وزارة التعليـم العالـي والبحـث العلمـي

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FACULTY OF SCIENCES

LECTURE NOTES

Maths1
Analysis and Algebra1

For students of the common core sciences of matter

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Contents

Preface

This mathematics course goes beyond certain developments within the strict framework of the program usually covered in the first year of the undergraduate cycle of higher education. We wanted to make it a reference document that engineering students can use in the rest of their studies to deepen or review the notions of algebra or analysis used in the teaching of applied mathematics for the master's degree.
In this work, we have endeavored to give precise definitions and present rigorous reasoning without, however, seeking exhaustiveness. Furthermore, as far as possible, we have sought to motivate the concepts introduced and to illustrate them with examples, remarks and warnings in order to make learning more dynamic.

This manuscript constitutes the essential part of **Analysis 1** and **Algebra 1** which I gave first year **LMD** science of matter. We will enhance the course with applications and motivations from physics and chemistry.
It covers the essential elements of set theory, applications and relationships, internal laws, an introduction to general algebra such as groups, rings, fields. It then addresses the real functions of a real variable, in particular the notion of limit, its properties, the notion of continuity and differentiability of functions and finally we study the usual functions, these functions appear naturally in solving simple problems, especially those dealing with real-world topics in physics. It also covers an introduction of vector spaces given in the last chapter. We then formalize the abstract and fundamental concept in linear algebra as well as that of linear applications.
At the end of each party, we offer exercises with solutions.

Chapter 1

General algebra

1.1 Introduction

The primary purpose of this chapter is to review a number of topics from analysis, and some from algebra, that will be called upon in the following chapters. These are topics of a classical nature, such as appear in books on advanced calculus and linear algebra. For our treatment of modern analysis, we can distinguish three fundamental notions which will be particularly stressed in this chapter. These are
(a) set theory, of an elementary nature;
(b) the concept of a function;
(c) algebraic structures
On a number of occasions in this chapter, we will also take the time to discuss the relationship of modern analysis to classical analysis. We begin this now, assuming some knowledge of the points (a) to (c) just mentioned.
Modern analysis is not a new brand of mathematics that replaces the old brand. It is totally dependent on the time-honoured concepts of classical analysis, although in parts it can be given without reference to the specifics of classical analysis. For example, whereas classical analysis is largely concerned with functions of a real or complex variable, modern analysis is concerned with functions whose domains and ranges are far more general than just sets of real or complex numbers. In fact, these functions can have domains and ranges which are themselves sets of functions. A function of this more general type will be called an operator or mapping. Importantly, very often any set will do as the domain of a mapping, with no specific reference to the nature of its elements.
A set is a concept so basic to modern mathematics that it is not possible to give it a precise definition without going deeply into the study of mathematical logic.

Commonly, a set is described as any collection of objects but no attempt is made to say what a collection is or what an object is. We are forced in books of this type to accept sets as fundamental entities and to rely on an intuitive feeling for what a set is. The objects that together make up a particular set are called elements or members of that set. The list of possible sets is as long as the imagination is vivid, or even longer (we are hardly being precise here) since, importantly, the elements of a set may themselves be sets.
Later in next chapter we will be looking with some detail into the properties of certain sets of numbers. We are going to rely on the readers experience with numbers and not spend a great deal of time on the development of the real number system.

1.2 Definitions of sets

Definition 1.2.1 (**Set**)**.** A set is a collection of elements. In practice there is two ways of constructing or writing sets by giving the list of its elements, for example $\{0, 1, 2, 3, 5, 7, 8\}$, $\{\text{red, black, blue}\}$, or else a collection of elements that verify a property, for example $\{x \in \mathbb{R}| \quad 0 \leq x \leq 1\} = [0, 1]\,, \quad \{x \in \mathbb{R}| \quad |x - 2| < 2\}$.

Definition 1.2.2 (Empty set)**.** This is the set containing no element, we note it $\emptyset$, it can also be defined as

$$\emptyset := \{x \mid x \neq x\}.$$

It is necessary to make some comments regarding the definition of an empty set.

