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Hakim Maroua

Dedication

To my parents,

To my brothers and my sister,

To my husband.

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Glossary of Notation

TP	Transportation problem
STP	Solid transportation problem
TP4	Four index transportation problem
FCTP	Fixed charge transportation problem
FFCTP	Fuzzy fixed charge transportation problem
FFCTP4	Fuzzy four index fixed charge transportation problem
FBOFCTP4	Fuzzy bi-objective four index fixed charge transportation problem
$F(\mathbb{R})$	The set of fuzzy numbers
μ	The membership function
$\Re(\cdot)$	The ranking function of a fuzzy number
$\oplus$	Fuzzy addition operation
$\ominus$	Fuzzy subtraction operation
$\otimes$	Fuzzy multiplication operation
GA	Genetic algorithm
SA	Simulated Annealing
PSO	Particle Swarm Optimization
PSO-GA	Hybrid particle swarm optimization with genetic algorithm
$pbest$	The best personal position attained by the particle
$gbest$	The best global position attained by all particles
AL_{TP4C}	Algorithm introduced by R. Zitouni et al. to solve the capacitated four index transportation problem
AL_{TP4}	The non-capacitated version of AL_{TP4C}
Al$_{FBOFCTP4}$	Algorithm used to solve FBOFCTP4

Introduction

Operations research (OR) is a crucial aspect of decision-making; it can be described as a set of techniques used to determine the best solution for problems dealing with the operations of systems. The need to solve complex problems related to industry and business, as well as resource allocation, served as motivation for the development of OR as a discipline during World War II. OR is considered a powerful tool to facilitate the evaluation of many alternatives and enhance the decision-making process. It encompasses multiple areas, offering diverse tools and methodologies for addressing various decision-making issues; the most popular branches are linear programming, discrete programming, nonlinear programming, and multi-objective programming.

Combinatorial optimization is an important field of mathematical optimization related to operations research. It is used to model and solve optimization problems of various applications. This field is concerned with problems where the solution space is discrete, intending to determine an optimal solution from a set of candidate solutions for a given problem. The transportation problem is one of the most important subjects in the field of combinatorial optimization.

The transportation problem (TP) is one of the most prominent topics in operation research and combinatorial optimization, first formulated by Hitchcock in 1941, it involves reducing the overall cost within an expedition plan. Many linear programming problems can be modeled as transportation problems and have been recieved considerable attention, it has many recognized extensions issues under both crisp and fuzzy environments, among them, the solid transportation problem [17], the capacitated four-index transportation problem [51, 52, 53, 54], the fuzzy four-index transportation problem [19, 20], and the fixed charge transportation problem [21], etc.

The fixed charge transportation problem (FCTP) is a generalization of the well-known transportation problem, initiated by Hirsch and Dantzig in 1954 [21]. It involves shipping goods from localization sources to destination centers where the transportation cost comprises two components: direct cost and fixed cost. The fixed cost may represent the rental fees for land, landing charges at an airport, or the expenses associated with arranging goods within a manufacturing setting. Many distribution issues can be modeled as FCTP.

Due to the globalization of the socio-economic environment, companies impose the consideration of multiple objectives in optimization problems, rather than a single one; this is referred to as multi-objective optimization. Indeed, in real-world situations, we face many different transportation problems dealing with more than one objective, such as minimizing transportation cost and delivery time, which are not two independent problems from a practical perspective; this is called a bi-objective transportation problem. Several methods have been proposed to solve bi-objective transportation problems [4, 26].

In 2015, Adlakha and Khurana [26] introduced an approach for solving a bi-objective fixed-charge solid transportation problem in which two objectives are minimized: total transportation cost and transportation time. However, in real-world scenarios, the parameters of the problem are imprecise and not well defined. To overcome such situations, the fuzzy fixed-charge transportation problem with one or more objectives is used.

Our aim in this thesis is to explore two models of transportation problems: fuzzy four-index fixed-charge transportation problem and fuzzy bi-objective four-index fixed-charge transportation problem. To the best of our knowledge, such models have not been treated yet. To solve the first one, we propose an adaptation of some well-known metaheuristics and an approximation method. For the resolution of the second model, we propose an adaptation of the above-mentioned method [26] in a fuzzy context with four subscripts. This thesis is divided into three chapters.

In the first chapter, we present some fundamental concepts of linear programming, the four-index transportation problem, the basics of fuzzy mathematics as well as preliminaries on multi-objective optimization.

The second chapter introduces our first contribution. It consists in solving a fuzzy four-index fixed-charge transportation problem (FFCTP4) via an approximation method and four metaheuristic algorithms, namely genetic algorithm, simulated annealing, particle swarm optimization, and hybrid particle swarm optimization with genetic algorithm. Our proposed approximation method is based on an extension work of [6] under fuzziness with four subscripts. We terminate this chapter with a numerical comparative study to assess the performance of the proposed algorithms and identify the most suitable one for solving FFCTP4.

In the last chapter, we present our second contribution, which concerns the elaboration and numerical implementation of an algorithm denoted by $\text{Al}_{FBOFCTP4}$ for solving a fuzzy bi-objective four-index fixed-charge transportation problem in a fuzzy environment, based on A. Khurana et al.'s approach [26].

Chapter 1

Generalities and Preliminaries

This chapter outlines some basic and general notions relevant to our research. It is divided into four sections, organized as follows. In the first section, we introduce some basic concepts of linear programming. Section 2 presents the mathematical framework of the four-index transportation problem and its solution method. Section 3 provides a brief survey on multi-objective optimization. The fourth section is dedicated to generalities on fuzzy set theory.

1.1 Linear programming

Linear programming (LP) is a mathematical technique used to find the best solution for a problem whose objective function and set of constraints are linear. LP came to existence as a discipline in the 1940s; since then, LP has attracted the attention of many researchers, such as Kolmogorov, Kantorovich, Danzig, and Karmakar.

In 1939, Kantorovich showed that many production problems could be modeled as linear problems. In 1942, Hitchcock formulated the classical transportation problem. In 1947, Dantzig introduced the well-known simplex algorithm to solve linear problems. In the same year, the theory of duality was established by John Von Neumann. Later, many other effective methods were developed to solve linear programming problems. LP is considered a powerful tool in many organizations operations and has many applications in various fields such as economics, business, telecommunications, and manufacturing.

1.1.1 Linear programming problem

In a mathematical context, a Linear Programming Problem is expressed in its standard form as follows:

$$\min\ Z(x) = [c^T x : A\, x = b, x \geq 0], \tag{LP}$$

where $c, x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.

In the following, the feasible region of LP is defined by:

$$F = \{x \in \mathbb{R}^n_+ : Ax = b\}$$

Definition 1.1. If $x \in F$ then x is called a feasible solution for the problem LP.

Definition 1.2. Let $A \in \mathbb{R}^{m\times n}$ be a matrix with rank m. A regular submatrix $B \in \mathbb{R}^{m\times m}$ extracted from A is called a base.

We associate to the matrix A the following combination $A = [B \,|\, N]$.

$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}$ if we set the vector $x_N = 0$, then the vector

$x = \begin{pmatrix} B^{-1}\, x_B \\ 0 \end{pmatrix}$ satisfying $A\, x = b$ is called a basic solution.

The components of the vector x_N are called non-basic variables while the components of vector x_B are called basic variables.

Definition 1.3. A basic feasible solution $\hat{x}$ is called degenerate if one of the basic variables is equal to zero.

Definition 1.4. A solution $\hat{x}$ is called optimal if $\hat{x} \in F$ and $Z(\hat{x}) = c^T \hat{x} \leq Z(x) = c^T x, \forall x \in F$.

1.1.2 Notions of duality

Let us consider the dual of LP as follows:

$$\max_y\, [b^T y : A^T\, y \leq c, y \in \mathbb{R}^m], \tag{DP}$$

Lemma 1.5. *If $\hat{x}$ and $\hat{y}$ are two feasible solutions for LP and DP, respectively, then*

$$c^T \hat{x} \geq b^T \hat{y}.$$

Corollary 1.6. *If $\hat{x}$ and $\hat{y}$ are two feasible solutions for LP and DP, respectively, such that:*

$$c^T \hat{x} = b^T \hat{y},$$

then, $\hat{x}$ is the optimal solution of LP and $\hat{y}$ is the optimal solution of the problem DP.

Theorem 1.7. *If $\hat{x}$ is a feasible solution of LP, then it is optimal if and only if there exists a feasible solution $\hat{y}$ for DP such that:*

$$c^T \hat{x} = b^T \hat{y}.$$

The solution $\hat{y}$ of DP is also optimal.

Theorem 1.8. *If $\hat{x}$ and $\hat{y}$ are two feasible solutions for LP and DP, respectively, then $\hat{x}$ and $\hat{y}$ are optimal if and only if:*

$$(A\,\hat{x} - b)^T \hat{y} = 0 \text{ and } (c - A^T\,\hat{y})^T \hat{x} = 0.$$

1.1.3 Resolution of linear programming problem

In this subsection, we cite some methods used to solve linear programming problems.
Graphical method. This is one of the earliest optimization techniques used to find an optimal solution for linear problems with two variables, where the optimal solution is obtained at a vertex of the convex polyhedron. Such a method is not effective for problems where the number of variables is not limited.
Simplex method. It was originally introduced by George Dantzig in 1947 and comprises two phases: first, finding an initial basic feasible solution or declaring that the problem is infeasible, then moving from a vertex to another adjacent vertex while improving the objective function value until an optimal solution is reached.
Interior point methods. These techniques are addressed to solve both linear and

nonlinear problems. They start from an interior point of the feasible region and move through it to get an optimal solution using a specified strategy. They are mainly divided into three categories:

- Potential-reduction methods.
- Path-following methods.
- Primal-dual methods.

1.2 Four index transportation problem

1.2.1 Introduction

The Transportation Problem (TP) is one of the most significant issues in logistics and supply chain management that was first introduced by Hitchcock in 1942. It is a linear programming problem in which a product needs to be transported from source locations to destination centers, intending to minimize the overall transportation cost. Researchers have proposed many algorithms to solve TP and have divided it into four classes: 2-dimensional, 3-dimensional, 4-dimensional, and n-dimensional.

In 1955, Schell [44] formulated the Solid Transportation Problem (STP) in which three types of constraints are involved (origin, destination, means of transport). Next, Haley [17] developed an algorithm to solve STP; its inspiration comes from the modified distribution method. Pandian and Anuradha [37] proposed an algorithm to solve the STP based on the zero-point concept.

In addition, many researchers have also studied four-dimensional transportation problems (TP4) in different environment. Zitouni et al. [52, 53] provided an optimal solution for the capacitated four-index transportation problem. Later, Zitouni et al. [51] conducted a comparative study between the well-known simplex method and AL_{TP4C}. Their obtained results show the superiority of AL_{TP4C}. Skitsko et al. [46] solved the four-index transportation problem via genetic algorithm. Subsequently, authors in [19, 20] solved the fuzzy four index transportation problem using an exact algorithm along with some well-known metaheuristic algorithms. In 2021, Abd-Elazeem et al. [1] presented an approach to find a set of non-dominated solutions for the multi-criteria transportation problem with four indexes. Their approach consists of transforming the primary issue into TP4 and then separating the resulting problem into a series of two-dimensional transportation problems.

1.2.2 Four-index transportation problem

The four-index transportation problem is formulated mathematically as follows:

$$\begin{aligned}
&Minimize\, Z = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} c_{ijkl}\, x_{ijkl},\\
&\text{Subject to constraints}\\
&\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} = \hat{a}_i, \text{ for all } i = 1, ..., m,\\
&\sum_{i=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} = \hat{b}_j, \text{ for all } j = 1, ..., n,\\
&\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{q} x_{ijkl} = \hat{e}_k, \text{ for all } k = 1, ..., p,\\
&\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} x_{ijkl} = \hat{d}_l, \text{ for all } l = 1, ..., q,\\
&x_{ijkl} \geq 0, \text{ for all } i = 1:m; j = 1:n; k = 1:p, l = 1:q,
\end{aligned} \tag{TP4}$$

The above-mentioned problem can be written in the following linear form:

$$\begin{cases}
\min\ Z = c^T x,\\
\text{Subject to constraints:}\\
Ax = \beta,\\
x \geq 0,
\end{cases} \tag{1.1}$$

where

- $c = (c_{1111}, ..., c_{mnpq})^T \in \mathbb{R}^N$,
- $x = (x_{1111}, ..., x_{mnpq})^T \in \mathbb{R}^N$,

- $\beta = (\hat{a}_1, ..., \hat{a}_m, \hat{b}_1, ..., \hat{b}_n, \hat{e}_1, ..., \hat{e}_p, \hat{d}_1, ..., \hat{d}_q)^T \in \mathbb{R}^M$,
- $N = mnpq$ and $M = m + n + p + q$.

Transportation table

This table is used to organize the data of transportation, it is an array of M rows and N columns, besides two additional rows and one additional column. The values of c_{ijkl} and x_{ijkl} are reserved on the two additional rows, while the additional column contains the values of $\hat{a}_i$, $\hat{b}_j$, $\hat{e}_k$, and $\hat{d}_l$, respectively. The remaining cases are ones or zeros such that the entry of a case corresponding to line $\hat{a}_{i_0}$ and column $\hat{P}_{i_0jkl}$ is 1; otherwise, it is 0. The same process with $\hat{b}_{j_0}$, $\hat{e}_{k_0}$, and $\hat{d}_{l_0}$.

c_{1111}	c_{1211}	$\cdots$	c_{1npq}	c_{2111}	c_{2211}	$\cdots$	c_{2npq}	$\cdots$	c_{m111}	c_{m211}	$\cdots$	c_{mnpq}	
x_{1111}	x_{1211}	$\cdots$	x_{1npq}	x_{2111}	x_{2211}	$\cdots$	x_{2npq}	$\cdots$	x_{m111}	x_{m211}	$\cdots$	x_{mnpq}	
1	1	$\cdots$	1	0	0	0	0	0	0	0	0	0	$\hat{a}_1$
0	0	$\cdots$	0	1	1	$\ldots$	1	0	0	0	0	0	$\hat{a}_2$
$\vdots$		$\cdots$				$\vdots$						$\vdots$	
0	0	$\cdots$	0	0	0	0	0	0	1	1	$\cdots$	1	$\hat{a}_m$
1	0	$\cdots$	0	1	0	0	0	0	1	0	0	0	$\hat{b}_1$
0	1	$\cdots$	0	0	1	0	0	0	0	1	0	0	$\hat{b}_2$
$\vdots$		$\cdots$				$\vdots$						$\vdots$	
0	0	$\cdots$	1	0	0	0	1	0	0	0	0	1	$\hat{b}_n$
1	1	$\cdots$	0	1	1	0	0	0	1	1	0	0	$\hat{e}_1$
$\vdots$		$\cdots$				$\vdots$						$\vdots$	
0	0	$\cdots$	1	0	0	0	1	0	0	0	0	1	$\hat{e}_p$
1	1	$\cdots$	0	1	1	0	0	0	1	1	0	0	$\hat{d}_1$
$\vdots$		$\cdots$				$\vdots$						$\vdots$	
0	0	$\cdots$	1	0	0	0	1	0	0	0	0	1	$\hat{d}_q$

Table 1.1: Transportation table for TP4

1.2.3 Feasibility and optimality conditions

This subsection gives conditions that ensure the feasibility and optimality of a solution for the four index transportation problem TP4.

Theorem 1.9. *(Feasibility condition [54].) The four index transportation problem TP4 has a feasible solution if and only if*

$$\sum_{i=1}^{m} \hat{a}_i = \sum_{j=1}^{n} \hat{b}_j = \sum_{k=1}^{p} \hat{e}_k = \sum_{l=1}^{q} \hat{d}_l. \tag{1.2}$$

Theorem 1.10. *(Optimality criterion [54].) A feasible solution x of TP4 is considered optimal if and only if there exists a vector*

$$\hat{y} = (\hat{u}_1, ..., \hat{u}_m, \hat{v}_1, ..., \hat{v}_n, \hat{w}_1, ..., \hat{w}_p, \hat{t}_1, ..., \hat{t}_q) \in \mathbb{R}^M,$$

such that:

$$\begin{cases} \hat{u}_i + \hat{v}_j + \hat{w}_k + \hat{t}_l = c_{ijkl} \text{ if } x_{ijkl} = 0, \\ \hat{u}_i + \hat{v}_j + \hat{w}_k + \hat{t}_l \leq c_{ijkl} \text{ if } x_{ijkl} > 0. \end{cases}$$

1.2.4 Resolution of four index transportation problem

In this subsection, we present an algorithm for finding an optimal solution for the four-index transportation problem TP4.

AL_{TP4} algorithm

This algorithm is based on an adaptation of the algorithm AL_{TP4C} proposed by Zitouni and Keraghel [52] in a non-capacitated context. It comprises two phases:

Phase 1: Finding an initial basic feasible solution

This phase is used to determine an initial basic feasible solution for $(TP4)$.

For all (i, j, k, l) set $\beta_{ijkl} = 0$ (β_{ijkl} is an integer variable that takes one if x_{ijkl} has been found; otherwise, it is zero). Let $\hat{E} = \{(i, j, k, l), \beta_{ijkl} = 0\}$.

While $\hat{E}$ is not empty do

- Choose (i^*, j^*, k^*, l^*), such that $c_{i^*,j^*,k^*,l^*} = \min c_{ijkl}$.
- Take $x_{i^*j^*k^*l^*} = \min(\hat{a}_{i^*}, \hat{b}_{j^*}, \hat{e}_{k^*}, \hat{d}_{l^*})$ and $\beta_{i^*j^*k^*l^*} = 1$.

- Update $\hat{a}_{i^*}, \hat{b}_{j^*}, \hat{e}_{k^*}$, and $\hat{d}_{l^*}$ as follows:

  1. $\hat{a}_{i^*} = \hat{a}_{i^*} - x_{i^*j^*k^*l^*}$

     If $\hat{a}_{i^*} = 0$ then take $x_{i^*jkl} = 0$ and $\beta_{i^*jkl} = 1$, $\forall(j,k,l) \neq (j^*, k^*, l^*)$.

  2. $\hat{b}_{j^*} = \hat{b}_{j^*} - x_{i^*j^*k^*l^*}$

     If $\hat{b}_{j^*} = 0$ then take $x_{ij^*kl} = 0$ and $\beta_{ij^*kl} = 1$, $\forall(i,k,l) \neq (i^*, k^*, l^*)$.

  3. $\hat{e}_{k^*} = \hat{e}_{k^*} - x_{i^*j^*k^*l^*}$

     If $\hat{e}_{k^*} = 0$ then take $x_{ijk^*l} = 0$ and $\beta_{ijk^*l} = 1$, $\forall(i,j,l) \neq (i^*, j^*, l^*)$.

  4. $\hat{d}_{l^*} = \hat{d}_{l^*} - x_{i^*j^*k^*l^*}$

     If $\hat{d}_{l^*} = 0$ then take $x_{ijkl^*} = 0$ and $\beta_{ijkl^*} = 1$, $\forall(i,j,k) \neq (i^*, j^*, k^*)$.

  End While.

Phase 2: Determining an optimal solution

The second phase is used to improve a basic feasible solution until an optimum one is attained.

1. Initialization: At the beginning of this phase, we have an initial basic feasible solution $x^{(\hat{r})}$ and set $\hat{r} = 0$.

2. Determine the set of interesting quadruplet (i, j, k, l), denoted as $\hat{I}^{(\hat{r})}$.

$$\hat{I}^{(\hat{r})} = \{(i,j,k,l) : x^{(\hat{r})}_{ijkl} \text{ is a basic variable}\}.$$

3. Solve the following linear system for all $(i, j, k, l) \in \hat{I}^{(\hat{r})}$.

$$\hat{u}^{(\hat{r})}_i + \hat{v}^{(\hat{r})}_j + \hat{w}^{(\hat{r})}_k + \hat{t}^{(\hat{r})}_l = c_{ijkl}$$

   Where $i = 1 : m, j = 1 : n, k = 1 : p, l = 1 : q$.

4. For all $(i, j, k, l) \notin \hat{I}^{(\hat{r})}$, determine

$$\hat{\delta}^{(\hat{r})}_{ijkl} = c_{ijkl} - (\hat{u}^{(\hat{r})}_i + \hat{v}^{(\hat{r})}_j + \hat{w}^{(\hat{r})}_k + \hat{t}^{(\hat{r})}_l)$$

5. If $\hat{\delta}_{ijkl}^{(\hat{r})}) \geq 0, \forall (i,j,k,l) \notin \hat{I}^{(\hat{r})}$, then the solution $x^{(\hat{r})}$ is optimal.

