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ANALYSIS III
Intended to the students of second year Bachelor in Mathematics

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Contents

Introduction

These notes constitute the core of the semester-long course in Analysis III offered to second-year students in the Mathematics LMD program. They may also be useful for second-year students in the physics program as well as for students in the common engineering core. This course consists of five chapters: the first is dedicated to numerical series, the second to sequences and series of functions, and the third to power series, particularly important for the study of functions of complex variables. Fourier series, which are useful in the third year, especially in problems governed by parabolic equations, constitute the fourth chapter. Finally, the last chapter addresses the convergence of integrals and is complemented by an introduction to integrals dependent on one or more parameters. We hope that our dear students will derive greater benefit from these notes.

Chapter 1

NUMERICAL SERIES

1.1 Definitions

Let $(U_n)_{n\in\mathbb{N}^*}$ be a sequence of real or complex numbers. From this sequence, we define a new sequence
$(S_n)_{n\in\mathbb{N}^*}$ as follows:

$$\begin{aligned}
S_1 &= U_1 \\
S_2 &= U_1 + U_2 \\
&\dots \\
S_k &= U_1 + U_2 + \dots + U_k \\
&\dots \\
S_n &= \textstyle\sum_{k=1}^{n} U_k.
\end{aligned} \tag{1.1}$$

The general term S_n is called the partial sum of order n.

1. A numerical series, denoted by $\{(U_n)_{n\in\mathbb{N}^*},(S_n)_{n\in\mathbb{N}^*}\}$, is defined, where U_n is called the general term of the series.

2. The series is said to be **convergent** with a sum S if the following limit exists and is finite:
$$S = \lim_{n\to+\infty} S_n = \sum_{n=1}^{\infty} U_n, \tag{1.2}$$
Otherwise, i.e., if this limit is ∞ or does not exist, then the series is said to be divergent.

3. The remainder of order n for the series $\sum_{n=1}^{\infty} U_n$ is defined as the number
$$R_n = U_{n+1} + U_{n+2} + ... = \sum_{p=n+1}^{\infty} U_p. \tag{1.3}$$
The series converges if the sequence $(R_n)_{N\in\mathbb{N}^*}$ approaches 0 as $n \to +\infty$:
$$\sum_{n=1}^{\infty} U_n \text{ converge} \Leftrightarrow \forall\varepsilon > 0, \exists n_0 \in \mathbb{N}^*, \forall n \in \mathbb{N}^*, n \geq n_0 \Longrightarrow |R_n| < \varepsilon. \tag{1.4}$$
If S is this limit, this is equivalent to
$$\forall\varepsilon > 0, \exists n_0 \in \mathbb{N}^*, \forall n \in \mathbb{N}^*, n \geq n_0 \Longrightarrow |S - S_n| < \varepsilon. \tag{1.5}$$

Example 1 *Consider the series $a+ar+ar^2+..+ar^2...$, which is called a geometric series with the first term a and common ratio r. Assuming $r \neq 1$, we then calculate S_n:*
$$\begin{aligned} U_1 &= a, \\ U_2 &= ar, \\ &... \\ U_n &= ar^{n-1}. \end{aligned}$$
We have $S_n = a + ar + ... + ar^{n-1}$ and $rS_n = ar + ar^2 + ... + ar^{n-1} + ar^n$, By

subtraction, $S_n - rS_n = a - ar^2$*, so*

$$S_n = \frac{a\left(1-r^n\right)}{1-r}.$$

Discussion:

- If $|r| < 1$, $\lim\limits_{n\to+\infty} r^n = 0 \Longrightarrow S = \lim\limits_{n\to+\infty} S_n = \dfrac{a}{1-r}$, then the series $\sum_{n=1}^{\infty} U_n$ is convergent.

- If $|r| > 1, r^n \to +\infty$ as $n \to +\infty$, so $\lim\limits_{n\to+\infty} S_n = \pm\infty$ and in this case, the geometric series is divergent.

- If $r = 1$, the series is $a+a+...a+...$, and in this case $S_n = na \Longrightarrow \lim\limits_{n\to+\infty} S_n = +\infty$ and the series diverges.

- If $r = -1$, we have the series $a - a + a - a + ...$, so

$$S_n = \begin{cases} 0 \text{ if } n \text{ is even,} \\ a \text{ if } n \text{ is odd,} \end{cases}$$

The sequence $\left(S_n\right)_{n\in\mathbb{N}^*}$ has no limit; therefore, the series $\sum_{n=1}^{\infty} \left(-1\right)^{n-1} a$ diverges.

1.2 Operations on Series

Let $\left(U_n\right)_{n\in\mathbb{N}^*}$ and $\left(V_n\right)_{n\in\mathbb{N}^*}$ be two numerical series, then:

1. If $\sum_{n=1}^{\infty} U_n$ converges and $\sum_{n=1}^{\infty} U_n$ converges, then the sum series $\sum_{n=1}^{\infty} \left(U_n + V_n\right)$ also converges.

2. If one of the series converges and the other diverges, then the series $\sum_{n=1}^{\infty}(U_n+V_n) = \sum_{n=1}^{\infty} U_n + V_n$ diverges.

3. If both series diverge, there is no general statement about the sum (as one may have a sum of $+\infty$ and the other $-\infty$).

   The series $\sum_{n=1}^{\infty} U_n$ and $\sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} aU_n$, where $a \in \mathbb{R}^*$ are of the same nature.

1.2.1 Necessary Condition for Convergence

Suppose that the series $\sum_{n=1}^{\infty} U_n$ converges to S :

$$S = \lim_{n\to+\infty} S_n = \sum_{n=1}^{\infty} U_n,$$

so $S = \lim\limits_{n\to+\infty} S_{n-1}$, but $S_n - S_{n-1} = U_n$, which yields, in the limit:

$$\begin{aligned}
\lim_{n\to+\infty}(S_n - S_{n-1}) &= \lim_{n\to+\infty} U_n \\
\Longrightarrow \lim_{n\to+\infty} S_n - \lim_{n\to+\infty} S_{n-1} &= \lim_{n\to+\infty} U_n \\
\Longrightarrow \lim_{n\to+\infty} U_n &= 0
\end{aligned}$$

so $\sum_{n=1}^{\infty} U_n$ converges $\Longrightarrow \lim\limits_{n\to+\infty} U_n = 0$.We also have, by contrapositive reasoning:

$$\lim_{n\to+\infty} U_n \neq 0 \Longrightarrow \sum_{n=1}^{\infty} U_n \text{ diverge.} \tag{1.6}$$

Example 2 $\sum_{n=1}^{\infty} \frac{e^n}{n}$.

Let $U_n = \frac{e^n}{n}$, then $\lim\limits_{n\to+\infty} U_n = +\infty$, so the series $\sum_{n=1}^{\infty} \frac{e^n}{n}$diverge.

Remark 3 *A series can diverge even if its general term tends towards* 0*.*

1.2.2 The harmonic series diverges.

It is the series

$$1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}+...=\sum_{n=1}^{\infty}\frac{1}{n}. \tag{1.7}$$

Note that $\lim_{n\to+1}\frac{1}{n}=0$. However, as we will see, this series diverges:

$$1+\frac{1}{2}+\left\{\frac{1}{3}+\frac{1}{4}\right\}+\left\{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right\}+\left\{\frac{1}{9}+\frac{1}{10}+...+\frac{1}{15}+\frac{1}{16}\right\}+....$$

Let's then introduce the following auxiliary series:

$$1+\frac{1}{2}+\left\{\frac{1}{4}+\frac{1}{4}\right\}+\left\{\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right\}+\left\{\frac{1}{16}+\frac{1}{16}+.....+\frac{1}{16}\right\}+...$$

By comparison, we have:

$$S_2\geq 1+\frac{1}{2},$$
$$S_4\geq 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}=1+2.\frac{1}{2},$$
$$S_8\geq 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+4.\frac{1}{2}.$$

By induction

$$S_{2^n}\geq 1+\frac{n}{2}.$$

Since $\lim\limits_{n\to+\infty}\left(1+\frac{n}{2}\right)=+\infty$, then $\lim\limits_{n\to+\infty}S_{2^n}=+\infty$ and thus $\lim\limits_{n\to+\infty}S_n=+\infty$, Therefore, the harmonic series diverges.

Another demonstration:

Let $f(x) = \frac{1}{x}, x > 0$, then $U_n = f(x) = \frac{1}{n}$. According to the graph of the function f, we have the inequalities

$$\begin{aligned} 1 &> \int_1^2 \frac{dx}{x}, \\ \tfrac{1}{2} &> \int_2^3 \frac{dx}{x}, \\ &\cdots\cdot \\ \tfrac{1}{n} &> \int_{n-1}^n \frac{dx}{x}. \end{aligned}$$

By addition:

$$1 + \frac{1}{2} + ... + \frac{1}{n} > \int_1^{n+1} \frac{dx}{x}, \tag{1.8}$$

Meaning $S_n > \ln(n+1)$, nd by taking the limit, we have $\lim_{n \to +\infty} S_n = +\infty$, thus the harmonic series diverges.

1.3 Criteria for Convergence of Series with Positive Terms

1.3.1 Comparison of Series

Theorem 4 *Let $\sum_{n=1}^{\infty} U_n$ and $\sum_{n=1}^{\infty} V_n$ dbe two series with positive terms such that*

$$U_n \leq V_n, \forall_n \geq 1 \tag{1.9}$$

then

$$\sum_{n=1}^{\infty} V_n \text{ converge} \Rightarrow \sum_{n=1}^{\infty} U_n \text{ converge.}$$

Proof. Let S_n and G_n be the respective partial sums of the series $\sum_{n=1}^{\infty} U_n$ and $\sum_{n=1}^{\infty} V_n$. Then (1.9) $\Longrightarrow \forall_{n \in \mathbb{N}^*} : S_n \leq G_n \leq G$ because $\{G_n\}$ is increasing, so the sequence $\{S_n\}$ is bounded by G, Moreover, $\{S_n\}_{n\geq 1}$ is an increasing sequence as the terms U_n are positive. ■

Conclusion 5 *the sequence $\{S_n\}_{n\geq 1}$ is increasing and bounded, so it is convergent. We also have: if $U_n \geq V_n \quad \forall n \geq 1$, and if the series $\sum_{n=1}^{\infty} V_n$ diverges, then the series $\sum_{n=1}^{\infty} U_n$ also diverges..*

Example 6 *Study the nature of the series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\sqrt{n}}{n+1}$*

Let $U_n = \left(\frac{1}{2}\right)^n \frac{\sqrt{n}}{n+1}$, then

$$\forall_n \in \mathbb{N}^* : U_n \leq \left(\frac{1}{2}\right)^n \frac{1}{n+1} \leq \left(\frac{1}{2}\right)^n .$$

As the series with general term $V_n = \left(\frac{1}{2}\right)^n$ converges, the same is true for the series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\sqrt{n}}{n+1}$.

