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INTRODUCTION TO NORMED AND HILBERT SPACES
Intended to the students of the third year Bachelor in Mathematics

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Contents

General introduction

This manuscript provides an introductory exploration of the fundamental concepts and properties of normed and linear spaces. It is intended for third-year Bachelor of Mathematics students. It is divided into several chapters and sections, each covering specific topics. It also includes exercises to reinforce understanding of the concepts presented.

Normed and linear spaces are essential mathematical structures used in various fields such as functional analysis, linear algebra, and mathematical physics. This manuscript serves as a comprehensive guide for students to develop a solid understanding of these spaces, their properties, and their applications. By studying this subject, students can lay a strong foundation for advanced mathematical studies and applications in various scientific disciplines.

Chapter one introduces the basic concepts of metric spaces. It covers definitions and examples of metric spaces, open and closed balls, convergence, continuity, and open and closed sets.

Chapter two focuses on normed linear spaces. It discusses norms over E, distance associated with a norm, properties of the norm, equivalent norms, and Banach spaces. It also explores finite- dimensional normed linear spaces, continuous linear applications, and the product of two normed spaces.

The third chapter delves into Hilbert spaces, which are special types of normed linear spaces. It introduces the scalar product, orthogonality, the Hilbert projection theorem, the theorem of F. Riesz, and orthonormal systems in a Hilbert space.

Overall, this manuscript provides a systematic and comprehensive introduction to normed and linear spaces. It covers essential concepts, definitions, and properties, allowing students to develop a solid foundation in this area of mathematics. By studying this manuscript, students can gain the necessary knowledge to understand and apply normed and linear spaces in various mathematical and scientific contexts.

A table of contents is provided to help you easily navigate the course.

We hope that this course will allow you to develop an in-depth understanding of normed and linear spaces. Good study!

Rachid CHEURFA and Djamel DEGHOUL

Chapter 1

Elements of topology of metric spaces

In this chapter, we recall some notions and properties about metric spaces useful for the undertanding of the next chapters of this course.

1.1 Definitions and examples

Definition 1 *A metric space (X, d) consists of a non-empty set X and a function $d : X \times X \to [0, \infty)$ such that*

i) ***(Separation)*** $\forall x, y \in X, d(x, y) = 0 \iff x = y$.

ii) ***(Symmetry)*** *For all* $x, y \in X, d(x, y) = d(y, x)$.

iii) ***(Triangle inequality)*** *For all* $x, y, z \in X, d(x, z) \leq d(x, y) + d(y, z)$. *then, d is called a metric (or a distance) on X.*

Then, d is called a metric (or a distance) on X.

Example 2 *Let X be the set of all continuous functions $f, g\colon [a,b] \to \mathbb{R}$, we denote it by $X = C([a,b],\mathbb{R})$, we can see that the applications $d_i, i = 1,2,3$ defined respectively by*

$$\begin{aligned} d_1(f,g) &= \sup\{|f(x)-g(x)|\colon x \in [a,b]\} \\ d_2(f,g) &= \int_a^b |f(x)-g(x)|dx \\ d_3(f,g) &= \left(\int_a^b |f(x)-g(x)|dx\right)^{\frac{1}{2}} \end{aligned}$$

are metrics on $X = C([a,b],\mathbb{R})$

Example 3 *The sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is bounded if there exists a real number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.*
- Show that $d(\{x_n\},\{y_n\}) = \sup\{|x_n - y_n|, n \in \mathbb{N}\}$ is a metric of X (see Worksheet of exercises 1).

1.2 Open and closed balls

Definition 4 *Let (X,d) a metric space, $a \in X$ and $r > 0$. The open ball centred at a with radium r is the set $B(a,r) = \{x \in X, d(x,a) < r\}$.*
The closed ball centred at a with radium r is the set $\overline{B(a,r)} = \{x \in X, d(x,a) \leq r\}$.
A subset E of X is called open if any point a of E is a center of an open ball completely included in E, this means that E open $\implies \forall a \in E, \exists r > 0, B(a,r) \subset E$.

1.3 Convergence and continuity

Definition 5 *Let (X, d) be a metric space. A sequence $\{x_n\}$ in X converges to a point $a \in X$ if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $d(x_n, a) < \epsilon$ for all $n \geq N$. We write $\lim_{n\to\infty} = a$ or $x_n \to a$. So, it is clear that we have the equivalence $\{x_n\}$ converges to a if and only if $\lim_{n\to\infty} d(x_n, a) = 0$.*

Definition 6 *Assume that $(X, d_1), (Y, d_2)$ are two metric spaces. A function $f : X \to Y$ is said to be continuous at a point $a \in X$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $d_1(x, a) < \delta, d_2(f(x), f(y)) < \epsilon$. This definition means that we can get the distance between $f(x)$ and $f(a)$ smaller than ϵ by choosing x such that the distance between x and a is smaller than δ.*
Geometrically, this means that for any open ball $B(f(a), \epsilon)$ around $f(a)$, there is an open ball $B(a, \delta)$ around a such that $f(B(a, \delta)) \subset B(f(a), \epsilon)$

The next proposition is very usefull for studying continuity.

Proposition 7 *Let $f : X \to Y$. The following are equivalent:*

i) f is continuous at a point $a \in X$.

ii) For every sequence $\{x_n\}$ converging to a, the sequence $\{f(x_n)\}$ converges to $f(a)$.

Proof. See Worksheet of exercises 1. ■

1.4 Open and closed sets

We recall the definitions of open and closed balls:

$$B(a, r) = \{x \in X : d(x, a) < r\}.$$
$$\overline{B(a, r)} = \{x \in X : d(x, a) \leq r\}.$$

Let $A \subseteq X$ and x a point of X. We have the following possibilities:

(i) There is a ball $B(x, r)$ around x which is contained in A. In this case x is called an interior point of A.

(ii) There is a ball $B(x, r)$ around x which is contained in the complement A^c. In this case x is called an exterior point of A.

(iii) All balls $B(x, r)$ around x contain points in A as well as points in the complement A^c. In this case x is a boundary point of A.

Definition 8 *A subset A of a metric space is open if it does not contain any of its boundary points, and it is closed if it contains all its boundary points.*

Proposition 9 *A subset A of a metric space X is open if and only if it consists only of interior points, i.e., for all $a \in A$ there is a ball $B(a, r)$ around a which is contained in A.*

Proposition 10 *A subset A of a metric space X is closed if and only if its complement A^c is open.*

Lemma 11 *All open balls $B(a, r)$ are open sets, while all closed balls $B(a, r)$ are closed sets.*

Proof. We prove the statement for open balls. Let $x \in B(a, r)$, we must show that there is a ball $B(x, \epsilon)$ around x which is contained in $B(a, r)$. If we choose $\epsilon = r - d(x, a)$, we choose $\epsilon = r - d(x, a)$, we see that by the triangle inequality, we have

$$\begin{aligned} \text{for } y \in B(x, \epsilon); d(y, a) \leq d(y, x) + d(x, a) &< r - d(x, a) + d(x, a) \\ &< r \end{aligned}$$

thus $d(y, a) < r$, hence $B(x, \epsilon) \subset B(a, r)$.

■ Using sequences, we can show that

Proposition 12 *Assume that F is a subset od a metric space X. The following are equivalent:*

i) F is closed

ii) If x_n is a convergent sequence of elements in F, then the limit $a = \lim_{n\to\infty} x_n$ remains in F (exercise 3).

Worksheet of exercises 1

Exercise 1 In the space X of bounded real sequence, show that for every sequences $\{x_n\}_{n\in\mathbb{N}}\,,\ \{y_n\}_{n\in\mathbb{N}} \in X, d\left(\{x_n\},\{y_n\}\right) = \sup\left\{|x_n - y_n|\,,\ n\in\mathbb{N}\right\}$ defines a metric on X.

Exercise 2 Let $(X,d)\,,\ (Y,d')$ two metric spaces and $f : X \to Y$ an application. Show that the following are equivalent:

i) f is continuous at a point $a\in X$,

ii) for every sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to a, the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $f(a)$.

Exercise 3 Assume that F is a subset of a metric space $(X,\ d)$. Show that the following assertions are equivalent:

i) F is closed,

ii) if $\{x_n\}_{n\in\mathbb{N}}$ is an arbitrary convergent sequence of elements in F, then $a = \lim\limits_{n\to\infty} x_n \in F$.

Exercise 4 Let A a non-empty subset of $(E,\ d)$. For $x\in E$, we define the distance between x and A by $d(x,\ A) = \inf\limits_{y\in A} d(x,\ y)$.
Show that

1) $d(x,\ A) = 0 \Leftrightarrow x\in\overline{A}$,

2) the application $x\to d(x,A)$ is uniformly continuous.
(Indication: show that for any $(x,x')\in E\times E : |d(x,\ A) - d(x',A)| \le d(x,\ x')$).

Exercise 5 (Left to the students) Extension of equalities. Let f,g be two continuous functions from $(E,\ d)$ to $(E',\ d')$.

1 Show that the subset $A = \{x \in E : f(x) = g(x)\ \}$ is closed in E.

2 Suppose that A is dense in E, then we have $f(x) = g(x)$ for every $x \in E$.

Chapter 2

Normed linear spaces

In what follows, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let E be a vector space over the field $\mathbb{K}$.

2.1 Norms over E

Definition 13 *Let $N\colon E \to \mathbb{R}^+$ a function. We said that N is a norm in E (or over E) if the following conditions are satisfied :*

(1) For any $x \in E$, if $N(x) = 0 \Longrightarrow x = 0$ (condition of separation or nonnegativity of N)

(2) For any $\lambda \in \mathbb{K}$ and $x \in E : N(\lambda x) = |\lambda| N(x)$ (N homogeneous).

(3) Triangle inequality: $\forall x, y \in E$ we have: $N(x + y) \leq N(x) + N(y)$.

If all these conditions are fullfilled, E equipped with a norm N is called a normed linear space or a normed space.

Remark 14 - *The norm on E is often denoted $\|.\|$.*
- *The condition* (1) *is equivalent via contrapositive to:*
if $x \neq 0$, then $N(x) \neq 0$ or to
if $x \neq 0$, then $N(x) > 0$.

2.1.1 Examples

Example 15 *Let $E = \mathbb{K}$. We know that $\mathbb{K}$ is a linear space over itself, with*

$$N(x) = |x|.$$

Example 16 $E = \mathbb{K}^n, n \in \mathbb{N}^*, x = (x_1, x_2, \dots, x_n) \in E$. *We know that $\mathbb{K}$ is a linear space over itself, for N where*

$$N(x_1, x_2, \dots, x_n) = (\sum_{i=1}^{n} |x|^2)^{\frac{1}{2}}$$

is the Euclidean norm on E.

Example 17 $E = \mathbb{K}^n$. *For $x = (x_1, x_2, \dots, x_n) \in E$, we define*

$$\|(x_1, x_2, \dots, x_n)\|_\infty = \max_{1 \le i \le n} |x_i|$$

the infinite norm.

If $n = 2$: $\|(x_1, x_2)\| = \max(|x|, |y|)$.

Example 18 *Generalisation of the Euclidean norm: Let $p \in \mathbb{R}$ with $p \ge 1$. For any $x = (x_1, x_2, \dots, x_n) \in E$, we define $\|.\|_p$ by: $\|(x_1, x_2, \dots, x_n)\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$.*

We can show that $\|.\|_p$ is a norm over $E = \mathbb{K}^n$.

For $0 < p < 1, \|.\|_p$ is not a norm because of the fact that the triangle inequality is not true.

Example 19 *Let X be a non-empty set and $E = \beta(X) = \{f : X \to \mathbb{K}, such\ thatf is\ bounded\}$ be the linear space of functions defined and bounded over X. So for $f \in X, \sup\{|f(x)|, x \in E\} < \infty$ (finite).*

Let $\|f\|_\infty = \sup_{x \in X} |f(x)|$. We can show that $\|.\|_\infty$ is a norm over E.

Example 20 *Let the bounded and closed interval* $[a, b]$ *of* $\mathbb{R}$ *and* $\mathbb{K} = \mathbb{R}$. *Let* $E_0 = C([a, b]) = \{f : [a, b] \to \mathbb{R}, \text{such that} f \text{is continuous}\}$. *We know that any function* f *defined and continuous over* $[a, b]$ *is bounded. We put then* $\|f\|_\infty = \sup_{x \in [a,b]} |f(x)| < \infty$ *(finite), where* $\|.\|_\infty$ *is well defined since* $\|f\|_\infty < \infty$ *for all* $f \in E_0$. $\|.\|_\infty$ *is a norm over* $E_0 = C([a, b])$.

Remark 21 E_0 *is a subspace of* $E = \beta([a, b])$.

Example 22 *Let* l^∞ *be the linear space of bounded real sequences, that is:*

$$l^\infty = \{(x_n)n \geq 1, \sup_{n \geq 1} |x_n| < \infty\}.$$

For $(x_n)_{n \in \mathbb{N}} \in l^\infty$, *we define* $\|(x_n)_n\|_\infty$ *by:* $\|(x_n)_n\|_\infty = \sup_{n \geq 1} |x_n|$. *We can show that* $\|.\|_\infty$ *is a norm over* l^∞.

Example 23 *Let* $E = C([a, b]), f \in E$.

$$\|f\|_1 = \int_a^b |f(x)| dx.$$

$\|.\|_1$ *is a norm called the integral norm. More generally, if* $p \geq 1$, *we define for* $f \in E$,

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}}.$$

We can show that f is a norm. Also, when $0 < p < 1$, we can show that $\|.\|_p$ is not a norm (the triangle inequality is not always true).