Théorème 1.2.3. *(a) All empty sets are equal.*
(b) The empty is the set containihg no elements.
(c) The only set with no elements is the empty set.

Definition 1.2.4 (Relationships between elements and sets)**.** If x indicates one of the elements of the set E, we say that x **belongs** to E and we note $x \in E$. If x is not an element of E, we say that x **does not belong** to E and we note $x \notin E$.

Definition 1.2.5 (Finite set)**.** We say that the set E is **finite** if the number of elements of E is finite. The number of elements of E is called the **cardinal** of E denoted $\text{Card}(E)$.

Example 1.2.6. $E = \{0, 1, 2, 3, 5, 7, 8\} \Rightarrow \text{Card}(E) = 7,\ E = \emptyset \Rightarrow \text{Card}(E) = 0$,
$\mathbb{N}$ is not a finite set.

Definition 1.2.7 (Subset (Inclusion))**.** We say that a set A is **included** in another set B (which we note $A \subset B$), if all the elements of A are also in B. We also call that A is a subset of B.

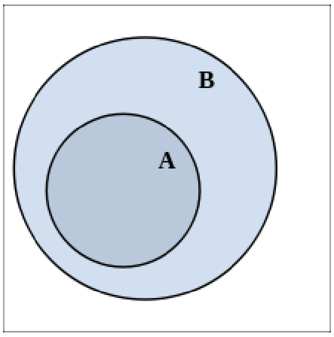


Figure 1.1: Inclusion

Definition 1.2.8 (Set of parts of E)**.** Let E be a finite set, we call **set of parts** of E, or (power set) the set of all subsets that set E could contain denoted $\mathcal{P}(E)$ and we have $\mathrm{Card}(\mathcal{P}(E)) = 2^{\mathrm{Card}(E)}$.

Example 1.2.9. $E = \{1, 2, 3\}$ and as $\mathrm{Card}(E) = 3$, then $\mathrm{Card}(\mathcal{P}(E)) = 2^3 = 8$, we have

$$\mathcal{P}(E) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, E\}.$$

Definition 1.2.10 (Equal Sets)**.** Two sets are **equal** if they have the same elements, in particular $(E \subset F \text{ and } F \subset E) \Leftrightarrow (E = F)$

Example 1.2.11. $E = \left\{x \in \mathbb{R} \mid \quad x^2 - 3x + 2 = 0\right\}, \quad F = \{1, 2\}$

1- $E \subset F$ because if $x \in E$, we have $x^2 - 3x + 2 = 0 \Rightarrow x = 1$, or $x = 2$. then $x \in F$.

2-$F \subset E$ because if $x \in F$, we have $\begin{cases} x = 1 \Rightarrow 1 - 3(1) + 2 = 0 \\ \text{or} \\ x = 2 \Rightarrow 2^2 - 3(1) + 2 = 0 \end{cases}$ then $x \in E$.

Finally we conclude that $E = F$.

Definition 1.2.12 (Complement Set)**.** Let E be a set, and A a part of E. The set of elements of E which do not belong to A is called complementary of A in E, and is denoted C_A^E. We also note CA if there is no ambiguity (and sometimes also A^c or $\overline{A}$).

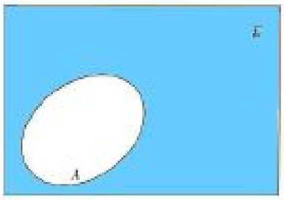


Figure 1.2: Complement

Definition 1.2.13 (Union of Sets). Let A and B be two parts of E. The union of A and B is the set, denoted $A \cup B$, consisting of the elements belonging to at least one of the sets A and B.

$$A \cup B = \{x \in E / x \in A \text{ or } x \in B\}$$

The «or » is not exclusive: x can belong to A and B at the same time.