1.3.2 Alembert criterion

Theorem 7 *Let $\sum_{n=1}^{\infty} U_n$ be a series with positive terms. Suppose $\lim\limits_{n\to+\infty} \frac{U_{n+1}}{U_n} = l$, then:*

a) *if $l < 1$, then the series converges,*

b) *if $l > 1$,then the series diverges,*

c) *if $l = 1$, no conclusion can be drawn.*

Proof.

a) If $l < 1, \exists r \in \mathbb{R}_+^*$: $l < r < 1$.Therefore,

$$\exists N \in \mathbb{N}^* \text{ such that } \forall n > N : \frac{U_{n+1}}{U_n} < r.$$

We obtain successively:

$$\begin{aligned} U_{n+1} & \quad < rU_n, \\ U_{n+2} & < U_{n+1} < r^2 U_n, \\ U_{n+3} & < rU_{n+2} < r^3 U_n, \\ ... \end{aligned}$$

By adding term by term, we will have

$$\begin{aligned} U_{n+1} + U_{n+2} + U_{n+3} + ... & < rU_n + r^2 U_n + r^3 U_n ... \\ & = U_n \left(r + r^2 + r^3 + ... \right). \end{aligned}$$

The bounding series is convergent because it is a geometric series with a ratio $r < 1$, Therefore, the series $U_{n,+1} + U_{n+2} + ...$ is convergent. We deduce that the series $\sum_{n=1}^{\infty} U_n$ also converges.

b) If $l > 1$, then

$$\exists \varepsilon > 0 \text{ such that } 1 < 1 + \epsilon < l.$$

For this ε, all the elements of the sequence $\left\{ \frac{U_{n+1}}{U_n} \right\}$ are $> 1 + \varepsilon$ except for a finite number, i.e.,

$$\exists N \in \mathbb{N}^* : \frac{U_{N+1}}{U_N} > 1 + \varepsilon \qquad \forall n \geq N,$$

By giving N the successive values $N, N+1, N+2$, we will have:

$$\frac{U_{N+1}}{U_N} > 1 + \varepsilon \Longrightarrow U_{N+1} > (1 + \varepsilon) U_N$$

$$\frac{U_{N+2}}{U_{N+1}} > 1 + \varepsilon \Longrightarrow U_{N+2} > (1 + \varepsilon) U_{N+1} > (1 + \varepsilon)^2 U_N,$$

Thus, the series $U_{N+1} + U_{N+2} + ...$ is bounded below by the series $(1+\varepsilon) U_N + (1+\varepsilon)^2 U_N + ...$which is a divergent geometric series because its ratio is $1+\varepsilon > 1$. This implies that the series $U_{N+1} + U_{N+2} + ...$is divergent, and consequently, the series $\sum_{n=1}^{\infty} U_n$ is also divergent.

c) If $l = 1$: There are examples that lead to convergence and others to divergence, but this is demonstrated using other means.

■

1.3.3 Cauchy criterion

Theorem 8 *Let $\sum_{n=1}^{\infty} U_n$ be a numerical series with positive terms, and let $l = \lim\limits_{n \to +\infty} \sqrt[n]{U_n} = \lim\limits_{n \to +\infty} (U_n)^{\frac{1}{n}}$. Then, we have the cases:*

a) *if $l < 1$, the series converges,*

b) *if $l > 1$, the series diverges,*

c) *if $l = 1$, no conclusion can be drawn.*

Proof.

a) For $l < 1$, use a similar technique as in the Alembert criterion. For any $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n > N : \left| \sqrt[n]{U_n} - l \right| < \varepsilon \Longrightarrow l - \varepsilon < \sqrt[n]{U_n} < l + \varepsilon$, so $\forall n > N$ $U_n < l + \varepsilon$. We can always choose ε small enough so that $l + \varepsilon$ is < 1. For suchε : $\forall n > N \quad U_n < (l+\varepsilon)^n < 1$, so

$$U_{N+1} < (l+\varepsilon)^{N+1},$$

$$U_{N+2} < (l+\varepsilon)^{N+2},$$

the geometric series $(l+\varepsilon)^{N+1}+(l+\varepsilon)^{N+2}+\ldots$ converges. Therefore, the series $U_{N+1}+U_{N+2}+\ldots$ is convergent as it is bounded by a convergent geometric series. Hence, the series $U_1+U_2+\ldots+U_N+U_{N+1}+\ldots$ is also convergent.

b) If $l>1:\exists N\in\mathbb{N}^*:\forall n>N:\sqrt[n]{U_n}>l-\varepsilon$ and we can always choose ε such that $l-\varepsilon>1$. Then

$$\forall n>N:U_n>(l-\varepsilon)^n>1.$$

Since $\sum_{n=1}^{\infty}(l+\varepsilon)^n$ diverges (geometric series with ratio $l-\varepsilon>1$), then $\sum_{n=1}^{\infty}U_n$ diverges $\Longrightarrow$ $\sum_{n=1}^{\infty}U_n$ diverges.

c) If $l=1$ no conclusion can be drawn: There is a possibility of convergence or divergence depending on the series under consideration

■

1.3.4 Comparison with an Integral:

Theorem 9 *Let $\sum_{n=1}^{\infty}U_n$ be a series with positive terms and $\{U_n\}_{n\in\mathbb{N}^*}$ a non-decreasing sequence. Let f be a continuous, non-decreasing function on an interval of the form $[a,+\infty[$ such that $\forall n\in\mathbb{N}^*$ $f(n)=U_n$, Then, the series $\sum_{n=1}^{\infty}U_n$ nd the integral $\int_1^{+\infty}f(x)dx$ shave the same nature.*

Proof. Considering the graph of this function for x between n and $n+1$, and assigning values $1,2,\ldots,$ to n, we have:

$$\begin{aligned}
U_1 &= U_1(2-1) && > \textstyle\int_1^2 f(x)dx,\\
U_2 &= U_2(2-1) && > \textstyle\int_2^3 f(x)dx,\\
&\ldots\\
U_n &= U_n(n+1-n) && > \textstyle\int_n^{n+1} f(x)dx.
\end{aligned}$$

By addition:

$$U_1 + U_2 + ... + U_n = S_n > \int_1^{n+1} f(x)dx. \tag{1.10}$$

On the other hand, we also have:

$$\begin{aligned}
&U_2 = U_2\,(2-1) && > \textstyle\int_1^2 f(x)dx,\\
&U_3 = U_2\,(3-1) && > \textstyle\int_2^3 f(x)dx,\\
&...\\
&U_{n+1} = U_n\,(n+1-n) && > \textstyle\int_n^{n+1} f(x)dx.
\end{aligned}$$

By addition:

$$U_2 + U_3 + ... + U_{n+1} = S_n > \int_1^{n+1} f(x)dx,$$

Namely:

$$S_{n+1} - U_1 < \int_1^{n+1} f(x)dx. \tag{1.11}$$

Two cases may arise:

1. If $\int_1^{+\infty} f(x)dx$ converges, then according to (1.11)

$$S_{n+1} < \int_1^{n+1} f(x)dx + U_1 < \int_1^{+\infty} f(x)dx + U_1 \Longrightarrow S_n < \int_1^{+\infty} f(x) + U_1.$$

   The sequence $\{S_n\}_{n\geq 1}$ is bounded, and since it is also increasing, it converges, i.e., $\sum_1^{+\infty} U_n$ converges.

2. If $\int_1^{+\infty} f(x)dx = +\infty$, then according to (1.10)

$$S_n > \int_1^{+\infty} f(x)dx \ , \forall n \geq 1. \tag{1.12}$$

   By taking the limit: $\sum_{n=1}^{\infty} U_n > \int_1^{+\infty} f(x)dx \Longrightarrow \sum_{n=1}^{\infty} U_n$ diverges.

■

1.3.5 Equivalence criterion

Let $\sum_{n=1}^{\infty} U_n$ and $\sum_{n=1}^{\infty} V_n$ be two series with positive terms such that for $n \in V(+\infty) : U_n \sim V_n$, then $\sum_{n=1}^{\infty} U_n$ and $\sum_{n=1}^{\infty} V_n$ have the same nature.

1.3.6 Another comparison criterion:

Let $\sum_{n=1}^{\infty} U_n$ and $\sum_{n=1}^{\infty} V_n$ dbe two series with positive terms. By calculating $\lim_{n \longrightarrow +\infty} \frac{U_n}{V_n}$, three cases arise:

- 1st case: If $\lim_{n \longrightarrow +\infty} \frac{U_n}{V_n} = 0$, then $\sum_{n=1}^{\infty} V_n$ converges $\Rightarrow \sum_{n=1}^{\infty} U_n$ converges.
- 2nd case: If $\lim_{n \longrightarrow +\infty} \frac{U_n}{V_n} = +\infty$, then $\sum_{n=1}^{\infty} V_n$ diverges $\Rightarrow \sum_{n=1}^{\infty} U_n$ diverges.
- 3rd case: If $\lim_{n \longrightarrow +\infty} \frac{U_n}{V_n} = l \implies \sum_{n=1}^{\infty} U_n$ and $\sum_{n=1}^{\infty} V_n$ have the same nature.

In general, one takes $V_n = \frac{1}{n^{\alpha}}$and calculates $\lim_{n \longrightarrow +\infty} \frac{U_n}{V_n} = \lim_{n \longrightarrow +\infty} n^{-\alpha} U_n$ to conclude.

1.4 Series with Arbitrary Terms

Let $\sum_{n=1}^{\infty} U_n$ be a numerical series that can take positive and negative values.

Definition 10 *We say that the series*$\sum_{n=1}^{\infty} U_n$ *converges absolutely if the series of positive terms* $\sum_{n=1}^{\infty} |U_n|$ *converges.*

Theorem 11 *Every absolutely convergent series is convergent.*

Proof. Suppose that the series $\sum_{n=1}^{\infty} U_n$converges absolutely. Consider $\sum_{n=1}^{\infty} U_n$, and show that its remainder $|R_n| \longrightarrow 0$ as $n \longrightarrow +\infty$. Now,

$$|R_n| = |U_{n+1} + U_{n+2} + ...| \leq |U_{n+1}| + |U_{n+2}| + ... \tag{1.13}$$

but $\lim_{n\to+\infty}(|U_{n+1}|+|U_{n+2}|+...)=0$ as it is the remainder of the convergent series $\sum_{n=1}^{\infty}|U_n|$, so $\lim_{n\to+\infty}|R_n|=0 \Longrightarrow \sum_{n=1}^{\infty}U_n$ converges. ■

Remark 12 *As we will see from the study of alternating series, the reverse is not always true, i.e., there are convergent series that are not absolutely convergent.*

Alternating Series:

These are series of the following form:

$$\sum_{n=1}^{\infty}(-1)^n U_n = U_1 - U_2 + U_3 - U_4 + ..., \text{ where all } U_i \text{ are} \geq 0. \tag{1.14}$$

Theorem 13 (Leibniz's Criterion) *If the sequence* $\{U_n\}_{n\geq1}$ *is decreasing, and* $\lim_{n\to+\infty}U_n=0$, *then the series* $\sum_{n=1}^{\infty}(-1)^n U_n$ *converges, and its sum* S *is positive with* $S\leq U_1$.

Proof. Consider the sum S_{2n}, then

$$S_{2n}=(U_1-U_2)+(U_3-U_4)+...+(U_{2n-1}-U_{2n})$$

By hypothesis, all these parentheses are positive, so S_{2n} is also positive, and the sequence $\{S_{2n}\}$ is thus increasing. Let's show that it is bounded. We write S_{2n} as follows:

$$S_{2n}=U_1-(U_2-U_3)-(U_4-U_5)-...-(U_{2n-2}-U_{2n-1})-U_{2n},$$

which means that $S_{2n}<U_1$, so the sequence $\{S_{2n}\}$ is increasing and bounded, so it converges: $\lim_{n\to+\infty}S_{2n}=S$.

Now consider the sequence $\{S_{2n+1}\}_{n\geq1}$. We have

$$S_{2n+1}=S_{2n}+U_{2n+1}\Longrightarrow \lim_{n\to+\infty}S_{2n+1}=\lim_{n\to+\infty}S_{2n}+\lim_{n\to+\infty}U_{2n+1}=S$$

The subsequences $\{S_{2n}\}$ and $\{S_{2n+1}\}$ are convergent, so they converge to S,. As $\{S_{2n}\} = \{S_{2n}\} \cup \{S_{2n+1}\}$ the sequence $\{S_n\}$ lso converges to S. Since $S_{2n+1} = S_{2n} + U_{2n+1} \leq U_1 - U_{2n-1} \leq U_1$, then $S_n \leq U_1$ for all $\forall n \geq 1$ c.q.f.d. ■

Remark 14 *We have* $|R_n| \leq U_{n+1}$.