Example 24 *For* $p \geq 1$, *we introduce the linear space by* $l^p = \{(x_n)_n, \sum_{i \geq 1} |x_n|^p < \infty\}$ *and for* $(x_n)_n \geq 1 \in l^p, \|(x_n)_n\|_p = \left(\sum_{n=1}^{\infty} |x_n| \right)^{\frac{1}{p}}$. $\|.\|_p$ *is a norm on* l^p. *Also, when* $0 < p < 1, \|.\|_p$ *is not a norm for the same reason.*

Example 25 *Let V be an open of $\mathbb{C}$.*

$$O(V) = \{f : V \to \mathbb{C}, f \ \text{holomorphic on} \, V\}.$$

Let K be a non-empty interior compact of V. If f is holomorphic on V, it is continuous and therefore $\sup_K |f(x)|$ is finite.
Put $\|f\|_K = \sup_{x\in K} |f(x)|$, then $\|.\|_K$ is a norm on $O(V)$ and $(O(V), \|.\|_K)$ is a $\mathbb{C}-$normed linear space.

Example 26 *More generally, let S be a compact topological space and let $C(S)$ be the linear space of all functions defined and continuous on S and having values in $\mathbb{K}$.*
If $f \in C(S)$, then $\sup_{x\in S} |f(x)| < \infty$ (finite).
Put for $f \in C(S)$, $\|f\|_S = \sup_{x\in S} |f(x)|$, where $\|.\|_S$ is a norm on $C(S)$.

Example 27 *Let $l^\infty = \{(x_n)_{n\geq 1}, \sup_{n\geq 1} |x_n| < \infty\}$, for $(x_n)_n \in l^\infty$, we put $\|(x_n)_n\|_\infty = \sup_{n\geq 1} |x_n|$, then $\|.\|_\infty$ is a norm on l^∞.*

Example 28 *For $p \geq 1$, let*

$$l^\infty = \{(x_n)_n, \sum_{n\geq 1} |x_n|^p < \infty\}.$$

For $(x_n)_n \in l^p$, $\|(x_n)_n\|_p = (\sum |x_n|^p)^{\frac{1}{p}}$.
$\|.\|_p$ is a norm on l^p. For $0 < p < 1$, $\|.\|_p$ is not a norm because the triangular inequality is not verified.

Example 29 *Let X be a non-empty set and $E = \beta(X) = \{f : X \to \mathbb{K}, f \, \text{bounded}\}$ that is E is the space of functions defined and bounded over X. So for $f \in X$, $\sup\{|f(x)|, x \in E\} < \infty$, we define $\|f\|_\infty$ by $\|f\|_\infty = \sup_{x\in X} |f(x)|$. We can show that $\|.\|_\infty$ is a norm over E.*

Example 30 *Let* $[a, b]$ *be a bounded and closed interval of* $\mathbb{R}$ *and* $\mathbb{K} = \mathbb{R}$ *and* $E_0 = C([a, b]) = \{f : [a, b] \to \mathbb{R}, f\, contiuous\}$. *We know that any continuous function over* $[a, b]$ *is bounded. We put then* $\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|$. $\|.\|_\infty$ *is well defined since* $\|f\|_\infty < \infty$ *for all* $f \in E_0$.
$\|.\|_\infty$ *is a norm over* $E_0 = C([a, b])$.

2.2 Distance (or metric) associated to a norm

Let $(E, \|.\|)$ be a $\mathbb{K}-$linear space. For $x, y \in E$, we define the application d:

$$\begin{aligned} d : E \times E &\to \mathbb{R}^+ \\ (x, y) &\longmapsto d(x, y) = \|x - y\| \end{aligned}$$

then d defines a (metric) distance on E, since:

(1) $d(x, y) = \|x - y\| = \|y - x\| = d(y, x)$.

(2) $d(x, y) = 0 \Leftrightarrow \|x - y\| = 0 \Leftrightarrow x - y = 0 \Leftrightarrow x = y$.

(3) For $z \in E, d(x, y) = \|x - y\| = \|(x - z) + (z - y)\| \leq \|x - z\| + \|z - y\| = d(x, z) + d(z, y)$.

In this manner, (E, d) becomes a metric space. The topology associated to the distance d is called the topology associated to the norm $\|.\|$.

2.3 Properties of the norm:

- $\forall x, y \in E : |\|x\| - \|y\|| \leq \|x - y\|$. We have

$$\begin{aligned} \|x\| &= \|x - y + y\| \leq \|x - y\| + \|y\| \\ &\Longleftrightarrow \|x\| - \|y\| \leq \|x - y\|. \end{aligned} \tag{2.1}$$

By changing x by y and y by x, we get

$$\|y\| - \|x\| \leq \|y - x\| = \|x - y\|. \tag{2.2}$$

By combining (2.1) and (2.2) , we obtain

$$\begin{aligned} -(\|x\| - \|y\|) &\leq \|x\| - \|y\| \leq \|x - y\| \\ |\|x\| - \|y\|| &\leq \|x - y\| \end{aligned} \tag{2.3}$$

Remark 31 • *From the inequality* (2.3)*, the application* $x \longmapsto \|x\|$ *is uniformly continuous.*

- $\forall x \in E, \forall r > 0$*, the ball* $B(x,r), x \in E, r > 0$ *is a convex set of* E*.*
- *The applications* $x \longmapsto x + y$ *(*y *fixed) and* $x \longmapsto \lambda x$ *(for* $\lambda \in \mathbb{K}$*, fixed), respectively called translation of vector* y *and dilation of ratio* λ*, are uniformly continuous.*

Application 1. Let $x, y \in \mathbb{R}_+^*, p, q \in [1, +\infty[$ such that $1/p + 1/q = 1$, and $a_1, \ldots, a_n, b_1, \ldots, b_n$ $2n$ strictly positive real numbers.

1) Show that $xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$.

2) We suppose in this question that $\sum\limits_{i=1}^{n} a_i^p = \sum\limits_{i=1}^{n} b_i^q = 1$. Show that $\sum\limits_{i=1}^{n} a_i b_i \leq 1$ 3) Deduce from that the **Hölder inequality**:

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q}$$

4) It is further assumed that $p > 1$.

Deduce from the Hölder inequality, the **Minkowski inequality**:

$$\left(\sum_{i=1}^{n} (a_i + b_i)^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} a_i{}^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i{}^q\right)^{1/q}.$$

5) We define for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$. Prove that $\|.\|_p$ is a norm on $\mathbb{R}^n$.

Solution

1) Using the concavity of the logarithm function ln, we have:

$$\ln(\tfrac{1}{p}x^p + \tfrac{1}{q}y^q) \geq \tfrac{1}{p}\ln(x^p) + \tfrac{1}{q}\ln(y^q) = \ln(xy).$$

By passing to the exponential in both sides, we deduce the desired result.

2) If we add up the n equations: $a_i b_i \leq \frac{1}{p}a_i^p + \frac{1}{q}b_i^q$, we get

$$\sum_{i=1}^n a_i b_i \leq \frac{1}{p}\sum_{i=1}^n a_i^p + \frac{1}{q}\sum_{i=1}^n b_i^q = \frac{1}{p} + \frac{1}{q} = 1.$$

3) Let $\alpha_i = \dfrac{a_i}{\left(\sum_{i=1}^n {a_i}^p\right)^{1/p}}$ and $\beta_i = \dfrac{b_i}{\left(\sum_{i=1}^n {b_i}^q\right)^{1/q}}$. we have

$$\sum_{i=1}^n \alpha_i^p = \sum_{i=1}^n \left(\frac{a_i}{\left(\sum_{i=1}^n {a_i}^p\right)^{1/p}}\right)^p = \sum_{i=1}^n \left(\frac{a_i^p}{\left(\sum_{i=1}^n {a_i}^p\right)}\right) = 1$$

we have also $\sum_{i=1}^n \beta_i^p = 1$.

Hence, according to the previous question, $\sum_{i=1}^n \alpha_i \beta_i \leq 1$. It is then sufficient to replace α_i and β_i by their values to obtain the formula.

4) We write $(a_i + b_i)^p$ as $(a_i + b_i)^p = a_i(a_i + b_i)^{p-1} + b_i(a_i + b_i)^{p-1}$.
so $\sum_{i=1}^n (a_i + b_i)^p = \sum_{i=1}^n a_i(a_i + b_i)^{p-1} + \sum_{i=1}^n b_i(a_i + b_i)^{p-1}$

We then apply Hölder's inequality to each of the two sums, with the coefficients p and $q = \frac{p-1}{p}$, we obtain:

$$\left(\sum_{i=1}^n (a_i + b_i)^p\right)^{1/p} \leq \left(\sum_{i=1}^n {a_i}^p\right)^{1/p}\left(\sum_{i=1}^n |a_i + b_i|^{p(q-1)}\right)^{\frac{p-1}{p}} + \left(\sum_{i=1}^n {b_i}^q\right)^{1/q}\left(\sum_{i=1}^n |a_i + b_i|^{p(q-1)}\right)^{\frac{p-1}{p}}$$

$$\leq \left[\left(\sum_{i=1}^n {a_i}^p\right)^{1/p} + \left(\sum_{i=1}^n {b_i}^q\right)^{1/q}\right]\left(\sum_{i=1}^n (a_i + b_i)^{p(q-1)}\right)^{\frac{p-1}{p}}$$

We divid the both sides of this inequality by $\left(\sum_{i=1}^n |a_i + b_i|^{p(q-1)}\right)^{\frac{p-1}{p}}$, we get the desired inequality.

We remark that this resullt is true for $p = 1$.

5) This inequality means that the the previous inequality is very easily translated by saying that$\|.\|_p$

satisfyies the triangular inequality $\|x+y\|_p \leq \|x\|_p + \|y\|_p$. the first properties are simply verified. Thus, $\|.\|_p$ defines a norm on $\mathbb{R}^n$.

2.4 Equivalent norms

Definition 32 *Let E be a $\mathbb{K}-$linear space, $\|.\|_1$ and $\|.\|_2$ are two norms defined over E. The norm $\|.\|_1$ is said to be finer or stronger than the norm $\|.\|_2$ if the topology τ_1 associated to $\|.\|_1$ is stronger than the topology τ_2 associated to $\|.\|_2$, so $\tau_2 \subset \tau_1$, that is any open of τ_2 is an open of τ_1.*

Proposition 33 *τ_1 is stronger than τ_2 if and only if any open ball with respect to $\|.\|_2$ is an open of $\|.\|_1$.*

Proof.
Let V_2 an open of τ_2 and $x \in V_2$. An arbitrary element V_2 open means that there exists $r_x > 0$ such that $B_2(x, r_x) = \{y, \|y-x\|_2 < r_x\} \subset V_2$. We can put V_2 on the form $V_2 = \cup_{x \in V_2} B_2(x, r_x)$, the $B_2(x, r_x)$ is an open of τ_2. So if we suppose that $B_2(x, r_x) \in \tau_1$, i.e., $B_2(x, r_x)$ is an open of τ_1, we deduce that V_2 is a union of opens of τ_1 and is therefore an open of τ_1. ■

Proposition 34 *The norm $\|.\|_1$ is stronger (or finer) than the norm $\|.\|_2$ if only if $\exists c > 0, \forall x \in E$: $\|x\|_2 \leq c\|x\|_1$.*

Proof.
($\Rightarrow$) Let the ball $B_2(0,1)$ an open of $\tau_2 \implies B_2(0,1) \in \tau_1$. As $0 \in B_2(0,1), \exists \epsilon > 0$ such that $B_1(0,\epsilon) \subset B_2(0,\epsilon) \implies \forall y$, we have if

$$\|y\|_1 < \epsilon \implies \|y\|_2 < 1. \tag{2.4}$$

Let $\delta < \epsilon$ and $z = \frac{\delta y}{\|y\|_1}$

$$\|z\|_1 = \left\| \frac{\delta y}{\|y\|_1} \right\|_1 = \delta < \epsilon \implies \left\| \frac{\delta y}{\|y\|_1} \right\|_2 < 1 \text{ (from (2.4))} \tag{2.5}$$

$$\implies \frac{\delta}{\|y\|_1}\|y\|_2 < 1, \tag{2.6}$$

$$\implies \|y\|_2 < \frac{1}{\delta}\|y\|_1, \tag{2.7}$$

hence $c = \frac{1}{\delta}$.

($\Leftarrow$) Let us show that any open ball $B_2(x, \rho)$ of τ_2 is an open of τ_1. Let $y \in B_2(x, \rho) \implies \|y - x\|_2 < \rho$, we have to show that $\exists \epsilon > 0, B_1(y, \epsilon) \subset B_2(x, \rho)$, or $\forall z \in B_1(y, \epsilon)$: $\|z - y\|_1 < \epsilon \Rightarrow \|z - x\|_2 < \rho$, but $\|z - x\|_2 \leq \|z - y\|_2 + \|y - x\|_2$. To get $\|z - x\|_2 \leq \rho$, it suffices that $\|z - y\|_2 + \|y - x\|_2 \leq \rho$, i.e., it suffices that $\|z - y\|_2 \leq \rho - \|y - x\|_2$. Since $\|z - y\|_2 \leq c\|z - y\|_1$, so to have $\|z - y\|_2 \leq \rho - \|y - x\|_2$, it suffices that $c\|z - y\|_1 \leq \rho - \|y - x\|_2$, and thus $\|z - y\|_1 \leq \frac{\rho - \|y - x\|_2}{c}$.

