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QUENCHING PHENOMENA AND REACTION-DIFFUSION SYSTEMS IN BIOLOGY AND MEDICINE

Presented by

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QUENCHING PHENOMENA AND REACTION-DIFFUSION SYSTEMS IN BIOLOGY AND MEDICINE

Doctoral Thesis

Presented by :
Miss. Samiha DJEMAI

Supervisor :
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بسم الله الرحمن الرحیم

It is my conviction that pure mathematical construction enables us
to discover the concepts and the laws connecting them,
which gives us the key to the understanding of nature ...
In a certain sense, therefore,
I hold it true that pure thought can grasp reality,
as the ancients dreamed.

Albert Einstein.

DEDICATION

To the soul of my dear mother,
To my dear father,
To all my family,
To all my friends.

ACKNOWLEDGEMENTS

First of all, I thank God the Almighty for having given me the patience, the will and the energy to follow and carry out this humble work.

I would like to extend my deepest gratitude and appreciation to my thesis director, Professor Salim Mesbahi, for his exceptional supervision, invaluable advice, and unwavering support throughout the years of my research.

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participate in academic discussions has broadened my horizons and enriched my overall academic experience.

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Once again, I express my heartfelt gratitude to each of you for your time, expertise, and dedication. I consider myself fortunate to have benefited from your wisdom and guidance.

Samiha

Setif, March 10th, 2024

العنوان : ظواهر التبريد وأنظمة تفاعل-انتشار في علم الأحياء والطب.

ملخص : تستكشف هذه الأطروحة موضوع أنظمة تفاعل- انتشار وتطبيقاتها في علم الأحياء والطب والهندسة الحيوية. ويتكون من تسعة فصول تغطي جوانب مختلفة من الموضوع. تبدأ الأطروحة بمقدمة عامة تؤكد على جوهر الموضوع والقضايا المحددة التي تم استكشافها في البحث. يقدم الفصل الأول نظرة عامة حول أنظمة تفاعل- انتشار وتطبيقاتها في التخصصات العلمية المختلفة. ويركز الفصل الثاني على مفهوم "التبريد" وتطبيقاته في علم الأحياء والطب والهندسة الحيوية. ويقدم الفصل الثالث أنظمة تفاعل- انتشار مع شروط أولية. ويناقش الفصل الرابع المعادلات التفاضلية الجزئية شبه الخطية من الدرجة الأولى. تتعمق الفصول الخامس والسادس والسابع في أنظمة تفاعل- انتشار الشاذة وظاهرة التبريد. ويتناول الفصل الثامن نظام تفاعل- انتشار شبه خطي ذي رتبة كيفية. ويركز الفصل الأخير على فئة محددة من المعادلات التفاضلية الجزئية شبه الخطية من الدرجة الأولى. تقدم الأطروحة الأطر النظرية والأساليب الرياضية والتطبيقات العملية، مما يساهم في التقدم في أنظمة تفاعل- انتشار.

كلمات مفتاحية : أنظمة تفاعل - انتشار، معادلات مكافئة شاذة، ظاهرة التبريد، وجود شامل، حلول موجبة، معادلات تفاضلية جزئية شبه خطية من الدرجة الأولى.

Title : QUENCHING PHENOMENA AND REACTION-DIFFUSION SYSTEMS IN BIOLOGY AND MEDICINE.

Abstract : This thesis explores the topic of reaction-diffusion systems (RDSs) and their applications in biology, medicine, and bioengineering. It consists of nine chapters that cover various aspects of the subject. The thesis begins with a general introduction that emphasizes the essence of the subject and the specific issues explored in the research. The first chapter provides an overview of RDSs and their applications in different scientific disciplines. The second chapter focuses on the concept of "Quenching" and its applications in biology, medicine, and bioengineering. The third chapter introduces RDSs with initial conditions. The fourth chapter discusses first order quasilinear PDEs. Chapters five, six and seven delve into singular RDSs and the quenching phenomenon. The eighth chapter examines a quasilinear RDS of arbitrary order. The last chapter focuses on a specific class of first order quasilinear PDEs. The thesis presents theoretical frameworks, mathematical methods, and practical applications, contributing to advancements in RDSs.

Keywords : Reaction-diffusion systems, singular parabolic equations, quenching phenomenon, global existence, positive solutions, quasilinear first order PDE.

Titre : PHÉNOMÈNES DE TREMPE ET SYSTÈMES DE RÉACTION-DIFFUSION EN BIOLOGIE ET MÉDECINE.

Résumé : Cette thèse explore le thème des systèmes de réaction-diffusion (SRDs) et leurs applications en biologie, médecine et bio-ingénierie. Il se compose de neuf chapitres qui couvrent divers aspects du sujet. La thèse commence par une introduction générale qui met l’accent sur l’essence du sujet et les enjeux spécifiques explorés dans la recherche. Le premier chapitre donne un aperçu des SRDs et de leurs applications dans différentes disciplines scientifiques. Le deuxième chapitre se concentre sur le concept de "Quenching" et ses applications en biologie, médecine et bio-ingénierie. Le troisième chapitre présente les SRDs avec conditions initiales. Le quatrième chapitre traite des EDPs quasi-linéaires du premier ordre. Les chapitres cinq, six et sept se penchent sur les SRDs singuliers et sur le phénomène de trempe. Le huitième chapitre examine un SRD quasi-linéaire d'ordre arbitraire. Le dernier chapitre se concentre sur une classe spécifique d’EDPs quasi-linéaires du premier ordre. La thèse présente des cadres théoriques, des méthodes mathématiques et des applications pratiques, contribuant aux progrès des SRDs.

Mots-clés : Systèmes de réaction-diffusion, équations paraboliques singulières, phénomène de quenching, existence globale, solutions positives, EDP quasi-linéaire du premier ordre.

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GENERAL INTRODUCTION

Introduction

Reaction-diffusion systems are mathematical models used to describe the dynamics of chemical reactions and the diffusion of substances in space and time. These systems have been widely applied in various fields, including biology and medicine, to understand complex biological phenomena and help explain patterns and processes observed in living organisms.

The concept of reaction-diffusion systems dates back to the early 20th century when the Belgian mathematician Paul Langevin and the Russian Adolf Fick independently formulated the mathematical equations describing the diffusion of substances. Later, in the 1950s and 1960s, the British mathematician Alan Turing made significant contributions to the field by proposing reaction-diffusion models to explain biological pattern formation.

Turing's seminal work introduced the idea that complex patterns and structures, such as the formation of animal coat patterns or the development of organs during embryogenesis, could arise from simple chemical reactions combined with diffusion processes. He demonstrated that the interaction between two or more chemical substances (referred to as morphogens) with different diffusion rates and reaction kinetics could give rise to spatial patterns. This concept, known as Turing patterns, provided a theoretical framework for understanding how intricate structures could emerge from local interactions.

Since Turing's groundbreaking work, reaction-diffusion systems have found numerous applications in biology, ecology, and medicine. Here are a few notable examples:

(i) Biology:

- Embryonic Development: Reaction-diffusion systems, such as the famous Turing model, help explain how spatial patterns and cellular differentiation emerge during embryonic development. They provide insights into the formation of structures like fingers, feathers, and organs.

- Cell Signaling: Reaction-diffusion processes regulate cell signaling, allowing cells to communicate and coordinate their activities. Signaling molecules, such as morphogens, diffuse through tissues and establish concentration gradients that guide cell fate determination and tissue patterning.

- Neuronal Activity: Reaction-diffusion dynamics are vital for generating and propagating electrical signals in neurons. They underlie phenomena like action potential propagation, synaptic activity, and the establishment of spatial patterns in neuronal circuits.

(ii) Ecology:

- Population Dynamics: Reaction-diffusion models are used to study the spatial spread and interactions of species in ecological systems. They help understand the formation of species distributions, population dynamics, and the effects of spatial heterogeneity on ecological communities.
- Pattern Formation in Ecosystems: Reaction-diffusion mechanisms contribute to the formation of spatial patterns in ecosystems, such as vegetation patterns in arid landscapes or the zonation of species in intertidal zones. These patterns arise due to the interplay of biological processes and diffusive interactions.

(iii) Medicine:

- Morphogenesis and Tissue Engineering: Reaction-diffusion models aid in understanding tissue morphogenesis and guiding tissue engineering approaches. They help design strategies to control cell behavior, tissue growth, and pattern formation, with applications in regenerative medicine and organ development.
- Cancer Biology: Reaction-diffusion processes are relevant to tumor growth and invasion. Models incorporating reaction-diffusion dynamics contribute to understanding tumor heterogeneity, the formation of tumor boundaries, and the spread of cancer cells.
- Drug Delivery: Reaction-diffusion models inform drug delivery strategies by predicting how drugs diffuse and distribute within tissues. They help optimize drug release systems, such as implants or nanoparticles, to achieve desired therapeutic concentrations.

These examples highlight the broad importance of reaction-diffusion systems in elucidating biological phenomena, ecological patterns, and disease processes. They provide valuable insights into the dynamics and organization of complex systems, guiding experimental design, and aid in the development of therapeutic interventions.

Reaction-diffusion systems have emerged as a highly significant research topic, captivating the attention of researchers across diverse disciplines such as mathematics, physics, biology, chemistry, and beyond. The wide range of applications of reaction-diffusion systems in various fields underscores their importance. This multifaceted nature of these systems served as a compelling motivation for me to select reaction-diffusion systems as the focal point of my doctoral thesis in mathematics. This choice not only allows me to delve into captivating mathematical concepts, but also provides an avenue to explore the intricate interdisciplinary connections that arise in the study of these systems. By delving into this research area, I will have the opportunity to tackle challenging problems, unravel the underlying mechanisms of pattern formation and dynamics, and contribute to both the theoretical understanding and practical applications of reaction-diffusion systems. This choice offers an intellectually stimulating and enriching research path that has the potential to shape my expertise and open doors to various academic fields and professional opportunities.

In our study, we focused on investigating the intriguing phenomenon of quenching and its wide-ranging applications in the fields of biology, medicine, and bioengineering. Quenching has emerged as a fundamental diffusion phenomenon with significant practical implications and benefits. Exploring this field allowed us to uncover valuable insights and achieve noteworthy results, which we subsequently documented in two scientific papers that were published in renowned specialized journals. These publications not only contribute to the existing body of knowledge but also highlight the significance of quenching in various disciplines. Moreover, our research endeavors involved presenting a multitude of applications and models, showcasing the versatility and relevance of quenching across different domains. By thoroughly studying this phenomenon and its applications, we have contributed to the advancement of scientific understanding and opened up avenues for further research and exploration in this exciting and impactful field.

In addition to studying the phenomenon of quenching, another significant aspect of our thesis revolved around investigating a distinct class of reaction-diffusion systems that find diverse and wide-ranging applications in population dynamics, environmental studies, and related fields. Our research in this area led to a substantial contribution, which formed the basis of a third scientific papers published in a prestigious mathematics journal. This publication not only highlights our novel findings but also underscores the importance of this class of reaction-diffusion systems in various scientific domains. The scientific paper was strongly supported by numerous applications, showcasing the practical relevance and real-world impact of our research. By delving into this aspect of reaction-diffusion systems, we have not only expanded our understanding of mathematical principles but have also provided valuable insights that can inform decision-making processes and contribute to the development of sustainable solutions in areas such as population dynamics.

Furthermore, our research endeavors extended to addressing a significant area of study, leading to the publication of a fourth scientific paper that focuses on the solution of first-order quasi-linear partial differential equations. This category of equations holds immense interest for researchers due to its ability to model various phenomena across all scientific domains without exception. The published paper sheds light on innovative techniques and methodologies developed to tackle these equations and provides valuable insights into their analytical solutions. Through this research, we have contributed to a wider understanding of mathematical modeling and its applications in diverse scientific fields. The implications of this research extend beyond mathematics, as these equations serve as valuable tools for describing and analyzing complex phenomena in physics, engineering, biology, economics, and numerous other disciplines. Our findings and methodologies provide researchers with powerful tools and approaches to address and comprehend the intricate dynamics and behaviors inherent in these systems, thereby facilitating advancements and breakthroughs across a wide range of scientific endeavors.

Furthermore, our research efforts encompassed the exploration of additional aspects within the realm of reaction-diffusion systems, specifically focusing on periodic models that exhibit a recurring pattern. These models hold great significance due to their wide-ranging applications in the fields of biology and ecology, where periodic phenomena often play a crucial role. The outcomes of our investigations in this area have yielded noteworthy results, which have been compiled into two scientific articles that have been submitted for publication. These articles present novel findings and insights into the behavior and dynamics of periodic reaction-diffusion systems, shedding light on their underlying mechanisms and highlighting their relevance to biological and ecological processes. By delving into this research domain, we aim to contribute to the existing body of knowledge, providing researchers and practitioners with valuable tools and understanding to further unravel the complexities of periodic patterns in natural systems. The forthcoming publication of these articles will disseminate our findings and stimulate further scientific discourse in the field, enabling researchers to build upon our work and explore new avenues of investigation in the context of reaction-diffusion systems with periodic behavior.

Layout of the thesis

This thesis revolves around the captivating theme of reaction-diffusion systems and their extensive applications in the fields of biology, medicine, and bioengineering. Within its pages, the thesis is thoughtfully divided into eight comprehensive chapters, each of which delves into various aspects of this fascinating subject. Before embarking on this intellectual journey, the thesis commences with a general introduction that not only sheds light on the intricate nature of reaction-diffusion systems but also highlights the fundamental problems that are addressed throughout the research. This introduction serves as a doorway into the world of the thesis, setting the stage for the subsequent chapters. Now, let us embark on a brief overview of the content presented in each chapter.

The first chapter provides a comprehensive overview of reaction-diffusion systems and their applications in various scientific disciplines. It explains that reaction-diffusion systems are mathematical models used to describe the dynamic behavior of interacting substances undergoing diffusion and chemical reactions. The chapter emphasizes key concepts such as diffusion, reaction, mathematical modeling using partial differential equations, Turing patterns and instability, and the wide range of applications in chemistry, biology, physics, and materials science. It also introduces famous examples of reaction-diffusion systems and discusses their experimental realizations and computational simulations. Practical applications of reaction-diffusion systems are highlighted in fields such as chemical engineering, biophysics, neuroscience, ecological modeling, and materials science. Furthermore, it introduces fundamental concepts and theories that lay the foundation for understanding subsequent chapters and topics.

The second chapter serves as an introductory chapter to Chapters Five, Six and Seven. It provides a detailed exploration of the concept of "quenching" and its diverse applications, as well as its significance in different areas of science and engineering. It examines two categories of reaction-diffusion equations that involve singularities in the reaction term: “blowup” and “quenching”. The chapter also highlights the role of quenching in biology and medicine, including fluorescence quenching, reactive oxygen species quenching, and quenching of biological signals and processes. Additionally, the impact of quenching on cellular processes such as cryopreservation, protein folding, enzyme inactivation, cellular metabolism, and gene expression analysis is explored. The chapter is also concerned with the importance of quenching in bioengineering, such as the manufacture of contact lenses.

The third chapter serves as an introductory chapter to the eighth chapter, It provides an introduction to reaction-diffusion systems with initial conditions. It highlights the significance of potential analysis in understanding parabolic problems and reaction-diffusion systems. The chapter covers various applications of these systems in biology, ecology, medicine, and physics, referencing prior studies in the field. It explores the use of reaction-diffusion systems to study embryonic pattern formation, morphogenesis, tissue development, population spread, drug diffusion in tissues, chemical reactions, and pattern formation in physical systems. The chapter also introduces the specific reaction-diffusion system that will be the focus of the seven chapter.

The fourth chapter delves into the fundamental definitions and concepts related to first-order linear and quasilinear partial differential equations (PDEs). It provides a clear exposition of the classical method used for solving these equations, shedding light on the underlying principles and techniques involved. Additionally, the chapter introduces the method of finding surfaces orthogonal to a given system of surfaces, showcasing its relevance and applicability in the context of this class of equations. It also provides insightful examples and applications that demonstrate the practical significance of these mathematical concepts in various real-world scenarios. By presenting these definitions, methods, and applications, the chapter equips readers with a solid foundation for understanding and working with first-order linear and quasilinear PDEs, thus making it easier for them to understand the content of Chapter 9.

The fifth chapter of the thesis investigates a singular parabolic reaction-diffusion system with positive Dirichlet boundary conditions, which are commonly utilized in the modeling of the quenching phenomenon. The primary objective of this study is to establish conditions that guarantee both finite-time quenching and the global existence of solutions for the system under consideration. By addressing this problem, the paper provides a theoretical framework for analyzing and simulating quenching phenomena, thereby offering valuable insights into the behavior and properties of the systems involved. The results obtained through this research are of significant importance and have been published in a reputable mathematics journal, indicating their academic rigor and potential impact on the field. These findings contribute to our understanding of the dynamics of singular parabolic reaction-diffusion systems with positive Dirichlet boundary conditions, shedding light on the fundamental mechanisms underlying the quenching phenomenon and paving the way for further advancements in this area of study.

In this chapter, we are interested in the study of the following reaction-diffusion system with general singular terms and positive Dirichlet boundary conditions that can be applied to the quenching phenomenon:

$$\begin{cases} (u_1)_t - \Delta u_1 = -f_1(u_2) & \text{in } (0,T)\times\Omega, \\ \vdots & \vdots \\ (u_{m-1})_t - \Delta u_{m-1} = -f_{m-1}(u_m) & \text{in } (0,T)\times\Omega, \\ (u_m)_t - \Delta u_m = -f_m(u_1) & \text{in } (0,T)\times\Omega, \\ u_1 = u_2 = \cdots = u_m = 1 & \text{on } (0,T)\times\partial\Omega, \\ u_1(0,x) = u_{10}(x),\ldots,\ u_m(0,x) = u_{m0}(x) & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with smooth boundary. The functions f_j ($1 \leq j \leq m$) are positive on $(0,1]$. The initial data satisfy

$$\begin{cases} u_{10},\ u_{20},\ldots,\ u_{m0} \in C^2(\Omega)\cap C^1\left(\overline{\Omega}\right), \\ u_{j0} = 1, \text{ for all } 1 \leq j \leq m, \text{ on } \partial\Omega, \\ 0 < u_{j0} \leq 1, \text{ for all } 1 \leq j \leq m, \text{ in } \overline{\Omega}. \end{cases}$$

The sixth chapter delves into the investigation of the quenching phenomenon in reaction-diffusion systems and emphasizes its significance in various fields. The central focus of this chapter revolves around the analysis of a specific type of parabolic singular reaction-diffusion model that incorporates positive Dirichlet boundary conditions. The main objective is to establish the sufficiency of certain conditions that lead to quenching within a finite time frame while also demonstrating the global existence of solutions for the system. What sets this paper apart is the simplicity of the conditions imposed on the nonlinearity, which allows for a wide range of possibilities when choosing it. This simplicity facilitates the application of the model to numerous singular reaction-diffusion phenomena encountered in practical scenarios. To further strengthen the findings, the paper presents various real-world applications in the fields of bioengineering and life sciences, highlighting the practical relevance of quenching phenomena in these domains. Our findings were published in a specialized and reputable scientific journal.

In this chapter, we will mathematically investigate a problem that aligns with the previously discussed concept of quenching. Our focus lies on examining a reaction-diffusion model that incorporates singular nonlinearity and positive Dirichlet boundary conditions:

$$\begin{cases} u_t - \Delta u = -f(u) & \text{in } (0,T)\times\Omega \\ u = 1 & \text{on } (0,T)\times\partial\Omega \\ u(0,x) = u_0(x) & \text{in } \Omega \end{cases}$$

with

$$\begin{cases} u_0 \in C^2(\Omega)\cap C^1\left(\overline{\Omega}\right) \\ u_0 = 1, \text{ on } \partial\Omega, \\ 0 < u_0 \leq 1, \text{ in } \overline{\Omega} \\ \Delta u_0 - f(u_0) < 0, \text{ in } \Omega \end{cases}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^N$ ($N \geq 2$), and f is a positive function on $(0,1]$.

The seventh chapter investigates a singular degenerate reaction-diffusion model characterized by natural growth concerning the gradient, employing Schauder's fixed point theorem. The model is significant for its applications in biology, medicine, and life sciences, particularly in understanding phenomena such as the spread of infectious diseases and tumor dynamics. We demonstrate how the singular terms in the model can represent critical thresholds, influencing population behaviors like extinction or uncontrolled growth. By analyzing these systems, the research provides insights into ecological interactions and the conditions under which biological processes are suppressed or enhanced. This mathematical framework not only advances theoretical understanding but also supports practical applications, offering valuable tools for conservation strategies and medical treatments. Ultimately, the paper emphasizes the interconnectedness of mathematics and life sciences, highlighting the potential for improved modeling techniques to address complex biological and medical challenges.

We are interested in the following nonlinear singular degenerated reaction-diffusion system having natural growth with respect to the gradient

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mathbf{div}\,(a(t,x)\nabla u) + d(t,x)\dfrac{|\nabla u|^p}{u^\gamma} = f(t,x) & \text{in } Q_T \\ u(t,x) = 0 & \text{on } \Gamma \\ u(0,x) = u_0(x) & \text{in } \Omega \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N > 2$, and Q is the cylinder $(0,T)\times\Omega$, $T > 0$, $\Gamma = (0,T)\times\partial\Omega$, $2 < p < N$, $0 < \gamma < 1$, $a(t,x)$ and $d(t,x)$ are two bounded measurable functions satisfying

$$\begin{aligned} 0 &< \alpha_1 \le a(t,x) \le \alpha_2 \\ 0 &< \beta_1 \le d(t,x) \le \beta_2 \end{aligned}$$

where α_1, α_2, β_1 and β_2 are fixed real numbers such that $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$. On the function f, we assume that it is non-negative and not identically zero, and that it belongs to the Lebesgue space $L^m(Q_T)$ with $m > 1$. Moreover, the initial data $u_0 \in L^\infty(\Omega)$ satisfies the following condition of strict positivity

$$\exists D_\omega > 0, \forall \omega \subset\subset \Omega : u_0 \ge D_\omega$$

The eighth chapter presents a study on a quasilinear parabolic reaction-diffusion system of arbitrary order with initial conditions. The main focus is on establishing the existence of continuous positive solutions using potential analysis techniques. The novelty of the research lies in the fact that the system is of arbitrary order, which extends and generalizes previous findings. The chapter discusses the mathematical model, provides relevant references and applications in various fields, and outlines the obstacles and challenges encountered in the analysis. Significant findings, with extensive applications in fields such as biology, medicine, and bioengineering, have been achieved and published in a renowned specialized journal.

We will therefore be interested in studying the existence of positive solutions to the following reaction-diffusion system:

$$\begin{cases} -\dfrac{\partial u_i}{\partial t} + \Delta u_i = \lambda_i p_{i+1}(x,t) f_{i+1}(u_{i+1}) & , 1 \leq i \leq 2m-1 \\ -\dfrac{\partial u_{2m}}{\partial t} + \Delta u_{2m} = \lambda_{2m} p_1(x,t) f_1(u_1) & \\ u_i(x,0) = \varphi_i(x),\ x \in \mathbb{R}^n & , 1 \leq i \leq 2m \end{cases}$$

where $(x,t) \in \mathbb{R}^n \times (0,\infty)$, $n \geq 3$, and for $1 \leq i \leq 2m$, $m \in \mathbb{N}^*$; the functions $\varphi_i : \mathbb{R}^n \to [0,\infty)$ are continuous, the constants λ_i are nonnegative, the functions $f_i : (0,\infty) \to [0,\infty)$ are continuous and nondecreasing, $p_i : \mathbb{R}^n \times (0,\infty) \to [0,\infty)$ refers to measurable functions that fulfill a relevant assumption connected to the parabolic Kato class $\mathbf{P}^\infty(\mathbb{R}^n)$ introduced by Zhang. They refer to a class of functions that satisfy certain growth and regularity conditions in the context of parabolic equations. It is proposed as a natural generalization of the Kato class in the study of elliptic equations. Specifically, these functions typically have controlled growth rates and possess sufficient smoothness properties that allow for the rigorous analysis of parabolic problems.

The ninth chapter focuses on a specific class of first-order quasilinear partial differential equations with three real variables. We present a solving method inspired by the Lagrange approach and provide a comprehensive examination of the method through practical exercises. We have shown that solutions to this class of equations can be obtained by finding the linearly independent first integrals of the associated characteristic system. The chapter highlights the importance of this class of quasi-linear equations in understanding real-world phenomena and their applications in various scientific disciplines, especially biology and bioengineering. The presented method provides a valuable solution approach and contributes to ongoing research in this area. The results obtained were published in a famous mathematics journal, which proves the importance of our paper.

In this chapter, we restrict our work to a class of first order quasilinear PDEs with three real variables, whose general form is

$$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = S$$

where $u = u(x,y,z)$ is a smooth vector field in a domain Ω of $\mathbb{R}^3$. The functions P, Q, R, S are linear of (x,y,z,u), and a_k, b_k, c_k, d_k, e_k are real numbers for all $1 \leq k \leq 4$.

The thesis concludes with a comprehensive summary and provides insights into future perspectives. Additionally, it includes a well-curated list of bibliographic references that thoroughly cover the subject matter discussed in the thesis. This ensures that readers have access to relevant sources for further exploration and research in the field.

CHAPTER 1

GENERALITIES AND BASIC CONCEPTS

In this chapter, a comprehensive overview of reaction-diffusion systems is presented, along with an explanation of how to model them using partial differential equations. The chapter emphasizes the diverse applications of reaction-diffusion systems in fields such as chemistry, biology, physics, and materials science. Furthermore, it introduces fundamental concepts and theories that lay the foundation for understanding subsequent chapters and topics.

1.1 Reaction-diffusion systems

In this section, our primary source of reference will be the book written by Mesbahi [155], which provides comprehensive insights into the subject matter at hand. Mesbahi's book offers a detailed analysis and presents a wealth of information that will serve as a solid foundation for our exploration. However, in order to ensure a more comprehensive and well-rounded understanding, we also recommend referring to the works of Alaa and Mesbahi et *al*. [9]-[14], [140], [141], [27], [28], [61]-[65], [109], [111], [142]-[159], [183], [182], [190]-[194], Their contributions in the field have been significant and complement the insights provided by Mesbahi's book. By consulting multiple sources, including Alaa's works, the collective research of Mesbahi et *al*., Lions [128], Murray [161, 162], Murray and Oster [163, 164, 170], Pao et *al*. [171]-[175], Pierre [179], Volpert [213]. we can gain a broader perspective and delve into various aspects of the topic, enhancing our comprehension and enriching our analysis.

Reaction-diffusion systems hold immense importance in modeling diffusion phenomena across various disciplines like biology, medicine, and environmental science. These systems combine the principles of diffusion, where molecules passively move from high to low concentration areas, with chemical reactions that transform these molecules. This allows for a more comprehensive understanding of how molecules interact and distribute within a space. In biology, reaction-diffusion models can describe processes like nerve impulse propagation, where ions flow and react across nerve cells. Similarly, they can depict morphogenesis, the formation of complex shapes in developing organisms, where signaling molecules diffuse and trigger localized cell differentiation. In medicine, these models can simulate drug delivery within the body, accounting for the diffusion of drugs and their interactions with tissues and target cells. This helps optimize drug design and treatment strategies. Environmental science utilizes reaction-diffusion models to understand ecological phenomena. For instance, they can depict predator-prey interactions, where predator and prey populations diffuse and react through encounters, influencing ecosystem dynamics. By incorporating both diffusion and reactions, reaction-diffusion systems provide a powerful tool for unraveling the complexities of diffusion-driven processes in these diverse fields.

The chemical or biological reactions that occur in reaction-diffusion systems can be of different types, for example:

- Autocatalytic reactions: A substance transforms into itself, which can lead to the formation of complex structures.

- Inhibition reactions: One substance inhibits the transformation of another substance.

- Competition reactions: Two substances compete for the same resources.

Reaction-diffusion systems have several advantages over pure diffusion models:

- Better precision: They take into account chemical or biological reactions that can influence the distribution of substances.

- Understanding interactions: They provide a better understanding of the interactions between different substances and the biological processes involved.

- Behavior prediction: They allow predicting the behavior of substances under different conditions.

1.1.1 Key points about reaction-diffusion systems

(i) **Diffusion:** Diffusion refers to the process by which particles or substances move from an area of higher concentration to an area of lower concentration, driven by the random motion of molecules. It is a fundamental mechanism in various natural and artificial systems, including chemistry, physics, biology, and engineering.

(ii) **Reaction:** Reactions refer to chemical processes or interactions between different species that occur concurrently with diffusion. These reactions can have a significant impact on the behavior and dynamics of the system.

(iii) **Mathematical Modeling:** Reaction-diffusion systems are typically described using partial differential equations (PDEs). The PDEs incorporate terms for diffusion and reaction, representing the rates at which substances spread and interact with each other.

(iv) **Turing Patterns:** One of the remarkable features of reaction-diffusion systems is their ability to generate spatial patterns. The interplay between diffusion and chemical reactions can lead to the spontaneous formation of complex patterns, such as spots, stripes, spirals, and labyrinthine structures. These patterns often emerge through a process known as a Turing instability, where small perturbations in a homogeneous state amplify and give rise to spatially organized structures.

(v) **Turing Instability:** The stability analysis of reaction-diffusion systems plays a crucial role in understanding the behavior of the system. Stability conditions determine whether the system's patterns will persist or undergo changes over time. Certain parameter regimes can lead to instability, resulting in pattern formation or the formation of localized structures such as waves or fronts.

(vi) **Applications:** Reaction-diffusion systems have broad applications in various fields. In chemistry, they help explain chemical reactions and diffusion processes. In biology, they are used to model developmental processes, morphogenesis, and spatially structured populations. Reaction-diffusion models are also applied in physics to study phenomena like wave propagation, self-organization, and nonlinear dynamics.

(vii) **Famous Examples:** The most well-known example of a reaction-diffusion system is the Gray-Scott model, which exhibits the formation of intricate patterns resembling spots and stripes. Other notable examples include the Belousov-Zhabotinsky reaction, which displays traveling waves of chemical activity, and the Fisher-Kolmogorov equation, which describes the spread of populations in ecological systems.

(viii) **Mathematical Formulation:** Reaction-diffusion systems are typically described using systems of partial differential equations (PDEs). The most common form of these PDEs is the reaction-diffusion equation, which combines terms for diffusion and reaction. The general form of these systems is

$$\frac{\partial u}{\partial t} = \mathbf{div}\ (D(t,x,u,\nabla u).\nabla u) + f(t,x,u,\nabla u) \quad , \ x \in \Omega, \ t \geq 0$$

where $u = u(t,x) = (u_1,\ldots,u_m) : \mathbb{R}^+ \times \Omega \to \mathbb{R}^m$ is a vector of variables. f is a linear or nonlinear vector function, which is called the reaction terms, it is a regular application (at least locally Lipschitzian). $D : \mathbb{R}^+ \times \Omega \times \mathbb{R}^m \times \mathbb{R}^{mN} \to \mathbb{R}^m$ is a regular function. When $D = (D_{ij})$ is a square matrix it is called the diffusion matrix, in this case $\mathbf{div}(D(t,x,u,\nabla u).\nabla u) = D\Delta u$ are the broadcast terms. D_{ij} characterize the diffusion of u_i in u_j. In this case we have what is called diffusion crossing between the densities u_i (cross diffusion).

It should be noted here that:

- Diffusion coefficients can represent molecular diffusions or a few random movements of individuals in a population and they are not always positive. The positivity of these coefficients means that the flow of matter is from the more concentrated media to the less concentrated. It is possible that the organisms attract themselves towards their species and the movement is then in the direction of the concentration gradient, that is to say from the least concentrated to the most concentrated; and in this case, the diffusion coefficient is negative.

- The reaction terms are the result of any interaction between the components of u; u can be a vector of chemical concentrations, and f is the effect of chemical reactions of these concentrations, or the components of u can be densities of plant or animal populations, and f represents the effect of relationships (of competition or symbiosis) between predators and prey. If the reaction term $f_i > 0$, there is a source or mass production for the i-th species. Otherwise $f_i < 0$, there is mass annihilation.

- The diffusion coefficient D either constant if the region Ω is a homogeneous medium, and be regionalized (depends on the position x) if the region Ω is a heterogeneous medium.

The equation is placed on an open domain $\Omega \subset \mathbb{R}^N$, with some appropriate boundary conditions and initial conditions. may be bounded or unbounded.

(ix) **Experimental Realizations:** Reaction-diffusion systems have been observed and studied in various experimental setups. One classic example is the Belousov-Zhabotinsky reaction, which involves the oscillatory reaction of chemicals and exhibits traveling waves of chemical activity. Other experimental realizations include the use of chemical reactions on gel surfaces or in microfluidic devices to observe pattern formation and wave propagation.

(x) **Computational Simulations:** Due to the complexity of reaction-diffusion systems, computational simulations play a crucial role in studying their behavior. Numerical methods, such as finite difference methods, finite element methods, or lattice-based approaches, are commonly employed to solve the reaction-diffusion equations and simulate the evolution of the system over time.

(xi) **Multicomponent and Multiscale Systems:** Reaction-diffusion systems can be extended to involve multiple interacting substances or components, giving rise to multicomponent reaction-diffusion systems. Additionally, reaction-diffusion models can be coupled with other physical processes, such as advection or other transport mechanisms, to study systems with multiple scales or complex dynamics.

1.1.2 Diffusion and its significance in reaction-diffusion systems

In the context of reaction-diffusion systems, diffusion plays a crucial role in the spatial distribution and temporal dynamics of chemical species. Reaction-diffusion systems are characterized by the interplay between chemical reactions and the diffusion of reactants and products. The diffusion of molecules allows for the spread of chemical species throughout a medium, influencing the rates and patterns of reaction.

Here are some key points that highlight the significance of diffusion in reaction-diffusion systems:

(i) **Homogenization of concentration**: Diffusion tends to equalize concentration gradients over time. In a reaction-diffusion system, molecules move from regions of higher concentration to regions of lower concentration, leading to the homogenization of the chemical species. This process helps establish a uniform distribution of reactants and products, which is important for the overall dynamics of the system.

(ii) **Spatial patterning**: Diffusion can give rise to spatial patterns in reaction-diffusion systems. When there are spatial variations in the initial distribution of reactants, diffusion can amplify these differences and generate complex patterns. This phenomenon is exemplified by the famous Turing patterns, where diffusing substances interact through chemical reactions to produce stable spatial patterns.

(iii) **Reaction kinetics**: Diffusion influences the rates of chemical reactions. The rate at which reactants encounter each other depends on their diffusion coefficients, which determine how quickly they diffuse through the medium. Diffusion can either promote or hinder the reaction rates depending on the specific system and the properties of the reactants.

(iv) **Oscillations and waves**: Diffusion can facilitate the propagation of oscillatory behavior and waves in reaction-diffusion systems. Oscillatory chemical reactions, such as the Belousov-Zhabotinsky reaction, rely on the diffusion of reactants to spread the oscillations throughout the medium. Waves in reaction-diffusion systems, such as the traveling fronts in combustion processes, are also governed by diffusion.

(v) **Spatiotemporal control**: By manipulating the diffusion properties of molecules, it is possible to control the spatiotemporal dynamics of a reaction-diffusion system. Modulating the diffusion coefficients or introducing diffusion barriers can influence the formation and stability of patterns, alter reaction rates, and enable precise control over the system's behavior.

1.1.3 Reaction and its significance in reaction-diffusion systems

Reactions in reaction-diffusion systems play a crucial role in pattern formation, nonlinear dynamics, spatially localized structures, controlling reaction rates, and enabling self-organization. Understanding the interplay between diffusion and reactions is essential for comprehending the behavior and emergent properties of complex systems, and it has implications in various scientific disciplines and practical applications. Here's a discussion on the significance of reactions in reaction-diffusion systems:

(i) **Pattern formation**: Reactions are essential for the emergence of patterns in reaction-diffusion systems. When there are spatial variations in the concentration of reactants, the reactions can amplify these differences, leading to the formation of complex spatial patterns. This phenomenon is exemplified by the reaction-diffusion model proposed by Alan Turing, which demonstrated how a simple system of reacting and diffusing chemicals can give rise to intricate patterns.

(ii) **Nonlinear dynamics**: Reactions introduce nonlinearity into the system, which can lead to a rich variety of dynamic behaviors. Nonlinear reaction kinetics can give rise to oscillations, bifurcations, and chaotic dynamics in reaction-diffusion systems. These dynamic behaviors can have important implications for the stability, robustness, and self-organization of the system.

(iii) **Spatially localized structures**: Reactions can lead to the formation of spatially localized structures, such as fronts, interfaces, or localized concentrations known as solitons. These structures can arise due to the interplay between diffusion and reaction rates, resulting in stable boundaries or interfaces that separate different regions of the system. Spatially localized structures are relevant in various fields, including chemical waves, combustion fronts, and biological morphogenesis.

(iv) **Reaction rates and kinetics**: The rates of reactions can strongly influence the overall dynamics of the system. The reaction rates determine the speed at which reactants are consumed or produced, which affects the temporal evolution of the concentrations. The kinetics of the reactions, including factors such as reaction orders and rate constants, determine the relationship between reactant concentrations and the reaction rate. Understanding the kinetics is crucial for predicting and controlling the behavior of reaction-diffusion systems.

(v) **Feedback and self-organization**: Reactions in reaction-diffusion systems can create feedback loops, where the concentrations of reactants influence the reaction rates, and in turn, the reaction rates affect the concentrations. This feedback can give rise to self-organization phenomena, where local interactions between molecules at the microscopic level lead to the emergence of ordered structures or patterns at the macroscopic level. Self-organization is a fundamental concept in the study of complex systems and is relevant in fields such as developmental biology, chemical engineering, and materials science.

1.1.4 Advection and Convection reaction-diffusion systems

An advection-reaction-diffusion system and a convection-reaction-diffusion system are both powerful tools for modeling how a quantity, like a chemical or heat, changes within a system due to three main processes: movement, reactions, and random spreading. However, there's a subtle but important difference in how they handle the movement aspect.

Advection Reaction-Diffusion System:

- Advection refers to the transportation of a substance by bulk motion of the fluid/medium it is contained in.

- In an advection reaction-diffusion system, the movement of the substance occurs only due to diffusion and reaction kinetics. Diffusion is the random movement of particles from a region of higher concentration to one of lower concentration.

- There is no bulk fluid flow term present. The concentration of the substance changes only due to diffusion and local reactions.

- Examples include chemical reactions where reactants diffuse through a static medium and react upon contact.

Convection Reaction-Diffusion System:

- Convection refers to the transportation of a substance by bulk fluid motion in addition to diffusion.

- In a convection reaction-diffusion system, the movement occurs due to diffusion, reaction kinetics, and an additional convection/advection term due to bulk fluid flow.

- The bulk flow of the fluid transports the substance in addition to diffusion.

- Examples include chemical reactions in stirred tanks, where reactants are mixed by mechanical stirring in addition to diffusion.

Key Difference:

- In an advection reaction-diffusion system, movement is purely due to diffusion and reactions.

- In a convection reaction-diffusion system, movement occurs due to diffusion, reactions, AND an additional convection/advection term accounting for bulk fluid transport.

- The presence of a convection term to account for bulk fluid motion is what distinguishes a convection reaction-diffusion system from an advection reaction-diffusion system.

1.1.5 Fick's laws of diffusion

Fick's laws of diffusion describe the fundamental principles governing the process of diffusion. They were formulated by Adolf Fick in the mid-19th century and provide a mathematical framework for understanding how particles or substances move through a medium.

1.1.5.1 Fick's first law of diffusion

Fick's first law states that the rate of diffusion of a substance is directly proportional to the concentration gradient.

Mathematically, Fick's first law can be expressed as

$$J = -d\nabla\varrho$$

where J is the diffusion flux, which represents the amount of substance that flows across a unit area per unit time, d is the diffusion coefficient or diffusion constant, which represents the ability of the substance to diffuse through the medium, $\nabla\varrho$ is the concentration gradient, which represents the spatial variation in the concentration of the substance. The negative sign indicates that diffusion occurs from regions of higher concentration to regions of lower concentration, in accordance with the second law of thermodynamics.

1.1.5.2 Fick's second law of diffusion

Fick's second law describes how the concentration of a diffusing substance changes over time.

Mathematically, Fick's second law can be expressed as

$$\frac{\partial\varrho}{\partial t} = d\Delta\varrho$$

where $\frac{\partial\varrho}{\partial t}$ is the rate of change of concentration with respect to time, $\Delta\varrho$ is the Laplacian of the concentration, which represents the spatial variation of the concentration.

Fick's second law states that the rate of change of concentration is proportional to the second derivative of the concentration with respect to position. This equation describes how diffusion spreads the substance and leads to the equalization of concentration gradients over time.

It's important to note that Fick's laws assume that the diffusion process is driven by concentration gradients and that the medium is homogeneous and isotropic. In more complex scenarios, such as with varying diffusion coefficients or heterogeneous media, modified forms of Fick's laws or additional terms may be required to accurately describe the diffusion process.

Fick's laws of diffusion have widespread applications in various fields, including physics, chemistry, biology, and engineering. They provide a foundation for understanding and modeling diffusion processes, enabling predictions and analysis of transport phenomena in diverse systems. We find more details Coirier [51], Duvaut [69], Royis [188] and Salençon [195].

1.1.6 Reaction-diffusion systems in practical applications

Reaction-diffusion systems have practical applications in various fields. We mention here some of them.

(i) **Chemical Engineering:** Reaction-diffusion models are employed in chemical engineering to optimize reaction conditions and understand the behavior of chemical reactors. By simulating the diffusion and reaction processes, engineers can determine the optimal design parameters, such as reactor size, catalyst distribution, and operating conditions, to achieve desired reaction rates and product yields.

(ii) **Biophysics and Biological Systems:** Reaction-diffusion systems are extensively used in modeling biological systems. They help explain various biological phenomena, including embryonic development, morphogenesis, pattern formation in tissues and organs, and the formation of spatially organized structures in biological systems. By simulating reaction-diffusion processes, researchers gain insights into the underlying mechanisms of biological processes.

(iii) **Neuroscience:** In neuroscience, reaction-diffusion models are employed to study the dynamics of neural networks and the propagation of signals in the brain. They provide insights into the formation of spatial patterns in neural activity, such as the formation of cortical maps or the generation of traveling waves in neural circuits. Understanding these patterns is crucial for unraveling the mechanisms of brain function.

(iv) **Ecological Modeling:** Reaction-diffusion systems find applications in ecological modeling and population dynamics. They help understand the spread of species, the formation of spatial patterns in populations, and the interaction between different species in ecosystems. By incorporating diffusion and reaction processes, researchers can study the dynamics of species interactions, the formation of ecological patterns, and the effects of spatial heterogeneity on biodiversity.

(v) **Materials Science:** Reaction-diffusion models are valuable in materials science and engineering. They are used to study the growth of thin films, the formation of crystal patterns, and the self-assembly of nanostructures. By understanding the diffusion and reaction processes during material synthesis, researchers can design and control the formation of specific patterns and structures with desired properties.

(vi) **Chemical Pattern Generation:** Reaction-diffusion systems have practical applications in generating chemical patterns for various purposes, such as in microfluidic devices or lab-on-a-chip technologies. By carefully designing the reaction kinetics and diffusion properties, researchers can create precise spatial patterns of chemical concentrations, which are useful for applications like controlled drug release, chemical sensing, and microscale patterning.

(vii) **Computer Graphics and Animation:** Reaction-diffusion systems are utilized in computer graphics and animation to generate realistic and visually appealing patterns and textures. By simulating reaction-diffusion processes, computer graphics artists can create intricate and natural-looking patterns, such as fire, smoke, or natural textures like skin or fur.

1.1.7 Derivation of reaction-diffusion systems

The derivation of reaction-diffusion systems involves formulating mathematical equations that describe the diffusion and chemical reactions occurring in a system. The specific derivation process depends on the nature of the system being modeled and the assumptions made. The below steps provide a general framework for deriving reaction-diffusion systems, but the details can differ based on the specific context and system under consideration..

(i) **Identify the Components:** First, we need to identify the components or substances involved in the system. These can be chemical species, biological entities, or any other relevant entities. Assign variables to represent the concentrations or densities of these components.

(ii) **Diffusion:** Diffusion describes the spreading and transport of substances due to random molecular motion. Assume that each component diffuses independently and at a rate determined by a diffusion coefficient. Introduce the diffusion term into the equations to account for this process. The diffusion term is typically proportional to the Laplacian of the concentration variable.

(iii) **Chemical Reactions:** Introduce the chemical reactions that occur between the components. The reaction terms describe how the concentrations of the components change due to these reactions. The form of the reaction terms depends on the specific reaction kinetics. Common reaction terms involve linear or nonlinear functions of the concentrations, representing different reaction rates or mechanisms.

(iv) **Combine Diffusion and Reactions:** Combine the diffusion and reaction terms to obtain the final equations. The resulting equations are typically partial differential equations (PDEs) that describe the spatiotemporal evolution of the concentrations.

(v) **Boundary and Initial Conditions:** Specify the boundary and initial conditions for the system. These conditions define the concentrations or other relevant properties of the components at the system boundaries and the initial time. The specific conditions depend on the system being modeled and the experimental or theoretical context.

(vi) **Analysis and Numerical Methods:** Analyze the derived reaction-diffusion equations to gain insights into the system's behavior. Depending on the complexity of the equations, analytical solutions may be challenging to obtain. In such cases, numerical methods, such as finite difference methods, finite element methods, or other computational techniques, can be used to simulate and solve the equations numerically. These simulations provide a way to study the spatiotemporal dynamics and patterns that emerge from the reaction-diffusion system.

