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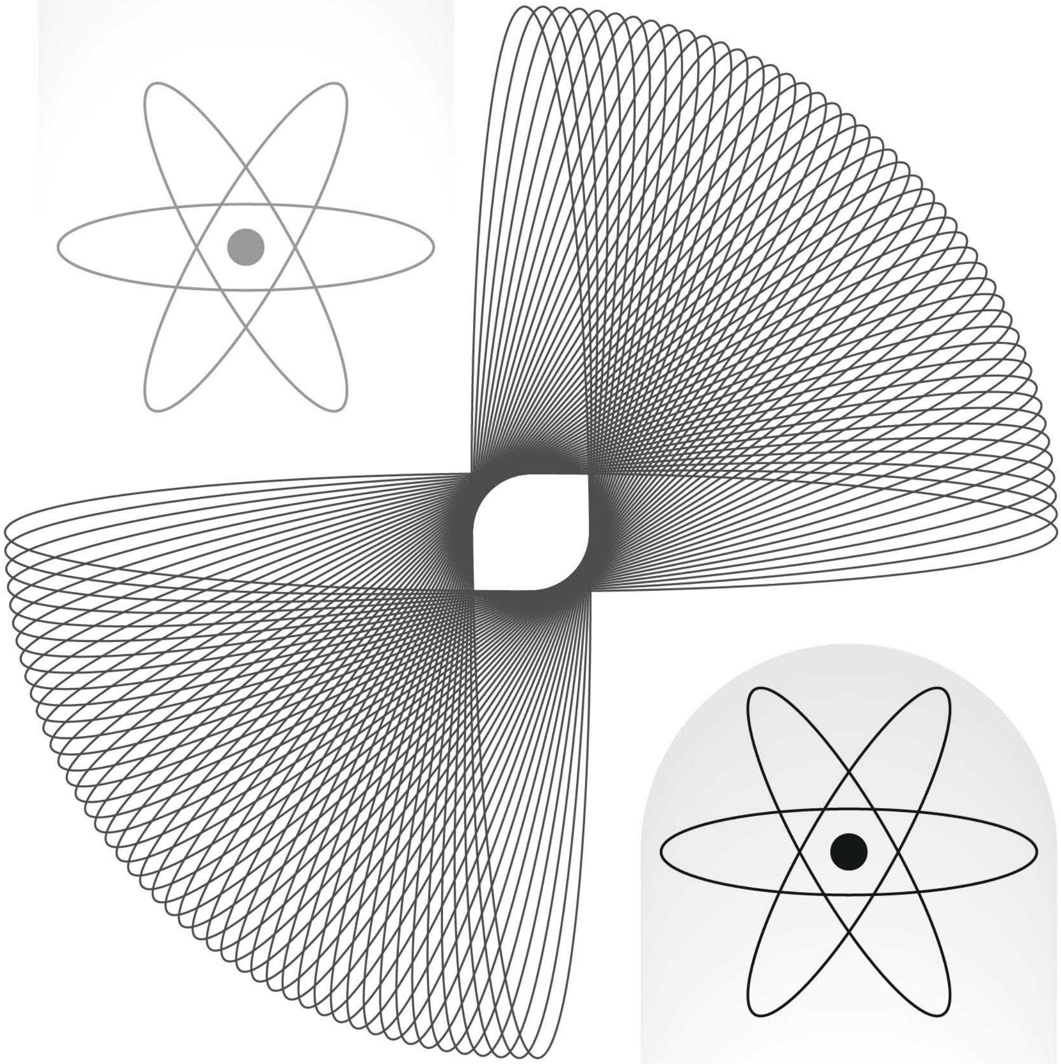
SCIENCES FACULTY
PHYSICS DEPARTEMENT

PHYSICS 1
Particle Mechanics



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CHAPTER 1

UNITS AND DIMENSIONAL ANALYSIS

I. MATHEMATICAL REMAINDER

A. UNITS AND DIMENSIONAL ANALYSIS

A.1 Introduction

A physical quantity is defined as any property of nature that can be quantified through measurement or calculation, and whose different possible values are expressed using a number accompanied by a unit of measurement.

Examples: time, length, and mass…

Dimensional analysis is the use of a set of units to establish the form of an equation, or more often, to check that the answer to a calculation as a guard against many simple errors.

A.2 Dimension of a Physical Quantity

The dimension of a quantity represents its physical nature and can be measured. In fact, A quantity can have the dimension of length, area, energy, mass, and so on. However, the concept of unit is arbitrary does not imply any particular choice of unit. For example, length can be expressed in meters, inches, angstroms, and so on.

Mathematically, the dimension of a physical quantity is the powers to which the fundamental (or base) quantities like mass, length and time etc. have to be raised to represent the quantity.

A.3 International system of units (SI)

The International System of Units, internationally known by the abbreviation SI (for Système International). It is based on seven fundamental quantities associated with a corresponding *base unit*. In terms of dimension, these base units are considered independent and are used to define all other quantities.

A.4 Fundamental (Base) Quantities

Base quantity	Symbol	Description	SI base unit	dimension
Amount of substance	n	The quantity proportional to the number of particles in a sample, with the Avogadro constant as the proportionality constant	mole (mol)	**N**
Length	l	The one-dimensional extent of an object	metre (m)	**L**
Time	t	The duration of an event	second (s)	**T**
Mass	m	A measure of resistance to acceleration	kilogram (kg)	**M**
Temperature	T	Average kinetic energy per degree of freedom of a system	kelvin (K)	**Θ**
Electric Current	I	Rate of flow of *electrical* charge per unit time	ampere (A)	**I**
Luminous intensity	I_v	Wavelength-weighted power of emitted light per unit solid angle	candela (cd)	**J**

A.5 Derived Quantities

Derived quantities are quantities that are calculated from two or more measurements. They cannot be measured directly. They can only be computed.

Derived quantity	Symbol	Description	SI derived unit	dimension
Area	A	Extent of a surface	m^2	$\boldsymbol{L^2}$
Acceleration	$\vec{a}$	Rate of change of velocity per unit time: the second time derivative of position	m/s^2	$\boldsymbol{L\,T^{-2}}$

A.6 cgs Measurement System

cgs measurement system is a metric system of measurement that uses the centimeter, gram and second respectively for length, mass and time. For example, the units of force and work are the "dyne" and "erg." However, MKS is the system of units based on measuring lengths in meters, mass in kilograms, and time in seconds. MKS is generally used in engineering and beginning physics, where the so-called cgs system (based on the centimeter, gram, and second) is commonly used in theoretic physics.

A.7 Dimensional Equation

The dimensional formula of any quantity is the expression showing the powers to which the fundamental units are to be raised to obtain one unit of a derived quantity. If Q is any physical quantity, the expression representing its dimensional formula is given by:

$$[\boldsymbol{Q}] = \boldsymbol{M}^{\alpha_1}.\boldsymbol{L}^{\alpha_2}.\boldsymbol{T}^{\alpha_3}.\boldsymbol{I}^{\alpha_4}.\boldsymbol{\theta}^{\alpha_5}.\boldsymbol{I_V}^{\alpha_6}.\boldsymbol{N}^{\alpha_7}$$

Where α_i are positive or negative integers.

N.B

- The dimensional equation of a dimensionless quantity Q reduces to $[Q]$ = 1.
- An angle is a dimensionless quantity: $[\theta]$ = 1.

A.8 Uses of Dimensional Equations

The dimensional equations have got the following uses:

- To check the correctness of a physical relation.
- To derive the relation between various physical quantities.
- To convert the value of physical quantity from one system of units to another system.
- To find the dimension of constants in a given relation.

A.9 Dimensional Analysis Rules

- $[Q^a] = [Q]^a$
- $[Q_1.Q_2] = [Q_1].[Q_2]$
- $[Q'(X)] = \left[\frac{dQ}{dX}\right] = \frac{[Q]}{[X]}$
- $\frac{d^nQ}{dX^n} = \frac{[\mathrm{Q}]}{[\mathrm{X}]^{\mathrm{n}}}$

- The argument of trigonometric functions (sin, cos, tan), exponential function (exp), and logarithmic functions (ln and log) must be dimensionless.
- We can only add quantities of the same dimension.
- An equation is homogeneous if both sides have the same dimension.

Example1

Write the dimensional equation for the following quantities and deduce their unit in the International System based on the base units.

1. Mass density ρ.
2. Force magnitude F = $\|\vec{F}\|$.
3. Energy E.
4. Angular frequency ω.

Solution

1. Mass density is the ratio of mass to volume: $\rho = m/V$, therefore $[\rho] = M\,L^{-3}$. Mass density is expressed in kg·m^{-3}

2. $F = ma \quad \Rightarrow \quad [F] = [ma] = M.\frac{L}{T^2}$. The force is expressed in $kg.m.s^{-2}$

3. We can use the expression for kinetic energy: $E = \frac{1}{2}mv^2$.

Therefore, $[E] = \left[\frac{1}{2}\right][m].[v]^2 = M\,L^2\,T^{-2}$.

Energy is expressed in joules (J), where 1 J = 1 kg·m^2·s^{-2}.

4. Angular frequency is the derivative of angle with respect to time: $\omega = \frac{d\theta}{dt}$. It is expressed in radians per second (rad·s^{-1}). Since an angle is a dimensionless quantity, [ω] = T^{-1}. Angular frequency has the dimension of frequency.

Example 2

verify the homogeneity of the relation $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ representing the oscillation frequency f of a solid-spring system, where m is the mass of the solid and k is the spring constant, and the elastic force $\vec{F}$ is related to the elongation $\overrightarrow{\Delta l}$ by the relation $\vec{F} = -k\overrightarrow{\Delta l}$

Solution

The relation $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ is homogeneous means that both sides of the equation have the same dimensions.

On the left-hand side, the frequency f is measured in hertz (Hz), which is the unit of time inverse (s^{-1}).

$$[f] = T^{-1}$$

On the right-hand side, we have $\frac{1}{2\pi}\sqrt{\frac{k}{m}}$.

From the relation $\vec{F} = -k\overrightarrow{\Delta l}$, it is clear that : $[F] = [k]L$.

since $[F] = M\,LT^{-2}$, we deduce that: $[k] = MT^{-2}$,

then $\left[\frac{1}{2\pi}\sqrt{\frac{k}{m}}\right] = \left[\sqrt{\frac{k}{m}}\right] = \sqrt{\frac{[k]}{[m]}} = \sqrt{\frac{MT^{-2}}{M}}$

The two sides of the relation have the same dimensions: the relation is homogenous

B. VECTORS

B.1 Vector and Scalar Quantities

B.1.1 *Vector Quantities*

Vector quantities are physical quantities that possess both magnitude and direction. They are represented by vectors ($\vec{Q} = |Q|\vec{u}\ unit$), which are mathematical objects containing information about the length (magnitude) and the direction of the quantity. Examples of vector quantities include displacement, velocity, force, acceleration, and momentum. Vector quantities cannot be fully described with just a single numerical value; their complete representation requires both magnitude and direction.

B.1.2 *Scalar Quantities*

Scalar quantities are physical quantities that have only magnitude and no direction. They are described completely by a numerical value and a unit ($Q = Q\ unit$), representing measurements such as length, mass, temperature, and time. Scalar quantities can be added, subtracted, multiplied, and divided using ordinary arithmetic operations. However, they do not involve any information about the direction in which the quantity is acting.

B.2 Vector Representation

Geometrically, the vector is commonly represented by a directed line segment whose length represents the magnitude and whose orientation in space represents the direction.

Algebraically, in an orthonormal coordinate system (*o, x, y, z*) for example, the vector can be expressed in terms of its components as:

$$\vec{A} = A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k} \quad or \quad \vec{A}\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

And its magnitude: $\|\vec{A}\| = |\vec{A}| = A = \sqrt{(A_x)^2 + \left(A_y\right)^2 + (A_z)^2}$

The vector joining the initial point $C(x_1, y_1, z_1)$ and the terminal point $D(x_2, y_2, z_2)$; is expressed as: $\overrightarrow{CD} = (x_2 - x_1)\vec{\imath} + (y_2 - y_1)\vec{\imath} + (z_2 - z_1)\vec{\imath}$

B.3 Unit Vector

A unit vector is a vector whose magnitude (norm) is equal to one (the number one).

A vector $\vec{A}$ parallel to the unit vector $\vec{u}$ can be expressed as: $\vec{A} = \|\vec{A}\|.\vec{u}$

The unit vector in the direction of a vector $\vec{V}$ is given by: $\vec{u} = \frac{\vec{A}}{\|\vec{A}\|}$

Example

The unit vector in the direction of the vector $\vec{V} = \sqrt{2}\vec{\imath} - 3\vec{\jmath} + 5\vec{k}$ is given by:

$$\vec{u} = \frac{\vec{V}}{\|\vec{V}\|} = \frac{\sqrt{2}\vec{\imath} - 3\vec{\jmath} + 5\vec{k}}{\sqrt{2+9+25}} = \frac{\sqrt{2}\vec{\imath} - 3\vec{\jmath} + 5\vec{k}}{6} = \frac{\sqrt{2}}{6}\vec{\imath} - \frac{1}{2}\vec{\jmath} + \frac{5}{6}\vec{k}$$

N.B: it is clear that $|\vec{u}| = 1$

B.4 Basis set

A vector basis set is a set of linearly independent vectors that span a vector space. In other words, it's a collection of vectors that can be combined in various linear combinations to represent any vector within that space.

Mathematically, if we have a vector space V and a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, then any vector v in V can be represented as:

$$v = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$$

where $c_1, c_2, \ldots, c_n$ are coefficients (scalars). The vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, are the vector basis set for the vector space V.

Note

The vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, must be linearly independent (no vector in the set can be expressed as a linear combination of the others) and they span the entire vector space (any vector in the space can be expressed as a linear combination of the basis vectors).

Example

The cartesian basis set is: $\{\vec{\imath}, \vec{\jmath}, \vec{k}\}$

B.5 Vector Operations

B.5.1 Vector Addition:

The sum of two vectors is a third vector. There are three methods: two geometric methods involving drawing, and one algebraic method.

1st Method: Place the vectors endpoint to origin and connect them with a vector from the origin of the first vector to the endpoint of the last vector. This resulting vector is called the resultant.

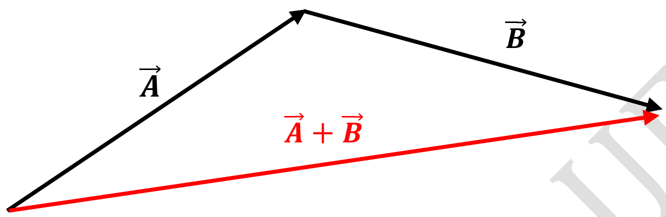


Figure 1.Vector Addition Illustration

2nd Method: To add two vectors with the same origin (point of application), one forms a parallelogram with the two vectors, and the diagonal of the parallelogram represents the resultant.

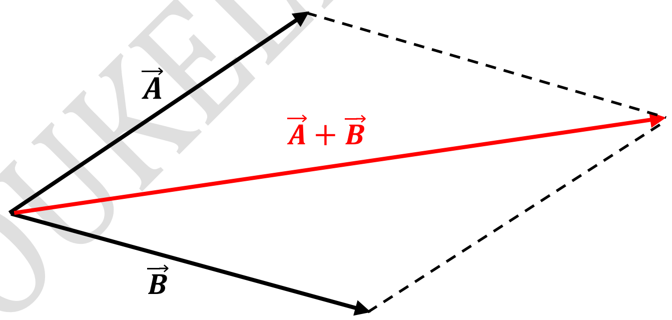


Figure 2. Vector Addition Allustration

3rd Method: The algebraic method involves adding the components of the vectors to find the components of the resultant.

Let:

$$\vec{A} = A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k}$$

$$\vec{B} = B_x\vec{\imath} + B_y\vec{\jmath} + B_z\vec{k}$$

The sum $\vec{C} = \vec{A} + \vec{B}$ is given by:

$$\vec{C} = \vec{A} + \vec{B} = (A_x + B_x)\vec{\imath} + (A_y + B_y)\vec{\jmath} + (A_z + B_Z)\vec{k}$$

Chasles Relation

This relation permits the calculation of the sum of two or more vectors. It is given by:

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

In this relation, AB, BC, and AC represent the displacement vectors from point A to B, from point B to C, and from point A to C, respectively.

B.5.2 Vector Subtraction

Geometrically, using the relation $\overrightarrow{AB} = -\overrightarrow{BA}$, subtracting vectors $\vec{A}$ and $\vec{B}$ becomes very simple. In fact, to subtract the vector $\vec{B}$ from the vector $\vec{A}$, we reverse the direction of the vector $\vec{B}$ (changing the sign of its components) and then add it to vector $\vec{A}$.

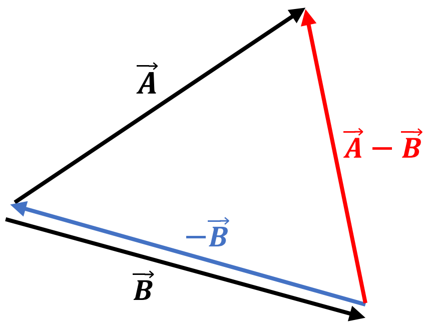


Figure 3.Vector Subtraction Illustration

Algebraically, the subtraction of $\vec{B}$ from $\vec{A}$ can be expressed as follows:

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B}) = (A_x + B_x)\vec{\imath} + (A_y + B_y)\vec{\jmath} + (A_z + B_Z)\vec{k}$$

N.B: The subtraction of vectors is not commutative (The result depends on the order of subtraction).

B.5.3 Scalar Multiple

Let $\vec{A}$ be a vector and λ be a scalar (a real number). Then the **scalar multiple** $\lambda\vec{A}$ is the vector whose magnitude is $|\lambda||\vec{A}|$ and whose direction is

- the same as $\vec{A}$ if λ is positive,
- undefined if λ is zero (the answer is the zero vector),
- the same as $(-\vec{A})$ if λ is negative.

Properties

- $\lambda(\mu\vec{A}) = (\lambda\mu)\vec{A}$ (Associative law)
- $\lambda(\vec{A}+\vec{B}) = \lambda\vec{A}+\lambda\vec{B}$ (Distributive law)
- $(\lambda+\mu)\vec{A} = \lambda\vec{A}+\mu\vec{B}$ (Distributive law)

B.5.4 Vector Product (Multiplication)

There are two main types of vector products: ***The Dot Product*** (also known as the scalar product) and ***The Cross Product*** (also known as the vector product). Both operations involve two vectors and yield different results.

B.5.4.1 Dot Product (Scalar Product)

a. *Geometrically*

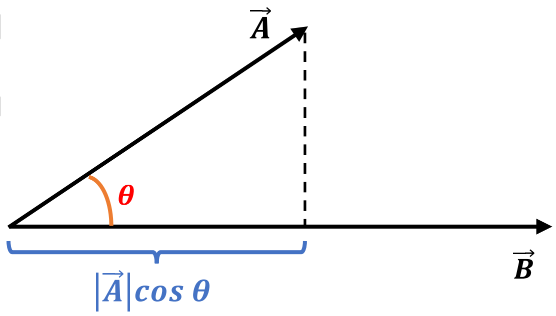


Figure 4. Dot Product Illustration

The dot product of two vectors $\vec{A}$ and $\vec{B}$ is a scalar quantity obtained by multiplying the magnitudes of the vectors and the cosine of the angle between them. Mathematically, it is represented as:

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$$

Where:

$|\vec{A}|$ and $|\vec{B}|$ are the magnitudes (lengths) of vectors $\overrightarrow{A}$ and $\overrightarrow{B}$, respectively, and θ is the angle between the two vectors.

b. Algebraically

The dot product of two vectors $\vec{A}$ and $\vec{B}$ is given by the sum of the products of their corresponding components.

$\vec{A} = A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k}$

$\vec{B} = B_x\vec{\imath} + B_y\vec{\jmath} + B_z\vec{k}$

$\vec{A} \cdot \vec{B} = A_x.B_x + A_y.B_y + A_z.B_z$

c. Dot Product Properties

- The dot product result is a scalar value,
- The dot product is commutative $(\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A})$,
- $\vec{A}.\vec{B} = 0 \;\Leftrightarrow \begin{cases} \vec{A} = \vec{0} \\ \vec{B} = \vec{0} \\ \vec{A} \perp \vec{B} \end{cases}$

B.5.4.2 Cross Product (Vector Product)

a. Geometrically

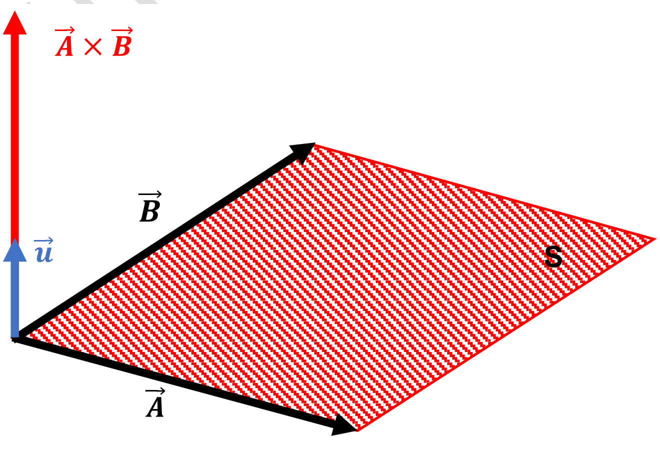


Figure 5. Cross Product Illustration

The cross product of two vectors $\vec{A}$ and $\vec{B}$ is a vector quantity perpendicular to both $\vec{A}$ and $\vec{B}$. The magnitude of the cross product is equal to the product of the magnitudes of $\vec{A}$ and $\vec{B}$ multiplied by the sine of the angle between them. Mathematically, it is represented as:

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta \, \vec{u}$$

Where:

$|\vec{A}|$ and $|\vec{B}|$: are the magnitudes of vectors $\overrightarrow{A}$ and $\overrightarrow{B}$, respectively,

θ: is the angle between the two vectors,

$\vec{u}$: is the unit vector perpendicular to both $\vec{A}$ and $\vec{B}$, following the right-hand rule.

b. Algebraically

The cross product of two vectors $\vec{A}$ and $\vec{B}$ is a vector that is perpendicular to both $\vec{A}$ and $\vec{A}$. It is defined as:

$$\vec{A} \times \vec{B} = \vec{c} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y.B_z - A_z.B_y)\vec{\imath} + (A_z.B_x - A_x.B_z)\vec{\jmath} + (A_x.B_y - A_y.B_x)\vec{k}$$

c. Cross Product Properties

- The cross-product results in a vector value,
- The cross product is not commutative: $\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$,
- $|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin\theta = S$,
- $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ (Anti-commutative law)
- $\vec{A} \times \overrightarrow{(B + C)} = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$ (Distributive law)
- $(\lambda\vec{A}) \times \vec{B} = \lambda(\vec{A} \times \vec{B})$ (Associative with scalar multiple)
- $\vec{A} \times \vec{B} = \vec{0} \quad \Leftrightarrow \begin{cases} \vec{A} = \vec{0} \\ \vec{B} = \vec{0} \\ \vec{A} \parallel \vec{B} (colinear) \end{cases}$
- $\overrightarrow{(A \times B)} \times \vec{C} \neq \vec{A} \times \overrightarrow{(B \times C)}$ (Non-associative)
- If $\{\vec{\imath}, \vec{\jmath}, \vec{k}\}$ is a standard basis then

  $\vec{\imath} \times \vec{\jmath} = \vec{k} \quad \vec{\jmath} \times \vec{k} = \vec{\imath} \quad \vec{k} \times \vec{\imath} = \vec{\jmath}$

  $\vec{\imath} \times \vec{\imath} = \vec{\jmath} \times \vec{\jmath} = \vec{k} \times \vec{k} = \vec{0}$

Note

- Since the vector product is anti-commutative, the *order of the terms in vector products must be preserved*.
- The vector product is not associative.

B.5.4.3 Tripple Scalar Product (Mixed Product)

is defined as the dot product of one of the vectors with the cross product of the other two.

$$\vec{A}.(\vec{B} \times \vec{C})$$

Geometrically, the scalar triple product is the volume of the parallelepiped defined by the three given vectors.

Properties

- Unchanged under a circular shift
  $\vec{A}.(\vec{B} \times \vec{C}) = \vec{B}.(\vec{C} \times \vec{A}) = \vec{C}.(\vec{A} \times \vec{B})$
- Swapping the positions of the operators without re-ordering the operands, leaves the triple product unchanged.
  $\vec{A}.(\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}).\vec{C}$
- Swapping any two of the three operands negates the triple product
  $\vec{A}.(\vec{B} \times \vec{C}) = -\vec{A}.(\vec{C} \times \vec{B}) = -\vec{B}.(\vec{A} \times \vec{C}) = \vec{C}.(\vec{B} \times \vec{A})$
- The scalar triple product can also be understood as the determinant.

$$\vec{A}.(\vec{B} \times \vec{C}) = \det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

B.6 Application

In a Cartesian coordinate system (O, x, y, z) we consider three vectors:

$\overrightarrow{V_1} = 3\vec{\imath} - 4\vec{\jmath} + 4\vec{k}$

$\overrightarrow{V_2} = 2\vec{\imath} + 3\vec{\jmath} - 4\vec{k}$

$\overrightarrow{V_3} = 5\vec{\imath} - \vec{\jmath} + 3\vec{k}$

a) Calculate the magnitude of $\overrightarrow{V_1}, \overrightarrow{V_2}, \overrightarrow{V_3}$

b) Calculate the components and the magnitude of the vectors:

$\vec{A} = \overrightarrow{V_1} + \overrightarrow{V_2} + \overrightarrow{V_3}$

$\vec{B} = 2\overrightarrow{V_1} - \overrightarrow{V_2} + \overrightarrow{V_3}$

c) Determine the unit vector pointing to the direction (carried by) of the vector:

$\vec{C} = \overrightarrow{V_1} + \overrightarrow{V_3}$

d) Calculate the scalar product $\overrightarrow{V_1}.\overrightarrow{V_3}$ and deduce the angle between the two vectors.

e) Calculate the cross product $\overrightarrow{V_2} \times \overrightarrow{V_3}$

B.7 Solution

a) $\|\overrightarrow{V_1}\| = \sqrt{3^2 + (-4)^2 + 4^2} = \sqrt{41}$

$\|\overrightarrow{V_2}\| = \sqrt{2^2 + 3^2 + (-4)^2} = \sqrt{29}$

$\|\overrightarrow{V_3}\| = \sqrt{5^2 + (-1)^2 + (3)^2} = \sqrt{35}$

b) $\vec{A} = 10\vec{\imath} - 2\vec{\jmath} + 3\vec{k}$

$\vec{B} = 9\vec{\imath} - 15\vec{\jmath} + 15\vec{k}$

c) $\vec{C} = 8\vec{\imath} - 5\vec{\jmath} + 7\vec{k} \Rightarrow \qquad \overrightarrow{u_C} = \frac{\vec{C}}{\|\vec{C}\|} = \frac{8}{\sqrt{138}}\vec{\imath} - \frac{5}{\sqrt{138}}\vec{\jmath} + \frac{7}{\sqrt{138}}\vec{k}$

d) $\overrightarrow{V_1}.\overrightarrow{V_3} = 31$

$$cos\alpha = \frac{\overrightarrow{V_1}.\overrightarrow{V_3}}{V_1.V_3} = \frac{31}{\sqrt{41}\sqrt{35}} \approx 0.176 \qquad \Rightarrow \qquad \alpha \approx arcos 0.176 = 79.96°$$

e) $\overrightarrow{V_2} \wedge \overrightarrow{V_3} = 5\vec{\imath} - 26\vec{\jmath} - 17\vec{k}$

C. ERRORS AND UNCERTAINTIES

C.1 Introduction

Understanding and quantifying uncertainty and errors is crucial in scientific research, engineering, and decision-making processes. It allows researchers and professionals to communicate the reliability and limitations of their results and conclusions, and to make informed judgments about the significance of their findings. It also helps assess the accuracy and reliability of measurements, experimental data, and numerical simulations.