We can therefore take $\epsilon = \frac{\rho - \|y - x\|_2}{c}$. ■

Proposition 35 *τ_1 is stronger (finer) than τ_2 if and only if any sequence converging with respect to τ_1 converges with respect to τ_2.*

Proposition 36 *$\|.\|_1$ is stronger than $\|.\|_2$ if and only if any continuous function f on (E, τ_1) where $f: E \to \mathbb{K}$, still continuous on (E, τ_2).*

Equivalent norms: Let $\|.\|_1$ and $\|.\|_2$ two norms on a $\mathbb{K}-$linear space E. $\|.\|_1$ is said to be equivalent to $\|.\|_2$ if and only if $\tau_1 = \tau_2$, i.e., they generate the same topology on E.

Theorem 37 *$\|.\|_1$ is equivalent to $\|.\|_2$ if and only if $\exists A > 0, B > 0$ such that:*

$$\forall x \in E\colon A\|x\|_1 \leq \|x\|_2 \leq B\|x\|_1.$$

Proof.

$\|.\|_1$ and $\|.\|_2$ are equivalent if and only if $\tau_1 = \tau_2 \iff \tau_2 \subset \tau_1$ and $\tau_1 \subset \tau_2$. $\tau_2 \subset \tau_1 \iff \|.\|_1$ is stronger than $\|.\|_2 \iff \exists B > 0$ such that $\forall x \in E$: $\|x\|_2 \leq B\|x\|_1$ and $\tau_1 \subset \tau_2 \iff \|.\|_2$ is stronger than $\|.\|_1 \iff \exists c > 0, \forall x \in E$: $\|x\|_1 \leq c\|x\|_2 \iff \exists A > 0, \forall x \in E$: $A\|x\|_1 \leq \|x\|_2$ where $A = \frac{1}{c}$.

Combining both: $A\|x\|_1 \leq \|x\|_2 \leq B\|x\|_1$. ■

Corollary 38 *Two norms are equivalent if and only if any convergent sequence with respect to the first norm converges with respect to the second and conversly.*

Application 2. We consider in $E = C^1([0,1], \mathbb{R})$ and N_1, N_2: $E \to \mathbb{R}_+$ defined for any $f \in E$ by

$$N_1(f) = |f(0)| + 2\int_0^1 |f'(t)|dt$$

$$N_2(f) = 2|f(0)| + \int_0^1 |f'(t)|dt.$$

1. Show that N_1 and N_2 are norms on E.

2. Are N_1 and N_2 equavalents?

Solution.We show that N_1 is a norm, the proof is the same for N_1.

• If $N_1(f)0$ then $f(0) = 0$ and $\int_0^1 |f'(t)|dt = 0$.

Sinc $|f'|$ is continuous and nonegative function on $[0,1]$, the condition $\int_0^1 |f'(t)|dt$ for any $t \in [0,1]$ i.e $f' = 0$, so f is constant.

Sinc $f(0) = 0$ we have $f = 0$.

•For any $\lambda \in \mathbb{R}$, and $f, g \in E$, we have

$$\begin{aligned} N_1(\lambda f) &= |\lambda f(0)| + \int_0^1 |\lambda f'(t)|dt \\ &= |\lambda|.|f(0)| + |\lambda| \int_0^1 |f'(t)|dt \ . \\ &= |\lambda| N_1(f). \end{aligned}$$

and also

$$\begin{aligned} N_1(f+g) &= |(f+g)(0)| + \int_0^1 |\lambda(f+g)'(t)|dt \\ &= |f(0)+g(0)| + \int_0^1 |f'(t)+g'(t)|dt \\ &\leqslant |f(0)| + |g(0)| + \int_0^1 (|f'(t)| + |g'(t)|)dt. \\ &\leqslant |f(0)| + \int_0^1 |f'(t)|dt + |g(0)| + \int_0^1 |g'(t)|dt. \\ &\leqslant N_1(f) + N_1(g). \end{aligned}$$

To conclude, we remark that $N_1 \leqslant 2N_2$ and $N_2 \leqslant 2N_1$ the two norms are equivalents.

2.5 Banach spaces

Let $(E, \|.\|)$ be a $\mathbb{K}$-linear space and d the assocoated distance. Is (E, d) a complete metric space?

The answer is no in general.

Example 39 *We consider the space $E = C([0,1], \mathbb{R})$ of continuous functions on $[0,1]$ in E, let the sequence $\{f_n\}_{n\mathbb{N}^*}$ of continuous functions defined by*

$$f_n(m) = \begin{cases} 1, & if\, x \in [-1, -\frac{1}{n}] \\ -nx, & if\, x \in [-\frac{1}{n}, \frac{1}{n}] \\ -1, & if\, x \in [\frac{1}{n}, 1]. \end{cases} \quad (2.8)$$

We define on E, the norm

$$\|f\|_1 = \int_{-1}^{1} |f(x)|dx.$$

We can show that $(f_n)_n$ is a Cauchy sequence and that its limit, which is obtained by making $n \to +\infty$:

$$f(x) = \begin{cases} 1, & x \in [-1, 0[\\ 0, & x = 0 \\ -1, & x \in [0, 1], \end{cases} \quad (2.9)$$

pointwise convergence:
we show that $\|f_n - f\|_1 \to 0$*, and but* f *is not continuous, i.e,* $f \notin E$*.*

Definition 40 *A* $\mathbb{K}-$*normed space* E *is said to be a Banach space if* (E, d) *(where* d *denotes the distance associated with the norm) is a complete metric space.*

Examples of Banach spaces

1. $E = \mathbb{K}^n = \mathbb{R}^n$ or $\mathbb{C}^n$.

2. $E = C([a, b])$ where for $f \in E$, $\|f\|_\infty = \sup\limits_{x\in[a,b]} |f(x)|$.

3. $E = \beta(X) = \{f : X \to \mathbb{K}/f \text{ bounded}\}$.

4. $l^p = \{(x_n)_{n\geq1}/ \sum\limits_{n\geq1} |x_n|^p < +\infty\}, (p \geq 1)$.

5. $l^\infty = \{(x_n)_{n\geq1}/ \sup|x_n| < \infty\}$.

6. $c_0 = \{(x_n)_{n\geq1}/ \lim\limits_{n\to\infty} x_n = 0\}$.

7. $L^p(X, \mu)$ where $p > 1$ and μ is a measure on X.

- A linear subspace F of a Banach space E (over $\mathbb{K}$) is a Banach space if and only if F is closed.

- If $(E, \|.\|_1)$ is a Banach space and $(E, \|.\|_2)$ is a normed space such that $\|.\|_1$ is stronger than $\|.\|_2$, then $(E, \|.\|_2)$ is a Banach space. In particular, if $\|.\|_1$ and $\|.\|_2$ are equivalent, then $(E, \|.\|_1)$ is a Banach space if and only if $(E, \|.\|_2)$ is also a Banach space.

2.6 Finite-dimensional normed linear spaces

Let E be a finite-dimensional linear space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) with norm $\|.\|$. The space $(E, \|.\|)$ is said to be a finite-dimensional normed linear space.

- Every normed linear space $(E, \|.\|)$ is a Banach space.
- On a finite-dimensional linear space, all norms are equivalent and therefore the space has one and only one topology derived from a norm (defined by a norm)
- In a finite-dimensional normed linear space, any closed and bounded part is compact.
- Any normed linear space of finite dimension n ($\dim(E) = n$) is isomorphic to $\mathbb{K}^n$. Consequently, all normed spaces of the same finite dimension are isomorphic to each other:
  By isomorphism between normed spaces, we mean a linear application that is a homeomorphism.

2.7 Continuous linear applications

Definition 41 *Let E and F be linear spaces on $\mathbb{K}$ and $u\colon E \to F$ an application defined from E into F. We say that u is linear (linear space morphism) if and only if:*

$$\forall \lambda \in \mathbb{K}, \forall x, y \in E : u(\lambda x + y) = \lambda u(x) + u(y).$$

If $(E, \|.\|_1)$ and $(F, \|.\|_2)$ are two normed spaces, we can talk about the continuity of u.

In this case we have the following:
Suppose that u is continuous at a point $x_0 \in E$. Since E and F are metric spaces, this is equivalent to saying that:
$\forall (x_n)_n, x_n \in E$ and $x_n \to x_0$, thus $u(x_n) \to u(x_0)$.

Proposition 42 *If u is continuous at a point $x_0 \in E$, it is then continuous everywhere.*

Proof.
Assume u is continuous at point x_0. Let $x \in E$. Let us show that u is continuous at the point x. Let $(y_n)_n$ be a sequence in E converging to x and show that $u(y_n))_n$ converges to $u(x)$.
As $y_n \to x$ and let $x_n = y_n - x + x_0$, $\|x_n - x_0\|_1 = \|y_n - x\| \to 0$, thus $x_n \to x_0$ and this results that $u(x_n) \to u(x_0)$ in F, or u is additive (because linear).

$$\begin{aligned} u(x_n) &= u(y_n) - u(x) + u(x_0) \to +u(x_0) \\ &\Longrightarrow \|u(y_n) - u(x) + u(x_0) - u(x_0)\|_2 \to 0 \\ &\Longrightarrow \|u(y_n) - u(x)\|_2 \to 0. \end{aligned}$$

and this shows that $u(y_n) \to u(x)$, where u is continuous at point x. So u is continuous everywhere. ■

Corollary 43 *u is continuous on E if and only if u is continuous at the origin.*

Proposition 44 *u is continuous if and only if*

$$\exists c > 0, \textit{such that} \, \forall x \in E \colon \|u(x)\|_2 \le c\|u(x)\|_1. \tag{2.10}$$

Proof.

If (2.10) is satisfied and if $x_n \to 0$ in E, thus:

$$\forall \epsilon > 0, \exists n_0 / \forall n \geq n_0 \colon \|x_n\|_1 \leq \frac{\epsilon}{c},$$

where $\|u(x_n)\|_2 \leq c\|x_n\|_1 = \epsilon$, and this $\forall n \geq n_0$. Therefore $u(x_n) \to 0 = u(0)$.

Reciprocally, if u is continuous at point 0, then $u^{-1}(B[0,1])$ is a neighborhood of 0 in E, since $B[0,1]$ is a neighborhood of $u(0) = 0$ in F and u is assumed to be continuous at point 0 of E.

There exists then a $J > 0$, such that

$$\begin{aligned} \|x\|_1 \leq J \quad &\Longrightarrow \quad x \in u^{-1}(B[0,1]) \\ &\Longrightarrow \quad u(x) \in B[0,1] \\ &\Longrightarrow \quad \|u(x)\|_2 \leq 1. \end{aligned}$$

Or if $x \neq$ and $x \in E$, we have:

$$\left\| \frac{\delta x}{\|x\|_1} \right\|_1 = \delta,$$

and this implies that:

$$\begin{aligned} &\left\| u \frac{\delta x}{\|x\|_1} \right\|_2 \leq 1 \\ \Longrightarrow \quad &\left\| \frac{\delta}{\|x\|_1} u(x) \right\|_2 \leq 1 \\ \Longrightarrow \quad &\frac{\delta}{\|x\|_1} \|u(x)\|_2 \leq 1 \\ \Longrightarrow \quad &\|u(x)\|_2 \leq \frac{1}{\delta} \|x\|_1, \end{aligned}$$

and it is enough to put $c = \frac{1}{\delta}$. ■

Proposition 45 *u is continuous if and only if*

$$\sup\left\{\frac{\|u(x)\|_2}{\|x\|_1}, x \in E \textit{ and } x \neq 0\right\}$$

is finite.

Proof.

($\Longrightarrow$) If u is continuous, therefore, according to the Proposition 44

$$\exists c > 0, \text{such that } \forall x \in E, \|u(x)\|_2 \leq c\|x\|_1,$$

and thus

$$x \neq 0 \implies \frac{\|u(x)\|_2}{\|x\|_1} \leq c.$$

Consequently, the set $\{\frac{\|u(x)\|_2}{\|x\|_1}/x \in E, \text{and } x \neq 0\}$ is bounded by the constant c. Therefore, $\sup\left\{\frac{\|u(x)\|_2}{\|x\|_1}, x \in E, x \neq 0\right\} < \infty$.

($\Longleftarrow$) is evident. ■

Proposition 46 *u is continous if and only if*

$$s = \sup\{\|u(x)\|_2/x \in E \textit{ and } \|x\|_1 \leq 1\} \textit{ is finite.}$$

Proof.

s is finite if and only if the set $\{\|u(x)\|_2/x \in E$ and $\|x \leq 1\}$ is bounded . If u is continuous, thus, according to the Proposition 44: $\exists c > 0$, such that $\forall x \in E$: $\|u(x)\|_2 \leq c\|u(x)\|_1$. Since

$$\begin{aligned} \|x\|_1 \leq 1 \quad \text{then} \quad & \|x\|_2 \leq c \\ \Longrightarrow \quad & \{\|u(x)\|_2, x \in E \text{ and } \|x\|_1\} \end{aligned}$$

is bounded, therefore s is finite.

Conversely, if $\{\|u(x)\|_2, x \in E$ and $\|x\|_1\}$ is bounded, this implies that

$$\exists c > 0 \text{ such that } \forall x \in E\colon \|x\|_1 \leq 1 \implies \|u(x)\|_2 \leq c.$$

But for $x \neq 0$ and $\left\| \frac{x}{\|x\|_1} \right\| = 1$, we get

$$\left\| u\left(\frac{x}{\|x\|_1}\right) \right\|_2 \leq c,$$

so $\|u(x)\|_2 \leq c\|x\|_1$ and u is continuous. ■

Proposition 47 *u is continuous if and only if $\{\|u(x)\|_2, x \in E$ and $\|x\|_1 = 1\}$ is bounded.*

Proof.

$(\Longrightarrow)$ If u is continuous then $\exists c > 0$ such that $\|x\|_1 = 1$, so $\|x\|_1 \leq 1$ and we get $\|u(x)\|_2 \leq c$.

