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Analytical and Numerical Treatment of Nonlinear Fractional Differential Equations Involving Caputo Fractional Operator

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بسم الله الرحمن الرحيم

شكر وتقدير

أولا و قبل كل شيء أود أن أشكر وأحمد الله العظيم الحميد، الذي منحني الصبر والمثابرة لإنجاز وإتمام هذه الأطروحة.

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لا أجد كلمات للتعبير عن امتناني للجنة التحكيم لقراءتهم للمخطوطة وأيضا لوقتهم الثمين وجهودهم المكرسة لتقييم عملي.

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Dedication

I dedicate this work,

To every researcher who loves science, he illuminated the mind of others with his knowledge.

To those who are unmatched by anybody in the universe and to whom God has Instructed us to show homage, who have always encouraged, helped and lavished me with supplications, and happiness, to whom I owe a great debt of gratitude to my dear mother and father.

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To my best friend, owner of the words that helped me achieve success.

To all my kind friends and all my beautiful family.

To every person who his advices have lighted my way and all who made an effort to help me.

Table of contents

Introduction

In mathematics, fractional calculus is a branch of analysis, which studies the generalization of derivation and integration from integer order (ordinary) to non-integer order (fractional). Fractional derivation theory is a subject almost as old as classical calculus as we know it today, its origins dating back to the late 17^{th} century, the time when Isaac Newton and Gottfried Wihelm Leibniz developed the foundations of differential and integral calculus. In particular, Leibniz introduced the symbol $\frac{d^n f}{dt^n}$ to denote the n^{th} derivative of a function f when he announced in a letter to Guillaume de L'Hôpital dated September 30, 1695, with the implicit assumption that $n \in \mathbb{N}$, L'Hôpital replied: What does $\frac{d^n f}{dt^n}$ mean if $n = \frac{1}{2}$?. Leibniz replied: "This would lead to a paradox from which one day we will be able to draw useful consequences" [**51**]. This letter from L'Hôpital is today accepted as the first incident of what we call fractional derivation, and the fact that L'Hôpital specifically requested for $n = \frac{1}{2}$, i.e. a fraction (rational number), actually gave rise to the name of this field of mathematics.

Systems described by fractional order models using fractional differential equations based on the non-integer derivative have attracted the interest of the scientific community. Engineers have only realized the importance of non-integer order differential equations in the last three decades, especially when they observed that the description of some systems is more exact than when the fractional derivative is used.

The credit for the first conference is given to B. Ross who organized this conference at the University of New Haven in June 1974 under the title "Fractional Calculus and Its Applications". For the first study, another credit is given to K. B. Oldham and J. Spanier [**51**] who published a book in 1974 after a joint collaboration, started in 1968 and devoted to the presentation of methods and applications of fractional calculus in physics and engineering. Since then, fractional calculus has gained popularity and significant consideration mainly due to the numerous applications in various fields of applied sciences and engineering where it has been noticed that the behavior of a large number of physical systems can be described using the fractional derivative which provides an excellent instrument for the description of many properties of materials and processes **[9]**,**[10]**,**[21]**,[**45**],[**46**],[**53**].

In recent years, nonlinear fractional differential equations have attracted the attention of many researchers due to a wide range of applications in many fields of physics, fluid mechanics, electrochemistry, viscoelasticity, nonlinear control theory, nonlinear biological systems, hydrodynamics and other fields of science and engineering **[2]**,**[7]**,**[23]**,**[24]**,**[39]**,**[41]**,**[42]**,**[50]**. In all these scientific fields, it is important to find exact or approximate solutions to these problems. There is therefore a strong interest in developing methods for solving problems related to nonlinear fractional differential equations. The exact solutions of these problems are sometimes too complicated to achieve by classical techniques due to the complexity of the nonlinear parts involving them.

The main objective of this thesis is to present new analytical and numerical methods for solving nonlinear fractional differential equations where the fractional derivative is in the sense of Caputo.

This thesis is divided into five chapters as follows:

In the first chapter, we present some examples of applications of the theory of fractional calculus in certain scientific fields, then we recall the basic notions related to the theory of fractional calculus that will be needed in the rest of this work such as the gamma function and the Mittag-Leffler function which plays an important role in the theory of fractional differential equations as well as the fixed point theorem. Two approaches (Riemann-Liouville and Caputo) to the generalization of the notions of derivation will then be considered.

In the second chapter, we will address the question of the existence and uniqueness of the solution for a Cauchy problem of a frational differential equation with fractional derivative in the sense of Caputo of the form

$$\begin{cases} {}^{C}D^{\alpha}y(t) = f(t, y(t)), t \in \Omega = [0, T], n-1 < \alpha < n, \\ y^{(k)}(0) = b_k \in \mathbb{R}, \quad k = 0, 1, ..., n-1, n = [\alpha] + 1. \end{cases}$$

The existence and uniqueness of a continuous solution is established by transforming this problem to an equivalent integral equation, whose solution is identified with a fixed point of a contracting operator (under certain sufficient assumptions on the function f) in a suitably chosen functional space. We conclude this chapter with two illustrative examples.

In the third chapter, we describe some semi-analytical methods: the Adomian decomposition method (ADM), the homotopy perturbation method (HPM), the variational iteration method (VIM), the new iterative method (NIM), then we study the convergence of each of these methods. These methods are applied to classical nonlinear differential equations (of integer order).

In the fourth chapter, we study the ADM, HPM, VIM and NIM for applications on nonlinear fractional differential equations. Furthermore, we present different numerical examples to illustrate the efficiency and accuracy of these methods.

In the fifth chapter, we propose a new hybrid method called Khalouta differential transform method which is a combination of two powerful methods: Khalouta transform method and differential transform method to solve a certain class of nonlinear fractional differential equations namely nonlinear fractional Liénard equation of the form

$$^{C}D^{\alpha}y(t) + ay(t) + by^{3}(t) + cy^{5}(t), t > 0,$$

with the initial conditions

$$y(0) = y_0, y'(0) = y_1,$$

where $^{C}D^{\alpha}$ is the fractional derivative operator in the sense of the Caputo of order α with $1 < \alpha \leq 2$, and a, b, c, y_0, and y_1 are constants.

Furthermore, we prove the convergence theorem of this method under appropriate conditions. Then we provide two numerical examples to show the efficiency and precision of the proposed method.

Finally, we end our work with a general conclusion, where the validity and reliability of such research is highlighted, also we propose some perspectives on the subject.

Chapter 1

Basic concepts of fractional calculus

This chapter will be devoted to the basic definitions and concepts related to fractional calculus such as specific functions for fractional integration, fractional derivation, and other concepts that we will need in the rest of our work. We will begin by presenting some examples of applications of the theory of fractional calculus in certain scientific fields.

1.1 Applications of fractional systems

Fractional systems are increasingly appearing in various fields of research. However, the progressive interest in these systems is their applications in fundamental and applied sciences. It can be noted that for the majority of the fields presented below, fractional operators are used to take into account memory effects. Let us mention the works **[36]**,**[60]** which group together various applications of fractional calculus.

1.1.1 Automatic

In automatic, a few authors have used control laws introducing fractional derivatives. Podlubny [**57**] showed that the best method to ensure efficient control of fractional systems is the use of fractional controllers. He proposes a generalization of traditional controllers PID. The CRONE group, founded by Oustaloup in the 70s, applies these methods to many industrial systems: spectroscope, car suspension [**54**], robot-picker,electro-hydraulic plow, car battery, etc...

1.1.2 Physics

One of the most remarkable applications of fractional calculus in physics was in the context of classical mechanics. Fred Riewe [**59**] showed that the Lagrangian containing time derivatives of fractional orders leads to an equation of motion with nonconservative forces such as friction. This result is remarkable since frictional forces and nonconservative forces are essential in the usual macroscopic variational treatment, and therefore, in the most advanced methods of classical mechanics. Fred Riewe generalized the usual calculus of variations to the Lagrangian that depends on fractional derivatives [**58**] in order to deal with the usual nonconservative forces. On the other hand, several approaches have been developed to generalize the principle of least action and the Euler-Lagrange equation to the case of fractional derivatives [**5**],[**6**].

1.1.3 Mechanics of continuous media

The deformation of continuous media (solid or liquid) is often described using two tensors,that of the deformations noted ε_{ij} and that of the constraints σ_{ij}. Some materials, such as polymers (erasers, rubber,...), exhibit an intermediate behavior between viscous and elastic characteristics, called viscoelastic. Such systems can be modeled using the following relation between the two tensors

$$\sigma_{ij} = E\varepsilon_{ij}(t) + \eta D^{\alpha}\varepsilon_{ij}(t), 0 < \alpha < 1.$$

This law is justified by Bagley and Torvik in [**9**],[**10**] (for $\alpha = \frac{1}{2}$). In [**55**], the introduction of fractional derivatives in the case of polymers is motivated by the following analysis: due to the length of the fibers, the applied deformations take time to be communicated from step by step (the length of the wound fibers being much greater than the geometric distance). They are progressively damped and induce memory effects (the state at time t will depend on previous states). If the constraint decreases as $t^{-(1+\alpha)}$, it can induce a fractional derivative of order α. This operator thus makes it possible to give a simple macroscopic description (requiring only a few parameters) of complex microscopic phenomena. A presentation of viscoelasticity via fractional derivation is given in [**22**].

1.1.4 Acoustic

For some wind musical instruments visco-thermal losses can be modeled effectively using time fractional derivatives **[35]**.

1.2 Functional spaces

In this part, we present a preliminary in which we recall fundamental notions and results of the theory of functional analysis which represent an essential tool in the theory of fractional calculus.

1.2.1 Spaces of integrable functions

Definition 1.2.1 *[13] Let* $\Omega = [0, T]$ $(0 < T < +\infty)$ *a finite interval of* $\mathbb{R}$ *and* $1 \leq p \leq \infty$.

1) For $1 \leq p < \infty$, *the space* $L^p(\Omega)$ *is the space of real functions* y *on* Ω *such that* y *is measurable and*

$$\int_0^T |y(t)|^p \, dt < \infty.$$

2) For $p = \infty$, *the space* $L^\infty(\Omega)$ *is the space of measurable functions* y *bounded almost everywhere (a.e) on* Ω.

Theorem 1.2.1 *[13] Let* $\Omega = [0, T]$ $(0 < T < +\infty)$ *a finite interval of* $\mathbb{R}$.

1) For $1 \leq p < \infty$, *the space* $L^p(\Omega)$ *is Banach space with the norm*

$$\|y\|_p = \left(\int_0^T |y(t)|^p \, dt \right)^{1/p} < \infty.$$

2) The space $L^\infty(\Omega)$ *is Banach space with the norm*

$$\|y\|_\infty = \inf \left\{ M \geq 0 : |y(t)| \leq M \ \ a.e \ on \ \Omega \right\}.$$

1.2.2 Spaces of continuous and absolutely continuous functions

Definition 1.2.2 *[43] Let $\Omega = [0, T]$ $(0 < T < +\infty)$ a finite interval of $\mathbb{R}$ and $n \in \mathbb{N}$.*

We denote by $C^n(\Omega)$ the space of functions y which have their derivatives of order less than or equal to n continues on Ω, equipped with the norm

$$\|y\|_{C^n(\Omega)} = \sum_{k=0}^{n} \left\|y^{(k)}\right\|_{C(\Omega)} = \sum_{k=0}^{n} \max_{t\in\Omega} \left|y^{(k)}(t)\right|, \ n \in \mathbb{N}.$$

In particular if $n = 0, C^0(\Omega) = C(\Omega)$ the space of continuous functions y on Ω equipped with the norm

$$\|y\|_{C(\Omega)} = \max_{t\in\Omega} |y(t)| .$$

Definition 1.2.3 *[43] Let $\Omega = [0, T]$ $(0 < T < +\infty)$ a finite interval of $\mathbb{R}$.*

We denote by $AC(\Omega)$ the space of primitive functions of integrable functions i.e.

$$AC(\Omega) = \left\{ y / \exists \varphi \in L^1(\Omega) : y(t) = c + \int_0^t \varphi(s) ds \right\},$$

and we call $AC(\Omega)$ the space of absolutely continuous functions on Ω.

Definition 1.2.4 *[43] For $n \in \mathbb{N}^*$ we denote by $C^n_\mu(\Omega)$ the space of functions y which have continuous derivatives on Ω up to order $(n-1)$ and such that $y^{(n-1)} \in AC(\Omega)$ i.e.*

$$AC^n(\Omega) = \{y : \Omega \longrightarrow \mathbb{C}, \ y^{(k)} \in C(\Omega), k \in \{0, 1, 2...n-1\}, \ y^{(n-1)} \in AC(\Omega)\}.$$

In particular if $AC^1(\Omega) = AC(\Omega)$.

A characterization of the functions of this space is given by the following lemma.

Lemma 1.2.1 *[43] A function $f \in AC^n(\Omega), n \in \mathbb{N}^*$, if and only if it is represented in the form*

$$y(t) = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} y^{(n)}(\tau) d\tau + \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} t^k.$$

1.2.3 Spaces of continuous functions with weight

Definition 1.2.5 *[43] Let $\Omega = [0, T]$ $(0 < T < +\infty)$ a finite interval of $\mathbb{R}$ and $\mu \in \mathbb{C}$ $(0 \leq \operatorname{Re}(\mu) < 1)$.*

We denote by $C_\mu(\Omega)$ the space of functions y defined on Ω such that the function $t^\mu y(t) \in C(\Omega)$ i.e.

$$C_\mu(\Omega) = \{y : \Omega \longrightarrow \mathbb{C},\ (.)^\mu y(.) \in C(\Omega)\},$$

equipped with the norm

$$\|y\|_{C_\mu(\Omega)} = \|t^\mu y(t)\|_{C(\Omega)} = \max_{t\in\Omega} |t^\mu y(t)|.$$

The space $C_\mu(\Omega)$ is called the space of continuous functions with weight.

In particular, $C_0(\Omega) = C(\Omega)$.

Definition 1.2.6 *[43] For $n \in \mathbb{N}^*$ we denote by $C_\mu^n(\Omega)$ the space of functions y which have continuous derivatives on Ω up to order $(n-1)$, such that $y^{(n)} \in C_\mu(\Omega)$, i.e.*

$$C_\mu^n(\Omega) = \{y : \|y\|_{C_\mu^n(\Omega)} = \sum_{k=0}^{n-1} \left\|y^{(k)}\right\|_{C(\Omega)} + \left\|y^{(n)}\right\|_{C_\mu(\Omega)}\}.$$

In particular $C_\mu^0(\Omega) = C_\mu(\Omega)$.

1.2.4 Banach fixed point theorem

Definition 1.2.7 *Let X be a Banach space,and $T : X \longrightarrow X$ a continuous map, we say that T is contracting if T is Lipschitzian with ratio $K < 1$, i.e.*

$$\exists k < 1 : \forall u, v \in X : \|T(u) - T(v)\| \leq K \|u - v\|.$$

Theorem 1.2.2 ***(Banach)** [27] Let X be a Banach space and $T : X \longrightarrow X$ a contracting operator, then T admits a unique fixed point, i.e. $\exists! y^* \in X$ such that*

$$Ty^* = y^*.$$

Furthermore, if $T^k, k \in \mathbb{N}$ is a sequence of operators defined by

$$T^1 = T \text{ and } T^k = TT^{k-1}, k \in \mathbb{N} \setminus \{1\},$$

then for all $y_0 \in X$ the sequence $\left\{T^k y_0\right\}_{k=0}^{\infty}$ converges to the fixed point u^* and we have

$$\lim_{k\to\infty} \left\|T^k y_0 - y^*\right\| = 0.$$

1.3 Specific functions for fractional derivation

In this part, we present the Gamma, Beta and Mittag-Leffler functions. These functions play a very important role in the theory of fractional calculus and its applications.

1.3.1 Gamma function

One of the basic functions of fractional calculus is **Euler's Gamma function** $\Gamma(z)$ which naturally extends the factorial to positive real numbers (and even to complex numbers with positive real parts).

Definition 1.3.1 *[56] For $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > 0$. The Gamma function $\Gamma(z)$ is defined by the following integral*

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt, \tag{1.3.1}$$

with $\Gamma(1) = 1$, $\Gamma(0^+) = +\infty$, $\Gamma(z)$ is a monotonic and strictly decreasing function for $0 < z < 1$.

An important property of the Gamma function $\Gamma(z)$ is the following recurrence relation

$$\Gamma(z+1) = z\Gamma(z), \;\; Re(z) > 0, \tag{1.3.2}$$

that it can be demonstrated by integration by parts

$$\Gamma(z+1) = \int_0^{+\infty} e^{-t} t^{z} dt = \left[-e^{-t} t^{z}\right]_0^{+\infty} + z \int_0^{+\infty} e^{-t} t^{z-1} dt = z\Gamma(z).$$

Euler's Gamma function generalizes the factorial because $\Gamma(n+1) = n!$, $\forall n \in \mathbb{N}$, indeed $\Gamma(1) = 1$ and using (1.3.2) we obtain

$$\begin{aligned}
\Gamma(2) &= 1\Gamma(1) = 1!, \\
\Gamma(3) &= 2\Gamma(2) = 2.1! = 2!, \\
\Gamma(4) &= 3\Gamma(3) = 3.2! = 3!, \\
&\vdots \\
\Gamma(n+1) &= n\Gamma(n) = n(n-1)! = n!.
\end{aligned}$$

1.3.2 Beta function

It is one of the basic functions of fractional calculus. This function plays an important role when combined with the Gamma function.

Definition 1.3.2 *[56] The Beta function is a type of Euler integral defined for complex numbers z and w by*

$$B(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}dt, \mathrm{Re}(z)>0, \mathrm{Re}(w)>0. \tag{1.3.3}$$

The Beta function is related to the Gamma function by the following relation

$$B(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, Re(z)>0, Re(w)>0, \tag{1.3.4}$$

it follows from (1.3.4) that

$$B(z,w)=B(w,z), Re(z)>0, Re(w)>0.$$

1.3.3 Mittag-Leffler function

The Mittag-Leffler function plays a very important role in the theory of differential equations of integer order. It is also widely used in the search for solutions of differential equations of fractional order, this function was introduced by G.M. Mittag-Leffler **[47]**,**[48]**.

Definition 1.3.3 *[56] For $z\in\mathbb{C}$, the Mittag-Leffler function $E_\alpha(z)$ is defined as follows*

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \alpha>0. \tag{1.3.5}$$

In particular

$$E_1(z)=e^z,\ E_2(z)=\cosh(\sqrt{z}).$$

This function can be generalized for two parameters to give

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \alpha>0,\ \beta>0. \tag{1.3.6}$$

1.4 Fractional integrals and derivatives

The aim of this part is to introduce the two most important approaches to fractional calculus: in the Riemann-Liouville sense and in the Caputo sense, including some of their properties as well as the relationship between these two approaches.

1.4.1 Fractional integral in the Riemann-Liouville sense

The notion of fractional integral of order $\alpha \in \mathbb{C}$ $(Re(\alpha) > 0)$, according to the Riemann-Liouville approach, generalizes the famous formula (attributed to Cauchy) of integral repeated $n-$times.

Let y a continuous function on the interval $[0, T], T > 0$. A primitive of y is given by the expression

$$I^1 y(t) = \int_0^t y(\tau) d\tau.$$

For a second primitive and according to Fubini's theorem we will have

$$\begin{aligned} I^2 y(t) &= \int_0^t I^1 y(u) du = \int_0^t \left(\int_0^u y(\tau) d\tau \right) du = \int_0^t \left(\int_\tau^t du \right) y(\tau) d\tau \\ &= \int_0^t (t-\tau)\, y(\tau) d\tau. \end{aligned}$$

By repeating $n-$times, we arrive at the n^{th} primitive of the function y in the form

$$I^n y(t) = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} y(\tau) d\tau, t > 0, n \in \mathbb{N}^*. \tag{1.4.1}$$

This formula is called Cauchy's formula, and since the generalization of the factorial by the Gamma function $\Gamma(n) = (n-1)!$, Riemann realized that the right-hand side of (1.4.1) could make sense even when n takes a non-integer value, he defined the fractional integral as follows

Definition 1.4.1 *[43],[56] Let $y \in L^1([0,T]), T > 0$. The Riemann-Liouville fractional integral of the function y of order $\alpha \in \mathbb{C}$ $(\mathrm{Re}(\alpha) > 0)$ denoted I^α is defined by*

$$I^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} y(\tau) d\tau, \ t > 0, \tag{1.4.2}$$

where $\Gamma(\alpha)$ is the Gamma function given by (1.3.1).

Theorem 1.4.1 *[43],[56] If $y \in L^1([0,T]), T > 0$, then $I^\alpha y$ exists for almost all $t \in [0,T]$ and moreover $I^\alpha y \in L^1([0,T])$.*

Proof. By introducing (1.4.2) and then using Fubini's theorem, we find

$$\begin{aligned}
\int_0^t |I^\alpha y(t)| \, dt &\leq \frac{1}{\Gamma(\alpha)} \int_0^T \int_0^t (t-\tau)^{\alpha-1} |y(\tau)| \, d\tau dt \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^T |y(\tau)| \left(\int_\tau^T (t-\tau)^{\alpha-1} \, dt \right) d\tau \\
&\leq \frac{1}{\alpha\Gamma(\alpha)} \int_0^T |y(\tau)| \, (T-\tau)^\alpha d\tau \\
&\leq \frac{T^\alpha}{\Gamma(\alpha+1)} \int_0^T |y(\tau)| \, d\tau.
\end{aligned}$$

Since $y \in L^1([0,T])$, the last quantity is finite, which establishes the result. ■

Example 1.4.1. The integral of $y(t) = t^\beta$ in the Riemann-Liouville sense.

Let $\alpha > 0, \beta > -1$, then we have

$$I^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \tau^\beta d\tau. \tag{1.4.3}$$

By performing the variable change

$$\tau = ts,$$

where $s=0$ when $\tau=0$ and $s=1$ when $\tau=t$ and $d\tau=tds$, then (1.4.3) becomes

$$\begin{aligned}I^{\alpha}y(t) &= \frac{1}{\Gamma(\alpha)}\int_0^1 (t-ts)^{\alpha-1}(ts)^{\beta}\,tds\\ &= \frac{1}{\Gamma(\alpha)}\int_0^1 [t(1-s)]^{\alpha-1}t^{\beta+1}s^{\beta}ds\\ &= \frac{t^{\alpha+\beta}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}s^{\beta}ds\\ &= \frac{t^{\alpha+\beta}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}s^{(\beta+1)-1}ds.\end{aligned}$$

Using the definition of the Beta function (1.3.3) then the relation (1.3.4), we arrive at

$$\begin{aligned}I^{\alpha}y(t) &= \frac{t^{\alpha+\beta}}{\Gamma(\alpha)}B(\alpha,\beta+1)\\ &= \frac{t^{\alpha+\beta}}{\Gamma(\alpha)}\frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\\ &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}.\end{aligned}$$

So the fractional integral in the Rieman-Liouville sense of the function $y(t)=t^{\beta}$ is given by

$$I^{\alpha}t^{\beta}=\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}. \tag{1.4.4}$$

In particular, relation (1.4.4) shows that the fractional integral in the Riemann-Liouville sense of order α of a constant $c\in\mathbb{R}$ is given by

$$I^{\alpha}c=\frac{c}{\Gamma(\alpha+1)}t^{\alpha}.$$

Proposition 1.4.1 *[43],[56] Let $\alpha,\beta\in\mathbb{C}$ ($\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$), for any function $y\in L^1([0,T]), T>0$ we have*

$$I^{\alpha}\left(I^{\beta}y(t)\right)=I^{\alpha+\beta}y(t)=I^{\beta}\left(I^{\alpha}y(t)\right),$$

for almost all $t\in[0,T]$. If in addition $y\in C([0,T])$, then this identity is true $\forall t\in[0,T]$.