1.1.8 Boundary and initial conditions

Boundary and initial conditions play a crucial role in determining the behavior and outcomes of a reaction-diffusion system. These conditions provide essential information about the initial state of the system and its interaction with the surrounding environment. Here's how boundary and initial conditions affect the behavior of a reaction-diffusion system:

(i) **Initial Conditions:** The initial conditions specify the concentrations or other relevant properties of the components at the starting time ($t = 0$). They define the initial distribution of the substances within the system. The choice of initial conditions can significantly impact the subsequent evolution of the system. Different initial conditions can lead to different patterns, stability, and dynamics. For example, starting with a homogeneous distribution of substances may result in a uniform state, while introducing small perturbations or concentration gradients can lead to the formation of spatial patterns.

(ii) **Boundary Conditions:** Boundary conditions specify the behavior of the system at its boundaries, defining how the components interact with the external environment or other regions of the system. The choice of boundary conditions influences the flow of substances into or out of the system and can significantly impact the spatial and temporal behavior of the reaction-diffusion system. Different types of boundary conditions include Dirichlet boundary conditions (specifying the concentrations or values at the boundaries), Neumann boundary conditions (specifying the flux or derivative at the boundaries), or periodic boundary conditions (imposing periodicity in the system).

(iii) **Stability and Pattern Formation:** The choice of boundary and initial conditions can affect the stability and pattern formation in a reaction-diffusion system. For example, certain combinations of conditions can lead to the formation of stable spatial patterns, such as spots, stripes, or labyrinthine structures. The stability and characteristics of these patterns depend on the interplay between diffusion, reaction kinetics, and the boundary conditions. Small changes in the boundary or initial conditions can result in different patterns or even destabilize existing patterns.

(iv) **Spatial and Temporal Dynamics:** Boundary and initial conditions influence the spatial and temporal dynamics of a reaction-diffusion system. They determine how the concentrations of the components evolve over time and how they propagate or spread within the system. Different combinations of conditions can lead to phenomena such as wave propagation, wavefront interactions, spiral waves, or the emergence of stationary or oscillatory patterns. The choice of conditions can also affect the speed, direction, and stability of these dynamics.

(v) **Model Validation and Comparison:** The choice of appropriate boundary and initial conditions is crucial for validating and comparing reaction-diffusion models with experimental observations or real-world systems. By carefully selecting conditions that mimic the experimental setup or the natural environment, researchers can assess the model's ability to capture the observed behavior. Matching the boundary and initial conditions to experimental conditions helps validate and refine the model, improving its predictive power and relevance.

1.1.9 Common types of initial conditions

In reaction-diffusion systems, the choice of initial conditions determines the starting state of the system at time $t = 0$. The specific type of initial conditions used depends on the nature of the system being modeled and the research question at hand. Here are some common types of initial conditions employed in reaction-diffusion systems:

(i) **Homogeneous Initial Conditions:** Homogeneous initial conditions assume that the concentrations of the components are uniformly distributed throughout the system at the beginning. In this case, all spatial points have the same initial concentration values. Homogeneous initial conditions are often used as a baseline to study the system's behavior and its evolution from a uniform state.

(ii) **Perturbed or Spatially Varying Initial Conditions:** Perturbed initial conditions introduce small deviations or variations from the homogeneous state. These perturbations can be random or designed to create specific patterns or concentration gradients. Perturbed initial conditions are frequently used to study the formation and evolution of spatial patterns in reaction-diffusion systems.

(iii) **Concentration Gradients:** Concentration gradients involve setting up initial conditions where the concentrations of the components vary across space. These gradients can be linear, exponential, or any other desired profile. Concentration gradients are useful for investigating phenomena such as chemotaxis, cell migration, or the establishment of morphogenic gradients during embryonic development.

(iv) **Multiple Concentration Peaks or Clusters:** In some cases, researchers may want to simulate a system with multiple distinct concentration peaks or clusters at the initial time. These initial conditions represent situations where there are pre-existing localized concentrations of the components. Such conditions can be relevant for modeling cell aggregates, colonies, or the presence of specific chemical sources within the system.

(v) **Experimental Data-Based Initial Conditions:** In certain cases, initial conditions can be derived from experimental measurements or data. For example, if concentration profiles are available from experimental observations, these profiles can be used as initial conditions to simulate and compare the model's behavior with the experimental data.

1.1.10 Solving reaction-diffusion systems

Solving reaction-diffusion systems involves finding solutions to the PDEs that describe the spatiotemporal evolution of the concentrations or densities of the components. Depending on the complexity of the system and the desired level of detail, different solution methods can be employed. Here are some common approaches for solving reaction-diffusion systems:

(i) **Analytical Solutions:** Analytical solutions provide exact mathematical expressions for the concentrations or densities of the components as functions of space and time. Obtaining analytical solutions is challenging and often limited to simplified reaction-diffusion systems with specific assumptions and boundary conditions. Simple linear systems or systems with specific reaction kinetics may allow for analytical solutions. However, analytical solutions are not always feasible for complex, nonlinear systems.

(ii) **Numerical Methods:** Numerical methods are widely used for solving reaction-diffusion systems. These methods discretize the continuous PDEs into a set of discrete equations that can be solved iteratively. Common numerical methods include:

(ii.a) Finite Difference Methods: Finite difference methods approximate the derivatives in the PDEs using finite difference approximations. The spatial domain is divided into a grid, and the concentrations are computed at each grid point using difference equations. The system of difference equations is solved iteratively in time.

(ii.b) Finite Element Methods: Finite element methods discretize the spatial domain into a set of finite elements. The concentrations are approximated within each element using basis functions, and the PDEs are transformed into a system of algebraic equations. This system is then solved numerically.

(ii.c) Spectral Methods: Spectral methods use basis functions, such as Fourier series or orthogonal polynomials, to represent the concentrations. The PDEs are transformed into a system of ordinary differential equations (ODEs), which can be solved using numerical ODE solvers.

(iii) **Computational Modeling Software:** There are various computational modeling software packages available that offer built-in solvers for reaction-diffusion systems. These software tools provide user-friendly interfaces, pre-defined templates for reaction-diffusion equations, and efficient numerical solvers.

(iv) **Simulation and Visualization:** Once the numerical solutions are obtained, simulations can be performed to visualize the spatiotemporal dynamics of the reaction-diffusion system. Visualization techniques, such as contour plots, color maps, or 3D representations, can help understand the patterns, wave propagation, and stability of the concentrations.

For the mathematical analysis of RDS, see, for example, the works of Alaa and Mesbahi et *al*. [9]-[14], [140], [141], [27], [28], [61]-[65], [109], [111], [142]-[159], [183], [182], [190]-[194], Lions [128], Pao et *al*. [171]-[175], Pierre [179], where we find also many models studied by different techniques.

1.1.11 Experimental techniques used to study reaction-diffusion systems

Experimental techniques play a crucial role in the study of reaction-diffusion systems, enabling researchers to observe and characterize the dynamics and patterns that emerge from these complex systems. Here are some commonly used experimental techniques:

(i) **Microscopy**: Microscopy techniques, such as optical microscopy, confocal microscopy, and fluorescence microscopy, are widely used to visualize and track the spatial and temporal behavior of chemical species in reaction-diffusion systems. These techniques provide high-resolution imaging, allowing researchers to directly observe concentration profiles, pattern formation, and the evolution of structures over time.

(ii) **Chemical imaging**: Chemical imaging techniques, such as fluorescence imaging, Raman spectroscopy, and mass spectrometry imaging, provide spatially resolved information about the distribution and concentration of chemical species in reaction-diffusion systems. These techniques allow for the mapping of concentration gradients, identification of reaction products, and the analysis of chemical composition within the system.

(iii) **Microfluidics**: Microfluidics involves the manipulation of small volumes of fluids in microscale channels or devices. Microfluidic platforms can be used to create controlled environments for studying reaction-diffusion systems. They enable precise control of flow rates, gradients, and mixing, facilitating the observation and manipulation of diffusion and reaction processes. Microfluidic devices also allow for the generation of spatially varying conditions, such as concentration gradients or chemical patterning.

(iv) **Time-lapse imaging**: Time-lapse imaging involves capturing consecutive images or videos of reaction-diffusion systems at regular time intervals. This technique enables the observation of dynamic processes, such as the propagation of waves, the formation of patterns, or the growth and evolution of structures over time. Time-lapse imaging provides valuable insights into the temporal behavior and kinetics of reaction-diffusion systems.

(v) **Fluorescence resonance energy transfer (FRET)**: FRET is a technique used to study molecular interactions and diffusion in reaction-diffusion systems. It involves the transfer of energy between two fluorophores—one acting as a donor and the other as an acceptor—when they are in close proximity. FRET can be used to measure distances, diffusion rates, and molecular interactions, providing information about the spatial organization and dynamics of chemical species.

(vi) **Electrochemical techniques**: Electrochemical techniques, such as cyclic voltammetry and chronoamperometry, can be employed to study reaction-diffusion systems involving electroactive species. These techniques involve applying controlled electrical potentials and measuring the resulting current or potential changes. Electrochemical methods allow for the investigation of diffusion coefficients, reaction rates, and electrochemical processes in reaction-diffusion systems.

(vii) **Pattern analysis and quantification**: In addition to direct observation, image analysis and quantification methods are used to analyze and characterize patterns and structures in reaction-diffusion systems. These techniques involve extracting relevant information from experimental images, such as pattern wavelengths, spatial correlations, and morphological properties. Image analysis tools, statistical methods, and pattern recognition algorithms can aid in the quantitative analysis of experimental data.

By combining these experimental techniques with mathematical modeling and simulation, researchers can gain a comprehensive understanding of the dynamics, patterns, and underlying mechanisms in reaction-diffusion systems. This interdisciplinary approach allows for the validation and refinement of theoretical models and provides insights into the complex behavior of these systems in real-world settings.

1.2 Functional spaces

Functional spaces are mathematical spaces that consist of functions with certain properties or characteristics. These spaces are defined by specifying a set of functions and a set of operations or properties that the functions must satisfy. Functional spaces play a crucial role in various branches of mathematics and mathematical analysis, providing a framework for studying and analyzing functions with specific properties. We can find more specific details in Brezis [35], Kavian [112], Lions [128] and Pao [174].

Here are some commonly encountered functional spaces:

1.2.1 L^p spaces

L^p spaces are a family of function spaces that are defined based on the integrability properties of functions. They play a fundamental role in measure theory, functional analysis, and various areas of mathematics. L^p spaces provide a framework for studying the size, convergence, and properties of functions with respect to a measure.

Here are key points about L^p spaces:

(i) **Definition:** Let (Ω, Σ, μ) be a measure space, where Ω is a set, Σ is a sigma-algebra of subsets of Ω, and μ is a measure defined on Σ. For $1 \leq p < \infty$ the L^p space, denoted as $L^p(\Omega, \Sigma, \mu)$ or simply $L^p(\Omega)$, consists of functions $f : \Omega \to \mathbb{R}$ (or $\mathbb{C}$) such that the p-th power of the absolute value (or modulus) of f is measurable and has a finite integral, i.e.,

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ (or } \mathbb{C}\text{) measurable with } \int_\Omega |f|^p \, d\mu < \infty \right\}$$

(ii) **Norm:** The L^p norm of a function f in $L^p(\Omega)$ is defined as

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p \, d\mu \right)^{\frac{1}{p}}$$

The norm measures the size or magnitude of a function in the L^p space. It satisfies the properties of a norm, such as non-negativity, homogeneity, and the triangle inequality.

(iii) **Special Cases:** Some specific L^p spaces are commonly encountered:

- $L^1(\Omega)$ consists of functions with integrable absolute value, i.e., $\int_\Omega |f| \, d\mu < \infty$, equipped with the norm

$$\|f\|_{L^1(\Omega)} = \int_\Omega |f| \, d\mu$$

- $L^2(\Omega)$ is the space of square integrable functions, i.e., functions for which the integral of the square of the absolute value is finite, $\int_\Omega |f|^2 d\mu < \infty$, equipped with the norm

$$\|f\|_{L^2(\Omega)} = \left(\int_\Omega |f|^2 d\mu\right)^{\frac{1}{2}}$$

$L^2(\Omega)$ is a Hilbert space endowed with the scalar product

$$(f,g) = \int_\Omega f(x)g(x)dx$$

- $L^\infty(\Omega)$ consists of essentially bounded functions, i.e., functions for which there exists a constant M such that $|f(x)| \leq M$ almost everywhere on Ω, i.e.,

$$L^\infty(\Omega) = \{f : \Omega \longrightarrow \mathbb{R} \text{ measurable}, \ \exists M > 0, \ |f| \leq M \text{ a.e. on } \Omega\},$$

it is a complete vector space for the norm

$$\|f\|_{L^\infty(\Omega)} = \underset{x\in\Omega}{\text{supess}}\,|f(x)| = \inf\{M > 0 \ , \ |f| \leq M \text{ a.e. on } \Omega\}$$

- We define the spaces $L^p(0,T,\Omega)$, $1 \leq p < \infty$, and $L^\infty(0,T,\Omega)$ as follows:

$$L^p(0,T,\Omega) = \left\{f : [0,T] \longrightarrow \Omega \text{ measurable}, \int_0^T \|f\|_\Omega^p dt < \infty\right\}$$

equipped with the norm

$$\|f\|^p_{L^p(0,T,\Omega)} = \int_0^T \|f\|^p_\Omega dt$$

$$L^\infty(0,T,\Omega) = \{f : [0,T] \longrightarrow \Omega \text{ measurable}, \ \sup_{t\in(0,T)} \text{ess}\,\|f\|_\Omega < \infty\}$$

equipped with the norm

$$\|f\|_{L^\infty(0,T,\Omega)} = \sup_{t\in(0,T)} \text{ess}\,\|f\|_\Omega$$

Of course, we have

$$L^p(0,T,L^p(\Omega)) \equiv L^p((0,T)\times\Omega), \ 1 \leq p \leq \infty$$

(iv) **Inclusion Relations:** L^p spaces exhibit inclusion relations. Specifically, for $1 \leq p \leq q \leq \infty$, if Ω is a finite measure space or a σ-finite measure space, then $L^q(\Omega)$ is a subset of $L^p(\Omega)$.

(v) **Completeness:** $L^p(\Omega)$ are Banach spaces, which means they are complete with respect to the L^p norm. This completeness property is crucial for the study of convergence and the existence of solutions to certain equations and problems.

(vi) **Duality:** L^p spaces are dual to each other in certain cases. For $1 < p < \infty$, the dual space of $L^p(\Omega)$ is $L^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$. This duality relationship allows for the study of functionals and operators defined on L^p spaces.

(vii) **Applications:** L^p spaces have widespread applications in various fields of mathematics, including analysis, probability theory, harmonic analysis, functional analysis, and PDEs. They provide a mathematical framework for studying convergence, approximation, interpolation, and the behavior of functions with respect to a measure.

1.2.2 Harmonic functions

In mathematics, a harmonic function is a real-valued function that satisfies the Laplace's equation. Harmonic functions are widely studied in various branches of mathematics, including complex analysis, partial differential equations, and potential theory. They have many important properties and applications in physics, engineering, and other fields. Let's explore the definitions and main properties of harmonic functions:

Definition 1.1. The functions, harmonic in domain V, are the functions u satisfying the Laplace equation in this domain, i.e., $\Delta u = 0$.

These are some of the key definitions and properties of harmonic functions u. They provide a foundation for studying and understanding various aspects of these functions and their applications in mathematics and science.

(i)

$$\int_S \frac{\partial u}{\partial v} dS = 0$$

i.e., the integral of the normal derivative of the harmonic function on the surface of the domain is equal to zero.

(ii) The value of the harmonic function u at any point inside the domain is expressed through the values of this function and its normal derivative on the surface of the domain by the formula

$$u(A) = \frac{1}{4\pi} \int_S \left[\frac{1}{r} \frac{\partial u}{\partial v} - u \frac{\partial}{\partial v} \left(\frac{1}{r} \right) \right] dS$$

(iii) The value of the harmonic function u in the centre A of the sphere S_R with radius R is equal to the mean arithmetic value of the function on the surface of sphere

$$u(A) = \frac{1}{4\pi R^2} \int_{S_R} u dS$$

(iv) From (iii) we obtain the maximum principle: a function, harmonic inside the domain and continuous up to its boundary, reaches its highest and smallest values at the boundary of the domain.

i.e., A harmonic function cannot have a local maximum or minimum in the interior of its domain unless it is constant. This property is a consequence of the mean value property and has important implications for the behavior of solutions to certain physical problems.

(v) Uniqueness: Given certain boundary conditions, a harmonic function is uniquely determined within its domain. This property is crucial in solving boundary value problems involving Laplace's equation.

To delve deeper into the subject of harmonic functions and explore their applications, you may refer to relevant sources such as Agoshkov et *al*. [6] for comprehensive details and additional information.

1.3 Fundamental theorems

In this section, we introduce a selection of crucial theorems that played a significant role in deriving our results. We will rely mainly on Apostol [19], Brezis [34, 35], Lions [128], Pucci [180]. These references have been carefully chosen due to their relevance, credibility, and extensive coverage of the topic.

1.3.1 Dini's Theorem

The Dini's theorem, also known as the Dini's test or Dini's criterion, is a fundamental result in real analysis that relates the pointwise convergence of a sequence of functions to its uniform convergence on a compact set. It is named after the Italian mathematician Ulisse Dini, who first stated and proved the theorem in the late 19th century.

Dini's theorem is as follows.

Theorem 1.1. *Suppose K is a compact subset of a metric space (X,d), and $f_n : K \to \mathbb{R}$ is a decreasing sequence of continuous functions which converges pointwise to a continuous real valued function f on K. Then, f_n converges uniformly to f.*

Dini's Theorem has many applications, including the following.

(i) **Approximation Theory:** Dini's theorem provides a powerful tool for approximating functions. It guarantees that if a sequence of functions converges pointwise to a function and the convergence is accompanied by a certain type of monotonicity and uniform convergence, then the convergence is uniform. This result is often used to establish the uniform convergence of various approximation methods, such as polynomial interpolation and Fourier series approximation.

(ii) **Existence of Solutions:** Dini's theorem is used in the study of partial differential equations and integral equations to establish the existence of solutions. By constructing appropriate sequences of functions that satisfy the conditions of Dini's theorem, one can show the existence of a solution to certain classes of equations.

(iii) **Asymptotic Analysis:** Dini's theorem is also employed in asymptotic analysis, which deals with the behavior of functions as certain parameters tend to infinity or some other limiting values. It can be applied to prove the uniform convergence of certain sequences of functions, which is essential for establishing the validity of asymptotic expansions.

(iv) **Optimization and Control Theory:** Dini's theorem has applications in optimization and control theory, particularly in the study of optimal control problems. It is used to prove the existence of optimal solutions and to establish the convergence of optimization algorithms.

1.3.2 Contraction mapping theorem

Definition 1.2. Let (X,d) be a metric space. A mapping $T : X \to X$ is a contraction mapping, or contraction, if there exists a constant c, with $0 \le c < 1$, such that

$$d(T(x),T(y)) \le cd(x,y) \tag{1.1}$$

for all $x, y \in X$.

Thus, a contraction maps points closer together. In particular, for every $x \in X$ and $r > 0$, all points y in the ball $B_r(x)$ are mapped into a ball $B_s(Tx)$, with $s < r$. It follows from (1.1) that a contraction mapping is uniformly continuous.

If $T : X \to X$, then a point $x \in X$ such that

$$T(x) = x \tag{1.2}$$

is called a fixed point of T. The contraction mapping theorem states that a strict contraction on a complete metric space has a unique fixed point. The contraction mapping theorem is only one example of what are more generally called fixed point theorems. For example, the Schauder fixed point theorem states that a continuous mapping on a convex, compact subset of a Banach space has a fixed point.

In general, the condition that c is strictly less than one is needed for the uniqueness and the existence of a fixed point. For example, if $X = \{0,1\}$ is the discrete metric space with metric determined by $d(0,1) = 1$. then the map T defined by $T(0) = 1$, $T(1) = 0$ satisfies (1.1) with $c = 1$. but T does not have any fixed points. On the other hand, the identity map on any metric space satisfies (1.1) with $c = 1$ and every point is a fixed point.

It is worth noting that (1.2), and hence its solutions, do not depend on the metric d. Thus, if we can find any metric on X such that X is complete and T is a contraction on X, then we obtain the existence and uniqueness of a fixed point. It may happen that X is not complete in any of the metrics for which one can prove that T is a contraction. This can be an indication that the solution of the fixed point problem does not belong to X, but to a larger space, namely the completion of X with respect to a suitable metric d.

Theorem 1.2. *If $T : X \to X$ is a contraction mapping on a complete metric space (X,d), then there is exactly one solution $r \in X$ of (1.2).*

Proof. The proof is constructive, meaning that we will explicitly construct a sequence converging to the fixed point. Let x_0 be any point in X. We define a sequence (x_n) in X by

$$x_{n+1} = Tx_n \quad \text{for } n \geq 0$$

To simplify the notation, we often omit the parentheses around the argument of a map. We denote the n-th iterate of T by T^n, so that $x_n = T^n x_0$.

First, we show that (x_n) is a Cauchy sequence. If $n \geq m \geq 1$, then from (1.1) and the triangle inequality, we have

$$\begin{aligned}
d(x_n,x_m) &= d\left(T^n x_0, T^m x_0\right) \\
&\leq c^m d\left(T^{n-m} x_0, x_0\right) \\
&\leq c^m \left[d\left(T^{n-m} x_0, T^{n-m-1} x_0\right) + d\left(T^{n-m-1} x_0, T^{n-m-2} x_0\right) + \cdots + d(Tx_0,x_0)\right] \\
&\leq c^m \left[\sum_{k=0}^{n-m-1} c^k\right] d(x_1,x_0) \\
&\leq c^m \left[\sum_{k=0}^{\infty} c^k\right] d(x_1,x_0) \\
&\leq \left(\frac{c^m}{1-c}\right) d(x_1,x_0),
\end{aligned}$$

implies that (x_n) is Cauchy. Since X is complete, (x_n) converges to a limit $x \in X$. The fact that the limit x is a fixed point of T follows from the continuity of T :

$$Tx = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x.$$

Finally, if x and y are two fixed points, then

$$0 \leq d(x,y) = d(Tx,Ty) \leq c d(x,y).$$

Since $c < 1$ we have $d(x,y) = 0$ so $x = y$ and the fixed point is unique. ■

1.3.3 Maximum principles for parabolic equation

The maximum principle is a fundamental concept in the study of partial differential equations (PDEs) and plays a crucial role in several areas of analysis. Its importance stems from the fact that it provides valuable information about the behavior and properties of solutions to PDEs. Here are some key reasons why the maximum principle is significant in the study of PDEs:

(i) **Uniqueness of Solutions:** The maximum principle is often used to establish uniqueness results for solutions of PDEs. It states that if a solution attains its maximum (or minimum) at an interior point of the domain, then the solution must be constant throughout the domain. This property is essential in proving that a PDE has a unique solution, which is a fundamental requirement in many mathematical models.

(ii) **Boundedness:** The maximum principle allows us to establish bounds on solutions to PDEs. If a solution is bounded from above (or below) at an initial time or on the boundary of the domain, then the solution remains bounded throughout the domain and at all subsequent times. This property is crucial for ensuring the well-posedness of PDEs and for understanding the qualitative behavior of solutions.

(iii) **Stability and Convergence**: The maximum principle is closely related to stability and convergence results for numerical methods and approximation schemes used to solve PDEs. By establishing maximum principles for discrete approximations of PDEs, one can ensure that the numerical solution remains bounded and converges to the true solution as the grid size or time step approaches zero. This is essential for reliable and accurate numerical simulations.

(iv) **Existence of Solutions**: In certain cases, the maximum principle can be used to prove the existence of solutions to PDEs. By constructing suitable super/sub-solutions and utilizing the comparison principle, one can establish the existence of solutions that satisfy certain properties or boundary conditions. This is particularly useful when dealing with nonlinear and nonlocal PDEs, where direct methods may not be applicable.

(v) **Qualitative Behavior**: The maximum principle provides insights into the qualitative behavior of solutions. For example, it can reveal information about the spreading or decay rates of solutions, the location of maximum or minimum points, and the overall shape of the solution profile. Understanding these qualitative properties is crucial for interpreting and analyzing solutions in various scientific and engineering applications.

In general, the maximum principle serves as a potent tool in PDE analysis, allowing researchers to establish properties such as uniqueness, boundedness, stability, existence, and qualitative behavior of solutions. It plays a fundamental role in comprehending and analyzing PDEs and has broad implications across diverse fields including mathematics, physics, and engineering.

There are different versions of the maximum principle depending on the specific context, but I'll provide a general overview. We consider parabolic operators of the form

$$Lu = -\frac{\partial u}{\partial t} + \sum_{i,j=0}^{N} a_{i,j}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=0}^{N} b_i(t,x)\frac{\partial u}{\partial x_i} + c(t,x)u \tag{1.3}$$

for $(t,x) \in (0,T) \times \Omega$ where Ω is an open, path-wise connected subset of $\mathbb{R}^N$. We set

$$\begin{aligned} D &= (0,T] \times \Omega \\ Q &= (0,T] \times \Omega \\ \Sigma &= (0,T] \times \partial\Omega \cup \{0\} \times \Omega \end{aligned}$$

We shall always assume a_{ij}, b_i, c bounded on $\overline{D}$, $a_{ij} = a_{ji}$ and we call L in (1.3) uniformly parabolic if

$$\sum_{i,j=0}^{N} a_{i,j}(t,x)\xi_i\xi_j \geq \lambda_0 \|\xi\|^2$$

with $\lambda_0 > 0$ uniformly for all $\xi \in \mathbb{R}^N$ and $(t,x) \in \overline{D}$.

Theorem 1.3 (Weak maximum principle)**.** *Let L be uniformly parabolic. Assume that Ω is bounded, $u \in C^2(D) \cap C^0\left(\overline{D}\right)$ with $Lu \geq 0$, let $c(t,x) = 0$ in D. Then*

$$\max_{\overline{D}} u = \max_{\Sigma} u$$

Theorem 1.4 (Strong maximum principle)**.** *Let Ω be an open, path-wise connected subset of $\mathbb{R}^N$. Let L be uniformly parabolic. Assume $u \in C^2(D) \cap C^0\left(\overline{D}\right)$ with $Lu \geq 0$, let $M = \sup_D u$. Assume $u = M$ at a point $(t_0, x_0) \in D$ and that one of the following conditions holds: (i) $c = 0$ and $M \in \mathbb{R}$, (ii) $c \leq 0$ and $M \geq 0$, (iii) $M = 0$ and c is arbitrary. Then $u = M$ on $[0, t_0] \times \overline{\Omega}$.*

Remark 1.1. *By considering $-u$, we can show corresponding results about $M = \inf_D u$. The only assumptions changing are $Lu \leq 0$ and $M \leq 0$ in (ii).*

1.3.4 Comparison principle for semi-linear equations

The comparison principle is used to establish comparison results between two solutions of a semi-linear equation. It states that if one solution is greater than or equal to another solution at some initial time or on the boundary of the domain, and the equation satisfies certain conditions (such as monotonicity or concavity), then the same inequality holds at all subsequent times or throughout the domain. The comparison principle is a powerful tool for establishing various properties of solutions, including existence, uniqueness, and qualitative behavior.

Definition 1.3. The semilinear operator

$$Lu = -\frac{\partial u}{\partial t} + \sum_{i,j=0}^{N} a_{i,j}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} + F\left(t, x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\right) \tag{1.4}$$

is called uniformly parabolic, if

$$\sum_{i,j=0}^{N} a_{i,j}(t,x)\xi_i\xi_j \geq \lambda_0 \|\xi\|^2$$

with $\lambda_0 > 0$ uniformly for all $\xi \in \mathbb{R}^N$ and $(t,x) \in \overline{D}$. In (1.4), we shall assume that F is continuously differentiable and that $a_{ij} = a_{ji}$ are bounded on $\overline{D}$.

Theorem 1.5. *Let L be uniformly parabolic. Let $u,v \in C^2(D) \cap C^0\left(\overline{D}\right)$ with Ω open, path-wise connected and bounded such that*

$$\begin{cases} Lv \geq Lu & \text{in } D \\ v(0,x) \geq u(0,x) & \text{in } \Omega \\ v(t,x) \geq u(t,x\,) & \text{on } (0,T] \times \partial\Omega. \end{cases}$$

Then $v \geq u$ in D. If $u(t,x\,) = v(t,x)$ for some $(t,x\,) \in (0,T] \times \Omega$ then $u \equiv v$ in $[0,t] \times \overline{\Omega}$.

Remark 1.2. *(i) The result can be extended to certain fully nonlinear problems.*

(ii) We can replace the boundary condition on $[0,T] \times \partial\Omega$ with isolating Neumann data

$$\frac{\partial v}{\partial n} \geq \frac{\partial u}{\partial n}$$

for $\partial\Omega$ of class C^2 and $u,v \in C^2(D) \cap C^1\left(\overline{D}\right)$.

1.4 Green's function

The Green's function is a concept in mathematics and physics that plays a fundamental role in solving certain types of differential equations. It provides a mathematical representation of the response of a system to an impulse or localized source term. The history of the Green's function dates back to 1828 , when George Green published work in which he sought solutions of Poisson's equation $\Delta u = f$ for the electric potential u defined inside a bounded volume with specified boundary conditions on the surface of the volume. He introduced a function now identified as what Riemann later coined the "Green's function".

In a region Ω with boundary $\partial\Omega$, let

$$Lu = f(\mathbf{x}) \quad \text{in } \Omega \tag{1.5}$$

$$Bu = 0 \text{ in } \partial\Omega \tag{1.6}$$

represent, respectively, a linear second order PDE and linear boundary initial conditions, such that for each continuous f, problem (1.5)−(1.6) has a unique solution. Then $G(\mathbf{x},\xi)$ is the Green's function for the problem if this unique solution is given by

$$u(\mathbf{x}) = \int_{\Omega} G(x,\xi) f(\xi)\, d_{\xi}\Omega$$

We attach a subscript to the volume element to emphasize that the integration is with respect to the ξ-variables.

Example 1.1. *The initial value problem for the heat equation*

$$\begin{cases} v_t(x,t) - v_{xx}(x,t) = 0 & , x \in \mathbb{R},\ t > 0 \\ v(x,0) = f(x) & , x \in \mathbb{R} \end{cases}$$

has the unique solution

$$v(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} \exp\left[-\frac{(x-\xi)^2}{4t}\right] f(\xi)\, d\xi$$

It then follows from Duhamel's principle that the problem

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = f(x) & , x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x) & , x \in \mathbb{R} \end{cases} \tag{1.7}$$

has the unique solution

$$u(x,t) = \int_0^t v(x,t-\tau)\, d\tau = \int_0^t \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi(t-\tau)}} \exp\left[-\frac{(x-\xi)^2}{4(t-\tau)}\right] f(\xi)\, d\xi\, d\tau \tag{1.8}$$

From (1.8), we infer that the Green's function for problem (1.7) is

$$G(x,t;\xi,\tau) = \frac{1}{\sqrt{4\pi(t-\tau)}} \exp\left[-\frac{(x-\xi)^2}{4(t-\tau)}\right] \ ,\ t > \tau > 0 \tag{1.9}$$

It is seen that the Green's function (1.9) exhibits singular behavior as $\mathbf{x} = (x,t)$ approaches $\xi = (\xi,\tau)$. This holds true for Green's functions in general, and is reflected in the fact that $G(\mathbf{x},\xi)$ for (1.5)−(1.6) satisfies, as a function of $\mathbf{x}$, the PDE

$$L\sigma = \delta(\mathbf{x} - \xi) \tag{1.10}$$

which is (1.5) with $f(\mathbf{x})$ replaced by $\delta(\mathbf{x}-\xi)$. We call a solution of (1.10) a singularity solution for $L[]$. The essence, then, of the Green's function method is to represent $f(\mathbf{x})$ in (1.5) as a sum of delta functions, thereby obtaining $u(\mathbf{x})$ as the "sum" of the corresponding singularity solutions adjusted to obey (1.6).

Green's functions play a fundamental role in the study of partial differential equations (PDEs) and are widely used in various areas of mathematical physics and engineering. Here are some key reasons why Green's functions are important in the study of PDEs:

(i) **Existence and Uniqueness**: Green's functions can be used to establish the existence and uniqueness of solutions to certain types of PDEs. By constructing a Green's function, which is a solution to a PDE with a specific type of forcing term, one can then represent the general solution of the PDE as a convolution integral involving the Green's function and the given forcing term. This representation allows for the characterization of solutions and the establishment of existence and uniqueness results.

(ii) **Boundary Value Problems**: Green's functions are particularly useful in solving boundary value problems for linear PDEs. By exploiting the properties of Green's functions, one can find solutions that satisfy specific boundary conditions. The construction of Green's functions provides a systematic and general approach to solving boundary value problems, allowing for the determination of solutions in complex domains with diverse boundary conditions.

(iii) **Integral Representations**: Green's functions provide integral representations for solutions of PDEs. These representations can simplify the analysis and computation of solutions, as they often involve convolutions or integrals that can be evaluated more easily than solving the PDE directly. The use of Green's functions allows for the transformation of PDE problems into integral equations, which can be amenable to analytical or numerical techniques.

(iv) **Operator Inversion**: Green's functions enable the inversion of differential operators. By considering the adjoint operator associated with a given PDE, one can construct the corresponding Green's function. This Green's function can then be used to invert the differential operator, allowing for the determination of solutions from given data or boundary conditions. This inversion process is essential in many applications, such as inverse problems and parameter identification.

(v) **Approximation and Numerical Methods**: Green's functions provide a theoretical foundation for approximation and numerical methods for solving PDEs. By approximating the Green's function or using its properties, one can develop efficient numerical algorithms, such as boundary element methods or finite element methods, for the discretization and solution of PDEs. Green's functions also play a role in the analysis of convergence and stability properties of these numerical methods.

(vi) **Physical Interpretation**: Green's functions have important physical interpretations. In many cases, the Green's function represents the response of a system to a unit impulse or point source. Therefore, it provides insights into the behavior and characteristics of the underlying physical phenomenon described by the PDE. Green's functions allow for the understanding of propagation, diffusion, and wave phenomena, among others, in terms of the fundamental solutions of the associated PDEs.

In Duffy's work [68], we can discover additional intricacies and distinctive features regarding Green's functions and their practical applications.

CHAPTER

2

BLOW-UP AND QUENCHING PHENOMENA

This chapter provides a detailed exploration of the concept of quenching and its diverse applications, as well as its significance in different areas of science and engineering. It examines two categories of reaction-diffusion equations that involve singularities in the reaction term: "blowup," where the reaction tends towards infinity as the variable approaches infinity, and "quenching," where the reaction tends towards infinity at a specific value of the variable. The chapter also delves into the role of quenching in biology and medicine.

2.1 Quenching phenomenon and applications

2.1.1 What is Quenching?

Quenching refers to a rapid cooling process used in materials science and metallurgy to achieve specific properties in a material, such as increased hardness or improved strength. During quenching, a heated material is rapidly cooled by immersing it in a quenching medium, which can be a liquid, gas, or even a solid. Each material has a certain hardenability, which is determined based on its chemical composition and its previous history (whether it has received other heat treatments). We find in Liščić et *al*. [129] everything related to this topic.

In the study of quenching and tempering, several key aspects are illustrated through

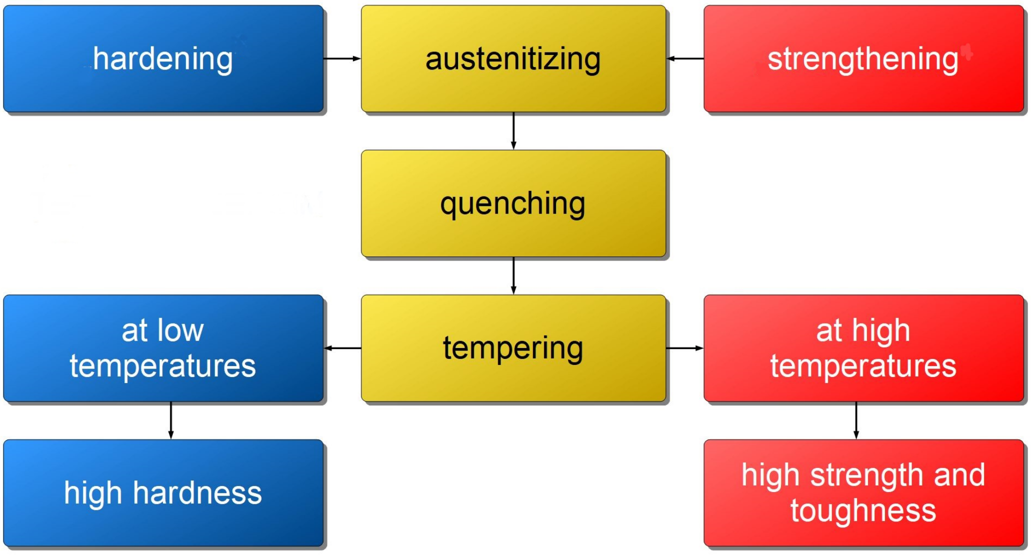

FIGURE 2.1. Flowchart of quenching and tempering.

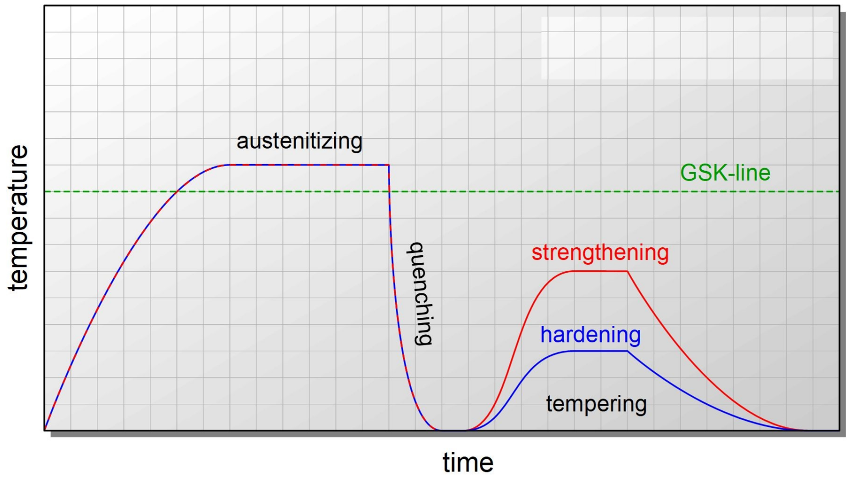

FIGURE 2.2. Temperature curve during quenching and tempering.

various figures. Figure 2.1 presents a comprehensive flowchart that outlines the process of quenching and tempering. This flowchart serves as a valuable visual aid in understanding the overall procedure. Additionally, Figure 2.2 provides a temperature curve that depicts the variations in temperature throughout the quenching and tempering process. Lastly, Figure 2.3 showcases the microstructural changes that occur during quenching. This figure highlights the transformations that take place at a microscopic level, shedding light on the structural modifications that arise from the quenching process.

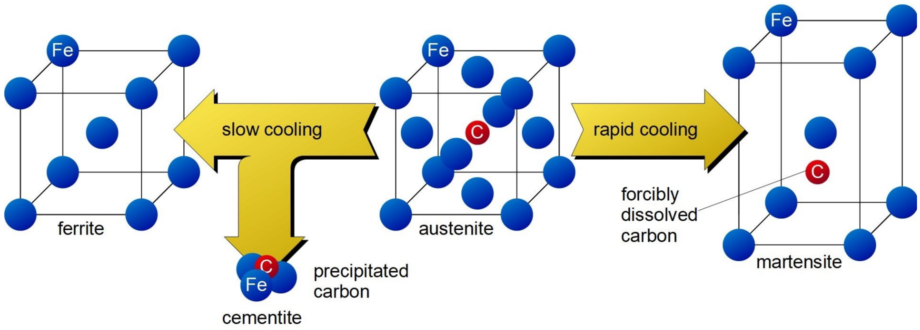

FIGURE 2.3. Microstructural change during quenching.

FIGURE 2.4. Quenching by water.

There are several types of quenching media commonly used:

(i) **Water**: Water is one of the most commonly used quenching media due to its high cooling capacity. It provides rapid cooling rates, but it can also cause severe distortion or cracking in some materials. See Figure 2.4.

(ii) **Oil**: Different types of oils, such as mineral oil or vegetable oil, are often used as quenching media. Compared to water, oil provides a slower and more controlled cooling rate, which helps reduce the risk of distortion or cracking. See Figure 2.5.

(iii) **Polymer solutions**: Polymer solutions, such as water-based polymer solutions or polymer quenchants, offer a controlled and uniform cooling rate. They can be tailored to specific applications and materials. See Figure 2.6.

(iv) **Salt solutions**: Salt solutions, such as brine solutions, are used for specialized quenching applications. They provide a controlled cooling rate and are often used for specific materials or processes. See Figure 2.7.

(v) **Gas**: In some cases, gases like nitrogen or helium are used as quenching media. Gas quenching allows for rapid cooling without direct contact, which is useful for

FIGURE 2.5. Quenching by oil.

FIGURE 2.6. Quenching by Polymer solutions.

FIGURE 2.7. Quenching by salt solutions.

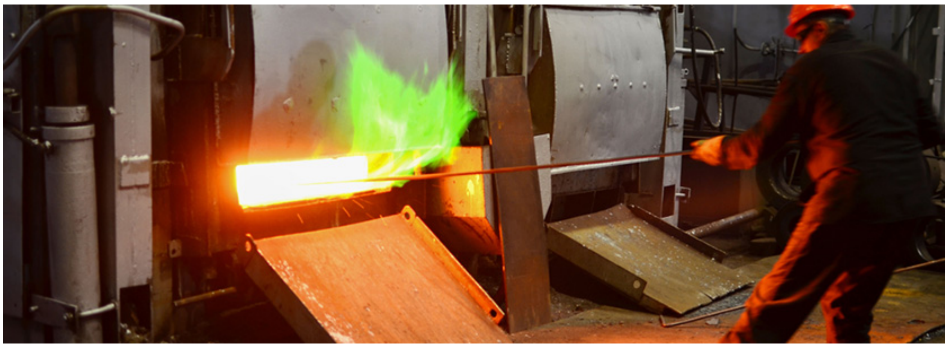

FIGURE 2.8. Quenching by gas.

materials that are sensitive to contact with liquids. See Figure 2.8.

The quenching process typically involves several steps:

(i) **Heating**: The material is heated to a specific temperature, known as the austenitizing temperature, to transform its microstructure.

(ii) **Immersion**: The heated material is quickly immersed in the quenching medium, ensuring rapid cooling.

(iii) **Cooling**: The quenching medium extracts heat from the material, causing it to cool rapidly. The cooling rate and quenching media selection are critical factors that influence the resulting material properties.

(iv) **Stabilization**: After quenching, the material may undergo a stabilization process, such as tempering or annealing, to relieve internal stresses and improve the material's overall stability and strength.

The choice of quenching medium and the specific quenching process parameters depend on factors such as the material type, desired properties, and the application requirements. It is essential to carefully control the quenching process to achieve the desired material characteristics while minimizing the risk of distortion or cracking.

2.1.2 Quenching in biology and medicine

In biology, medicine, and ecology, the term "quenching" is used to describe various phenomena related to the extinguishing or suppression of biological processes, signals, or activities. Understanding and manipulating quenching phenomena are important for studying cellular processes, diagnosing diseases, and maintaining ecological balance. Here are a few examples of how quenching is used in these fields:

(i) **Fluorescence Quenching**: Fluorescence quenching refers to the reduction or suppression of fluorescence emission in fluorophores or fluorescent molecules. It occurs when an external agent or molecule interacts with the excited state of the fluorophore, resulting in the dissipation of energy and a decrease in fluorescence intensity. Fluorescence quenching is widely used in molecular biology, biochemistry, and medical diagnostics for studying molecular interactions, protein-protein interactions, and detecting biomarkers.

(ii) **Reactive Oxygen Species (ROS) Quenching**: Reactive oxygen species, such as superoxide radicals and hydrogen peroxide, are highly reactive molecules involved in various physiological and pathological processes. Quenching refers to the process of neutralizing or reducing the levels of ROS in cells or tissues. This can be achieved through the action of antioxidants, enzymes, or other molecules that scavenge or convert ROS into less reactive forms. ROS quenching is important for maintaining cellular homeostasis and preventing oxidative damage to biomolecules.

(iii) **Quenching of Biological Signals**: In cellular signaling and communication, quenching can refer to the termination or suppression of signaling pathways or signals. For example, G-protein-coupled receptors (GPCRs) can undergo desensitization or quenching, where their ability to transmit signals is reduced or turned off in response to prolonged stimulation. This helps prevent overstimulating and allows cells to reset their responsiveness to subsequent signals.

(iv) **Quenching of Biological Processes**: Quenching can also refer to the inhibition or suppression of biological processes or activities. For instance, in the context of photosynthesis, quenching mechanisms regulate the excess energy absorbed by chlorophyll molecules, preventing damage to the photosynthetic machinery. In ecology, quenching may describe the suppression of specific ecological processes or interactions, such as the reduction of predation or competition, through various mechanisms.

2.1.3 Role of quenching in cellular processes

Quenching, as a rapid cooling process, can play a significant role in cellular processes by affecting cellular behavior, metabolism, and various physiological responses. Here are some ways in which quenching impacts cellular processes:

(i) **Cryopreservation**: Quenching is a crucial step in cryopreservation, a process used to preserve cells and tissues at extremely low temperatures. By rapidly cooling cells, quenching prevents the formation of ice crystals, which can cause cellular damage. This enables the successful preservation of cells for long-term storage, transplantation, and research purposes.