$(\Longleftarrow)$ If $\{\|u(x)\|_2, x \in E$ and $\|x\|_1 = 1\}$ is bounded by $c > 0$, thus: $\|x\|_1 = 1$, this implies $\|u(x)\|_2 \leq c$.

$$\text{For } x \neq 0, \left\| \frac{x}{\|x\|_1} \right\| = 1 \implies \left\| u(\frac{x}{\|x\|_1}) \right\|_2 \leq c$$
$$\implies \|u(x)\|_2 \leq c\|x\|_1,$$

which means that u is continuous. ■

Theorem 48 *Let $u\colon E \to F$ linear. Then u is continuous everywhere on E if and only if one of the following equivalent conditions is fulfilled :*

1. *u is continuous at any point x_0.*
2. *$\left\{\frac{\|u(x)\|_2}{\|x\|_1}, x \in E \text{ and } x \neq 0\right\}$ is bounded.*
3. *$\{\|u(x)\|_2, x \in E$ and $\|x\|_1 \leq 1\}$ is bounded.*
4. *$\{\|u(x)\|_2, x \in E$ and $\|x\|_1 = 1\}$ is bounded.*

We have seen that $u\colon E \to F$ is linear and continuous if and only if:

$$\sup\left\{\frac{\|u(x)\|_2}{\|x\|_1}, x \in E \text{ and } x \neq 0\right\} \text{ is finite.}$$

Or or it is easy to check that the set of continuous linear applications on E is a linear subspace of the linear space of all linear applications (continuous or no).

Notice by $\mathcal{L}(E,F)$ the linear space of continuous linear applications on E with values in F. Thus, $u \in \mathcal{L}(E,F)$ if and only if

$$s = \sup\left\{\frac{\|u(x)\|_2}{\|x\|_1}, x \in E \text{ and } x \neq 0\right\} < \infty.$$

We put $\|u\| = s$ and we show that $\|.\|$ makes of $\mathcal{L}(E,F)$ a normed space. We must therefore show that $\|.\|$ is a norm on E.

Proof.

(i) $\|u\| = 0 \implies s = 0 \implies \forall x \in E, x \neq 0\colon \frac{\|u(x)\|_2}{\|x\|_1} = 0.$
$\implies \|u(x)\|_2 = 0 \implies u(x) = 0 \implies u = 0.$

(ii) Let $\lambda \in \mathbb{K}$ and $u \in \mathcal{L}(E,F)$

$$\begin{aligned}
\|\lambda u\| &= \sup\left\{\frac{\|u(x)\|_2}{\|x\|_1}, x \in E \text{ and } x \neq 0\right\} \\
&= \sup\left\{\frac{|\lambda|\|u(x)\|_2}{\|x\|_1}, x \in E \text{ and } x \neq 0\right\} \\
&= |\lambda| \sup\left\{\frac{\|u(x)\|_2}{\|x\|_1}, x \in E \text{ and } x \neq 0\right\} \\
&= |\lambda| s = |\lambda|\|u\|.
\end{aligned}$$

(iii) Let u_1 an $u_2 \in \mathcal{L}(E,F)$. If $x \in E$ and $x \neq 0$.

$$\begin{aligned}
\|(u_1+u_2)(x)\|_2 &\leq \|u_1(x)\|_2 + \|u_2(x)\|_2 \\
&\implies \frac{\|(u_1+u_2)(x)\|_2}{\|x\|_1} \leq \frac{\|u_1(x)\|_2}{\|x\|_1} + \frac{\|u_2(x)\|_2}{\|x\|_1} \\
&\leq \|u_1\| + \|u_2\|,
\end{aligned}$$

where $\|u_1\| + \|u_2\|$ is a majorant of $\left\{\frac{\|(u_1+u_2)(x)\|_2}{\|x\|_1}, x \in E \text{ and } x \neq 0\right\}$, thus $\|u_1+u_2\| \leq \|u_1\| + \|u_2\|$. Consequently, the application $\|.\|$ is a norm on $\mathcal{L}(E,F)$, hence $(\mathcal{L}(E,F), \|.\|)$ is a normed linear space on $\mathbb{K}$.

■

Proposition 49 *Let u linear and continuous and*

$$\begin{aligned} s &= \sup\left\{\frac{\|u(x)\|_2}{\|x\|_1}, x \in E \textit{ and } x \neq 0\right\} \\ s_1 &= \sup\{\|u(x)\|_2, x \in E \textit{ and } \|x\|_1 \leq 1\} \\ s_2 &= \sup\{\|u(x)\|_2, x \in E \textit{ and } \|x\|_1 = 1\}. \end{aligned}$$

Thus $s = s_1 = s_2 = \|u\|$.

The proof is left to the students.

Theorem 50 *If F is a Banach space, thus $\mathcal{L}(E, F)$ is Banach space. (Even if E is not a Banach space).*

For the proof, see the Worksheet of exercises 3.

Proposition 51 *Let $u\colon E \to F$ be a continuous linear application. Then the set*

$$\ker u = \{x, u(x) = 0\} = u^{-1}\{0\},$$

is a closed linear subspace.

Proposition 52 *Let E, F, G be three normed spaces on $\mathbb{K}$ and u, v such that*

$$E \xrightarrow{u} F \xrightarrow{v} G,$$

are continuous linear applications. Then $v \circ u\colon E \to G$ is a continuous linear application and $\|v \circ u\| \leq \|v\|\|u\|$.

Proof.

$$\begin{aligned}\|v\circ u\| &= \sup\{\|v\circ u(x)\|, \|x\|=1\}\\ &\Longrightarrow \|v\circ u(x)\| = \|v(u(x))\| \leq \|v\|\|u(x)\|\\ &\leq \|v\|\|u\|\|x\| = \|v\|\|u\|\\ &\Longrightarrow \sup\{\|v\circ u(x)\|, \|x\|=1\} \leq \|v\|\|u\|\\ &\Longrightarrow \|v\circ u\| \leq \|v\|\|u\|.\end{aligned}$$

■

Theorem 53 (Closed graph theorem) *Let $u\colon E\to F$ linear and $G(u) = \{(x,u(x)), x\in E\}$ the graph of u. Therefore if u is continuous, $G(u)$ is closed and reciprocally if $G(u)$ is closed, u is continuous.*

Proposition 54 *Let E be a normed linear space and F a linear subspace of E. Thus $\bar{F} = Adh(F)$ is a closed linear subspace of E.*

Theorem 55 (Open application theorem) *E, F two Banach spaces on $\mathbb{K}$ and $u\colon E\to F$ linear and continuous. If u is surjective $(\forall y\in F, \exists x\in E$ such that $y = u(x)), u$ is an open application $(\forall V$ open in $E,)$, thus $u(V)$ is an open of F.*

Theorem 56 (Banach theorem) *$u\colon E\to F$ linear and continuous if u is bijective, thus u^{-1} is a linear application and continuous.*

Theorem 57 (Borélian graph theorem) *$u\colon E\to F$ linear. E and F Banach spaces. If $G(u) = \{(x,u(x)) \notin x\in E\}$ is a Borélian subset, (belongs to the Borélian tribe of $E\times F$) thus u is continuous.*

Remark 58 *The Borélian graph theorem is stronger than the closed graph theorem, because if $G(u)$ is closed, then it is Borélian, since every closed of $E \times F$ is a Borélian subset. However, not every Borélian is closed.*

Application 3. Let $E = C^{\infty}([0,1])$. We consider the application of derivation $D : E \longrightarrow E$, $f \longrightarrow f'$. Show that for any norm N defined on E, the application $D : (E, N) \longrightarrow (E, N)$ is discontinuous.

Solution. Let $a \in \mathbb{R}$, the fuction $f_a(x) = e^{ax}$ is in E, and $Df_a = af_a$. but, if we suppose that D is continuous with respect to the norm N, there exists a constant $C > 0$ such that

$$N(D(f_a)) \leqslant CN(f_a)$$

for any $a \in \mathbb{R}$ we get in this case $\forall a \in \mathbb{R}$

$$|a|N(f_a) \leqslant CN(f_a) \implies |a| \leqslant C.$$

Which is obviousely impossile and D is discontinuous on (E, N).

2.8 Product of two normed spaces

$(E, \|.\|_1)$ and $(F, \|.\|_2)$ two normed linear spaces on $\mathbb{K}$. Let $G = E \times F$ the Cartesian product of E and F. We know that G is a $\mathbb{K}-$linear space. We seek to provide G with a norm N that keeps the two projections continuous $P_E \colon E \times F \to E, P_E(x, y) = x$ and $P_F \colon E \times F \to E, P_F(x, y) = y$. As P_E and P_F are linear, thus P_E is continuous if and only if $\exists c_1 > 0$, such that $\forall (x, y) \colon \|P_E(x, y)\|_1 \leq c_1 N(x, y)$ and $\exists c_2 \colon \|P_F(x, y)\|_2 \leq c_2 N(x, y)$. Where posing $c = \min(\frac{1}{c_1}, \frac{1}{c_2})$. We have:

$$N(x, y) \geq c \left(\|P_E(x, y)\|_1 + \|P_F(x, y)\|_2\right).$$

Or, it is clear that $(x, y) \to \|P_E(x, y)\|_1 + \|P_F(x, y)\|_2$ is a norm on G for which P_E and P_F are continuous. Therefore, we can define N by

$$N(x, y) = \|P_E(x, y)\|_1 + \|P_F(x, y)\|_2.$$

Theorem 59 *The functions N_1, N_p and N_∞ defined on G by: $(p \geq 1)$*

$$\begin{aligned} N_1(x, y) &= \|P_E(x, y)\|_1 + \|P_F(x, y)\|_2 = \|x\|_1 + \|y\|_2 \\ N_p(x, y) &= (\|x\|_1^p + \|y\|_2^p)^{\frac{1}{p}} \\ N_\infty(x, y) &= \max(\|x\|_1, \|y\|_2) \end{aligned}$$

are equivalent norms on G, so they define the same topology on G that keeps the two projections P_E and P_F continuous .

Proposition 60 *If E and F are two Banach spaces, thus G is a Banach space.*

2.9 Dual of a normed linear space

Definition 61 *Let E be a normed space on $\mathbb{K}$. The algebraic dual is by definition the linear space of linear applications from E^* into $\mathbb{K}$. The topological dual is by definition the linear space, denoted by E' of linear and continuous applications from E^* into $\mathbb{K}$ or it is the linear space of continuous linear forms. $E^{'}$ is a linear subspace of E^*.*

Notion of hyperplane If $\alpha \in \mathbb{E}^*$, thus $\ker \alpha$ is a maximal linear subspace of E with respect to the ensemblistic inclusion in the set of linear subspaces of E. So if $V \notin \ker \alpha \iff \alpha(V) \neq 0$. Therefore, $vect\{x, \ker \alpha\} = \ker \alpha \oplus \mathbb{K}V = E$. Any maximal linear subspace in the set of linear subspaces of E is called a "hyperplane".

Example 62 *The hyperplanes of* $\mathbb{R}^2$ *are the straight lines passing through* $(0,0)$.
The hyperplanes of $\mathbb{R}^3$ *are the planes passing through* $(0,0)$.

Theorem 63 *For any linear form* $\alpha \in E^*, \ker\alpha$ *is a hyperplane and reciprocally, for any hyperplane* H*, there exists a linear form* α *such that* $H = \ker\alpha = \{x \in E, \alpha(x) = 0\}$ *equation of* H.

Theorem 64 *If* H *is a hyperplane of* E*, where* E *is a normed linear space, thus* H *is either closed or dense in* E.

Proof.
If H is a hyperplane, we know that H is a linear subspace of E containing H and therfore, as H is maximal, then necessarily $\bar{H} = H$ or $\bar{H} = E$. ■

Remark 65 *If* $\dim E < \infty$*, all linear spaces of* E *are closed and* E *can not contain dense subspaces. Thus, the dense linear subspaces exist only in infinite dimensional linear spaces.*

Theorem 66 *If* $\alpha \in E^*$ *where* E *is a normed linear space, then*
α *is continuous if and only if* $\ker\alpha$ *is closed.*
α *is discontinuous if and only if* $\ker\alpha$ *is dense.*

Proposition 67 *If* $E = \mathbb{K}^n$*, thus the lineair forms of* E *are the functions* $\alpha \in E^*$ *defined by*

$$(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \alpha(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} c_i x_i,$$

where c_i *are the constants and thus the hyperplanes of* E *for the equations*

$$H = \{(x_1, x_2, \ldots, x_n), \sum_{i=1}^{n} c_i x_i = 0\}.$$

- The topological dual E' *is a Banach space because* $E' = \mathcal{L}(E, \mathbb{K})$ *and* $\mathbb{K}$ *is a* $\mathbb{K}-$*Banach space (*$\mathbb{K}$ *is complet).*

Examples of continuous linear forms

1. Let $E = C([0,1]), \|.\|$ and $g \in E$. Let $\alpha \colon E \to \mathbb{R}$ defined by:
   $f \in E \colon \alpha(f) = \int_0^1 g(x)f(x)dx, \alpha$ is linear and continuous because:
   $|\alpha(f)| \leq \|g\|_\infty \|f\|_1$, where $\alpha \colon (E, \|.\|_1) \to \mathbb{R}$ is continuous.
   $\ker \alpha = \{f \in E, \int_0^1 g(x)f(x)dx = 0\}$.

2. The linear form "Evaluation at a point", $e \colon E \to \mathbb{K}$ and let $x_0 \in E, f \in E \colon e(f) = e(x_0)$.
   $e \colon (E, \|.\|_\infty) \to \mathbb{R}$ is linear and continuous and $\|e\| = \sup\{\frac{|f|}{\|f\|_\infty}, f \neq 0\} = 1$.

