \ w^m \ \text{ are bounded in } \ L^\infty(0, T; H_0^1(0, L)), \tag{3.29}$$

$$u_t^m, \ \varphi_t^m, \ w_t^m \ \text{ are bounded in } \ L^\infty(0, T; L^2(0, L)). \tag{3.30}$$

Step 3: second a priori estimate. Taking the derivative of the approximate equations in (3.13) with respect to t, we get

$$\begin{cases} \rho(u_{ttt}^m(t), v_i) + \alpha(u_{xt}^m(t), v_{ix}) - \lambda(\varphi_t^m(t) - u_t^m(t), v_i) + \mu(u_{tt}^m(t), v_i) = 0, \\ \rho_1(\varphi_{ttt}^m(t), v_i) + K(\varphi_{xt}^m(t) + \psi_t^m(t), v_{ix}) + \lambda(\varphi_t^m(t) - u_t^m(t), v_i) \\ \qquad + \gamma(\varphi_{tt}^m(t), v_i) + \beta(w_{xtt}^m, v_i) = 0, \\ b(\psi_{xt}^m(t), v_{ix}) + K(\varphi_{xt}^m(t) + \psi_t^m(t), v_i) = 0, \\ \rho_3(w_{ttt}^m(t), v_i) + \delta(w_{xt}^m(t), v_{ix}) + \beta(\varphi_{xtt}^m(t), v_i) + \kappa(w_{xtt}^m, v_{ix}) = 0. \end{cases} \tag{3.31}$$

Multiplying $(3.31)_1$, $(3.31)_2$, $(3.31)_3$ and $(3.31)_4$ by g''_{im}, $\hat{g}''_{im}$, f''_{im} and $\hat{f}''_{im}$, respectively, and summing up over i from 1 to m, it follows that

$$\begin{cases} \rho(u_{ttt}^m(t), u_{tt}^m(t)) + \alpha(u_{xt}^m(t), u_{xtt}^m(t)) - \lambda(\varphi_t^m(t) - u_t^m(t), u_{tt}^m(t)) \\ \qquad + \mu(u_{tt}^m(t), u_{tt}^m(t)) = 0, \\ \rho_1(\varphi_{ttt}^m(t), \varphi_{tt}^m(t)) + K(\varphi_{xt}^m(t) + \psi_t^m(t), \varphi_{xtt}^m(t)) + \lambda(\varphi_t^m(t) - u_t^m(t), \varphi_{tt}^m(t)) \\ \qquad + \gamma(\varphi_{tt}^m(t), \varphi_{tt}^m(t)) + \beta(w_{xtt}^m, \varphi_{tt}^m(t)) = 0, \\ b(\psi_{xt}^m(t), \psi_{xtt}^m(t)) + K(\varphi_{xt}^m(t) + \psi_t^m(t), \psi_{tt}^m(t)) = 0, \\ \rho_3(w_{ttt}^m(t), w_{tt}^m(t)) + \delta(w_{xt}^m(t), w_{xtt}^m(t)) + \beta(\varphi_{xtt}^m(t), w_{tt}^m(t)) + \kappa(w_{xtt}^m, w_{xtt}^m(t)) = 0. \end{cases} \tag{3.32}$$

From (3.32), we infer that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\rho\|u_{tt}^m(t)\|^2\right) + \frac{1}{2}\frac{d}{dt}\left(\alpha\|u_{tx}^m(t)\|^2\right) \\ &- \lambda\int_0^L (\varphi_t^m(t) - u_t^m(t))u_{tt}^m(t)dx + \mu\|u_{tt}^m(t)\|^2 = 0, \end{aligned} \tag{3.33}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi_{tt}^m(t)\|^2\right)+K\int_0^L(\varphi_{xt}^m(t)+\psi_t^m(t))\varphi_{xtt}^m(t)dx+\gamma\|\varphi_{tt}^m(t)\|^2\\&+\lambda\int_0^L(\varphi_t^m(t)-u_t^m(t))\varphi_{tt}^m(t)dx+\beta\int_0^L w_{xtt}^m(t)\varphi_{tt}^m(t)dx=0,\end{aligned}\tag{3.34}$$

$$\frac{1}{2}\frac{d}{dt}\left(b\|\psi_{xt}^m(t)\|^2\right)+K\int_0^L(\varphi_{xt}^m(t)+\psi_t^m(t))\psi_{tt}^m(t)dx=0,\tag{3.35}$$

$$\frac{1}{2}\frac{d}{dt}\left(\rho_3\|w_{tt}^m(t)\|^2\right)+\frac{1}{2}\frac{d}{dt}\left(\delta\|w_{xt}^m(t)\|^2\right)+\beta\int_0^L\varphi_{xt}^m(t)w_{tt}^m(t)dx+\kappa\|w_{xtt}^m(t)\|^2=0.\tag{3.36}$$

Adding up (3.33)–(3.36) and using integration by parts, we obtain that

$$\begin{aligned}&\frac{d}{dt}\Big(\frac{\rho}{2}\|u_{tt}^m(t)\|^2+\frac{\alpha}{2}\|u_{xt}^m(t)\|^2+\frac{\lambda}{2}\|\varphi_t^m(t)-u_t^m(t)\|^2+\frac{\rho_1}{2}\|\varphi_{tt}^m(t)\|^2\\&+\frac{K}{2}\|\varphi_{xt}^m(t)+\psi_t^m(t)\|^2+\frac{b}{2}\|\psi_{xt}^m(t)\|^2+\frac{\rho_3}{2}\|w_{tt}^m(t)\|^2+\frac{\delta}{2}\|w_{xt}^m(t)\|^2\Big)\\&+\mu\|u_{tt}^m(t)\|^2+\gamma\|\varphi_{tt}^m(t)\|^2+\kappa\|w_{xtt}^m(t)\|^2=0.\end{aligned}\tag{3.37}$$

Now, integrating (3.37) over $(0,t)$, yields

$$\mathcal{E}_2^m(t)+\mu\int_0^t\|u_{tt}^m(s)\|^2ds+\gamma\int_0^t\|\varphi_{tt}^m(s)\|^2ds+\kappa\int_0^t\|w_{xtt}^m(s)\|^2ds=\mathcal{E}_2^m(0),\tag{3.38}$$

where

$$\begin{aligned}\mathcal{E}_2^m(t)=\frac{1}{2}\Big(&\rho\|u_{tt}^m(t)\|^2+\alpha\|u_{xt}^m(t)\|^2+\lambda\|\varphi_t^m(t)-u_t^m(t)\|^2+\rho_1\|\varphi_{tt}^m(t)\|^2\\&+K\|\varphi_{xt}^m(t)+\psi_t^m(t)\|^2+b\|\psi_{xt}^m(t)\|^2+\rho_3\|w_{tt}^m(t)\|^2+\delta\|w_{xt}^m(t)\|^2\Big).\end{aligned}$$

Thus, there exists C independent of m such that

$$\mathcal{E}_2^m(t)\leqslant\mathcal{E}_2^m(0)\leqslant C,\ t\geqslant 0\tag{3.39}$$

and, consequently,

$$\begin{aligned}&\|u_{tt}^m(t)\|^2+\|u_{xt}^m(t)\|^2+\|\varphi_t^m(t)-u_t^m(t)\|^2+\|\varphi_{tt}^m(t)\|^2\\&+\|\varphi_{xt}^m(t)+\psi_t^m(t)\|^2+\|\psi_{xt}^m(t)\|^2+\|w_{tt}^m(t)\|^2+\|w_{xt}^m(t)\|^2\leqslant C\end{aligned}\tag{3.40}$$

and, for any $m\in\mathbb{N}$, we have

$$u_t^m,\ \varphi_t^m,\ \psi_t^m,\ w_t^m\ \text{ bounded in }\ L^\infty(0,T;H_0^1(0,L)),\tag{3.41}$$

$$u_{tt}^m,\ \varphi_{tt}^m,\ w_{tt}^m\ \text{ bounded in }\ L^\infty(0,T;L^2(0,L)).\tag{3.42}$$

Step 4: third a priori estimate. Replacing v_i by $-v_{ixx}$ in (3.13) and multiplying the resulting equations by g'_{im}, $\hat{g}'_{im}$, f'_{im} and $\hat{f}'_{im}$, respectively, summing over i from 1 to m, we obtain that

$$\begin{cases} -\rho(u^m_{tt}(t), u^m_{xxt}(t)) - \alpha(u^m_x(t), u^m_{xxxt}(t)) + \lambda(\varphi^m(t) - u^m(t), u^m_{xxt}(t)) \\ \qquad\qquad\qquad\qquad\qquad\qquad - \mu(u^m_t(t), u^m_{xxt}(t)) = 0, \\ -\rho_1(\varphi^m_{tt}(t), \varphi^m_{xxt}(t)) - K(\varphi^m_x(t) + \psi^m(t), \varphi^m_{xxxt}(t)) - \lambda(\varphi^m(t) - u^m(t), \varphi^m_{xxt}(t)) \\ \qquad\qquad\qquad\qquad\qquad - \gamma(\varphi^m_t(t), \varphi^m_{xxt}(t)) - \beta(w^m_{xt}(t), \varphi^m_{xxt}(t)) = 0, \\ -b(\psi^m_x(t), \psi^m_{xxxt}(t)) - K(\varphi^m_x(t) + \psi^m(t), \psi^m_{xxt}(t)) = 0, \\ -\rho_3(w^m_{tt}(t), w^m_{xxt}(t)) - \delta(w^m_x(t), w^m_{xxxt}(t)) - \beta(\varphi^m_{xt}(t), w^m_{xxt}(t)) \\ \qquad\qquad\qquad\qquad\qquad - \kappa(w^m_{xt}(t), w^m_{xxxt}(t)) = 0, \end{cases} \tag{3.43}$$

using integration by parts with the fact (3.12), gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\rho\|u^m_{xt}(t)\|^2\right) + \alpha\varsigma\left[u^m_x u^m_t\right]_0^L + \frac{1}{2}\frac{d}{dt}\left(\alpha\|u^m_{xx}(t)\|^2\right) \\ &- \lambda\int_0^L (\varphi^m_x(t) - u^m_x(t))u^m_{xt}(t)dx + \mu\|u^m_{xt}(t)\|^2 = 0, \end{aligned} \tag{3.44}$$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi^m_{xt}(t)\|^2\right) + K\varsigma\left[(\varphi^m_x + \psi^m)\varphi^m_t\right]_0^L + K\int_0^L (\varphi^m_{xx}(t) + \psi^m_x(t))\varphi^m_{xxt}(t)dx \\ &+ \gamma\|\varphi^m_{xt}(t)\|^2 + \lambda\int_0^L (\varphi^m_x(t) - u^m_x(t))\varphi^m_{xt}(t)dx - \beta\int_0^L w^m_{xt}(t)\varphi^m_{xxt}(t)dx = 0, \end{aligned} \tag{3.45}$$

$$b\varsigma\left[\psi^m_x\psi^m_t\right]_0^L + \frac{1}{2}\frac{d}{dt}\left(b\|\psi^m_{xx}(t)\|^2\right) - K\int_0^L (\varphi^m_x(t) + \psi^m(t))\psi^m_{xxt}(t)dx = 0, \tag{3.46}$$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\rho_3\|w^m_{xt}(t)\|^2\right) + \delta\varsigma\left[w^m_x w^m_t\right]_0^L + \frac{1}{2}\frac{d}{dt}\left(\delta\|w^m_{xx}(t)\|^2\right) \\ &+ \beta\int_0^L \varphi^m_{xxt}(t)w^m_{xt}(t)dx + \kappa\varsigma\left[w^m_{xt}w^m_t\right]_0^L + \kappa\|w^m_{xxt}(t)\|^2 = 0. \end{aligned} \tag{3.47}$$

Taking the sum of (3.44)–(3.47), one has

$$\begin{aligned} \mathcal{E}^m_3(t) = \frac{d}{dt}\Big(&\frac{\rho}{2}\|u^m_{xt}(t)\|^2 + \frac{\alpha}{2}\|u^m_{xx}(t)\|^2 + \frac{\lambda}{2}\|\varphi^m_x(t) - u^m_x(t)\|^2 + \frac{\rho_1}{2}\|\varphi^m_{xt}(t)\|^2 \\ &+ \frac{K}{2}\|\varphi^m_{xx}(t) + \psi^m_x(t)\|^2 + \frac{b}{2}\|\psi^m_{xx}(t)\|^2\frac{\rho_3}{2}\|w^m_{xt}(t)\|^2 + \frac{\delta}{2}\|w^m_{xx}(t)\|^2\Big) \\ &+ \mu\|u^m_{xt}(t)\|^2 + \gamma\|\varphi^m_{xt}(t)\|^2 + \kappa\|w^m_{xxt}(t)\|^2, \end{aligned} \tag{3.48}$$

a simple integrating over $(0, t)$, leads to

$$\mathcal{E}^m_3(t) + \mu\int_0^t \|u^m_{xt}(s)\|^2 ds + \gamma\int_0^t \|\varphi^m_{xt}(s)\|^2 ds + \kappa\int_0^t \|w^m_{xxt}(s)\|^2 ds = \mathcal{E}^m_3(0), \tag{3.49}$$

where

$$\mathcal{E}_3^m(t) = \frac{1}{2}\Big(\rho\|u_{xt}^m(t)\|^2 + \alpha\|u_{xx}^m(t)\|^2 + \lambda\|\varphi_x^m(t) - u_x^m(t)\|^2 + \rho_1\|\varphi_{xt}^m(t)\|^2 + K\|\varphi_{xx}^m(t) + \psi_x^m(t)\|^2 + b\|\psi_{xx}^m(t)\|^2 + \rho_3\|w_{xt}^m(t)\|^2 + \delta\|w_{xx}^m(t)\|^2\Big).$$

Then, there exists C, independent of m, such that

$$\mathcal{E}_3^m(t) \leqslant \mathcal{E}_3^m(0) \leqslant C,\ t \geqslant 0,$$

which entails that

$$\begin{aligned}&\|u_{xt}^m(t)\|^2 + \|u_{xx}^m(t)\|^2 + \|\varphi_x^m(t) - u_x^m(t)\|^2 + \|\varphi_{xt}^m(t)\|^2\\ &+ \|\varphi_{xx}^m(t) + \psi_x^m(t)\|^2 + \|\psi_{xx}^m(t)\|^2 + \|w_{xt}^m(t)\|^2 + \|w_{xx}^m(t)\|^2 \leqslant C.\end{aligned} \tag{3.50}$$

Therefore, for all $m \in \mathbb{N}$, (3.50) implies that

$$u^m,\ \varphi^m,\ \psi^m,\ w^m \text{ are bounded in } L^\infty(0,T; H^2(0,L) \cap H_0^1(0,L)). \tag{3.51}$$

Step 5: passage to the limit. Thanks to (3.41), (3.42) and (3.51), and up to a subsequence, we have

$$\begin{cases} u^m \rightharpoonup^* u \quad \text{in } L^\infty(0,T; H^2(0,L) \cap H_0^1(0,L)),\\ u_t^m \rightharpoonup^* u_t \quad \text{in } L^\infty(0,T; H_0^1(0,L)),\\ u_{tt}^m \rightharpoonup^* u_{tt} \quad \text{in } L^\infty(0,T; L^2(0,L)),\end{cases} \tag{3.52}$$

$$\begin{cases} \varphi^m \rightharpoonup^* \varphi \quad \text{in } L^\infty(0,T; H^2(0,L) \cap H_0^1(0,L)),\\ \varphi_t^m \rightharpoonup^* \varphi_t \quad \text{in } L^\infty(0,T; H_0^1(0,L)),\\ \varphi_{tt}^m \rightharpoonup^* \varphi_{tt} \quad \text{in } L^\infty(0,T; L^2(0,L)),\end{cases} \tag{3.53}$$

$$\begin{cases} \psi^m \rightharpoonup^* \psi \quad \text{in } L^\infty(0,T; H^2(0,L) \cap H_0^1(0,L)),\\ \psi_t^m \rightharpoonup^* \psi_t \quad \text{in } L^\infty(0,T; H_0^1(0,L)),\end{cases} \tag{3.54}$$

$$\begin{cases} w^m \rightharpoonup^* w \quad \text{in } L^\infty(0,T; H^2(0,L) \cap H_0^1(0,L)),\\ w_t^m \rightharpoonup^* w_t \quad \text{in } L^\infty(0,T; H_0^1(0,L)),\\ w_{tt}^m \rightharpoonup^* w_{tt} \quad \text{in } L^\infty(0,T; L^2(0,L)).\end{cases} \tag{3.55}$$

The embedding $H^2(0,L) \cap H_0^1(0,L)$ in $H_0^1(0,L)$ is compact. Note that if we let $B_1 = B_2 = H_0^1(0,L)$ and $B_0 = H^2(0,L) \cap H_0^1(0,L)$ in Aubin–Lions–Simon Theorem 2.1.24, then we get that the embedding of $W_{\infty,\infty}$ in $C(]0,T[, H_0^1(0,L))$ is compact, where

$$W_{\infty,\infty} = \left\{u^m :\ u^m \in L^\infty(]0,T[, H^2(0,L) \cap H_0^1(0,L)),\ u_t^m \in L^\infty(]0,T[, H_0^1(0,L))\right\}.$$

Now, from (3.52), we deduce that u^m is bounded in $W_{\infty,\infty}$ and, therefore, we can extract a subsequence (u^ν) of (u^m) such that

$$u^\nu \longrightarrow u \quad \text{in} \;\; C([0,T], H_0^1(0,L)). \tag{3.56}$$

Similarly, we obtain

$$\psi^\nu \longrightarrow \psi \quad \text{in} \;\; C([0,T], H_0^1(0,L)),$$

$$\varphi^\nu \longrightarrow \varphi \quad \text{in} \;\; C([0,T], H_0^1(0,L)),$$

$$w^\nu \longrightarrow w \quad \text{in} \;\; C([0,T], H_0^1(0,L)).$$

Since the embedding $H_0^1(0,L) \hookrightarrow L^2(0,L)$ is compact, going back again to the compactness Theorem 2.1.24 with $B_0 = H_0^1(0,L)$, $B_1 = B_2 = L^2(0,L)$ and $\chi = u_t^m$, gives

$$u_t^\nu \longrightarrow u_t \quad \text{in} \;\; C([0,T], L^2(0,L)), \tag{3.57}$$

where here (u_t^ν) is a subsequence of (u_t^m). Similarly, we find

$$\varphi_t^\nu \longrightarrow \varphi_t \quad \text{in} \;\; C([0,T], L^2(0,L)),$$

$$w_t^\nu \longrightarrow w_t \quad \text{in} \;\; C([0,T], L^2(0,L)).$$

On the other hand, note that, by using

$$-u_{xx}^\nu = \varsigma u^\nu \;\; \text{and} \;\; C([0,T], H_0^1(0,L)) \subset C([0,T], L^2(0,L)),$$

then (3.56) yields

$$u^\nu \longrightarrow u \quad \text{in} \;\; C([0,T], H^2(0,L) \cap H_0^1(0,L)).$$

Now, making use of (3.56) and (3.57), respectively, with the dominated convergence theorem, we arrive at

$$\|u^\nu - u\|_{C([0,T],H_0^1(0,L))} \xrightarrow{\nu\to\infty} 0$$

and

$$\|u_t^\nu - u_t\|_{C([0,T],L^2(0,L))} \xrightarrow{\nu\to\infty} 0.$$

Differentiating under the integral sign (Leibnitz's rule) gives

$$\|u_t^\nu - u_t\|_{C([0,T],H_0^1(0,L))} \xrightarrow{\nu\to\infty} 0$$

and

$$\|u_{tt}^\nu - u_{tt}\|_{C([0,T],L^2(0,L))} \xrightarrow{\nu\to\infty} 0,$$

which implies that

$$u_t^\nu \longrightarrow u_t \quad \text{in} \;\; C([0,T], H_0^1(0,L)),$$

$$u_{tt}^{\nu} \longrightarrow u_{tt} \quad \text{in } \; C([0,T], L^2(0,L)).$$

The same arguments are applied to $\varphi^{\nu}, \varphi_t^{\nu}, \varphi_{tt}^{\nu}, \psi^{\nu}, w^{\nu}, w_t^{\nu}$ and w_{tt}^{ν}. With these limits, we can pass to the limit of the terms of the approximate equations in (3.13) to get

$$\begin{gathered}
(u_{tt}^{\nu}(t), v_i) \longrightarrow (u_{tt}(t), v_i), \; (u_x^{\nu}(t), v_{ix}) \longrightarrow (u_x(t), v_{ix}), \\
(\varphi^{\nu}(t) - u^{\nu}(t), v_i) \longrightarrow (\varphi(t) - u(t), v_i), \; (u_t^{\nu}(t), v_i) \longrightarrow (u_t(t), v_i),
\end{gathered}$$

$$\begin{gathered}
(\varphi_{tt}^{\nu}(t), v_i) \longrightarrow (\varphi_{tt}(t), v_i), \; (\varphi_x^{\nu}(t) + \psi^{\nu}(t), v_{ix}) \longrightarrow (\varphi_x(t) + \psi(t), v_{ix}), \\
(\varphi^{\nu}(t) - u^{\nu}(t), v_i) \longrightarrow (\varphi(t) - u(t), v_i), \; (\varphi_t^{\nu}(t), v_i) \longrightarrow (\varphi_t(t), v_i), \\
(w_{xt}^{\nu}(t), v_i) \longrightarrow (w_{xt}(t), v_i), \\
(\psi_x^{\nu}(t), v_{ix}) \longrightarrow (\psi_x(t), v_{ix}), \; (\varphi_x^{\nu}(t) + \psi^{\nu}(t), v_i) \longrightarrow (\varphi_x(t) + \psi(t), v_i)
\end{gathered}$$

and

$$\begin{gathered}
(w_{tt}^{\nu}(t), v_i) \longrightarrow (w_{tt}(t), v_i), \; (w_x^{\nu}(t), v_{ix}) \longrightarrow (w_x(t), v_{ix}), \\
(\varphi_{xt}^{\nu}(t), v_i) \longrightarrow (\varphi_{xt}(t), v_i), \; (w_{xt}^{\nu}(t), v_{ix}) \longrightarrow (w_{xt}(t), v_{ix}).
\end{gathered}$$

From the above limits, we conclude that

$$\begin{cases}
\rho(u_{tt}(t), v_i) + \alpha(u_x(t), v_{ix}) - \lambda(\varphi(t) - u(t), v_i) + \mu(u_t(t), v_i) = 0, \\
\rho_1(\varphi_{tt}(t), v_i) + K(\varphi_x(t) + \psi(t), v_{ix}) + \lambda(\varphi(t) - u(t), v_i) \\
\qquad\qquad\qquad + \gamma(\varphi_t(t), v_i) + \beta(w_{xt}(t), v_i) = 0, \\
b(\psi_x(t), v_{ix}) + K(\varphi_x(t) + \psi(t), v_i) = 0, \\
\rho_3(w_{tt}(t), v_i) + \delta(w_x(t), v_{ix}) + \beta(\varphi_{xt}(t), v_i) + \kappa(w_{xt}(t), v_{ix}) = 0.
\end{cases}$$

Using the density of $\{v_i\}_{i\in\mathbb{N}}$ in $H^2(0,L) \cap H_0^1(0,L)$, we obtain

$$\begin{cases}
\rho(u_{tt}(t), v_1) + \alpha(u_x(t), v_{1x}) - \lambda(\varphi(t) - u(t), v_1) + \mu(u_t(t), v_1) = 0, \\
\rho_1(\varphi_{tt}(t), v_1) + K(\varphi_x(t) + \psi(t), v_{1x}) + \lambda(\varphi(t) - u(t), v_1) \\
\qquad\qquad\qquad + \gamma(\varphi_t(t), v_1) + \beta(w_{xt}(t), v_1) = 0, \\
b(\psi_x(t), v_{1x}) + K(\varphi_x(t) + \psi(t), v_1) = 0, \\
\rho_3(w_{tt}(t), v_1) + \delta(w_x(t), v_{1x}) + \beta(\varphi_{xt}(t), v_1) + \kappa(w_{xt}(t), v_{1x}) = 0,
\end{cases}$$

for any $v_1 \in H^2(0,L) \cap H_0^1(0,L)$. Therefore $(u, u_t, \varphi, \varphi_t, \psi, w, w_t)$ is a strong solution of (3.4).

Step 6: continuous dependence and uniqueness. Let $(u, u_t, \varphi, \varphi_t, \psi, w, w_t)$ and $(\tilde{u}, \tilde{u}_t, \tilde{\varphi}, \tilde{\varphi}_t, \tilde{\psi}, \tilde{w}, \tilde{w}_t)$ be strong solutions of problem (3.4). Then,

$$(U, U_t, \Lambda, \Lambda_t, X, \Theta, \Theta_t) = (u - \tilde{u}, u_t - \tilde{u}_t, \varphi - \tilde{\varphi}, \varphi_t - \tilde{\varphi}_t, \psi - \tilde{\psi}, w - \tilde{w}, w_t - \tilde{w}_t)$$

satisfies

$$\rho U_{tt}(x,t) - \alpha U_{xx}(x,t) - \lambda(\Lambda - U)(x,t) + \mu U_t(x,t) = 0, \tag{3.58}$$

$$\rho_1 \Lambda_{tt}(x,t) - K(\Lambda_x + X)_x(x,t) + \lambda(\Lambda - U)(x,t) + \gamma\Lambda_t(x,t) + \beta\Theta_{xt}(x,t) = 0, \tag{3.59}$$

$$-bX_{xx}(x,t) + K(\Lambda_x + X)(x,t) = 0, \tag{3.60}$$

$$\rho_3\Theta_{tt}(x,t) - \delta\Theta_{xx}(x,t) + \beta\Lambda_{xt}(x,t) - \kappa\Theta_{xxt}(x,t) = 0, \tag{3.61}$$

with the initial data

$$U(0) = u(0) - \tilde{u}(0),\ U_t(0) = u_t(0) - \tilde{u}_t(0),\ \Lambda(0) = \varphi(0) - \tilde{\varphi}(0),\ \Lambda_t(0) = \varphi_t(0) - \tilde{\varphi}_t(0),$$
$$X(0) = \psi(0) - \tilde{\psi}(0),\ \Theta(0) = w(0) - \tilde{w}(0),\ \Theta_t(0) = w_t(0) - \tilde{w}_t(0).$$

Repeating exactly the same arguments used to obtain the estimate (3.6), we get

$$\frac{d\tilde{E}(t)}{dt} = -\mu\|U_t\|^2 - \gamma\|\Lambda_t\|^2 - \kappa\|\Theta_{xt}\|^2. \tag{3.62}$$

Integrating (3.62) over $(0,t)$, we conclude that

$$\tilde{E}(t) \leqslant C_T\tilde{E}(0),\ t \in [0,T],$$

for some constant $C_T > 0$, which proves the continuous dependence of the solutions on the initial data. In particular, the strong solution is unique.

To demonstrate the existence of weak solutions, Let us consider the initial data $u_0, \varphi_0, \psi_0, w_0 \in H_0^1(0,L)$ and $u_1, \varphi_1, w_1 \in L^2(0,L)$ in the approximate problem (3.13). Then by density, we have

$$\begin{aligned} u_0^m, \varphi_0^m, \psi_0^m, w_0^m &\to u_0, \varphi_0, \psi_0, w_0 \ \text{ in } H_0^1(0,L), \\ u_1^m, \varphi_1^m, w_1^m &\to u_1, \varphi_1, w_1 \ \text{ in } L^2(0,L). \end{aligned} \tag{3.63}$$

Repeating the same steps used in the first estimate, following the same procedure already used in the uniqueness of strong solutions for $(U^m, U_t^m, \Lambda^m, \Lambda_t^m, X^m, \Theta^m, \Theta_t^m) = (u^m - \tilde{u}^m, u_t^m - \tilde{u}_t^m, \varphi^m - \tilde{\varphi}^m, \varphi_t^m - \tilde{\varphi}_t^m, \psi^m - \tilde{\psi}, w^m - \tilde{w}^m, w_t^m - \tilde{w}_t^m)$ and taking into account the convergences (3.63), we deduce that there exists u, φ, ψ, and w such that

$$u^m \longrightarrow u \quad \text{in } C([0,T], H_0^1(0,L)),$$

$$\psi^m \longrightarrow \psi \quad \text{in } C([0,T], H_0^1(0,L)),$$

$$\varphi^m \longrightarrow \varphi \quad \text{in } C([0,T], H_0^1(0,L)),$$

$$w^m \longrightarrow w \quad \text{in } C([0,T], H_0^1(0,L)),$$

$$u_t^m \longrightarrow u_t \quad \text{in } C([0,T], L^2(0,L)),$$

$$\varphi_t^m \longrightarrow \varphi_t \quad \text{in} \;\; C([0,T],L^2(0,L)),$$

$$w_t^m \longrightarrow w_t \quad \text{in} \;\; C([0,T],L^2(0,L)).$$

The uniqueness is obtained making use of the well-known regularization procedure, as presented in [97, Chapter 3, Section 8.2]. □

3.4 Exponential stability

The main focus of this section is to prove the following stability theorem for regular solution, since the same occurs for weak solution using standard density arguments.

Theorem 3.4.1. *With the regularity stated in Theorem 3.3.1, the energy $E(t)$ decays exponentially as time approaches infinity, that is, there exists two positive constants, σ_0 and σ_1, such that*

$$E(t) \leqslant \sigma_0 e^{-\sigma_1 t} \quad \forall t \geqslant 0. \tag{3.64}$$

We begin by proving some important lemmas that will be essential to establish Theorem 3.4.1.

Lemma 3.4.2. *The functional*

$$I_1(t) = \int_0^L \left(\rho u_t u + \frac{\mu}{2}u^2\right) dx + \int_0^L \left(\rho_1 \varphi_t \varphi + \frac{\gamma}{2}\varphi^2\right) dx$$

satisfies

$$\begin{aligned} \frac{dI_1(t)}{dt} \leqslant & -\alpha \int_0^L u_x^2 dx - \lambda \int_0^L (\varphi - u)^2 dx - \frac{K}{2}\int_0^L (\varphi_x + \psi)^2 dx \\ & - \frac{b}{2}\int_0^L \psi_x^2 dx + \rho \int_0^L u_t^2 dx + \rho_1 \int_0^L \varphi_t^2 dx + C_1 \int_0^L w_{xt}^2 dx. \end{aligned} \tag{3.65}$$

Proof. Multiplying $(3.4)_1$ by u, using the fact that $u_{tt}u = \dfrac{\partial}{\partial t}(u_t u) - u_t^2$, we obtain

$$\rho u_{tt} u - \alpha u_{xx} u - \lambda\,(\varphi - u)\,u + \mu u_t u = 0.$$

That is,

$$\rho \frac{\partial}{\partial t}(u_t u) - \rho u_t^2 - \alpha u_{xx} u - \lambda\,(\varphi - u)\,u + \mu u_t u = 0.$$

Integrating by parts, we get

$$\frac{d}{dt}\int_0^L \left(\rho u_t u + \frac{\mu}{2}u^2\right) dx - \rho \int_0^L u_t^2 dx + \alpha \int_0^L u_x^2 dx - \lambda \int_0^L (\varphi - u) u dx = 0. \tag{3.66}$$

Similarly, multiplying $(3.4)_2$ by φ, using the fact that $\varphi_{tt}\varphi = \dfrac{\partial}{\partial t}(\varphi_t\varphi) - \varphi_t^2$, we have

$$\rho_1\varphi_{tt}\varphi - K\left(\varphi_x+\psi\right)_x\varphi + \lambda\left(\varphi - u\right)\varphi + \gamma\varphi_t\varphi + \beta w_{xt}\varphi = 0;$$

i.e.

$$\rho_1\frac{\partial}{\partial t}(\varphi_t\varphi) - \rho_1\varphi_t^2 - K\left(\varphi_x+\psi\right)_x\varphi + \lambda\left(\varphi - u\right)\varphi + \gamma\varphi_t\varphi + \beta w_{xt}\varphi = 0.$$

Integrating by parts, one obtains

$$\begin{aligned}\frac{d}{dt}\int_0^L\Big(\rho_1\varphi_t\varphi+\frac{\gamma}{2}\varphi\Big)\,dx - \rho_1\int_0^L\varphi_t^2dx + K\int_0^L(\varphi_x+\psi)\varphi_x dx&\\ +\lambda\int_0^L\left(\varphi-u\right)\varphi dx + \beta\int_0^L w_{xt}\varphi dx = 0.&\end{aligned}\tag{3.67}$$

Now, multiplying $(3.4)_3$ by ψ, it results that

$$b\int_0^L\psi_x^2dx + K\int_0^L(\varphi_x+\psi)\psi dx = 0.\tag{3.68}$$

Adding (3.66)–(3.68), we arrive at

$$\begin{aligned}&\frac{d}{dt}\int_0^L\Big(\rho u_t u+\frac{\mu}{2}u^2\Big)\,dx - \rho\int_0^L u_t^2dx + \alpha\int_0^L u_x^2dx - \lambda\int_0^L(\varphi-u)udx\\
&\frac{d}{dt}\int_0^L\Big(\rho_1\varphi_t\varphi+\frac{\gamma}{2}\varphi\Big)\,dx - \rho_1\int_0^L\varphi_t^2dx + K\int_0^L(\varphi_x+\psi)\varphi_x dx\\
&+\lambda\int_0^L\left(\varphi-u\right)\varphi dx + \beta\int_0^L w_{xt}\varphi dx + b\int_0^L\psi_x^2dx + K\int_0^L(\varphi_x+\psi)\psi dx\\
=&\frac{d}{dt}\Big[\int_0^L\Big(\rho u_t u+\frac{\mu}{2}u^2\Big)\,dx + \int_0^L\Big(\rho_1\varphi_t\varphi+\frac{\gamma}{2}\varphi\Big)\,dx\Big] - \rho\int_0^L u_t^2dx + \alpha\int_0^L u_x^2dx\\
&+\lambda\int_0^L(\varphi-u)^2dx - \rho_1\int_0^L\varphi_t^2dx + K\int_0^L(\varphi_x+\psi)^2dx + \beta\int_0^L w_{xt}\varphi dx + b\int_0^L\psi_x^2dx.\end{aligned}$$

Then, we deduce that

$$\begin{aligned}\frac{dI_1(t)}{dt} =&\rho\int_0^L u_t^2dx - \alpha\int_0^L u_x^2dx - \lambda\int_0^L(\varphi-u)^2dx + \rho_1\int_0^L\varphi_t^2dx\\
&- K\int_0^L(\varphi_x+\psi)^2dx + \beta\int_0^L w_t\varphi_x dx - b\int_0^L\psi_x^2dx.\end{aligned}\tag{3.69}$$

The fact that $\varphi_x = (\varphi_x+\psi)-\psi$ leads to

$$\beta\int_0^L w_t\varphi_x dx = \beta\int_0^L w_t(\varphi_x+\psi)dx - \beta\int_0^L w_t\psi dx\tag{3.70}$$

and Young's and Poincaré's inequalities yield

$$\beta \int_0^L w_t(\varphi_x + \psi)dx \leqslant \frac{K}{2} \int_0^L (\varphi_x + \psi)^2 dx + \frac{\beta^2 c_p}{2K} \int_0^L w_{xt}^2 dx \tag{3.71}$$

and

$$-\beta \int_0^L w_t \psi dx \leqslant \frac{b}{2} \int_0^L \psi_x^2 dx + \frac{\beta^2 c_p^2}{2b} \int_0^L w_{xt}^2 dx, \tag{3.72}$$

where c_p is a Poincaré constant. By plugging (3.71) and (3.72) into (3.69),

$$\begin{aligned}\frac{dI_1(t)}{dt} \leqslant &\rho \int_0^L u_t^2 dx - \alpha \int_0^L u_x^2 dx - \lambda \int_0^L (\varphi - u)^2 dx + \rho_1 \int_0^L \varphi_t^2 dx \\ &- \frac{K}{2} \int_0^L (\varphi_x + \psi)^2 dx + \left(\frac{\beta^2 c_p}{2K} + \frac{\beta^2 c_p^2}{2b} \right) \int_0^L w_{xt} dx - \frac{b}{2} \int_0^L \psi_x^2 dx.\end{aligned}$$

Hence, (3.65) is obtained with $C_1 = \max \left\{ \frac{\beta^2 c_p}{2K}, \frac{\beta^2 c_p^2}{2b} \right\}$. □

Lemma 3.4.3. *The functional*

$$I_2(t) = \rho_3 \int_0^L w_t w dx + \frac{\kappa}{2} \int_0^L w_x^2 dx + \beta \int_0^L \varphi_x w dx$$

satisfies

$$\frac{dI_2(t)}{dt} \leqslant -\delta \int_0^L w_x^2 dx + \frac{b}{4} \int_0^L \psi_x^2 dx + \frac{K}{4} \int_0^L (\varphi_x + \psi)^2 dx + C_2 \int_0^L w_{xt}^2 dx. \tag{3.73}$$

Proof. Differentiating I_2, we get

$$\begin{aligned}\frac{dI_2(t)}{dt} =&\rho_3 \int_0^L w_{tt} w dx + \rho_3 \int_0^L w_t^2 dx + \kappa \int_0^L w_{xt} w_x dx \\ &+ \beta \int_0^L \varphi_{xt} w dx + \beta \int_0^L \varphi_x w_t dx,\end{aligned}$$

using $(3.4)_4$, integration by parts and recalling the boundary conditions, we obtain

$$\begin{aligned}\frac{dI_2(t)}{dt} =& - \delta \int_0^L w_x^2 dx - \beta \int_0^L \varphi_{xt} w dx - \kappa \int_0^L w_{xt} w_x + \rho_3 \int_0^L w_t^2 dx \\ &+ \kappa \int_0^L w_{xt} w_x dx + \beta \int_0^L \varphi_{xt} w dx + \beta \int_0^L \varphi_x w_t dx,\end{aligned}$$

then,we conclude that

$$\frac{dI_2(t)}{dt} = \rho_3 \int_0^L w_t^2 dx - \delta \int_0^L w_x^2 dx + \beta \int_0^L w_t \varphi_x dx. \tag{3.74}$$

It follows, by (3.70) and Young's and Poincaré's inequalities, that

$$\beta \int_0^L w_t(\varphi_x + \psi)dx \leqslant \frac{K}{4} \int_0^L (\varphi_x + \psi)^2 dx + \frac{\beta^2 c_p}{K} \int_0^L w_{xt}^2 dx, \tag{3.75}$$

$$-\beta \int_0^L w_t \psi dx \leqslant \frac{b}{4} \int_0^L \psi_x^2 dx + \frac{\beta^2 c_p^2}{b} \int_0^L w_{xt}^2 dx \tag{3.76}$$

and

$$\rho_3 \int_0^L w_t^2 dx \leqslant \rho_3 c_p \int_0^L w_{xt}^2 dx. \tag{3.77}$$

By substituting (3.75), (3.76) and (3.77) into (3.74), we arrive at

$$\begin{aligned} \frac{dI_2(t)}{dt} \leqslant &-\delta \int_0^L w_x^2 dx + \frac{b}{4} \int_0^L \psi_x^2 dx + \frac{K}{4} \int_0^L (\varphi_x + \psi)^2 dx \\ &+ \left(\frac{\beta^2 c_p^2}{b} + \frac{\beta^2 c_p}{K} + \rho_3 c_p \right) \int_0^L w_{xt}^2 dx \end{aligned}$$

and, hence, obtain (3.73) with $C_2 = \max\left\{ \frac{\beta^2 c_p^2}{b}, \frac{\beta^2 c_p}{K}, \rho_3 c_p \right\}$. □

Let us introduce now the Lyapunov functional

$$L(t) = NE(t) + I_1(t) + I_2(t), \tag{3.78}$$

where N is a positive constant to be fixed later.

Lemma 3.4.4. *Let (u, φ, ψ, w) be a solution of (3.4). Then there exist two positive constants, ξ_1 and ξ_2, such that the functional energy E is equivalent to functional L; That is,*

$$\xi_1 E(t) \leqslant L(t) \leqslant \xi_2 E(t). \tag{3.79}$$

Proof. Since

$$\begin{aligned} |L(t) - NE(t)| &= |I_1(t) + I_2(t)| \\ &= \Big| \int_0^L \left(\rho u_t u + \frac{\mu}{2} u^2 \right) dx + \int_0^L \left(\rho_1 \varphi_t \varphi + \frac{\gamma}{2} \varphi^2 \right) dx \\ &\quad + \rho_3 \int_0^L w_t w dx + \frac{\kappa}{2} \int_0^L w_x^2 dx + \beta \int_0^L \varphi_x w dx \Big|, \end{aligned}$$

then, we have

$$
\begin{aligned}
|L(t)-NE(t)| \leqslant & \rho \int_0^L |u_t u|\,dx + \frac{\mu}{2}\int_0^L u^2 dx + \rho_1 \int_0^L |\varphi_t \varphi|\,dx + \frac{\gamma}{2}\int_0^L \varphi^2 dx \\
& + \rho_3 \int_0^L |w_t w|\,dx + \frac{\kappa}{2}\int_0^L w_x^2 dx + |\beta| \int_0^L |\varphi_x w|\,dx.
\end{aligned}
$$

By using $\varphi_x = (\varphi_x + \psi) - \psi$, $u = -(\varphi - u) + \varphi$, and Poincarés inequality, we get

$$
\begin{aligned}
|L(t)-NE(t)| \leqslant & \rho \int_0^L |u_t \varphi - u_t(\varphi - u)|\,dx + \frac{\mu}{2}\int_0^L u^2 dx \\
& + \rho_1 c_p \int_0^L |\varphi_t(\varphi_x+\psi) - \varphi_t \psi|\,dx + 2\gamma c_p \int_0^L |(\varphi_x+\psi)\psi|\,dx \\
& + \rho_3 \int_0^L |w_t w|\,dx + \frac{\kappa}{2}\int_0^L w_x^2 dx + |\beta| \int_0^L |(\varphi_x+\psi)w - \psi w|\,dx,
\end{aligned}
$$

Young's and Poincarés inequalities lead to

$$
\begin{aligned}
|L(t)-NE(t)| \leqslant & \rho \int_0^L u_t^2 dx + \frac{\rho}{2}\int_0^L (\varphi-u)^2 dx + \frac{\mu c_p}{2}\int_0^L u_x^2 dx + \rho_1 c_p \int_0^L \varphi_t^2 dx \\
& + \frac{\rho_1 c_p}{2}\int_0^L (\varphi_x+\psi)^2 dx + \frac{\rho_1 c_p}{2}\int_0^L \psi_x^2 dx + (\rho+\gamma)c_p \int_0^L (\varphi_x+\psi)^2 dx \\
& + (\rho+\gamma)c_p^2 \int_0^L \psi_x^2 dx + \frac{\rho_3}{2}\int_0^L w_t^2 dx + \frac{\rho_3 c_p}{2}\int_0^L w_x^2 dx + \frac{\kappa}{2}\int_0^L w_x^2 dx \\
& + \frac{|\beta|}{2}\int_0^L (\varphi_x+\psi)^2 dx + |\beta|\, c_p \int_0^L w_x^2 dx + \frac{|\beta|\, c_p}{2}\int_0^L \psi_x^2 dx;
\end{aligned}
$$

thus,

$$
\begin{aligned}
|L(t)-NE(t)| \leqslant & \rho \int_0^L u_t^2 dx + \frac{\mu c_p}{2}\int_0^L u_x^2 dx + \frac{\rho}{2}\int_0^L (\varphi-u)^2 dx \\
& + \rho_1 c_p \int_0^L \varphi_t^2 dx + \left(\frac{\rho_1 c_p}{2} + (\rho+\gamma)c_p + \frac{|\beta|}{2}\right)\int_0^L (\varphi_x+\psi)^2 dx \\
& + \left(\frac{\rho_1 c_p}{2} + (\rho+\gamma)c_p^2 + \frac{|\beta|\, c_p}{2}\right)\int_0^L \psi_x^2 dx + \frac{\rho_3}{2}\int_0^L w_t^2 dx \\
& + \left(\frac{\rho_3 c_p}{2} + \frac{\kappa}{2} + |\beta|\, c_p\right)\int_0^L w_x^2 dx.
\end{aligned}
$$

Therefore, we obtain, for some $\xi > 0$, that

$$
|L(t)-NE(t)| \leqslant \xi E(t),
$$

which yields

$$
(N-\xi)E(t) \leqslant L(t) \leqslant (N+\xi)E(t).
$$

The estimate (3.79) follows by choosing N large enough. $\square$

We are now in position to prove Theorem 3.4.1.

Proof. (of Theorem 3.4.1). Taking the derivative of $L(t)$ and using Lemmas 3.4.2 and 3.4.3 and the energy dissipation law (3.6), it follows that

$$\begin{aligned}\frac{dL(t)}{dt} \leqslant & - N\mu \int_0^L u_t^2 dx - N\gamma \int_0^L \varphi_t^2 dx - N\kappa \int_0^L w_{xt}^2 dx \\ & - \alpha \int_0^L u_x^2 dx - \lambda \int_0^L (\varphi - u)^2 dx - \frac{K}{2} \int_0^L (\varphi_x + \psi)^2 dx \\ & - \frac{b}{2} \int_0^L \psi_x^2 dx + \rho \int_0^L u_t^2 dx + \rho_1 \int_0^L \varphi_t^2 dx + C_1 \int_0^L w_{xt}^2 dx \\ & - \delta \int_0^L w_x^2 dx + \frac{b}{4} \int_0^L \psi_x^2 dx + \frac{K}{4} \int_0^L (\varphi_x + \psi)^2 dx + C_2 \int_0^L w_{xt}^2 dx.\end{aligned} \tag{3.80}$$

Combining similar terms, we obtain

$$\begin{aligned}\frac{dL(t)}{dt} \leqslant & - \alpha \int_0^L u_x^2 dx - \lambda \int_0^L (\varphi - u)^2 dx \\ & - \frac{K}{4} \int_0^L (\varphi_x + \psi)^2 dx - \frac{b}{4} \int_0^L \psi_x^2 dx - \delta \int_0^1 w_x^2 dx \\ & - [N\mu - \rho] \int_0^L u_t^2 dx - [N\gamma - \rho_1] \int_0^L \varphi_t^2 dx \\ & - [N\kappa - C_1 - C_2] \int_0^1 w_{xt}^2 dx.\end{aligned}$$

Next, we choose N large enough so that (3.79) remains valid and, further,

$$\begin{cases} N\mu - \rho > 0, \\ N\gamma - \rho_1 > 0, \\ N\kappa - C_1 - C_2 > 0. \end{cases}$$

Thus, for some $\zeta_1 > 0$, we have

$$\frac{dL(t)}{dt} \leqslant -\zeta_1 \int_0^L \left\{ u_t^2 + u_x^2 + (\varphi - u)^2 + \varphi_t^2 + (\varphi_x + \psi)^2 + \psi_x^2 + w_{xt}^2 + w_x^2 \right\} dx.$$

On account of (3.5) and Poincaré's inequality, we can write that

$$\frac{dL(t)}{dt} \leqslant -\zeta_2 E(t), \quad \forall\, t \geqslant 0, \tag{3.81}$$

for some $\zeta_2 > 0$. Combining (3.81) with (3.79), we get

$$\frac{dL(t)}{dt} \leqslant -\lambda_1 L(t), \quad \forall\, t \geqslant 0. \tag{3.82}$$

A simple integration of (3.82) over $(0, t)$ leads to

$$L(t) \leqslant L(0)e^{-\lambda_1 t}, \quad \forall\, t \geqslant 0.$$

Again, recalling (3.79), we easily see that

$$\xi_1 E(t) \leqslant L(t) \leqslant L(0)e^{-\lambda_1 t} \leqslant \xi_2 E(0)e^{-\lambda_1 t}, \quad \forall\, t \geqslant 0,$$

then

$$\xi_1 E(t) \leqslant \xi_2 E(0)e^{-\lambda_1 t}, \quad \forall\, t \geqslant 0.$$

Therfore, the theorem is proved with $\lambda_0 = \dfrac{\xi_2}{\xi_1}E(0)$. □

3.5 Numerical approximation

Now, we present a numerical analysis of the problem studied theoretically in Sections 3.3 and 3.4.

3.5.1 Description of the discrete problem

We acquire a weak form associated to the continuous problem by multiplying equations (3.4) with the test functions $\bar{u}$, $\bar{\varphi}$, $\bar{\psi}$, $\bar{w} \in H_0^1(0, L)$, respectively. Let $\xi = u_t$, $\Phi = \varphi_t$, $\Psi = \psi_t$, and $\vartheta = w_t$. Applying integration by parts and using the boundary conditions, we find that

$$\begin{cases} \rho(\xi_t, \bar{u}) + \alpha(u_x, \bar{u}_x) - \lambda(\varphi - u, \bar{u}) + \mu(\xi, \bar{u}) = 0, \\ \rho_1(\Phi_t, \bar{\varphi}) + K(\varphi_x + \psi, \bar{\varphi}_x) + \lambda(\varphi - u, \bar{\varphi}) + \gamma(\Phi, \bar{\varphi}) + \beta(\vartheta_x, \bar{\varphi}) = 0, \\ b(\psi_x, \bar{\psi}_x) + K(\varphi_x + \psi, \bar{\psi}) = 0, \\ \rho_3(\vartheta_t, \bar{w}) + \delta(w_x, \bar{w}_x) + \beta(\Phi_x, \bar{w}) + \kappa(\vartheta_x, \bar{w}_x) = 0. \end{cases} \tag{3.83}$$

In order to define the discrete initial conditions, assuming that they are smooth enough, we set

$$u_h^0 = P_h^0 u_0,\ \xi_h^0 = P_h^0 u_1,\ \varphi_h^0 = P_h^0 \varphi_0,\ \Phi_h^0 = P_h^0 \varphi_1,\ \psi_h^0 = P_h^0 \psi_0,\ w_h^0 = P_h^0 w_0,\ \vartheta_h^0 = P_h^0 w_1.$$

When using the backward-Euler scheme in time, the fully finite-element approximation of the variational problem (3.83) consists to find $\xi_h^n, \Phi_h^n, \psi_h^n, \vartheta_h^n \in S_h^0$ such that, for $n = 1, \ldots, N$ and for all $\bar{u}_h$, $\bar{\varphi}_h$, $\bar{\psi}_h$, $\bar{w}_h \in S_h^0$,

$$\begin{cases} \frac{\rho}{\Delta t}(\xi_h^n - \xi_h^{n-1}, \bar{u}_h) + \alpha(u_{hx}^n, \bar{u}_{hx}) - \lambda(\varphi_h^n - u_h^n, \bar{u}_h) + \mu(\xi_h^n, \bar{u}_h) = 0, \\ \frac{\rho_1}{\Delta t}(\Phi_h^n - \Phi_h^{n-1}, \bar{\varphi}_h) + K(\varphi_{hx}^n + \psi_h^n, \bar{\varphi}_{hx}) + \lambda(\varphi_h^n - u_h^n, \bar{\varphi}_h) \\ \qquad\qquad + \gamma(\Phi_h^n, \bar{\varphi}_h) + \beta(\vartheta_{hx}^n, \bar{\varphi}_h) = 0, \\ b(\psi_{hx}^n, \bar{\psi}_{hx}) + K(\varphi_{hx}^n + \psi_h^n, \bar{\psi}_h) = 0, \\ \frac{\rho_3}{\Delta t}(\vartheta_h^n - \vartheta_h^{n-1}, \bar{w}_h) + \delta(w_{hx}^n, \bar{w}_{hx}) + \beta(\Phi_{hx}^n, \bar{w}_h) + \kappa(\vartheta_{hx}^n, \bar{w}_{hx}) = 0, \end{cases} \tag{3.84}$$

where

$$u_h^n = u_h^{n-1} + \Delta t\, \xi_h^n,\ \varphi_h^n = \varphi_h^{n-1} + \Delta t\, \Phi_h^n,\ w_h^n = w_h^{n-1} + \Delta t\, \vartheta_h^n.$$

By using the well-known Lax–Milgram lemma and the assumptions imposed on the constitutive parameters, it is easy to obtain that the fully discrete problem (3.84) has a unique solution.

3.5.2 Study of the discrete energy

The next result is a discrete version of the energy decay property (3.6) satisfied by the continuous solution.

Theorem 3.5.1. *Let the discrete energy be given by*

$$\begin{aligned} E^n = \frac{1}{2}\Big(&\rho\|\xi_h^n\|^2 + \alpha\|u_{hx}^n\|^2 + \lambda\|\varphi_h^n - u_h^n\|^2 + \rho_1\|\Phi_h^n\|^2 \\ &+ K\|\varphi_{hx}^n + \psi_h^n\|^2 + b\|\psi_{hx}^n\|^2 + \rho_3\|\vartheta_h^n\|^2 + \delta\|w_{hx}^n\|^2\Big). \end{aligned} \tag{3.85}$$

Then, the decay property

$$\frac{E^n - E^{n-1}}{\Delta t} \leqslant 0$$

holds for $n = 1, 2, \ldots, N$.

Proof. Taking $\bar{u}_h = \xi_h^n$, $\bar{\varphi}_h = \Phi_h^n$, $\bar{\psi}_h = \Psi_h^n$, and $\bar{w}_h = \vartheta_h^n$ in (3.84) with the fact that $(a - b, a) = \frac{1}{2}\left(\|a - b\|^2 + \|a\|^2 - \|b\|^2\right)$, it results that

$$\frac{\rho}{2\Delta t}\left(\|\xi_h^n - \xi_h^{n-1}\|^2 + \|\xi_h^n\|^2 - \|\xi_h^{n-1}\|^2\right) + \alpha(u_{hx}^n, \xi_{hx}^n) - \lambda(\varphi_h^n - u_h^n, \xi_h^n) + \mu\|\xi_h^n\|^2 = 0, \tag{3.86}$$

$$\begin{aligned} \frac{\rho_1}{2\Delta t}&\left(\|\Phi_h^n - \Phi_h^{n-1}\|^2 + \|\Phi_h^n\|^2 - \|\Phi_h^{n-1}\|^2\right) + K(\varphi_{hx}^n + \psi_h^n, \Phi_{hx}^n) \\ &+ \lambda(\varphi_h^n - u_h^n, \Phi_h^n) + \gamma\|\Phi_h^n\|^2 + \beta(\vartheta_{hx}^n, \Phi_h^n) = 0, \end{aligned} \tag{3.87}$$

$$b(\psi_{hx}^n, \Psi_{hx}^n) + K(\varphi_{hx}^n + \psi_h^n, \Psi_h^n) = 0 \tag{3.88}$$

and

$$\frac{\rho_3}{2\Delta t}\left(\|\vartheta_h^n - \vartheta_h^{n-1}\|^2 + \|\vartheta_h^n\|^2 - \|\vartheta_h^{n-1}\|^2\right) + \delta(w_{hx}^n, \vartheta_{hx}^n) + \beta(\Phi_{hx}^n, \vartheta_h^n) + \kappa\|\vartheta_{hx}^n\|^2 = 0. \tag{3.89}$$

Summing up equations (3.86)–(3.89) and keeping in mind that

$$\begin{aligned}
K(\varphi_{hx}^n + \psi_h^n, \Phi_{hx}^n + \Psi_h^n) &= \frac{K}{\Delta t}(\varphi_{hx}^n + \psi_h^n, \varphi_{hx}^n + \psi_h^n - (\varphi_{hx}^{n-1} + \psi_h^{n-1})) \\
&= \frac{K}{2\Delta t}\Big(\|\varphi_{hx}^n + \psi_h^n - (\varphi_{hx}^{n-1} + \psi_h^{n-1})\|^2 + \|\varphi_{hx}^n + \psi_h^n\|^2 - \|\varphi_{hx}^{n-1} + \psi_h^{n-1}\|^2\Big) \\
&\geqslant \frac{K}{2\Delta t}\left(\|\varphi_{hx}^n + \psi_h^n\|^2 - \|\varphi_{hx}^{n-1} + \psi_h^{n-1}\|^2\right),
\end{aligned}$$

$$\begin{aligned}
\lambda(\varphi_h^n - u_h^n, \Phi_h^n - \xi_h^n) &= \frac{\lambda}{\Delta t}(\varphi_h^n - u_h^n, \varphi_h^n - u_h^n - (\varphi_h^{n-1} - u_h^{n-1})) \\
&= \frac{\lambda}{2\Delta t}\Big(\|\varphi_h^n - u_h^n - (\varphi_h^{n-1} - u_h^{n-1})\|^2 + \|\varphi_h^n - u_h^n\|^2 - \|\varphi_h^{n-1} - u_h^{n-1}\|^2\Big) \\
&\geqslant \frac{\lambda}{2\Delta t}\left(\|\varphi_h^n - u_h^n\|^2 - \|\varphi_h^{n-1} - u_h^{n-1}\|^2\right),
\end{aligned}$$

$$\begin{aligned}
\alpha(u_{hx}^n, \xi_{hx}^n) &= \frac{\alpha}{\Delta t}(u_{hx}^n, u_{hx}^n - u_{hx}^{n-1}) \\
&= \frac{\alpha}{2\Delta t}\left(\|u_{hx}^n - u_{hx}^{n-1}\|^2 + \|u_{hx}^n\|^2 - \|u_{hx}^{n-1}\|^2\right) \\
&\geqslant \frac{\alpha}{2\Delta t}\left(\|u_{hx}^n\|^2 - \|u_{hx}^{n-1}\|^2\right),
\end{aligned}$$

$$\begin{aligned}
b(\psi_{hx}^n, \Psi_{hx}^n) &= \frac{b}{\Delta t}(\psi_{hx}^n, \psi_{hx}^n - \psi_{hx}^{n-1}) \\
&= \frac{b}{2\Delta t}\left(\|\psi_{hx}^n - \psi_{hx}^{n-1}\|^2 + \|\psi_{hx}^n\|^2 - \|\psi_{hx}^{n-1}\|^2\right) \\
&\geqslant \frac{b}{2\Delta t}\left(\|\psi_{hx}^n\|^2 - \|\psi_{hx}^{n-1}\|^2\right)
\end{aligned}$$

and

$$\begin{aligned}
\delta(w_{hx}^n, \vartheta_{hx}^n) &= \frac{\delta}{\Delta t}(\psi_{hx}^n, \psi_{hx}^n - \psi_{hx}^{n-1}) \\
&= \frac{\delta}{2\Delta t}\left(\|w_{hx}^n - w_{hx}^{n-1}\|^2 + \|w_{hx}^n\|^2 - \|w_{hx}^{n-1}\|^2\right) \\
&\geqslant \frac{\delta}{2\Delta t}\left(\|w_{hx}^n\|^2 - \|w_{hx}^{n-1}\|^2\right),
\end{aligned}$$

we find

$$\begin{aligned}
&\frac{\rho}{2\Delta t}\left(\|\xi_h^n-\xi_h^{n-1}\|^2+\|\xi_h^n\|^2-\|\xi_h^{n-1}\|^2\right)+\frac{\alpha}{2\Delta t}\left(\|u_{hx}^n\|^2-\|u_{hx}^{n-1}\|^2\right)\\
&\quad+\frac{\lambda}{2\Delta t}\left(\|\varphi_h^n-u_h^n\|^2-\|\varphi_h^{n-1}-u_h^{n-1}\|^2\right)+\frac{\rho_1}{2\Delta t}\left(\|\Phi_h^n-\Phi_h^{n-1}\|^2+\|\Phi_h^n\|^2-\|\Phi_h^{n-1}\|^2\right)\\
&\quad+\frac{K}{2\Delta t}\left(\|\varphi_{hx}^n+\psi_h^n\|^2-\|\varphi_{hx}^{n-1}+\psi_h^{n-1}\|^2\right)+\frac{\rho_3}{2\Delta t}\left(\|\vartheta_h^n-\vartheta_h^{n-1}\|^2+\|\vartheta_h^n\|^2-\|\vartheta_h^{n-1}\|^2\right)\\
&\quad+\frac{b}{2\Delta t}\left(\|\psi_{hx}^n\|^2-\|\psi_{hx}^{n-1}\|^2\right)+\frac{\delta}{2\Delta t}\left(\|w_{hx}^n\|^2-\|w_{hx}^{n-1}\|^2\right)+\mu\|\xi_h^n\|^2+\gamma\|\Phi_h^n\|^2+\kappa\|\vartheta_{hx}^n\|^2\leqslant 0.
\end{aligned}$$

By discarding $\|\xi_h^n-\xi_h^{n-1}\|^2$, $\|\Phi_h^n-\Phi_h^{n-1}\|^2$, $\|\vartheta_h^n-\vartheta_h^{n-1}\|^2$, $\|\xi_h^n\|^2$, $\|\Phi_h^n\|^2$ and $\|\vartheta_{hx}^n\|^2$, we deduce that

$$\begin{aligned}
&\frac{\rho}{2\Delta t}\left(\|\xi_h^n\|^2-\|\xi_h^{n-1}\|^2\right)+\frac{\alpha}{2\Delta t}\left(\|u_{hx}^n\|^2-\|u_{hx}^{n-1}\|^2\right)\\
&\quad+\frac{\lambda}{2\Delta t}\left(\|\varphi_h^n-u_h^n\|^2-\|\varphi_h^{n-1}-u_h^{n-1}\|^2\right)+\frac{\rho_1}{2\Delta t}\left(\|\Phi_h^n\|^2-\|\Phi_h^{n-1}\|^2\right)\\
&\quad+\frac{K}{2\Delta t}\left(\|\varphi_{hx}^n+\psi_h^n\|^2-\|\varphi_{hx}^{n-1}+\psi_h^{n-1}\|^2\right)+\frac{b}{2\Delta t}\left(\|\psi_{hx}^n\|^2-\|\psi_{hx}^{n-1}\|^2\right)\\
&\quad+\frac{\rho_3}{2\Delta t}\left(\|\vartheta_h^n\|^2-\|\vartheta_h^{n-1}\|^2\right)+\frac{\delta}{2\Delta t}\left(\|w_{hx}^n\|^2-\|w_{hx}^{n-1}\|^2\right)\leqslant 0;
\end{aligned}$$

hence,

$$\begin{aligned}
&\frac{1}{\Delta t}\Big[\frac{1}{2}\Big(\rho\|\xi_h^n\|^2+\alpha\|u_{hx}^n\|^2+\lambda\|\varphi_h^n-u_h^n\|^2+\rho_1\|\Phi_h^n\|^2+K\|\varphi_{hx}^n+\psi_h^n\|^2+b\|\psi_{hx}^n\|^2\\
&\quad+\rho_3\|\vartheta_h^n\|^2+\delta\|w_{hx}^n\|^2\Big)-\frac{1}{2}\Big(\rho\|\xi_h^{n-1}\|^2+\alpha\|u_{hx}^{n-1}\|^2+\lambda\|\varphi_h^{n-1}-u_h^{n-1}\|^2\\
&\quad+\rho_1\|\Phi_h^{n-1}\|^2+K\|\varphi_{hx}^{n-1}+\psi_h^{n-1}\|^2+b\|\psi_{hx}^{n-1}\|^2+\rho_3\|\vartheta_h^{n-1}\|^2+\delta\|w_{hx}^{n-1}\|^2\Big)\Big]\leqslant 0,
\end{aligned}$$

which proves the intended result. □

3.5.3 Error estimate

We now state and prove some a priori error estimates for the difference between the exact solution and the numerical solution.

The linear convergence of the numerical method is outlined in the following Theorem.

Theorem 3.5.2. *Suppose that the solution to the continuous problem (3.4) is regular enough, that is*

$$\begin{gathered}
u,\varphi,w\in H^3(0,T;L^2(0,L))\cap H^2(0,T;H^1(0,L))\cap W^{1,\infty}(0,T;H^2(0,L)),\\
\psi\in H^1(0,T;H^2(0,L)).
\end{gathered}$$

Then, the following error estimates

$$\begin{aligned}
&\|\xi_h^n-\xi(t_n)\|^2+\|u_{hx}^n-(u(t_n))_x\|^2+\|\varphi_h^n-u_h^n-(\varphi(t_n)-u(t_n))\|^2\\
&+\|\Phi_h^n-\Phi(t_n)\|^2+\|\varphi_{hx}^n+\psi_h^n-((\varphi(t_n))_x+\psi(t_n))\|^2+\|\psi_{hx}^n-(\psi(t_n))_x\|^2\\
&+\|\vartheta_h^n-\vartheta(t_n)\|^2+\|w_{hx}^n-(w(t_n))_x\|^2\leqslant C(h^2+(\Delta t)^2),
\end{aligned}$$

holds, where C is independent of Δt and h.