(ii) **Protein Folding**: Quenching can influence protein folding, a critical process for protein structure and function. Rapidly cooling a protein sample can "trap" it in a specific conformation, preserving its structure at a particular moment in time. This allows researchers to study protein folding pathways and dynamics, aiding in the understanding of protein structure-function relationships and related cellular processes.

(iii) **Enzyme Inactivation**: Quenching can be employed to rapidly halt enzymatic activity. By rapidly cooling a sample, enzymatic reactions can be stopped, preserving the reaction state at a specific time point. This enables the analysis of enzyme kinetics, substrate specificity, and the study of metabolic pathways.

(iv) **Cellular Metabolism**: Quenching can be used to rapidly stop cellular metabolic processes. By quickly cooling cells, enzymatic reactions and metabolic pathways are halted, allowing for the analysis of metabolite levels, enzyme activities, and metabolic fluxes at a specific moment. This provides insights into cellular metabolism and its regulation, aiding in the understanding of metabolic pathways and their roles in cellular processes.

(v) **Gene Expression Analysis**: Quenching is employed in techniques such as the rapid freezing of cells or tissues for gene expression analysis. By quickly cooling cells, transcriptional activity is halted, preserving the gene expression profile at a specific time point. This allows researchers to study gene expression patterns and regulatory networks in various cellular processes.

2.1.4 Quenching in contact lenses manufacturing

Contact lenses are a complex product that relies on various factors from a materials science perspective. To meet the specific demands of different users, it is crucial to place a stronger emphasis on understanding and optimizing specific properties. One key aspect is the choice of material for the contact lens, as it directly influences wear time and comfort for the wearer. These properties are not only influenced by the inherent characteristics of the material but also by the manufacturing processes involved, such as plasma processing. By considering both material-based properties and manufacturing techniques, a comprehensive approach can be adopted to enhance the performance and comfort of contact lenses. See Figure 2.9.

In contact lens manufacturing, there are certain processes that can be related to the concept of quenching in the context of contact lens manufacturing. Here are a few

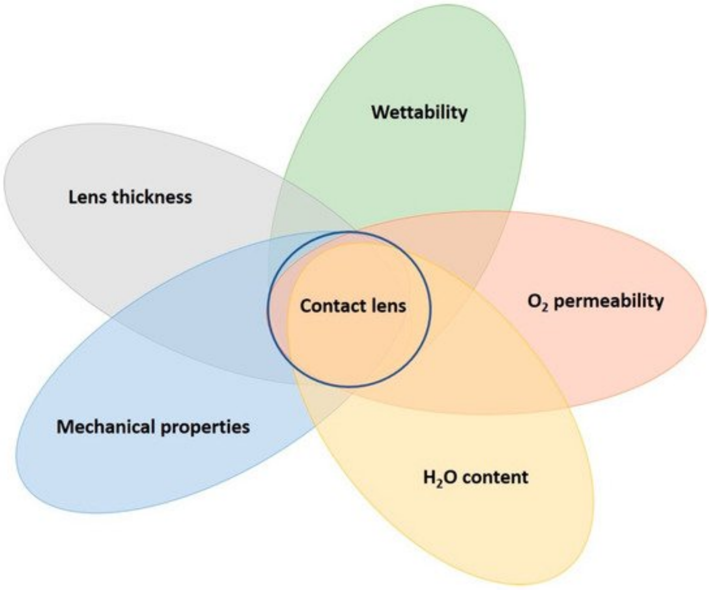

FIGURE 2.9. Contact lenses from a materials science.

relevant processes:

(i) **Heat Treatment**: After the contact lenses are formed, they usually undergo a heat treatment process to achieve their final physical properties. This process involves exposing the lenses to elevated temperatures, which initiates and completes the polymerization or crosslinking of the lens material. The heat treatment step can be considered analogous to the heating phase in quenching, where the lenses are subjected to high temperatures to induce specific changes in their structure.

(ii) **Quenching and Annealing**: After the lenses are heated, they need to be cooled down to stabilize their structure and reduce any residual stresses. Controllable quenching rates are employed to ensure the lenses solidify and retain their desired optical and mechanical properties. Additionally, an annealing process may follow the quenching phase, where the lenses are held at a specific temperature for a certain period to further relieve any internal stresses and enhance their stability.

(iii) **Surface Treatments**: In some cases, contact lenses may undergo surface treatments to modify their properties. This can involve the use of chemical agents or plasma treatment to alter the lens surface properties, such as increasing wettability or improving the deposition of certain coatings. These treatments can be considered quenching-like processes since they involve the rapid modification of the surface properties of the lenses.

2.1.5 Quenching in bioengineering

In the field of bioengineering, the term “quenching” can be used to describe certain phenomena and processes. Here are a few examples of how quenching is relevant in bioengineering:

(i) **Protein Quenching**: In bioengineering, the term “protein quenching” is often used to refer to the rapid inactivation or suppression of protein activity. This can involve the use of chemical agents or physical methods to halt or reduce the function of specific proteins. Protein quenching is employed in various applications, such as controlling enzymatic activity, preventing protein degradation, or modulating cellular signaling pathways.

(ii) **Cellular Quenching**: Cellular quenching refers to the process of rapidly stopping or inhibiting cellular activity. This can be achieved through the addition of specific agents or treatments that interrupt cellular processes or induce cell death. Cellular quenching is often used in bioengineering experiments or applications where the temporary or permanent cessation of cellular function is desired, such as in tissue engineering, drug screening, or cell-based assays.

(iii) **Quenching of Biochemical Reactions**: Quenching can also be used to describe the rapid termination or inhibition of biochemical reactions. For example, in enzymatic assays or reactions, quenching agents are often added to stop the reaction and stabilize the reaction products. This helps prevent further enzymatic activity and allows for the accurate measurement or analysis of reaction intermediates or end products.

(iv) **Quenching of Fluorescence**: Fluorescence quenching, as mentioned earlier, is also relevant in bioengineering. It involves the reduction of fluorescence emission from fluorophores or fluorescent molecules due to energy transfer or interactions with other molecules in the system. Fluorescence quenching techniques are widely used in bioengineering research to study molecular interactions, protein-protein binding, and biomolecular dynamics.

2.1.6 Protein quenching in bioengineering

Here are a few examples of how protein quenching is used in various bioengineering applications:

(i) **Enzyme Inhibition**: Protein quenching is often employed to inhibit the activity of specific enzymes. This can be useful in drug development and therapeutic interventions. By identifying small molecules or inhibitors that can bind to and quench the activity of target enzymes, bioengineers can design drugs that selectively block enzymatic pathways involved in disease processes. This approach is commonly used in the development of enzyme inhibitors for conditions such as cancer, cardiovascular diseases, and metabolic disorders.

(ii) **Protein Stabilization**: Protein quenching techniques can be used to stabilize proteins and prevent their degradation or denaturation. By introducing certain chemical agents or modifications, bioengineers can inhibit the activity of proteases or other factors that contribute to protein degradation. This is particularly important in areas such as protein therapeutics, where the stability of recombinant proteins needs to be maintained for effective delivery and therapeutic efficacy.

(iii) **Signal Transduction Modulation**: Protein quenching can be used to modulate signal transduction pathways within cells. By selectively targeting and inhibiting specific proteins involved in signaling cascades, bioengineers can manipulate cellular responses and regulate biological processes. This approach is valuable in areas such as synthetic biology and tissue engineering, where precise control of cellular behavior and function is required.

(iv) **Protein-Protein Interaction Studies**: Protein quenching techniques are commonly employed in studies aimed at understanding protein-protein interactions. By selectively quenching the fluorescence of specific proteins or fluorescently labeled proteins, researchers can investigate the dynamics and kinetics of protein-protein binding events. This allows for the characterization of protein interactions, the identification of binding partners, and the study of protein complex formation.

(v) **Biomaterial Coating and Surface Modification**: Protein quenching can be used to modify the surfaces of biomaterials to prevent unwanted protein adsorption or fouling. By quenching the activity of proteins that promote adhesion or biofilm formation, bioengineers can design coatings or surface modifications that resist protein binding and enhance the biocompatibility of biomaterials. This is particularly important in applications such as implantable medical devices, where protein fouling can lead to complications and device failure.

2.1.7 Advancements and innovations in quinching-based technologies

Advancements and innovations in quenching-based technologies have led to significant breakthroughs in various fields. Quenching, a process that involves rapid cooling of materials, plays a crucial role in enhancing the properties and performance of a wide range of materials and components. Here are some notable advancements and innovations in quenching-based technologies:

(i) **Quenching in Metallurgy**: In the field of metallurgy, advancements in quenching techniques have revolutionized material properties. Innovative quenching methods, such as high-pressure gas quenching and cryogenic quenching, have been developed to achieve improved hardness, strength, and wear resistance in metals. These techniques have found applications in the aerospace, automotive, and manufacturing industries, enabling the production of high-performance materials and components. See Figure 2.10.

(ii) **Quenching in Heat Treatment**: Heat treatment processes, such as hardening, tempering, and annealing, heavily rely on quenching to achieve desired material properties. Advancements in quenching technologies, such as precision-controlled cooling rates and tailored quenching mediums, have enabled precise control over material microstructures and mechanical properties. This has resulted in enhanced strength, toughness, and fatigue resistance in heat-treated components. See Figure 2.11.

(iii) **Quenching in Additive Manufacturing**: Additive manufacturing, or 3D printing, has seen remarkable advancements with the integration of quenching-based technologies. Rapid quenching of printed parts during the manufacturing process enables the formation of fine-grained microstructures, improved mechanical properties, and reduced residual stresses. This has expanded the possibilities for producing complex geometries and functional components with superior performance. See Figure 2.12.

(iv) **Quenching in Biomaterials**: Quenching techniques have also been applied to biomaterials, such as biomedical implants and prosthetics. Innovations in quenching-based surface treatments, such as plasma electrolytic oxidation and laser quenching, have been developed to improve the biocompatibility, corrosion resistance, and wear resistance of biomaterials. These advancements have contributed to the development of longer-lasting and more reliable medical devices. See Figure 2.13.

(v) **Quenching in Electronics**: Quenching-based technologies have found applications in the electronics industry, particularly in the rapid cooling of electronic components and semiconductors. Enhanced quenching methods, such as liquid immersion cooling and vapor phase quenching, have been employed to dissipate heat generated by high-power electronic devices, enabling efficient thermal management and improved device performance.

FIGURE 2.10. Quenching in Metallurgy.

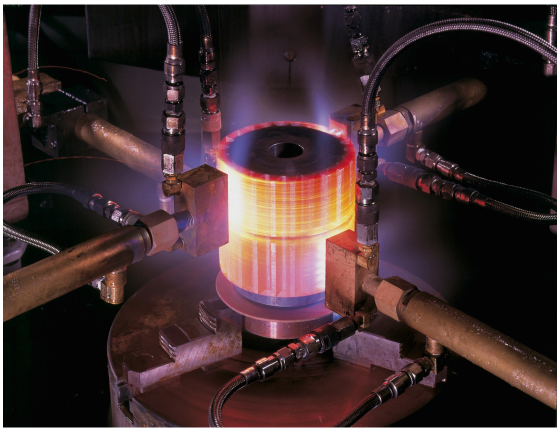

FIGURE 2.11. Quenching in Heat Treatment.

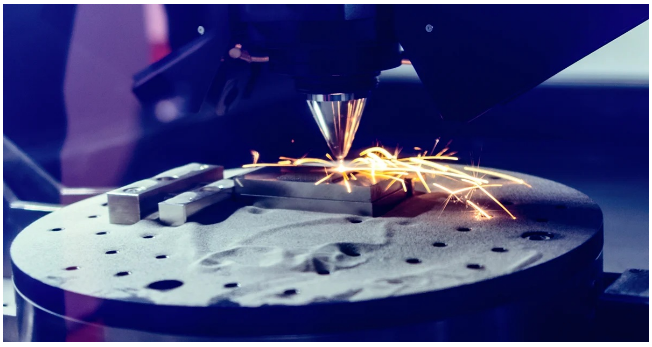

FIGURE 2.12. Quenching in Additive Manufacturing.

FIGURE 2.13. Quenching in Biomaterials.

2.2 Quenching reaction diffusion models

2.2.1 Quorum-quenching microbial infections

In [66], Dong et *al*. discuss the concept of quorum quenching as a potential strategy for controlling microbial infections. They explain that many pathogens rely on cell-to-cell communication, known as quorum sensing, to coordinate their activities during infection. By disrupting this communication, quorum quenching can prevent the expression of virulence genes and the formation of biofilms. The paper highlights that quorum-quenching mechanisms are widespread in various organisms and have been used to develop new antimicrobial strategies. It also mentions the discovery of quorum-sensing signal degradation enzymes in mammals, which may play a role in host defense against infectious diseases. Overall, the paper emphasizes the importance of studying quorum quenching for the development of novel approaches to combat infectious diseases.

2.2.2 Nonphotochemical quenching

In [209], Troiano et *al*. investigate the mechanisms of nonphotochemical quenching (NPQ) in the light-harvesting complex stress-related protein LHCSR3 from the green alga Chlamydomonas reinhardtii. The study identifies two distinct quenching processes within LHCSR3: pH-dependent quenching and zeaxanthin-dependent quenching. The pH-dependent quenching is controlled by protonated residues that sense a pH drop, while zeaxanthin-dependent quenching occurs even at neutral pH. Both quenching processes play a role in preventing the formation of damaging reactive oxygen species and provide different induction and recovery kinetics for photoprotection. The findings shed light on the molecular mechanisms of NPQ and contribute to the understanding of photoprotection in green algae.

2.2.3 Quenching for Microalgal Metabolomics

The paper of Kapoore and Vaidyanathan [110] investigates the impact of different quenching parameters on intracellular metabolite leakage in Chlamydomonas reinhardtii, a green alga. The study explores various factors such as quenching solvents, methanol concentration, buffer additives, quenching time, and solvent-to-sample ratio. The researchers measured the recovery of twelve metabolite classes using gas chromatography mass spectrometry (GC-MS) and established a mass balance to understand the fate of metabolites during quenching treatments. They found that conventional 60% methanol resulted in significant loss of intracellular metabolites and that increasing methanol concentration or quenching time increased leakage. They recommend quenching with 60% aqueous methanol supplemented with 70 mM HEPES at -40°C with a 1:1 sample-to-quenching solvent ratio for higher recovery and reduced metabolite leakage.

2.2.4 Fluorescent Quenching in Ophthalmology

This paper of Nichols et al. [166] investigates the contribution of evaporation to tear thinning in the tear film. The study aims to determine if a high concentration of fluorescein in the tear film would show a greater reduction in fluorescent intensity compared to a low concentration due to self-quenching at high concentrations. Tear film thickness, thinning rate, and fluorescent intensity were measured in healthy subjects using a modified spectral interferometer. The results show that there were no differences in tear film thickness or thinning rates between low and high fluorescein concentration conditions. However, the rate of fluorescent decay was significantly faster in the high concentration group, suggesting that evaporation plays a primary role in normal tear thinning between blinks.

2.2.5 Flame Enhancement and Quenching in Fluid Flows

In [212], Vladimirova et al. study the effects of turbulent combustion in flows where fuel and air are mixed before ignition, which is an important subject in scientific and industrial fields. The study examines how the movement of the flow affects the reaction process and alters the speed of the flame. Both experimental and theoretical approaches are used to understand this phenomenon. It is observed that moderate turbulence enhances the flame, but excessive movement of the flow can cause the flame to extinguish. Detailed analytical theories for combustion in fluid phases are currently considered impractical, so researchers rely on heuristic reasoning and simplified models, especially in well-understood situations such as the flamelet regime. The analysis often begins with an approximation based on geometric optics, using Huygens' principle to explain how the flame propagates.

The objective of this study is to examine a commonly used mathematical model, known as the scalar reaction diffusion equation with passive advection, which is widely employed in the field of combustion research:

$$\frac{\partial T}{\partial t} = u \cdot \nabla T = \kappa \nabla^2 T + \frac{1}{\tau} R(T) \tag{2.1}$$

Here T is the normalized temperature, $0 \leq T \leq 1$, u is the fluid velocity, which we assume is incompressible, κ is thermal diffusivity, and τ is the typical reaction time. In the absence of fluid velocity (2.1) admits flat propagation front with laminar burning velocity of the order of $v_0 \sim \sqrt{\frac{\kappa}{\tau}}$ and characteristic thickness of the order of $\delta \sim \sqrt{\kappa \tau}$. The model (2.1) can be derived from a more complete system under assumptions of constant density and unity Lewis number (the ratio of material and tem perature diffusivity), as shown, for instance, in Clavin and Williams [50]. The equation (2.1) has a more general applicability than the geometrical optics approximation; moreover, the geometrical optics limit can be obtained from (2.1) in a certain parameter range.

We will consider reaction rates $R(T)$ of two types, KPP (Kolmogorov, Petrovskii, Piskunov), see Fisher [82] and Kolmogorov et *al.* [119], and ignition. The KPP type is characterized by the condition that the function $R(T)$ is positive and convex on the interval $0 < T < 1$. This reaction type is used often in problems on population dynamics (see e.g. Aronson and Weinberger [22] and Fife [74]), but is relevant in combustion modelling, for example in some autocatalyctic reactions, see Hanna et *al.* [100]. A reaction term of ignition type is characterized by the presence of critical ignition temperature, such that the function $R(T)$ is identically 2 zero below ignition temperature. This type of reaction term is used widely to model combustion processes (see e.g. Volpert et *al.* [214] and Zeldovich et *al.* [221]), in particular approximating the behavior of Arrhenius-type chemical reactions which vanish rapidly as temperature approaches zero.

The main goal is to understand how the shape and strength of fluid flow impact the combustion process. We are motivated by recent analytical research that has established precise limits on combustion enhancement and quenching. The principal objective is to test and expand upon these findings. The authors examined two types of flows: shear flows, which are part of a larger category called "percolating," and cellular flows, characterized by closed streamlines that form isolated cells. For both types of flows, they investigated both flame enhancement and quenching phenomena.

In the study of flame enhancement, they examined the initial temperature in the form of a laminar front, propagating as a traveling wave distorted by the imposed flow. The focus is on establishing relations between flame propagation speed v and flow properties, particularly for large advection velocities.

In the investigation of quenching phenomena, we consider the presence of a non-zero initial temperature within a finite region. To examine this, we utilize an ignition-type reaction term, as quenching is not possible with the KPP-type source term. Our focus is on determining the critical size W° of the initial hot region. When the size is below this critical value, the flame is extinguished solely due to diffusion, without the influence of advection. This occurs as the temperature decreases below a certain threshold, causing the reaction to cease before a steady traveling wave configuration can be established. However, when advection is present, the initially hot region is stretched by the fluid flow, enabling quenching through diffusion. The objective is to gain insights into how the geometry and intensity of the flow impact the size of the initial hot region W that can be quenched.

2.2.6 Prey-predator model: Invasion and co-extinction waves

In [67], Ducrot and Langlais investigates a mathematical model describing the dynamics of prey-predator interactions in a fragile environment. The authors focus on the existence of traveling wave solutions for predator invasion and the possibility of co-extinction of both species. The model considered is the following

$$\begin{cases} B_t(t,x) - d\Delta B(t,x) = B(t,x)g(B(t,x)) - C(t,x) \\ C_t(t,x) - \Delta C(t,x) = rC(t,x)\left(1 - \frac{C(t,x)}{B(t,x)}\right) \end{cases}$$

The model consists of a singular reaction-diffusion system that describes the densities of prey B and predators C over time t and space x. The system includes diffusion coefficients, growth rates, and a logistic growth function for the prey. The authors consider both a spatially homogeneous problem and a spatially structured problem with initial data and no-flux boundary conditions.

The main goal of the study is to find traveling wave solutions for predator invasion in the system. The authors use numerical simulations to support their findings and propose a regularization procedure to analyze the existence of weak traveling wave solutions. They also investigate the possibility of finite-time quenching, where both populations vanish after the predator invasion wave.

The paper presents assumptions and main results, including the definition of weak traveling wave solutions and the conditions for their existence. The authors analyze the regularized Holling-Tanner system and provide nonexistence results. They also discuss the qualitative properties of traveling wave solutions and their implications.

Overall, the article contributes to the understanding of the dynamics of prey-predator interactions in fragile environments and provides insights into the existence and properties of traveling wave solutions in the studied system.

2.3 Quenching and Blow-up problems

In this section, our focus is on reaction-diffusion equations of the form

$$u_t - \Delta u = f(u, \nabla u, t, x) \tag{2.2}$$

where the solution $u = u(t,x)$ is a real-valued function defined for $(t,x) \in [0 \times T] \times \Omega$, with $0 < T \leq \infty$ and Ω being a bounded subset of $\mathbb{R}^N$. In a more general setting, the diffusion term in the equation can take the form of $A(u)$, where A is a second-order elliptic operator that may be nonlinear and degenerate. We focus solely on the case where the diffusion term is equivalent to the Laplacian. We assume Cauchy-Dirichlet data, meaning that u is given on the boundary $\partial\Omega$ and at the initial time $t = 0$. We are particularly interested in equations where f exhibits some form of singularity with respect to u. These situations can be broadly classified into two categories:

In the first case $f \to \infty$ as $u \to \infty$, this behavior can be observed in functions such as $f(u) = e^u$ or $f(u, u_x) = u^p - u_x^2$. These functions find applications in various fields like combustion theory, population genetics, and population dynamics. The main interest is a possibility that there are solutions which can tend to infinity in finite time. This phenomenon is known as “blowup”.

In another case we have reaction terms that satisfy $f \to \infty$ as $u \to K$ for some $K \in [0, \infty)$. An example of such a function is $f(u) = \frac{-1}{u}$. This type of reaction-diffusion equation with a singular reaction term is commonly encountered in the study of electric current transients in polarized ionic conductors. It can also be seen as a special case of models in chemical catalyst kinetics (Langmuir-Hinshelwood model) or enzyme kinetics. Researchers have extensively studied this equation under certain assumptions, which imply that the solution $u(t,x)$ approaches K in finite time. As the reaction term tends to infinity, the smooth solution ceases to exist, and this phenomenon is referred to as "quenching" or "extinction".

2.3.1 Reaction-diffusion equations

The main focus of our investigation lies in the reaction terms, denoted as $f = f(u)$, which do not explicitly depend on $\nabla u, t$ or x. To represent the semigroup generated by the operator Δ with Dirichlet boundary conditions in a specific function space, we use the notation $e^{t\Delta}$. In this context, the variation of constants formula for equation (2.2) can be expressed as follows:

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} f(u(s))\,ds. \tag{2.3}$$

To establish both local existence and uniqueness for equation (2.2), one method is to utilize the contraction mapping principle described in (2.3). A crucial requirement for this approach is that the reaction term f must exhibit local Lipschitz continuity.

The solution obtained can be continued locally, and in certain cases, it exists for all subsequent time intervals (global existence). However, for specific combinations of f and u_0, there exists a finite time $T < \infty$ such that $\|u(t)\|_\infty \to \infty$, as $t \to T$. This phenomenon is called blowup.

In the case where $f = 0$ in (2.2), the equation reduces to the linear diffusion or heat equation. For example, if we impose the condition $u = 0$ on the boundary and $u_0 \in C(\Omega)$, the solution can be expressed in closed form as $u(t) = e^{t\Delta}u_0$. By examining this expression, we can easily verify various qualitative properties of $u(t)$. In particular, we observe that $u(t)$ exists globally, and blowup does not occur.

This example clearly illustrates that a (possible) blowup in equation (2.2) arises as a result of the cumulative effect of the nonlinearity $f(u)$. This observation is elementary in the context of ordinary differential equations. Specifically, if we set $\Delta u = 0$ and $f(u) = u^p$ with $p > 1$ in (2.2), and examine the corresponding ODE:

$$u' = u^p,\ t > 0;\ u(0) = 1, \tag{2.4}$$

we find that the solution is given by $u(t) = [1 + (1-p)t]^{\frac{1}{1-p}}$. Therefore, the solution is smooth for $t \in \left(0, \frac{1}{1-p}\right)$, and $u(t) \to \infty$ as $t \to \frac{1}{1-p}$, i.e., u blows up.

The situation becomes significantly more intricate for partial differential equations such as equation (2.2). In general, it is not feasible to obtain explicit solutions for these equations, making it challenging to analyze the potential occurrence of blowup.

The pioneering work on the blowup problem for the reaction-diffusion equation was conducted by Fujita. He investigated the Cauchy problem for the equation $u_t - \Delta u = u^{1+\alpha}$, where $\alpha > 0$, and established that if $0 < N\alpha < 2$ (N being the spatial dimension), then the initial value problem did not possess any nontrivial global solutions. However, if $N\alpha > 2$, nontrivial global solutions did exist, provided that the initial values were sufficiently small. Following the publication of Fujita's paper, extensive research has been dedicated to studying the blowup phenomenon in Levine reaction-diffusion equations. For a comprehensive overview, one can refer to the review Galaktionov and Vazquez [84] and Levine [126], as well as the references cited therein.

Another scenario in which the reaction-diffusion equation lacks a global (smooth) solution is when the reaction term exhibits singularity for finite values of u. A common example is equation (2.2) with $f(u) = -u^{-p}$, where $p > 0$. In this case, it is possible that there exists a time T such that $\inf_{x\in\Omega} u \to 0$, as $t \to T$. Consequently, the reaction term blows up, leading to the cessation of a smooth solution. This phenomenon is known as quenching (or extinction, as referred to in some papers by Galaktionov and Vazquez [84]).

Similar to the blowup problem, the occurrence of quenching is also attributed to a nonlinear reaction term. However, we can establish that quenching is impossible under certain conditions using the parabolic Harnack's inequality. For example, if we have a uniformly elliptic operator as the diffusion term in (2.2) and $f \equiv 0$ with the boundary condition $u = 1$, then quenching cannot occur.

In the case of ordinary differential equations we can demonstrate quenching by a simple example. We replace the term u^p in (2.4) by $-u^{-p}$, $p > 0$, and solve it to get $u(t) = [1 + (1-p)t]^{\frac{1}{1-p}}$. From this we obtain that the solution is smooth for $t \in \left(0, \frac{1}{1-p}\right)$, and $u(t) \to \infty$ as $t \to \frac{1}{1-p}$, i.e., u quenches. As we delve into the examination of quenching in the context of the PDE (2.2), we find that the diffusion term Δu acts as a barrier against quenching. Consequently, the analysis of quenching becomes more challenging due to the resistance offered by the diffusion term.

Although there are some similarities between the blowup and quenching problems, there exists a qualitative difference between them. In the blowup problem, the solution $u(t)$ becomes unbounded, whereas in the quenching problem, some derivative of the solution $u(t)$ blows up. Typically, it is the time derivative u_t that blows up in quenching problems, which adds to the complexity of these equations. The changes in the solution with respect to time occur at an increasingly rapid rate. Consequently, traditional analysis methods such as the contraction mapping principle used in (2.3) do not provide significant insights into the qualitative properties of the solution near the quenching point, as the size of the time steps tends to zero.

The quenching problem was first addressed in a seminal paper by Kawarada [113]. This paper served as a catalyst for extensive research on the quenching problem conducted by numerous authors. The subsequent studies explored various aspects such as the existence and nonexistence of quenching solutions, the structure or size of quenching points, as well as the asymptotic behavior of solutions in space and time near the quenching points.

2.3.2 An overview of results

In this subsection, we provide an overview of the results obtained in this context. Additionally, for further information, one can refer to Galaktionov and Vazquez [84], Kawohl [114] and Levine [126]. Our focus is centered on examining the quenching and blowup problems for equations of the form (2.2). These types of equations find numerous applications in the fields of physics, chemistry, and biology, as highlighted in references Grinrod et *al.* [95], Henry [103] and Sperp [206]. In the following, we present several scenarios in which blowup or quenching behavior can potentially occur.

(i) The theory of combustion and population genetics (see Galaktionov and Vazquez [84], Souplet [205] and Zeldovich *et al.* [221] and related sources) utilizes the theory of blowup and quenching. In this context, there exist two well-known scalar models. The first one is the exponential reaction model, characterized by the equation (2.2) with $f(u) = \delta e^u$. This model, also referred to as the Frank-Kamenetsky equation see Zeldovich *et al.* [221], plays a significant role in combustion theory. For example, it is employed to describe the combustion of one-dimensional solid fuels, which can be represented by

$$\begin{cases} T_t = T_{xx} + \delta\varepsilon \exp\left(\frac{T-1}{\varepsilon T}\right) \\ c_t = -\varepsilon\Gamma\delta c \exp\left(\frac{T-1}{\varepsilon T}\right) \end{cases}$$

where T and c represent respectively the fuel temperature and concentration, and δ, Γ, ε are positive physical constants. Typically, ε represents the inverse of the activation energy. If we assume $0 < \varepsilon \ll 1$, and look for solutions in the form $T = 1 + \varepsilon u + \ldots$ and $c = 1 - \varepsilon C_1 + \ldots$, we are led to $u_t = u_{xx} + \delta e^u$ and $(C_i)_t = \Gamma\delta e^u$. The other classical blowup equation is (2.2) with $f(u) = u^p$.

(ii) Population dynamics were investigated in a study to Souplet [205], where equation (2.2) was initially introduced with the function $f(u) = |u|^{p-1}u - b|\nabla u|^q$ $(p > 1, q \geq 1)$ (see Chipot and Weissler [42]). The focus of this model was to analyze the impact of the gradient damping term $b|\nabla u|^q$ on potential blowup behavior. The term $f_1 = |u|^{p-1}u$ represents the birth rate, while the term $-b|\nabla u|^q$ represents the death rate within the population. Specifically, the dissipative gradient term symbolizes the action of a predator that eliminates individuals during their movements. Alternatively, the birth rate can also be described using an exponential term, such as i.e., $f_1(u) = e^u$ or $f_1(u) = ue^u$.

(iii) The investigation of the diffusion equation arising from polarization phenomena in ionic conductors was discussed in Kawarada [113] and related works (see references therein). Kawarada's Paper [113] specifically examined equation (2.2) in one spatial dimension, where $f(u) = \frac{1}{1-u}$ and $u \equiv 0$ on the parabolic boundary. In this context, "quenching" refers to the scenario where u approaches 1. It is worth noting that these equations are typically expressed in a form where the singularity occurs at $u = 0$, i.e., $f(u) = -\frac{1}{u}$.

(iv) This case can be considered as a limit of models used in chemical catalyst kinetics, such as the Langmuir Hinshelwood model, or in enzyme kinetics (see Fila et *al.* [75] and Phillips [178], as well as other relevant sources). In this particular case, the function $f = f(u,\varepsilon)$ is smooth for $\varepsilon > 0$, and $f(u,\varepsilon) \to f(u)$, as $\varepsilon \to 0$, where $f(u)$ is negative for $u > 0$ and singular at $u = 0$. Specifically, the reaction term is denoted by $f = f(u)\chi(\{u > 0\})$, which emphasizes that the reaction stops when $u = 0$.

(v) The issue of a superconducting vortex intersecting with the boundary separating a vacuum and a superconducting material was explored in previous research (see Chapman et *al.* [38], and Merle and Zaag [139]). In the paper to Merle and Zaag [139], a vortex line at time $t \geq 0$ is represented as

$$L(t) = \{(x,y,z) = (x,0,u(t,x)) \, | \, x \in \Omega\},$$

where $\Omega = (-1,1)$ or $\Omega = \mathbb{R}$, and $u > 0$ is a regular function. The physical derivation leads to the satisfaction of (2.2) by $u(x,t)$ with a reaction term $f(u) = e^{-u}H_0 - F_0(u)$, Here, H_0 denotes the applied constant magnetic field, and F_0 is a regular function that satisfies $F_0(u) \sim \frac{1}{u}$ and $F_0'(u) \sim -\frac{1}{u_2}$ as $u \to 0$. In this model, a vortex reconnecting with the boundary (the plane $z = 0$) corresponds to quenching.

(vi) When investigating the movement of the interface between liquids and solids in the context of phase transitions (see Fila et *al.* [78] and Kawohl [115]), the equation involved has a nonlinear diffusion term. This gives rise to the following form of the equation:

$$u_t - \nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = -\frac{1}{u}.$$

(vii) When examining detonation theory (as referenced in Galaktionov et *al.* [85] and other relevant literature), both the diffusion term and the reaction term in the equation are nonlinear. Consequently, the equation can be expressed as follows:

$$u_t - \ln\left(\frac{e^{cuu_{xx}}-1}{cuu_{xx}}\right) = \ln(u) - \frac{1}{2}u_x^2$$

where c is a nondimensional positive constant representing the chemical prop erties. This equation was studied in Galaktionov et *al.* [85] with Neumann boundary conditions.

Models **(i)** and **(ii)** exhibit the potential for the function u to experience a blow-up phenomenon within a finite time frame. Correspondingly, in models **(iii)**-**(vii)** it is possible that u quenches in finite time.

Now, we will provide a concise overview of the equations (2.2) and discuss the relevant literature on the topic. The equation (2.2) has been the subject of extensive research, as evidenced by numerous books, see Evans [72], Grinrod [95], Henry [103], Krylov [121], Lieberman [127], Smoller [203] and Sperp [206] and other referenced works. The studies conducted have yielded various outcomes, including the investigation of existence, uniqueness, continuous dependence, stability, smoothness, and asymptotic properties of solutions, among others.

Henry [103] has explored the geometric theory of equations (2.2). In this context, a fundamental strategy is to reformulate the partial differential equation as an ordinary differential equation within a Banach space, which includes unbounded operators. As a result of this approach, the theory of C_0-semigroups has emerged (see Goldstein [91] and Pazy [176]). The main objective is to establish the necessary and sufficient conditions for ensuring the well-posedness of the problem. When considering linear equations, whether they are homogeneous or nonhomogeneous, employing a semigroup approach allows for an explicit solution to the problem. However, when dealing with nonlinear equations, as mentioned earlier, the use of fixed-point theorems becomes necessary to establish the existence of solutions.

2.4 Earlier results

Let's consider the problem

$$\begin{cases} u_t - \Delta u = f(u) & , t \in (0,T),\ x \in \Omega \\ u(t,x) = 1 & , t \in (0,T),\ x \in \partial\Omega \\ u(0,x) = u_0(x) & , x \in \Omega \end{cases} \tag{2.5}$$

with an initial function $u_0(x)$ satisfying $0 < u_0(x) \le 1$ and $u_0 = 1$ on the boundary. Here T represents a positive constant. We assume that the reaction term $f(u)$ is singular at $u = 0$ in the sense that $\lim_{u\to 0} f(u) = -\infty$. For $u > 0$, we assume that $f(u)$ is smooth and to satisfy $(-1)^k f^{(k)}(u) < 0$; $k = 0,1,2$.

It is widely known, as referenced in (Sperp [206] p.34, Th.3.3.), that the problem (2.5) has a unique local solution within the set $(0,t_\varepsilon)\times\ \Omega$. This solution can be extended to $(0,t)\times\Omega$, where

$$T = \inf_\tau \left\{ \tau \ge 0 \mid \lim \sup_{t\to\tau, x\in\Omega} \left(u(t,x) + \frac{1}{u(t,x)} \right) = \infty \right\}$$

Furthermore, it is established, as mentioned in (Sperp [206] p.41, Th. 3.8.), that $u(t,x)$ is a C_∞-function with respect to to t and x_i in $(t,x) \in (0,t)\times\Omega$.

Definition 2.1. We say that a is a quenching point and T is a quenching time for $u(t,x)$, if there exists a sequence $\{(t_n,x_n)\}$ with $t_n \to T$ and $x_n \to a$, such that $u(t_n,x_n) \to 0$ as $n \to \infty$.

Definition 2.2. We say that b is a blowup point and T is a blowup time for $u(t,x)$, if there exists a sequence $\{(t_n,x_n)\}$ with $t_n \to T$ and $x_n \to b$, such that $u(t_n,x_n) \to \infty$ as $n \to \infty$.

Below **(i)** to **(iv)**, we will introduce the most important typical research subjects for blowup and quenching problems:

(i) *What are necessary and sufficient conditions for blowup or quenching?*

Kawarada [113] investigated the quenching problem in his initial paper. He focused on studying equation (2.5) withh specific parameter values: $f(u) = \frac{-1}{u}$, $N = 1$ and $u_0 = 1$. Acker and Walter [5] subsequently demonstrated that when Ω is sufficiently large, then u quenches in finite time. This phenomenon occurs specifically for singularities of this type, when $u_0 = 1$. An essential aspect of their research lies in proving that, for suitably large Ω, problem (2.5) lacks a stationary solution.

Even if equation (2.5) possesses a stationary solution, quenching can still occur under certain conditions. In such cases, it becomes crucial for the initial function to have values that are sufficiently close to zero. Acker and Kawohl [4] have provided a proof that when the initial value u_0 is small enough, then u quenches in finite time. This result applies specifically when $N = 1$ and $\int_0^1 f(s)ds = \infty$.

Levine [125] examines the stationary states of problem (2.5) and establishes that when the initial value u_0 is small enough, quenching occurs within a finite time frame. This result holds true for the specific case where $N = 1$ and $f(u) = -u^{-p}$ $(p > 0)$. In the proof, the initial function u_0 is compared to the smallest stationary solution, and it is demonstrated that this smallest stationary solution is unstable.

(ii) *Regarding the set of blowup or quenching points, it is worth considering whether these phenomena can occur throughout an entire interval or if they are limited to specific points. In other words, is it possible for blowup or quenching to happen continuously over an interval, or do they occur only at isolated points?*

Acker and Kawohl [4] proved that quenching occurs in the interval $(0,T)$ for functions $f(u)$ that satisfy $(-1)^k f^{(k)}(u) < 0$; $k = 0,1,2$. They considered Ω as a ball in $\mathbb{R}^N$ and assumed that the initial function u_0 satisfies $\Delta u_0 + f(u_0) \leq 0$ and $(\Delta u_0 + f(u_0))_r \geq 0$. The argument is based on the inequality $u_{rt} \geq 0$, which is proved by the maximum principle.

Deng and Levine [59] proved in $\mathbb{R}^N$ under certain assumptions on $f(u)$ and u_0 that the set of quenching points is a compact subset of Ω. They use the method developed in Friedman and McLeod [83], where the corresponding blowup problem has been studied.

Guo [98] established that the set of quenching points is a discrete subset of Ω for the specific case where $f(u) = -u^{-p}$, $(p > 0)$,and $u_0'' + f(u_0) \leq 0$ $(N = 1)$. The proof relied on Angenent's result [18] concerning certain parabolic equations.

(iii) *What kind of asymptotic behavior do solutions obey near the blowup or quenching points?*

It is evident that either u_t or Δu in the equation will blow up when quenching occurs. Concerning the asymptotic behavior of solutions near a quenching point, several results have been established, considering different assumptions on u_0, $f(u)$ and N.

Chan and Kwong [37] established that when quenching occurs, the derivative u_t blows up (Note that $f(u)$ does not need to be a power singularity). Their result specifically applies to the case where $\int_0^1 f(u)du = \infty$. Deng and Levine [59] extended this theorem to reaction terms that are less singular. Additionally, Fila and Kawohl [77] proved that if

$f(u) = -u^{-p}$, $(p > 0)$. Then

$$\begin{aligned} \min_{x\in\Omega} u(t,x) &\leq [(1+p)(T-t)]^{\frac{1}{1+p}} \\ u(t,x) &\geq C_1(T-t)^{\frac{1}{1+p}} \end{aligned}$$

in a neighborhood of the quenching point $(t < T)$, which provides upper and lower bounds for $u(t,x)$. However, it's important to note that the upper bound is only valid at one specific point with respect to x.

Friedman and McLeod [83] proved that if $f(u) = u^p$ and $N \geq 1$ in (2.5), then

$$\begin{aligned} \min_{x\in\Omega} u(t,x) &\leq \frac{c}{(T-t)^{\frac{1}{1+p}}} \\ u(t,x) &\leq \frac{c}{(T-t)^{\frac{1}{1+p}}} \end{aligned}$$

in a neighborhood of the blowup point $(t < T)$.

Guo [98] proved that, in the case $N = 1$ and $p \geq 3$, if $f(u) = -u^{-p}$, $(p > 0)$. Then for any quenching point (a, T),

$$\lim_{t\uparrow T} u(t,x)(T-t)^{-\frac{1}{1+p}} = (1+p)^{\frac{1}{1+p}}$$

uniformly, when $|x-a| \leq C\sqrt{T-t}$ for any positive constant C. Fila and Hulshof [79] extended this result to $p \geq 1$. For the weaker singularities $0 < p < 1$, the proof is done in Guo [96].

Giga and Kohn [89, 90] proved that if $f(u) = u^p$ in (2.5) and (a, T) be the blowup point. Then

$$\lim_{t\uparrow T} (T-t)^{-\frac{1}{1+p}}\, u\left(a + y\sqrt{T-t}, t\right) = (p-1)^{-\frac{1}{p-1}},$$

uniformly, when $|y| \leq C$. When $f(u) = e^u$ in (2.5), then

$$\lim_{t\uparrow T} u\left(a + y\sqrt{T-t}, t\right) + \ln(T-t) = 0,$$

uniformly for $|y| \leq C$. The extension of this result to higher space dimensions has been worked out in Guo [97] $(p \geq 1)$ and in Fila et *al.* [76] $(p > 0)$. See also Bebernes and Eberly [29], Bebernes et *al.* [30] and Yuen [220].

Fila and Kawohl [77] proved that if $f(u) = -u^{-p}$ $(p > 0)$ and let $(0, T)$ be the quenching point $(r = |x|,\ N \geq 1)$. Then

$$\begin{aligned} &u(T,r) \leq \left[\tfrac{(p+1)}{2(1-p)}\right]^{\frac{1}{1+p}} r^{\frac{2}{1+p}} && \text{for } 0 < p < 1 \\ &u(T,r) \geq C_\varepsilon r^{\frac{\varepsilon+2}{1+p}} && \text{for } 0 < p,\ t \in [0,T] \end{aligned}$$

See also the corresponding results for the blowup problem in A. Friedman and McLeod [83].

Flippas and Guo [80] proved that if $(0,T)$ be the quenching point for equation (2.5), when $f(u)=-u^{-p}$ $(p>0)$ and $N=1$. Then for given $C>0$ as $t\to T$, either

$$\begin{aligned} &(T-t)^{\frac{-1}{1+p}}u(t,x)-(1+p)^{\frac{1}{1+p}} \\ =\ &\frac{(1+p)^{\frac{1}{1+p}}}{2p(-\log(T-t))}\left(\frac{x^2}{2(T-t)}-1\right)+o\left(\frac{1}{-\log(T-t)}\right) \end{aligned}$$

or else, for some integer $m\geq 3$ and some constant $c\neq 0$

$$\begin{aligned} &(T-t)^{\frac{-1}{1+p}}u(t,x)-(1+p)^{\frac{1}{1+p}} \\ =\ &c(T-t)^{\left(\frac{m}{2}-1\right)}h_m\left(\frac{x}{\sqrt{T-t}}\right)+o\left((T-t)^{\left(\frac{m}{2}-1\right)}\right) \end{aligned}$$

where the convergence takes place in $C^k\left(|x|<C\sqrt{T-t}\right)$ for any $k\geq 0$. h_m is the Hermite polynomial of order m.

They also proved in Filippas and Guo [80] that if $(0,T)$ be the quenching point for equation (2.5), when $f(u)=-u^{-p}$ $(p>0)$ and $N=1$. Then

$$u(T,x)=\left[\frac{(1+p)^2}{8p}\right]^{\frac{1}{1+p}}\left(\frac{|x|^2}{|\log|x||}\right)^{\frac{1}{1+p}}(1+o(1)),\ \text{as } |x|\to 0$$

Note that the latter result is also proved independently in Merle and Zaag [139], where, in addition, the stability of quenching problems is studied. See also Filippas and Kohn [81], Herrero and Velazquez [104] and Velazquez [211].

In [198], Salin considered the problem

$$\begin{cases} u_t-u_{xx}=\log(\alpha u) & t\in(0,T),\ x\in(-\ell,\ell) \\ u(0,x)=u_0(x) & x\in(-\ell,\ell) \\ u(t,\pm\ell,t)=1 & t\in(0,T) \end{cases} \tag{2.6}$$

where $\alpha\in(0,1)$ and $u_0\in(0,1]$. The reaction term $f(u)=-u^{-p}$ in (2.5) is here replaced by the weaker logarithmic singularity $f(u)=\log(\alpha u)$. It is shown in Salin [198] that quenching is possible, i.e., we have: for ℓ large enough, the solution $u(t,x)$ of (2.6) quenches in finite time. Moreover, if $u_0''(x)+\log(\alpha u_0(x))\leq 0$, then the set of quenching points is finite.

We can find other very important results in Salin's works [196–200].

(iv) *What can be proved on solutions after blowup or quenching?*

Because u_t blows up, equation (2.5) does certainly not have a strong solution for all $t>0$. The answer to (iv) therefore depends essentially on the concept of solution that one employs and also on how singular the reaction term is. It is interesting to know: (a) Whether the solution $u(t,x)$ can have nontrivial continuations when $t > T$? or (b) Is $u(t,x)$ identically zero, when $t > T$ (complete quenching)? Note here that $f = f(u)\chi(\{u > 0\})$.