Proof. Let us first assume that $y \in L^1([0,T])$ we have

$$\begin{aligned} I^\alpha\left(I^\beta y(t)\right) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} I^\beta y(\tau)d\tau \\ &= \frac{1}{\Gamma(\beta)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\int_0^t (\tau-\zeta)^{\beta-1}y(\zeta)d\zeta d\tau. \end{aligned}$$

According to Theorem 1.4.1, the integrals appearing in the previous equality exist for almost all $t \in [0,T]$, and thus Fubini's theorem allows us to establish

$$I^\alpha\left(I^\beta y(t)\right) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t \left(\int_\zeta^t (t-\tau)^{\alpha-1}(\tau-\zeta)^{\beta-1}d\tau\right) y(\zeta)d\zeta.$$

By making the variable change

$$\tau = \zeta + (t-\zeta)s,$$

where $s=0$ when $\tau=\zeta$ and $s=1$ when $\tau=t$ and $d\tau=(t-\zeta)ds$, we obtain

$$I^\alpha\left(I^\beta y(t)\right) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t (t-\zeta)^{\alpha+\beta-1}\left(\int_0^1 (1-s)^{\alpha-1}s^{\beta-1}ds\right) y(\zeta)d\zeta.$$

Finally, taking into account the definition of the Beta function (1.3.3) then the relation (1.3.4), we obtain

$$I^\alpha\left(I^\beta y(t)\right) = \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-\zeta)^{\alpha+\beta-1} y(\zeta)d\zeta = \left(I^{\alpha+\beta}y\right)(t).$$

Now suppose that $y \in C([0,T])$, then (by the theorems on integrals depending on parameters) $I^\alpha y \in C([0,T])$, and consequently

$$I^{\alpha+\beta}y, I^\alpha I^\beta y \ \in C([0,T]).$$

Thus, from the above, the two continuous functions $I^{\alpha+\beta}y, I^\alpha I^\beta y$ coincide almost everywhere on $[0,T]$, so they must coincide everywhere on $[0,T]$. ■

The following theorem provides a result concerning the inversion of the limit and the fractional integral.

Theorem 1.4.2 *[43],[56] Let $\alpha \in \mathbb{C}$ $(\mathrm{Re}(\alpha) > 0)$ and $(y_k)_{k=1}^{+\infty}$ be a sequence of continuous and simply convergent functions on $[0, T]$. Then we can invert the fractional integral in the Riemann-Liouville sense and the limit sign as follows*

$$\left[I^{\alpha}(\lim_{k \longrightarrow +\infty} y_k) \right] (t) = \lim_{k \longrightarrow +\infty} I^{\alpha} y_k(t).$$

Proof. Let $y_k \longrightarrow y$ simply converge and

$$\begin{aligned} |I^{\alpha} y_k(t) - I^{\alpha} y(t)| &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |y_k(\tau) - y(\tau)| \, d\tau \\ &\leq \frac{\|y_k - y\|_{\infty}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \, d\tau \\ &\leq \frac{\|y_k - y\|_{\infty}}{\Gamma(\alpha)} \frac{1}{\alpha} t^{\alpha} \\ &\leq \frac{t^{\alpha}}{\alpha \Gamma(\alpha)} \|y_k - y\|_{\infty} \\ &\leq \frac{T^{\alpha}}{\Gamma(\alpha+1)} \|y_k - y\|_{\infty} \underset{k \longrightarrow +\infty}{\longrightarrow} 0. \end{aligned}$$

Hence the desired result. ∎

1.4.2 Fractional derivative in the Riemann-Liouville sense

Definition 1.4.2 *[43],[56] Let $y \in L^1([0, T]), T > 0$ be an integrable function on $[0, T]$, the fractional derivative in the Riemann-Liouville sense of the function y of order $\alpha \in \mathbb{C}$ $(\mathrm{Re}(\alpha) > 0)$ denoted $D^{\alpha} y$ is defined by*

$$\begin{aligned} D^{\alpha} y(t) &= D^n I^{n-\alpha} y(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t (t-\tau)^{n-\alpha-1} y(\tau) d\tau, t > 0. \end{aligned} \qquad (1.4.5)$$

where $n - 1 < [\mathrm{Re}(\alpha)] < n$ and $[\mathrm{Re}(\alpha)]$ is the integer part of $\mathrm{Re}(\alpha)$.

In particular, if $\alpha = 0$, then

$$D^0 y(t) = \frac{1}{\Gamma(1)} \left(\frac{d}{dt} \right) \int_0^t y(\tau) d\tau = y(t).$$

If $\alpha = n \in \mathbb{N}$, then

$$D^n y(t) = \frac{1}{\Gamma(1)} \left(\frac{d^{n+1}}{dt^{n+1}}\right) \int_0^t y(\tau)d\tau = \frac{d^n}{dt^n} y(t) = y^{(n)}(t).$$

Consequently the fractional derivative in the Riemann-Liouville sense coincides with the classical derivative for $\alpha \in \mathbb{N}$.

If furthermore $0 < \alpha < 1$, then $n = 1$, hence

$$D^\alpha y(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} y(\tau)d\tau, \ t > 0.$$

Example 1.4.2. The derivative of $y(t) = t^\beta$ in the Riemann-Liouville sense.

Let $\alpha > 0$ such that $n-1 < \alpha < n$ and $\beta > -1$, according to (1.4.5) and the relation (1.4.4), (See Example 1.4.1) we have

$$D^\alpha t^\beta = D^n I^{n-\alpha} t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+n-\alpha+1)} D^n t^{\beta+n-\alpha}. \tag{1.4.6}$$

Taking into account

$$\begin{aligned} D^n t^{\beta+n-\alpha} &= (\beta+n-\alpha)(\beta+n-\alpha-1).......(\beta-\alpha+1)t^{\beta-\alpha} \\ &= \frac{\Gamma(\beta+n-\alpha+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}. \end{aligned} \tag{1.4.7}$$

We substitute the result (1.4.7) into formula (1.4.6) to get

$$\begin{aligned} D^\alpha t^\beta &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+n-\alpha+1)} \frac{\Gamma(\beta+n-\alpha+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}. \end{aligned}$$

So the fractional derivative in the Riemann-Liouville sense of the function $y(t) = t^\beta$ is given by

$$D^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}. \tag{1.4.8}$$

In particular, if $\beta = 0$ and $\alpha > 0$, the Riemann-Liouville fractional derivative of a constant function $y(t) = c \in \mathbb{R}$ is non-zero, its value is

$$D^\alpha c = \frac{c}{\Gamma(1-\alpha)} t^{-\alpha}.$$

It is easy to establish the following result.

Lemma 1.4.1 *Let $n-1<\alpha<n, n=[\alpha]+1$ and y be a function that satisfies*

$$D^{\alpha}y(t)=0,$$

then

$$y(t)=\sum_{j=0}^{n-1}c_j\frac{\Gamma(j+1)}{\Gamma(j+1+\alpha-n)}t^{j+\alpha-n}, \forall c_0,c_1,...,c_{n-1}\in\mathbb{R}.$$

In particular, if $0<\alpha<1$, then

$$y(t)=\frac{c}{\Gamma(\alpha)}t^{\alpha-1}, \forall c\in\mathbb{R}.$$

Proof. Let $D^{\alpha}y(t)=0$, according to (1.4.5) we have

$$D^{\alpha}y(t)=D^{n}I^{n-\alpha}y(t)=0.$$

And consequently

$$I^{n-\alpha}y(t)=\sum_{j=0}^{n-1}c_jt^j. \tag{1.4.9}$$

Now, applying operator I^{α} to equation (1.4.9) gives

$$I^{n}y(t)=\sum_{j=0}^{n-1}c_jI^{\alpha}t^j.$$

Using relation (1.4.4) (See Example 1.4.1), we obtain

$$I^{n}y(t)=\sum_{j=0}^{n-1}c_j\frac{\Gamma(j+1)}{\Gamma(j+\alpha+1)}t^{j+\alpha}. \tag{1.4.10}$$

Applying the operator D^n to equation (1.4.10) gives

$$y(t)=\sum_{j=0}^{n-1}c_j\frac{\Gamma(j+1)}{\Gamma(j+\alpha+1)}D^{n}t^{j+\alpha}.$$

Finally, the classical derivation and use of the formula

$$D^{n}t^{\alpha}=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)}t^{\alpha-n},$$

gives

$$y(t)=\sum_{j=0}^{n-1}c_j\frac{\Gamma(j+1)}{\Gamma(j+\alpha+1)}\frac{\Gamma(j+\alpha+1)}{\Gamma(j+\alpha-n+1)}t^{j+\alpha-n}.$$

Finally we get

$$y(t)=\sum_{j=0}^{n-1} c_j \frac{\Gamma(j+1)}{\Gamma(j+1+\alpha-n)} t^{j+\alpha-n}.$$

This completes the proof of the Lemma. ■

The following proposition establishes a sufficient condition for the existence of the fractional derivative.

Proposition 1.4.2 *[43] Let $\alpha \geq 0$ and $n=[\alpha]+1$. If $y \in AC^n([0,T]), T>0$, then the fractional derivative $D^\alpha y$ exists almost everywhere on $[0,T]$ moreover, it is represented in the form*

$$D^\alpha y(t)=\sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{\Gamma(k-\alpha+1)} t^{k-\alpha}+\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} y^{(n)}(\tau) d\tau. \tag{1.4.11}$$

1.4.3 Some properties of fractional derivation in the sense of Riemann-Liouville

The Riemann-Liouville derivation operator has the properties summarized in the following propositions.

Proposition 1.4.3 *[43],[56] For $n-1<\alpha \leq n$, $m-1<\beta \leq m$ we have*

1) The Riemann-Liouville operator is linear

$$D^\alpha(\lambda y+z)(t)=\lambda(D^\alpha y)(t)+(D^\alpha z)(t), \lambda \in \mathbb{R}.$$

2) In general

$$D^\alpha(D^\beta y(t)) \neq D^\beta(D^\alpha y(t)).$$

Proof. 1) Let $y, z \in L^1([0,T])$, $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} D^\alpha(\lambda y+z)(t) &= D^n I^{n-\alpha}(\lambda y(t)+z(t)) \\ &= \lambda D^n I^{n-\alpha}((y+z)(t)). \end{aligned}$$

Since the n^{th} derivative and the integral are linear, then

$$\begin{aligned} D^{\alpha}\left(\lambda y+z\right)(t) &= \lambda D^{n} I^{n-\alpha} y(t)+D^{n} I^{n-\alpha} z(t) \\ &= \lambda\left(D^{\alpha} y\right)(t)+\left(D^{\alpha} z\right)(t). \end{aligned}$$

2) We have

$$D^{\alpha}(D^{\beta}y(t))=D^{\alpha+\beta}y(t)-\sum_{k=0}^{m-1}D^{\beta-k}y(0)\frac{t^{-\alpha-k}}{\Gamma(1-\alpha-k)},$$

and

$$D^{\beta}(D^{\alpha}y(t))=D^{\alpha+\beta}y(t)-\sum_{k=0}^{n-1}D^{\alpha-k}y(0)\frac{t^{-\beta-k}}{\Gamma(1-\beta-k)}.$$

Hence, the two fractional derivative operators commute only if $\alpha=\beta$ and $D^{\alpha-k}y(0)=0$, for all $k=1,2,...,n$ and $D^{\beta-k}y(0)=0$, for all $k=1,2,...,m$.

This completes the proof. ■

Proposition 1.4.4 *[43],[56] Let $\alpha,\beta>0$ such that $n-1<\alpha\leq n$, $m-1<\beta\leq m$ with $n,m\in\mathbb{N}^{*}$.*

1) For $y\in L^{1}([0,T]),T>0$, the equality

$$D^{\alpha}\left(I^{\alpha}y(t)\right)=y(t),$$

is true for almost all $t\in[0,T]$.

2) If $\alpha>\beta>0$, then for $y\in L^{1}([0,T]),T>0$, the relation

$$D^{\beta}(I^{\alpha}y(t))=I^{\alpha-\beta}y(t),$$

is true almost everywhere on $t\in[0,T]$.

In particular, when $\beta=k\in\mathbb{N}$ and $\alpha>k$, then

$$D^{k}(I^{\alpha}y(t))=I^{\alpha-k}y(t),$$

3) If $\beta\geq\alpha>0$ and the fractional derivative $D^{\beta-\alpha}y$ exists, then we have

$$D^{\beta}(I^{\alpha}y(t))=D^{\beta-\alpha}y(t).$$

4) For $\alpha>0$, $k\in\mathbb{N}^{*}$. If the fractional derivatives $D^{\alpha}y$ and $D^{k+\alpha}y$ exist, then

$$D^{k}(D^{\alpha}y(t))=D^{k+\alpha}y(t).$$

Proof. 1) Using (1.4.5) and Proposition (1.4.1), we have for $n = [\alpha] + 1$

$$D^{\alpha}(I^{\alpha}y(t)) = D^{n}I^{n-\alpha}I^{\alpha}y(t) = y(t), \text{ a.e on } [0,T].$$

2) From (1.4.5) and Proposition (1.4.1) we obtain

For $\alpha > \beta > 0$, then $n \geq m$, we have

$$\begin{aligned} D^{\beta}(I^{\alpha}y(t)) &= D^{n}I^{n-\beta}(I^{\alpha}y)(t) \\ &= D^{n}\left(I^{n+\alpha-\beta}y\right)(t) \\ &= D^{n}I^{n}\left(I^{\alpha-\beta}y\right)(t) \\ &= \left(I^{\alpha-\beta}y\right)(t). \end{aligned}$$

3) For $\beta \geq \alpha > 0$, we have

$$\begin{aligned} D^{\beta}(I^{\alpha}y(t)) &= D^{m}I^{m-\beta}(I^{\alpha}y)(t) \\ &= D^{m}I^{m-(\beta-\alpha)}y(t) \\ &= \left(D^{\beta-\alpha}y\right)(t), \end{aligned}$$

exists for $i-1 \leq \beta-\alpha < i$ and $i \leq m$.

4) We have for $\alpha > 0$, $k \in \mathbb{N}^{*}$

$$\begin{aligned} D^{k}(D^{\alpha}y(t)) &= D^{k}D^{n}I^{n-\alpha}y(t) \\ &= D^{k+n}I^{n-\alpha+k-k}y(t) \\ &= D^{k+n}I^{k+n-(k+\alpha)}y(t) \\ &= (D^{k+\alpha}y)(t). \end{aligned}$$

This completes the proof. ■

1.4.4 Fractional derivative in the sense of Caputo

Although the fractional derivation in the sense of Riemann-Liouville has played an important role in the development of fractional calculus, several authors including Caputo (1967-1969) have reported that this definition needs to be revised **[14]**, because the problems applied in viscoelasticity, solid mechanics and rheology, require initial conditions physically interpretable by classical derivatives such as $y(0)$, $y'(0)$, etc..., which is not the case in modeling by the Riemann-Liouville approach which requires knowledge of the initial conditions of the fractional derivatives.

Despite the fact that initial value problems with such initial conditions can be solved mathematically, the solution of this problem was proposed by M. Caputo in his definition which he adapted with F. Mainardi in the structure of the theory of viscoelasticity **[15]**.

In this part we give the definition of the fractional derivative in the sense of Caputo as well as some essential properties.

Let $[0,T]$ be a finite interval of $\mathbb{R}$ and let I^α and D^α be the fractional integration and derivation operators given by (1.4.2) and (1.4.5) respectively.

Definition 1.4.3 *[43] The Caputo fractional derivative* ${}^C D^\alpha y(t)$ *of order* $\alpha \in \mathbb{C}$ $(\mathrm{Re}(\alpha) \geq 0)$ *on the interval* $[0,T]$, *is defined via the Riemann-Liouville fractional derivative by*

$$ {}^C D^\alpha y(t) = D^\alpha \left(y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} t^k \right), \tag{1.4.12}$$

where

$$ n = [\mathrm{Re}(\alpha)] + 1 \textit{ for } \alpha \notin \mathbb{N} \textit{ and } n = \alpha \textit{ for } \alpha \in \mathbb{N}. \tag{1.4.13}$$

If $\alpha = 0$, then

$$ {}^C D^0 y(t) = y(t). $$

In particular, when $0 < Re(\alpha) < 1$, the relation (1.4.12) takes the form:

$$ {}^C D^\alpha y(t) = D^\alpha \left[y(t) - y(0) \right]. $$

The Caputo fractional derivative (1.4.12) is defined for functions $y(t)$ for which the Riemann-Liouville fractional integral (1.4.2) exists, in particular it is defined for functions $y \in AC^n([0,T])$. We have the following theorem.

Theorem 1.4.3 *[43] Let* $\mathrm{Re}(\alpha) \geq 0$ *and let* n *be given by (1.4.13). If* $y \in AC^n([0,T])$, *then the Caputo fractional derivative* ${}^C D^\alpha y(t)$ *exists almost everywhere on* $[0,T]$.

1) If $\alpha \notin \mathbb{N}$, then ${}^C D^\alpha y(t)$ is given by

$$\begin{aligned} {}^C D^\alpha y(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} y^{(n)}(\tau) d\tau, t > 0 \\ &= I^{n-\alpha} D^n y(t) \end{aligned} \tag{1.4.14}$$

In particular, when $0 < \mathrm{Re}(\alpha) < 1$ and $y \in AC([0,T])$, then

$$\begin{aligned} {}^C D^\alpha y(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} y'(\tau) d\tau, t > 0 \\ &= I^{1-\alpha} y'(t). \end{aligned} \tag{1.4.15}$$

2) If $\alpha \in \mathbb{N}$, then

$${}^C D^\alpha y(t) = y^{(n)}(t).$$

Proof. According to Definition 1.4.3, we have

$$\begin{aligned} {}^C D^\alpha y(t) &= D^\alpha \left(y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} t^k \right) \\ &= D^n I^{n-\alpha} \left(y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} t^k \right). \end{aligned}$$

By

$$\begin{aligned} D^\alpha y(t) &= D^n I^{n-\alpha} y(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t (t-\tau)^{n-\alpha-1} y(\tau) d\tau, t > 0, \end{aligned}$$

we put

$$Y(t) = I^{n-\alpha} \left(y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} t^k \right).$$

According to (1.4.2), we have

$$Y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \left(y(\tau) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} \tau^k \right) d\tau.$$

By integration by part we get

$$\begin{aligned} Y(t) &= \frac{1}{\Gamma(n-\alpha)}\int\limits_0^t (t-\tau)^{n-\alpha-1}\left(y(\tau)-\sum_{k=0}^{n-1}\frac{y^{(k)}(0)}{k!}\tau^k\right)d\tau \\ &= \frac{1}{\Gamma(n-\alpha)}\left\{-\frac{(t-\tau)^{n-\alpha}}{n-\alpha}\left(y(\tau)-\sum_{k=0}^{n-1}\frac{y^{(k)}(0)}{k!}\tau^k\right)\Bigg|_{\tau=0}^{\tau=t}\right. \\ &\quad \left.+\int\limits_0^t \frac{(t-\tau)^{n-\alpha}}{n-\alpha}D\left(y(\tau)-\sum_{k=0}^{n-1}\frac{y^{(k)}(0)}{k!}\tau^k\right)d\tau\right\} \\ &= \frac{1}{\Gamma(n-\alpha+1)}\int\limits_0^t (t-\tau)^{n-\alpha+1-1}D\left(y(\tau)-\sum_{k=0}^{n-1}\frac{y^{(k)}(0)}{k!}\tau^k\right)d\tau \\ &= I^{n-\alpha+1}D\left(y(t)-\sum_{k=0}^{n-1}\frac{y^{(k)}(0)}{k!}t^k\right). \end{aligned}$$

By repeating this process $n-$times, we find

$$\begin{aligned} Y(t) &= I^{n-\alpha+n}D^n\left(y(t)-\sum_{k=0}^{n-1}\frac{y^{(k)}(0)}{k!}t^k\right) \\ &= I^nI^{n-\alpha}D^n\left(y(t)-\sum_{k=0}^{n-1}\frac{y^{(k)}(0)}{k!}t^k\right), \end{aligned}$$

where $\sum\limits_{k=0}^{n-1}\frac{y^{(k)}(0)}{k!}t^k$ is a polynomial of degree $n-1$, therefore

$$Y(t) = I^nI^{n-\alpha}D^ny(t).$$

So

$$\begin{aligned} {}^CD^\alpha y(t) &= D^nY(t) \\ &= D^nI^nI^{n-\alpha}D^ny(t) \\ &= I^{n-\alpha}D^ny(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\int\limits_0^t (t-\tau)^{n-\alpha-1}y^{(n)}(\tau)d\tau. \end{aligned}$$

This completes the proof. ∎

Theorem 1.4.4 *[43] Let* $\mathrm{Re}(\alpha)\geq 0$ *and let* n *be given by (1.4.13) and* $y\in C^n([0,T])$. *Then the Caputo fractional derivative* ${}^CD^\alpha y(t)$ *is continuous on* $[0,T], T>0$.

1) If $\alpha \notin \mathbb{N}$, then ${}^{C}D^{\alpha}y(t)$ is given by (1.4.14). In particular, it takes the form (1.4.15) for $0 < \alpha < 1$.

2) If $\alpha \in \mathbb{N}$, then

$$ {}^{C}D^{\alpha}y(t) = y^{(n)}(t). $$

Example 1.4.3. The derivative of $y(t) = t^{\beta}$ in the sense of Caputo.

Let n be an integer and $0 \leq n-1 < \alpha < n$ with $\beta > n-1$, then according to (1.4.14) we have

$$ y^{(n)}(\tau) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)}\tau^{\beta-n}; \tag{1.4.16} $$

and

$$ {}^{C}D^{\alpha}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)}\int_{0}^{t}(t-\tau)^{n-\alpha-1}\tau^{\beta-n}d\tau. \tag{1.4.17} $$

By making the change of variable

$$ \tau = ts, $$

where $s = 0$ when $\tau = 0$ and $s = 1$ when $\tau = t$ and $d\tau = tds$, then (1.4.17) becomes

$$ \begin{aligned} {}^{C}D^{\alpha}t^{\beta} &= \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)}\int_{0}^{t}(t-\tau)^{n-\alpha-1}\tau^{\beta-n}d\tau \\ &= \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)}t^{\beta-\alpha}\int_{0}^{1}(1-s)^{n-\alpha-1}s^{\beta-n}ds. \end{aligned} $$

Using the definition of the Beta function (1.3.3) and then the relation (1.3.4) we arrive at

$$ \begin{aligned} {}^{C}D^{\alpha}t^{\beta} &= \frac{\Gamma(\beta+1)B(n-\alpha,\beta-n+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)}t^{\beta-\alpha} \\ &= \frac{\Gamma(\beta+1)\Gamma(n-\alpha)\Gamma(\beta-n+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)\Gamma(\beta-\alpha+1)}t^{\beta-\alpha} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}. \end{aligned} $$

Therefore the fractional derivative in the Caputo sense of the function $y(t) = t^{\beta}$ is given by

$$ {}^{C}D^{\alpha}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}. \tag{1.4.18} $$

In particular, the use of formula (1.4.12) or (1.4.14) to calculate the fractional derivative in the Caputo sense of order $\alpha > 0$ of a constant $c \in \mathbb{R}$ expresses that this derivative is zero, i.e.

$$^{C}D^{\alpha}c = 0.$$

1.4.5 Some properties of fractional derivation in the sense of Caputo

Fractional derivatives in the sense of Caputo have the properties summarized in the following propositions.