6. Else determine the quadruplet $(\hat{i}_0, \hat{j}_0, \hat{k}_0, \hat{l}_0)$ such that:

$$\hat{\delta}_{\hat{i}_0\hat{j}_0\hat{k}_0\hat{l}_0}^{(\hat{r})} = \min\{\hat{\delta}_{ijkl}^{(\hat{r})} : \hat{\delta}_{ijkl}^{(\hat{r})}) < 0\}$$

7. Solve the following system in order to construct a cycle $\hat{\mu}^{(\hat{r})}$.

$$\sum \hat{\lambda}_{ijkl}^{(\hat{r})} \hat{P}_{ijkl} = -\hat{P}_{\hat{i}_0\hat{j}_0\hat{k}_0\hat{l}_0}, \forall (i,j,k,l) \in \hat{I}^{(\hat{r})}.$$

8. Determine $\hat{\theta}$

$$\hat{\theta} = \min\{\frac{x_{ijkl}^{(\hat{r})}}{-\hat{\lambda}_{ijkl}^{(\hat{r})}}, \text{ such that } \hat{\lambda}_{ijkl}^{(\hat{r})} < 0\} = \hat{\theta}_{\hat{i}_s\hat{j}_s\hat{k}_s\hat{l}_s}^{(\hat{r})}$$

9. Determine the new basic solution.

$$x^{(\hat{r}+1)} = \{x_{ijkl}^{(\hat{r})} + \hat{\lambda}_{ijkl}\hat{\theta} : (i,j,k,l) \in \hat{\mu\hat{r}}\} \cup \{x_{ijkl}^{(\hat{r})} : (i,j,k,l) \notin \hat{\mu}^{\hat{r}}\}.$$

10. Repeat from 2) to 9) until an optimality criterion is verified.

1.3 Multi-Objective Optimization

Optimization is a crucial field of mathematics focused on identifying the best solution for a given problem. It can be categorized into two main types: single-objective and multi-objective optimization (MOO). Over the past two decades, MOO has garnered significant interest and has been utilized in various domains.

Multi-objective optimization is an essential field within multi-criteria decision-making, which addresses optimization issues with several conflicting objectives. It was introduced initially by Vilfredo Pareto, and since then many papers have emerged in this field. Multi-objective optimization has numerous applications in a variety of domains including economics, logistics, engineering, etc. Minimizing the total transportation cost and delivery time when transporting goods, minimizing both fuel consumption and emissions of pollutants, and maximizing performance while buying a car are few of examples of multi-objective problems that arise in our real-world situations with two and three criteria, respectively.

In this section, we aim to introduce an essential background of multi-objective optimization. We initially present the mathematical formulation of a multi-objective optimization problem, some basic definitions related to the notion of dominance, and the resolution approaches.

1.3.1 Problem formulation

A multi-objective optimization problem is stated mathematically as follows:

$$\begin{cases} \text{“min ”} g(x) = (g_1(x), g_2(x), ..., g_m(x)), \\ \text{Subject to constraints} \\ h(x) = 0, f(x) \leq 0, q(x) \geq 0, \\ x \in \chi \text{ and } m \geq 2, \end{cases}$$

where

- $x = (x_1, x_2, ..., x_n)$ is the decision variable vector.
- χ denotes the set of feasible solutions (decision space) ($\chi \subset \mathbb{R}^n$).
- $g = (g_1, ..., g_m)$ is the vector of objective functions to be minimized, and m is the number of objective functions.
- $h : \mathbb{R}^n \to \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}$ and $q : \mathbb{R}^n \to \mathbb{R}$.
- $g(\chi) \subset \mathbb{R}^m$ denotes the objective space.

1.3.2 Dominance notions

The notion of dominance plays an important role in identifying the set of non-dominated solutions for a multi-objective optimization problem.

In all the following, let us consider $x^{(1)}, x^{(2)} \in \chi$ as two decision vectors and U, V as their objective vectors, respectively,

$$x^{(1)} = (x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}) \text{ and } x^{(2)} = (x_1^{(2)}, \hat{x}_2^{(2)}, \ldots, x_n^{(2)}),$$

$$U = g(x^{(1)}) = (u_1, u_2, \ldots, u_m),$$

$$V = g(x^{(2)}) = (v_1, v_2, \ldots, v_m).$$

Definition 1.11. We say that $x^{(1)}$ weakly dominates $x^{(2)}$ and we write $x^{(1)} \preceq x^{(2)}$ if

$$\forall i \in \{1, \ldots, m\}, u_i \leq v_i.$$

Definition 1.12. We say that $x^{(1)}$ strongly dominates $x^{(2)}$ and we write $x^{(1)} \prec x^{(2)}$ if

$$\forall i \in \{1, \ldots, m\}, u_i \leq v_i \text{ and } \exists i \in \{1, \ldots, m\}, u_i < v_i.$$

Definition 1.13. We say that $x^{(1)}$ strictly dominates $x^{(2)}$ and we write $x^{(1)} \prec\prec x^{(2)}$ if

$$\forall i \in \{1, \ldots, m\}, u_i < v_i.$$

Definition 1.14. If neither $x^{(1)}$ weakly dominates $x^{(2)}$ nor $x^{(2)}$ weakly dominates $x^{(1)}$. Then $x^{(1)}$ and $x^{(2)}$ are called incomparable and we write $x^{(1)} \parallel x^{(2)}$.

Definition 1.15. A decision vector $x \in \chi$ is called Pareto optimal solution if

$$\nexists\, y \in \chi \text{ such that } y \prec x.$$

Definition 1.16. The set of Pareto optimal solutions PO and the Pareto front PF are defined as follows:

$$PO = \{x \in \chi \mid \nexists\, y \in \chi \text{ such that } y \prec x\},$$

$$PF = \{g(x) \in \mathbb{R}^m \text{ such that } x \in PO\}.$$

1.3.3 Classification of multi-objective optimization methods

The literature on multi-objective optimization provides different techniques to solve multi-objective problems, which are generally categorized into three main groups which are: Scalarization methods, Pareto-based methods, and Non-Pareto-based methods.

Scalarization methods

Scalarization techniques aim to convert the primary problem into a single-objective problem. The best-known techniques in this category are the Weighted Sum technique, ϵ-Constraints method, Goal Programming, and the Min-Max method.

a) **Weighted Sum technique**

This is considered one of the first optimization techniques to solve multi-objective problems. It converts the primary problem into an aggregate objective function g, representing the sum of all weighted objective functions. Each objective function is assigned a weighting factor as follows:

$$\begin{cases} \min\ g(x) = \sum_{i=1}^{m} \hat{\lambda}_i g_i(x), \\ \text{subject to} \\ h(x) = 0, f(x) \leq 0, q(x) \geq 0, \\ x \in \chi, \hat{\lambda}_i > 0, \text{ and } \sum_{i=1}^{m} \hat{\lambda}_i = 1. \end{cases}$$

b) **ϵ-constraints technique**

This is a scalarization optimization technique that was proposed by Haimes et al. (1977) [15]. It consists of minimizing only one objective function chosen by the decision-maker and transforming the remaining functions into constraints.

$$\begin{cases} \min\ g_j(x), \\ \text{subject to} \\ g_k(x) < \epsilon_k, k \neq j, k = 1, \ldots, m-1, \\ h(x) = 0, f(x) \leq 0, q(x) \geq 0, \\ x \in \chi \text{ and } \epsilon \in \mathbb{R}^{m-1}. \end{cases}$$

c) **Goal programming technique**

This technique consists of minimizing the deviation between goals and the asperation level of all objectives as follows:

$$\begin{cases} \min \displaystyle\sum_{k=1}^{m} ({d_k}^{+} - {d_k}^{-}), \\ \text{Subject to} \\ g_k(x) - {d_k}^{+} + {d_k}^{-} = \hat{g}_k, k = 1, \ldots, m, \\ {d_k}^{+} - {d_k}^{-} \geq 0, \\ h(x) = 0, f(x) \leq 0, q(x) \geq 0, \\ x \in \chi, \end{cases}$$

where ${d_k}^{+}$, ${d_k}^{-}$, $\hat{g}_k$ are the positive deviations, negative deviations, and the goal corresponding to the objective function $g_k(x)$, respectively.

d) **Min-Max technique**

This technique minimize the maximum of relative deviations between goals and the aspiration level of all objectives as follows:

$$\begin{cases} \min \lambda, \\ \text{Subject to} \\ g_k(x) - g_k^* \leq \lambda,\ k = 1, \ldots, m, \\ h(x) = 0, f(x) \leq 0, q(x) \geq 0, \\ x \in \chi, \end{cases}$$

where g_k^* is the ideal value of the objective function $g_k(x)$ that can be obtained by solving each problem independently and λ represents the maximum relative deviation between $g_k(x)$ and g_k^*.

Pareto-based methods

These techniques are widely employed to solve multi-objective optimization problems (MOOPs), comprising two phases. The first one is used to find the set of Pareto optimal solutions for MOOPs based on the dominance concept. The second phase looks at maintaining diversity in the population. Such algorithms are highly effective exclusively for multi-objective issues characterized by a limited number of objective functions. The best-known algorithm in this category are: MOGA (Multi Objective Genetic Algorithm), NSGA (Non dominated Sorting Genetic Algorithm), and SPEA (Strength Pareto Evolutionary Algorithm).

Non-Pareto based methods

These techniques tackle multi-objective problems by treating each objective function independently from the others. They are straightforward to comprehend and can be implemented in various programming languages easily. One of the most widely recognized algorithms in this category is the Vector Evaluated Genetic Algorithm (VEGA).

1.4 Preliminaries on fuzzy set theory

1.4.1 Fuzzy logic

In 1965, L. Zadeh introduced a novel concept known as fuzzy logic, which has since evolved into an important area of research. This field is considered an extension of traditional Boolean Logic and is based on the theory of fuzzy sets. Fuzzy logic incorporates the concept of degrees of truth, allowing a statement to exist in a state other than true or false. This stands in contrast to classical logic, which restricts statements to only two states: entirely true or entirely false. Moreover, the way of thinking in fuzzy logic is more intuitive; it allows the modeling natural phenomena and representation of vague information with the aid of fuzzy set notions such as membership functions.

Since the 199Os, fuzzy logic has experienced remarkable growth and many papers have been published regarding its applications across various domains such as optimization, decision-making, and control of fuzzy systems.

Fuzzy subset

In classical set theory, an element has a boolean value: 1 if the element belongs to the crisp set and 0 if it does not. In contrast to fuzzy logic, an element can partially belong to a fuzzy set and is assigned a value between 0 and 1, known as the membership grade. So, a fuzzy subset is defined by its membership function, which represents the characteristic function in classical logic.

Membership function

In fuzzy logic, the membership function generalizes the characteristic function of classical logic. It assigns to each element a value between 0 and 1. This value indicates the degree of membership, also known as the membership grade.
In the sequel, consider V a universe of discourse denoted by its elements x and μ a membership function defined on V, expressed as: $\mu : V \longrightarrow [0\,1]$.

Definition 1.17. A fuzzy set $\tilde{S}$ of the universe of discourse V is defined by the couples

$$\tilde{S} = \{(x, \mu_{\tilde{S}}(x)) \mid x \in V\},$$

where $\mu_{\tilde{S}}$ denotes the membership function that assigns to each element x of V a value in the interval $[0\ 1]$.

Definition 1.18. A fuzzy set $\tilde{S}$ defined on the set on real numbers $\mathbb{R}$ is termed a fuzzy number if it satisfies the following conditions:

1. Normality: $\exists\ \hat{x}_0 \in \mathbb{R}, \mu_{\tilde{S}}(\hat{x}_0) = 1$.

2. Convexity: $\forall \hat{x}_1, \hat{x}_2 \in \mathbb{R}$ and $\forall\, \hat{t} \in [0\ 1]$, the following inequality holds:

$$\mu_{\tilde{S}}(\hat{t}\hat{x}_1 + (1 - \hat{t})\hat{x}_2) \geq \min(\mu_{\tilde{S}}(\hat{x}_1), \mu_{\tilde{S}}(\hat{x}_2)).$$

3. Piecewise continuity: $\mu_{(\tilde{S})}(x)$ is piecewise continuous.

1.4.2 Basic notions of fuzzy sets

Let $\tilde{S}$ be a fuzzy set on V, defined by its membership function $\mu_{\tilde{S}}$ and let α be a real number within the range $[0,1]$. We can establish the following crisp sets:

- **α-cut of $\tilde{S}$:** It is the set of all elements x which membership grades equal or exceeding α.

$$^{\alpha}\tilde{S} = \{x \in V : \mu_{\tilde{S}}(x) \geq \alpha\}.$$

- **Strong α-cut of $\tilde{S}$:** It is the set of all elements x of V which membership grades are strictly superior than α.

$$^{\alpha+}\tilde{S} = \{x \in V : \mu_{\tilde{S}}(x) > \alpha\}.$$

- **Support of $\tilde{S}$:** It is the set of all elements x which membership grades are greater than zero.

$$^{0+}\tilde{S} = Supp(\tilde{S}) = \{x \in V : \mu_{\tilde{S}}(x) > 0\}.$$

- **Core of $\tilde{S}$:** It is the set of all elements x which membership grades are equal to one.

$$^{1}\tilde{S} = Core(\tilde{S}) = \{x \in V : \mu_{\tilde{S}}(x) = 1\}.$$

Operations on fuzzy sets

Consider two fuzzy sets $\tilde{S}_1$ and $\tilde{S}_2$ on V defined by their respective membership functions $\mu_{\tilde{S}_1}$ and $\mu_{\tilde{S}_2}$. Based on membership functions operations, we can establish the following set operations: equality, intersection, and union.

- **Equality:** $\tilde{S}_1 = \tilde{S}_2$ iff $\mu_{\tilde{S}_1}(x) = \mu_{\tilde{S}_2}(x), \forall x \in V$.

- **Inclusion:** $\tilde{S}_1 \subset \tilde{S}_2$ iff $\mu_{\tilde{S}_1}(x) < \mu_{\tilde{S}_2}(x), \forall x \in V$.

- **Intersection:** The intersection of $\tilde{S}_1$ and $\tilde{S}_1$ is the fuzzy set $\tilde{S}_3 = \tilde{S}_1 \cap \tilde{S}_2$, carachterized by:

$$\tilde{S}_3 = \{(x, \mu_{\tilde{S}_3}(x)) : x \in V\}, \text{ where } \mu_{\tilde{S}_3}(x) = \min\{\mu_{\tilde{S}_1}(x), \mu_{\tilde{S}_2}(x)\}.$$

- **Union:** The union of $\tilde{S}_1$ and $\tilde{S}_2$ is the fuzzy set $S_4 = S_1 \cup S_2$, given by:

$$\tilde{S}_4 = \{(x, \mu_{\tilde{S}_4}(x)) : x \in V\}, \text{ where } \mu_{\tilde{S}_4}(x) = \max\{\mu_{\tilde{S}_1}(x), \mu_{\tilde{S}_2}(x)\}.$$

- **Complement:** The complement of $\tilde{S}$ is the fuzzy set $\tilde{S}^c$, given by:

$$\tilde{S}^c = \{(x, \mu_{\tilde{S}^c}(x)) : x \in V\}, \text{ where } \mu_{\tilde{S}^c}(x) = 1 - \mu_{\tilde{S}}(x).$$

Properties of fuzzy sets

Let $\tilde{S}_1$, $\tilde{S}_2$ and $\tilde{S}_3$ be three fuzzy sets on V. We have the following properties:

- **Commutativity**

$$\tilde{S}_1 \cap \tilde{S}_2 = \tilde{S}_2 \cap \tilde{S}_1,$$
$$\tilde{S}_1 \cup \tilde{S}_2 = \tilde{S}_2 \cup \tilde{S}_1.$$

- **Associativity**

$$\tilde{S}_1 \cap (\tilde{S}_2 \cap \tilde{S}_3) = (\tilde{S}_1 \cap \tilde{S}_2) \cap \tilde{S}_3,$$
$$\tilde{S}_1 \cup (\tilde{S}_2 \cup \tilde{S}_3) = (\tilde{S}_1 \cup \tilde{S}_2) \cup \tilde{S}_3.$$

- **Distributivity**

$$\tilde{S}_1 \cup (\tilde{S}_2 \cap \tilde{S}_3) = (\tilde{S}_1 \cup \tilde{S}_2) \cap (\tilde{S}_1 \cup \tilde{S}_3),$$
$$\tilde{S}_1 \cap (\tilde{S}_2 \cup \tilde{S}_3) = (\tilde{S}_1 \cap \tilde{S}_2) \cup (\tilde{S}_1 \cap \tilde{S}_3).$$

1.4.3 Fuzzy numbers

This subsection mentions the most popular types of fuzzy numbers: trapezoidal and triangular fuzzy numbers.

Trapezoidal fuzzy number

A trapezoidal fuzzy number is a fuzzy set that can be represented as $\tilde{S} = (a_1, a_2, a_3, a_4)$ where $a_1 \leq a_2 \leq a_3 \leq a_4$ and its membership function is as follows:

$$\mu_{\tilde{S}}(x) = \begin{cases} 0 & if \;\; x < a_1, \\ \dfrac{x - a_1}{a_2 - a_1} & if \;\; a_1 \leq x \leq a_2, \\ 1 & if \;\; s_2 \leq x \leq s_3, \\ \dfrac{x - a_4}{a_3 - a_4} & if \;\; a_3 \leq x \leq a_4, \\ 0 & if \;\; x \geq a_4. \end{cases}$$

Triangular fuzzy number

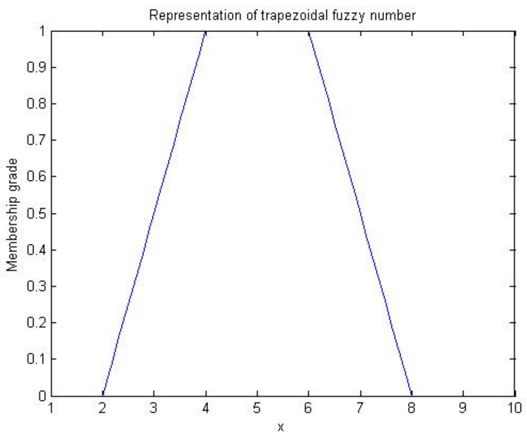

Figure 1.1: Representation of trapezoidal fuzzy number.

A triangular fuzzy number is a special case of a trapezoidal fuzzy number where $a_2 = a_3$. It can be represented as $\tilde{S} = (a_1, a_2, a_4)$ with $a_1 \leq a_2 \leq a_4$ and its membership function is as follows:

$$\mu_{\tilde{S}}(x) = \begin{cases} 0 & if \;\; x < a_1, \\ \dfrac{x - a_1}{a_2 - a_1} & if \;\; a_1 \leq x \leq a_2, \\ \dfrac{x - a_4}{a_2 - a_4} & if \;\; a_2 \leq x \leq a_4, \\ 0 & if \;\; x \geq a_4. \end{cases}$$

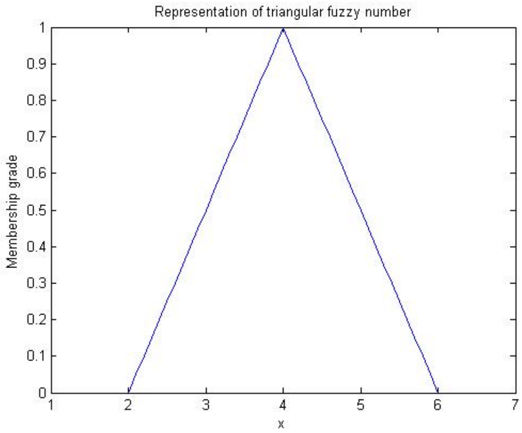

Figure 1.2: Representation of triangular fuzzy number.

Ranking Function

The function $\Re : F(\mathbb{R}) \longrightarrow \mathbb{R}$ is called the ranking function which is utilized to defuzzify a given fuzzy number into a crisp one where $F(\mathbb{R})$ denotes the set of fuzzy numbers.

For any trapezoidal fuzzy number represented as $\tilde{S} = (a_1, a_2, a_3, a_4)$, its ranking function can be determined via the following formula:

$$\Re(\tilde{S}) = \frac{a_1 + a_2 + a_3 + a_4}{4}.$$

For any triangular fuzzy number represented as $\tilde{S} = (a_1, a_2, a_4)$, its ranking function can be determined via the following formula:

$$\Re(\tilde{S}) = \frac{a_1 + 2\,a_2 + a_4}{4}.$$

Let $\tilde{S}_1$ and $\tilde{S}_2$ be two fuzzy sets. We have the following properties:

$$\tilde{S}_1 <_{\Re} \tilde{S}_2 \Longleftrightarrow \Re(\tilde{S}_1) < \Re(\tilde{S}_2),$$

$$\tilde{S}_1 >_{\Re} \tilde{S}_2 \Longleftrightarrow \Re(\tilde{S}_1) > \Re(\tilde{S}_2),$$

$$\tilde{S}_1 =_{\Re} \tilde{S}_2 \Longleftrightarrow \Re(\tilde{S}_1) = \Re(\tilde{S}_2).$$

Arithmetic operations

Let us consider two trapezoidal fuzzy numbers $\tilde{S}_1 = (a_1, a_2, a_3, a_4)$, $\tilde{S}_2 = (a'_1, a'_2, a'_3, a'_4)$ and a scalar $\hat{\lambda}$. We can establish the following arithmetic operations: addition, subtraction, and multiplication.

- **Addition**

$$\tilde{S}_1 \oplus \tilde{S}_2 = (a_1 + a'_1, a_2 + a'_2, a_3 + a'_3, a_4 + a'_4).$$

- **Subtraction**

$$\tilde{S}_1 \ominus \tilde{S}_2 = (a_1 - a'_4, a_2 - a'_3, a_3 - a'_2, a_4 - a'_1).$$

- **Multiplication**

$$\tilde{S}_1 \otimes \tilde{S}_2 = \begin{cases} \Re(\tilde{S}_2)\,(a_1, a_2, a_3, a_4), & \text{if } \Re(\tilde{S}_2) \geq 0, \\ \Re(\tilde{S}_2)\,(a_4, a_3, a_2, a_1), & \text{if } \Re(\tilde{S}_2) < 0. \end{cases}$$

- **Scalar multiplication**

$$\hat{\lambda}\,\tilde{S}_1 = \begin{cases} (\hat{\lambda}\,a_1, \hat{\lambda}\,a_2, \hat{\lambda}\,a_3, \hat{\lambda}\,a_4), & \text{if } \hat{\lambda} \geq 0, \\ (\hat{\lambda}\,a_4, \hat{\lambda}\,a_3, \hat{\lambda}\,a_2, \hat{\lambda}\,a_1), & \text{if } \hat{\lambda} < 0. \end{cases}$$

Chapter 2

Fuzzy Four-index Fixed-Charge Transportation Problem

2.1 Introduction

The fixed-charge transportation problem (FCTP) is one of the most significant problems in optimization, initiated by Hirsch and Danzig in 1954 [21]. It is a variation of the well-known Hitchcock problem. FCTP involves sending a product from source locations to destination locations, incorporating two types of costs: variable costs and fixed costs, where the purpose is to minimize the overall cost of transport. It is a mixed-integer problem and can be modeled as a distribution problem. This is one of the most interesting problems that has attracted numerous researchers.