Proof. As a first step, let us set

$$z^n=u_h^n-P_h^0u(t_n),\ \hat{z}^n=\xi_h^n-P_h^0\xi(t_n),$$

$$e^n=\varphi_h^n-P_h^0\varphi(t_n),\ \hat{e}^n=\Phi_h^n-P_h^0\Phi(t_n),$$

$$y^n=\psi_h^n-P_h^0\psi(t_n),\ \hat{y}^n=\Psi_h^n-P_h^0\Psi(t_n),$$

and

$$\varrho^n=w_h^n-P_h^0w(t_n),\ \hat{\varrho}^n=\vartheta_h^n-P_h^0\vartheta(t_n).$$

Substituting in the scheme (3.84) and taking $\bar{u}_h=\hat{z}^n$, $\bar{\varphi}_h=\hat{e}^n$, $\bar{\psi}_h=\hat{y}^n$, and $\bar{w}=\hat{\varrho}^n$, we infer

$$\begin{cases}
\dfrac{\rho}{\Delta t}(\hat{z}^n+P_h^0\xi(t_n)-(\hat{z}^{n-1}+P_h^0\xi(t_{n-1})),\hat{z}^n)+\alpha(z_x^n+(P_h^0u(t_n))_x,\hat{z}_x^n)\\
\qquad-\lambda(e^n+P_h^0\varphi(t_n)-(z^n+P_h^0u(t_n)),\hat{z}^n)+\mu(\hat{z}^n+P_h^0\xi(t_n),\hat{z}^n)=0,\\
\dfrac{\rho_1}{\Delta t}(\hat{e}^n+P_h^0\Phi(t_n)-(\hat{e}^{n-1}+P_h^0\Phi(t_{n-1})),\hat{e}^n)\\
\qquad+K(e_x^n+(P_h^0\varphi(t_n))_x+y^n+P_h^0\psi(t_n),\hat{e}_x^n)\\
\qquad+\lambda(e^n+P_h^0\varphi(t_n)-(z^n+P_h^0u(t_n)),\hat{e}^n)\\
\qquad+\gamma(\hat{e}^n+P_h^0\Phi(t_n),\hat{e}^n)+m(\hat{\varrho}_x^n+(P_h^0\vartheta(t_n))_x,\hat{e}^n)=0,\\
b(y_x^n+(P_h^0\psi(t_n))_x,\hat{y}_x^n)+K(e_x^n+(P_h^0\varphi(t_n))_x+y^n+P_h^0\psi(t_n),\hat{y}^n)=0,\\
\dfrac{\rho_3}{\Delta t}(\hat{\varrho}^n+P_h^0\vartheta(t_n)-(\hat{\varrho}^{n-1}+P_h^0\vartheta(t_{n-1}),\hat{\varrho}^n)+\delta(\varrho_x^n+(P_h^0w(t_n))_x,\hat{\varrho}^n)\\
\qquad+m(\hat{e}_x^n+(P_h^0\Phi(t_n))_x,\hat{\varrho}^n)+\kappa(\hat{\varrho}_x^n+(P_h^0\vartheta(t_n))_x,\hat{\varrho}_x^n)=0,
\end{cases}$$

then, we arrive at

$$\begin{aligned}
&\frac{\rho}{2\Delta t}\left(\|\hat{z}^n-\hat{z}^{n-1}\|^2+\|\hat{z}^n\|^2-\|\hat{z}^{n-1}\|^2\right)+\frac{\rho}{\Delta t}(P_h^0\xi(t_n)-P_h^0\xi(t_{n-1}),\hat{z}^n)\\
&\quad+\alpha(z_x^n,\hat{z}_x^n)+\alpha(P_h^0u(t_n))_x,\hat{z}_x^n)-\lambda(e^n-z^n,\hat{z}^n)\\
&\quad-\lambda(P_h^0\varphi(t_n)-P_h^0u(t_n),\hat{z}^n)+\mu\|\hat{z}^n\|^2+\mu(P_h^0\xi(t_n),\hat{z}^n)=0,
\end{aligned}\tag{3.90}$$

$$
\begin{aligned}
&\frac{\rho_1}{2\Delta t}\left(\|\hat{e}^n-\hat{e}^{n-1}\|^2+\|\hat{e}^n\|^2-\|\hat{e}^{n-1}\|^2\right)+\frac{\rho_1}{\Delta t}(P_h^0\Phi(t_n)-P_h^0\Phi(t_{n-1}),\hat{e}^n)\\
&\quad+K(e_x^n+y^n,\hat{e}_x^n)+K((P_h^0\varphi(t_n))_x+P_h^0\psi(t_n),\hat{e}_x^n)+\lambda(e^n-z^n,\hat{e}^n)\\
&\quad+\lambda(P_h^0\varphi(t_n)-P_h^0u(t_n),\hat{e}^n)+\gamma\|\hat{e}^n\|^2+\gamma(P_h^0\Phi(t_n),\hat{e}^n)+\beta(\hat{\varrho}_x^n,\hat{e}^n)\\
&\quad+\beta((P_h^0\vartheta(t_n))_x,\hat{e}^n)=0,
\end{aligned}
\tag{3.91}
$$

$$
b(y_x^n,\hat{y}_x^n)+b((P_h^0\psi(t_n))_x,\hat{y}_x^n)+K(e_x^n+y^n,\hat{y}^n)+K((P_h^0\varphi(t_n))_x+P_h^0\psi(t_n),\hat{y}^n)=0,
\tag{3.92}
$$

$$
\begin{aligned}
&\frac{\rho_3}{2\Delta t}\left(\|\hat{\varrho}^n-\hat{\varrho}^{n-1}\|^2+\|\hat{\varrho}^n\|^2-\|\hat{\varrho}^{n-1}\|^2\right)+\frac{\rho_3}{\Delta t}(P_h^0\vartheta(t_n)-P_h^0\vartheta(t_{n-1}),\hat{\varrho}^n)\\
&\quad+\delta(\varrho_x^n,\hat{\varrho}_x^n)+\delta((P_h^0w(t_n))_x,\hat{\varrho}_x^n)+\beta(\hat{e}_x^n,\hat{\varrho}^n)+\beta((P_h^0\Phi(t_n))_x,\hat{\varrho}^n)\\
&\quad+\kappa\|\hat{\varrho}_x^n\|^2+\kappa((P_h^0\vartheta(t_n))_x,\hat{\varrho}_x^n)=0.
\end{aligned}
\tag{3.93}
$$

Let $\bar{u}=\hat{z}^n$, $\bar{\varphi}=\hat{e}^n$, $\bar{\psi}=\hat{y}^n$, $\bar{w}=\varrho^n$, in the weak form (3.83). We combine the resulting equations with (3.90)–(3.93) to obtain

$$
\begin{aligned}
&\frac{\rho}{2\Delta t}\left(\|\hat{z}^n-\hat{z}^{n-1}\|^2+\|\hat{z}^n\|^2-\|\hat{z}^{n-1}\|^2\right)+\alpha(z_x^n,\hat{z}_x^n)-\lambda(e^n-z^n,\hat{z}^n)+\mu\|\hat{z}^n\|^2\\
&=\rho(\xi_t(t_n)-\frac{P_h^0\xi(t_n)-P_h^0\xi(t_{n-1})}{\Delta t},\hat{z}^n)+\alpha(u_x(t_n)-(P_h^0u(t_n))_x,\hat{z}_x^n)\\
&\quad-\lambda(\varphi(t_n)-u(t_n)-(P_h^0\varphi(t_n)-P_h^0u(t_n)),\hat{z}^n)+\mu(\xi(t_n)-P_h^0\xi(t_n),\hat{z}^n),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\rho_1}{2\Delta t}\left(\|\hat{e}^n-\hat{e}^{n-1}\|^2+\|\hat{e}^n\|^2-\|\hat{e}^{n-1}\|^2\right)+\gamma\|\hat{e}^n\|^2\\
&\quad+K(e_x^n+y^n,\hat{e}_x^n)+\lambda(e^n-z^n,\hat{e}^n)+\beta(\hat{\varrho}_x^n,\hat{e}^n)\\
&=\rho_1(\Phi_t(t_n)-\frac{P_h^0\Phi(t_n)-P_h^0\Phi(t_{n-1})}{\Delta t},\hat{e}^n)\\
&\quad+K(\varphi_x(t_n)+\psi(t_n)-((P_h^0\varphi(t_n))_x+P_h^0\psi(t_n)),\hat{e}_x^n)\\
&\quad+\lambda(\varphi(t_n)-u(t_n)-(P_h^0\varphi(t_n)-P_h^0u(t_n)),\hat{e}^n)\\
&\quad+\gamma(\Phi(t_n)-P_h^0\Phi(t_n),\hat{e}^n)+\beta(\vartheta_x(t_n)-(P_h^0\vartheta(t_n))_x,\hat{e}^n),
\end{aligned}
$$

$$
\begin{aligned}
b(y_x^n,\hat{y}_x^n)+K(e_x^n+y^n,\hat{y}^n)=&\,b(\psi_x(t_n)-(P_h^0\psi(t_n))_x,\hat{y}_x^n)\\
&+K(\varphi_x(t_n)+\psi(t_n)-((P_h^0\varphi(t_n))_x+P_h^0\psi(t_n)),\hat{y}^n)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\rho_3}{2\Delta t}\left(\|\varrho^n-\varrho^{n-1}\|^2+\|\varrho^n\|^2-\|\varrho^{n-1}\|^2\right)+\delta(\varrho_x^n,\hat{\varrho}_x^n)+\beta(\hat{e}_x^n,\hat{\varrho}^n)+\kappa\|\hat{\varrho}_x^n\|^2\\
&=\rho_3(\vartheta_t(t_n)-\frac{P_h^0\vartheta(t_n)-P_h^0\vartheta(t_{n-1})}{\Delta t},\hat{\varrho}^n)+\delta(w_x(t_n)-(P_h^0w(t_n))_x,\hat{\varrho}_x^n)\\
&\quad+\beta(\Phi_x(t_n)-(P_h^0\Phi(t_n))_x,\hat{\varrho}^n)+\kappa(\vartheta_x(t_n)-(P_h^0\vartheta(t_n))_x,\hat{\varrho}_x^n).
\end{aligned}
$$

We sum up the last four equations, to obtain that

$$
\begin{aligned}
&\frac{\rho}{2\Delta t}\left(\|\hat{z}^n-\hat{z}^{n-1}\|^2+\|\hat{z}^n\|^2-\|\hat{z}^{n-1}\|^2\right)+\mu\|\hat{z}^n\|^2\\
&+\frac{\rho_1}{2\Delta t}\left(\|\hat{e}^n-\hat{e}^{n-1}\|^2+\|\hat{e}^n\|^2-\|\hat{e}^{n-1}\|^2\right)+\gamma\|\hat{e}^n\|^2\\
&+\frac{\rho_3}{2\Delta t}\left(\|\hat{\varrho}^n-\hat{\varrho}^{n-1}\|^2+\|\hat{\varrho}^n\|^2-\|\hat{\varrho}^{n-1}\|^2\right)+\kappa\|\hat{\varrho}_x^n\|^2\\
&+\lambda(e^n-z^n,\hat{e}^n-\hat{z}^n)+K(e_x^n+y^n,\hat{e}_x^n+\hat{y}^n)\\
&+b(y_x^n,\hat{y}_x^n)+\alpha(z_x^n,\hat{z}_x^n)+\delta(\varrho_x^n,\hat{\varrho}_x^n)\\
=&\rho(\xi_t(t_n)-\frac{P_h^0\xi(t_n)-P_h^0\xi(t_{n-1})}{\Delta t},\hat{z}^n)\\
&+\mu(\xi(t_n)-P_h^0\xi(t_n),\hat{z}^n)+\alpha(u_x(t_n)-(P_h^0u(t_n))_x,\hat{z}_x^n)\\
&+\lambda(\varphi(t_n)-u(t_n)-(P_h^0\varphi(t_n)-P_h^0u(t_n)),\hat{e}^n-\hat{z}^n)\\
&+\rho_1(\Phi_t(t_n)-\frac{P_h^0\Phi(t_n)-P_h^0\Phi(t_{n-1})}{\Delta t},\hat{e}^n)\\
&+\gamma(\Phi(t_n)-P_h^0\Phi(t_n),\hat{e}^n)\\
&+K(\varphi_x(t_n)+\psi(t_n)-((P_h^0\varphi(t_n))_x+P_h^0\psi(t_n)),\hat{e}_x^n+\hat{y}^n)\\
&+b(\psi_x(t_n)-(P_h^0\psi(t_n))_x,\hat{y}_x^n)\\
&+\rho_3(\vartheta_t(t_n)-\frac{P_h^0\vartheta(t_n)-P_h^0\vartheta(t_{n-1})}{\Delta t},\hat{\varrho}^n)\\
&+\delta(w_x(t_n)-(P_h^0w(t_n))_x,\hat{\varrho}_x^n)+\kappa(\vartheta_x(t_n)-(P_h^0\vartheta(t_n))_x,\hat{\varrho}_x^n).
\end{aligned}
\tag{3.94}
$$

Now, by using the definitions of $\hat{z}^n$, $\hat{e}^n$, $\hat{y}^n$ and $\hat{\varrho}^n$, we get the following estimates:

$$
\begin{aligned}
(e^n-z^n,\hat{e}^n-\hat{z}^n)&=(e^n-z^n,\Phi_h^n-P_h^0\Phi(t_n)-(\xi_h^n-P_h^0\xi(t_n))\\
&=(e^n-z^n,\frac{e^n-e^{n-1}}{\Delta t}-\frac{z^n-z^{n-1}}{\Delta t})\\
&\quad+(e^n-z^n,\frac{P_h^0\varphi(t_n)-P_h^0\varphi(t_{n-1})}{\Delta t}-P_h^0\Phi(t_n))\\
&\quad-(e^n-z^n,\frac{P_h^0u(t_n)-P_h^0u(t_{n-1})}{\Delta t}-P_h^0\xi(t_n))\\
&=\frac{1}{2\Delta t}\left(\|e^n-z^n-(e^{n-1}-z^{n-1})\|^2+\|e^n-z^n\|^2-\|e^{n-1}-z^{n-1}\|^2\right)\\
&\quad+(e^n-z^n,\frac{P_h^0\varphi(t_n)-P_h^0\varphi(t_{n-1})}{\Delta t}-P_h^0\Phi(t_n))\\
&\quad-(e^n-z^n,\frac{P_h^0u(t_n)-P_h^0u(t_{n-1})}{\Delta t}-P_h^0\xi(t_n)),
\end{aligned}
\tag{3.95}
$$

$$
\begin{aligned}
(e_x^n + y^n, \hat{e}_x^n + \hat{y}^n) &= (e_x^n + y^n, \Phi_{hx}^n - (P_h^0 \Phi(t_n))_x + \Psi_h^n - P_h^0 \Psi(t_n)) \\
&= (e_x^n + y^n, \frac{e_x^n - e_x^{n-1}}{\Delta t} + \frac{y^n - y^{n-1}}{\Delta t}) \\
&\quad + (e_x^n + y^n, \frac{(P_h^0 \varphi(t_n))_x - (P_h^0 \varphi(t_{n-1}))_x}{\Delta t} - (P_h^0 \Phi(t_n))_x) \\
&\quad + (e_x^n + y^n, \frac{P_h^0 \psi(t_n) - P_h^0 \psi(t_{n-1})}{\Delta t} - P_h^0 \Psi(t_n)) \\
&= \frac{1}{2\Delta t} \left(\|e_x^n + y^n - (e_x^{n-1} + y^{n-1})\|^2 + \|e_x^n + y^n\|^2 - \|e_x^{n-1} + y^{n-1}\|^2 \right) \\
&\quad + (e_x^n + y^n, \frac{(P_h^0 \varphi(t_n))_x - (P_h^0 \varphi(t_{n-1}))_x}{\Delta t} - (P_h^0 \Phi(t_n))_x) \\
&\quad + (e_x^n + y^n, \frac{P_h^0 \psi(t_n) - P_h^0 \psi(t_{n-1})}{\Delta t} - P_h^0 \Psi(t_n)),
\end{aligned} \tag{3.96}
$$

$$
\begin{aligned}
(z_x^n, \hat{z}_x^n) &= (z_x^n, \xi_{hx}^n - (P_h^0 \xi(t_n))_x) \\
&= (z_x^n, \frac{z_x^n - z_x^{n-1}}{\Delta t} + \frac{(P_h^0 u(t_n))_x - (P_h^0 u(t_{n-1}))_x}{\Delta t} - (P_h^0 \xi(t_n))_x) \\
&= (z_x^n, \frac{z_x^n - z_x^{n-1}}{\Delta t}) + (z_x^n, \frac{(P_h^0 u(t_n))_x - (P_h^0 u(t_{n-1}))_x}{\Delta t} - (P_h^0 \xi(t_n))_x) \\
&= \frac{1}{2\Delta t} \left(\|z_x^n - z_x^{n-1}\|^2 + \|z_x^n\|^2 - \|z_x^{n-1}\|^2 \right) \\
&\quad + (z_x^n, \frac{(P_h^0 u(t_n))_x - (P_h^0 u(t_{n-1}))_x}{\Delta t} - (P_h^0 \xi(t_n))_x),
\end{aligned} \tag{3.97}
$$

$$
\begin{aligned}
(y_x^n, \hat{y}_x^n) &= (y_x^n, \Psi_{hx}^n - (P_h^0 \Psi(t_n))_x) \\
&= (y_x^n, \frac{y_x^n - y_x^{n-1}}{\Delta t} + \frac{(P_h^0 \psi(t_n))_x - (P_h^0 \psi(t_{n-1}))_x}{\Delta t} - (P_h^0 \Psi(t_n))_x) \\
&= (y_x^n, \frac{y_x^n - y_x^{n-1}}{\Delta t}) + (y_x^n, \frac{(P_h^0 \psi(t_n))_x - (P_h^0 \psi(t_{n-1}))_x}{\Delta t} - (P_h^0 \Psi(t_n))_x) \\
&= \frac{1}{2\Delta t} \left(\|y_x^n - y_x^{n-1}\|^2 + \|y_x^n\|^2 - \|y_x^{n-1}\|^2 \right) \\
&\quad + (y_x^n, \frac{(P_h^0 \psi(t_n))_x - (P_h^0 \psi(t_{n-1}))_x}{\Delta t} - (P_h^0 \Psi(t_n))_x)
\end{aligned} \tag{3.98}
$$

and

$$
\begin{aligned}
(\varrho_x^n, \hat{\varrho}_x^n) &= (\varrho_x^n, \vartheta_{hx}^n - (P_h^0 \vartheta(t_n))_x) \\
&= (\varrho_x^n, \frac{\varrho_x^n - \varrho_x^{n-1}}{\Delta t} + \frac{P_h^0 w(t_n))_x - (P_h^0 w(t_{n-1}))_x}{\Delta t} - (P_h^0 \vartheta(t_n))_x) \\
&= (\varrho_x^n, \frac{\varrho_x^n - \varrho_x^{n-1}}{\Delta t}) + (\varrho_x^n, \frac{P_h^0 w(t_n))_x - (P_h^0 w(t_{n-1}))_x}{\Delta t} - (P_h^0 \vartheta(t_n))_x) \\
&= \frac{1}{2\Delta t} \left(\|\varrho_x^n - \varrho_x^{n-1}\|^2 + \|\varrho_x^n\|^2 - \|\varrho_x^{n-1}\|^2 \right) \\
&\quad + (\varrho_x^n, \frac{(P_h^0 w(t_n))_x - (P_h^0 w(t_{n-1}))_x}{\Delta t} - (P_h^0 \vartheta(t_n))_x).
\end{aligned} \tag{3.99}
$$

Inserting (3.95)–(3.99) into (3.94), then discarding the positive terms:

$$\|\hat{z}^n - \hat{z}^{n-1}\|^2,\ \|z_x^n - z_x^{n-1}\|^2,\ \|e^n - z^n - (e^{n-1} - z^{n-1})\|^2,\ \|\hat{e}^n - \hat{e}^{n-1}\|^2$$

$\|e_x^n + y^n - (e_x^{n-1} + y^{n-1})\|^2,\ \|y_x^n - y_x^{n-1}\|^2,\ \|\hat{\varrho}^n - \hat{\varrho}^{n-1}\|^2,\ \|\varrho_x^n - \varrho_x^{n-1}\|^2,\ \|\hat{z}^n\|^2,\ \|\hat{e}^n\|^2$ and $\|\hat{\varrho}_x^n\|^2$,

we arrive at

$$\begin{aligned}
&\frac{\rho}{2\Delta t}\left(\|\hat{z}^n\|^2 - \|\hat{z}^{n-1}\|^2\right) + \frac{\alpha}{2\Delta t}\left(\|z_x^n\|^2 - \|z_x^{n-1}\|^2\right)\\
&+ \frac{\lambda}{2\Delta t}\left(\|e^n - z^n\|^2 - \|e^{n-1} - z^{n-1}\|^2\right) + \frac{\rho_1}{2\Delta t}\left(\|\hat{e}^n\|^2 - \|\hat{e}^{n-1}\|^2\right)\\
&+ \frac{K}{2\Delta t}\left(\|e_x^n + y^n\|^2 - \|e_x^{n-1} + y^{n-1}\|^2\right) + \frac{b}{2\Delta t}\left(\|y_x^n\|^2 - \|y_x^{n-1}\|^2\right)\\
&+ \frac{\rho_3}{2\Delta t}\left(\|\hat{\varrho}^n\|^2 - \|\hat{\varrho}^{n-1}\|^2\right) + \frac{\delta}{2\Delta t}\left(\|\varrho_x^n\|^2 - \|\varrho_x^{n-1}\|^2\right)\\
\leqslant& \rho(\xi_t(t_n) - \frac{P_h^0\xi(t_n) - P_h^0\xi(t_{n-1})}{\Delta t}, \hat{z}^n) + \alpha(u_x(t_n) - (P_h^0 u(t_n))_x, \hat{z}_x^n)\\
&- \alpha(z_x^n, \frac{(P_h^0 u(t_n))_x - (P_h^0 u(t_{n-1}))_x}{\Delta t} - (P_h^0\xi(t_n))_x)\\
&+ \lambda(\varphi(t_n) - u(t_n) - (P_h^0\varphi(t_n) - P_h^0 u(t_n)), \hat{e}^n - \hat{z}^n)\\
&- \lambda(e^n - z^n, \frac{P_h^0\varphi(t_n) - P_h^0\varphi(t_{n-1})}{\Delta t} - P_h^0\Phi(t_n))\\
&+ \lambda(e^n - z^n, \frac{P_h^0 u(t_n) - P_h^0 u(t_{n-1})}{\Delta t} - P_h^0\xi(t_n))\\
&+ \mu(\xi(t_n) - P_h^0\xi(t_n), \hat{z}^n) + \rho_1(\Phi_t(t_n) - \frac{P_h^0\Phi(t_n) - P_h^0\Phi(t_{n-1})}{\Delta t}, \hat{e}^n)\\
&+ K(\varphi_x(t_n) + \psi(t_n) - ((P_h^0\varphi(t_n))_x + P_h^0\psi(t_n)), \hat{e}_x^n + \hat{y}^n)\\
&- K(e_x^n + y^n, \frac{(P_h^0\varphi(t_n))_x - (P_h^0\varphi(t_{n-1}))_x}{\Delta t} - (P_h^0\Phi(t_n))_x)\\
&- K(e_x^n + y^n, \frac{P_h^0\psi(t_n) - P_h^0\psi(t_{n-1})}{\Delta t} - P_h^0\Psi(t_n)) + \gamma(\Phi(t_n) - P_h^0\Phi(t_n), \hat{e}^n)\\
&+ b(\psi_x(t_n) - (P_h^0\psi(t_n))_x, \hat{y}_x^n) - b(y_x^n, \frac{(P_h^0\psi(t_n))_x - (P_h^0\psi(t_{n-1}))_x}{\Delta t} - (P_h^0\Psi(t_n))_x)\\
&+ \rho_3(\vartheta_t(t_n) - \frac{P_h^0\vartheta(t_n) - P_h^0\vartheta(t_{n-1})}{\Delta t}, \hat{\varrho}^n) + \delta(w_x(t_n) - (P_h^0 w(t_n))_x, \hat{\varrho}_x^n)\\
&- \delta(\varrho_x^n, \frac{(P_h^0 w(t_n))_x - (P_h^0 w(t_{n-1}))_x}{\Delta t} - (P_h^0\vartheta(t_n))_x) + \kappa(\vartheta_x(t_n) - (P_h^0\vartheta(t_n))_x, \hat{\varrho}_x^n).
\end{aligned}$$

Finally, let

$$Z_n = \rho\|\hat{z}^n\|^2 + \alpha\|z_x^n\|^2 + \lambda\|e^n - z^n\|^2 + \rho_1\|\hat{e}^n\|^2 + K\|e_x^n + y^n\|^2 + b\|y_x^n\|^2 + \rho_3\|\hat{\varrho}^n\|^2 + \delta\|\varrho_x^n\|^2.$$

Using Young's inequality, we easily find that

$$
\begin{aligned}
Z_n - Z_{n-1} \leqslant 2C\Delta t\Big(Z_n &+ \left\| \xi_t(t_n) - \frac{P_h^0\xi(t_n) - P_h^0\xi(t_{n-1})}{\Delta t} \right\|^2 + \|u_x(t_n) - (P_h^0 u(t_n))_x\|^2 \\
&+ \left\| \frac{(P_h^0 u(t_n))_x - (P_h^0 u(t_{n-1}))_x}{\Delta t} - (P_h^0\xi(t_n))_x \right\|^2 \\
&+ \|\varphi(t_n) - u(t_n) - (P_h^0\varphi(t_n) - P_h^0 u(t_n)\|^2 \\
&+ \left\| \frac{P_h^0\varphi(t_n) - P_h^0\varphi(t_{n-1})}{\Delta t} - P_h^0\Phi(t_n) \right\|^2 \\
&+ \left\| \frac{P_h^0 u(t_n) - P_h^0 u(t_{n-1})}{\Delta t} - P_h^0\xi(t_n) \right\|^2 + \left\|\xi(t_n) - P_h^0\xi(t_n)\right\|^2 \\
&+ \left\| \Phi_t(t_n) - \frac{P_h^0\Phi(t_n) - P_h^0\Phi(t_{n-1})}{\Delta t} \right\|^2 \\
&+ \|\varphi_x(t_n) + \psi(t_n) - ((P_h^0\varphi(t_n))_x + P_h^0\psi(t_n))\|^2 \\
&+ \left\| \frac{(P_h^0\varphi(t_n))_x - (P_h^0\varphi(t_{n-1}))_x}{\Delta t} - (P_h^0\Phi(t_n))_x \right\|^2 \\
&+ \left\| \frac{P_h^0\psi(t_n) - P_h^0\psi(t_{n-1})}{\Delta t} - P_h^0\Psi(t_n) \right\|^2 \\
&+ \|\Phi(t_n) - P_h^0\Phi(t_n)\|^2 + \|\psi_x(t_n) - (P_h^0\psi(t_n))_x\|^2 \\
&+ \left\| \frac{(P_h^0\psi(t_n))_x - (P_h^0\psi(t_{n-1}))_x}{\Delta t} - (P_h^0\Psi(t_n))_x \right\|^2 \\
&+ \left\| \vartheta_t(t_n) - \frac{P_h^0\vartheta(t_n) - P_h^0\vartheta(t_{n-1})}{\Delta t} \right\|^2 + \|w_x(t_n) - (P_h^0 w(t_n))_x\|^2 \\
&+ \left\| \frac{(P_h^0 w(t_n))_x - (P_h^0 w(t_{n-1}))_x}{\Delta t} - (P_h^0\vartheta(t_n))_x \right\|^2 \\
&+ \|\vartheta_x(t_n) - (P_h^0\vartheta(t_n))_x\|^2 \Big).
\end{aligned}
$$

As a consequence, we have

$$
Z_n - Z_{n-1} \leqslant 2C\Delta t(Z_n + R_n), \tag{3.100}
$$

where the residual R_n is the sum of the approximation errors. Summing the previous inequality over n, it follows that

$$
Z_n - Z_0 \leqslant 2C\Delta t \sum_{j=0}^{n} (Z_j + R_j)
$$

and, making use of Taylor's expansion in time and (2.5) to estimate the time and the space error, we get that

$$
2C\Delta t \sum_{j=0}^{n} R_j \leqslant C(h^2 + (\Delta t)^2).
$$

Since $Z_0 = 0$, we end up with

$$Z_n \leqslant 2C\Delta t \sum_{j=1}^{n} Z_n + C(h^2 + (\Delta t)^2).$$

The result follows by applying a discrete version of Gronwall's inequality and taking into account that $n\Delta t \leqslant T$. □

3.6 Simulations

In our simulations, we select the following values:

$$L = 1,\ h = 0.01,\ \Delta t = h/2,\ \alpha = 6,\ \rho_1 = 2,\ K = 365,$$

$$\rho = \lambda = \mu = \gamma = \beta = b = \rho_3 = \delta = \kappa = 1,$$

taking as initial conditions

$$u_0(x) = u_1(x) = \varphi_0(x) = \varphi_1(x) = \psi_0(x) = w_0(x) = w_1(x) = \sin(\pi x).$$

The evolution of u, φ, ψ and w are represented in Figures 3.1, 3.2, 3.3 and 3.4, respectively.

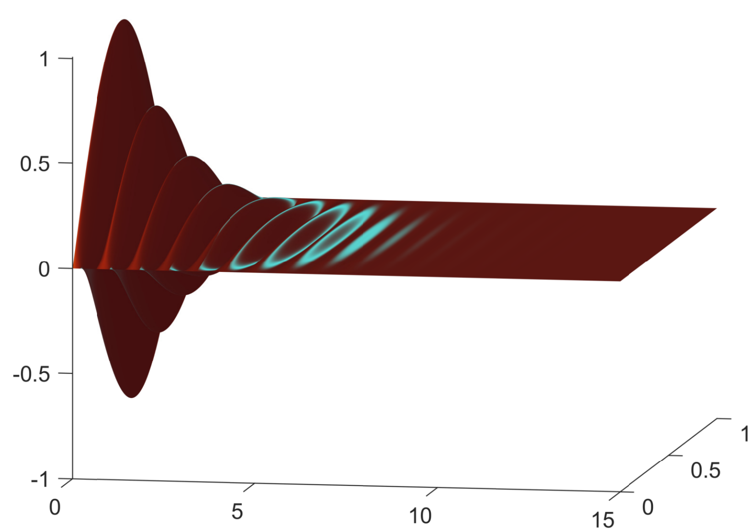


Figure 3.1 The evolution in time and space of u.

The results at point $x = 0.6$ are displayed in Figures 3.5, 3.6 and 3.7.

The decay of energy with respect to time is shown in Figures 3.8, 3.9 and 3.10.

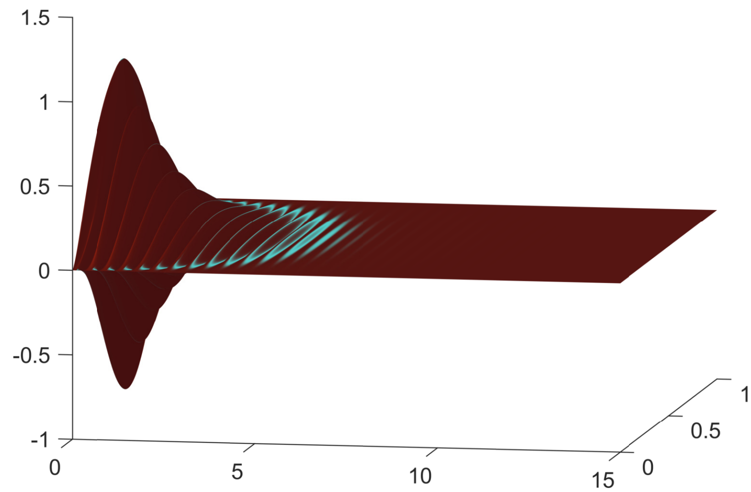


Figure 3.2 The evolution in time and space of φ.

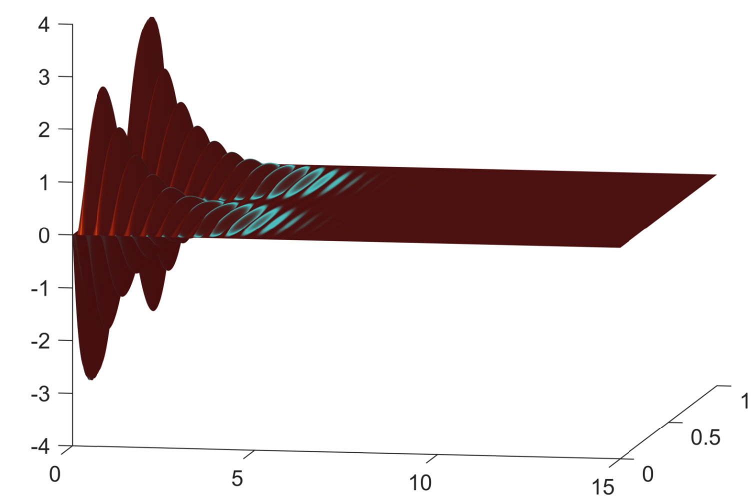


Figure 3.3 The evolution in time and space of ψ.

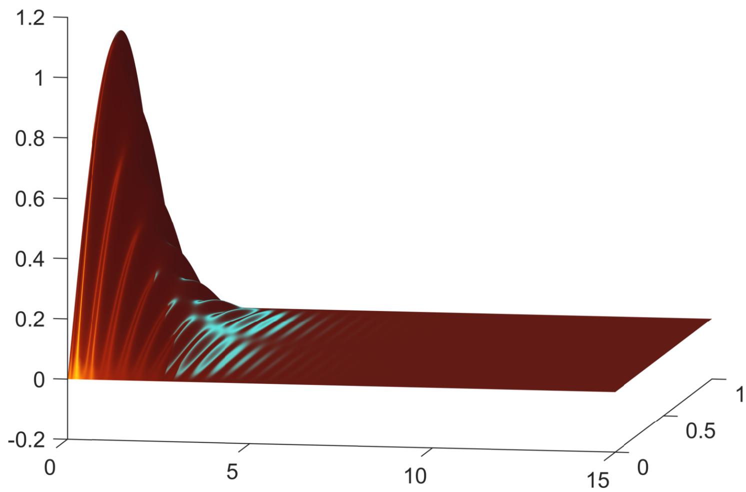


Figure 3.4 The evolution in time and space of w.

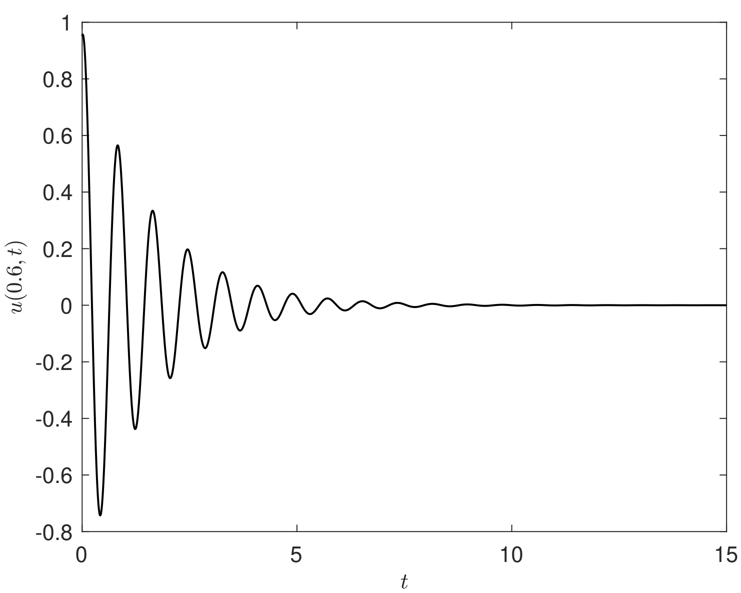


Figure 3.5 The evolution in time of u at $x = 0.6$.

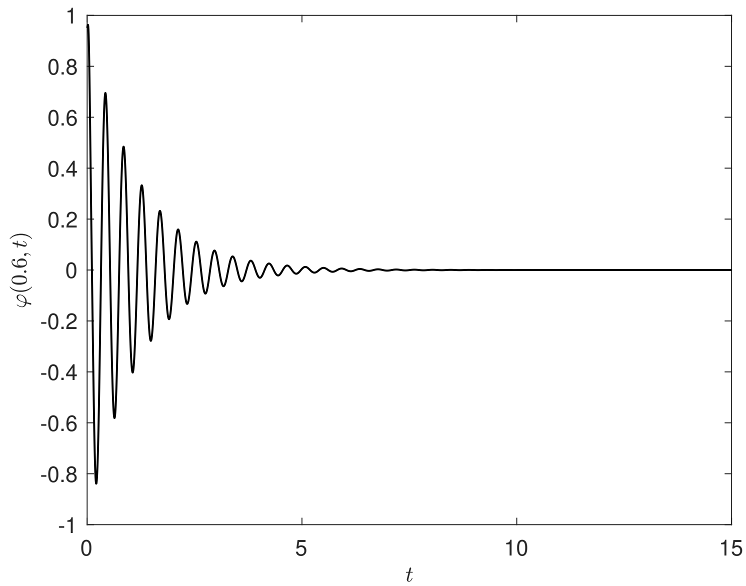


Figure 3.6 The evolution in time of φ at $x = 0.6$.

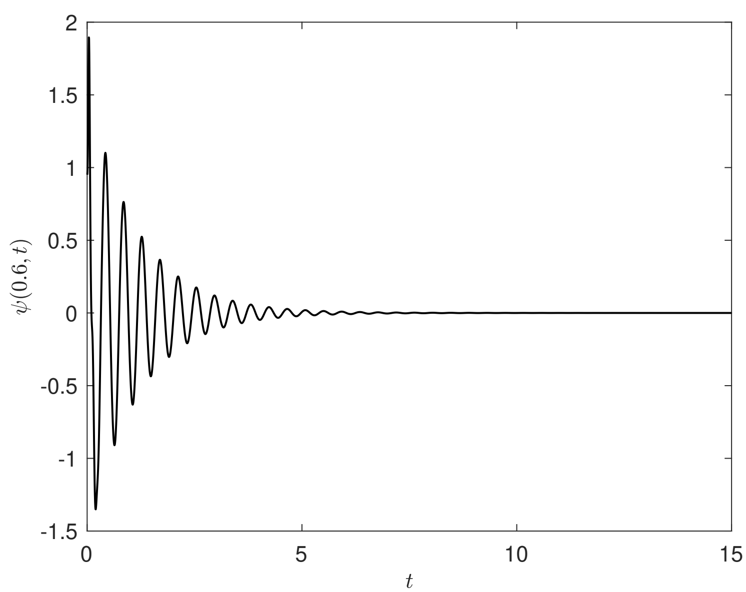


Figure 3.7 The evolution in time of ψ at $x = 0.6$.

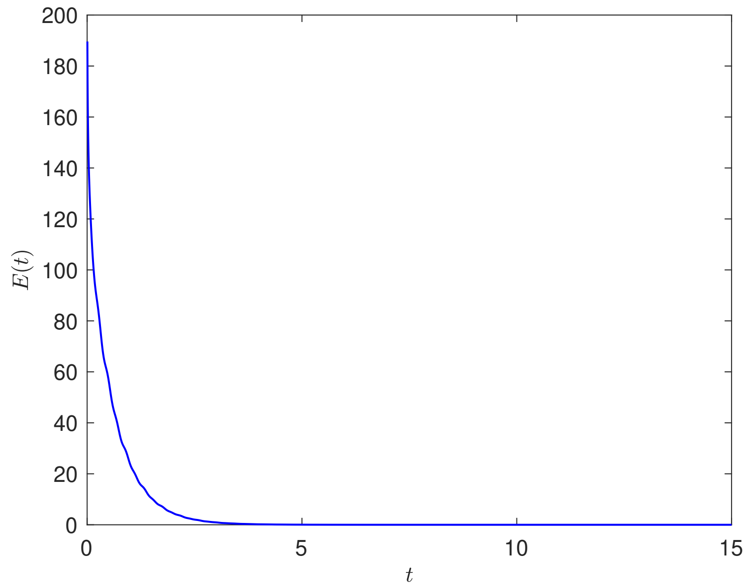


Figure 3.8 The evolution in time of E.

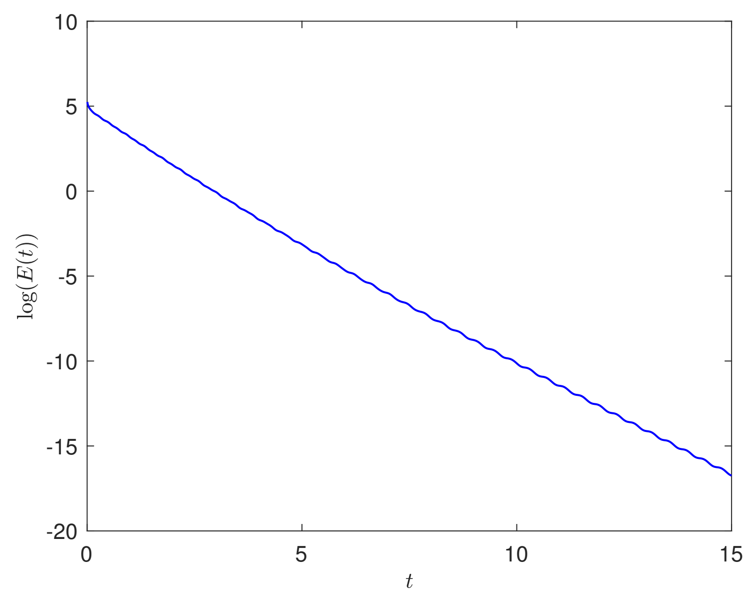


Figure 3.9 The evolution in time of $\log(E(t))$.

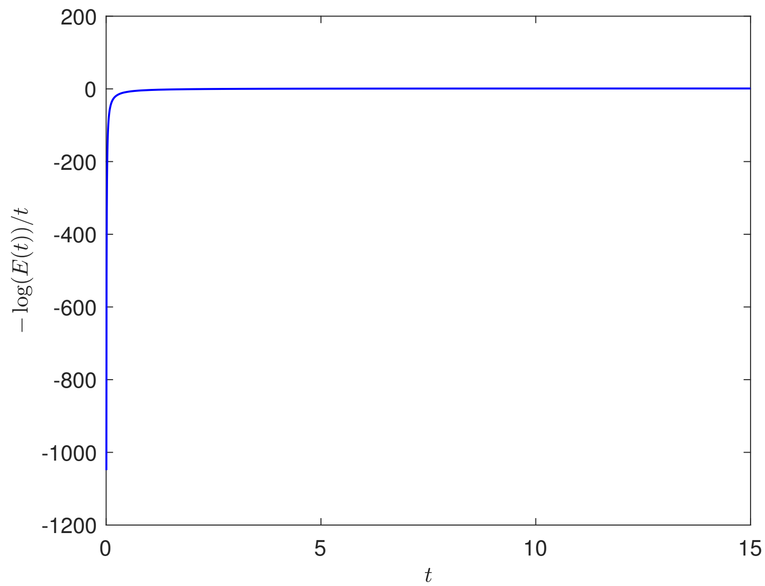


Figure 3.10 The evolution in time of $-\log(E(t))/t$.

Following this, we carried out a numerical simulation to evaluate the accuracy of the error estimate. We solved the modified problem

$$\begin{cases} \rho u_{tt} - \alpha u_{xx} - \lambda(\varphi - u) + \mu u_t = \mathcal{F}_1, \\ \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x + \lambda(\varphi - u) + \gamma\varphi_t + \beta w_{xt} = \mathcal{F}_2, \\ -b\psi_{xx} + K(\varphi_x + \psi) = \mathcal{F}_3, \\ \rho_3 w_{tt} - \delta w_{xx} + \beta\varphi_{xt} - \kappa w_{xxt} = \mathcal{F}_4, \end{cases} \tag{3.101}$$

the functions $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$, and the initial data are derived from the exact solution

$$u(x,t) = 0.01tx^2(x-1)^2, \varphi(x,t) = e^t \sin(\pi x),$$
$$\psi(x,t) = e^t x \cos(0.5\pi x), w(x,t) = 2e^t \sin(\pi x).$$

The calculated errors at time $T = 1.2$ are presented in Table 3.1, where the $Error$ is defined as

$$\begin{aligned} Error = \Big(& \|\xi_h^n - \xi(t_n)\|^2 + \|u_{hx}^n - (u(t_n))_x\|^2 + \|\varphi_h^n - u_h^n - (\varphi(t_n) - u(t_n))\|^2 \\ & + \|\Phi_h^n - \Phi(t_n)\|^2 + \|\varphi_{hx}^n + \psi_h^n - ((\varphi(t_n))_x + \psi(t_n))\|^2 + \|\psi_{hx}^n - (\psi(t_n))_x\|^2 \\ & + \|\vartheta_h^n - \vartheta(t_n)\|^2 + \|w_{hx}^n - (w(t_n))_x\|^2 \Big)^{\frac{1}{2}}. \end{aligned}$$

It can be observed that the errors decrease by a factor of approximately 2 when the discretization parameters are halved. The linear convergence rate is also evident in the curves illustrated in Figure 3.11.

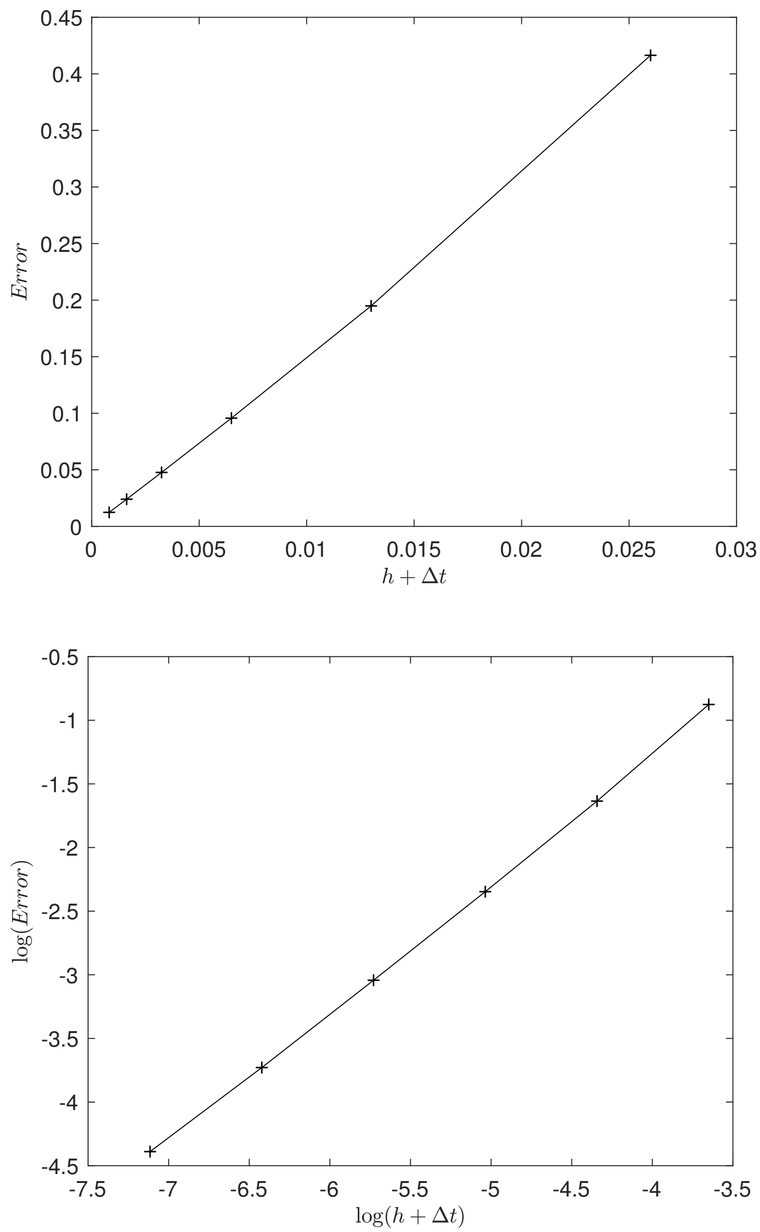


Figure 3.11 The evolution of the error depending on $h + \Delta t$.

Table 3.1 Computed errors when $T = 1.2$.

M	Δt	$Error$
40	1.00×10^{-3}	4.164×10^{-1}
80	5.00×10^{-4}	1.949×10^{-1}
160	2.50×10^{-4}	9.567×10^{-2}
320	1.25×10^{-4}	4.770×10^{-2}
640	6.25×10^{-5}	2.402×10^{-2}
1280	3.125×10^{-5}	1.241×10^{-2}

Chapter 4

Analysis of a laminated beam with dual-phase-lag thermoelasticity

4.1 Introduction

Laminated beams play a crucial role in engineering due to their wide-ranging applications in building and construction of various structures. In recent years, significant attention has been given by researchers to investigating the well-posedness and asymptotic stability of these beams, particularly under the influence of thermal effects. For instance, Apalara [11] investigated a thermoplastic laminated beam with structural damping and second sound given by

$$\begin{cases} \rho\omega_{tt} + G(\psi - \omega_x)_x = 0, \\ I_\rho(3s-\psi)_{tt} - D(3s-\psi)_{xx} - G(\psi - \omega_x) + \delta\theta_x = 0, \\ I_\rho s_{tt} - Ds_{xx} + G(\psi - \omega_x) + \dfrac{4}{3}\gamma s + \dfrac{4}{3}\beta s_t = 0, \\ \rho_3\theta_t + q_x + \delta(3s-\psi)_{xt} = 0, \\ \tau q_t + \alpha q + \theta_x = 0, \end{cases} \tag{4.1}$$

and proved the well-posedness and established both exponential and polynomial stability results depending on the stability number

$$\chi_\tau = \left(1 - \frac{\tau\rho_3 G}{\rho}\right)\left(\frac{D}{I_\rho} - \frac{G}{\rho}\right) - \frac{\tau G\delta^2}{\rho I_\rho}.$$

To extend the previous results, the same author in [10] examined system (1.11) with thermal effects in the slip rather than frictional damping, without any additional damping (internal or boundary) term, which has the form

$$\begin{cases} \rho\omega_{tt} + G(\psi - \omega_x)_x = 0, \\ I_\rho(3s-\psi)_{tt} - D(3s-\psi)_{xx} - G(\psi - \omega_x) = 0, \\ I_\rho s_{tt} - Ds_{xx} + G(\psi - \omega_x) + \dfrac{4}{3}\gamma s + \dfrac{4}{3}\beta s_t + \delta\theta_x = 0, \\ \rho_3\theta_t - \alpha\theta_{xx} + \delta s_{xt} = 0, \end{cases} \tag{4.2}$$

and demonstrated that this unique dissipation is sufficiently strong to exponentially stabilize the system, provided the wave speeds are equal. Feng [59] considered the following system

$$\begin{cases} \rho\omega_{tt} + G(\psi - \omega_x)_x + \delta\theta_x = 0, \\ I_\rho(3s-\psi)_{tt} - D(3s-\psi)_{xx} - G(\psi - \omega_x) = 0, \\ I_\rho s_{tt} - Ds_{xx} + G(\psi - \omega_x) + \frac{4}{3}\gamma s + \frac{4}{3}\beta s_t = 0, \\ \rho_3\theta_t + q_x + \delta\omega_{xt} = 0, \\ \tau q_t + \alpha q + \theta_x = 0 \end{cases} \tag{4.3}$$

and established the global well-posedness and showed that, under a new stability number, denoted by

$$\chi = \tau\delta^2 D - (D\rho - GI_\rho)\left(\frac{\tau\rho_3 D}{I_\rho} - 1\right),$$

the system is exponentially stable when $\chi = 0$ and polynomially stable when $\chi \neq 0$. For further results on thermal effects, we refer the reader to [101, 111] for the classical and second sound heat effects, and to [100] for thermoelasticity of type III. In these studies, the authors established both exponential and polynomial decay results, subject to certain restrictions on the system parameters.

Recently, Bresse system was analyzed within the dual-phase-lag thermoelastic theory from both mathematical (existence) and numerical points of view in [14]. By using the multiplier method, Bouraoui et al. [25] considered a Bresse system and proved that the system is dissipative under the condition (1.7) and exponentially stable by introducing a new stability number.

Considering the observations mentioned above, a natural question arises: Can a laminated beam system be exponentially stabilized using the dual-phase-lag heat conduction? To provide an answer to this question, we analyze system (1.11) with the presence of thermal effect as described by the dual-phase-lag theory (1.6), which has the form

$$\begin{cases} \rho\omega_{tt} + G(\psi - \omega_x)_x + \delta\left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)_x = 0, \\ I_\rho(3s-\psi)_{tt} - D(3s-\psi)_{xx} - G(\psi - \omega_x) = 0, \\ I_\rho s_{tt} - Ds_{xx} + G(\psi - \omega_x) + \frac{4}{3}\gamma s + \frac{4}{3}s_t = 0, \\ \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)_t - \kappa(\tau_\theta\theta_{xt} + \theta_x)_x + \delta\theta^0\omega_{xt} = 0, \end{cases} \tag{4.4}$$

together with the following boundary conditions

$$\begin{cases} \omega_x(0,t) = \omega_x(1,t) = \psi(0,t) = \psi(1,t) = 0, \ t \geq 0, \\ s(0,t) = s(1,t) = \theta(0,t) = \theta(1,t) = 0, \ t \geq 0, \end{cases} \tag{4.5}$$

and initial conditions

$$\begin{cases} \omega\left(x,0\right)=\omega_{0}\left(x\right), \ \omega_{t}\left(x,0\right)=\omega_{1}\left(x\right), \ \psi\left(x,0\right)=\psi_{0}\left(x\right), \\ \psi_{t}\left(x,0\right)=\psi_{1}\left(x\right), \ s\left(x,0\right)=s_{0}\left(x\right), \ s_{t}\left(x,0\right)=s_{1}\left(x\right), \\ \theta(x,0)=\theta_{0}(x), \ \theta_{t}(x,0)=\theta_{1}(x), \ \theta_{tt}(x,0)=\theta_{2}(x), \end{cases} \tag{4.6}$$

where $(x,t)\in(0,1)\times(0,\infty)$, $\delta>0$ represents the coupling coefficient depending on the material properties and θ^0 is a constant reference temperature assumed to be positive.

The chapter structure is outlined as follows. In Section 4.2, we prove the well-posedness of the system under the assumption (1.7) by using the Lumer–Philips theorem 2.3.7. Section 4.3 contains the statement and proof of some technical lemmas. In Section 4.4, we introduce a new stability number, denoted by

$$\chi=\delta^{2}\theta^{0}D+(D\rho-GI_{\rho})\left(\frac{2\kappa\tau_{\theta}}{\tau_{q}^{2}}-\frac{D}{I_{\rho}}\right), \tag{4.7}$$

and show that the system is exponentially stable when $\chi=0$ and polynomially stable when $\chi\neq 0$. The proof of stability results is based on the multiplier method, considering the assumption (1.7). In section 4.5, we introduce a finite-element-discretization approach to numerically solve the given problem. Discrete stability results and a priori error estimates are obtained. Finally, in section 4.6, we present numerical simulations carried out using MATLAB.

4.2 Well-posedness

In this section, we apply the semigroup theory to provide an existence and uniqueness result for the problem (4.4)–(4.6).

From equation $(4.4)_1$ and the boundary conditions (4.5), it is straightforward to verify that

$$\frac{d^{2}}{dt^{2}}\int_{0}^{1}\omega\left(x,t\right)dx=\frac{G}{\rho}\int_{0}^{1}(\psi-\omega_{x})_{x}dx+\frac{\delta}{\rho}\int_{0}^{1}\left(\frac{\tau_{q}^{2}}{2}\theta_{tt}+\tau_{q}\theta_{t}+\theta\right)_{x}dx=0. \tag{4.8}$$

Solving (4.8) with initial data of ω yields

$$\int_{0}^{1}\omega\left(x,t\right)dx=t\int_{0}^{1}\omega_{1}\left(x\right)dx+\int_{0}^{1}\omega_{0}\left(x\right)dx.$$

Thus, if we let

$$\bar{\omega}\left(x,t\right)=\omega\left(x,t\right)-t\int_{0}^{1}\omega_{1}\left(x\right)dx-\int_{0}^{1}\omega_{0}\left(x\right)dx,$$

we arrive at

$$\int_{0}^{1}\bar{\omega}\left(x,t\right)dx=0, \ \forall t\geq 0.$$

Consequently, applying Poincaré's inequality for $\bar{\omega}$ is justified, and a simple substitution reveals that $(\bar{\omega}, \psi, s, \theta)$ satisfies the system (4.4) and the boundary conditions (4.5). For convenience, from now on, we will work with $\bar{\omega}$ but write ω.

We first introduce the vector function

$$U = (\omega, \nu, 3s - \psi, (3\varphi - u), s, \varphi, \theta, \vartheta, \xi)^T,$$

where $\nu = \omega_t, u = \psi_t, \varphi = s_t, \vartheta = \theta_t$ and $\xi = \theta_{tt}$ then the system (4.4)–(4.6) can be written as an evolutionary equation

$$\begin{cases} U_t = \mathcal{A}U, \ t > 0, \\ U(0) = U_0 = (\omega_0, \nu_0, 3s_0 - \psi_0, 3\varphi_0 - u_0, s_0, \varphi_0, \theta_0, \vartheta_0, \xi_0)^T. \end{cases} \tag{4.9}$$

where $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is a linear operator defined by

$$\mathcal{A}U = \begin{pmatrix} \nu \\ -\frac{G}{\rho}(\psi_x - \omega_x)_x - \dfrac{\delta\tau_q^2}{2\rho}\xi_x - \dfrac{\delta\tau_q}{\rho}\vartheta_x - \dfrac{\delta}{\rho}\theta_x \\ 3\varphi - u \\ \frac{D}{I_\rho}(3s - \psi)_{xx} + \frac{G}{I_\rho}(\psi - \omega_x) \\ \varphi \\ \frac{D}{I_\rho}s_{xx} - \frac{G}{I_\rho}(\psi_x - \omega_x) - \frac{4}{3}\frac{\gamma}{I_\rho}s - \frac{4}{3}\frac{1}{I_\rho}\varphi \\ \vartheta \\ \xi \\ -\dfrac{2}{\tau_q}\xi - \dfrac{2}{\tau_q^2}\vartheta + \dfrac{2\kappa\tau_\theta}{\tau_q^2}\vartheta_{xx} + \dfrac{2\kappa}{\tau_q^2}\theta_{xx} - \dfrac{2\delta\theta^0}{\tau_q^2}\nu_x \end{pmatrix},$$

with domain

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{r} U \in \mathcal{H} \ / \ \omega \in H_*^2 \cap H_*^1; 3s - \psi, s \in H^2 \cap H_0^1; \nu \in H_*^1; \\ 3\varphi - u, \varphi, \theta, \vartheta, \xi \in H_0^1(0,1); \theta + \tau\vartheta \in H^2(0,1) \end{array} \right\},$$

where

$$L_*^2(0,1) = \{\phi \in L^2(0,1) \ / \ \int_0^1 \phi(x)dx = 0\}, H_*^1(0,1) = H^1(0,1) \cap L_*^2(0,1),$$
$$H_*^2(0,1) = \{\phi \in H^2(0,1) \ / \ \phi_x(1) = \phi_x(0) = 0\}.$$

The energy space $\mathcal{H}$ is given by

$$\mathcal{H} = H_*^1 \times L_*^2(0,1) \times H_0^1 \times L^2(0,1) \times H_0^1 \times L^2(0,1) \times H_0^1 \times H_0^1 \times L^2(0,1).$$

It is a Hilbert space equipped with the inner product

$$\begin{aligned}\langle U,\widetilde{U}\rangle_{\mathcal{H}} =&\rho\theta^0\int_0^1 \nu\tilde{\nu}dx + I_\rho\theta^0\int_0^1 (3\varphi-u)(3\tilde{\varphi}-\tilde{u})dx\\&+G\theta^0\int_0^1(\psi-\omega_x)(\tilde{\psi}-\tilde{\omega}_x)dx+3I_\rho\theta^0\int_0^1\varphi\tilde{\varphi}dx\\&+D\theta^0\int_0^1(3s-\psi)_x(3\tilde{s}-\tilde{\psi})_xdx+4\gamma\theta^0\int_0^1 s\tilde{s}dx\\&+3D\theta^0\int_0^1 s_x\tilde{s}_xdx+\kappa\frac{\tau_q^2}{2}\int_0^1(\theta_x\tilde{\vartheta}_x+\vartheta_x\tilde{\theta}_x)dx\\&+\int_0^1\left(\frac{\tau_q^2}{2}\xi+\tau_q\vartheta+\theta\right)\left(\frac{\tau_q^2}{2}\tilde{\xi}+\tau_q\tilde{\vartheta}+\tilde{\theta}\right)dx\\&+\kappa(\tau_\theta+\tau_q)\int_0^1\theta_x\tilde{\theta}_xdx+\kappa\tau_\theta\frac{\tau_q^2}{2}\int_0^1\vartheta_x\tilde{\vartheta}_xdx,\end{aligned}\tag{4.10}$$

for any

$$U=(\omega,\nu,3s-\psi,3\varphi-u,s,\varphi,\theta,\vartheta,\xi)^T,\ \widetilde{U}=(\tilde{\omega},\tilde{\nu},3\tilde{s}-\tilde{\psi},3\tilde{\varphi}-\tilde{u},\tilde{s},\tilde{\varphi},\tilde{\theta},\tilde{\vartheta},\tilde{\xi})^T\in\mathcal{H}.$$

We can now state the following well-posedness result.