C.2 Error Definition

An error, in the context of measurement and scientific analysis, refers to the discrepancy between the measured or observed value of a quantity and its true or expected value. Errors can arise from various sources such as:

- limitations in measurement instruments,
- human errors during data collection,
- environmental factors,
- and inherent uncertainties in the phenomena being studied.

Errors can be:

- systematic (consistent and predictable)
- random (unpredictable and erratic).

Managing and understanding errors is crucial in obtaining accurate and reliable results in scientific research and various practical applications.

In mathematical terms, the error (often denoted as δx or ε) associated with a measured or observed quantity x is defined as the difference between the measured value x and the true value x_0:

Error=Measured Value−True Value

This can also be expressed as:

$$\boldsymbol{\delta x = x - x_0}$$

Where:

- x is the measured value.
- x_0 is the true value.

The error can be positive or negative, depending on whether the measured value is greater or smaller than the true value. In many cases, errors are expressed as absolute values to indicate the magnitude of the difference regardless of its direction.

C.3 Uncertainty

Uncertainty refers to the lack of complete precision or exactness in a measurement, calculation, prediction, or any other quantitative value. It indicates the range of possible values within the true value of a quantity is expected to lie, taking into account the limitations of the measurement process or the inherent variability of the system.

Uncertainty can arise from various sources, including limitations in measurement instruments, variations in experimental conditions, errors in data collection, and inherent randomness in certain physical processes. Uncertainty is often quantified by using measures such as standard deviation, error bars, confidence intervals, and uncertainty intervals.

C.4 Absolute uncertainty

Absolute uncertainty, also known as absolute error, refers to the actual magnitude of the difference between a measured or calculated value and the true or expected value of a quantity. It quantifies how much an individual measurement or calculation deviates from the true value.

Mathematically, the absolute uncertainty is expressed as:

$$\boldsymbol{\Delta x = sup\{|\delta x_i|\}}$$

This means that the exact value of a measurement is found between two known limit values:

$$x - \Delta x \leq x_0 \leq x + \Delta x$$

The result of measurement is written:

$$\boldsymbol{x = x_0 \mp \Delta x}$$

Note

Absolute uncertainty is often expressed with the same units as the quantity being measured or calculated.

C.5 Significant figures

Significant figures, also known as significant digits, are the digits in a numerical value that contribute to its precision and accuracy. They indicate the reliability of the measurement or calculation and help convey the uncertainty associated with the value. The rules for determining significant figures are as follows:

- *Experimental uncertainties should almost always be rounded to one significant figure.*

If we measure the acceleration of gravity g, it would be absurd to state the result like:

$$g = 9.82 \mp 0.02385 \quad m/s^2$$

The uncertainty should be rounded to $\Delta g = 0.02\, m/s^2$ and the measurement result should be rewritten as:

$$g = 9.82 \mp 0.02\, m/s^2$$

- *The last significant figure in any stated answer should usually be of the same order of magnitude (in the same decimal position) as the uncertainty.*

Example

- The answer 592.81 with an uncertainty of 0.3 should be rounded as: 592.8 ∓ 0.3
- The answer 592.81 with an uncertainty of 3 should be rounded as: 593 ∓ 3
- The answer 592.81 with an uncertainty of 30 should be rounded as: 590 ∓ 30

C.6 Relative uncertainty

Relative uncertainty, also known as relative error, is a measure of the error or uncertainty associated with a measurement or calculation, expressed as a ratio of the absolute uncertainty to the measured or calculated value itself. It provides a way to understand the error in relation to the size of the quantity being measured.

Mathematically, the relative uncertainty (RU) is calculated as:

$$RU = \frac{\Delta x}{x}$$

It is equal to the modulus of the logarithmic differential: $\frac{\Delta x}{x} = \left|\frac{dx}{x}\right|$

Relative uncertainty calculates the precision of the measurement. It is often expressed as a percentage (%).

$$RU\% = \frac{\Delta x}{x} \times 100$$

Example

if you measure the length of an object as 50 cm with an absolute uncertainty of 2 cm, the relative uncertainty is:

$$RU = \frac{\Delta x}{x} = \frac{2}{50} \times 100 = 4\ \%$$

This indicates that the measurement has a relative error of 4% compared to the measured value.

C.7 Uncertainty Calculations

Let $Q = Q(x, y, z)$ be a physical quantity.

Algebraic Method

- **Sum :** $Q = x + y \quad \Rightarrow \quad \Delta Q = \Delta x + \Delta y$
- **Subtraction :** $Q = x - y \quad \Rightarrow \quad \Delta Q = \Delta x + \Delta y$
- **Product:** $Q = x.y \quad \Rightarrow \quad \Delta Q = y\Delta x + x\Delta y \quad \Rightarrow \quad \frac{\Delta Q}{Q} = \frac{\Delta x}{x} + \frac{\Delta y}{y}$
- **Power (exponent):** $Q = x^n \quad \Rightarrow \quad \frac{\Delta Q}{Q} = |n|\frac{\Delta x}{x}$
- **Quotient :** $Q = \frac{x}{y} \quad \Rightarrow \quad \frac{\Delta Q}{Q} = \frac{\Delta x}{x} + \frac{\Delta y}{y}$

Differential method

To calculate the uncertainties on Q, we follow the steps below:

- We calculate the differential dQ of the quantity Q given as:

$$dQ = \frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy + \frac{\partial Q}{\partial z}dz$$

- The absolute uncertainty is :

$$\Delta Q = \left|\frac{\partial Q}{\partial x}\right|\Delta x + \left|\frac{\partial Q}{\partial y}\right|\Delta y + \left|\frac{\partial Q}{\partial z}\right|\Delta z$$

Logarithmic differential method

To calculate the uncertainties on Q, we follow the steps below:

- We calculate $\ln Q(x, y, z)$
- We calculate the logarithmic derivative of each term appearing in the expression of $\ln Q(x, y, z)$.

- After calculating the absolute value of each term, we obtain the relative uncertainty.

Example

The speed is given by the formula

$$v = \frac{s}{t}$$

Where:

s is the displacement

t is the time

let's calculate the relative uncertainty on speed using the logarithmic differential:

$$\ln v = \ln \frac{s}{t} = \ln s - \ln t$$

$$\frac{dv}{v} = \frac{ds}{s} - \frac{dt}{t} \qquad \Rightarrow \quad \frac{\Delta v}{v} = \frac{\Delta s}{s} + \frac{\Delta t}{t}$$

CHAPTER 2

PARTICLE KINEMATICS

II. PARTICLE KINEMATICS

II.1 Introduction

In practice, all studied objects in physics are in motion: elementary particles that makeup atoms, everyday objects, and celestial bodies... Therefore, it is essential to know the laws that govern the different types of motion. The branch of physics that studies motion is called mechanics, which itself, is subdivided into two sub-branches: kinematics and dynamics.

II.2 General Definitions

II.2.1 Kinematics

Kinematics is a branch of physics that deals with the description of motion, including the position, velocity, and acceleration of objects. It focuses on studying the relationships between the various parameters that characterize the movement of objects without necessarily delving into the causes of the motion, such as forces or interactions.

In kinematics, the primary concepts include:

Position: The location of an object in space with respect to a chosen reference point or coordinate system.

Displacement: The change in position of an object during its motion, represented as a vector quantity.

Velocity: The rate of change of displacement with respect to time. It is also a vector, having both magnitude and direction.

Acceleration: The rate of change of velocity with respect to time. Like velocity, it's also a vector.

Time: The parameter that quantifies the duration of motion.

By analyzing these parameters, kinematics allows us to describe the motion of objects and predict their future positions, velocities, and accelerations without considering the forces or factors causing the motion. It provides the foundation for understanding and describing various types of motion, such as uniform motion, uniformly accelerated motion, and projectile motion.

II.2.2 Point Particle

A "particle" or "point particle" is a concept used in physics to represent an object with mass but without any physical dimensions. In other words, it is an idealization of a physical object where its size, shape, and internal structure are ignored, and only its mass and certain attributes like position and velocity are considered.

Particles are often used in simplified models to study the behavior of objects in various physical phenomena. They are treated as mathematical points in space, and their motion and interactions are described using principles of classical mechanics or other relevant theories, depending on the context.

But, even a body of *finite* dimensions may be treated as a “particle” if its motion is *purely translational* (that is, if the body is not rotating).

II.2.3 Frame of Reference

To locate the position of a point particle in space, we need to define a spatial frame of reference. This involves choosing an origin O and a basis $(\vec{\imath}, \vec{\jmath}, \vec{k})$, for example. The trihedron $(\vec{\imath}, \vec{\jmath}, \vec{k})$ represents a *Cartesian coordinate system*.

Consequently, a frame of reference is a set of coordinates that can be used to determine positions and velocities of objects in that frame; different frames of reference move relative to one another. Therefore, all motion is relative to the reference frame used.

II.2.4 Usual reference frames

II.2.4.1 Galilean (inertial) Reference Frame

In classical physics and special relativity, a Galilean frame of reference is a frame of reference that is not undergoing any acceleration. It is a frame in which an isolated physical object (an object with zero net force acting on it) is perceived to move with a constant velocity or, equivalently, it is a frame of reference in which Newton's first law of motion holds. According to the durations considered in the experiments, the phenomena linked to their non-Galilean character remain negligible compared to the Galilean one.

II.2.4.2 Earth-centered Inertial (ECI)

It consists of an origin at the center of mass of the earth and three axes pointing to three stars considered as fixed. Point on the ground and three axes (generally, one vertical axis and two axes in the horizontal plane). It is not rotating, with respect to the stars; useful to describe motion of celestial bodies and spacecraft.

It is also used to describe the movements of objects on a small scale that surround us.

II.2.4.4 Earth-centered, Earth-fixed Reference Frame (ECEF)

It is also known as ***Geocentric Reference Frame.*** It has its origin at the center of mass of the earth and three axes pointing towards sufficiently far stars that can be considered fixed. It remains fixed with respect to Earth's surface in its rotation, and then rotates with respect to stars. It is not inertial, accelerated and rotating with respect to the stars. It is useful to describe motion of objects on Earth surface.

II.2.4.5 Reference Frame of Kepler (Heliocentric Reference Frame)

It consists of the center of the Sun and three axes pointing towards sufficiently distant stars that can be considered fixed. This reference frame is used to describe movements on the scale of the solar system (e.g., planetary motion).

II.2.4.5 Copernican Reference Frame

The Copernican reference frame is centered on the center of mass of the solar system, and its axes point towards three distant stars.

II.3 Motion Characteristics

The study of an object's motion can be carried out in the following ways:

Algebraic*:* Using the time equations of motion.

Vectorial*:* Using position vector $\overrightarrow{OP}$, velocity $\vec{v}$ and acceleration $\vec{a}$.

II.3.1 Time Equations

Mathematically, an object remains in rest (in a chosen reference frame) if its coordinates x, y, and z are independent of time, and it is in motion if its coordinates vary with time. The variations of the coordinates of an object with respect to time are called the time equations of motion. Generally, they are given by: $\begin{cases} x(t) \\ y(t) \\ z(t) \end{cases}$

II.3.2 Path (Trajectory)

The path of a moving point P in a given reference frame is the curve formed by the set of successive positions of P in that reference frame.

- The path of a moving point depends on the chosen reference frame.
- In Cartesian coordinates, the path is obtained by eliminating time from the time dependent equations.

II.3.3 Position Vector:

The position P of a point particle at time t is represented in a reference frame by a position vector $\overrightarrow{OP} = \vec{r}$, which join the origin of the reference frame to the position of the particle.

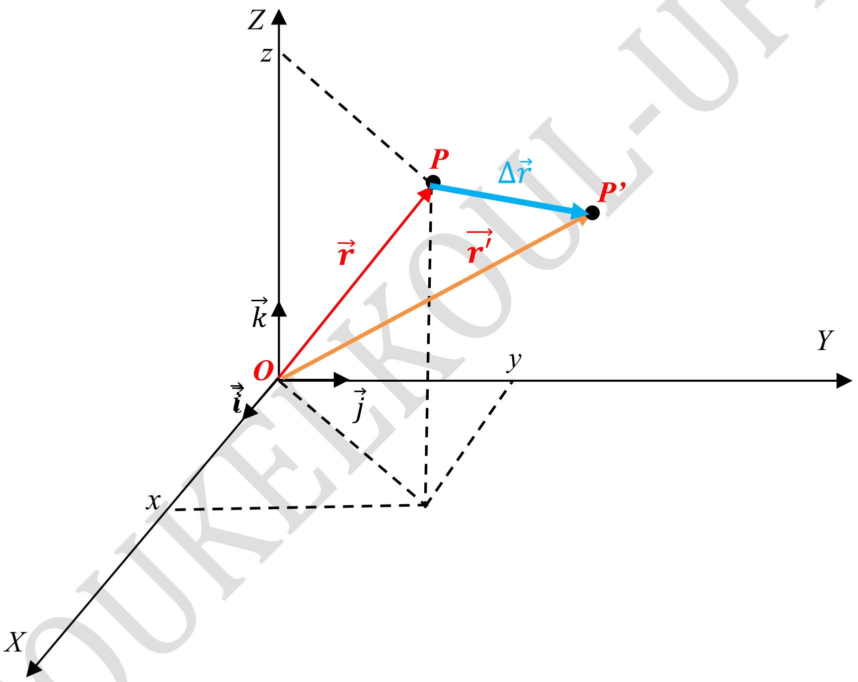


Figure 2-1: Illustration of the Position Vector

In Cartesian coordinates, it is given by the expression:

$$\vec{r}(t) = \overrightarrow{OP} = x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}$$

$$|\vec{r}(t)| = |\overrightarrow{OP}| = \sqrt{x(t)^2 + y(t)^2 + z(t)^2}$$

- The position vector $\vec{r}(t)$ is a function of time.
- The vector change $\Delta\vec{r} = \vec{r}' - \vec{r}$ is called *displacement vector*.
- The distance between P and P' is $PP' = |\Delta\vec{r}| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$

II.3.4 Velocity

II.3.4.1 Average Velocity

Let's consider an object (point particle) moving from the position P_1 at time t_1 to the position P_2 at time t_2 on the trajectory C.

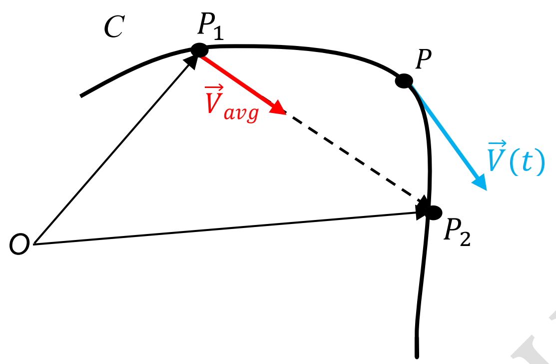


Figure2-2: Schematic Illustration of the Average (Red) and Instantaneous (Blue) Velocity

The average velocity of the object between times t_1 and t_2, where $\Delta t = t_2 - t_1$ is given by:

$$\vec{V}_{avg} = \frac{\overrightarrow{OP_2} - \overrightarrow{OP_1}}{t_2 - t_1} = \frac{\overrightarrow{P_1P_2}}{\Delta t}$$

$\overrightarrow{P_1P_2}$ is the displacement vector.

$\vec{V}_{avg}$ is in the direction of (carried by) the displacement vector.

II.3.4.2 Instantaneous Velocity

The instantaneous velocity of an object is the velocity at a specific moment in time. It represents the object's speed and direction at that exact instant. Instantaneous velocity. To calculate instantaneous velocity, you take the derivative of the object's position with respect to time.

$$\vec{V}(t) = \lim_{t_1 \to t_2} \frac{\overrightarrow{OP_2} - \overrightarrow{OP_1}}{t_2 - t_1} = \lim_{\Delta t \to 0} \frac{\overrightarrow{P_1P_2}}{\Delta t} = \frac{d\overrightarrow{OP}}{dt} = \frac{d\vec{r}}{dt}$$

The instantaneous velocity vector $\vec{V}(t)$ is carried by the tangent to the path C at the point P and is always oriented in the direction of motion.

In a Cartesian coordinate system, the instantaneous velocity at time t is given from the position vector $\vec{r} = \overrightarrow{OP}$ by the following relation:

$$\vec{V}(t) = \frac{d\overrightarrow{OP}}{dt} = \frac{d\vec{r}}{dt} = \frac{dx(t)}{dt}\vec{\imath} + \frac{dy(t)}{dt}\vec{\jmath} + \frac{dz(t)}{dt}\vec{k} = \dot{x}(t)\vec{\imath} + \dot{y}(t)\vec{\jmath} + \dot{z}(t)\vec{k}$$

It is the derivative of the position vector with respect to time

$$\vec{V}(t) = \begin{cases} V_x = \dot{x}(t) \\ V_y = \dot{y}(t) \\ V_z = \dot{z}(t) \end{cases} \quad \Rightarrow \quad \|\vec{V}(t)\| = \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2 + (\dot{z}(t))^2}$$

Note:

- $\|\vec{V}(t)\|$ represents the speed.
- In practice, the relation between instantaneous velocity and average velocity is given by: $\vec{V}(t) = \vec{V}_{avg}\Big|_{t_1}^{t_2}$ where $t = \frac{t_1+t_2}{2}$
- This relation is valid when t_1 and t_2 are very close to each other.

II.3.5 Acceleration Vector

II.3.5.1 Average Acceleration Vector

The average acceleration vector of an object over a specific time interval is defined as the change in velocity divided by the change in time during that interval.

Mathematically, if an object's initial velocity at time t_1 is $\vec{V}(t_1)$ and its final velocity at time t_2 is $\vec{V}(t_2)$, the average acceleration vector is given by:

$$\vec{a}_{avg} = \frac{\vec{V}(t_2) - \vec{V}(t_1)}{t_2 - t_1} = \frac{\Delta\vec{V}}{\Delta t}$$

The average acceleration vector is a vector quantity and takes into account both the magnitude and direction of the change in velocity over the specified time interval. It represents the average rate at which the object's velocity changes during that time interval.

II.3.5.2 Instantaneous Acceleration Vector

The instantaneous acceleration of an object at a specific moment in time is defined as the rate of change of its velocity with respect to time at that instant. In other words, it represents how quickly the velocity of the object is changing at that particular moment.

Mathematically, the instantaneous acceleration vector is given by the derivative of the velocity vector with respect to time:

$$\vec{a}(t) = \lim_{t_1 \to t_2} \frac{\vec{V}(t_2) - \vec{V}(t_1)}{t_2 - t_1} = \lim_{\Delta t \to 0} \frac{\Delta\vec{V}}{\Delta t} = \frac{d\vec{V}(t)}{dt}$$

In cartisian coordinates

$$\vec{a}(t) = \frac{d\vec{V}(t)}{dt} = \frac{dV_x}{dt}\vec{\imath} + \frac{dV_y}{dt}\vec{\jmath} + \frac{dV_z}{dt}\vec{k} = \frac{d^2x(t)}{dt^2}\vec{\imath} + \frac{d^2y(t)}{dt^2}\vec{\jmath} + \frac{d^2z(t)}{dt^2}\vec{k}$$

$$\vec{a}(t) = \frac{d\vec{V}(t)}{dt} = \ddot{x}\vec{\imath} + \ddot{y}\vec{\jmath} + \ddot{z}\vec{k}$$

$$|\vec{a}(t)| = \sqrt{(\ddot{x}(t))^2 + (\ddot{y}(t))^2 + (\ddot{z}(t))^2}$$

The acceleration vector is always directed towards the concave part of the trajectory to enable the object to follow the curve and maintain its motion on this curved path.

II.3.6 Change of Speed

- $\vec{a}.\vec{V} > 0$: the motion is accelerated,
- $\vec{a}.\vec{V} < 0$: the motion is retarded,
- $\vec{a}.\vec{V} = 0 \ (\vec{a} \perp \vec{V})$: the speed of the moving object is constant in time, even though the direction of the velocity is changing. Thus, the motion is uniform.
- $V = cte$: the motion is uniform.

Exercise

The time equations of a point particle M in a Cartesian reference frame (O,x,y) are given by: $\begin{cases} x(t) = 2cos\frac{t}{3} \\ y(t) = \sqrt{3}sin\frac{t}{3} \end{cases}$

1. Find the trajectory (path) of the point M.

2. Write the expressions for $\vec{V}$ and $\vec{a}$ the velocity and acceleration vectors of M.

3. Deduce the relationship between the acceleration $\vec{a}$ and the position $\overrightarrow{OM}$.

4. Give the period T of the motion.

5. Give the position of the moving body when the speed is equal to $\frac{2}{3}m/s$

N.B: $\cos\,\alpha = cos\,\beta \quad \Rightarrow \quad \alpha = \beta + 2\pi$

Solution

The time equations of a point particle M in a Cartesian reference frame (O,x,y) are given by: $\begin{cases} x(t) = 2cos\frac{t}{3} \\ y(t) = \sqrt{3}sin\frac{t}{3} \end{cases}$

1) Trajectory of the point *M*.

$$\begin{cases} \frac{x(t)}{2} = cos\frac{t}{3} \\ \frac{y(t)}{\sqrt{3}} = sin\frac{t}{3} \end{cases} \quad \Rightarrow \quad \frac{x^2}{4}+\frac{y^2}{3} = 1$$ The trajectory is an ellipse.

2) Velocity an acceleration of M

a. Velocity $\vec{V}$

$$\vec{V}(t) = \frac{d\overrightarrow{OM}}{dt} = \begin{cases} V_x = \dot{x}(t) = -\frac{2}{3}sin\frac{t}{3} \\ V_y = \dot{y}(t) = \frac{\sqrt{3}}{3}cos\frac{t}{3} \end{cases} \quad \Longrightarrow \vec{V}(t) = \left(-\frac{2}{3}sin\frac{t}{3}\right)\vec{\imath}+\left(\frac{\sqrt{3}}{3}cos\frac{t}{3}\right)\vec{\jmath}$$

b. Acceleration $\vec{a}$

$$\vec{a}(t) = \frac{d\vec{V}}{dt} = \begin{cases} a_x = \ddot{x}(t) = -\frac{2}{9}cos\frac{t}{3} \\ a_y = \ddot{y}(t) = -\frac{\sqrt{3}}{9}sin\frac{t}{3} \end{cases} \quad \Longrightarrow \vec{a}(t) = -\left(\frac{2}{9}cos\frac{t}{3}\right)\vec{\imath}-\left(\frac{\sqrt{3}}{9}sin\frac{t}{3}\right)\vec{\jmath}$$

3) Relationship between Acceleration and Position.

$$\vec{a}(t) = -\left(\frac{2}{9}cos\frac{t}{3}\right)\vec{\imath}-\left(\frac{\sqrt{3}}{9}sin\frac{t}{3}\right)\vec{\jmath} = -\frac{1}{9}\left[\left(2\,cos\frac{t}{3}\right)\vec{\imath}+\left(\sqrt{3}sin\frac{t}{3}\right)\vec{\jmath}\right] = -\frac{1}{9}\overrightarrow{OM}$$

4) Motion Period

$$x(t+T) = x(t) \quad \Leftrightarrow \quad 2cos\frac{t+T}{3} = 2cos\frac{t}{3}$$

Using the following trigonometric relationship: $\cos\alpha = \cos\beta \quad \Longrightarrow \quad \alpha = \beta+2\pi$, we get:

$$\frac{t+T}{3} = \frac{t}{3}+2\pi \quad \Longrightarrow \quad \frac{T}{3} = 2\pi \quad \Longrightarrow T = 6\pi\ s$$

5) Position of the moving body when the speed is equal to $\frac{2}{3}m/s$

$$\vec{V}(t) = -\left(\frac{2}{3}sin\frac{t}{3}\right)\vec{\imath}+\left(\frac{\sqrt{3}}{3}cos\frac{t}{3}\right)\vec{\jmath} \quad \Longrightarrow \left\|\vec{V}(t)\right\| = \frac{1}{3}\sqrt{4\,sin^2\frac{t}{3}+3cos^2\frac{t}{3}}$$

$$= \frac{1}{3}\sqrt{sin^2\frac{t}{3}+3}$$

$$\left\|\vec{V}(t)\right\| = \mathbf{\frac{2}{3}} \quad \Leftrightarrow \quad \frac{1}{3}\sqrt{sin^2\frac{t}{3} + 3} = \mathbf{\frac{2}{3}}$$

$$\Rightarrow \sqrt{sin^2\frac{t}{3} + 3} = \mathbf{2} \quad \Rightarrow \quad sin^2\frac{t}{3} + 3 = 4$$

$$\Rightarrow \quad sin\frac{t}{3} = \pm 1 \quad \Rightarrow \quad \begin{cases} \dfrac{t}{3} = \dfrac{\pi}{2}\ s \\ \dfrac{t}{3} = \dfrac{3\pi}{2}\ s \end{cases}$$

So, the position for this speed is given by:

$$M_1 \begin{cases} x_1(t) = 2cos\frac{\pi}{2} = 0 \\ y_1(t) = \sqrt{3}sin\frac{\pi}{2} = \sqrt{3} \end{cases} \quad \text{and} \quad M_2 \begin{cases} x_2(t) = 2cos\frac{3\pi}{2} = 0 \\ y_2(t) = \sqrt{3}sin\frac{3\pi}{2} = -\sqrt{3} \end{cases}$$

II.4 Motion Characteristics in different Coordinate Systems

The characteristics of an object's motion can vary depending on the coordinate system used to describe the motion. Here are some characteristics of motion in common coordinate systems:

II.4.1 Polar Coordinates

Polar coordinates are a two-dimensional coordinate system used to describe the position of a point in a plane. Instead of using Cartesian (x, y) coordinates, polar coordinates use two parameters to represent a point P as (ρ, θ), where:

$\rho = \|\overrightarrow{OP}\|$ ($\rho > 0$) : the radial distance (coordinate).