In Galaktionov and Vazquez [86], the analysis focuses on question (iv) for a broader range of singularities compared to the ones examined by Fila et *al*. [75] and Phillips [178]. The authors Galaktionov and Vazquez [86] make a significant contribution by establishing the essential conditions that determine complete quenching, taking into account the function $f(u)$. Specifically, in the case of power singularities, Galaktionov and Vazquez [86] proved that if $f(u) = -u^{-p}$ in (2.5) and $N = 1$. Then

(a) Complete quenching occurs if and only if $p \geq 1$.

(b) If $0 < p < 1$, then the solution of (2.5) has a non-trivial continuation after the quenching time T.

CHAPTER 3

REACTION-DIFFUSION SYSTEMS WITH INITIAL CONDITIONS

This Chapter establishes the fundamental principles necessary to comprehend Chapter 7, emphasizing the significance of potential analysis in comprehending parabolic problems and reaction-diffusion systems. It delivers a succinct introduction to Kato classes, enabling the examination of elliptic and parabolic equations with lower-order singular terms, while providing a historical perspective on the model that will receive primary attention in Chapter 7. Furthermore, the chapter explores a specific reaction-diffusion system and showcases its wide-ranging applications in biology, ecology, medicine, and physics. Numerous prior studies are referenced throughout the chapter.

3.1 Introduction

Reaction-diffusion systems with initial conditions are mathematical models that describe the dynamics of interacting substances undergoing both diffusion and chemical reactions. These systems involve partial differential equations (PDEs) that capture the spatiotemporal evolution of the concentrations of the substances involved.

In biology, reaction-diffusion systems with initial conditions have been widely used to study various biological phenomena, including embryonic pattern formation (see Figure 3.1), morphogenesis (see Figure 3.2), and tissue development (see Figure 3.3). One of

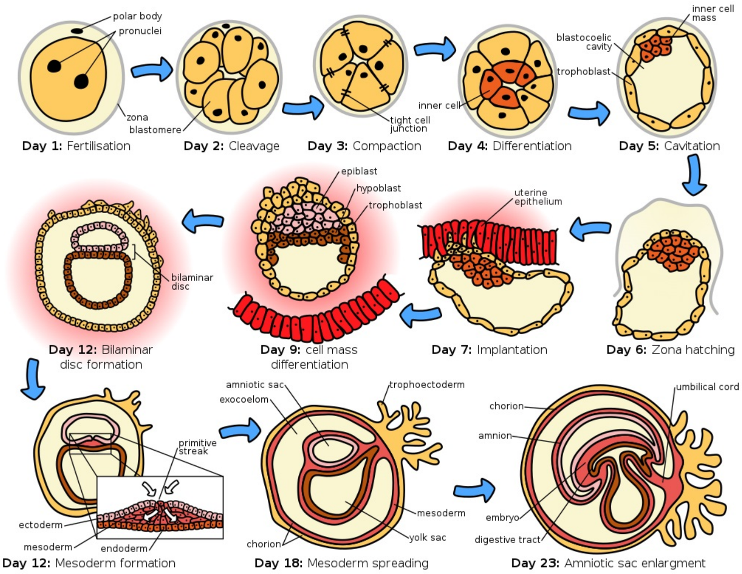

FIGURE 3.1. Embryonic pattern formation.

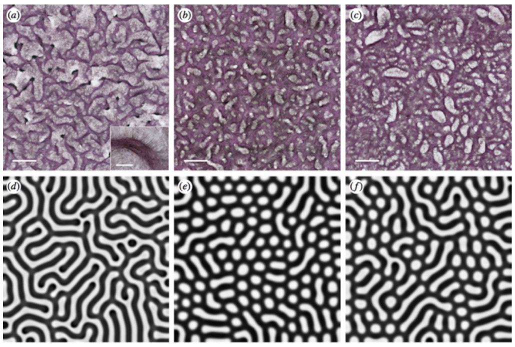

FIGURE 3.2. Morphogenesis.

the most famous examples is the [210] model , proposed by Alan Turing in 1952, which explains how spatial patterns can arise from homogeneous initial conditions through the interaction of diffusing substances with activator-inhibitor dynamics. This model has been applied to understand the formation of patterns in biological structures such as animal coat markings (see Figure 3.4), plant morphogenesis (see Figure 3.5), fish skin pigmentation (see Figure 3.6), patterns in butterflies (see Figure 3.7), development of digits in vertebrate limbs (see Figure 3.8), and the fingerprints (see Figure 3.9).

In ecology, reaction-diffusion systems have been used to study the spread of populations, such as the spreading of invasive species or the propagation of diseases. By incorporating diffusion and reaction terms into the models, researchers can simulate

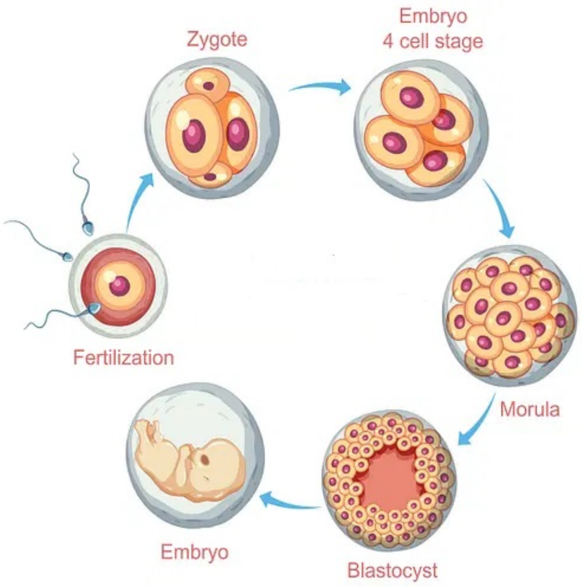

FIGURE 3.3. Tissue development.

FIGURE 3.4. Animal coat markings.

the spatial expansion of populations and investigate factors that influence their dynamics. These models can help in understanding the impact of habitat fragmentation, environmental gradients, and dispersal mechanisms on population distribution and persistence.

In medicine, reaction-diffusion systems have been employed to study the diffusion and reaction of drugs within biological tissues. By modeling the transport of drugs and their interactions with targeted receptors or enzymes, researchers can simulate drug

FIGURE 3.5. Plant morphogenesis.

FIGURE 3.6. Fish skin pigmentation.

distribution patterns and optimize treatment strategies. Reaction-diffusion models have been used, for example, to investigate drug delivery in cancer treatment, where the spatial distribution of drugs within tumors can play a crucial role in their effectiveness.

In science, reaction-diffusion systems have broader applications beyond biology and medicine. They have been used to study chemical reactions, patterns formation in physical systems (see Figure 3.10), patterns formation in bacterial mixtures (see Figure 3.11), self-organization phenomena (see Figure 3.12), and patterns in nature (see Figure 3.13). Reaction-diffusion models can provide insights into complex spatiotemporal behaviors that emerge from simple local interactions, leading to the formation of intricate patterns and structures (see Figure 3.14).

FIGURE 3.7. Patterns in butterflies.

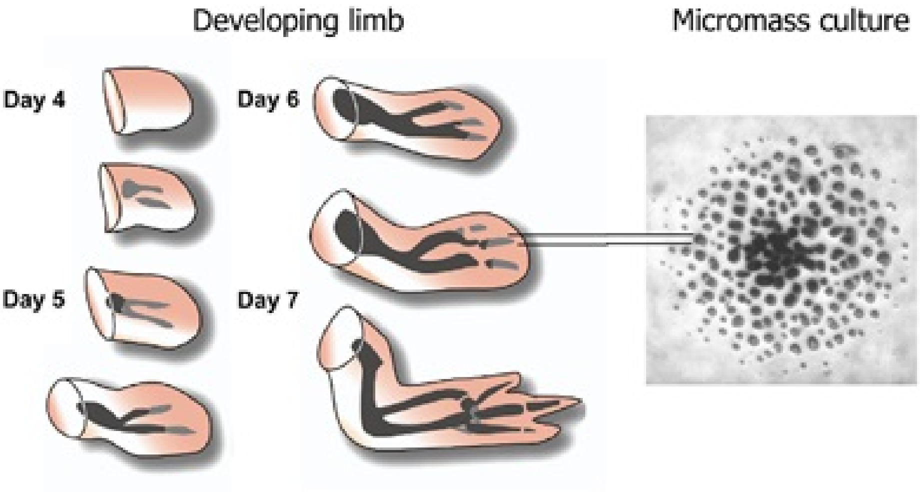

FIGURE 3.8. Development of digits in vertebrate limbs.

In Murray's books [161, 162] we find many models and examples in this vein.

This chapter serves as an introduction to Chapter 8, focusing on a distinct class of reaction-diffusion systems with initial conditions that have attracted great interest among researchers due to their many applications in the fields of biology and environmental sciences. Our main objective will be to study the following reaction-diffusion

FIGURE 3.9. Fingerprints.

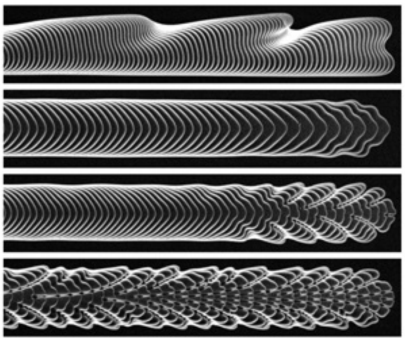

FIGURE 3.10. Patterns formation in physical systems.

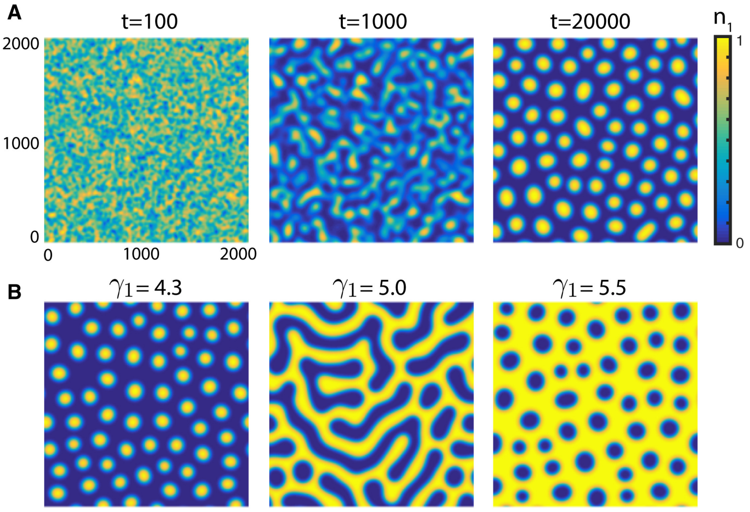

FIGURE 3.11. Patterns formation in bacterial mixtures.

system.

$$\begin{cases} -\dfrac{\partial u_i}{\partial t} + \Delta u_i = \lambda_i p_{i+1}(x,t) f_{i+1}(u_{i+1}) & , 1 \le i \le 2m-1 \\ -\dfrac{\partial u_{2m}}{\partial t} + \Delta u_{2m} = \lambda_{2m} p_1(x,t) f_1(u_1) & \\ u_i(x,0) = \varphi_i(x),\ x \in \mathbb{R}^n & , 1 \le i \le 2m \end{cases} \tag{3.1}$$

FIGURE 3.12. Self-organization phenomena.

FIGURE 3.13. Patterns in nature.

where $(x,t) \in \mathbb{R}^n \times (0,\infty)$, $n \geq 3$, and for $1 \leq i \leq 2m$, $m \in \mathbb{N}^*$; the functions $\varphi_i : \mathbb{R}^n \to [0,\infty)$ are continuous, the constants λ_i are nonnegative, the functions $f_i : (0,\infty) \to [0,\infty)$ are continuous and nondecreasing, $p_i : \mathbb{R}^n \times (0,\infty) \to [0,\infty)$ refers to measurable functions that fulfill a relevant assumption connected to the parabolic Kato class $\mathbf{P}^\infty(\mathbb{R}^n)$ introduced in Zhang [223], see Appendix B.

In their work [182], Redjouh et *al*. investigate this quasi-linear parabolic reaction-diffusion system of arbitrary order with initial conditions. The content of paper [182] forms the basis of Chapter 8 of this thesis.

These systems have garnered interest due to their wide-ranging applications in fields such as biology, ecology, medicine, bioengineering, biochemistry, and physics. Notably, there is a wealth of examples and references related to these applications in the works of Abdellaoui [2], Alqahtani [17], Ghanmi et *al*. [87, 88], Gontara et *al*. [92, 93], Haq et

FIGURE 3.14. Intricate patterns and structures.

al. [101], Lair and Wood [124], Mâagli et *al*. [130, 131], Murray [161, 162], Nisar et *al*. [167], Yadav et *al*. [218], Zhang [222, 223], and the corresponding references therein.

3.2 Potential analysis

Potential analysis is a branch of mathematics that focuses on the study of harmonic functions, their properties, and their applications in various fields. It has a rich history and plays a crucial role in many areas of science and engineering.

The origins of potential analysis can be traced back to the 18th and 19th centuries, when mathematicians like Laplace, Poisson, Lagrange, and Gauss made significant contributions to the development of the theory. Laplace's equation and Poisson's equation, which are fundamental equations in potential theory, emerged during this period. However, the field gained significant development and recognition in the 19th century with the contributions of mathematicians like Augustin-Louis Cauchy, Carl Gustav Jacob Jacobi, and Bernhard Riemann.

The importance of potential analysis lies in its ability to describe and understand various physical phenomena. It provides a mathematical framework to analyze the behavior of quantities such as electric potential, temperature distribution, fluid flow, and gravitational potential. By studying harmonic functions and their properties, potential analysis helps solve real-world problems and make predictions about the behavior of physical systems.

One of the key issues addressed by potential analysis is the study of parabolic problems. Parabolic partial differential equations (PDEs) describe processes that evolve over time, such as heat conduction, diffusion, and population dynamics. Potential analysis provides techniques to study the behavior and solutions of parabolic PDEs, enabling insights into the evolution of these processes and their long-term behavior.

Another important area where potential analysis is applied is in the study of reaction-diffusion systems. These systems arise in various biological, chemical, and ecological contexts, where the interaction between diffusion and chemical reactions plays a crucial role. Potential analysis provides tools to analyze the dynamics and stability of such systems, helping to understand pattern formation, spatial organization, and other complex behaviors.

In this context, we will employ various potential analysis techniques. To explore additional concepts and information in this field, we suggest consulting Armitage and Gardiner's work [21].

The most important contents of a potential analysis include:

(i) **Harmonic Functions:** Potential analysis focuses on the study of harmonic functions, which are solutions to Laplace's equation. Understanding the properties and behavior of harmonic functions is fundamental to potential analysis.

(ii) **Green's Functions:** Green's functions are fundamental solutions to certain partial differential equations, such as the Poisson equation. They play a crucial role in potential analysis by providing a way to represent solutions and investigate the properties of the underlying equations.

(iii) **Dirichlet Problem:** The Dirichlet problem is a central concept in potential analysis. It involves finding a harmonic function that satisfies prescribed boundary conditions. The study of the existence, uniqueness, and regularity of solutions to the Dirichlet problem is an important aspect of potential analysis.

(iv) **Potential Theory:** Potential theory deals with the study of potentials, which are functions related to the distribution of charges or energy in a given domain. It encompasses various concepts and techniques used to analyze potentials, including the maximum principle, mean value property, and representation formulas.

(v) **Applications:** Potential analysis has wide-ranging applications in various fields, including physics, engineering, probability theory, and geometric function theory. It is employed in understanding phenomena such as electrostatics, heat conduction, fluid flow, and random processes.

3.3 Kato Classes

In recent years, there have been many results in the study of elliptic equations with singular lower order terms in the Kato class. We recall that

Definition 3.1. For $n \geq 3$, a function $V \in L^1_{Loc}(\mathbb{R}^n)$ is said to belong to the Kato class K_n if it satisfies the following condition

$$\lim_{r\to 0}\sup_{x} \int_{|x-y|<r} \frac{|V(y)|}{|x-y|^{n-2}} dy = 0.$$

The Kato class elliptic K_n is used to analyze elliptic equations with lower-order singular terms. It plays a significant role in establishing results such as Harnack inequalities and properties of solutions to elliptic equations within this class. The work by Aizenman and Simon [8] demonstrated that the Kato class elliptic is the natural replacement for the Lebesgue class L^p with $p > \frac{n}{2}$, to ensure important properties of solutions to elliptic equations.

As part of the natural generalization of the Kato class from the elliptic case to the parabolic case, we consider the following parabolic equation

$$Hu(t,x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(t,x)\frac{\partial u}{\partial x_j}\right)(t,x) - V(t,x)u(t,x) - \frac{\partial u}{\partial t}(t,x) = 0 \tag{3.2}$$

in a bounded domain $Q = D \times [0,T] \subset \mathbb{R}^{n+1}$. Here a_{ij} are bounded measurable functions and the matrix $a = \left(a_{ij}(t,x)\right)$ is positive definite uniformly in (t,x). This means the existence of a number $\lambda > 1$ such that $\lambda^{-1}I \leq a \leq \lambda I$.

For the function V we will impose the following condition which will be called condition K. Let

$$N_h(V) = \sup_{t,x} \int_{t-h}^{t}\int_D |V(y,s)| \frac{1}{(t-s)^{\frac{n}{2}}} \exp\left(-\alpha\frac{|x-y|^2}{t-s}\right) ds dy, \tag{3.3}$$

$$N_h^*(V) = \sup_{s,y} \int_{s}^{s+h}\int_D |V(x,t)| \frac{1}{(t-s)^{\frac{n}{2}}} \exp\left(-\alpha\frac{|x-y|^2}{t-s}\right) dt dx, \tag{3.4}$$

where α is a fixed positive constant and $V(s,y)$ is regarded as zero when (s,y) is outside of Q.

Definition 3.2. We say that V satisfies condition K if

$$\lim_{h\to 0} N_h(V) = \lim_{h\to 0} N_h^*(V) = 0. \tag{3.5}$$

The space of all L^1_{Loc} functions satisfying condition K will be called the parabolic Kato class, denoted by $\mathbf{P}^\infty(\mathbb{R}^n)$.

The Kato class parabolic $\mathbf{P}^\infty(\mathbb{R}^n)$ plays a significant role in the analysis of parabolic equations with lower-order singular terms. It is used to establish results such as Harnack inequalities and other properties of solutions within this class. The Kato class parabolic $\mathbf{P}^\infty(\mathbb{R}^n)$ is an extension of the Kato class K_n in the elliptic case, providing a natural framework for studying parabolic equations with lower-order singularities.

Proposition 3.1. *If the function V is independent of t, then V belongs to the parabolic Kato class if and only if it belongs to the Kato class in the elliptic case, i.e.,*

$$\lim_{r\to 0}\sup_{x}\int_{B_r(x)} \frac{|V(y)|}{|x-y|^{n-2}}dy = 0,\ n \geq 3.$$

For a more comprehensive understanding of Kato classes and their applications, I recommend referring to Appendix A of this thesis, as well as exploring the references Aizenman and Simon [8] and Zhang [223]. These sources provide further insights and in-depth discussions on Kato classes, offering valuable information and perspectives on their significance and relevance in the respective field of study.

3.4 History and earlier results

We will present a historical overview of the model (3.1) that will be the central focus of Chapter 6. This overview will include a discussion of significant works closely related to the model's content, along with references to these works.

In [222], Zhang discussed the existence and the asymptotic behavior of solutions of the parabolic problem

$$\begin{cases} \Delta u - \dfrac{\partial u}{\partial t} = q(x,t)u^{p+1} & ,\ (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x) & ,\ x \in \mathbb{R}^n,\ n \geq 3, \end{cases} \tag{3.6}$$

and arrived at the following result:

Theorem 3.1. *(See [222]) Assume $p > 0$, $q \in \mathbf{P}^\infty(\mathbb{R}^n)$. For any $M > 1$, there is a constant $b_0 > 0$ such that for each nonnegative $u_o \in C^2(\mathbb{R}^n)$ satisfying $\|u_0\|_{L^\infty(\mathbb{R}^n)} \leq b_0$, there exists* *a positive and continuous solution* u *of* (3.6) *such that*

$$M^{-1}\int_{\mathbb{R}^n} G(x,t,y,0)u_0(y)dy \le u(x,t) \le M\int_{\mathbb{R}^n} G(x,t,y,0)u_0(y)dy$$

for all $(x,t)\in\mathbb{R}^n\times(0,\infty)$.

G denotes the fundamental solution of the heat equation $\Delta u-\dfrac{\partial u}{\partial t}=0$ in $\mathbb{R}^n\times(0,\infty)$, defined by

$$G(x,t,y,s)=\frac{1}{[4\pi(t-s)]^{\frac{n}{2}}}\exp\left(-\frac{|x-y|^2}{4(t-s)}\right) \text{ for all } t>s \text{ and } x,y\in\mathbb{R}^n$$

In [131], Maâgli et *al.* studied the existence of positive solutions for the following nonlinear parabolic boundary value problem

$$\text{(3.7)}\qquad \begin{cases} \Delta u - u\varphi(.,u) - \dfrac{\partial u}{\partial t} = 0 & \text{in } \mathbb{R}^n\times(0,\infty) \\ u(x,0)=u_0(x) & \text{in } \mathbb{R}^n,\ n\ge 3, \end{cases}$$

with initial condition $u_0(x)$ not necessarily bounded function. The nonlinearity φ satisfies an appropriate condition related to the parabolic Kato class $\mathbf{P}^{\infty}(\mathbb{R}^n)$.

Under certain assumptions about the initial value and the nonlinearity, They arrived at the following result:

Theorem 3.2. *There exists a positive continuous solution* u *in* $\mathbb{R}^n\times(0,\infty)$ *for problem* (3.7) *satisfying for each* $t>0$ *and* $x\in\mathbb{R}^n$

$$cPu_0(x,t)\le u(x,t)\le Pu_0(x,t)$$

where $c\in(0,1)$.

$(P_t)_{t>0}$ denote the Gauss semigroup defined for all nonnegative measurable function Φ on $\mathbb{R}^n$ by

$$P_t\Phi(x)=P\Phi(x,t)=\int_{\mathbb{R}^n} G(x,t,y,0)\Phi(y)dy,\ t>0,\ x\in\mathbb{R}^n$$

In [133], Mâatoug and Riahi studied the following problem

$$\text{(3.8)}\qquad \begin{cases} \Delta u + q(x,t)u^p - \dfrac{\partial u}{\partial t} = 0 & \text{in } D\times(0,\infty) \\ u=0 & \text{on } \partial D\times(0,\infty) \\ u(x,0)=u_0(x) & \text{in } D \end{cases}$$

with D is a domain of $\mathbb{R}^n$. Mâatoug and Riahi in [133] introduced a parabolic functional class $J_\infty(D)$ and required the function $q(x,t)$ in (3.8) to satisfy some conditions related to this class while allowing global existence and continuity for solutions of problem (3.8). They arrived at the Theorem below.

Let h be the positive solution of the homogeneous Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \\ \lim\limits_{|x|\to\infty} u(x) = 1 & \end{cases}$$

Theorem 3.3. *Suppose $p > 0$ and q be a nonnegative function defined on $D \times (0,\infty)$ such that the function $(x,t) \mapsto h(x)^{p-1} q(x,t)$ is in the class $J_\infty(D)$. Then there exist positive constants α, β and c such that for each nonnegative function $u_0 \in C^2(D)$ satisfying for all $x \in D$, $u_0(x) \le \alpha h(x)$, there exists a continuous solution u of (3.8) such that for all $(x,t) \in D \times (0,\infty)$*

$$\int_D G(x,t,y,0)u_0(y)dy \le u(x,t) \le \beta \int_D G(x,t,y,0)u_0(y)dy.$$

Here $G(x,t,y,s)$ denotes the heat kernel in $D \times (0,\infty)$ with Dirichlet boundary condition.

In [130], Mâagli et *al*. are interested in the problem

$$\begin{cases} \Delta u - u\varphi(.,u) - \dfrac{\partial u}{\partial t} = 0 & \text{in } \mathbb{R}^n_+ \times (0,\infty),\ n \ge 2 \\ u = 0 & \text{on } \partial\mathbb{R}^n_+ \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n_+ \end{cases} \tag{3.9}$$

where D is the half-space $\mathbb{R}^n_+ = \{x = (x_1,,x_n) : x_n > 0\}$, $p \ge 1$, $q(x,t)$ is a measurable function in $D \times (0,\infty)$ and u_0 is a nonnegative measurable function defined on $\mathbb{R}^n_+$ and satisfying some properties which allows u_0 to be not bounded.

In the simplest case that $u\varphi((x,t),u) = p(x,t)u^p$, the global existence of positive solutions of problem (3.9) has been shown in [134] where Mâatoug and Riahi require u_0 to be bounded of class $C^2\left(\mathbb{R}^n_+\right)$ and $q(x,t)$ to belong to a certain functional class. Later, in [135] Mâatoug and Riah generalised their result in [134] by considering a more large class called parabolic Kato class in $\mathbb{R}^n_+$ denoted by $\mathbf{P}^\infty\left(\mathbb{R}^n_+\right)$.

In [92]**,** Gontara and Turki studied the existence and asymptotic behavior of positive continuous solutions for the followin nonlinear parabolic problem

$$\left\{\begin{array}{ll} \Delta u-\dfrac{\partial u}{\partial t}=\lambda p(x,t)g(v) & \text{in } \mathbb{R}^n\times(0,\infty) \\ \Delta v-\dfrac{\partial v}{\partial t}=\mu q(x,t)f(v) & \text{in } \mathbb{R}^n\times(0,\infty) \\ u(x,0)=\varphi(x)\ ,\ v(x,0)=\psi(x) & \text{in } \mathbb{R}^n,\ n\geq 3 \end{array}\right. \tag{3.10}$$

where the initial conditions φ, $\psi:\mathbb{R}^n\to[0,\infty)$ are continuous, the constants λ, μ are nonnegative, f , $g:(0,\infty)\to[0,\infty)$ are nondecreasing and continuous, p, $q:\mathbb{R}^n\to[0,\infty)$ are measurable functions and satisfy an appropriate hypothesis related to the parabolic Kato class $\mathbf{P}^{\infty}(\mathbb{R}^n)$.

Theorem 3.4. *There exist two constants λ_0 and μ_0 such that for each $\lambda\in[0,\lambda_0)$ and each $\mu\in[0,\mu_0)$, problem (3.10) has a positive continuous solution (u,v) satisfying for each $t>0$, $x\in\mathbb{R}^n$*

$$\left\{\begin{array}{l} \left(1-\frac{\lambda}{\lambda_0}\right)P\varphi(x,t)\leq u(x,t)\leq P\varphi(x,t) \\ \left(1-\frac{\mu}{\mu_0}\right)P\psi(x,t)\leq v(x,t)\leq P\psi(x,t) \end{array}\right.$$

In [183], Redjouh and Mesbahi studied problem (3.10) but in the case of four coupled equations. They showed the existence of continuous positive solutions using potential analysis techniques.

The elliptic case where the problem formed by a single elliptic equation has been studied by several authors such as Gontara et *al*. [93], Cîrstea and Radulescu [44–48]**.**

The elliptical case of (3.10) was studied in Cîrstea and Radulescu [49], Lair and Wood [124], and Ghanmi et *al*. [87]. In [49], Cirstea and Radulescu consider the problem

$$\left\{\begin{array}{l} \Delta u=p(x)g(v) \\ \Delta v=q(x)f(u)\ ,\ x\in\mathbb{R}^n,\ n\geq 3 \end{array}\right. \tag{3.11}$$

where f, g are positive and nondecreasing functions on $(0,\infty)$ and p, q are nonnegative locally holder and radially symmetric functions in $\mathbb{R}^n$. They established the existence of positive entire solutions for (3.11) provided that

$$\lim_{t\to+\infty}\frac{g(cf(t))}{t}\ \text{for all } c>0$$

Moreover, they proved that if

$$\int_0^{\infty}tp(t)dt=\int_0^{\infty}tq(t)dt=\infty$$

then all positive entire radial solutions of (3.11) blow-up at infinity. However, if p and q satisfy the following condition

$$\int_0^\infty t\left[p(t)+q(t)\right]dt<\infty$$

then all positive entire radial solutions of (3.10) are bounded.

In [124], Lair and Wood studied the system (3.11) in the case when $f(u)=u^{\beta}$, $g(v)=v^{\alpha}$, $\alpha>0$, $\beta>0$ and p, q are nonnegative continuous and not necessarily radial. They showed that entire positive bounded solutions exist if p and q satisfy at infinity the following decay condition

$$p(x)+q(x)\leq C\left|x\right|^{-(2+\delta)}$$

for some positive constant δ.

In [87], Ghanmi et *al*. considered the following system

$$\begin{cases} \Delta u=\lambda p(x)g(v)\ ,\ x\in D\\ \Delta v=\mu q(x)f(u)\ ,\ x\in D\\ u\mid_{\partial D}=a\varphi\ ,\ v\mid_{\partial D}=b\psi\\ \lim\limits_{x\in D,\ |x|\to\infty}u(x)=\alpha \text{ and } \lim\limits_{x\in D,\ |x|\to\infty}v(x)=\beta \text{ if } D \text{ is unbounded}\end{cases}\tag{3.12}$$

where D is a $C^{1,1}$ domain in $\mathbb{R}^n$ $(n\geq3)$ with compact boundary. λ and μ are nonnegative constants, the functions $f,g:(0,\infty)\mapsto[0,\infty)$ are continuous and nondecreasing, the functions p, q are measurable and nonnegative in D belonging to a certain elliptic Kato class $K(D)$ introduced and studied in Bachar et *al*. [24] and Mâagli and Zbiri [132]. The initial conditions φ, ψ are nonnegative continuous functions and the constants a, b, α, β are nonnegative and satisfy $a+\alpha>0$, $b+\beta>0$. For two constants λ_0, μ_0 assumed to be positive and by applying a potential theory approach, Ghanmi et *al*. proved in [87] the following theorem:

Theorem 3.5. *For each* $\lambda\in[0,\lambda_0)$ *and each* $\mu\in\left[0,\mu_0\right)$, *problem* (3.12) *has a positive continuous bounded solution* (u,v) *such that*

$$\begin{cases}\left(1-\frac{\lambda}{\lambda_0}\right)\left[\alpha h+aH_D\varphi\right]\leq u\leq\alpha h+aH_D\varphi\\ \left(1-\frac{\mu}{\mu_0}\right)\left[\beta h+bH_D\psi\right]\leq v\leq\beta h+bH_D\psi\end{cases}$$

where $H_D\varphi$ *denotes the unique harmonic function* u *in* D *with boundary value* φ *and satisfying further* $\lim\limits_{|x|\to\infty}u(x)=0$ *whenever* D *is unbounded.*

In [88], Ghanmi et *al.* are concerned with the entire radially symmetric solutions of nonlinear Schrödinger elliptic systems in anisotropic media. In terms of the growth of the variable potential functions, they establish conditions such that the solutions are either bounded or blow up at infinity. In order to discuss the existence of positive radial solutions to this class of nonlinear systems, They are concerned with the following system of nonlinear differential equations

$$\left\{\begin{array}{l} \frac{1}{A}\left(Au'\right)' - p(t)g(v) = 0 \;\; , \;\; t \in (0,\infty) \\ \frac{1}{B}\left(Bv'\right)' - p(t)g(u) = 0 \;\; , \;\; t \in (0,\infty) \\ u(0) = a > 0 \; , \; v(0) = b > 0 \\ \lim\limits_{t\to 0} A(t)u'(t) = 0 \; , \; \lim\limits_{t\to 0} B(t)v'(t) = 0 \end{array}\right. \tag{3.13}$$

where the continuous functions A, $B : [0,\infty) \to [0,\infty)$ are differentiable and positive on $(0,\infty)$ and satisfy certain assumptions.

Problem (3.13) in the case of arbitrary order was studied by Mesbahi in [152] by applying some results and techniques of functional analysis such as Schauder's fixed point theorem and potential theory tools.

We can find in the references mentioned in this paragraph and references given there other results linked to similar problems and in the same context whether in the parabolic or elliptical case.

CHAPTER

4

FIRST-ORDER QUASI-LINEAR PDES AND THEIR APPLICATIONS

The primary focus of this chapter is on introducing first order quasilinear PDEs and exploring their applications. Consequently, this chapter serves as an introductory section for the concluding chapter of this thesis. It discusses Lagrange's form for first order PDEs and the method of characteristics for solving them. Applications presented include determining systems of surfaces orthogonal to a given set, the important Hamilton-Jacobi equation in physics, and modeling the birth-death processes of bacteria using first order PDEs. First order equations are also significant in areas like stochastic processes, fluid dynamics, and solid mechanics. Overall, the chapter explores the theory and wide-ranging relevance of first order PDEs across various domains of mathematics and science.

4.1 Introduction

The field of partial differential equations continues to captivate mathematicians due to the many unsolved problems that remain. Despite tremendous progress over decades of study and application of advanced mathematical concepts, PDEs still hold immense complexity that challenges us to develop new insights and innovative solution methods. Each equation that defies conventional techniques serves as a reminder of how much is yet to be understood about systems governed by continuous changes across time and space. It stimulates ever greater curiosity to uncover the hidden truths just beyond our current reach. The enduring attraction of PDEs stems from their representation of real-world phenomena across diverse domains from biology to astronomy. Through modeling with PDEs, we have gained comprehension of varied disciplines beyond mathematics. Our modern technologies likewise rely on applications of PDE theory. There is no question that PDEs constitute one of the most vibrant areas of ongoing research, as their widespread utility across sciences inspires continued investigation. The bounty of unanswered questions and opportunities for breakthroughs ensures PDEs will remain a wellspring of exciting mathematical problems demanding the innovative perspectives of future scholars. It is a testament to both the power and limitations of human understanding confronted with nature's complexities. See Chowdhury et *al*. [43], Kruzhkov [120], Rhee et *al*. [184], Sneddon [204] and references given there.

4.2 First order quasilinear PDEs

We restrict our work to a class of first order quasilinear PDEs whose general form is

$$\xi_1 p_1 + \cdots + \xi_n p_n = R \tag{4.1}$$

where $\xi_j = \xi_j(x,u)$ $(1 \le j \le n)$ and R are given functions of n independent variables $x_1, \dots, x_n$, and a dependent variable $u = u(x_1, \dots, x_n)$, $p_j = \dfrac{\partial u}{\partial x_j}$, and u is a smooth vector field in a domain Ω of $\mathbb{R}^3$. We wish to find a relation between $x = (x_1, \dots, x_n)$ and u involving an arbitrary function. Lagrange presented the initial comprehensive theory of equations of this nature, which led to equation (4.1) being commonly known as Lagrange's equation.

We recall that a PDE is said to be quasilinear, if it is linear with respect to all the highest order derivatives of the unknown function. A smooth function $u = u(x_1, \dots, x_n)$ is a solution of equation (4.1), if and only if u is constant along the phase curves of the field u, i.e., it is the first integral of the associated characteristic system

$$\frac{dx_1}{\xi_1} = \cdots = \frac{dx_n}{\xi_j} = \frac{du}{R} \tag{4.2}$$

Lagrange's method of characteristics reduces the problem of solving PDE (4.1) to the characteristic system (4.2). For further information about this method and how to apply it, see Ince [106], Kruzhkov [120], Rhee et *al*. [184] and Sneddon [204].

The method of solving quasilinear equations of the form (4.1) is contained in:

Theorem 4.1. *The general solution of a quasilinear PDE* (4.1) *is* $F\left(\varphi_1,\dots\varphi_n\right)=0$, *where* F *is an arbitrary function and* $\varphi_j(x_1,\dots,x_n,u)=c_j$ $(1\le j\le n)$ *are linearly independent first integrals of the associated characteristic system* (4.2).

Proof. We first note that if the solutions of system (4.2) are

$$\varphi_j(x_1,\dots,x_n,u)=c_j,\ 1\le j\le n$$

then the n equations

$$\sum_{i=1}^{n}\frac{\partial\varphi_j}{\partial x_i}dx_i+\frac{\partial\varphi_j}{\partial u}du=0,\ \text{for } 1\le j\le n$$

must be compatible with equations (4.2). In other words, we must have

$$\sum_{i=1}^{n}p_i\frac{\partial\varphi_j}{\partial x_i}+R\frac{\partial\varphi_j}{\partial u}=0 \tag{4.3}$$

Solving the set of n equations (4.3) for p_j, we find that

$$\frac{p_j}{\frac{\partial(\varphi_1,\dots\varphi_n)}{\partial(x_1,\dots,x_{i-1},u,x_{i+1},\dots,x_n)}}=\frac{R}{\frac{\partial(\varphi_1,\dots\varphi_n)}{\partial(x_1,\dots,x_n)}},\ \text{for } 1\le j\le n \tag{4.4}$$

$$\frac{\partial\left(\varphi_1,\dots\varphi_n\right)}{\partial(x_1,\dots,x_n)}=\begin{vmatrix}\frac{\partial u_1}{\partial x_1}&\frac{\partial u_1}{\partial x_2}&\cdots&\frac{\partial u_1}{\partial x_n}\\ \frac{\partial u_2}{\partial x_1}&\frac{\partial u_2}{\partial x_2}&\cdots&\frac{\partial u_2}{\partial x_n}\\ \vdots&\vdots&&\vdots\\ \frac{\partial u_n}{\partial x_1}&\frac{\partial u_n}{\partial x_2}&\cdots&\frac{\partial u_n}{\partial x_n}\end{vmatrix}$$

Consider now the relation

$$\Phi\left(\varphi_1,\dots\varphi_n\right)=0 \tag{4.5}$$

Differentiating it with respect to x_j, we obtain the equation

$$\sum_{i=1}^{n}\left(\frac{\partial\Phi}{\partial\varphi_i}\frac{\partial\varphi_i}{\partial x_j}+\frac{\partial\varphi_i}{\partial u}\frac{\partial u}{\partial x_j}\right)=0$$

and there are n such equations, one for each value of j. Eliminating the n quantities $\frac{\partial \Phi}{\partial \varphi_1}, \dots, \frac{\partial \Phi}{\partial \varphi_n}$ from these equations, we obtain the relation

$$\frac{\partial\left(\varphi_1, \dots \varphi_n\right)}{\partial\left(x_1, \dots, x_n\right)} + \sum_{i=1}^{n} \frac{\partial u}{\partial x_j} \frac{\partial\left(\varphi_1, \dots, \varphi_{i-1}, \varphi_i, \varphi_{i+1}, \dots, \varphi_n\right)}{\partial\left(x_1, \dots, x_{i-1}, u, x_{i+1}, \dots, x_n\right)} = 0 \tag{4.6}$$

Substituting from equations (4.4) into equation (4.6), we see that the function U defined by the relation (4.5) is a solution of equation (4.1), as we desired to show. ■

In Kruzhkov [120], Mesbahi [156], Reinhard [185], Sneddon [204], we find a proof of this theorem, in addition to many other important theorems and properties relevant within this context

4.3 The significance of first order PDEs

First order partial differential equations (PDEs) are encountered in various fields of study, including stochastic process theory, mathematical physics, and solid mechanics. In stochastic process theory, the Fokker-Planck equation is a prominent example of a first order PDE. This equation describes the evolution of the probability density function associated with a stochastic process. It is widely used in the study of diffusion processes and Brownian motion. Oksendal [168] covers the Fokker-Planck equation as a first order PDE arising in stochastic processes.

In mathematical physics, the Hamilton-Jacobi equation is another important first order PDE. It arises in classical mechanics and plays a fundamental role in the formulation of Hamiltonian dynamics. The equation describes the evolution of a characteristic function associated with the motion of a dynamical system.

The Hopf equation, also known as Burgers' equation without viscosity, is a first order PDE that finds applications in diverse contexts. It is utilized in the study of gas dynamics without pressure and in describing the velocity field of a medium composed of non-interacting particles in the absence of external forces. The Hopf equation is of particular interest in fluid dynamics and can provide insights into the behavior of flows in various physical systems. See Rugina [189] for an application to wave propagation in soft tissues, and see Aris [20] for a discussion of first order PDEs such as the continuity equation and the Navier-Stokes equations in fluid dynamics.

In solid mechanics, the mass conservation equation is commonly encountered as a first order PDE. This equation describes the motion of a fluid, whether it is a liquid or a gas, in the absence of sources and sinks. It captures the fundamental principle of mass conservation and is used to model fluid movement in a variety of scenarios, such as the flow of fluids in pipes or channels and the movement of biological fluids of different types. See Marsden and Hughes [137] and the references cited there.

The transport equation is a prototype of first order PDEs and appears in different scientific disciplines. It describes the transport or advection of a quantity, such as a fluid or a scalar field, by a velocity field. The transport equation has widespread applications, including the modeling of fluid infiltration through porous media, such as sand. In this context, the equation helps understand how fluids flow under the influence of gravity alone, without the presence of sources or sinks.

Moreover, the transport equation is relevant in the mathematical modeling of collective movements observed at various levels of biological organization, such as traffic-like behavior in biological systems. For example, molecular motor proteins like kinesin and dynein, which are responsible for intracellular transport in eukaryotic cells, can experience traffic jams that have implications for cellular function and disease. The transport equation provides a mathematical framework to study and analyze these phenomena, aiding in the understanding of complex biological processes. See Evans [72] and Murray [161, 162].

In summary, first order PDEs play a crucial role in diverse scientific domains. They arise in stochastic process theory, mathematical physics, solid mechanics, and other areas of research. Understanding and solving these equations contributes to our comprehension of fundamental physical and biological phenomena, enabling us to make predictions and develop practical applications in various fields. See as well Han and Park [99] and Gorgone and Oliveri [94].

4.4 Applications

4.4.1 Surfaces orthogonal to a given system of surfaces

An interesting application of the theory of linear PDEs of the first order is to the determination of the systems of surfaces orthogonal to a given system of surfaces. If we have a collection of surfaces whose positions in space are defined by a linear PDE, we can use the theory of such equations to derive the differential equation that the orthogonal surfaces must satisfy. Specifically, we take the given differential equation for the reference surfaces and apply the necessary conditions such that the tangent planes

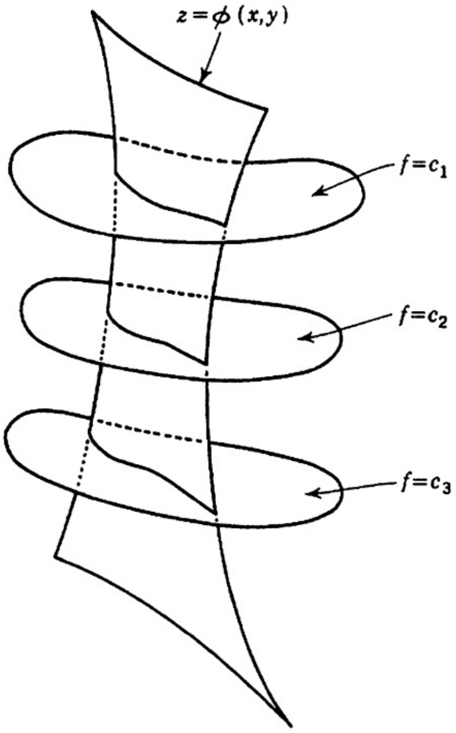

FIGURE 4.1. Surfaces orthogonal to a given system of surfaces.

of any two surfaces, one from each system, are perpendicular at every point where they intersect. This leads to a new differential equation whose solution gives the orthogonal surfaces. Solving this related equation allows us to systematically determine whether the entire system of surfaces is fully orthogonal to the original given surfaces. This demonstrates how linear partial differential equations can be used to analyze geometric relationships between surfaces and derive new surfaces with desired properties based on their interactions with other surfaces. See Reinhard [185], Sneddon [204].

Suppose we are given a one-parameter family of surfaces characterized by the equation

$$f(x,y,z)=c \tag{4.7}$$

We want to determine a system of surfaces that intersect the given surfaces in a perpendicular manner, with the surfaces from each system meeting at 90 degree angles. (See Figure 4.1).

The normal at the point (x,y,u) to the surface of system (4.7) which passes through that point is the direction given by the direction ratios

$$(P,Q,R) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \tag{4.8}$$

If the surface with equation

$$z = \phi(x,y) \tag{4.9}$$

cuts each surface of the given system orthogonally, then its normal at the point (x,y,z) which is in the direction

$$\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$$

is perpendicular to the direction (P,Q,R) of the normal to the surface of the set (4.7) at that point. We therefore have the linear PDE

$$P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R \tag{4.10}$$

for the determination of surfaces (4.9). Substituting from equations (4.8), we see that this equation is equivalent to

$$\frac{\partial f}{\partial x}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial z}{\partial y} = \frac{\partial f}{\partial z}$$

Conversely, any solution of the linear PDE (4.10) is orthogonal to every surface of the system characterized by equation (4.7), for (4.10) simply states that the normal to any solution of (4.10) is perpendicular to the normal to that member of system (4.7) which passes through the same point.

Linear equation (4.10) is therefore the general PDE determining the surfaces orthogonal to members of system (4.7); i.e., the surfaces orthogonal to system (4.7) are the surfaces generated by the integral curves of system

$$\frac{dx}{f_x(x,y,z)} = \frac{dy}{f_y(x,y,z)} = \frac{dz}{f_z(x,y,z)} \tag{4.11}$$

Example 4.1. *Consider the family of one-parameter surfaces characterized by the equation*

$$z(x+y) = c(3z+1)$$

In this case

$$f(z) = \frac{z(x+y)}{(3z+1)}$$

so that system (4.11) take the form

$$\frac{dx}{z(3z+1)} = \frac{dy}{z(3z+1)} = \frac{dz}{(x+y)}$$

which has the following two first integrals

$$\begin{aligned} x-y &= c_1 \\ x^2+y^2-2z^3-z^2 &= c_2 \end{aligned}$$

Thus any surface which is orthogonal to the given surfaces has equation of the form

$$x^2+y^2-2z^3-z^2 = \psi(x-y)$$

where ψ is an arbitrary function.