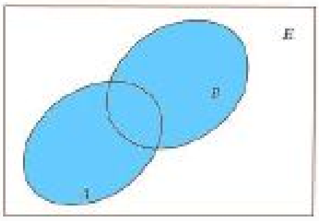


Figure 1.3: Union

Definition 1.2.14 (Intersection). Let A and B be two parts of the set E. The intersection of A and B is the set, denoted $A \cap B$, consisting of the elements belonging to both A and B. When the intersection of A and B is the empty set, we say that the sets A and B are disjoint.

$$A \cap B = \{x \in E / x \in A \text{ and } x \in B\}$$

Definition 1.2.15 (Set Difference). For $A, B \subset E$, Set difference which is denoted by $A - B$, lists the elements in set A that are not present in set B .

Proposition 1.2.16. *Let A, B, C be parts of a set E.*

- *$A \cap B = B \cap A$ and $A \cup B = B \cup A$ (**Commutative Property**).*

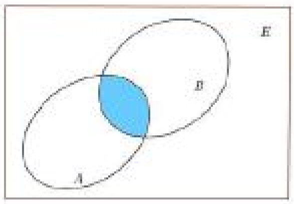


Figure 1.4: Intersection

- $(A \cap B) \cap C = A \cap (B \cap C)$ *and* $(A \cup B) \cup C = A \cup (B \cup C)$ *(**Associative Property**).*

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ *and* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ *(**Distributive Property**).*

- $C^E_{A \cup B} = C^E_A \cap C^E_B$ *and* $C^E_{A \cap B} = C^E_A \cup C^E_B$.

- $C^E \left[C^E_A \right] = A, \quad A \subset B \Leftrightarrow C^E_B \subset C^E_A$.

- $A \cap \emptyset = \emptyset, \quad A \cap A = A, \quad A \subset B \Leftrightarrow A \cap B = A.$

- $A \cup \emptyset = A, \quad A \cup A = A, \quad A \subset B \Leftrightarrow A \cup B = B.$

Definition 1.2.17 (Cartesian Product of Sets)**.** Let E and F be two sets. If $x \in E$ and $y \in F$ we can make a new element called couple and denoted (x, y). All of these pairs is called the **Cartesian product** of E and F and is noted

$$E \times F = \{(x, y) \ / \quad x \in E \text{ and } \quad y \in F\}.$$

Example 1.2.18. 1-$E = \{1, 2\}, \quad F = \{3, 5\}$

$$E \times F = \{(1, 3), \ (1, 5), \ (2, 3), \ (2, 5)\}$$

2- $[0, 2] \times \mathbb{R} = \{(x, y) \ / \quad 0 \leq x \leq 2 \text{ and } \quad y \in \mathbb{R}\}$

1.3 Functions

The concept of function is fundamental in modern analysis. (It is equally important in classical analysis but may be given a restricted meaning there, as we remark below.)

A function is often described as a rule which associates with an element in one set a unique element in another set; we will give a definition which avoids the undefined term rule. In this definition we will include all associated terms and notations that will be required.

Definition 1.3.1 (Function)**.** We call **Functions** of a set E in a set F, any correspondence f between the elements of E and those of F.

Definition 1.3.2 (Domain of definition of f)**.** The domain of a function f is the set D_f of elements $x \in E$ corresponds a unique element $y = f(x) \in F$.

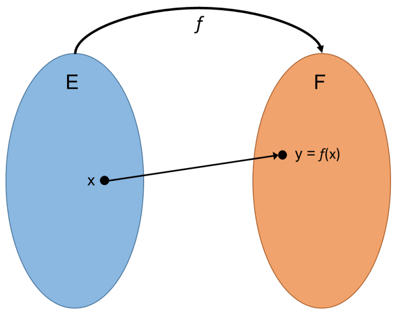


Figure 1.5: Sagittal diagram

- $y = f(x)$ is called **image** of x and x is a **predecessor** of y.
- E is called **starting set** and F **the arrival set** of the function f. We write

$$f : E \to F$$
$$x \to f(x)$$

For example The image of 1 is 1, the image of 2 is 1,

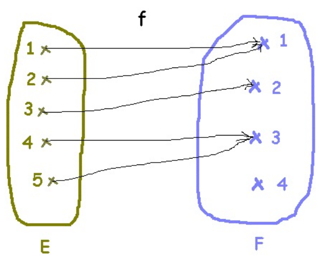


Figure 1.6: Direct image

1 has two antecedents: 1 and 2. 2 has antecedent: 3.
3 has two antecedents: 4 and 5.
4 has no antecedent.