$\theta = (\vec{\imath}, \overrightarrow{OP})$: the polar angle

the radial distance ρ and the polar angle θ (the angle between the positive x-axis and the line segment OP, measured counterclockwise) with $\theta \in [0, 2\pi]$

the basis set is defined by the unit vectors $(\vec{u}_\rho, \vec{u}_\theta)$

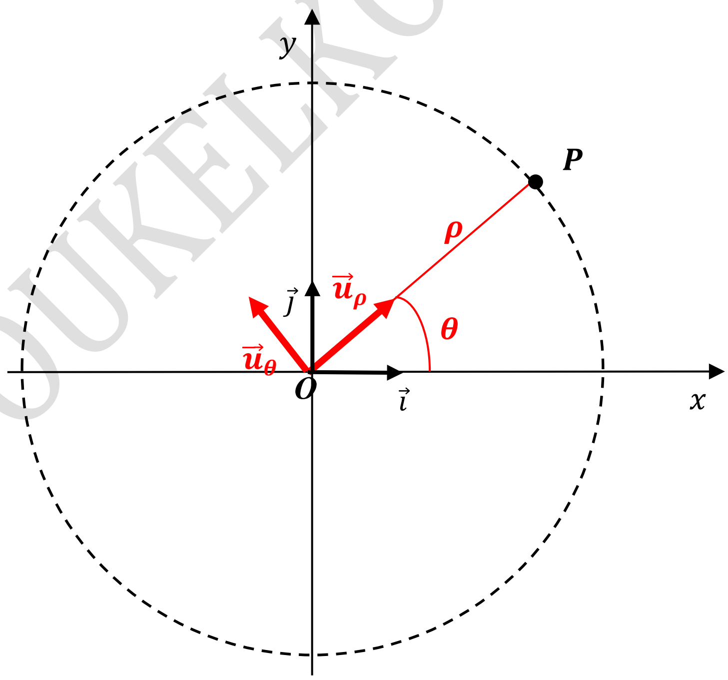


Figure 2-3: Polar Coordinates

II.4.1.1 Conversion between Cartesian and Polar Coordinates

The conversion between Cartesian and polar coordinates is given by the following equations:

$$\begin{cases} \vec{u}_\rho = \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath} \\ \vec{u}_\theta = -\sin\theta\,\vec{\imath} + \cos\theta\,\vec{\jmath} \end{cases} \qquad \begin{cases} x = \rho.\cos\theta \\ y = \rho.\sin\theta \end{cases} \Rightarrow \begin{cases} \rho = \sqrt{x^2+y^2} \\ \theta = \arctan\frac{y}{x} \\ \theta = \arccos\frac{x}{\sqrt{x^2+y^2}} \\ \theta = \arcsin\frac{y}{\sqrt{x^2+y^2}} \end{cases}$$

Example

a) in an orthonormal coordinate system, the cartesian coordinates of a point A are given by: $x = -3$ and $y = 4$.

Calculate the polar coordinates of the point A (the angle must be given in degrees).

b) The polar coordinates of a point B are $\left(2, \frac{3\pi}{4}\right)$.

Calculate the cartesian coordinates $(x\,, y)$ of this point.

Solution

a) $\rho = \sqrt{x^2+y^2} = \sqrt{9+16} = 5$

$$\theta = \arctan\frac{y}{x} = \arctan\frac{4}{-3} = -0.92rad == -53.13\,deg$$

Thus, the polar coordinates of the point A are $(5, -53.13°)$

b) $\begin{cases} x = \rho.\cos\theta = 2\cos\frac{3\pi}{4} = -\sqrt{2} \\ y = \rho.\sin\theta = 2\sin\frac{3\pi}{4} = \sqrt{2} \end{cases}$ $\qquad B\left(-\sqrt{2}, \sqrt{2}\right)$

II.4.1.2 Motion Characteristics in the Polar Coordinates System

Position Vector

$\vec{r} = \overrightarrow{OP} = \rho\vec{u}_\rho$

Velocity :

$$\vec{V} = \frac{d\overrightarrow{OP}}{dt} = \frac{d}{dt}\left[\rho\vec{u}_\rho\right] = \dot{\rho}\vec{u}_\rho + \rho\frac{d\vec{u}_\rho}{dt}$$

$$\frac{d\vec{u}_\rho}{dt} = \frac{d\vec{u}_\rho}{d\theta}\frac{d\theta}{dt} = \dot{\theta}(-\sin\theta\,\vec{\imath} + \cos\theta\,\vec{\jmath}) = \dot{\theta}\vec{u}_\theta$$

Thus: $\qquad \vec{V} = \dot{\rho}\vec{u}_\rho + \rho\dot{\theta}\vec{u}_\theta$

$\vec{V} = \begin{cases} V_r = \dot{\rho} & : radial\ velocity \\ V_\theta = \rho\dot{\theta} & : angular\ velocity \end{cases}$ $\Rightarrow \|\vec{V}\| = \sqrt{(\dot{\rho})^2 + (r\dot{\theta})^2}$

Acceleration Vector

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}\left[\dot{\rho}\vec{u}_\rho + \rho\dot{\theta}\vec{u}_\theta\right] = \ddot{\rho}\vec{u}_\rho + \dot{\rho}\frac{d\vec{u}_\rho}{dt} + \dot{\rho}\dot{\theta}\vec{u}_\theta + \rho\ddot{\theta}\vec{u}_\theta + \rho\dot{\theta}\frac{d\vec{u}_\theta}{dt}$$

$$\frac{d\vec{u}_\rho}{dt} = \dot{\theta}\vec{u}_\theta \quad et \quad \frac{d\vec{u}_\theta}{dt} = -\dot{\theta}\vec{u}_\rho$$

$$\vec{a} = \left(\ddot{\rho} - \rho\dot{\theta}^2\right)\vec{u}_\rho + \left(\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta}\right)\vec{u}_\theta$$

Thus $\vec{a}\begin{cases} a_\rho = \ddot{\rho} - \rho\dot{\theta}^2 : radial\ acceleration \\ a_\theta = \rho\ddot{\theta} + 2\dot{\rho}\dot{\theta} : angular\ acceleration \end{cases}$

$$\|\vec{a}\| = \sqrt{\left(\ddot{\rho} - \rho\dot{\theta}^2\right)^2 + \left(\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta}\right)^2}$$

II.5.2 Cylindrical Coordinates

Cylindrical coordinates are a type of coordinate system used to represent points in three-dimensional space. It is obtained by extending the polar coordinate system (in the *xy*-plane) with a third axis: the *oz*-axis with its cartesian coordinate z (called the height or elevation). Any point specified by these coordinates belongs to a cylinder with axis *oz* and radius ρ.

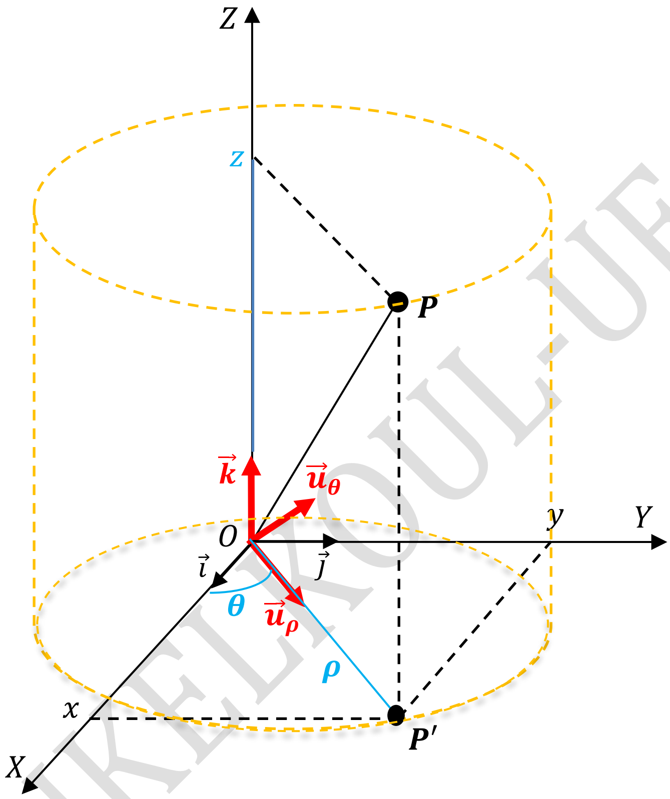


Figure 2-4: Cylindrical Coordinates

In this system, a point P is specified by three coordinates (ρ, θ, z) with:

- ρ: radial distance ($\rho > 0$). It represents the distance of the point from the origin to the projection P' of the point P in the *xy*-plane ($\rho = OP'$). It is similar to the polar coordinate ρ.
- θ: azimuthal angle. It is given by: $\theta \equiv \left(\vec{\imath}, \overrightarrow{OP'}\right)$ $et\ \theta \in [0, 2\pi]$.

  The angle θ is measured in the counterclockwise direction from the positive x-axis, similar to polar coordinates.
- z: (height) the vertical position of the point along the *oz*-axis. It is the coordinate that completes the 3D representation, making it a cylindrical system.
- Basis set: $(\vec{u}_\rho, \vec{u}_\theta, \vec{k})$

II.4.2.1 Conversion between Cartesian and Cylindrical Coordinates

The conversion between Cartesian and cylindrical coordinates is given by the following equations:

$$\begin{cases} \vec{u}_\rho = \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath} \\ \vec{u}_\theta = -\sin\theta\,\vec{\imath} + \cos\theta\,\vec{\jmath} \\ \vec{u}_z = \vec{k} \end{cases} \qquad \begin{cases} x = \rho.\cos\theta \\ y = \rho.\sin\theta \\ z = z \end{cases} \Rightarrow \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \theta = \arctan\frac{y}{x} \\ \theta = \arccos\frac{x}{\sqrt{x^2+y^2}} \\ \theta = \arcsin\frac{y}{\sqrt{x^2+y^2}} \end{cases}$$

II.4.2.2 Motion Characteristics in the Cylindrical Coordinates System

II.4.2.2.1 Position Vector

$$\vec{r} = \overrightarrow{OP} = \overrightarrow{OP'} + \overrightarrow{P'P} = \rho\vec{u}_\rho + z\vec{k}$$

$$\|\overrightarrow{OP}\| = \sqrt{\rho^2 + z^2} = \sqrt{x^2 + y^2 + z^2}$$

II.4.2.2.2 Velocity

$$\vec{V} = \frac{d\vec{r}}{dt} = \frac{d\overrightarrow{OP}}{dt} = \frac{d}{dt}\left[\rho\vec{u}_\rho + z\vec{k}\right] = \dot{\rho}\vec{u}_\rho + \rho\frac{d\vec{u}_\rho}{dt} + \dot{z}\vec{k}$$

$$\frac{d\vec{u}_\rho}{dt} = \frac{d\vec{u}_\rho}{d\theta}\frac{d\theta}{dt} = \dot{\theta}(-sin\theta\vec{\imath} + cos\theta\vec{\jmath}) = \dot{\theta}\vec{u}_\theta$$

Thus: $\vec{V} = \dot{\rho}\vec{u}_\rho + \rho\dot{\theta}\vec{u}_\theta + \dot{z}\vec{k}$

$$\vec{V} = \begin{cases} V_\rho = \dot{\rho} \;:\textit{radial velocity} \\ V_\theta = \rho\dot{\theta} \;:\textit{angular (azimuthal) velocity} \\ V_z = \dot{z} \;:\textit{vertical velocity} \end{cases} \Rightarrow \|\vec{V}\| = \sqrt{(\dot{\rho})^2 + \left(\rho\dot{\theta}\right)^2 + (\dot{z})^2}$$

II.4.2.2.2 Acceleration Vector

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}\left[\dot{\rho}\vec{u}_\rho + \rho\dot{\theta}\vec{u}_\theta\right] = \ddot{\rho}\vec{u}_\rho + \dot{\rho}\frac{d\vec{u}_\rho}{dt} + \dot{\rho}\dot{\theta}\vec{u}_\theta + \rho\ddot{\theta}\vec{u}_\theta + \rho\dot{\theta}\frac{d\vec{u}_\theta}{dt} + \ddot{z}\,\vec{k}$$

$$\vec{a} = \ddot{\rho}\vec{u}_\rho + \dot{\rho}\frac{d\vec{u}_\rho}{dt} + \dot{\rho}\dot{\theta}\vec{u}_\theta + \rho\ddot{\theta}\vec{u}_\theta + \rho\dot{\theta}\frac{d\vec{u}_\theta}{dt}$$

$$\frac{d\vec{u}_\rho}{dt} = \dot{\theta}\vec{u}_\theta \quad \textit{and} \quad \frac{d\vec{u}_\theta}{dt} = -\dot{\theta}\vec{u}_\rho\vec{a} = \left(\ddot{\rho} - \rho\dot{\theta}^2\right)\vec{u}_\rho + \left(\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta}\right)\vec{u}_\theta + \ddot{z}\,\vec{k}$$

$$\vec{a}\begin{cases} a_\rho = \ddot{\rho} - \rho\dot{\theta}^2 \textit{: radial acceleration} \\ a_\theta = \rho\ddot{\theta} + 2\dot{\rho}\dot{\theta} \;:\textit{angular (azimuthal) acceleration} \\ a_z = \ddot{z} :\textit{vertical acceleration} \end{cases}$$

$$\|\vec{a}\| = \sqrt{\left(\ddot{\rho} - \rho\dot{\theta}^2\right)^2 + \left(\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta}\right)^2 + (\ddot{z})^2}$$

II.4.4.1 Spherical Coordinates

The spherical coordinate system is a three-dimensional coordinate system that uses three parameters to specify the position of a point in space.

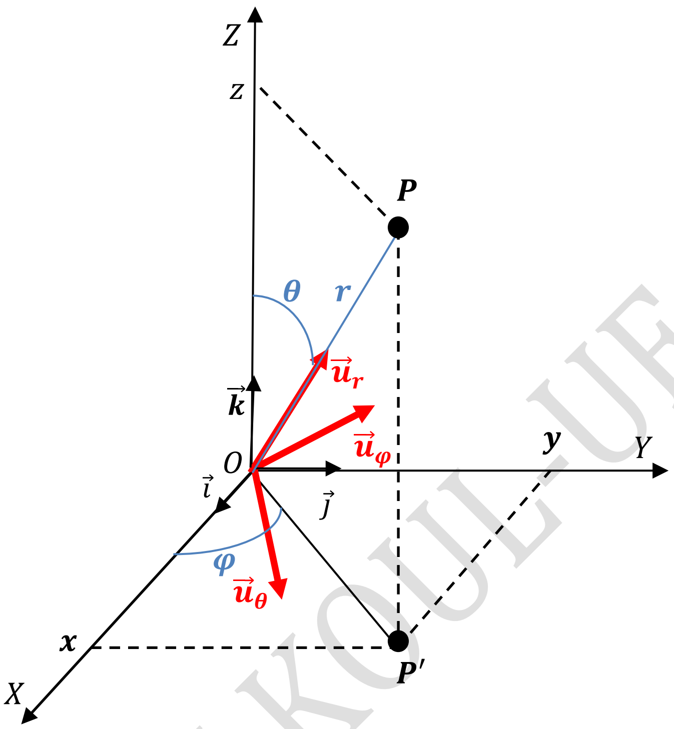


Figure 2-5: Spherical Coordinates

The spherical coordinates are defined as:

- $r = \|\overrightarrow{OP}\|$: radius
- $\theta = (\vec{k}, \overrightarrow{OP})$ $(\theta \in [0, \pi])$: polar angle. It is called latitude or zenith.
- $\varphi = (\vec{\imath}, \overrightarrow{OP'})$ $(\varphi \in [0, 2\pi])$: azimuthal angle (P' is the projection of the point P on the (oxy) plane). It is called longitude
- Basis set: $(\vec{u}_r, \vec{u}_\theta, \vec{u}_\varphi)$

N.B

Spherical coordinates are particularly useful for describing positions on a sphere and are commonly used in fields such as physics, astronomy, and engineering, especially when dealing with problems involving spherical objects or symmetrical systems.

II.4.3.1 Conversion between Cartesian and Spherical Coordinates

$$\begin{cases} \vec{u}_r = \sin\theta\,.\cos\varphi\,\vec{i} + \sin\theta\,.\sin\varphi\,\vec{j} + \cos\theta\,\vec{k} \\ \vec{u}_\theta = \cos\theta\,.\cos\varphi\,\vec{i} + \cos\theta\,.\sin\varphi\,\vec{j} - \sin\theta\,\vec{k} \\ \vec{u}_\varphi = -\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j} \end{cases}$$

$$\begin{cases} x = r.\sin\theta\,.\cos\varphi \\ y = r.\sin\theta\,.\sin\varphi \\ z = r.\cos\theta \end{cases} \quad \Rightarrow \quad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arccos\frac{z}{\sqrt{x^2+y^2+z^2}} \\ \varphi = \arctan\frac{y}{x} \end{cases}$$

Example

Let M be a point in a Cartesian coordinate system given by $M\left(\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{2}\right)$

Find the spherical coordinates of point M.

Solution

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} = \sqrt{\left(\frac{\sqrt{6}}{4}\right)^2 + \left(\frac{\sqrt{2}}{4}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = 1 \\ \theta = \arccos\frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{\pi}{4} \\ \varphi = \arctan\frac{y}{x} = \frac{\pi}{6} \end{cases}$$

The spherical coordinates of the point M are: $M\left(1, \frac{\pi}{4}, \frac{\pi}{6}\right)$

II.4.3.2 Motion characteristics in the Spherical Coordinates System

III.4.2.2.2 Position Vector

$\vec{r} = \overrightarrow{OP} = r\vec{u}_r$

$\|\overrightarrow{OP}\| = \sqrt{x^2+y^2+z^2}$

II.4.3.2.2 Velocity

$$\vec{V} = \frac{d\overrightarrow{OP}}{dt} = \frac{d}{dt}[r\vec{u}_r] = \dot{r}\vec{u}_r + r\frac{d\vec{u}_r}{dt}$$

$$\frac{d\vec{u}_r}{dt} = \frac{d\vec{u}_r}{d\theta}\frac{d\theta}{dt} + \frac{d\vec{u}_r}{d\varphi}\frac{d\varphi}{dt}$$

$$= \dot{\theta}\left(\cos\theta\cos\varphi\,\vec{i} + \cos\theta\sin\varphi\,\vec{j} - \sin\theta\,\vec{k}\right) + \dot{\varphi}(-\sin\theta\sin\varphi\,\vec{i} + \sin\theta\cos\varphi\,\vec{j})$$

$$\frac{d\vec{u}_r}{dt} = \dot{\theta}\vec{u}_\theta + \dot{\varphi}\sin\theta\,\vec{u}_\varphi$$

Therefore: $\vec{V} = \dot{r}\vec{u}_r + r\dot{\theta}\vec{u}_\theta + r\dot{\varphi}\sin\theta\,\vec{u}_\varphi$

$$\vec{V}\begin{cases} V_r = \dot{r}: & \text{radial velocity} \\ V_\theta = r\dot{\theta}: & \text{polar velocity} \\ V_\varphi = r\dot{\varphi}\sin\theta: & \text{azimuthal velocity}\end{cases} \Rightarrow \quad \|\vec{V}\| = \sqrt{(\dot{r})^2 + \left(r\dot{\theta}\right)^2 + (r\dot{\varphi}\sin\theta)^2}$$

II.4.3.2.3 Acceleration Vector

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}\left[\dot{r}\vec{u}_r + r\dot{\theta}\vec{u}_\theta + r\dot{\varphi}\sin\theta\,\vec{u}_\varphi\right]$$

$$\vec{a} = \ddot{r}\vec{u}_r + \dot{r}\frac{d\vec{u}_r}{dt} + \dot{r}\dot{\theta}\vec{u}_\theta + r\ddot{\theta}\vec{u}_\theta + r\dot{\theta}\frac{d\vec{u}_\theta}{dt} + \dot{r}\dot{\varphi}\sin\theta\,\vec{u}_\varphi + r\ddot{\varphi}\sin\theta\,\vec{u}_\varphi + r\dot{\varphi}\dot{\theta}\cos\theta\,\vec{u}_\varphi + r\dot{\varphi}\sin\theta\frac{d\vec{u}_\varphi}{dt}$$

$$\frac{d\vec{u}_r}{dt} = \dot{\theta}\vec{u}_\theta + \dot{\varphi}\sin\theta\,\vec{u}_\varphi$$

$$\frac{d\vec{u}_\theta}{dt} = \frac{d\vec{u}_\theta}{d\theta}\frac{d\theta}{dt} + \frac{d\vec{u}_\theta}{d\varphi}\frac{d\varphi}{dt}$$

$$= \dot{\theta}\left(-\sin\theta\,.\cos\varphi\,\vec{\imath} - \sin\theta\,.\sin\varphi\,\vec{\jmath} - \sin\theta\,\vec{k}\right) + \dot{\varphi}(-\cos\theta\,.\sin\varphi\,\vec{\imath} + \cos\theta\,.\cos\varphi\,\vec{\jmath}\,)$$

$$\frac{d\vec{u}_\theta}{dt} = -\dot{\theta}\vec{u}_r + \dot{\varphi}\cos\theta\,\vec{u}_\varphi$$

$$\frac{d\vec{u}_\varphi}{dt} = \frac{d\vec{u}_\varphi}{d\varphi}\frac{d\varphi}{dt} = \dot{\varphi}(-\cos\varphi\,\varphi\vec{\imath} - \sin\varphi\,\vec{\jmath}) = -\dot{\varphi}\sin\theta\,\vec{u}_r - \dot{\varphi}\cos\theta\,\vec{u}_\theta$$

Thus

$$\vec{a} = \ddot{r}\vec{u}_r + \dot{r}(\dot{\theta}\vec{u}_\theta + \dot{\varphi}\sin\theta\,\vec{u}_\varphi) + \dot{r}\dot{\theta}\vec{u}_\theta + r\ddot{\theta}\vec{u}_\theta + r\dot{\theta}(-\dot{\theta}\vec{u}_r + \dot{\varphi}\cos\theta\,\vec{u}_\varphi) + \dot{r}\dot{\varphi}\sin\theta\,\vec{u}_\varphi + r\ddot{\varphi}\sin\theta\,\vec{u}_\varphi + r\dot{\varphi}\dot{\theta}\cos\theta\,\vec{u}_\varphi + r\dot{\varphi}\sin\theta\,(-\dot{\varphi}\sin\theta\,\vec{u}_r - \dot{\varphi}\cos\theta\,\vec{u}_\theta)$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2 sin^2\theta)\,\vec{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2\sin\theta\,.\cos\theta\,)\vec{u}_\theta + (r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta)\vec{u}_\varphi$$

Therefore

$$\vec{a}\begin{cases} a_r = \ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2 sin^2\,\theta \\ a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta \\ a_\varphi = r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta \end{cases} \Rightarrow \quad \|\vec{a}\| = \sqrt{{a_r}^2 + {a_\theta}^2 + {a_\varphi}^2}$$

Example

The time-dependent equations of a moving object (point particle) P in the cartesian frame reference, are given by: $\begin{cases} x(t) = R\cos\theta(t) \\ y(t) = R\sin\theta(t) \\ z(t) = h\theta(t) \end{cases}$. where R, h and $\dot{\theta}$ are constants and $\theta \geq 0$.

1. Write the trajectory (path) equation

2. Determine the velocity and the acceleration vector in:

a) Cartesian coordinate system

b) Cylindrical coordinate system

Solution

1. Trajectory equation

$$\begin{cases} x^2 + y^2 = R^2 \\ \quad and \\ z(t) = h\theta \end{cases}$$

The trajectory is a helix with a pitch of h.

2. Velocity an acceleration

a) Cartesian coordinate system

The position Vector

$$\overrightarrow{OP} = x\vec{\imath} + y\vec{\jmath} + z\vec{k} = Rcos\theta\vec{\imath} + Rsin\theta\vec{\jmath} + h\theta\vec{k}$$

Velocity

$$\vec{V} = \frac{d\overrightarrow{OP}}{dt} = -R\dot{\theta}sin\theta\vec{\imath} + R\dot{\theta}cos\theta\vec{\imath} + h\dot{\theta}\vec{k}$$

$$\vec{V}\begin{cases} V_x = -R\dot{\theta}sin\theta \\ V_y = R\dot{\theta}cos\theta \\ V_z = h\dot{\theta} \end{cases}$$

Acceleration Vector $\vec{a}$

$$\vec{a} = \frac{d\vec{V}}{dt} = -R\dot{\theta}^2cos\theta\vec{\imath} - R\dot{\theta}^2sin\theta\vec{\imath}$$

$$\vec{a}\begin{cases} a_x = -R\dot{\theta}^2cos\theta \\ a_y = -R\dot{\theta}^2sin\theta \\ a_z = 0 \end{cases}$$

b) Cylindrical coordinate system

The Position Vector

$$\overrightarrow{OP} = \rho\overrightarrow{u_\rho} + z\vec{k}$$

$$\rho = \sqrt{x^2 + y^2} = R$$

Therefore

$$\overrightarrow{OP} = R\overrightarrow{u_\rho} + h\theta\vec{k}$$

Velocity

$$\vec{V} = \frac{d\overrightarrow{OP}}{dt} = R\dot{\theta}\overrightarrow{u_\theta} + h\dot{\theta}\vec{k}$$

$$\vec{V}\begin{cases} V_\rho = 0 \\ V_\theta = R\dot{\theta} \\ V_z = h\dot{\theta} \end{cases}$$

Acceleration Vector $\vec{a}$

$$\vec{a} = \frac{d\vec{V}}{dt} = -R\dot{\theta}^2\overrightarrow{u_\rho}$$

$$\vec{a}\begin{cases} a_\rho = -R\dot{\theta}^2 \\ a_\theta = 0 \\ a_z = 0 \end{cases}$$

II.4.4 Intrinsic (Curvilinear) Coordinates

In the case of a curvilinear movement, it is useful to use the curvilinear abscissa to identify the position of the material point. For this, a point A of the oriented trajectory is fixed. The curvilinear abscissa s(t) is then defined as being the curvilinear distance from the fixed-point A to the point P(t) occupied by the material point at time t:

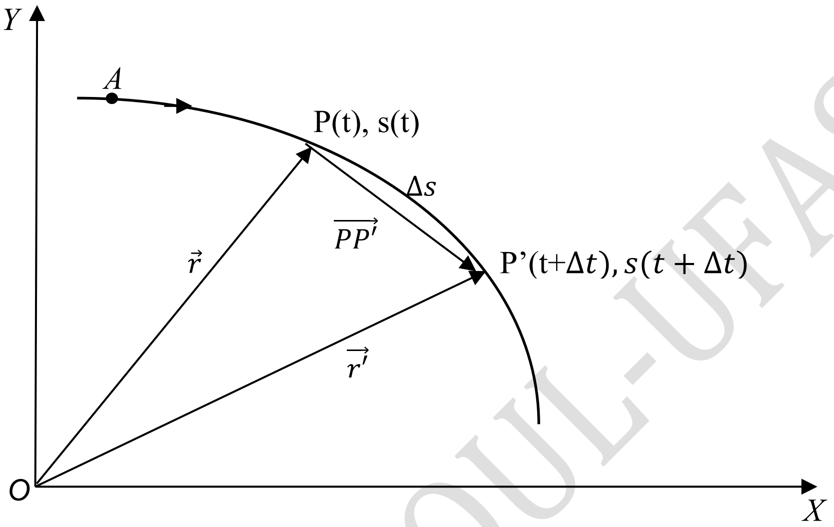


Figure 2-6: Curvilinear Abscissa

II.4.4.1.1 Motion Characteristics in the Intrinsic Coordinates System

$s(t) = \widehat{AP}$: curvilinear abscissa

$\widehat{PP'} = s'^{(t)} - s(t) = \Delta s$: curved displacement

$(\vec{u}_T, \vec{u}_N)$: Frenet basis set where:

$\vec{u}_T = \frac{d\overrightarrow{OP}}{ds}$: tagential unit vector.