In real-world applications, we may face many situations where the parameters of transportation problems are defined imprecisely due to a lack of information. To overcome such situations, Zadeh introduced fuzzy set theory, which is the foundation of fuzzy logic.

2.2 Literature Review

Various methods have been introduced for the resolution of the FCTP, which fall into three major categories: exact, heuristic, and metaheuristic algorithms. In 1968, Murty [35] provided an exact solution for FCTP using a ranking extreme point technique. It is shown that Murty's algorithm is efficient for fixed charge problems in which fixed costs are small compared with variable costs. In 1971, Gray [14] developed an exact algorithm that involves separating the FCTP into two subproblems and solving each subproblem independently. Other methodologies utilizing branch-and-bound algorithms have been suggested, including [36, 47]. It turns out that some existing exact approaches are not useful for large scale instances like [7, 33, 41]. In 2014, Roberti et al. [39] described an exact method for obtaining an optimal solution for FCTP using a new integer formulation. In another work, Mingozzi and Roberti [32] introduced a branch-and-cut-and-price approach with embedded lower bounds based on a pseudo-polynomial number of equations. It is demonstrated that their technique is significantly quicker than current exact methods.

In addition, many researchers have focused on solving FCTPs using heuristic and metaheuristic algorithms instead of exact ones. In 1961, Balinski [6] discussed the mathematical formulation of the problem and proposed the first approximation method that consists of transforming the FCTP into a linear transportation problem and then solving the resulting problem using the available transportation algorithms. Next, Denzler [10] presented an approximation method called the fixed charge simplex algorithm. Denzler's method extends the well-known simplex method, incorporating a new criterion to select the vector entering the base. Since then, many methods have been provided to yield an optimal solution for FCTP, such as [9, 40, 42, 49]. In [48], a heuristic algorithm based on the tabu search method has been introduced to solve the same problem. Later, Adlakha and Kowalski [**?**] addressed small fixed charge transportation problems and presented a simple heuristic algorithm for solutions. Subsequently, Adlakha et al. [3] solved the FCTP using a branching method. In 2012, El-Sherbiny et al. [11] treated the FCTP using a hybrid algorithm. In 2019, Balaji et al. [5] treated the fixed charge transportation problem with truckload constraint. After that, Yousefi et al. [50] proposed three metaheuristics and an approximation method to provide an optimal solution for FCTP.

In 2023, Kartli et al. [24] described a new heuristic approach to determine an approximate solution for FCTP, then they compared the results obtained from their method with st-GA and pb-GA. The fuzzy FCTP has been tackled using diverse metaheuristics in [29, 34, 43]. In 2014, Mahmoodirad et al. [30] proposed an algorithm to deal with the fuzzy fully FCTP based on an extension of Balinski's approximation in a fuzzy context. Later, Pop et al. [38] investigated a two-stage fixed-charge transportation problem and suggested a hybrid algorithm for the resolution, which integrates a local search process with the steady-state genetic algorithm.

As mentioned above, many researchers have studied FCTPs with two or three indexes. However, none of them have treated the four index FCTP. This model is more realistic, especially when the parameters of the problem are not defined exactly. In addition, it is very difficult to determine an exact solution to such a non-deterministic problem and the use of metaheuristics requires designing a suitable method to encode and decode the candidate solution. These challenges serve as motivations for treating the FCTP with four indices in a fuzzy context.

This chapter introduces four metaheuristic algorithms to solve the fuzzy four-index fixed-charge transportation problem (FFCTP4). The suggested metaheuristic algorithms are the Genetic Algorithm (GA), Simulated Annealing (SA), Particle Swarm Optimization (PSO), and a hybrid Particle Swarm Optimization with the Genetic Algorithm (PSO-GA), and for this, a new priority-based decoding procedure is developed. For the purpose of comparison, we develop an approximation method that consists of transforming the nonlinear FFCTP4 in an adequate way into a linear transportation problem and solving it using an extended version of the least cost cell and MODI methods.

2.3 Preliminaries on metaheuristics

In this section, we present an overview of some metaheuristic algorithms which are genetic algorithms, simulated annealing, and particle swarm optimization.

2.3.1 Metaheuristics

Metaheuristics refer to a category of probabilistic optimization algorithms, that initially appeared during the 19880s. They are inspired by biology (genetic algorithms), physics (simulated annealing), and swarm intelligence (particle swarm optimization) which are widely utilized to address complex optimization issues. Metaheuristic algorithms incorporate two key elements: exploration and exploitation, for exploration, the so-called diversification indicates the capability of an algorithm to uncover a varied range of solutions, distributed across various regions of the search space. Exploitation indicates the search for the best solution across a set of local and global solutions. It has been shown that the harmonization between the two aforementioned key elements is crucial for the efficiency and robustness of metaheuristics. Such algorithms present both advantages and drawbacks, some of which we mention below.

Advantages

- Metaheuristic algorithms belong to the category of global optimization algorithms, employing some randomness degree.
- Metaheuristics serve as robust and efficient algorithms, providing near-optimal solutions within a reasonable execution time.
- Metaheuristic algorithms are easy to understand and implement.
- Metaheuristics can be extended to solve multi-objective optimization problems in various environments.

Drawbacks

- Metaheuristic algorithms can provide an approximate solution without guaranteeing its optimality.
- The performance of metaheuristics is related to the choice of parameters.

- Tuning of parameters of metaheuristics may require a long time.
- Some metaheuristic algorithms get stuck in local optima, resulting in premature convergence.

Now, we provide a depth overview of three metaheuristic algorithms that will be utilized to solve the fuzzy four-index fixed-charge transportation problem.

2.3.2 Genetic Algorithm

Genetic Algorithm (GA) is a population-based algorithm and refers to a class of stochastic optimization algorithms. It was originally introduced by J. Holland [23] in 1975, and its inspiration comes from natural selection, which employs the concept of survival of the fittest. GA is considered one of the best-known and most powerful techniques used to generate approximate solutions for various optimization problems.

Terminology: Vocabulary and Definition

In this subsection, we provide some definitions of the key terminology used in genetic algorithm to clarify the concepts.

Chromosome: It comprises a sequence of genes with specific values assigned from a fixed alphabet.

Individual: It is a potential solution to the problem we aim to solve, represented as a chromosome.

Encoding Scheme: It consists of representing the information of a given problem into a sequence of strings. The literature provides many encoding scheme techniques, among them binary, octal, hexadecimal, permutation, value-based, and tree.

Evaluation: It is used to determine the fitness value of each individual within the population.

Selection. This is the key process in genetic algorithms. It involves choosing some individuals to perform genetic operations such as mutation and crossover. Various selection methods are discussed in the literature, among them tournament selection, roulette wheel selection, and ranking selection.

a) **Roulette wheel selection.** It is considered one of the most popular selection strategies that is used to select certain individuals to perform recombination operator. Each individual within the population is assigned a fitness value associated with probability of selection, shown as follows:

$$p_s(x_i) = \frac{F(x_i)}{\sum_{j=1}^{N} F(x_j)},$$

where N represents the size of population, $F(x_j)$ denotes the fitness value of an individual x_j, and $p_s(x_i)$ denotes the probability of selection of an individual x_j.

b) **Tournament selection.** It is a method used to randomly choose two individuals; the individual with the higher fitness value is then selected. This process is repeated multiple times until a predefined number of individuals is obtained.

c) **Ranking selection.** In this method, each individual within the population is assigned a rank based on its fitness value. Then, the individuals are selected according to their rank.

Crossover: It is used to combine the genetic materials of two or more individuals to produce novel offspring. The literature on genetic algorithms provides several kinds of crossover such as one-point crossover, two-point, and uniform.

Mutation: It is a process that involves changes in certain genes, leading to diversity within a population.

Termination criterion: It is a crucial element in genetic algorithms to guarantee the optimality of the solution found. We distinguish two types of termination criterion:

- Termination is performed after a predefined number of generations.
- Termination is performed when the population stops evolving over multiple generations.

Mechanism of Genetic Algorithm

To solve an optimization issue via GA, it begins by generating an initial population of N individuals. Each individual within the population is evaluated using some measure of fitness. Then, a set of individuals are selected to undergo recombination operators in order to generate novel offspring. After that, the best chromosomes are copied to the the subsequent generation and the whole process is repeated over numerous generations. The steps of GA are shown in figure 2.1.

2.3.3 Simulated Annealing

Simulated Annealing (SA) is a single-solution-based algorithm and refers to the class of probabilistic optimization techniques that is used to yield an approximate solution for combinatorial optimization issues. It was initially proposed by Kirkpatrick [27] and inspired by the annealing process in metallurgy. Moreover, an annealing process is composed of two primary phases:

- The initial phase is the solid melt-out, achieved by elevating the solid to an exceedingly high temperature.
- The second phase aims to gradually cool the solid until it attains the minimum energy state.

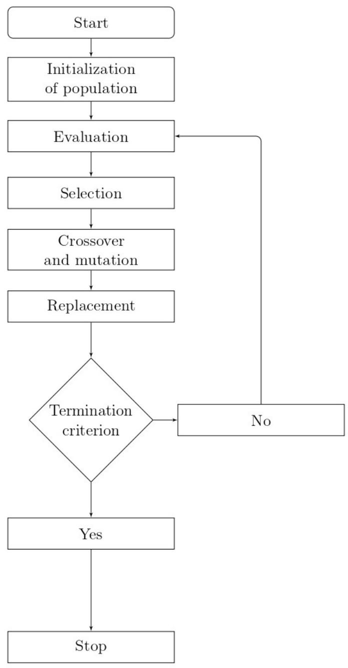

Figure 2.1: Flowchart of Genetic Algorithm.

Mechanism of simulated annealing

To address an optimization problem using SA, it starts with the generation of an initial solution S (configuration) at a high temperature T. Then, SA searches for another solution S^* in the neighborhood of the current solution. After that, each solution is evaluated, if the new solution is better than the current one, then S is replaced by S^*. Otherwise, S is replaced by S^* with probability $P(T, S, S^*) = e^{\frac{f(S^*)-f(S)}{T}}$, and the temperature is reduced via a predetermined cooling schedule. The whole process continues until a termination criterion is verified. The steps of SA are shown in figure 2.2.

2.3.4 Particle Swarm Optimization

Particle Swarm Optimization (PSO) is a population-based algorithm that is inspired from the social behavior of animals like fish and birds. The basic concepts of (PSO)

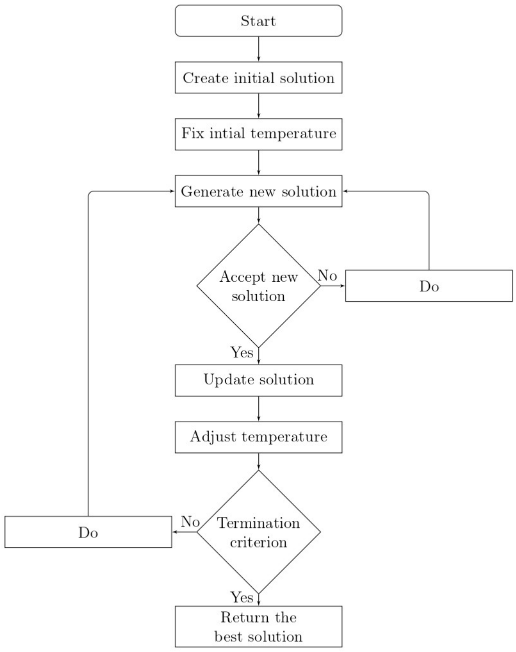

Figure 2.2: Flowchart of simulated annealing algorithm.

were introduced by Kennedy and Eberhart [25] in 1955. PSO is widely applied yield an approximate solution for combinatorial optimization problems. This algorithm Operates on a swarm of particles in which each particle's memory and learning experiences are used to change the search pattern to find food cooperatively.

Mechanism of particle swarm optimization algorithm

To address an optimization problem via PSO, it starts with the generation of a swarm of particles. A particle within the swarm represents a candidate solution for a given optimization issue. In a n-dimensional search space, the particle i of the swarm at time t is represented with its position x_i^t and its velocity v_i^t. This particle memorizes its best-visited position (referred to as $Pbest_i$) and the best position achieved by its neighbors (referred to as $Gbest$). At each iteration, the particle's velocity is updated, which is then used to determine a new position for the particle. The equations for the velocity and position adjustment are given as follows:

$$\begin{cases} v_i^{t+1} + = wv_i^t + c_1r_1(Pbest_i - x_i^t) + c_2r_2(Gbest - x_i^t), \\ x_i^{t+1} = x_i^t + v_i^{t+1}, \end{cases}$$

where w is the inertia weight, v_i^t is the velocity of particle i at time t, x_i^t is the position of particle i at time t, c_1, c_2 are acceleration coefficients, r_1, r_2 are random numbers in the range $[0\ 1]$., $Pbest_i$ is the best solution achieved by the particle i, and $Gbset$ is the best solution ever achieved by all particles. From a sociological viewpoint, three influence components exist on the updated velocity formula which are: inertia component, cognitive component, and social component.

- In the inertia component (wv_i^t), a particle tends to follow its way.
- In the cognitive component ($c_1r_1(Pbest_i - x_i^t)$), a particle tends to move toward its best visited position.
- In the social component ($c_2r_2(Gbest - x_i^t)$), a particle memorizes the learning experience of its neighbors and tends to move toward the best position attained by all particles.

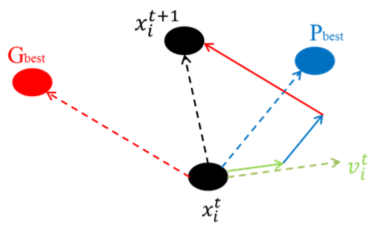

Figure 2.3: Representation of direction of particle in search space.

The steps of PSO are shown in figure 2.4.

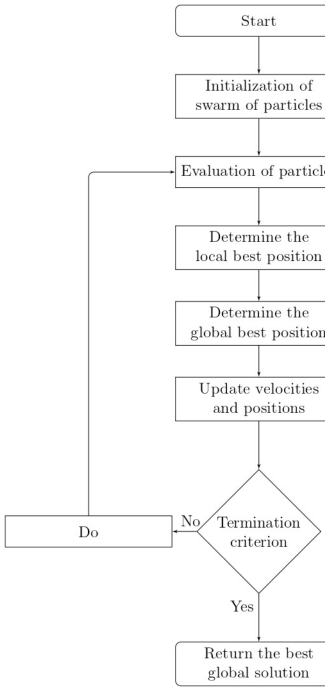

Figure 2.4: Flowchart of the particle swarm optimization algorithm.

2.4 Problem position

2.4.1 Economical interpretation

Let

- $O_1, \ldots, O_m$, be m origin nodes of supplies $\hat{a}_1, ..., \hat{a}_m$, at respective.
- $D_1, \ldots, D_n$, be n destination nodes of demands $\hat{b}_1, ..., \hat{b}_n$, at respective.
- $S_1, \ldots, S_p$, be p types of vehicle of reserved charges $\hat{e}_1, ..., \hat{e}_p$, at respective.
- $H_1, \ldots, H_q$, be q types of goods of quantities $\hat{d}_1, ..., \hat{d}_l$, at respective.
- x_{ijkl}: be the quantity of product of type H_l transported from the origin O_i to the destination D_j using the vehicle of type S_k.
- $\tilde{c}_{ijkl}$: be the unit fuzzy variable cost of transport of the quantity x_{ijkl}.

- $\tilde{f}_{ijkl}$: be the unit fuzzy fixed cost of transport of the quantity x_{ijkl}.

  We have $\hat{a}_i > 0, \hat{b}_j >_0, \hat{e}_k >, \hat{d}_l > 0, \tilde{c}_{ijkl} \geq_{\Re} 0$, and $\tilde{f}_{ijkl} \geq_{\Re} 0$, $\forall (i,j,k,l)$.

2.4.2 Problem formulation

The mathematical formulation of the fuzzy four index fixed charge transportation problem is given as follows:

$$\text{minimize}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q}\tilde{c}_{ijkl}\otimes x_{ijkl}\oplus\tilde{f}_{ijkl}\otimes y_{ijkl}\right\} \tag{2.1}$$

Subject to constraints

$$\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} = \hat{a}_i, \text{ for all } i = 1, ..., m, \tag{2.2}$$

$$\sum_{i=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} = \hat{b}_j, \text{ for all } j = 1, ..., n, \tag{2.3}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{q} x_{ijkl} = \hat{e}_k, \text{ for all } k = 1, ..., p, \tag{2.4}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} x_{ijkl} = \hat{d}_l, \text{ for all } l = 1, ..., q, \tag{2.5}$$

$$y_{ijkl} = \begin{cases} 1, \text{if } \; x_{ijkl} > 0, \\ 0, \text{if } \; x_{ijkl} = 0, \end{cases} \tag{2.6}$$

$$x_{ijkl} \geq 0, \text{ for all } i = 1:m; j = 1:n; k = 1:p, l = 1:q. \tag{2.7}$$

By generalizing the feasible condition in [52], we establish the following theorem:

Theorem 2.1 (Feasibility condition [52]). *The fuzzy four index fixed charge transportation problem has a feasible solution if and only if*

$$\sum_{i=1}^{m}\hat{a}_i = \sum_{j=1}^{n}\hat{b}_j = \sum_{k=1}^{p}\hat{e}_k = \sum_{l=1}^{q}\hat{d}_l.$$

Theorem 2.2 (Optimality criterion). *Let $x = (x_{ijkl})$ be a basic feasible solution of FFCTP4 with a basic matrix B. Then, x is optimal if and only if:*

$$\begin{cases} \theta_{ijkl}(\tilde{u}_i \oplus \tilde{v}_j \oplus \tilde{w}_k \oplus \tilde{t}_l) \oplus DF_{ijkl} \geq_{\Re} 0, \;\; \forall (i,j,k,l) \notin B, \\ \theta_{ijkl}(\tilde{u}_i \oplus \tilde{v}_j \oplus \tilde{w}_k \oplus \tilde{t}_l) \oplus DF_{ijkl} =_{\Re} 0, \;\; \forall (i,j,k,l) \in B. \end{cases} \tag{2.8}$$

Note that: $\tilde{u}_i, \tilde{v}_j, \tilde{w}_k, \text{and} \tilde{t}_l$: are the dual variables that can be determined as follows:

$$\begin{cases} \tilde{u}_i \oplus \tilde{v}_j \oplus \tilde{w}_k \oplus \tilde{t}_l = \tilde{c}_{ijkl}, \ \forall (i,j,k,l) \in B \\ \tilde{u}_i \oplus \tilde{v}_j \oplus \tilde{w}_k \oplus \tilde{t}_l \leq_{\Re} \tilde{c}_{ijkl}, \ \forall (i,j,k,l) \notin B \end{cases} \tag{2.9}$$

θ_{ijkl} is the value at which a non-basic variable enters the base.

Proof

Let $\tilde{Z} =_{\Re} Z_{(RP_1)} \oplus \tilde{F}$ with $\tilde{F} = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} \sum_{l=1}^{q} \tilde{f}_{ijkl} \otimes y_{ijkl}$.

Let x_θ be a basic feasible solution associated with the base B obtained by entering a non-basic variable into the base B with value θ_{ijkl} which undergoes a change $D\tilde{F}_{ijkl}$ in the total cost.

Let $\tilde{Z}_\theta$ be the objective value of the FBOFCT4 corresponding to the solution x_θ. We have:

$$\tilde{Z}_\theta =_{\Re} \tilde{Z}_{(RP_1)} \oplus [\theta_{ijkl}(\tilde{c}_{ijkl} \ominus (\tilde{u}_i \oplus \tilde{v}_j \oplus \tilde{w}_k \oplus \tilde{t}_l)) \oplus D\tilde{F}_{ijkl} \oplus \tilde{F}],$$

$$\tilde{Z}_\theta =_{\Re} Z_{\tilde{(RP_1)}} \oplus \tilde{F} \oplus [\theta_{ijkl}(\tilde{c}_{ijkl} \ominus (\tilde{u}_i \oplus \tilde{v}_j \oplus \tilde{w}_k \oplus \tilde{t}_l)) \oplus D\tilde{F}_{ijkl}],$$

$$\tilde{Z}_\theta =_{\Re} \tilde{Z} \oplus [\theta_{ijkl}(\tilde{c}_{ijkl} \ominus (\tilde{u}_i \oplus \tilde{v}_j \oplus \tilde{w}_k \oplus \tilde{t}_l)) \oplus D\tilde{F}_{ijkl}].$$

It is clear that if $\theta_{ijkl}(\tilde{c}_{ijkl} \ominus (\tilde{u}_i \oplus \tilde{v}_j \oplus \tilde{w}_k \oplus \tilde{t}_l)) \oplus D\tilde{F}_{ijkl} \leq_{\Re} 0$ for some non basic variable, then the solution x is not optimal (because $\tilde{Z}_\theta \leq \tilde{Z}$).