Example 15 $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

$U_n = \frac{1}{n}$ *is the general term of a series with positive terms where* $U_{n+1} = \frac{1}{n+1} < U_n = \frac{1}{n}$ *and* $\lim\limits_{n \to +\infty} U_n = 0 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ *converges.*

The series $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ *is divergent, i.e., the series* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ *is not absolutely convergent. Such a series is called semi-convergent.*

1.4.1 Abel's criterion

This criterion is a generalization of Leibniz's criterion. We have

Theorem 16 *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be two numerical sequences satisfying the following three conditions:*

1. *The sequence* $\{B_n\}_{n=1}^{\infty}$ *is bounded, where* $B_n = \sum_{k=1}^{n} b_k$,
2. *The sequence* $\{a_n\}_{n=1}^{\infty}$ *is decreasing, and* $\lim\limits_{n \to +\infty} U_n = 0$, *then the series* $\sum_{1}^{+\infty} a_n b_n$ *converges.*

Remark 17 *If* $b_n = (-1)^n$, *Abel's criterion reduces to Leibniz's criterion.*

Example 18 *Let's study the series of terms* $U_n = e^{in\alpha} a_n$ *where* $\alpha \neq \pi k,\ k \in \mathbb{Z}$

Here, $b_n = e^{in\alpha}$,*so let* $B_n = \sum_{k=1}^{n} e^{ik\alpha}$ *be the partial sum corresponding to the series*

$$\sum_{n=1}^{+\infty} b_n = \sum_{n=1}^{+\infty} e^{in\alpha}.$$

Thus, for $\alpha \neq \pi\mathbb{Z}$*, we have*

$$B_n = e^{ix}\frac{e^{i(n+1)\alpha} - 1}{e^{i\alpha} - 1}$$

As $|B_n| < \frac{2}{\left|e^{i\frac{\alpha}{2}}\right|\left|e^{i\frac{\alpha}{2}} - e^{-i\frac{\alpha}{2}}\right|} < \frac{1}{\left|\sin\frac{\alpha}{2}\right|} \Longrightarrow$ *the sequence* $\{B_n\}$ *is bounded.*

For$\{a_n\}$ *sa decreasing sequence such that* $\lim\limits_{n\longrightarrow+\infty} a_n \longrightarrow 0$*, we have* $\sum_{n=1}^{\infty} e^{in\alpha}a_n$ *converges.*

Special cases: *the series* $\sum_{n=1}^{+\infty} \frac{e^{in\alpha}}{n^s}, 0 < s \leq 1, \alpha \in \mathbb{R}\backslash\{\pi\mathbb{Z}\}$, $\sum_{n=1}^{+\infty} \frac{\cos n\alpha}{n^s}$ *et* $\sum_{n=1}^{+\infty} \frac{\sin n\alpha}{n^s}, n \geq 1, 0 < s \leq 1, \alpha \in \mathbb{R}\backslash\{\pi\mathbb{Z}\}$ *are all convergent.*

For $\alpha > 1$ *all these series are evidently convergent.*

Chapter 2

SEQUENCES AND SERIES OF FUNCTIONS

2.1 Sequences of functions.

Consider a sequence of functions $f_n : \ (a, \ \ b) \to \mathbb{R}$ such that $x \to f_n(x)$, possibly with $a = -\infty$, or $b = +\infty$. In the following, $(\alpha, \ \beta)$ denotes an interval $\subseteq (a, \ \ b)$.

2.1.1 Simple convergence

Suppose that for each $x \in (\alpha, \ \beta)$, the sequence $\{f_n(x)\}$ has a limit. This limit is then a function f of x defined on $(\alpha, \ \beta)$. We say that the sequence of functions $\{f_n\}$ **simply converges** (S.C) to f on $(\alpha, \ \beta)$:

$$f_n \to f \text{ simply on } (\alpha, \ \beta) \Leftrightarrow \forall x \in (\alpha, \ \beta) : \lim_{n \to +\infty} f_n(x) = f(x)$$

This is equivalent to:

$$\forall x \in (\alpha, \ \beta), \forall \varepsilon > 0, \exists N(x, \varepsilon) \in \mathbb{N}, \forall n \in \mathbb{N} : n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon. \quad (2.1)$$

Example 19 **a)** *Let* $f_n : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f_n(x) = \frac{nx^2}{1+nx^2},\ n \in \mathbb{N}.$$

- *For* $x \neq 0$, $\lim\limits_{n \to +\infty} f_n(x) = 1$.
- *For* $x = 0$, *we have* $f_n(0) = 0 \Rightarrow \lim\limits_{n \to +\infty} f_n(0) = 0$.

Thus, the sequence $\{f_n\}$ *S.C to the function* f *defined by* $x \longmapsto f(x) = \begin{cases} 1 \ si\ x \neq 0, \\ 0 \ si\ \ x = 0. \end{cases}$

b) *Let* $f_n : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f_n(x) = n\pi e^{-nx^2} + x.$$

$\forall x \in \mathbb{R}$, *we have* $\lim\limits_{n \to +\infty} f_n(x) = x$. *Thus, the sequence* $\{f_n\}$ *S.C to the function* f *defined by* $x \longmapsto f(x) = x$, $\forall x \in \mathbb{R}$.

c) *Consider the sequence of functions*

$$f_n(x) = \frac{\sin nx}{n},\ \ x \in \mathbb{R}.$$

We have $\forall x \in \mathbb{R}$, $\lim\limits_{n \to +\infty} f_n(x) = 0$, *Therefore, the sequence of functions* $\{f_n\}$ *simply converges to the zero function* $f(x) = 0$, $\forall x \in \mathbb{R}$.

d) *Consider the sequence of functions*

$$f_n(x) = 2nx^2 e^{-n^2x^2},\ \ x \in \mathbb{R}.$$

This S.C sequence towards the null function. Nevertheless, we have

$$\int_0^1 f_n(x)\,dx = 1 - e^{-n^2} \Rightarrow \lim_{n\to+\infty}\left(\int_0^1 f_n(x)\,dx\right) = 1;$$

but $\int_0^1 \left(\lim_{n\to+\infty} f_n(x)\right)dx = \int_0^1 0dx = 0,$ *i,e.,*

$$\lim_{n\to+\infty}\left(\int_0^1 f_n(x)\,dx\right) \neq \int_0^1 \left(\lim_{n\to+\infty} f_n(x)\right)dx.$$

e) *Let's return to example c). We note that* $f_n^r(x) = \cos nx$ *and* $\lim_{n\to+\infty} f_n'(x)$ *does not exist in general and yet* $f'(x) = 0$*; in other words:*

$$\frac{d}{dx}\left(\lim_{n\to+\infty} f_n(x)\right) \neq \lim_{n\to+\infty}\left(\frac{d}{dx} f_n(x)\right).$$

Finally, returning to the first example, the functions f_n are all continuous on $[0,\ 1]$ but the function f is not.

Conclusion 20 *In general, simple convergence does not preserve continuity and does not allow the symbols* $\frac{d}{dx}$ *and* $\int$ *with* $\lim_{n\to+\infty}$*. To remedy this defect of simple convergence, we introduce a new type of convegence for sequences of functions.*

2.1.2 Uniform convergence

Definition 21 *We say that the sequence of functions* $\{f_n\}$ ***converges uniformly*** *(C.U) on* $(\alpha,\ \beta)$ *to the function* f *to mean:*

$$\forall\varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N}, \forall x \in (\alpha,\ \beta), \forall n \in \mathbb{N} : n \geq N(\varepsilon) \Rightarrow |f_n(x) - f(x)| < \varepsilon. \quad (2.2)$$

This time, $N(\varepsilon)$ depends only on ε and does not depend on x.

It is clear that

$$C.U \text{ on } (\alpha,\ \beta) \Rightarrow C.S \text{ on } (\alpha,\ \beta).$$

Example 22 *Let's revisit example **a).** We have uniform convergence of the sequence $\{f_n\}$ to f on $[c,\ +\infty[$ for a given $c > 0$. Indeed, on this interval:*

$$|f_n(x) - f(x)| = \left|\frac{nx^2}{1+nx^2} - 1\right| = \frac{1}{1+nx^2}.$$

To get $\frac{1}{1+nx^2} < \varepsilon$ for all $x \geq c$. it suffices to have $\frac{1}{1+nc^2} < \varepsilon$, i.e., $nc^2 > \frac{1}{\varepsilon} - 1 \Rightarrow n > \frac{1-\varepsilon}{nc^2}$. Therefore, for $n \geq N(\varepsilon) = \left[\frac{1-\varepsilon}{nc^2}\right] + 1$, we have $|f_n(x) - f(x)| < \varepsilon$.

Can we get U.C on $[0,\ c[$? If yes, then we will have U.C on $[0,\ +\infty[$.

Suppose this is the case; then $\forall\varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N}, \forall n \geq \mathbb{N}$ et $\forall x \in [0,\ c[$: $|f_n(x) - f(x)| < \varepsilon$.

Choose n such that $x = \frac{1}{\sqrt{n}} \in [0,\ c[$, which is possible; it suffices that $n > c^2$. Then, for this choice of n and x:

$$\left|f_n\left(\frac{1}{\sqrt{n}}\right) - f\left(\frac{1}{\sqrt{n}}\right)\right| = \frac{1}{2} \text{does not tend to } 0 \text{ as } n \to +\infty.$$

Therefore, there is no uniform convergence on $[0,\ c[$ and consequently not on $[0,\ +\infty[$.

Remark 23 *In the definition of U.C, it is specified that N is independent of $x \in (\alpha,\ \beta)$, but N obviously depends on the interval $(\alpha,\ \beta)$ itself.*

Stating that $\{f_n\} \to f$ uniformly on $(\alpha,\ \beta)$ is equivalent to saying that

$\forall\varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N}, \forall x \in (\alpha,\ \beta), \forall n, m \in \mathbb{N} : n > m \geq N(\varepsilon) \Rightarrow |f_n(x) - f_m(x)| < \varepsilon$.

Interpretation of U.C using a norm:

Let E be the vector space over $\mathbb{R}$ of functions defined on $(\alpha,\ \beta)$, taking real and bounded values on $(\alpha,\ \beta)$. The norm $\|.\|_\infty$ on E is defined as: if $f \in E$

$$\|f\|_\infty = \sup_{x\in(\alpha,\ \beta)} |f(x)|. \tag{2.3}$$

Stating that the sequence $\{f_n\} \to f$ uniformly on $(\alpha,\ \beta)$ means that

$$\|f_n - f\|_\infty = \sup_{x\in(\alpha,\ \beta)} |f_n(x) - f(x)| \to 0 \text{ if } n \to +\infty. \tag{2.4}$$

2.2 Series of functions

Let the sequence of functions $U_n : (\alpha,\ \beta) \to \mathbb{R}$, where eventually $a = -\infty$, or $b = +\infty$. Consider the series

$$U_1(x) + U_2(x) + ...U_n(x) + ... \tag{2.5}$$

and the sequence of partial sums:

$$S_1(x) = U_1(x), S_n(x) = U_1(x) + U_2(x) + ...U_n(x). \tag{2.6}$$

This series is called a **series of functions**. The set D_c of real numbers x for which this series converges is called **the domain of convergence**.