$\vec{u}_N = R_c \frac{d\vec{u}_T}{ds}$: normal unit vector.

R_c : The radius of curvature of the trajectory at the considered point.

II.6 Velocity

$$\vec{V} = \frac{d\overrightarrow{OP}}{dt} = \frac{d\overrightarrow{OP}}{ds}\frac{ds}{dt} = \frac{ds}{dt}\vec{u}_T$$

$$\vec{V} = \|\vec{V}\|\vec{u}_T \quad \Rightarrow \quad \|\vec{V}\| = \left|\frac{ds}{dt}\right|$$

II.4.4.1.2 Acceleration Vector

$$\vec{a}=\frac{d\vec{V}}{dt}=\frac{d}{dt}\left(\|\vec{V}\|\vec{u}_T\right)=\frac{d\|\vec{V}\|}{dt}\vec{u}_T+\|\vec{V}\|\frac{d\vec{u}_T}{dt}$$

$$\frac{d\vec{u}_T}{dt}=\frac{d\vec{u}_T}{ds}\frac{ds}{dt}=V.\frac{1}{R_c}\vec{u}_N$$

Thus :

$$\vec{a}=\frac{d\|\vec{V}\|}{dt}\vec{u}_T+\frac{V^2}{R_c}\vec{u}_N=a_T\vec{u}_T+a_N\vec{u}_N$$

$$\vec{a}\begin{cases}a_T=\frac{dV}{dt}: tangential\ \ component\\ a_N=\frac{V^2}{R_c}: normal\ component\end{cases} \qquad (\vec{a}_T\perp\vec{a}_N)\ \Rightarrow\ \|\vec{a}\|=\sqrt{a_T^2+a_N^2}$$

Important Notes

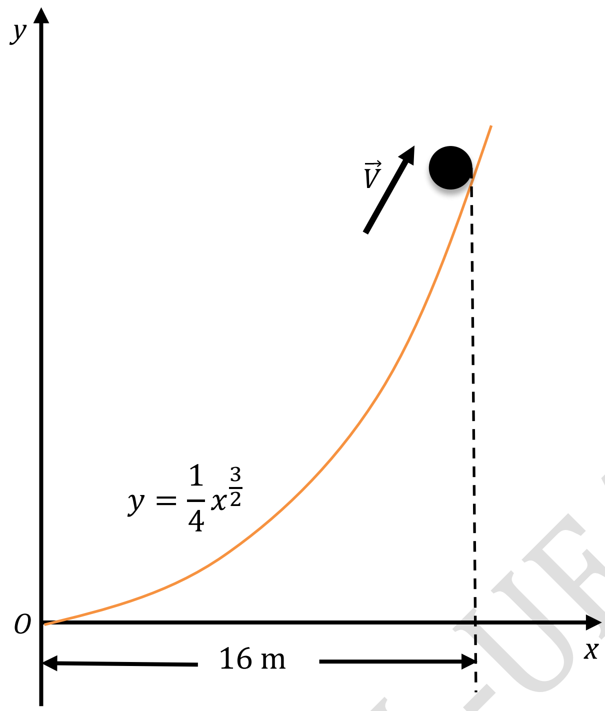
- The intrinsic (also called *n*-*t*) coordinate system is *fixed* on the particle, therefore *moves* with the particle,
- When described using n-t components, the velocity has one component, $\vec{V}=V\vec{u}_T$ (tangential),
- a_T is the change in *speed* and a_N is the change in the *direction* of the velocity, therefore:

  $$V=\frac{ds}{dt} \qquad a_T=\frac{dV}{dt} \qquad a_T ds=VdV$$
- a_N, also known as the *centripetal acceleration*, always points to the concave side of the path,
- If the path is known by its cartesian equation: $y=f(x)$, then the curvature radius is given by: $R_C=\frac{[1+(dy/dx)^2]^{\frac{3}{2}}}{[d^2/dx^2]}$

Example

An object considered as a point mass, travels along a curved path with the cartesian equation: $y=\frac{1}{4}x^{\frac{3}{2}}$, as shown. If at the point shown its speed is 28.8 *m/s* and is increasing at 8 *m/s²*.

- Determine the direction of its velocity.
- Determine the direction and the magnitude of its acceleration at this point.



Solution

To determine the direction of the velocity, let's consider the tangent axis $\vec{T}$ and the normal axis $\vec{N}$ at the considered point. The velocity points to the same direction of the $\vec{T}$ axis (the velocity is tangent to the path). The direction of $\vec{T}$ axis is determined via the angle θ between the $\vec{T}$ axis and the horizon as shown.

The equation of the path is given by: $y = \frac{1}{4}x^{\frac{3}{2}}$

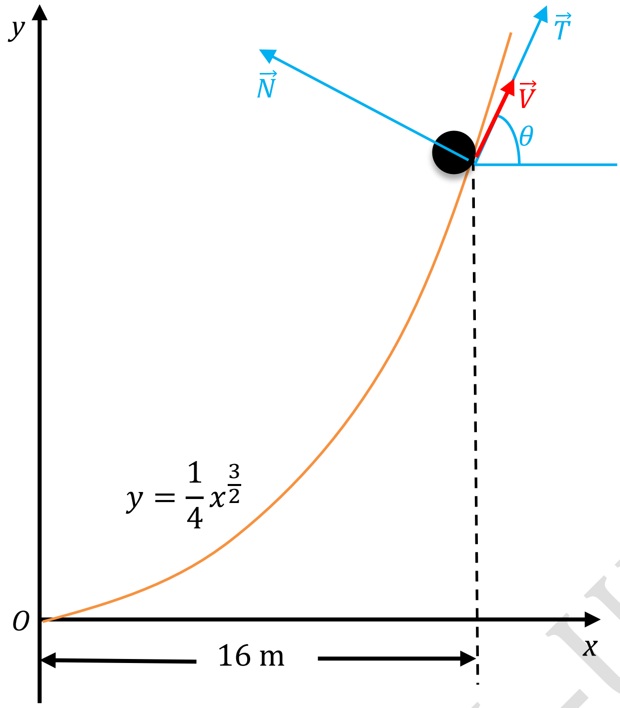
So its slope is

$$slope = \frac{dy}{dx} = \frac{3}{8}x^{\frac{1}{2}}$$

At $x = 16\ m$ the slope equals to:

$$slope = \frac{dy}{dx} = \frac{3}{8}16^{\frac{1}{2}} = 1.5 = \tan\theta$$

Therefore: $\theta = \arctan 1.5 = 56.3°$



$$\vec{a} = \frac{d\|\vec{V}\|}{dt}\vec{u}_T + \frac{V^2}{R_c}\vec{u}_N = a_T\vec{u}_T + a_N\vec{u}_N$$

$$a_T = \frac{dV}{dt} = 8\ m/s^2$$

$$a_N = \frac{V^2}{R_c}$$

$$R_C = \frac{[1 + (dy/dx)^2]^{\frac{3}{2}}}{[d^2/dx^2]}$$

$$\frac{dy}{dx} = \frac{3}{8}x^{\frac{1}{2}} \quad \Rightarrow \frac{d^2y}{dx^2} = \frac{3}{16}x^{-\frac{1}{2}}$$

At $x = 16\ m$: $\frac{dy}{dx} = 1.5$ and $\frac{d^2y}{dx^2} = \frac{3}{64}$

Substituting both in the radius curvature equation, we get:

$R_C = 125\ m$

Therefore:

$$a_T = \frac{dV}{dt} = 8\ m/s^2$$

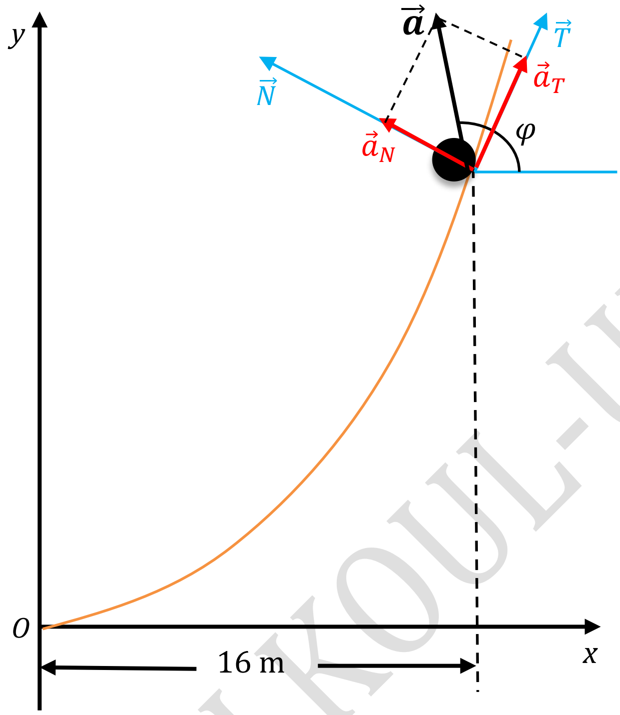
$$a_N = \frac{V^2}{R_c} = \frac{28.8^2}{125} = 6.64\ m/s^2$$

The magnitude of the acceleration at the considered point given by:

$$\|\vec{a}\| = \sqrt{a_T^2 + a_N^2} = 10.4\ \|\vec{a}\|\ m/s^2$$

The acceleration is represented in the figure below. Its direction is also characterized by the angle φ given by:

$$\varphi = \theta + \arctan\frac{a_T}{a_N} = 56.3° + \arctan\frac{6.64}{8} = 96°$$



II.5 Types of Motion

II.5.1 Rectilinear Motion

The simplest type of motion is *rectilinear motion*, i.e., motion along a straight line. Such a line could be, e.g., the x-axis on which we have defined a positive orientation in the direction of the unit vector $\vec{\imath}$, as well as a point O (an *origin*) at which x=0.

Uniform Rectilinear Motion

- Straight path (Trajectory)
- $V = constant$ and $a = 0$
- $a = \frac{dV(t)}{dt} \Rightarrow dV = adt \;\Rightarrow \int_{V_0}^{V} dV = V - V_0 = \int_0^t adt = 0 \Rightarrow V = V_0$
- $V(t) = \frac{dx}{dt} \;\Rightarrow dx = Vdt \;\Rightarrow \int_{x_0}^{x} dx = x - x_0 = \int_0^t Vdt \;\; \Rightarrow x = Vt + x_0$

Uniformly Accelerated Rectilinear Motion

- Straight path (Trajectory)
- a = constant $\neq$ 0
- $a = \frac{dV}{dt} \Rightarrow \int_{V_0}^{V} dV = V - V_0 = \int_0^t adt = at \;\Rightarrow\; V = at + V_0$
- $V = \frac{dx}{dt} \Rightarrow \int_{x_0}^{x} dx = x - x_0 = \int_0^t Vdt = \frac{1}{2}at^2 + V_0t \;\Rightarrow x = \frac{1}{2}at^2 + V_0t + x_0$
- $\frac{dV}{a} = \frac{dx}{V} \;\Rightarrow VdV = adx \;\Rightarrow \int_{V_0}^{V} VdV = \int_{x_0}^{x} adx \;\Rightarrow \frac{1}{2}\left(V^2 - {V_0}^2\right) = a(x - x_0)$

  $\Rightarrow\; V^2 - {V_0}^2 = 2a(x - x_0)$

II.5.2 Simple Harmonic Rectilinear Motion

- Straight path (Trajectory)
- $x(t) = Acos(\omega t + \varphi)$: A is the amplitude which swings between -A an +A
- $\varphi(t) = \omega t + \varphi$: instantaneous phase
- $\omega = \frac{2\pi}{T} = 2\pi f$: the pulsation with $[\omega] = rd/s$,
- f: the frequency with $[f] = Hz$ and
- T: the period with $[T] = s$

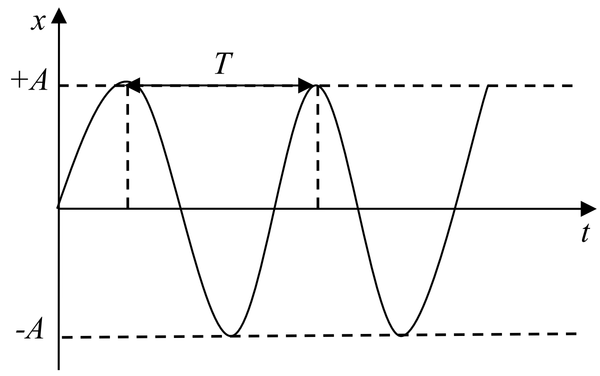


Figure 2-7: Simple Harmonic Rectilinear Motion

Therefore

- $\dot{x}(t) = -A\omega sin(\omega t + \varphi)$
- $\ddot{x}(t) = -A\omega^2 cos(\omega t + \varphi) = -\omega^2 x(t)$
- $\ddot{x} + \omega^2 x = 0$: differential equation of motion

II.5.3 Circular Motion

Circular motion is plane motion with constant radius of curvature *ρ=R*. The trajectory of the moving object is a circle of radius R (Figure below).

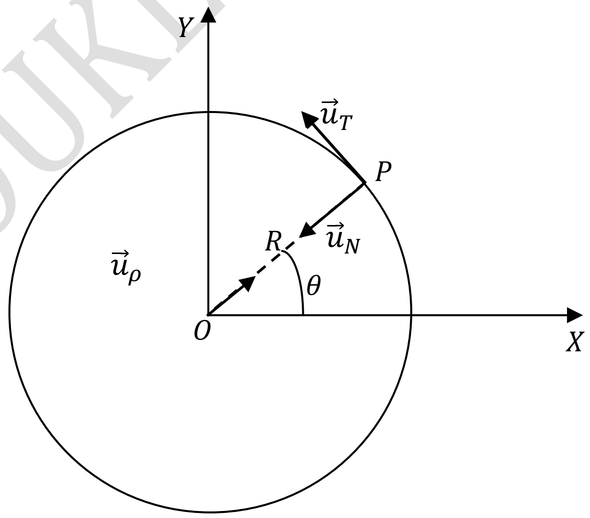


Figure 2-8: Circular Motion

Characteristics

- $ds = Rd\theta$
- $\vec{V}(t) = \frac{ds}{dt}\vec{u}_T = R.\frac{d\theta}{dt}\vec{u}_T \quad \Rightarrow \quad V = R\omega$

  Where $\omega = \frac{d\theta}{dt}$ is the angular velocity
- If the circular motion is occuring around the z-axis, we define the rotation vector $\vec{\omega}$ as:

  $\vec{\omega} = \omega\vec{k} = \frac{d\theta}{dt}\vec{k}$
- $\vec{a}(t) = \frac{dV}{dt}\vec{u}_T + \frac{V^2}{R}\vec{u}_N = R\frac{d\omega}{dt}\vec{u}_T + R\omega^2\vec{u}_N$

Important Notes

- It can be demonstrated that: $\frac{d\overrightarrow{OP}}{dt} = \vec{\omega}\wedge\overrightarrow{OP}$
- In the case of uniform circular motion, the angular velocity is constant, i.e. that the angular acceleration is zero: $\omega = cte \quad \Rightarrow \quad \frac{d\omega}{dt} = 0 \quad \Rightarrow \vec{a}(t) = R\omega^2\vec{u}_N$

II.5.4 Motion with Central Acceleration

A motion with central acceleration refers to the movement of an object in a circular or curved path where the acceleration is directed toward the center of the path. In other words, the object's velocity changes, causing it to move along a curved trajectory, while experiencing a continuous acceleration directed radially inward or outward from the center of curvature.

$$\vec{a}//\overrightarrow{CM} \quad \Leftrightarrow \quad \vec{a}\times\overrightarrow{CM} = \vec{0}$$

This type of motion is commonly observed in circular orbits and other curved paths where a central force, such as gravity, provides the acceleration that keeps the object in its path.

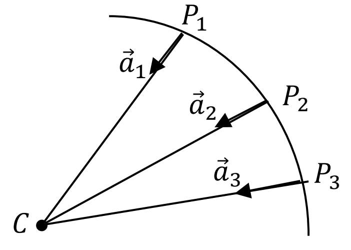


Figure 2-9: Motion with Central Acceleration

II.5.5 Erratic Motion

An erratic motion is said to occur when any one of the motion parameters like displacement (s), velocity (v), acceleration (a), etc. is not continuous. These discontinuous functions show a jump or sharp bends. The concept of this motion is understood when these motion parameters are plotted against time.

The analyzis of an erratic motion requires that:

- Each continuous segment of motion should be analyzed separately.
- The end condition of the previous segment is known to be the initial condition for the current segment.

Mathematically, the erratic motion is described by separate equations:

$$s(t)\begin{cases}\text{equation 1}\\ \text{equation 2}\\ \text{equation 3}\\ \quad \dots\end{cases} \qquad V(t)\begin{cases}\text{equation 1}\\ \text{equation 2}\\ \text{equation 3}\\ \quad \dots\end{cases} \qquad a(t)\begin{cases}\text{equation 1}\\ \text{equation 2}\\ \text{equation 3}\\ \quad \dots\end{cases}$$

Note:

- This type of motion is also called ‘uncontrolled motion’.
- This class of motion is known to be unpredictable.

Example

An object travels along straight path. $s_0 = 0$ and $V_0 = 0$. Its acceleration (in m/s²) as a function of time is given by: $a(t)\begin{cases}0.4t & 0 \le t < 10s\\ 2.4 & 10 \le t < 20s\\ 0 & 20 \le t < 30s\end{cases}$

Find V(t) and s(t) expressions.

Speed V(t)

1) $0 \le \boldsymbol{t} < \mathbf{10}\boldsymbol{s}$

$$a(t) = \frac{dV(t)}{dt} \quad \Rightarrow \quad \int_0^V dV(t) = \int_o^t a dt$$

$$\int_0^V dV(t) = \int_o^t 0.4t dt \qquad \Rightarrow \qquad V(t) = 0.2t^2$$

At $t = 10s \ \Rightarrow \ V = 20 \ m/s$

2) $\mathbf{10} \le \boldsymbol{t} < \mathbf{20}\boldsymbol{s}$

$$a(t) = \frac{dV(t)}{dt} \quad \Rightarrow \quad \int_{20}^V dV(t) = \int_{10}^t a dt$$

$$\int_{20}^{V} dV(t) = \int_{10}^{t} 2.4dt \quad \Rightarrow \quad V(t) - 20 = 2.4t - 24$$

$$V(t) = 2.4t - 4$$

At $t = 20s \;\Rightarrow\; V = 44\,m/s$

3) $20 \leq t < 30s$

$$a(t) = \frac{dV(t)}{dt} \quad \Rightarrow \quad \int_{44}^{V} dV(t) = \int_{20}^{t} adt$$

$$\int_{44}^{V} dV(t) = \int_{20}^{t} 0dt \quad \Rightarrow \quad V(t) - 44 = 0 \quad \Rightarrow V(t) = 44$$

$$V(t)\begin{cases} 0.2t^2 & 0 \leq t < 10s \\ 2.4t - 4 & 10 \leq t < 20s \\ 44 & 20 \leq t < 30s \end{cases}$$

Displacement s(t)

1) $0 \leq t < 10s$

$$V(t) = \frac{ds(t)}{dt} \quad \Rightarrow \quad \int_{0}^{S} ds(t) = \int_{o}^{t} Vdt$$

$$\int_{0}^{s} ds(t) = \int_{o}^{t} 0.2t^2dt \quad \Rightarrow \quad s(t) = 0.067t^3$$

At $t = 10s \;\Rightarrow\; s = 67m$

2) $10 \leq t < 20s$

$$V(t) = \frac{ds(t)}{dt} \quad \Rightarrow \quad \int_{67}^{S} ds(t) = \int_{10}^{t} Vdt$$

$$\int_{67}^{S} ds(t) = \int_{10}^{t} (2.4t - 4)dt \quad \Rightarrow \quad s(t) - 67 = 1.2t^2 - 4t - 120 + 40$$

$s(t) = 1.2t^2 - 4t - 13$

At $t = 20s \;\Rightarrow\; s = 387\,m$

3) $20 \leq t < 30s$

$$V(t) = \frac{ds(t)}{dt} \quad \Rightarrow \quad \int_{387}^{S} ds(t) = \int_{20}^{t} V dt$$

$$\int_{387}^{S} ds(t) = \int_{20}^{t} 44 dt \quad \Rightarrow \quad s(t) - 387 = 44t - 880$$

$$s(t) = 44t - 493$$

$$s(t) \begin{cases} 0.067t^3 & 0 \leq t < 10s \\ 1.2t^2 - 4t - 13 & 10 \leq t < 20s \\ 44t - 493 & 20 \leq t < 30s \end{cases}$$

II.6 Relative Motion

II.6.1 Introduction

Motion is the displacement of an object with respect to a fixed point in space at a given moment. Therefore, motion and rest are two relative concepts that depend on the position of the object relative to the chosen reference point.

Physically:

Absolute Motion: is the motion that occurs in an absolute reference frame which is considered "fixed".

Relative Motion: is the motion that occurs in a relative reference frame, which itself undergoes a motion (translation or rotation) with respect to the absolute reference frame.

II.6.2 Composition of motions

We consider two reference frames: $R(O,\vec{\imath},\vec{\jmath},\vec{k})$ and $R'\left(O',\vec{\imath}',\vec{\jmath}',\vec{k}'\right)$, moving relative to each other. We assume that R is fixed and is referred to as the *absolute reference frame*. The reference frame R' is called the *relative reference frame* because it is in motion relative to R. We study the motion of a point particle P with respect to both reference frames.

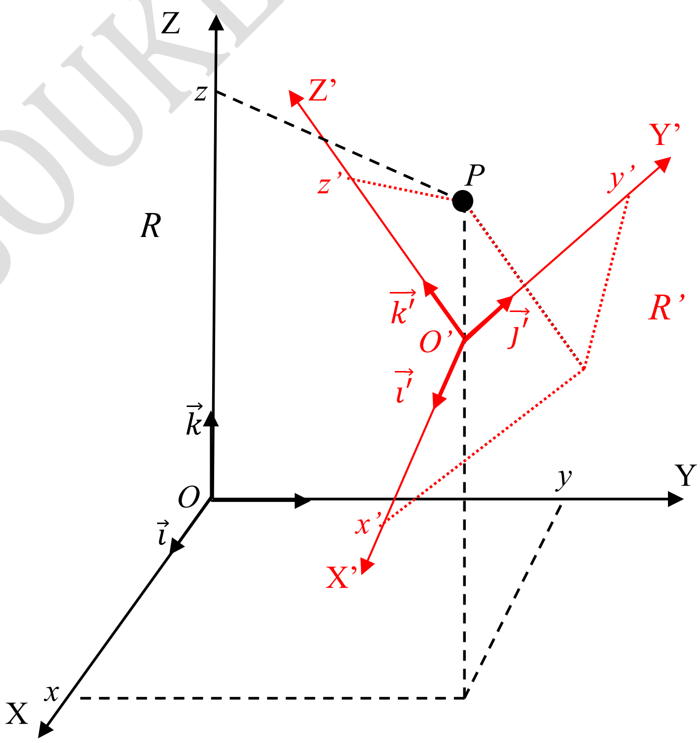


Figure3. Illustration of Relative Motion

II.6.3 Characteristics of Motion

a) Absolute Motion

The motion of P with respect to R reference frame is called absolute motion.

Absolute Position Vector

$\overrightarrow{OP} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

Absolute Velocity

$\vec{V}_a = \left.\frac{d\overrightarrow{OP}}{dt}\right|_R = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k} = \dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \dot{z}\vec{k}$

Where $\left.\frac{d\vec{\imath}}{dt}\right|_R = \left.\frac{d\vec{\jmath}}{dt}\right|_R = \left.\frac{d\vec{k}}{dt}\right|_R = \vec{0}$

Absolute Acceleration Vector

$\vec{a}_a = \left.\frac{d\vec{V}_a}{dt}\right|_R = \frac{d^2x}{dt^2}\vec{\imath} + \frac{d^2x}{dt^2}\vec{\jmath} + \frac{d^2x}{dt^2}\vec{k} = \ddot{x}\vec{\imath} + \ddot{y}\vec{\jmath} + \ddot{z}\vec{k}$

b) Relative Motion

The motion of P relative to R' reference frame is called relative motion.

Relative Position Vector

$\overrightarrow{O'P} = x'\overrightarrow{\imath'} + y'\overrightarrow{\jmath'} + z'\overrightarrow{k'}$

N.B

The relative unit vectors (base) are moving (direction change) in R and fixed in R'.

Relative Velocity

$$\vec{V}_r = \left.\frac{d\overrightarrow{O'M}}{dt}\right|_{R'} = \frac{dx'}{dt}\overrightarrow{\imath'} + \frac{dy'}{dt}\overrightarrow{\jmath'} + \frac{dz'}{dt}\overrightarrow{k'} = \dot{x}'\overrightarrow{\imath'} + \dot{y}'\overrightarrow{\jmath'} + \dot{z}'\overrightarrow{k'}$$

where: $\left.\frac{d\overrightarrow{\imath'}}{dt}\right|_{R'} = \left.\frac{d\overrightarrow{\jmath'}}{dt}\right|_{R'} = \left.\frac{d\overrightarrow{k'}}{dt}\right|_{R'} = \vec{0}$

Relative Acceleration Vector

$$\vec{a}_r = \left.\frac{d\vec{V}_r}{dt}\right|_{R'} = \frac{d^2x'}{dt^2}\overrightarrow{\imath'} + \frac{d^2x'}{dt^2}\overrightarrow{\jmath'} + \frac{d^2x'}{dt^2}\overrightarrow{k'} = \ddot{x}'\overrightarrow{\imath'} + \ddot{y}'\overrightarrow{\jmath'} + \ddot{z}'\overrightarrow{k'}$$

The motion of the relative reference frame with respect to the absolute reference frame is reduced to the composition of a rectilinear translational motion and a rotational motion.

II.6.4 Position Vectors Composition

From the figure above, we get:

$\overrightarrow{OP} = \overrightarrow{OO'} + \overrightarrow{O'P}$

II.6.5 Velocity Vectors Composition

The composition of velocities consists of finding the relationship between $\vec{V}_a$ and $\vec{V}_r$.

According to the figure above:

$$\vec{V}_a = \left.\frac{d\overrightarrow{OP}}{dt}\right|_R = \left.\frac{d\overrightarrow{OO'}}{dt}\right|_R + \left.\frac{d\overrightarrow{O'P}}{dt}\right|_R = \left.\frac{d\overrightarrow{OO'}}{dt}\right|_R + \left.\frac{d}{dt}\right|_R \left(x'\vec{\imath'} + y'\vec{\jmath'} + z'\vec{k'}\right)$$

$$\vec{V}_a = \left.\frac{d\overrightarrow{OO'}}{dt}\right|_R + \frac{dx'}{dt}\vec{\imath'} + x'\frac{d\vec{\imath'}}{dt} + \frac{dy'}{dt}\vec{\jmath'} + y'\frac{d\vec{\jmath'}}{dt} + \frac{dz'}{dt}\vec{k'} + z'\frac{d\vec{k'}}{dt}$$

$$\vec{V}_a = \left(\frac{dx'}{dt}\vec{\imath'} + + \frac{dy'}{dt}\vec{\jmath'} + \frac{dz'}{dt}\vec{k'}\right) + \left(\left.\frac{d\overrightarrow{OO'}}{dt}\right|_R + x'\frac{d\vec{\imath'}}{dt} + y'\frac{d\vec{\jmath'}}{dt} + z'\frac{d\vec{k'}}{dt}\right)$$

Therefore: $\vec{V}_a = \vec{V}_r + \vec{V}_e$

Where
$$\begin{cases} \vec{V}_a = \left.\frac{d\overrightarrow{OP}}{dt}\right|_R \quad absolute\ velocity \\ \vec{V}_r = \left.\frac{d\overrightarrow{O'P}}{dt}\right|_{R'} \quad relative\ velocity \\ \vec{V}_e = \underbrace{\left.\frac{d\overrightarrow{OO'}}{dt}\right|_R}_{\substack{Translation\ velocity \\ of\ R'\ with\ respect\ to\ à\ R}} + \underbrace{x'\frac{d\vec{\imath'}}{dt} + y'\frac{d\vec{\jmath'}}{dt} + z'\frac{d\vec{k'}}{dt}}_{\substack{Rotation\ velocity\ of\ R' \\ with\ respect\ to\ R}} \quad driven\ velocity \end{cases}$$

Special Cases

a) *R'* in Translation Motion with respect to *R*

In this case, the relative unit vectors (base) $\left(\vec{\imath'}, \vec{\jmath'}, \vec{k'}\right)$ are fixed with respect to R.