Proposition 1.4.5 *[43],[56] Let $\alpha \in \mathbb{C}$ such that $n-1 < \mathrm{Re}(\alpha) < n$, $n \in \mathbb{N}^*$and let the two functions $y(t)$ and $z(t)$ such that $^{C}D^{\alpha}y(t)$ and $^{C}D^{\alpha}z(t)$ exist. Caputo's fractional derivative is linear operator*

$$^{c}D^{\alpha}\left(\lambda y + z\right)(t) = \lambda\left(^{c}D^{\alpha}y\right)(t) + \left(^{c}D^{\alpha}z\right)(t),\ \ \lambda \in\ \mathbb{R}.$$

Proof. We have from (1.4.14)

$$\begin{aligned} ^{C}D^{\alpha}\left(\lambda y + z\right)(t) &= I^{n-\alpha}D^{n}\left(\left(\lambda y + z\right)(t)\right) \\ &= \lambda I^{n-\alpha}D^{n}\left(\left(y + z\right)(t)\right). \end{aligned}$$

Since the n^{th} derivative and the integral are linear, then

$$\begin{aligned} ^{C}D^{\alpha}\left(\lambda y + z\right)(t) &= \lambda I^{n-\alpha}D^{n}y(t) + I^{n-\alpha}D^{n}z(t) \\ &= \lambda\left(^{c}D^{\alpha}y\right)(t) + \left(^{c}D^{\alpha}z\right)(t). \end{aligned}$$

Hence the result. ■

Proposition 1.4.6 *[43],[56] Suppose that $n-1 < \mathrm{Re}(\alpha) < n$, $m, n \in \mathbb{N}^*$ and let the function $y(t)$ such that $^{C}D^{\alpha}y(t)$ exists, then*

$$^{C}D^{\alpha}D^{m}y(t) =^{C} D^{\alpha+m}y(t) \neq D^{m}\ ^{C}D^{\alpha}y(t).$$

1.4.6 Relation between the Riemann-Liouville approach and that of Caputo

The following theorem establishes the link between the fractional derivative in the sense of Caputo and that in the sense of Riemann-Liouville.

Theorem 1.4.5 *[43] Let* $\mathrm{Re}(\alpha) > 0$ *with* $n-1 < \mathrm{Re}(\alpha) < n$ $(n \in \mathbb{N}^*)$ *and let* y *be a function that the fractional derivatives* ${}^{C}D^{\alpha}y\,(t)$ *and* $D^{\alpha}y\,(t)$ *exist, then*

$$ {}^{C}D^{\alpha}y(t) = D^{\alpha}y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{\Gamma(k-\alpha+1)} t^{k-\alpha}. $$

Proof. We consider the limited development in Taylor series of the function y in $t = 0$

$$ \begin{aligned} y(t) &= y(0) + \frac{y'(0)}{1!}t + \frac{y''(0)}{2!}t^2 + ... + \frac{y^{(n-1)}(0)}{(n-1)!}t^{n-1} + R_{n-1} \\ &= \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{\Gamma(k+1)} t^k + R_{n-1}, \end{aligned} $$

with

$$ R_{n-1} = \int_0^t \frac{y^{(n)}(\tau)}{(n-1)!}(t-\tau)^{n-1} d\tau. $$

Using the properties of $n^{th}-$order integration, we have

$$ R_{n-1} = \frac{1}{\Gamma(n)} \int_a^t y^{(n)}(\tau)(t-\tau)^{n-1} d\tau = I^n y^{(n)}(t). $$

Using the linearity of the Riemann-Liouville operator and relation (1.4.8), we have

$$ \begin{aligned} D^{\alpha}y(t) &= D^{\alpha}\left(\sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{\Gamma(k+1)} t^k + R_{n-1} \right) \\ &= \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{\Gamma(k+1)} D^{\alpha}t^k + D^{\alpha}R_{n-1} \\ &= \sum_{k=0}^{n-1} \frac{y^{(k)}(0)\Gamma(k+1)}{\Gamma(k-\alpha+1)\Gamma(k+1)} t^{k-\alpha} + D^{\alpha}I^n y^{(n)}(t) \\ &= \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{\Gamma(k-\alpha+1)} t^{k-\alpha} + I^{n-\alpha} y^{(n)}(t) \\ &= \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{\Gamma(k-\alpha+1)} t^{k-\alpha} + {}^{C}D^{\alpha}y(t). \end{aligned} $$

So

$$^{C}D^{\alpha}y(t) = D^{\alpha}y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{\Gamma(k-\alpha+1)} t^{k-\alpha}.$$

Hence the result. ■

Remark 1.4.1 *If $y^{(k)}(0) = 0$ for $k = 0, 1, 2, ..., n-1$, we will have*

$$^{C}D^{\alpha}y(t) = D^{\alpha}y(t).$$

Composition with the fractional integration operator

The fractional derivative operator in the Caputo sense is a left inverse of the Riemann-Liouville fractional integration operator but not a right inverse because

If y is a continuous function on $[0, T]$, we have

$$^{C}D^{\alpha}\left(I^{\alpha}y(t)\right) = y(t) \text{ and } I^{\alpha}\left({}^{C}D^{\alpha}y(t)\right) = y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} t^{k}. \qquad (1.4.19)$$

The main advantage of the Caputo approach is that the initial conditions of fractional differential equations with Caputo derivatives accept the same form as for differential equations of integer order.

Chapter 2

Fractional differential equations in the sense of Caputo

In this chapter, we are interested in the question of existence and uniqueness of the solution for a Cauchy problem of a fractional differential equation in the sense of Caputo.

Definition 2.0.4 *[43] Let* $\alpha > 0, \alpha \notin \mathbb{N}$, $n = [\alpha] + 1$ *and* $f(.,y) : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$, *then*

$$^{C}D^{\alpha}y(t) = f(t, y(t)),\ t \in \Omega = [0, T]\,, \tag{2.1.1}$$

is called a fractional differential equation in the sense of Caputo, and in this case we use the initial conditions as

$$y^{(k)}(0) = b_k \in \mathbb{R}, k = 0, 1, ..., n-1, \tag{2.1.2}$$

where $^{C}D^{\alpha}$ *is the Caputo fractional derivative operator of order* α *with* $n-1 < \alpha < n,\ f(.,y) : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ *is a continuous function with respect to* $t \in \Omega$ *for all* $y \in \mathbb{R}$ *and* $[\alpha]$ *is the integer part of* α *.*

2.1 Equivalence result between the Cauchy problem and the Volterra integral equation

In this part, we prove an equivalence result between the Cauchy problem and a Volterra integral equation in the space $C^{n-1}(\Omega)$. Based on this result, the existence and uniqueness of the solution of the Cauchy problem considered are proven.

Theorem 2.1.1 *[43] Let $\alpha > 0$ with $\alpha \in \mathbb{N}$, $n = [\alpha] + 1$ and $f(.,y) : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function with respect to $t \in \Omega$ for all $y \in \mathbb{R}$.*

Then $y \in C^{n-1}(\Omega)$ is a solution of the Cauchy problem (2.1.1)-(2.1.2) if and only if y is a solution of the following Volterra integral equation

$$y(t) = \sum_{k=0}^{n-1} \frac{b_k}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau, 0 \le t \le T. \tag{2.1.3}$$

Proof. Let $\alpha > 0$ with $\alpha \notin \mathbb{N}$, $n = [\alpha] + 1$.

1) Suppose that $y \in C^{n-1}(\Omega)$ is a solution of problem (2.1.1)-(2.1.2). Since $f(.,y) \in C(\Omega)$ for all $y \in \mathbb{R}$, then by (2.1.1) we have ${}^C D^\alpha y(t) \in C(\Omega)$. Using relation (1.4.19), we obtain

$$I^\alpha \left({}^C D^\alpha y(t)\right) = y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} t^k.$$

So, we have

$$\begin{aligned} y(t) &= \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} t^k + I^\alpha \left({}^C D^\alpha y(t)\right) \\ &= \sum_{k=0}^{n-1} \frac{b_k}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau. \end{aligned}$$

2) Assume that $y \in C^{n-1}(\Omega)$ is a solution of the Volterra integral equation (2.1.3). By differentiating (2.1.3) $k-$times($k = 1, ..., n-1$) and using Proposition (1.4.4), we obtain for all $k = 0, 1, ..., n-1$,

$$y^{(k)}(t) = \sum_{j=k}^{n-1} \frac{b_j}{(j-k)!} t^{j-k} + \frac{1}{\Gamma(\alpha-k)} \int_0^t (t-\tau)^{\alpha-k-1} f(\tau, y(\tau)) d\tau.$$

With the change of variable $\tau = ts$, we obtain

$$y^{(k)}(t) = \sum_{j=k}^{n-1} \frac{b_j}{(j-k)!} t^{j-k} + \frac{t^{\alpha-k}}{\Gamma(\alpha-k)} \int_0^1 (1-s)^{\alpha-k-1} f(ts, y(ts)) ds.$$

By passing to the limit $t \longrightarrow 0$ and using the continuity of f, we obtain the relation (2.1.2). On the other hand, by applying the Riemann-Liouville fractional derivative operator D^α on the Volterra integral equation (2.1.3) and with (2.1.2), we obtain

$$D^\alpha \left(y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} t^k \right) = D^\alpha I^\alpha f(t, y(t)).$$

According to Definition 1.4.3 and Proposition 1.4.4, we obtain equation (2.1.1).

This completes the proof of the Theorem. ■

Corollaire 2.1.1 *Let* $0 < \alpha < 1$ *and* $f(.,y) : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ *be a continuous function with respect to* $t \in \Omega$ *for all* $y \in \mathbb{R}$. *Then* $y \in C(\Omega)$ *is a solution to the following Cauchy problem*

$$\begin{cases} {}^{C}D^{\alpha}y(t) = f(t, y(t)) \\ y(0) = b \end{cases},$$

if and only if y *is a solution to the following Volterra integral equation*

$$y(t) = b + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau.$$

2.2 Result of existence and uniqueness of the solution

Now, we will show the existence and uniqueness of the solution of the Cauchy problem (2.1.1)-(2.1.2) in the space of functions $C^{n-1,\alpha}(\Omega)$ defined by

$$C^{n-1,\alpha}(\Omega) = \left\{ y \in C^{n-1}(\Omega), {}^{C}D^{\alpha}y \in C(\Omega), n = [\alpha] + 1 \right\}.$$

To establish the existence and uniqueness of the solution to the Cauchy problem (2.1.1)-(2.1.2), we need the following lemma.

Lemma 2.2.1 *If* $\alpha > 0$ *with* $\alpha \notin \mathbb{N}$ *and* $n = [\alpha] + 1$,*then the fractional integration operator* $I^{\alpha} : C(\Omega) \longrightarrow C^{n-1}(\Omega)$ *in the sense of Riemann-Liouville is bounded, i.e.*

$$\|I^{\alpha}g\|_{C^{n-1}(\Omega)} \leq M \|g\|_{C(\Omega)}, M = \sum_{k=0}^{n-1} \frac{T^{\alpha-k}}{\Gamma(\alpha-k+1)}. \tag{2.2.1}$$

Proof. Let $g \in C(\Omega)$. Using Proposition 1.4.4, we obtain

$$D^k I^{\alpha} g(t) = I^{\alpha-k} g(t), \text{ for all } k = 0, 1, ..., n-1.$$

For all $t \in \Omega$, we have

$$\begin{aligned}
\|I^\alpha g\|_{C^{n-1}(\Omega)} &= \sum_{k=0}^{n-1} \left\|D^k I^\alpha g\right\|_{C(\Omega)} = \sum_{k=0}^{n-1} \left\|I^{\alpha-k} g\right\|_{C(\Omega)} \\
&\leq \sum_{k=0}^{n-1} \frac{\|g\|_{C(\Omega)}}{\Gamma(\alpha-k)} \int_0^t (t-\tau)^{\alpha-k-1}\, d\tau \\
&\leq \sum_{k=0}^{n-1} \frac{\|g\|_{C(\Omega)}}{(\alpha-k)\,\Gamma(\alpha-k)} t^{\alpha-k} \\
&\leq \sum_{k=0}^{n-1} \frac{T^{\alpha-k}}{\Gamma(\alpha-k+1)} \|g\|_{C(\Omega)}.
\end{aligned}$$

We put

$$M = \sum_{k=0}^{n-1} \frac{T^{\alpha-k}}{\Gamma(\alpha-k+1)},$$

we obtain

$$\|I^\alpha g\|_{C^{n-1}(\Omega)} \leq M \|g\|_{C(\Omega)}.$$

Which proves the lemma. ■

Theorem 2.2.1 *[43] Let $\alpha > 0$ with $\alpha \notin \mathbb{N}, n = [\alpha] + 1$ and G an open of $\mathbb{R}$. Assume that $f : \Omega \times G \longrightarrow \mathbb{R}$ a function such that*

1. *For all fixed $y \in G$, $f(.,y) \in C(\Omega)$.*

2. *The function $f : \Omega \times G \longrightarrow \mathbb{R}$ verifies the Lipschitz condition with respect to y, i.e. there exists $L > 0$ such that*

$$|f(t,y_1) - f(t,y_2)| \leq L\,|y_1 - y_2|\,, \text{ for any } t \in \Omega, y_1, y_2 \in G. \tag{2.2.2}$$

If

$$L \sum_{k=0}^{n-1} \frac{T^{\alpha-k}}{\Gamma(\alpha-k+1)} < 1, \tag{2.2.3}$$

then, the Cauchy problem (2.1.1)-(2.1.2) admits a unique solution $y \in C^{n-1,\alpha}(\Omega)$.

Proof. We start by showing the existence of a unique solution $y \in C^{n-1}(\Omega)$ of the problem (2.1.1)-(2.1.2). According to Theorem 2.1.1, it suffices to prove the existence of a unique solution $y \in C^{n-1}(\Omega)$ of the Volterra integral equation (2.1.3). For this, we use Theorem 1.2.2 of the Banach fixed point in the space $C^{n-1}(\Omega)$ with the following norm

$$\|y_1 - y_2\|_{C^{n-1}(\Omega)} = \sum_{k=0}^{n-1} \left\| y_1^{(k)} - y_2^{(k)} \right\|_{C(\Omega)}. \tag{2.2.4}$$

We rewrite the integral equation (2.1.3) in the following form

$$y(t) = (Ty)(t),$$

where

$$(Ty)(t) = y_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau, \tag{2.2.5}$$

with

$$y_0(t) = \sum_{j=0}^{n-1} \frac{b_j}{j!} t^j. \tag{2.2.6}$$

To apply Banach's Theorem, we must show

1. If $y \in C^{n-1}(\Omega)$ then $Ty \in C^{n-1}(\Omega)$.
2. for each $y_1, y_2 \in C^{n-1}(\Omega)$, we have

$$\|Ty_1 - Ty_2\|_{C^{n-1}(\Omega)} \leq w \|y_1 - y_2\|_{C(\Omega)}, \text{ with } 0 < w < 1. \tag{2.2.7}$$

Let $y \in C^{n-1}(\Omega)$. By differentiating (2.2.5) $k-$times $(k = 1, ..., n-1)$ and using Proposition 1.4.4, we obtain for all $k = 0, 1, ..., n-1$,

$$(Ty)^{(k)}(t) = y_0^{(k)}(t) + \frac{1}{\Gamma(\alpha-k)} \int_0^t (t-\tau)^{\alpha-k-1} f(\tau, y(\tau)) d\tau, \tag{2.2.8}$$

with

$$y_0^{(k)}(t) = \sum_{j=k}^{n-1} \frac{b_j}{(j-k)!} t^{j-k}.$$

For all $k = 0, 1, ..., n-1$, the first term on the right of (2.2.8) is a continuous function on $[0, T]$, and by Lemma 2.2.1, the second term is continuous on $[0, T]$. So, we have

$$\left\| \frac{1}{\Gamma(\alpha-k)} \int_0^t (t-\tau)^{\alpha-k-1} f(\tau, y(\tau)) d\tau \right\|_{C^{n-1}(\Omega)} \leq \frac{T^{\alpha-k}}{\Gamma(\alpha-k+1)} \|f(\tau, y(\tau))\|_{C(\Omega)}, \tag{2.2.9}$$

for all $k = 0, 1, ..., n-1$. Therefore $Ty \in C^{n-1}(\Omega)$.

Using (2.2.4), (2.2.8) and (2.2.9) and the Lipschitz condition (2.2.2), we have

$$\begin{aligned}
\|Ty_1 - Ty_2\|_{C^{n-1}(\Omega)} &= \sum_{k=0}^{n-1} \left\| (Ty_1)^{(k)} - (Ty_2)^{(k)} \right\|_{C(\Omega)} \\
&\leq \sum_{k=0}^{n-1} \left\| \frac{1}{\Gamma(\alpha-k)} \int\limits_0^t (t-\tau)^{\alpha-k-1} \left[f(\tau, y_1(\tau)) - f(\tau, y_2(\tau)) \right] d\tau \right\|_{C(\Omega)} \\
&\leq \sum_{k=0}^{n-1} \frac{T^{\alpha-k}}{\Gamma(\alpha-k+1)} \left\| f(\tau, y_1(\tau)) - f(\tau, y_2(\tau)) \right\|_{C(\Omega)} \\
&\leq L \sum_{k=0}^{n-1} \frac{T^{\alpha-k}}{\Gamma(\alpha-k+1)} \left\| y_1(\tau) - y_2(\tau) \right\|_{C(\Omega)} .
\end{aligned}$$

Using (2.2.3), we obtain (2.2.7), with

$$w = L \sum_{k=0}^{n-1} \frac{T^{\alpha-k}}{\Gamma(\alpha-k+1)}.$$

Consequently, according to Banach's fixed point theorem 1.2.2, there exists a unique solution $y^* \in C^{n-1}(\Omega)$ of the Volterra integral equation (2.1.3) on the interval $[0, T]$.

By Banach's theorem 1.2.2, the solution $y^*(t)$ is a limit of the convergent sequence $(T^n y_0^*)(t)$

$$\lim_{n\to\infty} \|T^n y_0^* - y^*\|_{C^{n-1}(\Omega)} = 0. \tag{2.2.10}$$

We take

$$y_0^* = y_0,$$

with $y_0(t)$ defined by (2.2.6).

According to (2.2.5), the sequence $\left(T^n y_0^*\right)(t)$ is defined by the recurrence formula

$$\left(T^n y_0^*\right)(t) = y_0(t) + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-\tau)^{\alpha-1} f\left(\tau, \left(T^{n-1} y_0^*\right)(\tau)\right) d\tau, n = 1, 2, ...$$

Noting $y_n(t) = T^n y_0^*(t)$, then the previous relation takes the following form

$$y_n(t) = y_0(t) + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-\tau)^{\alpha-1} f(\tau, y_{n-1}(\tau)) d\tau, n \in \mathbb{N},$$

and (2.2.10) becomes

$$\lim_{n \longrightarrow \infty} \|y_n - y\|_{C^{n-1}(\Omega)} = 0. \tag{2.2.11}$$

Then using (2.1.1) and the Lipschitz condition, we have

$$\begin{aligned}
\left\|{}^C D^\alpha y_n -{}^C D^\alpha y\right\|_{C(\Omega)} &= \|f(t, y_n(t)) - f(t, y(t))\|_{C(\Omega)} \\
&\leq L\,\|y_n - y\|_{C(\Omega)} \\
&\leq L\,\|y_n - y\|_{C(\Omega)}\,.
\end{aligned}$$

From (2.2.11), we obtain

$$\lim_{n \longrightarrow \infty} \left\|{}^C D^\alpha y_n -{}^C D^\alpha y\right\|_{C(\Omega)} = 0.$$

Then ${}^C D^\alpha y \in C(\Omega)$ and therefore $y \in C^{n-1,\alpha}(\Omega)$.

This completes the proof of the Theorem. ■

Example 2.2.1. *Consider the following Cauchy problem for the fractional differential equation*

$$\begin{cases} {}^C D^\alpha y(t) = t^2 - 1 \\ y(0) = 1 \end{cases} , t \in \Omega = [0, 1]\,, \tag{2.2.12}$$

where ${}^C D^\alpha$ *is the Caputo fractional derivative operator of order* $\alpha = 1/2$.

We search for a continuous function $y : [0, 1] \longrightarrow \mathbb{R}$ *satisfying (2.2.12).*

By solving the problem (2.2.12), we get

$$\begin{aligned}
y(t) &= 1 + \frac{1}{\Gamma(1/2)} \int_0^t (t-\tau)^{-1/2}\,\tau^2 d\tau - \frac{1}{\Gamma(1/2)} \int_0^t (t-\tau)^{-1/2}\,d\tau \\
&= 1 + \frac{16}{15\sqrt{\pi}} t^{5/2} - \frac{2}{\sqrt{\pi}} t^{1/2}.
\end{aligned}$$

Example 2.2.2. *Consider the following Cauchy problem for the fractional differential equation*

$$\begin{cases} {}^C D^\alpha y(t) = t^2 - 2t + 1 \\ y(1) = 0 \end{cases} , t \in \Omega = [1, 2]\,, \tag{2.2.13}$$

where ${}^C D^\alpha$ *is the Caputo fractional derivative operator of order* $\alpha = 1/3$.

We search for a continuous function $y : [1, 2] \longrightarrow \mathbb{R}$ *satisfying (2.2.13).*

By solving the problem (2.2.13), we get

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(1/3)} \int_{1}^{t} (t-\tau)^{-2/3} (\tau - 1)^{2} \, d\tau \\ &= \frac{27}{14\Gamma(1/3)} (t-1)^{7/3}. \end{aligned}$$

Chapter 3

Semi-analytical methods and their convergence

In this chapter, we present some semi-analytical methods: the Adomian decomposition method (ADM), homotopy perturbation method (HPM), variational iteration method (VIM), and new iterative method (NIM), then we study their convergence.

3.1 Adomian decomposition method (ADM)

The Adomian decomposition is a semi-analytical method developed by the American mathematician George Adomian [**3**] during the second part of the 20^{th} century. It is used for solving a wide range of problems including the mathematical models involved, namely algebraic, differential, integral, integro-differential, ordinary differential equations of higher order and partial differential equations. The advantage of this method is that it allows to solve the problem considered by a direct scheme and gives the solution in the form of an infinite series, which converges rapidly to the exact solution if it exists [**17**].

3.1.1 Description of the method

To illustrate the basic ideas of this method, consider the following functional equation

$$Fy = g, \tag{3.1.1}$$

where F represents a nonlinear ordinary or partial differential operator comprising linear and nonlinear terms and g is a known function. The linear part is generally decomposed into $L+R$, where L is an easily invertible differential operator and R represents the remainder of the linear operator. Under these conditions, the previous equation can be written in the following form

$$Ly + Ry + Ny = g, \tag{3.1.2}$$

with N a nonlinear operator.

We can write equation (3.1.2) as follows

$$Ly = g - Ry - Ny. \tag{3.1.3}$$

Multiplying equation (3.1.3) by L^{-1}, we obtain

$$L^{-1}(Ly) = L^{-1}g - L^{-1}(Ry) - L^{-1}(Ny), \tag{3.1.4}$$

where $L^{-1} = \int\int ... \int (.)\,(dt)^n$ is the inverse of the operator L.

Since

$$L^{-1}(Ly) = y - \phi,$$

and ϕ is the constant of integration.