Theorem 24 *The remainder $R_n(x)$ of a convergent series of functions tends to* 0 *as $n \to +\infty$.*

Proof. It suffices to use the fact that $R_n(x) = S(x) - S_n(x)$. Now, for all $x \in D_c$:

$$\lim_{n\to+\infty} S_n(x) = S(x) \Rightarrow \lim_{n\to+\infty} R_n(x) = 0.$$

■

2.3 Simple, uniform and normal convergence of a series of functions

Definition 25 a) *We say that the series of functions* $\sum_{n=0}^{+\infty} U_n(x)$ ***simple convergence*** *(S.C) to its sum* $S(x)$ *on an interval* $I = (\alpha,\ \beta)$, *to express that the numerical series* $\sum_{n=0}^{+\infty} U_n(x)$ *is convergent :*

$$\forall x \in I : \lim_{n\to+\infty} S_n(x) \underset{n\to+\infty}{\to} S(x) = \sum_{n=0}^{+\infty} U_n(x), \tag{2.7}$$

the relation (2.7) *is equivalent to*

$$\forall x \in I, \forall \varepsilon > 0, \exists N(x,\epsilon) \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N(x,\epsilon) \Rightarrow |S_n(x) - S(x)| < \varepsilon. \tag{2.8}$$

b) *We say that the series of functions* $\sum_{n=0}^{+\infty} U_n(x)$ ***converges uniformly*** *(C.U) to its sum* $S(x)$ *on* I *when the sequence of partial sums* $\{S_n\}_{n\geq1}$ ***converges uniformly*** *on* I *to the function* $S(x) = \sum_{n=0}^{+\infty} U_n(x)$, *which is equivalent to:*

$$\forall \varepsilon > 0, \exists N(\epsilon) \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N(\epsilon) \Rightarrow |S_n(x) - S(x)| < \varepsilon\ \forall x \in I, \tag{2.9}$$

In a more concise and practical form, this is equivalent to:

$$\lim_{n\to+\infty} \|S_n - S\|_\infty = 0,\ \text{où}\ \|S_n - S\|_\infty = \sup_{x\in(\alpha,\ \beta)} |S_n(x) - S(x)|. \tag{2.10}$$

Using Cauchy sequences, uniform convergence is equivalent to:

$$\forall \varepsilon > 0, \exists N(\epsilon) \in \mathbb{N}, \forall n \in \mathbb{N}, \forall m \in \mathbb{N}, m > n \geq N(\epsilon) \Rightarrow |S_m(x) - S_n(x)| < \varepsilon \ \forall x \in I. \tag{2.11}$$

c) *Sufficient Conditions for Uniform Convergence*

Let's first give the following definition:

Definition 26 *We say that the series* $\sum_{n=0}^{+\infty} U_n(x)$ *is bounded on the interval* I*, if there exists a convergent numerical series with positive terms* $\sum_{n=0}^{+\infty} v_n$ *such that*

$$\forall n \in \mathbb{N} : |U_n(x)| \leq v_n \ \forall x \in I. \tag{2.12}$$

We also say that the series $\sum_{n=0}^{+\infty} U_n(x)$ ***converges normally*** *on* I*.*

Theorem 27 (C.N) $\sum_{n=0}^{+\infty} U_n(x)$ ***converges normally*** *on* $I \Rightarrow \sum_{n=0}^{+\infty} U_n(x)$ ***converges uniformly*** *on* I*.*

Proof. Suppose that the series $\sum_{n=0}^{+\infty} U_n(x)$ converges normally on I, then there exists a sequence $\{v_n\}_{n\geq 0}$ such that $\forall n \in \mathbb{N} : |U_n(x)| \leq v_n$. Therefore, we have:

$$\begin{aligned} &|U_{n+1}(x)| \leq v_{n+1}, \\ &|U_{n+2}(x)| \leq v_{n+2}, \\ &\ldots \end{aligned}$$

By summing term by term, we get:

$$\sum_{k=n+1}^{+\infty} |U_k(x)| \leq \sum_{k=n+1}^{+\infty} v_k \Rightarrow \left| \sum_{k=n+1}^{+\infty} U_k(x) \right| \leq \sum_{k=n+1}^{+\infty} v_k,$$

This inequality involves the remainders of both series $\sum_{n=0}^{+\infty} U_n(x)$ and $\sum_{n=0}^{+\infty} v_n$. As the latter series converges, its remainder $\sum_{k=n+1}^{+\infty} v_k \underset{n\to+\infty}{\to} 0$, so we can write:

$$\forall\varepsilon > 0, \exists N(\epsilon) \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N(\epsilon) \Rightarrow \left| \sum_{k=n+1}^{+\infty} v_k \right| < \varepsilon$$

so also that

$$\forall\varepsilon > 0, \exists N(\epsilon) \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N(\epsilon) \Rightarrow \left| \sum_{k=n+1}^{+\infty} U_k(x) \right| < \varepsilon,\ \forall x \in I,$$

This precisely means that we have uniform convergence of the series of functions $\sum_{n=0}^{+\infty} U_n(x)$ on I. ■

Corollary 28 *Every normally convergent series is absolutely convergent.*

Theorem 29 *The series* $\sum_{n=0}^{+\infty} U_n(x)$ *is normally convergent on* I *if and only if* ***the numerical series*** $\sum_{n=0}^{+\infty} \|U_n\|_\infty$, *where* $\|U_n\|_\infty = \sup_{x\in I} |U_n(x)|$, *is convergent.*

Proof. If the series $\sum_{n=0}^{+\infty} U_n(x)$ is normally convergent on I, there exists a convergent numerical series with positive terms $\sum_{n=0}^{+\infty} v_n$ such that $\forall n \in \mathbb{N} : |U_n(x)| \leq v_n\ \forall x \in I$. Now, $\forall n \in \mathbb{N} : \|U_n\|_\infty \leq v_n$, and therefore, the numerical series $\sum_{n=0}^{+\infty} \|U_n\|_\infty$ is convergent.

The converse is obvious; one can simply take $v_n = \|U_n\|_\infty$. ■

Theorem 30 (Integration) *If the series of functions* $\sum_{n=0}^{+\infty} U_n(x)$ *converges uniformly on* I*, and if, moreover, all functions* $n \to U_n(x)$ *are continuous on* I*, then its sum* $S(x)$ *is a continuous function on* I*.*

Proof. Let x_0 be any element in I. In terms of the remainder $R_n(x) = U_{n+1}(x) + U_{n+2}(x) + ...$, the difference $S(x) - S(x_0)$ is expressed as:

$$S(x) - S(x_0) = S_n(x) - S_n(x_0) + R_n(x) - R_n(x_0).$$

Taking absolute values, we have successively:

$$\begin{aligned} |S(x) - S(x_0)| &\leq |S_n(x) - S_n(x_0)| + |R_n(x) - R_n(x)| \\ &\leq |S_n(x) - S_n(x_0)| + |R_n(x)| + |R_n(x)|. \end{aligned}$$

Now, S_n is continuous at $x_0 \Leftrightarrow \forall \varepsilon > 0, \exists \delta_1(\varepsilon) > 0 \,/\, |x - x_0| < \delta_1 \Rightarrow |S_n(x) - S_n(x_0)| < \varepsilon/3$

But $\sum_{n=0}^{+\infty} U_n(x)$ converges uniformly on I so there exists $\exists N(\varepsilon)$ such that we have both for x and x_0 :

$$\forall \varepsilon > 0, \exists N_1(\varepsilon) \in \mathbb{N} \,/\, n > N(\varepsilon) \Rightarrow |R_n(x)| < \varepsilon/3 \text{ and } |R_n(x_0)| < \varepsilon/3.$$

Combining these, we get for $|S(x) - S(x_0)|$ and for $n > N(\varepsilon)$

$$\forall \varepsilon > 0, \exists \delta_1(\varepsilon) > 0 \,/\, |x - x_0| < \delta_1 \Rightarrow |S(x) - S(x_0)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,$$

which means that the sum $S(x)$ is continuous at x_0. As x_0 is arbitrary in the interval I, $S(x)$ is therefore continuous on I. c.q.f.d. ■

We present above, without proof, the two theorems.

Theorem 31 (Integration) *Suppose the series of functions* $\sum_{n=0}^{+\infty} U_n(x)$ *converges uniformly on* I *to its sum* $S(x)$*. If, for every* $n \in \mathbb{N}$*, the functions* $x \to U_n(x)$ *are integrable over* $I = [\alpha, \ \beta]$*, then*

i) *the function* $x \rightarrow S(x)$ *is integrable on* I,

ii) *the numerical series with general term* $v_n = \int\limits_{\alpha}^{\beta} U_n(x)\,dx$ *is convergent, and moreover, we have*

$$\forall \alpha_0, \alpha_1 \in [\alpha,\ \beta] : \sum_{n=0}^{+\infty} \int_{\alpha_0}^{\alpha_1} U_n(x)\,dx = \int_{\alpha_0}^{\alpha_1} \left(\sum_{n=0}^{+\infty} U_n(x) \right) dx,$$

iii) *the series of functions with general term* $w_n(x) = \int\limits_{x_0}^{x} U_n(t)\,dt$ *is uniformly convergent on* $[\alpha,\ \beta]$ *to its sum* $\sum\limits_{n=0}^{+\infty} \int\limits_{x_0}^{x} U_n(t)\,dt$ *and we have:*

$$\sum_{n=0}^{+\infty} \int_{x_0}^{x} U_n(t)\,dt = \int_{x_0}^{x} \left(\sum_{n=0}^{+\infty} U_n(t) \right) dt. \tag{2.14}$$

Theorem 32 (Differentiation) *Let* $\sum\limits_{n=0}^{+\infty} U_n(x)$, $x \in \mathbb{N}$ *be a series of functions such that:*

a) $\forall n \in \mathbb{N}$, *the function* $x \rightarrow U_n(x)$ *is a* C^1 *function on* $[\alpha,\ \beta]$;

b) *the derived series* $\sum\limits_{n=0}^{+\infty} \frac{d}{dx} U_n(x)$ *converges uniformly on* $[\alpha,\ \beta]$;

c) $\exists x_0 \in [\alpha,\ \beta]$ *such that the numerical series* $\sum\limits_{n=0}^{+\infty} U_n(x_0)$ *is convergent,*

then we have:

i) $\sum\limits_{n=0}^{+\infty} U_n(x)$ *converges uniformly on* $[\alpha,\ \beta]$,

ii) *the function* $x \to S(x) = \sum_{n=0}^{+\infty} U_n(x)$ *is differentiable on* $[\alpha,\ \beta]$, *and we have*

$$\frac{d}{dx}\left(\sum_{n=0}^{+\infty} U_n(x)\right) = \sum_{n=0}^{+\infty} \frac{d}{dx} U_n(x). \tag{2.15}$$

At α *and* β, *it involves semi-differentiability.*

Theorem 33 (Leibniz's Theorem for Alternating Series) *Let* $\{U_n\}_{n\in\mathbb{N}^*}$ *be a sequence of non-negative functions defined on an interval* $[\alpha,\ \beta]$ *satisfying:*

i) $\forall n \in \mathbb{N}^*, \forall x \in [\alpha,\ \beta] : U_{n+1} \leq U_n$;

ii) *the sequence* $\{U_n\}_{n\in\mathbb{N}^*}$ *converges to the zero function on* $[\alpha,\ \beta]$;

then the series $\sum_{n=0}^{+\infty} (-1)^n U_n(x)$ *converges uniformly on* $[\alpha,\ \beta]$.