Theorem 4.2.1. *Let $U_0\in\mathcal{H}$ and assume that (1.7) holds. Then, there exists a unique solution $U\in\mathcal{C}(\mathbb{R}_+,\mathcal{H})$, of problem (4.9). Moreover, if $U_0\in\mathcal{D}(\mathcal{A})$, then*

$$U\in\mathcal{C}(\mathbb{R}_+,\mathcal{D}(\mathcal{A}))\cap\mathcal{C}^1(\mathbb{R}_+,\mathcal{H}).$$

Proof. First, we show that $\mathcal{A}$ is dissipative. Using the inner product given in (4.10), for any $U\in\mathcal{D}(\mathcal{A})$, we obtain

$$\begin{aligned}\langle\mathcal{A}U,U\rangle_{\mathcal{H}}=&\rho\theta^0\int_0^1\left[-\frac{G}{\rho}(\psi_x-\omega_x)_x-\frac{\delta\tau_q^2}{2\rho}\xi_x-\frac{\delta\tau_q}{\rho}\vartheta_x-\frac{\delta}{\rho}\theta_x\right]\nu dx+4\gamma\theta^0\int_0^1\varphi s dx\\&+3I_\rho\theta^0\int_0^1\left[\frac{D}{I_\rho}(3s-\psi)_{xx}+\frac{G}{I_\rho}(\psi-\omega_x)\right](3\varphi-u)dx+3D\theta^0\int_0^1\varphi_x s_x dx\\&+D\theta^0\int_0^1(3\varphi-u)_x(3s-\psi)_xdx+G\theta^0\int_0^1(u-\nu_x)(\psi-w_x)dx\\&+\frac{\tau_q^2}{2}\int_0^1\left[-\frac{2}{\tau_q}\xi-\frac{2}{\tau_q^2}\vartheta+\frac{2\kappa\tau_\theta}{\tau_q^2}\vartheta_{xx}+\frac{2\kappa}{\tau_q^2}\theta_{xx}-\frac{2\delta\theta^0}{\tau_q^2}\nu_x\right]\left(\frac{\tau_q^2}{2}\xi+\tau_q\vartheta+\theta\right)dx\\&+\tau_q\int_0^1\xi\left(\frac{\tau_q^2}{2}\xi+\tau_q\vartheta+\theta\right)dx+\int_0^1\vartheta\left(\frac{\tau_q^2}{2}\xi+\tau_q\vartheta+\theta\right)dx\\&+\kappa(\tau_\theta+\tau_q)\int_0^1\vartheta_x\theta_xdx+\kappa\tau_\theta\frac{\tau_q^2}{2}\int_0^1\xi_x\vartheta_xdx+\kappa\frac{\tau_q^2}{2}\int_0^1\vartheta_x^2dx+\kappa\frac{\tau_q^2}{2}\int_0^1\theta_x\xi_xdx.\end{aligned}$$

Integrating by parts, we infer

$$
\begin{aligned}
\langle \mathcal{A}U, U\rangle_{\mathcal{H}} =& -4\theta^0 \int_0^1 \varphi^2 dx - \kappa\tau_\theta \int_0^1 \vartheta_x \left(\frac{\tau_q^2}{2}\xi + \tau_q\vartheta + \theta\right)_x dx \\
& - \kappa \int_0^1 \theta_x \left(\frac{\tau_q^2}{2}\xi + \tau_q\vartheta + \theta\right)_x dx + \kappa(\tau_\theta + \tau_q)\int_0^1 \vartheta_x\theta_x dx \\
& + \kappa\tau_\theta \frac{\tau_q^2}{2}\int_0^1 \xi_x\vartheta_x dx + \kappa\frac{\tau_q^2}{2}\int_0^1 \vartheta_x^2 dx + \kappa\frac{\tau_q^2}{2}\int_0^1 \theta_x\xi_x dx \\
=& -4\theta^0 \int_0^1 \varphi^2 dx - \kappa\tau_\theta\frac{\tau_q^2}{2}\int_0^1 \xi_x\vartheta_x dx - \kappa\tau_\theta\tau_q \int_0^1 \vartheta_x^2 dx \\
& - \kappa\tau_\theta \int_0^1 \vartheta_x\theta_x dx - \kappa\frac{\tau_q^2}{2}\int_0^1 \theta_x\xi_x dx - \kappa\tau_q \int_0^1 \theta_x\vartheta_x dx \\
& - \kappa\int_0^1 \theta_x^2 dx + \kappa(\tau_\theta + \tau_q)\int_0^1 \vartheta_x\theta_x dx + \kappa\tau_\theta\frac{\tau_q^2}{2}\int_0^1 \xi_x\vartheta_x dx \\
& + \kappa\frac{\tau_q^2}{2}\int_0^1 \vartheta_x^2 dx + \kappa\frac{\tau_q^2}{2}\int_0^1 \theta_x\xi_x dx \\
=& -4\theta^0 \int_0^1 \varphi^2 dx - \kappa\int_0^1 \theta_x^2 dx - \kappa\tau_q\left(\tau_\theta - \frac{\tau_q}{2}\right)\int_0^1 \vartheta_x^2 dx \leqslant 0.
\end{aligned}
$$

Therefore, the operator $\mathcal{A}$ is dissipative. Given that $\mathcal{A}$ is dissipative, it is enough to demonstrate that $\mathcal{A}$ is maximal. In other words, we need to prove that $(Id - \mathcal{A})$ is surjective. That is, for any $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9)^T \in \mathcal{H}$, we have to find $V = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9)^T \in \mathcal{D}(\mathcal{A})$ such that

$$
(Id - \mathcal{A})V = F,
$$

which implies

$$
\begin{cases}
v_1 - v_2 = f_1, \\
\rho v_2 - G v_{1xx} - G v_{3x} + 3G v_{5x} + \delta\frac{\tau_q^2}{2} v_{9x} + \delta\tau_q v_{8x} + \delta v_{7x} = \rho f_2, \\
v_3 - v_4 = f_3, \\
I_\rho v_4 - D v_{3xx} - 3G v_5 + G v_3 + G v_{1x} = I_\rho f_4, \\
v_5 - v_6 = f_5, \\
(I_\rho + \frac{4}{3})v_6 - D v_{5xx} - G v_3 - G v_{1x} + (3G + \frac{4}{3}\gamma)v_5 = I_\rho f_6, \\
v_7 - v_8 = f_7, \\
v_8 - v_9 = f_8, \\
\left(\frac{\tau_q^2}{2} + \tau_q\right) v_9 + v_8 - \kappa\tau_\theta v_{8xx} - \kappa v_{7xx} + \delta\theta^0 v_{2x} = \frac{\tau_q^2}{2} f_9.
\end{cases}
\tag{4.11}
$$

From (4.11), it follows

$$v_2 = v_1 - f_1, \ \ v_4 = v_3 - f_3, \ \ v_6 = v_5 - f_5, \ \ v_8 = v_7 - f_7, \ \ v_9 = v_8 - f_8. \tag{4.12}$$

Plugging (4.12) into $(4.11)_2$, $(4.11)_4$, $(4.11)_6$ and $(4.11)_9$, we end up with

$$\begin{cases} \rho(v_1 - f_1) - Gv_{1xx} - Gv_{3x} + 3Gv_{5x} + \delta\dfrac{\tau_q^2}{2}(v_{7x} - f_{7x} - f_{8x}) \\ \qquad\qquad\qquad + \delta\tau_q(v_{7x} - f_{7x}) + \delta v_{7x} = \rho f_2, \\ I_\rho(v_3 - f_3) - Dv_{3xx} - 3Gv_5 + Gv_3 + Gv_{1x} = I_\rho f_4, \\ (I_\rho + \frac{4}{3})(v_5 - f_5) - Dv_{5xx} - Gv_3 - Gv_{1x} + (3G + \frac{4}{3}\gamma)v_5 = I_\rho f_6, \\ \left(\dfrac{\tau_q^2}{2} + \tau_q\right)(v_7 - f_7 - f_8) + (v_7 - f_7) - \kappa\tau_\theta(v_{7xx} - f_{7xx}) \\ \qquad\qquad\qquad - \kappa v_{7xx} + \delta\theta^0(v_{1x} - f_{1x}) = \dfrac{\tau_q^2}{2} f_9, \end{cases} \tag{4.13}$$

and then, multiplying $(4.13)_1$, $(4.13)_2$, $(4.13)_3$ by θ^0 and $(4.13)_4$ by $\left(\dfrac{\tau_q^2}{2} + \tau_q + 1\right)$, respectively, we obtain

$$\begin{cases} \rho\theta^0 v_1 - G\theta^0 v_{1xx} - G\theta^0 v_{3x} + 3G\theta^0 v_{5x} + \delta\theta^0\left(\dfrac{\tau_q^2}{2} + \tau_q + 1\right) v_{7x} = h_1, \\ I_\rho\theta^0 v_3 - D\theta^0 v_{3xx} - 3G\theta^0 v_5 + G\theta^0 v_3 + G\theta^0 v_{1x} = h_2, \\ (3I_\rho + 9G + 4\gamma + 4)\theta^0 v_5 - 3D\theta^0 v_{5xx} - 3G\theta^0 v_3 - 3G\theta^0 v_{1x} = h_3, \\ \left(\dfrac{\tau_q^2}{2} + \tau_q + 1\right)^2 v_7 - \kappa(\tau_\theta + 1)\left(\dfrac{\tau_q^2}{2} + \tau_q + 1\right) v_{7xx} \\ \qquad\qquad + \delta\theta^0\left(\dfrac{\tau_q^2}{2} + \tau_q + 1\right) v_{1x} = h_4, \end{cases} \tag{4.14}$$

where

$$\begin{cases} h_1 = \rho\theta^0(f_1 + f_2) + \delta\theta^0\left(\tau_q + \dfrac{\tau_q^2}{2}\right) f_{7x} + \delta\theta^0\dfrac{\tau_q^2}{2} f_{8x} \ \in L^2_*(0,1), \\ h_2 = I_\rho\theta^0(f_3 + f_4) \ \in L^2(0,1), \\ h_3 = (3I_\rho + 4)\theta^0 f_5 + 3I_\rho\theta^0 f_6 \ \in L^2(0,1), \\ h_4 = \left(\dfrac{\tau_q^2}{2} + \tau_q + 1\right)\left[\delta\theta^0 f_{1x} + \left(\dfrac{\tau_q^2}{2} + \tau_q + 1\right) f_7\right] \\ \qquad + \left(\dfrac{\tau_q^2}{2} + \tau_q + 1\right)\left[\left(\dfrac{\tau_q^2}{2} + \tau_q\right) f_8 + \dfrac{\tau_q^2}{2} f_9\right] \in H^{-1}(0,1). \end{cases}$$

To solve (4.14), we consider the following variational formulation

$$B((v_1,v_3,v_5,v_7),(\widetilde{v}_1,\widetilde{v}_3,\widetilde{v}_5,\widetilde{v}_7)) = \Gamma(\widetilde{v}_1,\widetilde{v}_3,\widetilde{v}_5,\widetilde{v}_7), \tag{4.15}$$

where

$$B : (H_*^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1))^2 \to \mathbb{R}$$

is the bilinear form defined by

$$\begin{aligned} B((v_1,v_3,v_5,v_7),(\widetilde{v}_1,\widetilde{v}_3,\widetilde{v}_5,\widetilde{v}_7)) =& \rho\theta^0\int_0^1 v_1\widetilde{v}_1dx + \delta\theta^0\left(\frac{\tau_q^2}{2}+\tau_q+1\right)\int_0^1 v_{7x}\widetilde{v}_1dx \\ &+ I_\rho\theta^0\int_0^1 v_3\widetilde{v}_3dx + D\theta^0\int_0^1 v_{3x}\widetilde{v}_{3x}dx \\ &+ (3I_\rho+4\gamma+4)\theta^0\int_0^1 v_5\widetilde{v}_5dx + \left(\frac{\tau_q^2}{2}+\tau_q+1\right)^2\int_0^1 v_7\widetilde{v}_7dx \\ &+ \delta\theta^0\left(\frac{\tau_q^2}{2}+\tau_q+1\right)\int_0^1 v_{1x}\widetilde{v}_7dx + 3D\theta^0\int_0^1 v_{5x}\widetilde{v}_{5x}dx \\ &+ \kappa(\tau_\theta+1)\left(\frac{\tau_q^2}{2}+\tau_q+1\right)\int_0^1 v_{7x}\widetilde{v}_{7x}dx \\ &+ G\theta^0\int_0^1(-v_{1x}-v_3+3v_5)(-\widetilde{v}_{1x}-\widetilde{v}_3+3\widetilde{v}_5)dx, \end{aligned}$$

and $\Gamma : (H_*^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)) \to \mathbb{R}$ is the linear functional given by

$$\Gamma(\widetilde{v}_1,\widetilde{v}_3,\widetilde{v}_5,\widetilde{v}_7) = \int_0^1 h_1\widetilde{v}_1dx + \int_0^1 h_2\widetilde{v}_3dx + \int_0^1 h_3\widetilde{v}_5dx + \langle h_4,\widetilde{v}_7\rangle_{H^{-1}\times H_0^1}.$$

Now, for $\mathcal{V} = H_*^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)$, equipped with the norm

$$\|(v_1,v_3,v_5,v_7)\|_{\mathcal{V}}^2 = \|v_1\|_2^2 + \|(-v_{1x}-v_3+3v_5)\|_2^2 + \|v_7\|_2^2 + \|v_{3x}\|_2^2 + \|v_{5x}\|_2^2 + \|v_{7x}\|_2^2,$$

and through integration by parts, we have

$$\begin{aligned} B((v_1,v_3,v_5,v_7),(v_1,v_3,v_5,v_8)) = & \rho\theta^0\int_0^1 v_1^2dx + I_\rho\theta^0\int_0^1 v_3^2dx + D\theta^0\int_0^1 v_{3x}^2dx \\ &+ (3I_\rho+4\gamma+4)\theta^0\int_0^1 v_5^2dx + \left(\frac{\tau_q^2}{2}+\tau_q+1\right)^2\int_0^1 v_7^2dx \\ &+ 3D\theta^0\int_0^1 v_{5x}^2dx + \kappa(\tau_\theta+1)\left(\frac{\tau_q^2}{2}+\tau_q+1\right)\int_0^1 v_{7x}^2dx \\ &+ G\theta^0\int_0^1(-v_{1x}-v_3+3v_5)^2dx, \end{aligned}$$

then

$$\begin{aligned} B((v_1,v_3,v_5,v_7),(v_1,v_3,v_5,v_8)) &\geqslant \rho\theta^0\int_0^1 v_1^2dx + G\theta^0\int_0^1(-v_{1x}-v_3+3v_5)^2dx \\ &\quad + \left(\frac{\tau_q^2}{2}+\tau_q+1\right)^2\int_0^1 v_7^2dx + D\theta^0\int_0^1 v_{3x}^2dx \\ &\quad + 3D\theta^0\int_0^1 v_{5x}^2dx + \kappa(\tau_\theta+1)\left(\frac{\tau_q^2}{2}+\tau_q+1\right)\int_0^1 v_{7x}^2dx \\ &\geqslant M_0\|(v_1,v_3,v_5,v_7)\|_{\mathcal{V}}^2, \end{aligned}$$

for some $M_0 > 0$. Thus B is coercive. Moreover, applying Cauchy-Schwarz inequality we find for a positive costant M_1 that

$$\begin{aligned} |B((v_1,v_3,v_5,v_7),(\widetilde{v}_1,\widetilde{v}_3,\widetilde{v}_5,\widetilde{v}_7))|^2 \leqslant M_1^2\Big(&\|v_1\|\|\widetilde{v}_1\| + \|v_{7x}\|\|\widetilde{v}_1\| + \|v_3\|\|\widetilde{v}_3\| \\ &+ \|v_{3x}\|\|\widetilde{v}_{3x}\| + \|v_5\|\|\widetilde{v}_5\| + \|v_7\|\|\widetilde{v}_7\| \\ &+ \|v_{1x}\|\|\widetilde{v}_7\| + \|v_{5x}\|\|\widetilde{v}_{5x}\| + \|v_{7x}\|\|\widetilde{v}_{7x}\| \\ &+ \|-v_{1x}-v_3+3v_5\|\|-\widetilde{v}_{1x}-\widetilde{v}_3+3\widetilde{v}_5\|\Big)^2, \end{aligned}$$

consequently, we have

$$\begin{aligned} |B((v_1,v_3,v_5,v_7),(\widetilde{v}_1,\widetilde{v}_3,\widetilde{v}_5,\widetilde{v}_7))|^2 \leqslant 2M_1^2\Big(&\|v_1\|^2\|\widetilde{v}_1\|^2 + \|v_{7x}\|^2\|\widetilde{v}_1\|^2 + \|v_3\|^2\|\widetilde{v}_3\|^2 \\ &+ \|v_{3x}\|^2\|\widetilde{v}_{3x}\|^2 + \|v_5\|^2\|\widetilde{v}_5\|^2 + \|v_7\|^2\|\widetilde{v}_7\|^2 \\ &+ \|v_{1x}\|^2\|\widetilde{v}_7\|^2 + \|v_{5x}\|^2\|\widetilde{v}_{5x}\|^2 + \|v_{7x}\|^2\|\widetilde{v}_{7x}\|^2 \\ &+ \|-v_{1x}-v_3+3v_5\|^2\|-\widetilde{v}_{1x}-\widetilde{v}_3+3\widetilde{v}_5\|^2\Big). \end{aligned}$$

In the light of ($\|v_1\|^2 \leqslant \|(v_1,v_3,v_5,v_7)\|_{\mathcal{V}}^2,\ \|\widetilde{v}_1\|^2 \leqslant \|(\widetilde{v}_1,\widetilde{v}_3,\widetilde{v}_5,\widetilde{v}_7)\|_{\mathcal{V}}^2,\ \ldots$), we arrive at

$$|B((v_1,v_3,v_5,v_7),(\widetilde{v}_1,\widetilde{v}_3,\widetilde{v}_5,\widetilde{v}_7))|^2 \leqslant 20M_1^2\|(v_1,v_3,v_5,v_7)\|_{\mathcal{V}}^2\|(\widetilde{v}_1,\widetilde{v}_3,\widetilde{v}_5,\widetilde{v}_7)\|_{\mathcal{V}}^2$$

and then, we deduce that

$$|B((v_1,v_3,v_5,v_7),(\widetilde{v}_1,\widetilde{v}_3,\widetilde{v}_5,\widetilde{v}_7))| \leqslant \zeta_1\|(v_1,v_3,v_5,v_7)\|_{\mathcal{V}}\|(\widetilde{v}_1,\widetilde{v}_3,\widetilde{v}_5,\widetilde{v}_7)\|_{\mathcal{V}},$$

where $\zeta_1 = 2\sqrt{5}M_1$. Similarly, we can find

$$|\Gamma(\widetilde{v}_1,\widetilde{v}_3,\widetilde{v}_5,\widetilde{v}_7)|^2 \leqslant \Big(\|h_1\|\|\widetilde{v}_1\| + \|h_2\|\|\widetilde{v}_3\| + \|h_3\|\widetilde{v}_5\| + \|h_4\|\|\widetilde{v}_7\|\Big)^2,$$

then, there exists a positive constant M_2 such that

$$|\Gamma(\widetilde{v}_1, \widetilde{v}_3, \widetilde{v}_5, \widetilde{v}_7)|^2 \leqslant 2\Big(\|h_1\|^2\|\widetilde{v}_1\|^2 + \|h_2\|^2\|\widetilde{v}_3\|^2 + \|h_3\|^2\|\widetilde{v}_5\|^2 + \|h_4\|^2\|\widetilde{v}_7\|^2\Big),$$

with the same previous arguments, it follows that

$$|\Gamma(\widetilde{v}_1, \widetilde{v}_3, \widetilde{v}_5, \widetilde{v}_7)|^2 \leqslant 8M_2\|(\widetilde{v}_1, \widetilde{v}_3, \widetilde{v}_5, \widetilde{v}_7)\|^2_{\mathcal{V}},\ M_2 > 0.$$

Therefore, we have

$$\Gamma(\widetilde{v}_1, \widetilde{v}_3, \widetilde{v}_5, \widetilde{v}_7) \leqslant \zeta_2\|(\widetilde{v}_1, \widetilde{v}_3, \widetilde{v}_5, \widetilde{v}_7)\|_{\mathcal{V}},$$

where $\zeta_2 = 2\sqrt{2M_2}$. Hence, both B and Γ are bounded. As a result, by applying the Lax-Milgram theorem, problem (4.15) has a unique solution

$$(v_1, v_3, v_5, v_7) \in H^1_*(0,1) \times H^1_0(0,1) \times H^1_0(0,1) \times H^1_0(0,1).$$

The above result, together with (4.12) leads to

$$v_2 \in H^1_*(0,1) \text{ and } v_4, v_6, v_8, v_9 \in H^1_0(0,1).$$

Now, if $(\widetilde{v}_1, \widetilde{v}_3, \widetilde{v}_5, \widetilde{v}_7)$ = $(\widetilde{v}_1, 0, 0, 0)$ then (4.15) reduces to

$$\begin{aligned} G\theta^0\int_0^1 v_{1x}\widetilde{v}_{1x}dx =& -\rho\theta^0\int_0^1 v_1\widetilde{v}_1dx + G\theta^0\int_0^1 v_{3x}\widetilde{v}_1dx - 3G\theta^0\int_0^1 v_{5x}\widetilde{v}_1dx \\ & - \delta\theta^0\left(\frac{\tau_q^2}{2} + \tau_q + 1\right)\int_0^1 v_{7x}\widetilde{v}_1dx + \int_0^1 h_1\widetilde{v}_1dx,\ \forall\widetilde{v}_1 \in H^1_*(0,1). \end{aligned} \tag{4.16}$$

The regularity theory cannot be applied directly here because $\widetilde{v}_1 \in H^1_*(0,1)$. Therefore, let $\widehat{v}_1 \in H^1_0(0,1)$ and take

$$\widetilde{v}_1 = \widehat{v}_1(x) - \int_0^1 \widehat{v}_1(s)ds, \tag{4.17}$$

which implies $\widetilde{v}_1 \in H^1_*(0,1)$. Substituting (4.17) into (4.16) results in

$$G\theta^0\int_0^1 v_{1x}\widehat{v}_{1x}dx = \int_0^1 \widehat{h}_1\widehat{v}_1dx,$$

where

$$\begin{aligned} \widehat{h}_1 =& -\rho\theta^0 v_1 + G\theta^0 v_{3x} - 3G\theta^0 v_{5x} - \delta\theta^0\left(\frac{\tau_q^2}{2} + \tau_q + 1\right) \\ & + \rho\theta^0(f_1 + f_2) + \delta\theta^0\left(\tau_q + \frac{\tau_q^2}{2}\right)f_{7x} + \delta\theta^0\frac{\tau_q^2}{2}f_{8x}\ \in L^2_*(0,1). \end{aligned}$$

Thus

$$v_1 \in H^2(0,1)$$

and

$$-G\theta^0 v_{1xx} = -\rho\theta^0 v_1 + G\theta^0 v_{3x} - 3G\theta^0 v_{5x} - \delta\theta^0 \left(\frac{\tau_q^2}{2} + \tau_q + 1\right)$$
$$+ \rho\theta^0(f_1 + f_2) + \delta\theta^0 \left(\tau_q + \frac{\tau_q^2}{2}\right) f_{7x} + \delta\theta^0 \frac{\tau_q^2}{2} f_{8x}.$$

In view of $f_1 = v_1 - v_2$, $f_7 = v_7 - v_8$ and $f_8 = v_8 - v_9$, we obtain

$$\rho v_2 - G v_{1xx} - G v_{3x} + 3G v_{5x} + \delta \frac{\tau_q^2}{2} v_{9x} + \delta\tau_q v_{8x} + \delta v_{7x} = \rho f_2,$$

which solves $(4.11)_2$. Furthermore, since $-G\theta^0 v_{1xx} = \widehat{h}_1$, it follows that

$$-G\theta^0 \int_0^1 v_{1xx}\Phi dx = \int_0^1 \widehat{h}_1 \Phi dx, \ \forall \Phi \in H^1(0,1)$$

and by using integration by parts, we get

$$G\theta^0 v_{1x}(1)\Phi(1) - G\theta^0 v_{1x}(0)\Phi(0) + G\theta^0 \int_0^1 v_{1x}\Phi_x dx = \int_0^1 \widehat{h}_1 \Phi dx, \ \forall \Phi \in H^1(0,1).$$

Given that $H_*^1 \subset H^1$, we arrive at

$$G\theta^0 v_{1x}(1)\widetilde{v}_1(1) - G\theta^0 v_{1x}(0)\widetilde{v}_1(0) + G\theta^0 \int_0^1 v_{1x}\widetilde{v}_{1x} dx = \int_0^1 \widehat{h}_1 \widetilde{v}_1 dx, \ \forall \widetilde{v}_1 \in H_*^1(0,1).$$

In light of (4.16), we have

$$G\theta^0 v_{1x}(1)\widetilde{v}_1(1) - G\theta^0 v_{1x}(0)\widetilde{v}_1(0) = 0.$$

As $\widetilde{v}_1$ is arbitrary, $v_{1x}(0) = v_{1x}(1) = 0$. Hence,

$$v_1 \in H_*^2(0,1).$$

Next, by taking $(\widetilde{v}_1, \widetilde{v}_3, \widetilde{v}_5, \widetilde{v}_7)$ as $(0, \widetilde{v}_3, 0, 0)$ or $(0, 0, \widetilde{v}_5, 0)$ in (4.15), we find

$$D\theta^0 \int_0^1 v_{3x}\widetilde{v}_{3x} dx = \int_0^1 \widehat{h}_2 \widetilde{v}_3 dx, \ \forall \widetilde{v}_3 \in H_0^1(0,1),$$

or

$$3D\theta^0 \int_0^1 v_{5x}\widetilde{v}_{5x} dx = \int_0^1 \widehat{h}_3 \widetilde{v}_5 dx, \ \forall \widetilde{v}_5 \in H_0^1(0,1),$$

where

$$\widehat{h}_2 = -I_\rho \theta^0 v_3 + G\theta^0(-v_{1x} - v_3 + 3v_5) + I_\rho \theta^0 (f_3 + f_4) \ \in L^2(0,1),$$

$$\widehat{h}_3 = -(3I_\rho + 4\gamma + 4)\theta^0 v_5 - 3G\theta^0(-v_{1x} - v_3 + 3v_5) + (3I_\rho + 4)\theta^0 f_5 + 3I_\rho \theta^0 f_6 \ \in L^2(0,1).$$

Therefore, the theory of elliptic regularity implies that

$$v_3,\ v_5 \in H^2(0,1)$$

and

$$-D\theta^0 v_{3xx} = -I_\rho \theta^0 v_3 + G\theta^0(-v_{1x} - v_3 + 3v_5) + I_\rho \theta^0 (f_3 + f_4),$$

$$-3D\theta^0 v_{5xx} = -(3I_\rho + 4\gamma + 4)\theta^0 v_5 - 3G\theta^0(-v_{1x} - v_3 + 3v_5) + (3I_\rho + 4)\theta^0 f_5 + 3I_\rho \theta^0 f_6.$$

Since $f_3 = v_3 - v_4$ and $f_5 = v_5 - v_6$, we end up with

$$I_\rho v_4 - Dv_{3xx} - 3Gv_5 + Gv_3 + Gv_{1x} = I_\rho f_4,$$

$$(I_\rho + \frac{4}{3})v_6 - Dv_{5xx} - Gv_3 - Gv_{1x} + (3G + \frac{4}{3}\gamma)v_5 = I_\rho f_6.$$

These give $(4.11)_4$ and $(4.11)_6$.

Finally, if $(\widetilde{v}_1, \widetilde{v}_3, \widetilde{v}_5, \widetilde{v}_7)$ = $(0,0,0,\widetilde{v}_7)$ in (4.15), then for any $\widetilde{v}_7 \in H_0^1(0,1)$, we obtain

$$\left(\frac{\tau_q^2}{2} + \tau_q + 1\right)^2 \int_0^1 v_7 \widetilde{v}_7 dx + \delta\theta^0 \left(\frac{\tau_q^2}{2} + \tau_q + 1\right) \int_0^1 v_{1x} \widetilde{v}_7 dx$$
$$+ \kappa(\tau_\theta + 1) \left(\frac{\tau_q^2}{2} + \tau_q + 1\right) \int_0^1 v_{7x} \widetilde{v}_{7x} dx = \int_0^1 h_4 \widetilde{v}_7 dx.$$

This, in turn, yields

$$\kappa \int_0^1 [(\tau_\theta + 1)v_{7x} - \tau_\theta f_{7x}] \widetilde{v}_{7x} dx = \int_0^1 \widehat{h}_4 \widetilde{v}_7 dx,$$

where

$$\widehat{h}_4 = -\left(\frac{\tau_q^2}{2} + \tau_q + 1\right) v_7 - \delta\theta^0 v_{1x} + \delta\theta^0 f_{1x}$$
$$+ \left(\frac{\tau_q^2}{2} + \tau_q + 1\right) f_7 + \left(\frac{\tau_q^2}{2} + \tau_q\right) f_8 + \frac{\tau_q^2}{2} f_9 \ \in L^2(0,1).$$

Hence

$$[(\tau_\theta + 1)v_7 - \tau_\theta f_7] \in H^2(0,1).$$

Given that $f_1 = v_1 - v_2$, $f_7 = v_7 - v_8$ and $f_8 = v_8 - v_9$, then

$$\tau v_8 + v_7 \in H^2(0,1)$$

and we have

$$\left(\frac{\tau_q^2}{2} + \tau_q\right) v_9 + v_8 - \kappa\tau_\theta v_{8xx} - \kappa v_{7xx} + \delta\theta^0 v_{2x} = \frac{\tau_q^2}{2} f_9,$$

wich solves $(4.11)_9$.

As a consequence $V \in \mathcal{D}(\mathcal{A})$ and $\mathcal{A}$ is a maximal dissipative operator. Hence by Lumer-Philips' theorem 2.3.7, we deduce that $\mathcal{A}$ generates a C_0-semigroup of contractions $S(t) = e^{t\mathcal{A}}$ on $\mathcal{H}$. According to semi-group theory, the unique solution of (4.9) is $U(t) = e^{t\mathcal{A}}U_0$, satisfying the conditions of the Theorem 4.2.1. □

4.3 Technical lemmas

In this section, we state and prove some essential lemmas needed to construct a suitable Lyapunov functional, which is used to establish our stability results for the system (4.4)–(4.6).

Lemma 4.3.1. *Let $(\omega, 3s-\psi, s, \theta)$ be the solution of (4.4)–(4.6) and assume that (1.7) holds. Then the energy functional E, defined by*

$$\begin{aligned} E(t) = \frac{1}{2}\int_0^1 &\{\rho\theta^0\omega_t^2 + I_\rho\theta^0[(3s-\psi)_t]^2 + 3I_\rho\theta^0 s_t^2 + 3D\theta^0 s_x^2 + 4\gamma\theta^0 s^2 \\ &+ D\theta[(3s-\psi)_x]^2 + G\theta^0(\psi-\omega_x)^2 + \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)^2 \\ &+ \kappa(\tau_\theta+\tau_q)\theta_x^2 + \kappa\tau_\theta\frac{\tau_q^2}{2}\theta_{xt}^2 + \kappa\tau_q^2\theta_x\theta_{xt}\}dx, \end{aligned} \tag{4.18}$$

satisfies

$$\frac{dE(t)}{dt} = -4\theta^0\int_0^1 s_t^2 dx - \kappa\tau_q\left(\tau_\theta - \frac{\tau_q}{2}\right)\int_0^1 \theta_{xt}^2 dx - \kappa\int_0^1 \theta_x^2 dx \leqslant 0. \tag{4.19}$$

Proof. Multiplying the equations of system (4.4) by $\theta^0\omega_t$, $\theta^0(3s-\psi)_t$ and $\theta^0 s_t$, respectively, integrating by parts, using the boundary conditions (4.5), we arrive at

$$\rho\theta^0\int_0^1 \omega_{tt}\omega_t dx + G\theta^0\int_0^1(\psi-\omega_x)_x\omega_t dx - \delta\theta^0\int_0^1\left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)\omega_{xt}dx = 0,$$

$$\begin{aligned} I_\rho\theta^0\int_0^1(3s-\psi)_{tt}(3s-\psi)_t dx &+ D\theta^0\int_0^1(3s-\psi)_x(3s-\psi)_{xt}dx \\ &- G\theta^0\int_0^1(\psi-\omega_x)(3s-\psi)_t dx = 0 \end{aligned}$$

and

$$3I_\rho\theta^0\int_0^1 s_{tt}s_t dx + 3D\theta^0\int_0^1 s_x s_{xt}dx + 3G\theta^0\int_0^1(\psi-\omega_x)s_t dx + 4\gamma\theta^0\int_0^1 s s_t dx + 4\theta^0\int_0^1 s_t^2 dx = 0,$$

then, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 \rho\theta^0\omega_t^2 dx - G\theta^0\int_0^1(\psi-\omega_x)\omega_{xt}dx\\ = \delta\theta^0\int_0^1\left(\frac{\tau_q^2}{2}\theta_{tt}+\tau_q\theta_t+\theta\right)\omega_{xt}dx,\end{aligned} \tag{4.20}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1\Big\{I_\rho\theta^0[(3s-\psi)_t]^2 + D\theta^0[(3s-\psi)]_x^2\Big\}dx\\ - G\theta^0\int_0^1(\psi-\omega_x)(3s-\psi)_t dx = 0\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1\Big\{3I_\rho\theta^0 s_t^2 + 3Ds_x^2 + 4\gamma\theta^0 s^2\Big\}dx\\ + 3G\theta^0\int_0^1(\psi-\omega_x)s_t dx = -4\theta^0\int_0^1 s_t^2 dx.\end{aligned} \tag{4.22}$$

Note that

$$\begin{aligned}&G\theta^0\int_0^1\Big\{3(\psi-\omega_x)s_t - (\psi-\omega_x)\omega_{xt} - (\psi-\omega_x)(3s-\psi)_t\Big\}dx\\ =&3G\theta^0\int_0^1(\psi-\omega_x)s_t dx - G\theta^0\int_0^1(\psi-\omega_x)\omega_{xt}dx\\ &- 3G\theta^0\int_0^1(\psi-\omega_x)s_t dx + G\theta^0\int_0^1(\psi-\omega_x)\psi_t dx\\ =&G\theta^0\int_0^1(\psi-\omega_x)(\psi-\omega_x)_t dx = \frac{1}{2}\frac{d}{dt}\int_0^1(\psi-\omega_x)^2 dx.\end{aligned} \tag{4.23}$$

Taking the sum of the resulting equations (4.20)–(4.22) with (4.23), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1\Big\{\rho\theta^0\omega_t^2 + I_\rho\theta^0[(3s-\psi)_t]^2 + 3I_\rho\theta^0 s_t^2 + 3D\theta^0 s_x^2\\ + 4\gamma\theta^0 s^2 + D\theta^0[(3s-\psi)_x]^2 + G\theta^0(\psi-\omega_x)^2\Big\}dx\\ = -4\theta^0\int_0^1 s_t^2 dx + \delta\theta^0\int_0^1\left(\frac{\tau_q^2}{2}\theta_{tt}+\tau_q\theta_t+\theta\right)\omega_{xt}dx.\end{aligned} \tag{4.24}$$

Next, multiplying the last equation in (4.4) by $\left(\frac{\tau_q^2}{2}\theta_{tt}+\tau_q\theta_t+\theta\right)$ and integrating by parts we obtain

$$\begin{aligned}&\int_0^1\left(\frac{\tau_q^2}{2}\theta_{tt}+\tau_q\theta_t+\theta\right)_t\left(\frac{\tau_q^2}{2}\theta_{tt}+\tau_q\theta_t+\theta\right)dx+\kappa\tau_\theta\frac{\tau_q^2}{2}\int_0^1\theta_{xt}\theta_{xtt}dx\\&\quad+\kappa\tau_\theta\tau_q\int_0^1\theta_{xt}^2dx+\kappa\tau_\theta\int_0^1\theta_{xt}\theta_xdx-\kappa\frac{\tau_q^2}{2}\int_0^1\theta_{xx}\theta_{tt}dx+\kappa\tau_q\int_0^1\theta_x\theta_{xt}dx\\&\quad+\kappa\int_0^1\theta_x^2dx+\delta\theta^0\int_0^1\left(\frac{\tau_q^2}{2}\theta_{tt}+\tau_q\theta_t+\theta\right)w_{xt}dx=0,\end{aligned}$$

therefore, we arrive at

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1\Big\{\left(\frac{\tau_q^2}{2}\theta_{tt}+\tau_q\theta_t+\theta\right)^2+\kappa(\tau_\theta+\tau_q)\theta_x^2+\kappa\tau_\theta\frac{\tau_q^2}{2}\theta_{xt}^2\Big\}dx\\&=-\delta\theta^0\int_0^1\left(\frac{\tau_q^2}{2}\theta_{tt}+\tau_q\theta_t+\theta\right)\omega_{xt}dx-\kappa\tau_\theta\tau_q\int_0^1\theta_{xt}^2dx\\&\quad+\kappa\frac{\tau_q^2}{2}\int_0^1\theta_{xx}\theta_{tt}dx-\kappa\int_0^1\theta_x^2dx.\end{aligned}\tag{4.25}$$

Note that

$$\kappa\frac{\tau_q^2}{2}\int_0^1\theta_{xx}\theta_{tt}=-\kappa\frac{\tau_q^2}{2}\frac{d}{dt}\int_0^1\theta_x\theta_{xt}dx+\kappa\frac{\tau_q^2}{2}\int_0^1\theta_{xt}^2dx.\tag{4.26}$$

Plugging (4.26) into (4.25) gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1\Big\{\left(\frac{\tau_q^2}{2}\theta_{tt}+\tau_q\theta_t+\theta\right)^2+\kappa(\tau_\theta+\tau_q)\theta_x^2+\kappa\tau_\theta\frac{\tau_q^2}{2}\theta_{xt}^2+\kappa\tau_q^2\theta_x\theta_{xt}\Big\}dx\\&=-\delta\theta^0\int_0^1\left(\frac{\tau_q^2}{2}\theta_{tt}+\tau_q\theta_t+\theta\right)\omega_{xt}dx-\kappa\tau_q\left(\tau_\theta-\frac{\tau_q}{2}\right)\int_0^1\theta_{xt}^2dx-\kappa\int_0^1\theta_x^2dx,\end{aligned}\tag{4.27}$$

coupled with the estimate (4.24), we deduce (4.18) and (4.19). □

In the sequel, we use c_i to denote a generic positive constant.

Lemma 4.3.2. *Let $(\omega,3s-\psi,s,\theta)$ be the solution of (4.4)–(4.6). Then the functional*

$$F_1(t)=I_\rho\int_0^1(3s-\psi)(3s-\psi)_tdx-\rho\int_0^1\omega_t\left(\int_0^x(3s-\psi)(y)dy\right)dx$$

satisfies, for any $\varepsilon_1 > 0$,

$$\begin{aligned}\frac{dF_1(t)}{dt} \leqslant & -\frac{D}{2}\int_0^1 [(3s-\psi)_x]^2 dx + c_1\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1 [(3s-\psi)_t]^2 dx \\ & + \varepsilon_1 \int_0^1 \omega_t^2 dx + c_1 \int_0^1 \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)^2 dx.\end{aligned} \tag{4.28}$$

Proof. Differentiating F_1 and using integration by parts, we have

$$\begin{aligned}\frac{dF_1(t)}{dt} =& I_\rho \int_0^1 [(3s-\psi)_t]^2 dx - \delta\int_0^1 \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)(3s-\psi) dx \\ & - D\int_0^1 [(3s-\psi)_x]^2 dx - \rho\int_0^1 \omega_t \left(\int_0^x (3s-\psi)_t(y) dy\right) dx.\end{aligned} \tag{4.29}$$

Applying Young's and poincaré's inequalities, we find

$$\begin{aligned}-\delta\int_0^1 \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)(3s-\psi) dx \leqslant & \frac{\delta^2 c_p}{2D}\int_0^1 \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)^2 dx \\ & + \frac{D}{2}\int_0^1 [(3s-\psi)_x]^2 dx.\end{aligned} \tag{4.30}$$

Young's and Cauchy-Schwarz inequalities yield

$$-\rho\int_0^1 \omega_t \left(\int_0^x (3s-\psi)_t(y) dy\right) dx \leqslant \varepsilon_1 \int_0^1 w_t^2 dx + \frac{\rho^2}{4\varepsilon_1}\int_0^1 [(3s-\psi)_t]^2 dx. \tag{4.31}$$

Estimate (4.28) follows by substituting (4.30) and (4.31) into (4.29). □

Lemma 4.3.3. *Let* $(\omega, 3s-\psi, s, \theta)$ *be the solution of (4.4)–(4.6). Then the functional*

$$F_2(t) = \rho\int_0^1 (\psi - \omega_x)\left(\int_0^x \omega_t(y) dy\right) dx$$

satisfies, for any $\varepsilon_2 > 0$,

$$\begin{aligned}\frac{dF_2(t)}{dt} \leqslant & -\frac{G}{2}\int_0^1 (\psi-\omega_x)^2 dx + \varepsilon_2 \int_0^1 [(3s-\psi)_t]^2 dx + c_2\int_0^1 s_t^2 dx \\ & + c_2(1+\frac{1}{\varepsilon_2})\int_0^1 \omega_t^2 dx + c_2\int_0^1 \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)^2 dx.\end{aligned} \tag{4.32}$$

Proof. We take the derivative of F_2, to find

$$\begin{aligned}\frac{dF_2(t)}{dt} =&\rho\int_0^1 \psi_t \left(\int_0^x \omega_t(y)dy\right)dx - G\int_0^1(\psi-\omega_x)^2dx \\ &+ \rho\int_0^1 \omega_t^2dx - \delta\int_0^1(\psi-\omega_x)\left(\frac{\tau_q^2}{2}\theta_{tt}+\tau_q\theta_t+\theta\right)dx.\end{aligned} \tag{4.33}$$

Young's inequality leads to

$$-\delta\int_0^1(\psi-w_x)\theta dx \leqslant \frac{\delta^2}{4G}\int_0^1\left(\frac{\tau_q^2}{2}\theta_{tt}+\tau_q\theta_t+\theta\right)^2dx + \frac{G}{2}\int_0^1(\psi-w_x)^2dx. \tag{4.34}$$

Using the fact that $\psi_t = 3s_t - (3s-\psi)_t$, we obtain

$$\begin{aligned}\rho\int_0^1 \psi_t\left(\int_0^x \omega_t(y)dy\right)dx =&3\rho\int_0^1 s_t\left(\int_0^x \omega_t(y)dy\right)dx \\ &- \rho\int_0^1 (3s-\psi)_t\left(\int_0^x \omega_t(y)dy\right)dx\end{aligned}$$

and then, Young's and Cauchy-Schwarz inequalities yield

$$3\rho\int_0^1 s_t\left(\int_0^x \omega_t(y)dy\right)dx \leqslant \frac{3\rho}{2}\left(\int_0^1 s_t^2dx + \int_0^1 w_t^2dx\right), \tag{4.35}$$

$$-\rho\int_0^1(3s-\psi)_t\left(\int_0^x \omega_t(y)dy\right)dx \leqslant \varepsilon_2\int_0^1[(3s-\psi)_t]^2dx + \frac{\rho^2}{4\varepsilon_2}\int_0^1 w_t^2dx. \tag{4.36}$$

Plugging (4.34)–(4.36) into (4.33), we conclude (4.32). □

Lemma 4.3.4. *Let $(\omega, 3s-\psi, s, \theta)$ be the solution of (4.4)–(4.6). Then the functional*

$$F_3(t) = -\rho\int_0^1\left(\frac{\tau_q^2}{2}\theta_{tt}+\tau_q\theta_t+\theta\right)\left(\int_0^x \omega_t(y)dy\right)dx$$

satisfies, for any $\varepsilon_3 > 0$,

$$\begin{aligned}\frac{dF_3(t)}{dt} \leqslant& -\frac{\rho\delta\theta^0}{2}\int_0^1 \omega_t^2dx + c_3(1+\frac{1}{\varepsilon_3})\int_0^1\left(\frac{\tau_q^2}{2}\theta_{tt}+\tau_q\theta_t+\theta\right)^2dx \\ &+ \varepsilon_3\int_0^1(\psi-\omega_x)^2dx + c_3\int_0^1(\theta_{xt}^2+\theta_x^2)dx.\end{aligned} \tag{4.37}$$

Proof. We differentiate F_3 and integrate by parts to achieve

$$\begin{aligned}\frac{dF_3(t)}{dt} =& -\rho\delta\theta^0 \int_0^1 \omega_t^2 dx + G\int_0^1 \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)(\psi - \omega_x)dx \\ &+ \rho\kappa \int_0^1 (\tau_\theta\theta_{xt} + \theta_x)\omega_t dx + \delta \int_0^1 \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)^2 dx.\end{aligned} \tag{4.38}$$

Using Young's inequality, we obtain

$$\rho\kappa \int_0^1 (\tau_\theta\theta_{xt} + \theta_x)w_t dx \leqslant \frac{\rho\kappa^2\tau_\theta^2}{\delta\theta^0}\int_0^1 \theta_{xt}^2 dx + \frac{\rho\kappa^2}{\delta\theta^0}\int_0^1 \theta_x^2 dx + \frac{\rho\delta\theta^0}{2}\int_0^1 w_t^2 dx \tag{4.39}$$

and

$$\begin{aligned}G\int_0^1 \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)(\psi - w_x)dx \leqslant& \frac{G^2}{4\varepsilon_3}\int_0^1 \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)^2 dx \\ &+ \varepsilon_3 \int_0^1 (\psi - w_x)^2 dx.\end{aligned} \tag{4.40}$$

By substituting (4.39) and (4.40) into (4.38), we get (4.37). □

Lemma 4.3.5. *Let $(\omega, 3s - \psi, s, \theta)$ be the solution of (4.4)–(4.6). Then the functional*

$$F_4(t) = 3I_\rho \int_0^1 s s_t dx + 2\int_0^1 s^2 dx$$

satisfies

$$\frac{dF_4(t)}{dt} \leqslant -3\gamma \int_0^1 s^2 dx - 3D\int_0^1 s_x^2 dx + c_4 \int_0^1 \left[s_t^2 + (\psi - \omega_x)^2\right] dx. \tag{4.41}$$

Proof. Direct computations, using integration by parts, yield

$$\frac{dF_4(t)}{dt} = 3I_\rho \int_0^1 s_t^2 dx - 3D\int_0^1 s_x^2 dx - 4\gamma\int_0^1 s^2 dx - 3G\int_0^1 s(\psi - \omega_x)dx. \tag{4.42}$$

Exploiting Young's inequality, we have

$$-3G\int_0^1 s(\psi - \omega_x)dx \leqslant \frac{9G^2}{4\gamma}\int_0^1 (\psi - w_x)^2 dx + \gamma\int_0^1 s^2 dx. \tag{4.43}$$

The substitution of (4.43) into (4.42) gives (4.41). □

Lemma 4.3.6. *Let $(\omega, 3s-\psi, s, \theta)$ be the solution of (4.4)–(4.6). Then the functional*

$$F_5(t) = -\int_0^1 \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t\right)\left(\frac{\tau_q^2}{2}\theta_t + \tau_q\theta\right)dx - \frac{\tau_q}{2}\int_0^1 \theta^2 dx$$

satisfies, for any $\varepsilon_4 > 0$,

$$\begin{aligned}\frac{dF_5(t)}{dt} \leqslant & -\frac{1}{2}\int_0^1 \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)^2 dx + 2\varepsilon_4\int_0^1 \omega_t^2 dx \\ & + c_5\left(1 + \frac{1}{\varepsilon_4}\right)\int_0^1 \left[\theta_{xt}^2 + \theta_x^2\right] dx.\end{aligned} \tag{4.44}$$

Proof. Differentiation of F_5, together with integration by parts, shows that

$$\begin{aligned}\frac{dF_5(t)}{dt} = & -\int_0^1 \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t\right)^2 dx + \frac{\tau_q^2}{2}\int_0^1 \theta_t^2 dx + \kappa\tau_\theta\frac{\tau_q^2}{2}\int_0^1 \theta_{xt}^2 dx \\ & + \kappa\tau_q\int_0^1 \theta_x^2 dx + \tau_q\kappa\left(\kappa\tau_\theta + \frac{\tau_q}{2}\right)\int_0^1 \theta_{xt}\theta_x dx \\ & - \delta\theta^0\frac{\tau_q^2}{2}\int_0^1 \omega_t\theta_{xt} dx - \delta\theta^0\tau_q\int_0^1 \omega_t\theta_x dx.\end{aligned}$$

Using Young's inequality, we infer

$$-\delta\theta^0\frac{\tau_q^2}{2}\int_0^1 \omega_t\theta_{xt} dx \leqslant \delta^2\theta^2\frac{\tau_q^2}{8\varepsilon_4}\int_0^1 \theta_{xt}^2 dx + \varepsilon_4\int_0^1 w_t^2 dx, \tag{4.45}$$

$$-\delta\theta^0\tau_q\int_0^1 \omega_t\theta_x dx \leqslant \delta^2\theta^2\frac{\tau_q^2}{4\varepsilon_4}\int_0^1 \theta_x^2 dx + \varepsilon_4\int_0^1 w_t^2 dx, \tag{4.46}$$

$$\tau_q\kappa\left(\kappa\tau_\theta + \frac{\tau_q}{2}\right)\int_0^1 \theta_{xt}\theta_x dx \leqslant +\frac{\tau_q\kappa}{2}\left(\kappa\tau_\theta + \frac{\tau_q}{2}\right)\int_0^1 (\theta_{xt}^2 + \theta_x^2) dx. \tag{4.47}$$

Clearly, we have

$$\int_0^1 \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)^2 dx \leqslant 2\int_0^1 \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t\right)^2 dx + 2\int_0^1 \theta^2 dx$$

and along with Poincaré's inequality, we arrive at

$$-\int_0^1 \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t\right)^2 dx \leqslant -\frac{1}{2}\int_0^1 \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)^2 dx + c_p\int_0^1 \theta_x^2 dx. \tag{4.48}$$

Previous inequalities lead to (4.44). □

Lemma 4.3.7. *Let $(\omega, 3s-\psi, s, \theta)$ be the solution of (4.4)–(4.6). Then the functional*

$$\begin{aligned} F_6(t) =& \delta\theta^0 G I_\rho \int_0^1 (3s-\psi)_t(\psi-\omega_x)dx - \delta\theta^0 D\rho \int_0^1 \omega_t (3s-\psi)_x dx \\ &+ (D\rho - GI_\rho)\int_0^1 \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)(3s-\psi)_t dx \\ &+ \kappa(D\rho - GI_\rho)\int_0^1 (\tau_\theta\theta_{xt} + \theta_x)(3s-\psi)_x dx \end{aligned}$$

satisfies, for any $\varepsilon_5 > 0$,

$$\begin{aligned} \frac{dF_6(t)}{dt} \leqslant & -\frac{\delta\theta^0 G I_\rho}{2}\int_0^1 [(3s-\psi)_t]^2 dx + 2\varepsilon_5 \int_0^1 [(3s-\psi)_x]^2 dx \\ &+ c_6 \int_0^1 \left[s_t^2 + (\psi-\omega_x)^2 + \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)^2 \right] dx \\ &+ \frac{c_6}{\varepsilon_5}\int_0^1 (\theta_{xt}^2 + \theta_x^2)dx + \chi \int_0^1 \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)_x (3s-\psi)_x dx, \end{aligned} \tag{4.49}$$

where

$$\chi = \delta^2\theta^0 D + (D\rho - GI_\rho)\left(\frac{2\kappa\tau_\theta}{\tau_q^2} - \frac{D}{I_\rho}\right).$$

Proof. Differentiating F_6 followed by integrating by parts, we arrive at

$$\begin{aligned} \frac{dF_6(t)}{dt} =& \delta\theta^0 G^2 \int_0^1 (\psi-\omega_x)^2 dx + \delta\theta^0 G I_\rho \int_0^1 (3s-\psi)_t \psi_t dx \\ &+ \kappa\left[1 - \frac{2\tau_\theta}{\delta\tau_q}(D\rho - GI_\rho)\right]\int_0^1 \theta_{xt}(3s-\psi)_x dx \\ &- \frac{2\kappa\tau_\theta}{\delta\tau_q^2}(D\rho - GI_\rho)\int_0^1 \theta_x (3s-\psi)_x dx \\ &+ \frac{G}{I_\rho}(D\rho - GI_\rho)\int_0^1 \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)(\psi-\omega_x)dx \\ &+ \chi\int_0^1 \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)_x (3s-\psi)_x dx. \end{aligned} \tag{4.50}$$

Then, by using $\psi_t = 3s_t - (3s-\psi)_t$ and Young's inequality, we get (4.49). □

4.4 Asymptotic behavior

Now, we state and prove our stability results by leveraging the lemmas from Section 4.3.

4.4.1 Exponential stability

In this subsection, we consider the case $\chi = 0$ and establish the exponential stability.

Theorem 4.4.1. *Assume that (1.7) holds, then, there exist positive constants λ_1 and λ_2 such the energy functional, given by (4.18), satisfies*

$$E(t) \leqslant \lambda_2 e^{-\lambda_1 t}, \ \forall\, t \geqslant 0. \tag{4.51}$$

Proof. Let $N,\ N_j,\ j = 1, ..., 6$, be positive constants to be chosen later. We define the Lyapunov functional by

$$L(t) = NE(t) + F_1(t) + N_2F_2(t) + N_3F_3(t) + F_4(t) + N_5F_5(t) + N_6F_6(t).\ \forall t \geqslant 0, \tag{4.52}$$

then, we have

$$\begin{aligned}
|L(t) - NE(t)| \leqslant & I_\rho \int_0^1 |(3s-\psi)(3s-\psi)_t| dx + \rho \int_0^1 \left|\omega_t \left(\int_0^x (3s-\psi)(y)dy\right)\right| dx \\
& + \rho N_2 \int_0^1 \left|(\psi-\omega_x)\left(\int_0^x \omega_t(y)dy\right)\right| dx + 3I_\rho \int_0^1 |ss_t| dx \\
& + 2\int_0^1 s^2 dx + \rho N_3 \int_0^1 \left|\left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)\left(\int_0^x \omega_t(y)dy\right)\right| dx \\
& + N_5 \int_0^1 \left|\left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t\right)\left(\frac{\tau_q^2}{2}\theta_t + \tau_q\theta\right)\right| dx + \frac{\tau_q}{2}N_5 \int_0^1 \theta^2 dx \\
& + \delta\theta^0 GI_\rho N_6 \int_0^1 |(3s-\psi)_t(\psi-\omega_x)| dx + \delta\theta^0 D\rho N_6 \int_0^1 |\omega_t(3s-\psi)_x| dx \\
& + |D\rho - GI_\rho| N_6 \int_0^1 \left|\left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)(3s-\psi)_t\right| dx \\
& + \kappa|D\rho - GI_\rho| N_6 \int_0^1 |(\tau_\theta\theta_{xt} + \theta_x)(3s-\psi)_x| dx.
\end{aligned}$$

By exploiting Young's, Cauchy–Schwarz, and Poincaré's inequalities, it easy to deduce that for some $\alpha > 0$,

$$|L(t) - NE(t)| \leqslant \alpha E(t).$$

Consequently,

$$(N-\alpha)E(t) \leqslant L(t) \leqslant (N+\alpha)E(t) \tag{4.53}$$

and by choosing N sufficiently large, there exist positive constants α_1 and α_2 such that the estimate (4.53) yields the relation

$$\alpha_1 E(t) \leqslant L(t) \leqslant \alpha_2 E(t),\ \forall\, t \geqslant 0. \tag{4.54}$$

On the other hand, by differentiating (4.52) and using (4.19), (4.28), (4.32), (4.37), (4.41), (4.44) and (4.49), we have

$$\begin{aligned}
\frac{dL(t)}{dt} \leqslant & -\left[\frac{D}{2}-2\varepsilon_5 N_6\right]\int_0^1 [(3s-\psi)_x]^2 dx \\
& -\left[\frac{\delta\theta^0 G I_\rho}{2}N_6-\varepsilon_2 N_2 - c_1(1+\frac{1}{\varepsilon_1})\right]\int_0^1 [(3s-\psi)_t]^2 dx \\
& -\left[\frac{G}{2}N_2-\varepsilon_3 N_3 - c_4 - c_6 N_6\right]\int_0^1 (\psi-\omega_x)^2 dx \\
& -\left[\frac{\rho\delta\theta^0}{2}N_3-\varepsilon_1 - c_2(1+\frac{1}{\varepsilon_2})N_2 - 2\varepsilon_4 N_5\right]\int_0^1 \omega_t^2 dx \\
& -\left[\frac{1}{2}N_5-c_1-c_2N_2-c_3(1+\frac{1}{\varepsilon_3})N_3-c_6N_6\right]\int_0^1 \left(\frac{\tau_q^2}{2}\theta_{tt}+\tau_q\theta_t+\theta\right)^2 dx \\
& -3\gamma\int_0^1 s^2 dx - 3D\int_0^1 s_x^2 dx + \left[4\theta^0 N - c_2 N_2 - c_4 - c_6 N_6\right]\int_0^1 s_t^2 dx \\
& -\left[\kappa\tau_q(\tau_\theta-\frac{\tau_q}{2})N - c_3N_3 - c_5(1+\frac{1}{\varepsilon_4})N_5-\frac{c_6}{\varepsilon_5}N_6\right]\int_0^1 \theta_{xt}^2 dx \\
& -\left[\kappa N - c_3N_3 - c_5(1+\frac{1}{\varepsilon_4})N_5-\frac{c_6}{\varepsilon_5}N_6\right]\int_0^1 \theta_x^2 dx.
\end{aligned}$$

By setting

$$\varepsilon_1 = 1,\ \varepsilon_2 = \frac{\delta\theta^0 G I_\rho N_6}{4N_2},\ \varepsilon_3 = \frac{GN_2}{4N_3},\ \varepsilon_4 = \frac{1}{N_5},\ \varepsilon_5 = \frac{D}{8N_6},$$

we obtain

$$\begin{aligned}
\frac{dL(t)}{dt} \leqslant & -\frac{D}{4}\int_0^1 [(3s-\psi)_x]^2 dx - \left[\frac{\delta\theta^0 G I_\rho}{4}N_6 - 2c_1\right]\int_0^1 [(3s-\psi)_t]^2 dx \\
& -\left[\frac{G}{4}N_2 - c_4 - c_6N_6\right]\int_0^1 (\psi-\omega_x)^2 dx \\
& -\left[\frac{\rho\delta\theta^0}{2}N_3 - 3 - c_2\left(1+\frac{4N_2}{\delta\theta^0 G I_\rho N_6}\right)N_2\right]\int_0^1 \omega_t^2 dx \\
& -\left[\frac{1}{2}N_5 - c_1 - c_2N_2 - c_3\left(1+\frac{4N_3}{GN_2}\right)N_3 - c_6N_6\right]\int_0^1 \left(\frac{\tau_q^2}{2}\theta_{tt}+\tau_q\theta_t+\theta\right)^2 dx \\
& -3\gamma\int_0^1 s^2 dx - 3D\int_0^1 s_x^2 dx + \left[4\theta^0 N - c_2N_2 - c_4 - c_6N_6\right]\int_0^1 s_t^2 dx \\
& -\left[\kappa\tau_q(\tau_\theta-\frac{\tau_q}{2})N - c_3N_3 - c_5(1+N_5)N_5 - \frac{8c_6}{D}N_6^2\right]\int_0^1 \theta_{xt}^2 dx \\
& -\left[\kappa N - c_3N_3 - c_5(1+N_5)N_5 - \frac{8c_6}{D}N_6^2\right]\int_0^1 \theta_x^2 dx.
\end{aligned} \tag{4.55}$$

At this point, we choose N_6 large enough such that

$$\frac{\delta\theta^0 GI_\rho}{4}N_6 - 2c_1 > 0,$$

then, we pick N_2 large enough so that

$$\frac{G}{4}N_2 - c_4 - c_6 N_6 > 0.$$

Fixing N_2 and N_6 allows us to select N_3 sufficiently large such that

$$\frac{\rho\delta\theta^0}{2}N_3 - 3 - c_2\left(1 + \frac{4N_2}{\delta\theta^0 GI_\rho N_6}\right)N_2 > 0,$$

then, we take N_5 large enough such that

$$\frac{1}{2}N_5 - c_1 - c_2N_2 - c_3\left(1 + \frac{4N_3}{GN_2}\right)N_3 - c_6N_6 > 0.$$

Finally, we choose N large enough so that (4.53) remains valid and

$$\begin{cases} 4\theta^0 N - c_2N_2 - c_4 - c_6N_6 > 0, \\ \kappa\tau_q(\tau_\theta - \frac{\tau_q}{2})N - c_3N_3 - c_5(1+N_5)N_5 - \frac{8c_6}{D}N_6^2 > 0, \\ \kappa N - c_3N_3 - c_5(1+N_5)N_5 - \frac{8c_6}{D}N_6^2 > 0. \end{cases}$$

By taking into consideration (4.18), we infer that there exists a positive constant α_3, such that

$$\frac{dL(t)}{dt} \leqslant -\alpha_3 E(t),\ \forall\, t \geqslant 0,$$

which, together with (4.54), gives

$$\frac{dL(t)}{dt} \leqslant -\frac{\alpha_3}{\alpha_2}L(t),\ \forall\, t \geqslant 0. \tag{4.56}$$

A simple integration of (4.56) over (0, t) yields

$$L(t) \leqslant L(0)e^{-\frac{\alpha_3}{\alpha_2}t},\ \forall\, t \geqslant 0. \tag{4.57}$$

Consequently, again from the relation (4.54), we have

$$E(t) \leqslant \frac{\alpha_2}{\alpha_1}E(0)e^{-\frac{\alpha_3}{\alpha_2}t},\ \forall\, t \geqslant 0, \tag{4.58}$$

then, we deduce that there exist two positive constants $\lambda_1 = \dfrac{\alpha_3}{\alpha_2}$, $\lambda_2 = \dfrac{\alpha_2}{\alpha_1}E(0)$ such that the energy estimate (4.51) follows. $\square$

4.4.2 Polynomial stability

In this subsection we consider the case $\chi \neq 0$ and establish a polynomial stability result.

Theorem 4.4.2. *Assume that (1.7) holds. Then, there exists a positive constant C such the energy functional given by (4.18), satisfies*

$$E(t) \leqslant \frac{C}{t}, \ \forall\, t > 0. \tag{4.59}$$

Proof. We introduce the second-order energy functional by

$$\begin{aligned}\mathcal{E}(t) &= E(\omega_t, \psi_t, s_t, \theta_t)\\ &= \frac{1}{2}\int_0^1 \Big\{\rho\theta^0\omega_{tt}^2 + I_\rho\theta^0[(3s-\psi)_{tt}]^2 + 3I_\rho\theta^0 s_{tt}^2 + 3D\theta^0 s_{xt}^2 + 4\gamma\theta^0 s_t^2\\ &\qquad + D[(3s-\psi)_{xt}]^2 + G(\psi_t - \omega_{xt})^2 + \left[\left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)_t\right]^2\\ &\qquad + \kappa(\tau_\theta + \tau_q)\theta_{xt}^2 + \kappa\tau_\theta\frac{\tau_q^2}{2}\theta_{xtt}^2 + \kappa\tau_q^2\theta_{xt}\theta_{xtt}\Big\}dx.\end{aligned} \tag{4.60}$$

As established in Lemma 4.3.1, similar computations show that $\mathcal{E}(t)$ is non-increasing and satisfies

$$\frac{d\mathcal{E}(t)}{dt} = -4\theta^0\int_0^1 s_{tt}^2 dx - \kappa\tau_q\left(\tau_\theta - \frac{\tau_q}{2}\right)\int_0^1 \theta_{xtt}^2 dx - \kappa\int_0^1 \theta_{xt}^2 dx \leqslant 0. \tag{4.61}$$

Taking the last term of (4.49),

$$\begin{aligned}\chi\int_0^1\left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)_x (3s-\psi)_x dx =& \frac{\tau_q^2}{2}\chi\int_0^1 \theta_{xtt}(3s-\psi)_x dx\\ &+ \tau_q\chi\int_0^1 \theta_{xt}(3s-\psi)_x dx\\ &+ \chi\int_0^1 \theta_x(3s-\psi)_x dx,\end{aligned}$$

and using Young's inequality, we obtain, for any $\varepsilon_5 > 0$, that

$$\begin{aligned}\chi\int_0^1\left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)_x (3s-\psi)_x dx \leqslant& \frac{c_0}{\varepsilon_5}\int_0^1 (\theta_{xtt}^2 + \theta_{xt}^2 + \theta_x^2)dx\\ &+ 3\varepsilon_5\int_0^1 [(3s-\psi)_x]^2 dx.\end{aligned}$$

Therefore, the derivative of F_6 satisfies

$$\begin{aligned}\frac{dF_6(t)}{dt} \leqslant &-\frac{\delta\theta^0 GI_\rho}{2}\int_0^1 [(3s-\psi)_t]^2dx + 5\varepsilon_5\int_0^1 [(3s-\psi)_x]^2dx \\ &+ c_6\int_0^1\left[s_t^2 + (\psi-\omega_x)^2 + \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)^2\right]dx \\ &+ \frac{c_6}{\varepsilon_5}\int_0^1 (\theta_{xtt}^2 + \theta_{xt}^2 + \theta_x^2)dx.\end{aligned} \tag{4.62}$$

Now, we define a new Lyapunov functional as follows

$$\mathcal{L}(t) = N(E(t)+\mathcal{E}(t)) + F_1(t) + N_2F_2(t) + N_3F_3(t) + F_4(t) + N_5F_5(t) + N_6F_6(t),\ \forall t \geqslant 0. \tag{4.63}$$

By differentiating (4.63) and using (4.19), (4.28), (4.32), (4.37), (4.41), (4.61), (4.44) and (4.62), we arrive at

$$\begin{aligned}\frac{d\mathcal{L}(t)}{dt} \leqslant &-\left[\frac{D}{2} - 5\varepsilon_5 N_6\right]\int_0^1 [(3s-\psi)_x]^2dx \\ &-\left[\frac{\delta\theta^0 GI_\rho}{2}N_6 - \varepsilon_2 N_2 - c_1(1+\frac{1}{\varepsilon_1})\right]\int_0^1 [(3s-\psi)_t]^2dx \\ &-\left[\frac{G}{2}N_2 - \varepsilon_3 N_3 - c_4 - c_6 N_6\right]\int_0^1 (\psi-\omega_x)^2dx \\ &-\left[\frac{\rho\delta\theta^0}{2}N_3 - \varepsilon_1 - c_2(1+\frac{1}{\varepsilon_2})N_2 - 2\varepsilon_4 N_5\right]\int_0^1 \omega_t^2dx \\ &-\left[\frac{1}{2}N_5 - c_1 - c_2N_2 - c_3(1+\frac{1}{\varepsilon_3})N_3 - c_6N_6\right]\int_0^1 \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)^2 dx \\ &- 3\gamma\int_0^1 s^2dx - 3D\int_0^1 s_x^2dx + \left[4\theta^0 N - c_2N_2 - c_4 - c_6N_6\right]\int_0^1 s_t^2dx \\ &-\left[\kappa\tau_q(\tau_\theta - \frac{\tau_q}{2})N - c_3N_3 - c_5(1+\frac{1}{\varepsilon_4})N_5 - \frac{c_6}{\varepsilon_5}N_6\right]\int_0^1 \theta_{xt}^2dx \\ &-\left[\kappa N - c_3N_3 - c_5(1+\frac{1}{\varepsilon_4})N_5 - \frac{c_6}{\varepsilon_5}N_6\right]\int_0^1 \theta_x^2dx \\ &-\left[\kappa\tau_q(\tau_\theta - \frac{\tau_q}{2})N - \frac{c_6}{\varepsilon_5}N_6\right]\int_0^1 \theta_{xtt}^2dx.\end{aligned}$$

By taking ε_1, ε_2, ε_3, ε_4 as in above and $\varepsilon_5 = \dfrac{D}{20N_6}$, we get

$$\begin{aligned}
\frac{d\mathcal{L}(t)}{dt} \leqslant & -\frac{D}{4}\int_0^1 [(3s-\psi)_x]^2 dx - \left[\frac{\delta\theta^0 GI_\rho}{4}N_6 - 2c_1\right]\int_0^1 [(3s-\psi)_t]^2 dx \\
& - \left[\frac{G}{4}N_2 - c_4 - c_6N_6\right]\int_0^1 (\psi-\omega_x)^2 dx \\
& - \left[\frac{\rho\delta\theta^0}{2}N_3 - 3 - c_2\left(1+\frac{4N_2}{\delta\theta^0 GI_\rho N_6}\right)N_2\right]\int_0^1 \omega_t^2 dx \\
& - \left[\frac{1}{2}N_5 - c_1 - c_2N_2 - c_3\left(1+\frac{4N_3}{GN_2}\right)N_3 - c_6N_6\right]\int_0^1 \left(\frac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)^2 dx \\
& - 3\gamma\int_0^1 s^2 dx - 3D\int_0^1 s_x^2 dx + \left[4\theta^0 N - c_2N_2 - c_4 - c_6N_6\right]\int_0^1 s_t^2 dx \\
& - \left[\kappa\tau_q(\tau_\theta - \frac{\tau_q}{2})N - \frac{20c_6}{D}N_6^2\right]\int_0^1 \theta_{xtt}^2 dx \\
& - \left[\kappa\tau_q(\tau_\theta - \frac{\tau_q}{2})N - c_3N_3 - c_5(1+N_5)N_5 - \frac{20c_6}{D}N_6^2\right]\int_0^1 \theta_{xt}^2 dx \\
& - \left[\kappa N - c_3N_3 - c_5(1+N_5)N_5 - \frac{20c_6}{D}N_6^2\right]\int_0^1 \theta_x^2 dx.
\end{aligned}$$

Again, with similar choices of constants N_2, N_3, N_5, N_6, coupled with a suitable selection of N so that

$$\begin{cases}
4\theta^0 N - c_2N_2 - c_4 - c_6N_6 > 0, \\
\kappa\tau_q(\tau_\theta - \frac{\tau_q}{2})N - \frac{20c_6}{D}N_6^2 > 0, \\
\kappa\tau_q(\tau_\theta - \frac{\tau_q}{2})N - c_3N_3 - c_5(1+N_5)N_5 - \frac{20c_6}{D}N_6^2 > 0, \\
\kappa N - c_3N_3 - c_5(1+N_5)N_5 - \frac{20c_6}{D}N_6^2 > 0,
\end{cases} \tag{4.64}$$

and comparing with (4.18), we have for some $\sigma > 0$,

$$\frac{d\mathcal{L}(t)}{dt} \leqslant -\sigma E(t),\ \forall\ t \geqslant 0. \tag{4.65}$$

Furthermore, by using Young's, Cauchy-Schwarz, Poincaré's inequalties, we have

$$|\mathcal{L}(t) - N(E(t)+\mathcal{E}(t))| \leqslant \beta E(t),$$

wich implies

$$(N-\beta)(E(t)+\mathcal{E}(t)) \leqslant \mathcal{L}(t) \leqslant (N+\beta)(E(t)+\mathcal{E}(t)). \tag{4.66}$$

Now, by taking N large enough with respect of (4.64). The inequality (4.66) gives for some positive constants β_1 and β_2,

$$\beta_1(E(t)+\mathcal{E}(t)) \leqslant \mathcal{L}(t) \leqslant \beta_2(E(t)+\mathcal{E}(t)) \tag{4.67}$$

and

$$\mathcal{L}(t) \sim E(t)+\mathcal{E}(t).$$

Integrating (4.65) over $(0,t)$ and applying (4.67) yield

$$\int_0^t E(s)ds \leqslant \frac{1}{\sigma}\left[\mathcal{L}(0)-\mathcal{L}(t)\right] \leqslant \frac{1}{\sigma}\mathcal{L}(0) \leqslant \frac{\beta_2}{\sigma}(E(0)+\mathcal{E}(0)).$$

Since E is non-increasing and using the fact that

$$\frac{d(tE(t))}{dt} = t\frac{dE(t)}{dt} + E(t) \leqslant E(t),$$

we obtain

$$tE(t) \leqslant \frac{\beta_2}{\sigma}(E(0)+\mathcal{E}(0)).$$

Finally, for $C = \frac{\beta_2}{\sigma}(E(0)+\mathcal{E}(0))$, the estimate (4.59) follows. □

4.5 Numerical approximation

Now, we provide a numerical analysis of the problem discussed theoretically in Sections 4.2, 4.3 and 4.4.