Worksheet of exercises 2

Exercise 1 Let $a > 0$ and $b > 0$ two fixed real numbers and $(x, y) \in \mathbb{R}^2$ and arbitrary vector. We define on $\mathbb{R}^2$, the applications $\|.\|_1$, $\|.\|_2$ and $\|.\|_\infty$ respectively by $\|(x, y)\|_1 = \sqrt{ax^2 + by^2}$, $\|(x, y)\|_2 = a|x| + b|y|$ and $\|(x, y)\|_\infty = Max(|x|, |y|)$.

1) Show that $\|.\|_1$, $\|.\|_2$ and $\|.\|_\infty$ are norms on the $\mathbb{R}-$linear space $\mathbb{R}^2$.

2) Extend the result for $\|.\|_p$, where $p \geq 1$.

3) Is the application $\|.\|$ defined by $\|(x, y)\| = |ax + by|$ a norm over $\mathbb{R}^2$?

4) Show that $\lim\limits_{p \to +\infty} \|(x, y)\|_p = \|(x, y)\|_\infty$.

Exercise 2 Let $(E, \|.\|)$ be a $\mathbb{K}$-normed linear space ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and d the metric associated to the norm of $\|.\|$.

1) Show that d has the following properties:

   (i) $\forall \lambda \in \mathbb{K}, \forall x, y \in \mathbb{E} : d(\lambda x, \lambda y) = d(x, y)$.

   (ii) $\forall z \in E$, $d(x + z, y + z) = d(x, y)$ (the translation invariance od the distance d).

2) Conversely, if d is a metric on E satisfying (i) and (ii), show that the application $\|.\|$ defined by $\|x\| = d(x, 0)$ is a norm over E.

Exercise 3 We consider the linear space $E = \mathcal{C}([0,\ 1], \mathbb{R})$ equipped with the applications $\|.\|_\infty$ and $\|.\|_1$ defined respectively by $\|f\|_\infty = \sup\limits_{x \in [0,\ 1]} |f(x)|$ and $\|f\|_1 = \int_0^1 |f(x)|\, dx$.

- Show that $\|.\|_\infty$ and $\|.\|_1$ are norms on E.

- Show that for every $f \in E$, $\|f\|_1 \leq \|f\|_\infty$.

- Using the sequence of functions from E defined by $f_n(x) = x^n$, prove that the norms $\|.\|_\infty$ and $\|.\|_1$ are not equivalent.

Exercise 4 Let E be a normed linear space.

1) Show that any closed ball $\overline{B(x_0,\ r)} = \{x \in E / \ \|x - x_0\| \leq r\}$ is a convex set.

2) Let F a subspace of E. Show that:

   a) the closure $\overline{E}$ of E is also subspace of E,

   b) the interior $\overset{\circ}{F}$ of E is empty.

Exercise 5 (Left to the students) Let E be a normed linear space over $\mathbb{R}$.

1) Show that the application $(x, y) \to x + y$ from $\mathbb{E} \times E$ to E is uniformly continuous.

2) Show that the application $(\lambda, y) \to x + y$ from $\mathbb{R} \times E$ to E is continuous but not uniformly continuous.

Worksheet of exercises 3

Exercice 1 We provide the space $E = \mathcal{C}([0.1], \mathbb{R})$ with the norm $\|.\|_\infty$. Show that the sequence of functions $(f_n)_n$ of E converges to f in E if and only if this sequence converges uniformly on $[0.1]$ to f.

Exercice 2 Let $E = \mathcal{C}([0.1], \mathbb{R})$ and let for $g \in E$, the function denoted N_g and defined on E by $N_g(f) = \|gf\|_\infty$. Find a necessary and sufficient condition for N_g to be a norm on E.

Exercice 3 Consider for $p > 1$, the $\mathbb{K}-$linear space l^p of the sequences $(x_n)_{n\in\mathbb{N}^*}$ such that $\sum\limits_{n\geq 1} |x_n|^p < +\infty$. We define on l^p the function $\|.\|_p$ by $\|(x_n)_n\|_p = \left(\sum\limits_{n\geq 1} |x_n|^p\right)^{\frac{1}{p}}$. Show that $\|.\|_p$ is a norm on l^p and that $\left(l^p, \|.\|_p\right)$ is a $\mathbb{K}-$Banach space.

Exercice 4 Consider $E = \mathcal{C}([0.1], \mathbb{R})$ with the integral norm $\|.\|_1$. For $x_0 \in [0.1]$, show that the linear form (called evaluation at point x_0) denoted e_{x_0} and defined from E into $\mathbb{R}$ by: $f \in E$, $e_{x_0}(f) = f(x_0)$ is discontinuous on E. Show that if we provide E with the norm $\|.\|_\infty$, then e_{x_0}, is continuous on E and deduce that the norms $\|.\|_1$ et $\|.\|_\infty$ are not equivalent.

Worksheet of exercises 4

Exercice 1 Let $(E, \|.\|_1)$ and $(F, \|.\|_2)$ two normed linear spaces on $\mathbb{K}$ and $u : E, \to F$ is a linear application. Show that :

(i) u is continuous on E if and only if it is continuous at 0.

(ii) u is continuous on E if and only if $\exists c > 0, \forall x \in E : \|u(x)\|_2 \leq c \|x\|_1$.

(iii) We define on E the application $\|.\|_u = \|u(x)\|_2$. What is the condition on u so that $\|.\|_u$ is a norm on E?

(iv) Which of the two norms $\|.\|_1$ and $\|.\|_u$ on E is finer than the other?

Exercice 2 Determine whether the linear application $T\colon (E, N_1) \to (F, N_2)$ is continuous in the following cases:

1) $E = C([0, 1], \mathbb{R})$ equipped with $\|.\|_1 = \int_0^1 |f(t)|dt$ and $T : (E, \|.\|_1) \to (E, \|.\|_1)$, $f \mapsto fg$ where $g \in E$ is a fixed.

2) The linear space of polynomials with real variables, equipped with the norm $\|.\|$ defined by

$$\left\| \sum_{k\geq 0} a_k X^k \right\| = \sum_{k\geq 0} |a_k| \text{ et } T : (E, \|.\|) \to (E, \|.\|),\ P \mapsto P'.$$

Exercice 3 An example of a linear application never continuous.
Let $E = C^\infty([0, 1], \mathbb{R})$. Consider the differential operator $D : E \to E, f \mapsto f'$. Show that, whatever the norm N, with which we equip E, D is never a continuous linear application of (E, N) in (E, N).
Indication: We can search for the eigenvalues and eigenvectors of D.
Exercice 4 Soient $(E, \|.\|_1)$ and $(F, \|.\|_2)$ two normed linear spaces on $\mathbb{K}$.
Show that if F is complete (Banach), then so is $L(E, F)$.

Chapter 3

Hilbert space

Hilbert spaces are special normed spaces such that when they are infinite-dimensional, they are the closest to the finite-dimensional ones. In fact their norms are defined by scalar products, allowing to maintain many geometric properties valid in finite dimension (scalar product-dependent properties).

3.1 Scalar product

Definition 68 *Let H be a linear space on $\mathbb{K}$. The application that we note $\langle , \rangle$ defined from $H \times H$ in $\mathbb{K}$ is said to be a scalar product if and only if:*

1) $\forall x \neq 0, \langle x, x \rangle > 0$ *($<,>$ is non-degenerate),*

2) for a fixed $y \in H$, the application $x \to \langle x, y \rangle$ is a linear form,

3) the application: $x \to \langle y, x \rangle$ is an anti-linear form, i. e.,:

$$x \to < y, x > \text{ is additive}.$$

(i) $\langle y, x_1 + x_2 \rangle = \langle y, x_1 \rangle + \langle y, x_2 \rangle$.

(ii) $\forall\lambda \in \mathbb{K}, \forall x \in H\colon \langle y, \lambda x\rangle = \bar{\lambda}\langle y, x\rangle$.

((i) and (ii) $\iff \langle y, x_1 + \lambda x_2\rangle = \langle y, x_1\rangle + \bar{\lambda}\langle y, x_2\rangle$*).*

4) $\langle x, y\rangle = \overline{\langle x, y\rangle}$.

The four conditions can be summarized as follows:
$\langle , \rangle$ is scalar product if and only if:

1) $\mathbb{K} = \mathbb{C}$, it is a linear form with respect to the first variable satisfying the condition: $\langle x, y\rangle = \overline{\langle x, y\rangle}$ and that is positive non-degenerate : (anti-symmetric)

$$\forall x \neq 0 \implies \langle x, x\rangle > 0,$$

which is also equivalent to $\langle x, x\rangle = 0 \implies x = 0$.

2) $\mathbb{K} = \mathbb{R}$, it is a linear form with respect to the first variable that is symmetric ($\langle x, y\rangle = \langle y, x\rangle$) and positive non-degenerate.

3) If $\mathbb{K} = \mathbb{C}$, H is called a complex pre-Hilbert space.
If $\mathbb{K} = \mathbb{R}$, it is said to be a real pre-Hilbert space.

3.1.1 Properties of scalar product

For $x, y \in H$, we have:

$$\begin{aligned}
\langle x+y, x+y\rangle &= \langle x, x+y\rangle + \langle y, x+y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, y\rangle + \langle y, x\rangle \\
&= \langle x, x\rangle + \langle y, y\rangle + \langle x, y\rangle + \langle y, x\rangle
\end{aligned}$$

or $\langle y, x\rangle = \overline{\langle x, y\rangle}$ and $\langle x, y\rangle + \langle y, x\rangle = \langle x, y\rangle + \overline{\langle x, y\rangle} = 2Re\langle x, y\rangle$ and thus:

$$\langle x+y, x+y\rangle = \langle x, x\rangle + \langle y, y\rangle + 2Re\langle x, y\rangle,$$

this applies to both cases $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$.

If $\mathbb{K} = \mathbb{R} \implies Re\langle x, y\rangle = \langle x, y\rangle$ and thus

$$\langle x + y, x + y\rangle = \langle x, x\rangle + \langle y, y\rangle + 2\langle x, y\rangle$$

$$\langle x - y, x - y\rangle = \langle x, x\rangle + \langle y, y\rangle - 2Re\langle x, y\rangle$$

Cauchy-Schwarz inequality

We know $\langle , \rangle$ is non-degenerate, i.e., $\forall x \in H\colon \langle x, x\rangle > 0$, where for $\lambda \in \mathbb{K}$ and $x, y \in H$, we have:

$$\langle x + \lambda y, x + \lambda y\rangle \geq 0.$$

Or $\langle x + \lambda y, x + \lambda y\rangle = \langle x, x\rangle + \langle \lambda y, \lambda y\rangle + 2Re\langle x, \lambda y\rangle$.

If $\lambda \in \mathbb{R}$, we have:

$$\lambda^2\langle y, y\rangle + 2Re\langle x, y\rangle\lambda + \langle x, x\rangle \geq 0.$$

This expression is a trinomial with real coefficients (of degree 2) which does not change sign when λ runs through $\mathbb{R}$. So, its discriminant $\Delta \leq 0$, but we have

$$\begin{aligned} \Delta &= 4Re^2\langle x, y\rangle - 4\langle x, x\rangle\langle y, y\rangle \leq 0 \\ &\implies Re^2\langle x, y\rangle \leq \langle x, x\rangle\langle y, y\rangle, \end{aligned}$$

replacing $y \equiv \langle x, y\rangle y$, we get

$$\begin{aligned} (Re\langle x, \langle x, y\rangle y\rangle)^2 &\leq \langle x, x\rangle\langle\langle x, y\rangle y, \langle x, y\rangle y\rangle \\ (Re\langle x, \langle x, y\rangle y\rangle)^2 &\leq \langle x, x\rangle\langle x, y\rangle\overline{\langle x, y\rangle}\langle y, y\rangle \\ |\langle x, y\rangle|^4 &\leq |\langle x, y\rangle|^2\langle x, x\rangle\langle y, y\rangle, \end{aligned}$$

if $|\langle x, y\rangle| \neq 0 \implies |\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$. This last inequality is called the "Cauchy-Schwarz inequality".

Theorem 69 *The application of H in $\mathbb{R}^+$ defined by: $\|x\| = \sqrt{\langle x, x\rangle}$ is a norm on H.*

Proof.

(i) $\|x\| = 0 \implies \langle x, x\rangle = 0 \implies x = 0.$

(ii) $\lambda \in \mathbb{K}, \|\lambda x\| = \sqrt{\langle \lambda x, \lambda x\rangle} = \sqrt{\lambda\overline{\lambda}\langle x, x\rangle} = \sqrt{|\lambda|^2\langle x, x\rangle} = |\lambda|\sqrt{\langle x, x\rangle} = |\lambda|\|x\|$

(iii) **Triangle inegality:** For $x, y \in H$, we have

$$\|x+y\|^2 = \langle x, x\rangle + \langle y, y\rangle + 2Re\langle x, y\rangle = \|x\|^2 + \|y\|^2 + 2Re\langle x, y\rangle.$$

As we have seen that $Re^2\langle x, y\rangle = |Re\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$, thus

$$\begin{aligned} |Re\langle x, y\rangle| &\leq \|x\|\|y\| \\ Re\langle x, y\rangle &\leq \|x\|\|y\|, \end{aligned}$$

where

$$\begin{aligned} \|x+y\|^2 &\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \\ \implies \|x+y\|^2 &\leq (\|x\| + \|y\|)^2 \\ \implies \|x+y\| &\leq \|x\| + \|y\|. \end{aligned}$$

Thus $(H, \|.\|)$ is a normed space on $\mathbb{K}$.
When $(H, \|.\|)$ is a complete metric space, we say that $(H, \|.\|)$ is a Hilbert space.