Therefore, equation (3.1.4) becomes

$$y = \phi + L^{-1}g - L^{-1}(Ry) - L^{-1}(Ny). \tag{3.1.5}$$

The ADM consists of finding the solution in the form of a series

$$y = \sum_{n=0}^{\infty} y_n, \tag{3.1.6}$$

then to decompose the nonlinear term Ny in the form of a series

$$Ny = \sum_{n=0}^{\infty} A_n. \tag{3.1.7}$$

The terms A_n are called Adomian polynomials and are obtained using the following relation

$$A_n(y_0, y_1, ..., y_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i y_i\right)\right]_{\lambda=0}, n = 0, 1, 2, ..., \tag{3.1.8}$$

where λ is a real parameter introduced for convenience.

Substituting equations (3.1.6) and (3.1.7) in (3.1.5), we obtain

$$\sum_{n=0}^{\infty} y_n = \phi + L^{-1}g - L^{-1}R\left(\sum_{n=0}^{\infty} y_n\right) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right), \tag{3.1.9}$$

from which we deduce

$$\begin{cases} y_0 = \phi + L^{-1}g, \\ y_1 = L^{-1}R\left(y_0\right) - L^{-1}\left(A_0\right), \\ y_2 = L^{-1}R\left(y_1\right) - L^{-1}\left(A_1\right), \\ \vdots \\ y_{n+1} = L^{-1}R\left(y_n\right) - L^{-1}\left(A_n\right). \end{cases} \tag{3.1.10}$$

It should be noted that this identification is not unique but it is the only one which allows to explicitly define the y_n. The relation (3.1.10) makes it possible to calculate all the terms of the series without ambiguity because the A_n depend only on $y_0, y_1, ..., y_n$..

In practice, it is almost impossible to calculate the sum of the series $\sum\limits_{n=0}^{\infty} y_n$ (except for very special cases). We are therefore generally satisfied with an approximate solution φ_n, in the form of a truncated series

$$\varphi_n = \sum_{i=0}^{n-1} y_i, n \geq 1.$$

The question that can be asked is how to determine the $(A_n)_{n\geq 0}$ and under what conditions the method converges.

3.1.2 Adomian polynomials

Definition 3.1.1 *Adomian polynomials are defined by the formula*

$$\begin{cases} A_0(y_0) = N(y_0) \\ A_n(y_0, y_1, ..., y_n) = \dfrac{1}{n!}\dfrac{d^n}{d\lambda^n}\left[N\left(\sum\limits_{i=0}^{\infty}\lambda^i y_i\right)\right]_{\lambda=0} \end{cases}. \tag{3.1.11}$$

The formula proposed by George Adomian for the calculation of Adomian polynomials $(A_n)_{n\geq 0}$ is as follows [**4**]

$$\begin{aligned}
A_0(y_0) &= N(y_0), \\
A_1(y_0, y_1) &= y_1 \frac{\partial}{\partial y} N(y_0), \\
A_2(y_0, y_1, y_2) &= y_2 \frac{\partial}{\partial y} N(y_0) + \frac{1}{2!} y_1^2 \frac{\partial^2}{\partial y^2} N(y_0), \\
A_3(y_0, y_1, y_2, , y_3) &= y_3 \frac{\partial}{\partial y} N(y_0) + y_1 y_2 \frac{\partial^2}{\partial y^2} N(y_0) + \frac{1}{3!} y_1^3 \frac{\partial^3}{\partial y^3} N(y_0), \\
&\vdots
\end{aligned}$$

This formula is written in the form

$$A_n = \sum_{\nu=0}^{n} c(\nu, n) N^{(\nu)}(y_0), n \geq 1,$$

where $c(\nu, n)$ represents the sum of all the products (divided by $m!$) of the ν terms y_i whose sum of indices i is equal to $n; m$ being the number of repetitions of the same terms in the product.

3.1.3 Convergence of the ADM

Important theorems have been given involving sufficient conditions for convergence. All these conditions relate to the nonlinear operator N.

Indeed, from the relation (3.1.10) we deduce

Theorem 3.1.1

$$\text{If } \sum_{n \geq 0} A_n < +\infty \text{ then } \sum_{n \geq 0} y_n < +\infty, \tag{3.1.12}$$

and vice versa.

The first proofs of convergence were cited by Yves Cherruault. They are based on the fixed point method.

Let us give the broad outlines of the demonstration (see **[16]** for more details).

Let us first note that the decompositional method applied to (3.1.1) reduces to the search for a sequence

$$S_n = y_1 + y_2 + ... + y_n,$$

with $S_0 = 0$ and verifying the following recurring relation

$$S_{n+1} = N(y_0 + S_n), S_0 = 0, y_0 = g, n = 0, 1, 2, ... \tag{3.1.13}$$

We deduce the following convergence result.

Theorem 3.1.2 *If the operator N is a contraction (i.e. verifies$\|N\| < \delta < 1$) then the sequence $(S_n)_{n\geq 0}$ satisfying the recurrence relation $S_{n+1} = N(y_0 + S_n)$ with $S_0 = 0, n \geq 0$ converges to S solution of $S = N(y_0 + S)$.*

Proof. From the relation (3.1.13), we have

$$\begin{aligned} \|S_n - S\| &= \|N(y_0 + S_n) - N(y_0 + S)\| \\ &\leq \|N\| \|S_n - S\| \leq \delta \|S_n - S\| \\ &\leq \delta^n \|S_1 - S\| . \end{aligned}$$

Hence the convergence of the sequence $(S_n)_{n\geq 0}$ to S. ■

In addition, we have:

$$\sum_{n\geq 0} A_n = \sum_{n\geq 0} y_n,$$

and since $\sum\limits_{n\geq 0} y_n$ is convergent according to Theorem 3.1.1, then we have the following result.

Corollaire 3.1.1 *If N is a contraction then the series of y_n and A_n are convergent. Moreover, $\sum\limits_{n\geq 0} y_n$ is solution of the equation*

$$Fy = g.$$

Example 3.1.1. *Consider the following nonlinear differential equation*

$$\begin{cases} y' + y^2 = 0, t \geq 0 \\ y(0) = 1 \end{cases} . \tag{3.1.14}$$

We have

$$Ly = y', Ry = 0 \text{ and } Ny = y^2,$$

with $L = \frac{d}{dt}(.)$.

L^{-1} represents a simple integration from 0 to t. We find

$$y = \sum_{n=0}^{\infty} y_n = y(0) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{3.1.15}$$

Adomian polynomials are

$$\begin{aligned}
A_0 &= y_0^2, \\
A_1 &= 2y_0y_1, \\
A_2 &= 2y_0y_2 + y_1^2, \\
A_3 &= 2y_0y_3 + 2y_1y_2, \\
&\vdots
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
y_0 &= 1, \\
y_1 &= -L^{-1}(A_0) = -t, \\
y_2 &= -L^{-1}(A_1) = t^2, \\
y_3 &= -L^{-1}(A_2) = -t^3, \\
y_4 &= -L^{-1}(A_3) = t^4, \\
&\vdots
\end{aligned}$$

By (3.1.15), we have the solution of (3.1.14) given by

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} y_n = 1 - t + t^2 - t^3 + t^4 - .... \\
&= \sum_{n=0}^{\infty} (-t)^n = \frac{1}{1+t}.
\end{aligned}$$

Example 3.1.2. *Consider the following nonlinear differential equation*

$$\begin{cases} y' - e^y = 0, t \geq 0 \\ y(0) = 0 \end{cases}. \tag{3.1.16}$$

We have

$$Ly = y', Ry = 0 \text{ and } Ny = e^y,$$

with $L = \frac{d}{dt}(.)$.
L^{-1} *represents a simple integration from* 0 *to* t*. We find*

$$y = \sum_{n=0}^{\infty} y_n = y(0) + L^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{3.1.17}$$

Adomian polynomials are

$$\begin{aligned}
A_0 &= e^{y_0},\\
A_1 &= y_1,\\
A_2 &= y_2 + \frac{1}{2!}y_1^2,\\
A_3 &= y_3 + y_1y_2 + \frac{1}{3!}y_1^3,\\
&\vdots
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
y_0 &= 0,\\
y_1 &= L^{-1}(A_0) = t,\\
y_2 &= L^{-1}(A_1) = \frac{t^2}{2},\\
y_3 &= L^{-1}(A_2) = \frac{t^3}{3},\\
y_4 &= L^{-1}(A_3) = \frac{t^4}{4},\\
&\vdots
\end{aligned}$$

By (3.1.17), we have the solution of (3.1.16) in the form of an infinite series given by

$$y = \sum_{n=0}^{\infty} y_n = t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4} + ...$$

3.2 Homotopy perturbation method (HPM)

The homotopy perturbation method was proposed and developed by Chinese mathematician Ji-Haun-He in 1999 **[29]**,**[30]**,**[31]**. This method has been widely used to solve nonlinear and initial-value boundary problems. The homotopy perturbation method is a powerful mathematical tool for studying a wide variety of problems appearing in different domains. It is successfully obtained by combining the theory of homotopy in topology with the theory of perturbation. The important feature of the homotopy perturbation method is that it provides an almost exact solution to a wide range of linear and nonlinear problems, without the need for unrealistic assumptions, linearization, discretization and the calculation of Adomian polynomials **[37]**.

3.2.1 Description of the method

To illustrate the basic concept of this method, we consider the following nonlinear differential equation

$$A(y) - f(r) = 0, r \in \Omega, \tag{3.2.1}$$

with boundary conditions

$$B\left(y, \frac{\partial y}{\partial \eta}\right) = 0, r \in \Gamma, \tag{3.2.2}$$

where A is a general differential operator, B is an operator defining the boundary conditions, $f(r)$ is a known analytic function, y is the unknown function and Γ is the boundary of the domain Ω.

In general, the operator A can be decomposed into two operators L and N, where L is a linear operator and N is a nonlinear operator. So equation (3.2.1) can be rewritten as follows

$$L(y) + N(y) - f(r) = 0.$$

We construct a homotopy $z(r,p) : \Omega \times [0,1] \longrightarrow \mathbb{R}$, which satisfies

$$H(z,p) = (1-p)\left[L(z) - L(y_0)\right] + p\left[A(z) - f(r)\right] = 0, p \in [0,1], r \in \Omega, \tag{3.2.3}$$

where

$$H(z,p) = L(z) - L(y_0) + pL(y_0) + p\left[N(z) - f(r)\right] = 0, \tag{3.2.4}$$

where $p \in [0,1]$ is the homotopy parameter and y_0 is an initial approximation of equation (3.2.1) that satisfies the boundary conditions (3.2.2).

From equations (3.2.3) and (3.2.4) we have

$$\begin{aligned} H(z,0) &= L(z) - L(y_0) = 0, \\ H(z,1) &= A(z) - f(r) = 0. \end{aligned}$$

Changing p from zero to unity transforms $y_0(r)$ into $y(r)$. In topology with this last property, the function $z(r,p)$ is called homotopy. According to the HPM, we can use the parameter p as a small parameter, and assume that the solutions of equations (3.2.3) and (3.2.4) can be written as a power series of p

$$z = z_0 + pz_1 + p^2 z_2 + p^3 z_3 + ... = \sum_{i=0}^{\infty} p^i z_i. \tag{3.2.5}$$

For $p = 1$, the approximate solution of equation (3.2.1) becomes

$$y = \lim_{p \longrightarrow 1} z = z_0 + z_1 + z_2 + z_3 + ... = \sum_{i=0}^{\infty} z_i. \tag{3.2.6}$$

3.2.2 Convergence analysis

In this part, we study the convergence of the HPM **[8]**,**[12]**.

We can rewrite the relation (3.2.4) as follows

$$L(z) - L(y_0) = p\left[f(r) - L(y_0) - N(z)\right]. \tag{3.2.7}$$

By replacing (3.2.5) in (3.2.7), we get

$$L\left(\sum_{i=0}^{\infty} p^i z_i\right) - L(y_0) = p\left[f(r) - L(y_0) - N\left(\sum_{i=0}^{\infty} p^i z_i\right)\right]. \tag{3.2.8}$$

So

$$\sum_{i=0}^{\infty} L(z_i) - L(y_0) = p\left[f(r) - L(y_0) - N\left(\sum_{i=0}^{\infty} p^i z_i\right)\right]. \tag{3.2.9}$$

According to the Maclaurin development of $N\left(\sum\limits_{i=0}^{\infty} p^i z_i\right)$ with respect to p, we have

$$N\left(\sum_{i=0}^{\infty} p^i z_i\right) = \sum_{i=n=0}^{\infty} \left(\frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{\infty} p^i z_i\right)\right)_{p=0} p^i. \tag{3.2.10}$$

According to **[26],** we have

$$\left(\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{\infty} p^i z_i\right)\right)_{p=0} = \left(\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{n} p^i z_i\right)\right)_{p=0}.$$

Then

$$N\left(\sum_{i=0}^{\infty} p^i z_i\right) = \sum_{n=0}^{\infty}\left(\frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{n} p^i z_i\right)\right)_{p=0} p^i.$$

Let's put

$$H_n(z_0, z_1, ..., z_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{n} p^i z_i\right)\right]_{p=0}, n = 0, 1, 2, ..., \tag{3.2.11}$$

where H_n are called He polynomials **[26].**

Then

$$N\left(\sum_{i=0}^{\infty} p^i z_i\right) = \sum_{i=0}^{\infty} H_i p^i. \tag{3.2.12}$$

By replacing (3.2.12) in (3.2.9), we obtain

$$\sum_{i=0}^{\infty} L(z_i) - L(y_0) = p\left[f(r) - L(y_0) - \sum_{i=0}^{\infty} H_i p^i\right]. \tag{3.2.13}$$

By identifying the terms with those of the same power of p, we find

$$\begin{aligned}
p^0 &: L(z_0) - L(y_0) = 0, \\
p^1 &: L(z_1) = f(r) - L(y_0) - H_0, \\
p^2 &: L(z_2) = -H_1, \\
p^3 &: L(z_3) = -H_2, \\
& \vdots \\
p^{n+1} &: L(z_{n+1}) = -H_n, \\
& \vdots
\end{aligned} \tag{3.2.14}$$

So, we conclude that

$$\begin{aligned}
p^0 &: z_0 = y_0, \\
p^1 &: z_1 = L^{-1}(f(r)) - y_0 - L^{-1}(H_0), \\
p^2 &: z_2 = -L^{-1}(H_1), \\
p^3 &: z_3 = -L^{-1}(H_2), \\
&\vdots \\
p^{n+1} &: z_{n+1} = -L^{-1}(H_n), \\
&\vdots
\end{aligned} \tag{3.2.15}$$

Theorem 3.2.1 *The solution of equation (3.2.1) obtained by the homotopy perturbation method is equivalent to the determination of S_n given by*

$$S_n = z_1 + z_2 + ... + z_n \text{ with } S_0 = 0, \tag{3.2.16}$$

using the iterative scheme

$$S_{n+1} = -L^{-1}N(S_n + z_0) - y_0 + L^{-1}(f(r)), \tag{3.2.17}$$

where

$$N\left(\sum_{i=0}^{n} z_i\right) = \sum_{i=0}^{n} H_i, n = 0, 1, 2, ... \tag{3.2.18}$$

Proof. For $n = 0$, according to (3.2.17), we have

$$\begin{aligned}
S_1 &= -L^{-1}N(S_0 + z_0) - y_0 + L^{-1}(f(r)) \\
&= -L^{-1}(H_0) - y_0 + L^{-1}(f(r)).
\end{aligned}$$

Then

$$z_1 = -L^{-1}(H_0) - y_0 + L^{-1}(f(r)).$$

For

$$\begin{aligned}
S_2 &= -L^{-1}N(S_1 + z_0) - y_0 + L^{-1}(f(r)) \\
&= -L^{-1}(H_1 + H_0) - y_0 + L^{-1}(f(r)) \\
&= -L^{-1}(H_1) + z_1.
\end{aligned}$$

According to $S_2 = z_1 + z_2$, we obtain

$$z_2 = -L^{-1}(H_1).$$

The proof of this theorem will be done by induction.

Suppose that

$$z_{k+1} = -L^{-1}(H_k) \text{ for } k = 1, 2, ..., n-1,$$

so

$$\begin{aligned} S_{n+1} &= -L^{-1}N(S_n + z_0) - y_0 + L^{-1}(f(r)) \\ &= -L^{-1}\left(\sum_{i=0}^{n} H_i\right) - y_0 + L^{-1}(f(r)) \\ &= -\sum_{i=0}^{n} L^{-1}(H_i) - y_0 + L^{-1}(f(r)) \\ &= z_1 + z_2 + ... + z_n - L^{-1}(H_n). \end{aligned}$$

Then from (3.2.16), we can find

$$z_{n+1} = -L^{-1}(H_n).$$

This result is identical to that of (3.2.15) obtained by the HPM. ■

Theorem 3.2.2 *Let B be a Banach space.*

1) $\sum_{i=0}^{\infty} z_i$ converges to $S \in B$, if

- $$\exists \lambda : 0 < \lambda < 1 \textit{ such as } \forall n \in \mathbb{N} \Longrightarrow \|z_n\| \leq \lambda \|z_{n-1}\|. \tag{3.2.19}$$

2) $S = \sum_{i=1}^{\infty} z_i$ verifies

$$S = -L^{-1}N(S + z_0) - y_0 + L^{-1}(f(r)). \tag{3.2.20}$$

Proof. 1) we have

$$\|S_{n+1} - S_n\| = \|z_{n+1}\| \leq \lambda \|z_n\| \leq \lambda^2 \|z_{n-1}\| \leq ... \leq \lambda^{n+1} \|z_0\|.$$

For $n, m \in \mathbb{N}$ with $n \geq m$, we have

$$\begin{aligned}
\|S_n - S_m\| &= \|S_n - S_{n-1} + S_{n-1} - S_{n-2} + ... + S_{m+1} - S_m\| \\
&\leq \|S_n - S_{n-1}\| + \|S_{n-1} - S_{n-2}\| + ... + \|S_{m+1} - S_m\| \\
&\leq \lambda^n \|z_0\| + \lambda^{n-1} \|z_0\| + ... + \lambda^{m+1} \|z_0\| \\
&\leq \left(\lambda^n + \lambda^{n-1} + ... + \lambda^{m+1}\right) \|z_0\| \\
&\leq \left(\lambda^{m+1} + ... + \lambda^n + ...\right) \|z_0\| \\
&\leq \lambda^{m+1} (1 + \lambda + ... + \lambda^n + ...) \|z_0\| \\
&\leq \frac{\lambda^{m+1}}{1-\lambda} \|z_0\| .
\end{aligned}$$

Thus

$$\lim_{n,m \longrightarrow \infty} \|S_n - S_m\| = 0.$$

$(S_n)_{n \geq 0}$ is a Cauchy sequence in the Banach space and it is convergent, i.e.

$$\exists S \in B, \text{ with} \lim_{n \longrightarrow \infty} S_n = \sum_{n=1}^{\infty} z_n = S.$$

2) From (3.2.17), we have

$$\begin{aligned}
\lim_{n \longrightarrow \infty} S_{n+1} &= -L^{-1} \lim_{n \longrightarrow \infty} N(S_n + z_0) - y_0 + L^{-1}(f(r)) \\
&= -L^{-1} \lim_{n \longrightarrow \infty} N\left(\sum_{i=0}^{n} z_i\right) - y_0 + L^{-1}(f(r)) \\
S &= -L^{-1} \lim_{n \longrightarrow \infty} \sum_{i=0}^{n} H_i - y_0 + L^{-1}(f(r)) \\
&= -L^{-1} \sum_{i=0}^{\infty} H_i - y_0 + L^{-1}(f(r)) .
\end{aligned}$$

By (3.2.18) and (3.2.12), for $p = 1$, it comes

$$\sum_{i=0}^{\infty} H_i = N\left(\sum_{i=0}^{\infty} z_i\right) .$$

Thus

$$\begin{aligned}
S &= -L^{-1} N\left(\sum_{i=0}^{\infty} z_i\right) - y_0 + L^{-1}(f(r)) \\
&= -L^{-1} N(S + z_0) - y_0 + L^{-1}(f(r)) .
\end{aligned}$$

■

Lemma 3.2.1 *Equation (3.2.20) is equivalent to*

$$L(y) + N(y) - f(r) = 0. \tag{3.2.21}$$

Proof. We write the equation (3.2.20) as follows

$$S + y_0 = -L^{-1}N\left(S + z_0\right) + L^{-1}\left(f(r)\right).$$

By applying the operator L to the previous equation, we obtain

$$L\left(S + y_0\right) = -N\left(S + z_0\right) + f(r).$$

As $y_0 = z_0$,we get

$$L\left(S + z_0\right) = -N\left(S + z_0\right) + f(r).$$

Let $y = S + z_0 = \sum_{i=0}^{\infty} z_i$, equation (3.2.21) becomes the original equation. The solution of equation (3.2.20) is the same as that of the solution of $A(y) - f(r) = 0$. ■

Definition 3.2.1 *For all $i \in \mathbb{N}$, we define*

$$\lambda_i = \begin{cases} \frac{\|z_{i+1}\|}{\|z_i\|}, \|z_i\| \neq 0 \\ 0, \qquad \|z_i\| = 0. \end{cases}.$$

In Theorem 3.2.2, $\sum_{i=0}^{\infty} z_i$ converges to the exact solution when $0 < \lambda_i < 1$.

If z_i and z_i' are obtained by two different homotopies, and $\lambda_i < \lambda_i'$ for each $i \in \mathbb{N}$, the convergence rate of $\sum_{i=0}^{\infty} z_i$ is greater than $\sum_{i=0}^{\infty} z_i'$.