4.5.1 Description of the discrete problem

First, to streamline the notation, let us denote $\nu=\omega_t$, $\varphi=s_t$, $u=\psi_t$, $\vartheta=\theta_t$, and $\xi=\theta_{tt}$. We obtain a weak form associated to the continuous problem by multiplying the evolution equations (4.4) by the test functions $\bar{\omega}\in H^1_*(0,1)$, $\bar{\psi}$, $\bar{s}$, $\bar{\theta}\in H^1_0(0,1)$, respectively. Through integration by parts and the use of boundary conditions (4.5), we get

$$\begin{cases} \rho(\nu_t,\bar{\omega}) - G(\psi-\omega_x,\bar{\omega}_x) - \delta(\frac{\tau_q^2}{2}\xi+\tau_q\vartheta+\theta,\bar{\omega}_x) = 0, \\ I_\rho(3\varphi_t-u_t,\bar{\psi}) + D(3s_x-\psi_x,\bar{\psi}_x) - G(\psi-\omega_x,\bar{\psi}) = 0, \\ 3I_\rho(\varphi_t,\bar{s}) + 3D(s_x,\bar{s}_x) + 3G(\psi-\omega_x,\bar{s}) + 4\gamma(s,\bar{s}) + 4(\varphi,\bar{s}) = 0, \\ (\frac{\tau_q^2}{2}\xi_t+\tau_q\xi+\vartheta,\bar{\theta}) + \kappa(\tau_\theta\vartheta_x+\theta_x,\bar{\theta}_x) + \delta\theta^0(\nu_x,\bar{\theta}) = 0. \end{cases} \tag{4.68}$$

Furthermore, assuming that the discrete initial conditions are sufficiently smooth, we set

$$\omega_h^0 = P_h^*\omega_0,\ \nu_h^0 = P_h^*\omega_1,\ \psi_h^0 = P_h^0\psi_0,\ u_h^0 = P_h^0\psi_1,$$

$$s_h^0 = P_h^0 s_0,\ \varphi_h^0 = P_h^0 s_1,\ \theta_h^0 = P_h^0\theta_0,\ \vartheta_h^0 = P_h^0\theta_1,\ \xi_h^0 = P_h^0\theta_2.$$

Using the backward-Euler scheme in time, the complete finite-element approximation of the variational problem (4.68) is to find $\nu_h^n \in S_h^*$, $u_h^n, \varphi_h^n, \xi_h^n \in S_h^0$ such that, for $n = 1, \dots, N$ and for all $\bar{\omega}_h \in S_h^*$, $\bar{\psi}_h, \bar{s}_h, \bar{\theta}_h \in S_h^0$,

$$\begin{cases} \dfrac{\rho}{\Delta t}(\nu_h^n - \nu_h^{n-1}, \bar{\omega}_h) - G(\psi_h^n - \omega_{hx}^n, \bar{\omega}_{hx}) - \delta(\dfrac{\tau_q^2}{2}\xi_h^n + \tau_q\vartheta_h^n + \theta_h^n, \bar{\omega}_{hx}) = 0, \\ \dfrac{I_\rho}{\Delta t}(3\varphi_h^n - u_h^n - (3\varphi_h^{n-1} - u_h^{n-1}), \bar{\psi}_h) + D(3s_{hx}^n - \psi_{hx}^n, \bar{\psi}_{hx}) - G(\psi_h^n - \omega_{hx}^n, \bar{\psi}_h) = 0, \\ \dfrac{3I_\rho}{\Delta t}(\varphi_h^n - \varphi_h^{n-1}, \bar{s}) + 3D(s_{hx}^n, \bar{s}_{hx}) + 3G(\psi_h^n - \omega_{hx}^n, \bar{s}_h) + 4\gamma(s_h^n, \bar{s}_h) + 4(\varphi_h^n, \bar{s}_h) = 0, \\ \dfrac{\tau_q^2}{2\Delta t}(\xi_h^n - \xi_h^{n-1}, \bar{\theta}_h) + \dfrac{\tau_q}{\Delta t}(\vartheta_h^n - \vartheta_h^{n-1}, \bar{\theta}_h) + \dfrac{1}{\Delta t}(\theta_h^n - \theta_h^{n-1}, \bar{\theta}_h) \\ \qquad\qquad + \kappa\tau_\theta(\vartheta_{hx}^n, \bar{\theta}_{hx}) + \kappa(\theta_{hx}^n, \bar{\theta}_{hx}) + \delta\theta^0(\nu_{hx}^n, \bar{\theta}_h) = 0, \end{cases} \tag{4.69}$$

where

$$\omega_h^n = \omega_h^{n-1} + \Delta t\ \nu_h^n,\ \psi_h^n = \psi_h^{n-1} + \Delta t\ u_h^n,\ s_h^n = s_h^{n-1} + \Delta t\ \varphi_h^n,$$
$$\theta_h^n = \theta_h^{n-1} + \Delta t\ \vartheta_h^n,\ \vartheta_h^n = \vartheta_h^{n-1} + \Delta t\ \xi_h^n.$$

The well-known Lax–Milgram lemma, along with the assumptions on the constitutive parameters, guarantees that the fully discrete problem (4.69) has a unique solution.

4.5.2 Study of the discrete energy

The following result is a discrete version of the energy decay property (4.19) that the continuous solution satisfies.

Theorem 4.5.1. *Let the discrete energy be given by*

$$\begin{aligned} E^n = \frac{1}{2}\Big(&\rho\theta^0\|\nu_h^n\|^2 + I_\rho\theta^0\|3\varphi_h^n - u_h^n\|^2 + 3I_\rho\theta^0\|\varphi_h^n\|^2 + 3D\theta^0\|s_{hx}^n\|^2 + 4\gamma\theta^0\|s_h^n\|^2 \\ &+ D\theta^0\|3s_{hx}^n - \psi_{hx}^n\|^2 + G\theta^0\|\psi_h^n - \omega_{hx}^n\|^2 + \|\frac{\tau_q^2}{2}\xi_h^n + \tau_q\vartheta_h^n + \theta_h^n\|^2 \\ &+ \kappa(\tau_\theta + \tau_q)\|\theta_{hx}^n\|^2 + \kappa\tau_\theta\frac{\tau_q^2}{2}\|\vartheta_{hx}^n\|^2 + \kappa\tau_q^2(\theta_{hx}^n, \vartheta_{hx}^n)\Big). \end{aligned} \tag{4.70}$$

Then, the decay property

$$\frac{E^n - E^{n-1}}{\Delta t} \leqslant 0,$$

holds for $n = 1, 2, \dots, N$.

Proof. Taking $\bar{\omega}_h = \nu_h^n$, $\bar{\psi}_h = 3\varphi_h^n - u_h^n$, $\bar{s}_h = \varphi_h^n$, and $\bar{\theta}_h = \dfrac{\tau_q^2}{2}\xi_h^n + \tau_q\vartheta_h^n + \theta_h^n$ in (4.69), it follows that

$$\begin{aligned}\frac{\rho\theta^0}{2\Delta t}\left(\|\nu_h^n - \nu_h^{n-1}\|^2 + \|\nu_h^n\|^2 - \|\nu_h^{n-1}\|^2\right) - G\theta^0(\psi_h^n - \omega_{hx}^n, \nu_{hx}^n) \\ - \delta\theta^0(\frac{\tau_q^2}{2}\xi_h^n + \tau_q\vartheta_h^n + \theta_h^n, \nu_{hx}^n) = 0,\end{aligned} \tag{4.71}$$

$$\begin{aligned}&\frac{I_\rho\theta^0}{2\Delta t}\left(\|3\varphi_h^n - u_h^n - (3\varphi_h^{n-1} - u_h^{n-1})\|^2 + \|3\varphi_h^n - u_h^n\|^2 - \|3\varphi_h^{n-1} - u_h^{n-1}\|^2\right) \\ &+ D\theta^0(3s_{hx}^n - \psi_{hx}^n, 3\varphi_{hx}^n - u_{hx}^n) - G\theta^0(\psi_h^n - \omega_{hx}^n, 3\varphi_h^n - u_h^n) = 0,\end{aligned} \tag{4.72}$$

$$\begin{aligned}&\frac{3I_\rho\theta^0}{2\Delta t}\left(\|\varphi_h^n - \varphi_h^{n-1}\|^2 + \|\varphi_h^n\|^2 - \|\varphi_h^{n-1}\|^2\right) + 3D\theta^0(s_{hx}^n, \varphi_{hx}^n) \\ &+ 3G\theta^0(\psi_h^n - \omega_{hx}^n, \varphi_h^n) + 4\gamma\theta^0(s_h^n, \varphi_h^n) + 4\theta^0(\varphi_h^n, \varphi_h^n) = 0\end{aligned} \tag{4.73}$$

and

$$\begin{aligned}&\frac{1}{2\Delta t}\Big(\|\frac{\tau_q^2}{2}\xi_h^n + \tau_q\vartheta_h^n + \theta_h^n - (\frac{\tau_q^2}{2}\xi_h^{n-1} + \tau_q\vartheta_h^{n-1} + \theta_h^{n-1})\|^2 \\ &+ \|\frac{\tau_q^2}{2}\xi_h^n + \tau_q\vartheta_h^n + \theta_h^n\|^2 - \|\frac{\tau_q^2}{2}\xi_h^{n-1} + \tau_q\vartheta_h^{n-1} + \theta_h^{n-1}\|^2\Big) \\ &+ \kappa\tau_\theta(\vartheta_{hx}^n, \frac{\tau_q^2}{2}\xi_{hx}^n + \tau_q\vartheta_{hx}^n + \theta_{hx}^n) + \kappa(\theta_{hx}^n, \frac{\tau_q^2}{2}\xi_{hx}^n + \tau_q\vartheta_{hx}^n + \theta_{hx}^n) \\ &+ \delta\theta^0(\nu_{hx}^n, \frac{\tau_q^2}{2}\xi_h^n + \tau_q\vartheta_h^n + \theta_h^n) = 0.\end{aligned} \tag{4.74}$$

By adding equations (4.71)–(4.74) and taking into account that

$$\begin{aligned}G\theta^0(\psi_h^n - \omega_{hx}^n, u_h^n - \nu_{hx}^n) &= \frac{G\theta^0}{\Delta t}(\psi_h^n - \omega_{hx}^n, \psi_h^n - \omega_{hx}^n - (\psi_h^{n-1} - \omega_{hx}^{n-1})) \\ &\geqslant \frac{G\theta^0}{2\Delta t}\left(\|\psi_h^n - \omega_{hx}^n\|^2 - \|\psi_h^{n-1} - \omega_{hx}^{n-1}\|^2\right),\end{aligned}$$

$$\begin{aligned}D\theta^0(3s_{hx}^n - \psi_{hx}^n, 3\varphi_{hx}^n - u_{hx}^n) &= \frac{D\theta^0}{\Delta t}(3s_{hx}^n - \psi_{hx}^n, 3s_{hx}^n - \psi_{hx}^n - (3s_{hx}^{n-1} - \psi_{hx}^{n-1})) \\ &\geqslant \frac{D\theta^0}{2\Delta t}\left(\|3s_{hx}^n - \psi_{hx}^n\|^2 - \|3s_{hx}^{n-1} - \psi_{hx}^{n-1}\|^2\right),\end{aligned}$$

$$\begin{aligned}3D\theta^0(s_{hx}^n, \varphi_{hx}^n) &= \frac{3D\theta^0}{\Delta t}(s_{hx}^n, s_{hx}^n - s_{hx}^{n-1}) \\ &\geqslant \frac{3D\theta^0}{2\Delta t}\left(\|s_{hx}^n\|^2 - \|s_{hx}^{n-1}\|^2\right),\end{aligned}$$

$$
\begin{aligned}
4\gamma\theta^0(s_h^n,\varphi_h^n) &= \frac{4\gamma\theta^0}{\Delta t}(s_h^n, s_h^n - s_h^{n-1}) \\
&\geqslant \frac{4\gamma\theta^0}{2\Delta t}\left(\|s_h^n\|^2 - \|s_h^{n-1}\|^2\right), \\
\kappa\tau_\theta\frac{\tau_q^2}{2}(\vartheta_{hx}^n,\xi_{hx}^n) &= \frac{\kappa\tau_\theta}{\Delta t}\frac{\tau_q^2}{2}(\vartheta_{hx}^n, \vartheta_{hx}^n - \vartheta_{hx}^{n-1}) \\
&\geqslant \frac{\kappa\tau_\theta}{2\Delta t}\frac{\tau_q^2}{2}\left(\|\vartheta_{hx}^n\|^2 - \|\vartheta_{hx}^{n-1}\|^2\right), \\
\kappa(\tau_\theta+\tau_q)(\theta_{hx}^n,\vartheta_{hx}^n) &= \frac{\kappa(\tau_\theta+\tau_q)}{\Delta t}(\theta_{hx}^n, \theta_{hx}^n - \theta_{hx}^{n-1}) \\
&\geqslant \frac{\kappa(\tau_\theta+\tau_q)}{2\Delta t}\left(\|\theta_{hx}^n\|^2 - \|\theta_{hx}^{n-1}\|^2\right)
\end{aligned}
$$

and

$$
\kappa\frac{\tau_q^2}{2}(\theta_{hx}^n,\xi_{hx}^n) = \kappa\frac{\tau_q^2}{2\Delta t}(\theta_{hx}^n,\vartheta_{hx}^n) - \kappa\frac{\tau_q^2}{2\Delta t}(\theta_{hx}^{n-1},\vartheta_{hx}^{n-1}) - \kappa\frac{\tau_q^2}{2}\|\vartheta_{hx}^n\|^2,
$$

we find

$$
\begin{aligned}
&\frac{\rho\theta^0}{2\Delta t}\left(\|\nu_h^n - \nu_h^{n-1}\|^2 + \|\nu_h^n\|^2 - \|\nu_h^{n-1}\|^2\right) + \frac{G\theta^0}{2\Delta t}\left(\|\psi_h^n - \omega_{hx}^n\|^2 - \|\psi_h^{n-1} - \omega_{hx}^{n-1}\|^2\right) \\
&+ \frac{I_\rho\theta^0}{2\Delta t}\left(\|3\varphi_h^n - u_h^n - (3\varphi_h^{n-1} - u_h^{n-1})\|^2 + \|3\varphi_h^n - u_h^n\|^2 - \|3\varphi_h^{n-1} - u_h^{n-1}\|^2\right) \\
&+ \frac{D\theta^0}{2\Delta t}\left(\|3s_h^n - \psi_{hx}^n\|^2 - \|3s_h^{n-1} - \psi_{hx}^{n-1}\|^2\right) + \frac{3I_\rho\theta}{2\Delta t}\left(\|\varphi_h^n - \varphi_h^{n-1}\|^2 + \|\varphi_h^n\|^2 - \|\varphi_h^{n-1}\|^2\right) \\
&+ \frac{3D\theta^0}{2\Delta t}\left(\|s_{hx}^n\|^2 - \|s_{hx}^{n-1}\|^2\right) + \frac{4\gamma\theta^0}{2\Delta t}\left(\|s_h^n\|^2 - \|s_h^{n-1}\|^2\right) + 4\theta^0\|\varphi_h^n\|^2 \\
&+ \frac{1}{2\Delta t}\Big(\|\frac{\tau_q^2}{2}\xi_h^n + \tau_q\vartheta_h^n + \theta_h^n - (\frac{\tau_q^2}{2}\xi_h^{n-1} + \tau_q\vartheta_h^{n-1} + \theta_h^{n-1})\|^2 + \|\frac{\tau_q^2}{2}\xi_h^n + \tau_q\vartheta_h^n + \theta_h^n\|^2 \\
&- \|\frac{\tau_q^2}{2}\xi_h^{n-1} + \tau_q\vartheta_h^{n-1} + \theta_h^{n-1}\|^2\Big) + \frac{\kappa(\tau_\theta+\tau_q)}{2\Delta t}\left(\|\theta_{hx}^n\|^2 - \|\theta_{hx}^{n-1}\|^2\right) + \frac{\kappa\tau_\theta}{2\Delta t}\frac{\tau_q^2}{2}\left(\|\vartheta_{hx}^n\|^2 - \|\vartheta_{hx}^{n-1}\|^2\right) \\
&+ \frac{\kappa\tau_q^2}{2\Delta t}((\theta_{hx}^n,\vartheta_{hx}^n) - (\theta_{hx}^{n-1},\vartheta_{hx}^{n-1})) + \kappa\tau_q\left(\tau_\theta - \frac{\tau_q}{2}\right)\|\vartheta_{hx}^n\|^2 + \kappa\|\theta_{hx}^n\|^2 \leqslant 0.
\end{aligned}
$$

By eliminating the positive terms

$$
\begin{gathered}
\|\nu_h^n - \nu_h^{n-1}\|^2,\ \|3\varphi_h^n - u_h^n - (3\varphi_h^{n-1} - u_h^{n-1})\|^2,\ \|\varphi_h^n - \varphi_h^{n-1}\|^2,\ \|\varphi_h^n\|^2, \\
\|\frac{\tau_q^2}{q}\xi_h^n + \tau_q\vartheta_h^n + \theta_h^n - (\frac{\tau_q^2}{q}\xi_h^{n-1} + \tau_q\vartheta_h^{n-1} + \theta_h^{n-1})\|^2,\ \kappa\tau_q\left(\tau_\theta - \frac{\tau_q}{2}\right)\|\vartheta_{hx}^n\|^2 \text{ and } \|\theta_{hx}^n\|^2,
\end{gathered}
$$

we conclude that

$$
\begin{aligned}
&\frac{\rho\theta^0}{2\Delta t}\left(\|\nu_h^n\|^2-\|\nu_h^{n-1}\|^2\right)+\frac{G\theta^0}{2\Delta t}\left(\|\psi_h^n-\omega_{hx}^n\|^2-\|\psi_h^{n-1}-\omega_{hx}^{n-1}\|^2\right)\\
&+\frac{I_\rho\theta^0}{2\Delta t}\left(\|3\varphi_h^n-u_h^n\|^2-\|3\varphi_h^{n-1}-u_h^{n-1}\|^2\right)+\frac{D\theta^0}{2\Delta t}\left(\|3s_h^n-\psi_{hx}^n\|^2-\|3s_h^{n-1}-\psi_{hx}^{n-1}\|^2\right)\\
&+\frac{3I_\rho\theta^0}{2\Delta t}\left(\|\varphi_h^n\|^2-\|\varphi_h^{n-1}\|^2\right)+\frac{3D\theta^0}{2\Delta t}\left(\|s_{hx}^n\|^2-\|s_{hx}^{n-1}\|^2\right)+\frac{4\gamma\theta^0}{2\Delta t}\left(\|s_h^n\|^2-\|s_h^{n-1}\|^2\right)\\
&+\frac{1}{2\Delta t}(\|\frac{\tau_q^2}{2}\xi_h^n+\tau_q\vartheta_h^n+\theta_h^n\|^2-\|\frac{\tau_q^2}{2}\xi_h^{n-1}+\tau_q\vartheta_h^{n-1}+\theta_h^{n-1}\|^2)\\
&+\frac{\kappa(\tau_\theta+\tau_q)}{2\Delta t}\left(\|\theta_{hx}^n\|^2-\|\theta_{hx}^{n-1}\|^2\right)+\frac{\kappa\tau_\theta}{2\Delta t}\frac{\tau_q^2}{2}\left(\|\vartheta_{hx}^n\|^2-\|\vartheta_{hx}^{n-1}\|^2\right)\\
&+\frac{\kappa\tau_q^2}{2\Delta t}((\theta_{hx}^n,\vartheta_{hx}^n)-(\theta_{hx}^{n-1},\vartheta_{hx}^{n-1}))\leqslant 0,
\end{aligned}
$$

which proves the desired result. □

4.5.3 Error estimate

We now present and demonstrate a priori error estimates that characterize the difference between the exact solution and the corresponding numerical solution.

The following theorem outlines the linear convergence of the numerical method.

Theorem 4.5.2. *Suppose that the solution to the continuous problem (4.4)-(4.6) is regular enough, that is*

$$\omega,\psi,s\in H^3(0,T;L^2(0,L))\cap H^2(0,T;H^1(0,L))\cap W^{1,\infty}(0,T;H^2(0,L)),$$

$$\theta\in H^4(0,T;L^2(0,L))\cap H^3(0,T;H^1(0,L))\cap H^2(0,T;H^2(0,L)).$$

Then, the following error estimates hold:

$$
\begin{aligned}
&\|\nu_h^n-\nu(t_n)\|^2+\|\psi_h^n-\omega_{hx}^n-(\psi(t_n)-\omega_x(t_n))\|^2+\|3\varphi_h^n-u_h^n-(3\varphi(t_n)-u(t_n))\|^2\\
&+\|3s_{hx}^n-\psi_{hx}^n-(3s_x(t_n)-\psi_x(t_n))\|^2+\|\varphi_h^n-\varphi(t_n)\|^2+\|s_{hx}^n-s_x(t_n)\|^2+\|s_h^n-s(t_n)\|^2\\
&+\|\xi_h^n+\vartheta_h^n+\theta_h^n-(\xi(t_n)+\vartheta(t_n)+\theta(t_n))\|^2+\|\theta_{hx}^n+\vartheta_{hx}^n-(\theta_x(t_n)+\vartheta_x(t_n))\|^2\\
&+\|\theta_{hx}^n-\theta_x(t_n)\|^2\leqslant C(h^2+(\Delta t)^2),
\end{aligned}
$$

where C is independent of Δt and h.

Remark 4.5.3. *Note that regular solutions can be obtained by taking regular enough initial data.*

Proof. To begin, we define

$$e^n=\omega_h^n-P_h^*\omega(t_n),\ \hat{e}^n=\nu_h^n-P_h^*\nu(t_n),\ r^n=\psi_h^n-P_h^0\psi(t_n),$$

$$\hat{r}^n=u_h^n-P_h^0u(t_n),\ y^n=s_h^n-P_h^0s(t_n),\ \hat{y}^n=\varphi_h^n-P_h^0\varphi(t_n),$$

and

$$z^n = \theta_h^n - P_h^0\theta(t_n),\ \hat{z}^n = \vartheta_h^n - P_h^0\vartheta(t_n),\ \hat{\varrho}^n = \xi_h^n - P_h^0\xi(t_n).$$

Substituting in the scheme (4.69) and selecting $\bar{\omega}_h = \hat{e}^n$, $\bar{\psi}_h = 3\hat{y}^n - \hat{r}^n$, $\bar{s}_h = \hat{y}^n$, and $\bar{\theta} = \frac{\tau_q^2}{2}\hat{\varrho}^n + \tau_q\hat{z}^n + z^n$, we obtain

$$\begin{aligned}
&\frac{\rho}{2\Delta t}\left(\|\hat{e}^n - \hat{e}^{n-1}\|^2 + \|\hat{e}^n\|^2 - \|\hat{e}^{n-1}\|^2\right) + \frac{\rho}{\Delta t}(P_h^*\nu(t_n) - P_h^*\nu(t_{n-1}), \hat{e}^n)\\
&- G(r^n - e_x^n, \hat{e}_x^n) - G(P_h^0\psi(t_n) - (P_h^*\omega(t_n))_x, \hat{e}_x^n) - \delta(\frac{\tau_q^2}{2}\hat{\varrho}^n + \tau_q\hat{z}^n + z^n, \hat{e}_x^n)\\
&- \delta(\frac{\tau_q^2}{2}P_h^0\xi(t_n) + \tau_q P_h^0\vartheta(t_n) + P_h^0\theta(t_n), \hat{e}_x^n) = 0,
\end{aligned} \tag{4.75}$$

$$\begin{aligned}
&\frac{I_\rho}{2\Delta t}\left(\|3\hat{y}^n - \hat{r}^n - (3\hat{y}^{n-1} - \hat{r}^{n-1})\|^2 + \|3\hat{y}^n - \hat{r}^n\|^2 - \|3\hat{y}^{n-1} - \hat{r}^{n-1}\|^2\right)\\
&+ \frac{I_\rho}{\Delta t}(3P_h^0\varphi(t_n) - P_h^0u(t_n) - (3P_h^0\varphi(t_{n-1}) - P_h^0u(t_{n-1})), 3\hat{y}^n - \hat{r}^n)\\
&+ D(3y_x^n - r^n, 3\hat{y}_x^n - \hat{r}_x^n) + D(3(P_h^0s(t_n))_x - (P_h^0\psi(t_n))_x, 3\hat{y}_x^n - \hat{r}_x^n))\\
&- G(r^n - e_x^n, 3\hat{y}^n - \hat{r}^n) - G(P_h^0\psi(t_n) - (P_h^*\omega(t_n))_x, 3\hat{y}^n - \hat{r}^n) = 0,
\end{aligned} \tag{4.76}$$

$$\begin{aligned}
&\frac{3I_\rho}{2\Delta t}\left(\|\hat{y}^n - \hat{y}^{n-1}\|^2 + \|\hat{y}^n\|^2 - \|\hat{y}^{n-1}\|^2\right) + \frac{3I_\rho}{\Delta t}(P_h^0\varphi(t_n) - P_h^0\varphi(t_{n-1}), \hat{y}^n)\\
&+ 3D(y_x^n, \hat{y}_x^n) + 3D((P_h^0s(t_n))_x, \hat{y}_x^n) + 3G(r_h^n - e_x^n, \hat{y}^n)\\
&+ 3G(P_h^0\psi(t_n) - (P_h^*\omega(t_n))_x, \hat{y}^n) + 4\gamma(y^n, \hat{y}^n)\\
&+ 4\gamma(P_h^0s(t_n), \hat{y}^n) + 4\|\hat{y}^n\|^2 + 4(P_h^0\varphi(t_n), \hat{y}^n) = 0
\end{aligned} \tag{4.77}$$

and

$$\begin{aligned}
&\frac{1}{2\Delta t}\Big(\|\frac{\tau_q^2}{2}\hat{\varrho}^n + \tau_q\hat{z}^n + z^n - (\frac{\tau_q^2}{2}\hat{\varrho}^{n-1} + \tau_q\hat{z}^{n-1} + z^{n-1})\|^2\\
&+\|\frac{\tau_q^2}{2}\hat{\varrho}^n + \tau_q\hat{z}^n + z^n\|^2 - \|\frac{\tau_q^2}{2}\hat{\varrho}^{n-1} + \tau_q\hat{z}^{n-1} + z^{n-1}\|^2\Big)\\
&+ \frac{\tau_q^2}{2}(\frac{P_h^0\xi(t_n) - P_h^0\xi(t_{n-1})}{\Delta t}, \frac{\tau_q^2}{2}\hat{\varrho}^n + \tau_q\hat{z}^n + z^n)\\
&+ \tau_q(\frac{P_h^0\vartheta(t_n) - P_h^0\vartheta(t_{n-1})}{\Delta t}, \frac{\tau_q^2}{2}\hat{\varrho}^n + \tau_q\hat{z}^n + z^n)\\
&+ (\frac{P_h^0\theta(t_n) - P_h^0\theta(t_{n-1})}{\Delta t}, \frac{\tau_q^2}{2}\hat{\varrho}^n + \tau_q\hat{z}^n + z^n)\\
&+ \kappa\tau_\theta(\hat{z}_x^n, \frac{\tau_q^2}{2}\hat{\varrho}_x^n + \tau_q\hat{z}_x^n + z_x^n) + \kappa\tau_\theta((P_h^0\vartheta(t_n))_x, \frac{\tau_q^2}{2}\hat{\varrho}_x^n + \tau_q\hat{z}_x^n + z_x^n)\\
&+ \kappa(z_x^n, \frac{\tau_q^2}{2}\hat{\varrho}_x^n + \tau_q\hat{z}_x^n + z_x^n) + \kappa((P_h^0\theta(t_n))_x, \frac{\tau_q^2}{2}\hat{\varrho}_x^n + \tau_q\hat{z}_x^n + z_x^n)\\
&+ \delta\theta^0(\hat{e}_x^n, \frac{\tau_q^2}{2}\hat{\varrho}^n + \tau_q\hat{z}^n + z^n) + \delta\theta^0((P_h^*\nu(t_n))_x, \frac{\tau_q^2}{2}\hat{\varrho}^n + \tau_q\hat{z}^n + z^n) = 0.
\end{aligned} \tag{4.78}$$

Taking $\bar{\omega} = \hat{e}^n$, $\bar{\psi} = 3\hat{y}^n - \hat{r}^n$, $\bar{s} = \hat{y}^n$, $\bar{\theta} = \frac{\tau_q^2}{2}\hat{\varrho}^n + \tau_q\hat{z}^n + z^n$ in the weak form (4.68), and combining the resulting equations with (4.75)–(4.78), we obtain

$$
\begin{aligned}
&\frac{\rho}{2\Delta t}\left(\|\hat{e}^n - \hat{e}^{n-1}\|^2 + \|\hat{e}^n\|^2 - \|\hat{e}^{n-1}\|^2\right) - G(r^n - e_x^n, \hat{e}_x^n) - \delta(\frac{\tau_q^2}{2}\hat{\varrho}^n + \tau_q\hat{z}^n + z^n, \hat{e}_x^n) \\
&\quad = -\delta(\frac{\tau_q^2}{2}\xi(t_n) + \tau_q\vartheta(t_n) + \theta(t_n) - (\frac{\tau_q^2}{2}P_h^0\xi(t_n) + \tau_q P_h^0\vartheta(t_n) + P_h^0\theta(t_n)), \hat{e}_x^n) \\
&\qquad - G(\psi(t_n) - \omega_x(t_n) - (P_h^0\psi(t_n) - (P_h^0\omega(t_n))_x), \hat{e}_x^n) + \rho(\nu_t(t_n) - \frac{P_h^*\nu(t_n) - P_h^*\nu(t_{n-1})}{\Delta t}, \hat{e}^n),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{I_\rho}{2\Delta t}\left(\|3\hat{y}^n - \hat{r}^n - (3\hat{y}^{n-1} - \hat{r}^{n-1})\|^2 + \|3\hat{y}^n - \hat{r}^n\|^2 - \|3\hat{y}^{n-1} - \hat{r}^{n-1}\|^2\right) \\
&\qquad + D(3y_x^n - r_x^n, 3\hat{y}_x^n - \hat{r}_x^n) - G(r^n - e_x^n, 3\hat{y}^n - \hat{r}^n) \\
&\quad = I_\rho(3\varphi_t(t_n) - u_t(t_n) - \frac{3P_h^0\varphi(t_n) - P_h^0u(t_n) - (3P_h^0\varphi(t_{n-1}) - P_h^0u(t_{n-1}))}{\Delta t}, 3\hat{y}^n - \hat{r}^n) \\
&\qquad + D(3s_x(t_n) - \psi_x(t_n) - (3(P_h^0 s(t_n))_x - (P_h^0\psi(t_n))_x), 3\hat{y}_x^n - \hat{r}_x^n) \\
&\qquad - G(\psi(t_n) - \omega_x(t_n) - (P_h^0\psi(t_n) - (P_h^*\omega(t_n))_x), 3\hat{y}^n - \hat{r}^n),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{3I_\rho}{2\Delta t}\left(\|\hat{y}^n - \hat{y}^{n-1}\|^2 + \|\hat{y}^n\|^2 - \|\hat{y}^{n-1}\|^2\right) + 3G(r_h^n - e_x^n, \hat{y}^n) + 3D(y_x^n, \hat{y}_x^n) + 4\gamma(y^n, \hat{y}^n) + 4\|\hat{y}^n\|^2 \\
&\quad = 3I_\rho(\varphi_t(t_n) - \frac{P_h^0\varphi(t_n) - P_h^0\varphi(t_{n-1})}{\Delta t}, \hat{y}^n) + 3D(s_x(t_n) - (P_h^0 s(t_n))_x, \hat{y}_x^n) \\
&\qquad + 3G(\psi(t_n) - \omega_x(t_n) - (P_h^0\psi(t_n) - (P_h^*\omega(t_n))_x), \hat{y}^n) \\
&\qquad + 4\gamma(s(t_n) - P_h^0 s(t_n), \hat{y}^n) + 4(\varphi(t_n) - P_h^0\varphi(t_n), \hat{y}^n),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2\Delta t}\Big(\|\frac{\tau_q^2}{2}\hat{\varrho}^n + \tau_q\hat{z}^n + z^n - (\frac{\tau_q^2}{2}\hat{\varrho}^{n-1} + \tau_q\hat{z}^{n-1} + z^{n-1})\|^2 + \|\frac{\tau_q^2}{2}\hat{\varrho}^n + \tau_q\hat{z}^n + z^n\|^2 \\
&\qquad - \|\frac{\tau_q^2}{2}\hat{\varrho}^{n-1} + \tau_q\hat{z}^{n-1} + z^{n-1}\|^2\Big) + \kappa\tau_\theta(\hat{z}_x^n, \frac{\tau_q^2}{2}\hat{\varrho}_x^n + \tau_q\hat{z}_x^n + z^n) \\
&\qquad + \kappa(z_x^n, \frac{\tau_q^2}{2}\hat{\varrho}_x^n + \tau_q\hat{z}_x^n + z_x^n) + \delta\theta^0(\hat{e}_x^n, \frac{\tau_q^2}{2}\hat{\varrho}^n + \tau_q\hat{z}^n + z^n) \\
&\quad = \frac{\tau_q^2}{2}(\xi_t(t_n) - \frac{P_h^0\xi(t_n) - P_h^0\xi(t_{n-1})}{\Delta t}, \frac{\tau_q^2}{2}\hat{\varrho}^n + \tau_q\hat{z}^n + z^n) \\
&\qquad + \tau_q(\vartheta_t(t_n) - \frac{P_h^0\vartheta(t_n) - P_h^0\vartheta(t_{n-1})}{\Delta t}, \frac{\tau_q^2}{2}\hat{\varrho}^n + \tau_q\hat{z}^n + z^n) \\
&\qquad + (\theta_t(t_n) - \frac{P_h^0\theta(t_n) - P_h^0\theta(t_{n-1})}{\Delta t}, \frac{\tau_q^2}{2}\hat{\varrho}^n + \tau_q\hat{z}^n + z^n) \\
&\qquad + \kappa\tau_\theta(\vartheta_x(t_n) - (P_h^0\vartheta(t_n))_x, \frac{\tau_q^2}{2}\hat{\varrho}_x^n + \tau_q\hat{z}_x^n + z_x^n) \\
&\qquad + \kappa(\theta_x(t_n) - (P_h^0\theta(t_n))_x, \frac{\tau_q^2}{2}\hat{\varrho}_x^n + \tau_q\hat{z}_x^n + z_x^n) + \delta\theta^0(\nu_x(t_n) - (P_h^*\nu(t_n))_x, \frac{\tau_q^2}{2}\hat{\varrho}^n + \tau_q\hat{z}^n + z^n).
\end{aligned}
$$

We add the last four equations to get

$$
\begin{aligned}
&\frac{\rho\theta^0}{2\Delta t}\left(\|\hat{e}^n-\hat{e}^{n-1}\|^2+\|\hat{e}^n\|^2-\|\hat{e}^{n-1}\|^2\right)+G\theta^0(r^n-e_x^n,\hat{r}^n-\hat{e}_x^n)\\
&+\frac{I_\rho\theta^0}{2\Delta t}\left(\|3\hat{y}^n-\hat{r}^n-(3\hat{y}^{n-1}-\hat{r}^{n-1})\|^2+\|3\hat{y}^n-\hat{r}^n\|^2-\|3\hat{y}^{n-1}-\hat{r}^{n-1}\|^2\right)\\
&+D\theta^0(3y_x^n-r_x^n,3\hat{y}_x^n-\hat{r}_x^n)+\frac{3I_\rho\theta^0}{2\Delta t}\left(\|\hat{y}^n-\hat{y}^{n-1}\|^2+\|\hat{y}^n\|^2-\|\hat{y}^{n-1}\|^2\right)+3D\theta^0(y_x^n,\hat{y}_x^n)\\
&+4\gamma\theta^0(y^n,\hat{y}^n)+4\theta^0\|\hat{y}^n\|^2+\frac{1}{2\Delta t}\Big(\|\frac{\tau_q^2}{2}\hat{\varrho}^n+\tau_q\hat{z}^n+z^n-(\frac{\tau_q^2}{2}\hat{\varrho}^{n-1}+\tau_q\hat{z}^{n-1}+z^{n-1})\|^2\\
&+\|\frac{\tau_q^2}{2}\hat{\varrho}^n+\tau_q\hat{z}^n+z^n\|^2-\|\frac{\tau_q^2}{2}\hat{\varrho}^{n-1}+\tau_q\hat{z}^{n-1}+z^{n-1}\|^2\Big)+\kappa\tau_\theta(\hat{z}_x^n,\frac{\tau_q^2}{2}\hat{\varrho}_x^n+\tau_q\hat{z}_x^n+z_x^n)\\
&+\kappa(z_x^n,\frac{\tau_q^2}{2}\hat{\varrho}_x^n+\tau_q\hat{z}_x^n+z_x^n)=\rho\theta^0(\nu_t(t_n)-\frac{P_h^*\nu(t_n)-P_h^*\nu(t_{n-1})}{\Delta t},\hat{e}^n)\\
&+G\theta^0(\psi(t_n)-\omega_x(t_n)-(P_h^0\psi(t_n)-(P_h^0\omega(t_n))_x),\hat{r}^n-\hat{e}_x^n)\\
&-\delta\theta^0\frac{\tau_q^2}{2}(\xi(t_n)-P_h^0\xi(t_n),\hat{e}_x^n)-\delta\theta^0\tau_q(\vartheta(t_n)-P_h^0\vartheta(t_n),\hat{e}_x^n)-\delta\theta^0(\theta(t_n)-P_h^0\theta(t_n),\hat{e}_x^n)\\
&+I_\rho\theta^0(3\varphi_t(t_n)-u_t(t_n)-\frac{3P_h^0\varphi(t_n)-P_h^0u(t_n)-(3P_h^0\varphi(t_{n-1})-P_h^0u(t_{n-1}))}{\Delta t},3\hat{y}^n-\hat{r}^n)\\
&+D\theta^0(3s_x(t_n)-\psi_x(t_n)-(3(P_h^0s(t_n))_x-(P_h^0\psi(t_n))_x),3\hat{y}_x^n-\hat{r}_x^n)\\
&+3I_\rho\theta^0(\varphi_t(t_n)-\frac{P_h^0\varphi(t_n)-P_hP_h^0\varphi(t_{n-1})}{\Delta t},\hat{y}^n)+3D\theta^0(s_x(t_n)-(P_h^0s(t_n))_x,\hat{y}_x^n)\\
&+4\gamma\theta^0(s(t_n)-P_h^0s(t_n),\hat{y}^n)+4\theta^0(\varphi(t_n)-P_h^0\varphi(t_n),\hat{y}^n)\\
&+\frac{\tau_q^2}{2}(\xi_t(t_n)-\frac{P_h^0\xi(t_n)-P_h^0\xi(t_{n-1})}{\Delta t},\frac{\tau_q^2}{2}\hat{\varrho}^n+\tau_q\hat{z}^n+z^n)\\
&+\tau_q(\vartheta_t(t_n)-\frac{P_h^0\vartheta(t_n)-P_h^0\vartheta(t_{n-1})}{\Delta t},\frac{\tau_q^2}{2}\hat{\varrho}^n+\tau_q\hat{z}^n+z^n)\\
&+(\theta_t(t_n)-\frac{P_h^0\theta(t_n)-P_h^0\theta(t_{n-1})}{\Delta t},\frac{\tau_q^2}{2}\hat{\varrho}^n+\tau_q\hat{z}^n+z^n)\\
&+\kappa\tau_\theta(\vartheta_x(t_n)-(P_h^0\vartheta(t_n))_x,\frac{\tau_q^2}{2}\hat{\varrho}_x^n+\tau_q\hat{z}_x^n+z_x^n)\\
&+\kappa(\theta_x(t_n)-(P_h^0\theta(t_n))_x,\frac{\tau_q^2}{2}\hat{\varrho}_x^n+\tau_q\hat{z}_x^n+z_x^n)\\
&+\delta\theta^0(\nu_x(t_n)-(P_h^*\nu(t_n))_x,\frac{\tau_q^2}{2}\hat{\varrho}^n+\tau_q\hat{z}^n+z^n)
\end{aligned}
\tag{4.79}
$$

and notice that for some positive constant c, we have

$$
\begin{aligned}
\kappa\tau_\theta(\hat{z}_x^n,\frac{\tau_q^2}{2}\hat{\varrho}_x^n+\tau_q\hat{z}_x^n+z_x^n)+\kappa(z_x^n,\frac{\tau_q^2}{2}\hat{\varrho}_x^n+\tau_q\hat{z}_x^n+z_x^n)\geqslant& c(z_x^n+\hat{z}_x^n,\hat{z}_x^n+\hat{\varrho}_x^n)\\
&+\kappa\tau_\theta(z_x^n,\hat{z}_x^n)+\kappa\|z_x^n\|^2.
\end{aligned}
\tag{4.80}
$$

Referring to the definitions of $\hat{e}^n$, $\hat{r}^n$, $\hat{y}^n$, $\hat{z}^n$ and $\hat{\varrho}^n$, we arrive at the following estimates:

$$\begin{aligned}(r^n - e_x^n, \hat{r}^n - \hat{e}_x^n) =& \frac{1}{2\Delta t}\left(\|r^n - e_x^n - (r^{n-1} - e_x^{n-1})\|^2 + \|r^n - e_x^n\|^2 - \|r^{n-1} - e_x^{n-1}\|^2\right) \\ &+ (r^n - e_x^n, \frac{P_h^0\psi(t_n) - P_h^0\psi(t_{n-1})}{\Delta t} - P_h^0 u(t_n)) \\ &- (r^n - e_x^n, \frac{(P_h^*\omega(t_n))_x - (P_h^*\omega(t_{n-1}))_x}{\Delta t} - (P_h^*\nu(t_n))_x),\end{aligned} \tag{4.81}$$

$$\begin{aligned}(3y_x^n - r_x^n, 3\hat{y}_x^n - \hat{r}_x^n) =& \frac{1}{2\Delta t}\Big(\|3y_x^n - r_x^n - (3y_x^{n-1} - r_x^{n-1})\|^2 \\ &+ \|3y_x^n - r_x^n\|^2 - \|3y_x^{n-1} - r_x^{n-1}\|^2\Big) \\ &+ (3y_x^n - r_x^n, \frac{3(P_h^0 s(t_n))_x - 3(P_h^0 s(t_{n-1}))_x}{\Delta t} - 3(P_h^0\varphi(t_n))_x) \\ &- (3y_x^n - r_x^n, \frac{(P_h^0\psi(t_n))_x - (P_h^0\psi(t_{n-1}))_x}{\Delta t} - (P_h^0 u(t_n))_x),\end{aligned} \tag{4.82}$$

$$\begin{aligned}(y^n, \hat{y}^n) =& \frac{1}{2\Delta t}\left(\|y^n - y^{n-1}\|^2 + \|y^n\|^2 - \|y^{n-1}\|^2\right) \\ &+ (y^n, \frac{P_h^0 s(t_n) - P_h^0 s(t_{n-1})}{\Delta t} - P_h^0\varphi(t_n)),\end{aligned} \tag{4.83}$$

$$\begin{aligned}(z_x^n + \hat{z}_x^n, \hat{z}_x^n + \hat{\varrho}_x^n) =& \frac{1}{2\Delta t}\left(\|z_x^n + \hat{z}_x^n - (z_x^{n-1} + \hat{z}_x^{n-1})\|^2 + \|z_x^n + \hat{z}_x^n\|^2 - \|z_x^{n-1} + \hat{z}_x^{n-1}\|^2\right) \\ &+ (z_x^n + \hat{z}_x^n, \frac{(P_h^0\theta(t_n))_x - (P_h^0\theta(t_{n-1}))_x}{\Delta t} - (P_h^0\vartheta(t_n))_x) \\ &+ (z_x^n + \hat{z}_x^n, \frac{(P_h^0\vartheta(t_n))_x - (P_h^0\vartheta(t_{n-1}))_x}{\Delta t} - (P_h^0\xi(t_n))_x)\end{aligned} \tag{4.84}$$

and

$$\begin{aligned}(z_x^n, \hat{z}_x^n) =& \frac{1}{2\Delta t}\left(\|z_x^n - z_x^{n-1}\|^2 + \|z_x^n\|^2 - \|z_x^{n-1}\|^2\right) \\ &+ (z_x^n, \frac{(P_h^0\theta(t_n))_x - (P_h^0\theta(t_{n-1}))_x}{\Delta t} - (P_h^0\vartheta(t_n))_x).\end{aligned} \tag{4.85}$$

Substituting (4.80)–(4.85) into (4.79), then eliminating the positive terms:

$$\begin{gathered}\|\hat{e}^n - \hat{e}^{n-1}\|,\ \|3\hat{y}^n - \hat{r}^n - (3\hat{y}^{n-1} - \hat{r}^{n-1})\|,\ \|\hat{y}^n - \hat{y}^{n-1}\|,\ \|\hat{y}^n\|,\ \|z_x^n\|, \\ \|\frac{\tau_q^2}{2}\hat{\varrho}^n + \tau_q\hat{z}^n + z^n - (\frac{\tau_q^2}{2}\hat{\varrho}^{n-1} + \tau_q\hat{z}^{n-1} + z^{n-1})\|,\ \|r^n - e_x^n - (r^{n-1} - e_x^{n-1})\|,\ \|y_x^n - y_x^{n-1}\|, \\ \|3y_x^n - r_x^n - (3y_x^{n-1} - r_x^{n-1})\|,\ \|y^n - y^{n-1}\|,\ \|z_x^n + \hat{z}_x^n - (z_x^{n-1} + \hat{z}_x^{n-1})\| \text{ and } \|z_x^n - z_x^{n-1}\|,\end{gathered}$$

we have

$$
\begin{aligned}
&\frac{\rho\theta^0}{2\Delta t}\left(\|\hat{e}^n\|^2-\|\hat{e}^{n-1}\|^2\right)+\frac{G\theta^0}{2\Delta t}\left(\|r^n-e_x^n\|^2-\|r^{n-1}-e_x^{n-1}\|^2\right)+\frac{\kappa\tau_\theta}{2\Delta t}\left(\|z_x^n\|^2-\|z_x^{n-1}\|^2\right)\\
&+\frac{I_\rho\theta^0}{2\Delta t}\left(\|3\hat{y}^n-\hat{r}^n\|^2-\|3\hat{y}^{n-1}-\hat{r}^{n-1}\|^2\right)+\frac{D\theta^0}{2\Delta t}\left(\|3y_x^n-r_x^n\|^2-\|3y_x^{n-1}-r_x^{n-1}\|^2\right)\\
&+\frac{3I_\rho\theta^0}{2\Delta t}\left(\|\hat{y}^n\|^2-\|\hat{y}^{n-1}\|^2\right)+\frac{3D\theta^0}{2\Delta t}\left(\|y_x^n\|^2-\|y_x^{n-1}\|^2\right)+\frac{4\gamma\theta^0}{2\Delta t}\left(\|y^n\|^2-\|y^{n-1}\|^2\right)\\
&+\frac{1}{2\Delta t}(\|\frac{\tau_q^2}{2}\hat{\varrho}^n+\tau_q\hat{z}^n+z^n\|^2-\|\frac{\tau_q^2}{2}\hat{\varrho}^{n-1}+\tau_q\hat{z}^{n-1}+z^{n-1}\|^2)+\frac{c}{2\Delta t}(\|z_x^n+\hat{z}_x^n\|^2-\|z_x^{n-1}+\hat{z}_x^{n-1}\|^2)\\
\leqslant\ &\rho\theta^0(\nu_t(t_n)-\frac{P_h^*\nu(t_n)-P_h^*\nu(t_{n-1})}{\Delta t},\hat{e}^n)+G\theta^0(\psi(t_n)-\omega_x(t_n)-(P_h^0\psi(t_n)-(P_h^*\omega(t_n))_x),\hat{r}^n-\hat{e}_x^n)\\
&-G\theta^0(r^n-e_x^n,\frac{P_h^0\psi(t_n)-P_h^0\psi(t_{n-1})}{\Delta t}-P_h^0u(t_n))\\
&+G\theta^0(r^n-e_x^n,\frac{(P_h^*\omega(t_n))_x-(P_h^*\omega(t_{n-1}))_x}{\Delta t}-(P_h^*\nu(t_n))_x)\\
&-\delta\theta^0\frac{\tau_q^2}{2}(\xi(t_n)-P_h^0\xi(t_n),\hat{e}_x^n)-\delta\theta^0\tau_q(\vartheta(t_n)-P_h^0\vartheta(t_n),\hat{e}_x^n)-\delta\theta^0(\theta(t_n)-P_h^0\theta(t_n),\hat{e}_x^n)\\
&+I_\rho\theta^0(3\varphi_t(t_n)-u_t(t_n)-\frac{3P_h^0\varphi(t_n)-P_h^0u(t_n)-(3P_h^0\varphi(t_{n-1})-P_h^0u(t_{n-1}))}{\Delta t},3\hat{y}^n-\hat{r}^n)\\
&+D\theta^0(3s_x(t_n)-\psi_x(t_n)-(3(P_h^0s(t_n))_x-(P_h^0\psi(t_n))_x),3\hat{y}_x^n-\hat{r}_x^n)\\
&-D\theta^0(3y_x^n-r_x^n,\frac{3(P_h^0s(t_n))_x-3(P_h^0s(t_{n-1}))_x}{\Delta t}-3(P_h^0\varphi(t_n))_x)\\
&+D\theta^0(3y_x^n-r_x^n,\frac{(P_h^0\psi(t_n))_x-(P_h^0\psi(t_{n-1}))_x}{\Delta t}-(P_h^0u(t_n))_x)\\
&+3I_\rho\theta^0(\varphi_t(t_n)-\frac{P_h^0\varphi(t_n)-P_h^0\varphi(t_{n-1})}{\Delta t},\hat{y}^n)\\
&+3D\theta^0(s_x(t_n)-(P_h^0s(t_n))_x,\hat{y}_x^n)-3D\theta^0(y_x^n,\frac{(P_h^0s(t_n))_x-(P_h^0s(t_{n-1}))_x}{\Delta t}-(P_h^0\varphi(t_n))_x)\\
&+4\gamma\theta^0(s(t_n)-P_h^0s(t_n),\hat{y}^n)+4\theta^0(\varphi(t_n)-P_h^0\varphi(t_n),\hat{y}^n)-4\gamma\theta^0(y^n,\frac{P_h^0s(t_n)-P_h^0s(t_{n-1})}{\Delta t}-P_h^0\varphi(t_n))\\
&+\frac{\tau_q^2}{2}(\xi_t(t_n)-\frac{P_h^0\xi(t_n)-P_h\xi(t_{n-1})}{\Delta t},\frac{\tau_q^2}{2}\hat{\varrho}^n+\tau_q\hat{z}^n+z^n)\\
&+\tau_q(\vartheta_t(t_n)-\frac{P_h^0\vartheta(t_n)-P_h^0\vartheta(t_{n-1})}{\Delta t},\frac{\tau_q^2}{2}\hat{\varrho}^n+\tau_q\hat{z}^n+z^n)\\
&+(\theta_t(t_n)-\frac{P_h^0\theta(t_n)-P_h^0\theta(t_{n-1})}{\Delta t},\frac{\tau_q^2}{2}\hat{\varrho}^n+\tau_q\hat{z}^n+z^n)\\
&+\kappa\tau_\theta(\vartheta_x(t_n)-(P_h^0\vartheta(t_n))_x,\frac{\tau_q^2}{2}\hat{\varrho}_x^n+\tau_q\hat{z}_x^n+z_x^n)+\kappa(\theta_x(t_n)-(P_h^0\theta(t_n))_x,\frac{\tau_q^2}{2}\hat{\varrho}_x^n+\tau_q\hat{z}_x^n+z_x^n)\\
&-c(z_x^n+\hat{z}_x^n,\frac{(P_h^0\vartheta(t_n))_x-(P_h^0\vartheta(t_{n-1}))_x}{\Delta t}-(P_h^0\xi(t_n))_x)\\
&-c(z_x^n+\hat{z}_x^n,\frac{(P_h^0\theta(t_n))_x-(P_h^0\theta(t_{n-1}))_x}{\Delta t}-(P_h^0\vartheta(t_n))_x)\\
&-\kappa\tau_\theta(z_x^n,\frac{(P_h^0\theta(t_n))_x-(P_h^0\theta(t_{n-1}))_x}{\Delta t}-(P_h^0\vartheta(t_n))_x)+\delta\theta^0(\nu_x(t_n)-(P_h^*\nu(t_n))_x,\frac{\tau_q^2}{2}\hat{\varrho}^n+\tau_q\hat{z}^n+z^n).
\end{aligned}
$$

Finally, let

$$
\begin{aligned}
Z_n =&\rho\theta^0\|\hat{e}^n\|^2+G\theta^0\|r^n-e_x^n\|^2+I_\rho\theta^0\|3\hat{y}^n-\hat{r}^n\|^2+D\theta^0\|3y_x^n-r_x^n\|^2+3I_\rho\theta^0\|\hat{y}^n\|^2\\
&+3D\theta^0\|y_x^n\|^2+4\gamma\theta^0\|y^n\|^2+\|\frac{\tau_q^2}{2}\hat{\varrho}^n+\tau_q\hat{z}^n+z^n\|^2+c\|z_x^n+\hat{z}_x^n\|^2+\kappa\tau_\theta\|z_x^n\|^2.
\end{aligned}
$$

Applying Young's inequality, we obtain

$$
\begin{aligned}
Z_n-Z_{n-1}\leqslant&2C\Delta t\Big(Z_n+\left\|\nu_t(t_n)-\frac{P_h^*\nu(t_n)-P_h^*\nu(t_{n-1})}{\Delta t}\right\|^2\\
&+\|\psi(t_n)-\omega_x(t_n)-(P_h^0\psi(t_n)-(P_h^0\omega(t_n))_x)\|^2\\
&+\left\|\frac{P_h^0\psi(t_n)-P_h^0\psi(t_{n-1})}{\Delta t}-P_h^0u(t_n)\right\|^2\\
&+\left\|\frac{(P_h^*\omega(t_n))_x-(P_h^*\omega(t_{n-1}))_x}{\Delta t}-(P_h^*\nu(t_n))_x\right\|^2\\
&+\|\xi(t_n)-P_h^0\xi(t_n)\|^2+\|\vartheta(t_n)-P_h^0\vartheta(t_n)\|^2+\|\theta(t_n)-P_h^0\theta(t_n)\|^2\\
&+\left\|3\varphi_t(t_n)-u_t(t_n)-\frac{3P_h^0\varphi(t_n)-P_h^0u(t_n)-(3P_h^0\varphi(t_{n-1})-P_h^0u(t_{n-1}))}{\Delta t}\right\|^2\\
&+\|3s_x(t_n)-\psi_x(t_n)-(3(P_h^0s(t_n))_x-(P_h^0\psi(t_n))_x)\|^2\\
&+\left\|\frac{3(P_h^0s(t_n))_x-3(P_h^0s(t_{n-1}))_x}{\Delta t}-3(P_h^0\varphi(t_n))_x\right\|^2\\
&+\left\|\frac{(P_h^0\psi(t_n))_x-(P_h^0\psi(t_{n-1}))_x}{\Delta t}-(P_h^0u(t_n))_x\right\|^2\\
&+\left\|\varphi_t(t_n)-\frac{P_h^0\varphi(t_n)-P_h^0\varphi(t_{n-1})}{\Delta t}\right\|^2+\|s_x(t_n)-(P_h^0s(t_n))_x\|^2\\
&+\left\|\frac{(P_h^0s(t_n))_x-(P_h^0s(t_{n-1}))_x}{\Delta t}-(P_h^0\varphi(t_n))_x\right\|^2\\
&+\|s(t_n)-P_h^0s(t_n)\|^2+\|\varphi(t_n)-P_h^0\varphi(t_n)\|^2+\left\|\frac{P_h^0s(t_n)-P_h^0s(t_{n-1})}{\Delta t}-P_h^0\varphi(t_n)\right\|^2\\
&+\left\|\xi_t(t_n)-\frac{P_h^0\xi(t_n)-P_h^0\xi(t_{n-1})}{\Delta t}\right\|^2+\left\|\vartheta_t(t_n)-\frac{P_h^0\vartheta(t_n)-P_h^0\vartheta(t_{n-1})}{\Delta t}\right\|^2\\
&+\left\|\theta_t(t_n)-\frac{P_h^0\theta(t_n)-P_h^0\theta(t_{n-1})}{\Delta t}\right\|^2+\|\vartheta_x(t_n)-(P_h^0\vartheta(t_n))_x\|^2\\
&+\|\theta_x(t_n)-(P_h^0\theta(t_n))_x\|^2+\left\|\frac{(P_h^0\vartheta(t_n))_x-(P_h^0\vartheta(t_{n-1}))_x}{\Delta t}-(P_h^0\xi(t_n))_x\right\|^2\\
&+\left\|\frac{(P_h^0\theta(t_n))_x-(P_h^0\theta(t_{n-1}))_x}{\Delta t}-(P_h^0\vartheta(t_n))_x\right\|^2+\|\nu_x(t_n)-(P_h^*\nu(t_n))_x\|^2\Big).
\end{aligned}
$$

Hence, we get

$$Z_n - Z_{n-1} \leqslant 2C\Delta t(Z_n + R_n),$$

where the residual R_n is the sum of the approximation errors. By summing this inequality over n, we end up with

$$Z_n - Z_0 \leqslant 2C\Delta t \sum_{j=1}^{n} (Z_j + R_j).$$

Then, using Taylor's expansion in time and (2.5) to estimate both the time and space errors, we find

$$2C\Delta t \sum_{j=1}^{n} R_j \leqslant C(h^2 + (\Delta t)^2).$$

Since $Z_0 = 0$, it follows that

$$Z_n \leqslant 2C\Delta t \sum_{j=1}^{n} Z_j + C(h^2 + (\Delta t)^2).$$

The result is derived by using a discrete version of Gronwall's inequality and considering that $n\Delta t \leqslant T$. □

4.6 Simulations

In this section, we conduct two tests and evaluate the error estimate numerically. The first test is performed when the stability is exponential, while the second test is carried out when the stability is polynomial. For both tests, we use the following data:

$$h = 0.01,\ \Delta t = h/2,\ I_\rho = \gamma = \tau_\theta = 1,\ D = 6,\ \tau_q = 0.5$$

and initial conditions

$$\omega_0(x) = \omega_1(x) = \cos(2\pi x),\ \psi_0(x) = \psi_1(x) = x^2(1-x)^2,$$

$$s_0(x) = s_1(x) = \sin(2\pi x),\ \theta_0(x) = \theta_1(x) = \theta_2(x) = \sin(\pi x).$$

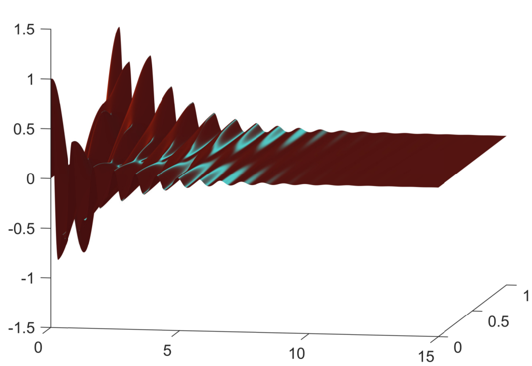
First test ($\chi = 0$). The first experiment was considered with the following values:

$$\rho = 1/6,\ \delta = 1,\ G = 0.5,\ \kappa = 0.25,\ \theta^0 = 1/3.$$

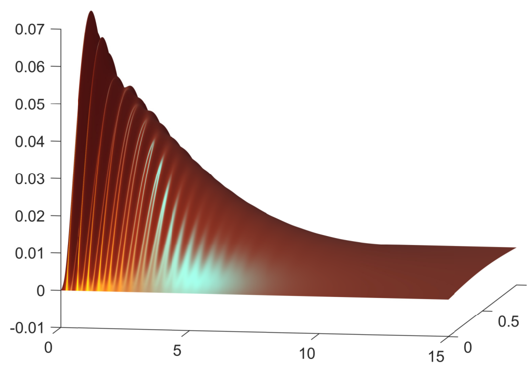
The 3D evolution of ω, ψ, s, and θ is illustrated in Figure 4.1.

Figure 4.2 presents the results at $x = 0.7$.

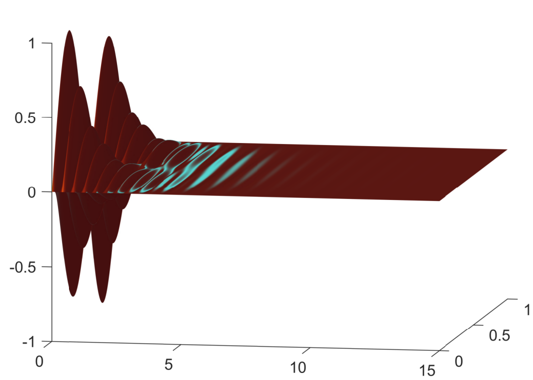
The decay of energy over time is depicted in Figure 4.3.



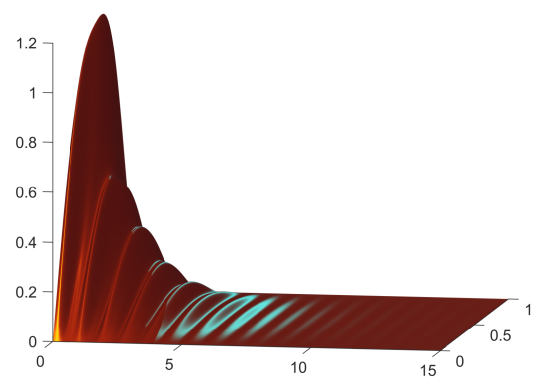
(a) $\omega(x,t)$



(b) $\psi(x,t)$



(c) $s(x,t)$



(d) $\theta(x,t)$

Figure 4.1 The evolution in time and space of ω, ψ, s and θ.

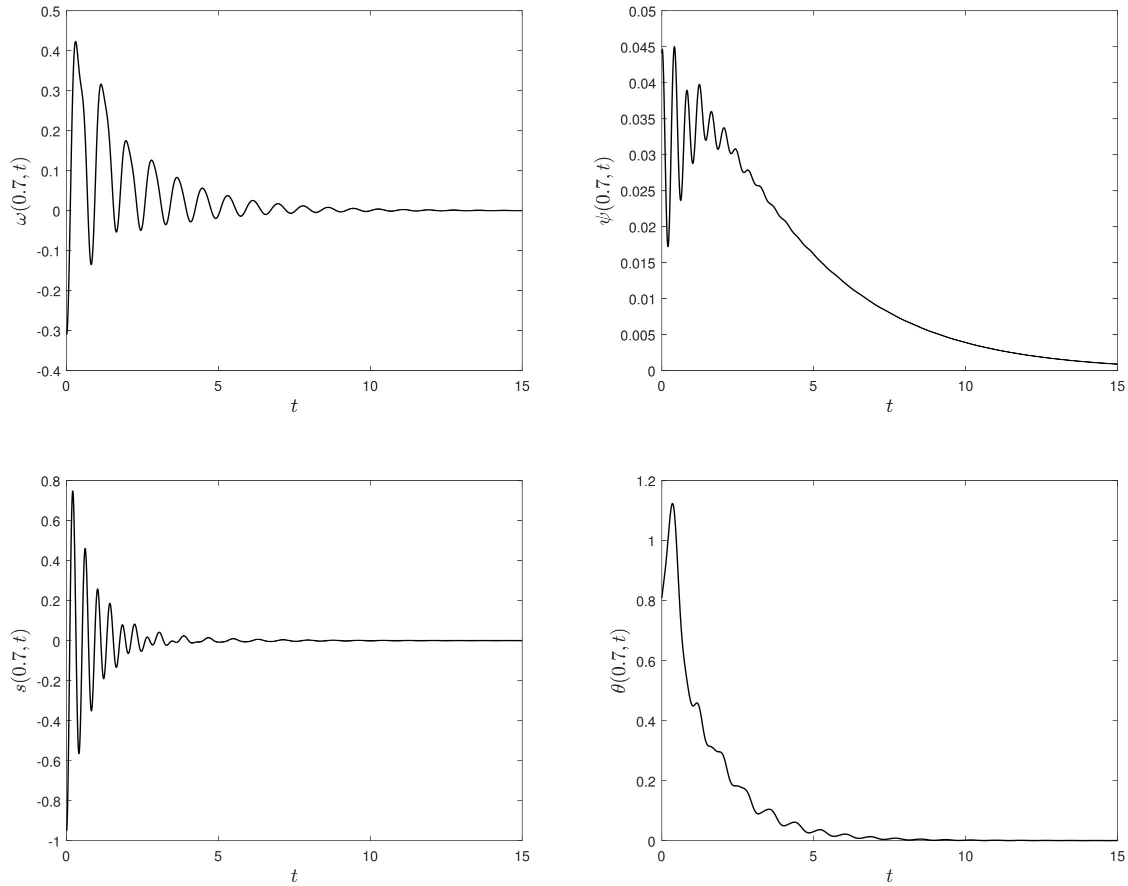


Figure 4.2 The evolution in time of ω, ψ, s and θ at $x = 0.7$.