Therefore:

- $\left.\frac{d\vec{\imath'}}{dt}\right|_R = \left.\frac{d\vec{\jmath'}}{dt}\right|_R = \left.\frac{d\vec{k'}}{dt}\right|_R = \vec{0}$
- $\vec{V}_e = \left.\frac{d\overrightarrow{OO'}}{dt}\right|_R \Rightarrow \vec{V}_a = \left.\frac{d\overrightarrow{O'P}}{dt}\right|_R + \left.\frac{d\overrightarrow{OO'}}{dt}\right|_R$ ($\vec{V}_e$ independante of P)

b) *R'* in Rotation with respect to *R*

If the reference frame R' is carrying out a rotation motion with respect to R with an angular velocity $\vec{\omega}$, the relative unit vectors are also rotating with the same angular velocity. Indeed, the time derivatives of the relative unit vectors satisfy the following relations:

$$\left.\frac{d\vec{\imath'}}{dt}\right|_R = \vec{\omega}\wedge\vec{\imath'} \qquad \left.\frac{d\vec{\jmath'}}{dt}\right|_R = \vec{\omega}\wedge\vec{\jmath'} \qquad \left.\frac{d\vec{k'}}{dt}\right|_R = \vec{\omega}\wedge\vec{k'}$$

$$\vec{V}_e = x'\frac{d\vec{\imath'}}{dt} + y'\frac{d\vec{\jmath'}}{dt} + y'\frac{d\vec{k'}}{dt} = x'(\vec{\omega}\wedge\vec{\imath'}) + y'(\vec{\omega}\wedge\vec{\jmath'}) + z'(\vec{\omega}\wedge\vec{k'})$$

$$= \vec{\omega}\wedge(x'\vec{\imath'}) + \vec{\omega}\wedge(y'\vec{\imath'}) + \vec{\omega}\wedge(z'\vec{\imath'}) = \vec{\omega}\wedge(x'\vec{\imath'} + y'\vec{\imath'} + z'\vec{\imath'})$$

$$\Rightarrow \quad \vec{V}_e = \vec{\omega}\wedge\overrightarrow{O'M}$$

Therefore:

$$\boldsymbol{\vec{V}_a = \left.\frac{d\overrightarrow{O'P}}{dt}\right|_R + \vec{\omega}\wedge\overrightarrow{O'P} = \vec{V}_r + \vec{\omega}\wedge\overrightarrow{O'P}}$$

c) *R'* in a Random Motion with respect to *R*

$$\vec{V}_a = \left.\frac{d\overrightarrow{O'P}}{dt}\right|_R + \left.\frac{d\overrightarrow{OO'}}{dt}\right|_R + \vec{\omega}\wedge\overrightarrow{O'P} = \vec{V}_r + \left.\frac{d\overrightarrow{OO'}}{dt}\right|_R + \vec{\omega}\wedge\overrightarrow{O'P}$$

II.6.6 Acceleration Vectors Composition

$$\vec{a}_a = \frac{d\vec{V}_a}{dt} = \frac{d}{dt}\left[\left.\frac{d\overrightarrow{OO'}}{dt}\right|_R + \frac{dx'}{dt}\vec{\imath'} + \frac{dy'}{dt}\vec{\jmath'} + \frac{dz'}{dt}\vec{k'} + x'\frac{d\vec{\imath'}}{dt} + +y'\frac{d\vec{\jmath'}}{dt} + +z'\frac{d\vec{k'}}{dt}\right]$$

$$\vec{a}_a = \left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_R + \frac{d^2x'}{dt^2}\vec{\imath'} + \frac{dx'}{dt}\frac{d\vec{\imath'}}{dt} + \frac{d^2y'}{dt^2}\vec{\jmath'} + \frac{dy'}{dt}\frac{d\vec{\jmath'}}{dt} + \frac{d^2z'}{dt^2}\vec{k'} + \frac{dz'}{dt}\frac{d\vec{k'}}{dt} + \frac{dx'}{dt}\frac{d\vec{\imath'}}{dt}$$

$$+ x'\frac{d^2\vec{\imath'}}{dt^2} + \frac{dy'}{dt}\frac{d\vec{\jmath'}}{dt} + y'\frac{d^2\vec{\jmath'}}{dt^2} + \frac{dz'}{dt}\frac{d\vec{k'}}{dt} + z'\frac{d^2\vec{k'}}{dt^2}$$

$$\vec{a}_a = \left(\frac{d^2x'}{dt^2}\vec{\imath'} + \frac{d^2y'}{dt^2}\vec{\jmath'} + \frac{d^2z'}{dt^2}\vec{k'}\right) + \left(\left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_R + x'\frac{d^2\vec{\imath'}}{dt^2} + y'\frac{d^2\vec{\jmath'}}{dt^2} + z'\frac{d^2\vec{k'}}{dt^2}\right)$$

$$+ 2\left(\frac{dx'}{dt}\frac{d\vec{\imath'}}{dt} + \frac{dy'}{dt}\frac{d\vec{\jmath'}}{dt} + \frac{dz'}{dt}\frac{d\vec{k'}}{dt}\right)$$

$$\boldsymbol{\vec{a}_a = \vec{a}_r + \vec{a}_e + \vec{a}_c}$$

With

$$\begin{cases} \vec{a}_a = \dfrac{d\vec{V}_a}{dt} = \dfrac{d^2\overrightarrow{OM}}{dt^2} & absolute\ acceleation \\ \vec{a}_r = \dfrac{d^2x'}{dt^2}\vec{\imath'} + \dfrac{d^2y'}{dt^2}\vec{\jmath'} + \dfrac{d^2z'}{dt^2}\overrightarrow{k'} & relative\ acceleration \\ \vec{a}_e = \left.\dfrac{d^2\overrightarrow{OO'}}{dt^2}\right|_R + x'\frac{d^2\vec{\imath'}}{dt^2} + y'\frac{d^2\vec{\jmath'}}{dt^2} + z'\frac{d^2\overrightarrow{k'}}{dt^2} & driven\ acceleration \\ \vec{a}_c = 2\left(\dfrac{dx'}{dt}\dfrac{d\vec{\imath'}}{dt} + \dfrac{dy'}{dt}\dfrac{d\vec{\jmath'}}{dt} + \dfrac{dz'}{dt}\dfrac{d\overrightarrow{k'}}{dt}\right) & Coriolis\ acceleration \end{cases}$$

$$\vec{a}_a = \vec{a}_r + \vec{a}_e + \vec{a}_c$$

In other words

$$\vec{a}_r = \left.\frac{d^2\overrightarrow{O'M}}{dt^2}\right|_{R'} = \frac{d\vec{V}_r}{dt}$$

$$\vec{a}_e = \left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_R + x'\frac{d}{dt}\left(\frac{d\vec{\imath'}}{dt}\right) + y'\frac{d}{dt}\left(\frac{d\vec{\jmath'}}{dt}\right) + z'\frac{d}{dt}\left(\frac{d\overrightarrow{k'}}{dt}\right)$$

$$\vec{a}_e = \left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_R + x'\frac{d}{dt}\left(\vec{\omega}\wedge\vec{\imath'}\right) + y'\frac{d}{dt}\left(\vec{\omega}\wedge\vec{\jmath'}\right) + z'\frac{d}{dt}\left(\vec{\omega}\wedge\overrightarrow{k'}\right)$$

$$\vec{a}_e = \left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_R + \frac{d}{dt}\left(\vec{\omega}\wedge x'\vec{\imath'}\right) + y'\frac{d}{dt}\left(\vec{\omega}\wedge y'\vec{\jmath'}\right) + z'\frac{d}{dt}\left(\vec{\omega}\wedge z'\overrightarrow{k'}\right)$$

$$\vec{a}_e = \left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_R + \frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'P} + \vec{\omega}\wedge\frac{d\overrightarrow{O'P}}{dt}$$

$$\vec{a}_e = \underbrace{\left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_R}_{translation} + \underbrace{\frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'P}}_{rotaion} + \underbrace{\vec{\omega}\wedge\left(\vec{\omega}\wedge\overrightarrow{O'P}\right)}_{centripital}$$

$$\vec{a}_c = 2\left(\frac{dx'}{dt}\frac{d\vec{\imath'}}{dt} + \frac{dy'}{dt}\frac{d\vec{\jmath'}}{dt} + \frac{dz'}{dt}\frac{d\overrightarrow{k'}}{dt}\right) = 2\left(\frac{dx'}{dt}\left(\vec{\omega}\wedge\vec{\imath'}\right) + \frac{dy'}{dt}\left(\vec{\omega}\wedge\vec{\jmath'}\right) + \frac{dz'}{dt}\left(\vec{\omega}\wedge\overrightarrow{k'}\right)\right)$$

$$\vec{a}_c = 2\left(\left(\vec{\omega}\wedge\frac{dx'}{dt}\vec{\imath'}\right) + \left(\vec{\omega}\wedge\frac{dy'}{dt}\vec{\jmath'}\right) + \left(\vec{\omega}\wedge\frac{dz'}{dt}\overrightarrow{k'}\right)\right)$$

$$\vec{a}_c = 2\vec{\omega}\wedge\left(\frac{dx'}{dt}\vec{\imath'} + \frac{dy'}{dt}\vec{\jmath'} + \frac{dy'}{dt}\overrightarrow{k'}\right)$$

$$\vec{a}_c = 2\vec{\omega}\wedge\vec{V}_r$$

Which means that: $\vec{a}_a = \left.\frac{d^2\overrightarrow{O'P}}{dt^2}\right|_{R'} + \left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_R + \frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'P} + \vec{\omega}\wedge\left(\vec{\omega}\wedge\overrightarrow{O'P}\right) + 2\vec{\omega}\wedge\vec{V}_r$

Note

The Coriolis acceleration (named after the French scientist Gaspard-Gustave Coriolis) allows us to interpret several phenomena such as:

- The movement of air masses and cyclones,
- The deflection of long-range projectiles,
- The deflection of the plane of motion of a pendulum,
- The slight eastward deflection during free fall, etc.

Example 1

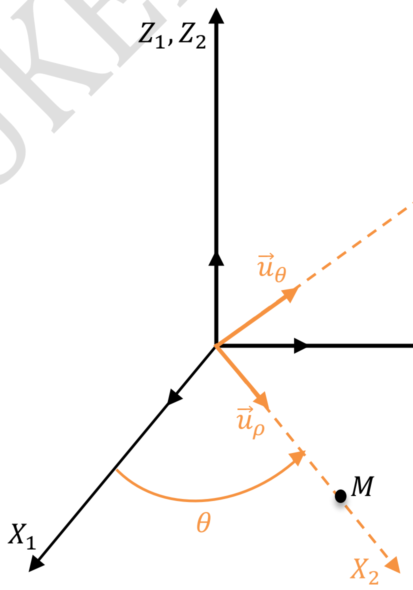
Snowflakes fall vertically at a speed of 8 m/s.

With what speed and at what angle (direction) do these snowflakes strike the windshield of a car moving at a speed of 50 km/h?

Example 2

In coordinate system *R1* (O, X_1, Y_1, Z_1), we consider a second coordinate system R_2 (O, X_2, Y_2, Z_2) that rotates around the Z_1 axis, such that the X_2 axis forms an angle θ with the X_1 axis (as shown in the figure below).

A point *P* moves along the X_1 axis. Its position is given by *r*.



Calculate

1. The relative velocity and acceleration of point *P*.
2. The driving velocity and acceleration of point *P*.
3. The Coriolis acceleration.
4. Deduce the velocity and acceleration of point P in coordinate system R_1 in polar coordinates.

Example 3

Consider, in a plane (P), an orthonormal coordinate system *xoy* and an object M moving in this plane.

At time t, its coordinates are defined by: $\begin{cases} x(t) = \sqrt{2}cos\frac{t}{2} \\ y(t) = 2\sqrt{2}sin\frac{t}{2} \end{cases}$

1. Find the equation of the trajectory (path).
2. Find at time t the coordinates of the velocity vector $\vec{V}$ and the acceleration vector $\vec{a}$ of this moving object. What is the relationship between $\overrightarrow{OM}$ and $\vec{a}$?
3. After how much time does the moving object pass through the same position on the curve?
4. Between times $t_1 = 0$ and $t_2 = 4\pi$, determine the positions of the moving object and the coordinates of $\vec{V}$ for which the magnitude of the acceleration is equal to $\sqrt{5}/4$.

Example 4

Consider a moving object M performing a motion in space. The equations of motion for this object are expressed in cylindrical coordinates as: $\begin{cases} \rho(t) = t(t+1) \\ \theta(t) = \frac{\pi}{6}t \\ z(t) = 2t \end{cases}$

1. Find the position vector of M.
2. Find the velocity vector of M.
3. Find the acceleration vector of M.
4. Deduce the position of M at t=2s in Cartesian, polar, and spherical coordinates.

Solutions

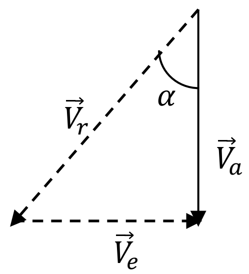
Example 1

$\vec{V}_a$: velocity of the snowflakes with respect to the ground.

$\vec{V}_e$: velocity of the car with respect to the ground.

$\vec{V}_r$: velocity of the snowflakes with respect to the car.

Vectorially:



$$\vec{V}_a = \vec{V}_r + \vec{V}_e \Rightarrow \vec{V}_r = \vec{V}_a - \vec{V}_e \quad = \quad \vec{V}_a = \vec{V}_r + \qquad (-\vec{V}_e) \quad \Rightarrow \quad V_r = \sqrt{{V_a}^2 + {V_e}^2}$$

$$V_e = 50\ km/h \quad = 13.9\ m/s$$

$$V_r = \sqrt{(13.9)^2 + (8)^2} = 16\ m/s$$

The direction of the relative velocity is determined via the angle α between this velocity $\vec{V}_r$ and the vertical axis.

$$\tan\alpha = \frac{V_e}{V_a} = \frac{13.9}{8} \Rightarrow \quad \alpha = 60.1°$$

Example 2

1. Relative velocity and acceleration of the point M.

The position vector in R_2 is: $\overrightarrow{OM} = \vec{r} = r\vec{u}_\rho$

So the relative velocity in R_2 is: $\vec{V}_r = \dot{r}\vec{u}_\rho$ (because $\vec{u}_\rho$, $\vec{u}_\theta, \vec{k}$ are invariable in R_2)

The relative acceleration in R_2 is: $\vec{a}_r = \ddot{r}\vec{u}_\rho$

2. The driven velocity is the velocity of R_2 with respect to R_1.

$$\vec{V}_e = \left.\frac{d\overrightarrow{OO\prime}}{dt}\right|_R + \vec{\omega}\wedge\overrightarrow{O'M}$$

With $\left.\frac{d\overrightarrow{OO\prime}}{dt}\right|_R = \vec{0}$ (The motion of R2 is rotational and $O \equiv O'$)

In this case: $\vec{\omega} = \frac{d\theta}{dt}\vec{k} = \dot{\theta}\vec{k}$

$$\vec{\omega}\wedge\overrightarrow{O'M} = \begin{vmatrix} \vec{u}_\rho & \vec{u}_\theta & \vec{k} \\ 0 & 0 & \dot{\theta} \\ r & 0 & 0 \end{vmatrix} = r\dot{\theta}\vec{u}_\theta$$

Therefore: $\vec{V}_e = r\dot{\theta}\vec{u}_\theta$

The driven acceleration: is the acceleration of R2 with respect to R1.

$$\vec{a}_e = \frac{d^2\overrightarrow{OO'}}{dt^2} + \frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M} + \vec{\omega}\wedge\left(\vec{\omega}\wedge\overrightarrow{O'M}\right)$$

$$\vec{a}_e = \frac{d^2\overrightarrow{OO'}}{dt^2} + \frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M} + \vec{\omega}\wedge\frac{d\overrightarrow{O'M}}{dt}$$

$$\frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M} = \begin{vmatrix}\vec{u}_\rho & \vec{u}_\theta & \vec{k}\\ 0 & 0 & \ddot{\theta}\\ r & 0 & 0\end{vmatrix} = r\ddot{\theta}\vec{u}_\theta$$

$$\vec{\omega}\wedge\frac{d\overrightarrow{O'M}}{dt} = \begin{vmatrix}\vec{u}_\rho & \vec{u}_\theta & \vec{k}\\ 0 & 0 & \dot{\theta}\\ 0 & r\dot{\theta} & 0\end{vmatrix} = -r\dot{\theta}^2\vec{u}_r$$

$$\frac{d^2\overrightarrow{OO'}}{dt^2} = \vec{0}$$

Therefore: $\vec{a}_e = -r\dot{\theta}^2\vec{u}_r + r\ddot{\theta}\vec{u}_\theta$

3. Coriolis acceleration:

$$\vec{a}_C = 2\vec{\omega}\wedge\vec{V}_r = 2\begin{vmatrix}\vec{u}_\rho & \vec{u}_\theta & \vec{k}\\ 0 & 0 & \dot{\theta}\\ \dot{r} & 0 & 0\end{vmatrix} = 2\dot{r}\dot{\theta}\vec{u}_\theta$$

4. The velocity and acceleration of the point M in R_1 are:

$$\vec{V}_a = \vec{V}_r + \vec{V}_e = \dot{r}\vec{u}_\rho + r\dot{\theta}\vec{u}_\theta$$

$$\vec{a}_a = \vec{a}_r + \vec{a}_e + \vec{a}_c = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{u}_\rho + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{u}_\theta$$

Example 3

a) Trajectory

To obtain the equation of the trajectory, it is sufficient to eliminate time between the parametric equations.

$$\begin{cases}\cos\dfrac{t}{2} = \dfrac{x}{\sqrt{2}}\\ \sin\dfrac{t}{2} = \dfrac{y}{2\sqrt{2}}\end{cases} \Rightarrow \quad \frac{x^2}{2} + \frac{y^2}{8} = 1$$ The trajectory is an ellipse.

$$\vec{V} = \begin{cases}V_x = \frac{dx}{dt} = -\frac{\sqrt{2}}{2}\sin\frac{t}{2}\\ V_y = \frac{dy}{dt} = \sqrt{2}\cos\frac{t}{2}\end{cases}$$

$$\vec{a} = \begin{cases} a_x = \dfrac{d^2x}{dt^2} = -\dfrac{\sqrt{2}}{4}\cos\dfrac{t}{2} \\ a_y = \dfrac{d^2y}{dt^2} = -\dfrac{\sqrt{2}}{2}\sin\dfrac{t}{2} \end{cases} \quad \Rightarrow \quad \vec{a} = -\frac{1}{4}\overrightarrow{OM}$$

Since the trajectory is an ellipse, the motion is periodic. Let T be the period of the motion, so: $\quad x(t) = x(t+T) \quad \Leftrightarrow \quad \sqrt{2}cos\frac{t}{2} = \sqrt{2}cos\left(\frac{t+T}{2}\right)$

It is known that: $\cos\alpha = cos\,(\alpha + 2\pi)$

$$cos\frac{t}{2} = cos\left(\frac{t+T}{2}\right) \quad \Rightarrow \quad \frac{T}{2} = 2\pi \quad \Rightarrow \quad T = 4\pi\ s$$

b) $\|\vec{a}\| = \frac{\sqrt{5}}{4} \quad \Leftrightarrow \quad \sqrt{\left(-\frac{\sqrt{2}}{4}cos\frac{t}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}sin\frac{t}{2}\right)^2} = \frac{\sqrt{5}}{4}$

$$a^2 = \frac{2}{16}cos^2\frac{t}{2} + \frac{2}{4}sin^2\frac{t}{2} = \frac{5}{16} \quad \Rightarrow \quad 2cos^2\frac{t}{2} + 8sin^2\frac{t}{2} = 5$$

$$2\left(1 - sin^2\frac{t}{2}\right) + 8sin^2\frac{t}{2} = 5 \quad \Rightarrow \quad 6sin^2\frac{t}{2} = 3$$

$$\sin\frac{t}{2} = \pm\frac{1}{\sqrt{2}} \quad ,t > 0 \quad \Rightarrow \frac{t}{2} = \begin{cases} \dfrac{\pi}{4} + 2k\pi \\ \dfrac{3\pi}{4} + 2k\pi \end{cases}$$

The coordinates of V are summarized in the table below:

t	V_x	V_y	x	y
$\frac{\pi}{2}$	$-\frac{1}{2}$	+1	*1*	*2*
$\frac{3\pi}{2}$	$-\frac{1}{2}$	*-1*	*-1*	*+2*

Example 4

c) Position vector

$$\overrightarrow{OM} = \rho\vec{u}_\rho = t(t+1)\vec{u}_\rho + 2t\vec{k}$$

a) Velocity

$$\vec{V} = \dot{\rho}\vec{u}_\rho + \rho\dot{\theta}\vec{u}_\theta + \dot{z}\vec{k} = (2t+1)\vec{u}_\rho + \frac{\pi}{6}(t^2+t)\vec{u}_\theta + 2\vec{k}$$

b) Acceleration vector

$$\vec{a} = (\ddot{\rho} - \rho\dot{\theta}^2)\vec{u}_\rho + (\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta})\vec{u}_\theta + \ddot{z}\,\vec{k} = \left[2 - \frac{\pi^2}{36}(t^2 + t)\vec{u}_\rho\right] + \frac{\pi}{3}[2t + 1]\vec{u}_\theta$$

c) Position of M

At $t = 2s$, we have: $\begin{cases} \rho = 6 \\ \theta = \frac{\pi}{3} \\ z = 4 \end{cases}$

In the Cartisian coordinate system (x, y, z)

$$\begin{cases} x = \rho.cos\theta \\ y = \rho.sin\theta \\ z = z \end{cases} \Rightarrow \begin{cases} x = 3 \\ y = 3\sqrt{3} \\ z = 4 \end{cases}$$

In polar coordinates (r, θ)

$$\begin{cases} r = 6 \\ \theta = \dfrac{\pi}{3} \end{cases}$$

In spherical coordinates: (r, θ, φ)

$$\begin{cases} r = \left\|\overrightarrow{OM}\right\| = \sqrt{\rho^2 + z^2} = \sqrt{52} \\ \varphi = \theta_{cylind} = \dfrac{\pi}{3} \\ \cos\theta = \dfrac{z}{r} \Rightarrow \theta = \arccos\dfrac{4}{\sqrt{52}} = 0.98\,rads \end{cases}$$

CHAPTER 3
PARTICLE DYNAMICS

III. PARTICLE DYNAMICS

III.1 Introduction

Dynamics is the branch of mechanics that studies the motion of material bodies in relation to the causes (forces) that produce it. It highlights the cause-and-effect principle. It predicts the motion of a body located in a specific field or medium.

III.2 Definitions

- **Material System:** A material system is a collection of material points. It has mass and occupies space.

- **Rigid Material System:** All material points constituting the system remain fixed relative to each other. This corresponds to the definition of a solid in mechanics.

- **Deformable Material System:** All systems not corresponding to the definition of a solid. For example, two solids without connections between them form a deformable system when each of the solids moves independently of the other.

- **Isolated Material System**: An "isolated system" refers to a physical system that does not exchange matter or energy with its surroundings. In other words, it is a closed system that is entirely self-contained and isolated from any external influences. This means that no matter can enter or leave the system, and no energy, such as heat or work, can be exchanged between the system and its surroundings.

- **Inertia:** It is the resistance exhibited by a material body when an attempt is made to change its state of motion (to move or stop a body). Inertia implies that an object at rest tends to stay at rest, and an object in motion tends to stay in motion with a constant velocity along a straight line, unless acted upon by an external force.

- **Mass:** The mass of a system is the quantity of matter it contains. It remains constant in the context of Newtonian mechanics.

- **Center of Inertia:** The center of inertia, also known as the center of mass, is a point within a body or system of particles that represents the average position of all the mass in the system. It's the point where the total mass of the object or system can be considered to be concentrated for the purposes of analyzing its motion.

  Mathematically, for a system of particles with masses $m_1,\ m_2,\ \ldots,\ m_n$ located at positions (x_1, y_1, z_1), (x_2, y_2, z_2), …, (x_n, y_n, z_n), the center of inertia $C(x_G, y_G, z_G)$, is calculated as:

$$\begin{cases} x_G = \dfrac{\sum_{i=1}^{i=n} m_i x_i}{\sum_{i=1}^{i=n} m_i} \\ y_G = \dfrac{\sum_{i=1}^{i=n} m_i y_i}{\sum_{i=1}^{i=n} m_i} \\ z_G = \dfrac{\sum_{i=1}^{i=n} m_i z_i}{\sum_{i=1}^{i=n} m_i} \end{cases} \Leftrightarrow \quad \overrightarrow{OG} = \frac{\sum m_i \overrightarrow{OM_i}}{\sum m_i}$$

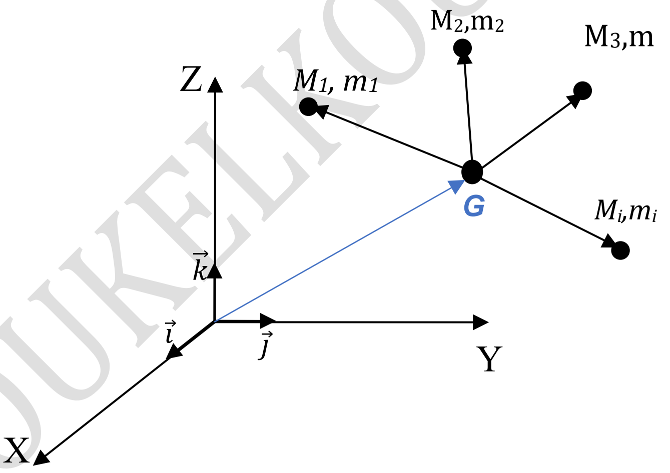


Figure 3-1: Illustration of the Center of Inertia

III.3 Momentum

Momentum is a fundamental concept in physics that represents the quantity of motion possessed by an object. It is a vector quantity, meaning it has both magnitude and direction. Mathematically, momentum ($\vec{p}$) is defined as the product of an object's mass (m) and its velocity (v):

$$\vec{p} = m \cdot \vec{v}$$

The momentum vector is collinear with the velocity of the point and depends on the reference frame in which the velocity is expressed.