Definition 1.3.3 (Application)**. An application** is a function of a set E in a set F such that $D_f = E$. In other words that f is a map if

$$\forall x, x' \in E, \quad x = x' \Rightarrow f(x) = f(x')$$

Example 1.3.4. The application $Id : E \to E$ such that

$$\forall x \in E, \quad Id(x) = x$$

is called identity application on E.

Definition 1.3.5 (Graph)**.** We call **graph** of an application $f : E \to F$, the set

$$\Gamma_f = \{(x, f(x)); \quad x \in E\}.$$

Definition 1.3.6 (Equality)**.** Two maps f, $g : E \to F$ are **equal** if and only if for all $x \in E$, $f(x) = g(x)$

Definition 1.3.7 (Composition)**.** Let $f : E \to F$ and $g : F \to G$ then $g \circ f : E \to G$ is the map defined by $g \circ f(x) = g(f(x))$

$$\underbrace{E \xrightarrow{f} F \xrightarrow{g} G}_{g\circ f}$$

Definition 1.3.8 (Restriction And extension of a application)**.** Given an application $f : E \to F$

1- We call **restriction** of f to a subset non-empty X of E, the application $g : X \to F$ such that

$$\forall x \in X, \quad g(x) = f(x).$$

We note $g = f_{\diagup X}$.

2- Given a set G such that $E \subset G$, we call **extension** of the application f to the set G, all application h of G in F such that f is the restriction of h to E.

Let E, F be two sets.

Definition 1.3.9 (Direct image of a part)**.** Let $A \subset E$ and $f : E \to F$, **the direct image** of A by f is the set

$$f(A) = \{f(x) \diagup x \in A\}.$$

$f(A)$ is a subset of F.

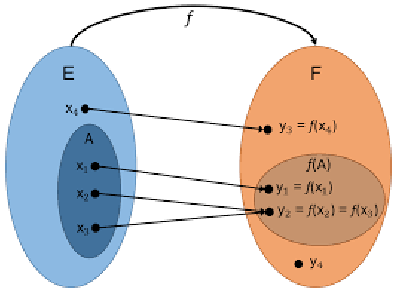


Figure 1.7: Direct image of a part

Definition 1.3.10 (Reciprocal image)**.** Let $B \subset F$ and $f : E \to F$, **the reciprocal image** of B by f is the set

$$f^{-1}(B) = \{x \in E \;/\; f(x) \in B\}.$$

$f^{-1}(B)$ is a subset of E.

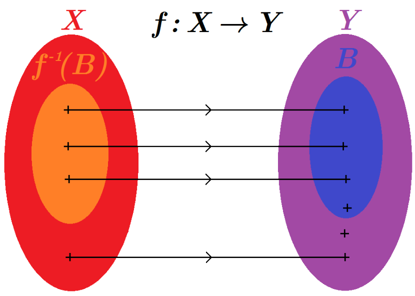


Figure 1.8: Reciprocal image of a part

Proposition 1.3.11. *Let $f : E \to F$, A_1, $A_2 \subset E$ and B_1, $B_2 \subset F$, then*

- $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.
- $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$.
- $f^{-1}(B_1 \cup B_2) \subset f^{-1}(B_1) \cup f^{-1}(B_2)$.
- $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.
- $f^{-1}(C^E_{B_1}) = C^E_{f^{-1}(B)}$

Definition 1.3.12 (Injective application)**.** Let E, F be two sets and $f : E \to F$ be a map. f is injective if every element of F has at most one antecedent, that is, if two distinct elements of E have distinct images:

$$\forall x_1 \in E \quad \forall x_2 \in E \qquad x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2) \tag{1}$$

which is equivalent to the following implication

$$\forall x_1 \in E \quad \forall x_2 \in E \qquad f(x_1) = f(x_2) \Longrightarrow x_1 = x_2 \tag{2}$$

Remark 1.3.13. In practice we must use (2) and not (1), it is easier to show that quantities are equal than to show that quantities are different.