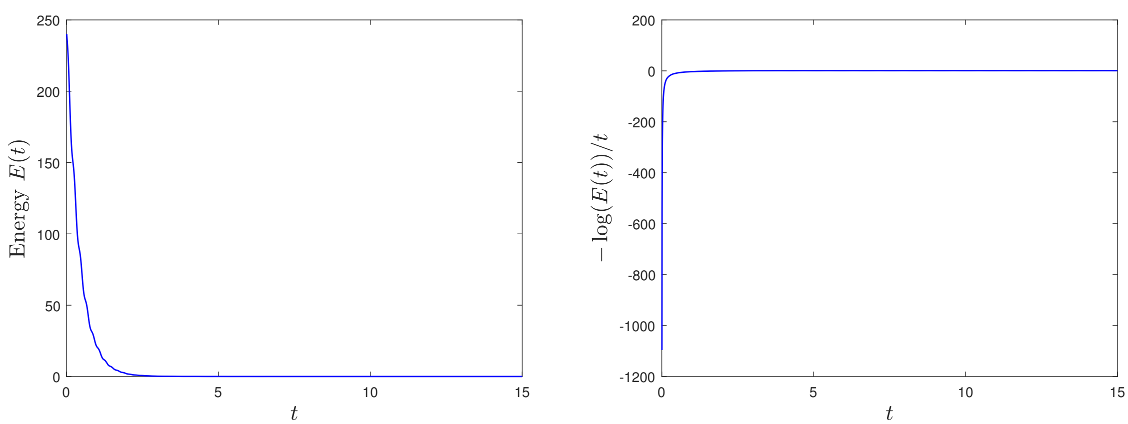
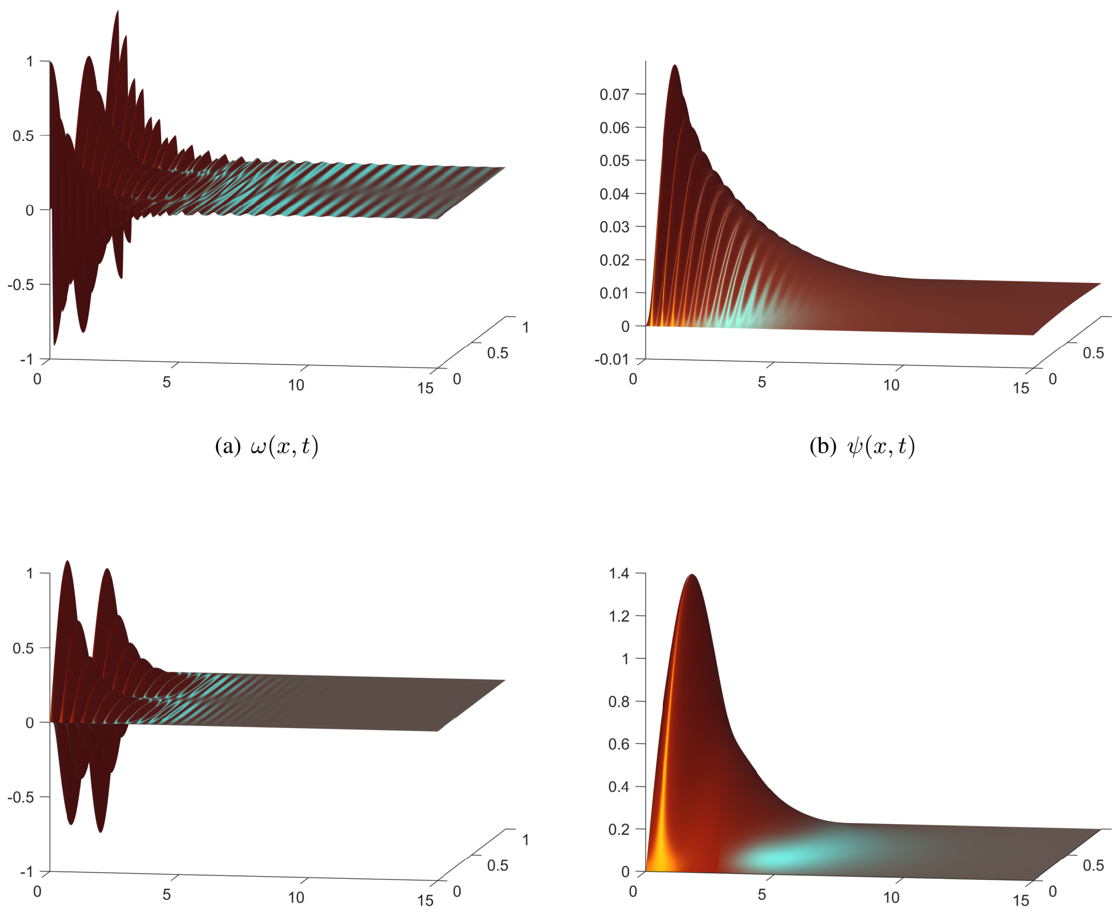


Figure 4.3 The evolution in time of the energy.

Second test ($\chi \neq 0$). We select the following entries:

$$\rho = \delta = 0.01,\ G = \kappa = 0.1,\ \theta^0 = 0.001.$$

The solutions ω, ψ, s and θ are graphed in 3D (see Figure 4.4).
The long time behavior of ω, ψ, s and θ at $x = 0.7$ are shown in Figure 4.5.
In Figure 4.6, a polynomial decay is achieved after time $t = 5$.



(a) $\omega(x,t)$ (b) $\psi(x,t)$ (c) $s(x,t)$ (d) $\theta(x,t)$

Figure 4.4 The evolution in time and space of ω, ψ, s and θ.

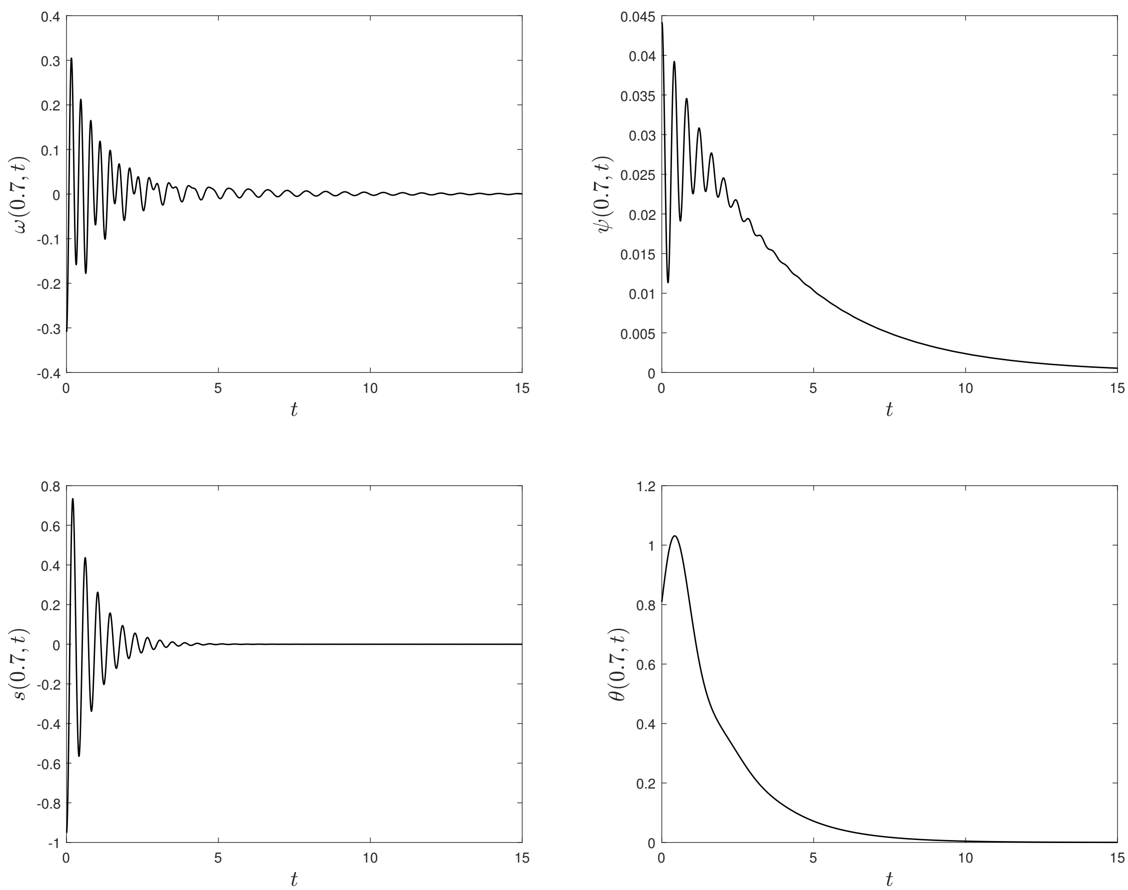


Figure 4.5 The evolution in time of ω, ψ, s and θ at $x = 0.7$.

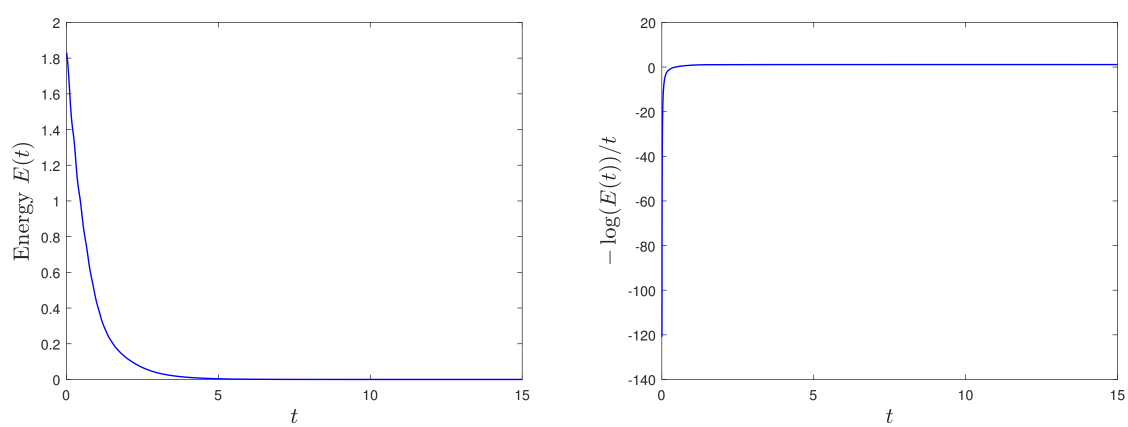


Figure 4.6 The evolution in time of the energy.

Numerical convergence. To illustrate the accuracy of the approximations, we examine the following academic problem

$$\begin{cases} \rho\omega_{tt} + G(\psi - \omega_x)_x + \delta\left(\dfrac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)_x = \mathcal{F}_1, \\ I_\rho(3s-\psi)_{tt} - D(3s-\psi)_{xx} - G(\psi - \omega_x) = \mathcal{F}_2, \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - \omega_x) + \dfrac{4}{3}\gamma s + \dfrac{4}{3}s_t = \mathcal{F}_3, \\ \left(\dfrac{\tau_q^2}{2}\theta_{tt} + \tau_q\theta_t + \theta\right)_t - \kappa(\tau_\theta\theta_{xt} + \theta_x)_x + \delta\theta^0\omega_{xt} = \mathcal{F}_4, \end{cases} \tag{4.86}$$

with the same entries of second test. The functions $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$, and the initial data are calculated from the exact solution

$$\omega(x,t) = 6e^t cos(\pi x),\ \psi(x,t) = tx^3(x-1)^3,\ s(x,t) = e^t x^2(x-1)^2,\ \theta(x,t) = 0.01e^t x(x-1).$$

The computed errors at time $T = 1$ are presented in Table 4.1, where the $Error$ is defined as

$$\begin{aligned} Error = &\Big(\|\nu_h^n - \nu(t_n)\|^2 + \|\psi_h^n - \omega_{hx}^n - (\psi(t_n) - \omega_x(t_n))\|^2 + \|3\varphi_h^n - u_h^n - (3\varphi(t_n) - u(t_n))\|^2 \\ &+ \|3s_{hx}^n - \psi_{hx}^n - (3s_x(t_n) - \psi_x(t_n))\|^2 + \|\varphi_h^n - \varphi(t_n)\|^2 + \|s_{hx}^n - s_x(t_n)\|^2 + \|s_h^n - s(t_n)\|^2 \\ &+ \|\xi_h^n + \vartheta_h^n + \theta_h^n - (\xi(t_n) + \vartheta(t_n) + \theta(t_n))\|^2 + \|\theta_{hx}^n + \vartheta_{hx}^n - (\theta_x(t_n) + \vartheta_x(t_n))\|^2 \\ &+ \|\theta_{hx}^n - \theta_x(t_n)\|^2\Big)^{\frac{1}{2}}. \end{aligned}$$

Observing the results, one can notice that the errors decrease when the discretization parameters are halved, achieving the linear convergence stated in Theorem 4.5.2. This behavior is also apparent in the curve shown in Figure 4.7.

Table 4.1 Computed errors when $T = 1$.

M	Δt	$Error$
20	1.00×10^{-3}	1.649
40	5.00×10^{-4}	8.243×10^{-1}
80	2.50×10^{-4}	4.132×10^{-1}
161	1.25×10^{-4}	2.091×10^{-1}
320	6.25×10^{-5}	1.095×10^{-1}
640	3.125×10^{-5}	6.371×10^{-2}

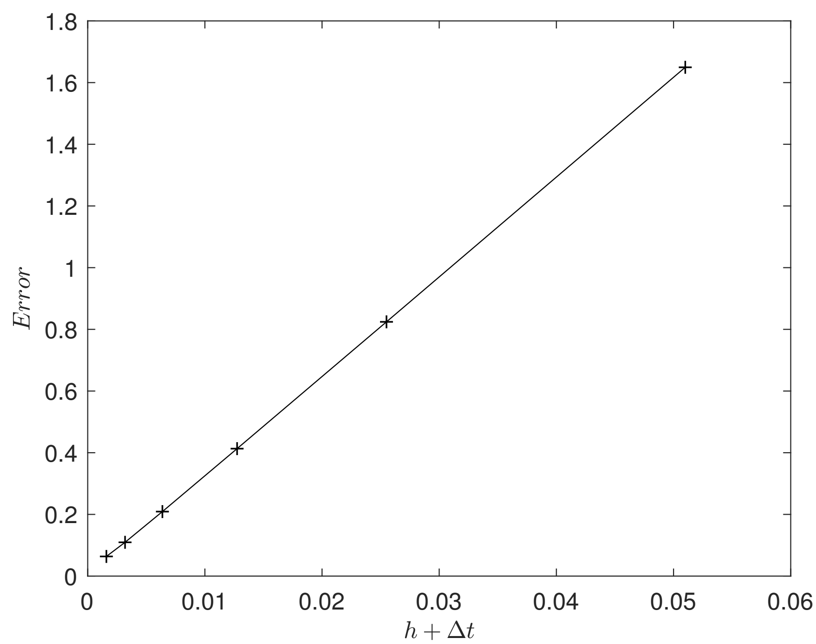


Figure 4.7 The evolution of the error depending on $h + \Delta t$.

Chapter 5

Long time behavior and numerical treatment of shear beam model subject to a delay

5.1 Introduction

As previously mentioned, Almeida Júnior et al. [89] and Ramos et al. [129] were among the first to study the well-posedness and stability properties of the Shear beam model. For the damped shear beam model shown below, the authors in [89] proved that the energy undergoes exponential decay no matter the wave speed is,

$$\begin{cases} \rho_1\varphi_{tt}\left(x,t\right)-K\left(\varphi_x+\psi\right)_x\left(x,t\right)+\mu\varphi_t\left(x,t\right)=0, \\ -b\psi_{xx}\left(x,t\right)+K\left(\varphi_x+\psi\right)\left(x,t\right)=0. \end{cases} \tag{5.1}$$

While the authors in [129] examined how feedback damping affected the rotational angle using semi-group techniques and proved that the system exhibits non-exponential stability. Aouragh et al. [9] applied the multiplier method to establish exponential stability for a nonlinear Shear beam system.

This chapter extends the findings from [89] to a Shear beam model with a delay, where we establish an exponential stability result. To be more specific, we consider the following Shear model system, which includes a damping term and an internal constant delay term in the transverse displacement:

$$\begin{cases} \rho_1\varphi_{tt}\left(x,t\right)-K\left(\varphi_x+\psi\right)_x\left(x,t\right)+\mu_1\varphi_t\left(x,t\right) \\ \qquad\qquad +\mu_2\varphi_t\left(x,t-\tau\right)=0,\ (x,t)\in(0,L)\times(0,\infty), \\ -b\psi_{xx}\left(x,t\right)+K\left(\varphi_x+\psi\right)\left(x,t\right)=0,\ (x,t)\in(0,L)\times(0,\infty), \\ \varphi(0,t)=\varphi(L,t)=\psi(0,t)=\psi(L,t)=0,\ t>0, \\ \varphi(x,0)=\varphi_0(x),\ \varphi_t(x,0)=\varphi_1(x),\ x\in(0,L), \\ \varphi_t(x,t-\tau)=f_0(x,t-\tau),\ (x,t)\in(0,L)\times(0,\tau), \end{cases} \tag{5.2}$$

where $\tau>0$ is a time delay, μ_1 is a positive constant, μ_2 is a real number, and the initial data $(\varphi_0,\varphi_1,f_0)$ belong to a suitable functional space.

The structure of this chapter is as follows: Section 2 presents preliminary considerations and our main results in Theorem 5.2.1. Sections 3 and 4 are dedicated of the main results. To demonstrate the global existence and uniqueness of the solution to problem (5.2), we employ the Faedo-Galerkin method combined with various estimates. Under the condition $|\mu_2| < \mu_1$, we use the energy method to construct a Lyapunov functional and establish an exponential decay. In Section 5, we propose a finite-element discretization of this problem. Discrete stability results and a priori error estimates are obtained. Finally, we present some numerical examples using MATLAB to demonstrate the accuracy of the algorithm and the behavior of the solution.

Through out this chapter we use the symbols C, C_i, to denote several positive constants.

5.2 Preliminaries and main results

In order to deal with the delay feedback term, we introduce as in [116, 136] the following new dependent variable

$$z(x,\rho,t) = \varphi_t(x, t-\tau\rho),\ x \in (0,L),\ \rho \in (0,1),\ t > 0,$$

then, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0,\ (x,\rho,t) \in (0,L) \times (0,1) \times (0,\infty).$$

Therefore, problem (5.2) can be rewritten as

$$\begin{cases} \rho_1 \varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) + \mu_1 \varphi_t(x,t) \\ +\mu_2 z(x,1,t) = 0,\ (x,t) \in (0,L) \times (0,\infty), \\ -b\psi_{xx}(x,t) + K(\varphi_x + \psi)(x,t) = 0,\ (x,t) \in (0,L) \times (0,\infty), \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0,\ (x,\rho,t) \in (0,L) \times (0,1) \times (0,\infty), \\ \varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0,\ t > 0, \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x),\ x \in (0,L), \\ z(x,\rho,0) = z_0(x,\rho) = f_0(x,-\tau\rho),\ (x,\rho) \in (0,L) \times (0,1). \end{cases} \tag{5.3}$$

In order to define the functional energy associated with the solution of the problem (5.3). We multiply the first equation in (5.3) by φ_t, the second equation by ψ_t and integrate by parts to get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\rho_1 \int_0^L \varphi_t^2 dx\right) + \frac{1}{2}\frac{d}{dt}\left(K \int_0^L \varphi_x^2 dx\right) + K\int_0^L \psi\varphi_{xt} dx \\ =& -\mu_1 \int_0^L \varphi_t^2 dx - \mu_2 \int_0^L \varphi_t z(x,1,t) dx, \end{aligned} \tag{5.4}$$

and

$$\frac{1}{2}\frac{d}{dt}\left(b\int_0^L \psi_x^2 dx\right) + \frac{1}{2}\frac{d}{dt}\left(K\int_0^L \psi^2 dx\right) + K\int_0^L \varphi_x\psi_t dx = 0. \tag{5.5}$$

Adding (5.4) and (5.5), we have

$$\frac{1}{2}\frac{d}{dt}\int_0^L \left(\rho_1\varphi_t^2 + K(\varphi_x+\psi)^2 + b\psi_x^2\right) dx = -\,\mu_1\int_0^L \varphi_t^2 dx - \mu_2\int_0^L \varphi_t z(x,1,t)dx. \tag{5.6}$$

Now, let ξ be a positive constant satisfying

$$\tau|\mu_2| < \xi < \tau(2\mu_1 - |\mu_2|). \tag{5.7}$$

Multiplying the last equation in (5.3) by $(\xi/\tau)z$ and integrating the result over $(0,L)\times(0,1)$, we obtain

$$\begin{aligned}\frac{\xi}{2}\frac{d}{dt}\int_0^L\int_0^1 z^2(x,\rho,t)d\rho\, dx &= -\frac{\xi}{2\tau}\int_0^L\int_0^1 \frac{\partial}{\partial\rho}z^2(x,\rho,t)d\rho\, dx \\ &= \frac{\xi}{2\tau}\int_0^L \left[z^2(x,0,t) - z^2(x,1,t)\right] dx.\end{aligned} \tag{5.8}$$

From (5.6) and (5.8), it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^L \left[\rho_1\varphi_t^2 + K(\varphi_x+\psi)^2 + b\psi_x^2\right] dx + \frac{\xi}{2}\frac{d}{dt}\int_0^L\int_0^1 z^2(x,\rho,t)d\rho\, dx \\ &= -\,\mu_1\int_0^L \varphi_t^2 dx - \mu_2\int_0^L \varphi_t z(x,1,t)dx + \frac{\xi}{2\tau}\int_0^L z^2(x,0,t)dx - \frac{\xi}{2\tau}\int_0^L z^2(x,1,t)dx,\end{aligned}$$

then,

$$E(t) = \frac{1}{2}\int_0^L \left[\rho_1\varphi_t^2 + K(\varphi_x+\psi)^2 + b\psi_x^2\right] dx + \frac{\xi}{2}\int_0^L\int_0^1 z^2(x,\rho,t)d\rho\, dx \tag{5.9}$$

and

$$\frac{dE(t)}{dt} = -\left(\mu_1 - \frac{\xi}{2\tau}\right)\int_0^L \varphi_t^2 dx - \mu_2\int_0^L \varphi_t z(x,1,t)dx - \frac{\xi}{2\tau}\int_0^L z^2(x,1,t)dx. \tag{5.10}$$

Exploiting Young's inequality, (5.10) can be rewritten as

$$\frac{dE(t)}{dt} \le -\left(\mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\int_0^L \varphi_t^2 dx - \left(\frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\int_0^L z^2(x,1,t)dx, \tag{5.11}$$

therefore, there exists $C \ge 0$ such that

$$\frac{dE(t)}{dt} \le -C\left(\int_0^L \varphi_t^2 dx + \int_0^L z^2(x,1,t)dx\right), \quad \forall\, t \ge 0. \tag{5.12}$$

Our main results is the following.

Theorem 5.2.1. *Assume that $|\mu_2| \leq \mu_1$ holds.*

1. *If the initial data $\varphi_0, \psi_0 \in H_0^1(0,L)$, $\varphi_1 \in L^2(0,L)$ and $f_0 \in L^2((0,L) \times (0,1))$. Then the problem (5.3) has a unique weak solution satisfying*

$$\begin{gathered} \varphi \in C([0,T], H_0^1(0,L)) \cap C^1([0,T], L^2(0,L)), \\ \psi \in C([0,T], H_0^1(0,L)),\ \varphi_t \in C([0,T], L^2(0,L)). \end{gathered} \tag{5.13}$$

2. *If the initial data $\varphi_0, \psi_0 \in H^2(0,L) \cap H_0^1(0,L)$, $\varphi_1 \in H_0^1(0,L)$ and $f_0 \in H_0^1((0,L) \times (0,1))$. Then the problem (5.3) has a unique strong solution satisfying*

$$\varphi,\ \psi \in L^\infty([0,T], H^2(0,L) \cap H_0^1(0,L)),\ \varphi_t \in L^\infty([0,T], H_0^1(0,L)). \tag{5.14}$$

3. *In both cases, with $|\mu_2| < \mu_1$, the energy $E(t)$ satisfies the following decay rate*

$$E(t) \leq \lambda_0 e^{-\lambda_1 t},\ \forall\, t \geq 0, \tag{5.15}$$

where λ_0 and λ_1 are positive constants.

5.3 Global well-posedness

In this section, we employ the Faedo-Galerkin method to construct a regular solution to the problem (5.3). The result extends to the weak solution, using a density argument.

Approximate problem

Let $\{v_i\}_{i\in\mathbb{N}}$ be the basis introduced in Chapter 3, Section 3.3. We define the function $\phi_i(x,\rho)$ by

$$\phi_i(x,0) = v_i(x), \tag{5.16}$$

then we can extend $\phi_i(x,0)$ by $\phi_i(x,\rho)$ over $H_0^1((0,L)\times(0,1))$ and for $m \in \mathbb{N}$, we denote

$$W_m = \{\phi_1,\ \phi_2,\ ...,\ \phi_m\}.$$

Given initial data φ_0, $\psi_0 \in H^2(0,L) \cap H_0^1(0,L)$, $\varphi_1 \in H_0^1(0,L)$ and $f_0 \in H_0^1((0,L)\times(0,1))$, we define the functions

$$\varphi^m(t) = \sum_{i=1}^m g_{im}(t) v_i,\ \psi^m(t) = \sum_{i=1}^m \hat{g}_{im}(t) v_i,\ z^m(x,\rho,t) = \sum_{i=1}^m f_{im}(t)\phi_i(x,\rho),$$

wich satisfy the following approximate problem

$$\begin{cases} \rho_1(\varphi_{tt}^m(t), v_i) + K(\varphi_x^m(t) + \psi^m(t), v_{ix}) + \mu_1(\varphi_t^m(t), v_i) + \mu_2(z^m(x,1,t), v_i) = 0, \\ b(\psi_x^m(t), v_{ix}) + K(\varphi_x^m(t) + \psi^m(t), v_i) = 0, \\ \tau(z_t^m(x,\rho,t), \phi_i) + (z_\rho^m(x,\rho,t), \phi_i) = 0, \end{cases} \tag{5.17}$$

with initial conditions

$$\varphi^m(0) = \varphi_0^m = \sum_{i=1}^m (\varphi_0, v_i) v_i \xrightarrow{m\to\infty} \varphi_0 \text{ in } H^2(0,L) \cap H_0^1(0,L), \tag{5.18}$$

$$\psi^m(0) = \psi_0^m = \sum_{i=1}^m (\psi_0, v_i) v_i \xrightarrow{m\to\infty} \psi_0 \text{ in } H^2(0,L) \cap H_0^1(0,L), \tag{5.19}$$

$$\varphi_t^m(0) = \varphi_1^m = \sum_{i=1}^m (\varphi_1, v_i) v_i \xrightarrow{m\to\infty} \varphi_1 \text{ in } H_0^1(0,L), \tag{5.20}$$

$$z^m(0) = z_0^m = \sum_{i=1}^m (z_0, \phi_i) \phi_i \xrightarrow{m\to\infty} f_0 \text{ in } H_0^1((0,L)\times(0,1)). \tag{5.21}$$

Substituting (φ^m, ψ^m, z^m) into (5.17), we obtain

$$\begin{cases} \rho_1 g_{im}'' + \mu_1 g_{im}' + K\varsigma_i g_{im} + \mu_2 \sum_{j=1}^m (\phi(x_j,1), v_i) f_{jm} = K \sum_{j=1}^m (v_{jx}, v_i) \hat{g}_{jm}, \\ (b\varsigma_i + K)\hat{g}_{im} + K \sum_{j=1}^m (v_{jx}, v_i) g_{jm} = 0, \\ \tau f_{im}' + \sum_{j=1}^m (\phi_{j\rho}, \phi_i) f_{jm} = 0. \end{cases} \tag{5.22}$$

Note that $(5.22)_2$ leads to

$$\hat{g}_{im} = -\frac{K}{b\varsigma_i + K} \sum_{j=1}^m (v_{jx}, v_i) g_{jm}, \tag{5.23}$$

then, (5.22) yields

$$\begin{cases} \rho_1 g_{im}'' + \mu_1 g_{im}' + K\varsigma_i g_{im} + \mu_2 \sum_{j=1}^m (\phi(x_j,1), v_i) f_{jm} \\ \qquad = -\sum_{j,k=1}^m \frac{K^2}{b\varsigma_j + K} (v_{jx}, v_i)(v_{kx}, v_j) \hat{g}_{km}, \\ \tau f_{im}' + \sum_{j=1}^m (\phi_{j\rho}, \phi_i) f_{jm} = 0. \end{cases} \tag{5.24}$$

By standard ordinary differential equations theory, the problem (5.24) has a unique solution $(g_{im}, f_{im}) \in C^2[0,T_m) \times C^1[0,T_m)$. Then, from (5.23), we infer that $\hat{g}_{im} \in C^2[0,T_m)$. Therefore, the approximate problem (5.17)-(5.21) has a unique local solution $(\varphi^m(t), \psi^m(t), z^m(t))$ in a maximal interval $[0,T_m)$ with $0 < T_m < T$. The estimate below will allow us to extend local solution $(\varphi^m(t), \psi^m(t), z^m(t))$ to the interval $[0,T]$, for any given $T > 0$.

A priori estimate 1

Multiplying the first equation of (5.17) by g'_{im}, the second by $\hat{g}'_{im}$ and the third by $(\xi/\tau)f_{im}$, summing up over i from 1 to m and taking the sum of the resulting equations, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\rho_1}{2}\|\varphi_t^m(t)\|^2 + \frac{K}{2}\|\varphi_x^m(t) + \psi^m(t)\|^2 + \frac{b}{2}\|\psi_x^m(t)\|^2\right) + \mu_1\|\varphi_t^m(t)\|^2 \\
&+ \mu_2\int_0^L z^m(x,1,t)\varphi_t^m(t)dx + \frac{d}{dt}\left(\frac{\xi}{2}\int_0^L\int_0^1 (z^m)^2(x,\rho,t)d\rho\, dx\right) \\
&- \frac{\xi}{2\tau}\|\varphi_t^m(t)\|^2 + \frac{\xi}{2\tau}\int_0^L (z^m)^2(x,1,t)dx = 0.
\end{aligned}
\tag{5.25}
$$

Integrating (5.25) over $(0,t)$, we find

$$
\begin{aligned}
&\mathcal{E}_m(t) + \left(\mu_1 - \frac{\xi}{2\tau}\right)\int_0^t \|\varphi_t^m(s)\|^2 ds + \frac{\xi}{2\tau}\int_0^t\int_0^L (z^m)^2(x,1,s)dx\, ds \\
&+ \mu_2\int_0^t\int_0^L z^m(x,1,s)\varphi_t^m(s)dx\, ds = \mathcal{E}_m(0),
\end{aligned}
\tag{5.26}
$$

where

$$
\begin{aligned}
\mathcal{E}_m(t) =& \frac{1}{2}\left(\rho_1\|\varphi_t^m(t)\|^2 + K\|\varphi_x^m(t) + \psi^m(t)\|^2 + b\|\psi_x^m(t)\|^2\right) \\
&+ \frac{\xi}{2}\int_0^L\int_0^1 (z^m)^2(x,\rho,t)d\rho\, dx.
\end{aligned}
\tag{5.27}
$$

Using Young's inequality, we have

$$
\begin{aligned}
\mu_2\int_0^t\int_0^L z^m(x,1,s)\varphi_t^m(s)dx\, ds \geq& -\frac{|\mu_2|}{2}\int_0^t\int_0^L (z^m)^2(x,1,s)dx\, ds \\
&- \frac{|\mu_2|}{2}\int_0^t \|\varphi_t^m(s)\|^2 ds,
\end{aligned}
\tag{5.28}
$$

which, together with (5.26), yields

$$
\begin{aligned}
&\mathcal{E}_m(t) + \left(\mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\int_0^t \|\varphi_t^m(s)\|^2 ds \\
&+ \left(\frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\int_0^t\int_0^L (z^m)^2(x,1,s)dx\, ds \leq \mathcal{E}_m(0).
\end{aligned}
\tag{5.29}
$$

The fact that
$$\left(\mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right) \geq 0 \quad \text{and} \quad \left(\frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right) \geq 0,$$
gives
$$\mathcal{E}_m(t) \leq \mathcal{E}_m(0). \tag{5.30}$$
In the case $\mu_1 = |\mu_2|$, for $\xi = \tau\mu_1$, we also have (5.30). Hence, in both cases, there exists C independent of m such that
$$\mathcal{E}_m(t) \leq C,\ t \geq 0.$$
Thus
$$\|\varphi_t^m(t)\|^2 + \|\varphi_x^m(t) + \psi^m(t)\|^2 + \|\psi_x^m(t)\|^2 + \int_0^L \int_0^1 (z^m)^2(x,\rho,t)d\rho\, dx \leq C,\ t \geq 0, \tag{5.31}$$
therefore, $T_m = T$, for all $T > 0$. The estimate (5.31) implies that, for any $m \in \mathbb{N}$,
$$\varphi^m,\ \psi^m \text{ are bounded in } L^\infty([0,T], H_0^1(0,L)), \tag{5.32}$$
$$\varphi_t^m \text{ is bounded in } L^\infty([0,T], L^2(0,L)), \tag{5.33}$$
$$z^m \text{ is bounded in } L^\infty([0,T], L^2((0,L)\times(0,1))). \tag{5.34}$$

A priori estimate 2

Differentiating the equations in (5.17) with respect to t, we get
$$\begin{cases} \rho_1(\varphi_{ttt}^m(t), v_i) + K(\varphi_{xt}^m(t) + \psi_t^m(t), v_{ix}) \\ +\mu_1(\varphi_{tt}^m(t), v_i) + \mu_2(z_t^m(x,1,t), v_i) = 0, \\ b(\psi_{xt}^m(t), v_{ix}) + K(\varphi_{xt}^m(t) + \psi_t^m(t), v_i) = 0, \\ \tau(z_{tt}^m(x,\rho,t), \phi_i) + (z_{\rho t}^m(x,\rho,t), \phi_i) = 0. \end{cases} \tag{5.35}$$
Multiplying the equations in (5.35) by $g_{im}''(t)$, $\hat{g}_{im}''(t)$ and $(\xi/\tau)f_{im}'$, respectively, and summing up over i from 1 to m, it follows that
$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi_{tt}^m(t)\|^2\right) + K\int_0^L (\varphi_{xt}^m(t) + \psi_t^m(t))\varphi_{xtt}^m(t)dx \\ &+ \mu_1\|\varphi_{tt}^m(t)\|^2 + \mu_2\int_0^L z_t^m(x,1,t)\varphi_{tt}^m(t)dx = 0, \end{aligned} \tag{5.36}$$

$$\frac{1}{2}\frac{d}{dt}\left(b\|\psi_{xt}^m(t)\|^2\right)+K\int_0^L(\varphi_{xt}^m(t)+\psi_t^m(t))\psi_{tt}^m(t)dx=0 \tag{5.37}$$

and

$$\frac{1}{2}\frac{d}{dt}\left(\xi\int_0^L\int_0^1(z_t^m)^2(x,\rho,t)d\rho\,dx\right)-\frac{\xi}{2\tau}\|\varphi_{tt}^m(t)\|^2+\frac{\xi}{2\tau}\int_0^L(z_t^m)^2(x,1,t)dx=0. \tag{5.38}$$

Taking the sum of (5.36)–(5.38), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\rho_1\|\varphi_{tt}^m(t)\|^2+K\|\varphi_{xt}^m(t)+\psi_t^m(t)\|^2+b\|\psi_{xt}^m(t)\|^2\\&+\xi\int_0^L\int_0^1(z_t^m)^2(x,\rho,t)d\rho\,dx\Big)+\left(\mu_1-\frac{\xi}{2\tau}\right)\|\varphi_{tt}^m(t)\|^2\\&+\mu_2\int_0^L z_t^m(x,1,t)\varphi_{tt}^m(t)dx+\frac{\xi}{2\tau}\int_0^L(z_t^m)^2(x,1,t)dx=0.\end{aligned} \tag{5.39}$$

Integrating the last equality over $(0,t)$, we get

$$\begin{aligned}&\mathcal{G}_m(t)+\left(\mu_1-\frac{\xi}{2\tau}\right)\int_0^t\|\varphi_{tt}^m(s)\|^2ds+\frac{\xi}{2\tau}\int_0^t\int_0^L(z_t^m)^2(x,1,s)dx\,ds\\&+\mu_2\int_0^t\int_0^L z_t^m(x,1,s)\varphi_{tt}^m(s)dx\,ds=\mathcal{G}_m(0),\end{aligned} \tag{5.40}$$

where

$$\begin{aligned}\mathcal{G}_m(t)=&\frac{1}{2}\left(\rho_1\|\varphi_{tt}^m(t)\|^2+K\|\varphi_{xt}^m(t)+\psi_t^m(t)\|^2+b\|\psi_{xt}^m(t)\|^2\right)\\&+\frac{\xi}{2}\int_0^L\int_0^1(z_t^m)^2(x,\rho,t)d\rho\,dx.\end{aligned} \tag{5.41}$$

Similarly to the first a priori estimate, we infer that there exists C independent of m such that

$$\mathcal{G}_m(t)\leq C,\ t\geq 0, \tag{5.42}$$

which implies

$$\|\varphi_{tt}^m(t)\|^2+\|\varphi_{xt}^m(t)+\psi_t^m(t)\|^2+\|\psi_{xt}^m(t)\|^2+\int_0^L\int_0^1(z_t^m)^2(x,\rho,t)d\rho\,dx\leq C$$

and for any $m\in\mathbb{N}$, we have

$$\varphi_t^m,\ \psi_t^m\ \text{ are bounded in }\ L^\infty([0,T],H_0^1(0,L)), \tag{5.43}$$

$$\varphi_{tt}^m\ \text{ is bounded in }\ L^\infty([0,T],L^2(0,L)), \tag{5.44}$$

$$z_t^m \text{ is bounded in } L^\infty([0,T], L^2((0,L)\times(0,1))). \tag{5.45}$$

A priori estimate 3

Replacing v_i by $-v_{ixx}$ in the first and second equations, and ϕ_i by $-\phi_{ixx}$ in the last equation of (5.17), then multiplying the resulting equations by $g'_{im}(t)$, $\hat{g}'_{im}(t)$ and $(\xi/\tau)f_{im}(t)$, respectively, and summing over i from 1 to m, we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi_{xt}^m(t)\|^2\right)+K\int_0^L(\varphi_x^m(t)+\psi^m(t))_x\varphi_{xxt}^m(t)dx\\&+\mu_1\|\varphi_{xt}^m(t)\|^2+\mu_2\int_0^L z_x^m(x,1,t)\varphi_{xt}^m(t)dx=0,\end{aligned} \tag{5.46}$$

$$\frac{1}{2}\frac{d}{dt}\left(b\|\psi_{xx}^m(t)\|^2\right)-K\int_0^L(\varphi_x^m(t)+\psi^m(t))\psi_{xxt}^m(t)dx=0 \tag{5.47}$$

and

$$\frac{1}{2}\frac{d}{dt}\left(\xi\int_0^L\int_0^1(z_x^m)^2(x,\rho,t)d\rho\, dx\right)-\frac{\xi}{2\tau}\|\varphi_{xt}^m(t)\|^2+\frac{\xi}{2\tau}\int_0^L(z_x^m)^2(x,1,t)dx=0. \tag{5.48}$$

Applying Young's inequality, the last term in (5.46) gives

$$\mu_2\int_0^L z_x^m(x,1,t)\varphi_{xt}^m(t)dx\geq-\frac{|\mu_2|}{2}\int_0^L(z_x^m)^2(x,1,t)dx-\frac{|\mu_2|}{2}\|\varphi_{xt}^m(t)\|^2. \tag{5.49}$$

Taking into acount (5.49), summing (5.46)-(5.48) and integrating over $(0,t)$, we arrive at

$$\begin{aligned}&\mathcal{H}_m(t)+\left(\mu_1-\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)\int_0^t\|\varphi_{xt}^m(s)\|^2ds\\&+\left(\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)\int_0^t\int_0^L(z_x^m)^2(x,1,s)dx\, ds\leq\mathcal{H}_m(0),\end{aligned} \tag{5.50}$$

where

$$\begin{aligned}\mathcal{H}_m(t)=&\frac{1}{2}\left(\rho_1\|\varphi_{xt}^m(t)\|^2+K\|\varphi_{xx}^m(t)+\psi_x^m(t)\|^2+b\|\psi_{xx}^m(t)\|^2\right)\\&+\frac{\xi}{2}\int_0^L\int_0^1(z_x^m)^2(x,\rho,t)d\rho\, dx,\end{aligned} \tag{5.51}$$

then,

$$\mathcal{H}_m(t)\leq\mathcal{H}_m(0).$$

Therefore, there exists C independent of m such that

$$\mathcal{H}_m(t)\leq C,\ t\geq 0$$

and, consequently,

$$\|\varphi_{xt}^m(t)\|^2 + \|\varphi_{xx}^m(t) + \psi_x^m(t)\|^2 + \|\psi_{xx}^m\|^2 + \int_0^L \int_0^1 (z_x^m)^2(x,\rho,t)d\rho\, dx \le C.$$

For all $m \in \mathbb{N}$, this estimate implies that

$$\varphi^m,\ \psi^m \text{ are bounded in } L^\infty([0,T], H^2(0,L)\cap H_0^1(0,L)), \tag{5.52}$$

$$z^m \text{ is bounded in } L^\infty([0,T], H_0^1((0,L)\times(0,1))). \tag{5.53}$$

Passage to the limit

In light of (5.32), (5.33), (5.34), (5.43), (5.44), (5.45), (5.52) and (5.53), and up to a subsequence, we have

$$\begin{cases} \varphi^m \rightharpoonup^* \varphi \quad \text{in } L^2([0,T], H^2(0,L)\cap H_0^1(0,L)), \\ \varphi_t^m \rightharpoonup^* \varphi_t \quad \text{in } L^2([0,T], H_0^1(0,L)), \\ \varphi_{tt}^m \rightharpoonup^* \varphi_{tt} \quad \text{in } L^2([0,T], L^2(0,L)), \\ \psi^m \rightharpoonup^* \psi \quad \text{in } L^2([0,T], H^2(0,L)\cap H_0^1(0,L)), \\ \psi_t^m \rightharpoonup^* \psi_t \quad \text{in } L^2([0,T], H_0^1(0,L)), \\ z^m \rightharpoonup^* z \quad \text{in } L^2([0,T], H_0^1((0,L)\times(0,1))), \\ z_t^m \rightharpoonup^* z_t \quad \text{in } L^2([0,T], L^2((0,L)\times(0,1))). \end{cases} \tag{5.54}$$

From the above limits we conclude that $(\varphi, \varphi_t, \psi, z)$ has the following regularity

$$\varphi, \psi \in L^\infty([0,T], H^2(0,L)\cap H_0^1(0,L)),\ \varphi_t \in L^\infty([0,T], H_0^1(0,L)),$$
$$z \in L^\infty([0,T], H_0^1((0,L)\times(0,1))).$$

Using the compact embedding ($H_0^1 \hookrightarrow L^2$ and $H^2 \cap H_0^1 \hookrightarrow H_0^1$) and Lions [98] (Chapter 1, Theorem 5.1), we get

$$\begin{cases} \varphi^m \longrightarrow \varphi \quad \text{in } L^2([0,T], H_0^1(0,L)), \\ \varphi_t^m \longrightarrow \varphi_t \quad \text{in } L^2([0,T], L^2(0,L)), \\ \psi^m \longrightarrow \psi \quad \text{in } L^2([0,T], H_0^1(0,L)), \\ z^m \longrightarrow z \quad \text{in } L^2([0,T], L^2((0,L)\times(0,1))). \end{cases} \tag{5.55}$$

With these limits, we arrive at

$$\int_0^t (\varphi_x^m(t)+\psi^m(t),v_{ix})ds \longrightarrow \int_0^t (\varphi_x(t)+\psi(t),v_{ix})ds,$$
$$\int_0^t (\varphi_t^m(t),v_i)ds \longrightarrow \int_0^t (\varphi_t(t),v_i)ds,$$
$$\int_0^t (z^m(x,1,t),v_i)ds \longrightarrow \int_0^t (z(x,1,t),v_i)ds,$$

$$\int_0^t (\psi_x^m(t),v_{ix})ds \longrightarrow \int_0^t (\psi_x(t),v_{ix})ds,$$
$$\int_0^t (\varphi_x^m(t)+\psi^m(t),v_i)\,ds \longrightarrow \int_0^t (\varphi_x(t)+\psi(t),v_i)\,ds,$$
$$\int_0^t (z_\rho^m(x,\rho,t),\phi_i)ds \longrightarrow \int_0^t (z_\rho(x,\rho,t),\phi_i)ds$$

and for any $t\in[0,T]$

$$(\varphi_t^m(t),v_i) \longrightarrow (\varphi_t^m(t),v_i),\ (z^m(x,\rho,t),\phi_i) \longrightarrow (z(x,\rho,t),\phi_i).$$

Integrating the equations of (5.17) over $(0,t)$, we obtain

$$\begin{cases} \rho_1(\varphi_t^m(t),v_i)-\rho_1(\varphi_1^m,v_i)+K\displaystyle\int_0^t(\varphi_x^m(t)+\psi^m(t),v_{ix})ds \\ \qquad +\mu_1\displaystyle\int_0^t(\varphi_t^m(t),v_i)ds+\mu_2\int_0^t(z^m(x,1,t),v_i)ds=0, \\ b\displaystyle\int_0^t(\psi_x^m(t),v_{ix})ds+K\int_0^t(\varphi_x^m(t)+\psi^m(t),v_i)\,ds=0, \\ \tau(z^m(x,\rho,t),\phi_i)-\tau(z_0^m,\phi_i)+\displaystyle\int_0^t(z_\rho^m(x,\rho,t),\phi_i)ds=0, \end{cases}$$

then, from the above limits, (5.20) and (5.21), we have

$$\begin{cases} \rho_1(\varphi_t(t),v_i)-\rho_1(\varphi_1,v_i)+K\displaystyle\int_0^t(\varphi_x(t)+\psi(t),v_{ix})ds \\ \qquad +\mu_1\displaystyle\int_0^t(\varphi_t(t),v_i)ds+\mu_2\int_0^t(z(x,1,t),v_i)ds=0, \\ b\displaystyle\int_0^t(\psi_x(t),v_{ix})ds+K\int_0^t(\varphi_x(t)+\psi(t),v_i)\,ds=0, \\ \tau(z(x,\rho,t),\phi_i)-\tau(z_0,\phi_i)+\displaystyle\int_0^t(z_\rho(x,\rho,t),\phi_i)ds=0. \end{cases}$$

Using the density of $\{v_i\}_{i\in\mathbb{N}}$ in $H^2(0,L)\cap H_0^1(0,L)$ and in $\{\phi_i\}_{i\in\mathbb{N}}$ in $H_0^1((0,L)\times(0,1))$, we get

$$\begin{cases} \rho_1(\varphi_t(t),v_1)-\rho_1(\varphi_1,v_1)+K\displaystyle\int_0^t(\varphi_x(t)+\psi(t),v_{1x})ds \\ \qquad +\mu_1\displaystyle\int_0^t(\varphi_t(t),v_1)ds+\mu_2\int_0^t(z(x,1,t),v_1)ds=0, \\ b\displaystyle\int_0^t(\psi_x(t),v_{1x})ds+K\int_0^t(\varphi_x(t)+\psi(t),v_1)\,ds=0, \\ \tau(z(x,\rho,t),\phi_1)-\tau(z_0,\phi_1)+\displaystyle\int_0^t(z_\rho(x,\rho,t),\phi_1)ds=0, \end{cases} \tag{5.56}$$

for any $(v_1,\phi_1)\in(H^2(0,L)\cap H_0^1(0,L))\times H_0^1((0,L)\times(0,1))$. Now, the differentiation of the equations in (5.56) leads to

$$\begin{cases} \rho_1(\varphi_{tt}(t),v_1)+K(\varphi_x(t)+\psi(t),v_{1x}) \\ \qquad +\mu_1(\varphi_t(t),v_1)+\mu_2(z(x,1,t),v_1)=0, \\ b(\psi_x(t),v_{1x})+K(\varphi_x(t)+\psi(t),v_1)=0, \\ \tau(z_t(x,\rho,t),\phi_1)+(z_\rho(x,\rho,t),\phi_1)=0, \end{cases} \tag{5.57}$$

for any $(v_1,\phi_1)\in(H^2(0,L)\cap H_0^1(0,L))\times H_0^1((0,L)\times(0,1))$. Therefore $(\varphi,\varphi_t,\psi,z)$ is a strong solution of (5.3).

It remains to show that the strong solution satisfies the initial conditions in (5.3). By using $(5.55)_1$, $(5.55)_2$ and Aubins-Lions-Simon Theorem 2.1.24, we deduce that we can extract a subsequence still denoted by (φ^m) such that

$$\varphi^m\longrightarrow\varphi(0)\quad\text{in }\ C([0,T],L^2(0,L)),$$

together with (5.57) lead to

$$\varphi(0)=\varphi_0.$$

Now, integrating $(5.57)_1$ over $(0,T)$ and taking $v_1=\zeta(t)v(x)$ where $v\in H^2(0,L)\cap H_0^1(0,L)$ and $\zeta\in C^\infty([0,T])$ with $\zeta(0)=1$ and $\zeta(T)=0$, we find

$$\begin{aligned} &\rho_1\int_0^T(\varphi_{tt}(t),\zeta(t)v(x))dt+K\int_0^T(\varphi_x(t)+\psi(t),\zeta(t)v_x(x))dt \\ &+\mu_1\int_0^T(\varphi_t(t),\zeta(t)v(x))dt+\mu_2\int_0^T(z(x,1,t),\zeta(t)v(x))dt=0, \end{aligned}$$

the integration by parts of the first term gives

$$\begin{aligned} \rho_1(\varphi_t(0),v(x))=&-\rho_1\int_0^T(\varphi_t(t),\zeta_t(t)v(x))dt+K\int_0^T(\varphi_x(t)+\psi(t),\zeta(t)v_x(x))dt \\ &+\mu_1\int_0^T(\varphi_t(t),\zeta(t)v(x))dt+\mu_2\int_0^T(z(x,1,t),\zeta(t)v(x))dt. \end{aligned} \tag{5.58}$$

On the other hand, multiplying $(5.17)_1$ by the function $\zeta(t)$, and integrate over $(0,T)$, we get

$$\begin{aligned}\rho_1(\varphi_1^m, v_i) =& -\rho_1\int_0^T(\varphi_t^m(t),\zeta_t(t)v_i)dt + K\int_0^T(\varphi_x^m(t)+\psi^m(t),\zeta(t)v_{ix})dt\\ &+\mu_1\int_0^T(\varphi_t^m(t),\zeta(t)v_i)dt+\mu_2\int_0^T(z^m(x,1,t),\zeta(t)v_i)dt,\end{aligned}$$

passing to the limit and keeping in mind (5.20) and (5.55), we infer

$$\begin{aligned}\rho_1(\varphi_1, v_i) =& -\rho_1\int_0^T(\varphi_t(t),\zeta_t(t)v_i)dt + K\int_0^T(\varphi_x(t)+\psi(t),\zeta(t)v_{ix})dt\\ &+\mu_1\int_0^T(\varphi_t(t),\zeta(t)v_i)dt+\mu_2\int_0^T(z(x,1,t),\zeta(t)v_i)dt,\end{aligned} \tag{5.59}$$

wich remains valid for all $v \in H^2(0,L)\cap H_0^1(0,L)$ by density argument. Comparing equations (5.58) and (5.59), we deduce that

$$\varphi_t(0) = \varphi_1.$$

Next, selecting $\phi_1 = \zeta(t)\phi(x)$ in the last equation of (5.57), where $\phi \in H_0^1((0,L)\times(0,1))$ and ζ chosen as previously introduced, and applying the same procedure as above, we obtain

$$z(0) = z_0 = f_0.$$

Continuous dependence and uniqueness

Let $(\varphi,\varphi_t,\psi,z)$ and $(\tilde\varphi,\tilde\varphi_t,\tilde\psi,\tilde z)$ be the strong solutions of the problem (5.3) with respect to initial data $(\varphi_0,\varphi_1,\psi_0,z_0)$ and $(\tilde\varphi_0,\tilde\varphi_1,\tilde\psi_0,\tilde z_0)$, respectively. Then, $(\Lambda,\Lambda_t,X,Z) = (\varphi-\tilde\varphi,\varphi_t-\tilde\varphi_t,\psi-\tilde\psi,z-\tilde z)$ verifies (5.3) and we have the following equations

$$\rho_1\Lambda_{tt}(x,t) - K(\Lambda_x+X)_x(x,t)+\mu_1\Lambda_t(x,t)+\mu_2 Z(x,1,t) = 0, \tag{5.60}$$

$$-bX_{xx}(x,t)+K(\Lambda_x+X)(x,t) = 0, \tag{5.61}$$

$$\tau Z_t(x,\rho,t)+Z_\rho(x,\rho,t) = 0, \tag{5.62}$$

with the initial data

$$\Lambda(0)=\varphi(0)-\tilde\varphi(0),\ \Lambda_t(0)=\varphi_t(0)-\tilde\varphi_t(0),\ X(0)=\psi(0)-\tilde\psi(0),\ Z(0)=z(0)-\tilde z(0).$$

Multiplying (5.60) by Λ_t, (5.61) by X_t and (5.62) by $(\xi/\tau)Z$, and then integrating the resulting expressions over $(0,L)$. Applying the same arguments used to derive the estimate (5.11) give

$$\frac{d\tilde E(t)}{dt} \le -\left(\mu_1-\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)\int_0^L\Lambda_t^2dx-\left(\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)\int_0^L Z^2(x,1,t)dx. \tag{5.63}$$

where

$$\tilde{E}(t) = \frac{1}{2}\int_0^L \left[\rho_1 \Lambda_t^2 + K(\Lambda_x + X)^2 + bX_x^2\right] dx + \frac{\xi}{2}\int_0^L \int_0^1 Z^2(x,\rho,t)d\rho\, dx.$$

Integrating (5.63) over $(0,t)$ to get

$$\tilde{E}(t) \leq \tilde{E}(0) + \left(\frac{\xi}{2\tau} + \frac{|\mu_2|}{2}\right)\int_0^t \|\Lambda_t(s)\|^2 ds,$$

this implies that for some constant $C > 0$

$$\tilde{E}(t) \leq \tilde{E}(0) + C\int_0^t \tilde{E}(t)ds$$

and by Gronwall inequality, we conclude that there exists a positive constant $C_T = e^{CT}$ such that for any $t \in [0,T]$

$$\tilde{E}(t) \leq C_T \tilde{E}(0),$$

this shows that the strong solution of the problem (5.3) depends continuously on the initial data. As a result, this solution is unique.

For weak solutions, the application of a regularization method, as in [97, Chapter 3, Section 8.2], enables the proof of the continuous dependence and uniqueness .

The proof of items (i) and (ii) of Theorem 5.2.1 is now complete.

5.4 Exponential stability

In this section, we establish the decay property, in the case $|\mu_2| < \mu_1$, of the solution for the system (5.3) with the regularity stated in Theorem 5.2.1. The same holds for weak solution through the application of standard density arguments.

Using the energy method, we construct a Lyapunov functional L that is equivalent to E. For this, we define several functionals that allow us to obtain the required estimates.

Lemma 5.4.1. *The functional*

$$I_1(t) = \rho_1 \int_0^L \varphi_t \varphi dx + \frac{\mu_1}{2}\int_0^L \varphi^2 dx$$

satisfies, for any ε_1*,*

$$\begin{aligned}\frac{dI_1(t)}{dt} \leq& -b\int_0^L \psi_x^2 dx - K\int_0^L (\varphi_x + \psi)^2 dx + C_1\int_0^L \varphi_t^2 dx \\ &+ \varepsilon_1 \int_0^L \varphi_x^2 dx + \frac{C_1}{\varepsilon_1}\int_0^1 z^2(x,1,t)dx.\end{aligned} \tag{5.64}$$

Proof. Multiplying $(5.3)_1$ by φ and using the fact that $\varphi_{tt}\varphi = \frac{\partial}{\partial t}(\varphi_t \varphi) - \varphi_t^2$, we end up with

$$\begin{aligned} &\frac{d}{dt}\int_0^L (\rho_1 \varphi_t \varphi + \frac{\mu_1}{2}\varphi^2)dx - \rho_1 \int_0^L \varphi_t^2 dx \\ &+ K\int_0^L (\varphi_x + \psi)\varphi_x dx + \mu_2 \int_0^L z(x,1,t)\varphi dx = 0. \end{aligned} \tag{5.65}$$

Similarly, multiplying $(5.3)_2$ by ψ, we get

$$b\int_0^L \psi_x^2 dx + K\int_0^L (\varphi_x + \psi)\psi dx = 0. \tag{5.66}$$

Adding up (5.65) and (5.66), we arrive at

$$\frac{dI_1(t)}{dt} = -\, b\int_0^L \psi_x^2 dx - K\int_0^L (\varphi_x+\psi)^2 dx + \rho_1 \int_0^L \varphi_t^2 dx - \mu_2 \int_0^L z(x,1,t)\varphi dx. \tag{5.67}$$

Application of Young's and Poincaré's inequalities yields

$$-\mu_2 \int_0^L z(x,1,t)\varphi dx \le \frac{\mu_2^2 c_p}{4\varepsilon_1}\int_0^L z^2(x,1,t)dx + \varepsilon_1 \int_0^L \varphi_x^2 dx. \tag{5.68}$$

By substituting (5.68) into (5.67), we obtain (5.64). □

Lemma 5.4.2. *The functional*

$$I_2(t) = \rho_1 \int_0^L \varphi_t \varphi dx$$

satisfies

$$\frac{dI_2(t)}{dt} \le -\frac{K}{2}\int_0^L \varphi_x^2 dx + C_2 \int_0^L (\varphi_t^2 + \psi_x^2 + z^2(x,1,t))dx. \tag{5.69}$$

Proof. Differentiating I_2, we get

$$\frac{dI_2(t)}{dt} = \rho_1 \int_0^L \varphi_{tt}\varphi dx + \rho_1 \int_0^L \varphi_t^2 dx,$$

then, by using $(5.3)_1$ and integrating by parts, we find

$$\begin{aligned} \frac{dI_2(t)}{dt} = &- K\int_0^L \varphi_x^2 dx - K\int_0^L \psi\varphi_x dx - \mu_1 \int_0^L \varphi_t \varphi dx \\ &- \mu_2 \int_0^L z(x,1,t)\varphi dx + \rho_1 \int_0^L \varphi_t^2 dx. \end{aligned} \tag{5.70}$$

Young's and Poincaré's inequalities lead to

$$-K\int_0^L \psi\varphi_x dx \le \frac{3Kc_p}{2}\int_0^L \psi_x^2 dx + \frac{K}{6}\int_0^L \varphi_x^2 dx, \tag{5.71}$$

$$-\mu_1\int_0^L \varphi_t\varphi dx \le \frac{3\mu_1^2 c_p}{2K}\int_0^L \varphi_t^2 dx + \frac{K}{6}\int_0^L \varphi_x^2 dx, \tag{5.72}$$

$$-\mu_2\int_0^L z(x,1,t)\varphi dx \le \frac{3\mu_2^2 c_p}{2K}\int_0^L z^2(x,1,t)dx + \frac{K}{6}\int_0^L \varphi_x^2 dx. \tag{5.73}$$

Plugging (5.71)-(5.73) into (5.70) gives (5.69). □

Lemma 5.4.3. *The functional*

$$I_3(t) = \int_0^L\int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)d\rho\, dx$$

satisfies

$$\frac{dI_3(t)}{dt} \le -2I_3(t) - \frac{C_3}{\tau}\int_0^L z^2(x,1,t)dx + \frac{1}{\tau}\int_0^L \varphi_t^2 dx. \tag{5.74}$$

Proof. Differentiating I_3 and using $(5.3)_3$, we have

$$\begin{aligned}
\frac{d}{dt}\left(\int_0^L\int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)d\rho\, dx\right) =& -\frac{2}{\tau}\int_0^L\int_0^1 e^{-2\tau\rho} z z_\rho(x,\rho,t)d\rho\, dx\\
=& -2\int_0^L\int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)d\rho\, dx\\
&-\frac{1}{\tau}\int_0^L\int_0^1 \frac{\partial}{\partial\rho}\left(e^{-2\tau\rho} z^2(x,\rho,t)\right)d\rho\, dx.
\end{aligned}$$

This latter estimate implies the existence of a constant C_3 for which inequality (5.74) is fulfilled. □

Now, we consider the following Lyapunov functional defined by

$$L(t) = NE(t) + N_1 I_1(t) + I_2(t) + I_3(t),$$

where N and N_1 are positive constants to be fixed later. Then taking into account the energy dissipation (5.12), it follows that

$$\begin{aligned}
\frac{dL(t)}{dt} \le& -KN_1\int_0^L (\varphi_x+\psi)^2 dx - [bN_1 - C_2]\int_0^L \psi_x^2 dx\\
&-\left[\frac{K}{2} - \varepsilon_1 N_1\right]\int_0^L \varphi_x^2 dx - \left[NC - C_1N_1 - C_2 - \frac{1}{\tau}\right]\int_0^L \varphi_t^2 dx\\
&-2I_3(t) - \left[NC + \frac{C_3}{\tau} - \frac{C_1}{\varepsilon_1}N_1 - C_2\right]\int_0^1 z^2(x,1,t)dx.
\end{aligned}$$

We begin by setting $\varepsilon_1 = \dfrac{K}{4N_1}$ to get

$$\begin{aligned}\frac{dL(t)}{dt} \leq & - KN_1 \int_0^L (\varphi_x + \psi)^2 dx - \frac{K}{4} \int_0^L \varphi_x^2 dx \\ & - [bN_1 - C_2] \int_0^L \psi_x^2 dx - \left[NC - C_1N_1 - C_2 - \frac{1}{\tau}\right] \int_0^L \varphi_t^2 dx \\ & - 2I_3(t) - \left[NC + \frac{C_3}{\tau} - \frac{4C_1}{K}N_1^2 - C_2\right] \int_0^L z^2(x,1,t)dx.\end{aligned} \tag{5.75}$$

Next, we choose N_1 large enough such that

$$bN_1 - C_2 > 0,$$

then, we pick N large enough so that

$$\begin{cases} NC - C_1N_1 - C_2 - \dfrac{1}{\tau} > 0, \\ NC + \dfrac{C_3}{\tau} - \dfrac{4C_1}{K}N_1^2 - C_2 > 0, \end{cases}$$

As a result, we infer that a constant $\eta_1 > 0$ exists such that

$$\frac{dL(t)}{dt} \leq -\eta_1 \int_0^L \left[\varphi_t^2 + (\varphi_x + \psi)^2 + \psi_x^2\right] dx - \eta_1 \int_0^L \int_0^1 z^2(x,\rho,t)d\rho\, dx.$$

Comparing with (5.9), we have for some $\eta_2 > 0$,

$$\frac{dL(t)}{dt} \leq -\eta_2 E(t),\ \forall\, t \geq 0. \tag{5.76}$$

On the other hand, it is easy to see there exist two positive constants β_1 and β_2 such that

$$\beta_1 E(t) \leq L(t) \leq \beta_1 E(t),\ \forall\, t \geq 0. \tag{5.77}$$

Combining (5.76) and (5.77), we conclude for some $\lambda_1 > 0$,

$$\frac{dL(t)}{dt} \leq -\lambda_1 L(t),\ \forall\, t \geq 0,$$

a simple integration over $(0, t)$ leads to

$$L(t) \leq L(0)e^{-\lambda_1 t},\ \forall\, t \geq 0.$$

Consequently, the equivalence of L and E, yields the estimate (5.15), and then the proof of Theorem 5.2.1 is now complete.

5.5 Numerical approximation

In this section, we propose a numerical approximation to the solution of the continuous problem (5.3), where we study the stability of the scheme and analysis of the error. After that, some numerical simulations are performed.

5.5.1 Stability of the scheme

To acquire the weak formulation, we multiply the equations in (5.3) by the test functions $\bar{\varphi}$, $\bar{\psi}$ and $\bar{z}$, respectively, then integrate by parts, where $\Phi = \varphi_t$, to obtain

$$\begin{cases} \rho_1(\Phi_t, \bar{\varphi}) + K(\varphi_x + \psi, \bar{\varphi}_x) + \mu_1(\Phi, \bar{\varphi}) + \mu_2(\Phi(t-\tau), \bar{\varphi}) = 0, \\ b(\psi_x, \bar{\psi}_x) + K(\varphi_x + \psi, \bar{\psi}) = 0, \\ \tau(z_t, \bar{z}) + (z_\rho, \bar{z}) = 0. \end{cases} \tag{5.78}$$

Here, $(.,.)$ is the inner product in $L^2(\Omega)$ where $\Omega = (0, L)$ for φ and ψ, and $\Omega = (0, L) \times (0, 1)$ for z.

The mesh of a given delay $\tau = M\Delta t$ is $t_n = n\Delta t,\ n = -M, -M+1, ..., 0,\ 0 < M < N$. We define the discrete space

$$S_h^* = \left\{\sigma_h \in H_0^1(I \times (0,1)) : \forall\, [x_i, x_{i+1}] \in \mathit{\Gamma}_h,\ \sigma_h|_{[x_i,x_{i+1}]} \in P_1([x_i, x_{i+1}])\right\}.$$

By using the implicit Euler scheme, the finite element approximation of the variational problem (5.78) is written as follows:

For $n = 1, ..., N$, find $(\Phi_h^n, \psi_h^n, z_h^n) \in S_h^0 \times S_h^0 \times S_h^*$, such that for all $\bar{\varphi}_h$, $\bar{\psi}_h$ and $\bar{z}_h$, we obtain

$$\begin{cases} \dfrac{\rho_1}{\Delta t}(\Phi_h^n - \Phi_h^{n-1}, \bar{\varphi}_h) + K(\varphi_{hx}^n + \psi_h^n, \bar{\varphi}_{hx}) \\ \quad +\mu_1(\Phi_h^n, \bar{\varphi}_h) + \mu_2(\Phi_h^{n-M}, \bar{\varphi}_h) = 0, \\ b(\psi_{hx}^n, \bar{\psi}_{hx}) + K(\varphi_{hx}^n + \psi_h^n, \bar{\psi}_h) = 0, \\ \dfrac{\tau}{\Delta t}(z_h^n - z_h^{n-1}, \bar{z}_h) + (z_{h\rho}^n, \bar{z}_h) = 0, \end{cases} \tag{5.79}$$

where

$$\varphi_h^n = \varphi_h^{n-1} + \Delta t\ \Phi_h^n.$$

Here φ_h^0, Φ_h^0, ψ_h^0 and f_h^0 are approximations to φ_0, φ_1, ψ_0 and f_0, respectively.