In terms of components, for an object moving in three-dimensional space with velocity components v_x, v_y and v_z along the x, y, and z axes respectively, the momentum components are given by:

$p_x = m.v_x$

$p_y = m.v_y$

$p_z = m.v_z$

For a material system consisting of *n* masses moving at velocities $\vec{v}_i$, the total momentum vector is the sum of the momentum vectors of each individual part (points) that makes up the system.

$$\vec{p}_{tot} = \sum_i^n m_i \vec{v}_i = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \ldots \ldots \vec{p}_n = \sum_i^n \vec{p}_i$$

III.4 Principle (law) of Conservation of Momentum.

The concept of momentum is crucial in understanding the behaviour of objects in motion, especially in interactions involving forces and collisions. In a closed (isolated) system where no external forces are acting, the total momentum is conserved, meaning the total momentum before an event (like a collision) is equal to the total momentum after the event. This principle is known as the *law of conservation of momentum*.

Example

Let's consider an isolated system composed of two particles such that:

At time t: $\vec{p}_{tot} = \vec{p}_1 + \vec{p}_2$

At time t': $\vec{p}'_{tot} = \vec{p}'_1 + \vec{p}'_2$

From the law of conservation of momentum, we can write:

$$\vec{p}_1 + \vec{p}_2 = \vec{p}'_1 + \vec{p}'_2$$

$$\vec{p}'_1 - \vec{p}_1 += -(\vec{p}'_2 - \vec{p}_2 +)$$

$$\Rightarrow \quad \Delta\vec{p}_1 = -\Delta\vec{p}_2$$

Therefore, in an isolated system, there is a balance of momentum.

III.5 Principles of Dynamics

Mechanics and several branches of physics are grounded in principles or postulates that are not proven. Among these, we have the principles of mechanics which form the foundation for studying the motion of material systems.

The principles of dynamics encompass the fundamental laws and concepts that govern the behaviour of motion in physical systems. These principles are derived from Newton's laws of motion and provide a framework for understanding the causes and effects of various forces acting on objects, as well as how these forces influence their motion and interactions.

III.5.1 Principle of Inertia

The principle of inertia, often referred to as Newton's First Law of Motion, states that an object at rest will remain at rest, and an object in uniform motion will continue in that motion, unless acted upon by an external force. In other words, an object will maintain its state of motion (whether at rest or moving at a constant velocity) unless a net force is applied to it. This principle is a foundational concept in classical mechanics and is essential for understanding the behaviour of objects in the absence of external influences.

Consequence

A Galilean reference frame is one in which the principle of inertia applies.

III.5.2 Newton’s Second Law of Motion

III.5.2.1 Force

In physics, force is a vector quantity that represents an interaction between objects or particles that causes a change in their motion or deformation. Force is what causes an object to accelerate, decelerate, change direction, or deform. It is a fundamental concept in classical mechanics and is described by Newton's second law of motion.

Forces can be categorized as:

1. Action-at-a-distance forces: These include gravitational forces, electromagnetic forces, and nuclear cohesive forces.
2. Contact forces: These encompass frictional forces and tension.

III.5.2.2 Fundamental Principle of Dynamics: Newton's Second Law

If a system S with mass m and center of inertia G moves within a Galilean reference frame and undergoes an uncompensated action (it's not isolated), then according to the principle of dynamics, the momentum of this system cannot remain constant over time. In other words, the Fundamental Principle of Dynamics (FPD) links the cause (uncompensated actions) to the observed effect (change in momentum).

The FPD is expressed as:

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v})$$

Special Cases

1. **For constant mass:**

$$\vec{F} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} = m\vec{a}$$

Notes

- If the resultant force is constant, then $\vec{a} = \frac{\vec{F}}{m} = \overrightarrow{cte}$, and the motion is uniformly accelerated rectilinear motion.
- the force on a particle is the same for all inertial observers (recall that inertial observers move with constant velocities relative to one another).

2. **For Variable Mass:**

$$\vec{F} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt}$$

III.5.3 Action-Reaction Principle: Newton's Third Law

For every action, there is an equal and opposite reaction. When two systems S1 and S2 interact with each other, regardless of the reference frame or their motion (or lack thereof), the force applied by the first system on the second is equal in magnitude and opposite in direction to the force applied by the second system on the first.

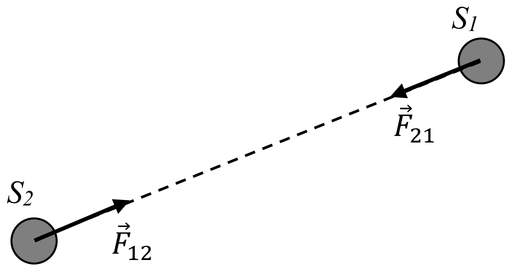


Figure 3-2: Illustration of Action-Reaction Law

We consider an isolated system made up of two particles:

$$\vec{p}_1 + \vec{p}_2 = \overrightarrow{cte} \quad \Rightarrow \quad \frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = \vec{0}$$

$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_1}{dt} = \vec{0} \quad \Rightarrow \quad \frac{d\vec{p}_1}{dt} = -\frac{d\vec{p}_1}{dt} \quad \Rightarrow \quad \vec{F}_1 = -\vec{F}_2$$

III.8.3 Force of Gravity

The force of gravity, also known as gravitational interaction force, refers to the attractive force exerted by a mass m_1 on another mass m_2 (figure below).

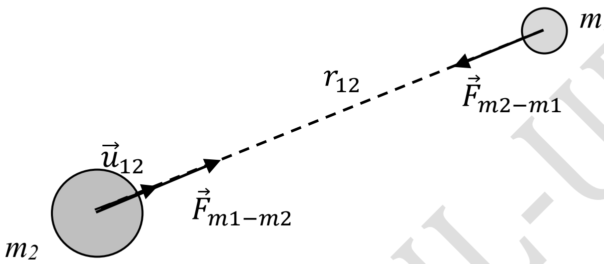


Figure 3-3: Illustration of the Force of Gravity.

According to Newton's law of universal gravitation, the force of gravity is directly proportional to the product of the masses of the objects and inversely proportional to the square of the distance between their centers. This force is always directed along the line joining the centers of the masses. Mathematically, it is expressed as:

$$\vec{F}_{m1-m2} = G\frac{m_1 . m_2}{r_{12}^2}\vec{u}_{12}$$

Where:

$G = 6.673 \times 10^{-11}\ Nm^2kg^{-2}$: the universal Gravity constant.

$\vec{u}_{12}$: the unit vector directed from m_1 to m_2.

- ***In the Vicinity of the Earth***

Near the surface of the Earth and in the absence of air resistance, all bodies fall toward the ground with a common acceleration $\vec{g}$, called the *acceleration of Gravity* and having a magnitude $g = 9.8\ m.s^{-2}$. The force of gravitational attraction between a body and the Earth is called *the weight* $\vec{w}$ of the body. If m is the mass of the body, then, by Newton's second law,

$$\vec{w} = m\vec{g}$$

- ***Far from the Earth***

For larger distances from the surface of the Earth, the value of g (hence also the weight of a body) varies as a function of the distance from the Earth. We call M and R the mass and the radius of the Earth, respectively, and we let h be a given height above the surface of the Earth. We would like to determine the value of g at this height.
According to *Newton's Law of Gravity*, the magnitude of the gravitational force on a body of mass m, located at a height h above the Earth, is:

$$\vec{w} = G\frac{M.m}{r^2}\vec{u}_r \quad \Rightarrow \quad w = G\frac{M.m}{(R+h)^2}$$

Taking into account that *w=mg,* we find that:

$$g = \frac{G.M}{(R+h)^2}$$

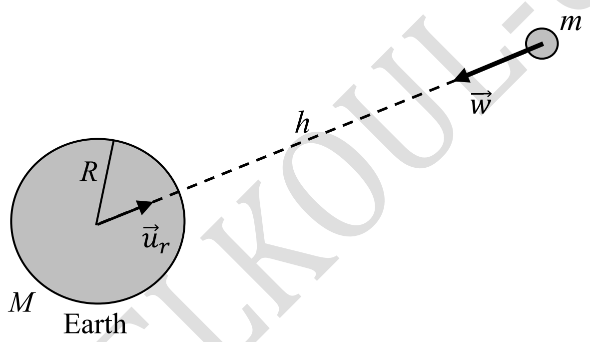


Figure 3-4: Illustration of Gravitational Force near the Earth.

III.7 Motion of Projectile

The motion of a projectile refers to the trajectory followed by an object that is projected into the air and subject only to the force of gravity (free fall motion). The object is launched with an initial velocity at a certain angle to the horizontal and then follows a curved path under the influence of gravity. The vertical and horizontal components of the object's motion are considered separately.

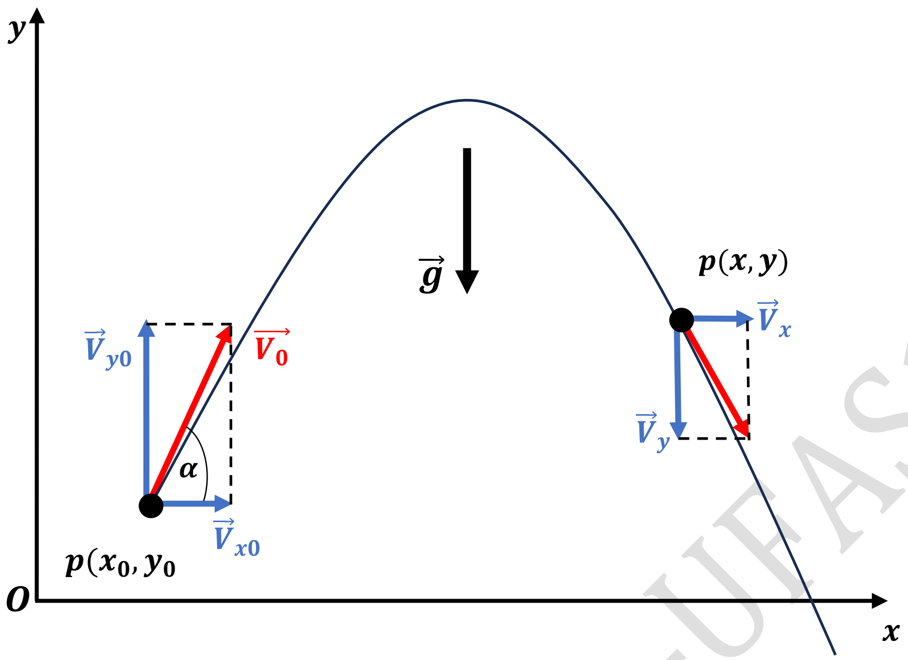


Figure 1-5: Illustration of Projectile Motion

The projectile motion is expressed as:

- **Horizontal Motion**

$$\begin{cases} a_x = 0 \\ V_x = V_{x0} = V_0 \cos\alpha = constant \\ x(t) = V_{x0}.t + x_0 = V_0 \cos\alpha \,.\, t + x_0 \end{cases}$$

- **Vertical Motion**

$$\begin{cases} a_y = -g \\ V_y = -gt + V_{y0} = -gt + V_0 \sin\alpha \\ y(t) = -\frac{1}{2}gt^2 + (V_{y0})t = y(t) = -\frac{1}{2}gt^2 + (V_0 \sin\alpha)t \end{cases}$$

- **Trajectory Equation**

$$y = -\frac{1}{2V^2{}_0}\frac{g}{\cos^2\alpha}x^2 + (\tan\alpha)x$$

- **Apogee**

The apogee, also known as the summit, refers to the highest point reached by a projectile during its motion. It is the point where the projectile's upward trajectory stops and its vertical velocity becomes zero before it starts descending under the influence of gravity. It is obtained from the motion equations as:

$$V_y = 0 \quad \Rightarrow \quad t = \frac{V_0 \sin\alpha}{g}$$

Therefore:

$$y_{max} = -\frac{g}{2}\left(\frac{V_0 \sin\alpha}{g}\right)^2 + V_0 \sin\alpha \left(\frac{V_0 \sin\alpha}{g}\right) = \frac{{V_0}^2 sin^2\alpha}{2g}$$

- **Range**

The range, also known as the horizontal distance or extent, refers to the distance covered horizontally by a projectile during its motion before it returns to the ground level. It is an important parameter that characterizes the extent of the projectile's trajectory. The range depends on factors such as the initial velocity, launch angle, and gravitational acceleration.

It corresponds to:

$$y = 0 \;\; \Rightarrow \;\; x_{max} = \frac{2V_0^2 \sin\alpha \cos\alpha}{g} = \frac{V_0^2 \sin 2\alpha}{g}$$

III.8 Contact Forces

Contact forces are the interactions that occur between objects when they are in direct physical contact with each other. These forces arise due to the interaction of molecules and atoms at the point of contact. Contact forces can include various types of interactions such as friction, normal reaction, tension, compression, and more. These forces play a crucial role in determining the behaviour and motion of objects when they are in contact with each other.

III.8.1 Support Reaction

The support reaction, also referred to as the normal force or contact force, is the force exerted by a surface or object on another object in contact with it. It acts perpendicular to the surface of contact and counterbalances the weight of the object pressing against it. The support reaction prevents the objects from passing through each other and ensures stability and equilibrium. The magnitude of the support reaction depends on factors such as the weight of the object and the nature of the contact surface.

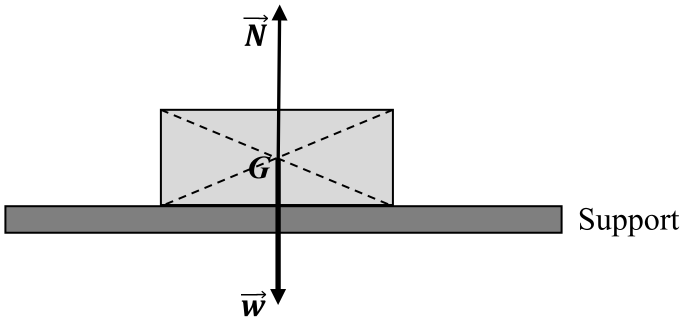


Figure 3-6: Support Reaction

The object is subjected to two external forces: its weight vector $\vec{w}$ and the reaction force $\vec{R}$.

The object is in equilibrium, so according to the fundamental principle of dynamics, we have:

$$\sum_i \vec{F}_i = \frac{d\vec{P}}{dt} = \vec{w} + \vec{N} = \vec{0} \quad \Rightarrow \quad \vec{N} = -\vec{w}$$

III.8.2 Frictional Forces

Sliding friction (or simply friction) is a force that tends to oppose the relative motion of two surfaces when they are in contact. It is a cumulative effect of a large number of microscopic interactions of electromagnetic origin, among the atoms or molecules of the two surfaces. Practically speaking, these surfaces belong to two bodies that are in contact with each other.

Friction forces can be divided into two main types:

1. Static Friction:

Static friction is the frictional force that opposes the initiation of motion between two surfaces (moving object + support) initially at rest relative to each other. This force is directed to the opposite direction of the motion. Let F be the force exerted to move the object on the support and f the force opposing the motion. If F is not large enough, f manages to balance it and the moving object remains at rest. We say that f is *static friction* and we denote it by f_s.

Obviously, $\begin{cases} \vec{F} + \vec{f_s} = \vec{0} \\ \vec{N} + \vec{w} = \vec{0} \end{cases}$

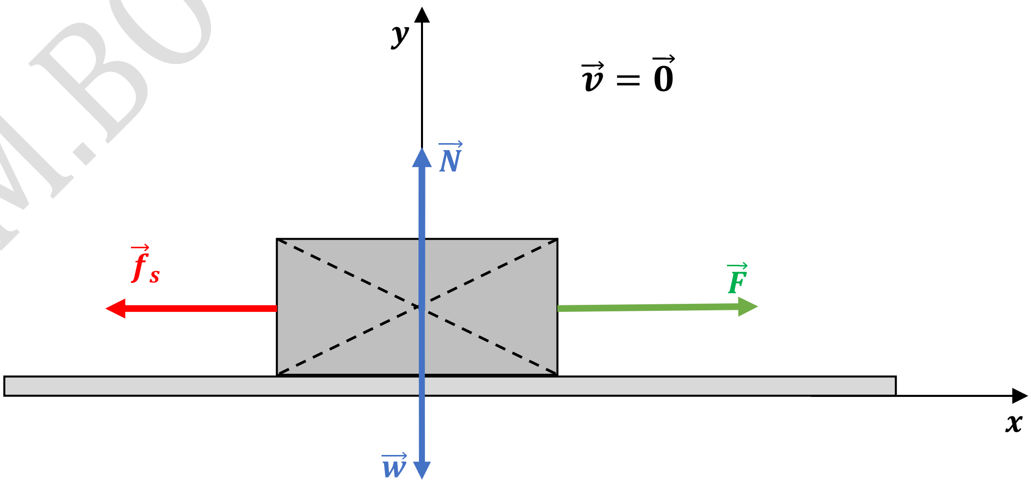


Figure 3-7: Static Friction

Depending on the applied force $\vec{F}$, the force $\vec{f}_s$ varies from zero (when $F = 0$) to a maximum value $f_{s,\max}$. The maximum static frictional force $f_{s,\max}$ is given by:

$$f_{s,max} = \mu_s . N$$

where μ_s is the coefficient of static friction between the surfaces, and N is the normal force between the surfaces.

Special Case

The force F makes an angle θ with the horizontal axis (figure below).

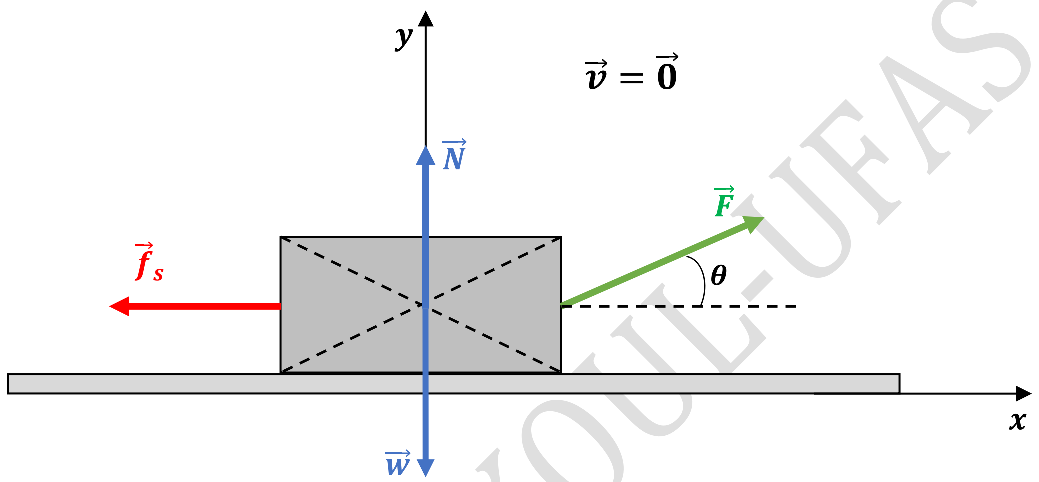


Figure 3-8*: *Static Friction with non-colinear Force

According to the FPD, we write:

$$\sum_i \vec{F_i} = \vec{0} \Rightarrow \vec{F} + \vec{w} + \vec{N} + \vec{f_s} = \vec{0}$$

By projecting onto the two horizontal and vertical axes, we obtain:

$$\begin{cases} (ox): F.\cos\theta - f_s = 0 \\ (oy): N + F.\sin\theta - w = 0 \end{cases}$$

$\vec{N}$ is the force that keeps the body at rest until the applied force $\vec{F}$ is strong enough to overcome it and detach the object from the surface. Before the object starts moving, the static friction force $\vec{f_s}$ reaches its maximum value defined by the law:

$$f_{s,max} = \mu_s . N \quad \Rightarrow \quad \mu_s = \frac{f_{s,max}}{N} = \frac{F.\cos\theta}{w - F.\sin\theta}$$

It is clear that if: $\theta = 0$

$$\mu_s = \frac{f_{s,max}}{N} = \frac{F}{w}$$

2. Kinetic Friction

The kinetic friction is the frictional force that acts when two surfaces are in relative motion to each other. It generally depends on the nature of the surfaces and their roughness.

Using the previous configuration, when $\vec{F}$ exceeds $\vec{f}_{s,max}$ in magnitude, the object is set in motion on the support, accelerating to the motion direction. The frictional force then decreases from $\vec{f}_{s,max}$ to a new, constant value $\vec{f}_k$ (also directed to the opposite direction of motion) that opposes the motion; it is called *kinetic friction*. It is given by:

$$f_k = \mu_k . N$$

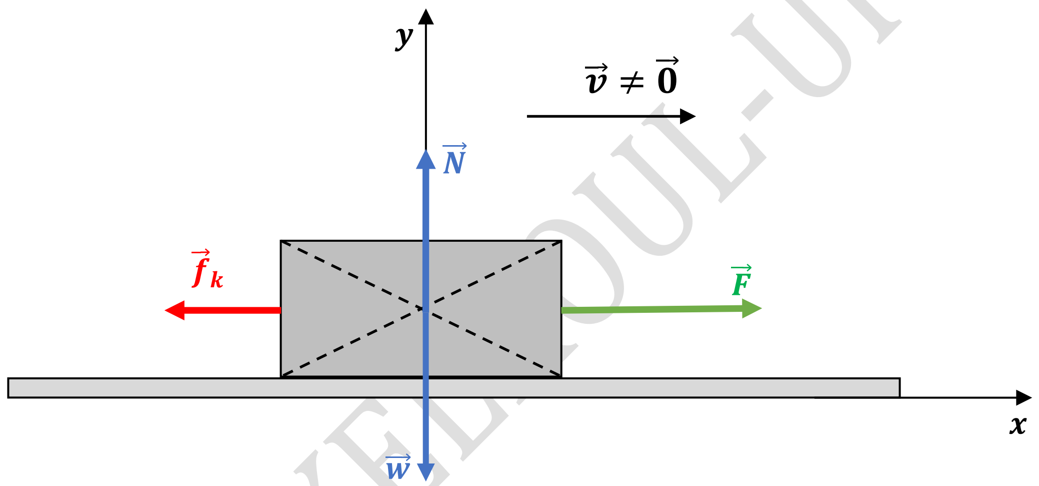


Figure 3-9*: *Kinetic Friction

If we assume that the object moves in the positive direction of the x-axis, then according to the FPD, we get:

$$\sum_i \vec{F}_i = m\vec{a} \;\Rightarrow\; \vec{F} + \vec{w} + \vec{N} + \vec{f}_k = m\vec{a}$$

By projecting onto the two horizontal and vertical axes, we obtain:

$$\begin{cases} F - f_k = ma & \Rightarrow f_k = F - ma \\ N - w = 0 & \Rightarrow N = w \end{cases}$$

$$\mu_k = \frac{f_k}{N} = \frac{F - ma}{w}$$

Special Case

The force F makes an angle θ with the horizontal axis (figure below).

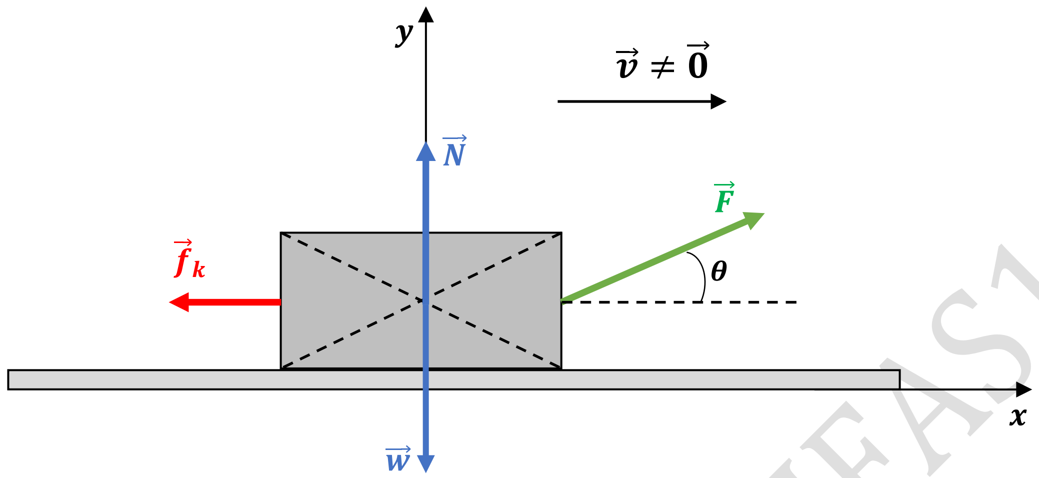


Figure 3-10: Static Friction with non-colinear Forces

According to the FPD, we write:

$$\sum_i \vec{F}_i = m\vec{a} \quad \Rightarrow \quad \vec{F} + \vec{w} + \vec{N} + \vec{f}_k = \vec{0}$$

By projecting onto the two horizontal and vertical axes, we obtain:

$$\begin{cases} (ox): F.\cos\theta - f_k = ma \\ (oy): N + F.\sin\theta - w = 0 \end{cases}$$

$$f_k = \mu_k.N \quad \Rightarrow \quad \mu_k = \frac{f_k}{N} = \frac{F.\cos\theta - ma}{w - F.\sin\theta}$$

It is clear that if: $\theta = 0$

$$\mu_k = \frac{f_k}{N} = \frac{F - ma}{w}$$

Important Notes

- $\mu_k < \mu_s$ (They are *dimensionless* quantities)
- The possible values of static friction are: $0 \leq f \leq f_{s,max} = \mu_s.N$
- True contact is never possible at the atomic level!

III.8.3 Viscous Friction Force

The viscous friction force, also known as drag force, is a resistive force that opposes the motion of an object through a fluid (liquid or gas). It arises due to the interactions between the object and the molecules of the fluid. Viscous friction force is proportional to the velocity of the object and the viscosity of the fluid. It is given by the equation:

$$\vec{F}_{friction} = -b\vec{v}$$

Where:

- $\vec{F}_{friction}$ is the viscous friction force.
- b is the coefficient of viscosity of the fluid.
- $\vec{v}$ is the velocity of the object relative to the fluid.

The negative sign indicates that the force opposes the direction of motion. Viscous friction force is particularly important at low speeds and is responsible for phenomena such as air resistance and the slowing down of objects moving through fluids.

III.8.4 Elastic Force

Elastic force, also known as spring force or Hookean force, is a force exerted by a material when it is stretched or compressed. It is proportional to the displacement from the equilibrium position and acts to restore the object to its original position. Mathematically, the elastic force $\vec{F}_{elastic}$ is often expressed using Hooke's Law:

The acceleration of this kind of motion is given as a function of the position vector:

$$\vec{a} = -\omega^2\overrightarrow{OP}$$

According to the FDP, we can write:

$$\vec{F} = m\vec{a} = -m\omega^2\overrightarrow{OP} = -k\overrightarrow{OP}$$

If the motion occurs along the Ox axis, the tension force is expressed as: $\vec{F} = -kx\vec{\imath}$

Where: $k = m\omega^2 = m\left(\frac{2\pi}{T}\right)^2$

III.8.5 Inertial Force

Inertial force, also known as fictitious force or pseudo force, is a force that appears to act on a mass in a non-inertial reference frame due to the acceleration of that frame. It's a mathematical construct used to explain the observed motion of objects in accelerating reference frames, such as when using a non-inertial coordinate system. This force is not associated with any physical interaction but rather emerges due to the choice of the reference frame. Examples of inertial forces include centrifugal force and Coriolis force in a rotating reference frame.