Example 3.2.1. *We consider the following nonlinear differential equation*

$$\begin{cases} y' + y^2 = 0, t \geq 0, t \in \Omega \\ y(0) = 1 \end{cases}. \tag{3.2.22}$$

According to the HPM, we can construct the following homotopy: $y : \Omega \times [0,1] \longrightarrow R$

$$(1-p)\left(z' - y_0'\right) + p(z' + z^2) = 0, p \in [0,1], t \in \Omega, \tag{3.2.23}$$

with

$$y_0 = 1.$$

The solution of equation (3.2.22), can be written in the form of a series

$$z = z_0 + pz_1 + p^2 z_2 + ... \tag{3.2.24}$$

Replacing (3.2.24) in (3.2.23) and identifying the terms with those of the same powers of p, we obtain

$$\begin{aligned} p^0 &: z_0' = y_0', \\ p^1 &: z_1' = -y_0 - z_0^2, z_1(0) = 0, \\ p^2 &: z_2' = -2z_0 z_1, z_2(0) = 0, \\ &\vdots \end{aligned}$$

Therefore, the first terms of the solution are given by

$$\begin{aligned} p^0 &: z_0 = 1, \\ p^1 &: z_1 = -t, \\ p^2 &: z_2 = t^2, \\ &\vdots \end{aligned}$$

So, the solution of equation (3.2.22) is

$$\begin{aligned} y &= \lim_{p \longrightarrow 1} z = z_0 + z_1 + z_2 + ... = 1 - t + t^2 - ... \\ &= \sum_{n=0}^{\infty} (-t)^n = \frac{1}{1+t}. \end{aligned}$$

Example 3.2.2. *We consider the following nonlinear differential equation*

$$\begin{cases} y' = 2y - y^2 + 1, t \geq 0, t \in \Omega \\ y(0) = 0. \end{cases} . \tag{3.2.25}$$

We search the solution with the HPM, we can construct the following homotopy: $y : \Omega \times [0, 1] \longrightarrow \mathbb{R}$

$$(1 - p)(z' - y_0') + p(z' - 2z + z^2 - 1) = 0, p \in [0, 1], t \in \Omega,$$

The solution of equation (3.2.25), can be written in the form of a series

$$z = z_0 + pz_1 + p^2 z_2 + ...$$

By identifying the terms with those of the same powers of p, *we obtain*

$$\begin{aligned} p^0 &: z_0' = y_0', \\ p^1 &: z_1' + y_0' + y_0^2 - 1 = 0, \\ p^2 &: z_2' + 2z_0z_1 = 0, \\ p^3 &: z_3' + z_1' + 2z_0z_1 = 0, \\ p^4 &: z_4' + z_1' + 2z_0z_3 + 2z_1z_2 = 0, \\ &\vdots \end{aligned}$$

Therefore, the first terms of the solution are given by

$$\begin{aligned} p^0 &: z_0 = t, \\ p^1 &: z_1 = \frac{1}{4}\left(-1 + e^t - 2t + 2t^2\right), \\ p^2 &: z_2 = \frac{1}{4}\left(t^2 - t^2e^t + 2t^3\right), \\ &\vdots \end{aligned}$$

Taking $p = 1$, *the approximate solution of equation (3.2.25) is given by*

$$y = z_0 + z_1 + z_2 + ...$$

which means that

$$y = t + \frac{1}{4}\left(-1 + e^t - 2t + 2t^2\right) + \frac{1}{4}\left(t^2 - t^2e^t + 2t^3\right) + ...$$

On the other hand, after using the Taylor expansion of e^t *in the vicinity of zero, the solution of equation (3.2.25) is given by*

$$\begin{aligned} y &= t + t^2 + \frac{1}{3}t^3 - \frac{1}{3}t^4 - \frac{7}{15}t^5 - \frac{7}{45}t^6 + \frac{53}{315}t^7 + \frac{71}{315}t^8 + ... \\ &= 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right). \end{aligned}$$

3.3 Variational iteration method (VIM)

The variational iteration method (VIM) was proposed and developed by Chinese mathematician Je-Haun-He in the early 1990s **[32]**,**[33]**,**[34]**, it was first proposed to solve problems in mechanics. This method has been used to solve a wide variety of linear and nonlinear problems with successive approximations that rapidly converge to the exact solution if it exists. The method is based on the optimal determination of the Lagrange multiplier by variational theory.

3.3.1 Description of the method

To illustrate the basic ideas of this method, we consider the following nonlinear differential equation

$$L(y) + N(y) = g(t), \tag{3.3.1}$$

where L is a linear differential operator, N is a nonlinear operator and g is a known function.

We can construct a functional correction according to the following variational iteration method

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda(\tau)\left[L\left(y_n(\tau)\right) + N(\widetilde{y_n}(\tau)) - g(\tau)\right] d\tau, n \geq 0, \tag{3.3.2}$$

where λ is a general Lagrange multiplier. The index n represents the n^{th} approximation and $\widetilde{y_n}(\tau)$ is considered as a restricted variation, i.e. $\delta\widetilde{y_n}(\tau) = 0$. To solve equation (3.3.1) by VIM, we must first determine the Lagrange multiplier λ which will be identified optimally via integration by parts. Then the successive approximations y_n of the solution $y(t)$ will be obtained by using the Lagrange multiplier and a well-chosen function y_0 (which must at least satisfy the initial conditions), consequently, the exact solution will be the limit

$$\lim_{n \longrightarrow \infty} y_n(t) = y(t).$$

3.3.2 Alternative approach to VIM

In this part, we present an alternative approach to VIM. This approach can be performed reliably and efficiently to solve the following nonlinear differential equation

$$Ly(t) + Ny(t) = g(t), t > 0, \tag{3.3.3}$$

with the initial conditions

$$y^{(k)}(0) = c_k, k = 0, 1, ..., m-1, \tag{3.3.4}$$

where L is a linear differential operator defined by $L = \dfrac{d^m}{dt^m}, m \in \mathbb{N}$, N is a nonlinear operator, g a known function and c_k are real numbers.

According to the variational iteration method, one can construct a functional correction formula for (3.3.3) as follows

$$y_{k+1}(t) = y_k(t) + \int_0^t \left[\lambda(\tau)\left(Ly_k(\tau) + N\widetilde{y_k}(\tau) - g(\tau)\right)\right] d\tau, \tag{3.3.5}$$

where λ is a general Lagrange multiplier, which can be optimally identified by variational theory. Here, we apply restricted variations to the term nonlinear Ny, in this case we can easily determine the multiplier.

Then make the functional correction (3.3.5) stationary by noting that $\delta\widetilde{y_k}(\tau) = 0$, the equation

$$\delta y_{k+1}(t) = \delta y_k(t) + \delta \int_0^t \left[\lambda(\tau)\left(Ly_k(\tau) - g(\tau)\right)\right] d\tau, \tag{3.3.6}$$

gives the following Lagrange multipliers

$$\begin{aligned} \lambda &= -1 \text{ for } m = 1, \\ \lambda &= \tau - t \text{ for } m = 2, \end{aligned}$$

and in general

$$\lambda = \frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1} \text{ for } m \geq 1. \tag{3.3.7}$$

Therefore, substituting (3.3.7) in the functional (3.3.5), we obtain the following iteration formula

$$y_{k+1}(t) = y_k(t) + \int_0^t \left[\frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1}\left(Ly_k(\tau) + Ny_k(\tau) - g(\tau)\right)\right] d\tau, \tag{3.3.8}$$

Now, we define the operator $A(y)$ as

$$A(y) = \int_0^t \left[\frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1}\left(Ly_k(\tau) + Ny_k(\tau) - g(\tau)\right)\right] d\tau, \tag{3.3.9}$$

and we define the components $z_k, k = 0, 1, 2, ...$, as follows

$$\begin{cases} z_0 = y_0, \\ z_1 = A(z_0), \\ z_2 = A(z_0 + z_1), \\ \vdots \\ z_{k+1} = A(z_0 + z_1 + ... + z_k). \end{cases} \quad (3.3.10)$$

So we have

$$y(t) = \lim_{k \longrightarrow \infty} y_k(t) = \sum_{k=0}^{\infty} z_k(t).$$

Finally, the solution of problem (3.3.3) can be deduced using (3.3.9) and (3.3.10), in the form of a series

$$y(t) = \sum_{k=0}^{\infty} z_k(t). \quad (3.3.11)$$

The initial approximation $z_0 = y_0$ can be chosen freely if it satisfies the initial conditions of the problem. The success of the method depends on the correct choice of the initial approximation z_0. In this alternative approach, we choose the initial approximation as follows

$$z_0 = \sum_{k=0}^{m-1} \frac{c_k}{k!} t^k. \quad (3.3.12)$$

3.3.3 Convergence analysis

In this part, we study the convergence of the variational iteration method, according to the alternative approach of VIM presented in the previous part **[52]**,**[61]**.

Theorem 3.3.1 *Let H be a Hilbert space and $A : H \longrightarrow H$, an operator defined by (3.3.9). The series solution $y(t) = \sum_{k=0}^{\infty} z_k(t)$ converges if $\exists 0 < \gamma < 1$ such as*

$$\| A(z_0 + z_1 + ... + z_{k+1}) \| \leq \gamma \| A(z_0 + z_1 + ... + z_k) \|,$$

i.e.

$$\| z_{k+1} \| \leq \gamma \| z_k \|, \forall k \in \mathbb{N} \cup \{0\}.$$

Proof. Let $(S_n)_{n\geq 0}$ be a sequence defined as follows

$$\begin{cases} S_0 = z_0, \\ S_1 = z_0 + z_1, \\ S_2 = z_0 + z_1 + z_2, \\ \vdots \\ S_n = z_0 + z_1 + z_2 ... + z_n. \end{cases}.$$

We show that $(S_n)_{n\geq 0}$ is a Cauchy sequence in Hilbert space H.

For this, we have

$$\|S_{n+1} - S_n\| = \|z_{n+1}\| \leq \gamma \|z_n\| \leq \gamma^2 \|z_{n-1}\| \leq ... \leq \gamma^{n+1} \|z_0\|.$$

For $n, m \in \mathbb{N}, n \geq m$, we have

$$\begin{aligned} \|S_n - S_m\| &= \|S_n - S_{n-1} + S_{n-1} - S_{n-2} + ... + S_{m+1} - S_m\| \\ &\leq \|S_n - S_{n-1}\| + \|S_{n-1} - S_{n-2}\| + ... + \|S_{m+1} - S_m\| \\ &\leq \gamma^n \|z_0\| + \gamma^{n-1} \|z_0\| + ... + \gamma^{m+1} \|z_0\| \\ &\leq \left(\gamma^n + \gamma^{n-1} + ... + \gamma^{m+1}\right) \|z_0\| \\ &\leq \left(\gamma^{m+1} + ... + \gamma^n + ...\right) \|z_0\| \\ &\leq \gamma^{m+1} (1 + \gamma + ... + \gamma^n + ...) \|z_0\| \\ &\leq \frac{\gamma^{m+1}}{1-\gamma} \|z_0\|, \end{aligned}$$

and since $0 < \gamma < 1$, we obtain

$$\lim_{n,m \longrightarrow \infty} \|S_n - S_m\| = 0.$$

Therefore, $(S_n)_{n\geq 0}$ is a Cauchy sequence in the Hilbert space H, and this implies that the series solution $y(t) = \sum_{k=0}^{\infty} z_k(t)$ converges. ■

Theorem 3.3.2 *If the series solution $y(t) = \sum_{k=0}^{\infty} z_k(t)$ converges, then it is an exact solution of the nonlinear problem (3.3.3).*

Proof. Suppose the series solution $y(t) = \sum_{k=0}^{\infty} z_k(t)$ converges, then we have

$$\lim_{k \longrightarrow \infty} z_k = 0,$$

$$\sum_{k=0}^{n} (z_{k+1} - z_k) = z_{n+1} - z_0,$$

So

$$\sum_{k=0}^{\infty} (z_{k+1} - z_k) = \lim_{k \longrightarrow \infty} (z_{n+1} - z_0) = -z_0. \tag{3.3.13}$$

By applying the operator $L = \dfrac{d^m}{dt^m}, m \in \mathbb{N}$, on both sides of equation (3.3.13) then the relation (3.3.12), we obtain

$$\sum_{k=0}^{\infty} L(z_{k+1} - z_k) = -L(z_0) = 0. \tag{3.3.14}$$

On the other hand, from relation (3.3.10), we have

$$L(z_{k+1} - z_k) = L(A(z_0 + z_1 + ... + z_k) - A(z_0 + z_1 + ... + z_{k-1})), k \geq 1.$$

Using the definition of the operator $A(y)$ defined by (3.3.9), we obtain

$$\begin{aligned} L(z_{k+1} - z_k) &= L\left(\int_0^t \left[\frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1}(L(z_0 + z_1 + ... + z_k)\right.\right. \\ &\quad -L(z_0 + z_1 + ... + z_{k-1}) + N(z_0 + z_1 + ... + z_k) \\ &\quad \left.\left. -N(z_0 + z_1 + ... + z_{k-1}))\right] d\tau\right), k \geq 1. \end{aligned} \tag{3.3.15}$$

Now, the operator $A(y)$ defined by (3.3.9), gives the integral of the m^{th} times of $Ly(t) + Ny(t) - g(t)$. Since the differential operator $L = \frac{d^m}{dt^m}$ of order m is the inverse of the integral operator m^{th} times, then equation (3.3.15) becomes

$$L(z_{k+1} - z_k) = L(z_k) + N(z_0 + z_1 + ... + z_k) - N(z_0 + z_1 + ... + z_{k-1}), k \geq 1.$$

Therefore, we have

$$\begin{aligned} \sum_{k=0}^{n} L(z_{k+1} - z_k) &= L(z_0) + N(z_0) - g(t) \\ &\quad + L(z_1) + N(z_0 + z_1) - N(z_0) \\ &\quad + L(z_2) + N(z_0 + z_1 + z_2) - N(z_0 + z_1) \\ &\quad \vdots \\ &\quad + L(z_n) + N(z_0 + z_1 + ... + z_n) - N(z_0 + z_1 + ... + z_{n-1}). \end{aligned}$$

So

$$\sum_{k=0}^{\infty} L\left(z_{k+1}-z_k\right) = L\left(\sum_{k=0}^{\infty} z_k\right) + N\left(\sum_{k=0}^{\infty} z_k\right) - g(t). \tag{3.3.16}$$

From (3.3.13) and (3.3.16), we can observe that $y(t) = \sum_{k=0}^{\infty} z_k(t)$ is an exact solution of the problem (3.3.3). ■

Example 3.3.1. *Consider the following nonlinear differential equation*

$$\begin{cases} y'(t) = y^2(t) + 1, 0 < t \leq 1, \\ y(0) = 0. \end{cases} \tag{3.3.17}$$

The functional correction of equation (3.3.17) according to the VIM, is given by:

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda(\tau)(y_n'(\tau) - (\widetilde{y_n})^2(\tau) - 1)d\tau.$$

From (3.3.7), the Lagrange multiplier $\lambda(\tau)$ can be identified as $\lambda(\tau) = -1$, hence the iteration formula can be obtained as follows

$$y_{n+1}(t) = y_n(t) - \int_0^t (y_n'(\tau) - (y_n)^2(\tau) - 1)d\tau. \tag{3.3.18}$$

According to formula (3.3.18), we obtain the first terms of the approximate solution

$$\begin{aligned} y_0(t) &= 0, \\ y_1(t) &= t, \\ y_2(t) &= t + \frac{1}{3}t^3, \\ y_3(t) &= t + \frac{1}{3}t^3 + \frac{2}{15}t^6 + \frac{1}{63}t^7, \\ &\vdots \end{aligned}$$

And as

$$y(t) = \lim_{n\longrightarrow\infty} y_n(t),$$

we can express the solution of equation (3.3.17) as a convergent series to the exact solution given by

$$\begin{aligned} y(t) &= t + \frac{1}{3}t^3 + \frac{2}{15}t^6 + \frac{1}{63}t^7 + ... \\ &= \tan(t). \end{aligned}$$

Example 3.3.2. *Consider the following linear differential equation*

$$\begin{cases} y''(t) + y(t) = 0, 0 < t \leq 1, \\ y(0) = 1, y'(0) = 0. \end{cases} \tag{3.3.19}$$

The functional correction of equation (3.3.19) according to the VIM, is given by

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda(\tau)(y_n''(\tau) + y_n(\tau))d\tau.$$

From (3.3.7), the Lagrange multiplier $\lambda(\tau)$ can be identified as $\lambda(\tau) = \tau - t$, hence the iteration formula can be obtained as follows

$$y_{n+1}(t) = y_n(t) + \int_0^t (\tau - t)(y_n''(\tau) + y_n(\tau))d\tau. \tag{3.3.20}$$

According to formula (3.3.20), we obtain the first terms of the approximate solution

$$\begin{aligned} y_0(t) &= 1, \\ y_1(t) &= 1 - \frac{1}{2!}t^2, \\ y_2(t) &= 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4, \\ y_3(t) &= 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6, \\ &\vdots \end{aligned}$$

And as

$$y(t) = \lim_{n \longrightarrow \infty} y_n(t),$$

we can express the solution of equation (3.3.19) as a convergent series to the exact solution given by

$$\begin{aligned} y(t) &= 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + ... \\ &= \lim_{n \longrightarrow \infty} \sum_{k=0}^{n} (-1)^k \frac{t^{2k}}{2k!} = \cos(t). \end{aligned}$$

3.4 New iterative method (NIM)

Recently, Daftardar-Gejji and Jafari [20] proposed a new technique for solving linear/nonlinear functional equations called new iterative method (NIM) or (DJM). The new iterative method has been widely used by many researchers for the treatment of linear and nonlinear ordinary and partial differential equations of integer and fractional order. The method converges to the exact solution if it exists by successive approximations. The advantage of this method is that it is easy to understand and apply, it provides best results and does not require any restrictive assumptions for nonlinear terms, unlike some existing techniques.

3.4.1 Description of the method

To illustrate the basic ideas of NIM, consider the following general functional equation

$$y = N(y) + f, \tag{3.4.1}$$

where N is a nonlinear operator of a Banach space $B \rightarrow B$ and f is a known function.

We search for a solution y of equation (3.4.1) in the form of a series

$$y = \sum_{i=0}^{\infty} y_i. \tag{3.4.2}$$

The nonlinear operator N can be decomposed as follows

$$N\left(\sum_{i=0}^{\infty} y_i\right) = N(y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\}. \tag{3.4.3}$$

From equations (3.4.2) and (3.4.3), equation (3.4.1) can be represented in the following form

$$\sum_{i=0}^{\infty} y_i = f + N(y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\}. \tag{3.4.4}$$

We define the recurrence relation

$$\begin{aligned} y_0 &= f, \\ y_1 &= N(y_0), \\ y_{n+1} &= N\left(\sum_{j=0}^{n} y_j\right) - N\left(\sum_{j=0}^{n-1} y_j\right), n = 1, 2, ... \end{aligned} \tag{3.4.5}$$

Then

$$y_1 + y_2 + ... + y_{n+1} = N\left(y_0 + y_1 + ... + y_n\right), n = 1, 2, ...,$$

and

$$y = \sum_{i=0}^{\infty} y_i = f + N\left(\sum_{i=1}^{\infty} y_i\right).$$

If N is a contraction, i.e.

$$\|N(x) - N(y)\| \leq k\|x - y\|, 0 < k < 1,$$

then, from (3.4.5), we have

$$\begin{aligned}
y_0 &= f, \\
\|y_1\| &= \|N\left(y_0\right)\| \leq k\|y_0\|, \\
\|y_2\| &= \|N\left(y_0 + y_1\right) - N\left(y_0\right)\| \leq k\|y_1\| \leq k^2\|y_0\| \\
\|y_3\| &= \|N\left(y_0 + y_1 + y_2\right) - N\left(y_0 + y_1\right)\| \leq k\|y_2\| \leq k^3\|y_0\| \\
&\vdots \\
\|y_{n+1}\| &= \|N\left(y_0 + y_1 + ... + y_n\right) - N\left(y_0 + y_1 + ... + y_{n-1}\right)\| \\
&\leq k\|y_n\| \leq k^{n+1}\|y_0\|, n = 0, 1, 2, ...,
\end{aligned}$$

and the series $\sum\limits_{i=0}^{\infty} y_i$ converges absolutely and uniformly to a solution of equation (3.4.1), which is unique from the point of view of Banach's fixed point theorem (1.3.8). For more details, you can see **[11]**

The $n-$term approximate solution of equation (3.4.1) is given by

$$y = \sum_{i=0}^{n-1} y_i = y_0 + y_1 + ... + y_{n-1}.$$

3.4.2 Convergence of NIM

Now, we analyze the convergence of the NIM to solve the general functional equation (3.4.1).

Let $e = y - y^*$, where y is the exact solution, y^* the approximate solution and e the error in the solution of (3.4.1), obviously e satisfies (3.4.1), i.e.

$$e = N(e) + f,$$

and the recurrence relation (3.4.5) becomes

$$\begin{aligned} e_0 &= f, \\ e_1 &= N\left(e_0\right), \\ e_{n+1} &= N\left(\sum_{j=0}^{n} e_j\right) - N\left(\sum_{j=0}^{n-1} e_j\right), n = 1, 2, ... \end{aligned}$$

If $\|N(x) - N(y)\| \leq k\|x - y\|, 0 < k < 1$, then

$$\begin{aligned} e_0 &= f, \\ \|e_1\| &= \|N\left(e_0\right)\| \leq k\|e_0\|, \\ \|e_2\| &= \|N\left(e_0 + e_1\right) - N\left(e_0\right)\| \leq k\|e_1\| \leq k^2\|e_0\| \\ \|e_3\| &= \|N\left(e_0 + e_1 + e_2\right) - N\left(e_0 + e_1\right)\| \leq k\|e_2\| \leq k^3\|e_0\| \\ &\vdots \\ \|e_{n+1}\| &= \|N\left(e_0 + e_1 + ... + e_n\right) - N\left(e_0 + e_1 + ... + e_{n-1}\right)\| \\ &\leq k\|e_n\| \leq k^{n+1}\|e_0\|, n = 0, 1, 2, ..., \end{aligned}$$

So $e_{n+1} \longrightarrow 0$ when $n \longrightarrow \infty$, which proves the convergence of the NIM to solve the general functional equation (3.4.1).

Example 3.4.1. *Consider the following nonlinear differential equation*

$$y'(t) + y^2(t) = 1, t > 0, \tag{3.4.6}$$

with initial condition

$$y(0) = 0. \tag{3.4.7}$$

By integrating equation (3.4.6) from 0 *to* t *and using the initial condition (3.4.7), we get*

$$\begin{aligned} y(t) &= t - \int_0^t y^2(\tau) d\tau \\ &= f(t) + N(y(t)), \end{aligned}$$

where

$$\begin{aligned} f(t) &= t, \\ N(y(t)) &= -\int_0^t y^2(\tau)d\tau. \end{aligned}$$

Applying NIM, we have the following first approximations

$$\begin{aligned} y_0(t) &= t, \\ y_1(t) &= N(y_0(t)) = -\frac{t^3}{3}, \\ y_2(t) &= N(y_1(t)+y_0(t)) - N(y_0(t)) = \frac{2}{15}t^5 - \frac{1}{63}t^7, \\ &\vdots \end{aligned}$$

and so on.

And since

$$y(t) = \sum_{i=0}^{\infty} y_i(t),$$

we can express the solution of equation (3.4.6) in the form of an infinite series which converges rapidly to the exact solution as

$$\begin{aligned} y(t) &= t - \frac{t^3}{3} + \frac{2}{15}t^5 - \frac{1}{63}t^7 + ... \\ &= \frac{e^{2t}-1}{e^{2t}+1}. \end{aligned}$$

Example 3.4.2. *Consider the following nonlinear differential equation*

$$y''(t) + 2y'(t) + y(t) + 8y^3(t) = 1 - 3t, \tag{3.4.8}$$

with initial conditions

$$y(0) = \frac{1}{2}, y'(0) = \frac{1}{2}. \tag{3.4.9}$$

By integrating both sides of equation (3.4.8) twice from 0 *to* t *and using the initial conditions (3.4.9), we get*

$$\begin{aligned} y(t) &= \frac{1}{2}+\frac{t}{2}+\frac{t^2}{2}-\frac{t^3}{3}-2\int_0^t y(\tau)d\tau-\int_0^t\int_0^t \left(y(\tau)+8y^3(\tau)\right)d\tau d\tau \\ &= \frac{1}{2}+\frac{t}{2}+\frac{t^2}{2}-\frac{t^3}{3}-2\int_0^t y(\tau)d\tau-\int_0^t (t-\tau)\left(y(\tau)+8y^3(\tau)\right)d\tau \\ &= f(t)+N(y(t)), \end{aligned}$$

where

$$\begin{aligned} f(t) &= \frac{1}{2}+\frac{t}{2}+\frac{t^2}{2}-\frac{t^3}{3}, \\ N(y(t)) &= -2\int_0^t y(\tau)d\tau-\int_0^t (t-\tau)\left(y(\tau)+8y^3(\tau)\right)d\tau. \end{aligned}$$

Applying NIM, we have the following first approximations

$$\begin{aligned} y_0(t) &= \frac{1}{2}+\frac{t}{2}+\frac{t^2}{2}-\frac{t^3}{3}, \\ y_1(t) &= N(u_0(t))=-t-\frac{5t^2}{4}-\frac{11t^3}{12}-\frac{7t^4}{24}+\frac{7t^5}{40}, \\ y_2(t) &= N(y_1(t)+y_0(t))-N(y_0(t))=t^2+2t^3+\frac{19t^4}{16}+\frac{31t^5}{80}-\frac{371t^6}{720}, \\ &\ \ \vdots \end{aligned}$$

and so on.