We introduce the following discrete energy

$$E^n = \frac{1}{2}\left(\rho_1\|\Phi_h^n\|^2 + K\|\varphi_{hx}^n + \psi_h^n\|^2 + b\|\psi_{hx}^n\|^2 + \xi\|z_h^n\|^2\right). \tag{5.80}$$

Theorem 5.5.1. *Assuming (5.7) holds, the discrete energy (5.80) satisfies*

$$\frac{E^n - E^{n-1}}{\Delta t} \leq 0, \quad n = 1, ..., N.$$

Proof. Taking $\bar{\varphi}_h = \Phi_h^n$, $\bar{\psi}_h = \Psi_h^n$ (where $\Psi = \psi_t$) and $\bar{z}_h = (\xi/\tau) z_h^n$ in the scheme, and considering the fact that

$$(a - b, a) = \frac{1}{2}\left(\|a-b\|^2 + \|a\|^2 - \|b\|^2\right), \tag{5.81}$$

we arrive at

$$\begin{cases} \frac{\rho_1}{2\Delta t}\left(\|\Phi_h^n - \Phi_h^{n-1}\|^2 + \|\Phi_h^n\|^2 - \|\Phi_h^{n-1}\|^2\right) \\ + K(\varphi_{hx}^n + \psi_h^n, \Phi_{hx}^n) + \mu_1\|\Phi_h^n\|^2 + \mu_2(\Phi_h^{n-M}, \Phi_h^n) = 0, \\ b(\psi_{hx}^n, \Psi_{hx}^n) + K(\varphi_{hx}^n + \psi_h^n, \Psi_h^n) = 0, \\ \frac{\xi}{2\Delta t}\left(\|z_h^n - z_h^{n-1}\|^2 + \|z_h^n\|^2 - \|z_h^{n-1}\|^2\right) + \frac{\xi}{\tau}(z_{h\rho}^n, z_h^n) = 0. \end{cases} \tag{5.82}$$

Young's inequality gives the following estimate

$$\mu_2(\Phi_h^{n-M}, \Phi_h^n) \geq -\frac{|\mu_2|}{2}\|\Phi_h^{n-M}\|^2 - \frac{|\mu_2|}{2}\|\Phi_h^n\|^2. \tag{5.83}$$

Thanks to (5.81), we observe that

$$\begin{aligned} K(\varphi_{hx}^n + \psi_h^n, \Phi_{hx}^n + \Psi_h^n) &= \frac{K}{\Delta t}(\varphi_{hx}^n + \psi_h^n, \varphi_{hx}^n + \psi_h^n - (\varphi_{hx}^{n-1} + \psi_h^{n-1})) \\ &\geq \frac{K}{2\Delta t}\left(\|\varphi_{hx}^n + \psi_h^n\|^2 - \|\varphi_{hx}^{n-1} + \psi_h^{n-1}\|^2\right) \end{aligned} \tag{5.84}$$

and

$$\begin{aligned} b(\psi_{hx}^n, \Psi_{hx}^n) &= \frac{b}{\Delta t}(\psi_{hx}^n, \psi_{hx}^n - \psi_{hx}^{n-1}) \\ &\geq \frac{b}{2\Delta t}\left(\|\psi_{hx}^n\|^2 - \|\psi_{hx}^{n-1}\|^2\right). \end{aligned} \tag{5.85}$$

Clearly we have

$$\frac{\xi}{\tau}(z_{h\rho}^n, z_h^n) = \frac{\xi}{2\tau}\|\Phi_h^{n-M}\|^2 - \frac{\xi}{2\tau}\|\Phi_h^n\|^2. \tag{5.86}$$

Now, adding up the equations in (5.82) together with (5.83), (5.84), (5.85) and (5.86), we find

$$\begin{aligned} &\frac{\rho_1}{2\Delta t}\left(\|\Phi_h^n - \Phi_h^{n-1}\|^2 + \|\Phi_h^n\|^2 - \|\Phi_h^{n-1}\|^2\right) + \left(\mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\|\Phi_h^n\|^2 \\ &+ \left(\frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\|\Phi_h^{n-M}\|^2 + \frac{K}{2\Delta t}\left(\|\varphi_{hx}^n + \psi_h^n\|^2 - \|\varphi_{hx}^{n-1} + \psi_h^{n-1}\|^2\right) \\ &+ \frac{b}{2\Delta t}\left(\|\psi_{hx}^n\|^2 - \|\psi_{hx}^{n-1}\|^2\right) + \frac{\xi}{2\Delta t}\left(\|z_h^n - z_h^{n-1}\|^2 + \|z_h^n\|^2 - \|z_h^{n-1}\|^2\right) \leq 0. \end{aligned}$$

Next, by discarding the positive terms

$$\|\Phi_h^n - \Phi_h^{n-1}\|^2,\ \|z_h^n - z_h^{n-1}\|^2,\ \left(\frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\|\Phi_h^{n-M}\|^2,\ \left(\mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\|\Phi_h^n\|^2,$$

we get

$$\begin{aligned}&\frac{\rho_1}{2\Delta t}\left(\|\Phi_h^n\|^2 - \|\Phi_h^{n-1}\|^2\right) + \frac{K}{2\Delta t}\left(\|\varphi_{hx}^n + \psi_h^n\|^2 - \|\varphi_{hx}^{n-1} + \psi_h^{n-1}\|^2\right)\\ &+ \frac{b}{2\Delta t}\left(\|\psi_{hx}^n\|^2 - \|\psi_{hx}^{n-1}\|^2\right) + \frac{\xi}{2\Delta t}\left(+\|z_h^n\|^2 - \|z_h^{n-1}\|^2\right) \le 0\end{aligned}$$

and, thus, the proof is complete. □

5.5.2 A priori error estimate

We now state and prove a priori error estimate for the numerical approximation, in which we find the convergence of the error.

Theorem 5.5.2. *Suppose that the solution (φ, ψ, z) of (5.3) is regular enough and (5.7) holds, then the following a priori estimate*

$$\begin{aligned}&\|\Phi_h^n - \Phi(t_n)\|^2 + \|\varphi_{hx}^n + \psi_h^n - ((\varphi(t_n))_x + \psi(t_n))\|^2\\ &+ \|\psi_{hx}^n - (\psi(t_n))_x\|^2 + \|z_h^n - z(t_n)\|^2 \le C(\Delta t^2 + h^2),\end{aligned}$$

is achieved, where C is independent of Δt and h.

Proof. First, we introduce the projection operator

$$P_h^* : H_0^1(I \times (0,1)) \longrightarrow S_h^*.$$

The operator P_h^* exhibits similar properties as defined previously in Chapter 2. To define the approximation of the initial data, we suppose that they are smooth enough and set

$$\varphi_h^0 = P_h\varphi_0,\ \Phi_h^0 = P_h\varphi_1,\ \psi_h^0 = P_h\psi_0,\ z_h^0 = P_h^* f_0.$$

Next, let us define

$$\begin{aligned}e^n = \varphi_h^n - P_h^0\varphi(t_n),\ \hat{e}^n = \Phi_h^n - P_h^0\Phi(t_n),\ y^n = \psi_h^n - P_h^0\psi(t_n),\\ \hat{y}^n = \Psi_h^n - P_h^0\Psi(t_n),\ r^n = z_h^n - P_h^* z(t_n).\end{aligned}$$

Several steps are required for the proof of this theorem

Step 1: Substitute in the scheme (5.79) and taking $\bar{\varphi}_h = \hat{e}^n$, $\bar{\psi}_h = \hat{y}^n$, $\bar{z}_h = (\xi/\tau)r^n$, we find

$$\begin{cases} \dfrac{\rho_1}{\Delta t}(\hat{e}^n + P_h^0\Phi(t_n) - (\hat{e}^{n-1} + P_h^0\Phi(t_{n-1})), \hat{e}^n) + K(e_x^n + (P_h^0\varphi(t_n))_x + y^n + P_h^0\psi(t_n), \hat{e}_x^n) \\ + \mu_1(\hat{e}^n + P_h^0\Phi(t_n), \hat{e}^n) + \mu_2(\hat{e}^{n-M} + P_h^0\Phi(t_{n-M}), \hat{e}^n) = 0, \\ b(y_x^n + (P_h^0\psi(t_n))_x, \hat{y}_x^n) + K(e_x^n + (P_h^0\varphi(t_n))_x + y^n + P_h^0\psi(t_n), \hat{y}^n) = 0, \\ \dfrac{\xi}{\Delta t}(r^n + P_h^* z(t_n) - (r^{n-1} + P_h^* z(t_{n-1})), r^n) + \dfrac{\xi}{\tau}(r_\rho^n + (P_h^* z(t_n))_\rho, r^n) = 0. \end{cases}$$

Applying (5.81), we obtain

$$\begin{cases} \dfrac{\rho_1}{2\Delta t}\left(\|\hat{e}^n - \hat{e}^{n-1}\|^2 + \|\hat{e}^n\|^2 - \|\hat{e}^{n-1}\|^2\right) + \dfrac{\rho_1}{\Delta t}(P_h^0\Phi(t_n) - P_h^0\Phi(t_{n-1}), \hat{e}^n) \\ + K(e_x^n + y^n, \hat{e}_x^n) + K((P_h^0\varphi(t_n))_x + P_h^0\psi(t_n), \hat{e}_x^n) + \mu_1\|\hat{e}^n\|^2 \\ + \mu_1(P_h^0\Phi(t_n), \hat{e}^n) + \mu_2(\hat{e}^{n-M}, \hat{e}^n) + \mu_2(P_h^0\Phi(t_{n-M}), \hat{e}^n) = 0, \\ b(y_x^n, \hat{y}_x^n) + b((P_h^0\psi(t_n))_x, \hat{y}_x^n) + K(e_x^n + y^n, \hat{y}^n) \\ + K((P_h^0\varphi(t_n))_x + P_h^0\psi(t_n), \hat{y}^n) = 0, \\ \dfrac{\xi}{2\Delta t}\left(\|r^n - r^{n-1}\|^2 + \|r^n\|^2 - \|r^{n-1}\|^2\right) + \dfrac{\xi}{\Delta t}(P_h^* z(t_n) - P_h^* z(t_{n-1}), r^n) \\ + \dfrac{\xi}{\tau}(r_\rho^n, r^n) + \dfrac{\xi}{\tau}((P_h^* z(t_n))_\rho, r^n) = 0. \end{cases} \tag{5.87}$$

Step 2: Now, let $\bar{\varphi} = \hat{e}^n$, $\bar{\psi} = \hat{y}^n$, $\bar{z} = (\xi/\tau)r^n$ in the week problem (5.78) and combine it with (5.87),

$$\begin{cases} \quad \dfrac{\rho_1}{2\Delta t}\left(\|\hat{e}^n - \hat{e}^{n-1}\|^2 + \|\hat{e}^n\|^2 - \|\hat{e}^{n-1}\|^2\right) \\ \quad + K(e_x^n + y^n, \hat{e}_x^n) + \mu_1\|\hat{e}^n\|^2 + \mu_2(\hat{e}^{n-M}, \hat{e}^n) \\ = \rho_1(\Phi_t - \dfrac{P_h^0\Phi(t_n) - P_h^0\Phi(t_{n-1})}{\Delta t}, \hat{e}^n) \\ \quad + K(\varphi_x + \psi - ((P_h^0\varphi(t_n))_x + P_h^0\psi(t_n)), \hat{e}_x^n) \\ \quad + \mu_1(\Phi - P_h^0\Phi(t_n), \hat{e}^n) + \mu_2(\Phi(t-\tau) - P_h^0\Phi(t_{n-M}), \hat{e}^n), \\ \quad b(y_x^n, \hat{y}_x^n) + K(e_x^n + y^n, \hat{y}^n) \\ = b(\psi_x - (P_h^0\psi(t_n))_x, \hat{y}_x^n) + K(\varphi_x + \psi - ((P_h^0\varphi(t_n))_x + P_h^0\psi(t_n)), \hat{y}^n), \\ \quad \dfrac{\xi}{2\Delta t}\left(\|r^n - r^{n-1}\|^2 + \|r^n\|^2 - \|r^{n-1}\|^2\right) + \dfrac{\xi}{\tau}(r_\rho^n, r^n) \\ = \xi(z_t - \dfrac{P_h^* z(t_n) - P_h^* z(t_{n-1})}{\Delta t}, r^n) + \dfrac{\xi}{\tau}(z_\rho - (P_h^* z(t_n))_\rho, r^n). \end{cases} \tag{5.88}$$

Step 3: Adding up the three equations of (5.88),

$$\begin{aligned}
&\frac{\rho_1}{2\Delta t}\left(\|\hat{e}^n-\hat{e}^{n-1}\|^2+\|\hat{e}^n\|^2-\|\hat{e}^{n-1}\|^2\right)+K(e_x^n+y^n,\hat{e}_x^n+\hat{y}^n)\\
&+\frac{\xi}{2\Delta t}\left(\|r^n-r^{n-1}\|^2+\|r^n\|^2-\|r^{n-1}\|^2\right)+\frac{\xi}{\tau}(r_\rho^n,r^n)\\
&+\mu_1\|\hat{e}^n\|^2+\mu_2(\hat{e}^{n-M},\hat{e}^n)+b(y_x^n,\hat{y}_x^n)\\
=&\rho_1(\Phi_t-\frac{P_h^0\Phi(t_n)-P_h^0\Phi(t_{n-1})}{\Delta t},\hat{e}^n)\\
&+K(\varphi_x+\psi-((P_h^0\varphi(t_n))_x+P_h^0\psi(t_n)),\hat{e}_x^n)\\
&+\mu_1(\Phi-P_h^0\Phi(t_n),\hat{e}^n)+\mu_2(\Phi(t-\tau)-P_h^0\Phi(t_{n-M}),\hat{e}^n)\\
&+b(\psi_x-(P_h^0\psi(t_n))_x,\hat{y}_x^n)+K(\varphi_x+\psi-((P_h^0\varphi(t_n))_x+P_h^0\psi(t_n)),\hat{y}^n)\\
&+\xi(z_t-\frac{P_h^*z(t_n)-P_h^*z(t_{n-1})}{\Delta t},r^n)+\frac{\xi}{\tau}(z_\rho-(P_h^*z(t_n))_\rho,r^n).
\end{aligned} \tag{5.89}$$

Step 4: In this step let $\nu=(e_x^n+y^n,\hat{e}_x^n+\hat{y}^n)$ and by the definition of $\hat{e}^n$ and $\hat{y}^n$, we end up with

$$\begin{aligned}
\nu&=(e_x^n+y^n,\Phi_{hx}^n-(P_h^0\Phi(t_n))_x+\Psi_h^n-P_h^0\Psi(t_n))\\
&=(e_x^n+y^n,\frac{e_x^n-e_x^{n-1}}{\Delta t}+\frac{y^n-y^{n-1}}{\Delta t})\\
&\quad+(e_x^n+y^n,\frac{(P_h^0\varphi(t_n))_x-(P_h^0\varphi(t_{n-1}))_x}{\Delta t}-(P_h^0\Phi(t_n))_x)\\
&\quad+(e_x^n+y^n,\frac{P_h^0\psi(t_n)-P_h^0\psi(t_{n-1})}{\Delta t}-P_h^0\Psi(t_n)),
\end{aligned}$$

then

$$\begin{aligned}
\nu=&\frac{1}{2\Delta t}\left(\|e_x^n+y^n-(e_x^{n-1}+y^{n-1})\|^2+\|e_x^n+y^n\|^2-\|e_x^{n-1}+y^{n-1}\|^2\right)\\
&+(e_x^n+y^n,\frac{(P_h^0\varphi(t_n))_x-(P_h^0\varphi(t_{n-1}))_x}{\Delta t}-(P_h^0\Phi(t_n))_x)\\
&+(e_x^n+y^n,\frac{P_h^0\psi(t_n)-P_h^0\psi(t_{n-1})}{\Delta t}-P_h^0\Psi(t_n)).
\end{aligned} \tag{5.90}$$

Similarly, we consider

$$\vartheta=(y_x^n,\hat{y}_x^n)=(y_x^n,\frac{y_x^n-y_x^{n-1}}{\Delta t}+\frac{(P_h^0\psi(t_n))_x-(P_h^0\psi(t_{n-1}))_x}{\Delta t}-(P_h^0\Psi(t_n))_x),$$

then

$$\vartheta = \frac{1}{2\Delta t}\left(\|y_x^n - y_x^{n-1}\|^2 + \|y_x^n\|^2 - \|y_x^{n-1}\|^2\right) + (y_x^n, \frac{(P_h^0\psi(t_n))_x - (P_h^0\psi(t_{n-1}))_x}{\Delta t} - (P_h^0\Psi(t_n))_x). \tag{5.91}$$

Young's inequality leads to

$$\mu_2(\hat{e}^{n-M}, \hat{e}^n) \geq -\frac{|\mu_2|}{2}\|\hat{e}^{n-M}\|^2 - \frac{|\mu_2|}{2}\|\hat{e}^n\|^2. \tag{5.92}$$

By direct computations, we have the following equality

$$\frac{\xi}{\tau}(r_\rho^n, r^n) = \frac{\xi}{2\tau}\|\hat{e}^{n-M}\|^2 - \frac{\xi}{2\tau}\|\hat{e}^n\|^2. \tag{5.93}$$

Inserting (5.90)-(5.93) into (5.89), then by taking into account that

$$\|\hat{e}^n - \hat{e}^{n-1}\|^2,\ \|e_x^n + y^n - (e_x^{n-1} + y^{n-1})\|^2,\ \|y_x^n - y_x^{n-1}\|^2,\ \|r^n - r^{n-1}\|^2,$$
$$\left(\frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\|\hat{e}^{n-M}\|^2,\ \text{and}\ \left(\mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right)\|\hat{e}^n\|^2$$

are positive terms, we arrive at

$$\begin{aligned}
&\frac{\rho_1}{2\Delta t}\left(\|\hat{e}^n\|^2 - \|\hat{e}^{n-1}\|^2\right) + \frac{K}{2\Delta t}\left(\|e_x^n + y^n\|^2 - \|e_x^{n-1} + y^{n-1}\|^2\right)\\
&\frac{b}{2\Delta t}\left(\|y_x^n\|^2 - \|y_x^{n-1}\|^2\right) + \frac{\xi}{2\Delta t}\left(\|r^n\|^2 - \|r^{n-1}\|^2\right)\\
\leq&\rho_1(\Phi_t - \frac{P_h^0\Phi(t_n) - P_h^0\Phi(t_{n-1})}{\Delta t}, \hat{e}^n)\\
&+ K(\varphi_x + \psi - ((P_h^0\varphi(t_n))_x + P_h^0\psi(t_n)), \hat{e}_x^n + \hat{y}^n)\\
&- K(e_x^n + y^n, \frac{(P_h^0\varphi(t_n))_x - (P_h^0\varphi(t_{n-1}))_x}{\Delta t} - (P_h^0\Phi(t_n))_x)\\
&- K(e_x^n + y^n, \frac{P_h^0\psi(t_n) - P_h^0\psi(t_{n-1})}{\Delta t} - P_h^0\Psi(t_n))\\
&+ \mu_1(\Phi - P_h^0\Phi(t_n), \hat{e}^n) + \mu_2(\Phi(t-\tau) - P_h^0\Phi(t_{n-M}), \hat{e}^n)\\
&+ b(\psi_x - (P_h^0\psi(t_n))_x, \hat{y}_x^n) - b(y_x^n, \frac{(P_h^0\psi(t_n))_x - (P_h^0\psi(t_{n-1}))_x}{\Delta t} - (P_h^0\Psi(t_n))_x)\\
&+ \xi(z_t - \frac{P_h^* z(t_n) - P_h^* z(t_{n-1})}{\Delta t}, r^n) + \frac{\xi}{\tau}(z_\rho - (P_h^* z(t_n))_\rho, r^n).
\end{aligned}$$

Step 5: let

$$Z_n = \rho_1\|\hat{e}^n\|^2 + K\|e_x^n + y^n\|^2 + b\|y_x^n\|^2 + \xi\|r^n\|^2$$

and by applying Young's inequality, we have

$$\begin{aligned}
Z_n - Z_{n-1} \leq & 2C\Delta t\Big(Z_n + \Big\|\Phi_t - \frac{P_h\Phi(t_n) - P_h^0\Phi(t_{n-1})}{\Delta t}\Big\|^2 \\
& + \|(\varphi_x + \psi - ((P_h^0\varphi(t_n))_x + P_h^0\psi(t_n))\|^2 \\
& + \Big\|\frac{(P_h^0\varphi(t_n))_x - (P_h^0\varphi(t_{n-1}))_x}{\Delta t} - (P_h^0\Phi(t_n))_x\Big\|^2 \\
& + \Big\|\frac{P_h^0\psi(t_n) - P_h^0\psi(t_{n-1})}{\Delta t} - P_h^0\Psi(t_n)\Big\|^2 + \|\Phi - P_h^0\Phi(t_n)\|^2 \\
& + \|\Phi(t-\tau) - P_h^0\Phi(t_{n-M})\|^2 + \|\psi_x - (P_h^0\psi(t_n))_x\|^2 \\
& + \Big\|\frac{(P_h^0\psi(t_n))_x - (P_h^0\psi(t_{n-1}))_x}{\Delta t} - (P_h^0\Psi(t_n))_x\Big\|^2 \\
& + \Big\|z_t - \frac{P_h^* z(t_n) - P_h^* z(t_{n-1})}{\Delta t}\Big\|^2 + \|z_\rho - (P_h^* z(t_n))_\rho\|^2\Big).
\end{aligned}$$

Collecting all these estimates, we observe that

$$Z_n - Z_{n-1} \leq 2C\Delta t(Z_n + R_n), \tag{5.94}$$

where the residual R_n is the sum of the approximation errors. Summing (5.94) over n, it follows that

$$Z_n - Z_0 \leq 2C\Delta t\sum_{j=1}^{n}(Z_j + R_j),$$

we then combine time error, which is estimated by using Taylor's expansion in time, and space error, which can be bounded from (2.5) to obtain

$$2C\Delta t\sum_{j=1}^{n} R_j \leq C(\Delta t^2 + h^2)$$

and since $Z_0 = 0$, we deduce

$$Z_n \leq 2C\Delta t\sum_{j=1}^{n} Z_j + C(\Delta t^2 + h^2).$$

Finally, using a discrete version of Gronwall's inequality with the fact that $n\Delta t \leq T$ to get the desired result. □

5.6 Simulations

This section describes the procedure used to find the numerical solution as well as the results of some numerical simulations.

Like in [16, 50], to solve the system (5.2) we use an iterative algorithm. Assuming that Φ_h^{n-1} is known and setting

$$\varphi_h^{n,0} = \varphi_h^{n-1},\ \Phi_h^{n,0} = \Phi_h^{n-1},\ \psi_h^{n,0} = \psi_h^{n-1},$$

the system is solved iteratively:

$$\begin{cases} \dfrac{\rho_1}{\Delta t}(\Phi_h^{n,l} - \Phi_h^{n-1}, \bar{\varphi}_h) + K(\varphi_{hx}^{n,l} + \psi_h^{n,l}, \bar{\varphi}_{hx}) \\ +\mu_1(\Phi_h^{n,l}, \bar{\varphi}_h) + \mu_2(\Phi_h^{n-M,l}, \bar{\varphi}_h) = 0, \\ b(\psi_{hx}^{n,l}, \bar{\psi}_{hx}) + K(\varphi_{hx}^{n,l} + \psi_h^{n,l}, \bar{\psi}_h) = 0, \end{cases} \tag{5.95}$$

where, for $l = 1,\ 2,\ \ldots,$

$$\varphi_h^{n,l} = \varphi_h^{n-1} + \Delta t\ \Phi_h^{n,l}.$$

Problem (5.95) consists of two, uncoupled, linear systems of algebraic equations, that have a unique solution. First, we compute $\psi_h^{n,l}$, then $\Phi_h^{n,l}$. A tolerance $tol = 10^{-7}$ is used to stop the iterative procedure.

For our simulations, we choose the following entries:

$$L = 1,\ h = 0.01,\ \Delta t = h/2,\ \tau = 0.1T,\ \rho_1 = 2,\ K = 365,\ b = 1.$$

The initial values are

$$\varphi_0(x) = \varphi_1(x) = \psi_0(x) = x(1-x),$$

and the delay condition is

$$f_0(x, t-\tau) = x(1-x)\cos(t-\tau).$$

Test 1. In the first experiment, we choose $\mu_1 = 1$ and $\mu_2 = 0.1$.

Test 2. We run an experiment with $\mu_1 = 2$ and $\mu_2 = -1$.

The evolution of φ and ψ are represented in 3D (see Figures 5.1, 5.2, 5.8 and 5.9 respectively).

Figures 5.3, 5.4, 5.10 and 5.11 show the displacement and the angular rotation at the point $x = 0.5$.

The decay of the energy with respect to time is shown in Figures 5.5, 5.6, 5.7, 5.12, 5.13 and 5.14.

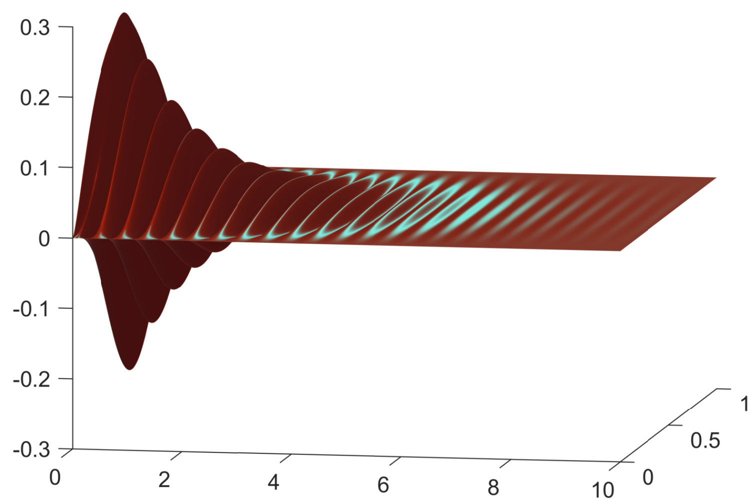


Figure 5.1 Test 1: The evolution in time and space of φ.

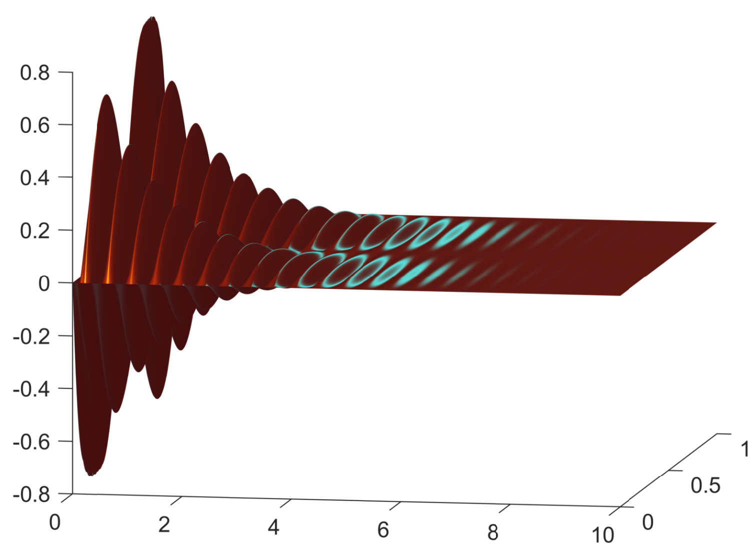


Figure 5.2 Test 1: The evolution in time and space of ψ.

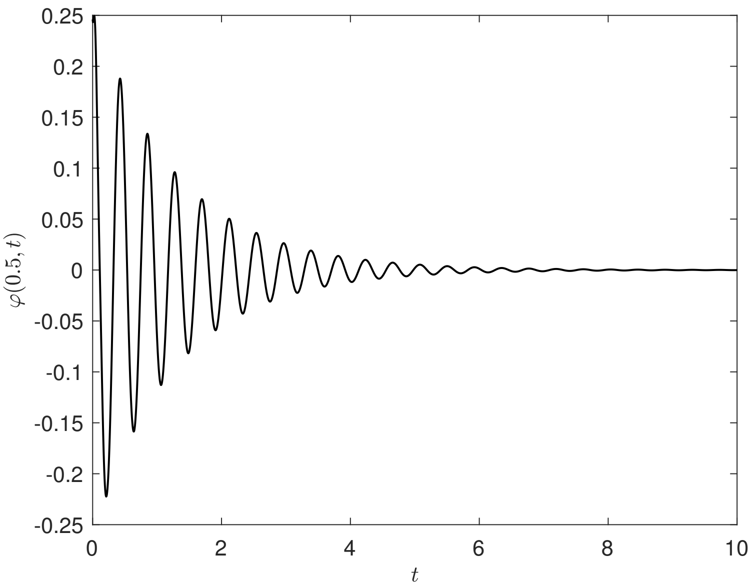


Figure 5.3 Test 1: The evolution in time of φ at $x = 0.5$.

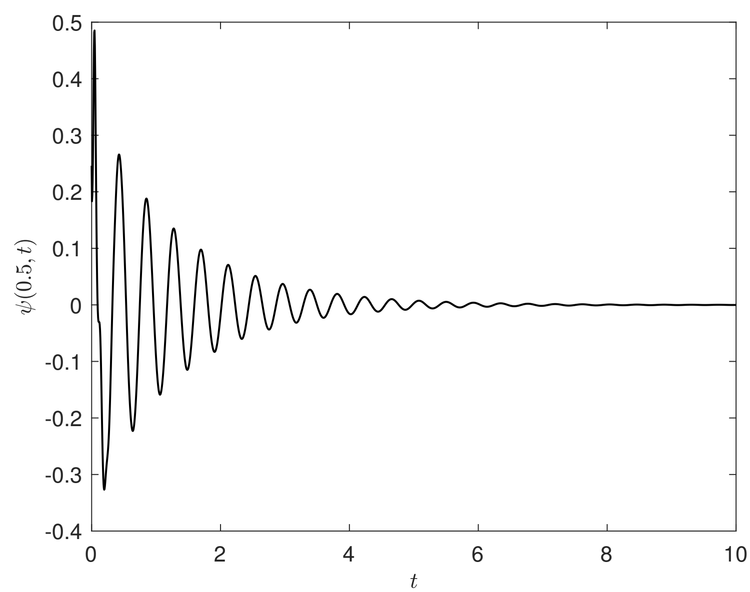


Figure 5.4 Test 1: The evolution in time of ψ at $x = 0.5$.

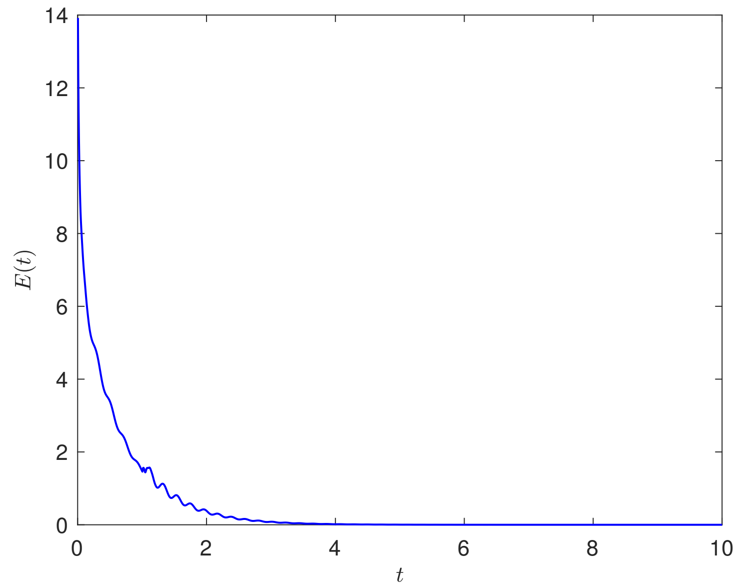


Figure 5.5 Test 1: The evolution in time of E.

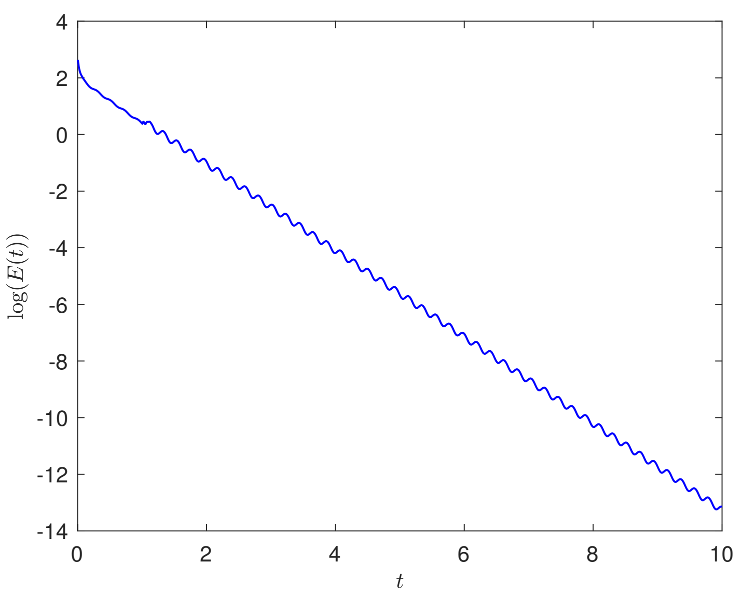


Figure 5.6 Test 1: The evolution in time of $\log(E(t))$.

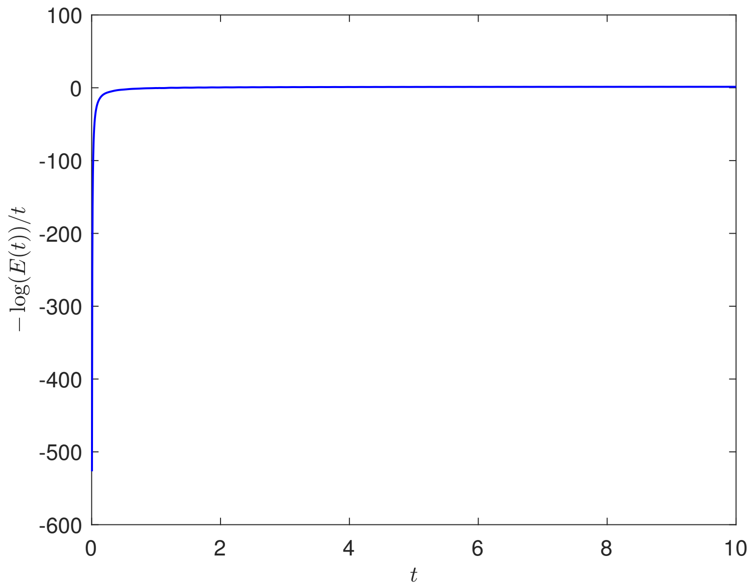


Figure 5.7 Test 1: The evolution in time of $-\log(E(t))/t$.

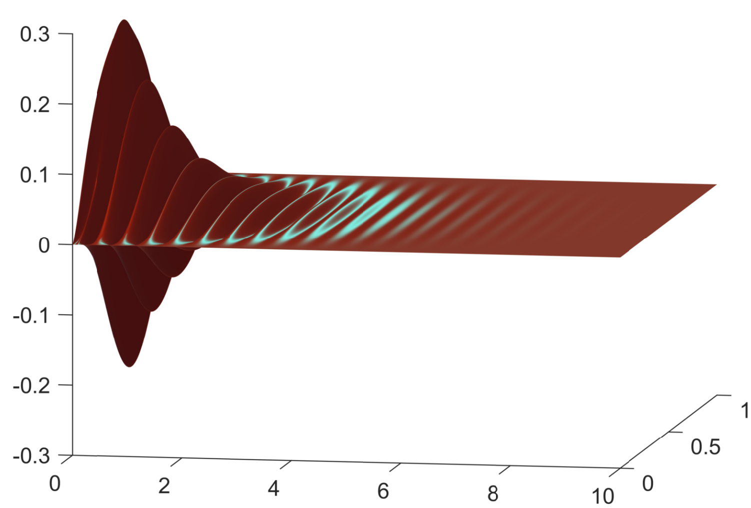


Figure 5.8 Test 2: The evolution in time and space of φ.

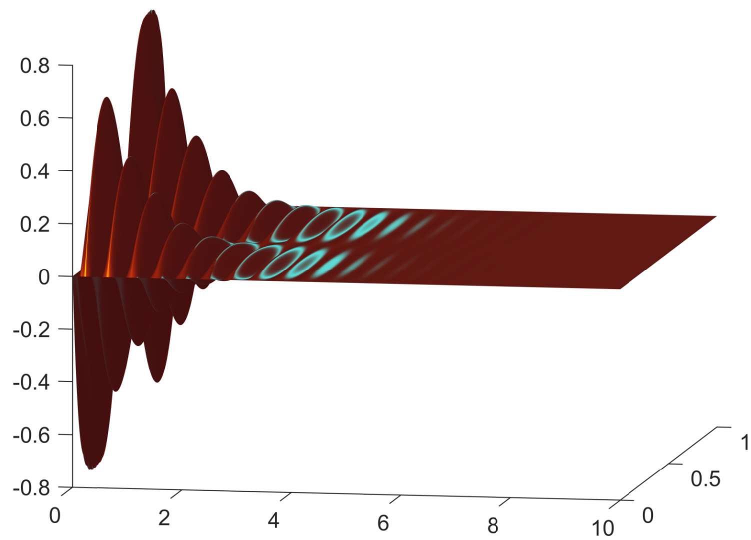


Figure 5.9 Test 2: The evolution in time and space of ψ.

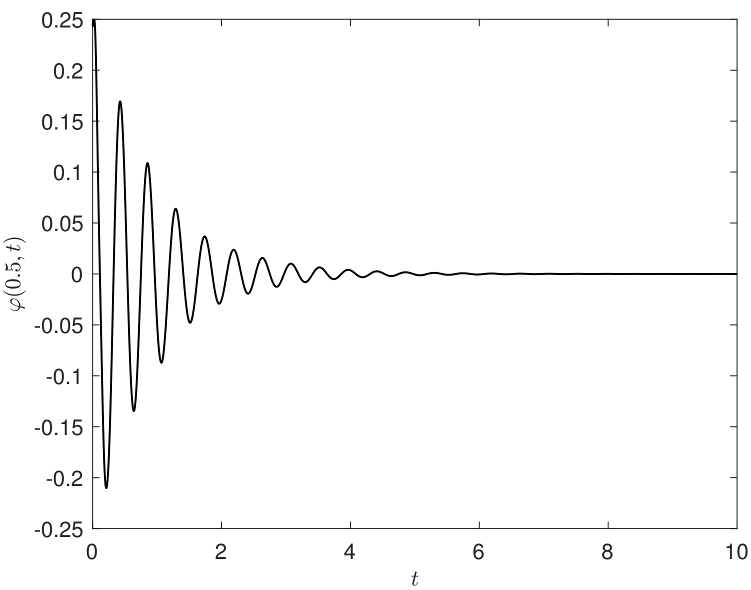


Figure 5.10 Test 2: The evolution in time of φ at $x = 0.5$.

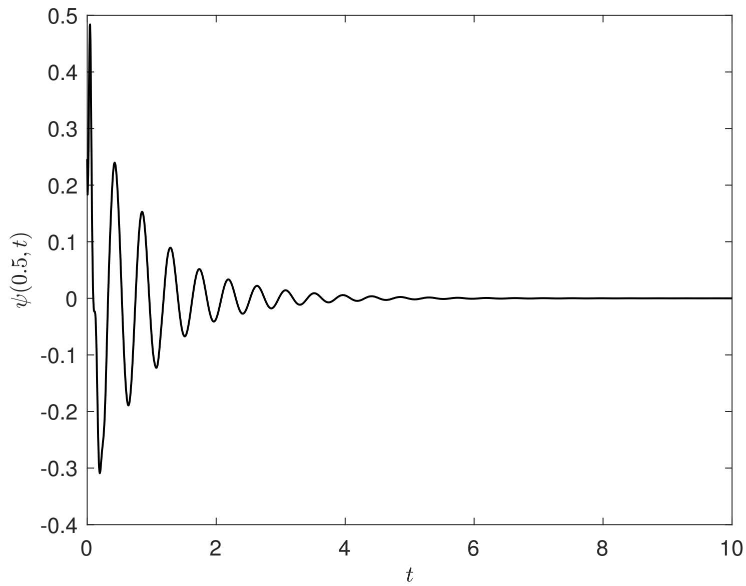


Figure 5.11 Test 2: The evolution in time of ψ at $x = 0.5$.

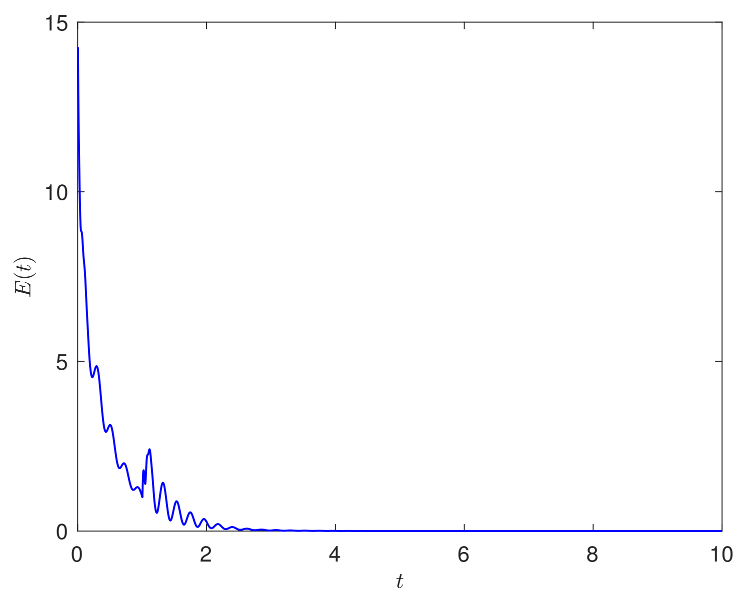


Figure 5.12 Test 2: The evolution in time of E.

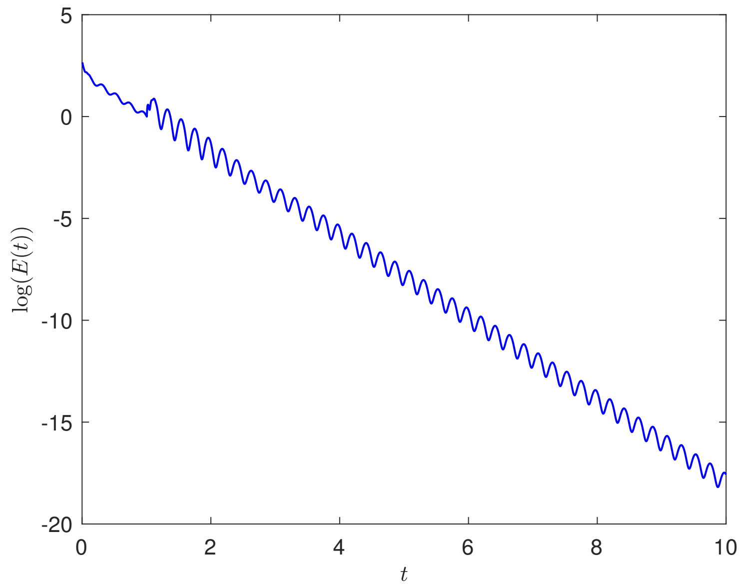


Figure 5.13 Test 2: The evolution in time of $\log(E(t))$.

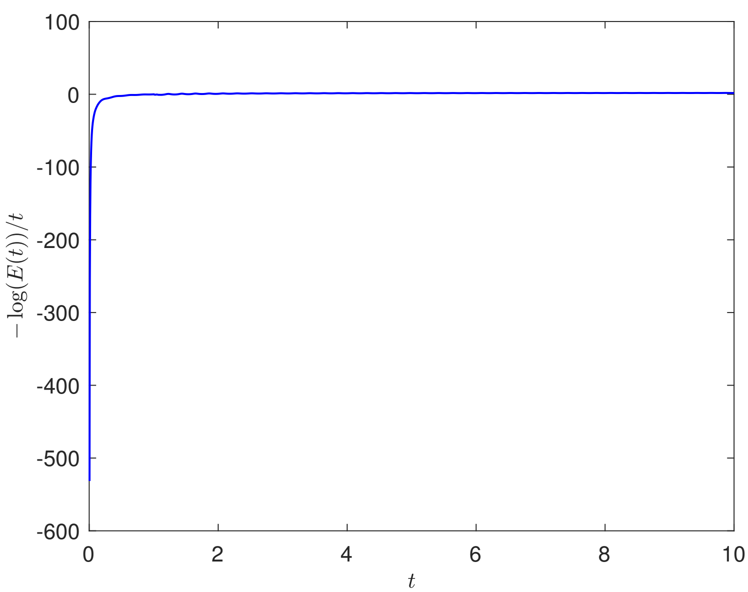


Figure 5.14 Test 2: The evolution in time of $-\log(E(t))/t$.

Chapter 6

A memory-type porous thermoelastic system with microtemperatures effects and delay term in the internal feedback: Well-Posedness, Stability and Numerical Results

6.1 Introduction

This chapter addresses the system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} - \mu_1 u_t - \mu_2 u_t(x,t-\tau) + b\varphi_x - \gamma\theta_x, \\ J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi - dw_x + m\theta - \displaystyle\int_0^\infty g(s)\varphi_{xx}(t-s)ds, \\ c\theta_t = k_0\theta_{xx} - \gamma u_{tx} - m\varphi_t - k_1 w_x, \\ \alpha w_t = k_2 w_{xx} - k_3 w - k_1\theta_x - d\varphi_{tx}, \end{cases} \tag{6.1}$$

where $(x,t) \in (0,1) \times (0,\infty)$. The functions u, φ, θ, w represent the displacement of the solid elastic material, the volume fraction, the temperature difference and the microtemperature vector, respectively. The relaxation function g is positive and decreasing. The parameters ρ and J , which are strictly positive constants, denote the mass density and the product of the mass density with the equilibrated inertia, respectively, while $\tau > 0$ indicates the time delay. The constants c, μ, μ_1, μ_2, δ, γ, ξ, m, d, k_1, k_2, k_3, α are positive, and b is a non-zero real number. This system is subjected to the following boundary condition

$$\begin{cases} u_x(0,t) = u_x(1,t) = \varphi(0,t) = \varphi(1,t) = 0,\ t > 0, \\ \theta(0,t) = \theta(1,t) = w_x(0,t) = w_x(1,t) = 0,\ t > 0, \end{cases} \tag{6.2}$$

and initial conditions

$$\begin{cases} u(x,0) = u_0(x), u_t(x,0) = u_1(x), \varphi(x,-t) = \varphi_0(x,t),\ x \in (0,1),\ t \geqslant 0, \\ \varphi_t(x,0) = \varphi_1(x), \theta(x,0) = \theta_0(x), w(x,0) = w_0(x),\ x \in (0,1), \\ u_t(x,t-\tau) = f_0(x,t-\tau),\ (x,t) \in (0,1) \times (0,\tau). \end{cases} \tag{6.3}$$

The initial data u_0, u_1, $\varphi_0(.,0)$, φ_1, θ_0, w_0 and $f_0(x, t-\tau)$ belongs to a suitable functional space.

The evolution of the porous material is governed by a linear damped wave equation. This naturally led to the exploration of multiple couplings. Quintanilla [126] made the first contribution in this research direction. Since then, numerous articles have been published examining how various mechanisms influence the entire system to either exponential or slow decay. To clarify this, Casas and Quintanilla [31,32] demonstrated that when porous dissipation is combined with temperatures (or microtemperatures), the system decays exponentially. Magańa and Quintanilla [105] showed that viscoelastic damping and temperature resulted in slow decay over time, but when coupled with porous damping or microtemperatures, the system decays exponentially.

Recently, problem (6.1) has been studied in [135] and proved that, in absence of the infinite memory and the delay terms, the dissipation given only with the microtemperatures is sufficient to get an exponential stability in the case when $\chi_1 = 0$, where

$$\chi_1 = \frac{\mu}{\rho} - \frac{\delta}{J} - \frac{\gamma^2}{c\rho},$$

Khochemane in [90] for $\mu_1 = \mu_2 = 0$, proved a general decay estimate for the solutions of system (6.1).

The main goal of this chapter is is to extend the results in [90, 135] to the case where $g \neq 0$ and $\mu_i \neq 0, i = 1, 2$. Under the the condition

$$\mu\xi > b^2, \tag{6.4}$$

we prove the well-posedness and establish a general energy decay result from which the usual exponential and polynomial types of decay are only special cases. More specifically, we discuss the two cases separately: the case $\mu_2 < \mu_1$ and the case $\mu_2 = \mu_1$. Furthermore, our result is dependent on the kernel of the infinite memory term where the relaxation function g satisfies the following assumptions:

(H1) $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a C^1 function satisfying

$$g(0) > 0, \quad \delta - \int_0^\infty g(s)ds = l > 0, \quad \int_0^\infty g(s)ds = g_0. \tag{6.5}$$

(H2) There exists a non-increasing differential function $\zeta(t) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfying

$$g'(t) \leqslant -\zeta(t)\, g(t),\ t \geqslant 0 \quad \text{and} \quad \int_0^\infty \zeta(t)dt = +\infty, \tag{6.6}$$

which allows us with the construction of an appropriate Lyapunov functional to estimate the energy of the system.

Remark 6.1.1. *There are many functions satisfying (H1) and (H2). Here are three examples of such functions, assuming that $a, b > 0$ and $a < \delta b$*

$$g_1(t) = ae^{-bt},\ g_1'(t) = -\zeta(t)g_1(t),\ \textit{where } \zeta(t) = b.$$

$$g_2(t) = \frac{a}{(1+t)^{b+1}},\ g_2'(t) = -\zeta(t)g_2(t),\ \textit{where } \zeta(t) = \frac{b+1}{1+t}.$$

$$g_3(t) = \frac{a}{(e+t)[\ln(e+t)]^{b+1}},\ g_3'(t) = -\zeta(t)g_3(t),\ \textit{where } \zeta(t) = \frac{1}{e+1} + \frac{b+1}{(e+1)\ln(e+t)}.$$

It is worth noting that in the past decades, there has been a significant interest among scientists in problems related to viscoelastic damping with a memory or past history term. The decay results obtained depend on the rate of decay of the relaxation function, which is exponential for g that satisfies: $g'(t) \leqslant -\xi g(t)$ forall $t \geqslant 0$ and some positive constant ξ. However, it has been proved that only a polynomial decay result occurs for relaxation functions that satisfy $g'(t) \leqslant -\xi g^p(t)$, $\forall t \geqslant 0$ and $1 < p \leqslant \frac{3}{2}$, (see [72, 91, 120, 137–139]). Considerable effort has been dedicated to broadening the range of admissible relaxation functions that lead to either strong or slow decay. Messaoudi and Mustafa [109] proposed that the relaxation function g satisfies the inequality $g'(t) \leqslant -\xi(t)g(t), \forall t \geqslant 0$ and established a more general decay result, with exponential and polynomial decay rates as special cases.

The chapter is organized as follows. In Section 2, we introduce some transformations needed in our work. In Section 3, we use the semigroup method to prove the well-posedness of problem (1). In Section 4, under the condition that $\mu_2 \leqslant \mu_1$, we prove a general stability. Finally, some numerical simulations are obtained using MATLAB

6.2 Preliminaries

In this section, we present the materials necessary for proving our results.

In order to simplify calculations, as proposed in [72] we define, for all $\omega \in L^2(0,1)$,

$$(g \circ \omega)(t) = \int_0^1 \int_0^\infty g(s)(\omega(t) - \omega(t-s))^2 ds\, dx,$$

$$g \odot \omega = \int_0^1 \left(\int_0^\infty g(s)(\omega(t) - \omega(t-s)) ds \right)^2 dx.$$

Lemma 6.2.1. *If assumption (H1) holds, then there is a positive constant k such that*

$$g \odot \omega \leqslant k\ g \circ \omega_x \quad \textit{for all} \quad \omega \in H_0^1(0,1).$$

Proof. Note that

$$g \odot \omega = \int_0^1 \left(\int_0^\infty g(s)(\omega(t) - \omega(t-s))ds \right)^2 dx$$
$$= \int_0^1 \left(\int_0^\infty g^{\frac{1}{2}}(s) g^{\frac{1}{2}}(s)(\omega(t) - \omega(t-s))ds \right)^2 dx.$$

By using Cauchy Schwarz, we get

$$\begin{aligned} g \odot \omega &\leqslant \int_0^1 \left(\left(\int_0^\infty (g^{\frac{1}{2}}(s))^2 ds \right)^{\frac{1}{2}} \left(\int_0^\infty \left(g^{\frac{1}{2}}(s)(\omega(t) - \omega(t-s)) \right)^2 ds \right)^{\frac{1}{2}} \right)^2 dx \\ &\leqslant \int_0^\infty g(s) ds \int_0^1 \int_0^\infty g(s)(\omega(t) - \omega(t-s))^2 ds \, dx. \end{aligned} \tag{6.7}$$

Then, Poincaré's inequality leads to

$$g \odot \omega \leqslant g_0 c_p \int_0^1 \int_0^\infty g(s)(\omega_x(t) - \omega_x(t-s))^2 ds \, dx = k \, g \circ w_x,$$

where $k = g_0 c_p$. ☐

Applying Lemma 6.2.1, we obtain the following inequalities

$$\int_0^1 \left(\int_0^\infty g(s)(\varphi(t) - \varphi(t-s))ds \right)^2 dx \leqslant d_1 (g \circ \varphi_x)(t), \tag{6.8}$$

$$\int_0^1 \left(\int_0^\infty g'(s)(\varphi(t) - \varphi(t-s))ds \right)^2 dx \leqslant -d_2 (g' \circ \varphi_x)(t), \tag{6.9}$$

where $d_1 = g_0 c_p$ and $d_2 = g(0) c_p$. Now the use of (6.7) with g', φ_x instead of g, φ gives

$$\int_0^1 \left(\int_0^\infty g'(s)(\varphi_x(t) - \varphi_x(t-s))ds \right)^2 dx \leqslant -g(0)(g' \circ \varphi_x)(t). \tag{6.10}$$

Similarly, the inequality (6.7) yields

$$\int_0^1 \left(\int_0^\infty g(s)(\varphi_x(t) - \varphi_x(t-s))ds \right)^2 dx \leqslant g_0 (g \circ \varphi_x)(t). \tag{6.11}$$

To demonstrate the dissipative nature of system (6.1)-(6.3), it is useful to introduce, as in [103, 120], the relative history of φ

$$\eta^t = \varphi(x,t) - \varphi(x,t-s), \; (x,t,s) \in (0,1) \times \mathbb{R}_+ \times \mathbb{R}_+. \tag{6.12}$$

Taking the derivative of (6.12), we find

$$\begin{cases} \eta_t^t + \eta_s^t - \varphi_t = 0,\ (x,t,s) \in (0,1) \times \mathbb{R}_+ \times \mathbb{R}_+, \\ \eta^t(0,s) = \varphi(0,t) - \varphi(0,t-s) = 0,\ t,s > 0, \\ \eta^t(1,s) = \varphi(1,t) - \varphi(1,t-s) = 0,\ t,s > 0, \\ \eta^t(x,0) = 0,\ \eta^0(x,s) = \eta_0(x,s),\ (x,t,s) \in (0,1) \times \mathbb{R}_+ \times \mathbb{R}_+. \end{cases} \tag{6.13}$$

Next as in [116, 136], we introduce another new dependent variable

$$z(x,p,t) = u_t(x, t - \tau p),\ (x,p,t) \in (0,1) \times (0,1) \times \mathbb{R}_+,$$

a simple differentiation shows that z satisfies

$$\tau z_t(x,p,t) + z_p(x,p,t) = 0,\ (x,p,t) \in (0,1) \times (0,1) \times \mathbb{R}_+.$$

Therefore, problem (6.1)-(6.3) can be rewritten as

$$\begin{cases} \rho u_{tt} = \mu u_{xx} - \mu_1 u_t - \mu_2 z(x,1,t) + b\varphi_x - \gamma\theta_x,\ \text{in } (0,1) \times (0,\infty), \\ J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi - dw_x + m\theta - \displaystyle\int_0^\infty g(s)\varphi_{xx}(t-s)ds,\ \text{in } (0,1) \times (0,\infty), \\ c\theta_t = k_0\theta_{xx} - \gamma u_{tx} - m\varphi_t - k_1 w_x,\ \text{in } (0,1) \times (0,\infty), \\ \alpha w_t = k_2 w_{xx} - k_3 w - k_1\theta_x - d\varphi_{tx},\ \text{in } (0,1) \times (0,\infty), \\ \eta_t^t + \eta_s^t = \varphi_t,\ x \in (0,1), s, t > 0, \\ \tau z_t(x,p,t) + z_p(x,p,t) = 0,\ \text{in } (0,1) \times (0,1) \times (0,\infty), \end{cases} \tag{6.14}$$

with the following initial data

$$\begin{cases} u(x,0) = u_0(x), u_t(x,0) = u_1(x), \varphi(x,-t) = \varphi_0(x,t),\ x \in (0,1), \\ \varphi_t(x,0) = \varphi_1(x), \theta(x,0) = \theta_0(x), w(x,0) = w_0(x),\ x \in (0,1), \\ \eta^t(x,0) = 0, \eta^0(x,s) = \eta_0(x,s),\ x \in (0,1),\ t > 0, \\ z(x,0,t) = u_t(x,t),\ x \in (0,1),\ t > 0, \\ z(x,1,t) = f_0(x, t-\tau),\ (x,t) \in (0,1) \times (0,\tau), \end{cases} \tag{6.15}$$

and boundray conditions

$$\begin{cases} u_x(0,t) = u_x(1,t) = \varphi(0,t) = \varphi(1,t) = 0,\ t > 0, \\ \theta(0,t) = \theta(1,t) = w_x(0,t) = w_x(1,t) = 0,\ t > 0, \\ \eta^t(0,s) = \eta^t(1,s) = 0,\ t,s > 0. \end{cases} \tag{6.16}$$

To apply Poincaré's inequality to u and w, we need the following transformations:

Integrating the first and fourth equations in (6.14) over $(0,1)$ and using the boundary conditions, we get

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)dx + \frac{\mu_1}{\rho}\frac{d}{dt}\int_0^1 u(x,t)dx + \frac{\mu_2}{\rho}\int_0^1 z(x,1,t)dx = 0, \ \forall t \geqslant 0$$

and

$$\alpha\frac{d}{dt}\int_0^1 wdx + k_3\int_0^1 wdx = 0, \ \forall t \geqslant 0.$$

If we take $X(t) = \int_0^1 u(x,t)dx$ and $V(t) = \int_0^1 w(x,t)dx$, we have the initial value problems

$$\begin{cases} X_{tt}(t) + \dfrac{\mu_1}{\rho}X_t(t) = -\dfrac{\mu_2}{\rho}\int_0^1 z(x,1,t)dx, \\ X(0) = \int_0^1 u_0(x)dx, \ X_t(0) = \int_0^1 u_1(x)dx. \end{cases} \tag{6.17}$$

$$\begin{cases} \alpha V_t(t) + k_3 V(t) = 0, \\ V(0) = \int_0^1 w_0(x)dx. \end{cases} \tag{6.18}$$

Solving these ODEs yields

$$\int_0^1 u(x,t)dx = a_1 + a_2\exp(\frac{\mu_1}{\rho}t) - \frac{\mu_2}{\mu_1}t\int_0^1 f_0(x,t-\tau)dx,$$

$$\int_0^1 w(x,t)dx = \exp(-\frac{k_3}{\alpha}t)\int_0^1 w_0(x)dx,$$

where

$$\begin{aligned} a_1 &= \frac{\mu_1}{\mu_1-\rho}\int_0^1 u_0(x)dx + \frac{\rho}{\rho-\mu_1}\left[\int_0^1 u_1(x)dx + \frac{\mu_2}{\mu_1}\int_0^1 f_0(x,-\tau)dx\right], \\ a_2 &= \frac{\rho}{\rho-\mu_1}\left[\int_0^1 u_0(x)dx - \int_0^1 u_1(x)dx - \frac{\mu_2}{\mu_1}\int_0^1 f_0(x,t-\tau)dx\right]. \end{aligned}$$

After this, we use the following change of variables

$$\begin{aligned} \bar{u} &= u - \left(a_1 + a_2\exp(\frac{\mu_1}{\rho}t) - \frac{\mu_2}{\mu_1}t\int_0^1 f_0(x,t-\tau)dx\right), \\ \bar{w} &= w - \left(\exp(-\frac{k_3}{\alpha}t)\int_0^1 w_0(x)dx\right), \end{aligned}$$

to get

$$\int_0^1 \bar{u}dx = \int_0^1 \bar{w}dx = 0, \ \forall t \geqslant 0.$$

A simple substitution reveals that $(\bar{u}, \varphi, \theta, \bar{w}, \eta^t, z)$ satisfies (6.14), with initial data for $\bar{u}$ and $\bar{w}$ specified as:

$$\bar{u}_0(x) = u_0(x) - (a_1 + a_2),\ \bar{u}_1(x) = u_1(x) - \left(a_1 + a_2\frac{\mu_1}{\rho}\right),$$
$$\bar{w}_0(x) = w_0(x) - \int_0^1 w_0(x)dx,\ \bar{w}_1(x) = w_1(x) + \frac{k_3}{\alpha}\int_0^1 w_0(x)dx.$$

For convenience, we will use u and w to refer to $\bar{u}$ and $\bar{w}$. We also introduce the following spaces,

$$L^2_*(0,1) = \left\{\phi \in L^2(0,1) : \int_0^1 \phi(x)dx = 0\right\},\ H^1_*(0,1) = H^1(0,1) \cap L^2_*(0,1)$$

and

$$H^2_*(0,1) = \{\phi \in H^2(0,1) :\ \phi_x(0) = \phi_x(1) = 0\}.$$

It is well known that Poincaré's inequality can be applied to elements of $H^1_*(0,1)$, which means that:

$$\exists\, c_p > 0 \quad \text{such that} \quad \int_0^1 v^2 dx \leqslant c_p \int_0^1 v_x^2 dx \quad \forall v \in H^1_*(0,1).$$

6.3 Well-posedness of the problem

This section presents the existence and uniqueness results for the system (6.14)-(6.16) using the semigroup theory.

By using the notation (6.12), the second equation of (6.14) becomes

$$\begin{aligned} &J\varphi_{tt} - \delta\varphi_{xx} + bu_x + \xi\varphi + dw_x - m\theta \\ &- \int_0^\infty g(s)\eta^t_{xx}(x,s)ds + \int_0^\infty g(s)\varphi_{xx}(x,t)ds = 0. \end{aligned} \tag{6.19}$$

The last term of the above equation gives

$$\int_0^\infty g(s)\varphi_{xx}(x,t)ds = \varphi_{xx}(x,t)\int_0^\infty g(s)ds.$$

According to (H1), the equation (6.19) can be formulated as

$$J\varphi_{tt} - l\varphi_{xx} + bu_x + \xi\varphi + dw_x - m\theta - \int_0^\infty g(s)\eta^t_{xx}(x,s)ds = 0.$$

We consider the energy space $\mathcal{H}$ given by

$$H^1_*(0,1) \times L^2_*(0,1) \times H^1_0(0,1) \times L^2(0,1) \times L^2(0,1) \times L^2_*(0,1) \times L_g \times L^2_*(0,1),$$

such that

$$L_g = \left\{ f : \mathbb{R}_+ \longrightarrow H_0^1(0,1), \int_0^1 \left(\int_0^\infty g(s) f_x^2 ds \right) dx < \infty \right\},$$

is provided with the following inner product

$$\langle f_1, f_2 \rangle_{L_g} = \int_0^1 \left(\int_0^\infty g(s) (f_1)_x (f_2)_x ds \right) dx.$$

We equip $\mathcal{H}$ with the inner product

$$\begin{aligned} \langle U, \tilde{U} \rangle_{\mathcal{H}} =& \rho \int_0^1 \nu \tilde{\nu} dx + \mu \int_0^1 u_x \tilde{u}_x dx + J \int_0^1 \psi \tilde{\psi} dx + b \int_0^1 (u_x \tilde{\varphi} + \tilde{u}_x \varphi) dx \\ &+ \xi \int_0^1 \varphi \tilde{\varphi} dx + l \int_0^1 \varphi_x \tilde{\varphi}_x dx + \alpha \int_0^1 w \tilde{w} dx + c \int_0^1 \theta \tilde{\theta} dx \\ &+ \langle \eta^t, \tilde{\eta}^t \rangle_{L_g} + \kappa \int_0^1 \int_0^1 z(x,p) \tilde{z}(x,p) dp \, dx, \end{aligned} \tag{6.20}$$

for $U = (u, \nu, \varphi, \psi, \theta, w, \eta^t, z)^T, \ \tilde{U} = (\tilde{u}, \tilde{\nu}, \tilde{\varphi}, \tilde{\psi}, \tilde{\theta}, \tilde{w}, \tilde{\eta}^t, \tilde{z})^T \in \mathcal{H}$, It is easy to check that $\mathcal{H}$, with respect to (6.20), forms a Hilbert space. Note that κ is a positive constant satisfying

$$\tau \mu_2 \leqslant \kappa \leqslant \tau(2\mu_1 - \mu_2), \text{ if } \quad \mu_2 \leqslant \mu_1. \tag{6.21}$$

With the vector function $U = (u, \nu, \varphi, \psi, \theta, w, \eta^t, z)^T$, where $\nu = u_t$ and $\psi = \varphi_t$, system (6.14)–(6.16) can be rewritten as follows:

$$\begin{cases} U_t = AU, \ t > 0, \\ U(x,0) = U_0(x) = (u_0, u_1, \varphi_0(.,0), \varphi_1, \theta_0, w_0, \eta_0, f_0(x, -\tau))^T, \end{cases}$$

where the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is defined by

$$AU = \begin{pmatrix} \nu \\ \frac{\mu}{\rho} u_{xx} - \frac{\mu_1}{\rho} \nu - \frac{\mu_2}{\rho} z(x,1,t) + \frac{b}{\rho} \varphi_x - \frac{\gamma}{\rho} \theta_x \\ \psi \\ \frac{l}{J} \varphi_{xx} - \frac{b}{J} u_x - \frac{\xi}{J} \varphi - \frac{d}{J} w_x + \frac{m}{J} \theta + \frac{1}{J} \int_0^\infty g(s) \eta^t_{xx}(x,s) ds \\ \frac{k_0}{c} \theta_{xx} - \frac{\gamma}{c} \nu_x - \frac{m}{c} \psi - \frac{k_1}{c} w_x \\ \frac{k_2}{\alpha} w_{xx} - \frac{k_3}{\alpha} w - \frac{k_1}{\alpha} \theta_x - \frac{d}{\alpha} \psi_x \\ -\eta^t_s + \psi \\ -(1/\tau) z_p \end{pmatrix}.$$

with domain

$$\mathcal{D}(\mathcal{A}) = \Big\{U \in \mathcal{H}:\ u \in H^2_*(0,1)\cap H^1_*(0,1); \nu \in H^1_*(0,1); \varphi \in H^2(0,1)\cap H^1_0(0,1);$$
$$\psi \in H^1_0(0,1); \theta \in H^2(0,1)\cap H^1_0(0,1); w \in H^2_*(0,1)\cap H^1_*(0,1); \eta^t \in L_g;$$
$$z \in H^1_*(0,1)\Big\}.$$

Now, we state the following well-posedness result.