So, the solution x is optimal if and only if $\theta_{ijkl}(\tilde{c}_{ijkl} \ominus (\tilde{u}_i \oplus \tilde{v}_j \oplus \tilde{w}_k \oplus \tilde{t}_l)) \oplus D\tilde{F}_{ijkl} \geq_{\Re} 0, \ \forall (i,j,k,l) \notin B$.

2.5 Application of metaheuristics to solve FFCTP4

In this section, we present four metaheuristic algorithms for generating an approximate solution for FFCTP4. The proposed metaheuristics are: genetic algorithm, simulated annealing, particle swarm optimization, and hybrid particle swarm optimization with genetic algorithm.

2.5.1 Genetic algorithm to solve FFCTP4

Encoding scheme and initialization

The representation of the chromosome (candidate solution) involves randomly generating a permutation of $m + n + p + q$ digits between 1 and $m + n + p + q$. The gene on the chromosome has two types of information. The value that the gene takes, known as priority, is the first type of information. The second type of information is the position of the gene in the chromosome that represents the node (source, destination, type of product, means of transport).

To decode the solution, we propose an adaptation of the priority-based decoding procedure employed in [29] in fuzzy environment with four indices. The steps of the priority-based decoding procedure are listed as shown in figure 2.5.

Input:
Enter the problem's dimensions: m, n, p, q, variable costs $\tilde{c}_{ijkl}$,
fixed costs $\tilde{f}_{ijkl}$ and the quantities $\hat{a}, \hat{b}, \hat{e}$ and $\hat{d}$.
Create a chromosome $v(i+j+k+l)$.

Output: the amount of goods x_{ijkl} .

Iterations

Step 1: $x_{ijkl} = 0$, for each i, j, k and l.
Step 2: Choose a maximum number of chromosome $\{v(t) : t = 1, ..., m+n+p+q\}$.
Step 3: Save the position of maximum number chosen and name it T.
Step 4: **If** $T \geq m$ then $i^* = T$.

$$j^* = \arg minUc_{i^*jkl} = \{\tilde{c}_{i^*jkl} + \frac{\tilde{f}_{i^*jkl}}{\min(\hat{a}_{i^*}, \hat{b}_j, \hat{e}_k, \hat{d}_l)}, v(m+j) \neq 0\}.$$

$$k^* = \arg minUc_{i^*jkl} = \{\tilde{c}_{i^*jkl} + \frac{\tilde{f}_{i^*jkl}}{\min(\hat{a}_{i^*}, \hat{b}_j, \hat{e}_k, \hat{d}_l)}, v(m+n+k) \neq 0\}.$$

$$l^* = \arg minUc_{i^*jkl} = \{\tilde{c}_{i^*jkl} + \frac{\tilde{f}_{i^*jkl}}{\min(\hat{a}_{i^*}, \hat{b}_j, \hat{e}_k, \hat{d}_l)}, v(m+n+p+l) \neq 0\}.$$

Elseif $T > m$ and $(T \leq m+n)$ then $j^* = T - m$.

$$i^* = \arg minUc_{ij^*kl} = \{\tilde{c}_{ij^*kl} + \frac{\tilde{f}_{ij^*kl}}{\min(\hat{a}_i, \hat{b}_{j^*}, \hat{e}_k, \hat{d}_l)}, v(i) \neq 0\}.$$

$$k^* = \arg minUc_{ij^*kl} = \{\tilde{c}_{ij^*kl} + \frac{\tilde{f}_{ij^*kl}}{\min(\hat{a}_i, \hat{b}_{j^*}, \hat{e}_k, \hat{d}_l)}, v(m+n+k) \neq 0\}.$$

$$l^* = \arg minUc_{ij^*kl} = \{\tilde{c}_{ij^*kl} + \frac{\tilde{f}_{ij^*kl}}{\min(\hat{a}_i, \hat{b}_{j^*}, \hat{e}_k, \hat{d}_l)}, v(m+n+p+l) \neq 0\}.$$

Elseif $T > m+n$ and $(T \leq m+n+p)$ then $k^* = T - m - n$.

$$i^* = \arg minUc_{ijk^*l} = \{\tilde{c}_{ijk^*l} + \frac{\tilde{f}_{ijk^*l}}{\min(\hat{a}_i, \hat{b}_j, \hat{e}_{k^*}, \hat{d}_l)}, v(i) \neq 0\}.$$

$$j^* = \arg minUc_{ijk^*l} = \{\tilde{c}_{ijk^*l} + \frac{\tilde{f}_{ijk^*l}}{\min(\hat{a}_i, \hat{b}_j, \hat{e}_{k^*}, \hat{d}_l)}, v(m+j) \neq 0\}.$$

$$l^* = \arg minUc_{ijk^*l} = \{\tilde{c}_{ijk^*l} + \frac{\tilde{f}_{ijk^*l}}{\min(\hat{a}_i, \hat{b}_j, \hat{e}_{k^*}, \hat{d}_l)}, v(m+n+p+l) \neq 0\}.$$

Elseif $T > m+n+p$ and $(T \leq m+n+p+q)$ then $l^* = T - m - n - p$.

$$i^* = \arg minUc_{ijkl^*} = \{\tilde{c}_{ijkl^*} + \frac{\tilde{f}_{ijkl^*}}{\min(\hat{a}_i, \hat{b}_j, \hat{e}_k, \hat{d}_{l^*})}, v(i) \neq 0\}.$$

$$j^* = \arg minUc_{ijkl^*} = \{\tilde{c}_{ijkl^*} + \frac{\tilde{f}_{ijkl^*}}{\min(\hat{a}_i, \hat{b}_j, \hat{e}_k, \hat{d}_{l^*})}, v(m+j) \neq 0\}.$$

$$k^* = \arg minUc_{ijkl^*} = \{\tilde{c}_{ijkl^*} + \frac{\tilde{f}_{ijkl^*}}{\min(\hat{a}_i, \hat{b}_j, \hat{e}_k, \hat{d}_{l^*})}, v(m+n+k) \neq 0\}.$$

End if
Step 5: $x_{i^*j^*k^*l^*} = min(\hat{a}_{i^*}, \hat{b}_{j^*}, \hat{e}_{k^*}, \hat{d}_{l^*})$
and update the availabilities of $\hat{a}_{i^*}, \hat{b}_{j^*}, \hat{e}_{k^*}$, and $\hat{d}_{l^*}$) as follows:
$\hat{a}_{i^*} = \hat{a}_{i^*} - x_{i^*j^*k^*l^*}$.
$\hat{b}_{j^*} = \hat{b}_{j^*} - x_{i^*j^*k^*l^*}$.
$\hat{e}_{k^*} = \hat{e}_{k^*} - x_{i^*j^*k^*l^*}$.
$\hat{d}_{l^*} = \hat{d}_{l^*} - x_{i^*j^*k^*l^*}$.
Step 6: **If** $\hat{a}_{i^*} = 0$ **then** $v(i^*) = 0$;If $\hat{b}_{j^*} = 0$ then $v(m+j^*) = 0$;
If $\hat{e}_{k^*} = 0$ **then** $v(m+n+k^*) = 0$; **If** $\hat{d}_{l^*} = 0$ **then** $v(m+n+p+l^*) = 0$;
remove the selected priority.
Step 7: **If** $\exists i \leq m$, $v(i) \neq 0$ go to step 2. **Else** calculate the total transportation cost.

Figure 2.5: Priority-based encoding scheme for FFCTP4.

Numerical example

To clarify the above suggested priority-based decoding procedure, we give a numerical example of a fuzzy four index fixed charge transportation problem (FFCTP4), whose dimensions $(m = n = p = q = 2)$, and the values of $\hat{a}_i, \hat{b}_j, \hat{e}_k, \hat{d}_l, \tilde{c}_{ijkl}$ and $\tilde{f}_{ijkl}$ are shown in table 2.1, table 2.2, and table 2.3, at respective.

$\hat{a}_1$	$\hat{a}_2$	$\hat{b}_1$	$\hat{b}_2$	$\hat{e}_1$	$\hat{e}_2$	$\hat{d}_1$	$\hat{d}_2$
35	32	39	28	34	33	28	39

Table 2.1: Table of quantities of $\hat{a}_i, \hat{b}_j, \hat{e}_k$ and $\hat{d}_l$

$\tilde{c}_{1111}$	$\tilde{c}_{1112}$	$\tilde{c}_{1121}$	$\tilde{c}_{1122}$	$\tilde{c}_{1211}$	$\tilde{c}_{1212}$	$\tilde{c}_{1221}$	$\tilde{c}_{1222}$
$(6, 11, 17)$ $\Re = 11.25$	$(5, 17, 18)$ $\Re = 14.25$	$(5, 10, 10)$ $\Re = 8.75$	$(4, 11, 12)$ $\Re = 9.5$	$(7, 10, 17)$ $\Re = 11$	$(6, 11, 16)$ $\Re = 11$	$(6, 10, 17)$ $\Re = 10.75$	$(14, 18, 19)$ $\Re = 17.25$
$\tilde{c}_{2111}$	$\tilde{c}_{2112}$	$\tilde{c}_{2121}$	$\tilde{c}_{2122}$	$\tilde{c}_{2211}$	$\tilde{c}_{2212}$	$\tilde{c}_{2221}$	$\tilde{c}_{2222}$
$(7, 8, 17)$ $\Re = 10$	$(2, 2, 11)$ $\Re = 4.25$	$(3, 3, 18)$ $\Re = 6.75$	$(2, 2, 18)$ $\Re = 6$	$(10, 13, 14)$ $\Re = 10.5$	$(2, 10, 14)$ $\Re = 9$	$(7, 13, 15)$ $\Re = 12$	$(3, 10, 19)$ $\Re = 13$

Table 2.2: Matrix of variable costs

$\tilde{f}_{1111}$	$\tilde{f}_{1112}$	$\tilde{f}_{1121}$	$\tilde{f}_{1122}$	$\tilde{f}_{1211}$	$\tilde{f}_{1212}$	$\tilde{f}_{1221}$	$\tilde{f}_{1222}$
(2,4,10) $\Re = 27$	(4,6,10) $\Re = 22.25$	(1,3,8) $\Re = 18.5$	(4,9,10) $\Re = 16.5$	(2,5,6) $\Re = 16.75$	(1,8,9) $\Re = 18.25$	(4,9,10) $\Re = 20.25$	(1,8,10) $\Re = 25$
$\tilde{f}_{2111}$	$\tilde{f}_{2112}$	$\tilde{f}_{2121}$	$\tilde{f}_{2122}$	$\tilde{f}_{2211}$	$\tilde{f}_{2212}$	$\tilde{f}_{2221}$	$\tilde{f}_{2222}$
(1,5,8) $\Re = 10.5$	(2,3,9) $\Re = 11.75$	(1,6,9) $\Re = 19.25$	(1,7,10) $\Re = 17$	(1,6,8) $\Re = 29$	(7,8,10) $\Re = 25.75$	(1,2,5) $\Re = 23.25$	(3,4,8) $\Re = 16.5$

Table 2.3: Matrix of fixed costs

We generate a chromosome denoted by ch_1 which is permutation of $m+n+p+q = 8$ digits.

4	7	8	6	1	5	2	3

Table 2.4: Representation of chromosome ch_1.

Now, we apply our suggested decoding procedure to determine the solution x_1 corresponding to chromosome ch_1 as shown in table 2.5. The solution x_1 associated to the chromosome Ch_1 is given as follows:

$x_1 = \{x_{2112} = 32, x_{1121} = 7, x_{1211} = 2, x_{1221} = 19, x_{1222} = 2\}$.

Iter	$V(i+j+k+l)$	$\hat{a}_1$	$\hat{a}_2$	$\hat{b}_1$	$\hat{b}_2$	$\hat{e}_1$	$\hat{e}_2$	$\hat{d}_1$	$\hat{d}_2$	(i,j,k,l)	x_{ijkl}
1	[4 7 $\underline{8}$ 6 1 5 2 3]	35	32	39	28	34	33	28	39	$(2,1,1,2)$	32
2	[4 0 $\underline{8}$ 6 1 5 2 3]	35	0	7	28	2	33	28	7	$(1,1,2,1)$	7
3	[4 0 0 $\underline{6}$ 1 5 2 3]	28	0	0	28	2	26	21	7	$(1,2,1,1)$	2
4	[4 0 0 $\underline{6}$ 0 5 2 3]	26	0	0	26	0	26	19	7	$(1,2,2,1)$	19
5	[4 0 0 $\underline{6}$ 0 5 0 3]	7	0	0	7	0	7	0	7	$(1,2,2,2)$	7

Table 2.5: Trace table of the decoding procedure for FFCTP4.

The value of the objective function associated with x_1 is:
$\tilde{Z} = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q}(\tilde{c}_{ijkl}\, x_{ijkl} \oplus \tilde{f}_{ijkl}\, y_{ijkl}) =_{\Re} (265, 408, 860)$. We have $\Re(\tilde{Z}) = 485.25$.
In all the following, we denote:

N_p	the size of population
N_g	the maximum number of generations
p_c	crossover rate
p_m	mutation rate
$nmax$	the number of neighborhood solutions
$Iter_{max}$	the maximum number of iterations in SA
T_0	initial temperature

GA to FFCTP4

Input:

Fix N_p, N_g, p_c and p_m.

Enter the problem's dimension: m, n, p, q, values of variable costs $\tilde{c}_{ijkl}$, values of fixed costs $\tilde{f}_{ijkl}$, the values $\hat{a}, \hat{b}, \hat{e}$ and $\hat{d}$.

Initialization

Set $t := 0$.

Generate an initial random population P_0 of N_p individuals and evaluate them.

$cross = round(N_P \times p_c)$.

$$N_{cr} = \begin{cases} \dfrac{cross}{2}, & \text{if } cross \text{ is even,} \\ \dfrac{cross-1}{2}, & \text{if } cross \text{ is odd,} \end{cases}.$$

$N_{mu} = round(N_p \times p_m)$.

Crossover

While $t < N_g$ **do**

for $k \leftarrow 1$ to N_{cr} **do**

Choose two chromosomes x_{i_1} and x_{i_2} via the roulette wheel selection method.

Apply the two point crossover on x_{i_1} and x_{i_2}.

end for

Mutation

for $k \leftarrow 1$ to N_{mu} **do**

select a chromosome x_{j_1} via the roulette wheel selection method.

Generate a random number r in the range $[0\,1]$.

If $r \leq p_m$ **then**.

Apply swap mutation.

End if

End for

Determine the fitness value of new chromosomes.

Retain N_p best chromosomes from the population of parents and offspring for the next generation.

$t := t + 1$.

End while

2.5.2 Simulated Annealing Algorithm to solve FFCTP4

The steps of SA algorithm to FFCTP4 are given as follows:

Input:

Fix T_0, n_{max}, $\alpha = 0.92$ and $Iter_{max}$.

Enter the dimensions of problem: m, n, p, q, matrix of variable costs $\tilde{c}_{ijkl}$, matrix of fixed costs $\tilde{f}_{ijkl}$, the quantities of $\hat{a}, \hat{b}, \hat{e}$ and $\hat{d}$.

Initialization

$t := 0$.

Generate an initial solution x at random and evaluate it.

Set $x' := x$

While $(t < Iter_{max})$ **do**.

$n := 1$.

While $(n \leq n_{max})$ **do**.

Generate a new solution x'' in the neighborhood of the solution x' . (x'' can be obtained by applying swap mutation on solution x').

Determine the fitness value of x'', denoted by $F(x'')$;

If $((F(x'') - F(x')) \leq 0)$ **then**

$x' := x''$.

$n := n + 1$.

Else

Generate a random number r in the range $[0\ 1]$.

If $r \leq exp(\dfrac{-(F(x'') - F(x'))}{T})$**then**

$x' := x''$.

$n := n + 1$.

End if.

End if.

End while

$T := \alpha \times T$.

$t := t + 1$.

End while

2.5.3 Particle swarm optimization algorithm to solve FFCTP4

The proposed PSO algorithm to solve a fuzzy fixed charge transportation problem is as follows:

PSO algorithm to FFCTP4

Input:

Fix N_p, N_g.

Enter the dimension of the problem: m, n, p, q, the values of variable costs $\tilde{c}_{ijkl}$, the values of fixed costs $\tilde{f}_{ijkl}$, the values $\hat{a}, \hat{b}, \hat{e}$ and $\hat{d}$.

Fix the acceleration coefficient $c_1 = c_2 = 1.5$ and the interia whieght $w = 0.7$.

Initialization.

$t := 0$.

Generate a random population of N_p particles $x_1^t, ..., x_{N_p}^t$.

Generate particle velocity $v_1^t, v_2^t, ..., v_{N_p}^t$.

For $k := 1$ to N_p **do**

Determine the fitness value of particle $F(x_k^t)$.

$Pbest_k := x_k^0$.

End for

Determine the global best particle as follows:

$Gbest = \arg min\{F(x_1^t), F(x_2^t), ..., F(x_{N_p}^t)\}$.

While $(t < N_g)$ **do**

For $(k := 1$ to $N_p)$ **do**

Update the particle velocity based on the following equation.

$v_k^{t+1} = w * v_k^t + r_1 c_1(Pbest_k - x_k^t) + r_2 * c_2(Gbest - x_k^t)$.

r_1 and r_2 are two random numbers in the range $[0\ 1]$. Where $r_1 + r_2 = 1$.

Update the particle position as follows.

$x_k^{t+1} = x_k^t + v_k^{t+1}$.

Determine the fitness value of each particle $F(x_k^{t+1})$.

Update personal best particle $Pbest(k)$.

End for

Update global best particle.

$t := t + 1$.

End while.

2.5.4 Hybrid particle swarm optimization with genetic algorithm to solve FFCTP4

Several researchers have mentioned the capabilities of GA in exploitation and PSO in exploration. In this subsection, we introduce a new hybrid PSO-GA that integrates the mutation operator of the genetic algorithm in PSO to obtain a better solution. It comprises two phases:

Phase 1:

- Generate a swarm of particles.
- Update the position of particles based on velocities.

Phase 2:

- For each particle, choose a random number within the range $[0\ 1]$.
- If the chosen number is less than the mutation rate, then apply the mutation operator.

PSO-GA algorithm to FFCTP4

Input:

Fix N_p, N_g, p_m,

Enter the dimension of problem: m, n, p, q, the values of variable costs $\tilde{c}_{ijkl}$, the values of fixed costs $\tilde{f}_{ijkl}$, the values $\hat{a}, \hat{b}, \hat{e}$ and $\hat{d}$.

Fix $c_1 = c_2 = 1.5$ and the inertia whieght $w = 0.7$.

Initialization.

$t := 0$

Generate a random population of N_p particles $x_1^t, ..., x_{N_p}^t$.

Generate particle velocity $v_1^t, v_2(t), ..., v_{N_p}^t$.

For ($k := 1$ to N_p) **do**

Determine the fitness value of particle $F(x_k^t)$.

$Pbest_k := x_k^0$.

End for

Determine the global best particle based on the following equation.

$Gbest = \arg min\{F(x_1^t), F(x_2^t), ..., F(x_{N_p}^t)\}$.

While ($t < N_g$) **do**

For ($k := 1$ to N_p) **do**

Update the particle velocity.

Update the particle position.

Generate a random number r in the range $[0\ 1]$.

If ($r \leq p_m$) **then**

Apply swap mutation on particle x_k^{t+1}.

End if

Determine the fitness value of each particle $F(x_k^{t+1})$.

Update personal best particle $Pbest(k)$.

End for

Update global best particle.

$t := t + 1$.

End while

2.6 Approximation method to solve FFCTP4

In this section, we introduce an approximation method to resolve FFCTP4. This method adapts Balinski's approximation [6] to a fuzzy context with four indices and involves converting the FFCTP4 into a relaxed transportation problem by ignoring the integer constraints. We then solve the resulting problem using an algorithm that consists of two phases [52].

2.6.1 Relaxed transportation problem

The mathematical formulation of the relaxed transportation problem is given as follows:

$$Minimize\ \tilde{Z} = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} \tilde{c}'_{ijkl}\, x_{ijkl},$$

Subject to constraints

$$\begin{aligned} &\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} = \hat{a}_i, \text{ for all } i = 1, ..., m,\\ &\sum_{i=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} = \hat{b}_j, \text{ for all } j = 1, ..., n,\\ &\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{q} x_{ijkl} = \hat{e}_k, \text{ for all } k = 1, ..., p,\\ &\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} x_{ijkl} = \hat{d}_l, \text{ for all } l = 1, ..., q,\\ &x_{ijkl} \geq 0, \text{ for all } i = 1:m; j = 1:n; k = 1:p, l = 1:q, \end{aligned} \tag{RTP}$$

with $\tilde{c}'_{ijkl} =_{\Re} \tilde{c}_{ijkl} \oplus \dfrac{\tilde{f}_{ijkl}}{min(\hat{a}_i, \hat{b}_j, \hat{e}_k, \hat{d}_l)}$.

To solve the relaxed transportation problem RTP, we apply the following algorithm which is consisting of two phases.

- The first phase is used to generate an initial basic feasible solution for the relaxed transportation problem.

- The second phase is used to test the optimality of a solution or to improve it until an optimum one is obtained.

2.6.2 Resolution of relaxed transportation problem

The resolution algorithm is as follows:

Phase 1

In this phase, we generate an initial basic feasible solution for the relaxed transportation problem RTP based on an adaptation of the least-cost cell method under fuzziness with four indexes.