4.4.2 Hamilton-Jacobi equation

The most important first order PDE occurring in mathematical physics is the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H\left(q_1,\ldots,q_n;\frac{\partial S}{\partial q_1},\ldots,\frac{\partial S}{\partial q_n}\right) = 0 \tag{4.12}$$

appropriate to the Hamiltonian $H(q_1,\ldots,q_n,p_1,\ldots,p_n)$ of a dynamical system of n generalized coordinates $q_1,\ldots,q_n$ and the conjugate momenta $p_1,\ldots,p_n$.

The Hamilton-Jacobi equation, as expressed in equation (4.12) is one of the most pivotal first order PDEs that arises in mathematical physics. It relates the partial derivatives of a function S, known as the Hamilton's characteristic function, with respect to time and the generalized coordinates of a dynamical system. The function S contains complete information about the trajectory and evolution of the dynamical system. Remarkably, the Hamilton-Jacobi equation allows deriving the trajectory of a system from S alone, without needing to know its momenta or velocities explicitly. It involves the Hamiltonian H, which encapsulates the total energy of the system as a function of the generalized coordinates q_j and their conjugate momenta p_j. Solving the Hamilton-Jacobi equation permits finding S and thereby obtaining integral curves in the phase space, whose projections onto the coordinate space give the trajectory of the system. Moreover, knowledge of S is equivalent to solving the system's Hamilton's equations of motion.

Therefore, this first order PDE underlies classical analytical mechanics and serves as a link between the Lagrangian and Hamiltonian formulations. It has extraordinarily wide applications, from classical to quantum dynamics. In general, the solution methods for this nonlinear PDE have greatly expanded our understanding of both integrable and chaotic dynamical systems in nature. For more details, see Abraham [3].

This is an equation in which the dependent variable S is absent. Hence the following characteristic system

$$\frac{dt}{1}=\frac{dq_1}{\frac{\partial H}{\partial p_1}}=\cdots=\frac{dq_n}{\frac{\partial H}{\partial p_n}}=\frac{dp_1}{-\frac{\partial H}{\partial q_1}}=\cdots=\frac{dp_n}{-\frac{\partial H}{\partial q_n}} \tag{4.13}$$

i.e., they are equivalent to the Hamiltonian equations of motion

$$\frac{dq_j}{dt}=\frac{\partial H}{\partial p_j},\ \frac{dp_j}{dt}=-\frac{\partial H}{\partial q_j},\ 1\le j\le n \tag{4.14}$$

A modified form of equation (4.12) is obtained by writing

$$S=-W_t+S_1$$

We then find that

$$H\left(q_1,\ldots,q_n;\frac{\partial S_1}{\partial q_1},\ldots,\frac{\partial S_1}{\partial q_n}\right)=W \tag{4.15}$$

Suppose, for example, that a system with two degrees of freedom has Hamiltonian

$$H=\frac{Pp_{x^2}+Qq_{y^2}}{2(X+Y)}+\frac{\xi+\eta}{X+Y} \tag{4.16}$$

where P,X,ξ are functions of x alone and Q,Y,η are functions of y alone. Then equation (4.15) becomes

$$\frac{1}{2}\left(Pp_x+Qq_y\right)+\left(\xi+\eta\right)-W(X+Y)=0$$

Then one of the characteristic equations is

$$\frac{dx}{Pp_x}+\frac{dp_x}{\frac{1}{2}P'p_x+\xi'-WX'}=0$$

with solution

$$p_x=\{2(WX-\xi+a)\}^{\frac{1}{2}}$$

where a is an arbitrary constant. Similarly we could have shown that

$$q_y=\left\{2\left(WX-\eta+b\right)\right\}^{\frac{1}{2}}$$

where b is an arbitrary constant. Thus since p_x, is a function of x alone and q_y is a function of y alone, we have

$$S = -Wt + \int \{2(WX - \xi + a)\}^{\frac{1}{2}} dx + \int \{2\left(WX - \eta + b\right)\}^{\frac{1}{2}} dy$$

showing that a solution of the Hamilton- Jacobi equation can always be found for a Hamiltonian of the form (4.16).

4.4.3 Fokker-Planck equation

First order PDEs arise frequently in the theory of stochastic processes. One such equation is the Fokker-Planck equation

$$\frac{\partial p}{\partial t} = \beta \frac{\partial}{\partial x}(px) + D\frac{\partial^2 p}{\partial x^2} \tag{4.17}$$

which reduces in the case $D = 0$ to the first order linear equation

$$\frac{\partial p}{\partial t} = \beta x \frac{\partial p}{\partial x} + \beta P \tag{4.18}$$

Equation (4.17) is known as the Fokker-Planck equation, which governs the time evolution of the probability density function P of stochastic systems under Brownian motion. It contains diffusive and drift terms represented by the diffusion coefficient D and mobility parameter β respectively. In the special case where $D = 0$, it reduces to the first order linear PDE in (4.18). These equations have direct physical interpretations in terms of probabilistic distributions. For instance, P could denote the probability that a harmonically bound particle in a thermal bath or fluctuating electrical signal assumes a particular position or deflection value x at time t. Solving such equations helps characterize the dynamical behavior of random systems. However, it is important to note the Fokker-Planck equation (4.17) relies on the underlying stochastic process possessing a Gaussian distribution and following the Markoff property, meaning future states are independent of past states given the present state. This restrictive assumption may break down for non-Gaussian noise or systems with memory effects. Nonetheless, within this framework, these first-order PDEs provide a powerful tool for modeling diffusion and transport in noisy environments. See Oksendal [168] for additional clarification.

4.4.4 Birth and death processes connected with bacteria

First order partial differential equations play an extremely significant role in modeling and understanding birth and death processes, which are fundamental phenomena in fields such as population dynamics and cellular biology. Probably the most well-known application is in studying bacterial reproduction and mortality. Bacteria multiplying via binary fission and facing risks of death can be mathematically described as a continuous-time Markov process with birth and death rates. The time evolution of the probability of having a certain bacterial population size n at time t then satisfies a first order linear PDE. Solving this allows for predicting population distributions over generations. More complex models incorporate factors like limited resources, aging, or damage accumulation. These increase realism but yield quasilinear or nonlinear first order equations that are challenging to solve. Nonetheless, such equations have provided key insights into bacterial evolution, antibiotic resistance spread, disease outbreaks, and cellular regulation mechanisms. They also serve as prototypes for other branching processes across the sciences. Due to their biological importance, first order PDEs emerging from birth and death dynamics remain an extremely active area of research with ongoing discoveries about non-equilibrium dynamics in living systems. See L.J.S. Allen [15] and E. Allen [16].

Suppose, for example, that at time t there are exactly n live bacteria and that:

(i) The probability of a bacterium dying in time $(t, t+\delta t)$ is $\mu_n \delta t$;

(ii) The probability of a bacterium reproducing in time $(t, t+\delta t)$ is $\lambda_n \delta t$;

(iii) The probability of the number of bacteria remaining constant in time $(t, t+\delta t)$ is $\left(1-\lambda_n \delta t - \mu_n \delta t\right)$;

(iv) The probability of more than one birth or death occurring in time $(t, t+\delta t)$ is zero.

If we assume $P_n(t)$ is the probability of there being n bacteria at time t, then these assumptions lead to the equation

$$P_n(t+\delta t) = \lambda_{n-1} P_{n-1}(t)\delta t + \mu_{n+1} P_{n+1}(t)\delta t + \left\{1-\lambda_n \delta t - \mu_n \delta t\right\} P_n(t)$$

which is equivalent to

$$\frac{\partial P_n}{\partial t} = \lambda_{n-1} P_{n-1}(t) + \mu_{n+1} P_{n+1}(t) - \left(\lambda_n + \mu_n\right) P_n(t) \tag{4.19}$$

In the general case λ_n, μ_n would depend on n and t; if we assume that the probability of the birth or death of a bacterium is proportional to the number present, we write

$$\lambda_n = n\lambda \quad \text{and} \quad \mu_n = n\mu \tag{4.20}$$

where λ and μ are constants, and equation (4.19) reduces to

$$\frac{\partial P_n}{\partial t} = \lambda(n-1) P_{n-1}(t) - \left(\lambda+\mu\right) n P_n(t) + \mu(n+1) P_{n+1}(t)$$

and if we introduce a generating function $\Phi(t,z)$ defined by the relation

$$\Phi(t,z) = \sum_{n=0}^{\infty} P_n(t) z^n$$

we see that this last equation is equivalent to the first order linear equation

$$\frac{\partial \Phi}{\partial t} = (z-1)\left(\lambda z - \mu\right)\frac{\partial \Phi}{\partial z}$$

which is easily solved

$$\Phi = f\left(\frac{\mu - \lambda z}{1-z} e^{-(\lambda-\mu)t}\right) \tag{4.21}$$

where the function f is arbitrary. If there are m bacteria present at $t = 0$, then $\Phi - z^m$, so that

$$z^m = f\left(\frac{\mu - \lambda z}{1-z}\right)$$

from which it follows that

$$f(\xi) = \left(\frac{\mu - \xi}{\lambda - \xi}\right)^m$$

Hence at time t

$$\Phi = \left(\frac{\mu\left(1 - e^{(\lambda-\mu)t}\right) - z\left(\lambda - \mu e^{(\lambda-\mu)t}\right)}{\mu - \lambda e^{(\lambda-\mu)t} - \lambda z\left(1 - e^{(\lambda-\mu)t}\right)}\right)^m$$

$P_n(t)$ is the coefficient of z^n in the power series expansion of this function. If it $\lambda < \mu$, then $\Phi \to 1$ as $t \to \infty$, so that the probability of ultimate extinction is unity.

CHAPTER
5

SINGULAR REACTION-DIFFUSION SYSTEM ARISING FROM QUENCHING

In this paper, we study a singular parabolic reaction-diffusion system with positive Dirichlet boundary conditions. It is shown that certain conditions are sufficient to guarantee finite-time quenching and global existence of solutions. This system appears in the modeling of the quenching phenomena.

The content presented in this chapter has been documented in a publication [64] titled “Singular Reaction-Diffusion System Arising From Quenching” in the journal *Nonlinear Dynamics and Systems Theory*, co-authored by S. Mesbahi.

5.1 Introduction

Quenching refers to the process of rapidly cooling a material from a high temperature to a lower temperature. This is done to alter the material’s physical or mechanical properties such as hardness or strength. The rapid cooling prevents the material from undergoing a gradual cooling process, which would allow the material to form larger crystals that could weaken the material’s structure. Quenching can be accomplished using different methods, including immersion in water, oil, or air, depending on the desired outcome. The study of this important phenomenon began in 1975 with a paper by Kawarada [113], where he studied a model in one space dimension. That paper was an introduction to the large-scale studies of the quenching problem by many researchers in several scientific fields. For a detailed survey, we refer to Chan [36], Levine [126], Rouabah et *al*. [187], Zouaoui et *al*. [229].

By using reaction-diffusion models, researchers can simulate the behavior of quenching processes and predict the resulting microstructure and mechanical properties of the metal. This can help in the design of new quenching techniques and in the optimization of existing ones. For more research on the phenomenon of quenching via reaction-diffusion systems, we refer the readers to Bonis [57], Ji et *al*. [107], Mesbahi [143], Mu et *al*. [160], Pei and Li [177], Salin [196–198], Wang [215], Zheng and Song [225], Zheng and Wang [224] and the references therein, where we will also find, in addition to the results by Mesbahi [140] and [155], many theoretical and numerical methods frequently used to study such problems.

In biology, quenching is a process that involves the rapid cooling of a sample in order to interrupt or halt certain biological processes. This procedure has several uses, including stopping metabolic processes and preserving metabolite profile of a sample in metabolomics. Protein synthesis and degradation can also be stopped for protein level and modification analysis in cells or tissues, while RNA in cells or tissues can be preserved for the analysis of gene expression. Moreover, microbial cultures can be preserved for long-term storage or transport by rapidly cooling them to halt growth and metabolic activity.

Quenching has many applications in medicine. One common medical application of quenching is cryotherapy, where extreme cold to treat disease or injury is used. This can include using liquid nitrogen to freeze and destroy cancerous tissue, or the use of ice packs to reduce swelling and inflammation. Another application is controlling the release of drugs from drug delivery systems. Rapid cooling of the system can halt or slow down drug release, enabling sustained release over time. Furthermore, quenching can aid in the preservation of biological samples such as blood or tissue samples for analysis or storage. Rapid cooling can prevent degradation of the sample and preserve its integrity for later use.

Quenching is also an important process in the manufacture of contact lenses. Typically, after the lenses are shaped, they undergo thermal quenching by being immersed in cold water. This process helps in solidifying their structure and preventing any deformation or distortion during handling and further processing. Furthermore, it enhances the mechanical and optical features of the lenses making them stronger, more resistant to damage, and long-lasting. Chemical quenching is also used by manufacturers to adjust the properties of the lenses. For instance, to crosslink the polymer chains in the lenses or to enhance their strength and flexibility. For better understanding, we refer to Barka et *al.* [26], Khurshid et *al.* [117].

In this work, we are interested in the study of the following reaction-diffusion system with general singular terms and positive Dirichlet boundary conditions that can be applied to the quenching phenomenon:

$$\begin{cases} (u_1)_t - \Delta u_1 = -f_1(u_2) & \text{in } (0,T)\times\Omega, \\ \vdots & \vdots \\ (u_{m-1})_t - \Delta u_{m-1} = -f_{m-1}(u_m) & \text{in } (0,T)\times\Omega, \\ (u_m)_t - \Delta u_m = -f_m(u_1) & \text{in } (0,T)\times\Omega, \\ u_1 = u_2 = \cdots = u_m = 1 & \text{on } (0,T)\times\partial\Omega, \\ u_1(0,x) = u_{10}(x), \ldots,\ u_m(0,x) = u_{m0}(x) & \text{in } \Omega, \end{cases} \tag{5.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary. The functions f_j $(1 \leq j \leq m)$ are positive on $(0,1]$. The initial data satisfy

$$\begin{cases} u_{10},\ u_{20}, \ldots,\ u_{m0} \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right), \\ u_{j0} = 1, \text{ for all } 1 \leq j \leq m, \text{ on } \partial\Omega, \\ 0 < u_{j0} \leq 1, \text{ for all } 1 \leq j \leq m, \text{ in } \overline{\Omega}. \end{cases} \tag{5.2}$$

The rest of this chapter is organized as follows. In the next section, we state our main results. In the third section, we prove some important preliminary results. The fourth section is devoted to the proof of the main results. The chapter ends with a concluding remarks and perspectives.

5.2 Statement of Main Results

5.2.1 Assumptions

For this model, the finite-time quenching phenomena are caused by singular nonlinearities in the absorption terms of (5.1).

Definition 5.1. We say the solution $(u_1, \ldots, u_m)$ of problem (5.1) quenches if $(u_1, \ldots, u_m)$ exists in the classical sense and is positive for all $0 \leq t < T$, and also satisfies

$$\inf_{t\to T} \min_{x\in[0,1]} \{(u_1(t,x), \ldots, u_m(t,x)\} = 0$$

In this case, T is called quenching time.

To study problem (5.1), we also assume that the positive functions $f_j : (0,1] \to (0,+\infty)$, $1 \le j \le m$, satisfy the following simple assumptions which allow them to be chosen from a wide range:

(H_1) The functions f_j, $1 \le j \le m$, are locally Lipschitz on $(0,1]$,

(H_2) $f_j'(s) < 0$ on $(0,1]$ for all $1 \le j \le m$,

(H_3) $\lim\limits_{s\to 0^+} f_j(s) = +\infty$ for all $1 \le j \le m$.

In order to state our results more conveniently, we denote by φ the first eigenfunction associated with the first eigenvalue λ_1 of the problem

$$\begin{cases} \Delta\varphi + \lambda\varphi = 0 & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

normalized by $\int_\Omega \varphi(x)dx = 1$, with $\varphi(x) > 0$ in Ω.

5.2.2 The Main Results

The following theorem gives us a sufficient condition for finite-time quenching.

Theorem 5.1. *Under hypotheses $(H_1)-(H_3)$, the solution of problem (5.1) quenches in finite time for any initial data provided that λ_1 is small enough.*

Many quenching studies confirm that time-derivatives blow-up while the solution itself remains bounded. We refer, for example, to Chan [36] and Kawarada [113]. Throughout this chapter, without any special explanation, we assume that the initial data $u_{10}, \ldots, u_{m0}$ satisfy

$$\Delta u_{10} - f_1(u_{20}) < 0, \ldots, \ \Delta u_{m0} - f_m(u_{10}) < 0 \text{ in } \Omega. \tag{5.3}$$

Thus, the global existence of solutions can be described by the following theorem.

Theorem 5.2. *If the diameter of Ω is small enough and the initial data satisfies $0 < \varepsilon \le u_{10}, \ldots, u_{m0} \le 1$ in $\overline{\Omega}$, then under hypotheses $(H_1)-(H_3)$, the solution of problem (5.1) does not quench in finite time. In this case, we say that the solution $(u_1, \ldots, u_m)$ exists globally.*

5.3 Preliminary Results

We will prove two important lemmas which we will use to prove our main results.

Lemma 5.1. *Assume that the initial data satisfy* (5.3)*, then* $(u_1)_t,\dots,(u_m)_t<0$ *in* $(0,T)\times\Omega$.

Proof. Let $I_j(t,x)=\left(u_j\right)_t(t,x)$ for all $1\le j\le m$ and $(t,x)\in(0,T)\times\Omega$. Differentiating system (5.1) with respect to t, we have

$$\begin{cases} \frac{\partial}{\partial t}I_1=\Delta(u_1)_t-(u_2)_t\ f_1'(u_2) & \text{in } (0,T)\times\Omega,\\ \vdots & \vdots\\ \frac{\partial}{\partial t}(I_m)(x,t)=\Delta(u_m)_t-(u_1)_t f_m'(u_1) & \text{in } (0,T)\times\Omega,\\ I_1=I_2=\cdots=I_m=0 & \text{on } (0,T)\times\partial\Omega,\\ I_j(0,x)<0,\ \text{for all } 1\le j\le m & \text{in } \Omega, \end{cases}$$

which, after simplification, gives

$$\begin{cases} \frac{\partial}{\partial t}I_1-\Delta I_1=-I_2f_1'(u_2) & \text{in } (0,T)\times\Omega,\\ \vdots & \vdots\\ \frac{\partial}{\partial t}I_m-\Delta I_m=-I_1f_m'(u_1) & \text{in } (0,T)\times\Omega,\\ I_1=I_2=\cdots=I_m=0 & \text{on } (0,T)\times\partial\Omega,\\ I_j(0,x)<0,\ \text{for all } 1\le j\le m & \text{in } \Omega. \end{cases} \tag{5.4}$$

By the comparison principle, we have, for all $(t,x)\in(0,T)\times\Omega$,

$$I_j(t,x)=\left(u_j\right)_t(t,x)<0 \text{ for all } 1\le j\le m.$$

This shows that $u_1,\dots,u_m$ are strictly decreasing in time. ■

Now, we consider the radial solutions of problem (5.1) on $\Omega=Br=\left\{x\in\mathbb{R}^N:|x|<R\right\}$.

Lemma 5.2. *Let* $(u_1,\dots,u_m)$ *be the global solution of problem* (5.1) *with* $(u_{10},\dots,u_{m0})\equiv(1,\dots,1)$*,* $u_1,\dots,u_m\ge b$ *in* $(0,\infty)\times\overline{B}_R$ *for some* $b\in(0,1)$*. Then* $(u_1,\dots,u_m)$ *approaches* *uniformly from above to a solution* $(U_1,\dots,U_m)$ *of the steady-state problem*

$$\text{(5.5)}\qquad \begin{cases} \Delta U_1 = f(U_2) & \textit{in } B_R,\\ \vdots & \vdots\\ \Delta U_{m-1} = f(U_m) & \textit{in } B_R,\\ \Delta U_m = f(U_1) & \textit{in } B_R,\\ U_1 = U_2 = \cdots = U_m = 1 & \textit{on } \partial B_R. \end{cases}$$

Proof. Since $(1,\dots,1)$ is a strict super-solution of problem (5.1), by Lemma 5.1, we have $(u_1)_t,\dots,(u_m)_t < 0$ in $(0,\infty)\times B_R$. Define the functions

$$Q_j(t,x) = \int_{B_R} G(x,y)u_j(t,y)dy, \text{ in } (0,\infty)\times B_R, \text{ for all } 1\le j\le m,$$

where $G(x,y)$ is Green's function associated with the operator $-\Delta$ on B_R under Dirichlet boundary conditions. Hence

$$\begin{aligned} \frac{\partial}{\partial t}(Q_1) &= \int_{B_R} G(x,y)(u_1)_t(t,y)dy\\ &= \int_{B_R} G(x,y)\Delta u_1(t,y)dy - \int_{B_R} G(x,y)f_1(u_2(t,y))dy,\\ &\ \ \vdots\\ \frac{\partial}{\partial t}(Q_m) &= \int_{B_R} G(x,y)(u_m)_t(t,y)dy\\ &= \int_{B_R} G(x,y)\Delta u_m(t,y)dy - \int_{B_R} G(x,y)f_m(u_1(t,y))dy, \end{aligned}$$

this gives us

$$\begin{aligned} \frac{\partial}{\partial t}(Q_1) &= 1 - u_1(x,y) - \int_{B_R} G(x,y)f_1(u_2(t,y))dy,\\ &\ \ \vdots\\ \frac{\partial}{\partial t}(Q_m) &= 1 - u_m(x,y) - \int_{B_R} G(x,y)f_m(u_1(t,y))dy, \end{aligned}$$

It follows from $\left(u_j\right)_t < 0$ for all $1\le j\le m$, that

$$G(x,y)f_1(u_2(t,y)),\dots,\ G(x,y)f_{m-1}(u_m(t,y)) \text{ and } G(x,y)f_m(u_1(t,y))$$

are nondecreasing with respect to t. According to the monotone convergence theorem with

$$b \le U_j(x) = \lim_{t\to 0} u_j(t,x) \text{ for all } 1 \le j \le m,$$

we have

$$\begin{aligned}
\lim_{t\to 0}\frac{\partial}{\partial t}(Q_1) &= 1 - U_1(x) - \int_{B_R} G(x,y) f_1(U_2(y))\,dy,\\
&\vdots\\
\lim_{t\to 0}\frac{\partial}{\partial t}(Q_m) &= 1 - U_m(x) - \int_{B_R} G(x,y) f_m(U_1(y))\,dy.
\end{aligned}$$

Furthermore, since $Q_1,\dots,Q_m$ are bounded, $(Q_1)_t,\dots,(Q_m)_t \le 0$, and by $(u_1)_t,\dots,(u_m)_t < 0$, we have

$$\lim_{t\to 0}\frac{\partial}{\partial t}\left(Q_j\right) = 0 \text{ for all } 1 \le j \le m,$$

which yields

$$\begin{aligned}
U_1(x) &= 1 - \int_{B_R} G(x,y) f_1(U_2(y))\,dy,\\
&\vdots\\
U_m(x) &= 1 - \int_{B_R} G(x,y) f_m(U_1(y))\,dy,
\end{aligned}$$

and therefore $(U_1,\dots,U_m)$ is a solution of problem (5.5), and the uniform convergence is ensured by Dini's theorem. ■

5.4 Proofs of the Main Results

Proof of Theorem 5.1 Let $(u_1,\dots,u_m)$ be the solution of problem (5.1) with the maximal existence time T. By the maximum principle, we have $0 \le u_j \le 1$ for all $1 \le j \le m$, in $(0,T)\times\Omega$. Let

$$\psi_j(t) = \int_\Omega \left(1-u_j\right)\varphi\,dx \;\text{ for all } 1 \le j \le m,\; t \in [0,T) \tag{5.6}$$

and

$$\Psi(t)=\psi_1(t)+\cdots+\psi_m(t),\quad t\in[0,T). \tag{5.7}$$

By hypotheses $(H_1)-(H_3)$ and the corresponding Taylor expansions, we can easily get

$$f_1(u_2)\geq\delta(1-u_2)+c_1\ ,\ldots,\ f_m(u_1)\geq\delta(1-u_1)+c_m, \tag{5.8}$$

where δ, $c_1,\ldots,c_m$ are positive constants determined by $f_1(u_2),\ldots,f_m(u_1)$.

By a straight-forward computation and (5.8), we have

$$\begin{aligned}\psi_1'(t) &= -\int_\Omega \Delta u_1\varphi dx+\int_\Omega f_1(u_2)\varphi dx\\ &= \int_\Omega \Delta(1-u_1)\varphi dx+\int_\Omega f_1(u_2)\varphi dx\\ &\geq -\lambda_1\int_\Omega (1-u_1)\varphi dx+\delta\int_\Omega(1-u_2)\varphi dx+c_1\int_\Omega\varphi dx\\ &= -\lambda_1\psi_1(t)+\delta\psi_2(t)+c_1.\end{aligned}$$

In the same way, with $\psi_2(t),\ldots,\psi_m(t)$, we finally get the following inequalities:

$$\begin{aligned}\psi_1'(t) &\geq -\lambda_1\psi_1(t)+\delta\psi_2(t)+c_1,\\ &\ \ \vdots\\ \psi_m'(t) &\geq -\lambda_1\psi_m(t)+\delta\psi_1(t)+c_m.\end{aligned}$$

Using (5.7), we get

$$\Psi'(t)\geq(\delta-\lambda_1)\Psi(t)+C\ ,\ \text{with } C=c_1+\cdots+c_m. \tag{5.9}$$

Since $0\leq u_j\leq 1$ in $(0,T)\times\Omega$, then $0\leq 1-u_j\leq 1$ in $(0,T)\times\Omega$, which clearly implies by (5.6) that $0\leq\psi_j(t)\leq 1$ for all $1\leq j\leq m$, consequently, $0\leq\Psi(t)\leq m$. Since λ_1 is small enough, it is obvious that $(\delta-\lambda_1)\Psi(t)+C>0$. Then, by (5.9), we have

$$\frac{d\Psi}{(\delta-\lambda_1)\Psi(t)+C}\geq dt\ ,\ t\in[0,T),$$

which gives, by integration from 0 to T,

$$t \leq \begin{cases} \dfrac{1}{\delta-\lambda_1}\log\left(\dfrac{(\delta-\lambda_1)\Psi(t)+C}{(\delta-\lambda_1)\Psi(0)+C}\right) & \text{if } \delta \neq \lambda_1, \\ \dfrac{1}{C}(\Psi(t)-\Psi(0)) & \text{if } \delta = \lambda_1. \end{cases} \tag{5.10}$$

Now, letting $t \to T^-$ in (5.10) and combining $\lim\limits_{t\to T^-}\Psi(t) \leq m$, we get

$$T \leq \begin{cases} \dfrac{1}{\delta-\lambda_1}\log\left(\dfrac{m(\delta-\lambda_1)+C}{(\delta-\lambda_1)\Psi(0)+C}\right) & \text{if } \delta \neq \lambda_1, \\ \dfrac{1}{C}(m-\Psi(0)) & \text{if } \delta = \lambda_1. \end{cases} \tag{5.11}$$

Since $0 \leq \Psi(t) \leq m$, we can easily arrive at the positivity of the right-hand side of (5.11), which shows finite time quenching of the solutions in system (5.1). This ends the proof of Theorem 5.1. ■

Proof. Consider the auxiliary system

$$\begin{cases} \frac{\partial}{\partial t}\bar{u}_1 = \Delta\bar{u}_1 - f(\bar{u}_2) & \text{in } (0,T)\times\Omega, \\ \vdots & \vdots \\ \frac{\partial}{\partial t}\bar{u}_m = \Delta\bar{u}_m - f(\bar{u}_1) & \text{in } (0,T)\times\Omega, \\ \bar{u}_1 = \cdots = \bar{u}_m = 1 & \text{on } (0,T)\times\partial\Omega, \\ \bar{u}_1(0,x) = \cdots = \bar{u}_m(0,x) = 1 & \text{in } \overline{\Omega}. \end{cases}$$

By the comparison principle, we have $u_j \leq \bar{u}_j$ for all $1 \leq j \leq m$.

We first consider the following system:

$$\begin{cases} \Delta\bar{u}_1^* = f_1(1) & \text{in } B_R, \\ \vdots & \vdots \\ \Delta\bar{u}_m^* = f_m(1) & \text{in } B_R, \\ \bar{u}_1^* = \cdots = \bar{u}_m^* = 1 & \text{on } \partial B_R. \end{cases}$$

By Green's function, the solution is $\left(\bar{u}_1^*,\ldots,\bar{u}_m^*\right)$ denoted as follows:

$$\bar{u}_j^* = \frac{f_j(1)\left(|x|^2-R^2\right)}{2N} + 1\ ,\ 1 \leq j \leq m$$

and

$$\min \bar{u}_j^* = \frac{-f_j(1)R^2}{2N} + 1 \ , \ 1 \leq j \leq m.$$

Clearly, $\left(\bar{u}_1^*, \ldots, \bar{u}_m^*\right)$ is a super solution of (5.1). By Lemma 5.2, the solution $(u_1, \ldots, u_m)$ of (5.1) is global only if $\bar{u}_1^*, \ldots, \bar{u}_m^* > 0$. ■

5.5 Concluding Remarks and Perspectives

This contribution advances mathematical research on quenching phenomena. The results of this study can be used to study other singular reaction-diffusion phenomena. We managed to overcome some difficulties and achieved very important results. This leads us to think more about the problem and do further theoretical and numerical research under other conditions. These efforts will advance quenching technology and modeling in many scientific fields.

CHAPTER 6

QUENCHING REACTION-DIFFUSION SYSTEMS IN BIOENGINEERING AND LIFE SCIENCES

This research chapter revolves around investigating the phenomenon of quenching in reaction-diffusion systems and highlighting its significance. The primary focus is on analyzing a specific type of parabolic singular reaction-diffusion model that incorporates positive Dirichlet boundary conditions. The objective is to establish the sufficiency of certain conditions for quenching to occur within a finite time frame and to demonstrate the global existence of solutions. The novelty of this work lies in the simplicity of the conditions imposed on the nonlinearity. This simplicity allows us to choose it from a wide range of possibilities, thus facilitating the application of the model to numerous singular reaction-diffusion phenomena. To bolster our findings, we will present various real-world applications in the fields of bioengineering and life sciences, showcasing the practical relevance of quenching phenomena. Additionally, we are conducting a numerical study utilizing the Scilab program to complement our theoretical analysis. Finally, the chapter ends with a conclusion and some potential future perspectives for further research in this area.

The content presented in this chapter has been documented in a research paper [62] entitled "Quenching Reaction-Diffusion Systems in Bioengineering and Life Sciences" published in the *Engineering and Technology Journal*. The paper represents a collaborative effort involving S. Mesbahi and K. I. Saffidine.

6.1 Introduction

Quenching is a rapid cooling process that alters specific properties of materials by manipulating the cooling rate. The material is heated above the recrystallization temperature but below the melting point to allow grain restructuring, followed by controlled cooling to a predetermined temperature. The exploration of this significant phenomenon commenced in 1975 when Kawarada [113] published a paper that focused on a one-dimensional model. This publication served as an initial stepping stone for extensive investigations into the quenching problem conducted by numerous researchers across various scientific disciplines. The phenomenon of quenching is observed in many fields, such as metallurgy, biology, medicine, ecology, bioengineering, the manufacture of medical devices, instruments, and contact lenses, particularly in the context of polymerization processes. In this context, quenching refers to the rapid cooling or solidification of a material, typically a polymer, to halt or control a chemical reaction. We find sufficient information about this in Liščić et *al*. [129], Banasiak and Mokhtar-Kharroubi [25]. In the production of contact lenses, polymerization is a key step in forming the lens material. The polymerization reaction involves the conversion of monomers, small molecules with reactive groups, into a polymer network. This reaction is often initiated by heat or light, and it proceeds through a process known as reaction-diffusion. The monomers are dispersed in a liquid solution, and the polymerization reaction occurs as the monomers diffuse and react with each other. During the polymerization process, it is crucial to control the reaction to achieve the desired properties of the contact lens material, such as their mechanical strength, transparency, and water content. Quenching is employed as a means to stop the polymerization reaction at a specific stage. By rapidly cooling or solidifying the material, the diffusion of monomers and reaction products is effectively halted, preventing further polymerization. Quenching can be accomplished through various methods, such as immersion in a cooling bath or exposure to cold air or liquid nitrogen. The specific quenching technique used depends on the manufacturing process and the desired properties of the contact lenses. For some examples and additional details, see Barka et *al*. [26], Rouabah et *al*. [187].

Quenching in reaction-diffusion systems can manifest in different ways, depending on the specific characteristics of the system. In simultaneous quenching, the quenching of different components can occur simultaneously, meaning that the concentrations of all the substances involved in the reaction become zero at the same time. This type of quenching is often studied to understand the overall behavior of the system and the underlying mechanisms. Non-simultaneous quenching occurs when the concentrations of different substances in the system become zero at different times. This type of quenching can lead to more complex dynamics and is also the subject of research in reaction-diffusion systems. Different types of quenching rates can lead to different behaviors and can be studied to understand the system's dynamics. In many cases, the solution of a reaction-diffusion system quenches in finite time, meaning that the concentration of one or more substances becomes zero after a certain amount of time has passed. This type of quenching is often of interest in practical applications, as it can lead to the system reaching a steady state more quickly. Understanding these and other quenching species in reaction-diffusion systems is essential for studying their behavior and can have implications for various fields, such as engineering, chemistry, biology, ecology, and medicine. See Chan [36], Liščić et al. [129],

The spatial distribution of quenching, which refers to the location and extent of the regions where the concentration of one or more substances in a reaction-diffusion system becomes zero, can have various effects on the behavior of these systems. In some cases, it can lead to the formation of complex patterns, such as spots, stripes, or spirals. This is due to the interaction between the quenching regions and the diffusion of the substances involved in the reaction. The spatial distribution of quenching can also affect the stability of the system, as it can lead to the formation of unstable regions where the concentration of one or more substances becomes zero. Understanding the effect of quenching on stability is important for predicting the behavior of the system and for designing control strategies. See Cross and Greenside [52].

The phenomenon of quenching in bioengineering is the application of heat treatment quenching to biological or biomimetic materials. For example, quenching can be used to create bioactive glasses, which are materials capable of chemically bonding with living tissues. Bioactive glasses are obtained by melting a mixture of silicon, calcium, sodium, and phosphorus salts, and then rapidly cooling to prevent crystallization. Bioactive glasses can be used as implants, drug delivery agents, or tissue regeneration scaffolds. See Jones [108], Hench and Jones [102] and the references mentioned therein. Another example of quenching in bioengineering is the production of silk nanofibers. Silk is a natural protein produced by certain insects and spiders, which possesses exceptional mechanical properties. Silk can be dissolved in an aqueous solution, then stretched and rapidly cooled to form nanofibers. Silk nanofibers can be used as reinforcement materials, sensors, filters, or matrices for cell culture.

Therefore, quenching in bioengineering is a process that allows for the creation of innovative and high-performance materials, drawing inspiration from the principles of metallurgy. Quenching can modify the structure and properties of materials at the nanoscale or micrometer scale, depending on the temperature, cooling rate, and chemical composition parameters. In Kundu [123], we find a lot about this content.

Here are some phenomena that can be modeled using reaction-diffusion equations:

(i) Protein Folding and Unfolding: Quenching can refer to the rapid cooling or sudden change in environmental conditions that leads to the folding or unfolding of proteins. The process of protein folding involves the formation of a three-dimensional structure, while unfolding refers to the disruption of the native protein structure. Reaction-diffusion equations can be used to model the conformational changes and kinetics of protein folding and unfolding.

(ii) Cell Signaling and Receptor-Ligand Binding: Quenching can also describe the attenuation or termination of cell signaling pathways or the binding of ligands to receptors. In these cases, reaction-diffusion equations can be used to model the diffusion of signaling molecules or ligands, as well as their binding kinetics with receptors. The equations can capture the spatial and temporal dynamics of the signaling or binding process and provide insights into the quenching mechanisms.

(iii) Fluorescence Quenching in Biosensors: Quenching can occur in biosensors that rely on fluorescence-based detection. When a fluorescent molecule or fluorophore is in close proximity to a quencher molecule, the fluorescence emission can be attenuated. Reaction-diffusion equations can be employed to model the diffusion of the fluorescent and quencher molecules, as well as the quenching kinetics. This allows for the optimization of biosensor designs and the prediction of fluorescence quenching patterns.

It's important to note that these examples are just a few instances where reaction-diffusion equations can be applied to model quenching phenomena in bioengineering. The specific equations and parameters would depend on the particular system and the underlying mechanisms involved. We find many details, real models, and techniques used to study such problems in Barka et *al*. [26], Berestycki et *al*. [32], de Bonis [57], Ji et *al*. [107], Kawarada [113], Mesbahi [143, 155], Murray [161, 162], Pei and Li [177], Salin [196], Wang [215], Zhou et *al*. [227], Zhu et *al*. [228], as well as in the sources mentioned there.

In this chapter, we will mathematically investigate a problem that aligns with the previously discussed concept of quenching. Our focus lies on examining a reaction-diffusion model that incorporates singular nonlinearity and positive Dirichlet boundary conditions:

$$\begin{cases} u_t - \Delta u = -f(u) & \text{in } (0,T)\times\Omega \\ u = 1 & \text{on } (0,T)\times\partial\Omega \\ u(0,x) = u_0(x) & \text{in } \Omega \end{cases} \tag{6.1}$$

with

$$\begin{cases} u_0 \in C^2(\Omega)\cap C^1\left(\overline{\Omega}\right) \\ u_0 = 1, \text{ on } \partial\Omega, \\ 0 < u_0 \leq 1, \text{ in } \overline{\Omega} \\ \Delta u_0 - f(u_0) < 0, \text{ in } \Omega \end{cases} \tag{6.2}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^N$ $(N \geq 2)$, and f is a positive function on $(0,1]$.

The remaining sections of this chapter are structured as follows. In the subsequent section, we will showcase real-life applications of the investigated model in the fields of bioengineering and biology. Following that, in the third section, we will present our main result and demonstrate them in detail. The fourth section focuses on the numerical study conducted. Finally, the chapter concludes with concluding remarks and perspectives.

6.2 Real-life applications

We can explore several examples of quenching phenomena that can be effectively modeled using reaction-diffusion equations. For more in-depth information and comprehensive details about these examples, we can refer to the works of Chenna et *al*. [40], Hussain et *al*. [105], Purich [181], Suckart et *al*. [208], Williams [217], Zhou et *al*. [227], and in the sources mentioned there.

(i) The extinguishing of a chemical flame: When a flame is quenched, it undergoes a rapid transition from a state of combustion to an extinguished state due to the removal of one or more essential components for sustaining the flame, such as fuel or oxygen. This quenching process can be described by a singular reaction-diffusion equation that models the chemical reactions and diffusion of species involved in the combustion process. The equation used to model flame quenching depends on the specific combustion chemistry and the reaction mechanism of the fuel being burned. However, a simplified example can be represented by the reaction-diffusion equation (6.1) with a single species, such as the fuel concentration (denoted as u), and a reaction term that accounts for the combustion reaction. In this case, $-f(u)$ is the reaction term that describes the combustion reaction. The specific form of $f(u)$ depends on the combustion chemistry and can involve nonlinear terms to capture the reaction kinetics. Modeling flame quenching using reaction-diffusion equations aids in understanding the dynamics of combustion processes, optimizing fire safety measures, and designing efficient fire suppression systems.

(ii) The diffusion-limited quenching of reactive oxygen species (ROS) by antioxidants: Reactive oxygen species, such as hydrogen peroxide H_2O_2 and superoxide radicals $O_2^{\bullet -}$, are highly reactive molecules that can cause oxidative damage to cells and tissues. Antioxidants are substances that can neutralize or scavenge these ROS, protecting cells from oxidative stress. The reaction-diffusion equation (6.1) can describe the diffusion of ROS and antioxidants and the quenching process. The quenching effect is modeled by a reaction term that becomes singular when the concentration of ROS is high and the concentration of antioxidants is low. In this case, $-f(u)$ dominates and leads to a decrease in the ROS concentration. This behavior represents the quenching of ROS by antioxidants, where the antioxidants act as scavengers to neutralize the reactive species.

(iii) The enzyme inhibition: Enzymes play a crucial role in biological processes by catalyzing chemical reactions. However, enzymes can be inhibited by various factors, such as inhibitors or regulatory molecules, leading to a decrease in their activity. The reaction-diffusion equation (6.1) can be used to model the spatial distribution of the enzyme concentration and the inhibitor concentration, as well as the quenching effect due to enzyme inhibition. The quenching effect is modeled by a reaction term that becomes singular when the concentration of the inhibitor is high. In this case, the reaction term $-f(u)$ dominates and leads to a decrease in the enzyme activity. This behavior represents the quenching of enzyme activity due to inhibition.

(iv) The quenching in fluorescence resonance energy transfer (FRET) assays: FRET is a widely used technique to study molecular interactions and proximity in biological systems. It relies on the transfer of energy between a donor fluorophore and an acceptor fluorophore, which are typically attached to biomolecules of interest. However, the fluorescence signal can be quenched when the donor and acceptor fluorophores come into close proximity or interact with each other. The reaction-diffusion equation (6.1) can describe the diffusion of the donor and acceptor fluorophores and the quenching process in FRET assays. The quenching effect is modeled by a reaction term that becomes singular when the concentration of the acceptor fluorophore is high. In this case, term $-f(u)$ dominates and leads to a decrease in the fluorescence signal of the donor fluorophore. This behavior represents the quenching of fluorescence in FRET assays, where the proximity or interaction between the donor and acceptor fluorophores results in energy transfer and a reduction in the donor fluorescence intensity.

(v) The diffusion-limited quenching of free radicals by antioxidants: Free radicals, such as reactive oxygen species (ROS), are highly reactive molecules that can cause damage to cellular components and contribute to various diseases. Antioxidants are molecules that can neutralize free radicals by donating an electron, thereby reducing their reactivity and preventing cellular damage. The reaction-diffusion equation (6.1) can describe the diffusion of free radicals and antioxidants and the quenching process. When the concentration of free radicals is sufficiently high and the concentration of antioxidants is low, the reaction term $-f(u)$ dominates and leads to a decrease in the concentration of free radicals. This behavior represents the quenching of free radicals by antioxidants, where the antioxidants act as scavengers to neutralize the reactive species and reduce their harmful effects.

6.3 Statement of main results

The finite-time quenching phenomenon is caused by singular nonlinearity in the absorption term of (6.1). Therefore, we present the concept of a quenched solution to our problem.

6.3.1 Assumptions

Definition 6.1. A solution u of problem (6.1) – (6.2) is said to be quenche if u exists in the classical sense and is positive on $[0,T)$, and also satisfies $\inf\limits_{t\to T}\min\limits_{x\in[0,1]} u(t,x)=0$. In this case, T is called quenching time.

To study problem (6.1) – (6.2), we also assume that the positive function $f:(0,1]\to(0,+\infty)$ satisfies the following two hypotheses (H_1) and (H_2):

(H_1) f is a strictly decreasing and locally Lipschitzian function on $(0,1]$.

(H_2) $\lim\limits_{s\to 0^+} f(s)=+\infty$.

To make our findings more easily understandable, we denote by φ the first eigenfunction associated with the first eigenvalue λ_1 of the problem.

$$\begin{cases} \Delta\varphi + \lambda\varphi = 0 & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

normalized by $\int_\Omega \varphi(x)dx = 1$, with $\varphi(x) > 0$ in Ω.

6.3.2 The main results

An adequate condition for finite-time quenching is provided by the following theorem.

Theorem 6.1. *Under hypotheses $(H_1)-(H_2)$, the solution of problem (6.1)−(6.2) quenches in finite time for any initial data provided that λ_1 is small enough.*

Proof. Let u be the solution of problem (6.1) − (6.2) with the maximal existence time T. By the maximum principle, we have $0 \le u \le 1$ in $(0,T)\times\Omega$. Let

$$\xi(t) = \int_\Omega (1-u)\varphi(x)dx \ , \ t \in [0,T) \tag{6.3}$$

By Assumptions $(H_1) - (H_2)$ and the corresponding Taylor expansions, we get easily

$$f(u) \ge \delta(1-u) + \beta \tag{6.4}$$

where δ and β are positive constants. By a straight-forward computation and (6.4), we obtain

$$\begin{aligned} \frac{d\xi}{dt} &= -\int_\Omega u_t.\varphi(x)dx \\ &= -\int_\Omega \Delta u.\varphi(x)dx + \int_\Omega f(u)\varphi(x)dx \\ &= \int_\Omega \Delta(1-u).\varphi(x)dx + \int_\Omega f(u)\varphi(x)dx \\ &\ge -\lambda_1\int_\Omega (1-u)\varphi(x)dx + \delta\int_\Omega (1-u)\varphi(x)dx + \beta\int_\Omega \varphi(x)dx \\ &= (\delta-\lambda_1)\xi(t) + \beta. \end{aligned}$$

Since $0 \le u \le 1$ then $0 \le 1-u \le 1$ in $(0,T)\times\Omega$. This is confirmed by (6.3) that $0 \le \xi(t) \le 1$. Since λ_1 is small enough, it is obvious that $(\delta-\lambda_1)\xi(t)+\beta > 0$. This gives us

$$\frac{d\xi}{(\delta-\lambda_1)\xi(t)+\beta} \ge dt\ ,\ t\in[0,T),$$

which gives, by integration from 0 to T,

$$T \le \begin{cases} \dfrac{1}{\delta-\lambda_1}\log\left(\dfrac{(\delta-\lambda_1)\xi(T)+\beta}{(\delta-\lambda_1)\xi(0)+\beta}\right) & \text{if } \delta\neq\lambda_1 \\ \dfrac{1}{\beta}(\xi(t)-\xi(0)) & \text{if } \delta=\lambda_1 \end{cases} \tag{6.5}$$

Now, letting $t\to T^-$ in (6.5) and combining $\lim\limits_{t\to T^-}\xi(t)\le 1$, we obtain

$$T \le \begin{cases} \dfrac{1}{\delta-\lambda_1}\log\left(\dfrac{(\delta-\lambda_1)+\beta}{(\delta-\lambda_1)\xi(0)+\beta}\right) & \text{if } \delta\neq\lambda_1 \\ \dfrac{1}{\beta}(1-\xi(0)) & \text{if } \delta=\lambda_1 \end{cases} \tag{6.6}$$

The positivity of the right-hand side of (6.6), which illustrates the finite time quenching of the solutions of problem $(6.1)-(6.2)$, can be readily reached since $0\le\xi(t)\le 1$. This is what is required. ■

Remark 6.1. *Many quenching studies confirm that time-derivatives blow-up while the solution itself remains bounded. We refer, for example, to Chan [36] and Kawarada [113].*

The global existence of solutions can be described by the following theorem.