And since

$$y(t)=\sum_{i=0}^{\infty} y_i(t),$$

we can express the solution of equation (3.4.8) in the form of an infinite series which converges rapidly to the exact solution as

$$\begin{aligned} y(t) &= \frac{1}{2}\left(1-t+\frac{t^2}{2}-\frac{t^3}{6}+\frac{t^4}{24}-...\right) \\ &= \frac{1}{2}e^{-t}. \end{aligned}$$

Chapter 4

On the solution of nonlinear fractional differential equations

In this chapter, we demonstrate the applicability and efficiency of ADM, HPM, VIM, and NIM for solving nonlinear fractional differential equations where the fractional derivative is in the Caputo sense. To achieve our goal, we present some different illustrative examples.

Consider a general nonlinear fractional differential equation

$$^{C}D^{\alpha}y(t) = Ly(t) + Ny(t) + g(t), t > 0, \tag{4.1.1}$$

with the initial conditions

$$y^{(k)}(0) = y_k, k = 0, 1, 2, ..., n-1, \tag{4.1.2}$$

where $^{C}D^{\alpha}$ is the Caputo fractional derivative of order α with $n-1 < \alpha \leq n, n \in \mathbb{N}^{*}, L$ and N respectively are the linear and nonlinear differential operator, and $g(t)$ is the source term.

4.1 Application of the ADM

To verify the application of the ADM, first we apply the Riemann-Liouville fractional integral operator I^{α} to both sides of equation (4.1.1), we obtain

$$I^{\alpha}\left[{}^{C}D^{\alpha}y(t)\right] = I^{\alpha}\left[Ly(t) + Ny(t) + g(t)\right].$$

Using the relation (1.4.19) and the initial conditions (4.1.2), we get

$$y(t) = \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} t^k + I^\alpha [g(t)] + I^\alpha [Ly(t) + Ny(t)]. \tag{4.1.3}$$

Now we represent the solution as an infinite series

$$y(t) = \sum_{n=0}^{\infty} y_n(t). \tag{4.1.4}$$

and the nonlinear term can be decomposed as

$$Ny(t) = \sum_{n=0}^{\infty} A_n, \tag{4.1.5}$$

where A_n are the Adomian polynomials of $y_0, y_1, y_2, .., y_n$, which represents the nonlinear term $Ny(t)$ and it can be calculated by the formula (3.1.11).

Using equations (4.1.4) and (4.1.5), we can rewrite equation (4.1.3) as

$$\sum_{n=0}^{\infty} y_n(t) = \sum_{k=0}^{n-1} y_k \frac{t^k}{k!} + I^\alpha [g(t)] + I^\alpha \left[L \sum_{n=0}^{\infty} y_n(t) + \sum_{n=0}^{\infty} A_n \right], \tag{4.1.6}$$

By comparing both sides of equation (4.1.6), we obtain

$$\begin{aligned} y_0(t) &= \sum_{k=0}^{n-1} y_k \frac{t^k}{k!} + I^\alpha [g(t)], \\ y_1(t) &= I^\alpha [Ly_0(t) + A_0], \\ y_2(t) &= I^\alpha [Ly_1(t) + A_1], \\ y_3(t) &= I^\alpha [Ly_2(t) + A_2], \\ &\vdots \end{aligned}$$

and so on.

Similarly, we can obtain the recursive relation in general form for $n \geq 1$ and defined as

$$\begin{aligned} y_0(t) &= \sum_{k=0}^{n-1} y_k \frac{t^k}{k!} + I^\alpha [g(t)], \\ y_{n+1}(t) &= I^\alpha [Ly_n(t) + A_n]. \end{aligned}$$

Finally, the approximate solution is defined as follows

$$y(t) = \sum_{n=0}^{\infty} y_n(t).$$

Example 4.1.1. *We consider the following nonlinear fractional logistic equation*

$$^{C}D^{\alpha}y(t) = \frac{1}{2}y(t)(1-y(t)), \tag{4.1.7}$$

with the initial condition

$$y(0) = \frac{1}{2}. \tag{4.1.8}$$

where $^{C}D^{\alpha}$ *is the Caputo fractional derivative of order* α *with* $0 < \alpha \leq 1$.

Following the description of the ADM presented in part 4.1, gives

$$\sum_{n=0}^{\infty} y_n(t) = \frac{1}{2}I^{\alpha}\left(\sum_{n=0}^{\infty} y_n(t) - \sum_{n=0}^{\infty} A_n\right), \tag{4.1.9}$$

where A_n *are the Adomian polynomials which represents the nonlinear term* $y^2(t)$.

According to formula (3.1.11), the first terms of the Adomian polynomials are given by

$$\begin{aligned}
A_0 &= y_0^2,\\
A_1 &= 2y_0y_1,\\
A_2 &= 2y_0y_2 + y_1^2,\\
A_3 &= 2y_0y_3 + 2y_1y_2,\\
A_4 &= 2y_0y_4 + 2y_1y_3 + y_2^2,\\
&\vdots
\end{aligned}$$

and so on.

Comparing both sides of equation (4.1.9), we have

$$\begin{aligned}
y_0(t) &= \frac{1}{2},\\
y_1(t) &= \frac{1}{2}I^{\alpha}\left(y_0(t) - A_0\right) = \frac{1}{8}\frac{1}{\Gamma(\alpha+1)}t^{\alpha},\\
y_2(t) &= \frac{1}{2}I^{\alpha}\left(y_1(t) - A_1\right) = 0,\\
y_3(t) &= \frac{1}{2}I^{\alpha}\left(y_2(t) - A_2\right) = -\frac{1}{128}\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}t^{3\alpha},\\
y_4(t) &= \frac{1}{2}I^{\alpha}\left(y_3(t) - A_3\right) = 0,\\
y_5(t) &= \frac{1}{2}I^{\alpha}\left(y_4(t) - A_4\right) = \frac{1}{1024}\frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha},\\
&\vdots
\end{aligned}$$

and so on.

Hence, the approximate solution of equations (4.1.7)-(4.1.8) is given by

$$y(t) = \frac{1}{2} + \frac{1}{8}\frac{1}{\Gamma(\alpha+1)}t^{\alpha} - \frac{1}{128}\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}t^{3\alpha} + \frac{1}{1024}\frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha} + ... \tag{4.1.10}$$

By substituting $\alpha = 1$ into equation (4.1.10), we have

$$y(t) = \frac{1}{2} + \frac{1}{8}t - \frac{1}{384}t^3 + \frac{1}{15360}t^5 + ...$$

The closed form solution will be as follows

$$y(t) = \frac{\exp\left(\frac{1}{2}t\right)}{1+\exp\left(\frac{1}{2}t\right)}.$$

which is the exact solution of the nonlinear Logistic equation in the classical case.

Example 4.1.2. *We consider the following nonlinear fractional Bratu equation*

$$^{C}D^{\alpha}y(t) - 2\exp\left(y(t)\right) = 0, \tag{4.1.11}$$

with the initial conditions

$$y(0) = y'(0) = 0. \tag{4.1.12}$$

where $^{C}D^{\alpha}$ is the Caputo fractional derivative of order α with $1 < \alpha \leq 2$.

Following the description of the ADM presented in part 4.1, gives

$$\sum_{n=0}^{\infty} y_n(t) = 2I^{\alpha}\left(\sum_{n=0}^{\infty} A_n\right), \tag{4.1.13}$$

where A_n are the Adomian polynomials which represents the nonlinear term $\exp\left(y(t)\right)$.

According to formula (3.1.11), the first terms of the Adomian polynomials are given by

$$\begin{aligned}
A_0 &= \exp\left(y_0\right), \\
A_1 &= y_1\exp\left(y_0\right), \\
A_2 &= \left(y_2 + \frac{y_1^2}{2!}\right)\exp\left(y_0\right), \\
A_3 &= \left(y_3 + y_1y_3 + \frac{y_1^3}{3!}\right)\exp\left(y_0\right), \\
&\vdots
\end{aligned}$$

and so on.

Comparing both sides of equation (4.1.13), we have

$$\begin{aligned} y_0(t) &= 0, \\ y_1(t) &= 2I^{\alpha}(A_0) = \frac{2}{\Gamma(\alpha+1)}t^{\alpha}, \\ y_2(t) &= 2I^{\alpha}(A_1) = \frac{4}{\Gamma(2\alpha+1)}t^{2\alpha}, \\ y_3(t) &= 2I^{\alpha}(A_2) = \frac{4\left[2\Gamma^2(\alpha+1)+\Gamma(2\alpha+1)\right]}{\Gamma(3\alpha+1)\Gamma^2(\alpha+1)}t^{3\alpha}, \\ &\vdots \end{aligned}$$

and so on.

Hence, the approximate solution of equations (4.1.11)-(4.1.12) is given by

$$y(t) = \frac{2}{\Gamma(\alpha+1)}t^{\alpha} + \frac{4}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{4\left[2\Gamma^2(\alpha+1)+\Gamma(2\alpha+1)\right]}{\Gamma(3\alpha+1)\Gamma^2(\alpha+1)}t^{3\alpha} + \ldots \quad (4.1.14)$$

By substituting $\alpha = 2$ into equation (4.1.14), we have

$$y(t) = t^2 + \frac{1}{6}t^4 + \frac{2}{45}t^6 + \ldots$$

The closed form solution will be as follows

$$y(t) = -2\ln(\cos t).$$

which is the exact solution of the nonlinear Bratu equation in the classical case.

4.2 Application of the HPM

To illustrate the application of HPM, consider the general nonlinear fractional differential equation (4.1.1) with initial conditions (4.1.2).

Following the same steps as above, see part 4.1, we have

$$y(t) = \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!}t^k + I^{\alpha}\left[g(t)\right] + I^{\alpha}\left[Ly(t) + Ny(t)\right]. \quad (4.2.1)$$

Now, applying the HPM, we can assume that the solution can be expressed as a power series in p as given below

$$y(t)=\sum_{n=0}^{\infty}p^{n}y_{n}(t), \tag{4.2.2}$$

where the homotopy parameter p is considered as a small parameter $p\in[0,1]$.

The nonlinear terms can be decomposed as

$$Ny(t)=\sum_{n=0}^{\infty}p^{n}H_{n}(y), \tag{4.2.3}$$

where $H_n(y)$ are He's polynomials, and it can be calculated by the relation (3.2.11).

Substituting (4.2.2) and (4.2.3) into (4.2.1), we get

$$\sum_{n=0}^{\infty}p^{n}y_{n}(t)=\sum_{k=0}^{n-1}y_{k}\frac{t^{k}}{k!}+I^{\alpha}\left[g(t)\right]+I^{\alpha}\left[L\sum_{n=0}^{\infty}p^{n}y_{n}(t)+\sum_{n=0}^{\infty}p^{n}H_{n}(y)\right], \tag{4.2.4}$$

Comparing the coefficient of like powers of p, on both sides in equation (4.2.4), the following approximations are obtained

$$\begin{aligned}
p^{0} &: y_{0}(t)=\sum_{k=0}^{n-1}y_{k}\frac{t^{k}}{k!}+I^{\alpha}\left[g(t)\right],\\
p^{1} &: y_{1}(t)=I^{\alpha}\left[Ly_{0}(t)+H_{0}(y)\right],\\
p^{2} &: y_{2}(t)=I^{\alpha}\left[Ly_{1}(t)+H_{1}(y)\right],\\
p^{3} &: y_{2}(t)=I^{\alpha}\left[Ly_{2}(t)+H_{2}(y)\right],\\
&\vdots\\
p^{n} &: y_{n}(t)=I^{\alpha}\left[Ly_{n-1}(t)+H_{n-1}(y)\right].
\end{aligned}$$

Then, the solution of equations (4.1.1) and (4.1.2) can be defined as follows

$$y(t)=\lim_{p\longrightarrow 1}\sum_{n=0}^{\infty}p^{n}y_{n}(t)=\sum_{n=0}^{\infty}y_{n}(t).$$

Example 4.2.1. *We consider the following nonlinear fractional Riccati equation*

$$^{C}D^{\alpha}y(t)+y^{2}(t)=1, \tag{4.2.5}$$

with the initial condition

$$y(0)=0. \tag{4.2.6}$$

where ${}^{C}D^{\alpha}$ *is the Caputo fractional derivative of order* α *with* $0 < \alpha \leq 1$.

According to the description of the HPM presented in part 4.2, we have

$$\sum_{n=0}^{\infty} p^n y_n(t) = I^{\alpha}[1] - I^{\alpha}\left[\sum_{n=0}^{\infty} p^n H_n(y)\right], \tag{4.2.7}$$

where $H_n(y)$ *are He's polynomials of the nonlinear term* $y^2(t)$.

From the relation (3.2.11), the first terms of the He's polynomials are given by

$$\begin{aligned} H_0 &= y_0^2, \\ H_1 &= 2y_0y_1, \\ H_2 &= 2y_0y_2 + y_1^2, \\ H_3 &= 2y_0y_3 + 2y_1y_2, \\ &\vdots \end{aligned} \tag{4.2.8}$$

and so on.

Comparing the coefficient of like powers of p, *on both sides in equation (4.2.7) and using (4.2.8), gives*

$$\begin{aligned} p^0 &: \ y_0(t) = I^{\alpha}[1] = \frac{1}{\Gamma(\alpha+1)}t^{\alpha}, \\ p^1 &: \ y_1(t) = -I^{\alpha}[H_0(y)] = -\frac{\Gamma(2\alpha+1)}{\alpha^2\Gamma(3\alpha+1)}t^{3\alpha}, \\ p^2 &: \ y_2(t) = -I^{\alpha}[H_1(y)] = \frac{16\Gamma(2\alpha)\Gamma(4\alpha)}{\alpha\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha}, \\ p^3 &: \ y_3(t) = -I^{\alpha}[H_2(y)] = -\frac{\left[32\alpha^2\Gamma^2(2\alpha)\Gamma(4\alpha)\Gamma(3\alpha+1) + \Gamma^2(2\alpha+1)\Gamma(5\alpha+1)\right]\Gamma(6\alpha+1)}{\alpha^4\Gamma^2(3\alpha+1)\Gamma(5\alpha+1)\Gamma(7\alpha+1)}t^{7\alpha}, \\ &\vdots \end{aligned}$$

and so on.

Therefore, the approximate solution of equations (4.2.5)-(4.2.6) is given by

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\alpha+1)}t^{\alpha} - \frac{\Gamma(2\alpha+1)}{\alpha^2\Gamma(3\alpha+1)}t^{3\alpha} + \frac{16\Gamma(2\alpha)\Gamma(4\alpha)}{\alpha\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha} \\ &\quad - \frac{\left[32\alpha^2\Gamma^2(2\alpha)\Gamma(4\alpha)\Gamma(3\alpha+1) + \Gamma^2(2\alpha+1)\Gamma(5\alpha+1)\right]\Gamma(6\alpha+1)}{\alpha^4\Gamma^2(3\alpha+1)\Gamma(5\alpha+1)\Gamma(7\alpha+1)}t^{7\alpha} + ... \end{aligned} \tag{4.2.9}$$

Setting $\alpha = 1$ *in equation (4.2.9), then we have*

$$y(t) = t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + ...$$

The closed form of the solution can be easily written as

$$y(t) = \frac{\exp(2t) - 1}{\exp(2t) + 1},$$

which is the exact solution of the nonlinear Riccati equation in the classical case.

Example 4.2.2. *We consider the following nonlinear fractional differential equation*

$$^{C}D^{\alpha}y(t) = 1 + 2y(t) + 2y^2(t), \tag{4.2.10}$$

with the initial conditions

$$y(0) = y'(0) = 0. \tag{4.2.11}$$

where $^{C}D^{\alpha}$ *is the Caputo fractional derivative of order* α *with* $1 < \alpha \leq 2$.

According to the description of the HPM presented in part 4.2, we have

$$\sum_{n=0}^{\infty} p^n y_n(t) = I^{\alpha}[1] + 2I^{\alpha}\left[\sum_{n=0}^{\infty} p^n y_n(t) + \sum_{n=0}^{\infty} p^n H_n(y)\right], \tag{4.2.12}$$

where $H_n(y)$ *are He's polynomials of the nonlinear term* $y^2(t)$.

From the relation (3.2.11), the first terms of the He's polynomials are given by

$$\begin{aligned} H_0 &= y_0^2, \\ H_1 &= 2y_0y_1, \\ H_2 &= 2y_0y_2 + y_1^2, \\ H_3 &= 2y_0y_3 + 2y_1y_2, \\ &\vdots \end{aligned} \tag{4.2.13}$$

and so on.

Comparing the coefficient of like powers of p, *on both sides in equation (4.2.12) and using (4.2.13), gives*

$$\begin{aligned} p^0 &: \; y_0(t) = I^{\alpha}[1] = \frac{1}{\Gamma(\alpha+1)}t^{\alpha}, \\ p^1 &: \; y_1(t) = 2I^{\alpha}[y_0(t) + H_0(y)] = \frac{2}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{2\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha}, \\ p^2 &: \; y_2(t) = 2I^{\alpha}[y_1(t) + H_1(y)] = \frac{4}{\Gamma(3\alpha+1)}t^{3\alpha} + \frac{4\left[2\Gamma(\alpha+1)\Gamma(3\alpha+1) + \Gamma^2(2\alpha+1)\right]}{\Gamma^2(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)}t^{4\alpha} \\ &\qquad + \frac{8\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha}, \\ &\vdots \end{aligned}$$

and so on.

Therefore, the approximate solution of equations (4.2.10)-(4.2.11) is given by

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\alpha+1)}t^{\alpha} + \frac{2}{\Gamma(2\alpha+1)}t^{2\alpha} + 2\left(\frac{2\Gamma^2(\alpha+1)+\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right)t^{3\alpha} \\ &+4\left(\frac{2\Gamma(\alpha+1)\Gamma(3\alpha+1)+\Gamma^2(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)}\right)t^{4\alpha} + \frac{8\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha} + ... \end{aligned} \tag{4.2.14}$$

Setting $\alpha = 2$ in equation (4.2.14), then we have

$$y(t) = t^2 + \frac{1}{12}t^4 + \frac{1}{45}t^6 + ...$$

The closed form of the solution can be easily written as

$$y(t) = \ln(\sec t),$$

which is the exact solution of the nonlinear differential equation (4.2.10) in the classical case.

4.3 Application of the VIM

To verify the application of VIM, consider the general nonlinear fractional differential equation (4.1.1) with initial conditions (4.1.2).

Following the same steps mentioned in part 4.1, we get

$$\begin{aligned} y(t) &= \sum_{k=0}^{n-1}\frac{y^{(k)}(0)}{k!}t^k + I^{\alpha}\left[g(t)\right] + I^{\alpha}\left[Ly(t)+Ny(t)\right] \\ &= G(t) + I^{\alpha}\left[Ly(t)+Ny(t)\right]. \end{aligned} \tag{4.3.1}$$

where

$$G(t) = \sum_{k=0}^{n-1}\frac{y^{(k)}(0)}{k!}t^k + I^{\alpha}\left[g(t)\right],$$

is a term arising from the source term and the prescribed initial conditions.

Applying $\frac{d}{dt}$ to both sides of equation (4.3.1), we get

$$\frac{dy(t)}{dt} - \frac{d}{dt}I^{\alpha}\left[Ly(x,t)+Ny(x,t)\right] - \frac{dG(t)}{dt} = 0.$$

According to the VIM (see Chapter 3), a functional correction can be constructed as follows

$$y_{n+1}(t) = y_n(t) - \int_0^t \left[\frac{dy_n(\tau)}{d\tau} - \frac{d}{d\tau} I^\alpha \left[Ly_n(\tau) + Ny_n(\tau) \right] - \frac{dG(\tau)}{d\tau} \right] d\tau. \quad (4.3.2)$$

Or alternatively

$$y_{n+1}(t) = G(t) + I^\alpha \left[Ly_n(t) + Ny_n(t) \right].$$

Let us remember that

$$y(t) = \lim_{n \longrightarrow \infty} y_n(t).$$

According to the previous limit, we can obtain the exact solution if it exists or obtain an approximate solution for the equation considered (4.1.1).

Example 4.3.1. *Consider the following nonlinear fractional Riccati differential equation*

$$^{C}D^\alpha y = 2y(t) - y^2(t) + 1, \quad (4.3.3)$$

with the initial conditions

$$y(0) = 0. \quad (4.3.4)$$

where $^{C}D^\alpha$ *is the Caputo fractional derivative of order* α *with* $0 < \alpha \leq 1$.

According to the VIM, we can construct a functional correction as follows

$$y_{n+1}(t) = y_n(t) - \int_0^t \left[\frac{dy_n(\tau)}{d\tau} - \frac{d}{d\tau} I^\alpha \left[2y_n(\tau) - y_n^2(\tau) + 1 \right] \right] d\tau. \quad (4.3.5)$$

Or alternatively

$$y_{n+1}(t) = I^\alpha(1) + I^\alpha \left[2y_n(t) - y_n^2(t) \right]. \quad (4.3.6)$$

Using the iteration formula (4.3.5) or (4.3.6), the first terms are given by

$$
\begin{aligned}
y_0(t) &= \frac{1}{\Gamma(\alpha+1)}t^{\alpha},\\
y_1(t) &= \frac{1}{\Gamma(\alpha+1)}t^{\alpha} + \frac{2}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{4\Gamma^2(\alpha+1)-\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha},\\
y_2(t) &= \frac{1}{\Gamma(\alpha+1)}t^{\alpha} + \frac{2}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{4\Gamma^2(\alpha+1)-\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha},\\
&+ \left(\frac{8\Gamma^2(\alpha+1)-2\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(4\alpha+1)} - \frac{4\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)}\right)t^{4\alpha},\\
&\vdots
\end{aligned}
$$

and so on.

So, the approximate solution of equations (4.3.3)-(4.3.4) is given by

$$
\begin{aligned}
y(t) &= \frac{1}{\Gamma(\alpha+1)}t^{\alpha} + \frac{2}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{4\Gamma^2(\alpha+1)-\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha}\\
&+ \left(\frac{8\Gamma^2(\alpha+1)-2\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(4\alpha+1)} - \frac{4\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)}\right)t^{4\alpha} + .. \quad (4.3.7)
\end{aligned}
$$

Taking $\alpha = 1$ in equation (4.3.7), then we have

$$y(t) = t + t^2 + \frac{1}{3}t^3 - \frac{1}{3}t^4 + ...$$

Therefore, the solution in closed form is as follows

$$y(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right),$$

which is the exact solution of the nonlinear Riccati equation in the classical case.

Example 4.3.2. *Consider the following nonlinear fractional logistic differential equation*

$$^{C}D^{\alpha}y(t) = \frac{1}{4}y(t)(1-y(t)), \quad (4.3.8)$$

with the initial condition

$$y(0) = \frac{1}{3}. \quad (4.3.9)$$

where $^{C}D^{\alpha}$ is the Caputo fractional derivative of order α with $0 < \alpha \leq 1$.