While $\hat{E}$ is not empty do

1. Take $x_{ijkl} = 0$, $\beta_{ijkl} = 0$, $\forall(i, j, k, l)$, and $\hat{E}_b = \varnothing$.
   $\hat{E} = \{(i, j, k, l), \beta_{ijkl} = 0\}$, β_{ijkl} is a boolean variable returns 1 if x_{ijkl} has been determined and 0 in the opposite case.

2. Choose (i^*, j^*, k^*, l^*), such that $\tilde{c}'_{i^*,j^*,k^*,l^*} = \min \tilde{c}'_{ijkl}$.

3. Take $x_{i^*j^*k^*l^*} = \min(\hat{a}_{i^*}, \hat{b}_{j^*}, \hat{e}_{k^*}, \hat{d}_{l^*})$, $\beta_{i^*j^*k^*l^*} = 1$, and add (i^*, j^*, k^*, l^*) to $\hat{E}_b$.

4. Update $\hat{a}_{i^*}, \hat{b}_{j^*}, \hat{e}_{k^*}$, and $\hat{d}_{l^*}$ as follows:

   (a) $\hat{a}_{i^*} = \hat{a}_{i^*} - x_{i^*j^*k^*l^*}$
   If $\hat{a}_{i^*} = 0$ then set $x_{i^*jkl} = 0$ and $\beta_{i^*jkl} = 1$, $\forall(j, k, l) \neq (j^*, k^*, l^*)$.

   (b) $\hat{b}_{j^*} = \hat{b}_{j^*} - x_{i^*j^*k^*l^*}$
   If $\hat{b}_{j^*} = 0$ then set $x_{ij^*kl} = 0$ and $\beta_{ij^*kl} = 1$, $\forall(i, k, l) \neq (i^*, k^*, l^*)$

   (c) $\hat{e}_{k^*} = \hat{e}_{k^*} - x_{i^*j^*k^*l^*}$
   If $\hat{e}_{k^*} = 0$ then set $x_{ijk^*l} = 0$ and $\beta_{ijk^*l} = 1$, $\forall(i, j, l) \neq (i^*, j^*, l^*)$.

   (d) $\hat{d}_{l^*} = d_{l^*} - x_{i^*j^*k^*l^*}$
   If $\hat{d}_{l^*} = 0$ then set $x_{ijkl^*} = 0$ and $\beta_{ijkl^*} = 1$, $\forall(i, j, k) \neq (i^*, \bar{j}, k^*)$.

End while.

The above phase provides an initial basic feasible solution that can be degenerate or non degenerate. To handle the degeneracy case, we can use the procedure introduced by Zitouni et al. in [19, 52].

Phase 2

In this phase, we extend that phase 2 of AL_{PT4C} [51, 52] under uncertainty with four indices to find an optimal solution to the relaxed transportation problem.

1. Set $\hat{r} = 0$.

2. Determine the set of interesting quadruplets (i, j, k, l), denoted as $\hat{I}^{(\hat{r})}$.

3. $\forall\,(i, j, k, l) \in \hat{I}^{(\hat{r})}$, solve the following linear system.

$$\tilde{\hat{u}}_i^{(\hat{r})} \oplus \tilde{\hat{v}}_j^{(\hat{r})} \oplus \tilde{\hat{w}}_k^{(\hat{r})} \oplus \tilde{\hat{t}}_l^{(\hat{r})} =_{\Re} \tilde{c}'_{ijkl}.$$

   Where $i = 1 : m, j = 1 : n, k = 1 : p, l = 1 : q$.

4. Determine $\tilde{\hat{\delta}}_{ijkl}^{(\hat{r})}, \forall (i, j, k, l) \notin \hat{I}^{(\hat{r})}$.

$$\tilde{\hat{\delta}}_{ijkl}^{(\hat{r})} = \tilde{c}'_{ijkl} \ominus (\tilde{\hat{u}}_i^{(\hat{r})} \oplus \hat{v}_j^{(r)} \oplus \tilde{\hat{w}}_k^{(\hat{r})} \oplus \tilde{\hat{t}}_l^{(r)})$$

5. If $\Re(\tilde{\hat{\delta}}_{ijkl}^{(\hat{r})}) \geq 0, \forall (i, j, k, l) \notin \hat{I}^{(\hat{r})}$ then the solution is optimal.

(a) Else choose the quadruplet $(\hat{i}_0, \hat{j}_0, \hat{k}_0, \hat{l}_0)$ such that:

$$\tilde{\hat{\delta}}_{\hat{i}_0\hat{j}_0\hat{k}_0\hat{l}_0}^{(\hat{r})} = \min\{\tilde{\hat{\delta}}_{ijkl}^{(\hat{r})} : \Re(\tilde{\hat{\delta}}_{ijkl}^{(\hat{r})}) < 0\}.$$

(b) Solve the following system in order to construct a cycle $\mu^{(\hat{r})}$.

$$\sum \hat{\lambda}_{ijkl}^{(\hat{r})} \hat{P}_{ijkl} = -\hat{P}_{\hat{i}_0\hat{j}_0\hat{k}_0\hat{l}_0}, \forall (i, j, k, l) \in \hat{I}^{(\hat{r})}$$

(c) Determine $\hat{\theta}$

$$\hat{\theta} = \min\{\frac{x_{ijkl}^{(r)}}{-\hat{\lambda}_{ijkl}^{(\hat{r})}} = \hat{\theta}_{\hat{i}_s\hat{j}_s\hat{k}_s\hat{l}_s}^{(\hat{r})}\} \text{ with } \hat{\lambda}_{ijkl}^{(\hat{r})} < 0$$

(d) Determine the new set of basic solution
$x^{(\hat{r}+1)} = \{x_{ijkl}^{(\hat{r})} + \hat{\lambda}_{ijkl}\hat{\theta} : (i, j, k, l) \in \mu^{(\hat{r})}\} \cup \{x_{ijkl}^{(\hat{r})} : (i, j, k, l) \notin \mu^{(\hat{r})}\}$.

(e) Repeat from 2) to 5).

2.7 Experiment results and comparative study

To assess the performance of the suggested algorithms $GA, SA, PSO, PSO-GA$, numerous numerical problems of FFCTP4 of different sizes are solved. The results obtained are given in table 2.6.

On the other hand, 10 instances are generated for each problem of sizes (from 8×16 until 80×1600000) and 4 instances are generated for each problem of sizes (from 98×360000 until 120×810000). Table 2.7 shows the average computation time (ACT) for different problem sizes.

Notation:

- $M \times N$: the size of the problem, where $M = m+n+p+q$ and $N = mnpq$.
- OF: value of objective function obtained using $GA, SA, PSO, PSO-GA$ and our suggested approximation approach.
- $\Re(\text{OF})$: the ranking function of the value of objective function which is obtained $GA, SA, PSO, PSO-GA$ and our suggested approximation approach.
- $ACT_{GA}, ACT_{SA}, ACT_{PSO}, ACT_{PSO-GA}, ACT_{appro}$: Average computation time in seconds of the problems tested using $GA, SA, PSO, PSO-GA$ and our proposed approximation method.

Size	Metaheuristics								Approximation method	
	GA		SA		PSO		PSO-GA			
	OF	$\Re$(OF)	OF	$\Re$(OF)	OF	$\Re$(OF)	OF	$\Re$(OF)	OF	$\Re$(OF)
8×16	(404, 524, 877)	582.25	(404, 524, 877)	582.25	(404, 524, 877)	582.25	(404, 524, 877)	582.25	(396, 512, 847)	566.75
12×81	(293, 615, 1390)	728.25	(291, 604, 1369)	717	(322, 634, 1332)	730.5	(291, 604, 1369)	717	(302 563, 1189)	654.25
14×144	(195 411, 777)	448.5	(199, 398, 804)	449.75	(203, 441, 762)	461.75	(172, 452, 641)	429.25	(164 344 702)	388.5
16×256	(295, 531, 1086)	610.75	(295, 531, 1086)	610.75	(295, 531, 1086)	610.75	(198, 461, 951)	517.75	(220, 430, 981)	515.25
18×400	(234, 468, 1146)	579	(173, 539, 1137)	597	(234, 468, 1146)	579	(234, 468, 1146)	579	(197, 380, 1067)	506
26×1764	(366, 892, 1687)	959.25	(301, 813, 1755)	920.5	(412, 837, 1715)	950.25	(408, 865, 1729)	866.75	(288, 656, 1315)	728.75
34×5184	(524, 998 1933)	1113.25	(532, 1009, 1840)	1097.5	(570, 983, 1958)	1123.5	(501, 933, 1865)	1058	(360 694, 1385)	783.25
44×14400	(633, 1155, 2364)	1326.75	(617, 1133, 2173)	1264	(581, 1141, 2336)	1299.75	(531, 986, 2032)	1133.75	(458, 867, 1633)	956.25
64×65536	(920, 1748, 3271)	1921.75	(930, 1596, 3142)	1816	(949, 1685, 3147)	1866.5	(698, 1290.5, 2641.5)	1480.125	(624 1076, 2045)	1205.25
68×83521	(2373, 4723, 8515)	5083.5	(2147, 4429, 9450)	5113.75	(2167, 4683, 8723)	5064	(2225, 4538, 8715)	5004	(1477, 3151, 6373)	3538
74×116964	(2093, 4270, 8622)	4813.75	(1946, 4297, 8495)	4758.75	(2017, 4379, 7900)	4668.75	(1817, 3786, 7975)	4341	(1446, 3017, 5930)	3352.5
80×160000	(999, 1692, 3577)	1990	(939, 3551, 3551)	1940	(1006, 1728, 3502)	1991	(1008, 1770, 3181)	1932.5	(678, 1012, 2078)	1195
98×360000	(1071, 1852, 3549)	2081	(1055, 1788, 3857)	2122	(1046, 1772, 3587)	2044.25	(954, 1610, 3559)	1933.25	(689, 1022, 2266)	1249.75
120×810000	(1363, 2306, 4768)	2685.75	(1409 2485, 4662)	2760.25	(1428, 2294, 4611)	2656.75	(1352, 2297, 4302)	2562	(974, 1320, 2597)	1552.75

Table 2.6: GA, SA, PSO, PSO-GA and approximation method in solving FFCTP4.

We have also determined the efficiency of each metaheuristic algorithm using the following formula: Efficiency $= \left(1 - \dfrac{solution - best\ solution}{best\ solution}\right) \times 100$, where $best\ solution$ is the value of objective function obtained by our proposed approach. Our proposed approach serves as the benchmark for comparison. The results obtained are illustrated in figure 2.6.

Size	No	ACT_{GA}	ACT_{SA}	ACT_{PSO}	ACT_{PSO-GA}	ACT_{appro}
8×16	10	0.0459	0.0375	0.0241	0.0223	0.0100
12×81	10	0.1064	0.0830	0.0482	0.0479	0.0240
14×144	10	0.1241	0.1018	0.0696	0.0693	0.0442
16×256	10	0.2191	0.1722	0.1241	0.1197	0.0847
18×400	10	0.3501	0.2737	0.1741	0.1731	0.1439
26×1764	10	1.9881	1.7729	1.1243	1.0009	1.4381
34×5184	10	10.1156	6.2440	4.0779	3.9084	7.8977
44×14400	10	27.4769	21.4156	13.7045	13.5592	45.8607
64×65536	10	224.9079	158.7148	111.5810	90.1386	358.4540
68×83521	10	304.3514	184.8799	144.4186	119.8770	408.9620
74×116964	10	402.0149	280.0092	223.1880	207.2398	726.9327
80×160000	10	597.1493	415.3154	306.9305	289.8708	1.0083e+03
98×360000	4	1.5841e+03	1.2399e+03	1.1958e+03	822.6544	3.5981e+03
120×810000	4	5.3419e+03	3.8557e+03	2.7217e+03	2.4990e+03	8.2459e+03

Table 2.7: ACT of GA, SA, PSO, PSO-GA and our approximation method for different sizes of FFCTP4.

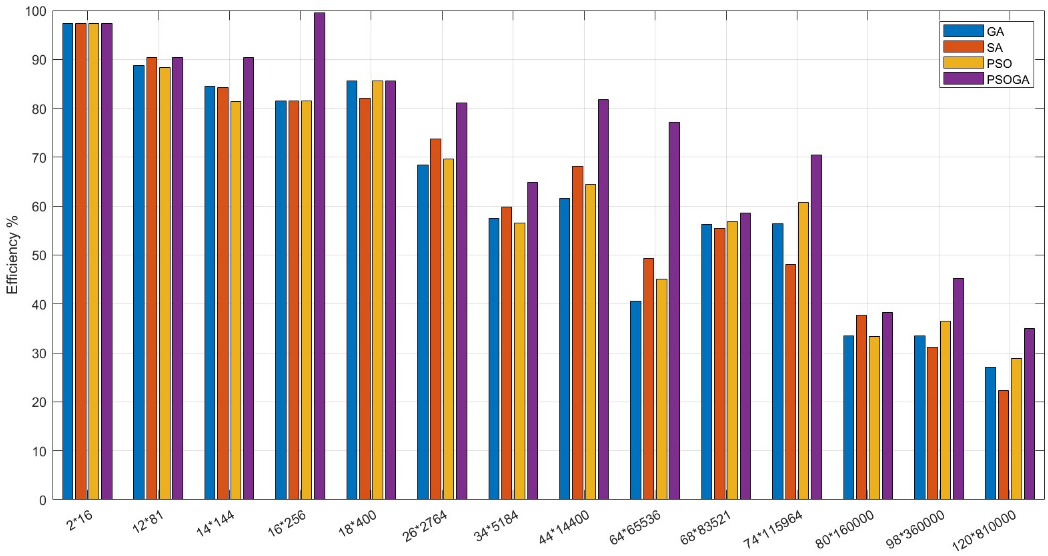

Figure 2.6: Efficiency of metaheuristic algorithms.

Comments

- Our approximation approach outperforms the metaheuristics that were proposed in terms of solution quality. Interestingly, it gives an upper bound for the primary issue, FFCTP4.
- The solutions obtained by GA, SA, PSO or PSO-GA are close to the approximation method's solution.
- For small instances, we remark that the average computation time in the proposed approximation approach is sensibly low compared to the average com-

putation time in metaheuristics.

- For medium- and large-sized instances, we note the superiority of metaheuristic algorithms in solving FFCTP4 in terms of ACT.

- (PSO-GA) has shown superior performance in comparison to GA, SA and PSO with respect to ACT and the quality of the solution. On the other hand, GA has shown the worst performance.

- Through rigorous testing, it has been demonstrated that PSO-GA is particularly effective for solving FFCTP4, producing impressive results in a relatively short amount of time as shown in figure 2.7.

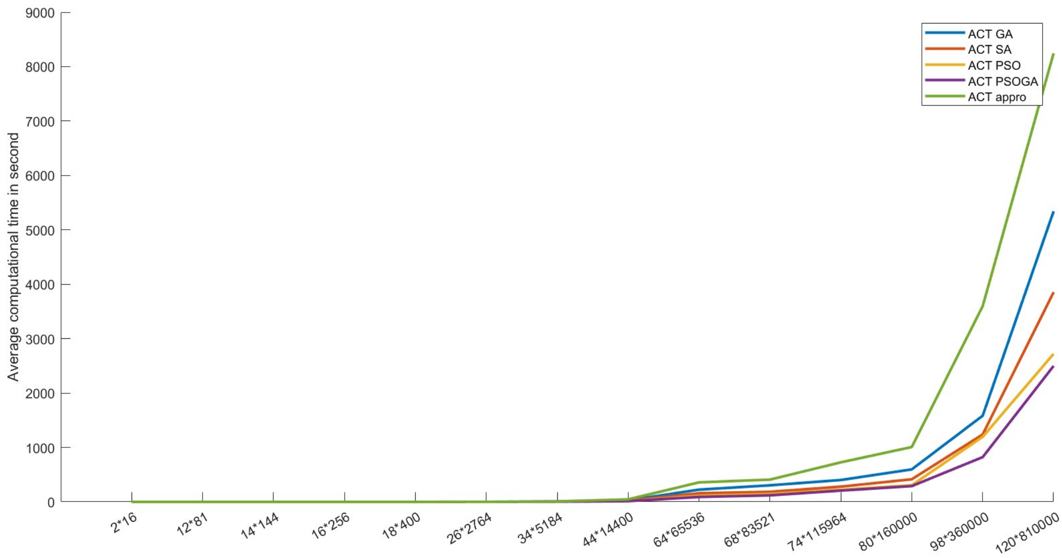

Figure 2.7: Average computational time of the proposed algorithms for different problem sizes.

2.8 Conclusion

In this chapter, we have solved a fuzzy four-index fixed charge transportation problem under uncertainty via four metaheuristics along with an approximation method. We opted for GA, SA, PSO and PSO-GA, due to their ability in treating various combinatorial optimization problems. The obtained results show that our proposed metaheuristic algorithms provide good solutions in an acceptable time. Besides, the hybrid particle swarm optimization with genetic algorithm has shown its success and its advantage in terms of efficiency, robustness and speed (execution time) compared to other metaheuristics.

Chapter 3

Fuzzy Bi-Objective Four-index Fixed-Charge Transportation Problem

3.1 Introduction

Transportation issues play a vital role in logistic and supply chain management for improving the quality of services and sustainable development of companies. This problem initiated by Hitchcock has known several extensions including multi-index transportation problems, fixed charge transportation problems, and multi-objective transportation problems. From a practical perspective, the transportation problem that consists of minimizing overall cost and delivery time cannot be considered as two independent problems, if one wishes to obtain a solution that simultaneously minimizes cost and time. Moreover, we face many practical transportation problems that deal with two objectives, minimizing cost and time, which is called a bi-objective transportation problem.

Currently, researchers focus on solving the fixed charge transportation problem with two or more objectives in a fuzzy environment in which the parameters of the problem are completely or partially represented by fuzzy numbers.

3.2 Literature Review

Several authors have dealt with multi-objective fixed charge transportation problems (MOFCTP) in various environments. In 2001, Ahuja and Arora [4] suggested an exact algorithm to resolve the bi-criterion fixed charge solid transportation problem. In

2010, Kumar et al. [28] treated the bi-objective fixed charge solid transportation issue, where all problem's parameters are considered to be trapezoidal fuzzy numbers. Next, Khurana and Adlakha [26] introduced an novel technique for solving the same issue addressed in [4]. Their suggested technique performed well than Ahuja and Arora's algorithm [4].

Furthermore, Singh et al. [45] presented an approach to find the set of non-dominated solutions for fuzzy BOFCTP. Their approach is based on an adaptation in a fuzzy context of Voguel approximation and MODI methods. Three strategies have been proposed in [12] to get the best solution for the intuitionist fixed charge transportation issue with several objectives. Moreover, Haque et al. [18] tackled a nonlinear fixed charge solid transportation subject to budget constraints where the problem's parameters are closed interval. After that, Biswas and Pal [8] performed a comparative study between three metaheuristics to identify the most suitable algorithm to solve the capacitate MOFCTP.

Currently, Mardanya and Roy [31] developed a new algorithm for solving a multi-index solid transportation problem with several objectives under fuzziness. Ghosh et al. [13] addressed a type-2 zigzag uncertain fixed charge solid transportation problem with multiple objectives. It is to say that all findings detailed in this chapter have been published in the journal Kybernetika [16].

3.3 Problem position

This problem can be interpreted in the same way as section **2.4.1** by simply adding the following point.
Let $\tilde{t}_{ijkl}$ be the fuzzy time associated to the transportation of the quantity $\tilde{x}_{ijkl}$.

3.3.1 Problem formulation

The mathematical formulation of the fuzzy fully bi-objective four-index fixed-charge transportation problem (FBOFCTP4) is as follows:

$$\text{Minimize } \tilde{Z} =_{\Re} \Big\{ \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q}(\tilde{c}_{ijkl} \otimes \tilde{x}_{ijkl}) \oplus (\tilde{f}_{ijkl} \otimes \tilde{y}_{ijkl}), \max[\tilde{t}_{ijkl} : \tilde{x}_{ijkl} >_{\Re} \tilde{0}] \Big\}$$

Subject to constraints

$$\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} \tilde{x}_{ijkl} =_{\Re} \tilde{\tilde{a}}_i, \text{ for all } i = 1, ..., m, \tag{3.1}$$

$$\sum_{i=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} \tilde{x}_{ijkl} =_{\Re} \tilde{\tilde{b}}_j, \text{ for all } j = 1, ..., n, \tag{3.2}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{q} \tilde{x}_{ijkl} =_{\Re} \tilde{\tilde{e}}_k, \text{ for all } k = 1, ..., p, \tag{3.3}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} \tilde{x}_{ijkl} =_{\Re} \tilde{\tilde{d}}_l, \text{ for all } l = 1, ..., q, \tag{3.4}$$

$$\tilde{y}_{ijkl} =_{\Re} \begin{cases} \tilde{1}, \text{if } \tilde{x}_{ijkl} >_{\Re} \tilde{0}, \\ \tilde{0}, \text{if } \tilde{x}_{ijkl} =_{\Re} \tilde{0}, \end{cases} \tag{3.5}$$

$$\tilde{x}_{ijkl} \geq_{\Re} 0, \text{ for all } i = 1 : m; j = 1 : n; k = 1 : p, l = 1 : q. \tag{3.6}$$

We have $\tilde{\tilde{a}}_i > 0, \tilde{\tilde{b}}_j >_0, \tilde{\tilde{e}}_k >, \tilde{\tilde{d}}_l > 0, \tilde{c}_{ijkl} \geq_{\Re} 0$, $\tilde{f}_{ijkl} \geq_{\Re} 0$ and $\tilde{t}_{ijkl} \geq_{\Re} 0$, $\forall(i,j,k,l)$. Let $\hat{E} = \{(i,j,k,l) \mid i = 1,\dots,m;\ j = 1,\dots,n;\ k = 1,\dots,p;\ l = 1,\dots,q\}$. For each quadruplets$(i,j,k,l) \in \hat{E}$, we associate a vector $\hat{P}_{ijkl} \in \mathbb{R}^M$, where $M = m + n + p + q$. The vector $\hat{P}_{ijkl}$ contains four non-zero components, positioned on the rows corresponding to i, $m + j$, $m + n + k$, and $m + n + p + l$, all sharing a common value of one. We denote the collection of these vectors as a matrix $\hat{A}$. Notably, the matrix

$\hat{A}$ has a rank of $m+n+p+q-3$.