Theorem 6.2. *If the diameter of Ω is small enough and the initial data satisfies $0<\varepsilon\le u_0\le 1$ in $\overline{\Omega}$, then under hypotheses $(H_1)-(H_2)$, the solution of problem $(6.1)-(6.2)$ does not quench in finite time. In this case, we say that the solution u exists globally.*

Proof. Consider the auxiliary problem

$$\begin{cases} \bar{u}_t = \Delta\bar{u} - f(\bar{u}) & \text{in } (0,T)\times\Omega \\ \bar{u} = 1 & \text{on } (0,T)\times\partial\Omega \\ \bar{u}(0,x) = 1 & \text{in } \overline{\Omega} \end{cases}$$

According to the comparison principle, we have $u \leq \bar{u}$.

Now, we consider the following problem:

$$\begin{cases} \Delta \bar{u}^* = f(1) & \text{in } B_\rho \\ \bar{u}^* = 1 & \text{on } \partial B_\rho \end{cases}$$

with

$$B_\rho = \left\{ x \in \mathbb{R}^N : |x| < \rho \right\} \text{ and } \rho \geq \left(\frac{2N}{f(1)} \right)^{\frac{1}{2}}$$

By Green's function, the solution is $\bar{u}^*$ denoted as follows:

$$\bar{u}^* = \frac{f(1)\left(|x|^2 - \rho^2\right)}{2N} + 1$$

and

$$\min \bar{u}^* = \frac{-f(1)\rho^2}{2N} + 1$$

Clearly, $\bar{u}^*$ is a super solution of (6.1) – (6.2). Therefore, the solution u of (6.1) – (6.2) is global only if $\bar{u}^* > 0$. ■

6.4 Conclusion and perspectives

(i) This research work focuses on understanding and studying quenching phenomena, which involve rapid changes in temperature or physical conditions and can lead to complex phenomena in different systems. The study contributes to a deeper understanding of the mathematical principles underlying quenching processes, advancing research in this area.

(ii) The results obtained in this study have wider implications beyond quenching phenomena alone. They can be applied to the study of other singular reaction-diffusion phenomena. This opens up new avenues for investigating and understanding various physical and chemical processes where similar mathematical principles may be at play.

(iii) The study's findings and insights provide a foundation for further theoretical and numerical investigations under different conditions. This deeper dive into the problem aims to achieve additional progress in the field and expand our knowledge. This deeper dive into the problem aims to push the boundaries of current understanding and potentially uncover new phenomena or mechanisms.

(iv) The primary goal of the research is to contribute to the advancement of quenching technology and modeling in diverse scientific disciplines. Quenching phenomena have practical implications in fields such as bioengineering, biology, and many others. By deepening the understanding of quenching through mathematical research, the study can potentially lead to improved techniques, methodologies, and models that benefit these scientific fields. The advancements may enable better control and optimization of quenching processes, which can have practical applications in various industries.

CHAPTER 7

ON A SINGULAR DEGENERATE REACTION-DIFFUSION MODEL APPLIED TO QUENCHING AND BIOLOGY

In this chapter, we are interested in studying a singular nonlinear degenerate reaction-diffusion model having a natural growth with respect to the gradient. Our approach uses Schauder's fixed point theorem. This type of problem has numerous important applications across multiple disciplines, such as biology, ecology and medicine. By employing rigorous mathematical techniques, we aim to advance the theoretical understanding of this type of nonlinear degenerate reaction-diffusion problems and lay the groundwork for further developments and real-world implementations.

The content of this chapter is based on a research paper [63] titled "On a Singular Degenerate Reaction-Diffusion Model Applied to Quenching and Biology", published in the *Algerian Journal of Sciences*. This paper is the result of a collaborative effort with S. Mesbahi.

7.1 Introduction

Singular degenerated reaction-diffusion systems are a class of partial differential equations that exhibit a unique mathematical structure. These systems are characterized by the presence of a singular or degenerate term in the equation, which can lead to complex behavior and interesting applications. In the context of quenching, singular degenerated reaction-diffusion systems can be used to model phenomena such as the quenching of flames or the extinction of biological populations. The singular term in the equation can represent a critical threshold or a phase transition, where the system's behavior changes dramatically. For example, consider a model for the spread of a fire in a forest. The reaction-diffusion equation can describe the dynamics of the fire, with the singular term representing the critical temperature at which the fire is extinguished. By analyzing the properties of the singular degenerated reaction-diffusion system, researchers can gain insights into the conditions under which the fire will be quenched and the factors that influence the quenching process. Similarly, in the context of biology, singular degenerated reaction-diffusion systems can be used to model the dynamics of biological populations, such as the spread of infectious diseases or the growth of tumors. The singular term in the equation can represent a critical population density or a phase transition in the system's behavior, such as the onset of extinction or the transition to a state of uncontrolled growth.

The analysis of singular degenerated reaction-diffusion systems often involves techniques from nonlinear analysis, such as the study of free boundaries, the analysis of steady-state solutions, and the investigation of the asymptotic behavior of the solutions. These systems can exhibit rich and complex dynamics, including the formation of patterns, the occurrence of bifurcations, and the existence of multiple stable states. The applications of singular degenerated reaction-diffusion systems span various fields, including ecology, epidemiology, materials science, and even social sciences. By understanding the mathematical properties of these systems and their connection to real-world phenomena, researchers can develop more accurate models and gain insights that inform practical applications. We find numerous real applications in biology, medicine and ecology in the works of Mesbahi et *al*. Mesbahi et *al*. [62, 64, 140, 143, 155], and also in DiBendetto [60], Murray [161, 162] and corresponding references therein.

The problem we will study here fits into this context, and we find that it has many important applications in biology, medicine, the environment, and many other interesting scientific fields. The analysis of these systems often involves advanced mathematical techniques. By understanding the rich and complex dynamics exhibited by these systems, researchers can develop more accurate models that provide valuable insights in various scientific domains. We are interested in the following nonlinear singular degenerated reaction-diffusion system having natural growth with respect to the gradient

$$
\begin{cases}
\dfrac{\partial u}{\partial t} - \mathbf{div}\ (a(t,x)\nabla u) + d(t,x)\dfrac{|\nabla u|^p}{u^\gamma} = f(t,x) & \text{in } Q_T \\
u(t,x) = 0 & \text{on } \Gamma \\
u(0,x) = u_0(x) & \text{in } \Omega
\end{cases}
\tag{7.1}
$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N > 2$, and Q_T is the cylinder $(0,T)\times\Omega$, $T > 0$, $\Gamma = (0,T)\times\partial\Omega$, $2 < p < N$, $0 < \gamma < 1$, $a(t,x)$ and $d(t,x)$ are two bounded measurable functions satisfying

$$
0 < \alpha_1 \leq a(t,x) \leq \alpha_2 \tag{7.2}
$$

$$
0 < \beta_1 \leq d(t,x) \leq \beta_2 \tag{7.3}
$$

where α_1, α_2, β_1 and β_2 are fixed real numbers such that $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$. On the function f, we assume that it is non-negative and not identically zero, and that it belongs to the Lebesgue space $L^m(Q_T)$ with $m > 1$. Moreover, the initial data $u_0 \in L^\infty(\Omega)$ satisfies the following condition of strict positivity

$$
\exists D_\omega > 0, \forall \omega \subset\subset \Omega : u_0 \geq D_\omega
$$

We find a detailed history of this problem and its applications in numerous areas in Benkirane et *al*. [31], Boccardo et *al*. [33], Dall'Aglio et *al*. [53–55], De Bonis and Giachetti [58], El Hadfi et *al*. [70], El Ouardy and El Hadfi [71], Keller and Choen [116], Magliocca [136], Martínez-Aparicio and Petitta [138], Nachman and Callegari [165], Youssfi et *al*. [219] and references therein.

The problem described in (7.1) presents several key difficulties that must be addressed. Firstly, the presence of the lower-order term introduces complications. The natural growth term in the equation depends on the gradient, which adds mathematical complexity. Additionally, the singularity in the equation depends on the variable u, further complicating the analysis. Perhaps most challenging is the need to prove the strict positivity of the solution within the interior of the parabolic cylinder. Establishing this positivity property is a non-trivial task. To overcome these difficulties, we must approximate the singular problem (7.1) by another non-singular one, and we show that this problem admits a non-negative solution (the proof is based on the application of Schauder's fixed point theorem) and that this solution is strictly positive in the interior of the parabolic cylinder (the proof is based on the use of the intrinsic Harnack inequality).

Below, we will use $|E|$ to denote the Lebesgue measure of a subset $E \subset \mathbb{R}^N$. The Hölder conjugate exponent of $q > 1$ is $q' = \frac{q}{q-1}$, while the Sobolev conjugate exponent of p for $1 \le p < N$ is $\frac{Np}{N-p}$. For a fixed $k > 0$, we define the truncation function T_k as follows:

$$T_k(s) = \max\{-k, \min\{s, k\}\}$$

To streamline notation, we will use C to represent values that may vary from step to step or even within the same step, depending on some parameters. However, C will remain constant with respect to the indices of any sequences introduced.

7.2 Statement of the main result

First, we have to clarify in which sense we want to solve our problem.

Definition 7.1. A weak solution to problem (7.1) is a function $u \in L^1\left(0,T;W_0^{1,1}(\Omega)\right)$ such that for every $\omega \subset\subset \Omega$ there exists c_ω such that $u \ge c_\omega > 0$ in $(0,T)\times\omega$, $\left(a(t,x)|\nabla u|^{p-1}\right) \in L^1(Q_T)$, $\dfrac{|\nabla u|^p}{u^\gamma} \in L^1\left(0,T;L^1_{loc}(\Omega)\right)$. Furthermore, we have that

$$-\int_{Q_T} u\frac{\partial\phi}{\partial t}dtdx + \int_{Q_T} a(t,x)\nabla u.\nabla\phi dtdx$$
$$+\int_{Q_T} d(t,x)\frac{|\nabla u|^p}{u^\gamma}\phi dtdx = \int_{Q_T} f\phi dtdx + \int_\Omega u_0(x)\phi(0,x)dx$$

for every $\phi \in C_c^1([0,T)\times\Omega)$.

Now, we can state the main result of this chapter, it is the following theorem.

Theorem 7.1. *Let* $0<\gamma<1$, $\lambda = \dfrac{p(N+1+\delta)}{p(N+1+\delta)-N\delta}$ *and* $\delta = \min\{\gamma, 1-q\}$. *Assume that* a *satisfy* (7.2), d *satisfy* (7.3) *and* $f \in L^m(Q_T)$ *with* $1 < m < \frac{N}{p+1}$. *Then there exists a solution* u *of problem* (7.1) *in the sense of Definition 7.1 verify the following regularity:*

(i) If $\lambda \le m < \dfrac{N}{p+1}$, *then* $u \in L^p\left(0,T;W_0^{1,p}(\Omega)\right) \cap L^\sigma(Q_T)$ *with*

$$\sigma = m\frac{N(p-\delta)+p}{N-pm+p}$$

(ii) If $1 < m < \lambda$, *then* $u \in L^s\left(0,T;W_0^{1,s}(\Omega)\right) \cap L^\sigma(Q_T)$ *with*

$$s = m\frac{N(p-\delta)+p}{N+1-\delta(m-1)}$$

7.3 Approximating Scheme

Let $0<\varepsilon<1$. We approximate problem (7.1) by the following nonlinear and non-singular problem

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t}-\textbf{div}\,(a(t,x)\nabla u_\varepsilon)+d(t,x)\dfrac{u_\varepsilon|\nabla u_\varepsilon|^p}{(\varepsilon+|u_\varepsilon|)^{\gamma+1}}=f_\varepsilon(t,x) & \text{in } Q_T \\ u_\varepsilon(t,x)=0 & \text{on } \Gamma \\ u_\varepsilon(0,x)=u_{\varepsilon 0}(x) & \text{in } \Omega \end{cases} \tag{7.4}$$

where

$$f_\varepsilon=\frac{f}{1+\varepsilon f} \quad \text{and } f_\varepsilon\in L^\infty(Q_T)$$

such that

$$\|f_\varepsilon\|_{L^m(Q_T)}\le\|f\|_{L^m(Q_T)} \quad \text{and} \quad f_\varepsilon\to f \text{ strongly in } L^m(Q_T),\ m>1 \tag{7.5}$$

and

$$u_{\varepsilon 0}(x)=\frac{u_0(x)}{1+\varepsilon u_0(x)}\in L^\infty(\Omega)$$

such that

$$\|u_{\varepsilon 0}\|_{L^\infty(\Omega)}\le\|u_0\|_{L^\infty(\Omega)} \quad \text{and } u_{\varepsilon 0}\to u_0 \text{ strongly in } L^1(\Omega)$$

Problem (7.4) admits weak solutions $u_\varepsilon\in L^p\left(0,T;W_0^{1,p}(\Omega)\right)\cap L^\infty(Q_T)$, as shown in references Dall'Aglio and Orsina [56] and Lions [128]. Additionally, the solution of problem (7.4) is continuous in time, meaning $u_\varepsilon\in C\left([0,T];L^1_{loc}(\Omega)\right)$. Since the right side of (7.4) is non-negative, u_ε is also non-negative.

Lemma 7.1. *Let u_ε be solutions to problem (7.4). Then*

$$\int_{Q_T} d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(\varepsilon+u_\varepsilon)^{\gamma+1}}\le|Q_T|^{1-\frac{1}{m}}\|f\|_{L^m(Q_T)}+\|u_0\|_{L^1(\Omega)}$$

Proof. See DiBendetto [60]. ■

Remark 7.1. *According to Lemma 7.1, and since*

$$\int_{Q_T} d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(\varepsilon+u_\varepsilon)^{\gamma+1}}\ge 0 \quad \textit{and} \quad f\in L^1(Q_T)$$

one has that

$$\int_{Q_T}\left|d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(\varepsilon+u_\varepsilon)^{\gamma+1}}-f\right|\le 2|Q_T|^{1-\frac{1}{m}}\|f\|_{L^m(Q_T)}+\|u_0\|_{L^1(\Omega)}<C$$

Lemma 7.2. *Let the assumptions of Theorem 7.1 be in force. Then the solution u_ε of (7.4) satisfy the following estimate:*

(i) If $\lambda \leq m < \dfrac{N}{p+1}$, then u_ε is uniformly bounded in the space

$$L^p\left(0,T;W_0^{1,p}(\Omega)\right) \cap L^\sigma(Q_T)$$

(ii) If $1 < m < \lambda$, then u_ε is uniformly bounded in the space

$$L^s\left(0,T;W_0^{1,s}(\Omega)\right) \cap L^\sigma(Q_T)$$

where s and σ are defined in Theorem 7.1.

7.4 Proof of the main result

Now we can prove Theorem 7.1.

Proof of Theorem 7.1 In view of Lemma 7.2,

$$u_\varepsilon \rightharpoonup u \text{ weakly in } L^\delta\left(0,T;W_0^{1,\delta}(\Omega)\right),\ \forall \delta < s < p \text{ and a.e. in } Q_T$$

By Remark 7.1, $f_\varepsilon - d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} \in L^1(Q_T)$ and from Lemma 7.2, we have $a(t,x)\nabla u_\varepsilon$ is bounded in $L^\rho(Q_T)$ for all $1 \leq \rho < \frac{s}{p-1} < p$. Then **div** $(a(t,x)\nabla u_\varepsilon)$ is bounded in the space $L^{\rho'}(Q_T) \subset L^{p'}(Q_T) \subset L^{p'}\left(0,T;W_0^{-1,p'}(\Omega)\right)$, and then $\frac{\partial u_\varepsilon}{\partial t}$ is bounded in the space $L^{p'}\left(0,T;W_0^{-1,p'}(\Omega)\right) + L^1(Q_T)$. Using the compactness results in Simon [202], we obtain

$$u_\varepsilon \to u \text{ strongly in } L^1(Q_T) \text{ and a.e. in } Q_T. \tag{7.6}$$

We can use the same proof as in Abdellaoui and Redwane [1], we obtain

$$T_k(u_\varepsilon) \to T_k(u) \text{ strongly in } L^p\left(0,T;W_0^{1,p}(\Omega)\right) \tag{7.7}$$

and also we have

$$\nabla u_\varepsilon \to \nabla u \text{ in a.e. in } Q_T \tag{7.8}$$

On the other hand, recalling (7.2), (7.6), (7.8), Lemma 7.2 and the dominated convergence theorem implies that the sequence $a(t,x)\nabla u_\varepsilon$ converges weakly to $a(t,x)\nabla u$ in $L^\rho(Q_T)$ for every $1 \le \rho < \frac{s}{p-1}$. Therefore, for every $\varphi \in C_c^1(\Omega \times [0,T))$,

$$\lim_{\varepsilon \to 0} \int_Q a(t,x)\nabla u_\varepsilon \nabla\varphi = \int_Q a(t,x)\nabla u \nabla\varphi \tag{7.9}$$

Now we prove that

$$d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} \to d(t,x)\frac{|\nabla u|^p}{u^\gamma} \quad \text{, strongly locally in } L^1(Q_T)$$

For any measurable compact subset E of Q_T, we have

$$\begin{aligned}\int_E d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} &= \int_{E\cap\{u_\varepsilon\le k\}} d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} \\ &\quad + \int_{E\cap\{u_\varepsilon> k\}} d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} \\ &\le \int_{E\cap\{u_\varepsilon\le k\}} d(t,x)\frac{|\nabla u_\varepsilon|^p}{u_\varepsilon^\gamma} \\ &\quad + \int_{E\cap\{u_\varepsilon> k\}} d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}}\end{aligned}$$

We get

$$\int_E d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} \le \frac{1}{c_\varepsilon^\gamma}\int_E d(t,x)|\nabla T_k(u_\varepsilon)|^p + \int_{E\cap\{u_\varepsilon> k\}} d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}}$$

Let $\nu > 0$ be fixed. For $k > 1$, we use $T_1(u_\varepsilon - T_{k-1}(u_\varepsilon))$ as a test function in (7.4), yielding

$$\begin{aligned}&\int_0^T\int_\Omega \frac{\partial u_\varepsilon}{\partial t}T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) + \int_{Q_T} a(t,x)\nabla u_\varepsilon T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) \\ &+ \int_{Q_T} d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}}T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) \\ = &\int_{Q_T} f_\varepsilon T_1(u_\varepsilon - T_{k-1}(u_\varepsilon))\end{aligned}$$

Recalling (7.2) and the fact $u_\varepsilon \ge 0$, we can write

$$\begin{aligned}&\int_\Omega S_1(u_\varepsilon(T)) + \alpha_1\int_{\{k-1\le u_\varepsilon\le k\}} |\nabla u_\varepsilon|^p \\ &+ \int_{Q_T} d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}}T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) \\ \le &\int_{Q_T} f_\varepsilon T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) + \int_{Q_T} S_1(u_0),\end{aligned}$$

where

$$S_1(u_\varepsilon(T)) = \int_0^{u_\varepsilon(T)} T_1(s - T_{k-1}(s))\,ds.$$

It is easy to see that $S_1(u_\varepsilon(T)) \geq 0$ a.e. in Ω. After the first and second non-negative terms of the previous inequality are removed, we arrive at

$$\begin{aligned} & \int_{Q_T} d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) \\ \leq & \int_{Q_T} f_\varepsilon T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) + \int_{Q_T} S_1(u_0) \\ = & \int_{Q_T} f_\varepsilon T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) \int_\Omega \int_0^{u_0} T_1(s - T_{k-1}(s))\,ds. \end{aligned} \tag{7.10}$$

Since $T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) \geq 0$,

$$T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) = \begin{cases} 0 & \text{if } u_\varepsilon \leq k-1 \\ 1 & \text{if } u_\varepsilon > k \end{cases}$$

recalling the condition (7.3) and the fact that $u_\varepsilon > 0$, we have

$$\begin{aligned} & \int_{Q_T} d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) \\ = & \int_{Q_T\cap\{u_\varepsilon>k\}} d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) \\ & + \int_{Q_T\cap\{u_\varepsilon\leq k\}} d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) \\ = & \int_{Q_T\cap\{u_\varepsilon>k\}} d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} \\ & + \int_{Q\cap\{u_\varepsilon\leq k\}} d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) \\ \geq & \int_{E\cap\{u_\varepsilon>k\}} d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} \end{aligned}$$

and

$$\begin{aligned}
&\int_{Q_T} f_\varepsilon T_1(u_\varepsilon - T_{k-1}(u_\varepsilon))\\
=\ &\int_{Q_T\cap\{u_\varepsilon\le k-1\}} f_\varepsilon T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) + \int_{Q_T\cap\{k-1<u_\varepsilon\le k\}} f_\varepsilon T_1(u_\varepsilon - T_{k-1}(u_\varepsilon))\\
&+\int_{Q\cap\{u_\varepsilon>k\}} f_\varepsilon T_1(u_\varepsilon - T_{k-1}(u_\varepsilon))\\
=\ &\int_{Q_T\cap\{k-1<u_\varepsilon\le k\}} f_\varepsilon T_1(u_\varepsilon - T_{k-1}(u_\varepsilon)) + \int_{Q_T\cap\{u_\varepsilon>k\}} f_\varepsilon\\
=\ &\int_{Q_T\cap\{u_\varepsilon\le k-1\}} f + \int_{Q_T\cap\{k-1<u_\varepsilon\le k\}} f + \int_{Q_T\cap\{u_\varepsilon>k\}} f
\end{aligned}$$

also we have

$$\int_\Omega S_1(u_0) = \int_\Omega \int_0^{u_0} T_1(s - T_{k-1}(s))ds = \int_\Omega \int_{[0,u_0]\cap\{s>k-1\}} T_1(s - T_{k-1}(s))ds$$

Therefore, from (7.10) combined with the two later inequalities and the above equality, we obtain

$$\begin{aligned}
&\int_{E\cap\{u_\varepsilon>k\}} d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}}\\
\le\ &\int_{Q\cap\{u_\varepsilon\ge k-1\}} f + \int_E \int_{[0,u_0]\cap\{s>k-1\}} T_1(s - T_{k-1}(s))ds
\end{aligned}$$

It follows from $f\in L^m(Q)$ and $T_1(s-T_{k-1}(s))\in L^1(\Omega)$ that

$$\int_{E\cap\{u_\varepsilon>k\}} d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} \to 0 \text{ as } k\to\infty$$

Then, there exists $k_0>1$ such that

$$\int_{E\cap\{u_\varepsilon>k\}} d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} \le \frac{\nu}{2},\ \forall k>k_0,\ \forall\varepsilon\in(0,T) \tag{7.11}$$

Since from (7.7) $(T_k(u_\varepsilon)\to T_k(u))$ strongly in $L^p\left(0,T;W_0^{1,p}(\Omega)\right)$, then there exists ε_ν, θ_ν such that $|E|\le\theta_\nu$, and we have

$$\frac{1}{c_\varepsilon^\gamma}\int_E d(t,x)|\nabla T_k(u_\varepsilon)|^p \le \frac{\nu}{2},\ \forall\varepsilon\le\varepsilon_\nu \tag{7.12}$$

The estimates (7.11) and (7.12) imply that $d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}}$ is equi-integrable. This fact, together with the a.e. convergence of this term to $d(t,x)\frac{|\nabla u|^p}{u^\gamma}$, implies by the Vitali

Theorem that

$$d(t,x)\frac{u_\varepsilon|\nabla u_\varepsilon|^p}{(u_\varepsilon+\varepsilon)^{\gamma+1}} \to d(t,x)\frac{|\nabla u|^p}{u^\gamma}\text{ , strongly locally in } L^1(Q_T) \tag{7.13}$$

Let $\varphi \in C_c^1(\Omega\times[0,T))$, taking φ test function in problem (7.4), by (7.5), (7.6), (7.9) and (7.13), we can let $\varepsilon \to 0$ yielding

$$\begin{aligned} &-\int_{Q_T} u\frac{\partial\varphi}{\partial t} + \int_{Q_T} a(t,x)\nabla u\nabla\varphi + \int_{Q_T} d(t,x)\frac{|\nabla u|^p}{u^\gamma}\varphi \\ = &\int_{Q_T} f\varphi + \int_\Omega u_0(x)\varphi(x,0) \end{aligned}$$

Thus, Theorem 7.1 is proved. ■

CHAPTER

8

EXISTENCE RESULT OF CONTINUOUS POSITIVE SOLUTIONS FOR A REACTION-DIFFUSION SYSTEM

TTis chapter mainly seeks to contribute to the study of a quasilinear parabolic reaction-diffusion system of arbitrary order with initial conditions. Using potential analysis techniques, we establish the existence of continuous positive solutions. The originality lies in the fact that our system is of arbitrary order, which represents a significant advancement as our main result extends and generalizes previous findings.

The content of this chapter was published in a research paper [182] entitled “Existence result of continuous positive solutions for a reaction-diffusion system” in the journal *Partial Differential Equations in Applied Mathematics* in collaboration with S. Mesbahi and M. Redjouh.

8.1 Introduction

Our research aims to investigate a specific class of reaction-diffusion systems that has captured considerable attention from numerous researchers, thus underscoring their importance and relevance. In this chapter, we will examine a model that possesses extensive applications across disciplines such as biology, ecology, medicine, bioengineering, biochemistry, physics, and various other fields of knowledge. Notably, we can find numerous models and references related to these applications in the works of Abdellaoui [2], Alqahtani [17], Ghanmi et *al*. [87, 88], Gontara et *al*. [92, 93], Haq et *al*. [101], Lair and Wood [124], Mâagli et *al*. [130, 131], Murray [161, 162], Nisar et *al*. [167], Yadav et *al*. [218], Zhang [222, 223], and the corresponding references therein. These sources also provide valuable mathematical techniques that we employ in addressing similar problems. The originality of our study lies in the system we intend to investigate, which possesses arbitrary order. This distinctive feature allows our main result to surpass the conclusions drawn from previous studies conducted on models with only one or two equations. The difficulties we encounter are similar to those in [92], [130], [131], [222] and [223]. However, our study introduces additional obstacles arising from the unique characteristics and underlying assumptions of the model. Nevertheless, we are confident of successfully overcoming these obstacles.

In this study, we will employ some potential analysis techniques. For further insights and information in this field, we recommend referring to Armitage and Gardiner [21]. A range of effective methods for investigating such problems can be found in the works of Aibinu et *al*. [7], Kleefeld et *al*. [118], Kumar and Nisar [122], Mesbahi et *al*. [143, 145, 146, 155, 159], Osman [169], Reisch and Langemann [186], Williams [216], Zhou and Wei [226], and the references provided therein.

We will therefore be interested in studying the existence of positive solutions to the following reaction-diffusion system:

$$\begin{cases} -\dfrac{\partial u_i}{\partial t} + \Delta u_i = \lambda_i p_{i+1}(x,t) f_{i+1}(u_{i+1}) & ,\ 1 \le i \le 2m-1 \\ -\dfrac{\partial u_{2m}}{\partial t} + \Delta u_{2m} = \lambda_{2m} p_1(x,t) f_1(u_1) & \\ u_i(x,0) = \varphi_i(x),\ x \in \mathbb{R}^n & ,\ 1 \le i \le 2m \end{cases} \tag{8.1}$$

where $(x,t) \in \mathbb{R}^n \times (0,\infty)$, $n \ge 3$, and for $1 \le i \le 2m$, $m \in \mathbb{N}^*$; the functions $\varphi_i : \mathbb{R}^n \to [0,\infty)$ are continuous, the constants λ_i are nonnegative, the functions $f_i : (0,\infty) \to [0,\infty)$ are continuous and nondecreasing, $p_i : \mathbb{R}^n \times (0,\infty) \to [0,\infty)$ refers to measurable functions that fulfill a relevant assumption connected to the parabolic Kato class $\mathbf{P}^\infty(\mathbb{R}^n)$ introduced in Zhang [223], see Appendix B. They refer to a class of functions that satisfy certain growth and regularity conditions in the context of parabolic equations. It is proposed as a natural generalization of the Kato class in the study of elliptic equations. Specifically, these functions typically have controlled growth rates and possess sufficient smoothness properties that allow for the rigorous analysis of parabolic problems.

Definition 8.1. (See [223]) A Borel measurable function q in $\mathbb{R}^{n+1}$ belongs to the Kato class $\mathbf{P}^{\infty}(\mathbb{R}^n)$ if for all $c>0$,

$$\lim_{\epsilon\to 0}\sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}}\int_{t-\epsilon}^{t+\epsilon}\int_{B(x,\sqrt{\epsilon})}G_c(x,|t-s|,y,0)|q(y,s)|dyds=0$$

and

$$\sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}}\int_{-\infty}^{+\infty}\int_{\mathbb{R}^n}G_c(x,|t-s|,y,0)|q(y,s)|dyds<\infty$$

where

$$G_c(x,t,y,s)=\frac{1}{(t-s)^{\frac{n}{2}}}\exp(-c\frac{|x-y|^2}{t-s})\quad\text{for } t>s\ \text{ and } x,\ y\in\mathbb{R}^n$$

Throughout this chapter, G denotes the fundamental solution of the heat equation $\Delta u-\dfrac{\partial u}{\partial t}=0$ in $\mathbb{R}^n\times(0,\infty)$, defined by

$$G(x,t,y,s)=\frac{1}{[4\pi(t-s)]^{\frac{n}{2}}}\exp\left(-\frac{|x-y|^2}{4(t-s)}\right)\ \text{for all } t>s \text{ and } x,y\in\mathbb{R}^n$$

The chapter is structured as follows: In the next section, we lay out the assumptions that form the foundation of our problem and state the main result. Following that, we present essential technical results that will serve as the basis for our subsequent analyses. The fourth section is devoted to the proof of the main result by using a potential theory approach that we deem appropriate. Our approach towards our goal is to prove the convergence of the approximate problem to some solution of problem (8.1), aligning with our ultimate goal. Lastly, we conclude the chapter by summarizing our findings and proposing potential directions for future research.

8.2 Statement of the main result

8.2.1 Assumptions

Below, $(P_t)_{t>0}$ denote the Gauss semigroup defined for all nonnegative measurable function Φ on $\mathbb{R}^n$ by

$$P_t\Phi(x)=\int_{\mathbb{R}^n}G(x,t,y,0)\Phi(y)dy,\ t>0,\ x\in\mathbb{R}^n.$$

A function ω is said to satisfy the condition (H_0) if it is nonnegative superharmonic, locally bounded in $\mathbb{R}^n$ and the mapping $(x,t)\mapsto P\omega(x,t)$ is continuous in $\mathbb{R}^n\times(0,\infty)$, with $P\Phi(x,t)=P_t\Phi(x)$.

Remark 8.1. *(i) The mapping $t \mapsto P_t\omega$ is nonincreasing.*

(ii) The mapping $(x,t) \mapsto P_t\Phi(x)$ is lower semicontinuous on $\mathbb{R}^n \times (0,\infty)$ and it becomes continuous if Φ is furthermore bounded.

(iii) All Superharmonic bounded functions in $\mathbb{R}^n$ satisfy (H_0).

Now, we fix $2m$ nonnegative superharmonic functions θ_i $(1 \le i \le 2m)$ satisfying condition (H_0) and we make the following assumptions for all $1 \le i \le 2m$:

(H_1) There exist $2m$ real constants $c_i > 1$ such that for all $1 \le i \le 2m$ and $x \in \mathbb{R}^n$, we have

$$\frac{1}{c_i}\theta_i(x) \le \varphi_i(x) \le c_i\theta_i(x) \text{ and } \lim_{t\to 0} P_t\varphi_i(x) = \varphi_i(x)$$

(H_2) The functions $f_i : (0,\infty) \to [0,\infty)$ $(1 \le i \le 2m)$ are continuous and nondecreasing.

(H_3) p_i $(1 \le i \le 2m)$ are nonnegative measurable functions and

$$\begin{cases} (\tilde{p}_{i+1})_c = \dfrac{p_{i+1}f_{i+1}(cP\theta_{i+1})}{P\theta_i}, \ 1 \le i \le 2m-1 \\ (\tilde{p}_1)_c = \dfrac{p_1 f_1(cP\theta_1)}{P\theta_{2m}} \end{cases} \quad \text{for all } c > 0$$

are elements of Kato class $\mathbf{P}^\infty(\mathbb{R}^n)$.

Another important assumption that we must add concerning the functions p_i, is the fixing of $2m$ superharmonic functions θ_i on $\mathbb{R}^n$ satisfying condition (H_0).

8.2.2 The main result

The main result of this chapter can be summarized by the following theorem:

Theorem 8.1. *Under assumptions $(H_1)-(H_3)$, there exist $2m$ constants λ_i^0 $(1 \le i \le 2m)$ such that for all $\lambda_i \in [0,\lambda_i^0)$ $(1 \le i \le 2m)$ problem (8.1) has a positive continuous solution $(u_1,\dots,u_{2m})$ in $(\mathbb{R}^n \times (0,\infty))^{2m}$ satisfying for all $t > 0$ and $x \in \mathbb{R}^n$*

$$(1-\frac{\lambda_i}{\lambda_i^0})P\varphi_i(x,t) \le u_i(x,t) \le P\varphi_i(x,t)\ ,\ 1 \le i \le 2m \tag{8.2}$$

8.3 Preliminary results

In this section, we will showcase a set of results derived from Mâagli et *al*. [131] and Zhang [222, 223], which we will employ as evidence to establish our main result. In [222], Zhang studied the scalar problem:

$$\begin{cases} \Delta u - \dfrac{\partial u}{\partial t} = q(x,t)u^{p+1} & , (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x) & , x \in \mathbb{R}^n,\ n \geq 3, \end{cases} \tag{8.3}$$

and arrived at the following result:

Theorem 8.2. *(See [222]) Assume $p > 0$, $q \in \mathbf{P}^\infty(\mathbb{R}^n)$. For any $M > 1$, there is a constant $b_0 > 0$ such that for each nonnegative $u_o \in C^2(\mathbb{R}^n)$ satisfying $\|u_0\|_{L^\infty(\mathbb{R}^n)} \leq b_0$, there exists a positive and continuous solution u of (8.3) such that*

$$M^{-1}\int_{\mathbb{R}^n} G(x,t,y,0)u_0(y)dy \leq u(x,t) \leq M\int_{\mathbb{R}^n} G(x,t,y,0)u_0(y)dy$$

for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$.

Proposition 8.1. *(See [131]) (i) $L^\infty(\mathbb{R}^n) \otimes L^1(\mathbb{R}) \subset \mathbf{P}^\infty(\mathbb{R}^n)$.*

(ii) Let $1 \leq p < \infty$ and $q \geq 1$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then for $\sigma > \dfrac{np}{2}$ and $\tau < \dfrac{2}{p} - \dfrac{n}{\sigma} < v$, we have

$$\frac{L^\sigma(\mathbb{R}^n)}{|\cdot|^\tau (1+|\cdot|)^{v-\tau}} \otimes L^q(\mathbb{R}) \subset \mathbf{P}^\infty(\mathbb{R}^n)$$

(iii) $\mathbf{P}^\infty(\mathbb{R}^n) \subset L^1_{loc}(\mathbb{R}^{n+1})$.

We define for all measurable function Φ on $\mathbb{R}^n \times (0,\infty)$, the potential

$$V\Phi(x,t) = \int_0^t \int_{\mathbb{R}^n} G(x,t,y,s)\Phi(y,s)dyds = \int_0^t P_{t-s}(\Phi(.,s))(x)ds$$

Proposition 8.2. *(See [131]) Let q be a nonnegative function in $\mathbf{P}^\infty(\mathbb{R}^n)$, then there exists a positive constant α_q such that for each superharmonic function v in $\mathbb{R}^n$, we have*

$$V(qPv)(x,t) \leq \alpha_q Pv(x,t) \ \text{ for } (x,t) \in \mathbb{R}^n \times (0,\infty)$$

Proposition 8.3. *(See [131]) Let v be a superharmonic function in $\mathbb{R}^n$ satisfying (H_0) and q be a nonnegative function in $\mathbf{P}^\infty(\mathbb{R}^n)$. Then the family of functions*

$$\left\{(x,t)\to Vf(x,t)=\int_0^t\int_{\mathbb{R}^n}G(x,t,y,s)f(y,s)dyds,\ |f|\le qPv\right\}$$

is equicontinuous in $\mathbb{R}^n\times[0,\infty)$. Moreover, for all $x\in\mathbb{R}^n$ we have $\lim\limits_{t\to0}Vf(x,t)=0$ uniformly on f.

Now, we can prove the following important proposition:

Proposition 8.4. *Under assumption (H_1), the functions $P\varphi_i$ $(1\le i\le 2m)$ are continuous in $\mathbb{R}^n\times(0,\infty)$.*

Proof. Let c_i $(1\le i\le 2m)$ be the constant given in (H_1). We write for all $t>0$ and $x\in\mathbb{R}^n$

$$c_iP_t\theta_i(x)=P_t(c_i\theta_i-\varphi_i)(x)+P_t\varphi_i(x),\ 1\le i\le 2m$$

Based on the initial assumption (H_0), the mappings $(x,t)\mapsto P\theta_i(x,t)$ $(1\le i\le 2m)$ are continuous in $\mathbb{R}^n\times(0,\infty)$. Additionally, considering the functions $(x,t)\mapsto P_t(c_i\theta_i-\varphi_i)(x)$ $(1\le i\le 2m)$ and $(x,t)\mapsto P_t\varphi_i(x)$ $(1\le i\le 2m)$, we observe that they exhibit lower semicontinuity in $\mathbb{R}^n\times(0,\infty)$. Consequently, we can conclude that the mappings $(x,t)\mapsto P_t\varphi_i(x)$ $(1\le i\le 2m)$ are continuous in $\mathbb{R}^n\times(0,\infty)$. ■

8.4 Proof of the main result

Let the following quantities:

$$\text{(8.4)}\qquad\left\{\begin{array}{ll}\lambda_i^0=\inf\limits_{(x,t)\in\mathbb{R}^n\times(0,\infty)}\dfrac{P\varphi_i}{V(p_{i+1}f_{i+1}(P\varphi_{i+1}))}, & 1\le i\le 2m-1\\ \lambda_{2m}^0=\inf\limits_{(x,t)\in\mathbb{R}^n\times(0,\infty)}\dfrac{P\varphi_{2m}}{V(p_1f_1(P\varphi_1))} & \end{array}\right.$$

Proposition 8.5. *Suppose that assumptions $(H_1)-(H_3)$ are satisfied, then the constants λ_i^0 $(1\le i\le 2m)$ defined by (8.4) are positive.*

Proof. Assumption (H_1) implies that $\varphi_i \le c_i\theta_i$ $(1 \le i \le 2m)$. From the fact that f_i $(1 \le i \le 2m)$ are nondecreasing and p_i $(1 \le i \le 2m)$ are nonnegative, we have then

$$V(p_i f_i(P\varphi_i)) \le V(p_i f_i(c_i P\theta_i)),\ 1 \le i \le 2m.$$

Therefore, based on assumption (H_3) and the stated Proposition 8.2, there exist $(\tilde{p}_i)_{c_i} \in \mathbf{P}^{\infty}(\mathbb{R}^n)$ $(1 \le i \le 2m)$ and a positive constants $\alpha_{(\tilde{p}_i)_{c_i}}$ $(1 \le i \le 2m)$ such that for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$ and $1 \le i \le 2m-1$, we obtain

$$V(p_{i+1}f_{i+1}(P\varphi_{i+1}))(x,t) \le V((\tilde{p}_{i+1})_{c_{i+1}}P\theta_i)(x,t) \le \alpha_{(\tilde{p}_{i+1})_{c_{i+1}}}P\theta_i(x,t)$$

and

$$V(p_1 f_1(P\varphi_1))(x,t) \le V((\tilde{p}_1)_{c_1}P\theta_{2m})(x,t) \le \alpha_{(\tilde{p}_1)_{c_1}}P\theta_{2m}(x,t).$$

Using again (H_1), we find for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$ and $1 \le i \le 2m-1$

$$\begin{cases} \dfrac{P\varphi_i(x,t)}{V(p_{i+1}f_{i+1}(P\varphi_{i+1}))(x,t)} \ge \dfrac{\dfrac{1}{c_i}P\theta_i(x,t)}{\alpha_{(\tilde{p}_{i+1})_{c_{i+1}}}P\theta_i(x,t)} = \dfrac{1}{c_i\alpha_{(\tilde{p}_{i+1})_{c_{i+1}}}} > 0 \\ \dfrac{P\varphi_{2m}(x,t)}{V(p_1 f_1(P\varphi_1))(x,t)} \ge \dfrac{\dfrac{1}{c_{2m}}P\theta_{2m}(x,t)}{\alpha_{(\tilde{p}_1)_{c_1}}P\theta_{2m}(x,t)} = \dfrac{1}{c_{2m}\alpha_{(\tilde{p}_1)_{c_1}}} > 0, \end{cases}$$

which implies that $\lambda_i^0 > 0$ for all $1 \le i \le 2m$. ■

Proof of Theorem 8.1 Let $\lambda_i \in [0, \lambda_i^0)$ $(1 \le i \le 2m)$. We define the sequences $(u_k^i)_{k \ge 0}$ (where i is an index, not power) by

$$\begin{cases} u_0^{2i} = P\varphi_{2i} & ,\ 1 \le i \le m \\ u_k^{2i-1} = P\varphi_{2i-1} - \lambda_{2i-1}V(p_{2i}f_{2i}(u_k^{2i})) & ,\ 1 \le i \le m \\ u_{k+1}^{2i} = P\varphi_{2i} - \lambda_{2i}V(p_{2i+1}f_{2i+1}(u_k^{2i+1})) & ,\ 1 \le i \le m-1 \\ u_{k+1}^{2m} = P\varphi_{2m} - \lambda_{2m}V(p_1 f_1(u_k^1)) & \end{cases}$$

We are determined to prove for all $k \in \mathbb{N}$ and $1 \le i \le m$

(8.5) $$0 < \left(1 - \frac{\lambda_{2i-1}}{\lambda_{2i-1}^0}\right)P\varphi_{2i-1} \le u_k^{2i-1} \le u_{k+1}^{2i-1} \le P\varphi_{2i-1}$$

and

$$0<(1-\frac{\lambda_{2i}}{\lambda_{2i}^0})P\varphi_{2i}\le u_{k+1}^{2i}\le u_k^{2i}\le P\varphi_{2i} \tag{8.6}$$

According to (8.4), we have for all $(x,t)\in\mathbb{R}^n\times(0,\infty)$

$$\begin{cases} \lambda_i^0 V(p_{i+1}f_{i+1}(P\varphi_{i+1}))(x,t)\le P\varphi_i(x,t),\ 1\le i\le 2m-1 \\ \lambda_{2m}^0 V(p_1f_1(P\varphi_1))(x,t)\le P\varphi_{2m}(x,t). \end{cases} \tag{8.7}$$

From (8.7), we have for all $1\le i\le m$

$$\begin{aligned} u_0^{2i-1} &= P\varphi_{2i-1}-\lambda_{2i-1}V(p_{2i}f_{2i}(P\varphi_{2i})) \\ &\ge P\varphi_{2i-1}-\frac{\lambda_{2i-1}}{\lambda_{2i-1}^0}P\varphi_{2i-1}=(1-\frac{\lambda_{2i-1}}{\lambda_{2i-1}^0})P\varphi_{2i-1}>0 \end{aligned}$$

then

$$\begin{cases} u_1^{2i}-u_0^{2i}=-\lambda_{2i}V(p_{2i+1}f_{2i+1}(u_0^{2i+1}))\le 0,\ 1\le i\le m-1 \\ u_1^{2m}-u_0^{2m}=-\lambda_{2m}V(p_1f_1(u_0^1))\le 0. \end{cases}$$

Since f_{2i} $(1\le i\le m)$ are nondecreasing, we obtain

$$u_1^{2i-1}-u_0^{2i-1}=\lambda_{2i-1}V(p_{2i}(f_{2i}(u_0^{2i})-f_{2i}(u_1^{2i})))\ge 0,\ 1\le i\le m$$

Now, since $u_0^{2i}>0$ $(1\le i\le m)$ are nonnegative which implies $u_0^{2i-1}\le P\varphi_{2i-1}$, and f_{2i-1} $(1\le i\le m)$ are nondecreasing, we deduce from (8.7) that

$$\begin{cases} u_1^{2i}=P\varphi_{2i}-\lambda_{2i}V(p_{2i+1}f_{2i+1}(u_0^{2i+1}))\ge(1-\frac{\lambda_{2i}}{\lambda_{2i}^0})P\varphi_{2i}>0,\ 1\le i\le m \\ u_1^{2m}=P\varphi_{2m}-\lambda_{2m}V(p_1f_1(u_0^1))\ge(1-\frac{\lambda_{2m}}{\lambda_{2m}^0})P\varphi_{2m}>0, \end{cases}$$

which gives us

$$u_1^{2i-1}\le P\varphi_{2i-1},\ 1\le i\le m.$$

Finally, we find

$$\begin{cases} 0<(1-\frac{\lambda_{2i-1}}{\lambda_{2i-1}^0})P\varphi_{2i-1}\le u_0^{2i-1}\le u_1^{2i-1}\le P\varphi_{2i-1},\ 1\le i\le m \\ 0<(1-\frac{\lambda_{2i}}{\lambda_{2i}^0})P\varphi_{2i}\le u_1^{2i}\le u_0^{2i}\le P\varphi_{2i},\ 1\le i\le m. \end{cases}$$

By induction, we suppose that (8.5) and (8.6) hold for k. Since f_{2i-1} $(1 \leq i \leq m)$ are nondecreasing and $u_{k+1}^{2i-1} \leq P\varphi_{2i-1}$ $(1 \leq i \leq m)$, we have for all $1 \leq i \leq m-1$

$$\begin{cases} u_{k+2}^{2i} - u_{k+1}^{2i} = \lambda_{2i} V(p_{2i+1}(f_{2i+1}(u_k^{2i+1}) - f_{2i+1}(u_{k+1}^{2i+1}))) \leq 0 \\ u_{k+2}^{2m} - u_{k+1}^{2m} = \lambda_{2m} V(p_1(f_1(u_k^1) - f_1(u_{k+1}^1))) \leq 0 \end{cases}$$

and

$$\begin{cases} u_{k+2}^{2i} = P\varphi_{2i} - \lambda_{2i} V(p_{2i+1} f_{2i+1}(u_{k+1}^{2i+1})) & \geq P\varphi_{2i} - \lambda_{2i} V(p_{2i+1} f_{2i+1}(P\varphi_{2i+1})) \\ & \geq (1 - \frac{\lambda_{2i}}{\lambda_{2i}^0}) P\varphi_{2i} > 0 \\ u_{k+2}^{2m} = P\varphi_{2m} - \lambda_{2m} V(p_1 f_1(u_{k+1}^1)) & \geq P\varphi_{2m} - \lambda_{2m} V(p_1 f_1(P\varphi_1)) \\ & \geq (1 - \frac{\lambda_{2m}}{\lambda_{2m}^0}) P\varphi_{2m} > 0. \end{cases}$$

Using (8.7), we have

$$0 < (1 - \frac{\lambda_{2i}}{\lambda_{2i}^0}) P\varphi_{2i} \leq u_{k+2}^{2i} \leq u_{k+1}^{2i} \leq P\varphi_{2i}, \ 1 \leq i \leq m.$$

Now, using that f_{2i} are nondecreasing, we have

$$u_{k+2}^{2i-1} - u_{k+1}^{2i-1} = \lambda_{2i-1} V(p_{2i}(f_{2i}(u_{k+1}^{2i}) - f_{2i}(u_{k+2}^{2i}))) \geq 0, \ 1 \leq i \leq m.$$

Since $u_{k+2}^{2i} \geq 0$, we obtain

$$0 < (1 - \frac{\lambda_{2i-1}}{\lambda_{2i-1}^0}) P\varphi_{2i-1} \leq u_{k+1}^{2i-1} \leq u_{k+2}^{2i-1} \leq P\varphi_{2i-1}, \ 1 \leq i \leq m.$$

Therefore, the sequences $(u_k^i)_{k \geq 0}$ converge respectively to u_i $(1 \leq i \leq 2m)$ satisfying (8.2).