According to the VIM, we can construct a functional correction as follows

$$y_{n+1}(t) = y_n(t) - \int_0^t \left[\frac{dy_n(\tau)}{d\tau} - \frac{1}{4}\frac{d}{d\tau} I^\alpha \left[y_n(\tau) - y_n^2(\tau) \right] \right] d\tau. \qquad (4.3.10)$$

Or alternatively

$$y_{n+1}(t) = \frac{1}{4} I^\alpha \left[y_n(t) - y_n^2(t) \right]. \qquad (4.3.11)$$

Using the iteration formula (4.3.10) or (4.3.11), the first terms are given by

$$\begin{aligned}
y_0(t) &= \frac{1}{3}, \\
y_1(t) &= \frac{1}{3} + \frac{1}{18}\frac{1}{\Gamma(\alpha+1)} t^\alpha, \\
y_2(t) &= \frac{1}{3} + \frac{1}{18}\frac{1}{\Gamma(\alpha+1)} t^\alpha + \frac{1}{216}\frac{1}{\Gamma(2\alpha+1)} t^{2\alpha}, \\
y_3(t) &= \frac{1}{3} + \frac{1}{18}\frac{1}{\Gamma(\alpha+1)} t^\alpha + \frac{1}{216}\frac{1}{\Gamma(2\alpha+1)} t^{2\alpha} + \frac{\Gamma^2(\alpha+1) - 2\Gamma(2\alpha+1)}{2592\Gamma(3\alpha+1)\Gamma^2(\alpha+1)} t^{3\alpha}, \\
&\vdots
\end{aligned}$$

and so on.

So, the approximate solution of equations (4.3.8)-(4.3.9) is given by

$$y(t) = \frac{1}{3} + \frac{1}{18}\frac{1}{\Gamma(\alpha+1)} t^\alpha + \frac{1}{216}\frac{1}{\Gamma(2\alpha+1)} t^{2\alpha} + \frac{\Gamma^2(\alpha+1) - 2\Gamma(2\alpha+1)}{2592\Gamma(3\alpha+1)\Gamma^2(\alpha+1)} t^{3\alpha} + ... \qquad (4.3.12)$$

Taking $\alpha = 1$ in equation (4.3.12), then we have

$$y(t) = \frac{1}{3} + \frac{1}{18} t + \frac{1}{432} t^2 - \frac{1}{5184} t^3 + ...$$

Therefore, the solution in closed form is as follows

$$y(t) = \frac{\exp\left(\frac{1}{4}t\right)}{1 + \exp\left(\frac{1}{4}t\right)},$$

which is the exact solution of the nonlinear logistic equation in the classical case.

4.4 Application of the NIM

To demonstrate the applicability of the NIM,consider the general nonlinear fractional differential equation (4.1.1) with initial conditions (4.1.2).

Following the same steps mentioned in part 4.3, we get

$$y(t) = G(t) + I^{\alpha}\left[Ly(t) + Ny(t)\right]. \tag{4.4.1}$$

where $G(t)$ is a term arising from the source term and the prescribed initial conditions.

Suppose that

$$\begin{aligned} f(t) &= G(t), \\ R(y(t)) &= I^{\alpha}\left[Ly(t)\right], \\ M(y(t)) &= I^{\alpha}\left[Ny(t)\right]. \end{aligned}$$

Thus, equation (4.4.1) can be written in the following form

$$y(t) = f(t) + R(y(t)) + M(y(t)), \tag{4.4.2}$$

where f is a known function, R and M are respectively the linear and nonlinear operator of y.

The solution of equation (4.4.2) can be written in the form of a series

$$y(t) = \sum_{i=0}^{\infty} y_i(t).$$

The nonlinear operator M is decomposed into (see Chapter 3)

$$M\left(\sum_{i=0}^{\infty} y_i\right) = M\left(y_0\right) + \sum_{i=1}^{\infty}\left\{M\left(\sum_{j=0}^{i} y_j\right) - M\left(\sum_{j=0}^{i-1} y_j\right)\right\},$$

and since R is linear, then we have

$$R\left(\sum_{i=0}^{\infty} y_i\right) = \sum_{i=0}^{\infty} R\left(y_i\right).$$

Therefore, equation (4.4.2) can be represented in the following form

$$\sum_{i=0}^{\infty} y_i = f(t) + \sum_{i=0}^{\infty} R\left(y_i\right) + \sum_{i=1}^{\infty}\left\{M\left(\sum_{j=0}^{i} y_j\right) - M\left(\sum_{j=0}^{i-1} y_j\right)\right\}.$$

We define the recurrence relation as follows

$$\begin{aligned} y_0 &= f, \\ y_1 &= R\left(y_0\right) + M\left(y_0\right), \\ y_{n+1} &= R\left(y_n\right) + M\left(\sum_{j=0}^{n} y_j\right) - M\left(\sum_{j=0}^{n-1} y_j\right), n = 1, 2, ... \end{aligned} \tag{4.4.3}$$

we have

$$y_1 + y_2 + ... + y_{n+1} = R(y_0 + y_1 + ... + y_n) + M(y_0 + y_1 + ... + y_n), n = 1, 2, ...,$$

and

$$y = \sum_{i=0}^{\infty} y_i = f + R\left(\sum_{i=0}^{\infty} y_i\right) + M\left(\sum_{i=1}^{\infty} y_i\right).$$

Example 4.4.1. *Consider the following nonlinear fractional Riccati equation*

$$^{C}D^{\alpha}y(t) + y^2(t) = 1, \tag{4.4.4}$$

with the initial condition

$$y(0) = 0. \tag{4.4.5}$$

where $^{C}D^{\alpha}$ is the Caputo fractional derivative of order α with $0 < \alpha \leq 1$.

By applying the technique described in part 4.4, the equation (4.4.4) is equivalent to the integral equation

$$y(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)} - I^{\alpha}\left[y^2(t)\right].$$

Let $M(y(t)) = -I^{\alpha}\left[y^2(t)\right]$.

In view of recurrence relation (4.4.3), we have the following first approximations

$$\begin{aligned} y_0 &= \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\ y_1 &= M(y_0) = -\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha}, \\ y_2 &= M(y_0+y_1) - M(y_0) = \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha} - \frac{\Gamma^2(2\alpha+1)\Gamma(6\alpha+1)}{\Gamma^4(\alpha+1)\Gamma^2(3\alpha+1)\Gamma(7\alpha+1)}t^{7\alpha}, \\ &\vdots \end{aligned}$$

Therefore, the approximate solution of equations (4.4.4)-(4.4.5) is given by

$$\begin{aligned} y(t) &= \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha} + \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha} \\ &\quad - \frac{\Gamma^2(2\alpha+1)\Gamma(6\alpha+1)}{\Gamma^4(\alpha+1)\Gamma^2(3\alpha+1)\Gamma(7\alpha+1)}t^{7\alpha} + ... \end{aligned} \tag{4.4.6}$$

For $\alpha = 1$, the solution (4.4.6) becomes

$$y(t) = t - \frac{1}{2}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + ...$$

The closed form of the solution can be easily written as

$$y(t) = \frac{\exp(2t) - 1}{\exp(2t) + 1},$$

which is the exact solution of the nonlinear Riccati equation in the classical case.

Example 4.4.2. *Consider the following nonlinear fractional differential equation*

$${}^{C}D^{\alpha}y(t) = \pi^2 + \pi^2 y(t) + \frac{\pi^2}{2}y^2(t), \tag{4.4.7}$$

with the initial condition

$$y(0) = 0, y'(0) = \pi. \tag{4.4.8}$$

where ${}^{C}D^{\alpha}$ *is the Caputo fractional derivative of order* α *with* $1 < \alpha \leq 2$.

By applying the technique described in part 4.4, the equation (4.4.7) is equivalent to the integral equation

$$y(t) = \pi t + \frac{\pi^2}{\Gamma(\alpha+1)}t^{\alpha} + \pi^2 I^{\alpha}\left[y(t)\right] + \frac{\pi^2}{2}I^{\alpha}\left[y^2(t)\right].$$

Let $R(y(t)) = I^{\alpha}[y(t)]$ and $M(y(t)) = I^{\alpha}[y^2(t)]$.

In view of recurrence relation (4.4.3), we have the following first approximations

$$\begin{aligned}
y_0 &= \pi t + \frac{\pi^2}{\Gamma(\alpha+1)}t^{\alpha}, \\
y_1 &= R(y_0) + M(y_0) = \frac{\pi^3}{\Gamma(\alpha+2)}t^{\alpha+1} + \frac{\pi^4}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{\pi^4}{\Gamma(\alpha+3)}t^{\alpha+2} + \frac{\pi^5\Gamma(\alpha+2)}{\Gamma(\alpha+1)\Gamma(2\alpha+2)}t^{2\alpha+1} \\
&\quad + \frac{\pi^6\Gamma(2\alpha+1)}{2\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha}, \\
&\ \vdots
\end{aligned}$$

and so on.

Therefore, the approximate solution of equations (4.4.7)-(4.4.8) is given by

$$\begin{aligned}
y(t) &= \pi t + \frac{\pi^2}{\Gamma(\alpha+1)}t^{\alpha} + \frac{\pi^3}{\Gamma(\alpha+2)}t^{\alpha+1} + \frac{\pi^4}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{\pi^4}{\Gamma(\alpha+3)}t^{\alpha+2} \\
&\quad + \frac{\pi^5\Gamma(\alpha+2)}{\Gamma(\alpha+1)\Gamma(2\alpha+2)}t^{2\alpha+1} + \frac{\pi^6\Gamma(2\alpha+1)}{2\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha} + \ldots
\end{aligned} \tag{4.4.9}$$

For $\alpha = 2$, *the solution (4.4.9) becomes*

$$y(t) = \pi t + \frac{\pi^2}{2}t^2 + \frac{\pi^3}{6}t^3 + \frac{\pi^4}{12}t^4 + \frac{\pi^5}{40}t^5 + \frac{\pi^6}{240}t^6 + \ldots$$

The closed form of the solution can be easily written as

$$y(t) = -\ln(1 - \sin \pi t),$$

which is the exact solution of the nonlinear differential equation (4.4.7) in the classical case.

Chapter 5

New combination method for solving nonlinear fractional Lienard equation

In this chapter, we propose a new method for solving a particular class of nonlinear fractional differential equations, namely nonlinear fractional Lienard equation where the fractional derivative is in the sense of Caputo. This method is called Khalouta differential transform method (KHDTM) and is a combination of two powerful methods: Khalouta transform method and differential transform method.

We will begin by providing an overview of the proposed equation, then give the definition and some basic results on the properties of the Khalouta transform and the differential transform method, and then present the basic principles of this method as well as some applications to the nonlinear fractional Lienard equation.

5.1 Lienard equation

The Lienard equation is a nonlinear second order differential equation proposed by Alfred-Marie Lienard [44] and is given by

$$y''(t) + f(y)y'(t) + g(y) = h(t), \tag{5.1.1}$$

where $f(y)y'(t)$ is the damping force, $g(y)$ is the restoring force, and $h(t)$ is the external force.

The Lienard equation (5.1.1) is a generalization of the damped pendulum equation or spring–mass system. Since this equation can be applied to describe the oscillating circuits, therefore it is used in the development of radio and vacuum-tube technology. For different choices of the variable coefficients $f(y), g(y)$, and $h(t)$, the Lienard equation is used in several phenomena. For example, the choices $f(y) = \varepsilon(y_2 - 1), g(y) = y$, and $h(t) = 0$ this equation becomes the Van der Pol equation as a nonlinear model of electronic oscillation, see [28],[62]

Several researchers have studied the exact solution of particular cases of Lienard equation. For example, Zhaosheng Feng [25] investigated the exact solution of

$$y''(t) + ay'(t) + by^3(t) + cy^5(t) = 0, \tag{5.1.2}$$

He found that one of the solutions of equation (5.1.2), is given by

$$y(t) = \sqrt{-\frac{2a}{b}(1 + \tanh(\sqrt{-a}t))},$$

when $b^2/4 - 4ac/3 = 0, b > 0$, and $a < 0$.

The objective of the present chapter is to propose a hybrid numerical method using Khalouta transform method and differential transform method in order to solve the nonlinear fractional Lienard equation in the form

$$^{C}D^{\alpha}y(t) + ay'(t) + by^3(t) + cy^5(t) = 0, \tag{5.1.3}$$

with the initial conditions

$$y(0) = y_0, y'(0) = y_1, \tag{5.1.4}$$

where $^{C}D^{\alpha}$ is the fractional derivative operator in the sense of the Caputo of order α with $1 < \alpha \leq 2$ and a, b, c, y_0, and y_1 are real constants.

5.2 Khalouta transform

Integral transform methods have their origins dating back to the XIX^e century with the work of Joseph Fourier and Oliver Heaviside. The fundamental idea is to represent a function $f(t)$ in terms of the transformation $F(z)$

$$F(z) = \int_{-\infty}^{+\infty} K(z,t) f(t) dt, \tag{5.2.1}$$

where the functions $K(z,t)$ are called kernel of the transform, z is a real (complex) number independent of t. Note that when $K(z,t)$ is $e^{-zt}, tJ_n(zt)$, and $t^{z-1}(zt)$, then equation (5.2.1) gives respectively, the Laplace transform, the Hankel transform and the Mellin transform.

Now, we introduce the definition and properties of the Khalouta transform that we will need in this chapter.

Definition 5.2.1 *[40] The Khalouta transform of the function $y : [0,\infty) \to \mathbb{R}$ of exponential order is defined over the set of functions*

$$\mathcal{S} = \left\{ y(t) : \exists K, \vartheta_1, \vartheta_2 > 0, |y(t)| < K \exp\left(\vartheta_j |t|\right),\ \text{if } t \in (-1)^j \times [0,\infty) \right\},$$

by the following integral

$$\mathbb{KH}\left[y(t)\right] = \mathcal{K}(s,\gamma,\eta) = s \int_0^{\infty} \exp\left(-st\right) y(\gamma\eta t) dt. \tag{5.2.2}$$

This is equivalent to

$$\begin{aligned} \mathbb{KH}\left[y(t)\right] &= \mathcal{K}(s,\gamma,\eta) = \frac{s}{\gamma\eta} \int_0^{\infty} \exp\left(-\frac{st}{\gamma\eta}\right) y(t) dt \qquad (5.2.3) \\ &= \lim_{\sigma\to\infty} \frac{s}{\gamma\eta} \int_0^{\sigma} \exp\left(-\frac{st}{\gamma\eta}\right) y(t) dt, \end{aligned}$$

where $s, \gamma, \eta > 0$ are the Khalouta transform variables, σ is a real number and the integral is taken along the line $t = \sigma$.

Theorem 5.2.1 *The inverse Khalouta transform of the function $y(t)$ is given by*

$$\mathbb{KH}^{-1}\left[\mathcal{K}(s,\eta,\gamma)\right] = y(t),\ \text{for } t \geq 0.$$

This is equivalent to

$$y(t) = \mathbb{KH}^{-1}\left[\mathcal{K}(s,\eta,\gamma)\right] = \frac{1}{2\pi i} \int_{\varphi - i\infty}^{\varphi + i\infty} \frac{1}{s} \exp\left(\frac{st}{\gamma\eta}\right) \mathcal{K}(s,\eta,\gamma) ds,$$

where φ is a real constant and the integral is taken along $s = \varphi$ in the complex plane $s = u + iv$.

Proof. To prove this Theorem, see [40]. ∎

Theorem 5.2.2 *If the function $y(t)$ is a piecewise continuous in each finite interval $t \in [0, A]$ and is of exponential order B for $t > A$, then the Khalouta transform $\mathcal{K}(s, \eta, \gamma)$ of the function $y(t)$ defined by (5.2.2) or (5.2.3) exists.*

Proof. To prove this Theorem, see [40]. ∎

Some basic properties of the Khalouta transform are given as follows [40]

Property 1: Let $\mathcal{K}_1(s, \gamma, \eta)$ and $\mathcal{K}_2(s, \gamma, \eta)$ be the Khalouta transforms of $y_1(t)$ and $y_2(t)$ respectively. For all constants of c_1 and c_2, then

$$\begin{aligned} \mathbb{KH}\left[c_1 y_1(t) + c_2 y_2(t)\right] &= c_1 \mathbb{KH}\left[y_1(t)\right] + c_2 \mathbb{KH}\left[y_2(t)\right] \\ &= c_1 \mathcal{K}_1(s, \gamma, \eta) + c_2 \mathcal{K}_2(s, \gamma, \eta). \end{aligned}$$

Property 2: Let $\mathcal{K}(s, \gamma, \eta)$ be the Khalouta transform of $y(t)$, then

$$\mathbb{KH}\left[y^{(n)}(t)\right] = \frac{s^n}{\gamma^n \eta^n} \mathcal{K}(s, \gamma, \eta) - \sum_{k=0}^{n-1} \left(\frac{s}{\gamma\eta}\right)^{n-k} y^{(k)}(0), n \geq 1.$$

Property 3: Let $\mathcal{K}_1(s, \gamma, \eta)$ and $\mathcal{K}_2(s, \gamma, \eta)$ be the Khalouta transforms of $y_1(t)$ and $y_2(t)$ respectively, then the Khalouta transform of the convolution of $y_1(t)$ and $y_2(t)$ is given by

$$\mathbb{KH}\left[(y_1 * y_2)(t)\right] = \int_0^\infty y_1(t) y_2(t-\tau) d\tau = \frac{\gamma\eta}{s} \mathcal{K}_1(s, \gamma, \eta) \mathcal{K}_2(s, \gamma, \eta).$$

Property 4: the Khalouta transforms for some basic functions.

$$\begin{aligned} \mathbb{KH}(1) &= 1, \\ \mathbb{KH}(t) &= \frac{\gamma\eta}{s}, \\ \mathbb{KH}\left(\frac{t^n}{n!}\right) &= \frac{\gamma^n \eta^n}{s^n}, n = 0, 1, 2, ... \\ \mathbb{KH}\left[\frac{t^\alpha}{\Gamma(\alpha+1)}\right] &= \frac{\gamma^\alpha \eta^\alpha}{s^\alpha}, \alpha > -1, \end{aligned}$$

Now, we present our results regarding the Khalouta transform of the Riemann-Liouville fractional integral and the Caputo fractional derivative [18].

Theorem 5.2.3 *If $\mathcal{K}(s,\gamma,\eta)$ is the Khalouta transform of the function $y(t)$, then the Khalouta transform of Riemann-Liouville fractional integral of order $\alpha > 0$, is given by*

$$\mathbb{KH}\left[I^{\alpha}y(t)\right] = \frac{\gamma^{\alpha}\eta^{\alpha}}{s^{\alpha}}\mathcal{K}(s,\gamma,\eta).$$

Proof. Applying the Khalouta transform to both sides of equation (1.4.2), we get

$$\begin{aligned}\mathbb{KH}\left[I^{\alpha}y(t)\right] &= \mathbb{KH}\left[\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-\tau)^{\alpha-1}y(\tau)d\tau\right]\\ &= \mathbb{KH}\left[\frac{1}{\Gamma(\alpha)}t^{\alpha-1}*y(t)\right].\end{aligned}$$

Then, using Properties (3) and (4), we get

$$\begin{aligned}\mathbb{KH}\left[I^{\alpha}y(t)\right] &= \frac{\gamma\eta}{s}\mathbb{KH}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right]\mathbb{KH}\left[y(t)\right]\\ &= \frac{\gamma\eta}{s}\frac{\gamma^{\alpha-1}\eta^{\alpha-1}}{s^{\alpha-1}}\mathcal{K}(s,\gamma,\eta)\\ &= \frac{\gamma^{\alpha}\eta^{\alpha}}{s^{\alpha}}\mathcal{K}(s,\gamma,\eta).\end{aligned}$$

The theorem is proved. ∎

Theorem 5.2.4 *If $\mathcal{K}(s,\gamma,\eta)$ is the Khaoluta transform of the function $y(t)$, then the Khalouta transform of the Caputo fractional derivative of order $n-1 < \alpha \leq n, n \in \mathbb{Z}^{+}$, is given by*

$$\mathbb{KH}\left[{}^{C}D^{\alpha}y(t)\right] = \frac{s^{\alpha}}{\gamma^{\alpha}\eta^{\alpha}}\mathcal{K}(s,\gamma,\eta) - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{\alpha-k}y^{(k)}(0).$$

Proof. First, we take

$$v(t) = y^{(n)}(t). \tag{5.2.4}$$

Thus, equation (1.4.14), can be written as follows

$$\begin{aligned}{}^{C}D^{\alpha}y(t) &= \frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}(t-\tau)^{n-\alpha-1}y^{(n)}(\tau)d\tau,\\ &= \frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}(t-\tau)^{n-\alpha-1}v(\tau)d\tau\\ &= I^{n-\alpha}v(t). \qquad (5.2.5)\end{aligned}$$

Applying the Khaoluta transform on both sides of equation (5.2.3) and using Theorem 5.2.3, we get

$$\mathbb{KH}\left[{}^{C}D^{\alpha}y(t)\right] = \mathbb{KH}\left[I^{n-\alpha}v(t)\right] = \frac{\gamma^{n-\alpha}\eta^{n-\alpha}}{s^{n-\alpha}}\mathcal{V}(s,\gamma,\eta), \tag{5.2.6}$$

where $\mathcal{V}(s,\gamma,\eta)$ is the Khaloua transform of the function $v(x)$.

Applying the Khaoluta transform on both sides of equation (5.2.5) and using Property 2, we get

$$\begin{aligned}\mathbb{KH}\left[v(t)\right] &= \mathbb{KH}\left[y^{(n)}(t)\right], \\ \mathcal{V}(s,\gamma,\eta) &= \frac{s^{n}}{\gamma^{n}\eta^{n}}\mathcal{K}(s,\gamma,\eta) - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{n-k} y^{(k)}(0). \end{aligned} \tag{5.2.7}$$

Substituting equation (5.2.7) into equation (5.2.6), we get

$$\begin{aligned}\mathbb{KH}\left[{}^{C}D^{\alpha}y(t)\right] &= \frac{\gamma^{n-\alpha}\eta^{n-\alpha}}{s^{n-\alpha}}\left(\frac{s^{n}}{\gamma^{n}\eta^{n}}\mathcal{K}(s,\gamma,\eta) - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{n-k} y^{(k)}(0)\right) \\ &= \frac{s^{\alpha}}{\gamma^{\alpha}\eta^{\alpha}}\mathcal{K}(s,\gamma,\eta) - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{\alpha-k} y^{(k)}(0).\end{aligned}$$

The theorem is proved. ■

5.3 Differential transform method

In this part, we introduce the basic definitions and fundamental theorems of differential transform method are defined and proved in [19],[49].

Definition 5.3.1 *The differential transform of the function $y(t)$ is defined as*

$$Y(k) = \sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{d^{k}}{dt^{k}}y(t)\right]_{t=t_0}, \tag{5.3.1}$$

where $y(t)$ is the original function and $Y(k)$ the transformed function

Definition 5.3.2 *The inverse differential transform of $Y(k)$ is defined as*

$$y(t) = \sum_{k=0}^{\infty}Y(k)(t-t_0)^{k}. \tag{5.3.2}$$

Combining equations (5.3.1) and (5.3.2), we get

$$y(t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{d^k}{dx^k} y(t) \right]_{t=t_0} (t - t_0)^k. \tag{5.3.3}$$

In particular, for $t_0 = 0$, equation (5.3.3) becomes

$$y(t) = \frac{1}{k!} \left[\frac{d^k}{dt^k} y(t) \right]_{t=0} t^k.$$

From the above definitions, the fundamental operations of the differential transform method are given by the following theorems.