Theorem 6.3.1. *Assume that $\mu_2 \leqslant \mu_1$ and (6.4) holds, then for any $U_0 \in \mathcal{H}$ there exists a unique solution $U \in C(\mathbb{R}_+, \mathcal{H})$ of problem (6.14)-(6.16). Moreover, if $U_0 \in \mathcal{D}(\mathcal{A})$, then*

$$U \in C(\mathbb{R}_+,\ \mathcal{D}(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

Proof. We employ the semigroup technique to prove Theorem 6.3.1. Specifically, we prove that the operator $\mathcal{A}$ generates a C_0-semigroup on $\mathcal{H}$.

We first show that the operator $\mathcal{A}$ is dissipative. Using the inner product (6.20), for any $U \in \mathcal{D}(\mathcal{A})$, we get

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle_{\mathcal{H}} =& \rho\int_0^1 \left(\frac{\mu}{\rho}u_{xx} - \frac{\mu_1}{\rho}\nu - \frac{\mu_2}{\rho}z(x,1) + \frac{b}{\rho}\varphi_x - \frac{\gamma}{\rho}\theta_x\right)\nu dx + \mu\int_0^1 \nu_x u_x dx \\
&+ J\int_0^1 \left(\frac{l}{J}\varphi_{xx} - \frac{b}{J}u_x - \frac{\xi}{J}\varphi - \frac{d}{J}w_x + \frac{m}{J}\theta + \frac{1}{J}\int_0^\infty g(s)\eta^t_{xx}ds\right)\psi dx \\
&+ b\int_0^1 (\nu_x\varphi + u_x\psi)dx + \xi\int_0^1 \psi\varphi dx + l\int_0^1 \psi_x\varphi_x dx \\
&+ \alpha\int_0^1 \left(\frac{k_2}{\alpha}w_{xx} - \frac{k_3}{\alpha}w - \frac{k_1}{\alpha}\theta_x - \frac{d}{\alpha}\psi_x\right) w dx \\
&+ c\int_0^1 \left(\frac{k_0}{c}\theta_{xx} - \frac{\gamma}{c}\nu_x - \frac{m}{c}\psi - \frac{k_1}{c}w_x\right)\theta dx \\
&+ \langle -\eta^t_s + \psi, \eta^t\rangle_{L_g} - \frac{\kappa}{\tau}\int_0^1\int_0^1 z(x,p)z_p(x,p)dp\,dx \\
=& -\mu_1\int_0^1 \nu^2 dx - \mu_2\int_0^1 z(x,1)\nu dx + \int_0^1 \left(\int_0^\infty g(s)\eta^t_{xx}ds\right)\psi dx \\
&+ k_2\int_0^1 w_{xx}w dx - k_3\int_0^1 w^2 dx + k_0\int_0^1 \theta_{xx}\theta dx + \langle -\eta^t_s + \psi, \eta^t\rangle_{L_g} \\
&- \frac{\kappa}{\tau}\int_0^1\int_0^1 z(x,p)z_p(x,p)dp\,dx.
\end{aligned}$$

Take into consideration that

$$\begin{aligned}\langle -\eta_s^t+\psi,\eta^t\rangle_{L_g} =&\int_0^1\left(\int_0^\infty g(s)(-\eta_{sx}^t+\psi_x)\eta_x^t ds\right)dx\\ =&-\int_0^1\left(\int_0^\infty g(s)\eta_{sx}^t\eta_x^t ds\right)dx+\int_0^1\left(\int_0^\infty g(s)\eta_x^t ds\right)\psi_x dx\\ =&\frac{1}{2}\int_0^\infty g'(s)(\eta_x^t)^2 ds-\int_0^1\left(\int_0^\infty g(s)\eta_{xx}^t ds\right)\psi dx\end{aligned}$$

then,

$$\begin{aligned}\langle -\eta_s^t+\psi,\eta^t\rangle_{L_g} =&\frac{1}{2}\int_0^\infty g'(s)\left(\varphi_x(x,t)-\varphi_x(x,t-s)\right)^2 ds\\ &-\int_0^1\left(\int_0^\infty g(s)\eta_{xx}^t ds\right)\psi dx\\ =&\frac{1}{2}(g'\circ\varphi_x)(t)-\int_0^1\left(\int_0^\infty g(s)\eta_{xx}^t ds\right)\psi dx,\end{aligned}$$

hence,

$$\begin{aligned}\langle \mathcal{A}U,U\rangle_{\mathcal{H}} =&-\mu_1\int_0^1\nu^2 dx-\mu_2\int_0^1 z(x,1)\nu dx+\frac{1}{2}(g'\circ\varphi_x)(t)\\ &-k_3\int_0^1 w^2 dx-k_2\int_0^1 w_x^2 dx-k_0\int_0^1\theta_x^2 dx\\ &-\frac{\kappa}{\tau}\int_0^1\int_0^1 z(x,p)z_p(x,p)dp\,dx.\end{aligned}\tag{6.22}$$

Considering the last term of (6.22), we obtain

$$\begin{aligned}\int_0^1\int_0^1 z(x,p)z_p(x,p)dp\,dx =&\int_0^1\int_0^1\frac{1}{2}\frac{\partial}{\partial p}z^2(x,p)dp\,dx\\ =&\frac{1}{2}\int_0^1(z^2(x,1)-z^2(x,0))dx,\end{aligned}$$

hence, (6.22) simplifies to

$$\begin{aligned}\langle \mathcal{A}U,U\rangle_{\mathcal{H}} =&-\mu_1\int_0^1\nu^2 dx-\mu_2\int_0^1 z(x,1)\nu dx+\frac{1}{2}(g'\circ\varphi_x)(t)\\ &-k_3\int_0^1 w^2 dx-k_2\int_0^1 w_x^2 dx-k_0\int_0^1\theta_x^2 dx\\ &-\frac{\kappa}{2\tau}\int_0^1 z^2(x,1)dx+\frac{\kappa}{2\tau}\int_0^1\nu^2 dx.\end{aligned}\tag{6.23}$$

Applying Young's inequality to (6.23), we find

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \leqslant &- \mu_1 \int_0^1 \nu^2 dx + \frac{\mu_2}{2}\int_0^1 z^2(x,1)dx + \frac{\mu_2}{2}\int_0^1 \nu^2 dx \\ &- \frac{\kappa}{2\tau}\int_0^1 z^2(x,1)dx + \frac{\kappa}{2\tau}\int_0^1 \nu^2 dx + \frac{1}{2}(g' \circ \varphi_x)(t) \\ &- k_3 \int_0^1 w^2 dx - k_2 \int_0^1 w_x^2 dx - k_0 \int_0^1 \theta_x^2 dx,\end{aligned}$$

then

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \leqslant &\Big(-\mu_1 + \frac{\mu_2}{2} + \frac{\kappa}{2\tau}\Big)\int_0^1 \nu^2 dx + \Big(\frac{\mu_2}{2} - \frac{\kappa}{2\tau}\Big)\int_0^1 z^2(x,1)dx \\ &+ \frac{1}{2}(g' \circ \varphi_x)(t) - k_3\int_0^1 w^2 dx - k_2 \int_0^1 w_x^2 dx - k_0 \int_0^1 \theta_x^2 dx,\end{aligned} \tag{6.24}$$

After taking condition (6.21) into account, we can see that

$$\Big(-\mu_1 + \frac{\mu_2}{2} + \frac{\kappa}{2\tau}\Big) \leqslant 0 \quad \text{and} \quad \Big(\frac{\mu_2}{2} - \frac{\kappa}{2\tau}\Big) \leqslant 0.$$

As a result, the operator $\mathcal{A}$ is dissipative.

Next, we show that the operator $\mathcal{A}$ is maximal. It is sufficient to prove that $(I - A)$ is surjective. To this end, we take $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8) \in \mathcal{H}$, and prove that there exists a unique $U \in \mathcal{D}(\mathcal{A})$ such that

$$(I - A)U = F,$$

that is,

$$\begin{cases} u - \nu = f_1, \\ \rho\nu - \mu u_{xx} + \mu_1 \nu + \mu_2 z(x,1) - b\varphi_x + \gamma\theta_x = \rho f_2, \\ \varphi - \psi = f_3, \\ J\psi - l\varphi_{xx} + bu_x + \xi\varphi + dw_x - m\theta - \int_0^\infty g(s)\eta^t_{xx}(x,s)ds = Jf_4, \\ c\theta - k_0\theta_{xx} + \gamma\nu_x + m\psi + k_1 w_x = cf_5, \\ (\alpha + k_3)w - k_2 w_{xx} + k_1\theta_x + d\psi_x = \alpha f_6, \\ \eta^t + \eta^t_s - \psi = f_7, \\ z + \frac{1}{\tau} z_p = f_8. \end{cases} \tag{6.25}$$

From $(6.25)_7$ and (6.13) we obtain the following initial value problem

$$\begin{cases} \eta^t_s + \eta^t = \psi + f_7, \\ \eta^t(0) = 0, \end{cases} \tag{6.26}$$

solving (6.26), gives

$$\eta^t = e^{-s}\int_0^s e^\tau(\psi + f_7(\tau))d\tau. \tag{6.27}$$

The first and the third equations in (6.25) give

$$\begin{cases}\nu = u - f_1,\\ \psi = \varphi - f_3.\end{cases} \tag{6.28}$$

Following the same approach as in [116], the last equation in (6.25) is a first order ODE

$$\frac{1}{\tau}z_p + z = f_8,$$

subject to the initial condition

$$z(x,0) = u_t(x,t) = \nu(x),\ x \in (0,1), \tag{6.29}$$

then, we get

$$z(x,p) = \nu(x)e^{-\tau p} + \tau e^{-\tau p}\int_0^p f_8(x,\sigma)e^{\tau\sigma}d\sigma.$$

From (6.28), we see that

$$z(x,p) = u(x)e^{-\tau p} - f_1 e^{-\tau p} + \tau e^{-\tau p}\int_0^p f_8(x,\sigma)e^{\tau\sigma}d\sigma \tag{6.30}$$

and that

$$z(x,1) = u(x)e^{-\tau} + z_0(x), \tag{6.31}$$

where

$$z_0(x) = -f_1 e^{-\tau} + \tau e^{-\tau}\int_0^1 f_8(x,\sigma)e^{\tau\sigma}d\sigma.$$

Inserting (6.27), (6.28), (6.31) in $(6.25)_2$, $(6.25)_4$, $(6.25)_5$, $(6.25)_6$, we get

$$\begin{cases}\rho u - \rho f_1 - \mu u_{xx} + \mu_1 u + \mu_1 f_1 + \mu_2(u(x)e^{-\tau}\\ \qquad\qquad + z_0(x)) - b\varphi_x + \gamma\theta_x = \rho f_2,\\ J(\varphi - f_3) - l\varphi_{xx} + bu_x + \xi\varphi + dw_x - m\theta\\ -\int_0^\infty g(s)e^{-s}\left(\int_0^s e^\tau(f_7 - f_3)_{xx}d\tau\right)ds = Jf_4,\\ c\theta - k_0\theta_{xx} + \gamma u_x - \gamma f_{1x} + m\varphi - mf_3 + k_1 w_x = cf_5,\\ (\alpha + k_3)w - k_2 w_{xx} + k_1\theta_x + d(\varphi_x - f_{3x}) = \alpha f_6,\end{cases} \tag{6.32}$$

and hence

$$\begin{cases}\rho_1 u - \mu u_{xx} - b\varphi_x + \gamma\theta_x = h_1 \ \in L^2_*(0,1),\\ J_1\varphi - \left(l + \int_0^\infty g(s)(1-e^{-s})ds\right)\varphi_{xx} + bu_x \\ \qquad\qquad +dw_x - m\theta = h_2 \ \in L^2(0,1),\\ c\theta - k_0\theta_{xx} + \gamma u_x + m\varphi + k_1 w_x = h_3 \ \in L^2(0,1),\\ \alpha_1 w - k_2 w_{xx} + k_1\theta_x + d\varphi_x = h_4 \ \in L^2_*(0,1),\end{cases} \tag{6.33}$$

where

$$\begin{aligned} h_1 &= \rho(f_1+f_2) + \mu_1 f_1 - \mu_2 z_0(x),\\ h_2 &= J(f_3+f_4) + \int_0^\infty g(s)e^{-s}\left(\int_0^s e^\tau (f_7 - f_3)_{xx} d\tau\right) ds,\\ h_3 &= cf_5 + \gamma f_{1x} + mf_3,\\ h_4 &= \alpha f_6 + df_{3x},\\ \rho_1 &= \rho + \mu_1 + \mu_2 e^{-\tau},\ J_1 = J + \xi,\ \alpha_1 = \alpha + k_3. \end{aligned}$$

Let $\mathcal{V} = H^1_*(0,1) \times H^1_0(0,1) \times H^1_0(0,1) \times H^1_*(0,1)$. To solve (6.33), we consider

$$\mathcal{B}((u,\varphi,\theta,w),(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w})) = \mathcal{F}(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w}),\ \forall(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w}) \in \mathcal{V}, \tag{6.34}$$

where $\mathcal{B} : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}$ is the bilinear form defined by

$$\begin{aligned} \mathcal{B}((u,\varphi,\theta,w),(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w})) =& \rho_1 \int_0^1 u\tilde{u}dx + \mu\int_0^1 u_x\tilde{u}_x dx + J_1\int_0^1 \varphi\tilde{\varphi}dx\\ &+ \left(l + \int_0^\infty g(s)(1-e^{-s})ds\right)\int_0^1 \varphi_x\tilde{\varphi}_x dx + c\int_0^1 \theta\tilde{\theta}dx\\ &+ k_0\int_0^1 \theta_x\tilde{\theta}_x dx + \alpha_1\int_0^1 w\tilde{w}dx + k_2\int_0^1 w_x\tilde{w}_x dx\\ &+ \gamma\int_0^1 (u_x\tilde{\theta} + \tilde{u}\theta_x)dx + b\int_0^1 (u_x\tilde{\varphi} + \varphi\tilde{u}_x)dx\\ &+ d\int_0^1 (w_x\tilde{\varphi} + \tilde{w}\varphi_x)dx + k_1\int_0^1 (w_x\tilde{\theta} + \tilde{w}\theta_x)dx\\ &+ m\int_0^1 (\varphi\tilde{\theta} - \tilde{\varphi}\theta)dx. \end{aligned}$$

$\mathcal{F} : \mathcal{V} \longrightarrow \mathbb{R}$ is the linear function given by

$$\mathcal{F} = \int_0^1 h_1\tilde{u}dx + \int_0^1 h_2\tilde{\varphi}dx + \int_0^1 h_3\tilde{\theta}dx + \int_0^1 h_4\tilde{w}dx.$$

Recall that $\mu\xi > b^2$, it follows that

$$\mu u_x^2 + 2bu_x\varphi + J_1\varphi^2 > l_1 u_x^2 + l_2\varphi^2, \tag{6.35}$$

by means of the substitution

$$\begin{aligned}\mu u_x^2 + 2bu_x\varphi + J_1\varphi^2 =& \frac{1}{2}\left[\mu\left(u_x + \frac{b}{\mu}\varphi\right)^2 + J_1\left(\varphi + \frac{b}{J_1}u_x\right)^2\right] \\ &+ \frac{1}{2}\left[\left(\mu - \frac{b^2}{J_1}\right)u_x^2 + \left(J_1 - \frac{b^2}{\mu}\right)\varphi^2\right],\end{aligned}$$

where $l_1 = \frac{1}{2}\left(\mu - \frac{b^2}{J_1}\right) > 0$, $l_2 = \frac{1}{2}\left(J_1 - \frac{b^2}{\mu}\right) > 0$. Now, we equip $\mathcal{V}$ with the following norm:

$$\|(u,\varphi,\theta,w)\|_{\mathcal{V}}^2 = \|u\|^2 + \|u_x\|^2 + \|\varphi\|^2 + \|\varphi_x\|^2 + \|\theta\|^2 + \|\theta_x\|^2 + \|w\|^2 + \|w_x\|^2.$$

Then, for any $(u,\varphi,\theta,w) \in V$, we find that

$$\begin{aligned}\mathcal{B}((u,\varphi,\theta,w),(u,\varphi,\theta,w)) =& \rho_1\|u\|^2 + \mu\|u_x\|^2 + J_1\|\varphi\|^2 + \left(l + \int_0^\infty g(s)(1-e^{-s})ds\right)\|\varphi_x\|^2 \\ &+ 2b\int_0^1 \varphi u_x dx + c\|\theta\|^2 + k_0\|\theta_x\|^2 + \alpha_1\|w\|^2 + k_2\|w_x\|^2.\end{aligned}$$

Apply (6.35), we conclude that

$$\begin{aligned}|\mathcal{B}((u,\varphi,\theta,w),(u,\varphi,\theta,w))| \geqslant& \rho_1\|u\| + l_1\|u_x\|^2 + l_2\|\varphi\|_2^2 + \left(l + \int_0^\infty g(s)(1-e^{-s})ds\right)\|\varphi_x\|^2 \\ &+ c\|\theta\|^2 + k_0\|\theta_x\|^2 + \alpha_1\|w\|^2 + k_2\|w_x\|^2.\end{aligned}$$

Hence, there exists a constant $M_0 = \min\{\rho_1, l_1, l_2, \delta, c, k_0, \alpha_1, k_2\} > 0$ such that

$$|\mathcal{B}((u,\varphi,\theta,w),(u,\varphi,\theta,w))| \geqslant M_0\|(u,\varphi,\theta,w)\|_{\mathcal{V}}^2,$$

which means that $\mathcal{B}$ is coercive.

By using Cauchy-Schwarz inequality with $M_1 = \max\{\rho_1, \mu, J_1, \delta, c, k_0, \alpha_1, k_2, \gamma, b, d, k_1, m\}$, we get

$$\begin{aligned}|\mathcal{B}((u,\varphi,\theta,w),(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w}))|^2 \leqslant M_1^2\Big(&\|u\|\|\tilde{u}\| + \|u_x\|\|\tilde{u}_x\| + \|\varphi\|\|\tilde{\varphi}\| + \|\varphi_x\|\|\tilde{\varphi}_x\| \\ &+ \|\theta\|\|\tilde{\theta}\| + \|\theta_x\|\|\tilde{\theta}_x\| + \|w\|\|\tilde{w}\| + \|w_x\|\|\tilde{w}_x\| \\ &+ \|u_x\|\|\tilde{\theta}\| + \|\theta_x\|\|\tilde{u}\| + \|u_x\|\|\tilde{\varphi}\| + \|\varphi\|\|\tilde{u}_x\| \\ &+ \|w_x\|\|\tilde{\varphi}\| + \|\varphi_x\|\|\tilde{w}\| + \|w_x\|\|\tilde{\theta}\| + \|\theta_x\|\|\tilde{w}\| \\ &+ \|\theta\|\|\tilde{\varphi}\| + \|\varphi\|\|\tilde{\theta}\|\Big)^2\end{aligned}$$

Consequently, we obtain

$$|\mathcal{B}((u,\varphi,\theta,w),(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w}))|^2 \leqslant 2M_1^2\Big(\|u\|^2\|\tilde{u}\|^2+\|u_x\|^2\|\tilde{u}_x\|^2+\|\varphi\|^2\|\tilde{\varphi}\|^2 \\ +\|\varphi_x\|^2\|\tilde{\varphi}_x\|^2+\|\theta\|^2\|\tilde{\theta}\|^2+\|\theta_x\|^2\|\tilde{\theta}_x\|^2 \\ +\|w\|^2\|\tilde{w}\|^2+\|w_x\|^2\|\tilde{w}_x\|^2+\|u_x\|^2\|\tilde{\theta}\|^2 \\ +\|\theta_x\|^2\|\tilde{u}\|^2+\|u_x\|^2\|\tilde{\varphi}\|^2+\|\varphi\|^2\|\tilde{u}_x\|^2 \\ +\|w_x\|^2\|\tilde{\varphi}\|^2+\|\varphi_x\|^2\|\tilde{w}\|^2+\|w_x\|^2\|\tilde{\theta}\|^2 \\ +\|\theta_x\|^2\|\tilde{w}\|^2+\|\theta\|^2\|\tilde{\varphi}\|^2+\|\varphi\|^2\|\tilde{\theta}\|^2\Big),$$

taking into consideratin that $(\|u\|^2 \leqslant \|(u,\varphi,\theta,w)\|_{\mathcal{V}}^2),\ \|\tilde{u}\|^2 \leqslant \|(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w})\|_{\mathcal{V}}^2,...)$, leads to

$$|\mathcal{B}((u,\varphi,\theta,w),(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w}))|^2 \leqslant 36M_1^2\|(u,\varphi,\theta,w)\|_{\mathcal{V}}^2\|(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w}))\|_{\mathcal{V}}^2.$$

Therefore,

$$|\mathcal{B}((u,\varphi,\theta,w),(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w}))| \leqslant \zeta_1\|(u,\varphi,\theta,w)\|_{\mathcal{V}}\|(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w}))\|_{\mathcal{V}},$$

where $\zeta_1 = 6M_1$. Similarly, applying Cauchy-Schwarz inequality, we find

$$|\mathcal{F}(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w}))|^2 \leqslant (\|h_1\|\|\tilde{u}\|+\|h_2\|\|\tilde{\varphi}\|+\|h_3\|\|\tilde{\theta}\|+\|h_4\|\|\tilde{w}\|)^2 \\ \leqslant 2(\|h_1\|^2\|\tilde{u}\|^2+\|h_2\|^2\|\tilde{\varphi}\|^2+\|h_3\|^2\|^2\tilde{\theta}\|^2+\|h_4\|^2\|\tilde{w}\|^2),$$

thus, there exists a positive constant M_2 such that

$$|\mathcal{F}(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w}))|^2 \leqslant 2M_2(\|\tilde{u}\|^2+\|\tilde{\varphi}\|^2+\|\tilde{\theta}\|^2+\|\tilde{w}\|^2).$$

Using again the fact that $(\|\tilde{u}\|^2 \leqslant \|(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w}))\|_{\mathcal{V}}^2),\ \|\tilde{\varphi}\|^2 \leqslant \|(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w}))\|_{\mathcal{V}}^2,...)$, we have

$$|\mathcal{F}(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w}))|^2 \leqslant 8M_2\|(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w}))\|^2{}_{\mathcal{V}};$$

hence, we have

$$|\mathcal{F}(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w}))| \leqslant \zeta_2\|(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w}))\|_{\mathcal{V}},$$

where $\zeta = 2\sqrt{2M_2}$. As a result, the Lax-Milgram Theorem guarantees the existence of a unique $(u,\varphi,\theta,w) \in \mathcal{V}$ satisfying, (6.34). Then from (6.28), we deduce that

$$(\nu,\psi) \in H_*^1(0,1) \times H_0^1(0,1). \tag{6.36}$$

By setting $(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w}))$ to $(\tilde{u},0,0,0)$ then to $(0,0,0,\tilde{w})$ in (6.34), we obtain

$$\rho_1\int_0^1 u\tilde{u}dx+\mu\int_0^1 u_x\tilde{u}_xdx+\gamma\int_0^1\tilde{u}\theta_xdx+b\int_0^1\varphi\tilde{u}_xdx=\int_0^1 h_1\tilde{u}dx,\ \forall\tilde{u}\in H_*^1(0,1) \tag{6.37}$$

and

$$
\begin{aligned}
\alpha_1 \int_0^1 w\tilde{w}dx + k_2 \int_0^1 w_x \tilde{w}_x dx + d \int_0^1 \tilde{w}\varphi_x dx \\
+ k_1 \int_0^1 \tilde{w}\theta_x dx = \int_0^1 h_4 \tilde{w} dx,\ \forall \tilde{w} \in H_*^1(0,1).
\end{aligned} \tag{6.38}
$$

The regularity theory cannot be applied directly in this context because $u, w \in H_*^1(0,1)$. Therefore, let $\hat{u}, \hat{w} \in H_0^1(0,1)$ and define

$$
\tilde{u} = \hat{u} - \int_0^1 \hat{u}(s)ds, \tag{6.39}
$$

$$
\tilde{w} = \hat{w} - \int_0^1 \hat{w}(s)ds, \tag{6.40}
$$

which implies $\tilde{u}, \tilde{w} \in H_*^1(0,1)$. Substituting (6.39) into (6.37) and (6.40) into (6.38), we arrive at

$$
\mu \int_0^1 u_x \hat{u}_x dx = \int_0^1 g_1 \hat{u} dx,\ \forall \hat{u} \in H_0^1(0,1)
$$

and

$$
k_2 \int_0^1 w_x \hat{w}_x dx = \int_0^1 g_2 \hat{w} dx,\ \forall \hat{w} \in H_0^1(0,1),
$$

where

$$
g_1 = -\rho_1 u - \gamma\theta_x + b\varphi_x + h_1 \in L_*^2(0,1),
$$

and

$$
g_2 = -\alpha_1 w - d\varphi_x - k_1\theta_x + h_4 \in L_*^2(0,1).
$$

Consequently

$$
u, w \in H^2(0,1),
$$

and, we have

$$
-\mu u_{xx} = -\rho_1 u - \gamma\theta_x + b\varphi_x + \rho(f_1 + f_2) + \mu_1 f_1 - \mu_2 z_0(x)
$$

and

$$
-k_2 w_{xx} = -\alpha_1 w - d\varphi_x - k_1\theta_x + \alpha f_6 + d f_{3x}.
$$

Considering that $f_1 = u - \mu$, $f_3 = \varphi - \psi$ and $z_0(x) = z(x,1) - u(x)e^{-\tau}$, we obtain

$$
\begin{aligned}
-\mu u_{xx} = & - \rho_1 u - \gamma\theta_x + b\varphi_x + \rho(u - \nu) + \rho f_2 \\
& + \mu_1(u - \nu) - \mu_2(z(x,1) - u(x)e^{-\tau})
\end{aligned}
$$

and

$$
-k_2 w_{xx} = -\alpha_1 w - d\varphi_x - k_1\theta_x + \alpha f_6 + d(\varphi - \psi)_x.
$$

Then,

$$\rho\nu - \mu u_{xx} + \mu_1\nu + \mu_2 z(x,1) - b\varphi_x + \gamma\theta_x = \rho f_2,$$

$$(\alpha + k_3)w - k_2 w_{xx} + k_1\theta_x + d\psi_x = \alpha f_6.$$

These solve $(6.25)_2$ and $(6.25)_6$, respectively. Moreover, since $-\mu u_{xx} = g_1$ and $-k_2 w_{xx} = g_2$, we have

$$-\mu \int_0^1 u_{xx}\Psi dx = \int_0^1 g_1\Psi dx,\ \forall\Psi \in H^1(0,1)$$

and

$$-k_2 \int_0^1 w_{xx}\Phi dx = \int_0^1 g_2\Phi dx,\ \forall\Phi \in H^1(0,1).$$

So, by applying integration by parts, we get

$$\mu u_x(1)\Psi(1) - \mu u_x(0)\Psi(0) - \mu \int_0^1 u_x\Psi_x dx = \int_0^1 g_1\Psi dx,\ \forall\Psi \in H^1(0,1)$$

and

$$k_2 w_x(1)\Phi(1) - \mu w_x(0)\Phi(0) - k_2 \int_0^1 w_x\Phi_x dx = \int_0^1 g_2\Phi dx,\ \forall\Phi \in H^1(0,1).$$

The fact that $H^1_* \subset H^1$ leads to

$$\mu u_x(1)\tilde{u}(1) - \mu u_x(0)\tilde{u}(0) - \mu \int_0^1 u_x\tilde{u}_x dx = \int_0^1 g_1\tilde{u} dx,\ \forall\tilde{u} \in H^1_*(0,1),$$

$$k_2 w_x(1)\tilde{w}(1) - \mu w_x(0)\tilde{w}(0) - k_2 \int_0^1 w_x\tilde{w}_x dx = \int_0^1 g_2\tilde{w} dx,\ \forall\tilde{w} \in H^1_*(0,1).$$

Thanks to (6.37) and (6.38) , we find

$$\mu u_x(1)\tilde{u}(1) - \mu u_x(0)\tilde{u}(0) = 0,\ k_2 w_x(1)\tilde{w}(1) - \mu w_x(0)\tilde{w}(0) = 0.$$

Since $\tilde{u}$ and $\tilde{w}$ are arbitrary, it follows that

$$u_x(0) = u_x(1) = w_x(0) = w_x(1) = 0.$$

Therefore

$$u, w \in H^2_*(0,1).$$

Next, by taking $(\tilde{u}, \tilde{\varphi}, \tilde{\theta}, \tilde{w})$ as $(0, \tilde{\varphi}, 0, 0)$ then as $(0, 0, \tilde{\theta}, 0)$ in (6.34), we find

$$J_1 \int_0^1 \varphi\tilde{\varphi} dx + \left(l + \int_0^\infty g(s)(1 - e^{-s}) ds\right) \int_0^1 \varphi_x\tilde{\varphi}_x dx + b \int_0^1 u_x\tilde{\varphi} dx$$
$$+ d \int_0^1 w_x\tilde{\varphi} dx - m \int_0^1 \theta\tilde{\varphi} dx = \int_0^1 h_2\tilde{\varphi} dx,\ \forall\tilde{\varphi} \in H^1_0(0,1),$$

and

$$c\int_0^1 \theta\tilde{\theta}dx + k_0\int_0^1 \theta_x\tilde{\theta}_x dx + \gamma\int_0^1 u_x\tilde{\theta}dx + k_1\int_0^1 w_x\tilde{\theta}dx$$
$$+ m\int_0^1 \varphi\tilde{\theta}dx = \int_0^1 h_3\tilde{\theta}dx,\ \forall\tilde{\theta}\in H_0^1(0,1).$$

Thus,

$$\left(l+\int_0^\infty g(s)(1-e^{-s})ds\right)\int_0^1 \varphi_x\tilde{\varphi}_x dx = \int_0^1 g_3\tilde{\varphi}dx,\ \forall\tilde{\varphi}\in H_0^1(0,1),$$

and

$$k_0\int_0^1 \theta_x\tilde{\theta}_x dx = \int_0^1 g_4\tilde{\theta}dx,\ \forall\tilde{\theta}\in H_0^1(0,1),$$

where

$$g_3 = -J_1\varphi - bu_x + dw_x + m\theta + h_2 \in L^2(0,1),$$

and

$$g_4 = -c\theta - \gamma u_x - k_1 w_x - m\varphi + h_3 \in L^2(0,1).$$

Then, the elliptic regularity implies that

$$\varphi,\theta\in H^2(0,1)$$

and, consequently, we get

$$-\left(l+\int_0^\infty g(s)(1-e^{-s})ds\right)\varphi_{xx} = - J_1\varphi - bu_x + dw_x + m\theta + J(f_3+f_4)$$
$$+\int_0^\infty g(s)e^{-s}\left(\int_0^s e^{\tau}(f_7-f_3)_{xx}d\tau\right)ds$$

and

$$-k_0\theta_{xx} = -c\theta - \gamma u_x - k_1 w_x - m\varphi + cf_5 + \gamma f_{1x} + mf_3.$$

Given that $f_1 = u-\nu$, $f_3 = \varphi - \psi$, we end up with

$$\begin{aligned} J\psi - l\varphi_{xx} + bu_x + \xi\varphi + dw_x - m\theta =& Jf_4 + \varphi_{xx}\int_0^\infty g(s)(1-e^{-s})ds \\ &+ \int_0^\infty g(s)e^{-s}\left(\int_0^s e^{\tau}(f_7+\psi-\varphi)_{xx}d\tau\right)ds \\ =& Jf_4 + \varphi_{xx}\int_0^\infty g(s)(1-e^{-s})ds \\ &+ \int_0^\infty g(s)\left[e^{-s}\int_0^s e^{\tau}(\psi+f_7(\tau))d\tau\right]_x xds \\ &- \int_0^\infty g(s)\left[e^{-s}\int_0^s e^{\tau}d\tau\right]ds\ \varphi_{xx} \end{aligned}$$

and

$$-k_0\theta_{xx} = -c\theta - \gamma u_x - k_1 w_x - m\varphi + cf_5 + \gamma(u-\nu)_x + m(\varphi - \psi).$$

Thus, (6.27) gives

$$J\psi - l\varphi_{xx} + bu_x + \xi\varphi + dw_x - m\theta - \int_0^\infty g(s)\eta^t_{xx}(x,s)ds = Jf_4,$$

and

$$c\theta - k_0\theta_{xx} + \gamma\nu_x + m\psi + k_1 w_x = cf_5.$$

wich solve $(6.25)_4$ and $(6.25)_5$.

As a result, $U \in \mathcal{D}(\mathcal{A})$ and $\mathcal{A}$ is a maximal monotone operator, it follows from the Lumer-Phillip Theorem 2.3.7, that $\mathcal{A}$ is the infinitesimal generator of C_0-semigroup of contractions on $\mathcal{H}$ and Theorm 2.3.9 provides the well-posedness. □

6.4 General stability result for $\mu_2 \leqslant \mu_1$

In this section, we prove the decay result stated in Theorem 6.4.2 under the assumption $\mu_2 \leqslant \mu_1$ by constructing an appropriate Lyapunov functional which is equivalent to the solution energy.

As a first step, we define the following energy functional

Lemma 6.4.1. *Let $(u,\varphi,\theta,w,\eta^t,z)$ be a solution of the problem (6.14)–(6.16) and (6.21) holds. Then the energy functional is defined by*

$$\begin{aligned} E(t) =& \frac{1}{2}\int_0^1 \left(\rho u_t^2 + J\varphi_t^2 + \mu u_x^2 + l\varphi_x^2 + c\theta^2 + \xi\varphi^2 + \alpha w^2 + 2b\varphi u_x\right) dx \\ &+ \frac{1}{2}(g\circ\varphi_x)(t) + \frac{\kappa}{2}\int_0^1\int_0^1 z^2(x,p,t)dp\,dx \end{aligned} \tag{6.41}$$

and satisfies

$$\begin{aligned} \frac{dE(t)}{dt} \leqslant& -\left(\mu_1 - \frac{\kappa}{2\tau} - \frac{\mu_2}{2}\right)\int_0^1 u_t^2 dx - \left(\frac{\kappa}{2\tau} - \frac{\mu_2}{2}\right)\int_0^1 z^2(x,1,t)dx \\ &- k_0\int_0^1 \theta_x^2 dx - k_2\int_0^1 w_x^2 dx - k_3\int_0^1 w^2 dx + \frac{1}{2}(g'\circ\varphi_x)(t) \leqslant 0. \end{aligned}$$

Proof. Multiplying $(6.14)_1$, $(6.14)_2$, $(6.14)_3$, $(6.14)_4$ by u_t, φ_t, θ, w, respectively, using (6.12) and integrate by parts, we obtain

$$\begin{aligned}\rho\int_0^1 u_{tt}u_t dx =&\mu\int_0^1 u_{xx}u_t dx-\mu_1\int_0^1\varphi_t^2 dx-\mu_2\int_0^1 z(x,1,t)\varphi_t dx\\&+b\int_0^1\varphi_x u_t dx-\gamma\int_0^1\theta_x u_t dx\\=&-\mu\int_0^1 u_x u_{tx}dx-\mu_1\int_0^1\varphi_t^2 dx-\mu_2\int_0^1 z(x,1,t)\varphi_t dx\\&-b\int_0^1 u_{tx}\varphi dx+\gamma\int_0^1 u_{tx}\theta dx,\end{aligned}$$

$$\begin{aligned}J\int_0^1\varphi_{tt}\varphi_t dx =&\delta\int_0^1\varphi_{xx}\varphi_t dx-b\int_0^1 u_x\varphi_t dx-\xi\int_0^1\varphi\varphi_t dx\\&-d\int_0^1 w_x\varphi_t dx+m\int_0^1\theta\varphi_t dx-\int_0^1\varphi_t\int_0^\infty g(s)\varphi_{xx}(t-s)dsdx\\=&-\delta\int_0^1\varphi_x\varphi_{tx}dx-b\int_0^1 u_x\varphi_t dx-\xi\int_0^1\varphi\varphi_t dx+d\int_0^1\varphi_{tx}w dx\\&+m\int_0^1\theta\varphi_t dx-\int_0^1\varphi_t\int_0^\infty g(s)\varphi_{xx}(t-s)dsdx,\end{aligned}$$

$$\begin{aligned}c\int_0^1\theta_t\theta dx =&k_0\int_0^1\theta_{xx}\theta dx-\gamma\int_0^1 u_{tx}\theta dx-m\int_0^1\varphi_t\theta dx-k_1\int_0^1 w_x\theta dx\\=&-k_0\int_0^1\theta_x^2 dx-\gamma\int_0^1 u_{tx}\theta dx-m\int_0^1\varphi_t\theta dx-k_1\int_0^1 w_x\theta dx,\end{aligned}$$

$$\begin{aligned}\alpha\int_0^1 w_t w dx =&k_2\int_0^1 w_{xx}w dx-k_3\int_0^1 ww dx-k_1\int_0^1\theta_x w dx-d\int_0^1\varphi_{tx}w dx\\=&-k_2\int_0^1 w_x^2 dx-k_3\int_0^1 w^2 dx-k_1\int_0^1\theta_x w dx-d\int_0^1\varphi_{tx}w dx.\end{aligned}$$

From the above equalities, we deduce

$$\begin{aligned}&\frac{d}{2dt}\int_0^1(\rho u_t^2+\mu u_x^2)dx+b\int_0^1 u_{tx}\varphi dx\\&\quad=-\mu_1\int_0^1\varphi_t^2 dx-\mu_2\int_0^1 z(x,1,t)\varphi_t dx+\gamma\int_0^1 u_{tx}\theta dx,\end{aligned}\tag{6.42}$$

$$\begin{aligned}&\frac{d}{2dt}\int_0^1(J\varphi_t^2+\delta\varphi_x^2+\xi\varphi^2)dx+b\int_0^1 u_x\varphi_t dx\\&\quad=d\int_0^1\varphi_{tx}w dx+m\int_0^1\theta\varphi_t dx-\int_0^1\varphi_t\int_0^\infty g(s)\varphi_{xx}(t-s)dsdx,\end{aligned}\tag{6.43}$$

$$\frac{d}{2dt}\int_0^1 c\theta^2 dx=-k_0\int_0^1\theta_x^2 dx-\gamma\int_0^1 u_{tx}\theta dx-m\int_0^1\varphi_t\theta dx+k_1\int_0^1 w\theta_x dx,\tag{6.44}$$

$$\frac{d}{2dt}\int_0^1 \alpha w^2 dx = -\,k_2\int_0^1 w_x^2 dx - k_3\int_0^1 w^2 dx - k_1\int_0^1 \theta_x w dx - d\int_0^1 \varphi_{tx} w dx. \tag{6.45}$$

Summing up (6.42)–(6.45), we end up with

$$\begin{aligned}
&\frac{d}{2dt}\int_0^1 (\rho u_t^2 + \mu u_x^2)dx + b\int_0^1 u_{tx}\varphi dx + \frac{d}{2dt}\int_0^1 (J\varphi_t^2 + \delta\varphi_x^2 + \xi\varphi^2)dx\\
&\quad + b\int_0^1 u_x\varphi_t dx + \frac{d}{2dt}\int_0^1 c\theta^2 dx + \frac{d}{2dt}\int_0^1 \alpha w^2 dx\\
&\quad = -\,\mu_1\int_0^1 \varphi_t^2 dx - \mu_2\int_0^1 z(x,1,t)\varphi_t dx + \gamma\int_0^1 u_{tx}\theta dx + d\int_0^1 \varphi_{tx} w dx\\
&\quad + m\int_0^1 \theta\varphi_t dx - \int_0^1 \varphi_t\int_0^\infty g(s)\varphi_{xx}(t-s)dsdx - k_0\int_0^1 \theta_x^2 dx\\
&\quad - \gamma\int_0^1 u_{tx}\theta dx - m\int_0^1 \varphi_t\theta dx + k_1\int_0^1 w\theta_x dx - k_2\int_0^1 w_x^2 dx\\
&\quad - k_3\int_0^1 w^2 dx - k_1\int_0^1 \theta_x w dx - d\int_0^1 \varphi_{tx} w dx,
\end{aligned}$$

which simplies to

$$\begin{aligned}
&\frac{d}{2dt}\int_0^1 \left(\rho u_t^2 + J\varphi_t^2 + \mu u_x^2 + \delta\varphi_x^2 + c\theta^2 + \xi\varphi^2 + \alpha w^2 + 2b\varphi u_x\right) dx\\
&\quad = -\mu_1\int_0^1 u_t^2 dx - \mu_2\int_0^1 z(x,1,t)u_t dx - k_0\int_0^1 \theta_x^2 dx - k_2\int_0^1 w_x^2 dx\\
&\qquad - k_3\int_0^1 w^2 dx - \int_0^1 \varphi_t\left(\int_0^\infty g(s)\varphi_{xx}(t-s)ds\right) dx.
\end{aligned} \tag{6.46}$$

The last term in (6.46) is handled as follows:

$$\begin{aligned}
&-\int_0^1 \varphi_t\left(\int_0^\infty g(s)\varphi_{xx}(t-s)ds\right) dx\\
&= -\int_0^1 \varphi_t\left(\int_0^\infty g(s)(\varphi_{xx}(x,t) - \eta_{xx}^t(x,s))ds\right) dx\\
&= -\int_0^1 \varphi_t\left(\int_0^\infty g(s)\varphi_{xx}(x,t)ds\right) dx + \int_0^1 \varphi_t\left(\int_0^\infty g(s)\eta_{xx}^t(x,s)ds\right) dx\\
&= -\int_0^\infty g(s)ds\int_0^1 \varphi_t\varphi_{xx}(x,t)dx + \int_0^1 \varphi_t\left(\int_0^\infty g(s)\eta_{xx}^t(x,s)ds\right) dx\\
&= +\int_0^\infty g(s)ds\int_0^1 \varphi_{tx}\varphi_x(x,t)dx + \int_0^1 \varphi_t\left(\int_0^\infty g(s)\eta_{xx}^t(x,s)ds\right) dx\\
&= \int_0^\infty g(s)ds\ \frac{d}{2dt}\int_0^1 \varphi_x^2 dx + \int_0^1 \varphi_t\left(\int_0^\infty g(s)\eta_{xx}^t(x,s)ds\right) dx.
\end{aligned} \tag{6.47}$$

Now, by using $(6.14)_5$ and integrating by part, we have

$$
\begin{aligned}
&\int_0^1 \varphi_t \left(\int_0^\infty g(s)\eta_{xx}^t(x,s)ds \right) dx \\
=& \int_0^1 (\eta_t^t + \eta_s^t) \left(\int_0^\infty g(s)\eta_{xx}^t(x,s)ds \right) dx \\
=& \int_0^\infty g(s) \left(\int_0^1 \eta_t^t \eta_{xx}^t(x,s)dx \right) ds + \int_0^\infty g(s) \left(\int_0^1 \eta_s^t \eta_{xx}^t(x,s)dx \right) ds \\
=& -\int_0^\infty g(s) \left(\int_0^1 \eta_{tx}^t \eta_x^t(x,s)dx \right) ds - \int_0^\infty g(s) \left(\int_0^1 \eta_{sx}^t \eta_x^t(x,s)dx \right) ds \\
=& -\int_0^1 \left(\int_0^\infty g(s)\eta_{tx}^t \eta_x^t(x,s)ds \right) dx - \int_0^1 \left(\int_0^\infty g(s)\eta_{sx}^t \eta_x^t(x,s)ds \right) dx \\
=& -\int_0^1 \left(\int_0^\infty g(s)\frac{d}{2dt}(\eta_x^t)^2 ds \right) dx - \int_0^1 \left(\frac{1}{2}\left[g(s)(\eta_x^t)^2 \right]_0^\infty - \frac{1}{2}\int_0^\infty g'(s)(\eta_x^t)^2 ds \right) dx,
\end{aligned}
$$

then,

$$
\begin{aligned}
&\int_0^1 \varphi_t \left(\int_0^\infty g(s)\eta_{xx}^t(x,s)ds \right) dx \\
=& -\frac{d}{2dt}\int_0^1 \left(\int_0^\infty g(s)(\eta_x^t)^2 ds \right) dx + \frac{1}{2}\int_0^1 \left(\int_0^\infty g'(s)(\eta_x^t)^2 ds \right) dx \\
=& -\frac{d}{2dt}\int_0^1 \left(\int_0^\infty g(s)(\varphi_x(x,t) - \varphi_x(x,t-s))^2 ds \right) dx \\
&+ \frac{1}{2}\int_0^1 \left(\int_0^\infty g'(s)(\varphi_x(x,t) - \varphi_x(x,t-s))^2 ds \right) dx \\
=& -\frac{d}{2dt}(g \circ \varphi_x)(t) + \frac{1}{2}(g' \circ \varphi_x)(t),
\end{aligned} \tag{6.48}
$$

Combining (6.46), (6.47) and (6.48), with the fact that $\delta - \int_0^\infty g(s) = l$, we find

$$
\begin{aligned}
&\frac{d}{2dt}\int_0^1 \left(\rho u_t^2 + J\varphi_t^2 + \mu u_x^2 + l\varphi_x^2 + c\theta^2 + \xi\varphi^2 + \alpha w^2 + 2b\varphi u_x \right) dx \\
&+ \frac{d}{2dt}(g \circ \varphi_x)(t) = -\mu_1 \int_0^1 u_t^2 dx - \mu_2 \int_0^1 z(x,1,t)u_t dx - k_0 \int_0^1 \theta_x^2 dx \\
&\qquad - k_2 \int_0^1 w_x^2 dx - k_3 \int_0^1 w^2 dx + \frac{1}{2}(g' \circ \varphi_x)(t).
\end{aligned} \tag{6.49}
$$

Moreover, by multiplying the last equation in (6.14) by $(\kappa/\tau)z$, we have

$$
\begin{aligned}
&\tau z_t(x,\rho,t)(\kappa/\tau)z(x,\rho,t) - z_\rho(x,\rho,t)(\kappa/\tau)z(x,\rho,t) \\
&= \frac{\kappa}{2}\frac{d}{dt}z^2(x,\rho,t) - \frac{\kappa}{\tau}z_\rho(x,\rho,t)z(x,\rho,t) = 0.
\end{aligned}
$$

Integrating the result, we obtain

$$\begin{aligned}
\frac{\kappa}{2}\frac{d}{dt}\int_0^1\int_0^1 z^2(x,p,t)dp\;dx =& -\frac{\kappa}{\tau}\int_0^1\int_0^1 zz_\rho(x,\rho,t)d\rho dx \\
=& -\frac{\kappa}{2\tau}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}z^2(x,\rho,t)d\rho dx \\
=& \frac{\kappa}{2\tau}\int_0^1 \left(z^2(x,0,t)-z^2(x,1,t)\right)dx.
\end{aligned} \tag{6.50}$$

Adding up (6.49), (6.50) to get

$$\begin{aligned}
E(t) =& \frac{1}{2}\int_0^1 \left(\rho u_t^2 + J\varphi_t^2 + \mu u_x^2 + l\varphi_x^2 + c\theta^2 + \xi\varphi^2 + \alpha w^2 + 2b\varphi u_x\right)dx \\
&+ \frac{1}{2}(g\circ\varphi_x)(t) + \frac{\kappa}{2}\int_0^1\int_0^1 z^2(x,p,t)dp\;dx
\end{aligned}$$

and

$$\begin{aligned}
\frac{dE(t)}{dt} =& -\mu_1\int_0^1 u_t^2dx - \mu_2\int_0^1 z(x,1,t)u_tdx - k_0\int_0^1 \theta_x^2dx - k_2\int_0^1 w_x^2dx \\
&- k_3\int_0^1 w^2dx + \frac{1}{2}(g'\circ\varphi_x)(t) + \frac{\kappa}{2\tau}\int_0^1 \left(z^2(x,0,t)-z^2(x,1,t)\right)dx.
\end{aligned}$$

Thus, we arrive at

$$\begin{aligned}
\frac{dE(t)}{dt} =& -\left(\mu_1 - \frac{\kappa}{2\tau}\right)\int_0^1 u_t^2dx - \frac{\kappa}{2\tau}\int_0^1 z^2(x,1,t)dx \\
&- \mu_2\int_0^1 u_tz(x,1,t)dx - k_0\int_0^1 \theta_x^2dx - k_2\int_0^1 w_x^2dx \\
&- k_3\int_0^1 w^2dx + \frac{1}{2}(g'\circ\varphi_x)(t).
\end{aligned} \tag{6.51}$$

Young's inequality yields

$$-\mu_2\int_0^1 u_tz(x,1,t)dx \leqslant \frac{\mu}{2}\int_0^1 u_t^2dx + \frac{\mu}{2}\int_0^1 z^2(x,1,t)dx. \tag{6.52}$$

A combination of (6.51) and (6.52) leads to

$$\begin{aligned}
\frac{dE(t)}{dt} \leqslant& -\left(\mu_1 - \frac{\kappa}{2\tau} - \frac{\mu_2}{2}\right)\int_0^1 u_t^2dx - \left(\frac{\kappa}{2\tau} - \frac{\mu_2}{2}\right)\int_0^1 z^2(x,1,t)dx \\
&- k_0\int_0^1 \theta_x^2dx - k_2\int_0^1 w_x^2dx - k_3\int_0^1 w^2dx + \frac{1}{2}(g'\circ\varphi_x)(t) \leqslant 0.
\end{aligned}$$

In light of (6.21), the last inequality implies that the energie E is a non-increasing function. □

The general decay result reads as follows:

Theorem 6.4.2. *Assume that $\mu_2 \leqslant \mu_1$, (6.4) holds and g satisfies (H1) and (H2) and for any $U_0 \in \mathcal{H}$ satisfying, for some $\kappa_0 \geqslant 0$*

$$\int_0^1 \varphi_{0x}(x,s)dx \leqslant \kappa_0, \ \forall s > 0, \tag{6.53}$$

then, there exist constants λ_0 , $\gamma_0 > 0$ such that for all $t \in \mathbb{R}_+$ and for all $\lambda_1 \in]0, \gamma_0]$,

$$E(t) \leqslant \lambda_0 \left(1 + \int_0^t (g(s))^{1-\lambda_1} ds\right) e^{-\lambda_1 \int_0^t \zeta(s)ds} + \lambda_0 \int_t^\infty g(s)ds. \tag{6.54}$$

Remark 6.4.3 ([71]). *We observe that the exponential and polynomial decay estimates are only particular cases of (6.54). Specifically, exponential decay is achieved when $\zeta(t) \equiv a$, and polynomial decay occurs when $\zeta(t) = a(1+t)^{-1}$, where a is a constant.*

6.4.1 The case $\mu_2 < \mu_1$

There exists $C > 0$ such that the energy function, given by (6.41), satisfies

$$\begin{aligned} \frac{dE(t)}{dt} \leqslant & - C\left(\int_0^1 u_t^2 dx + \int_0^1 z^2(x,1,t)dx\right) - k_0 \int_0^1 \theta_x^2 dx \\ & - k_2 \int_0^1 w_x^2 dx - k_3 \int_0^1 w^2 dx + \frac{1}{2}(g' \circ \varphi_x)(t). \end{aligned} \tag{6.55}$$

To prove Theorem 6.4.2, we need the following lemmas.

Lemma 6.4.4. *Let $(u, \varphi, \theta, w, \eta^t, z)$ be a solution of (6.14)–(6.16), then the functional*

$$I_1(t) = \rho \int_0^1 u_t u dx + \frac{\mu_1}{2} \int_0^1 u^2 dx, \ t \geqslant 0,$$

satisfies,

$$\frac{dI_1(t)}{dt} \leqslant -\frac{\mu}{2} \int_0^1 u_x^2 dx + \rho \int_0^1 u_t^2 dx + C_0 \int_0^1 \left(z^2(x,1,t) + \varphi^2 + \theta_x^2\right) dx. \tag{6.56}$$

Proof. By taking the derivative of I_1 and integrating by part, we conclude that

$$\frac{dI_1(t)}{dt} = -\mu \int_0^1 u_x^2 dx + \rho \int_0^1 u_t^2 dx - \mu_2 \int_0^1 z(x,1,t) u dx - b \int_0^1 \varphi u_x dx + \gamma \int_0^1 \theta u_x dx. \tag{6.57}$$

Applying Young's inequality, we get

$$-b \int_0^1 \varphi u_x dx \leqslant \frac{3b^2}{2\mu} \int_0^1 \varphi^2 dx + \frac{\mu}{6} \int_0^1 u_x^2 dx. \tag{6.58}$$

Young's and Poincaré's inequalities lead to

$$-\mu_2 \int_0^1 z(x,1,t) u dx \leqslant \frac{3\mu_2^2 c_p}{2\mu} \int_0^1 z^2(x,1,t) dx + \frac{\mu}{6} \int_0^1 u_x^2 dx, \tag{6.59}$$

$$\gamma \int_0^1 \theta u_x dx \leqslant \frac{3\gamma^2 c_p}{2\mu} \int_0^1 \theta_x^2 dx + \frac{\mu}{6} \int_0^1 u_x^2 dx. \tag{6.60}$$

By substituting (6.58)–(6.60) into (6.57),

$$\begin{aligned} \frac{dI_1(t)}{dt} \leqslant &- \frac{\mu}{2} \int_0^1 u_x^2 dx + \rho \int_0^1 u_t^2 dx + \frac{3\mu_2^2 c_p}{2\mu} \int_0^1 z^2(x,1,t) dx \\ &+ \frac{3b^2}{2\mu} \int_0^1 \varphi^2 dx + \frac{3\gamma^2 c_p}{2\mu} \int_0^1 \theta_x^2 dx, \end{aligned}$$

where $C_0 = \max\left\{ \frac{3\mu_2^2 c_p}{2\mu}, \frac{3b^2}{2\mu}, \frac{3\gamma^2 c_p}{2\mu} \right\}$. Thus, we obtain (6.56).

□

As stated in [108], let χ represent the solution of

$$-\chi_{xx} = \varphi_x \quad \text{with} \quad \chi(0) = \chi(1) = 0, \tag{6.61}$$

then we get

$$\chi(x,t) = -\int_0^x \varphi(y,t) dy + x \left(\int_0^1 \varphi(y,t) dy \right).$$

We have the following inequalities.

Lemma 6.4.5. *The solution of (6.61) satisfies*

$$\int_0^1 \chi_x^2 dx \leqslant \int_0^1 \varphi^2 dx, \tag{6.62}$$

and

$$\int_0^1 \chi_t^2 dx \leqslant \int_0^1 \varphi_t^2 dx. \tag{6.63}$$

Proof. We multiply equation (6.61) by χ and integrate by part, we find

$$\int_0^1 \chi_x^2 dx = -\int_0^1 \varphi \chi_x dx.$$

Using Cauchy Schwarz inequality, we obtain

$$\|\chi_x\|^2 \leqslant \|\varphi\| \|\chi_x\| \Longrightarrow \int_0^1 \chi_x^2 dx \leqslant \int_0^1 \varphi^2 dx.$$

Next, we differentiate (6.61) and multiply the result by χ_t to have

$$-\int_0^1 \chi_{xxt}\chi_t dx = \int_0^1 \varphi_{xt}\chi_t dx;$$

hence, by similar calculations, we get

$$\|\chi_{xt}\|^2 \leqslant \|\varphi_t\|\|\chi_{xt}\| \Longrightarrow \int_0^1 \chi_{xt}^2 dx \leqslant \int_0^1 \varphi_t^2 dx,$$

the best possible estimate for the Poincaré constant gives

$$\int_0^1 \chi_t^2 dx \leqslant \int_0^1 \chi_{xt}^2 dx \leqslant \int_0^1 \varphi_t^2 dx.$$

□

Lemma 6.4.6. *Let $(u, \varphi, \theta, w, \eta^t, z)$ be a solution of (6.14)–(6.16), then the functional*

$$I_2(t) = J\int_0^1 \varphi_t\varphi dx + \frac{b\rho}{\mu}\int_0^1 u_t\chi dx, \ t \geqslant 0,$$

satisfies, for any $\varepsilon_1 > 0$,

$$\begin{aligned}\frac{dI_2(t)}{dt} \leqslant & -\frac{l}{2}\int_0^1 \varphi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^1 \varphi^2 dx \\ & + (J+\varepsilon_1)\int_0^1 \varphi_t^2 dx + C_1\left(1 + \frac{1}{\varepsilon_1}\right)\int_0^1 u_t^2 dx \\ & + C_1\int_0^1 \left(z^2(x,1,t) + \theta_x^2 + w_x^2\right) dx + \frac{3g_0}{l}(g\circ\varphi_x)(t).\end{aligned} \tag{6.64}$$

Proof. By differentiating I_2, integrating by part with (6.61), we get

$$\begin{aligned}\frac{dI_2(t)}{dt} = & -(\delta - g_0)\int_0^1 \varphi_x^2 dx - \xi\int_0^1 \varphi^2 dx - d\int_0^1 w_x\varphi dx + m\int_0^1 \theta\varphi dx \\ & + J\int_0^1 \varphi_t^2 dx - \frac{b\mu_1}{\mu}\int_0^1 u_t\chi dx - \frac{b\mu_2}{\mu}\int_0^1 z(x,1,t)\chi dx \\ & + \frac{b^2}{\mu}\int_0^1 \varphi^2 dx - \frac{b\gamma}{\mu}\int_0^1 \theta_x\chi dx + \frac{b\rho}{\mu}\int_0^1 u_t\chi_t dx \\ & - \int_0^1 \varphi_x\left(\int_0^\infty g(s)(\varphi_x(t) - \varphi_x(t-s)ds\right) dx.\end{aligned} \tag{6.65}$$

Using Young's inequality and (6.63), we observe that

$$\frac{b\rho}{\mu}\int_0^1 u_t\chi_t dx \leqslant \frac{b^2\rho^2}{4\varepsilon_1\mu^2}\int_0^1 u_t^2 dx + \varepsilon_1\int_0^1 \varphi_t^2 dx. \tag{6.66}$$

Applying Young's and Poincaré's inequalities, we have

$$-d\int_0^1 w_x\varphi dx \leqslant \frac{3d^2c_p}{l}\int_0^1 w_x^2dx + \frac{l}{12}\int_0^1 \varphi_x^2dx. \tag{6.67}$$

Young's and Poincaré's inequalities with (6.62) give the following estimations

$$m\int_0^1 \theta\varphi dx \leqslant \frac{3m^2c_p^2}{l}\int_0^1 \theta_x^2dx + \frac{l}{12}\int_0^1 \varphi_x^2dx, \tag{6.68}$$

$$-\frac{b\mu_1}{\mu}\int_0^1 u_t\chi dx \leqslant \frac{3b^2\mu_1^2c_p^2}{l\mu^2}\int_0^1 u_t^2dx + \frac{l}{12}\int_0^1 \varphi_x^2dx, \tag{6.69}$$

$$-\frac{b\mu_2}{\mu}\int_0^1 z(x,1,t)\chi dx \leqslant \frac{3b^2\mu_2^2c_p^2}{l\mu^2}\int_0^1 z^2(x,1,t)dx + \frac{l}{12}\int_0^1 \varphi_x^2dx, \tag{6.70}$$

$$-\frac{b\gamma}{\mu}\int_0^1 \theta_x\chi dx \leqslant \frac{3b^2\gamma^2c_p^2}{l\mu^2}\int_0^1 \theta_x^2dx + \frac{l}{12}\int_0^1 \varphi_x^2dx. \tag{6.71}$$

Young's inequality and (6.11) leads to

$$-\int_0^1 \varphi_x\left(\int_0^\infty g(s)(\varphi_x(t)-\varphi_x(t-s)ds\right)dx \leqslant \frac{l}{12}\int_0^1 \varphi_x^2dx + \frac{3g_0}{l}(g\circ\varphi_x)(t). \tag{6.72}$$

By substituting (6.66)–(6.72) into (6.65),

$$\begin{aligned}\frac{dI_2(t)}{dt} \leqslant & -\frac{l}{2}\int_0^1 \varphi_x^2dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^1 \varphi^2dx + (J+\varepsilon_1)\int_0^1 \varphi_t^2dx \\ & + \left(\frac{3b^2\mu_1^2c_p^2}{l\mu^2} + \frac{b^2\rho^2}{4\varepsilon_1\mu^2}\right)\int_0^1 u_t^2dx + \frac{3b^2\mu_2^2c_p^2}{l\mu^2}\int_0^1 z^2(x,1,t)dx \\ & + \left(\frac{3m^2c_p^2}{l} + \frac{3b^2\gamma^2c_p^2}{l\mu^2}\right)\int_0^1 \theta_x^2dx + \frac{3d^2c_p}{l}\int_0^1 w_x^2dx + \frac{3g_0}{l}(g\circ\varphi_x)(t),\end{aligned}$$

in view of $C_1 = \max\left\{\frac{3b^2\mu_1^2c_p^2}{l\mu^2}, \frac{b^2\rho^2}{4\mu^2}, \frac{3b^2\mu_2^2c_p^2}{l\mu^2}, \left(\frac{3m^2c_p^2}{l} + \frac{3b^2\gamma^2c_p^2}{l\mu^2}\right), \frac{3d^2c_p}{l}\right\}$, we obtain (6.64). □

Lemma 6.4.7. *Let $(u,\varphi,\theta,w,\eta^t,z)$ be a solution of (6.14)–(6.16), then the functional*

$$I_3(t) = \int_0^1\int_0^1 e^{-2\tau p}z^2(x,p,t)dp\,dx,\ t \geqslant 0,$$

satisfies

$$\frac{dI_3(t)}{dt} = -2I_3(t) - \frac{e^{-2\tau}}{\tau}\int_0^1 z^2(x,1,t)dx + \frac{1}{\tau}\int_0^1 u_t^2dx. \tag{6.73}$$

Proof. Differentiating I_3 and using the last equation in (6.14), we have

$$\begin{aligned}\frac{d}{dt}\left(\int_0^1\int_0^1 e^{-2\tau p}z^2(x,p,t)dp\,dx\right) =&2\int_0^1\int_0^1 e^{-2\tau p}zz_t(x,p,t)dp\,dx\\ =&-\frac{2}{\tau}\int_0^1\int_0^1 e^{-2\tau p}zz_p(x,p,t)dp\,dx\\ =&-2\int_0^1\int_0^1 e^{-2\tau p}z^2(x,p,t)dp\,dx\\ &-\frac{1}{\tau}\int_0^1\int_0^1 \frac{\partial}{\partial p}\left(e^{-2\tau p}z^2(x,p,t)\right)dp\,dx\\ =&-2I_3(t)-\frac{e^{-2\tau}}{\tau}\int_0^1 z^2(x,1,t)dx+\frac{1}{\tau}\int_0^1 u_t^2dx.\end{aligned}$$

□

Lemma 6.4.8. *Let $(u,\varphi,\theta,w,\eta^t,z)$ be a solution of (6.14)–(6.16), then the functional*

$$I_4(t)=c\alpha\int_0^1\theta\left(\int_0^x w(y)dy\right)dx,\ t\geqslant 0,$$

satisfies, for any $\varepsilon_2>0$,

$$\begin{aligned}\frac{dI_4(t)}{dt}\leqslant&-\frac{ck_1}{2}\int_0^1\theta^2dx+C_2\int_0^1\left(u_t^2+\theta_x^2+w_x^2\right)dx\\ &+(C_2+\varepsilon_2)\int_0^1\varphi_t^2dx+C_2\left(1+\frac{1}{\varepsilon_2}\right)\int_0^1 w^2dx.\end{aligned}\tag{6.74}$$

Proof. Direct computations, with integration by parts, give

$$\begin{aligned}\frac{dI_4(t)}{dt}=&-ck_1\int_0^1\theta^2dx+\alpha k_1\int_0^1 w^2dx+ck_2\int_0^1\theta w_xdx-cd\int_0^1\theta\varphi_tdx\\ &-\alpha k_0\int_0^1\theta_xwdx-ck_3\int_0^1\theta\left(\int_0^x w(y)dy\right)dx+\alpha\gamma\int_0^1 wu_tdx\\ &-\alpha m\int_0^1\varphi_t\left(\int_0^x w(y)dy\right)dx.\end{aligned}\tag{6.75}$$

Applying Young's inequality, we get

$$ck_2\int_0^1\theta w_xdx\leqslant\frac{3ck_2^2}{2k_1}\int_0^1 w_x^2dx+\frac{ck_1}{6}\int_0^1\theta^2dx,\tag{6.76}$$

$$-cd\int_0^1\theta\varphi_tdx\leqslant\frac{3cd^2}{2k_1}\int_0^1\varphi_t^2dx+\frac{ck_1}{6}\int_0^1\theta^2dx,\tag{6.77}$$

$$-\alpha k_0\int_0^1\theta_x w dx \leqslant c_1\left(\int_0^1(\theta_x^2+w^2)dx\right), \tag{6.78}$$

$$\alpha\gamma\int_0^1 wu_t dx \leqslant c_2\left(\int_0^1(u_t^2+w^2)dx\right). \tag{6.79}$$

Using Young's and Cauchy Schwarz inequalities, we obtain

$$-ck_3\int_0^1\theta\left(\int_0^x w(y)dy\right)dx \leqslant \frac{3ck_3^2}{2k_1}\int_0^1 w^2dx+\frac{ck_1}{6}\int_0^1\theta^2dx, \tag{6.80}$$

$$-\alpha m\int_0^1\varphi_t\left(\int_0^x w(y)dy\right)dx \leqslant \frac{\alpha^2m^2}{4\varepsilon_2}\int_0^1 w^2dx+\varepsilon_2\int_0^1\varphi_t^2dx. \tag{6.81}$$

Substituting (6.76)–(6.81) into (6.75), we arrive at

$$\begin{aligned}\frac{dI_4(t)}{dt}\leqslant&-\frac{ck_1}{2}\int_0^1\theta^2dx+c_2\int_0^1u_t^2dx+c_1\int_0^1\theta_x^2dx+\frac{3ck_2^2}{2k_1}\int_0^1w_x^2dx\\&+\left(\frac{3cd^2}{2k_1}+\varepsilon_2\right)\int_0^1\varphi_t^2dx+\left(c_1+c_2+\frac{3ck_3^2}{2k_1}+\frac{\alpha^2m^2}{4\varepsilon_2}\right)\int_0^1w^2dx,\end{aligned}$$

taking $C_2=\max\left\{c_1,c_2,\frac{3ck_2^2}{2k_1},\frac{3cd^2}{2k_1},c_1+c_2+\frac{3ck_3^2}{2k_1},\frac{\alpha^2m^2}{4}\right\}$, we obtain (6.74). □

Lemma 6.4.9. *Let $(u,\varphi,\theta,w,\eta^t,z)$ be a solution of (6.14)–(6.16), then the functional*

$$I_5(t)=-J\int_0^1\varphi_t\left(\int_0^\infty g(s)(\varphi(t)-\varphi(t-s))ds\right)dx,\ t\geqslant 0,$$

satisfies, for any ε_3 , $\varepsilon_4>0$,

$$\begin{aligned}\frac{dI_5(t)}{dt}\leqslant&-\frac{Jg_0}{2}\int_0^1\varphi_t^2dx+3\varepsilon_3\int_0^1\varphi_x^2dx\ +\varepsilon_4\int_0^1u_x^2dx+C_3\int_0^1(\theta_x^2+w^2)dx\\&-\frac{Jd_2}{2g_0}(g'\circ\varphi_x)(t)+C_3\left(1+\frac{1}{\varepsilon_3}+\frac{1}{\varepsilon_4}\right)(g\circ\varphi_x)(t).\end{aligned} \tag{6.82}$$

Proof. First we note that

$$\begin{aligned}\frac{\partial}{\partial t}\left(\int_0^\infty g(s)(\varphi(t)-\varphi(t-s))ds\right)=&\frac{\partial}{\partial t}\left(\int_{-\infty}^t g(t-s)(\varphi(t)-\varphi(s))ds\right)\\=&\int_{-\infty}^t g'(t-s)(\varphi(t)-\varphi(s))ds+\int_{-\infty}^t g(t-s)\varphi_t(t)ds\\=&\int_0^\infty g'(s)(\varphi(t)-\varphi(t-s))ds+\varphi_t(t)\int_0^\infty g(s)ds\\=&g_0\varphi_t(t)+\int_0^\infty g'(s)(\varphi(t)-\varphi(t-s))ds.\end{aligned}$$

Then, by differentiating I_5, we find

$$\begin{aligned}\frac{dI_5(t)}{dt} =& - J\int_0^1 \varphi_{tt}\left(\int_0^\infty g(s)(\varphi(t)-\varphi(t-s))ds\right)dx\\ &- J\int_0^1 \varphi_t\left(g_0\varphi_t(t)+\int_0^\infty g'(s)(\varphi(t)-\varphi(t-s))ds\right)dx,\end{aligned}$$

making use of $(6.14)_2$ and integrating by parts, we get

$$\begin{aligned}\frac{dI_5(t)}{dt} =&\delta\int_0^1 \varphi_x\left(\int_0^\infty g(s)(\varphi_x(t)-\varphi_x(t-s))ds\right)dx\\ &- J\int_0^1 \varphi_t\left(\int_0^\infty g'(s)(\varphi(t)-\varphi(t-s))ds\right)dx\\ &+ b\int_0^1 u_x\left(\int_0^\infty g(s)(\varphi(t)-\varphi(t-s))ds\right)dx\\ &+ \xi\int_0^1 \varphi\left(\int_0^\infty g(s)(\varphi(t)-\varphi(t-s))ds\right)dx\\ &- d\int_0^1 w\left(\int_0^\infty g(s)(\varphi_x(t)-\varphi_x(t-s))ds\right)dx\\ &- m\int_0^1 \theta\left(\int_0^\infty g(s)(\varphi(t)-\varphi(t-s))ds\right)dx - Jg_0\int_0^1 \varphi_t^2dx\\ &- \int_0^1\left(\int_0^\infty g(s)\varphi_x(x,t-s)ds\right)\left(\int_0^\infty g(s)(\varphi_x(t)-\varphi_x(t-s))ds\right)dx.\end{aligned} \tag{6.83}$$

Note that

$$\begin{aligned}&-\int_0^1\left(\int_0^\infty g(s)\varphi_x(t-s)ds\right)\left(\int_0^\infty g(s)(\varphi_x(t)-\varphi_x(t-s))ds\right)dx\\ =&\int_0^1\left(\int_0^\infty g(s)(\varphi_x(t)-\varphi_x(t-s))ds-g_0\varphi_x(t)\right)\left(\int_0^\infty g(s)(\varphi_x(t)-\varphi_x(t-s))ds\right)dx\\ =&\int_0^1\left(\int_0^\infty g(s)(\varphi_x(t)-\varphi_x(t-s))ds\right)^2dx - g_0\int_0^1\varphi_x(t)\left(\int_0^\infty g(s)(\varphi_x(t)-\varphi_x(t-s))ds\right)dx.\end{aligned}$$

Young's inequality, (6.8), (6.9) and (6.11) lead to

$$\delta\int_0^1 \varphi_x\left(\int_0^\infty g(s)(\varphi_x(t)-\varphi_x(t-s))ds\right)dx \leqslant \varepsilon_3\int_0^1\varphi_x^2dx+\frac{\delta^2g_0}{4\varepsilon_3}(g\circ\varphi_x)(t), \tag{6.84}$$

$$-J\int_0^1 \varphi_t\left(\int_0^\infty g'(s)(\varphi(t)-\varphi(t-s))ds\right)dx \leqslant \frac{Jg_0}{2}\int_0^1\varphi_t^2dx-\frac{Jd_2}{2g_0}(g'\circ\varphi_x)(t), \tag{6.85}$$

$$b\int_0^1 u_x\left(\int_0^\infty g(s)(\varphi(t)-\varphi(t-s))ds\right)dx \leqslant \varepsilon_4\int_0^1 u_x^2dx+\frac{b^2d_1}{4\varepsilon_4}(g\circ\varphi_x)(t), \tag{6.86}$$

$$-d\int_0^1 w\left(\int_0^\infty g(s)(\varphi_x(t)-\varphi_x(t-s))ds\right)dx \leqslant c_3\left(\int_0^1 w^2dx+(g\circ\varphi_x)(t)\right), \tag{6.87}$$

$$\begin{aligned}&\int_0^1\left(\int_0^\infty g(s)(\varphi_x(t)-\varphi_x(t-s))ds\right)^2dx\\ &-g_0\int_0^1\varphi_x(t)\left(\int_0^\infty g(s)(\varphi_x(t)-\varphi_x(t-s))ds\right)dx\\ &\leqslant c_4\left(1+\frac{1}{\varepsilon_3}\right)(g\circ\varphi_x)(t)+\varepsilon_3\int_0^1\varphi_x^2dx.\end{aligned} \tag{6.88}$$

By using Young's and Poincaré's inequalities and (6.8), we have

$$\xi\int_0^1\varphi\left(\int_0^\infty g(s)(\varphi(t)-\varphi(t-s))ds\right)dx\leqslant\varepsilon_3\int_0^1\varphi_x^2dx+\frac{\xi^2d_1}{4\varepsilon_3}(g\circ\varphi_x)(t), \tag{6.89}$$

$$-m\int_0^1\theta\left(\int_0^\infty g(s)(\varphi(t)-\varphi(t-s))ds\right)dx\leqslant c_5\left(\int_0^1\theta_x^2dx+(g\circ\varphi_x)(t)\right). \tag{6.90}$$

Substituting (6.84)–(6.90) into (6.83), we conclude

$$\begin{aligned}\frac{dI_5(t)}{dt}\leqslant&-\frac{Jg_0}{2}\int_0^1\varphi_t^2dx+3\varepsilon_3\int_0^1\varphi_x^2dx\ +\varepsilon_4\int_0^1u_x^2dx\\ &+c_3\int_0^1w^2dx+c_5\int_0^1\theta_x^2dx-\frac{Jd_2}{2g_0}(g'\circ\varphi_x)(t)\\ &+\left(c_3+c_4+c_5+\frac{\delta^2g_0}{4\varepsilon_3}+\frac{c_4}{\varepsilon_3}+\frac{\xi^2d_1}{4\varepsilon_3}+\frac{b^2d_1}{4\varepsilon_4}\right)(g\circ\varphi_x)(t),\end{aligned}$$

setting $C_3=\max\left\{c_3+c_4+c_5,\dfrac{\delta^2g_0}{4}+c_4+\dfrac{\xi^2d_1}{4},\dfrac{b^2d_1}{4}\right\}$, we obtain (6.82). □

Next, we define a Lyapunov functional L and prove that it is equivalent to the energy functional E.