We claim that

$$\text{(8.8)} \qquad \begin{cases} u_i = P\varphi_i - \lambda_i V(p_{i+1} f_{i+1}(u_{i+1})), \ 1 \leq i \leq 2m-1 \\ u_{2m} = P\varphi_{2m} - \lambda_{2m} V(p_1 f_1(u_1)). \end{cases}$$

Since $u_k^i \leq P\varphi_i$ $(1 \leq i \leq 2m)$ for all $k \in \mathbb{N}$, using assumptions (H_1), (H_3) and the fact that f_i $(1 \leq i \leq 2m)$ are nondecreasing, there exist $(\tilde{p}_i)_{c_i} \in \mathbf{P}^\infty(\mathbb{R}^n)$ $(1 \leq i \leq 2m)$ such that

$$\text{(8.9)} \qquad \begin{cases} p_{i+1} f_{i+1}(u_{i+1}) \leq p_{i+1} f_{i+1}(c_{i+1} P\theta_{i+1}) \leq (\tilde{p}_{i+1})_{c_{i+1}} P\theta_i, \ 1 \leq i \leq 2m-1 \\ p_1 f_1(u_1) \leq p_1 f_1(c_1 P\theta_1) \leq (\tilde{p}_1)_{c_1} P\theta_{2m} \end{cases}$$

and

$$\begin{cases} p_{i+1}\left|f_{i+1}(u_k^{i+1})-f_{i+1}(u_{i+1})\right| \leq 2(\tilde{p}_{i+1})_{c_{i+1}}P\theta_i,\ 1\leq i\leq 2m-1 \text{ for all } k\in\mathbb{N} \\ p_1\left|f_1(u_k^1)-f_1(u_1)\right| \leq 2(\tilde{p}_1)_{c_1}P\theta_{2m} \text{ for all } k\in\mathbb{N}. \end{cases}$$

Based on Proposition 8.3 and the application of Lebesgue's theorem, we can deduce that

$$\lim_{k\to\infty} V(p_if_i(u_k^i)) = V(p_if_i(u_i)),\ 1\leq i\leq 2m.$$

So, letting $k\to\infty$ in equations

$$\begin{cases} u_k^i = P\varphi_i-\lambda_iV(p_{i+1}f_{i+1}(u_k^{i+1})),\ 1\leq i\leq 2m-1 \\ u_k^{2m} = P\varphi_{2m}-\lambda_{2m}V(p_1f_1(u_k^1)), \end{cases}$$

we have (8.8). Next, we claim that $(u_1,\ldots,u_{2m})$ satisfies

$$\begin{cases} \Delta u_i-\dfrac{\partial u_i}{\partial t} = \lambda_ip_{i+1}f_{i+1}(u_{i+1}), \quad 1\leq i\leq 2m-1 \\ \Delta u_{2m}-\dfrac{\partial u_{2m}}{\partial t} = \lambda_{2m}p_1f_1(u_1). \end{cases} \tag{8.10}$$

Since θ_i $(1\leq i\leq 2m)$ satisfies (H_0) and $(\tilde{p}_i)_{c_i}\in\mathbf{P}^\infty(\mathbb{R}^n)$ $(1\leq i\leq 2m)$, using Proposition 8.1 (iii), we have

$$(\tilde{p}_i)_{c_i}P\theta_i\in L^1_{loc}(\mathbb{R}^n\times(0,\infty)),\ 1\leq i\leq 2m.$$

Moreover (8.9) and Proposition 8.3 implies that

$$p_if_i(u_i)\in L^1_{loc}(\mathbb{R}^n\times(0,\infty)),\ 1\leq i\leq 2m$$

and

$$V(p_if_i(u_i))\in C(\mathbb{R}^n\times(0,\infty))\subset L^1_{loc}(\mathbb{R}^n\times(0,\infty)),\ 1\leq i\leq 2m.$$

Now, through the application of the heat operator $\Delta-\dfrac{\partial}{\partial t}$ to (8.8), it becomes evident that the solution $(u_1,\ldots,u_{2m})$ is positively defined for (8.10). Moreover, considering the continuity of $V(p_if_i(u_i))$ $(1\leq i\leq 2m)$ in $\mathbb{R}^n\times(0,\infty)$, and employing Proposition 8.4, we can conclude from (8.8) that

$$(u_1,\ldots,u_{2m})\in(C(\mathbb{R}^n\times(0,\infty)))^{2m}.$$

This implies, according to assumption (H_1), (8.8) and Proposition 8.3 that

$$\lim_{t\to 0} u_i(x,t) = \lim_{t\to 0} P\varphi_i(x,t) = \varphi_i(x), \quad x \in \mathbb{R}^n.\ 1 \leq i \leq 2m$$

With this final step, we have successfully established the validity of Theorem 8.1. ■

8.5 Conclusion

The result of this research is to prove the existence of continuous positive solutions for a specific class of quasi-linear reaction-diffusion systems of arbitrary order, which makes our study an extension of previous studies carried out on systems with a specific number of equations. The approach taken to achieve the main objective of the research is mainly based on potential analysis techniques, and it yielded reliable results with less computational effort, as the approximate problem was proven to converge to a solution to our problem which is consistent with our final objective. The main result obtained demonstrates the effectiveness of this approach for solving similar problems. We were able to overcome certain difficulties and achieved very important results. This leads us to think more about the problem and study it in other circumstances. Furthermore, a numerical study of our problem can be carried out. These efforts will develop this important aspect of mathematics.

CHAPTER 9

SOLVING A CLASS OF QUASILINEAR FIRST ORDER PDES

This paper focuses on a specific class of partial differential equations in three real variables, which are first order and quasilinear. Our objective is to introduce a solving method for these equations, drawing inspiration from the Lagrangian approach. We will thoroughly examine all potential cases and clarify various aspects of the method using practical exercises specifically designed for this purpose.

The content presented in this chapter is the subject of a research paper [61] titled "Solving a class of quasilinear first order PDEs" published in the "*Proceedings of the Institute of Mathematics and Mechanics*" in collaboration with S. Mesbahi.

9.1 Introduction

The study of partial differential equations (which will be abbreviated as PDE in the following) is still an enthralling field, full of challenging equations that beg for new ideas and novel approaches. Many problems still exist, acting as continual reminders of the immense intricacy buried in these equations despite decades of research and mathematical prowess. With each unsolved equation, the need for fresh perspectives and ground-breaking concepts grows, fostering eagerness for the undiscovered truths. The field of PDEs stands as a testament to the unyielding spirit of human curiosity and the limitless potential for breakthroughs that lie just beyond the horizon.

Our comprehension of real-world phenomena and our technology today are largely based on PDEs. It is thanks to modeling through PDEs that we have been able to understand phenomena derived from other disciplines (such as biology, ecology, economics, physical sciences, astronomy, chemistry, etc.). There is no doubt that PDEs remain one of the most active areas of research due to their multiple applications in all areas of science. See Chowdhury et *al*. [43], Kruzhkov [120], Rhee et *al*. [184], Sneddon [204] and references given there.

In this chapter, we restrict our work to a class of first order quasilinear PDEs with three real variables, whose general form is

$$Pp+Qq+Rr=S \tag{9.1}$$

with

$$\begin{aligned} P &= a_1x+b_1y+c_1z+d_1u+e_1\\ Q &= a_2x+b_2y+c_2z+d_2u+e_2\\ R &= a_3x+b_3y+c_3z+d_3u+e_3\\ S &= a_4x+b_4y+c_4z+d_4u+e_4 \end{aligned}$$

where $p=\dfrac{\partial u}{\partial x}$, $q=\dfrac{\partial u}{\partial y}$, $r=\dfrac{\partial u}{\partial z}$, and $u=u(x,y,z)$ is a smooth vector field in a domain Ω of $\mathbb{R}^3$. The functions P, Q, R, S are linear of (x,y,z,u), and a_k, b_k, c_k, d_k, e_k are real numbers for all $1\le k\le 4$.

We recall that a PDE is said to be quasilinear, if it is linear with respect to all the highest order derivatives of the unknown function. A smooth function $u=u(x,y,z)$ is a solution of equation (9.1), if and only if u is constant along the phase curves of the field u, i.e., it is the first integral of the associated characteristic system

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{du}{S} \tag{9.2}$$

Lagrange's method of characteristics reduces the problem of solving PDE (9.1) to the characteristic system (9.2). For further information about this method and how to apply it, see Rhee et *al*. [184].

Theorem 9.1. *The general solution of PDE* (9.1) *is* $F\left(\varphi,\psi,\xi\right)=0$*, where* F *is an arbitrary function and* $\varphi(x,y,z)=c_1$*,* $\psi(x,y,z)=c_2$*,* $\xi(x,y,z)=c_3$ *are linearly independent first integrals of the associated characteristic system* (9.2)*.*

In Kruzhkov [120], Mesbahi [156], Reinhard [185], we find a proof of this theorem as well as several other important theorems and properties.

First order PDEs appear frequently in stochastic process theory, as the Fokker-Planck equation, and in mathematical physics, as the Hamilton-Jacobi equation. As other examples, we mention the Hopf equation (also known as Burgers' equation without viscosity), which is used in a variety of contexts, such as the dynamics of gases without pressure and in describing the velocity field of a medium consisting of particles moving without interaction in the absence of external forces. In solid mechanics, we often find the mass conservation equation, which describes the movement of a fluid (liquid or gas) when sinks and sources are absent. We also mention the transport equation, which is the prototype of PDEs of the first order. The model is used in various sciences, such as the model for fluid infiltration through sand, where the fluid flows under gravity alone without sources or sinks, see Chechkin et *al*. [39]. The transport equation also appears in the mathematical modeling of traffic-like collective movements at different levels of biological organization. Molecular motor proteins like kinesin and dynein, which are responsible for most intracellular transport in eukayotic cells, sometimes experience traffic jams, which manifest as disease, see Chowdhury et *al*. [43] and Schadschneider [201].

Below we will present a method for solving system (9.2) and consequently equation (9.1). We will discuss all possible cases, supporting each case with an illustrative example that will explain many aspects of the method used.

9.2 Method of solving

Suppose it is possible to find constants λ, μ, v, τ such that each ratio of system (9.2) is equal to

$$\frac{\lambda dx + \mu dy + v dz + \tau du}{\lambda P + \mu Q + vR + \tau S}$$

If λ, μ, v ,τ are constant multipliers, this expression will be an exact differential, if it is of the form

$$\frac{1}{\rho}\frac{\lambda dx + \mu dy + v dz + \tau du}{\lambda x + \mu y + vz + \tau u}$$

This brings us to the following system

$$\frac{\lambda dx+\mu dy+v dz+\tau du}{(a_1\lambda+a_2\mu+a_3v+a_4\tau)x+(b_1\lambda+b_2\mu+b_3v+b_4\tau)y+(c_1\lambda+c_2\mu+c_3v+c_4\tau)z+(d_1\lambda+d_2\mu+d_3v+d_4\tau)u} = \frac{1}{\rho}\frac{\lambda dx + \mu dy + v dz + \tau du}{\lambda x + \mu y + vz + \tau u} \tag{9.3}$$

and this is possible only if

$$\begin{cases} \lambda\left(a_1-\rho\right)+a_2\mu+a_3v+a_4\tau=0 \\ b_1\lambda+\left(b_2-\rho\right)\mu+b_3v+b_4\tau=0 \\ c_1\lambda+c_2\mu+\left(c_3-\rho\right)v+c_4\tau=0 \\ d_1\lambda+d_2\mu+d_3v+\left(d_4-\rho\right)\tau=0 \end{cases} \tag{9.4}$$

System (9.4) can be represented in the matrix form $AX=0$, with

$$A=\begin{pmatrix} a_1-\rho & a_2 & a_3 & a_4 \\ b_1 & b_2-\rho & b_3 & b_4 \\ c_1 & c_2 & c_3-\rho & c_4 \\ d_1 & d_2 & d_3 & d_4-\rho \end{pmatrix},\ X=\begin{pmatrix} \lambda \\ \mu \\ v \\ \tau \end{pmatrix}$$

where $\det A=0$ is required to obtain non-zero solutions for X instead of just the trivial zero solution, in order to provide meaningful insight about the system's behavior. Setting $\det A=0$ allows finding a particular non-zero solution for X that meets the chapter's objective, which leads to

$$\Psi\left(\rho\right)=\det A=0 \tag{9.5}$$

This polynomial has four roots in $\mathbb{C}$, which we may denote by ρ_1, ρ_2, ρ_3, ρ_4. We distinguish the following possible cases:

(i) $\rho_1\neq\rho_2\neq\rho_3\neq\rho_4\in\mathbb{R}$	(ii) $\rho_1\neq\rho_2\in\mathbb{R},\ \rho_3=\overline{\rho_4}\in\mathbb{C}$
(iii) $\rho_1=\overline{\rho_2}\neq\rho_3=\overline{\rho_4}\in\mathbb{C}$	(iv) $\rho_1\neq\rho_2=\rho_3=\rho_4\in\mathbb{R}$
(v) $\rho_1=\rho_2\neq\rho_3=\rho_4\in\mathbb{R}$	(vi) $\rho_1=\rho_2\neq\rho_3\neq\rho_4\in\mathbb{R}$
(vii) $\rho_1=\rho_2\in\mathbb{R},\ \rho_3=\overline{\rho_4}\in\mathbb{C}$	(viii) $\rho_1=\overline{\rho_2}=\rho_3=\overline{\rho_4}\in\mathbb{C}$
(ix) $\rho_1=\rho_2=\rho_3=\rho_4\in\mathbb{R}$	

In the following paragraphs, we will discuss all possible cases and provide an illustrative example for each. In everything that follows, we denote by c or c_j an arbitrary real constant.

9.2.1 Case (i): $\rho_1\neq\rho_2\neq\rho_3\neq\rho_4\in\mathbb{R}$

In this case, for any ρ_j, $1\leq j\leq 4$, There exist real constants $\left(\lambda_j,\mu_j,v_j,\tau_j\right)$ satisfying system (9.3), and thus we have four possible exact differentials, which gives us

$$\frac{\lambda_1dx+\mu_1dy+v_1dz+\tau_1du}{\rho_1(\lambda_1x+\mu_1y+v_1z+\tau_1u)}=\frac{\lambda_2dx+\mu_2dy+v_2dz+\tau_2du}{\rho_2(\lambda_2x+\mu_2y+v_2z+\tau_2u)}=\frac{\lambda_3dx+\mu_3dy+v_3dz+\tau_3du}{\rho_3(\lambda_3x+\mu_3y+v_3z+\tau_4u)}=\frac{\lambda_4x+\mu_4dy+v_4dz+\tau_4du}{\rho_4(\lambda_4x+\mu_4y+v_4z+\tau_4u)}$$

this admits as first integrals the following

$$\begin{array}{rcl}\left(\lambda_1 x+\mu_1 y+\nu_1 z+\tau_1 u\right)^{\rho_2}.\left(\lambda_2 x+\mu_2 y+\nu_2 z+\tau_2 u\right)^{-\rho_1} & = & c_1\\ \left(\lambda_2 x+\mu_2 y+\nu_2 z+\tau_2 u\right)^{\rho_3}.\left(\lambda_3 x+\mu_3 y+\nu_3 z+\tau_3 u\right)^{-\rho_2} & = & c_2\\ \left(\lambda_3 x+\mu_3 y+\nu_3 z+\tau_3 u\right)^{\rho_4}.\left(\lambda_4 x+\mu_4 y+\nu_4 z+\tau_4 u\right)^{-\rho_3} & = & c_3\end{array}$$

which are linearly independent, where c_1, c_2 and c_3 are arbitrary real constants. Hence the integral curves $F(c_1, c_2, c_3) = 0$, where F is an arbitrary real function.

Example 9.1. *Let the equation*

$$xp+(x+2y)q+(3z)r=z+4u$$

In this case, the polynomial $\Psi(\rho)$ *of* (9.5) *admits four different real roots* $\rho_1 = 1$, $\rho_2 = 2$, $\rho_3 = 3$, $\rho_4 = 4$. *The constants* $\left(\lambda_j, \mu_j, \nu_j, \tau_j\right)$ *associated respectively are* $(1,0,0,0)$, $(1,1,0,0)$, $(0,0,1,0)$, $(0,0,1,1)$. *The characteristic system associated with our equation becomes*

$$\frac{dx}{x}=\frac{d(x+y)}{2(x+y)}=\frac{dz}{3z}=\frac{d(z+u)}{4(z+u)}$$

which gives us the following linearly independent first integrals

$$x^2(x+y)^{-1}=c_1\ ,\ (x+y)^3 z^{-2}=c_2\ ,\ z^4(z+u)^{-3}=c_3$$

where c_1, c_2 *and* c_3 *are arbitrary real constants. Hence the integral curves* $F(c_1, c_2, c_3) = 0$, *where* F *is an arbitrary real function.*

9.2.2 Case (ii): $\rho_1 \neq \rho_2 \in \mathbb{R},\ \rho_3 = \overline{\rho_4} \in \mathbb{C}$

Suppose that the roots of the polynomial $\Psi(\rho)$ are $\rho_1 \neq \rho_2 \in \mathbb{R}$ and $\rho_3 = \rho_3' + i\rho_3'' = \overline{\rho_4} \in \mathbb{C}$. In this case, we can find constants $\left(\lambda_j, \mu_j, \nu_j, \tau_j\right)$, $1 \le j \le 4$, satisfying system (9.3). The characteristic system becomes

$$\frac{df}{\rho_1 f}=\frac{dg}{\rho_2 g}=\frac{dh}{\left(\rho_3'+i\rho_3''\right)h}=\frac{dk}{\left(\rho_3'-i\rho_3''\right)k} \tag{9.6}$$

where

$$\begin{array}{rcl} f & = & \lambda_1 x+\mu_1 y+\nu_1 z+\tau_1 u\\ g & = & \lambda_2 x+\mu_2 y+\nu_2 z+\tau_2 u\\ \bar{k} & = & h=\left(\lambda_3'+i\lambda_3''\right)x+\left(\mu_3'+i\mu_3''\right)y+\left(\nu_3'+i\nu_3''\right)z+\left(\tau_3'+i\tau_3''\right)u\end{array}$$

From the first equality of system (9.6), we have the following first integral

$$\left(\lambda_1 x+\mu_1 y+v_1 z+\tau_1 u\right)^{\rho_2}\cdot\left(\lambda_2 x+\mu_2 y+v_2 z+\tau_2 u\right)^{-\rho_1}=c_1$$

The second equality of system (9.6) gives us

$$E=g^{\left(\rho_3'+i\rho_3''\right)}\cdot h^{-\rho_2}=c$$

and from it, we get

$$\begin{aligned}\log E &= \rho_3'\log\left(\lambda_2 x+\mu_2 y+v_2 z+\tau_2 u\right)-\rho_2 z_1\\ &\quad +i\left(\rho_3''\log\left(\lambda_2 x+\mu_2 y+v_2 z+\tau_2 u\right)-\rho_2 z_2\right)\end{aligned}$$

where

$$\begin{aligned}z_1 &= \frac{1}{2}\log\left(\left(\lambda_3' x+\mu_3' y+v_3' z+\tau_3' u\right)^2+\left(\lambda_3'' x+\mu_3'' y+v_3'' z+\tau_3'' u\right)^2\right)\\ z_2 &= \arctan\left(\frac{\lambda_3'' x+\mu_3'' y+v_3'' z+\tau_3'' u}{\lambda_3' x+\mu_3' y+v_3' z+\tau_3' u}\right)\end{aligned}$$

which implies

$$E=\exp\left(\log(g)^{\rho_3'}-\rho_2 z_1\right)\cdot\exp\left[i\left(\log(g)^{\rho_3''}-\rho_2 z_2\right)\right]$$

We get the following first integral

$$E=(g)^{\rho_3'}\cdot e^{-\rho_2 z_1}\cos\left(\rho_3''\log(g)-\rho_2 z_2\right)=c_2$$

The third equality of system (9.6) gives us

$$E'=h^{\left(\rho_3'-i\rho_3''\right)}\cdot k^{-\left(\rho_3'+i\rho_3''\right)}=c$$

which gives

$$\log E'=2i\left(\rho_3' z_2-\rho_3'' z_1\right)$$

and since we are looking for real solutions, we take the following first integral

$$E'=\rho_3' z_2-\rho_3'' z_1=c_3$$

The three first integrals obtained are linearly independent, where c_1, c_2 and c_3 are arbitrary real constants. Hence the integral curves $F(c_1,c_2,c_3)=0$, where F is an arbitrary real function.

Example 9.2. *Let the equation*

$$(2x)p+(x+2y+2u)q+(z-y)r=z+u$$

The polynomial $\Psi(\rho)$ *admits for roots* $\rho_1=0$, $\rho_2=2$, $\rho_3=2-i=\overline{\rho_4}$. *The constants* $(\lambda_j,\mu_j,\upsilon_j,\tau_j)$ *associated respectively are*

$$(1,-2,-4,4),\ (1,0,0,0),\ (1+i,1-i,1+i,2),\ (1-i,1+i,1-i,2)$$

In this way, we can obtain the following linearly independent first integrals

$$\begin{aligned} x-2y-4z+4u &= c_1 \\ x^2\exp(-2z_1)\cdot\cos(1\log(x)-2z_2) &= c_2 \\ 2z_2-z_1 &= c_3 \end{aligned}$$

where

$$\begin{aligned} z_1 &= \frac{1}{2}\log\left[(x+y+z+2u)^2+(x-y+z+0u)^2\right] \\ z_2 &= \arctan\left(\frac{x-y+z+0u}{x+y+z+2u}\right) \end{aligned}$$

Hence the integral curves $F(c_1,c_2,c_3)=0$, *where* F *is an arbitrary real function.*

9.2.3 Case (iii): $\rho_1=\overline{\rho_2}\neq\rho_3=\overline{\rho_4}\in\mathbb{C}$

Let $\rho_1=\rho_1'+i\rho_1''=\overline{\rho_2}$ and $\rho_3=\rho_3'+i\rho_3''=\overline{\rho_4}\in\mathbb{C}$ be the roots of the polynomial $\Psi(\rho)$. In this case, we can find real constants $(\lambda_j,\mu_j,\upsilon_j,\tau_j)$, $1\le j\le 4$, satisfying system (9.3). The characteristic system becomes

$$\frac{df}{(\rho_1'+i\rho_1'')f}=\frac{dg}{(\rho_1'-i\rho_1'')g}=\frac{dh}{(\rho_3'+i\rho_3'')h}=\frac{dk}{(\rho_3'-i\rho_3'')k} \tag{9.7}$$

where

$$\begin{aligned} \bar{g} &= f=(\lambda_1'+i\lambda_1'')x+(\mu_1'+i\mu_1'')y+(\upsilon_1'+i\upsilon_1'')z+(\tau_1'+i\tau_1'')u \\ \bar{k} &= h=(\lambda_3'+i\lambda_3'')x+(\mu_3'+i\mu_3'')y+(\upsilon_3'+i\upsilon_3'')z+(\tau_3'+i\tau_3'')u \end{aligned}$$

As in the previous case, by applying the same steps to the first and third equality of system (9.7), we obtain the following first integrals

$$\rho_1' z_2 - \rho_1'' z_1 = c_1 \quad \text{and} \quad \rho_3' z_2' - \rho_3'' z_1' = c_2$$

where

$$\begin{array}{rcl} z_1 & = & \dfrac{1}{2}\log\left(\left(\lambda_1' x + \mu_1' y + \upsilon_1' z + \tau_1' u\right)^2 + \left(\lambda_1'' x + \mu_1'' y + \upsilon_1'' z + \tau_1'' u\right)^2\right) \\ z_2 & = & \arctan \dfrac{\lambda_1'' x + \mu_1'' y + \upsilon_1'' z + \tau_1'' u}{\lambda_1' x + \mu_1' y + \upsilon_1' z + \tau_1' u} \\ z_1' & = & \dfrac{1}{2}\log\left(\left(\lambda_3' x + \mu_3' y + \upsilon_3' z + \tau_3' u\right)^2 + \left(\lambda_3'' x + \mu_3'' y + \upsilon_3'' z + \tau_3'' u\right)^2\right) \\ z_2' & = & \arctan \dfrac{\lambda_3'' x + \mu_3'' y + \upsilon_3'' z + \tau_3'' u}{\lambda_3' x + \mu_3' y + \upsilon_3' z + \tau_3' u} \end{array}$$

For the second equality of system (9.7)**,** we get

$$E = g^{\left(\rho_3' + i\rho_3''\right)} \cdot h^{-\left(\rho_1' - i\rho_1''\right)} = c$$

which gives

$$\log E = \left(\rho_3' z_1 + \rho_3'' z_2 - \rho_1' z_1' - \rho_1'' z_2'\right) + i\left(\rho_3'' z_1 - \rho_3' z_2 + \rho_1'' z_1' - \rho_1' z_2'\right)$$

then

$$E = \exp\left(\rho_3' z_1 + \rho_3'' z_2 - \rho_1' z_1' - \rho_1'' z_2'\right) . \exp\left[i\left(\rho_3'' z_1 - \rho_3' z_2 + \rho_1'' z_1' - \rho_1' z_2'\right)\right]$$

and since we are looking for real solutions, we take the following first integral

$$\exp\left(\rho_3' z_1 + \rho_3'' z_2 - \rho_1' z_1' - \rho_1'' z_2'\right) . \cos\left(\rho_3'' z_1 - \rho_3' z_2 + \rho_1'' z_1' - \rho_1' z_2'\right) = c_3$$

As a result, the integral curve $F(c_1, c_2, c_3) = 0$, where c_1, c_2, c_3 are arbitrary real constants and F is an arbitrary real function.

Example 9.3. *Let the equation*

$$(x - y + z)p + (x + u)q + (y - u)r = x + z$$

The polynomial $\Psi(\rho)$ *admits for roots* $\rho_1 = -i = \overline{\rho_2}$, $\rho_3 = \frac{1}{2} - \frac{1}{2}i\sqrt{3} = \overline{\rho_4}$. *The constants* $\left(\lambda_j, \mu_j, \upsilon_j, \tau_j\right)$ *associated respectively are*

$$(1 - 2i, 2 + i, 2 - 4i, -5),\ (1 + 2i, 2 - i, 2 + 4i, -5),$$
$$\left(1 - \sqrt{3}i, 0, 1 - \sqrt{3}i, -2\right),\ \left(1 + \sqrt{3}i, 0, 1 + \sqrt{3}i, -2\right).$$

In this way, we can obtain the following linearly independent first integrals

$$\begin{array}{rcl} -z_1 & = & c_1 \\ \exp\left(\frac{1}{2}z_1+\frac{\sqrt{3}}{2}z_2-0z_1'+1z_2'\right).\cos\left(\frac{1}{2}z_1-\frac{\sqrt{3}}{2}z_2-1z_1'+0z_2'\right) & = & c_2 \\ \frac{1}{2}z_2'-\frac{\sqrt{3}}{2}z_1' & = & c_3 \end{array}$$

where

$$\begin{array}{rcl} z_1 & = & \frac{1}{2}\log\left((x+2y+2z-5u)^2+(2x-y+4z+0u)^2\right) \\ z_2 & = & \arctan\frac{2x-y+4z+0u}{x+2y+2z-5u} \\ z_1' & = & \frac{1}{2}\log\left((x+0y+z-2u)^2+\left(\sqrt{3}x+0y+\sqrt{3}z+0u\right)^2\right) \\ z_2' & = & \arctan\frac{\sqrt{3}x+0y+\sqrt{3}z+0u}{x+0y+z-2u} \end{array}$$

Hence the integral curves $F(c_1,c_2,c_3)=0$, *where* F *is an arbitrary real function.*

The remaining cases will only be treated through examples. This approach aims to streamline concepts and circumvent specific computational challenges arising from the nature of the equation to be resolved. By employing the same methodology as in prior cases, we arrive each time at an insufficient number of first integrals. This is the basic difficulty in this work, and to overcome it, the suitable approach has been identified, which we will explain through the examples that we will present later in this chapter. First, we proceed as in the previous examples, we will find at most two first integrals, then we work to complete them to obtain three linearly independent first integrals, which will make it possible to find the general solution of the proposed equation. We will treat each case separately with an illustrative example. It is worth noting that in each of the subsequent cases, obtaining three first integrals that are linearly independent is sufficient, and our objective is not to discover all potential first integrals. It is important to mention that all the equations we will examine below have been meticulously formulated to be compatible with all conceivable cases. We will develop the method used previously in the first three cases to obtain additional linearly independent first integrals. For this, suppose that it is possible to find constants $\left(\alpha_j,\beta_j,\gamma_j,\delta_j\right)$, $1\le j\le 4$, and ρ such that each ratio of (9.2) is equal to the following exact differential form

$$\frac{\phi_1}{\phi_2}=\frac{dD}{\rho D} \tag{9.8}$$

where dD denotes the total derivative of D.

We will work to clarify all possible cases through the examples that we will address below.

9.2.4 Case (iv): $\rho_1 \neq \rho_2 = \rho_3 = \rho_4 \in \mathbb{R}$

Example 9.4. *Let the equation*

$$xp + (x+2y)q + zr = z + u \tag{9.9}$$

The polynomial $\Psi(\rho)$ *admits a triple root* $\rho_1 = 1$ *and a simple root* $\rho_2 = 2$. *The constants* $(\lambda_j, \mu_j, \upsilon_j, \tau_j)$ *associated respectively are* $(1,0,-1,0)$ *and* $(1,1,0,0)$. *The characteristic system corresponding to equation* (9.9) *can therefore be written in the form*

$$\frac{dx}{x} = \frac{dy}{x+2y} = \frac{dz}{z} = \frac{du}{z+u} = \frac{d(x-z)}{x-z} = \frac{d(x+y)}{2(x+y)} \tag{9.10}$$

This leads to a single first integral

$$(x+y)^1 (x-z)^{-2} = c_1$$

We only need two more first integrals. For that, suppose it is possible to find constants $(\alpha_j, \beta_j, \gamma_j, \delta_j)$, $1 \le j \le 4$, *and* ρ *such that each ratio of* (9.10) *is of the form* (9.8), *with*

$$\begin{aligned} D &= (\alpha_1+\alpha_2)x^2 + 2\beta_2 y^2 + (\gamma_3+\gamma_4)z^2 + \gamma_4 u^2 + (\beta_1+\beta_2+2\alpha_2)xy \\ &\quad + (\gamma_1+\gamma_2+\alpha_3+\alpha_4)xz + (\delta_1+\delta_2+\alpha_4)xu + (2\gamma_2+\beta_3+\beta_4)yz \\ &\quad + (2\delta_2+\beta_4)yu + (\delta_3+\delta_4+\gamma_4)zu \end{aligned}$$

That's right, if

$$\begin{cases} \left(2-\rho\right)\alpha_1+2\alpha_2=0, & \gamma_1+\gamma_2+\left(1-\rho\right)\alpha_3+\alpha_4=0 \\ \left(1-\rho\right)\beta_1+\beta_2+2\alpha_2=0, & 2\gamma_2+\left(1-\rho\right)\beta_3+\beta_4=0 \\ \left(1-\rho\right)\gamma_1+\gamma_2+\alpha_3+\alpha_4=0, & \left(2-\rho\right)\gamma_3+2\gamma_4=0 \\ \left(1-\rho\right)\delta_1+\delta_2+\alpha_4=0, & \left(1-\rho\right)\delta_3+\delta_4+\gamma_4=0 \\ \beta_1+\beta_2+\left(2-\rho\right)\alpha_2=0, & \delta_1+\delta_2+\left(1-\rho\right)\alpha_4=0 \\ \left(4-\rho\right)\beta_2=0, & 2\delta_2+\left(1-\rho\right)\beta_4=0 \\ \left(2-\rho\right)\gamma_2+\beta_3+\beta_4=0, & \delta_3+\delta_4+\left(1-\rho\right)\gamma_4=0 \\ \left(2-\rho\right)\delta_2+\beta_4=0, & 2\gamma_4-\rho\delta_4=0 \end{cases}$$

This linear homogeneous system will have a non-trivial solution, if the determinant of its coefficient matrix is zero, i.e.,

$$\rho^{16}-23\rho^{15}+231\rho^{14}-1323\rho^{13}+4716\rho^{12}-10626\rho^{11}$$
$$+14312\rho^{10}-8800\rho^{9}-2880\rho^{8}+7776\rho^{7}-3456\rho^{6}=0$$

The roots of this polynomial are : 2 *(quadruple),* 4, $1+\sqrt{3}$, $1-\sqrt{3}$, 3 *(triple),* 0 *(of order 6). We can find several first integrals, but we only need two. For this, we will choose enough values of* ρ. *We can take two different values, and we can be satisfied with a root which is at least double, then we can take* $\rho\in\{2,3\}$. *If we substitute the values of* ρ *in* (9.8) *and solve the resulting system, we find* $\left(\alpha_1,\beta_1,\gamma_1,\delta_1,\alpha_2,\beta_2,\gamma_2,\delta_2,\alpha_3,\beta_3,\gamma_3,\delta_3,\alpha_4,\beta_4,\gamma_4,\delta_4\right)$ *as following*

$$\begin{aligned} (-1,0,1,0,0,0,0,0,1,0,2,0,0,0,0,0) \textit{ for } \rho &= 2 \\ (2,1,-1,0,1,0,-1,0,-1,-1,0,0,0,0,0,0) \textit{ for } \rho &= 3 \end{aligned}$$

These values would transform system (9.10) *into the following exact differential form*

$$\frac{d\left(x-z\right)}{x-z}=\frac{d\left(x+y\right)}{2\left(x+y\right)}=\frac{d\left(-x^2+2z^2+2xz\right)}{2\left(-x^2+2z^2+2xz\right)}=\frac{d\left(3x^2+3xy-3xz-3yz\right)}{3\left(3x^2+3xy-3xz-3yz\right)}$$

which gives us these two new first integrals

$$\begin{aligned} \left(-x^2+2z^2+2xz\right)^3.\left(3x^2+3xy-3xz-3yz\right)^{-2} &= c_2 \\ \left(3x^2+3xy-3xz-3yz\right)^4.\left(2x^2+4y^2+4xy\right)^{-3} &= c_3 \end{aligned}$$

Hence the integral curves $F(c_1,c_2,c_3)=0$*, where* F *is an arbitrary real function.*

9.2.5 Case (v): $\rho_1=\rho_2\neq\rho_3=\rho_4\in\mathbb{R}$

Example 9.5. *Let the equation*

$$(x+y-z)p+(2y)q+(2z+u)r=u \tag{9.11}$$

The polynomial $\Psi(\rho)$ *admits two double roots* $\rho_1=1$ *and* $\rho_2=2$*. The constants* $\left(\lambda_j,\mu_j,\upsilon_j,\tau_j\right)$ *associated respectively are* $(0,0,0,1)$ *and* $(0,-1,1,1)$*. Therefore, the characteristic system that corresponds to equation* (9.11) *can be expressed as follows*

$$\frac{dx}{x+y-z}=\frac{dy}{2y}=\frac{dz}{2z+u}=\frac{du}{u}=\frac{d(-y+z+u)}{2(-y+z+u)} \tag{9.12}$$

This leads to a single first integral

$$\frac{u^2}{-y+z+u}=c_1$$

From system (9.12)*, other first integrals can be extracted, but we will find this using our method for the sake of better understanding. Suppose it is possible to find constants* $\left(\alpha_j,\beta_j,\gamma_j,\delta_j\right)$*,* $1\leq j\leq 4$*, and* ρ *such that each ratio of* (9.12) *is of the form* (9.8)*, with*

$$\begin{aligned}
D &= (\alpha_1)x^2+\left(\beta_1+2\beta_2\right)y^2+\left(-\gamma_1+2\gamma_3\right)z^2+\left(\beta_1+\alpha_1+2\alpha_2\right)xy\\
&\quad+\left(\gamma_1-\alpha_1+2\alpha_3\right)xz+(\delta_1+\alpha_3+\alpha_4)xu+\left(\gamma_1-\beta_1+2\beta_3+2\gamma_2\right)yz\\
&\quad+\left(\delta_1+2\delta_2+\beta_3+\beta_4\right)yu+\left(-\delta_1+2\delta_3+\gamma_3+\gamma_4\right)zu+(\delta_3+\delta_4)u^2
\end{aligned}$$

That's right, if

$$\begin{cases} (2-\rho)\,\alpha_1 = 0, & \gamma_1 - \alpha_1 + (2-\rho)\,\alpha_3 = 0 \\ (1-\rho)\,\beta_1 + \alpha_1 + 2\alpha_2 = 0, & \gamma_1 - \beta_1 + (2-\rho)\,\beta_3 + 2\gamma_2 = 0 \\ (1-\rho)\,\gamma_1 - \alpha_1 + 2\alpha_3 = 0, & -2\gamma_1 + (4-\rho)\,\gamma_3 = 0 \\ (1-\rho)\,\delta_1 + \alpha_3 + \alpha_4 = 0, & -\delta_1 + (2-\rho)\,\delta_3 + \gamma_3 + \gamma_4 = 0 \\ \beta_1 + \alpha_1 + (2-\rho)\,\alpha_2 = 0, & \delta_1 + \alpha_3 + (1-\rho)\,\alpha_4 = 0 \\ 2\beta_1 + (4-\rho)\,\beta_2 = 0, & \delta_1 + 2\delta_2 + \beta_3 + (1-\rho)\,\beta_4 = 0 \\ \gamma_1 - \beta_1 + 2\beta_3 + (2-\rho)\,\gamma_2 = 0, & -\delta_1 + 2\delta_3 + \gamma_3 + (1-\rho)\,\gamma_4 = 0 \\ \delta_1 + (2-\rho)\,\delta_2 + \beta_3 + \beta_4 = 0, & 2\delta_3 + (2-\rho)\,\delta_4 = 0 \end{cases}$$

If the determinant of the coefficient matrix of this linear homogeneous system is zero, it indicates that the system will possess a non-trivial solution, i.e.,

$$\rho^{16} - 30\rho^{15} + 406\rho^{14} - 3272\rho^{13} + 17\,449\rho^{12} - 64\,658\rho^{11} + 169\,896\rho^{10}$$
$$-316\,592\rho^{9} + 410\,064\rho^{8} - 351\,648\rho^{7} + 179\,712\rho^{6} - 41\,472\rho^{5} = 0$$

This polynomial has as roots : 2 *(quintuple),* 3 *(quadruple),* 4 *(double),* 0 *(quintuple). We only need two additional first integrals, for this we will choose enough values of* ρ*. As in the previous case, we avoid* $\rho = 0$*, then we can take* $\rho \in \{2,3\}$*. If we substitute the values of* ρ *in* (9.8) *and solve the resulting system, we find* $(\alpha_1,\beta_1,\gamma_1,\delta_1,\alpha_2,\beta_2,\gamma_2,\delta_2,\alpha_3,\beta_3,\gamma_3,\delta_3,\alpha_4,\beta_4,\gamma_4,\delta_4)$ *as following*

$$\begin{aligned} (0,0,0,0,0,0,0,1,0,-1,0,0,0,1,0,2) \textit{ for } \rho &= 2 \\ (0,0,0,0,0,0,0,0,0,0,4,1,0,0,2,1) \textit{ for } \rho &= 3 \end{aligned}$$

These values would transform system (9.12) *into the following exact differential form*

$$\frac{du}{u} = \frac{d(-y+z+u)}{2(-y+z+u)} = \frac{d\left(2u^2\right)}{2\left(2u^2\right)} = \frac{d\left(8z^2+2u^2+8zu\right)}{3\left(8z^2+2u^2+8zu\right)}$$

which gives us these two first integrals

$$\begin{aligned} \left(2u^2\right)^3 \cdot \left(8z^2+2u^2+8zu\right)^{-2} &= c_2 \\ \left(8z^2+2u^2+8zu\right)^4 \cdot \left(3u^2+3yu+3zu\right)^{-3} &= c_3 \end{aligned}$$

Hence the integral curves $F(c_1,c_2,c_3) = 0$*, where* F *is an arbitrary real function.*

9.2.6 Case (vi): $\rho_1 = \rho_2 \neq \rho_3 \neq \rho_4 \in \mathbb{R}$

Example 9.6. *Let the equation*

$$xp + (x+2y)q + (3z)r = z + u \tag{9.13}$$

The characteristic system corresponding to equation (9.13) *is*

$$\frac{dx}{x} = \frac{dy}{x+2y} = \frac{dz}{3z} = \frac{du}{z+u} \tag{9.14}$$

The polynomial $\Psi(\rho)$ *admits one double root* $\rho_1 = \rho_2 = 1$ *and two simple roots* $\rho_3 = 2$, $\rho_4 = 3$. *The constants* $(\lambda_j, \mu_j, v_j, \tau_j)$ *associated respectively are* $(1,0,-1,2)$, $(1,1,0,0)$ *and* $(0,0,1,0)$. *Then, system* (9.14) *gives us*

$$\frac{d(x-z+2u)}{(x-z+2u)} = \frac{d(x+y)}{2(x+y)} = \frac{dz}{3z}$$

This leads to these two first integrals

$$(x-z+2u)^2.(x+y)^{-1} = c_1 \quad \textit{and} \quad (x+y)^3.(z)^{-2} = c_2$$

So, we need another first integral. For that, suppose it is possible to find constants $(\alpha_j, \beta_j, \gamma_j, \delta_j)$, $1 \le j \le 4$, *and* ρ *such that each ratio of* (9.14) *is of the form* (9.8), *with*

$$\begin{aligned} D \;=\; & (\alpha_1+\alpha_2)x^2 + (\beta_1+2\beta_2)y^2 + (3\gamma_3+\gamma_4)z^2 + \delta_4 u^2 + (3\delta_3+\gamma_4)zu \\ & + (\gamma_1+\gamma_2+3\alpha_3+\alpha_4)xz + (\gamma_1+2\gamma_2+3\beta_3+\beta_4)yz \\ & + (\beta_1+\beta_2+2\alpha_2+\alpha_1)xy + (\delta_1+2\delta_2+\beta_4)yu + (\delta_1+\delta_2+\alpha_4)xu \end{aligned}$$