During relative motion, the law of composition of accelerations is written as:

$$\vec{a}_a = \vec{a}_r + \vec{a}_e + \vec{a}_c$$

For an observer in the absolute Galilean reference frame, the equation of motion is:

$$\sum_i \vec{F}_i = m\vec{a}_a = m\frac{d\vec{V}_a}{dt}$$

For an observer in the non-Galilean relative reference frame, the equation of motion is:

$$\vec{F} = m\vec{a}_r = m\frac{d\vec{V}_r}{dt} = m(\vec{a}_a - \vec{a}_e - \vec{a}_c) = m\vec{a}_a - m(\vec{a}_e + \vec{a}_c)$$

$$m\vec{a}_r = m\frac{d\vec{V}_r}{dt} = \sum_i \vec{F}_i + \sum_i \vec{f}_i$$

Where:

- $\sum_i \vec{F}_i$ is the sum of external forces acting on the system.
- $\sum_i \vec{f}_i = -m(\vec{a}_e + \vec{a}_c)$ represents the inertial forces.

In this context, $\vec{a}_e$ is the acceleration of the reference frame relative to the absolute Galilean frame, and $\vec{a}_c$ is the Coriolis acceleration associated with the motion of the observer in a non-inertial reference frame. These inertial forces arise due to the choice of a non-inertial reference frame and are not actual forces caused by interactions.

III.9 Angular Momentum and Torque

Consider a particle M of mass m, moving along some curve in space. The instantaneous position of the particle is determined by the position vector $\vec{r}$ relative to the fixed origin O of an inertial reference frame. Let $\vec{v}$ be the velocity of the particle at some point of the trajectory. The momentum of the particle at this point is $\vec{p} = m\vec{v}$.

The *angular momentum* of the particle *relative to point O* is defined as the cross product

$$\vec{L}_{M/O} = \overrightarrow{OM} \wedge \vec{P} = \vec{r} \wedge \vec{P} = \vec{r} \wedge m\vec{v} = m(\vec{r} \times \vec{v})$$

The total angular momentum of a system composed of points $(M_1, M_2 \ldots, M_n)$ of respectively the masses $(m_1, m_2 \ldots, m_n)$ is given by:

$$\vec{L}_{tot} = \sum_{i=1}^{n} \vec{r}_i \wedge \vec{P}_i = m_i(\vec{r}_i \times \vec{v}_i) = \sum_{i=1}^{n} \vec{L}_i$$

Angular momentum is a vector quantity that depends on both the linear velocity of the point mass and its position relative to a chosen axis of rotation.

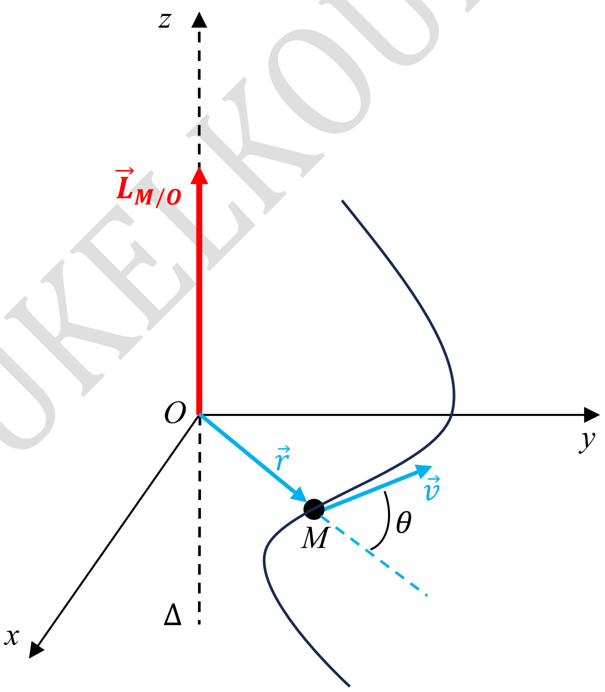


Figure 3-11: Illustration of the Angular Moment

Notes

- in contrast to the momentum, the angular momentum L is not an absolute quantity since its value depends on the choice of the reference point O.
- The vector $\vec{L}$ is perpendicular to the instantaneous plane defined by $\vec{r}$ and $\vec{v}$, its direction being determined by the right-hand rule.

- If θ is the angle between $\vec{r}$ and $\vec{v}$ (where $0 \leq \theta \leq 0$), the magnitude of the angular momentum is given by:

$$L = \|\vec{L}\| = m.r.v.\sin\theta$$

- If $\vec{u}$ is a unit vector normal to the plane of $\vec{r}$ and $\vec{v}$, we write: $\vec{L} = \mp|\vec{L}|.\vec{u}$
- In the case where the particle executes *circular motion* of radius R about O. We notice that $\theta = \pi/2$ and $r = l = R$. The angular momentum L of m with respect to O is a vector normal to the plane of the circle and directed in accordance with the direction of motion. By the relation $v = R\omega$, we have:

$$L = mRv = mR^2\omega$$

let $\vec{F}$ be a force acting on m at a point of the trajectory with position vector r (figure below). The *torque* of $\vec{F}$ *relative* to the origin O of the inertial frame is defined as the cross product

$$\vec{T} = \vec{r} \times \vec{F}$$

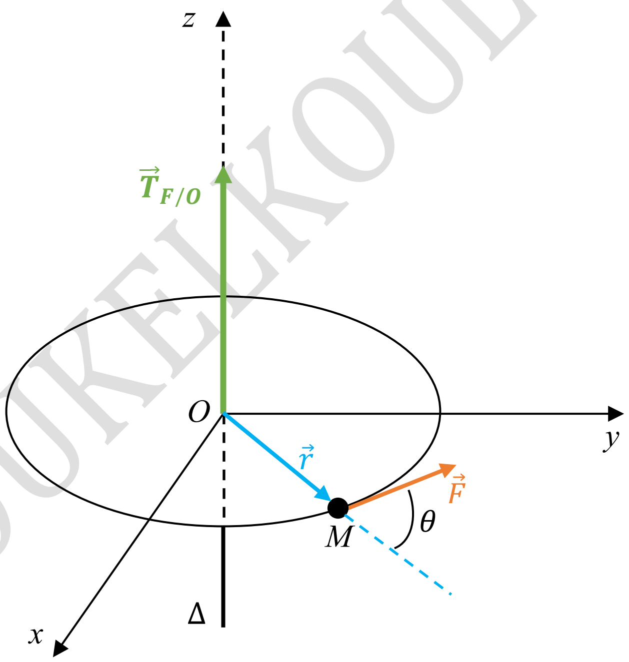


Figure 3-12: Illustration of the Force Torque

- The vector $\vec{T}$ is normal to the plane of $\vec{r}$ and $\vec{F}$,
- its direction being determined by the right-hand rule
- If θ is the angle between r and F ($0 \leq \theta \leq \pi$) and if r and F are the magnitude of the torque is:

$$T = \|\vec{T}\| = r.F.\sin\theta$$

- Also, if $\vec{u}$ is a unit vector normal to the plane of $\vec{r}$ and $\vec{F}$, we write: $\vec{T} = \mp|\vec{T}|.\vec{u}$

III.10 Angular Momentum Theorem

The angular momentum theorem, also known as the principle of conservation of angular momentum, states that the total angular momentum of an isolated system remains constant if no external torques act on the system. In other words, the angular momentum of a system remains unchanged unless an external torque is applied to it.

Mathematically, the angular momentum theorem can be stated as follows:

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{P} + \vec{r} \times \frac{d\vec{P}}{dt}$$

Note

upon differentiation, the order in which $\vec{r}$ and$\vec{P}$ appear must be preserved, since the cross product is not commutative.

$$\frac{d\vec{r}}{dt} \times \vec{P} = \vec{v} \times (m\vec{v}) = m(\vec{v} \times \vec{v}) = \vec{0}$$

on the other hand, we have:

$$\vec{r} \times \frac{d\vec{P}}{dt} = \vec{r} \times m\frac{d\vec{v}}{dt} = \vec{r} \times m\vec{a} = \vec{r} \times \sum \vec{F}_{ext} = \vec{r} \times \vec{F} = \vec{T}$$

Thus, finally,

$$\frac{d\vec{L}}{dt} = \vec{T}$$

In the case where $\vec{T} = \vec{0}$

$$\frac{d\vec{L}}{dt} = \vec{0} \quad \Rightarrow \ \vec{L} = \text{constant}$$

This leads us to the *principle of conservation of angular momentum*:

When the torque of the total force on a particle, relative to some point, is zero, the angular momentum of the particle relative to this point is constant in time.

In other words:

$$\frac{d\vec{L}}{dt} = \vec{0} \Rightarrow \vec{r} \times \sum \vec{F}_{ext} = \vec{0} \quad \Rightarrow \begin{cases} \sum \vec{F}_{ext} = \vec{0} \ \text{ isolated system} \\ \overrightarrow{OM} // \sum \vec{F}_{ext} \ \ \text{central force} \end{cases}$$

III.11 Central Forces

Consider a particle of mass m, moving on a curved path under the action of a total force $\vec{F}$. The instantaneous position of m is determined by the position vector $\vec{r}$ with respect to the origin O of an inertial reference frame. In general, the force $\vec{F}$ varies in space (and, in particular, along the path of m). This force is thus a function of $\vec{r}$. We say that the particle m is moving in a *force field* $\vec{F} = \vec{F}(\vec{r})$.

We consider the following cases:

1. the line of action of $\vec{F}$ always passes through O, regardless of the position of the particle m in space;

2. the magnitude of $\vec{F}$ depends only on the distance $r = |\vec{r}|$ of m from O.

By defining the unit vector $\vec{u}_r$ in the direction of $\vec{r}$,

$$\vec{u}_r = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{r}$$

we can express both these conditions mathematically as follows:

$$\vec{F} = F(r)\vec{u}_r = \frac{F(r)}{r}\vec{r}$$

where $F(r) = \mp \left|\vec{F}\right|$ is an algebraic value, the sign of which depends on the relative orientation of $\vec{F}$ with respect to $\vec{r}$. A force (more correctly, a force field) of is form is called a *central force* with center at O.

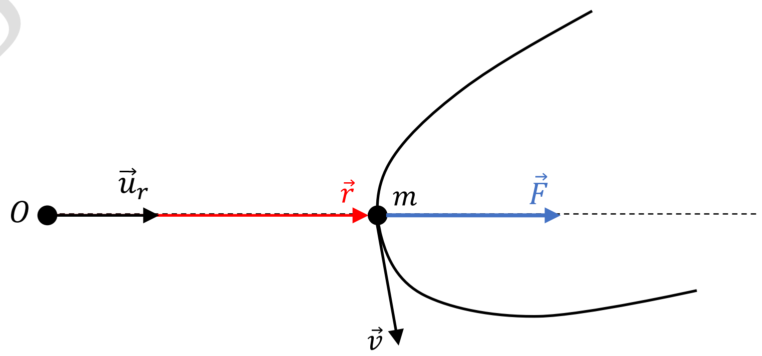


Figure 3-13: A Particle Subject to a Central Force

The motion of a particle m under the action of a central force has the following characteristics:

1. The angular momentum $\vec{L}$ of the particle, with respect to the center O of the force, is constant during the motion of m.
2. The motion takes place on a constant plane.

Indeed: The torque of $\vec{F}$ with respect to O is:

$$\vec{T} = \vec{r} \times \vec{F} = \vec{r} \times \frac{F(r)}{r}\vec{r} = \frac{F(r)}{r}\vec{r} \times \vec{r} = \vec{0}$$

III.12 Applications

Exercise 1

An object of mass m = 0.80 kg is moving on an a plopping plane (inclined plane) with an angle $\alpha = 30°$.

What is the required force to move the object:

a) Upward

b) Downward

In both cases, it is assumed that the object moves with uniform and accelerated motion with an acceleration of 0.01 m/s^2. The kinetic coefficient of friction is given as $\mu_k = 0.3$.

Exercise 2

A mass of 10.2 kg slides on a rough horizontal plane under the influence of a force with an intensity of 30 N. The direction of the force makes an angle of 45° with the horizon. The friction force is 20 N. Taking g = 9.8 m/s^2:

1. Calculate the normal force R.
2. Calculate the resultant force.
3. Calculate the acquired acceleration.

Exercise 3

A particle of mass m rotates around an axis perpendicular to the *xy*-plane. Its position is given at each instant by Cartesian coordinates.

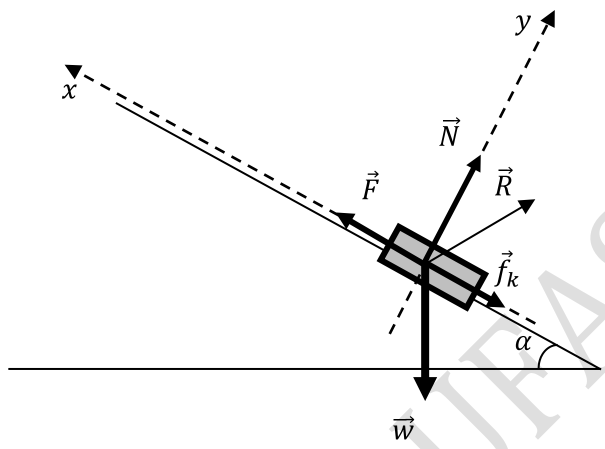
Calculate:

1. The moment of the weight with respect to O, then with respect to Oz, as a function of x, m, and g.
2. The angular momentum of the point M with respect to O, then with respect to Oz, as a function of x, y, $\dot{x}$, $\dot{y}$, and *m*.
3. Derive the equation of motion by applying the theorem of angular momentum.

III.13 Solutions

Exercice 1

a) Upward displacement (uniform motion)



Upward Motion

$$\sum_i \vec{F}_i = \vec{0}$$

$$\vec{F} + \vec{f}_k + \vec{w} + \vec{N} = \vec{0}$$

By projecting onto the axes

$$\begin{cases} ox: \quad F = f_k + w.\sin\alpha \\ oy: \quad N = w.\cos\alpha \end{cases}$$

$$f_k = \mu_k.N$$

$$F = \mu_k.N + w.\sin\alpha$$

$$F = \mu_k.w.\cos\alpha + w.\sin\alpha$$

$$F = mg(\mu_k.\cos\alpha + \sin\alpha)$$

Upward displacement (uniformly accelerated motion)

$$\sum_i \vec{F}_i = m\vec{a} \quad \Rightarrow \vec{F} + \vec{f}_k + \vec{w} + \vec{N} = m\vec{a}$$

By projecting onto the axes

$$\begin{cases} ox: \quad F - f_k - w.\sin\alpha = ma \\ oy: \quad N = w.\cos\alpha = mg.\cos\alpha \end{cases} \quad \Rightarrow \quad F = f_k + mg.\sin\alpha + ma$$

$$\Rightarrow \quad F = m(a + \mu_k.g.\cos\alpha + g.\sin\alpha)$$

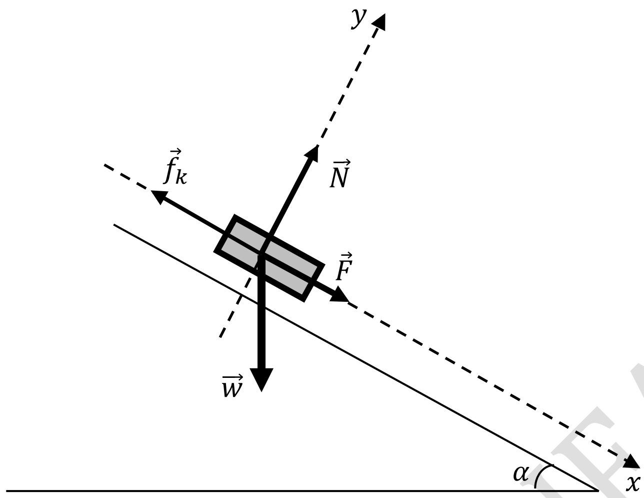
$$\Rightarrow \quad F = m[a + g(\mu_k.\cos\alpha + \sin\alpha)]$$

A.N

$F = 5.95\,N$ for $a = 0$

$$F = 6.03\,N \text{ for } a = 0.10\,m/s^2$$

Downward displacement (both cases).



Downward Motion

According to the FDP, we write:

$$\sum_i \vec{F}_i = m\vec{a} \quad \Rightarrow \vec{F} + \vec{P} + \vec{w} + \vec{f}_k = m\vec{a}$$

By projecting onto the axes

$$\begin{cases} ox: \ \ F + w.\sin\alpha - f_k = ma \\ oy: \ \ N = w.\cos\alpha = mg.\cos\alpha \end{cases} \Rightarrow \begin{cases} F = ma + f_k - w.\sin\alpha \\ f_k = \mu_k N = \mu_k mg.\cos\alpha \end{cases}$$

$$\Rightarrow \ F = m[a + g(\mu_k.cos\alpha - sin\alpha)]$$

In the case of a uniform motion, *a=0*

$$F = mg[\mu_k.\cos\alpha - \sin\alpha]$$

Exercice 2

1) Calculation of the normal force

$$\sum_i \vec{F}_i = m\vec{a} \quad \Rightarrow \vec{F} + \vec{N} + \vec{w} + \vec{f}_k = m\vec{a}$$

By projecting onto the axes

$$\begin{cases} ox: \ \ F.\cos\alpha - f_k = ma \\ oy: \ \ N = w = mg \end{cases} \quad \Rightarrow N = 10.2X9.8 = 99.96N$$

2) Calculation of the frictional force

$$f_k = \mu_k.N = 0.15X99.96 = 14.99\ N$$

3) Calculation of the resultant force

$$F_r = F.\cos\alpha - f_k = 6.22\ N$$

4) Calculation of the aquired acceleration

$$F_r = F.\cos\alpha - f_k = ma \Rightarrow a = \frac{F_r}{m} = 0.6\ m/s^2$$

Exercise 3

1) $\vec{T}_{\vec{w}/O} = \overrightarrow{OM} \times \vec{w}$ with $\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath}$ and $\vec{w} = mg\vec{\jmath}$

Thus

$$\vec{T}_{\vec{w}/O} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ x & y & 0 \\ 0 & mg & 0 \end{vmatrix} = xmg\vec{k}$$

$$T_{\vec{w}/\Delta} = \left(\overrightarrow{OM} \times \vec{w}\right).\vec{k} = \vec{T}_{\vec{w}/O}.\vec{k} = mgx$$

2) $\vec{L}_{M/O} = \overrightarrow{OM} \times \vec{p} = \overrightarrow{OM} \times m\vec{v} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ x & y & 0 \\ m\dot{x} & m\dot{y} & 0 \end{vmatrix} = m(x\dot{y} - y\dot{x})\vec{k}$

$$L_{M/\Delta} = \left(\overrightarrow{OM} \wedge \vec{p}\right).\vec{k} = m(x\dot{y} - y\dot{x})$$

3) Differential equation of motion

$$\frac{d\vec{L}_{M/O}}{dt} = \vec{T}_{\vec{w}/O} \Rightarrow \left\|\frac{d\vec{L}_{M/O}}{dt}\right\| = \left\|\vec{T}_{\vec{w}/O}\right\|$$

$$\Rightarrow \quad m(\dot{x}\dot{y} + x\ddot{y} - \dot{y}\dot{x} - y\ddot{x}) = mgx$$

$$\Rightarrow \quad (x\ddot{y} - y\ddot{x}) = gx$$

CHAPTER 4

WORK AND ENERGY

IV.1 WORK

IV.1.1 Introduction

Work and energy are fundamental concepts in physics that describe the interactions and changes that occur in a physical system. Work refers to the transfer of energy that results from the application of a force over a distance, causing a displacement. It is defined as the product of the force applied on an object and the distance it moves in the direction of the force.

Energy, on the other hand, is the capacity of a system to perform work. There are several forms of energy, including kinetic energy (energy due to motion), potential energy (energy due to position or configuration), and various other types like thermal, chemical, and electrical energy.

IV.1.2 Work of Constant Force on a Rectilinear Displacement

Let's consider a point particle G moving along a straight segment, from point A to point B, and subjected to a constant force $\vec{F}$ along the displacement (figure below).

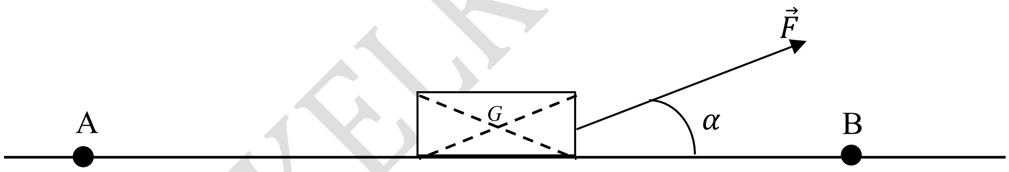


Figure 4-1: Work of a Constant Force

Mathematically, the work done by the constant force $\vec{F}$ during a rectilinear displacement AB is equal to the scalar product of the force vector and the displacement vector:

$$W_{A\to B} = \vec{F}.\overrightarrow{AB} = \|\vec{F}\|.\|\overrightarrow{AB}\|.\cos\alpha$$

Where:

$\overrightarrow{AB}$ is the displacement vector from point A to point B.

α: the angle between $\vec{F}$ and $\overrightarrow{AB}$.

The work of a force can be positive or negative depending on the direction of the force $\vec{F}$ with respect to the direction of the displacement $\overrightarrow{AB}$ (or velocity):

- If $\vec{F}$ is perpendicular to $\overrightarrow{AB}$, the work is zero: the force $\vec{F}$ does not contribute to the displacement of the object (does not produce work).

- When the force opposes the displacement, it is resistive, and the work is negative.
- When the force is in the same direction as the displacement, it is propelling, and the work is positive.
- in uniform motion the resultant force $\vec{F}$on a particle m does not produce work during the motion of m.

Work Unit

The unit of work in the International System (SI) is the joule (J). One joule is equal to one newton-meter (1 J = 1 N·m).

IV.1.3 Work of a Variable Force on an Arbitrary Displace

IV.1.3.1 Elementary Work

Consider a particle of mass m moving in a *force field* $\vec{F}(\vec{r},t)$ (or $\vec{F}(x,y,z,t)$) where $\vec{r}$ is the position vector of m relative to the origin of an inertial reference frame.

To calculate the work, we break down the path AB into a series of infinitesimal (straight-line) elementary displacements $d\vec{l}$ on which the force vector $\vec{F}$ can be considered constant.

The expression for the elementary work on such a displacement is written as:

$$dW = \vec{F}.d\vec{l}$$

Where $d\vec{l}$ is given in Cartesian coordinates by:

$$d\vec{l} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$$

To get the total work done by the force over the entire displacement AB, one simply needs to sum up the elementary works as we move from point A to point B.

$$W = dW_1 + dW_2 + \cdots = \vec{F}_1.d\vec{l}_1 + \vec{F}_2.d\vec{l}_2 + \cdots = \sum_i \vec{F}_i.d\vec{l}_i$$

Since the displacements are infinitesimal, this summation is continuous and we may replace the sum with an integral:

$$W_{A\to B}(\vec{F}) = \int_A^B dW = \int_A^B \vec{F}.d\vec{l}$$

IV.1.4 Examples of Work Calculation

IV.1.4.1 Work of a Constant Force: Weight of an Object

Let's consider a particle of a mass *m* moving from a point A at altitude Z_A to a point B at altitude Z_B, and let's calculate the work done by the weight of this object during this displacement (figure below).

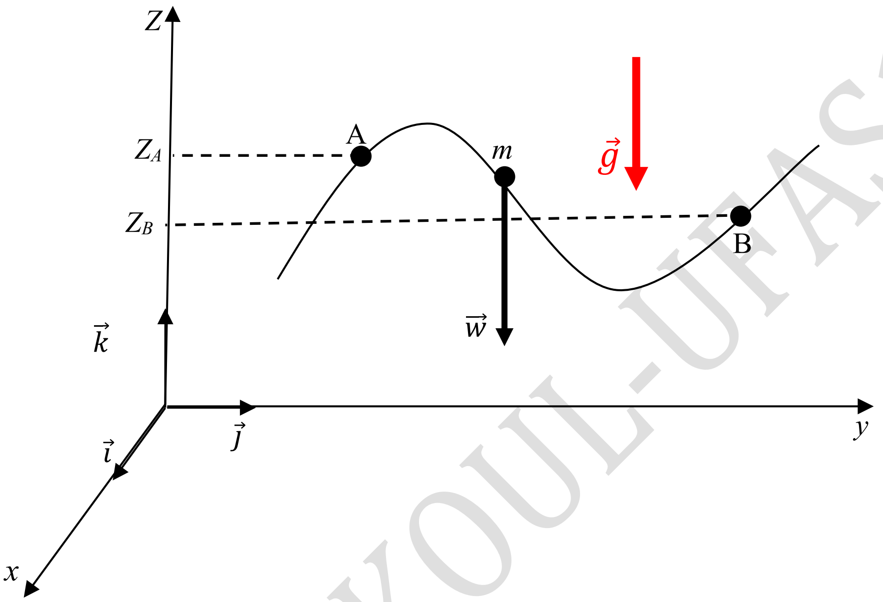


Figure 4-2: Work of the Weight Force

$$W_{A\to B}(\vec{w}) = \int_A^B dW = \int_A^B \vec{w}.d\vec{l}$$

The weight force is given by:

$$\vec{w} = -mg\vec{k} = \overrightarrow{cte}$$

Thus

$$W_{A\to B}(\vec{w}) = \vec{w}\int_A^B d\vec{l} = \vec{w}.\overrightarrow{AB}$$

Where :

$$\overrightarrow{AB} = (x_B - x_A)\vec{\imath} + (y_B - y_A)\vec{\imath} + (z_B - z_A)\vec{\imath}$$

$$\Rightarrow W_{A\to B}(\vec{P}) = mg(z_A - z_B)$$

Note

The work done by the weight force does not depend on the path taken but only on the initial position A and final position B (conservative force).

IV.1.5 Work of a Variable Force: Elastic Force

Let's consider a spring with a stiffness coefficient k, initially at its rest length l_0, with a mass m attached to its end as shown in the figure below. The spring and the mass are on a horizontal plane, and we are only interested in the tension force $\vec{T}$ exerted by the spring, which can be expressed as:

$$\vec{F}_{elastic} = -k(l - l_0)\vec{\imath} = -kx\vec{\imath}$$

When the spring is stretched or compressed by a displacement *x* from its rest length, the tension force $\vec{T}$ acts to restore the spring back to its equilibrium position.