Proof. According to Chapter I, we know that $\forall x \in [\alpha,\ \beta] : |R_n(x)| \leq U_{n+1}(x)$, but we also have $\forall x \in [\alpha,\ \beta]\ U_{n+1}(x) \leq \|U_{n+1}\|_\infty$. According to ii), we have

$$\forall \varepsilon > 0, \exists n_0(\varepsilon) \in \mathbb{N}^*\ /\ \forall n \in \mathbb{N}^* : n > n_0(\varepsilon) \Rightarrow \|U_n\| < \varepsilon.$$

By combining, we obtain

$$\forall \varepsilon > 0, \exists n_0(\varepsilon) \in \mathbb{N}^*/\forall n \in \mathbb{N}^* : n > n_0(\varepsilon) \Rightarrow \forall x \in [\alpha,\ \beta] : |R_n(x)| < \varepsilon,$$

or equivalently

$$\forall \varepsilon > 0, \exists n_0(\varepsilon) \in \mathbb{N}^*/\forall n \in \mathbb{N}^* : n > n_0(\varepsilon) \Rightarrow \forall x \in [\alpha,\ \beta] : \|R_n\| < \varepsilon,$$

meaning the uniform convergence of the series $\sum_{n=0}^{+\infty} (-1)^n U_n(x)$ on $[\alpha,\ \beta]$. ∎

2.4 Applications

By way of applications:

Exercise 34 *Show that the series given by the general term*

$$U_n(x) = x^{\frac{1}{2n+1}} - x^{\frac{1}{2n-1}}, \ \ n \in \mathbb{N}^*$$

is simply convergent but not uniformly convergent on $\mathbb{R}$.

Solution 35 *We find that this series simply converges to* $S(x)$ *such that*

$$S(x) = \begin{cases} -1 - x, \quad x < 0 \\ 0, \ x = 0 \\ 1 - x, \ x > 0. \end{cases}$$

Exercise 36 *Consider the series of functions with the general term*

$$U_n(x) = (-1)^n \ln\left(1 + \frac{x}{n(1+x)}\right), \ x \geq 0 \ \text{et} \ n \in \mathbb{N}^*.$$

1. *Show that this series converges simply on* $\mathbb{R}_+$.

2. *Show that this series converges uniformly on* $\mathbb{R}_+$.

3. *Is the convergence normal on* $\mathbb{R}_+$.

Solution 37 1. *Apply the alternating series criterion:* $\lim\limits_{n\to+\infty} U_n(x) = 0$ *and we have* $|U_{n+1}(x)| \leq |U_n(x)|$ *because*

$$\frac{x}{(n+1)(1+x)} \leq \frac{x}{n(1+x)} \Rightarrow 1 + \frac{x}{(n+1)(1+x)} \leq 1 + \frac{x}{n(1+x)},$$

Since the ln function is increasing, we get

$$|U_{n+1}(x)| \leq |U_n(x)|.$$

2. *We have*

$$|R_n(x)| \leq |U_{n+1}(x)| \leq \frac{x}{(n+1)(1+x)} \leq \frac{1}{n+1},$$

which comes from the fact that $\ln(1+t) \leq t$ *for* $t > -1$. *As* $\frac{1}{n+1} \to 0$ *if* $n \to +\infty$, *then* $|R_n(x)| \to 0$ *if* $n \to +\infty$ *independently of* x. *Therefore, the convergence is uniform.*

3. *There is no absolute convergence, hence no normal convergence.*

Chapter 3

POWER SERIES

3.1 Radius of Convergence of a Power Series

Definition 38 ***A power series***, *or series of power, is any series of functions of the form*

$$a_0 + a_1x + a_2x^2 + ...a_nx^n + ..., \tag{3.1}$$

where $a_0, a_1, a_2, ...a_n, ...$are constants called the ***coefficients*** *of the series.*

Theorem 39 (Abel)

1. *If the series converges for every $x_0 \neq 0$, it converges absolutely for every $x \in \left]-x_0, x_0\right[$.*

2. *If the series diverges for $x_1 \neq 0$, it diverges for every x such that $|x| \geq |x_1|$.*

Proof.

1. The series (3.1) converges at x_0 means that the numerical series $a_0 + a_1x + a_2x^2 + ...a_nx^n + ...$ converges, and therefore, $\lim_{n\to+\infty} a_nx^n = 0$, Thus:

$$\exists M > 0, \forall n \in \mathbb{N}, |a_nx^n| < M. \tag{3.2}$$

Consider the series

$$|a_0| + |a_1 x_0| \left|\frac{x}{x_0}\right| + \left|a_2 x_0^2\right| \left|\frac{x}{x_0}\right|^2 + ... + |a_n x_0^n| \left|\frac{x}{x_0}\right|^n + ... \qquad (3.3)$$

According to (3.2) , the series (3.3) has the upper bound

$$M + M\left|\frac{x}{x_0}\right| + M\left|\frac{x}{x_0}\right|^2 + ... + M\left|\frac{x}{x_0}\right|^n + ..., \qquad (3.4)$$

Now, $x \in \left]-x_0, x_0\right[\Rightarrow \left|\frac{x}{x_0}\right| < 1$, so the geometric series (3.4) is convergent. Therefore, the series (3.3) is also convergent, and consequently, the series (3.1) is absolutely convergent.

2. Assume the opposite, i.e., $\exists x$ such that $|x| > |x_1|$ for which the series (3.1) converges. Since $|x_1| < |x| \Rightarrow x_1 \in \left]-|x|, |x|\right[$. according to **part 1** of the theorem, the series converges (3.1) at x_1, which is a contradiction. Therefore, this series diverges for all x such that $|x| > |x_1|$.

■

Theorem 40 *To any power series* $\sum_{n=0}^{+\infty} a_n x^n$, *we can associate a positive real number* R, *epossibly zero or infinity, such that:*

a) *The series converges absolutely for every* x *such that* $|x| < R$;

b) *The series diverges for every* x *such that* $|x| > R$.

Proof.

a) Apply the d'Alembert criterion to the series $\sum_{n=0}^{+\infty} |a_n x^n|$ for $x \in \left]-R, R\right[$. Then

$$\lim_{n\to+\infty} \left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| . |x| .$$

There is convergence if $\lim\limits_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right|.\,|x|<1$, i.e., if $|x|<\lim\limits_{n\to+\infty}\left|\dfrac{a_n}{a_{n+1}}\right|$. The sought-after number R is thus

$$R=\lim_{n\to+\infty}\left|\frac{a_n}{a_{n+1}}\right|. \tag{3.5}$$

b) Of course, the series diverges for $\lim\limits_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right||x|>1$; hence, for $|x|>\lim\limits_{n\to+\infty}\left|\dfrac{a_n}{a_{n+1}}\right|$, i.e., for $|x|>R$.

The number R is called the radius of convergence of the power series $\sum\limits_{n=0}^{+\infty}a_nx^n$. With the Cauchy criterion, this number is calculated using the formula

$$R=\frac{1}{\lim\limits_{n\to+\infty}|a_n|^{1/n}}. \tag{3.6}$$

The series converges absolutely in the interval $(-R,R)$.

For $x=-R$ or $x=R$, we cannot conclude immediately; it will be necessary, of course, to study the convergence of the two numerical series $\sum\limits_{n=0}^{+\infty}a_nR^n$ and $\sum\limits_{n=0}^{+\infty}(-1)^n\,a_nR^n$.

■

Remark 41 *In the appendix at the end of this chapter, we provide the extension of the notion of power series to the complex variable, as well as that of the radius of convergence when the coefficients a_n are not given by a single formula.*

Theorem 42 *On any interval of the form $[-\alpha,\alpha]$ contained in the convergence domain $D_c=(-R,R)$, the power series $\sum\limits_{n=0}^{+\infty}a_nx^n$ is bounded.*

Proof. For any $x \in [-\alpha, \alpha]$, we have: $|a_n x^n| \leq |a_n| \alpha^n$. Now, the numerical series $\sum_{n=0}^{+\infty} |a_n| \alpha^n$ is convergent since $\alpha < R$, Therefore, the power series $\sum_{n=0}^{+\infty} a_n x^n$ is bounded, and we can deduce that it is uniformly convergent on the interval $[-\alpha, \alpha]$.

■

Theorem 43 *1. On any interval $[-\alpha, \alpha] \subset D_c$, the sum of a power series is a continuous function.*

2. If the bounds$-R$ and R of the convergence domain D_c belong to D_c, then

$$\int_0^x \left(\sum_{n=0}^{+\infty} a_n x^n \right) dx = \sum_{n=0}^{+\infty} \int_0^x a_n x^n dx \tag{3.7}$$

3. If $D_c = (-R, R)$ is the convergence domain of the power series $S(x) = \sum_{n=0}^{+\infty} a_n x^n$, then the derivative series

$$\varphi(x) = \sum_{n=0}^{+\infty} (n+1) a_{n+1} x^n \tag{3.8}$$

has the same convergence domain, and additionally, $\frac{d}{dx} S(x) = \varphi(x)$ or

Proof. Calculate the radius of convergence of the derivative series:

$$\begin{aligned} \lim_{n \to +\infty} \left| \frac{(n+1) a_{n+1}}{(n+2) a_{n+2}} \right| &= \lim_{n \to +\infty} \left| \frac{a_{n+1}}{a_{n+2}} \right| \\ &= \lim_{n \to +\infty} \left| \frac{a_n}{a_{n+1}} \right| \\ &= R. \end{aligned}$$

We thus have the same convergence domain, and the derivative series is therefore normally convergent within D_c. The series (3.1) is thus differentiable, and we can write:

$$\frac{d}{dx}\left(\sum_{n=0}^{+\infty} a_n x^n\right) = \sum_{n=0}^{+\infty} \frac{d}{dx}\left(a_n x^n\right) = \sum_{n=0}^{+\infty} (n+1)\, a_{n+1} x^n. \tag{3.9}$$

The series (3.8) can be further differentiated, and the process can be continued indefinitely.

In conclusion, the function $x \mapsto S(x)$ is C^∞ on any interval contained within the convergence domain. ■

Example 44 **a)** *The power series* $\sum_{n=0}^{+\infty} x^n$ *converges in* $D_c = \left]-1,1\right[$ *and*

$$\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}. \tag{3.10}$$

Differentiating both sides, we get

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{+\infty} n x^{n-1} = \sum_{n=0}^{+\infty} (n+1)\, x^n \tag{3.11}$$

within $D_c = \left]-1,1\right[$.

We also have in $\left]-1,1\right[$

$$\ln(1-x) = -\sum_{n=1}^{+\infty} \frac{x^n}{n}. \tag{3.12}$$

b) $\frac{1}{1+x^2} = \sum_{n=0}^{+\infty} (-1)^n x^{2n}$ *in* $\left]-1,1\right[$, *and integrating both sides, we get*

$$\arctan x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}. \tag{3.13}$$

Functions Expandable in Power Series

In the neighborhood of a point $x_0 \in \left]-R,R\right[$,we can affirm that

a) The sum of a power series is a C^∞ function on $]-R, R[$. Conversely, can we consider that a C^∞ function is the sum of a power series? The answer is yes under certain additional conditions. Indeed, let $f :]\alpha, \beta[\to \mathbb{R}$ be a C^∞ function. When we can associate a power series $\sum_{n=0}^{+\infty} a_n (x - x_0)^n$, with f, where a_n and $x_0 \in \mathbb{R}$ are such that we have

$$f(x) = \sum_{n=0}^{+\infty} a_n (x - x_0)^n, \forall x \in]x_0 - a, x_0 + a[\subset]\alpha, \beta[.$$

We say that f is expandable in a power series in $]x_0 - a, x_0 + a[$ around x_0. The sum $\sum_{n=0}^{+\infty} a_n (x - x_0)^n$ is then called the power series expansion of f. This expansion, when it exists, is unique because these coefficients a_n are uniquely determined by

$$a_n = \frac{f^{(n)}(x_0)}{n!}. \tag{3.14}$$

b) Existence condition:

Necessary condition:

f is expandable in a series in $]x_0 - a, x_0 + a[\Rightarrow f$ is C^∞in $]x_0 - a, x_0 + a[$ and $a_n = \dfrac{f^{(n)}(x_0)}{n!}$.