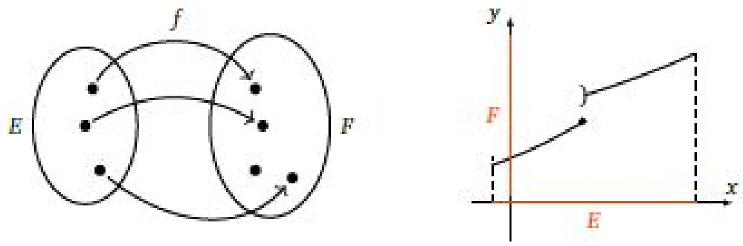


Figure 1.9: Injective application

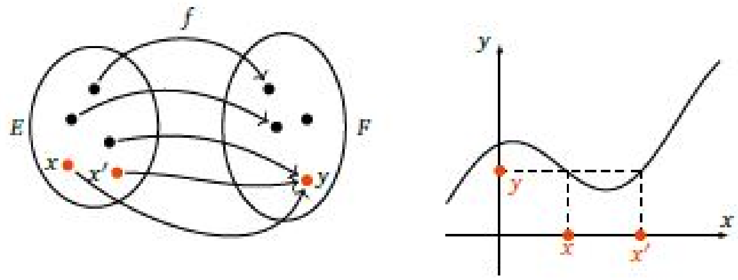


Figure 1.10: Non-injective application

Definition 1.3.14 (Surjective application)**.** Let E, F be two sets and $f : E \to F$ be a map. f is **surjective** if for all $y \in F$, there exists $x \in E$ such that $y = f(x)$. In other words

$$\forall y \in F, \quad \exists x \in E \text{ such that } y = f(x).$$

Another formulation: f is surjective if and only if $f(E) = F$.

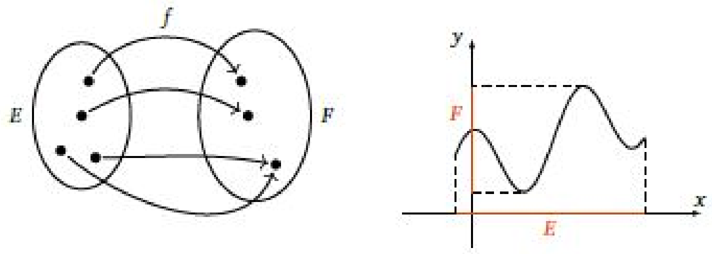


Figure 1.11: Surjective application

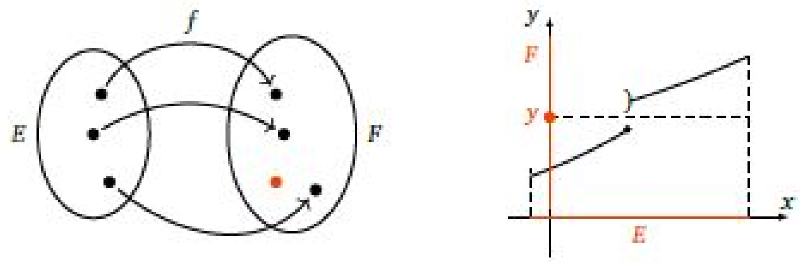


Figure 1.12: Non-surjective application

Remark 1.3.15. f is surjective if and only if every element y of F has at least one antecedent.

Definition 1.3.16 (Bijective application)**.** f is **bijective** if it is injective and surjective. That is equivalent to: for all $y \in F$ there exists a unique $x \in E$ such that $y = f(x)$. In other words

$$\forall y \in F, \quad \exists! x \in E \qquad [y = f(x)] .$$

Remark 1.3.17. The existence of x comes from the surjectivity and uniqueness from injectivity. In other words, every element of F has a unique antecedent by f.