■

Example 70 • *$H = \mathbb{R}^n, X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n), \langle X, Y\rangle = \sum_{i=1}^n x_i y_i$. $\langle , \rangle$ is a scalar product on H, because it is a non-degenerate positive symmetric bilinear form, since $\langle X, X\rangle = \sum_{i=1}^n x_i^2 \geq 0$, and if*

$$X \neq 0, \text{ there exists } i, 1 \leq i \leq n \mid x_i \neq 0,$$

so we have $\displaystyle\sum_{i=1}^n x_i^2 > 0$ which means that $\langle X, X\rangle > 0$.

$\|x\| = \sqrt{\displaystyle\sum_{i=1}^n x_i^2}$ is what we usually call the Euclidian norm.

- *$H = \mathbb{C}^n, X = (s_1, s_2, \ldots, s_n)$ and $Y = (t_1, t_2, \ldots, t_n), \langle X, Y\rangle = \displaystyle\sum_{j=1}^n s_j \overline{t_j}$. It is easy to check that $\langle X, Y\rangle$ is a scalar product on H.*

Parallelogram equality

For $x, y \in E$, we have

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2Re\langle x, y\rangle. \tag{3.1}$$

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2Re\langle x, y\rangle. \tag{3.2}$$

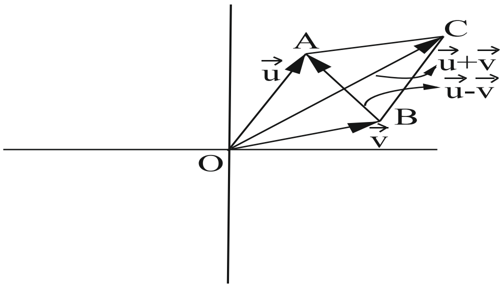
Therefore, (3.1) + (3.2) gives:

$$\|x + y\|^2 - \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2). \tag{3.3}$$

It is parallelogram equality.

Example 71 *$H = \mathbb{R}^2$ with an orthonormal coordinates $\{0, \overrightarrow{i}, \overrightarrow{j}\}$*

$$\begin{aligned} \overrightarrow{u} + \overrightarrow{v} &= \overrightarrow{OA} + \overrightarrow{OB} \\ &= \overrightarrow{OA} + \overrightarrow{BC} = \overrightarrow{OC} \end{aligned}$$



$$
\begin{aligned}
\vec{u} - \vec{v} &= \overrightarrow{OA} - \overrightarrow{OB} \\
&= \overrightarrow{OA} + \overrightarrow{BO} = \overrightarrow{BA}
\end{aligned}
$$

In the parallelogram $OACB$*, we have:* $OC^2 + AB^2 = 2(OA^2 + OB^2)$ *and this is why the equality* (3.3) *is called "Parallelogram equality".*
Reciprocally if $(H\|.\|)$ *is a normed space such as the norm satisfies parallelogram equality*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Therefore, the application $\langle , \rangle$ *defined by:* $\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 + \|x-y\|^2)$ *is a real scalar product on* H*.*
- The Cauchy-Schwarz inegality shows that the two linear forms:

$$
\begin{aligned}
x &\rightarrow \langle x, y\rangle \text{ where } y \text{ is fixed} \\
x &\rightarrow \langle y, x\rangle \text{ where } y \text{ is fixed}
\end{aligned}
$$

are continuous because

$$
\begin{aligned}
|\langle x, y\rangle| &\leq \|y\|\|x\| \\
|\langle x, y\rangle| &\leq \|y\|\|x\|.
\end{aligned}
$$

In addition, $\|\langle \quad , y\rangle\| \leq \|y\|$ *and as* $|\langle y, y\rangle| = \|y\|^2 = \|y\|\|y\|$*, thus* $\|\langle ., y\rangle\| = \|y\|$*, similarly* $\|\langle y, .\rangle\| = \|y\|$

3.2 Orthogonality

If $(H, \langle , \rangle)$ is a Pre-Hilbert space. Two vectors x and y are said to be orthogonal if and only if $\langle x, y\rangle = 0, x \neq 0$ and $y \neq 0$. We write $x \perp y$.
For x, y_1, y_2 such as $\langle x, y_1\rangle = \langle x, y_2\rangle = 0$ and for $\lambda \in \mathbb{K}$, we have:

$$\begin{aligned}\langle x, \lambda y_1 + y_2\rangle &= \langle x, \lambda y_1\rangle + \langle x, y_2\rangle \\ &= \overline{\lambda}\langle x, y_1\rangle + \langle x, y_2\rangle = 0\end{aligned}$$

- Let $\{x\}^{\perp} = \{y \in H, \langle x, y\rangle = 0\}$.
$\{x\}^{\perp}$ is a linear subspace of H.
$\{x\}^{\perp} = \ker\langle x, .\rangle = \ker\langle ., x\rangle$.
$\{x\}^{\perp}$ is closed, $\{0\}^{\perp} = H, H^{\perp} = \{0\}$.
The orthogonal of a part A of H: Let $A \subset H$ be a non-empty part of H

$$\begin{aligned}A^{\perp} &= \{y \in H, \forall x \in A\colon \langle x, y\rangle = 0\} \\ &= \cap\{x\}^{\perp},\end{aligned}$$

$A^{\perp}$ is a linear subspace because intersection of linear subspaces $\{x\}^{\perp}$ where x passes through A.
$A^{\perp}$ is closed because intersection of closed.

3.3 Hilbert projection theorem

Let C be a non-empty closed convex of a Hilbert space H and let $x \notin C$. As C is closed, thus $d(x, C) > 0$.

Theorem 72 (Hilbert orthogonal projection theorem) *There exists a vector x_c of the set C and only one such that $d(x, C) = d(x, x_c)$, or $d(x, C) = \|x - x_c\|$, or else $\inf_{y\in C} d(x, y) = \|x - x_c\|$, or in other words, the lower bound is reached at a single point of C.*

Proof. For $n \in \mathbb{N}^*$. Let $F_n = C \cap B[x, \delta + \frac{1}{n}]$, where $\delta = d(x, C)$. As $d(x, C) = \inf_{y \in C} d(x, y)$, thus
$\forall \geq 1 \implies \exists y \in d(x, y) \leq \delta + \frac{1}{n}$, this shows $\forall n \geq 1, F_n \neq \emptyset$.
F_n are closed because intersection of closed C and closed ball $B[x, \delta + \frac{1}{n}]$. As $\delta + \frac{1}{n+1} \leq \delta + \frac{1}{n}$, thus $B[x, \delta + \frac{1}{n+1}] \subset B[x, \delta + \frac{1}{n}]$ and threfore $F_{n+1} \subset F_n$. The sequence of closed $\{f_n\}_n$ is a nested sequence.
Let for $n \geq 1, \delta(F_n)$ is the diameter of F_n. By definition we have:

$$\delta(F_n) = \sup_{y,z \in F_n} d(y, z).$$

We show that $\delta(F_n) \to 0$ when $n \to \infty$. Let $y, z \in F_n$. The equality of the parallelogram gives:

$$\|y - z\|^2 = \|(y - x) - (z - x)\|^2,$$

and we have:

$$\|(y - x) - (z - x)\|^2 + \|(y - x) - (z - x)\|^2 = 2(\|y - x\|^2 + \|z - x\|^2),$$

where $\|y - z\|^2 = 2(\|y - x\|^2 + \|z - x\|^2) - \|y + z - 2x\|^2$ and therefore as $\|y - x\|^2 \leq (\delta + \frac{1}{n})^2$ and $\|z - x\|^2 \leq (\delta + \frac{1}{n})^2$, therefore:

$$\|y - z\|^2 \leq 4(\delta + \frac{1}{n})^2 - \|y + z - 2x\|^2,$$

or

$$\|y + z - 2x\| = 2\|\frac{y + z}{2} - x\| \implies \|\frac{y + z}{2} - x\|^2 = 4\|\frac{y + z}{2} - x\|^2$$

but $y, z \in F_n$ and F_n is convex because the intersection of the convex C and the closed ball $B[x\delta + \frac{1}{n}]$ is also convex, where $\frac{y+z}{2} \in F_n \implies \|\frac{y+z}{2} - x\| = d(x, \frac{y+z}{2}) \geq \delta = \inf_{y \in C} d(x, y)$. Therefore $\|\frac{y+z}{2} - x\|^2 \geq \delta^2$, where

$$\begin{aligned}
\|y - z\|^2 &\leq 4(\delta + \frac{1}{n})^2 - 4\delta^2 \\
\|y - z\|^2 &\leq \frac{4}{n^2} + \frac{8}{n}
\end{aligned}$$

and thus

$$(\sup_{y,z\in F_n}\|y-z\|)^2 \leq \frac{4}{n^2}+\frac{8}{n},$$

which implies that

$$\Longrightarrow \delta(F_n)^2 \leq \frac{4}{n^2}+\frac{8}{n}.$$

Consequently

$$\lim_{n\to\infty}\delta(F_n)=0.$$

We assume that C is complete, i.e., C is a complete metric space. If H is a Hilbert space, then it is complete, and since C is closed, then C is complete, i.e., any Cauchy sequence in C is convergent in C.
Since $(F_n)_n$ is a nested sequence of closed in C whose diameters $\delta(F_n)$ tend to zero when $n\to\infty$, and C is assumed to be a complete metric space, then the intersection $\bigcap_{n\geq 1} F_n$ is reduced to a single point. Let $x_c\in C$ such that $\bigcap_{n\geq 1} F_n=\{x_c\}$.

$$\begin{aligned}\forall n\geq 1 \Longrightarrow x_c\in F_n &\Longrightarrow \|x-x_c\|\leq \delta+\frac{1}{n}\\ \text{and as } x_c\in C &\Longrightarrow \|x-x_c\|\geq\delta\end{aligned}$$

and by passing to the limit on n, we deduce that $\|x-x_c\|=\delta \Longrightarrow d(x,y)=\|x-x_c\|=d(x,x_c)$.

■

We put $x_c=p(x)$ (p: projection), p is an application of H in C, defined by:

$x\in H$ then $p(x)$ is such that : $d(x,C)=\|x-p(x)\|$.

Characterization of the orthogonal projection

Theorem 73 *Let C be a closed convex of a Hilbert space H and let $x \in H$. Then $p(x)$ is the only point in C satisfying:*

$$\forall z \in C\colon Re\langle x - p(x), z - p(x)\rangle \leq 0.$$

Proof.

(1) If $p(x) \in C$ satisfies: $\forall z \in C\colon Re\langle x - p(x), z - p(x)\rangle \leq 0$, thus

$$\begin{aligned} \|x - z\|^2 &= \|(x - p(x)) - (z - p(x))\|^2 = \|x - p(x)\|^2 + \|z - p(x)\|^2 \\ &- 2Re\langle x - p(x), z - p(x)\rangle \geq \|x - p(x)\|^2. \end{aligned}$$

Uniqueness of $p(x)$: Assume that $p_1(x) \in C$ and $p_2(x) \in C$ satisfying the two two inequalities:

$$\forall z\mathbb{C}, Re\langle x - p_1(x), z - p_1(x)\rangle \leq 0 \tag{3.4}$$

and

$$Re\langle x - p_2(x), z - p_2(x)\rangle \leq .0 \tag{3.5}$$

In (3.4), we put $z = p_2(x)$ and in (3.5), we put $z = p_1(x)$. We get:

$$\begin{aligned} Re\langle x - p_1(x), p_2(x) - p_1(x)\rangle &\leq 0 \\ \text{and } Re\langle x - p_2(x), p_1(x) - p_2(x)\rangle &\leq 0, \end{aligned}$$

$$Re\langle x - p_1(x), p_2(x) - p_1(x)\rangle = Re\langle p_1(x) - x, p_1(x) - p_2(x) \leq 0\rangle \tag{3.6}$$

$$Re\langle x - p_2(x), p_1(x) - p_2(x)\rangle \leq 0 \tag{3.7}$$

$(3.6) + (3.7) \implies Re\langle p_1(x) - p_2(x), p_2(x) - p_1(x)\rangle \leq 0\rangle \implies \|p_1(x) - p_2(x)\|^2 = 0 \implies p_1(x) = p_2(x)$.

Reciprocally, if $z \in C\colon \|x - z\| \geq \|x - p(x)\|^2$, this is true for $\lambda z + (1 -$

$\lambda)p(x), \forall\lambda \in [0,1]\colon \|x - \lambda z - (1-\lambda)p(x)\|^2 \geq \|x - p(x)\|^2$.

$$\begin{aligned} \|(x-p(x)) - \lambda(z-p(x))\|^2 &= \|x-p(x)\|^2 + \lambda^2\|z-p(x)\|^2 \\ &- 2\lambda\langle x-p(x), z-p(x)\rangle \\ &\geq \|x-p(x)\|^2, \end{aligned}$$

where if $\lambda \neq 0$, we have: $\langle x-p(x), z-p(x)\rangle \leq \frac{\lambda}{2}\|z-p(x)\|^2 \implies \langle x-p(x), z-p(x)\rangle \leq \lim_{\lambda\to 0}\|z-p(x)\|^2 = 0$.

■

Remark 74 *1. In general, it is not at all easy to calculate $p(x)$, as calculating a lower bound is not always feasible. However, the characterization theorem is a convenient way of calculating $p(x)$, because it avoids the need to calculate a lower bound.*

2. If C is a closed convex of a Hilbert, then we have: $x \notin C \implies d(x,C) = d(x, Fr\ C)$ where $Fr\ C$ stands for the boundary of C. To calculate $p(x)$, it suffices to assume that $p(x) \in Fr\ C$.