Sufficient condition:

f is C^∞in $]x_0 - a, x_0 + a[$, and there exists a constant $M > 0$ such that

$$\forall n \in \mathbb{N} : \left| f^{(n)}(x) \right| \leq \frac{Mn!}{a^n} \Rightarrow f(x) = \sum_{n=0}^{+\infty} a_n (x - x_0)^n, \forall x \in]x_0 - a, x_0 + a[.$$

c) Examples:

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}, \forall x \tag{3.15}$$

$$\cosh x = \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} \text{ et } \cos x = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \forall x \tag{3.16}$$

$$\sinh x = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!} \text{ et } \sin x = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \forall x \tag{3.17}$$

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + ... \frac{m(m-1)...(m-n+1)}{n!}x^n + ... \tag{3.18}$$

$$\text{under the conditions: } \begin{cases} m \leq -1, \ D_c =]-1,1[\,; \\ -1 < m < 0, \ D_c =]-1,1[\,; \\ m \geq 0, \ D_c = [-1,1]\,. \end{cases}$$

3.2 Function C^∞ not developable in series

Let f be the function defined by

$$f(x) = \begin{cases} \exp\left(-\frac{1}{x^2}\right), & \text{si } x \in \mathbb{R}^* \\ 0, & \text{si } x = 0 \end{cases}$$

It is shown by induction that f is a C^∞ function on $\mathbb{R}^*$ and $f^{(n)}(x) = \frac{P_n(x)}{x^{3n}} \exp\left(-\frac{1}{x^2}\right)$, where P_n is a polynomial of degree $2n-2$. Therefore, $\lim_{x\to 0} f^{(n)}(x) = 0 \Rightarrow \forall n \in \mathbb{N}$: $f^{(n)}(0) = 0$. It follows that the Taylor series of f is the zero series. Consequently, there exists no real number $\alpha > 0$ such that on the interval $]-\alpha, \alpha[$, we have

$$\forall x \in]-\alpha, \alpha[: f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n$$

because f does not vanish at 0.

3.3 Additional information on power series

Accumulation Points and Radius of Convergence

Let $\{U_n\}_{n\in\mathbb{N}}$ be a numerical sequence. We say it has a **finite accumulation point** l if:

$\forall\varepsilon > 0$, there exists an infinite number of U_n in $]l-\varepsilon, l+\varepsilon[$.

Similarly, we say it has $+\infty$ (resp. $-\infty$) as an accumulation point if:

$\forall A > 0$, there exists an infinite number of U_n in $]A, +\infty[$ (resp. in $]-\infty, A[$).

The largest accumulation point of the sequence $\{U_n\}_{n\in\mathbb{N}}$ (denoted as L) is called the **upper limit** of the sequence. We write

$$L = \overline{\lim_{n\to+\infty}} U_n. \tag{3.19}$$

Considering the sequence $\left\{\sqrt[n]{|a_n|}\right\}$, we then have the general formula for calculating the radius of convergence

$$R = \frac{1}{\overline{\lim\limits_{n\to+\infty}} \sqrt[n]{|a_n|}}. \tag{3.20}$$

Example 45 *Determine the radius of convergence of the following power series:*

1. $z + z^2 + z^3 + z^5 + z^7 + z^{11} + z^{13} + ...z^p + ...,$ *p prime number;*

2. *Consider the sequence $\{\lambda_n\}$ defined by*

$$\lambda_n = \begin{cases} \frac{1}{p+\sqrt{p}}, \ si\ n = 3p \\ \frac{1}{p^p}, \ si\ n = 3p+1 \\ (-a)^p, \ si\ n = 3p+2n, \ a > 0 \ cste. \end{cases}$$

Solution 46 *1. This is a power series with* $a_n = \begin{cases} 0, & \text{si } n \text{ non premier;} \\ 1, & \text{si } n = p \text{ premier;} \end{cases}$

Therefore, $\sqrt[n]{|a_n|} = \begin{cases} 0, & \text{si } n \text{ non premier;} \\ 1, & \text{si } n = p \text{ premier;} \end{cases}$

and the sequence $\left\{ \sqrt[n]{|a_n|} \right\}$ *has two accumulation points, which are* 0 *and* 1. *Thus,* $\overline{\lim_{n\to+\infty}} \sqrt[n]{|a_n|} = 1 \Rightarrow R = 1.$

2. It is clear that

$$\sqrt[n]{|a_n|} = \sqrt[n]{|\lambda_n|} = \begin{cases} \dfrac{1}{\sqrt[3p]{p+\sqrt{p}}}, & \text{si } n = 3p \\ \dfrac{1}{p^{\frac{p}{3p+1}}}, & \text{si } n = 3p+1 \\ a^{\frac{p}{3p+1}}, & \text{si } n = 3p+2. \end{cases}$$

These three expressions tend to 1, 0 *and* $a^{1/3}$ *respectively as* $p \to +\infty \Rightarrow$

$$R = \frac{1}{\sup\{1, a^{1/3}\}}.$$

Extension of the Notion of Power Series to the Complex Variable

We consider power series $\sum_{n=0}^{+\infty} a_n z^n$ of the complex variable z. Its radius of convergence is calculated in the same way as in the case of the real variable, only the study at the boundary differs since the convergence domain D_c is now a disc in $\mathbb{C}$.

Example 47 *Study of the series* $\sum_{n=1}^{+\infty} \frac{z^n}{n}$.

It is clear that $R = 1$. *For* z *such that* $|z| = 1$, *we have* $z = \exp(i\theta)$. *The study at the boundary thus amounts to studying the numerical series* $\sum_{n=1}^{+\infty} \frac{\exp(in\theta)}{n}$ *for which we apply Abel's criterion:*

The sequence $\left\{\frac{1}{n}\right\}_{n\in N^*}$ *is positive, decreasing, and tends to* 0 *as* $n \to +\infty$ *while for* $\theta \neq \dfrac{k\pi}{2}, k \in \mathbb{Z}$*, the sequence* $\{\exp(in\theta)\}_{n\in N^*}$ *is uniformly bounded, i.e.,* $\exists M > 0,$

$$\forall N \in \mathbb{N}^*, \forall\theta \neq \theta \neq \frac{k\pi}{2}, k \in \mathbb{Z} : \left|\sum_{n=1}^{N} \exp(in\theta)\right| < M$$

The series $\displaystyle\sum_{n=1}^{+\infty} \frac{\exp(in\theta)}{n}$ *converges for the corresponding values of* z *and diverges for* z *of the form* $z = \exp\left(i\frac{k\pi}{2}\right)$.

Chapter 4

FOURIER SERIES

Definition 48 *A Fourier series is any series of functions of the form*

$$\begin{aligned} &\tfrac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + ... + a_n \cos nx + b_n \sin nx + ... \\ &= \tfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx, \end{aligned} \tag{4.1}$$

where the real constants a_0, a_n and b_n fot $n \geq 1$, are called the coefficients of the series.

Remark 49 *When the Fourier series converges, its sum is a function with a period of 2π.*

4.1 Determination of Fourier Coefficients

Let f be a $2\pi-$periodic function. Can it be represented by a convergent Fourier series (4.1) in the interval $(-\pi,\ \pi)$? In other words, we seek real numbers a_0, a_n and b_n, $n \geq 1$, to have

$$f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + ... + a_n \cos nx + b_n \sin nx + ... \tag{4.2}$$

It is clear that if the numerical series $\frac{|a_0|}{2} + \sum\limits_{n\geq 1} |a_n| + |b_n|$ converges, then the Fourier series is bounded on $(-\pi, \pi)$, Therefore, by integrating (4.2) on $(-\pi,\ \pi)$, we obtain successively:

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)dx &= \int_{-\pi}^{\pi} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \right) dx \\ &= \int_{-\pi}^{\pi} \frac{a_0}{2} dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx dx + b_n \int_{-\pi}^{\pi} \sin nx dx \right).\end{aligned} \tag{4.3}$$

but

$$\int_{-\pi}^{\pi} \cos nx dx = \left[\frac{\sin nx}{n} \right]_{-\pi}^{\pi} = 0 \forall n \geq 1,$$

and

$$\int_{-\pi}^{\pi} \sin nx dx = \left[\frac{-\cos nx}{n} \right]_{-\pi}^{\pi} = -\frac{1}{n} \left[\cos n\pi - \cos\left(-n\pi \right) \right] = 0 \forall n \geq 1$$

Therefore, (4.3) leads to $\int\limits_{-\pi}^{\pi} f(x)dx = \pi a_0$, so

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx. \tag{4.4}$$

To find the expressions for a_n and b_n, we will use the trigonometric identities:

$$\begin{aligned}\sin mx \cos nx &= \tfrac{1}{2} \left[\sin\left(m+n \right) x + \sin\left(m-n \right) x \right], \\ \cos mx \sin nx &= \tfrac{1}{2} \left[\cos\left(m+n \right) x + \cos\left(m-n \right) x \right], \\ \sin mx \cos nx &= \tfrac{1}{2} \left[\cos\left(m-n \right) x + \cos\left(m+n \right) x \right],\end{aligned} \tag{4.5}$$

which immediately give:

For $n \neq m$:

$$\begin{aligned}\int_{-\pi}^{\pi} \sin mx \cos nx dx &= 0,\\ \int_{-\pi}^{\pi} \cos mx \cos nx dx &= 0,\\ \int_{-\pi}^{\pi} \sin mx \sin nx dx &= 0,\end{aligned} \tag{4.6}$$

For $n = m : (n \geq 1)$

$$\int_{-\pi}^{\pi} \cos^2 (nx)\, dx = \frac{1}{2}\int_{-\pi}^{\pi} (1 - \cos (2nx))\, dx = \pi$$

To find $a_n$and b_n, we consider the two integrals respectively:

$$\begin{aligned}\int_{-\pi}^{\pi} f(x) \cos nx dx &= \int_{-\pi}^{\pi} \left[\left(\tfrac{a_0}{2} + \sum_{n\geq 1} a_n \cos mx + b_n \sin nx\right) \cos mx\right] dx\\ &= \tfrac{a_0}{2} \int_{-\pi}^{\pi} \cos mx dx + \sum_{n\geq 1} \int_{-\pi}^{\pi} a_n \cos nx \cos mx dx + \sum_{n\geq 1} b_n \int_{-\pi}^{\pi} \sin nx \cos mx dx\end{aligned}$$

According to the identities (4.5) and (4.6) above, we find

$$\int_{-\pi}^{\pi} f(x) \cos nx dx = a_n \int_{-\pi}^{\pi} \cos^2 (nx)\, dx,$$

which gives

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx dx. \tag{4.7}$$

To find the coefficients b_n, we consider $\int_{-\pi}^{\pi} f(x) \sin nx dx$, and we obtain

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx dx. \tag{4.8}$$

Remark 50 *If f is a periodic function, the same value is obtained by integrating on* *an arbitrary interval with a length equal to the period. Indeed, if* $T = 2\pi$ *for example, we have*

$$\forall x \in \mathbb{R} : \int_{-\pi}^{\pi} f(x)dx = \int_{\lambda}^{\lambda+2\pi} f(x)dx. \tag{4.9}$$

This is because the integral on the right decomposes into

$$\int_{\lambda}^{\lambda+2\pi} f(x)dx = \int_{\lambda}^{-\pi} f(x)dx + \int_{-\pi}^{\pi} f(x)dx + \int_{-\pi}^{\pi} f(x)dx + \int_{\pi}^{\lambda+2\pi} f(x)dx.$$

Now, if we substitute $t = x - 2\pi$ *in the last integral, then*

$$\int_{\pi}^{\lambda+2\pi} f(x)dx = \int_{-\pi}^{\lambda} f(2\pi + t)dt = -\int_{\lambda}^{-\pi} f(t)dt = \int_{-\pi}^{\pi} f(x)dx.$$