The purpose of this issue is to find $\tilde{x}_{ijkl}$ so that the overall cost of transport and the transportation time needs to be minimized simultaneously.

The following theorem is a generalization of the feasible condition introduced in [52].

Theorem 3.1 (Feasibility condition [52]). *The fuzzy bi-objective fixed-charge transportation problem with four indices has a feasible solution if and only if*

$$\sum_{i=1}^{m} \tilde{\hat{a}}_i = \sum_{j=1}^{n} \tilde{\hat{b}}_j = \sum_{k=1}^{p} \tilde{\hat{e}}_k = \sum_{l=1}^{q} \tilde{\hat{d}}_l.$$

3.4 Resolution of FBOFCTP4

In order to solve the above mentioned problem (FBOFCTP4), we divide it into two sub-problems, denoted by (P_1), (P_2). Then, For solving (P_1), we take into consideration the transportation problem that consists of direct cost only, denoted by (RP_1).

(P_1) : Minimize $\tilde{Z} =_{\Re} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q}(\tilde{c}_{ijkl} \otimes \tilde{x}_{ijkl}) \oplus (\tilde{f}_{ijkl} \otimes \tilde{y}_{ijkl})$, s. t. c. (3.1) – (3.6).

(P_2) : Minimize $\tilde{T} =_{\Re} \max[\tilde{t}_{ijkl} : \tilde{x}_{ijkl} > 0]$, s. t. c. (3.1) – (3.6).

(RP_1) : Minimize $\tilde{G} =_{\Re} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} \tilde{c}_{ijkl} \otimes \tilde{x}_{ijkl}$, s. t. c. (3.1) – (3.6).

Now, we describe our suggested method, denoted as $\text{Al}_{FBOFCTP4}$ to generate the set of Pareto optimal solutions for FFBOFCTP4.

3.4.1 Description of the proposed approach

This approach is composed of three primary steps, given as follows:

1. The first step is used to search for an optimal soltion for the solution to the problem (RP_1) with the use of of Algorithm 1 given below.

2. The solution provided at the end of step 1 is considered an initial basic feasible solution for the problem (P_1). Therefore, the second step is used to search for an optimum solution for (P_1) with the use of Algorithm 2 given below.

3. The third step is used to solve the problem (P_2) and identify the set of non-dominated solutions with the use of Algorithm 3 given below.

Algorithm 1

This algorithm is used to determine an optimal solution for (RP_1). It consists of two phases:

Phase 1

In this phase, we aim to find an initial basic feasible solution for (RP_1) via an adaptation of the least cost cell method under uncertainty.

Initialization

Let $\hat{E} = \{(i,j,k,l)$ such that $\beta_{ijkl} = 0\}$ where β_{ijkl} is a boolean variable returns 1 if $\tilde{x}_{ijkl}$ has been determined and 0 otherwise.

Let $\hat{I}_{(0)} = \{(i,j,k,l)$ such that $\tilde{x}_{ijkl}$ is a basic variable$\}$. At the beginning, take $\hat{I}^0 = \varnothing$.

While($\hat{E}$ is not empty) do

1. Determine $(i^*, j^*, k^*, l^*) \in \hat{E}$, where $\tilde{c}_{i^*,j^*,k^*,l^*} =_{\Re} \min \tilde{c}_{ijkl}$.

2. Take $\tilde{x}_{i^*j^*k^*l^*} =_{\Re} \min(\tilde{\hat{a}}_i^*, \tilde{\hat{b}}_{j^*}, \tilde{\hat{e}}_{k^*}, \tilde{\hat{d}}_{l^*})$ and $\beta_{i^*j^*k^*l^*} = 1$.

3. Take $\hat{I}^{(0)} = \hat{I}^{(0)} \cup \{(i^*j^*k^*l^*)\}$.

4. Adjust $\tilde{\hat{a}}_{i^*}, \tilde{\hat{b}}_{j^*}, \tilde{\hat{e}}_{k^*}$, and $\hat{d}_{l^*}$ as follows:

   (a) $\tilde{\hat{a}}_{i^*} =_{\Re} \tilde{\hat{a}}_{i^*} \ominus \tilde{x}_{i^*j^*k^*l^*}$
   If $\tilde{\hat{a}}_{i^*} =_{\Re} \tilde{0}$ then take $\tilde{x}_{i^*jkl} =_{\Re} \tilde{0}$ and $\beta_{i^*jkl} = 1, \forall (j,k,l) \neq (j^*, k^*, l^*)$.

   (b) $\tilde{\hat{b}}_{j^*} =_{\Re} \tilde{\hat{b}}_{j^*} \ominus \tilde{x}_{i^*j^*k^*l^*}$
   If $\tilde{\hat{b}}_{j^*} =_{\Re} \tilde{0}$ then take $\tilde{x}_{i\bar{j}kl} =_{\Re} \tilde{0}$ and $\beta_{i\bar{j}kl} = 1, \forall (i,k,l) \neq (\bar{i}, \bar{k}, \bar{l})$.

(c) $\tilde{\tilde{e}}_{k^*} = \tilde{\tilde{e}}_{k^*} \ominus \tilde{x}_{i^*j^*k^*l^*}$

If $\tilde{\tilde{e}}_{k^*} =_{\Re} \tilde{0}$ then take $\tilde{x}_{ijk^*l} =_{\Re} \tilde{0}$ and $\beta_{ijk^*l} = 1, \forall (i,j,l) \neq (i^*, j^*, l^*)$.

(d) $\tilde{\tilde{d}}_{l^*} =_{\Re} \tilde{\tilde{d}}_{l^*} \ominus \tilde{x}_{i^*j^*k^*l^*}$

If $\tilde{\tilde{d}}_{l^*} =_{\Re} \tilde{0}$ then take $\tilde{x}_{ijkl^*} =_{\Re} \tilde{0}$ and $\beta_{ijkl^*} = 1, \forall (i,j,k) \neq (i^*, j^*, k^*)$.

End While

Treating degeneracy

The phase 1 provides an initial feasible solution that can be degenerate or non-degenerate. Let $\hat{A}_x$ be a matrix of column vectors $\hat{P}_{ijkl}$ with $(i,j,k,l) \in \hat{I}^{(0)}$.

- If $rank(\hat{A}_x) = m + n + p + q - 3$, then the initial feasible solution is non-degenerate.
- If $rank(\hat{A}_x) < m + n + p + q - 3$, then the initial feasble solution is degenerate. We treat the degeneracy case via the procedure found in [19, 51].

Phase 2

In this phase, we search for an optimal solution for the problem (RP_1) via an adaptation of the phase 2 of AL$_{PT4C}$ [51].

1. Set $\hat{r} = 0$.
2. Determine the set of interesting quadruplets (i,j,k,l), denoted as $\hat{I}^{(\hat{r})}$.
3. Solve the following linear system
$$\tilde{\tilde{u}}_i^{(\hat{r})} \oplus \tilde{\tilde{v}}_j^{(\hat{r})} \oplus \tilde{\tilde{w}}_k^{(\hat{r})} \oplus \tilde{\tilde{t}}_l^{(\hat{r})} =_{\Re} \tilde{c}_{ijkl}, \forall (i,j,k,l) \in \hat{I}^{(\hat{r})}.$$
4. Determine $\tilde{\tilde{\delta}}_{ijkl}^{(\hat{r})}, \forall (i,j,k,l) \notin \hat{I}^{(\hat{r})}$.
$$\tilde{\tilde{\delta}}_{ijkl}^{(\hat{r})} =_{\Re} \tilde{c}_{ijkl} \ominus (\tilde{\tilde{u}}_i^{(\hat{r})} \oplus \tilde{\tilde{v}}_j^{(\hat{r})} \oplus \tilde{\tilde{w}}_k^{(\hat{r})} \oplus \tilde{\tilde{t}}_l^{(\hat{r})}).$$
5. **If** $\tilde{\tilde{\delta}}_{ijkl}^{(\hat{r})} \geq_{\Re} \tilde{0}, \forall (i,j,k,l) \notin \hat{I}^{(\hat{r})}$ **then** the solution is optimal.

Else, determine (i_0, j_0, k_0, l_0) such that:

$$\tilde{\hat{\delta}}^{(\hat{r})}_{\hat{\imath}_0\hat{\jmath}_0\hat{k}_0\hat{l}_0} = \min\{\tilde{\hat{\delta}}^{(\hat{r})}_{ijkl} : \Re(\tilde{\hat{\delta}}^{(\hat{r})}_{ijkl}) <_{\Re} \tilde{0}\}.$$

(a) Solve the following system to construct a cycle $\hat{\mu}^{(\hat{r})}$.

$$\sum \hat{\lambda}^{(\hat{r})}_{ijkl}\hat{P}_{ijkl} = -\hat{P}_{\hat{\imath}_0\hat{\jmath}_0\hat{k}_0\hat{l}_0}, \forall(i,j,k,l) \in \hat{I}^{(\hat{r})}.$$

(b) Determine $\hat{\theta}$

$$\hat{\theta} = \min\{\frac{\tilde{x}^{(\hat{r})}_{ijkl}}{-\hat{\lambda}^{(\hat{r})}_{ijkl}}\} = \hat{\theta}^{(\hat{r})}_{\hat{\imath}_s\hat{\jmath}_s\hat{k}_s\hat{l}_s}.$$

Where $\hat{\lambda}^{(\hat{r})}_{ijkl} < 0$.

(c) Update the set of basic solutions $\tilde{x}^{(\hat{r}+1)}$ as follows:

$$\tilde{x}^{(\hat{r}+1)} = \{\tilde{x}^{(\hat{r})}_{ijkl} \oplus \hat{\lambda}_{ijkl} \otimes \hat{\theta} : (i,j,k,l) \in \hat{\mu}^{(\hat{r})}\} \cup \{\tilde{x}^{(\hat{r})}_{ijkl} : (i,j,k,l) \notin \hat{\mu}^{(\hat{r})}\}.$$

(d) Take $\hat{r} = \hat{r} + 1$ and repeat from 2) to 5).

Algorithm 2

At the beginning of this algorithm, we have an optimal solution for the problem (RP_1).

1. Take $\hat{h} = 1$ and $\tilde{x}^{(\hat{1})} = \tilde{x}^{(opt)}_{(RP_1)}$.

2. Determine the set of interesting quadruplets (i,j,k,l), denoted as $\hat{I}^{(\hat{h})}$.

3. Determine the overall fixed charges, denoted as $\tilde{\hat{F}}^{(\hat{h})}$(current).

$$\tilde{\hat{F}}^{(\hat{h})}(\text{current}) =_{\Re} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} \tilde{f}_{ijkl} \otimes \tilde{y}_{ijkl}.$$

4. Solve the following system

$$\tilde{\hat{u}}^{(\hat{h})}_i \oplus \tilde{\hat{v}}^{(\hat{h})}_j \oplus \tilde{\hat{w}}^{(\hat{h})}_k \oplus \tilde{\hat{t}}^{(\hat{h})}_l =_{\Re} \tilde{c}_{ijkl}, \forall(i,j,k,l) \in \hat{I}^{(\hat{h})}.$$

5. Determine $\tilde{\hat{\delta}}^{(\hat{h})}_{ijkl}, \forall(i,j,k,l) \notin \hat{I}^{(\hat{h})}$.

$$\tilde{\hat{\delta}}_{ijkl}^{(\hat{h})} =_{\Re} \tilde{c}_{ijkl} \ominus (\tilde{\hat{u}}_i^{(\hat{h})} \oplus \tilde{\hat{v}}_j^{(\hat{h})} \oplus \tilde{\hat{w}}_k^{(\hat{h})} \oplus \tilde{\hat{t}}_l^{(\hat{h})}).$$

6. Solve the following system to determine all cycles $\hat{\mu}^{(\hat{h})}$.

$$\sum_{(i,j,k,l)\in \hat{I}^{(\hat{h})}} \hat{\lambda}_{ijkl}^{(\hat{h})} \hat{P}_{ijkl} = -\hat{P}_{i'j'k'l'}, \forall (i',j',k',l') \notin \hat{I}^{(\hat{h})}.$$

7. Determine $\hat{\theta}_{i'j'k'l'}^{(\hat{h})}, \forall (i',j',k',l') \notin \hat{I}^{(\hat{h})}$ as follows:

$$\hat{\theta}_{i'j'k'l'}^{(\hat{h})} = \min\{\frac{\tilde{x}_{ijkl}^{(\hat{h})}}{\hat{\lambda}_{ijkl}^{(\hat{h})}}, \hat{\lambda}_{ijkl}^{(\hat{h})} < 0\}.$$

8. Determine $\tilde{\hat{A}}_{ijkl}^{(\hat{h})}, \forall (i,j,k,l) \notin \hat{I}^{(\hat{h})}$.

$$\tilde{\hat{A}}_{ijkl}^{(\hat{h})} =_{\Re} \hat{\theta}_{ijkl}^{(\hat{h})} \tilde{\hat{\delta}}_{ijkl}^{(\hat{h})}.$$

9. Determine $D\tilde{\hat{F}}_{ijkl}^{(h)}, \forall (i,j,k,l) \notin \hat{I}^{(\hat{h})}$.

$$D\tilde{\hat{F}}_{ijkl}^{(h)} =_{\Re} \tilde{\hat{F}}_{ijkl}^{(\hat{h})}(\text{NB}) - \tilde{\hat{F}}^{(h)}(\text{current}).$$

$\tilde{\hat{F}}_{ijkl}^{(\hat{h})}$(NB) is the total fixed cost obtained by introducing variable $\tilde{x}_{ijkl}^{(\hat{h})}$.

10. Determine $\tilde{\hat{\Delta}}_{ijkl}^{(\hat{h})}, \forall (i,j,k,l) \notin \hat{I}^{(\hat{h})}$.

$$\tilde{\hat{\Delta}}_{ijkl}^{(\hat{h})} =_{\Re} \tilde{\hat{A}}_{ijkl}^{(\hat{h})} \oplus D\tilde{\hat{F}}_{ijkl}^{(\hat{h})}.$$

11. **If,** $\Re(\tilde{\hat{\Delta}}_{ijkl}^{(\hat{h})}) \geq_{\Re} \tilde{0}, \forall (i,j,k,l) \notin \hat{I}^{(\hat{h})}$ **then** the current solution is optimal.

12. Determine $(\hat{i}_0, \hat{j}_0, \hat{k}_0, l_0)$ such that:

$$\tilde{\hat{\Delta}}_{\hat{i}_0\hat{j}_0\hat{k}_0\hat{l}_0}^{(\hat{h})} =_{\Re} \min\{\tilde{\hat{\Delta}}_{ijkl}^{(\hat{h})} : \Re(\tilde{\hat{\Delta}}_{ijkl}^{(\hat{h})}) <_{\Re} \tilde{0}\}.$$

13. Solve the following system to determine a cycle $\hat{\mu}^{(\hat{h})}$.

$$\sum \hat{\lambda}_{ijkl}^{(\hat{h})} \hat{P}_{ijkl} = -\hat{P}_{\hat{i}_0\hat{j}_0\hat{k}_0\hat{l}_0}, \forall (i,j,k,l) \in \hat{I}^{(\hat{h})}.$$

14. Determine $\hat{\theta}$

$$\hat{\theta} =_{\Re} \min\{\frac{\tilde{x}^{(\hat{h})}_{ijkl}}{-\hat{\lambda}^{(\hat{h})}_{ijkl}}\} =_{\Re} \hat{\theta}^{(\hat{h})}_{\hat{i}_s\hat{j}_s\hat{k}_s\hat{l}_s}.$$

Where $\hat{\lambda}^{(\hat{h})}_{ijkl} < 0$.

15. Update the set of basic solution $\tilde{x}^{(\hat{h}+1)}$ as follows:

$$\tilde{x}^{(\hat{h}+1)} = \{\tilde{x}^{(\hat{h})}_{ijkl} \oplus \hat{\lambda}_{ijkl}\hat{\theta} : (i,j,k,l) \in \hat{\mu}^{(\hat{h})}\} \cup \{\tilde{x}^{(\hat{h})}_{ijkl} : (i,j,k,l) \notin \hat{\mu}^{(\hat{h})}\}$$

16. Take $\hat{h} = \hat{h} + 1$ and repeat from 1) to 10).

Algorithm 3

At the beginning of this algorithm, we have an optimal solution for the problem (P_1). Let $\tilde{\hat{M}} = (\hat{M}_1, \hat{M}_2, \hat{M}_3)$ be a large fuzzy number.

Take $\hat{t} = 1$, $(\hat{P}_1) = (\hat{P}_0) = (P_1)$, and $\tilde{c}^{(1)}_{ijkl} =_{\Re} \tilde{c}^{(0)}_{ijkl} =_{\Re} \tilde{c}_{ijkl}\, \forall (i,j,k,l)$.

1. Consider $(\hat{P}_{\hat{t}}) = (\hat{P}_{\hat{t}-1})$ with variable $\tilde{c}^{(\hat{t})}_{ijkl}$
2. Determine an optimal solution $\tilde{X}_{\hat{t}}$ to the problem $(\hat{P}_{\hat{t}})$. Let $\tilde{\hat{Z}}_{\hat{t}}$ be the optimal value associated with $\tilde{X}_{\hat{t}}$.
3. If $\tilde{\hat{Z}}_{\hat{t}} <_{\Re} \tilde{\hat{M}}$ then go to step 4. Else go to step 8 .
4. Determine $\tilde{\hat{T}}_{\hat{t}} =_{\Re} \max\{\tilde{t}_{ijkl} : \tilde{x}_{ijkl} >_{\Re} \tilde{0}$ according to $\tilde{X}^{(\hat{t})}\}$.
5. Take $\hat{t} = \hat{t} + 1$.
6. Define $\tilde{c}^{(\hat{t}+1)}_{ijkl} = \begin{cases} \tilde{c}_{ijkl} & \text{if } \Re(\tilde{t}_{ijkl}) < \Re(\tilde{\hat{T}}_{\hat{t}}), \\ \tilde{\hat{M}} & \text{if } \Re(\tilde{\hat{t}}_{ijkl}) \geq \Re(\tilde{\hat{T}}_{\hat{t}}). \end{cases}$
7. Take $\hat{t} = \hat{t} + 1$ and return to Step 1.
8. Take $\hat{t} = \hat{q} + 1$ and $\tilde{Z}_{\bar{q}+1} >_{\Re} \tilde{M}$.
9. Let $\hat{L}$ be the set of non-dominated solutions $\{\tilde{X}^{(1)}, \ldots, \tilde{X}^{(\hat{q})}\}$ with trade-off pairs: $(\tilde{\hat{Z}}_1, \tilde{\hat{T}}_1), (\tilde{\hat{Z}}_2, \tilde{\hat{T}}_2), (\tilde{\hat{Z}}_3, \tilde{\hat{T}}_3), \ldots, (\tilde{\hat{Z}}_{\hat{q}}, \tilde{\hat{T}}_{\hat{q}})$ where $\tilde{\hat{Z}}_1 <_{\Re} \tilde{\hat{Z}}_2 <_{\Re} \ldots <_{\Re} \tilde{\hat{Z}}_{\hat{q}}$ and $\tilde{\hat{T}}_1 >_{\Re} \tilde{\hat{T}}_2 >_{\Re} \ldots >_{\Re} \tilde{\hat{T}}_{\hat{q}}$.

10. Determine the distance $D_{\hat{r}}, \hat{r} = 1, \ldots, \hat{q}$ as follows:

    $D_{\hat{r}} = |\Re(\tilde{\hat{Z}}_{\hat{r}}) - \Re(\tilde{\hat{Z}}_1)| + |\Re(\tilde{\hat{T}}_{\hat{r}}) - \Re(\tilde{\hat{T}}_{\hat{q}})|.$

11. Determine $\hat{s}$ such that $D_{\hat{s}} = min\{D_{\hat{r}}, \hat{r} = 1, \ldots \hat{q}\}$.

    Therefore $(\tilde{\hat{Z}}_{\hat{s}}, \tilde{\hat{T}}_{\hat{s}})$ is the optimum trade-off pair.

Convergence of the Algorithm 3 Algorithm 3 converges after a finite number of iterations due to the choice of the variable cost $\tilde{c}_{ijkl}^{(t)}$ in step 4.

3.5 Computational results

In this section, we present a series of numerical results evaluating the performance of our algorithms (Algorithm 1, Algorithm 2, and Algorithm 3). These algorithms were implemented using Matlab R2010b, and the experiments were conducted on a personal computer running the Windows operating system. To assess the effectiveness of the proposed approach, we generated a variety of problems with different sizes. For each problem, we randomly created a set of data, including variable costs, fixed costs, and transportation time matrices. Denoted by:

- $M \times N$: the size of the problem, where $M = m + n + p + q$ and $N = mnpq$.
- it: the number of non-dominated solutions.
- $\tilde{x}_B^{(r)}$: the set of basic variables obtained at iteration r.
- $\tilde{x}_H^{(r)}$: the set of non-basic variable obtained at iteration r.