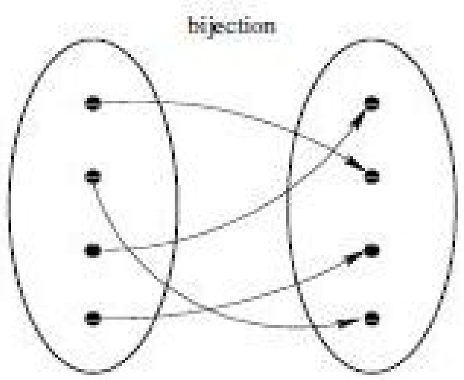


Figure 1.13: bijective application

Definition 1.3.18 (Reciprocal application)**.** Let E, F be sets. If $f : E \to F$ is bijective we define **the reciprocal map** by

$$f^{-1} : F \to E$$
$$y \to f^{-1}(y) = x$$

Example 1.3.19. Let f be the application from $]0, +\infty[$ to $]0.1[$ defined by

$$\forall x \in]0, +\infty[; \qquad f(x) = \frac{1}{\sqrt{x+1}}$$

Show that the map f is bijective and determine f^{-1}.

Solution • We show that f is injective.
Let $x, x' \in]0, +\infty[; \qquad f(x) = f(x') \implies \frac{1}{\sqrt{x+1}} = \frac{1}{\sqrt{x'+1}} \implies x = x'.$
Then f is injective.
• We show that f is surjective.
Let $y \in]0, 1[; \qquad y = f(x) \implies y = \frac{1}{\sqrt{x+1}} \implies x = \frac{1}{y^2} - 1.$ Then

$$\forall y \in]0, 1[; \quad \exists x \in]0, +\infty[; \qquad y = f(x).$$

Thus f is surjective.
As f is injective and surjective, then it is bijective and

$$f^{-1} :]0, 1[\to]0, +\infty[$$
$$y \to \frac{1}{y^2} - 1$$

1.4 Equivalence relation and Order relation

Definition 1.4.1 (Binary relation)**.** We call **binary relation**, any assertion between two objects, which may or may not be verified. We denote $x\mathcal{R}y$ and we read "x is related to y".

Definition 1.4.2. Given a binary relation $\mathcal{R}$ between the elements of a non-empty set E, we say that

1. **$\mathcal{R}$ is Reflexive**, $\forall x \in E$; $(x\mathcal{R}x)$ $\quad \overbrace{x \to x}^{x\mathcal{R}x}$
2. **$\mathcal{R}$ is Transitive**, $\forall x, y, z \in E$; $(x\mathcal{R}y)$ and $(y\mathcal{R}z) \Longrightarrow (x\mathcal{R}z)$ $\quad \overbrace{x \longrightarrow y \longrightarrow z}^{x\mathcal{R}z}$
3. **$\mathcal{R}$ is Symmetric**, $\forall x, y, \in E$; $(x\mathcal{R}y) \Longrightarrow (y\mathcal{R}x)$ $\quad \overbrace{x \leftrightarrow y}^{y\mathcal{R}x}$
4. **$\mathcal{R}$ is Anti-Symmetric**, $\forall x, y \in E$; $(x\mathcal{R}y)$ and $(y\mathcal{R}x) \Longrightarrow x = y$.

Definition 1.4.3 (Equivalence relations)**.** We say that a binary relation $\mathcal{R}$ on a set E is **an equivalence relation** if it is **Reflexive, Symmetric** and **Transitive**.

Definition 1.4.4 (Equivalence class and quotient set)**.** Let E be a set with an equivalence relation $\mathcal{R}$, and x an element of E. We call the equivalence class of x the part of E, denoted $\dot{x}$, consisting of the elements of E equivalent to x, this is the set $\dot{x} = \{y \in E; \quad x\mathcal{R}y\}$.