Definition 51 *We say that* f *is* ***piecewise monotonic*** *on the interval* $[a, b]$, *if it is possible to decompose it into points:* $x_0 = a < x_1 < x_2 < .... < x_n = b$; *such that on each interval* $(x_{i,}\ x_{i+1})$, *the function* f *is either increasing or decreasing. If* f *is monotonic and bounded, these possible discontinuity points are of the* ***1st kind****. To determine if* x_0 *is such a point, we have*

$$L_g = \lim_{x \overset{<}{\to} x_0} f(x) = f(x_0 - 0) \neq L_d = \lim_{x \overset{>}{\to} x_0} f(x) = f(x_0 + 0),$$

and L_g *and* L_d *are finite numbers.*

Dirichlet's Theorem: If the function f is periodic with a period of 2π, piecewise monotonic, and bounded, then its Fourier series converges everywhere. Its sum $S(x)$ is equal to $f(x)$ if f is continuous at x, while at discontinuity points, its sum is equal to the arithmetic mean of the left and right limits, i.e., if $x = x_0$ is a discontinuity point of f , then

$$S(x_0) = \frac{L_g \; + L_d}{2} = \frac{f(x_0-0)+f(x_0+0)}{2}. \tag{4.10}$$

Example 52 *Let f be a function with a period of 2π, given by $f(x) = x^2, -\pi \leq x \leq \pi$.*

For the calculation of coefficients:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2\pi^3}{3},$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx dx = \frac{4(-1)^n}{n^2},$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = 0.$$

As f is piecewise monotonic, bounded, and continuous, it is equal to its Fourier series:

$$x^2 = \frac{\pi^3}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx, \;\; x \in [-\pi, \pi].$$

For $x = 0$: $\quad 0 = \frac{\pi^3}{3} + 4\sum_{n\geq 1}\frac{(-1)^n}{n^2}$; *we deduce the sum*

$$\sum_{n\geq 1}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

For $x = \pi$, we obtain

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}. \tag{4.12}$$

4.2 Fourier Series of Functions with Arbitrary Period

Let f be a periodic function with period w. With the change of variable $x = \frac{Wt}{\pi}$, the new function $f(\frac{Wt}{\pi})$ ebecomes periodic with period 2π. In its Fourier series, it develops as follows:

$$f(\frac{Wt}{\pi}) = \frac{a_0}{2} + a_1 \cos t + b_1 \sin t + ... + a_n \cos nt + b_n \sin nt + ...,$$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos ntdt$ et $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin ntdt$

Returning to the variable x, we have: $t = \frac{\pi x}{W} \Longrightarrow dt = \frac{\pi}{W}dx$, then

$$a_n = \frac{1}{w}\int_{-W}^{W} f(x) \cos\left(\frac{nWx}{\pi}\right) dx \text{ and } b_n = \frac{1}{W}\int_{-W}^{W} f(x) \sin\left(\frac{nWx}{\pi}\right) dx,$$

and therefore,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{nWx}{\pi} + b_n \sin\frac{nWx}{\pi}\right). \tag{4.13}$$

4.3 Fourier Series of Even and Odd Functions

1. If f is an even function, then: $\int_{-\pi}^{\pi} f(x)dx = 2\int_{0}^{\pi} f(x)dx$, which gives:

   For $f(x) \cos nx$ even $\Longrightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nxdx = \frac{2}{\pi}\int_{0}^{\pi} f(x) \cos nxdx$,

   For $f(x) \sin nx$ odd $\Longrightarrow b_n = 0$.

2. If f is an odd function, i.e., $\int_{-\pi}^{\pi} f(x)dx = 0$,which gives:

   For $f(x) \cos nx$ odd $\Longrightarrow a_n = 0$, For $f(x)\sin nx$ even $\Longrightarrow b_n = \frac{2}{\pi}\int\limits_0^\pi f(x)\sin nx dx.$

4.4 Complex Form of Fourier Series

Let f be a periodic function with period 2π represented by its Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

Since $\cos nx = \frac{e^{inx}+e^{-inx}}{2}$ and $\sin nx = \frac{e^{inx}-e^{-inx}}{2i}$, then

$$\begin{aligned} f(x) &= \tfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left(\tfrac{e^{inx}+e^{-inx}}{2}\right) + b_n \left(\tfrac{e^{inx}-e^{-inx}}{2i}\right) \\ &= \tfrac{a_0}{2} + \sum_{n=1}^{\infty} \tfrac{a_n - ib_n}{2} e^{inx} + \tfrac{a_n + ib_n}{2} e^{-inx}. \end{aligned}$$

By letting $c_0 = \frac{a_0}{2}, c_n = \dfrac{a_n - ib_n}{2}$ and $c_{-n} = \dfrac{a_n + ib_n}{2}$, the Fourier series of f is written as:

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{inx} + c_{-n} e^{-inx},$$

or in a more compact form:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \tag{4.14}$$

which is the **complex form** of the Fourier series with

$$c_n = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f(x) e^{-inx} dx,\ n \in \mathbb{Z}. \tag{4.15}$$

4.5 Approximation of a Function by a Trigonometric Polynomial

Consider the function $f(x)$ on the interval $[a,b]$. If we approximate $f(x)$ by another function $g(x)$, the error can be evaluated by **the maximum deviation** δ_m given by:

$$\delta_m = \max_{a \leq x \leq b} |f(x) - g(x)|, \tag{4.16}$$

or more commonly and widely used, by **the mean square deviation** δ defined as:

$$\delta^2 = \frac{1}{b-a} \int_a^b (f(x) - g(x))^2 \, dx. \tag{4.17}$$

Now, consider f as a $2\pi-$periodic function. Among all trigonometric polynomials

$$P_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_n \sin kx \tag{4.18}$$

where a_0, a_k, b_k, $k = 1, 2, ...n$, are arbitrary real coefficients, it is shown that the polynomial that gives the best possible approximation (i.e., the smallest δ) is the one where a_0, a_n and b_n are the Fourier coefficients. This polynomial,

$\frac{a_0}{2} + a_1 \cos x + b_1 \sin x + ... + a_n \cos nx + b_n \sin nx$ is then called **the Fourier polynomial**. It is shown that:

$$\delta^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) dx - \left(\frac{a_0^2}{2} + \sum_{n=k}^{n} a_k^2 + b_k^2 \right), \tag{4.19}$$

which leads to **the Bessel's inequality**:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx \approx \frac{a_0^2}{2} + \sum_{k=1}^{n} a_k^2 + b_k^2, \tag{4.20}$$

and if $n \longrightarrow \infty$ ($\delta = 0$), it results in **the Parseval-Liapounov equality**:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2. \tag{4.21}$$

Example 53 *Let's revisit the previous example $f(x) = x^2$. Then $\int_{-\pi}^{\pi} x^4 dx = \frac{2}{5}\pi^5$, The Parseval-Liapounov equality reads:*

$$\frac{2}{5}\pi^4 = \frac{\left(\frac{2\pi^2}{3}\right)^2}{2} + \sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{n^2}\right)^2,$$

and after simplification, it yields the formula:

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}. \tag{4.22}$$

Chapter 5

IMPROPER INTEGRALS

5.1 Definitions

Definition 54 *A function f is said to be* ***locally integrable*** *on I if it is integrable on every interval $[a,b] \subseteq I$*

Definition 55 *Let f be a function defined on the interval $[a,b[= I$ (where $b = +\infty$) and locally integrable on I. We say that the integral $\int_a^b f(x)\,dx$* ***converges*** *at b if the function $F(x) = \int_a^x f(t)\,dt$ defined on $[a,b[$ has a finite limit as $x \to b$ (this finite limit is called the integral of f on $[a,b[$ and is denoted as $\int_a^b f(t)\,dt$); otherwise, $\int_a^b f(t)\,dt$ is said to be* ***divergent.***

- Let f be a function defined on $I =]a,b]$ (where $b = -\infty$) can be $\int_a^b f(x)\,dx$ and locally integrable on I. We say that the integral $F(x) = \int_x^b f(t)\,dt$ converges at a if the function $F(x) = \int\limits_x^b f(t)\,dt$ defined on $]a,b]$ ahas a finite limit as $x \to a$.

Example 56 $I = \int\limits_a^{+\infty} \frac{dx}{x^\alpha}$.

1. *For* $\alpha \neq 1$, $I = \int\limits_a^{+\infty} \frac{dx}{x^\alpha} = \left[\frac{1}{(\alpha-1)\,x^{\alpha-1}}\right]_1^{+\infty}$.

   If $\alpha > 1 \Rightarrow I = \frac{1}{\alpha-1}$, *so* I *converges.*

   If $\alpha < 1 \Rightarrow I$ *diverges.*

2. *For* $\alpha = 1$, $I = \int\limits_a^{+\infty} \frac{dx}{x} = [\ln x]_1^{+\infty} = +\infty$, *so* I *diverges.*

Example 57 *Study the convergence of the following integrals:*

a) $\int\limits_0^{+\infty} e^{-t}dt$:

We have $\int\limits_0^x e^{-t}dt = 1 - e^{-x}$, *and as* $\lim\limits_{x\to+\infty} e^{-x} = 0$, $\int\limits_0^{+\infty} e^{-t}dt$ *is convergent and equals* 1.

b) $\int\limits_0^{+\infty} \cos t dt$:

We have $\int\limits_0^x \cos t dt = \sin x$, *as* $\lim\limits_{x\to+\infty} \sin x$ *does not exist,* $\int\limits_0^{+\infty} \cos t dt$ *is divergent.*

c) $\int\limits_1^2 \frac{1}{t-1}dt$:

We have $\int\limits_x^2 \frac{1}{t-1}dt = -\ln(x-1)$ *if* $x > 1$. *As* $\lim\limits_{x \overset{>}{\to} 1} -\ln(x-1) = -\infty$, *the integral* $\int\limits_1^2 \frac{1}{t-1}dt$ *is divergent.*

- Let f be a function defined on $I =]a, b[$ except possibly at isolated points; a or b could be $-\infty$ or $+\infty$. Suppose that the function f is locally integrable on I. We say that the integral $\int\limits_a^b f(t)\,dt$ is convergent (at both a and b) if there exists $c \in]a, b[$ such that $\int\limits_a^c f(t)\,dt$ and $\int\limits_c^b f(t)\,dt$ are both convergent. By definition, we set

$$\int\limits_a^b f(t)\,dt = \int\limits_a^c f(t)\,dt + \int\limits_c^b f(t)\,dt.$$

Example 58 $I = \int\limits_0^2 \frac{dt}{t-1}$.

We have $I = \int\limits_0^1 \frac{dt}{t-1} + \int\limits_1^2 \frac{dt}{t-1}$

$\int\limits_0^1 \frac{dt}{t-1}$ *is divergent and* $\int\limits_1^2 \frac{dt}{t-1}$ *is divergent* $\Rightarrow I$ *is divergent.*

5.2 Absolute Convergence of Improper Integrals

Definition 59 *We say that the integral* $\int\limits_a^b f(t)\,dt$ *is* ***absolutely convergent*** *if* $\int\limits_a^b |f(t)|\,dt$ *is convergent.*

Theorem 60 $\int\limits_a^b f(t)\,dt$ *absolutely convergent* $\Rightarrow \int\limits_a^b f(t)\,dt$ *convergent. Indeed, since for all real* $t : -|f(t)| \leq f(t) \leq |f(t)|$ *, then* $\left|\int\limits_a^b f(t)\,dt\right| \leq \int\limits_a^b |f(t)|\,dt.$

As $\int\limits_a^b |f(t)|\,dt < +\infty$, *we have* $\left|\int\limits_a^b f(t)\,dt\right| < +\infty$, *so* $\int\limits_a^b f(t)\,dt$ *is convergent.*

Example 61 $\int\limits_0^{+\infty} e^{-t} \sin t dt$

We have $\int_0^{+\infty} |e^{-t}\sin t|\, dt = \int_0^{+\infty} e^{-t}|\sin t|\, dt \leq \int_0^{+\infty} e^{-t}dt < +\infty,$

so $\int_0^{+\infty} e^{-t}\sin t dt$ *is absolutely convergent, and therefore,* $\int_0^{+\infty} e^{-t}\sin t dt$ *is convergent.*

5.3 Some Convergence Criteria

Majorization Convergence Criterion: If f is positive, then the integral $\int_a^b f(t)\, dt$ converges at b if the function $F(x) = \int_a^x f(t)\, dt$ is bounded on $[a, b[$.