That's right, if

$$\left\{ \begin{array}{rr} 2\alpha_1+2\alpha_2=0, & \gamma_1+\gamma_2+3\alpha_3+\alpha_4=0 \\ \beta_1+\beta_2+2\alpha_2+\alpha_1=0, & \gamma_1+2\gamma_2+3\beta_3+\beta_4=0 \\ \gamma_1+\gamma_2+3\alpha_3+\alpha_4=0, & 6\gamma_3+2\gamma_4=0 \\ \delta_1+\delta_2+\alpha_4=0, & 3\delta_3+\gamma_4=0 \\ \beta_1+\beta_2+2\alpha_2+\alpha_1=0, & \delta_1+\delta_2+\alpha_4=0 \\ 2\beta_1+4\beta_2=0, & \delta_1+2\delta_2+\beta_4=0 \\ \gamma_1+2\gamma_2+3\beta_3+\beta_4=0, & 3\delta_3+\gamma_4=0 \\ \delta_1+2\delta_2+\beta_4=0, & 2\delta_4=0 \end{array} \right.$$

This system will have a non-trivial solution, if the determinant of its coefficient matrix is zero, i.e.,

$$\begin{gathered}\rho^{16}-35\rho^{15}+541\rho^{14}-4866\rho^{13}+28240\rho^{12}-110780\rho^{11}+298955\rho^{10}\\ -553425\rho^{9}+686695\rho^{8}-542426\rho^{7}+244980\rho^{6}-47880\rho^{5}=0\end{gathered}$$

Among the roots of this polynomial are the following : 1, 2, 3, 6, 0, $\frac{5}{2}\pm\frac{1}{2}\sqrt{5}$, $\frac{9}{2}\pm\frac{1}{2}\sqrt{5}$. *Just choose a single value for* ρ, *then we can take* $\rho=2$. *If we substitute this value of* ρ *in* (9.8) *and solve the resulting system, we find*

$$\left(\alpha_1,\beta_1,\gamma_1,\delta_1,\alpha_2,\beta_2,\gamma_2,\delta_2,\alpha_3,\beta_3,\gamma_3,\delta_3,\alpha_4,\beta_4,\gamma_4,\delta_4\right)$$

as following

$$(0,0,0,0,0,0,0,0,0,0,0,1,0,0,2,-1) \text{ for } \rho=2$$

Then system (9.14) *becomes*

$$\frac{d(x-z+2u)}{(x-z+2u)}=\frac{d(x+y)}{2(x+y)}=\frac{dz}{3z}=\frac{d\left(u^2+zu\right)}{2\left(u^2+zu\right)}$$

which gives us another first integral

$$(x-z+2u)^2.\left(u^2+zu\right)^{-1}=c_3$$

Hence the integral curves $F(c_1,c_2,c_3)=0$, *where* F *is an arbitrary real function.*

9.2.7 Case (vii): $\rho_1=\rho_2\in\mathbb{R},\ \rho_3=\overline{\rho_4}\in\mathbb{C}$

Example 9.7. *Let the equation*

$$(2x-y)p+(x+2y+2u)q+zr=z+u \tag{9.15}$$

The characteristic system corresponding to equation (9.15) *is*

$$\frac{dx}{2x-y}=\frac{dy}{x+2y+2u}=\frac{dz}{z}=\frac{du}{z+u} \tag{9.16}$$

The polynomial $\Psi(\rho)$ *admits one double root* $\rho_1 = \rho_2 = 1$ *and two complex roots conjugate* $\rho_3 = 2 + i = \overline{\rho_4}$. *The constants* $(\lambda_j, \mu_j, \upsilon_j, \tau_j)$ *associated respectively are* $(0,0,1,0)$, $(1+i, 1-i, 1+i, 2)$ *and* $(1-i, 1+i, 1-i, 2)$. *Then, each ratio of* (9.16) *is equal to*

$$\frac{dz}{z} = \frac{d((1-i)x+(1+i)+y(1-i)z+2u)}{(2+i)((1-i)x+(1+i)+y(1-i)z+2u)} = \frac{d((1+i)x+(1-i)+y(1+i)z+2u)}{(2-i)((1+i)x+(1-i)+y(1+i)z+2u)} \tag{9.17}$$

This leads to two first integrals, we can take the following

$$z^2 \exp(-z_1)\cos(1\log(g) - z_2) = c_1 \quad \text{and} \quad 2z_2 - z_1 = c_2$$

where

$$\begin{aligned} z_1 &= \frac{1}{2}\log\left((x+y+z+2u)^2 + (-x+y-z)^2\right) \\ z_2 &= \arctan\frac{-x+y-z}{x+y+z+2u} \end{aligned}$$

In order to obtain an additional first integral, we use the same procedure as in the previous examples. This approach also enables us to obtain other linearly independent first integrals, but we are limited to finding just one. For that, suppose it is possible to find constants $(\alpha_j, \beta_j, \gamma_j, \delta_j)$, $1 \le j \le 4$, *and* ρ *such that each ratio of* (9.16) *is of the form* (9.8), *with*

$$\begin{aligned} D &= (2\alpha_1+\alpha_2)x^2 + (-\beta_1+2\beta_2)y^2 + (2\gamma_2+\delta_3+\delta_4+\gamma_4)zu \\ &\quad + (\gamma_3+\gamma_4)z^2 + (2\delta_2+\delta_4)u^2 + (2\beta_1+\beta_2+2\alpha_2-\alpha_1)xy \\ &\quad + (-\delta_1+2\delta_2+2\beta_2+\beta_4)yu + (2\delta_1+\delta_2+2\alpha_2+\alpha_4)xu \\ &\quad + (-\gamma_1+2\gamma_2+\beta_3+\beta_4)yz + (2\gamma_1+\gamma_2+\alpha_3+\alpha_4)xz \end{aligned}$$

That's right, if

$$\begin{cases} (4-\rho)\,\alpha_1+2\alpha_2=0\,, & 2\gamma_1+\gamma_2+(1-\rho)\,\alpha_3+\alpha_4=0 \\ (2-\rho)\,\beta_1+\beta_2+2\alpha_2-\alpha_1=0\,, & -\gamma_1+2\gamma_2+(1-\rho)\,\beta_3+\beta_4=0 \\ (2-\rho)\,\gamma_1+\gamma_2+\alpha_3+\alpha_4=0\,, & (2-\rho)\,\gamma_3+2\gamma_4=0 \\ (2-\rho)\,\delta_1+\delta_2+2\alpha_2+\alpha_4=0\,, & 2\gamma_2+(1-\rho)\,\delta_3+\delta_4+\gamma_4=0 \\ 2\beta_1+\beta_2+(2-\rho)\,\alpha_2-\alpha_1=0\,, & 2\delta_1+\delta_2+2\alpha_2+(1-\rho)\,\alpha_4=0 \\ -2\beta_1+(4-\rho)\,\beta_2=0\,, & -\delta_1+2\delta_2+2\beta_2+(1-\rho)\,\beta_4=0 \\ -\gamma_1+(2-\rho)\,\gamma_2+\beta_3+\beta_4=0\,, & 2\gamma_2+\delta_3+\delta_4+(1-\rho)\,\gamma_4=0 \\ -\delta_1+(2-\rho)\,\delta_2+2\beta_2+\beta_4=0\,, & 4\delta_2+(2-\rho)\,\delta_4=0 \end{cases}$$

This system will have a non-trivial solution, if the determinant of its coefficient matrix is zero, i.e.,

$$\rho^{16}-31\rho^{15}+438\rho^{14}-3712\rho^{13}+20\,908\rho^{12}-81\,964\rho^{11}+227\,592\rho^{10}$$
$$-446\,368\rho^{9}+604\,096\rho^{8}-535\,360\rho^{7}+278\,400\rho^{6}-64\,000\rho^{5}=0$$

This polynomial has the following roots : 0, 1, 2, 4, $4\pm 2i$, $3\pm i$. *Just choose a single value for* ρ, *then we can take* $\rho=1$. *substitute it into* (9.8) *and solve the resulting system, we find*

$$(\alpha_1,\beta_1,\gamma_1,\delta_1,\alpha_2,\beta_2,\gamma_2,\delta_2,\alpha_3,\beta_3,\gamma_3,\delta_3,\alpha_4,\beta_4,\gamma_4,\delta_4)$$

as following

$$(0,0,0,0,0,0,0,0,0,0,-2,0,0,0,1,0)\ \textit{for}\ \rho=1$$

Then system (9.17) *becomes*

$$\frac{d\left(-z^2+zu\right)}{-z^2+zu}=\frac{dz}{z}=\frac{d((1-i)x+(1+i)+y(1-i)z+2u)}{(2+i)((1-i)x+(1+i)+y(1-i)z+2u)}=\frac{d((1+i)x+(1-i)+y(1+i)z+2u)}{(2-i)((1+i)x+(1-i)+y(1+i)z+2u)}$$

which gives us another first integral

$$\left(-z^2+zu\right).z^{-1}=c_3$$

Hence the integral curves $F(c_1,c_2,c_3)=0$, *where* F *is an arbitrary real function.*

9.2.8 Case (viii): $\rho_1 = \overline{\rho_2} = \rho_3 = \overline{\rho_4} \in \mathbb{C}$

Example 9.8. *Let the equation*

$$(x+y)p + (-x+y)q + (z+u)r = -z+u \tag{9.18}$$

The characteristic system corresponding to equation (9.18) is

$$\frac{dx}{x+y} = \frac{dy}{-x+y} = \frac{dz}{z+u} = \frac{du}{-z+u} \tag{9.19}$$

The polynomial $\Psi(\rho)$ admits two double conjugate complex roots $\rho_1 = 1-i = \overline{\rho_2} = \rho_3 = \overline{\rho_4}$. The constants $(\lambda_j, \mu_j, \upsilon_j, \tau_j)$ associated respectively are $(i,1,0,0)$ and $(-i,1,0,0)$. Each ratio of (9.19) is equal to

$$\frac{d(ix+y)}{(1-i)(ix+y)} = \frac{d(-ix+y)}{(1+i)(-ix+y)}$$

This leads to the following first integral

$$z_2 - z_1 = c_1$$

where

$$z_1 = \frac{1}{2}\log\left(y^2+x^2\right) \ \ \text{and} \ \ z_2 = \arctan\frac{x}{y}$$

We use the same procedure as before to obtain two additional first integrals. This approach also enables us to obtain other first integrals, but we are limited to finding only two. For that, suppose it is possible to find constants $(\alpha_j, \beta_j, \gamma_j, \delta_j)$, $1 \le j \le 4$, and ρ such that each ratio of (9.19) is of the form (9.8), with

$$\begin{aligned} D \ = \ & (\alpha_1-\alpha_2)x^2 + (\beta_1+\beta_2)y^2 + (\delta_3+\gamma_3+\gamma_4-\delta_4)zu \\ & + (\gamma_3-\gamma_4)z^2 + (\delta_1-\delta_2+\alpha_3+\alpha_4)xu + (\gamma_1-\gamma_2+\alpha_3-\alpha_4)xz \\ & + (\delta_1+\delta_2+\beta_3+\beta_4)yu + (\delta_3+\delta_4)u^2 + (\gamma_1+\gamma_2+\beta_3-\beta_4)yz \\ & + (\beta_1+\alpha_1-\beta_2+\alpha_2)xy \end{aligned}$$

That's right, if

$$\begin{cases} (2-\rho)\,\alpha_1-2\alpha_2=0, & \gamma_1-\gamma_2+(1-\rho)\,\alpha_3-\alpha_4=0\\ (1-\rho)\,\beta_1+\alpha_1-\beta_2+\alpha_2=0, & \gamma_1+\gamma_2+(1-\rho)\,\beta_3-\beta_4=0\\ (1-\rho)\,\gamma_1-\gamma_2+\alpha_3-\alpha_4=0, & (2-\rho)\,\gamma_3-2\gamma_4=0\\ (1-\rho)\,\delta_1-\delta_2+\alpha_3+\alpha_4=0, & (1-\rho)\,\delta_3+\gamma_3+\gamma_4-\delta_4=0\\ \beta_1+\alpha_1-\beta_2+(1-\rho)\,\alpha_2=0, & \delta_1-\delta_2+\alpha_3+(1-\rho)\,\alpha_4=0\\ 2\beta_1+(2-\rho)\,\beta_2=0, & \delta_1+\delta_2+\beta_3+(1-\rho)\,\beta_4=0\\ \gamma_1+(1-\rho)\,\gamma_2+\beta_2-\beta_4=0, & \delta_3+\gamma_3+(1-\rho)\,\gamma_4-\delta_4=0\\ \delta_1+(1-\rho)\,\delta_2+\beta_3+\beta_4=0, & 2\delta_3+(2-\rho)\,\delta_4=0 \end{cases}$$

This system will have a non-trivial solution, if the determinant of its coefficient matrix is zero, i.e.,

$$\rho^{16}-20\rho^{15}+193\rho^{14}-1173\rho^{13}+4957\rho^{12}-15232\rho^{11}+34650\rho^{10}$$
$$-58344\rho^{9}+71512\rho^{8}-61504\rho^{7}+34688\rho^{6}-11264\rho^{5}+1536\rho^{4}=0$$

Among the roots of this polynomial, we take 1 *and* 2*, and we substitute them in* (9.8)*. This allows us to find the constants*

$$(\alpha_1,\beta_1,\gamma_1,\delta_1,\alpha_2,\beta_2,\gamma_2,\delta_2,\alpha_3,\beta_3,\gamma_3,\delta_3,\alpha_4,\beta_4,\gamma_4,\delta_4)$$

as following

$$\begin{aligned} (0,0,1,1,0,0,-1,1,0,-2,0,0,1,1,0,0) \text{ for } \rho &= 1\\ (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1) \text{ for } \rho &= 2 \end{aligned}$$

Each ratio of (9.19) *is equal to*

$$\frac{d\,(xz+xu+-3yz+yu)}{1(xz+xu+-3yz+yu)}=\frac{d\left(u^2-zu\right)}{2\left(u^2-zu\right)}=\frac{d\,(-ix+y)}{(1+i)(-ix+y)}=\frac{d\,(ix+y)}{(1-i)(ix+y)}$$

which gives us the following first integrals

$$\begin{aligned} (xz+xu+-3yz+yu)^2\cdot\left(u^2-zu\right)^{-1} &= c_2\\ \left(u^2-zu\right)\cdot e^{-2z_1}\cos\left(\log\left(u^2-zu\right)-2z_2\right) &= c_3 \end{aligned}$$

Hence the integral curves $F(c_1,c_2,c_3)=0$*, where* F *is an arbitrary real function.*

9.2.9 Case (ix): $\rho_1 = \rho_2 = \rho_3 = \rho_4 \in \mathbb{R}$

Example 9.9. *Let the equation*

$$(2x)p + (2y)q + (2y+2z)r = -x+z+2u \tag{9.20}$$

The characteristic system corresponding to equation (9.20) *is*

$$\frac{dx}{2x} = \frac{dy}{2y} = \frac{dz}{2y+2z} = \frac{du}{-x+z+2u} \tag{9.21}$$

The polynomial $\Psi(\rho)$ *admits one quadruple root* $\rho = 2$. *The constants* (λ,μ,v,τ) *associated respectively are* $(1,-1,0,0)$. *Each ratio of* (9.21) *is equal to*

$$\frac{d(x-y)}{2(x-y)}$$

We use the same procedure as in the previous example to obtain other first integrals. For this, suppose that it is possible to find constants $(\alpha_j,\beta_j,\gamma_j,\delta_j)$, $1 \le j \le 4$, *and* ρ *such that each ratio of* (9.21) *is of the form* (9.8), *with*

$$\begin{aligned} D &= (2\alpha_1-\alpha_4)x^2 + (2\gamma_3+\gamma_4)z^2 + (\delta_4+2\delta_3+2\gamma_4)zu + (2\delta_1-\delta_4+2\alpha_4)xu \\ &\quad + (2\beta_1+2\alpha_2+2\alpha_3-\beta_4)xy + (2\gamma_1+2\alpha_3-\gamma_4+\alpha_4)xz + (2\delta_4)u^2 \\ &\quad + (2\gamma_3+\beta_4+2\beta_3+2\gamma_2)yz + (2\delta_3+2\delta_2+2\beta_4)yu + (2\beta_2+2\beta_3)y^2 \end{aligned}$$

That's right, if

$$\left\{ \begin{array}{rr} (4-\rho)\alpha_1 - 2\alpha_4 = 0, & 2\gamma_1 + (2-\rho)\alpha_3 - \gamma_4 + \alpha_4 = 0 \\ (2-\rho)\beta_1 + 2\alpha_2 + 2\alpha_3 - \beta_4 = 0, & 2\gamma_3 + \beta_4 + (2-\rho)\beta_3 + 2\gamma_2 = 0 \\ (2-\rho)\gamma_1 + 2\alpha_3 - \gamma_4 + \alpha_4 = 0, & (4-\rho)\gamma_3 + 2\gamma_4 = 0 \\ (2-\rho)\delta_1 - \delta_4 + 2\alpha_4 = 0, & \delta_4 + (2-\rho)\delta_3 + 2\gamma_4 = 0 \\ 2\beta_1 + (2-\rho)\alpha_2 + 2\alpha_3 - \beta_4 = 0, & 2\delta_1 - \delta_4 + (2-\rho)\alpha_4 = 0 \\ (4-\rho)\beta_2 + 4\beta_3 = 0, & 2\delta_3 + 2\delta_2 + (2-\rho)\beta_4 = 0 \\ 2\gamma_3 + \beta_4 + 2\beta_3 + (2-\rho)\gamma_2 = 0, & \delta_4 + 2\delta_3 + (2-\rho)\gamma_4 = 0 \\ 2\delta_3 + (2-\rho)\delta_2 + 2\beta_4 = 0, & (4-\rho)\delta_4 = 0 \end{array} \right.$$

This system will have a non-trivial solution, if the determinant of its coefficient matrix is zero, i.e.,

$$\rho^{16}-40\rho^{15}+720\rho^{14}-7676\rho^{13}+53632\rho^{12}-256256\rho^{11}+845824\rho^{10}$$
$$-1894400\rho^{9}+2719744\rho^{8}-2162688\rho^{7}+524288\rho^{6}+262144\rho^{5}=0$$

Among the roots of this polynomial, we take $\rho = 4$, and we substitute it in (9.8), This allows us to find the constants

$$\left(\alpha_1,\beta_1,\gamma_1,\delta_1,\alpha_2,\beta_2,\gamma_2,\delta_2,\alpha_3,\beta_3,\gamma_3,\delta_3,\alpha_4,\beta_4,\gamma_4,\delta_4\right)$$

as following

$$(1,1,0,0,1,1,0,0,0,0,0,0,0,0,0,0)$$
$$(1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)$$
$$(0,1,0,0,1,0,0,0,0,0,0,0,0,0,0,0)$$

These values would transform system (9.21) into the following exact differential form

$$\frac{d\left(2x^2+2y^2+4xy\right)}{4\left(2x^2+2y^2+4xy\right)}=\frac{d\left(2x^2\right)}{4\left(2x^2\right)}=\frac{d\left(4xy\right)}{4\left(4xy\right)}=\frac{d\left(x-y\right)}{2\left(x-y\right)}$$

which gives us these linearly independent first integrals

$$\begin{aligned}\left(2x^2+2y^2+4xy\right)\cdot\left(2x^2\right)^{-1} &= c_1\\ \left(2x^2\right)\cdot\left(4xy\right)^{-1} &= c_2\\ \left(4xy\right)\cdot\left(x-y\right)^{-2} &= c_3\end{aligned}$$

Hence the integral curves $F(c_1,c_2,c_3)=0$, where F is an arbitrary real function.

9.3 Conclusion

Our work introduces a versatile approach that can effectively solve a wide range of first order quasilinear equations, as described within the paper. The method's relevance transcends practical applications, encompassing various mathematical equations exhibiting quasilinearity. Through providing a comprehensive framework, our research establishes a solid groundwork for addressing such equations and ensuring their successful resolution, irrespective of their immediate real-world significance. The broad scope and applicability of our method highlight its importance and potential in advancing the field of mathematical analysis. As a practical application, we can use it to determine the surfaces orthogonal to a given system of surfaces and to solve the Hamilton-Jacobi equation, which is of great importance as a first order partial differential equation in mathematical physics. Additionally, it is worth noting that our methodology can be extended to tackle equations with two variables of the same type. This extension finds relevance in several renowned examples, including the transport equation, Maxwell's equation, and others.

In conclusion, we emphasize the challenging nature of unsolved PDEs and the need for fresh insights and innovative concepts to address them. We assert that the field of PDEs is rich with unresolved equations awaiting breakthroughs. In the near future, we will endeavor to study some classes of first order nonlinear PDEs.

APPENDIX A

CONCLUSION AND PERSPECTIVES

This thesis delves into the captivating realm of reaction-diffusion systems (RDSs) and their profound applications in the realms of biology and medicine.

Conclusion

- This research work focuses on understanding and studying quenching phenomena, which involve rapid changes in temperature or physical conditions and can lead to complex phenomena in different systems. The primary goal of the research is to contribute to the advancement of quenching technology and modeling in diverse scientific disciplines. By deepening the understanding of quenching through mathematical research, the study can potentially lead to improved techniques, methodologies, and models that benefit fields such as bioengineering, biology, and others.
- The results obtained in this study have broader implications beyond quenching phenomena alone. They can be applied to the study of other singular reaction-diffusion phenomena, opening up new avenues for investigating and understanding various physical and chemical processes where

similar mathematical principles may be at play. The study's findings and insights provide a foundation for further theoretical and numerical investigations under different conditions, aiming to achieve additional progress in the field and expand our knowledge. These efforts aim to push the boundaries of current understanding and potentially uncover new phenomena or mechanisms.

- **Specifically, the research proves the existence of continuous positive solutions for a specific class of quasi-linear reaction-diffusion systems of arbitrary order, extending previous studies conducted on systems with a specific number of equations. The approach taken to achieve this objective is mainly based on potential analysis techniques, yielding reliable results with less computational effort. The main result obtained demonstrates the effectiveness of this approach for solving similar problems. Overcoming certain difficulties, the study achieved significant results, prompting further exploration of the problem in different circumstances. Additionally, a numerical study of the problem can be conducted to deepen understanding. This would involve further exploration of the problem and could provide additional insights and results.**

- **This research contributes to the understanding and solving of a specific class of quasilinear first-order partial differential equations (PDEs). It emphasizes the challenging nature of unsolved PDEs and the need for fresh insights and innovative concepts to address them. The study provides a methodological approach and explores various aspects related to the solving process. In the future, we plan to study some classes of first-order nonlinear PDEs, further advancing the field of PDE research.**

Perspectives

Here are some potential directions for further research on reaction-diffusion systems that could build upon or complement the work conducted in this thesis:

- **Consider systems with spatial or time-dependent coefficients, to model non-homogeneity. This adds complexity but more realism.**

- **Explore mixed hyperbolic-parabolic or nonlocal systems coupling diffusion to advection/jump terms. Such couplings arise in many physical/biological settings.**
- **Extend theoretical analysis tools like potential theory to broader classes of equations, e.g. fully nonlinear or degenerate operators.**
- **Numerically simulate more complex, multispecies systems to visualize dynamics and aid refinement of theoretical results.**
- **Apply potential-based methods to open problems like proving finite-time blowup or quenching in certain models.**
- **Study singular irreversible reaction terms modeling quenching past critical points like phase transitions.**
- **Couple reaction-diffusion to other physics, e.g fluid dynamics, to understand multi-scale pattern formation.**

APPENDIX B

KATO'S CLASSES

This appendix draws inspiration from Zhang's work and provides additional information and explanations regarding Kato's class specifically for chapters three and seven.

In recent years, there have been many results in the study of elliptic equations with singular lower order terms in the Kato class. We recall that for $n \geq 3$, a function $V \in L^1_{Loc}(\mathbb{R}^n)$ is said to belong to the Kato class K_n provided that

$$\lim_{r\to 0} \sup_{x} \int_{|x-y|<r} \frac{|V(y)|}{|x-y|^{n-2}} dy = 0$$

In their celebrated work [8], using probabilistic ideas, Aizenman and Simon proved that the class K_n is the natural replacement of the Lebesgue class L^p, with $p > \frac{n}{2}$, in order for solutions of $-\Delta u + Vu = 0$ to satisfy a Harnack inequality. As a consequence, they obtained a modulus of continuity for such solutions. Subsequently, using PDE methods, the authors of the paper of Chiarenza et *al*. [41] generalized the results in Aizenman and Simon [8] to elliptic equations in divergence form with bounded, measurable coefficients. In both works an important role was played by an embedding result for the class K_n due to Schechter. In contrast to the elliptic case there is not much investigation (known to the author) on parabolic equations with lower order coefficients in a class parallel to the Kato class. In this Appendix we will take up this problem. As the reader will see, this task presents some notable differences from the elliptic situation. To clarify the point we mention that it is not clear whether a version of Schechter's embedding theorem holds in the parabolic setting. Therefore, we had to work around this obstacle. We benefited from the ideas in [73], in which Fabes and Stroock deduced the Harnack inequality for parabolic equations from lower and upper bounds of the fundamental solutions. In [223], Zhang succeeded in obtaining similar lower and upper bounds for the fundamental solutions of the parabolic equations with a singular lower order term in the parabolic Kato class, which is being proposed as a natural generalization of the Kato class in the elliptic case. Based on these bounds, a Harnack inequality was established. It is interesting to note that the results of the paper recapture those in Aizenman and Simon [8], Chiarenza et *al.* [41] when one deals with time-independent solutions.

We are interested in the parabolic equation

$$Hu(t,x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(t,x)\frac{\partial u}{\partial x_j}\right)(t,x) - V(t,x)u(t,x) - \frac{\partial u}{\partial t}(t,x) = 0$$

in a bounded domain $Q = D \times [0,T] \subset \mathbb{R}^{n+1}$. Here a_{ij} are bounded measurable functions and the matrix $a = \left(a_{ij}(t,x)\right)$ is positive definite uniformly in (t,x). This means the existence of a number $\lambda > 1$ such that $\lambda^{-1}I \le a \le \lambda I$.

For the function V we will impose the following condition which will be called condition K. Let

$$N_h(V) = \sup_{t,x} \int_{t-h}^{t} \int_D |V(y,s)| \frac{1}{(t-s)^{\frac{n}{2}}} \exp\left(-\alpha \frac{|x-y|^2}{t-s}\right) ds dy \tag{B.1}$$

$$N_h^*(V) = \sup_{s,y} \int_{s}^{s+h} \int_D |V(x,t)| \frac{1}{(t-s)^{\frac{n}{2}}} \exp\left(-\alpha \frac{|x-y|^2}{t-s}\right) dt dx \tag{B.2}$$

where α is a fixed positive constant and $V(s,y)$ is regarded as zero when (s,y) is outside of Q.

Definition B.1. We say that V satisfies condition K if

$$\lim_{h\to 0} N_h(V) = \lim_{h\to 0} N_h^*(V) = 0. \tag{B.3}$$

The space of all L^1_{Loc} functions satisfying condition K will be called the parabolic Kato class.

The parabolic Kato class is a natural generalization of the Kato class in the study of elliptic equations. Note that the parabolic Kato class depends on the parameter α. For our purpose α can be any positive number so that the following inequality holds:

$$\Gamma_0(t,x;s,y) \le \frac{C}{(t-s)^{\frac{n}{2}}} \exp\left(-\alpha \frac{|x-y|^2}{t-s}\right),$$

for some $C > 0$ and $0 < t-s \le T$. Here Γ_0 is the fundamental solution of the unperturbed operator H_0 defined by

$$H_0 u \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(t,x)\frac{\partial u}{\partial x_j}\right)(t,x) - \frac{\partial u}{\partial t}(t,x).$$

We need to give a few more notations and definitions.

Green's functions: We will use $G(t,x;s,y)$ and $G_0(t,x;s,y)$ to denote the weak Green's functions of operators H and H_0 for the initial Dirichlet problem on Q respectively. The precise definition of the weak Green's function is standard and can be found in Aronson [23]. The symbols $\nabla_x G$ and $\nabla_y G$ mean the gradient with respect to the first and the second space variables of G respectively.

Weak solutions: Given $f \in L^1(Q)$, a weak solution of $Hu = f$ in Q is a function u satisfying:

(a) $u \in C^0\left([0,T];L^2(D)\right) \cap L^2\left(0,T;W^{1,2}(D)\right)$

(b) $Vu \in L^1(Q)$

(c) $\int_0^T \int_D \left[-A\nabla u \nabla\phi - Vu\phi + u\phi_t\right] dtdx = \int_0^T \int_D f\phi dtdx$

for all $\phi \in C_0^\infty(Q)$.

Theorem B.1. *The weak Green's function of H for the initial Dirichlet problem exists and satisfies the following properties.*

(a)

$$\begin{aligned}\sup_{s,y} \|G(.,.;s,y)\|_{L^p(Q)} &\le C \\ \sup_{t,x} \|G(t,x;.,.)\|_{L^p(Q)} &\le C,\ 0 < p < \frac{n+2}{n}\end{aligned}$$

(b)

$$\begin{aligned}\sup_{t,x} \|\nabla G(t,x;.,.)\|_{L^p(Q)} &\le C \\ \sup_{s,y} \|\nabla G(.,.;s,y)\|_{L^p(Q)} &\le C,\ 0 < p < \frac{n+2}{n+1}\end{aligned}$$

Here the constant C depends on λ, p, Q and on V in terms of the rate of convergence of $(B.3)$.

A direct consequence of Theorem B.1 is the next:

Theorem B.2. *Suppose D is a bounded C^1 domain. Given $f \in C^0\left(\overline{D}\right)$ and $g \in L^\infty(Q)$, the following initial boundary value problem has a unique bounded weak solution.*

$$\begin{cases} Hu = g \text{ in } D \times [0,T], \\ u = 0 \text{ in } \partial D \times [0,T], \text{ and } \lim\limits_{t \to 0^+} u(t,x) = f(x). \end{cases}$$

Theorem B.3 (Harnack inequality)**.** *Let $0 < \alpha_2 < \beta_2 < \alpha_1 < \beta_1 < 1$ and $\delta \in (0,1)$ be given. Then there are $M > 0$ and $R_0 > 0$ such that for all $(s,x) \in \mathbb{R}^n \times \mathbb{R}$, all positive $R < R_0$ and all non-negative weak solutions u of $Hu = 0$ in $B(R,x) \times \left[s - R^2, s\right]$, one has*

$$\sup_{Q^-} u \leq M \inf_{Q^+} u$$

where

$$\begin{aligned} Q^- &= B(\delta R, x) \times \left[s - \beta_1 R^2, s\alpha_1 R^2\right] \\ Q^+ &= B(\delta R, x) \times \left[s - \beta_2 R^2, s\alpha_2 R^2\right] \end{aligned}$$

Functions satisfying condition K are in general more singular than $L^{p,q}$ functions when $\frac{n}{2p} + \frac{1}{q} < 1$. Moreover the parabolic Kato class we are proposing reduces to the Kato class in the elliptic case when the function V is independent of t.

We will provide some further analysis of Condition K and show that the condition is indeed the natural extension of the elliptic Kato class. Since much of the argument is elementary, we tend to be brief.

Proposition B.1. *The parabolic Kato class strictly contains the space $L^{p,q}(Q)$ with $\frac{n}{2p} + \frac{1}{q} < 1$.*

Proof. Proof. By Hölder's inequality we have

$$N_h(V) \leq C_h \|V\|_{L^{p,q}} \quad \text{and} \quad N_h^*(V) \leq C_h \|V\|_{L^{p,q}}.$$

Here C_h is a constant which depends on p, q and h and which goes to zero when $h \to 0$. We refer the reader to Sturm [207] for details. ■

Proposition B.2. *If the function V is independent of t, then V belongs to the parabolic Kato class if and only if it belongs to the Kato class in the elliptic case, i.e.,*

$$\lim_{r\to 0}\sup_{x}\int_{B_r(x)}\frac{|V(y)|}{|x-y|^{n-2}}dy=0,\ n\geq 3.$$

Proof. By direct calculation, when $n\geq 3$,

$$\begin{aligned}\int_{s}^{s+h}\Gamma_{\alpha}(t,x;s,y) &\leq \frac{C}{|x-y|^{n-2}},\\ \int_{s-h}^{s}\Gamma_{\alpha}(t,x;s,y) &\leq \frac{C}{|x-y|^{n-2}}\end{aligned}$$

where

$$\Gamma_{\alpha}(t,x;s,y)\leq\frac{C}{(t-s)^{\frac{n}{2}}}\exp\left(-\alpha\frac{|x-y|^2}{t-s}\right).$$

The result of the proposition immediately follows. ■

Proposition B.3. *Let $V'\equiv V(-t,x)$. Then*

$$N_h(V')=N_h^*(V')$$

In this sense N_h^* is a reflection of N_h.

Proof. From ($B.2$)

$$N_h(V')=\sup_{t,x}\int_{t-h}^{t}\int_{D}|V(y,-s)|\frac{1}{(t-s)^{\frac{n}{2}}}\exp\left(-\alpha\frac{|x-y|^2}{t-s}\right)dsdy,$$

Making the change of variables $s=-s$, we obtain

$$\begin{aligned}N_h(V') &= \sup_{t,x}\left[-\int_{-t+h}^{-t}\int_{D}|V(y,s)|\frac{1}{(t-s)^{\frac{n}{2}}}\exp\left(-\alpha\frac{|x-y|^2}{t-s}\right)dsdy\right]\\ &= \sup_{t,x}\int_{-t}^{-t+h}\int_{D}|V(y,s)|\frac{1}{(t-s)^{\frac{n}{2}}}\exp\left(-\alpha\frac{|x-y|^2}{t-s}\right)dsdy\\ &= N_h^*(V').\ q.e.d.\end{aligned}$$

■

Remark B.1. *We note that the parabolic Kato class depends on the constant α, which appears in ($B.1$) and ($B.2$). This is a new situation that does not happen in the elliptic case.*

APPENDIX C

KNOWN NAMES IN THE FIELD OF REACTION-DIFFUSION SYSTEMS

The purpose of this page is to present some mathematicians who, through their works, have contributed to the development of reaction-diffusion systems.

Alan Turing was a pioneering British mathematician, logician, and computer scientist. He was born on June 23, 1912, in London, England. Turing's work during World War II played a crucial role in breaking the German Enigma code, helping the Allies gain a significant advantage. Turing is widely regarded as the father of modern computer science and artificial intelligence. He laid the foundation for theoretical computer science by introducing the concept of a universal machine, now known as the Turing machine. His theoretical work on computation and algorithms established the basis for the development of modern computers. He also made significant contributions to the field of mathematical biology, particularly in the study of morphogenesis. Alan Turing died on June 7, 1954, at the age of 41.

V**ito Volterra** was an Italian mathematician and physicist born on May 3, 1860, in Ancona, Italy. He is renowned for his significant contributions to mathematical biology and mathematical physics. Volterra made important advancements in the fields of functional analysis and integral equations, particularly in the study of nonlinear systems. He developed a mathematical theory known as the Volterra integral equations, which have applications in various scientific disciplines. He also made notable contributions to economics and ecology. His ecological research focused on predator-prey relationships and the dynamics of biological populations. Vito Volterra's legacy continues to influence various scientific fields. His mathematical models and theories remain significant in the study of nonlinear systems, population dynamics, and ecology. Volterra died on October 11, 1940, in Rome, Italy.

A**lfred James Lotka** was an American mathematician, statistician, and demographer. He was born on March 2, 1880, in Lemberg, Austria-Hungary (now Lviv, Ukraine) and later immigrated to the United States. Lotka is known for his significant contributions to mathematical biology and population ecology. Lotka's groundbreaking work focused on the mathematical modeling of population dynamics and the study of biological systems. He developed the Lotka-Volterra equations, a set of differential equations that describe the interactions between predator and prey populations. Lotka also made significant contributions to the field of demography. He published numerous influential papers and books, including his notable work "Elements of Physical Biology" in 1925. He died on December 5, 1949, in New York City.

H**iroshi Kawarada** is the Professor of the Faculty of Science and Engineering at Waseda University, Japan, where he is the leader of nano-electronics and power-electronics, developing carbon-based material devices including diamonds for nanodevices, biosensors, microwaves, and power field effect transistors. He was appointed as a Board Member of the Japan Applied Physics Society for the years 2005–2007, the Director of the Research Organization for Nanoscience & Nanotechnology from 2009–2010, and he was also chosen as the Chairman of the New Diamond Forum from 2009–2014. Along with his career, he has published 9 books, given numerous invited talks, and has more than 300 scientific publications and over 30 patents. His articles, proceedings, and patents have been cited by about 9,000 (Google Scholar). He won the Minister of Education, Culture, Sports, Science, and Technology Award in 2016, the Applied Physics Society Fellow Award in 2010, and the Superconductivity Science and Technology Award in 2007.

Jacques-Louis Lions born on May 3, 1928, and passed away on May 17, 2001, was a renowned French mathematician who made significant contributions to the field of applied mathematics, particularly in the area of partial differential equations (PDEs). Lions was born in Grasse, France. He displayed exceptional mathematical talent from a young age and entered the prestigious École Normale Supérieure in Paris in 1948. There, he studied under renowned mathematicians such as Laurent Schwartz and Jean Leray. He obtained his doctorate in mathematics in 1954 under the guidance of Schwartz. Lions' research focused on the theory and applications of partial differential equations. His work greatly impacted the mathematical understanding and numerical analysis of these equations, providing new insights and techniques for their study. Lions received numerous honors and awards for his outstanding contributions. In 1986, he was awarded the Fields Medal, one of the highest honors in mathematics, for his work on nonlinear partial differential equations and their applications.

James Dickson Murray born on January 2, 1931, in Moffat, Scotland, is a highly regarded British mathematician and Professor Emeritus of Applied Mathematics at both the University of Washington and the University of Oxford. He has made significant contributions to a wide range of research areas, demonstrating remarkable breadth and depth in his work. One of Murray's notable achievements is his authoritative book "Mathematical Biology," which has established him as a leading figure in the field. Murray's research encompasses a diverse array of topics, showcasing his intellectual curiosity and versatility. For instance, he has made groundbreaking contributions to understanding the biomechanics of the human body during ejection seat launches from aircraft. His research has delved into understanding and preventing severe scarring, the formation of fingerprints, determining sex in biological systems, modeling animal coats, and analyzing the formation of territories in populations of interacting wolf-deer. He also has practical implications in fields such as medicine, ecology, and animal behavior.

A**mmar Youkana** is an Algerian mathematician, born on June 11, 1958, in Batna, Algeria. Youkana pursued his academic journey at the renowned Jacques-Louis Lions Laboratory, Pierre and Marie Curie University, Paris VI, France, where he obtained his degree in applied mathematics in 1986. Throughout his career, Youkana has focused on the study of reaction-diffusion systems. His work in this area has shed light on the behavior and properties of such systems, leading to valuable insights with applications in diverse fields. He has many important scientific papers in this field. Presently, Youkana serves as a distinguished mathematics professor in the Mathematics Department of the Mustapha Ben Boulaïd University of Batna. His contributions to the field and dedication to education have made him a respected figure within the academic community.

M**ichel Pierre** is an esteemed French mathematician, born in 1961 in France. He has held the position of Professor at ENS Cachan Bretagne since 1996 and is a distinguished researcher at IRMAR, specifically in the Digital Analysis team. His research focuses on improving models and studying reaction-diffusion systems, making notable contributions in both areas. Throughout his career, Pierre has demonstrated a remarkable dedication to advancing the understanding and application of mathematics. Pierre's research on reaction-diffusion systems has also been influential. His deep understanding of these systems has enabled him to contribute novel approaches and methods for their analysis and modeling. His work has enhanced our understanding of the dynamics and behavior of reaction-diffusion phenomena, with applications spanning diverse scientific disciplines. As an accomplished mathematician, Pierre has authored and co-authored a substantial number of scientific papers across various branches of applied mathematics. His publications showcase his breadth of knowledge and expertise, reflecting his contributions to the field and his commitment to advancing mathematical research. Among his students was the great mathematician Noureddine Alaa.

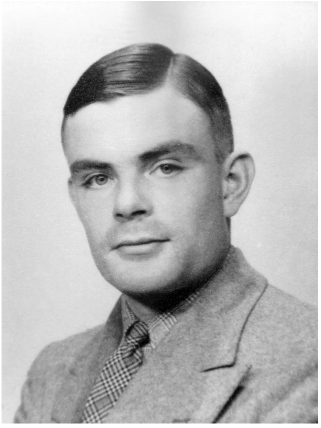

FIGURE C.1. Alan Turing.

FIGURE C.2. Vito Volterra.

FIGURE C.3. Alfred James Lotka.

FIGURE C.4. Hiroshi Kawarada.

FIGURE C.5. Jacques-Louis Lions.

FIGURE C.6. James Dickson Murray.

FIGURE C.7. Ammar Youkana.

FIGURE C.8. Michel Pierre.

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العنوان : ظواهر التبريد وأنظمة تفاعل-انتشار في علم الأحياء والطب.

ملخص : تستكشف هذه الأطروحة موضوع أنظمة تفاعل- انتشار وتطبيقاتها في علم الأحياء والطب والهندسة الحيوية. ويتكون من تسعة فصول تغطي جوانب مختلفة من الموضوع. تبدأ الأطروحة بمقدمة عامة تؤكد على جوهر الموضوع والقضايا المحددة التي تم استكشافها في البحث. يقدم الفصل الأول نظرة عامة حول أنظمة تفاعل- انتشار وتطبيقاتها في التخصصات العلمية المختلفة. ويركز الفصل الثاني على مفهوم "التبريد" وتطبيقاته في علم الأحياء والطب والهندسة الحيوية. ويقدم الفصل الثالث أنظمة تفاعل- انتشار مع شروط أولية. ويناقش الفصل الرابع المعادلات التفاضلية الجزئية شبه الخطية من الدرجة الأولى. تتعمق الفصول الخامس والسادس والسابع في أنظمة تفاعل- انتشار الشاذة وظاهرة التبريد. ويتناول الفصل الثامن نظام تفاعل- انتشار شبه خطي ذي رتبة كيفية. ويركز الفصل الأخير على فئة محددة من المعادلات التفاضلية الجزئية شبه الخطية من الدرجة الأولى. تقدم الأطروحة الأطر النظرية والأساليب الرياضية والتطبيقات العملية، مما يساهم في التقدم في أنظمة تفاعل- انتشار.

كلمات مفتاحية : أنظمة تفاعل - انتشار، معادلات مكافئة شاذة، ظاهرة التبريد، وجود شامل، حلول موجبة، معادلات تفاضلية جزئية شبه خطية من الدرجة الأولى.

Title : QUENCHING PHENOMENA AND REACTION-DIFFUSION SYSTEMS IN BIOLOGY AND MEDICINE.

Abstract : This thesis explores the topic of reaction-diffusion systems (RDSs) and their applications in biology, medicine, and bioengineering. It consists of nine chapters that cover various aspects of the subject. The thesis begins with a general introduction that emphasizes the essence of the subject and the specific issues explored in the research. The first chapter provides an overview of RDSs and their applications in different scientific disciplines. The second chapter focuses on the concept of "Quenching" and its applications in biology, medicine, and bioengineering. The third chapter introduces RDSs with initial conditions. The fourth chapter discusses first order quasilinear PDEs. Chapters five, six and seven delve into singular RDSs and the quenching phenomenon. The eighth chapter examines a quasilinear RDS of arbitrary order. The last chapter focuses on a specific class of first order quasilinear PDEs. The thesis presents theoretical frameworks, mathematical methods, and practical applications, contributing to advancements in RDSs.

Keywords : Reaction-diffusion systems, singular parabolic equations, quenching phenomenon, global existence, positive solutions, quasilinear first order PDE.

Titre : PHÉNOMÈNES DE TREMPE ET SYSTÈMES DE RÉACTION-DIFFUSION EN BIOLOGIE ET MÉDECINE.

Résumé : Cette thèse explore le thème des systèmes de réaction-diffusion (SRDs) et leurs applications en biologie, médecine et bio-ingénierie. Il se compose de neuf chapitres qui couvrent divers aspects du sujet. La thèse commence par une introduction générale qui met l’accent sur l’essence du sujet et les enjeux spécifiques explorés dans la recherche. Le premier chapitre donne un aperçu des SRDs et de leurs applications dans différentes disciplines scientifiques. Le deuxième chapitre se concentre sur le concept de "Quenching" et ses applications en biologie, médecine et bio-ingénierie. Le troisième chapitre présente les SRDs avec conditions initiales. Le quatrième chapitre traite des EDPs quasi-linéaires du premier ordre. Les chapitres cinq, six et sept se penchent sur les SRDs singuliers et sur le phénomène de trempe. Le huitième chapitre examine un SRD quasi-linéaire d'ordre arbitraire. Le dernier chapitre se concentre sur une classe spécifique d’EDPs quasi-linéaires du premier ordre. La thèse présente des cadres théoriques, des méthodes mathématiques et des applications pratiques, contribuant aux progrès des SRDs.

Mots-clés : Systèmes de réaction-diffusion, équations paraboliques singulières, phénomène de quenching, existence globale, solutions positives, EDP quasi-linéaire du premier ordre.