We call **quotient set** of E by the equivalence relation $\mathcal{R}$, the set of equivalence classes of all the elements of E. This set is noted

$$E_{/\mathcal{R}} = \left\{\dot{x}; \quad x \in E\right\}$$

Example 1.4.5. In $\mathbb{R}$ we define the relation $\mathcal{R}$ by

$$\forall x,\ y \in \mathbb{R}, \quad x\mathcal{R}y \Leftrightarrow x^2 = y^2$$

$\mathcal{R}$ is an equivalence relation.

1. **$\mathcal{R}$ is Reflexive**,

$$\forall x \in \mathbb{R}, \quad x^2 = x^2$$

then

$$\forall x \in \mathbb{R}, \quad x\mathcal{R}x$$

2. **$\mathcal{R}$ is symmetric,** $\forall x,\ y \in \mathbb{R}$;

$$\begin{aligned} x\mathcal{R}y &\Leftrightarrow x^2 = y^2 \\ &\Leftrightarrow y^2 = x^2 \\ &\Leftrightarrow y\mathcal{R}x \end{aligned}$$

3. $\mathcal{R}$ is Transitive, $\forall x, y, z \in \mathbb{R}$;

$$\begin{aligned} x\mathcal{R}y \text{ and } y\mathcal{R}z &\Leftrightarrow x^2 = y^2 \text{ and } y^2 = z^2 \\ &\Leftrightarrow x^2 = z^2 \\ &\Leftrightarrow x\mathcal{R}z \end{aligned}$$

Equivalence class of an element $x \in \mathbb{R}$

$$\begin{aligned} \dot{x} &= \{y \in \mathbb{R}; \quad x\mathcal{R}y\} \\ \dot{x} &= \left\{y \in \mathbb{R}; \quad x^2 = y^2\right\} \\ \dot{x} &= \{y \in \mathbb{R}; \quad x = y \text{ or } x = -y\} \\ \dot{x} &= \{x, -x\} \end{aligned}$$

Quotient set $\mathbb{R}_{/\mathcal{R}}$

$$\begin{aligned} \mathbb{R}_{/\mathcal{R}} &= \left\{\dot{x}; \quad x \in \mathbb{R}\right\} \\ \mathbb{R}_{/\mathcal{R}} &= \{\{x, -x\}; \quad x \in \mathbb{R}\}. \end{aligned}$$

Definition 1.4.6 (Order relation)**.** We say that a binary relation $\mathcal{R}$ on a set E is **an order relation** if it is **Reflexive, Antisymmetric** and **Transitive**. and we note $(E; \mathcal{R})$.

Remark 1.4.7. Order relations are often denoted $\preceq$. If $x \preceq y$, we say that x is less than or equal to y or that y is greater than or equal to x.

Definition 1.4.8. Consider an order relation on a set E.

1. We say that two elements x and y of E are **comparable** if

$$x \preceq y \text{ or } y \preceq x.$$

2. We say that is a relation **of total order**, or that E is **totally ordered**, if all the elements of E are two by two comparable. If not, we say that the relation is **a partial order relation** or that E is **partially ordered**.

Example 1.4.9. Let E be a set and $\mathcal{P}(E)$ be the set of parts of E. We consider on $\mathcal{P}(E)$, the binary relation $\subset$, then $\subset$ is an order relation on E

1. $\subset$ is Reflexive,

$$\forall A \subset \mathcal{P}(E); \quad A \subset A$$

2. $\subset$ is Transitive, $\forall A, B, C \subset \mathcal{P}(E)$;

$$A \subset B \text{ and } B \subset C \Rightarrow A \subset C$$

3. $\subset$ is anti-symmetric, $\forall A, B \subset \mathcal{P}(E)$;

$$A \subset B \text{ and } B \subset A \Rightarrow A = B$$

Example 1.4.10. The division $(\mathbb{N}; \backslash)$ is partially ordered

$$\exists p,\ q \in \mathbb{N}; \quad p \nmid q \text{ and } q \nmid p$$

1.5 Internal composition laws

Definition 1.5.1. Let E be a set. An **internal composition law (ICL)** on E is a map

$$\begin{aligned} * : E \times E &\to E \\ (a, b) &\to a * b, \end{aligned}$$

and we say that $a * b$ is the composite of a and b for the law $*$. A set E provided with an internal composition law constitutes an algebraic structure and denoted $(E, *)$.