Example 75 *Consider the disk defined by: $D = \{(x,y), (x-a)^2 + (y-b)^2 \leq r^2\}$, which is obviously a closed convex. Let $(X,Y) \notin C$, we have $d((X,Y),D) = d((X,Y), Fr\ D)$.*
Since $Fr\ D = \{(x,y), (x-a)^2 + (y-b)^2 = r^2\}$, we have to look for $(x_0,y_0) \in Fr\ D$, such that: $\min d((X,Y), Fr\ D)^2 = \|(X,Y) - (x_0,y_0)\|^2 = \min\{\|(X,Y) - (x,y)\|^2, (x,y) \in Fr\ D\}$. The existence and uniqueness of the point (x_0,y_0) is guaranteed by Hilbert's projection theorem. We can take:

$$x = a + r\cos t$$

$$y = b + r\sin t$$

$$\begin{aligned}\|(X,Y)-(x,y)\|^2 &= \|X-x,Y-y\|^2\\ &= (X-a-r\cos t)^2+(Y-b-r\sin t)^2 = F(t).\end{aligned}$$

It is about to find the minimum of F. If

$$F'(t) = 2r\sin(X-a-r\cos t)-2r\cos t(Y-b-r\sin t) \quad (3.8)$$

$$F'(t)=0 \iff \sin t(X-a)-\cos t(Y-b)=0. \quad (3.9)$$

If $X\neq a \implies \cos t\neq 0 \implies \frac{\sin t}{\cos t}=\frac{Y-b}{X-a} \implies t=t_0=\arctan\left(\frac{Y-b}{X-a}\right)$, *hence*

$$p(X,Y)=\left(a+r\cos\left(\arctan\left(\frac{Y-b}{X-a}\right)\right), b+r\sin\left(\arctan\left(\frac{Y-b}{X-a}\right)\right)\right).$$

$$\begin{cases} x_0 &= a+r\cos\left(\arctan\left(\frac{Y-b}{X-a}\right)\right)\\ y_0 &= b+r\sin\left(\arctan\left(\frac{Y-b}{X-a}\right)\right).\end{cases}$$

Where $\cos^2 t=\frac{1}{1+\tan^2 t} \implies \cos^2(\arctan x)=\frac{1}{1+x^2}$, *and* $\arctan x\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\implies$ $\cos(\arctan x)\geq 0$, *where*

$$\begin{aligned}\cos(\arctan x) &= \frac{1}{\sqrt{1+x^2}},\\ \textit{and } \sin(\arctan x) &= \frac{x}{\sqrt{1+x^2}},\end{aligned}$$

and finally

$$\begin{aligned}x_0 &= a+r\frac{(X-a)}{\sqrt{(X-a)^2+(Y-a)^2}}\\ y_0 &= b+r\frac{(Y-b)}{\sqrt{(X-a)^2+(Y-a)^2}}.\end{aligned}$$

3.4 Theorem of F. Riesz

The theorem of F. Riesz says that

Theorem 76 *Let H be a Hilbert space over $\mathbb{K}$ and φ a continuous linear form over H, then exists a unique element $a \in H$ such that $\forall x \in H \colon \varphi(x) = \langle x, a \rangle$.*

Proof. As φ is continuous then $\ker\varphi$ is closed hyperplane, or if $\varphi(b) = 1$ then $b \notin \ker\varphi, H = \ker\varphi \bigoplus \mathbb{K}.b$ and moreover we know that: $H = \ker\varphi \bigoplus \ker\varphi^{\perp}$. Let then be $x \in H$. On the one hand $\exists\lambda \in \mathbb{K}, \exists! x_1 \in \ker\varphi, x = x_1 + \lambda.b$ and on the other hand $\exists! \bar{x}_1 \in \ker\varphi$and $\exists! \bar{x}_2 \in \ker\varphi^{\perp}$ such that $x = \bar{x}_1 + \bar{x}_2$, but we have $\bar{x}_1 + \bar{x}_2 = x_1 + \lambda.b \implies \bar{x}_1 - x_1 = \lambda.b - \bar{x}_2$.

$$\left.\begin{array}{l} \lambda b - \bar{x}_2 \in \ker\varphi \\ \text{and as } \bar{x}_2 \in \ker\varphi \end{array}\right\} \text{then} \quad \lambda.b \in \ker\varphi^{\perp}$$

but $\dim\ker\varphi^{\perp} = 1$, thus $\ker\varphi^{\perp} = \mathbb{R}.b$ or $\varphi(x) = \varphi(x_1) + \lambda\varphi(b) \implies \lambda = \varphi(x)$. or $\langle x, b\rangle = \langle x_1 + \lambda b, b\rangle = \langle x_1, b\rangle + \lambda\langle b, b\rangle$.
Since $b \in \ker\varphi^{\perp}$ and $x_1 \in \ker\varphi \implies \langle x_1, b\rangle = 0$, where $\langle x, b\rangle = \lambda\|b\|^2 \implies \lambda = \langle x, \frac{b}{\|b\|^2}\rangle = \varphi(x)$. If we take $a = \frac{b}{\|b\|^2}$, we get:

$$\varphi(x) = \langle x, a\rangle.$$

■

Application of Riesz theorem

Let u be a continuous operator (continuous linear application) of H in H. For $x \in H$, let φ_y be the linear form defined by: $\varphi_y(x) = \langle u(x), y\rangle$, where $y \in H$ (it is clear that φ_y is is a linear form) and in addition $|\varphi_y(x)| = |\langle u(x), y\rangle| \leq \|u(x)\|\|y\| \leq (\|u\|\|y\|)\|x\| \implies \varphi_y$ is continuous.
By applying Riesz's theorem: $\exists!$ a unique element that we denote by $u^*(y)$ such that $\forall x \in H \colon \varphi_y(x) = \langle x, u^*(y)\rangle$, thus u^* is an application of H in H.
Let us prove that u^* is linear:

For $\lambda \in \mathbb{K}$ and $y_1, y_2 \in H$, we have

$$\begin{aligned} & \forall x \in H\colon \langle u(x), y_1\rangle = \langle x, u^*(y_1)\rangle \\ \Longrightarrow\quad & \langle u(x), \lambda y_1\rangle = \langle x, u^*(\lambda y_1)\rangle \\ \Longrightarrow\quad & \langle \overline{\lambda} u(x), y_1\rangle = \langle x, u^*(\lambda y_1)\rangle \\ \Longrightarrow\quad & \langle u(\overline{\lambda} x), y_1\rangle = \langle \overline{\lambda} x, u^*(y_1)\rangle = \langle x, \lambda u^*(y_1)\rangle \\ \Longrightarrow\quad & \forall x \in H\colon \langle x, u^*(\lambda y_1)\rangle = \langle x, \lambda u^*(y_1)\rangle \\ \Longrightarrow\quad & u^*(\lambda y_1) = \lambda u^*(y_1), \end{aligned}$$

and $\forall x \in H\colon \langle u(x), y_2\rangle = \langle x, u^*(y_2)\rangle$,
and $\forall x \in H\colon \langle u(x), y_1 + y_2\rangle = \langle x, u^*(y_1 + y_2)\rangle$, so

$$\begin{aligned} \langle x, u^*(y_1+y_2)\rangle &= \langle u(x), y_1 + y_2\rangle \\ &= \langle u(x), y_1\rangle + \langle u(x), y_2\rangle \\ &= \langle x, u^*(y_1)\rangle + \langle x, u^*(y_2)\rangle = \langle x, u^*(y_1+y_2)\rangle \end{aligned}$$

and thus

$$\forall x \in H\colon u^*(y_1 + y_2) = u^*(y_1) + u^*(y_2).$$

This shows that u is a linear operator of H in H. u^* is called the "adjoint operator of operator u".

- **Continuity of u^***

We have seen that the linear form $\varphi_y\colon x \to \langle x, y\rangle$ is continuous and that $\|\varphi_y\| = \|y\|$.

For $\|u^*\| = \sup\limits_{\|y\|\leq 1} u^*(y)\| = \sup\limits_{\|y\|\leq 1} \|\varphi_{u^*(y)}\|$ or

$$\begin{aligned} \|\varphi_{u^*(y)}\| &= \sup_{\|x\|\leq 1} \|\varphi_{u^*(y)}(x)\| \\ &= \sup_{\|x\|\leq 1} |x, u^*(y)| \\ &= \sup_{\|x\|\leq 1} |u(x), y| \\ &\leq \sup_{\|x\|\leq 1} (\|u(x)\|\|y\|) \leq \|y\| \sup_{\|x\|\leq 1} \|u(x)\| = \|y\|\|u\|, \end{aligned}$$

$$\begin{aligned} \text{so} \quad & \|u^*\| = \sup_{\|y\|\leq 1} \|\varphi_{u^*(y)}\| \leq \sup_{\|y\|\leq 1} (\|u\|\|y\|) \\ \Longrightarrow \quad & \|u^*\| \leq \|u\|, \end{aligned}$$

which means that u^* is continuous.

Proposition 77 *The following propositions summarize the main properties of the adjoint operator:*

a) $(u^*)^* = u$.

b) $\|u^*\| \leq \|u\|$.

c) u^* *commutes with* u,

d) $(u + v)^* = u^* + v^*$.

e) $(\lambda u)^* = \overline{\lambda} u^*$

f) $(u \circ v)^* = v^* \circ u^*$.

g) The $*$ *operator of* $\mathcal{L}(H)$ *in* $\mathcal{L}(H)$ *is isometric anti-linear.*

h) $I^* = I$ *where* $I(x) = x$.

i) If u *is invertible then* u^* *is invertible and* $(u^{-1})^* = (u^*)^{-1}$.

Proof.

a) $(u^*)^* = u$.

$$\begin{aligned} \langle u^*(x), y\rangle &= \langle x, (u^*)^*(y)\rangle \\ \overline{\langle y, u^*(x)\rangle} &= \overline{\langle u(y), x\rangle} \\ &= \langle x, u(y)\rangle \text{ and this } \forall x \in H \\ &\Longrightarrow (u^*)^*(y) = u(y) \end{aligned}$$

b) $\|u^*\| \leq \|u\|$ $\|u^*\| \leq \|u\|$

$\|u\| = \|(u^*)^*\| \leq \|u^*\|$ thus $\|u^*\| = \|u\|$.

c) u^* commutes with u

$$u \circ u^* = u^* \circ u.$$

For $x, y \in H$, we have:

$$\begin{aligned} \langle (u \circ u^*)(x), y \rangle &= \langle (u(u^*(x)), y \rangle \\ = \langle u^*(x), u^*(y) \rangle &= \overline{\langle u^*(y), u^*(x) \rangle} \\ = \langle u(u^*(x)), y \rangle & \end{aligned}$$

d) $(u + v)^* = u^* + v^*$

$$\begin{aligned} \langle (u+v)(x), y \rangle &= \langle x, (u+v)^*(y) \rangle \\ \langle u(x), y \rangle + \langle v(x), y \rangle &= \langle x, u^*(y) \rangle + \langle x, v^*(y) \rangle \\ &= \langle x, u^*(y) + v^*(y) \rangle \end{aligned}$$

e) $(\lambda u)^* = \overline{\lambda} u^*$

$$\begin{aligned} \langle x, (\lambda u)^*(y) \rangle &= \langle (\lambda u)(x), y \rangle = \lambda \langle u(x), y \rangle \\ &= \lambda \langle x, u^*(y) \rangle = \langle x, \overline{\lambda} u^*(y) \rangle \end{aligned}$$

f) $(u \circ v)^* = v^* \circ u^*$

$$\begin{aligned} \langle x, (u \circ v)^*(y) \rangle &= \langle (u \circ v)(x), y \rangle \\ \langle u(v(x)), y \rangle &= \langle v(x), u^*(y) \rangle \\ &= \langle x, v^*(u^*(y)) \rangle, \end{aligned}$$

where $(u \circ v)^*(y) = v^*(u^*(y)) = (v^* \circ u^*)(y)$.

i) If u is invertible then u^* is invertible and

$$(u^{-1})^* = (u^*)^{-1}$$

$$u^{-1} \circ u = I \implies (u^{-1} \circ u)^* = I$$

$$\implies u^* \circ (u^{-1})^* = I \implies u^* \text{ is right-invertible and}$$

$$u \circ u^{-1} = I \implies (u^{-1})^* \circ u^* = I$$

$$\implies u^* \text{ is left-invertible and}$$

$$(u^*)^{-1} = (u^{-1})^*.$$

■

3.4.1 Operators of special types

In this subsection, we give a little account about normal and self-adjoint operators:

Definition 78 (Normal operator) *An operator $u \in \mathcal{L}(H)$ is called normal if $u^* \circ u = u \circ u^*$.*

Proposition 79 *u is normal if and only if $\forall x, y \in H\colon \langle u^*(x), u^*(y)\rangle = \langle u(x), u(y)\rangle$.*

Definition 80 (Self-adjoint operator) *An operator is said to be self-adjoint if it is the adjoint of itself, i.e., if $u^* = u$.*
If $\mathbb{K} = \mathbb{C}$, thus u is self-adjoint if and only if $\forall x \in H\colon \langle u(x), x\rangle \in \mathbb{R}$.

Special cases and properties

Let u be a self-adjoint operator, then:

- If $\mathbb{K} = \mathbb{R}, u$ is said to be symmetric, and if $\mathbb{K} = \mathbb{C}, u$ is said to be hermitian, i.e., if $\forall x, y\colon \langle u(x), y\rangle = \langle x, u(y)\rangle$.