Lemma 6.4.10. *The Lyabunov functional defined by*

$$L(t)=NE(t)+I_1(t)+N_2I_2(t)+I_3(t)+I_4(t)+N_5I_5(t), \tag{6.91}$$

where N, N_2 and N_5 are positive real numbers to be chosen appropriately later, satisfies

$$\kappa_1E(t)\leqslant L(t)\leqslant\kappa_2E(t),\ \forall t\geqslant 0. \tag{6.92}$$

for two positive constants κ_1 and κ_2.

Proof. From (6.91) we have

$$|L(t)-NE(t)|=|I_1(t)+N_2I_2(t)+I_3(t)+I_4(t)+N_5I(t)|,$$

then, with the fact that $e^{-2\tau p} \leqslant 1$, we infer

$$
\begin{aligned}
|L(t)-NE(t)| \leqslant & \rho \int_0^1 |u_t u| dx + \frac{\mu}{2}\int_0^1 u^2 dx + JN_2 \int_0^1 |\varphi_t \varphi| dx + \frac{|b|\rho}{2\mu} N_2 \int_0^1 |u_t \chi| dx \\
& + \int_0^1 \int_0^1 z^2(x,p,t) dp\, dx + c\alpha \int_0^1 \left| \theta \left(\int_0^x w(y) dy \right) \right| dx \\
& + JN_5 \int_0^1 \left| \varphi_t \left(\int_0^\infty g(s)(\varphi(t)-\varphi(t-s)) ds \right) \right| dx.
\end{aligned}
$$

By using Young's and Poincaré's inequalities and (6.62), we get

$$
\begin{aligned}
|L(t)-NE(t)| \leqslant & \frac{\rho^2 c_p}{2l_1} \int_0^1 u_t^2 dx + \frac{l_1}{2}\int_0^1 u_x^2 dx + \frac{\mu c_p}{2}\int_0^1 u_x^2 dx + \frac{J^2 N_2^2}{2l_2}\int_0^1 \varphi_t^2 dx \\
& + \frac{l_2}{2}\int_0^1 \varphi^2 dx + \frac{b^2\rho^2 c_p^2}{8\mu^2 l} N_2^2 \int_0^1 u_t^2 dx + \frac{l}{2}\int_0^1 \varphi_x^2 dx + \frac{c\alpha}{2}\int_0^1 \theta^2 dx \\
& + \int_0^1 \int_0^1 z^2(x,p,t) dp\, dx + \frac{c\alpha}{2}\int_0^1 \left(\int_0^x w(y) dy \right)^2 dx \\
& + \frac{JN_5}{2}\int_0^1 \varphi_t^2 dx + \frac{JN_5}{2}\int_0^1 \left(\int_0^\infty g(s)(\varphi(t)-\varphi(t-s)) ds \right)^2 dx.
\end{aligned}
$$

Cauchy Schwarz inequality, (6.8) and (6.35) give

$$
\begin{aligned}
|L(t)-NE(t)| \leqslant & \left(\frac{\rho^2 c_p}{2l_1} + \frac{b^2\rho^2 c_p^2}{8\mu^2 l} N_2^2 \right) \int_0^1 u_t^2 dx + \left(\frac{\mu}{2} + \frac{\mu c_p}{2} \right) \int_0^1 u_x^2 dx \\
& + \left(\frac{J^2 N_2^2}{2l_2} + \frac{JN_5}{2} \right) \int_0^1 \varphi_t^2 dx + b \int_0^1 u_x \varphi dx + \frac{J_1}{2}\int_0^1 \varphi^2 dx \\
& + \frac{l}{2}\int_0^1 \varphi_x^2 dx + \int_0^1 \int_0^1 z^2(x,p,t) dp\, dx + \frac{c\alpha}{2}\int_0^1 \theta^2 dx \\
& + \frac{c\alpha}{2}\int_0^1 w^2 dx + \frac{JN_5 d_1}{2} (g \circ \varphi_x)(t).
\end{aligned}
$$

Thus the above inequality becomes, for some positive constant β,

$$
|L(t)-NE(t)| \leqslant \beta E(t),
$$

wich yields

$$
(N-\beta)E(t) \leqslant L(t) \leqslant (N+\beta)E(t).
$$

At this point, by choosing N sufficiently large, (6.92) occurs. □

By differentiating L and inserting (6.55), (6.56), (6.64), (6.73), (6.74), (6.82), we easily verify that

$$\begin{aligned}
\frac{dL(t)}{dt} \leqslant& \left(-\frac{\mu}{2}+\varepsilon_4 N_5\right)\int_0^1 u_x^2 dx \\
&- \left[NC-\rho-C_1\left(1+\frac{1}{\varepsilon_1}\right)N_2-\frac{1}{\tau}-C_2\right]\int_0^1 u_t^2 dx \\
&- 2I_3(t)-\left[NC+\frac{e^{-2\tau}}{\tau}-C_0-C_1N_2\right]\int_0^1 z^2(x,1,t)dx \\
&- \left[\left(\xi-\frac{b^2}{\mu}\right)N_2-C_0\right]\int_0^1 \varphi^2 dx-\left[\frac{l}{2}N_2-3\varepsilon_3N_5\right]\int_0^1 \varphi_x^2 dx \\
&- \left[\frac{Jg_0}{2}N_5-(J+\varepsilon_1)N_2-C_2-\varepsilon_2\right]\int_0^1 \varphi_t^2 dx \\
&- \frac{ck_1}{2}\int_0^1 \theta^2 dx-[Nk_0-C_0-C_1N_2-C_2-C_3N_5]\int_0^1 \theta_x^2 dx \\
&- \left[Nk_3-C_2\left(1+\frac{1}{\varepsilon_2}\right)-C_3N_5\right]\int_0^1 w^2 dx \\
&- [Nk_2-C_1N_2-C_2]\int_0^1 w_x^2 dx+\left[\frac{1}{2}N-\frac{Jd_2}{2g_0}N_5\right](g'\circ\varphi_x)(t) \\
&+ \left[\frac{3g_0}{l}N_2+C_3\left(1+\frac{1}{\varepsilon_3}+\frac{1}{\varepsilon_4}\right)N_5\right](g\circ\varphi_x)(t).
\end{aligned}$$

Now, setting $\varepsilon_1=\dfrac{Jg_0N_5}{4N_2}$, $\varepsilon_2=\dfrac{Jg_0N_5}{8}$, $\varepsilon_3=\dfrac{lN_2}{12N_5}$, $\varepsilon_4=\dfrac{\mu}{4N_5}$, we get

$$\begin{aligned}
\frac{dL(t)}{dt} \leqslant& -\frac{\mu}{4}\int_0^1 u_x^2 dx-\frac{l}{4}N_2\int_0^1 \varphi_x^2 dx-\frac{ck_1}{2}\int_0^1 \theta^2 dx \\
&- \left[NC-\rho-C_1\left(1+\frac{4N_2}{Jg_0N_5}\right)N_2-\frac{1}{\tau}-C_2\right]\int_0^1 u_t^2 dx \\
&- 2I_3(t)-\left[NC+\frac{e^{-2\tau}}{\tau}-C_0-C_1N_2\right]\int_0^1 z^2(x,1,t)dx \\
&- \left[\left(\xi-\frac{b^2}{\mu}\right)N_2-C_0\right]\int_0^1 \varphi^2 dx-\left[\frac{Jg_0}{8}N_5-JN_2-C_2\right]\int_0^1 \varphi_t^2 dx \\
&- [Nk_0-C_0-C_1N_2-C_2-C_3N_5]\int_0^1 \theta_x^2 dx \\
&- \left[Nk_3-C_2\left(1+\frac{8}{Jg_0N_5}\right)-C_3N_5\right]\int_0^1 w^2 dx \\
&- [Nk_2-C_1N_2-C_2]\int_0^1 w_x^2 dx+\left[\frac{1}{2}N-\frac{Jd_2}{2g_0}N_5\right](g'\circ\varphi_x)(t) \\
&+ \left[\frac{3g_0}{l}N_2+C_3\left(1+\frac{12N_5}{lN_2}+\frac{4N_5}{\mu}\right)N_5\right](g\circ\varphi_x)(t).
\end{aligned} \tag{6.93}$$

Following that, we select N_2 large enough so that

$$\left(\xi - \frac{b^2}{\mu}\right) N_2 - C_0 > 0.$$

When N_2 is fixed, we choose N_5 large enough to ensure that

$$\frac{Jg_0}{8} N_5 - JN_2 - C_2 > 0.$$

After fixing all the above constants, we select N to be sufficiently large such that (6.92) stays valid and,

$$\begin{cases} NC - \rho - C_1 \left(1 + \dfrac{4N_2}{Jg_0 N_5}\right) N_2 - \dfrac{1}{\tau} - C_2 > 0, \\ NC + \dfrac{e^{-2\tau}}{\tau} - C_0 - C_1 N_2 > 0, \\ Nk_0 - C_0 - C_1 N_2 - C_2 - C_3 N_5 > 0, \\ Nk_3 - C_2 \left(1 + \dfrac{8}{Jg_0 N_5}\right) - C_3 N_5 > 0, \\ Nk_2 - C_1 N_2 - C_2 > 0, \\ \dfrac{1}{2} N - \dfrac{Jd_2}{2g_0} N_5 > 0. \end{cases}$$

As a result, there exist two positive constants γ_1, γ_2 such that (6.93) takes the form

$$\begin{aligned} \frac{dL(t)}{dt} \leqslant & - \gamma_1 \int_0^1 (u_x^2 + u_t^2 + \varphi^2 + \theta^2 + \varphi_x^2 + \varphi_t^2 + w^2) dx \\ & - \gamma_1 \int_0^1 \int_0^1 z^2(x, p, t) dp \, dx + \gamma_2 (g \circ \varphi_x)(t). \end{aligned} \tag{6.94}$$

On the other hand, from equation (6.41), we have

$$\begin{aligned} E(t) \leqslant & \delta_1 \int_0^1 (u_x^2 + u_t^2 + \varphi^2 + \theta^2 + \varphi_x^2 + \varphi_t^2 + w^2) dx \\ & + \delta_1 \int_0^1 \int_0^1 z^2(x, p, t) dp \, dx + \delta_1 (g \circ \varphi_x)(t), \end{aligned}$$

by means of Young's inequality, wich implies that

$$\begin{aligned} & - \int_0^1 (u_x^2 + u_t^2 + \varphi^2 + \theta^2 + \varphi_x^2 + \varphi_t^2 + w^2) dx \\ & - \int_0^1 \int_0^1 z^2(x, p, t) dp \, dx - (g \circ \varphi_x)(t) \leqslant -\delta_2 E(t), \end{aligned} \tag{6.95}$$

where $\delta_2 \leqslant \dfrac{1}{\delta_1}$. The combination of (6.94) and (6.95) gives

$$\frac{dL(t)}{dt} \leqslant -\beta_1 E(t) + \beta_2 (g \circ \varphi_x)(t),\ \forall\, t \in \mathbb{R}_+. \tag{6.96}$$

Using (H_2), the fact that ζ is nonincreasing and (6.55), we obtain, for all $t \in \mathbb{R}_+$

$$\begin{aligned}
&\zeta(t) \int_0^1 \int_0^t g(s)(\varphi_x(x,t) - (\varphi_x(x,t-s))^2 ds dx \\
&\leqslant \int_0^1 \int_0^t \zeta(s) g(s)(\varphi_x(x,t) - (\varphi_x(x,t-s))^2 ds dx \\
&\leqslant -\int_0^1 \int_0^t g'(s)(\varphi_x(x,t) - (\varphi_x(x,t-s))^2 ds dx \\
&\leqslant -2\frac{dE(t)}{dt}.
\end{aligned} \tag{6.97}$$

On the other hand, the definition of E alongside its nonincreasing nature leads to

$$\int_0^1 \varphi_x^2(x,t) dx \leqslant \frac{2}{l} E(t) \leqslant \frac{2}{l} E(0),\ \forall\, t \in \mathbb{R}_+.$$

Hence, in view of (6.53), we get

$$\begin{aligned}
\int_0^1 (\varphi_x(x,t) - \varphi_x(x,t-s))^2 dx &\leqslant 2\int_0^1 \varphi_x^2(x,t) dx + 2\int_0^1 \varphi_x^2(x,t-s) dx \\
&\leqslant \frac{4}{l} E(0) + 2\kappa_0,\ \forall\, t,s \in \mathbb{R}_+.
\end{aligned}$$

Then, we deduce that

$$\begin{aligned}
&\zeta(t) \int_0^1 \int_t^\infty g(s)(\varphi_x(x,t) - (\varphi_x(x,t-s))^2 ds dx \\
&\leqslant \left(\frac{4}{l} E(0) + 2\kappa_0\right) \zeta(t) \int_t^\infty g(s) ds,\ \forall\, t \in \mathbb{R}_+.
\end{aligned} \tag{6.98}$$

Now multiplying (6.96) by $\zeta(t)$ and combning with (6.97) and (6.98), we obtain

$$\zeta(t)\frac{dL(t)}{dt} \leqslant -\beta_1 \zeta(t) E(t) - \beta_3 \frac{dE(t)}{dt} + \beta_4 h(t),\ \forall\, t \in \mathbb{R}_+, \tag{6.99}$$

where $\beta_3 = 2\beta_2$, $\beta_4 = \beta_2 \left(\dfrac{4}{l} E(0) + 2\kappa_0\right)$ and $h(t) = \zeta(t) \int_t^\infty g(s) ds$, which can be rewritten as

$$(\zeta(t) L(t) + \beta_3 E(t))' - \zeta'(t) L(t) \leqslant -\beta_1 \zeta(t) E(t) + \beta_4 h(t),\ \forall\, t \in \mathbb{R}_+.$$

Using the fact that $\zeta'(t) \leqslant 0$ and $L(t) \geqslant 0$, we have

$$(\zeta(t)L(t) + \beta_3 E(t))' \leqslant -\beta_1 \zeta(t) E(t) + \beta_4 h(t), \ \forall\, t \in \mathbb{R}_+.$$

By exploiting (6.92), it can easily be shown that

$$R(t) = \zeta(t)L(t) + \beta_3 E(t) \sim E(t),$$

hence, there exist two positive constants κ_3 and κ_4 such that

$$\kappa_3 E(t) \leqslant R(t) \leqslant \kappa_4 E(t), \ \forall\, t \in \mathbb{R}_+. \tag{6.100}$$

Conseqently, for some positive constant λ_1, we obtain

$$R'(t) \leqslant -\lambda_1 \zeta(t) R(t) + \beta_4 h(t), \ \forall\, t \in \mathbb{R}_+.$$

Then

$$\left(e^{\lambda_1 \int_0^t \zeta(s)ds} R(t)\right)' \leqslant \beta_4 e^{\lambda_1 \int_0^t \zeta(s)ds} h(t), \ \forall\, t \in \mathbb{R}_+.$$

Therefore, by integrating over $[0, T]$, with $T \geqslant 0$, we arrive at

$$R(T) \leqslant e^{-\lambda_1 \int_0^T \zeta(s)ds} \left(R(0) + \beta_4 \int_0^T e^{\lambda_1 \int_0^t \zeta(s)ds} h(t)dt \right),$$

which implies, thanks to (6.100), that

$$E(T) \leqslant \frac{1}{\kappa_3} e^{-\lambda_1 \int_0^T \zeta(s)ds} \left(\kappa_4 E(0) + \beta_4 \int_0^T e^{\lambda_1 \int_0^t \zeta(s)ds} h(t)dt \right). \tag{6.101}$$

Note that

$$e^{\lambda_1 \int_0^t \zeta(s)ds} h(t) = \frac{1}{\lambda_1} \left(e^{\lambda_1 \int_0^t \zeta(s)ds} \right)' \int_t^\infty g(s)ds, \ \forall\, t \in \mathbb{R}_+,$$

then, by integrating by parts, we get

$$\int_0^T e^{\lambda_1 \int_0^t \zeta(s)ds} h(t)dt = \frac{1}{\lambda_1} \left(e^{\lambda_1 \int_0^T \zeta(s)ds} \int_T^\infty g(s)ds - \int_0^\infty g(s)ds + \int_0^T e^{\lambda_1 \int_0^t \zeta(s)ds} g(t)dt \right),$$

thus, combining with (6.101), yields

$$\begin{aligned} E(T) \leqslant & \frac{1}{\kappa_3} \left(\kappa_4 E(0) e^{-\lambda_1 \int_0^T \zeta(s)ds} + \frac{\beta_4}{\lambda_1} \int_T^{+\infty} g(s)ds \right) \\ & + \frac{\beta_4}{\kappa_3 \lambda_1} e^{-\lambda_1 \int_0^T \zeta(s)ds} \int_0^T e^{\lambda_1 \int_0^t \zeta(s)ds} g(t)dt. \end{aligned} \tag{6.102}$$

Moreover, (H_2) entails that

$$\left(e^{\lambda_1 \int_0^t \zeta(s)ds}(g(t))^{\lambda_1}\right)' \leqslant 0,\ \forall\, t \in \mathbb{R}_+$$

and, then

$$e^{\lambda_1 \int_0^t \zeta(s)ds}(g(t))^{\lambda_1} \leqslant (g(0))^{\lambda_1},\ \forall\, t \in \mathbb{R}_+.$$

Therefore

$$\int_0^T e^{\lambda_1 \int_0^t \zeta(s)ds} g(t)dt \leqslant (g(0))^{\lambda_1} \int_0^T (g(t))^{1-\lambda_1}dt. \tag{6.103}$$

Finally, (6.54) is established by combining (6.102) and (6.103), where

$$\lambda_0 = \frac{1}{\kappa_3} \max\left\{\kappa_4 E(0), \frac{\beta_4}{\lambda_1}, \frac{\beta_4}{\lambda_1}(g(0))^{\lambda_1}\right\}.$$

6.4.2 The case $\mu_2 = \mu_1$

Now, we prove the general energy decay result for problem (6.14)-(6.16) in the case $\mu_2 = \mu_1$. From, (6.21) we can choose $\kappa = \tau\mu_2$, then the energy functional satisfies

$$\frac{dE(t)}{dt} \leqslant -k_0 \int_0^1 \theta_x^2 dx - k_2 \int_0^1 w_x^2 dx - k_3 \int_0^1 w^2 dx + \frac{1}{2}(g' \circ \varphi_x)(t) \leqslant 0. \tag{6.104}$$

In this case, we need some additional negative term of $\int_0^1 u_t^2 dx$. For this purpose we introduce the functional

$$I_6(t) = -c \int_0^1 \theta \left(\int_0^x u_t(y)dy\right) dx,\ t \geqslant 0,$$

then, the following result holds.

Lemma 6.4.11. *Let $(u, \varphi, \theta, w, \eta^t, z)$ be a solution of (6.14)-(6.16), then, for any ε_5 , ε_6 , $\varepsilon_7 > 0$,*

$$\begin{aligned}\frac{dI_6(t)}{dt} \leqslant & -\frac{\gamma}{2} \int_0^1 u_t^2 dx + \varepsilon_5 \int_0^1 u_x^2 dx + \varepsilon_6 \int_0^1 z^2(x,1,t)dx + \varepsilon_7 \int_0^1 \varphi_x^2 dx \\ & + C_4 \int_0^1 (\varphi_t^2 + w^2)dx + C_4 \left(1 + \frac{1}{\varepsilon_5} + \frac{1}{\varepsilon_6} + \frac{1}{\varepsilon_7}\right) \int_0^1 \theta_x^2 dx.\end{aligned} \tag{6.105}$$

Proof. A simple differentiation leads to

$$\frac{dI_6(t)}{dt} = -c \int_0^1 \theta_t \left(\int_0^x u_t(y)dy\right) dx - c \int_0^1 \theta \left(\int_0^x u_{tt}(y)dy\right) dx. \tag{6.106}$$

By using $(6.14)_1$, $(6.14)_3$ and integrating by part, we get

$$\begin{aligned}\frac{dI_6(t)}{dt} =& k_0 \int_0^1 \theta_x u_t dx - \gamma \int_0^1 u_t^2 dx + m \int_0^1 \varphi_t \left(\int_0^x u_t(y) dy \right) dx \\ & - k_1 \int_0^1 w u_t dx - \frac{c\mu}{\rho} \int_0^1 \theta u_x dx + \frac{c\mu_1}{\rho} \int_0^1 \theta \left(\int_0^x u_t(y) dy \right) dx \\ & + \frac{c\mu_2}{\rho} \int_0^1 \theta \left(\int_0^x z(x,1,t) dy \right) dx - \frac{cb}{\rho} \int_0^1 \theta \varphi dx - \frac{c\gamma}{\rho} \int_0^1 \theta^2 dx.\end{aligned} \tag{6.107}$$

Applying Young's, Cauchy Schwarz and Poincaré's inequalities to find

$$k_0 \int_0^1 \theta_x u_t dx \leqslant \frac{2k_0^2}{\gamma} \int_0^1 \theta_x^2 dx + \frac{\gamma}{8} \int_0^1 u_t^2 dx, \tag{6.108}$$

$$m \int_0^1 \varphi_t \left(\int_0^x u_t dy \right) dx \leqslant \frac{2m^2}{\gamma} \int_0^1 \varphi_t^2 dx + \frac{\gamma}{8} \int_0^1 u_t^2 dx, \tag{6.109}$$

$$-k_1 \int_0^1 w u_t dx \leqslant \frac{2k_1}{\gamma} \int_0^1 w^2 dx + \frac{\gamma}{8} \int_0^1 u_t^2 dx, \tag{6.110}$$

$$\frac{c\mu_1}{\rho} \int_0^1 \theta \left(\int_0^x u_t dy \right) dx \leqslant \frac{2c^2\mu_1^2 c_p}{\rho^2 \gamma} \int_0^1 \theta_x^2 dx + \frac{\gamma}{8} \int_0^1 u_t^2 dx, \tag{6.111}$$

$$-\frac{c\mu}{\rho} \int_0^1 \theta u_x dx \leqslant \frac{c^2\mu^2 c_p}{4\rho^2 \varepsilon_5} \int_0^1 \theta_x^2 dx + \varepsilon_5 \int_0^1 u_x^2 dx, \tag{6.112}$$

$$\frac{c\mu_2}{\rho} \int_0^1 \theta \left(\int_0^x z(x,1,t) dy \right) dx \leqslant \frac{c^2\mu_2^2 c_p}{4\rho^2 \varepsilon_6} \int_0^1 \theta_x^2 dx + \varepsilon_6 \int_0^1 z^2(x,1,t) dx, \tag{6.113}$$

$$-\frac{cb}{\rho} \int_0^1 \theta \varphi dx \leqslant \frac{c^2 b^2}{4\rho^2 \varepsilon_7} \int_0^1 \theta_x^2 dx + \varepsilon_7 \int_0^1 \varphi_x^2 dx, \tag{6.114}$$

$$\frac{c\gamma}{\rho} \int_0^1 \theta^2 dx \leqslant c_6 \int_0^1 \theta_x^2 dx. \tag{6.115}$$

Substituting (6.108)–(6.115) into (6.107), we arrive at

$$\begin{aligned}\frac{dI_6(t)}{dt} \leqslant & - \frac{\gamma}{2} \int_0^1 u_t^2 dx + \varepsilon_5 \int_0^1 u_x^2 dx + \varepsilon_6 \int_0^1 z^2(x,1,t) dx \\ & + \varepsilon_7 \int_0^1 \varphi_x^2 dx + \frac{2m^2}{\gamma} \int_0^1 \varphi_t^2 dx + \frac{2k_1}{\gamma} \int_0^1 w^2 dx \\ & + \left(\frac{2k_0^2}{\gamma} + \frac{2c^2\mu_1^2 c_p}{\rho^2\gamma} + \frac{c^2\mu^2 c_p}{4\rho^2\varepsilon_5} + \frac{c^2\mu_2^2 c_p}{4\rho^2\varepsilon_6} + \frac{c^2 b^2}{4\rho^2\varepsilon_7} + c_6 \right) \int_0^1 \theta_x^2 dx.\end{aligned}$$

By letting $C_4 = \max\left\{ \frac{2m^2}{\gamma}, \frac{2k_1}{\gamma}, \frac{2k_0^2}{\gamma} + \frac{2c^2\mu_1^2 c_p}{\rho^2\gamma} + c_6, \frac{c^2\mu^2 c_p}{4\rho^2}, \frac{c^2\mu_2^2 c_p}{4\rho^2}, \frac{c^2 b^2}{4\rho^2} \right\}$, we obtain (6.105). □

Now, we define the following Lyapunov functional

$$G(t) = NE(t) + I_1(t) + N_2I_2(t) + N_3I_3(t) + I_4(t) + N_5I_5(t) + N_6I_6(t), \tag{6.116}$$

where N, N_2, N_3, N_5 and N_6 are positive constants to be chosen appropriately later. For large N, we can verify that, for some $m_1, m_2 > 0$,

$$m_1E(t) \leqslant G(t) \leqslant m_2E(t), \ \forall t \geqslant 0. \tag{6.117}$$

Then, using (6.56), (6.64), (6.73), (6.74), (6.82), (6.104) and (6.105), we get for $\varepsilon_5 = \dfrac{\mu}{8N_6}$, $\varepsilon_6 = \dfrac{e^{-2\tau}N_3}{2\tau N_6}$, $\varepsilon_7 = \dfrac{lN_2}{8N_6}$,

$$\begin{aligned}
\frac{dG(t)}{dt} \leqslant & -\frac{\mu}{8}\int_0^1 u_x^2 dx - \frac{l}{8}N_2\int_0^1 \varphi_x^2 dx - \frac{ck_1}{2}\int_0^1 \theta^2 dx \\
& - \left[\frac{\gamma}{2}N_6 - \rho - C_1\left(1 + \frac{4N_2}{Jg_0N_5}\right)N_2 - \frac{1}{\tau}N_3 - C_2\right]\int_0^1 u_t^2 dx \\
& - 2N_3I_3(t) - \left[\frac{e^{-2\tau}}{2\tau}N_3 - C_0 - C_1N_2\right]\int_0^1 z^2(x,1,t)dx \\
& - \left[\left(\xi - \frac{b^2}{\mu}\right)N_2 - C_0\right]\int_0^1 \varphi^2 dx - \left[\frac{Jg_0}{8}N_5 - JN_2 - C_2 - C_4N_6\right]\int_0^1 \varphi_t^2 dx \\
& - \Big[(Nk_0 - C_0 - C_1N_2 - C_2 - C_3N_5 \\
& \qquad - C_4\Big(1 + \frac{8N_6}{\mu} + \frac{2\tau N_6}{e^{-2\tau}N_3} + \frac{8N_6}{lN_2}\Big)\Big]\int_0^1 \theta_x^2 dx \\
& - \left[Nk_3 - C_2\left(1 + \frac{8}{Jg_0N_5}\right) - C_3N_5 - C_4N_6\right]\int_0^1 w^2 dx \\
& - [Nk_2 - C_1N_2 - C_2]\int_0^1 w_x^2 dx \\
& + \left[\frac{3g_0}{l}N_2 + C_3\left(1 + \frac{12N_5}{lN_2} + \frac{4N_5}{\mu}\right)N_5\right](g\circ\varphi_x)(t) \\
& + \left[\frac{1}{2}N - \frac{Jd_2}{2g_0}N_5\right](g'\circ\varphi_x)(t).
\end{aligned} \tag{6.118}$$

Similarly to the proof of the case $\mu_2 < \mu_1$, by carefully selecting the constants, (6.118) takes the form

$$\frac{dG(t)}{dt} \leqslant -\beta_3E(t) + \beta_4(g\circ\varphi_x)(t),$$

where β_3 and β_4 are positive constants. The remainder of the proof goes exactly as in the case of $\mu_2 < \mu_1$, and thus we complete the proof of Theorem 6.4.2.

6.5 Numerical approximation

In this section, we introduce a scheme for the problem based on P1-finite element method in space and implicit Euler scheme for time discretization. Then, we show the evolution in time of the discrete energy and the approximation of the solutions u, φ, θ and w at point $x = 0.5$.

6.5.1 Description of the discrete problem

To obtain the week formulation, we multiply (6.1) by the test functions $\bar{u}, \bar{\varphi}, \bar{\theta}$ and $\bar{w}$, then integrate by part, where $\hat{u} = u_t$, $\hat{\varphi} = \varphi_t$ to obtain

$$\begin{cases} \rho(\hat{u}_t, \bar{u}) + \mu(u_x, \bar{u}_x) + \mu_1(\hat{u}, \bar{u}) + \mu_2(\hat{u}(x, t-\tau), \bar{u}) - b(\varphi_x, \bar{u}) + \gamma(\theta_x, \bar{u}) = 0, \\ J(\hat{\varphi}_t, \bar{\varphi}) + \delta(\varphi_x, \bar{\varphi}_x) + b(u_x, \bar{\varphi}) + \xi(\varphi, \bar{\varphi}) + d(w_x, \bar{\varphi}) - m(\theta, \bar{\varphi}) \\ \qquad\qquad - \int_0^\infty g(s)(\varphi_x(t-s), \bar{\varphi}_x) ds = 0, \\ c(\theta_t, \bar{\theta}) + k_0(\theta_x, \bar{\theta}_x) + \gamma(\hat{u}_x, \bar{\theta}) + m(\hat{\varphi}, \bar{\theta}) + k_1(w_x, \bar{\theta}) = 0, \\ \alpha(w_t, \bar{w}) + k_2(w_x, \bar{w}_x) + k_3(w, \bar{w}) + k_1(\theta_x, \bar{w}) + d(\hat{\varphi}_x, \bar{w}) = 0. \end{cases} \tag{6.119}$$

The mesh of a given delay $\tau = M\Delta t$ is $t_n = n\Delta t,\ n = -M, -M+1, ..., 0,\ 0 < M < N$. By using implicit Euler scheme, the finite element approximation to the variational problem (6.119) is written as follows:

For $n = 1, ..., N$, find $(\hat{u}_h^n, \hat{\varphi}_h^n, \theta_h^n, w_h^n) \in S_h^* \times S_h^0 \times S_h^0 \times S_h^*$, such that for all $\bar{u}_h, \bar{\varphi}_h, \bar{\theta}_h$ and $\bar{w}_h$ we obtain,

$$\begin{cases} \frac{\rho}{\Delta t}(\hat{u}_h^n - \hat{u}_h^{n-1} -, \bar{u}_h) + \mu(u_{hx}^n, \bar{u}_{hx}^n) + \mu_1(\hat{u}_h^n, \bar{u}_h) + \mu_2(\hat{u}_h^{n-M}, \bar{u}_h) \\ \qquad\qquad - b(\varphi_{hx}^n, \bar{u}_h) + \gamma(\theta_{hx}^n, \bar{u}_h) = 0, \\ \frac{J}{\Delta t}(\hat{\varphi}_h^n - \hat{\varphi}_h^{n-1}, \bar{\varphi}_h) + \delta(\varphi_{hx}^n, \bar{\varphi}_{hx}) + b(u_{hx}^n, \bar{\varphi}_h) + \xi(\varphi_h^n, \bar{\varphi}_h) + d(w_{hx}^n, \bar{\varphi}_h) \\ \qquad\qquad - m(\theta_h^n, \bar{\varphi}_h) - \Delta t \sum_{n_0=1}^{n} g(t_{n-n_0})(\varphi_{hx}^{n_0}, \bar{\varphi}_{hx}) = 0, \\ \frac{c}{\Delta t}(\theta_h^n - \theta_h^{n-1}, \bar{\theta}_h) + k_0(\theta_{hx}^n, \bar{\theta}_{hx}) + \gamma(\hat{u}_{hx}^n, \bar{\theta}_h) + m(\hat{\varphi}_h^n, \bar{\theta}_h) + k_1(w_{hx}^n, \bar{\theta}_h) = 0, \\ \frac{\alpha}{\Delta t}(w_h^n - w_h^{n-1}, \bar{w}_h) + k_2(w_{hx}^n, \bar{w}_{hx}) + k_3(w_h^n, \bar{w}_h) + k_1(\theta_{hx}^n, \bar{w}_h) + d(\hat{\varphi}_{hx}^n, \bar{w}_h) = 0, \end{cases}$$

where

$$u_h^n = u_h^{n-1} + \Delta t\ \hat{u}_h^n \quad \text{and} \quad \varphi_h^n = u_h^{n-1} + \Delta t\ \hat{\varphi}_h^n$$

and we introduce the discrete energy by

$$E^n = \frac{1}{2}\left(\rho\|\hat{u}_h^n\|^2 + J\|\hat{\varphi}_h^n\|^2 + \mu\|u_{hx}^n\|^2 + l\|\varphi_{hx}^n\|^2 + c\|\theta_h^n\|^2 + \xi\|\varphi_h^n\|^2 + \alpha\|w_h^n\|^2\right)$$
$$+ b\int_0^1 \varphi_h^n u_{hx}^n dx + \frac{\Delta t}{2}\int_0^1 \sum_{n_0=1}^{n} g(t_{n-n_0})(\varphi_{hx}^n - \varphi_{hx}^{n_0})^2 dx + \frac{\kappa}{2}\int_0^1\int_0^1 \left(\hat{u}_h^{n-Mp}\right)^2 dp dx.$$

Here u_h^0, $\hat{u}_h^0$, φ_h^0, $\hat{\varphi}_h^0$, θ_h^0 and w_h^0 are approximations to u_0, u_1, φ_0, φ_1, θ_0 and w_0, respectively.

6.5.2 An iterative algorithm

We use a fixed-point algorithm that we now describe. Assuming that $(\hat{u}^{n-1}, \hat{\varphi}^{n-1}, \theta^{n-1}, w^{n-1})$ is known, we first set:

$$u_h^{n,0} = u_h^{n-1},\ \hat{u}_h^{n,0} = \hat{u}_h^{n-1}, \varphi_h^{n,0} = \varphi_h^{n-1},\ \hat{\varphi}_h^{n,0} = \hat{\varphi}_h^{n-1},\ \theta_h^{n,0} = \theta_h^{n-1},\ w_h^{n,0} = w_h^{n-1}.$$

Next, we solve iteratively the problem:

Find $(\hat{u}_h^{n,l}, \hat{\varphi}_h^{n,l}, \theta_h^{n,l}, w_h^{n,l}) \in S_h^* \times S_h^0 \times S_h^0 \times S_h^*$ satisfying

$$\begin{aligned}
&\forall \bar{u}_h \in S_h^*,\\
&\qquad \frac{\rho}{\Delta t}(\hat{u}_h^{n,l} - \hat{u}_h^{n-1}, \bar{u}_h) + \mu(u_{hx}^{n-1}, \bar{u}_{hx}) + \mu\Delta t(\hat{u}_{hx}^{n,l}, \bar{u}_{hx})\\
&\qquad + \mu_1(\hat{u}_h^{n,l}, \bar{u}_h) + \mu_2(\hat{u}_h^{n-M}, \bar{u}_h) - b(\varphi_{hx}^{n,l}, \bar{u}_h) + \gamma(\theta_{hx}^{n,l}, \bar{u}_h) = 0,\\
&\forall \bar{\varphi}_h \in S_h^0,\\
&\qquad \frac{J}{\Delta t}(\hat{\varphi}_h^{n,l} - \hat{\varphi}_h^{n-1}, \bar{\varphi}_h) + \delta(\varphi_{hx}^{n-1}, \bar{\varphi}_{hx}) + \delta\Delta t(\hat{\varphi}_{hx}^{n,l}, \bar{\varphi}_{hx})\\
&\qquad + b(u_{hx}^{n-1}, \bar{\varphi}_h) + \xi(\varphi_h^{n-1}, \bar{\varphi}_h) + \xi\Delta t(\hat{\varphi}_h^{n,l}, \bar{\varphi}_h)\\
&\qquad + d(w_{hx}^{n,l}, \bar{\varphi}_h) - m(\theta_h^{n,l}, \bar{\varphi}_h) - \Delta t\sum_{m=1}^{n} g(t_{n-m})(\varphi_{hx}^m, \bar{\varphi}_{hx}) = 0,\\
&\forall \bar{\theta}_h \in S_h^0,\\
&\qquad \frac{c}{\Delta t}(\theta_h^{n,l} - \theta_h^{n-1}, \bar{\theta}_h) + k_0(\theta_{hx}^{n,l}, \bar{\theta}_{hx}) + \gamma(\hat{u}_{hx}^{n-1}, \bar{\theta}_h)\\
&\qquad + m(\hat{\varphi}_h^{n-1}, \bar{\theta}_h) + k_1(w_{hx}^{n-1}, \bar{\theta}_h) = 0,\\
&\forall \bar{w}_h \in S_h^*,\\
&\qquad \frac{\alpha}{\Delta t}(w_h^{n,l} - w_h^{n-1}, \bar{w}_h) + k_2(w_{hx}^{n,l}, \bar{w}_{hx})\\
&\qquad + k_3(w_h^{n,l}, \bar{w}_h) + k_1(\theta_{hx}^{n,l}, \bar{w}_h) + d(\hat{\varphi}_{hx}^{n-1}, \bar{w}_h) = 0,
\end{aligned} \tag{6.120}$$

where

$$u_h^{n,l} = u_h^{n-1} + \Delta t\ \hat{u}_h^{n,l} \quad \text{and} \quad \varphi_h^{n,l} = u_h^{n-1} + \Delta t\ \hat{\varphi}_h^{n,l}. \tag{6.121}$$

We now prove the well-posedness of (6.120).

Proposition 6.5.1. *For any data* $(u_0, u_1, \varphi_0, \varphi_1, \theta_0, w_0) \in H^1_* \times H^1_* \times H^1_0 \times H^1_0 \times H^1_0 \times H^1_*$, *for* $n = 1, ..., N$ *and any positive integer* l, *problem (6.120) has a unique solution when* Δt *is small enough.*

Proof. The initial conditions are given, we proceed by induction on n and l. At each step (n, l), problem (6.120) results into a square finite-dimensional linear system. So, assume that all data $(\hat{u}_h^{n-1}, \hat{\varphi}_h^{n-1}, \theta_h^{n-1}, w_h^{n-1})$ and also $(u_h^{n-1}, \varphi_h^{n-1})$ are zero. Note from (6.121) $u_h^{n,l} = \Delta t \hat{u}_h^{n,l}$ and $\varphi_h^{n,l} = \Delta t \hat{\varphi}_h^{n,l}$. It can thus be re-written

$$\begin{aligned}
\forall \bar{u}_h \in S_h^*, \quad & \frac{\rho}{\Delta t^2}(u_h^{n,l}, \bar{u}_h) + \mu(u_{hx}^{n,l}, \bar{u}_{hx}) + \frac{\mu_1}{\Delta t}(u_h^{n,l}, \bar{u}_h) + \frac{\mu_2}{\Delta t}(u_h^{n-M}, \bar{u}_h) \\
& \qquad - b(\varphi_{hx}^{n,l}, \bar{u}_h) + \gamma(\theta_{hx}^{n,l}, \bar{u}_h) = 0, \\
\forall \bar{\varphi}_h \in S_h^0, \quad & \frac{J}{\Delta t^2}(\varphi_h^{n,l}, \bar{\varphi}_h) + \delta t(\varphi_{hx}^{n,l}, \bar{\varphi}_{hx}) + \xi(\varphi_h^{n,l}, \bar{\varphi}_h) + d(w_{hx}^{n,l}, \bar{\varphi}_h) \\
& \qquad - m(\theta_h^{n,l}, \bar{\varphi}_h) - \Delta t \sum_{m=1}^{n} g(t_{n-m})(\varphi_{hx}^m, \bar{\varphi}_{hx}) = 0, \\
\forall \bar{\theta}_h \in S_h^0, \quad & \frac{c}{\Delta t}(\theta_h^{n,l}, \bar{\theta}_h) + k_0(\theta_{hx}^{n,l}, \bar{\theta}_{hx}) = 0, \\
\forall \bar{\theta}_h \in S_h^*, \quad & (\frac{\alpha}{\Delta t} + k_3)(w_h^{n,l}, \bar{w}_h) + k_2(w_{hx}^{n,l}, \bar{w}_{hx}) + k_1(\theta_{hx}^{n,l}, \bar{w}_h) = 0,
\end{aligned}$$

by taking $\bar{\theta}_h = \theta_h^{n,l}$ in the the third equation, we immediately derive that $\theta_h^{n,l}$ is zero. In the last line, taking $\bar{w}_h = w_h^{n,l}$, thus yields that $w_h^{n,l}$ is zero. On the other hand, taking $\bar{\varphi}_h = \varphi_h^{n,l}$ in the second equation implies that $\varphi_h^{n,l}$ is zero. Finally, taking $\bar{u}_h = u_h^{n,l}$ in the first equation gives that $u_h^{n,l}$ is zero. As a consequence, problem (6.120) has at most a solution, hence has a unique solution. □

It follows from the previous proof that problem (6.120) results into four uncoupled equations: Solve first the equation on $\theta_h^{n,l}$, next the equation on $w_h^{n,l}$, next the equation on $\hat{\varphi}_h^{n,l}$ and finally the equation on $\hat{u}_h^{n,l}$ until a finite number of times or the smaller l such that the difference between $(\hat{u}_h^{n,l}, \hat{\varphi}_h^{n,l}, \theta_h^{n,l}, w_h^{n,l})$ and $(\hat{u}_h^{n,l-1}, \hat{\varphi}_h^{n,l-1}, \theta_h^{n,l-1}, w_h^{n,l-1})$ in an appropriate norm becomes smaller than a given tolerance tol. We finally set:

$$\hat{u}_h^n = \hat{u}_h^{n,l}, \quad \hat{\varphi}_h^n = \hat{\varphi}_h^{n,l}, \quad \theta_h^n = \theta_h^{n,l}, \quad w_h^n = w_h^{n,l} \tag{6.122}$$

6.5.3 Numerical experiments

For the numerical experiments, we make 2 tests to illustrate the energy decay results. The first test is done when the case $\mu_2 < \mu_1$, the second test is done when the case $\mu_2 = \mu_1$. For both tests, we consider the following data:

$$\rho = J = \mu = \alpha = 1,\ c = 10^{-4},\ \delta = 3,\ g(t) = e^{-4t},\ l = \delta - (1/4),$$

$$b = \gamma = m = 1/2,\ \xi = d = k_1 = k_2 = k_3 = 1,\ \tau = 0.1T.$$

The discretization parameters are fixed equal to $h = 10^{-2}$, $\Delta t = 10^{-4}$. A tolerance $tol = 10^{-7}$ is used to stop the iterative procedure.

We work with the initial values

$$u_0(x) = w_0(x) = u_1(x) = \cos(2\pi x), \quad \varphi_0(x) = \varphi_1(x) = x(1-x), \quad \theta_0(x) = \sin(2\pi x)$$

and the delay condition

$$f_0(x, t-\tau) = \cos(2\pi x)\cos(t-\tau).$$

Test 1. For the first numerical test, we select the following entries:

$$\mu_1 = 1/2, \quad \mu_2 = 0.1, \quad k_0 = 0.1.$$

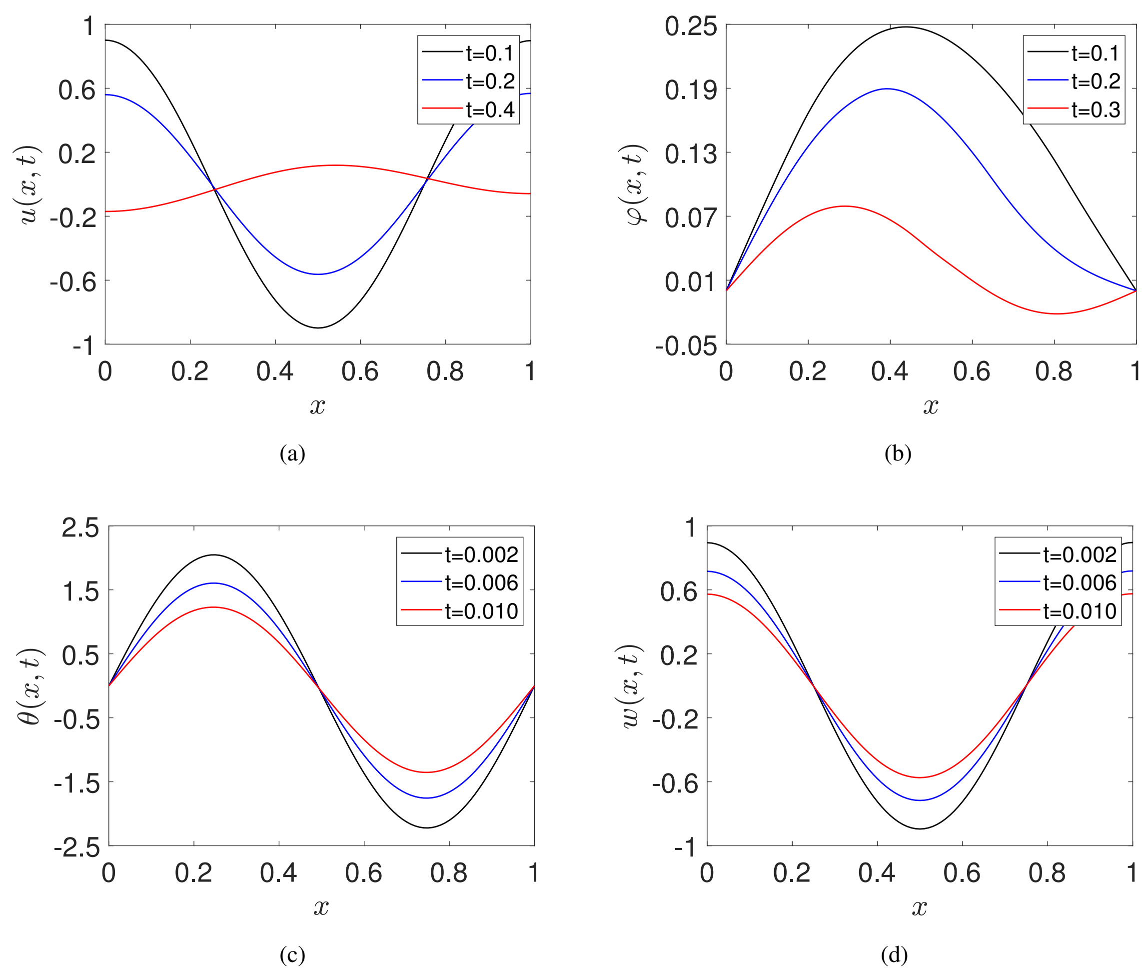
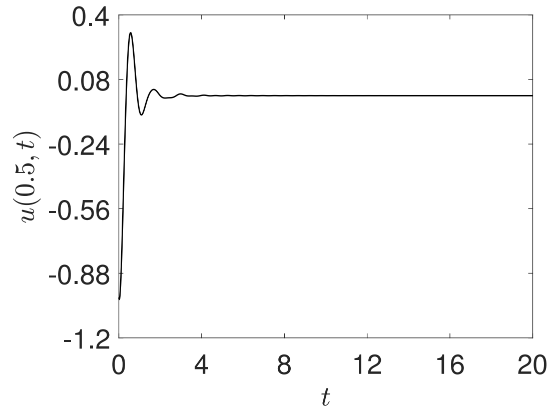
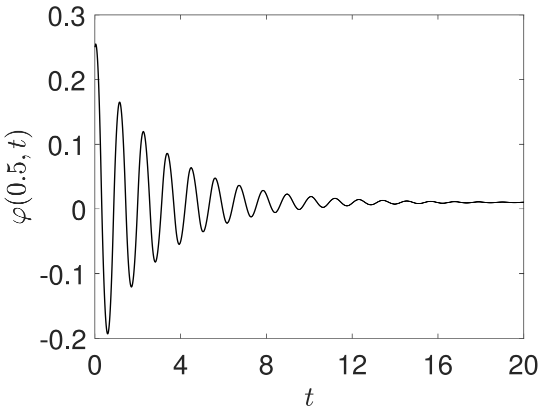


Figure 6.1 Test 1: The evolution in time of u, φ, θ and w.

Test 2. For the second numerical test, we choose the following entries:



(a)



(b)

Figure 6.2 Test 1: The evolution in time of u and φ at $x = 0.5$.

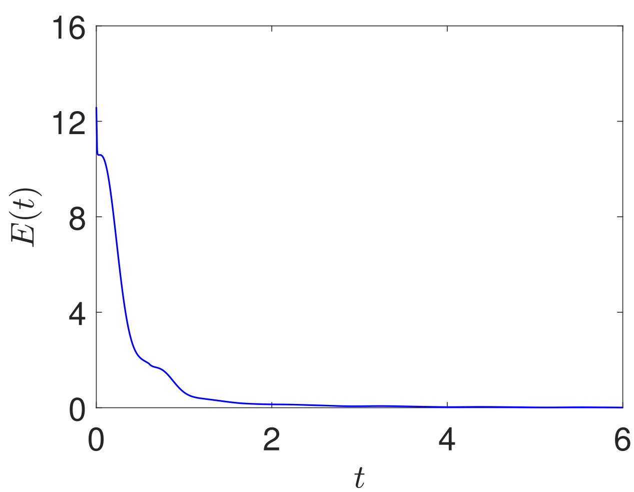


Figure 6.3 Test 1: The evolution in time of E.

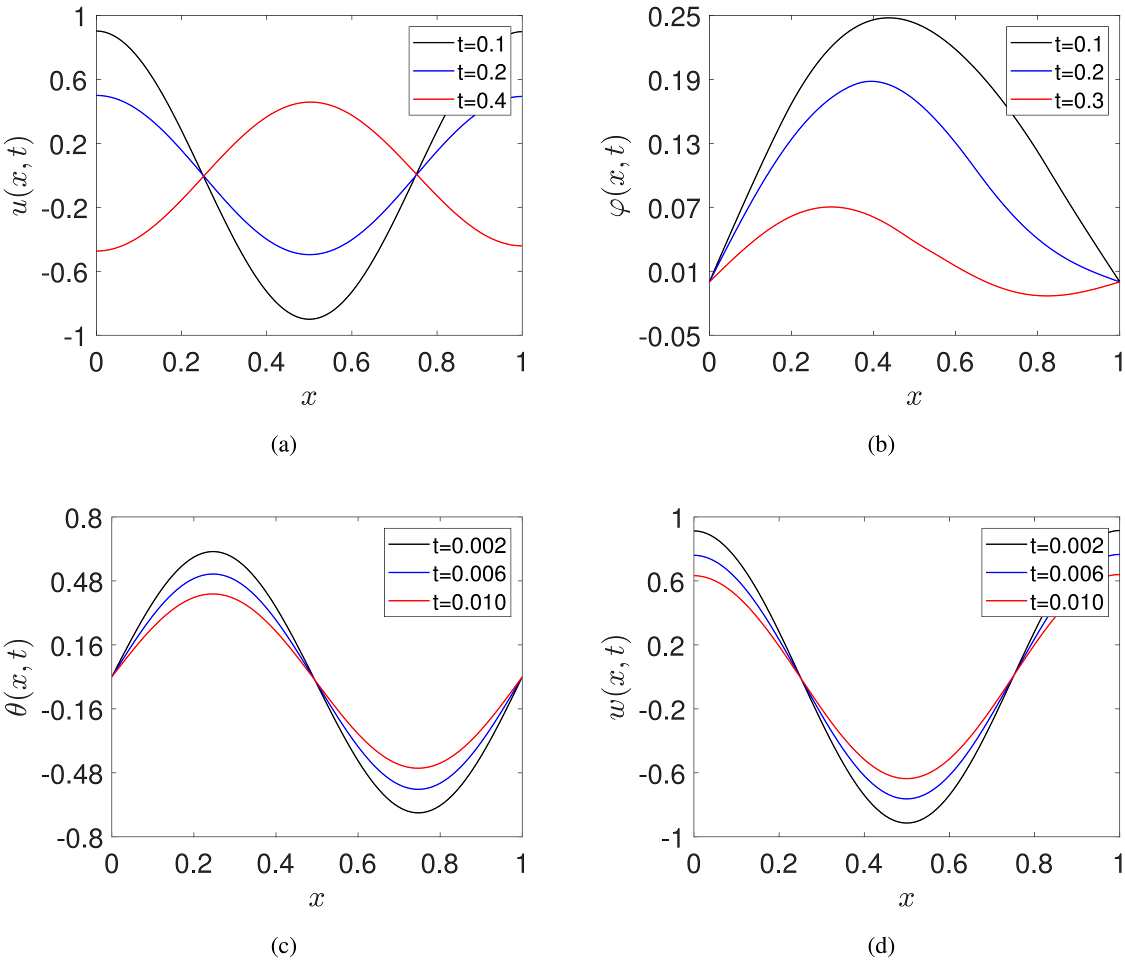


Figure 6.4 Test 2: The evolution in time of u, φ, θ and w.

$$\mu_1 = \mu_2 = 0.1, \quad k_0 = 1/3.$$

For each numerical test, the evolution in different values of time is shown in Figure 6.1 and 6.4 for the approximate solution (u, φ, θ, w). Figures 6.2 and 6.5 illustrate the cross section cuts for the numerical solution u and φ at the point $x = 0.5$, where we can see the effect of the damping term on the delay in the displacement. Notably, our simulations indicate that, u when $\mu_2 < \mu_1$ vanished faster than when the condition $\mu_2 = \mu_1$ is met.

Regarding the energy, we have two cases based on the condition $\mu_2 \leqslant \mu_1$. If $\mu_2 < \mu_1$, the discrete energy is presented in Figure 6.3, showcasing rapid decay to zero after time $t = 2$. If $\mu_2 = \mu_1$, exponential decay is shown in Figure 6.6 where we see that, at time $t = 4$.

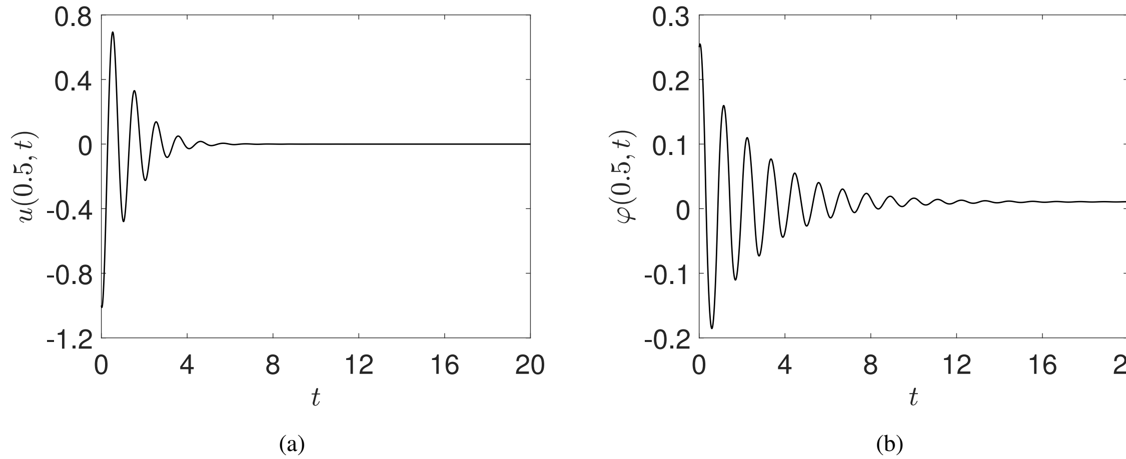


Figure 6.5 Test 2: The evolution in time of u and φ at $x = 0.5$.

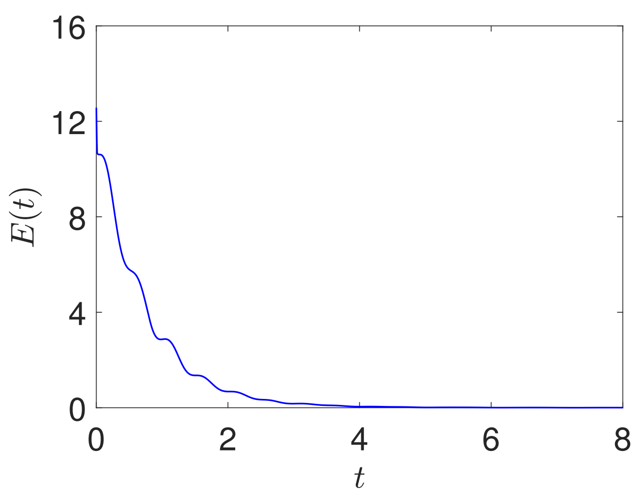


Figure 6.6 Test 2: The evolution in time of E.

Conclusion

This thesis has addressed key challenges in the study of thermoelastic systems, focusing on global existence and stability in time for various partial-differential-equation models. By leveraging rigorous mathematical methods and numerical analysis, this work has provided meaningful insights into the role of dissipation mechanisms in stabilizing mechanical systems.

We established well-posedness and stability results—ranging from exponential to polynomial and general decay rates—for systems such as thermoelastic Shear beams, laminated beams, and porous thermoelastic models. Complementing these findings, numerical simulations validated the theoretical results, by demonstrating the accuracy and effectiveness of the proposed methods.

Looking ahead, this study opens the door to several promising research directions, including the exploration of multi-dimensional systems, nonlinear dynamics, and more intricate feedback mechanisms. These avenues hold a potential for advancing both the theoretical understanding and practical applications of stabilization techniques in mechanical systems.

In summary, this thesis represents a step forward in the theoretical and numerical analysis of thermoelastic systems, contributing to the broader goal of developing stable, efficient, and reliable models for real-world applications.

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