Theorem 5.3.1 *Let $Y(k), Z(k)$ and $W(k)$ be the differential transforms of the functions $y(t), z(t)$ and $w(t)$ respectively, then*

(1) if

$$w(t) = \lambda y(t) + \mu z(t),$$

then

$$W(k) = \lambda Y(k) + \mu Z(k), \lambda, \mu \in \mathbb{R}.$$

(2) if

$$w(t) = y(t)z(t),$$

then

$$W(k) = \sum_{r=0}^{k} Y(r)Z(k-r).$$

(3) if

$$w(t) = y_1(t)y_2(t)...y_{n-1}(t)y_n(t),$$

then

$$W(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} ... \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} Y_1(k_1)Y_2(k_2-k_1) \times ... \times Y_{n-1}(k_{n-1}-k_{n-2})Y_n(k-k_{n-1}).$$

5.4 Description of the KHDTM

Theorem 5.4.1 *[18] Consider the following nonlinear fractional Lienard equation (5.1.3) with the initial conditions (5.1.4). The KHDTM gives the solution of (5.1.3)-(5.1.4) in the form of infinite series that rapidly converge to the exact solution as follows*

$$y(t) = \sum_{k=0}^{\infty} Y(k),$$

where $Y(k)$ is the differential transformed function of $y(t)$.

Proof. Consider the nonlinear fractional Lienard equation (5.1.3) with the initial conditions (5.1.4).

Computing the Khalouta transform to equation (5.1.3) and the use of the linearity property of Khalouta transform, we get

$$\mathbb{KH}\left[{}^{C}D^{\alpha}y(t)\right] + a\mathbb{KH}\left[y(t)\right] + b\mathbb{KH}\left[y^{3}(t)\right] + c\mathbb{KH}\left[y^{5}(t)\right] = 0.$$

Using Theorem 5.2.4, this gives

$$\mathbb{KH}\left[y(t)\right] = y(0) + \left(\frac{\gamma\eta}{s}\right)y'(0) - \frac{\gamma^{\alpha}\eta^{\alpha}}{s^{\alpha}}\mathbb{KH}\left[ay(t) + by^{3}(t) + cy^{5}(t)\right]. \tag{5.4.1}$$

Substituting the initial conditions of equation (5.1.4) into equation (5.4.1), we get

$$\mathbb{KH}\left[y(t)\right] = y_{0} + \left(\frac{\gamma\eta}{s}\right)y_{1} - \frac{\gamma^{\alpha}\eta^{\alpha}}{s^{\alpha}}\mathbb{KH}\left[ay(t) + by^{3}(t) + cy^{5}(t)\right]. \tag{5.4.2}$$

Taking the inverse Khalouta transform on both sides of equation (5.4.2), we obtain

$$y(t) = y_{0} + y_{1}t - \mathbb{KH}^{-1}\left[\frac{\gamma^{\alpha}\eta^{\alpha}}{s^{\alpha}}\mathbb{KH}\left[ay(t) + by^{3}(t) + cy^{5}(t)\right]\right]. \tag{5.4.3}$$

Now, by applying the differential transform method to equation (5.4.3), we get

$$\begin{aligned} Y(0) &= y_{0}, \\ Y(1) &= y_{1}t, \\ Y(k+2) &= -\mathbb{KH}^{-1}\left[\frac{\gamma^{\alpha}\eta^{\alpha}}{s^{\alpha}}\mathbb{KH}\left[(aY(k) + bA(k) + cB(k))\right]\right], k \geq 0 \end{aligned} \tag{5.4.4}$$

where $A(k)$ and $B(k)$ are the differential transform of the nonlinear terms $y^{3}(t)$ and $y^{5}(t)$, respectively.

The first few nonlinear terms are given by

$$\begin{aligned} A(0) &= Y^3(0), \\ A(1) &= 3Y^2(0)Y(1), \\ A(2) &= 3Y^2(0)Y(2) + 3Y(0)Y^2(1), \end{aligned}$$

and

$$\begin{aligned} B(0) &= Y^5(0), \\ B(1) &= 5Y^4(0)Y(1), \\ B(2) &= 5Y^4(0)Y(2) + 10Y^3(0)Y^2(1). \end{aligned}$$

Note that the recurrence formula (5.4.4) to the iterative terms of equations (5.1.3) and (5.1.4) is denoted KHDTM, and the k^{th} order solution for equations (5.1.3) and (5.1.4) is given as

$$S_k = \sum_{r=0}^{k} Y(r).$$

Thus, in the following theorem, we prove that the series solution (5.4.4) of equations (5.1.3) and (5.1.4) converges to the exact solution if $k \to \infty$, that is

$$y(t) = \lim_{k \to \infty} S_k = \sum_{r=0}^{\infty} Y(r). \tag{5.4.5}$$

■

5.5 Convergence of the KHDTM

Suppose that $\mathcal{B} = (C(\mathbb{R}^+), \|.\|)$ is the Banach space of all continuous functions on $\mathbb{R}^+$ with the norm

$$\|y(t)\|_{\mathcal{B}} = \sup_{t \in \mathbb{R}^+} |y(t)|.$$

Theorem 5.5.1 *[18] Let $Y(r)$ and $y(t)$ be defined in Banach space $\mathcal{B}$, then the series solution $\sum_{r=0}^{+\infty} Y(r)$ stated in equation (5.4.5) converges uniquely to the exact solution $y(t)$ of the nonlinear fractional Lienard equation (5.1.3), if there exists $0 < \theta < 1$ such that $\|Y(r)\| \leq \theta \|Y(r-1)\|, \forall r \in \mathbb{N} \cup \{0\}$.*

Proof. Let S_k be the sequence of partial sums of the series given by the recurrence formula (5.4.4), as

$$S_k = \sum_{r=0}^{k} Y(r).$$

We need to show that $\{S_k\}_{k=0}^{\infty}$ is a Cauchy sequence in Banach space $\mathcal{B}$.

For this purpose, we consider

$$\begin{aligned} \|S_{k+1} - S_k\| &\leq \|Y(r+1)\| \leq \theta \|Y(r)\| \\ &\leq \theta^2 \|Y(r-1)\| \leq ... \leq \theta^{n+1} \|Y(0)\| . \end{aligned} \quad (5.4.6)$$

For every, $n, m \in \mathbb{N}, n \geq m$, by using (5.4.6) and triangle inequality successively, we have

$$\begin{aligned} \|S_n - S_m\| &= \|S_n - S_{n-1} + S_{n-1} - S_{n-2} + ... + S_{m+1} - S_m\| \\ &\leq \|S_n - S_{n-1}\| + \|S_{n-1} - S_{n-2}\| + ... + \|S_{m+1} - S_m\| \\ &\leq \theta^n \|Y(0)\| + \theta^{n-1} \|Y(0)\| + ... + \theta^{m+1} \|Y(0)\| \\ &= \theta^{m+1} \left(1 + \theta + ... + \theta^{n-m-1}\right) \|Y(0)\| \\ &\leq \theta^{m+1} \left(\frac{1-\theta^{n-m}}{1-\theta}\right) \|Y(0)\| . \end{aligned}$$

Since $0 < \theta < 1$, we have $1 - \theta^{n-m} < 1$, then

$$\|S_n(X,t) - S_m(X,t)\| \leq \frac{\theta^{m+1}}{1-\theta} \|Y(0)\| . \quad (5.4.7)$$

So $\|S_n - S_m\| \to 0$ as $n, m \to \infty$ as $Y(0)$ is bounded.

Thus $\{S_k\}_{k=0}^{\infty}$ is a Cauchy sequence in Banach space and consequently it is converges to $y(t) \in \mathcal{B}$ such that

$$\lim_{k\to\infty} S_k = \sum_{r=0}^{\infty} Y(r) = y(t).$$

Now, suppose that the sequence $\{S_k\}_{k\geq 0}$ converges to two functions of $y_1(t), y_2(t) \in \mathcal{B}$, that is,

$$\lim_{k\to\infty} S_k = y_1(t) \text{ and } \lim_{k\to\infty} S_k = y_2(t). \quad (5.4.8)$$

Using the triangle inequality with (5.4.8), we get

$$\|y_1(t) - y_2(t)\| \leq \|y_1(t) - S_k\| + \|S_k - y_2(t)\| = 0 \text{ as } k \to \infty.$$

Hence we conclude that $y_1(t) = y_2(t)$.

The theorem is proved. ■

Theorem 5.5.2 *[18] The maximum absolute truncation error of the series solution given by the recurrence formula (5.4.4), is estimated to be*

$$\left\| y(t) - \sum_{l=0}^{N} Y(l) \right\| \leq \frac{\theta^{N+1}}{1-\theta} \|Y(0)\|.$$

Proof. From Theorem 5.5.1 and (5.4.7), we have

$$\|S_k - S_N\| \leq \frac{\theta^{N+1}}{1-\theta} \|Y(0)\|. \tag{5.4.9}$$

But we assume that $S_k = \sum_{l=0}^{k} Y(l)$ and since $k \to +\infty$, we obtain $S_k \to y(t)$, so (5.4.9) can be rewritten as

$$\|y(t) - S_N\| = \left\| y(t) - \sum_{l=0}^{N} Y(l) \right\| \leq \frac{\theta^{N+1}}{1-\theta} \|Y(0)\|.$$

The theorem is proved. ■

Corollaire 5.5.1 *If the series* $\sum_{r=0}^{\infty} Y(r)$ *converges then it is an exact solution of the nonlinear fractional Lienard equation (5.1.3) with initiales conditions (5.1.4).*

5.6 Illustrative examples

In this part, we provide two numerical examples of nonlinear fractional Lienard equations to evaluate the applicability, accuracy, and efficiency of the KHDTM.

Example 5.6.1. *Consider the nonlinear fractional Lienard equation*

$${}^{C}D^{\alpha} y(t) - y(t) + 4y^3(t) - 3y^5(t) = 0, t > 0, \tag{5.6.1}$$

with the initial conditions

$$y(0) = \frac{1}{\sqrt{2}}, y'(0) = \frac{1}{\sqrt{8}}, \tag{5.6.2}$$

where ${}^{C}D^{\alpha}$ *is the fractional derivative operator in the sense of the Caputo of order* α *with* $1 < \alpha \leq 2$.

If $\alpha = 2$, *equation (5.6.1) becomes the classical Lienard equation and its exact solution is of the form*

$$y(t) = \sqrt{\frac{1+\tanh(t)}{2}}.$$

According the description of the KHDTM presented in part 5.4, we have

$$y(t) = \sum_{r=0}^{\infty} Y(r),$$

and

$$\begin{aligned}
Y(0) &= \frac{1}{\sqrt{2}}, \\
Y(1) &= \frac{1}{\sqrt{8}}t, \\
Y(2) &= -\frac{1}{4\sqrt{2}}\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\
Y(3) &= -\frac{5}{4\sqrt{8}}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}, \\
&\vdots
\end{aligned}$$

and so on.

Hence, the approximate series solution of equations (5.6.1) and (5.6.2), is given as

$$\begin{aligned}
y(t) &= Y(0)+Y(1)+Y(2)+Y(3)+... \\
&= \frac{1}{\sqrt{2}}\left(1+\frac{1}{2}t-\frac{1}{4}\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{5}{8}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+...\right). \qquad (5.6.3)
\end{aligned}$$

When $\alpha = 2$, *the equation (5.6.3), becomes*

$$\begin{aligned}
u(x) &= \frac{1}{\sqrt{2}}\left(1+\frac{1}{2}t-\frac{1}{8}t^{2}-\frac{5}{48}t^{3}+...\right) \\
&= \sqrt{\frac{1+\tanh(t)}{2}},
\end{aligned}$$

which is the same exact solution as obtained using the modified fractional Taylor series method (MFTSM) [38].

Figure 1 shows the behavior of the exact solution and the KHDTM-solution for different values of α. Table 1 shows the numerical values of the KHDTM-solution, the exact solution, and the absolute error for different fractional values of α.

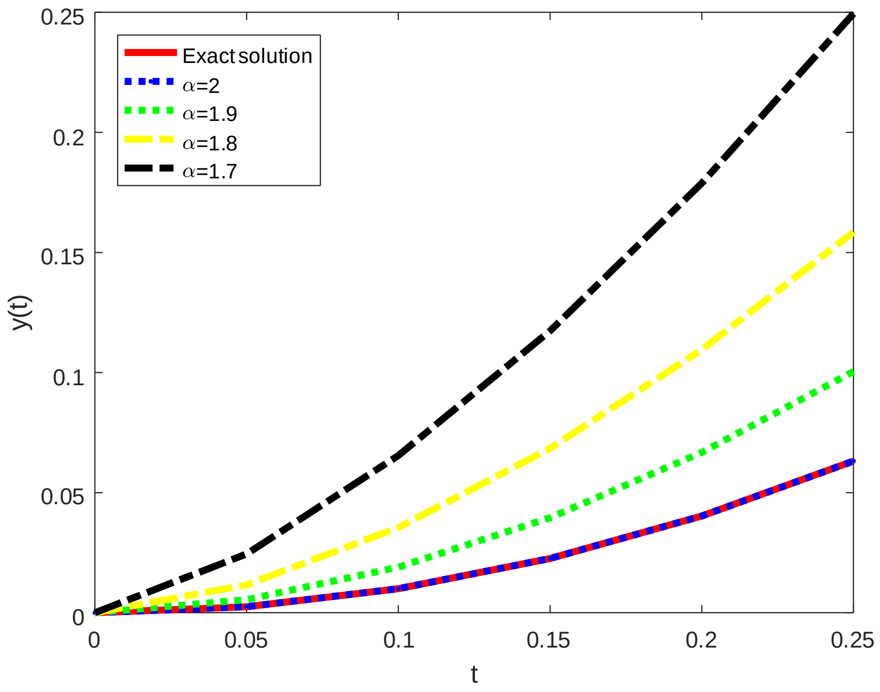

Figure 1 : The behavior of the approximate solutions using KHDTM and exact solution for equation (5.6.1)

t	$\alpha = 1.7$	$\alpha = 1.8$	$\alpha = 1.9$	$\alpha = 2$	*Exact solution*	$\lvert y_{exact} - y_{KHDTM}\rvert$
0.00	0.70711	0.70711	0.70711	0.70711	0.70711	0
0.02	0.71403	0.71408	0.71412	0.71414	0.71414	5.0793×10^{-9}
0.04	0.72075	0.72092	0.72103	0.72110	0.72110	8.2374×10^{-8}
0.06	0.72731	0.72762	0.72783	0.72799	0.72799	4.2249×10^{-7}
0.08	0.73371	0.73419	0.73454	0.73479	0.73479	1.3522×10^{-6}
0.1	0.73997	0.74064	0.74114	0.74151	0.74151	3.3415×10^{-6}

Table 1 : Numerical values of the approximate solutions using KHDTM and exact solution for equation (5.6.1)

Example 5.6.2. *Consider the nonlinear fractional Lienard equation*

$$^{C}D^{\alpha}y(t) - y(t) + 4y^{5}(t) + 3y^{5}(t) = 0, t > 0, \tag{5.6.4}$$

with the initial conditions

$$y(0) = \frac{1}{\sqrt{1+\sqrt{2}}}, y'(0) = 0, \tag{5.6.5}$$

where ${}^{C}D^{\alpha}$ *is the fractional derivative operator in the sense of the Caputo of order* α *with* $1 < \alpha \leq 2$.

If $\alpha = 2$, *equation (5.6.4) becomes the classical Lienard equation and its exact solution is of the form*

$$y(t) = \sqrt{\frac{\sec h^2(t)}{2\sqrt{2} + \left(1 - \sqrt{2}\right)\sec h^2(t)}}.$$

According the description of the KHDTM presented in part 5.4, we have

$$y(t) = \sum_{r=0}^{\infty} Y(r),$$

and

$$\begin{aligned}
Y(0) &= \frac{1}{\sqrt{1+\sqrt{2}}}, \\
Y(1) &= 0, \\
Y(2) &= -\left(\frac{4+2\sqrt{2}}{\left(3+2\sqrt{2}\right)\sqrt{1+\sqrt{2}}}\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\
Y(3) &= 0, \\
&\vdots
\end{aligned}$$

and so on.

Hence, the approximate series solution of equations (5.6.4)and (5.6.5), is given as

$$\begin{aligned}
y(t) &= Y(0) + Y(1) + Y(2) + Y(3) + ... \\
&= \frac{1}{\sqrt{1+\sqrt{2}}}\left(1 - \left(\frac{4+2\sqrt{2}}{3+2\sqrt{2}}\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + ...\right).
\end{aligned} \tag{5.6.6}$$

When $\alpha = 2$, *the equation (5.6.6), becomes*

$$\begin{aligned}
y(t) &= \frac{1}{\sqrt{1+\sqrt{2}}}\left(1 - \left(\frac{2+\sqrt{2}}{3+2\sqrt{2}}\right)t^2 + ...\right) \\
&= \sqrt{\frac{\sec h^2(t)}{2\sqrt{2} + \left(1 - \sqrt{2}\right)\sec h^2(t)}},
\end{aligned}$$

which is the same exact solution as obtained using the modified fractional Taylor series method (MFTSM) [38].

Figure 2 shows the behavior of the exact solution and the KHDTM-solution for different values of α. Table 2 shows the numerical values of the KHDTM-solution, the exact solution, and the absolute error for different fractional values of α.

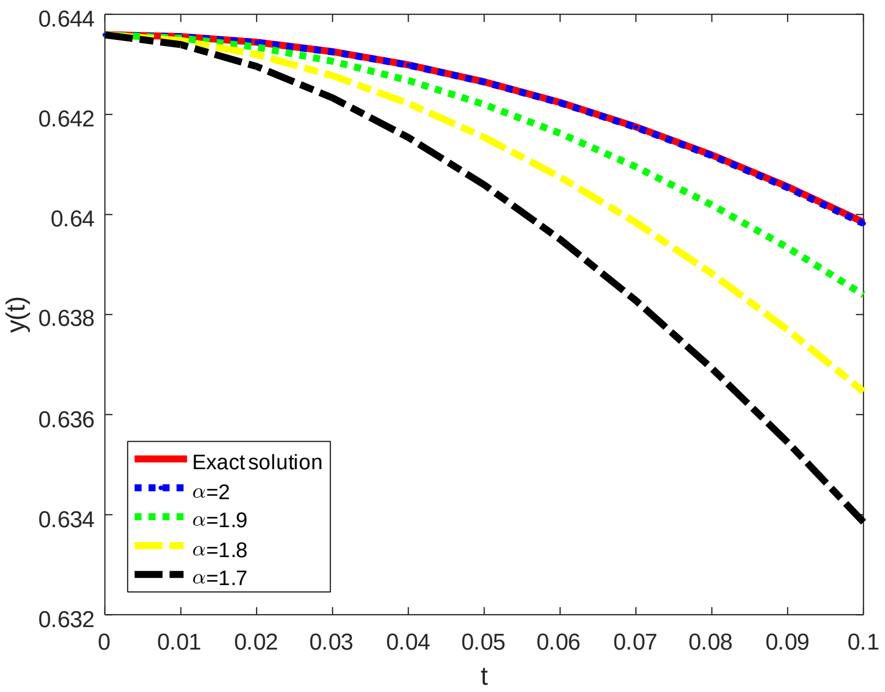

Figure 2 : The behavior of the approximate solutions using KHDTM and exact solution for equation (5.6.4)

t	$\alpha = 1.7$	$\alpha = 1.8$	$\alpha = 1.9$	$\alpha = 2$	*Exact solution*	$\lvert y_{exact} - y_{KHDTM}\rvert$
0.00	0.64359	0.64359	0.64359	0.64359	0.64359	0.0
0.02	0.64296	0.64320	0.64335	0.64344	0.64344	3.2888×10^{-8}
0.04	0.64154	0.64222	0.64268	0.64299	0.64299	5.2585×10^{-7}
0.06	0.63951	0.64075	0.64163	0.64224	0.64224	2.6590×10^{-6}
0.08	0.63693	0.63882	0.64019	0.64118	0.64118	8.3902×10^{-6}
0.1	0.63385	0.63647	0.63840	0.63982	0.63982	2.0441×10^{-5}

Table 2 : Numerical values of the approximate solutions using KHDTM and exact solution for equation (5.6.4)

Conclusion and research perspectives

In this thesis, the approximate and analytical solutions of nonlinear fractional differential equations in the sense of Caputo are studied by proposing a new method called the Khalouta differential transform method (KHDTM). To demonstrate the applicability and efficiency of the proposed method, it was applied to a special class of nonlinear fractional differential equations called nonlinear fractional Lienard equation, and the results showed that the approximate solutions obtained using this method agree excellently with the exact solutions. The main advantage of KHDTM is that it gives the solution in the form of an infinite series, which rapidly converges to the exact solution if it exists.

It can be concluded that the proposed method is very powerful and effective for finding approximate and analytical solutions of nonlinear fractional differential equations.

This field of research in the case of fractional differential equations is very interesting, therefore, the future prospects are:

1- Search for numerical and analytical methods for solving fractional differential equations, less expensive and more accurate than the proposed method in this thesis.

2- Apply the Khalouta differential transform method to solve fractional differential equations, but with other fractional derivative operators (in the sense of Riemann-Liouville, Grunwald-Letnikov, Caputo-Fabrizio, and in the sense of Hadamard).

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ملخص:

تلعب المعادلات التفاضلية الكسرية غير الخطية دورًا مهمًا في الرياضيات التطبيقية والفيزياء. من الصعب الحصول على الحل الدقيق لهذه المشكلات بسبب تعقيد المصطلحات غير الخطية المتضمنة. في العقود الأخيرة، كان هناك تطور كبير في التحليل العددي والحل الدقيق للمعادلات التفاضلية الكسرية غير الخطية. الهدف الرئيسي من هذه الأطروحة هو دراسة حلول المعادلات التفاضلية الكسرية غير الخطية التي تنطوي على عامل كابوتو الكسري من خلال اقتراح تقنية جديدة. لإثبات صحة وموثوقية هذه التقنية، يتم تطبيقها على العديد من الأمثلة العددية.

كلمات مفتاحية: المعادلات التفاضلية الكسرية ، المشتقة الكسرية لكابوتو، الحل التقريبي، الحل التحليلي.

Résumé:

Les équations différentielles fractionnaires non-linéaires jouent un rôle important en mathématiques appliquées et en physique. Il est difficile d'obtenir la solution exacte de ces problèmes en raison de la complexité des termes non-linéaires inclus. Au cours des dernières décennies, il y a eu un grand développement dans l'analyse numérique et la solution exacte des équations différentielles fractionnaires non-linéaires. L'objectif principal de cette thèse est d'étudier les solutions d'équations différentielles fractionnaires non-linéaires impliquant l'opérateur fractionnaire de Caputo en proposant une nouvelle technique. Pour démontrer la validité et la fiabilité de cette technique, elle est appliquée à plusieurs exemples numériques.

Mots clés: Équations différentielles fractionnaires, Dérivée fractionnaire de Caputo, Solution approximative, Solution analytique.

Abstract :

Nonlinear fractional differential equations play an important role in applied mathematics and physics. It is difficult to obtain the exact solution for these problems due to the complexity of the nonlinear terms included. In recent decades, there has been great development in the numerical analysis and exact solution for nonlinear fractional differential equations. The main objective of this thesis is to study the solutions of nonlinear fractional differential equations involving Caputo fractional operator by proposing new technique. To demonstrate the validity and reliability of this technique, it is applied to several numerical examples.

Key words: Fractional differential equations, Caputo fractional derivative, Approximate solution, Analytical solution.