- If $\mathbb{K} = C$ and if $u \in \mathcal{L}(H)$ thus

$$\frac{1}{2}\|u\| \leq \sup_{\|x\|\leq 1} |\langle u(x), x\rangle| \leq \|u\|.$$

- If $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and if u is self-adjoint , we have:

$$\|u\| = \sup_{\|x\|\leq 1} |\langle u(x), x\rangle|$$

Definition 81 (Partially unitary operator) *A linear operator u is said partially unitary if it preserves the norm: $\|u(x)\| = \|x\|, \forall x \in H$ (u is called an isometry).*

Definition 82 (Unitary operator) *A partially unitary operator u is said unitary when $\|u\| = 1$.*

Proposition 83 *u is a unitary operator if and only if $\forall x, y \in H\colon \langle u(x), u(y)\rangle = \langle x, y\rangle$.*

Proof.

$$(\implies)\|u(x) - u(y)\|^2 = \|u(x)\|^2 + \|u(y)\|^2 - 2Re\langle u(x), u(y)\rangle$$
$$= \|x\|^2 + \|y\|^2 - 2Re\langle x, y\rangle$$
$$\implies Re\langle u(x) - u(y)\rangle = Re\langle x, y\rangle$$
$$\implies \langle u(x) - u(y)\rangle = \langle x, y\rangle$$
$$\langle u(x) - u(y)\rangle = \langle u^*(u(x)), y\rangle = \langle x, y\rangle, \forall x, y \in H$$

thus $u \circ u = Id_H \implies u$ is left-invertible. Reciprocally, if $u^* \circ u = Id_H$, thus

$$\begin{aligned} \forall x, y \in H\colon \langle u^*(u(x)), y\rangle &= \langle x, y\rangle \\ \langle u(x), u(y)\rangle &= \langle x, y\rangle \end{aligned}$$

$\implies u$ maintains the norm. ■

Theorem 84 *u preserves the norm if and only if $u^* \circ u = Id_H$.*
If u preserves the norm, it is injective, we say that u is a unitary operator, and if u is surjective, i.e., if u is invertible and $u^{-1} = u^*$.

Remark 85 - *If $\dim H < \infty$, we know that any injective operator u of H in itself is surjective and vice versa. But if $\dim H = \infty$, an injective operator may not be surjective and vice versa. In addition if H is a Hilbert and u is left-invertible, u may not be right-invertible, i.e., it may not be surjective.*
As summary, we have
u is a unitary operator if and only if u is surjective preserving the norm if and only if $u^{-1} = u^*$.

3.5 Orthonormal system in a Hilbert space

Let H be a Hilbert space. A system of vectors $\{x_i\}_{i\in I}$ is said to be orthogonal if and only if $\forall i \neq j, \langle x_i, y_j\rangle = 0$.
$\{x_i\}_{i\in I}$ is therefore linearly independent (see Worksheet of exercises 5).
Let $E = Vect\{x_i\}_{i\in I}$: the smallest closed linear subspace containing the vectors x_i.
If $E = H$, the system x_i is said to be complete or total, or "a Hilbertian basis".
If H is separable, i.e., there exists a countable and dense subset D in H, which means also that H possesses a dense sequence. In this case, we show via the Gramm-Schmitt orthonormalization processus that H has an orthogonal (orthonormalized) system $\{x_i\}_{i\in I}, \|x_i\| = 1$ which is complete and countable.
Assume H is separable and let $\{x_n\}_{n\geq 1}$ be a complete orthonormal system.
Let $F_n = Vect\{x_1, x_2, \ldots, x_n\}, F_n$ is closed.
$m \geq n \implies F_n \subset F_m, F = \underset{n\geq 1}{\cup} F_n$ is a linear subspace of H and since $\{x_n\}_n$

is a complete system then $\overline{F} = H$. For $n \geq 1$, let p_n the projection of H onto F_n. For $x \in H, p_n(x) \in F_n$, i.e., $p_n(x) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$ or $x = p_n(x) + p_{F_n^\perp}(x), p_{F_n^\perp}(x) = x - p_n(x)$.

$$\langle x, x_j \rangle = \langle p_n(x), x_j \rangle + \underbrace{\langle p_{F_n^\perp}(x), x_j \rangle}_{\overset{\shortparallel}{0}}$$

$$\implies \langle x, x_j \rangle = \langle p_n(x), x_j \rangle = \langle \sum_{i=1}^{n} \alpha_i x_i, x_j \rangle$$

$$= \sum_{i=1}^{n} \alpha_i \langle x_i, x_j \rangle = \alpha_j \langle x_j, x_j \rangle = \alpha_j \|x_j\|^2 = \alpha_j.$$

Therefore $\alpha_j = \langle x, x_j \rangle$, so $p_n(x) = \sum_{i=1}^{n} \langle x, x_j \rangle x_i$, or $\|x\|^2 = \|p_n(x)\|^2 + \|p_{F_n^\perp}(x)\|^2$ thus $\|p_n(x)\|^2 \leq \|x\|^2$, or

$$\|p_n(x)\|^2 = \|\sum_{i=1}^{n} \langle x, x_i \rangle x_i\|^2 = \sum_{i=1}^{n} |\langle x, x_i \rangle|^2 \leq \|x\|^2.$$

This shows that the series $\sum_{n\geq 1} |\langle x, x_n \rangle|^2$ is convergent $\implies \sum_{n \geq 1} \langle x, x_n \rangle x_n$ converges normally and moreover:

$$\sum_{n \geq 1} |\langle x, x_n \rangle|^2 \leq \|x\|^2. \tag{3.10}$$

The inequality (3.10) is called Bessel inequality.

Let us show that the sequence $(p_n(x))_{n \geq 1}$ is convergent. To do this, we need only to show that $\|p_n(x) - p_m(x)\|^2 \underset{n,m \to \infty}{\longrightarrow} 0$, or if $m \geq n + 1$

$$\begin{aligned} \|p_n(x) - p_m(x)\|^2 &= \|\sum_{j=n+1}^{m} \langle x, x_j \rangle x_j\|^2 \\ = \sum_{j=n+1}^{m} |\langle x, x_j \rangle|^2 &= \left| \left(\sum_{j=1}^{m} |\langle x, x_j \rangle|^2 \right) - \left(\sum_{j=1}^{n} |\langle x, x_j \rangle|^2 \right) \right| \end{aligned}$$

or the second member tends towards 0 because the series $\sum_{n \geq 1} |\langle x, x_n \rangle|^2$ converges.

Let then $l = lim_{n\to\infty} p_n(x)$, for $m \geq n$, we have $F_m^\perp \subset F_n^\perp$ because $F_n \subset F_m$.

$$p_{F_m^\perp}(x) = x - p_m(x) \in F_m^\perp \subset F_n^\perp \implies x - p_m(x) \in F_n^\perp$$

where as $F_n^\perp$ is closed and $x - p_m(x) \in F_n^\perp$, therefore $\lim_{m\to\infty}(x - p_m(x)) \in F_n^\perp$.

$\implies x - \lim_{m\to\infty} p_m(x) \in F_n^\perp \implies x - l \in F_n^\perp$, where $x - l \in \bigcap\limits_{n\geq 1} F_n^\perp = (\cup F_n)^\perp = F^\perp$ and as F is dense in $H \implies F^\perp = \{0\} \implies x = l$ and thus $\lim_{n\to\infty} p_n(x) = x$.And so the sequence $\sum_{i=1}^n \langle x, x_i\rangle x_i$ is convergent, i.e., the the series $\sum_{n\geq 1}\langle x, x_n\rangle x_n$ is convergent and we have:

$$x = \sum_{n\geq 1}\langle x, x_n\rangle x_n \tag{3.11}$$

$$\implies \|x\|^2 = \sum_{n\geq 1}|\langle x, x_n\rangle|^2 \tag{3.12}$$

The equation (3.12) is called Parseval identity.

The numbers $\langle x, x_n\rangle$ are called Fourier coefficients of the element x. The egality (3.12) is the reason why the complete system $\{x_n\}_{n\geq 1}$ is called "Hilbertian basis".

Remark 86 *A Hilbertian basis is not an algebraic basis of H.*

Examples of complete systems

1. $H = l^2, e_1 = (1, 0, 0, \dots), \dots e_n = (0, 0, \dots, 1, 0, \dots, 0), \|e_n\| = 1.$

$$\langle e_n, e_m\rangle = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m. \end{cases}$$

As $X = (x_n)_n \in l^2 \iff \sum\limits_{n\geq 1}|x_n|^2 < \infty$. Let $y_n = \sum\limits_{i=1}^n x_i e_i = (x_1, x_2, \dots, x_n, 0, 0, \dots)$,

$\|x - y_n\|^2 = \sum_{m \geq n+1} |x_m|^2 \underset{n\to\infty}{\longrightarrow} 0$, where the sequence y_n converges to x and as $y_n \in F_n = Vect\{e_1, e_2, \ldots, e_n\}$, so $x \in \overline{\underset{n\geq 1}{\cup} F_n} = \overline{F}$, i.e., the $\{e_n\}_n$ system is complete, therefore $x = \sum_{n\geq 1} \langle x, e_n\rangle e_n = \sum_{n\geq 1} x_n e_n$

2. $H = L^2([-\overline{a}, \overline{a}])$ (equipped with Lebesgue measure).

   The system $1, \cos(nx), \sin(nx)(n = 1, 2, \ldots)$ is a complete orthogonal system.

$$\begin{aligned} \int_{-\overline{a}}^{\overline{a}} \cos(nx)\cos(mx)dx &= \frac{1}{2}\int_{-\overline{a}}^{\overline{a}} \cos((\frac{n+m}{2})x) + \cos((\frac{n-m}{2})x)dx \\ &= \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{else.} \end{cases} \end{aligned}$$

   The completeness of the system results from theorem of Weirstrass on the approximation of a continuous periodic function by trigonometric polynomials. The system: $\frac{1}{\sqrt{2\overline{a}}}, \cos nx$.

   The Fourier coefficients are:

$$\langle f, \frac{\cos nx}{\sqrt{\overline{a}}}\rangle = \frac{1}{\sqrt{\overline{a}}}\int_{-\overline{a}}^{\overline{a}} f(x)\cos(nx)dx$$

   and

$$\langle f, \frac{\sin nx}{\sqrt{\overline{a}}}\rangle = \frac{1}{\sqrt{\overline{a}}}\int_{-\overline{a}}^{\overline{a}} f(x)\sin(nx)dx$$

   and $a_0 = \langle f, \frac{1}{\sqrt{2\overline{a}}}\rangle = \frac{1}{\sqrt{2\overline{a}}}\int_{-\overline{a}}^{\overline{a}} f(x)dx$.

3. Legendre polynomials: $H = L^2([-1, 1]), p_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2 - 1)^n$.

$$\int_{-1}^{1} p_n(x)p_m(x)dx = \begin{cases} 0, & n \neq m \\ \frac{2}{2n+1,} & n = m \end{cases}$$

   The completeness results from theorem of Weirstrass on the uniform approximation of a continuous function on a closed interval by polynomials.

$$f \in H, f = \sum_{n \geq 0} c_n p_n(x).$$

where $c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) p_n(x) dx$. The series $\sum_{n \geq 0} c_n p_n(x)$ converges in quadratic mean to f:

$$\lim_{n \to \infty} \|f - \sum_{i=0}^{n} c_i p\|^2 = \lim_{n \to \infty} \int_{-1}^{1} (f(x) - \sum_{i=0}^{n} c_i p_i(x))^2 dx = 0.$$

Worksheet of exercises 5

Exercice 1: Let $(H, \langle , \rangle)$ be a pre-Hilbertian $\mathbb{K}$-space.

a) Let $(x_i)_{i=1,...p}$ be a family of orthogonal pairwise vectors of H. Show that they are linearly independent. Show that they are linearly independent.

b) Knowing that $\forall \lambda \in \mathbb{R}$, $\langle x - \lambda y, x - \lambda y \rangle \geq 0$, show that $\forall x \in H$ and $\forall y \in H$: $Re \langle x - \lambda y, x - \lambda y \rangle \leq \|x\| . \|y\|$.

c) Show that $|\langle x, y \rangle| \leq \|x\| . \|y\|$ (Cauchy-Schwarz inequality).

Exercice 2: Let $(H, \langle , \rangle)$ a $\mathbb{K}$-Hilbert space and F a closed linear subspace of H.

a) Show the equivalence $x \in F^{\perp} \Leftrightarrow \forall y \in F, \langle x - P_F(x), y \rangle = 0$ ($P_F(x)$ is the projection of x onto F).

b) Show that $\forall x \in H$, $x = P_F(x) + P_{F^{\perp}}(x)$.

c) Prove that $\|x\|^2 = \|P_F(x)\|^2 + \|P_F(x)\|^2$ (Pythagore inequality).

d) Show that P_F is a continuous linear operator from H into F and that $\|P_F\| = 1$.

e) Show that $\ker P_F = \{x \in H \ / \ P_F(x) = 0\} = F^{\perp}$.

Exercice 3: Let $(H, \langle , \rangle)$ a $\mathbb{K}$-Hilbert space and $(x_n)_{n\geq 1}$ an orthonormal system of H. Let us consider $F_n = vect\{x_1, x_2, ...x_n\}$, the subspace generated by the vectors $x_1, x_2, ...x_n$, P_n the orthogonal projection on F_n and $F = \bigcup\limits_{n\geq 1} F_n$.

1) Show that $\forall x \in F$, the series $\sum\limits_{n\geq 1} |\langle x, x_n \rangle|^2$ converges and that $\sum\limits_{n\geq 1} |\langle x, x_n \rangle|^2 \leq \|x\|^2$ (Bessel inequality).

2) Show then that any vector x of F can be written as $x = \sum_{n\geq 1} \langle x, x_n\rangle \, x_n$ and that $\|x\|^2 = \sum_{n\geq 1} |\langle x, x_n\rangle|^2$ (Parseval inequality).

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