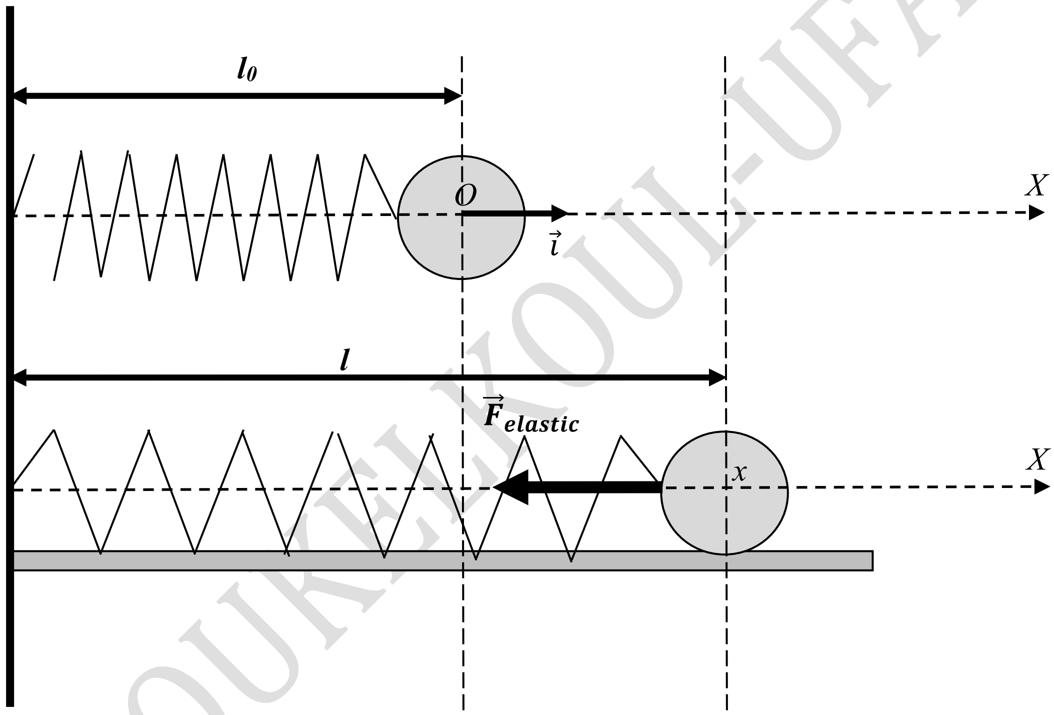


Figure 4-3: Work of Elastic Force

The work done by this variable force during this displacement can be calculated by integrating the force with respect to the displacement. The elementary work of this force is given by:

$$dW = \vec{F}_{elastic}.\,d\vec{l}$$

$$\overrightarrow{dl} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$$

$$\Rightarrow dW = -kxdx$$

When the spring is stretched or compressed from an initial length position x_1 to a final length position x_2, the work done by this variable force during this displacement can be calculated by integrating the force with respect to the displacement:

$$W_{x1\rightarrow x2}(\vec{F}_{elastic}) = \int_{x_1}^{x_2} dW = \int_{x_1}^{x_2} -kxdx = \frac{1}{2}k({x_1}^2 - {x_2}^2)$$

Note

The work done by the tension force does not depend on the path taken but only on the initial and final positions of the spring.

IV.1.6 Power

Power is the rate at which work is done or energy is transferred or converted. It is a measure of how quickly a certain amount of work is performed. Mathematically, power (P) is defined as the ratio of work (W) done to the time (t) taken to do that work:

$$P(t) = \frac{dW}{dt}$$

We can also write :

$$P(t) = \vec{F}.\frac{d\vec{l}}{dt} = \vec{F}.\vec{v}$$

where $\vec{v}$ is the instantaneous velocity of the particle.

The work produced by $\vec{F}$ in the time interval between t_1 and t_2 is:

$$dW = P(t).dt \quad \Rightarrow \quad W = \int_{t_1}^{t_2} P(t).dt = \int_{t_1}^{t_2} (\vec{F}.\vec{v})dt$$

The SI unit of power is the watt (W), which is equivalent to one joule per second (1 J/s).

$$1W = 1J.s^{-1} = 1kg.m^2.s^{-3}$$

It can also be expressed in other units such as horsepower (hp) or kilowatts (kW).

IV.2 ENERGY

IV.2.1 Kinetic Energy

Kinetic energy is the energy an object possesses due to its motion. It is a scalar quantity and depends on both the mass m of the object and its velocity v.

For a particle of mass m moving at a speed v, the formula for the kinetic energy is:

$$E_k = \frac{1}{2}mv^2$$

The SI unit of kinetic energy is the joule (J), which is equivalent to (kg·m²/s²). Kinetic energy is a fundamental concept in physics and is used to describe the energy associated with the motion of particles, objects, or systems.

If $p = mv$ is the magnitude of the particle's momentum, the above relation can also be written in the equivalent form:

$$E_k = \frac{p^2}{2m}$$

IV.2.2 Work-Energy Theorem

The work-energy theorem is a fundamental principle in physics that relates the work done by external forces on an object to the change in its kinetic energy. It states that:

The work of the total force on a particle (equal to the sum of works of all forces acting on the particle), in a displacement of the particle from one point of its trajectory to another, is equal to the change of the kinetic energy of the particle in this displacement.

Mathematically, the work-energy theorem can be expressed as:

$$\Delta E_k|_A^B = E_c(B) - E_c(A) = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 = \sum W_{A\to B}(\vec{F}_{ext})$$

Note

If the work of the resultant force on the particle is zero, the kinetic energy of the particle is constant and, the same is true for the particle's speed (case of uniform rectilinear motion, where the total force is zero, as well as in uniform curvilinear motion where the total force is perpendicular to the velocity).

IV.2.3 Potential Energy

IV.2.3.1 Conservative Forces

A conservative force $\vec{F}_C$ is a type of force in physics which has the property that the work done by this force on an object is independent of the path taken by the object. In other words, the work done by a conservative force depends only on the initial and final positions of the object, and not on the specific path it has followed. This property arises from the conservative nature of the force's associated field.

Examples of conservative forces include:

1. **Gravity:** Near the Earth's surface, the gravitational force is conservative. The work done by gravity to move an object between two points depends only on the difference in altitude between the points, not the followed path.
2. **Spring Force:** The force exerted by an ideal spring (Hooke's law) is conservative. The work done in compressing or stretching a spring is determined by the displacement from its equilibrium position.
3. **Electric Potential:** In electrostatics, the force between charges is conservative. The work done to move a charge between two points depends only on the difference in electric potential between the points.

IV.2.3.2 Non-conservative Forces

A non-conservative force $\vec{F}_{NC}$ is a force for which the work done in moving an object between two points depends on the path taken between those points. In other words, the work done by a non-conservative force is not solely determined by the initial and final positions of the object but also by the specific route or trajectory followed during the motion. Non-conservative forces often involve energy dissipation or conversion into other forms of energy.

Examples of non-conservative force include: friction, air resistance, and tension in a non-ideal string.

IV.2.3.3 Potential Energy

Potential energy E_p is a form of energy associated with the position or configuration of an object in a force field. It represents the potential of an object to do work due to its position or condition. Potential energy is a scalar quantity and is often measured in joules (J) in the International System of Units (SI).

Mathematically, the potential energy E_p is a state function of coordinates for which the integration between its two values taken at the initial and final states is equal to the work done by conservative forces.

IV.2.4 Potential Energy Theorem

The potential energy theorem states that the change in the potential energy of a system is equal to the negative of the work done by conservative forces during the system's displacement. In mathematical terms, it can be expressed as:

$$\Delta E_P|_A^B = E_P(B) - E_P(A) = -\sum W_{A\to B}(\vec{F}_C)$$

By explicitly defining work, we obtain the integral form of potential energy given by:

$$E_P(B) - E_P(A) = -\int_A^B \vec{F}_C . d\vec{l}$$

The differential form is then written as:

$$dE_p = -\vec{F}_{ext}^C . d\vec{l}$$

The local definition of the potential energy can be deduced from the differential form by:

$$dE_p = \overrightarrow{grad}E_p . \overrightarrow{dl} = -\vec{F}_C . d\vec{l}$$

Hence:

$$\overrightarrow{grad}E_p = -\vec{F}_C$$

This relationship shows that the gradient of potential energy is equal to the conservative external force. Consequentally, we can deduce that:

- The potential energy is calculated with respect to a *reference* (any point or any plane) where the value of E_p is assumed to be zero.
- The work of a conservative force $\vec{F}$along a *closed* path is *zero*.

In the case of a free fall for example:

$$E_p(x, y, z) = mgz$$

The conservative force responsible for the free fall is given by:

$$\vec{F}_C = -\overrightarrow{grad}E_p = -\left(\frac{\partial E_p}{\partial x}\vec{\imath} + \frac{\partial E_p}{\partial y}\vec{\imath} + \frac{\partial E_p}{\partial z}\vec{\imath}\right) = -mg\vec{k} = \vec{w}$$

IV.2.5 Examples of Potential Energy

IV.2.5.1 Gravitational Potential Energy

This is the energy that an object possesses due to its height above the reference point, usually the Earth's surface. The higher an object is above the ground, the greater its gravitational potential energy. It can be calculated using the formula:

$$dE_{PG} = -\vec{w}.d\vec{l} = mgdz$$

$$E_{PG} = \int mgdz = mgz + c$$

It is clear that the gravitational potential energy is defined up to a constant which is typically determined by choosing $E_{PG}(z = 0) = 0$ (at the surface of the Earth). Therefore, we can write:

$$E_{PG} = mgz = mgh$$

Where h is the altitude of the considered object.

IV.2.5.2 Elastic Potential Energy:

This is the energy stored in an elastic object, such as a spring, when it is stretched or compressed. It can be calculated in the same manner as gravitational energy. We can use the differential form of potential energy to express the formula for elastic potential energy:

$$dE_{Pe} = -\vec{T}.d\vec{l} = kxdx$$

$$E_{Pe} = \int kxdx = \frac{1}{2}kx^2 + c$$

It is evident that the elastic potential energy is zero in the absence of any deformation:

$$E_{Pe}(x = 0) = 0$$

Thus, the constant c is zero, and we can write:

$$E_{Pe} = \frac{1}{2}kx^2$$

IV.2.5.3 Chemical Potential Energy:

This is the energy stored in the chemical bonds of substances. When chemical reactions occur, this potential energy can be released as kinetic energy.

IV.2.5.4 Electric Potential Energy

This is the energy associated with the position of electric charges within an electric field. It's important in understanding the behaviour of charged particles.

IV.2.6 Equilibrium of a System

A system left to itself spontaneously evolves towards a state of equilibrium, which corresponds to a position where the potential energy is minimized.

Mathematically, the equilibrium of a system is expressed as:

Stable equilibrium

$$x = x_0 \iff \begin{cases} E_p(x = x_0) \text{ minimal} \\ \left.\dfrac{dE_p}{dx}\right|_{x=x_0} = 0 \\ \left.\dfrac{d^2E_p}{dx^2}\right|_{x=x_0} > 0 \end{cases}$$

Instable equilibrium

$$x = x_0 \iff \begin{cases} E_p(x = x_0) \text{ maximal} \\ \left.\dfrac{dE_p}{dx}\right|_{x=x_0} = 0 \\ \left.\dfrac{d^2E_p}{dx^2}\right|_{x=x_0} < 0 \end{cases}$$

IV.2.7 Mechanical Energy

Mechanical energy is the sum of kinetic energy and potential energy within a physical system. It represents the capacity of the system to perform work due to its motion and position. Mechanical energy is conserved in an isolated system when there are no non-conservative forces, such as friction or air resistance, acting on it. The total mechanical energy remains constant as long as only conservative forces are involved. It is given by the formula:

$$E_M = E_k + E_p$$

IV.2.8 Examples of mechanical energy expression:

Free fall of a particle:

$$E_M = \frac{1}{2}mv^2 + mgh$$

A spring:

$$E_M = \frac{1}{2}mV^2 + \frac{1}{2}kx^2$$

IV.2.9 Mechanical Energy Theorem

The mechanical energy of a system subjected to a conservative force (derived from a potential) is conserved over time.

$$E_M = E_k + E_p = \text{constant}$$

$$\Delta E_M = \Delta E_k + \Delta E_p = 0$$

In other words, the mechanical energy of an isolated system is conserved.

In the case of the presence of non-conservative forces $\vec{F}_{NC}$ (such as friction forces), the theorem of mechanical energy is expressed in the form:

$$\Delta E_M = \Delta E_k + \Delta E_p = \sum_i W_i(\vec{F}_{NC})$$

IV.2.10 Applications

Exercise 1

Consider the particle M in the xy-plane subjected to a force given by the expression:

$$\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$$

The particle moves from point A(0, 0) to point B(1,3) along two paths given by:

$y = 3x$ and $y = x^2$

Is the force $\overrightarrow{F}$ conservative?

Exercise 2

What is the initial velocity V_0 directed vertically upwards that needs an object to reach a height h from the surface of the Earth? (All friction is neglected).

Exercise 3

The potential energy of a body is given by: $E_p = 2x^2 - xy + yz$.

1. Find the expression for the force applied to this object.
2. Does the force derive from a potential?

Exercise 4

A boat of mass m, having reached cruising speed V_0, switches off its engines at time t = 0. The water exerts a frictional force proportional to the boat's speed V.

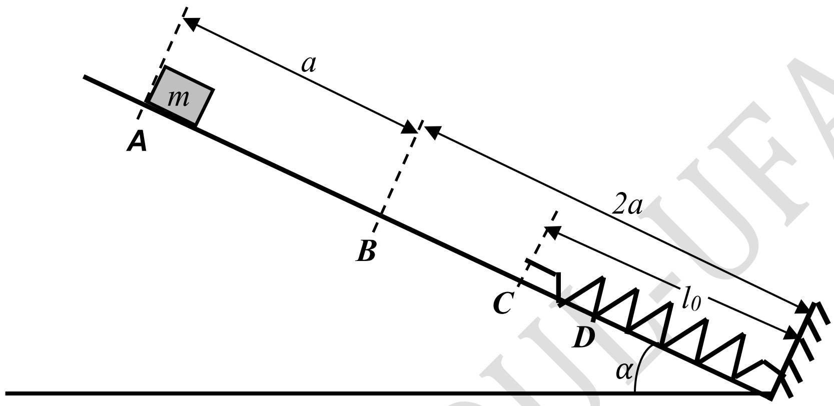
1. Using the fundamental relation of dynamics, give the expression for speed as a function of time. Where will the boat stop?
2. What is the work done by the force of friction between the moment when the boat switches off its engines and the moment when it stops? Compare it with the boat's kinetic energy at time t = 0.

Exercise 5

The body in the figure below has a mass m=5 kg. Starting from rest, it slides on a plane inclined at an angle α=60° to the horizontal, until it reaches a spring *R* of length l_0 =40cm, elasticity constant k = 5000 N m-1 and whose other end C is fixed at the end of the plane. It is assumed that a frictional force opposes the motion of the object on the segment AB=a, whose coefficient of kinetic friction is μ_k =0.2, then cancels out on the rest of the path BC =2a.

1. Calculate the friction force on segment AB.
2. Calculate the velocity acquired by the object at point B, then the velocity V with which the body hits the spring.
3. How much does the spring deform (compress)?
4. How much does the object rise up on the inclined plane when it is pushed upwards by the spring from the point of first impact, assuming frictionless ascent?

We give g=9.8 ms-2.



IV.2.11 Solutions

Exercise 1

- Along the first path: $\boldsymbol{y = 3x}$

$y = 3x \Rightarrow \vec{F} = (-8x^2)\vec{\imath} + 6x^2\vec{\jmath}$

$$dy = 3dx \Rightarrow d\vec{l} = dx\vec{\imath} + dy\vec{\jmath} = dx\vec{\imath} + 3dx\vec{\jmath}$$

$$dW_1 = \vec{F}.d\vec{l} = (-8x^2dx + 18x^2)dx = 10x^2dx$$

$$W_1 = \int_A^B \vec{F}.d\vec{l} = \int_0^1 10x^2dx = \frac{10}{3}x^3\Big|_0^1 = \frac{10}{3}\text{joules}$$

- Along the second path: $\boldsymbol{y = x^2}$

$y = x^2 \quad \Rightarrow \vec{F} = (x^2 - x^4)\vec{\imath} + 2x^3\vec{\jmath}$

$$dy = 2xdx \Rightarrow d\vec{l} = dx\vec{\imath} + dy\vec{\jmath} = dx\vec{\imath} + 2xdx\vec{\jmath}$$

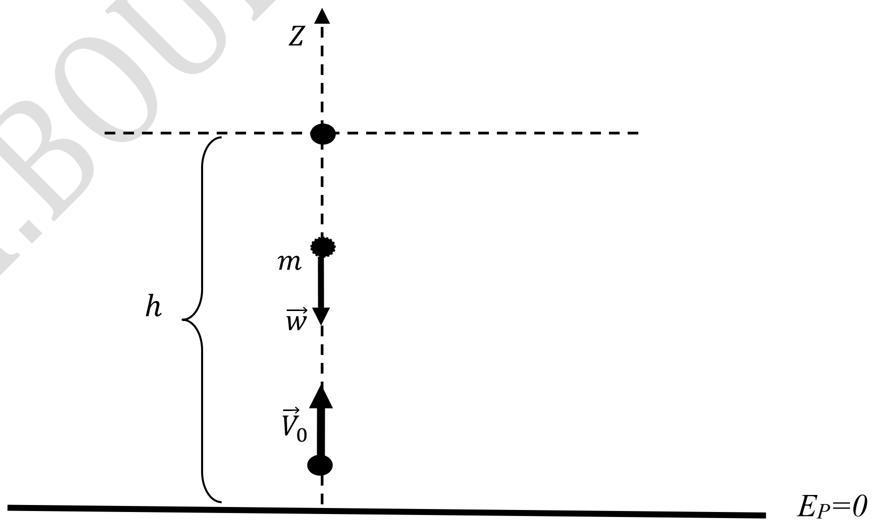
$$dW_2 = \vec{F}.d\vec{l} = (x^2 - x^4)dx + 4x^4dx = (x^2+3x^4)dx$$

$$W_2 = \int_A^B \vec{F}.d\vec{l} = \int_0^1 (x^2+3x^4)dx = \left(\frac{1}{3}x^3 + \frac{3}{4}x^4\right)\Big|_0^1 = \frac{13}{12}\ \text{joules}$$

$W_1 \neq W_2 \Rightarrow$ The force is non-conservative (it depends on the followed path).

Exercise 2

Applying the kinetic energy theorem between the earth's surface and the point reached by the projectile



$$\Delta E_k = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \sum W_{A\to B}(\vec{F}_{ext})$$

The body is subject only to the force of its own weight

$$\Delta E_k = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \sum W(\vec{w}) = -mgh$$

The body reaches its maximum height when v=0, hence

$$-\frac{1}{2}mv_0^2 = -mgh \Rightarrow v_0 = \sqrt{2gh}$$

Exercise 3

1) The expression of the potential energy is given by:

$$E_p = 2x^2 - xy + yz$$

We have :

$$\vec{F} = -\overrightarrow{grad}E_p \quad \Rightarrow \begin{cases} F_x = -\dfrac{\partial E_p}{\partial x} \\ F_y = -\dfrac{\partial E_p}{\partial y} \\ F_z = -\dfrac{\partial E_p}{\partial z} \end{cases} \Rightarrow \begin{cases} F_x = -4x + y \\ F_y = x - z \\ F_z = -y \end{cases}$$

$$\vec{F} = (-4x + y)\vec{\imath} + (x - z)\vec{\jmath} - y\vec{k}$$

2) Is the force derived from a potential?

$$\begin{cases} \dfrac{\partial F_x}{\partial y} = 1 \\ \dfrac{\partial F_y}{\partial x} = 1 \end{cases} \begin{cases} \dfrac{\partial F_y}{\partial z} = -1 \\ \dfrac{\partial F_z}{\partial y} = -1 \end{cases} \begin{cases} \dfrac{\partial F_x}{\partial z} = 0 \\ \dfrac{\partial F_z}{\partial x} = 0 \end{cases}$$

Thus, this force is conservative.

Exercise 4

Consider the boat system in the Galilean reference frame R. The external forces applied to the boat are:

$\vec{w}$: the weight of the boat;

$\vec{R}$: the thrust of the water on the boat;

$\vec{f}$: the force of friction.

The fundamental principle of dynamics gives:

$$\vec{w} + \vec{R} + \vec{f} = m\vec{a} = m\frac{d\vec{V}}{dt}$$

This leads to the following projection in the x direction of the boat's advance

$$m\frac{dV_x}{dt} = -kV_x$$

$$\frac{dV_x}{V_x} = -\frac{k}{m}dt \Rightarrow \ln V_x = -\frac{k}{m}t + C$$

The initial conditions of motion dictate that:

At $t = 0$

$$V = V_0 \ \Rightarrow C = ln\, V_0$$

$$\ln\frac{V_x}{V_0} = -\frac{k}{m}t \Rightarrow V_x = V_0 e^{-\frac{k}{m}t}$$

To find the stop position, we need to integrate the velocity, i.e.:

$$x = \int V_x dt = V_0 \int e^{-\frac{k}{m}t} dt = -\frac{mV_0}{k} e^{-\frac{k}{m}t} + C_1$$

At time t = 0 the boat was at x = 0, so :

$$C_1 = \frac{mV_0}{k} \Rightarrow \quad x = \frac{mV_0}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

The boat will stop after an infinite time at position: $x_a = \frac{mV_0}{k}$

2) By definition, the work of the friction force is given by :

$$W_{1\to2} = \int_1^2 \vec{f}.\overrightarrow{dl}$$

With

$$\vec{f} = -k\vec{V} = -kV_x\vec{\imath}$$

$$\overrightarrow{dl} = V_x\vec{\imath}dt$$

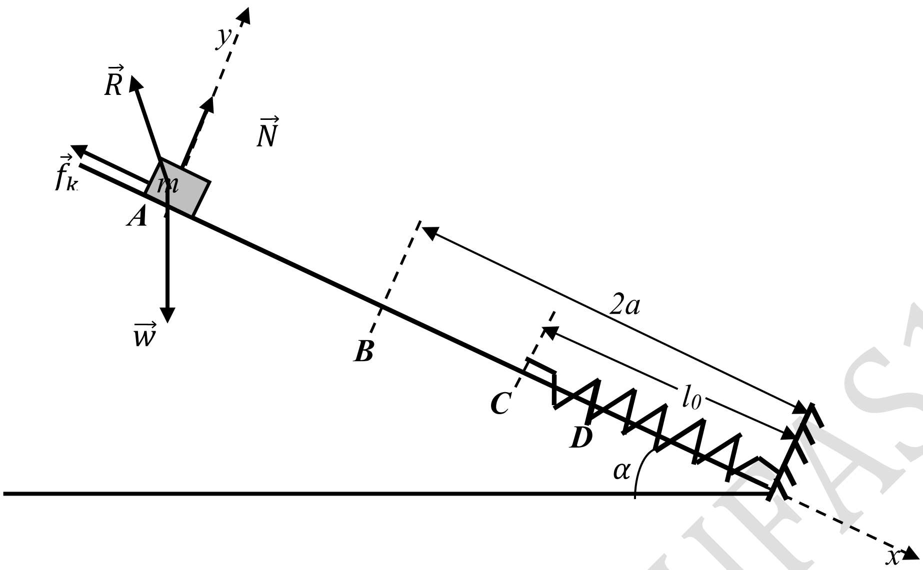
$$W_{1\to2}(\vec{f}) = \int_1^2 -k\, {V_x}^2 dt = -k{V_0}^2 \int_{t_1}^{t_2} e^{-2\frac{k}{m}t}\, dt$$

That is, between the origin of time and infinity:

$$W_{1\to2}(\vec{f}) = k{V_0}^2 \frac{m}{2k}\left[e^{-2\frac{k}{m}t}\right]_0^\infty = -\frac{1}{2}m{V_0}^2$$

which represents the opposite of the initial kinetic energy (this is logical, since only the frictional force is producing work).

Exercise 5



1) Calculation of Friction Force on Segment AB

According to the FPD

$$\sum_i \vec{F}_i = m\vec{a} \;\Rightarrow\; \vec{w} + \vec{R} = m\vec{a}$$

After projection

$$\begin{cases} w\sin\alpha - f_k = ma \\ N - w\cos\alpha = 0 \end{cases} \Rightarrow N = w\cos\alpha = mg\cos\alpha$$

$$f_k = \mu_k.N = \mu_k.m.g.cos\alpha = 4.9\ N$$

Speed at point B

Let's apply the kinetic energy theorem between points A and B

$$\Delta E_k|_A^B = E_k(B) - E_k(A) = \sum_i W_i(\vec{F}_{ext})$$

$$\frac{1}{2}mV_B^2 - \frac{1}{2}mV_A^2 = W(\vec{w}) + W(\vec{f}_k)$$

With

$$W(\vec{w}) = \vec{w}.\overrightarrow{AB} = \|\vec{w}\|.\left\|\overrightarrow{AB}\right\|.\cos(\vec{w}, \overrightarrow{AB}) = m.g.b.\sin\alpha$$

$$W(\vec{f}_k) = \vec{f}_k.\overrightarrow{AB} = \|\vec{f}_k\|.\left\|\overrightarrow{AB}\right\|.\cos(\vec{f}_k, \overrightarrow{AB}) = -\mu_k m.g.b.\cos\alpha$$

$$V_A = 0 \;\Rightarrow\; \tfrac{1}{2}mV_B^2 = m.g.b.\sin\alpha - \mu_k m.g.b.\cos\alpha$$

$$\Rightarrow\; V_B = \sqrt{2gb(\sin\alpha - \mu_g\cos\alpha} = 3.88m/s$$

The speed at which the body hits the spring

Let's apply the kinetic energy theorem between points B and C

$$\Delta E_k|_B^C = E_k(C) - E_k(B) = \sum_i W_i(\vec{F}_{ext})$$

$$\Rightarrow \frac{1}{2}mV_C^2 - \frac{1}{2}mV_B^2 = W(\vec{w}) = mg(2b - l_0)\sin\alpha$$

$$\Rightarrow V_C = V = \sqrt{2g(2b - l_0)\sin\alpha + v_B^2} = 4.6m/s$$

How much does the spring compress?

Applying the mechanical energy theorem between points C and D

$$\Delta E_M|_C^D = E_M(D) - E_M(C) = 0 \qquad (conservative\ system)$$

$$\Rightarrow E_M(D) = E_M(C) \qquad \Leftrightarrow \qquad E_k(D) + E_P(D) = E_k(C) + E_P(C)$$

$$\frac{1}{2}kx^2 = \frac{1}{2}mV^2 \qquad \Rightarrow \qquad x = V\sqrt{\frac{m}{k}} = 14.5cm$$

2) **How far is the spring pushed back?**

All the potential energy acquired by the spring during compression is transformed back into kinetic energy.

Let's apply the mechanical energy theorem between points D and C

$$\Delta E_M|_D^C = E_M(C) - E_M(D) = 0 \qquad (conservative\ system)$$

$$E_k(D) + E_P(D) = E_k(C) + E_P(C)$$

$$\frac{1}{2}kx^2 = \frac{1}{2}mV^2 \quad \Rightarrow \quad V = x\sqrt{\frac{k}{m}} = 4.6m/s$$

To calculate the distance covered by the body during ascent, we apply the kinetic energy theorem between the launch point C and the point E where it stops on the inclined plane.

$$\Delta E_C|_C^E = E_c(E) - E_c(C) = W(\vec{P})$$

$$\Rightarrow \quad -\frac{1}{2}mV_C^2 = -(mgsin\alpha)|CE|$$

Let : $CE = d$

$$\Rightarrow \quad d = \frac{V_C^2}{2gsin\alpha} = 1.23m$$

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REFERENCES

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