Example 3.2. *We explore an uncertain bi-objective fixed charge transportation problem with four indexes whose dimensions* $(m = n = p = q = 2)$, *and the values of* $\hat{a}_i, \hat{b}_j, \hat{e}_k, \hat{d}_l, \tilde{c}_{ijkl}, \tilde{f}_{ijkl}$ *and* $\tilde{t}_{ijkl}$ *are tabulated in tables 3.1, 3.2, 3.3, 3.4,*

$\tilde{\hat{a}}_1$	$\tilde{\hat{a}}_2$	$\tilde{\hat{b}}_1$	$\tilde{\hat{b}}_2$	$\tilde{\hat{e}}_1$	$\tilde{\hat{e}}_2$	$\tilde{\hat{d}}_1$	$\tilde{\hat{d}}_2$
$(2,5,15)$ $\Re = 6.75$	$(1,14,18)$ $\Re = 11.75$	$(,13,18)$ $\Re = 12.75$	$(1,4,14)$ $\Re = 5.75$	$(2,8,10)$ $\Re = 7$	$(9,12,13)$ $\Re = 11.5$	$(4,13,14)$ $\Re = 11$	$(3,4,19)$ $\Re = 7.5$

Table 3.1: Table of $\tilde{\hat{a}}_i, \tilde{\hat{b}}_j, \tilde{\hat{e}}_k$ and $\tilde{\hat{d}}_l$ quantities.

$\tilde{c}_{1111}$	$\tilde{c}_{1112}$	$\tilde{c}_{1121}$	$\tilde{c}_{1122}$	$\tilde{c}_{1211}$	$\tilde{c}_{1212}$	$\tilde{c}_{1221}$	$\tilde{c}_{1222}$
$(4,5,9)$ $\Re = 5.75$	$(1,7,10)$ $\Re = 6.25$	$(3,4,5)$ $\Re = 4$	$(2,6,10)$ $\Re = 6$	$(4,6,10)$ $\Re = 6.5$	$(2,4,7)$ $\Re = 4.25$	$(1,4,10)$ $\Re = 4.75$	$(3,6,9)$ $\Re = 6$
$\tilde{c}_{2111}$	$\tilde{c}_{2112}$	$\tilde{c}_{2121}$	$\tilde{c}_{2122}$	$\tilde{c}_{2211}$	$\tilde{c}_{2212}$	$\tilde{c}_{2221}$	$\tilde{c}_{2222}$
$(1,2,9)$ $\Re = 3.5$	$(1,4,5)$ $\Re = 3.5$	$(2,8,9)$ $\Re = 6.75$	$(1,8,9)$ $\Re = 6.5$	$(3,7,10)$ $\Re = 6.75$	$(2,5,9)$ $\Re = 5.25$	$(1,4,10)$ $\Re = 4.75$	$(7,8,9)$ $\Re = 8$

Table 3.2: Matrix of variable costs.

$\tilde{f}_{1111}$	$\tilde{f}_{1112}$	$\tilde{f}_{1121}$	$\tilde{f}_{1122}$	$\tilde{f}_{1211}$	$\tilde{f}_{1212}$	$\tilde{f}_{1221}$	$\tilde{f}_{1222}$
$(6,11,18)$ $\Re = 11.5$	$(1,17,19)$ $\Re = 13.5$	$(7,14,18)$ $\Re = 13.25$	$(3,13,14)$ $\Re = 10.75$	$(3,11,19)$ $\Re = 11$	$(10,12,17)$ $\Re = 12.75$	$(1,9,12)$ $\Re = 7.75$	$(2,5,17)$ $\Re = 7.25$
$\tilde{f}_{2111}$	$\tilde{f}_{2112}$	$\tilde{f}_{2121}$	$\tilde{f}_{2122}$	$\tilde{f}_{2211}$	$\tilde{f}_{2212}$	$\tilde{f}_{2221}$	$\tilde{f}_{2222}$
$(2,5,8)$ $\Re = 5$	$(7,10,13)$ $\Re = 10$	$(7,18,20)$ $\Re = 15.75$	$(1,8,16)$ $\Re = 8.25$	$(8,12,15)$ $\Re = 11.75$	$(8,10,16)$ $\Re = 11$	$(4,5,20)$ $\Re = 8.5$	$(6,12,13)$ $\Re = 10.75$

Table 3.3: Matrix of fixed costs.

$\tilde{t}_{1111}$	$\tilde{t}_{1112}$	$\tilde{t}_{1121}$	$\tilde{t}_{1122}$	$\tilde{t}_{1211}$	$\tilde{t}_{1212}$	$\tilde{t}_{1221}$	$\tilde{t}_{1222}$
$(6,9,10)$ $\Re = 8.5$	$(3,7,9)$ $\Re = 6.5$	$(1,3,9)$ $\Re = 4$	$(1,8,10)$ $\Re = 6.75$	$(1,3,5)$ $\Re = 3$	$(6,8,9)$ $\Re = 7.75$	$(2,7,9)$ $\Re = 6.25$	$(1,4,5)$ $\Re = 3.5$
$\tilde{t}_{2111}$	$\tilde{t}_{2112}$	$\tilde{t}_{2121}$	$\tilde{t}_{2122}$	$\tilde{t}_{2211}$	$\tilde{t}_{2212}$	$\tilde{t}_{2221}$	$\tilde{t}_{2222}$
$(4,8,9)$ $\Re = 7.25$	$(2,3,5)$ $\Re = 3.25$	$(1,3,6)$ $\Re = 3.25$	$(1,5,9)$ $\Re = 5$	$(1,6,8)$ $\Re = 5.25$	$(2,7,9)$ $\Re = 6.25$	$(2,8,9)$ $\Re = 6.75$	$(1,3,6)$ $\Re = 3.25$

Table 3.4: Matrix of transportation time.

The problem has a feasible solution because:

$$\sum_{i=1}^{2} \Re(\tilde{\hat{a}}_i) = \sum_{j=1}^{2} \Re(\tilde{\hat{b}}_j) = \sum_{1}^{2} \Re(tilde\hat{e}_k) = \sum_{1}^{2} \Re(\tilde{\hat{d}}_l) = 18.5.$$

To solve the primary problem, we divide it into two sub-problems (P_1) and (P_2) shown below.

(P_1) : *Minimize* $\tilde{Z} =_{\Re} \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2} (\tilde{c}_{ijkl} \otimes \tilde{x}_{ijkl}) \oplus (\tilde{f}_{ijkl} \otimes \tilde{y}_{ijkl})$, *s. t. c.* (3.1) – (3.6).

(P_2) : *Minimize* $\tilde{T} =_{\Re} \max[\tilde{t}_{ijkl} : \tilde{x}_{ijkl} >_{\Re} 0]$, *s. t. c.* (3.1) – (3.6).

Now, we apply our approach $Al_{FBOFCTP4}$ consisting of three steps.

Application of Al$_{BOFCTP4}$

we consider the relaxed transportation problem (RP_1) *consisting of variable cost only, given as follows:*

$$(RP_1) : \textit{Minimize } \tilde{Z} =_{\Re} \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{p} \sum_{l=1}^{2} (\tilde{c}_{ijkl}\, \tilde{x}_{ijkl}), \text{ s. t. c. } (3.1)-(3.6).$$

Application of Algorithm 1

We search for an optimal solution for the problem (RP_1) *via Algorithm 1.*

Phase 1

- *Take* $\hat{I} = \varnothing$.
- *Take* $\min \tilde{c}_{ijkl} =_{\Re} \tilde{c}_{2111} =_{\Re} (1, 2, 9)$.
- *Find* $\tilde{x}_{2111}$ *as follows:*

$$\tilde{x}_{2111} =_{\Re} \min(\tilde{\hat{a}}_2, \tilde{\hat{b}}_1, \tilde{\hat{e}}_1, \tilde{\hat{d}}_1)$$
$$\tilde{x}_{2111} =_{\Re} \min((1, 14, 18), (7, 13, 18), (2, 8, 10), (4, 13, 14))$$
$$\tilde{x}_{2111} =_{\Re} (2, 8, 10) \ \textit{ and } \ \beta_{2111} = 1.$$

- *Add* $(2, 1, 1, 1)$ *to* $\hat{I}$.
- *Adjust* $\tilde{\hat{a}}_2, \tilde{\hat{b}}_1, \tilde{\hat{e}}_1$, *and* $\tilde{\hat{d}}_1$ *as follows:*

$$\tilde{\hat{a}}_2 =_{\Re} (-9, 6, 16),\ \tilde{\hat{b}}_1 =_{\Re} (-3, 5, 16),\ \tilde{\hat{e}}_1 =_{\Re} (-8, 0, 8),\ \tilde{\hat{d}}_1 =_{\Re} (-6, 5, 12).$$

- $\forall (i, k, l) \neq (2, 1, 1), \tilde{x}_{ij1l} =_{\Re} 0$ *and* $\beta_{ij2l} = 1$.

The process continues over multiple iterations until all $\tilde{x}_{ijkl}$ *are determined.*

The initial basic feasible solution for the problem RP_1 *is given as follows:* $\tilde{x}^{(0)} = \tilde{x}_B^{(0)} \cup \tilde{x}_H^{(0)}$, *where:*

$$\tilde{x}_B^{(0)} = \{\tilde{x}_{2111} =_{\Re} (2, 8, 10), \tilde{x}_{1121} =_{\Re} (-6, 5, 12), \tilde{x}_{1122} =_{\Re} (-5, 0, 22), \tilde{x}_{1222} =_{\Re} (-32, 0, 36), \tilde{x}_{2222} =_{\Re} (-9, 6, 16)\}.$$

Test of degeneracy

The obtained solution is non-degenerate because the number of non-zero elements of $\tilde{x}_B^{(0)}$ *is equal to* $5 = M - 3$.

Phase 2

We find an optimal solution for the problem (RP_1) *by applying the steps of phase 2 (Algorithm 1).*

The obtained optimal solution is given as follows: $\tilde{x}^{opt} = \tilde{x}_B^{opt} \cup \tilde{x}_H^{opt}$ *where:*

$$\tilde{x}_B^{(opt)} = \{\tilde{x}_{1121} =_{\Re} (-39, 6, 51), \tilde{x}_{2221} =_{\Re} (-27, 7, 33), \tilde{x}_{1212} =_{\Re} (-18, 1, 17), \tilde{x}_{2112} =_{\Re} (-42, 7, 55), \tilde{x}_{1222} =_{\Re} (-66, -2, 72)\}.$$

Application of Algorithm 2

We use steps of Algorithm 2 to find an optimal solution to the problem (P_1).

- *Take* $\tilde{x}^{(1)} = \tilde{x}^{opt}_{(RP_1)}$. $\tilde{x}_B^{(1)} = \{\tilde{x}_{1121} =_{\Re} (-39, 6, 51), \tilde{x}_{2221} =_{\Re} (-27, 7, 33), \tilde{x}_{1212} =_{\Re} (-18, 1, 17), \tilde{x}_{2112} =_{\Re} (-42, 7, 55), \tilde{x}_{1222} =_{\Re} (-66, -2, 72)\}$.

- *The set of basic cells is as follows:*

$$\hat{I}^{(1)} = \{(1,1,2,1), (2,2,2,1), (1,2,1,2), (2,1,1,2), (1,2,2,2)\}.$$

- *Calculate* $\tilde{\hat{F}}^1(current)$ *as follows:*

$$\tilde{\hat{F}}^1(current) =_{\Re} \tilde{f}_{1121} \oplus \tilde{f}_{2221} \oplus \tilde{f}_{1212} \oplus \tilde{f}_{2112} \oplus \tilde{f}_{1222},$$

$$\tilde{\hat{F}}^1(current) =_{\Re} (30, 46, 85),$$

$$\Re(\tilde{\hat{F}}^1(current)) = 51.75.$$

- *The values of* $\tilde{\hat{\delta}}^{(1)}_{ijkl}, \hat{\theta}^{(1)}_{ijkl}, \tilde{\hat{A}}^{(1)}_{ijkl}, \tilde{DF}^{(1)}_{ijkl}$ *and* $\tilde{\hat{\Delta}}^{(1)}_{ijkl}$ *are tabulated in table 3.5.*

Based on the optimality test of the algorithm 2, it is shown that the solution $\tilde{x}^{(1)}$ *is not optimal. We need two other iterations to get the optimal one.*

The optimal solution is $\tilde{x}^{(opt)} = \tilde{x}_B^{(opt)} \cup \tilde{x}_H^{(opt)}$ *where:*

(i,j,k,l)	$\tilde{\hat{\delta}}^{(1)}_{ijkl}$	$\hat{\theta}^{(1)}_{ijkl}$	$\tilde{\hat{A}}^{(1)}_{ijkl}$	$\tilde{\hat{DF}}^{(1)}_{ijkl}$	$\tilde{\hat{\Delta}}^{(1)}_{ijkl}$
(1,1,1,1)	(1,3,7) $\Re = 3.5$	(-18,1,17) $\Re = 0.25$	(0.25,0.75,1.75) $\Re = 0.875$	(-59,-1,56) $\Re =$ -1.25	(-58.75,-0.25,57.75) $\Re =$ -0.275
(1,1,1,2)	(-2.5,2,7.5) $\Re =2.25$	(-36,2,34) $\Re =0.5$	(-1.25,1.5,3.75) $\Re = 1.375$	(-64,5,57) $\Re = 0.75$	(-65.25,6.5,60.75) $\Re = 2.125$
(1,1,2,2)	(-3.5,0,6.5) $\Re =0.75$	(-66,-2,72) $\Re =0.5$	(-1.75,0,3.25) $\Re =0.375$	(-54,8,52) $\Re =3.5$	(-55,9,55.25) $\Re =3.875$
(1,2,1,1)	(-2.5,4,8.5) $\Re = 3.5$	(-12,0,66) $\Re =0.16$	(-0.41,0.66,1.41) $\Re = 0.58$	(-62,-1,57) $\Re =$-1.75	(-62.41,-0.33,58.41) $\Re =$ -1.16
(1,2,2,1)	(-7.5,0,7.5) $\Re =0$	(-36,2,34) $\Re = 0.5$	(-3.75,0,3.75) $\Re = 0$	(-64,-3,50) $\Re =$ -5	(-67.75,-3,52.75) $\Re =$ -5
(2,1,1,1)	(-3.5,0,8.5) $\Re = 1.25$	(-36,2,34) $\Re =0.5$	(-1.75,0,4.25) $\Re = 0.625$	(-63,7,46) $\Re =$ -7.75	(-64.75,7,50.25) $\Re =$-7.125
(2,1,2,1)	(-4,5,4) $\Re = 2.75$	(-54,14,66) $\Re =10$	(-45,40,75) $\Re = 27.5$	(-52,13,55) $\Re =7.25$	(-97,53,130) $\Re = 34.75$
(2,1,2,2)	(-6,2,7) $\Re = 1.25$	(-66,-2,72) $\Re =0.5$	(-3,1,3.5) $\Re = 0.625$	(-56,3,54) $\Re = 1$	(-59,4,57.5) $\Re = 1.625$
(2,2,1,1)	(-5,5,10) $\Re = 3.75$	(-18,1,17) $\Re =0.25$	(-1.25,1.25,2.5) $\Re = 0.9375$	(-57,0,53) $\Re =$ -1	(-58.25,1.25,55.5) $\Re =$ -0.0625
(2,2,1,2)	(-6.5,1,5.5) $\Re = 0.25$	(-36,2,34) $\Re =0.5$	(-3.25,0.5,4.25) $\Re = 0.5$	(-57,-2,54) $\Re =$ -1.75	(-60.25,-1.5,58.25) $\Re =$ -1.25
(2,2,2,2)	(-3.5,2,7.5) $\Re = 2$	(-66,-2,72) $\Re =0.5$	(-1.75,1,3.75) $\Re = 1$	(-51,7,51) $\Re = 3.5$	(-52.75,8,54.75) $\Re = 4.5$

Table 3.5: Table of $\tilde{\hat{\delta}}^{(1)}_{ijkl}, \hat{\theta}^{(1)}_{ijkl}, \tilde{\hat{A}}^{(1)}_{ijkl}, \tilde{\hat{DF}}^{(1)}_{ijkl}$ and $\tilde{\hat{\Delta}}^{(1)}_{ijkl}$ quantities.

$$\tilde{x}_B^{(opt)} = \{\tilde{x}_{2111} =_{\Re} (-36,2,34), \tilde{x}_{1121} =_{\Re} (-56,5,69), \tilde{x}_{2221} =_{\Re} (-44,6,51), \tilde{x}_{2112} =_{\Re} (-59,6,73), \tilde{x}_{1222} =_{\Re} (-102,0,106)\}.$$

The optimal value of objective associated with the solution $\tilde{x}^{(opt)}$ is:

$$\tilde{Z} =_{\Re} \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2}(\tilde{c}_{ijkl} \otimes \tilde{x}_{ijkl}) \oplus (\tilde{f}_{ijkl} \otimes \tilde{y}_{ijkl}),$$

$$\tilde{Z} =_{\Re} (54, 114, 198.25). \Re(\tilde{Z}) = 120.0624.$$

Application of Algorithm 3

We identify the set of non-dominated solutions and trade-off pairs for the main problem with the use of Algorithm 3.

- *Set $(\hat{P}_1) = (P_1)$.*
- *Take $\tilde{\hat{X}}^{(1)} =_{\Re} \tilde{x}^{(opt)}_{(P_1)}$ and $\tilde{c}^{(1)}_{ijkl} = \tilde{c}_{ijkl}$ for all (i,j,k,l).*
- *The optimal solution of the problem (P_1) is $\tilde{\hat{X}}^{(1)} = \tilde{X}^{(1)}_B \cup \tilde{X}^{(1)}_H$ where:*

$$\tilde{X}_B^{(1)} = \{\tilde{x}_{2111} =_{\Re} (-36, 2, 34), \tilde{x}_{1121} =_{\Re} (-56, 5, 69), \tilde{x}_{2221} =_{\Re} (-44, 6, 51), \tilde{x}_{2112} =_{\Re} (-59, 6, 73), \tilde{x}_{1222} =_{\Re} (-102, 0, 106)\}.$$

- *The optimal cost corresponding to $\tilde{\hat{X}}^{(1)}$ is:*

$$\tilde{\hat{Z}}_1 = (54, 114, 198.25).$$

- *We have $\Re(\tilde{\hat{Z}}_1) = 120.0625$ and $\Re(\tilde{\hat{Z}}_1) < \Re(\tilde{\hat{M}})$.*

- *Determine $\tilde{\hat{T}}_1$ as follows:*

$$\tilde{\hat{T}}_1 =_{\Re} \max\{\tilde{t}_{ijkl} : \tilde{x}_{ijkl} >_{\Re} 0 \textit{ according to } \tilde{\hat{X}}^{(1)}\}.$$
$$\tilde{\hat{T}}_1 =_{\Re} (4, 8, 9), \Re(\tilde{\hat{T}}_1) = 7.25.$$

The first fuzzy cost-time trade off pair is:

$$(\tilde{\hat{Z}}_1, \tilde{\hat{T}}_1) = (54, 114, 198.25), (4, 8, 9)).$$

- *Take* $\tilde{c}_{ijkl}^{(2)} = \begin{cases} \tilde{c}_{ijkl} & \textit{if } \Re(\tilde{t}_{ijkl}) < \Re(\tilde{\hat{T}}_1), \\ \tilde{\hat{M}} & \textit{if } \Re(\tilde{t}_{ijkl}) \geq \Re(\tilde{\hat{T}}_1). \end{cases}$

- *Find an optimal solution to the problem $(\hat{P}_2)$ with variable costs $\tilde{c}_{ijkl}^{(2)}$.*

- *The optimal solution of the problem $(\hat{P}_2)$ is $\tilde{\hat{X}}^{(2)} = \tilde{\hat{X}}_B^{(2)} \cup \tilde{\hat{X}}_H^{(2)}$ where:*

$$\tilde{\hat{X}}_B^{(2)} = \{\tilde{x}_{1121} =_{\Re} (-3, 5, 16), \tilde{x}_{1221} =_{\Re} (-60, 2, 58), \tilde{x}_{2221} =_{\Re} (-70, 6, 77), \tilde{x}_{2112} =_{\Re} (2, 8, 10), \tilde{x}_{1222} =_{\Re} (-40, -2, 46)\}.$$

- *The optimal value of objective associated with the solution $\tilde{\hat{X}}^{(2)}$ is:*

$$\tilde{\hat{Z}}_2 = (52, 118, 200.75).$$

We have $\Re(\tilde{\hat{Z}}_2) = 122.1875$ then $\Re(\tilde{\hat{Z}}_2) < \Re(\tilde{\hat{M}})$.