Comparison Criterion: Let f and g be two positive functions, defined and locally integrable on $[a, b[$.

If there exists $M > 0$ such that $f(x) \leq M g(x)\ \forall x \in [a, b[$, then:

$$\int_a^b g(t)\, dt < +\infty \Rightarrow \int_a^b f(t)\, dt < +\infty.$$

Equivalence Criterion: Given two positive functions f and g defined and locally integrable on $[a, b[$. Let $l = \lim\limits_{\substack{x \to b \\ <}} \frac{f(x)}{g(x)}$. Then

- if $l = 0$, $\int_a^b g(x)\, dx$ converges $\Rightarrow \int_a^b f(x)\, dx$ converges.

- if $l = +\infty$, then if $\int_a^b f(x)\, dx$ diverges $\Rightarrow \int_a^b g(x)\, dx$ diverges

- if l is finite, both integrals are of the same nature.

Remark 62 *In applications, the function* $g(x) = \dfrac{1}{x^\alpha}$ *is frequently used.*

Example 63 a) *Convergence of* $\int_0^{+\infty} e^{-t^2}dt$ *(comparison with* e^{-t}*)*

b) *Show that* $\int_1^{+\infty} t^{\alpha-t}e^{-t^2}dt$ *converges for* $\alpha > 0$ $\left(\textit{comparison with } \frac{1}{t^2}\right)$

c) *Show that* $\int_1^{+\infty} t^{\alpha-t}e^{-t^2}dt$ *converges for* $\alpha > 0$ $(\textit{comparaison with } t^{\alpha-1})$

5.4 Reference Integrals

- Riemann Integrals:

$$\int_a^{+\infty} \frac{dt}{t^\alpha}, (a > 0), \text{ converges if } \alpha > 1.$$

$$\int_0^{a} \frac{dt}{t^\alpha}, (a > 0), \text{ converges if } \alpha < 1.$$

- Bertrand Integrals:

Let $\alpha, \beta \in \mathbb{R}$

$$\int_a^{+\infty} \frac{dt}{t^\alpha (\ln(t))^\beta}, (a > 0), \text{ converges if } (\alpha > 1) \text{ or } (\alpha = 1 \text{ and } \beta > 1).$$

- Gauss Integrals:

The integral $\int_0^{+\infty} e^{-t^2}dt$ converges and equals $\frac{\sqrt{\pi}}{2}$

- Dirichlet Integrals:

The integral $\int_0^{+\infty} \frac{\sin t}{t}dt$ converges and equals $\frac{\pi}{2}$.

- Fresnel Integrals:

The integrals $\int_0^{+\infty} \sin t^2 dt$ and $\int_0^{+\infty} \cos t^2 dt$ are convergent and equal $\frac{\pi}{2\sqrt{2}}$.

Let's show that the first integral converges:

$$\int_0^{+\infty} \sin t^2 dt = \int_0^{+\infty} 2t \frac{\sin t^2}{2t} dt.$$

We set $u = \frac{1}{2t} \to u' = \frac{1}{t^2}$

$$v' = \frac{\sin t^2}{2t} \to v = \sin t^2$$

$$\begin{aligned} \int_0^{+\infty} \sin t^2 dt &= \left.\frac{\sin t^2}{2t}\right]_0^{+\infty} - \int_0^{+\infty} \frac{\sin t^2}{t^2} dt = -\int_0^{+\infty} \frac{\sin t^2}{t^2} dt \\ &= -\int_0^{1} \frac{\sin t^2}{t^2} dt - \int_1^{+\infty} \frac{\sin t^2}{t^2} dt. \end{aligned}$$

The first integral converges because $\lim\limits_{t \to +\infty} \frac{\sin t^2}{t^2} = 1$.

For the second integral, we have $\left|\frac{\sin t^4}{t^2}\right| \leq \frac{1}{t^2}$. Since $\int_1^{+\infty} \frac{1}{t^2} dt$ is convergent, it follows that $\int_1^{+\infty} \frac{\sin t^2}{t^2} dt$ is convergent. In conclusion, $\int_0^{+\infty} \sin t^2 dt$ is convergent.

5.5 Integral depending on a parameter

5.5.1 Limit passage under the integral sign

We study $\lim\limits_{n \to +\infty} \int_I f_n(t)\, dt$.

If $I = [a, b]$, we know that if f_n are continuous and $f_n \to f$ (CU), then $\int_a^b f_n \to \int_a^b f dt$.

Let $\{f_n\}$ be a sequence of functions from I to $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).

1) f_n are piecewise continuous on I;

2) $f_n \to f$ simply, where f is piecewise continuous on I;

3) $\exists \varphi : I \to \mathbb{R}^+$ piecewise continuous on I and integrable, verifying $\forall n \in \mathbb{N}$ $|f_n(x)| \leq \varphi(x)$;

then the functions f_n are integrable on I, and thus $\int_I f_n \to \int_I f dt$.

Example 64 *Study* $\lim\limits_{n\to+\infty} \int_{-\infty}^{+\infty} \frac{1+2\sin\left(\frac{t}{n}\right)}{1+t^2} dt$.

Let $f : \mathbb{R} \to \mathbb{R}$, *defined by*

$$f_n(t) = \frac{1 + 2\sin\left(\frac{t}{n}\right)}{1+t^2}.$$

We have $f_n \to f$, *simply with* $f(t) = \frac{1}{1+t^2}$. *The functions* f_n *and* f *are piecewise continuous. Moreover,*

$$|f_n(t)| \leq \frac{3}{1+t^2} = \varphi(t).$$

By dominated convergence, the functions f_n *and* f *are integrable, and*

$$\lim_{n\to+\infty} \int_{-\infty}^{+\infty} \frac{1 + 2\sin\left(\frac{t}{n}\right)}{1+t^2} dt = \int_{-\infty}^{+\infty} \frac{1}{1+t^2} dt.$$

5.5.2 Continuity of a parameter-dependent integral

We study functions of the form

$$g : x \in X \to \int_I f(x,t)\, dt,$$

where X s often an interval in $\mathbb{R}$.

Continuity by Domination:

If $f : X \times I \to \mathbb{K}$ is such that

1. For all $x \in X, t \to f(x,t)$ is piecewise continuous on I;

2. For all $t \in I, x \to f(x,t)$ is piecewise continuous on X;

3. There exists $\varphi : I \to \mathbb{R}^+$, piecewise continuous and integrable, such that

$$\forall (x,t) \in X \times I, \ \ |f(x,t)| \leq \varphi(t);$$

then the function $g : x \to \int_I f(x,t)\,dt$ is well-defined and continuous on I.

Example 65 *Definition and continuity of* $g(x) = \int_0^{+\infty} \frac{e^{-Xt}}{1+t^2} dt$ *with* $x \in \mathbb{R}^+$

Consider $f(x,t) \to \frac{e^{-xt}}{1+t^2}$ *defined onr* $\mathbb{R}^+ \times [0, +\infty[$.

For all $t \in [0, +\infty[, x \to f(x,t)$ *is continuous on* $\mathbb{R}^+$.

For all $(x,t) \in \mathbb{R}^+ \times [0, +\infty[, |f(x,t)| \leq \varphi(t) = \frac{1}{1+t^2}$, *where* $\varphi : [0, +\infty[\to \mathbb{R}^+$ *is piecewise continuous on* $[0, +\infty[$ *because* $\varphi(t) \sim \frac{1}{t^2}$ *when* t *is very large.*

By domination, the function g *is well-defined and continuous on* $\mathbb{R}^+$.

5.5.3 Differentiation of a parameterized integral

We study functions of the form $g : x \in X \to \int_I f(x,t)\,dt$ where X is an interval in $\mathbb{R}$.

Definition 66 *Let* $f : (x,t) \to f(x,t)$ *be defined on* $X \times I$. *We say that* f *has a partial derivative* $\frac{\partial f}{\partial x}$ *if* $\forall t \in I$, *the function* $x \to f(x,t)$ *is differentiable. In this case, we define* $\frac{\partial f}{\partial x}(x,t) = \frac{d}{dx} f(x,t)$. *Let* $f : X \times I \to \mathbb{K}$ *be such that* f *has a partial derivative* $\frac{\partial f}{\partial x}$. *If, in addition,*

1. For all $x \in X$ $t \to \varphi(x,t)$ *is piecewise continuous and integrable on* I;

2. *For all* $x \in X$ $t \to f(x,t)$ *is piecewise continuous on* I;

3. *For all* $x \in X$ $t \to \frac{\partial f}{\partial x}(x,t)$ *is piecewise continuous on* I;

4. *For all* $x \in X$ $t \to \frac{\partial f}{\partial x}(x,t)$ *is continuous on* X;

5. *There exists* $\varphi : I \to \mathbb{R}^+$ *that is piecewise continuous and integrable, such that*

$$\forall (x,t) \in X \times I : \left|\frac{\partial f}{\partial x}(x,t)\right| \leq \varphi(t);$$

then the function $g : x \to \int\limits_I f(x,t)\, dt$ *is well-defined and* C^1 *on* X, *with* $g'(x) = \int\limits_I \frac{\partial f}{\partial x}(x,t)\, dt$.

Example 67 *Calculation of* $g(x) = \int\limits_0^{+\infty} e^{-t^2} \cos(xt)\, dt$ *with* $x \in \mathbb{R}$.

Let $f(x,t) = e^{-t^2} \cos(xt)$, *here* $X = \mathbb{R}, I = [0, +\infty[$. f *is defined on* $\mathbb{R} \times [0, +\infty[$ *and has a partial derivative* $\frac{\partial f}{\partial x}(x,t) = -te^{-t^2} \sin(xt)$,

For all $x \in \mathbb{R}$, $\frac{f(x,t)}{\frac{1}{t^2}} \to 0$ *as* $t \to +\infty$, *so* $t \to f(x,t)$ *is piecewise continuous on* $[0, +\infty[$,

For all $x \in \mathbb{R}$, $t \to \frac{\partial f}{\partial x}(x,t)$ *is piecewise continuous on* $[0, +\infty[$,

For all $t \in [0, +\infty[$, $x \to \frac{\partial f}{\partial x}(x,t)$ *is continuous on* $\mathbb{R}$,

Moreover, for all $(x,t) \in \mathbb{R} \times [0, +\infty[$: $\left|\frac{\partial f}{\partial x}(x,t)\right| \leq te^{-t^2} = \varphi(t)$ *where* $\varphi : [0, +\infty[\to \mathbb{R}$ *is piecewise continuous and integrable on* $[0, +\infty[$.

By domination, the function g *is* C^1, *and* $g'(x) = \int\limits_0^{+\infty} -te^{-t^2} \sin(xt)\, dt$.

Let $V' = -te^{-t^2}$ *and* $U = \sin(xt)$, *then*

$$g'(x) = \left[\frac{1}{2}e^{-t^2} \sin(xt)\right]_0^{+\infty} - \frac{x}{2} \int\limits_0^{+\infty} e^{-t^2} \cos(xt)\, dt,$$

resulting in the first-order linear differential equation for g

$$g^{'}(x) = -\frac{x}{2} g(x).$$

g *is a solution to a first-order linear differential equation with the initial condition* $g(0) = \frac{\sqrt{\pi}}{2}$, *and we obtain*

$$g(x) = \frac{\sqrt{\pi}}{2} e^{-\frac{1}{2}x^2}.$$

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