The second fuzzy cost-time trade-off pair is:

$$(\tilde{\hat{Z}}_2, \tilde{\hat{T}}_2) = ((52, 118, 200.75), (2, 8, 9)).$$

- *The process continues until all non-dominated solutions and the set of trade-off pairs are determined.*
- *After 4 other iterations, we find an optimal solution for the problem* $(\hat{P}_6)$ *with variable costs* $\tilde{c}^{(6)}_{ijkl}$.
- *The optimal solution of problem* $(\hat{P}_6)$ *is* $\tilde{\hat{X}}^{(6)} = \tilde{\hat{X}}^{(6)}_B \cup \tilde{\hat{X}}^{(6)}_H$ *where:*

$$\tilde{\hat{X}}^{(6)}_B = \{\tilde{x}_{1222} =_{\Re} (-39.5, 1.5, 47), \tilde{x}_{1211} =_{\Re} (-29.5, 4.5, 33), \tilde{x}_{2121} =_{\Re} (-38.5, 10.5, 49), \tilde{x}_{2112} =_{\Re} (-31, 3.5, 39.5), \tilde{x}_{1122} =_{\Re} (-19, -1, 25)\}.$$

We have $\Re(\tilde{\hat{Z}}_6) >_{\Re} \tilde{\hat{M}}$. *Therefore the algorithm ends here.*

- *We have five fuzzy cost-time trade-off pairs:*

$$(\tilde{\hat{Z}}_1, \tilde{\hat{T}}_1) = ((54, 1114, 198.25), (4, 8, 9)),$$
$$(\tilde{\hat{Z}}_2, \tilde{\hat{T}}_2) = ((52, 118, 200.75), (2, 8, 9)),$$
$$(\tilde{\hat{Z}}_3, \tilde{\hat{T}}_3) = ((50.25, 150.219.75), (2, 7, 9)),$$
$$(\tilde{\hat{Z}}_4, \tilde{\hat{T}}_4) = ((67, 158.5, 218.5), (1, 6, 8)),$$
$$(\tilde{\hat{Z}}_5, \tilde{\hat{T}}_5) = ((68, 173, 225), (1, 3, 9)).$$

- *The Ideal fuzzy cost-time trade-off pair is:*

$$(\tilde{\hat{Z}}_1, \tilde{\hat{T}}_5) = ((54, 1114, 198.25), (1, 3, 9)).$$

- *Now, we identify the optimum fuzzy cost-time trade-pair:*

$$D_1 = (\Re(\tilde{\hat{Z}}_1) - \Re(\tilde{\hat{Z}}_1)) + (\Re(\tilde{\hat{T}}_1) - \Re(\tilde{\hat{T}}_5)) = 3.25.$$
$$D_2 = (\Re(\tilde{\hat{Z}}_2) - \Re(\tilde{\hat{Z}}_1)) + (\Re(\tilde{\hat{T}}_2) - \Re(\tilde{\hat{T}}_5)) = 4.875$$
$$D_3 = (\Re(\tilde{\hat{Z}}_3) - \Re(\tilde{\hat{Z}}_1)) + (\Re(\tilde{\hat{T}}_3) - \Re(\tilde{\hat{T}}_5)) = 24.6875.$$
$$D_4 = (\Re(\tilde{\hat{Z}}_4) - \Re(\tilde{\hat{Z}}_1)) + (\Re(\tilde{\hat{T}}_4) - \Re(\tilde{\hat{T}}_5)) = 32.1875.$$
$$D_5 = (\Re(\tilde{\hat{Z}}_5) - \Re(\tilde{\hat{Z}}_1)) + (\Re(\tilde{\hat{T}}_5) - \Re(\tilde{\hat{T}}_5)) = 39.6875.$$
$$D^* = min\{D_1, D_2, D_3, D_4, D_5\} = D_1 = 3.25.$$

- *Therefore, the optimum fuzzy cost-time trade-off pair is:*

$$(\tilde{\hat{Z}}^*, \tilde{\hat{T}}^*) = ((54, 1114, 198.25), (4, 8, 9))$$

Example 3.3. *Consider the fuzzy bi-objective four index fixed charge transportation problem with $m = n = p = q = 2$.*

The values of $\tilde{a}_i$, $\tilde{\hat{b}}_j$, $\tilde{e}_k$ and $\tilde{\hat{d}}_l$:

$(7, 7, 59)$	$(39, 70, 98)$	$(24, 47, 62)$	$(17, 48, 64)$	$(52, 73, 97)$	$(2, 10, 40)$	$(13, 30, 64)$	$(19, 65, 71)$

The variable costs $\tilde{c}_{ijkl}$:

$(9, 13, 29)$	$(4, 5, 20)$	$(24, 26, 29)$	$(2, 3, 18)$	$(6, 14, 26)$	$(2, 10, 24)$	$(4, 6, 23)$	$(1, 28, 30)$
$(11, 11, 20)$	$(5, 5, 24)$	$(9, 9, 23)$	$(4, 15, 30)$	$(1, 3, 11)$	$(14, 16, 29)$	$(2, 2, 9)$	$(5, 6, 16)$

The fixed costs $\tilde{f}_{ijkl}$:

$(5, 18, 30)$	$(7, 8, 30)$	$((7, 8, 29))$	$(11, 34, 36)$	$(11, 18, 36)$	$(22, 31, 37)$	$(6, 21, 38)$	$(8, 9, 30)$
$(13, 29, 37)$	$(9, 30, 33)$	$(6, 9, 28)$	$(1, 6, 30)$	$(16, 18, 27)$	$(32, 37, 39)$	$(17, 22, 38)$	$(2, 6, 13)$

The transportation time $\tilde{t}_{ijkl}$:

$(4, 8, 9)$	$(7, 11, 12)$	$(3, 3, 11)$	$(5, 8, 15)$	$(5, 5, 7)$	$(7, 8, 10)$	$(3, 4, 6)$	$(7, 13, 15)$
$(3, 3, 5)$	$(4, 7, 8)$	$(6, 12, 15)$	$(2, 5, 10)$	$(1, 4, 13)$	$(3, 11, 13)$	$(1, 5, 12)$	$(8, 11, 14)$

The pairs of fuzzy cost-time trade off found after 0.305115 seconds are:

$(\tilde{\hat{Z}}_1, \tilde{\hat{T}}_1) = ((323, 472, 1664), (8, 11, 14))$,

$(\tilde{\hat{Z}}_2, \tilde{\hat{T}}_2) = ((355, 521, 1790), (7, 11, 12))$,

$(\tilde{\hat{Z}}_3, \tilde{\hat{T}}_3) = ((329, 537, 1796), (5, 8, 15))$,

$(\tilde{\hat{Z}}_4, \tilde{\hat{T}}_4) = ((384, 574, 1978), (7, 8, 10))$.

We have $\tilde{\hat{Z}}_5 >_{\Re} \tilde{\hat{M}}$. So, we terminate here.

The optimum pair is: $((323, 472, 1664), (8, 11, 14))$

Example 3.4. *Consider a fuzzy bi-objective fixed charge transportation problem with $m = 2$ and $n = p = q = 3$.*

Tha values of $\tilde{a}_i$, $\tilde{\hat{b}}_j$, $\tilde{e}_k$ and $\tilde{\hat{d}}_l$

$(25, 82, 93)$	$(20, 26, 35)$	$(36, 48, 62)$	$(72, 78, 91)$	$(34, 70, 90)$	$(31, 83, 89)$

$(40, 81, 95)$	$(48, 68, 87)$	$(61, 81, 89)$

the variable costs $\tilde{c}_{ijkl}$:

$(4, 25, 28)$	$(3, 19, 28)$	$(9, 17, 29)$	$(5, 29, 30)$	$(15, 25, 29)$	$(5, 13, 28)$	$(20, 24, 29)$	$(2, 26, 29)$
$(21, 23, 23)$	$(6, 12, 20)$	$(1, 9, 22)$	$(2, 3, 25)$	$(10, 21, 29)$	$(2, 12, 14)$	$(6, 23, 24)$	$(4, 15, 20)$
$(9, 22, 23)$	$(5, 20, 21)$	$(4, 15, 29)$	$(7, 11, 18)$	$(8, 16, 23)$	$(21, 27, 29)$	$(5, 5, 17)$	$(8, 8, 26)$

The fixed costs $\tilde{f}_{ijkl}$

$(8, 24, 38)$	$(27, 28, 33)$	$(10, 14, 16)$	$(10, 16, 40)$	$(2, 18, 32)$	$(5, 28, 40)$	$(7, 13, 18)$	$(18, 22, 36)$
$(7, 21, 30)$	$(28, 30, 33)$	$(6, 18, 22)$	$(20, 22, 28)$	$(2, 10, 20)$	$(8, 21, 33)$	$(3, 17, 30)$	$(4, 21, 23)$
$(12, 30, 35)$	$(27, 36, 39)$	$(4, 29, 38)$	$(11, 14, 36)$	$(6, 11, 29)$	$(11, 31, 33)$	$(18, 32, 36)$	$(9, 11, 32)$

The transportation time $\tilde{t}_{ijkl}$

$(4, 6, 12)$	$(5, 8, 12)$	$(4, 7, 10)$	$(5, 5, 7)$	$(7, 13, 15)$	$(2, 3, 7)$	$(9, 10, 15)$	$(9, 9, 13)$
$(1, 5, 10)$	$(11, 12, 14)$	$(1, 12, 12)$	$(1, 6, 11)$	$(2, 9, 12)$	$(2, 3, 8)$	$(2, 4, 9)$	$(2, 2, 15)$
$(8, 11, 15)$	$(5, 11, 15)$	$(1, 2, 7)$	$(1, 6, 9)$	$(5, 6, 11)$	$(2, 6, 8)$	$(5, 8, 14)$	$(1, 5, 12)$

The pairs of fuzzy cost-time trade off found after 1.423454 seconds are:

$(\tilde{\hat{Z}}_1, \tilde{\hat{T}}_1) = ((608, 1822, 3603), (1, 12, 12))$,

$(\tilde{\hat{Z}}_2, \tilde{\hat{T}}_2) = ((646, 1834, 3666), (5, 8, 14))$,

$(\tilde{\hat{Z}}_3, \tilde{\hat{T}}_3) = ((954, 1989, 3891), (5, 6, 11))$,

$(\tilde{\hat{Z}}_4, \tilde{\hat{T}}_4) = ((911, 2424, 4246), (1, 6, 11))$,

$(\tilde{\hat{Z}}_5, \tilde{\hat{T}}_5) = ((899, 2610, 4183), (5, 5, 7))$.

We have $\tilde{\hat{Z}}_6 >_{\Re} \tilde{\hat{M}}$. *So, we terminate here.*

The optimum pair is: $((608, 1822, 3603), (1, 12, 12))$.

Example 3.5. *Consider a fuzzy bi-objective four index fixed charge transportation problem with* $m = 3, n = 2, p = 3, q = 2$.

The values of $\tilde{\hat{a}}_i$, $\tilde{\hat{b}}_j$, $\tilde{\hat{e}}_k$ *and* $\tilde{\hat{d}}_l$

$(46, 69, 75)$	$(9, 23, 92)$	$(16, 54, 83)$	$(24, 86, 100)$	$(67, 79, 92)$	$(3, 46, 97)$

$(1, 54, 79)$	$(25, 58, 92)$	$(79, 82, 96)$	$(52, 73, 76)$

The variable costs $\tilde{c}_{ijkl}$:

$(4, 25, 28)$	$(3, 19, 28)$	$(9, 17, 29)$	$(5, 29, 30)$	$(15, 25, 29)$	$(5, 13, 28)$
$(20, 24, 29)$	$(2, 26, 29)$	$(21, 23, 23)$	$(6, 12, 20)$	$(1, 9, 22)$	$(2, 3, 25)$
$(10, 21, 29)$	$(2, 12, 14)$	$(6, 23, 24)$	$(14, 15, 20)$	$(9, 22, 23)$	$(5, 20, 21)$
$(4, 15, 29)$	$(7, 11, 18)$	$(8, 16, 23)$	$(21, 27, 29)$	$(5, 5, 17)$	$(8, 8, 26)$

$(8,25,28)$	$(6,8,11)$	$(11,15,19)$
$(17,18,25)$	$(9,23,28)$	$(12,18,23)$
$(2,3,16)$	$(4,24,29)$	$(1,15,18)$
$(5,11,24)$	$(5,10,16)$	$(8,19,20)$

The fixed costs $\tilde{f}_{ijkl}$*:*

$(7,12,26)$	$(16,24,30)$	$(2,2,5)$	$(10,36,36)$	$(4,30,32)$	$(17,22,32)$
$(1,16,36)$	$(5,30,35)$	$(1,12,39)$	$(1,5,10)$	$(2,30,39)$	$(19,22,29)$
$(1,2,30)$	$(7,10,14)$	$(24,25,33)$	$(2,16,24)$	$(1,15,22)$	$(8,12,34)$
$(29,33,33)$	$(27,31,38)$	$(2,12,24)$	$(6,8,19)$	$(8,14,34)$	$(5,15,35)$

$(18,21,37)$	$(6,19,37)$	$(11,22,34)$
$(3,13,25)$	$(10,16,27)$	$(2,2,21)$
$(2,21,29)$	$(20,30,38)$	$(25,38,38)$
$(12,20,23)$	$(16,21,34)$	$(3,20,27)$

The transportation time $\tilde{t}_{ijkl}$*:*

$(12,13,13)$	$(4,12,13)$	$(4,11,13)$	$(1,6,14)$	$(7,11,14)$	$(4,4,5)$
$(2,3,6)$	$(3,6,14)$	$(5,13,14)$	$(1,11,14)$	$(5,7,15)$	$(1,7,14)$
$(5,6,7)$	$(2,5,9)$	$(6,6,15)$	$(8,8,13)$	$(8,15,15)$	$(1,3,11)$
$(4,8,14)$	$(4,7,12)$	$(4,12,12)$	$(2,7,13)$	$(3,9,14)$	$(2,11,14)$

$(3,7,9)$	$(4,8,13)$	$(4,13,13)$
$(6,9,15)$	$(1,5,14)$	$(2,3,6)$
$(3,12,14)$	$(11,11,13)$	$(3,10,15)$
$(5,11,13)$	$(4,11,13)$	$(4,4,11)$

The pairs of fuzzy cost-time trade off found after 0.8878 seconds are:

$(\tilde{\hat{Z}}_1, \tilde{\hat{T}}_1) = ((920, 1423, 3158), (4, 13, 13))$,

$(\tilde{\hat{Z}}_2, \tilde{\hat{T}}_2) = ((910, 1425, 3200), (3, 12, 14))$,

$(\tilde{\hat{Z}}_3, \tilde{\hat{T}}_3) = ((894, 1577, 2969), (4, 11, 13))$,

$(\tilde{\hat{Z}}_4, \tilde{\hat{T}}_4) = ((746, 1847, 2994), (3, 10, 15))$,

$(\tilde{\hat{Z}}_5, \tilde{\hat{T}}_5) = ((793, 1913, 3077), (1, 11, 14))$,

$(\tilde{\hat{Z}}_6, \tilde{\hat{T}}_6) = ((1057, 2577, 3571), (5, 7, 15))$,

$(\tilde{\hat{Z}}_7, \tilde{\hat{T}}_7) = ((1247, 3032, 4085), (6, 6, 15))$,

$(\tilde{\hat{Z}}_8, \tilde{\hat{T}}_8) = ((1885, 3481, 4536), (2, 7, 13))$,

We have $\tilde{\hat{Z}}_9 >_{\Re} \tilde{\hat{M}}$. *So, we terminate here.*

The optimum pair is: $((920, 1423, 3158), (4, 13, 13))$.

The results of numerical tests of FBOFCTP4 are resumed in table 3.6.

Size $M \times N$	it	T(S)
8×16	4	0.3260
12×81	8	1.7472
14×144	11	4.7977
16×256	10	12.6657
18×400	10	24.3638
20×625	17	282.1060
22×900	14	38.7666
26×1764	8	4.9690e+03
34×5184	3	2.1151e+03
40×10000	3	5.2987e+03

Table 3.6: Al$_{FBOFCTP4}$ in solving FBOFCTP4.

Commentary

- Our tests indicate that our approach demonstrates stability and is capable of addressing a variety of problems across different sizes.
- In each problem addressed, the ranges of fixed costs and variable costs vary significantly from one another.
- Our experiments indicate that Al$_{FBOFCTP4}$ is efficient, offering an optimal solution in a shorter time, particularly for larger instances.
- The results obtained are not influenced by the number of indices. As a result, the proposed approach can be adapted to address bi-objective fixed-charge transportation problems with more than four indices.

3.6 Conclusion

In this chapter, we have considered a fuzzy bi-objective four-index fixed charge transportation problem where the aim is to minimize two conflicting objectives: total transportation cost and transportation time. To solve such a problem, we have proposed an approach denoted by Al$_{FBOFCTP4}$, based on an adaptation of the approach of [26] in a fuzzy environment with four indices. According to our experiments, the proposed approach has been proven to be effective, providing an optimal solution for FBOFCTP4 in a short time, especially for relatively large-size instances.

General conclusion and future work

In this thesis, we have considered two models related to transportation problems: the fuzzy four-index fixed-charge transportation problem (FFCTP4) and the fuzzy bi-objective four-index fixed-charge transportation problem (FBOFCTP4). The aforementioned issues have not been addressed previously. The parameters of each problem are represented as triangular fuzzy numbers, and a defuzzification tool is employed to extract real values for those parameters.

To solve the first model, we proposed four metaheuristics and one approximation method. The metaheuristics we proposed are: genetic algorithm (GA), particle swarm optimization (PSO), simulated annealing (SA) and a hybrid approach that combines the strengths of both PSO and GA, referred to as PSO-GA. Additionally, we introduce a representation method known as a priority-based encoding scheme, which relates to the structure of the problem.

Our proposed approximation method is an adaptation of Balinski's method [6] in a fuzzy context with four indexes. This method involves transforming the main problem (FFCTP4) into a relaxed transportation problem, which is then solved using an extended version of the least-cost cell and MODI methods. We conducted a comparative study between our approximation method and the proposed metaheuristics across various numerical tests of FFCTP4 of different sizes. The obtained results show that the proposed metaheuristic algorithms provide good solutions in an acceptable time. Besides, The hybrid particle swarm optimization with genetic algorithm has shown its success and its advantage in terms of efficiency, robustness and speed (execution time) compared to other metaheuristics.

To tackle the second model, we introduced an approach that is an extended work of Khurana and Adlakha [26] in a fuzzy context with four indexes. This approach separates the main problem into two subproblems and solves each problem independently using different algorithms. It then determines the set of efficient solutions while identifying the best solution among them. We conducted a numerical study across various numerical tests of FBOFCTP4 with different sizes. The results obtained are encouraging and demonstrate the effectiveness and robustness of our proposed method. Importantly, the results obtained are not influenced by the number of indices. Consequently, this approach holds the potential for extension to resolve bi-objective fixed charge transportation problems with additional indices across diverse environments.

Perspectives

Our perspectives can be summarized as follows:

- Extending both metaheuristic algorithms and approximation method used in this thesis to solve capacitated fixed charge transportation problems with more than four indices in various environments.
- Proposing other metaheuristic algorithms to solve the fuzzy four-index fixed charge transportation problem.
- Extending the approach used to solve the second model for addressing multi-objective fixed charge transportation problems in various environments.

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ملخص

نهتم في هذه الأطروحة بدراسة وحل نموذجين لمسألة النقل، يلبيان احتياجات المستخدمين في العالم الملموس ولم يسبق دراستهما من قبل وهما: مسألة النقل الضبابية ذات كلفة ثابتة وأربعة مؤشرات ومسألة النقل الضبابية ثنائية الهدف ذات كلفة ثابتة وأربعة مؤشرات. لحل النموذج الأول، نقترح امتدادا لطريقة بالانسكي وتكييفًا في سياق ضبابي وأربعة مؤشرات لبعض طرق الأدلة العليا (Metaheuristic) تتبع بدراسة عددية مقارنة بين هذه الخوارزميات. لحل النموذج الثاني، نقترح طريقة جديدة تعتمد أساسا على طريقة كورانا وآخرين وطريقة زيتوني وآخرين. قصد اختبار فعالية ومتانة هذه الطريقة، قمنا بإنجاز عدة تجارب عددية مختلفة الأحجام وكانت النتائجُ مرضية للغاية.

كلمات دلالية: البرمجة الخطية، الرياضيات الضبابية، مسألة النقل ذات كلفة ثابتة، مسألة النقل متعددة المؤشرات، مسألة النقل ثنائية الهدف، طرق الأدلة العليا.

Abstract:

In this thesis, we are interested in the study and resolution of two transportation models not previously addressed and responding to the needs of users in the concrete world: the fuzzy four-index fixed charge transportation problem and the fuzzy bi-objective four index fixed charge transportation problem. To solve the first model, we propose an extension of M. L. Balinski's method and some metaheuristics adapted to this problem, followed by a numerical comparative study between these algorithms. For solving the second model, we introduce a new approach based mainly on the method of Khurana et al. and that of Zitouni et al., followed by different numerical experiments in the aim of testing the effectiveness and the robustness of this approach. Results obtained are very satisfactory.

Keywords: Linear programming, Fuzzy mathematics, Fixed charge transportation problem, Multi-index transportation problem, Multi-objective optimization, Metaheuristics.

Résumé :

Dans cette thèse, nous nous sommes intéressée à l'étude et la résolution de deux modèles de transport non traités auparavant et répondant au besoin des utilisateurs dans le monde concret : le problème de transport flou avec charge fixe à quatre indices et le problème de transport flou bi-objectif avec charge fixe à quatre indices. Pour la résolution du premier modèle, nous proposons une extension de la méthode de M. L. Balinski et quelques métaheuristiques adaptés à ce problème, suivies par une étude numérique comparative entre ces algorithmes. Pour la résolution du deuxième modèle, nous introduisons une nouvelle approche basée essentiellement sur la méthode de Khurana et al. et celle de Zitouni et al., suivie par des différentes expérimentations numériques afin de tester l'efficacité et la robustesse de cette approche. Les résultats obtenus sont très satisfaisants.

Mots-Clés : Programmation linéaire, Mathématiques floues, Problème de transport avec charge fixe, Problème de transport à indices multiples, Optimisation multi-objectifs, Métaheuristiques.