Example 1.5.2.
1. The addition defined by $(a, b) \to a + b$ is an internal composition law in $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$.
2. The multiplication defined by $(a, b) \to a \times b$ is an internal composition law in $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$.
3. The composition defined by $(f, g) \to f \circ g$ is an internal composition law on the sets of applications from E to E.

Definition 1.5.3 (Usual properties of laws internal)**.** Let $*$ be an internal law on the set E.

- **Commutativity**: the law $*$ is commutative if

  $$\forall a, b \in E : a * b = b * a.$$

- **Associativity**: the law $*$ is associative if

  $$\forall a, b, c \in E : a * (b * c) = (a * b) * c.$$

- **Neutral element**: the law $*$ admits a neutral element $e \in E$ if

  $$\forall a \in E : a * e = e * a = a.$$

- **Symmetrical element:** An element $a' \in E$ is the symmetric of a in E, if

  $$a * a' = e = a' * a.$$

1.6 Group, Subgroups

Definition 1.6.1 (Groups)**.** A group $(G, *)$ is a set G to which is associated an operation $*$ (the law of composition) verifying the following four properties
1. For all $x, y \in G$, $x * y \in G$ ($*$ is a law of internal composition).
2. For all $x, y, z \in G$, $(x * y) * z = x * (y * z)$ (the law is associative).

3. There exists $e \in G$ such that $\forall x \in G$, $x * e = x$ and $e * x = x$ (e is the neutral element).

4. For all $x \in G$ there exists $x' \in G$ such that $x * x' = x' * x = e$ (x' is the inverse of x and is denoted x^{-1}), if in addition the operation checks for all $x, y \in G$, $x * y = y * x$, we say that G is a **commutative group** (or abelian).

Remark 1.6.2.

- The neutral element e is unique.
- An element $x \in G$ has only one inverse.

Definition 1.6.3 (Subgroups)**.** Let $(G, *)$ be a group. A part $H \subset G$ is a **subgroup** of G if

- $e \in H$,
- For all $x, y \in H$, we have $x * y \in H$,
- For all $x \in H$, we have $x^{-1} \in H$.

Remark 1.6.4. Note that a subgroup H is also a group $(H, *)$ with the law induced by that of G.

Example 1.6.5. $(\mathbb{R}_+^*, \times)$ is a subgroup of $(\mathbb{R}^*, \times)$. Indeed

- $1 \in \mathbb{R}_+^*$,
- If $x, y \in \mathbb{R}_+^*$ then $x \times y \in \mathbb{R}_+^*$,
- If $x \in \mathbb{R}_+^*$ then $x^{-1} = \dfrac{1}{x} \in \mathbb{R}_+^*$.

1.7 Rings, Sub-ring

Definition 1.7.1 (Ring)**.** We call **ring** a set A provided with two internal composition laws, an addition and a multiplication, satisfying the following axioms:

(P1) $(A, +)$ is a commutative group (Abelian). The neutral element for addition in a ring A is denoted by 0_A and is called **null element** of A.

Explicitly, for all $x, y, z \in A$ we have

1. $(x + y) + z = x + (y + z)$,
2. $x + y = y + x$,
3. $x + 0_A = 0_A + x = x$,
4. $x + (-x) = (-x) + x = 0_A$.

(P2) The multiplication $\times$ is associative on A.

So for all $x, y, z \in A$, we have: $x \times (y \times z) = (x \times y) \times z$.

(P3) The multiplication is distributive (on the left and right) compared to the addition on A and admits a neutral different from 0_A, denoted 1_A called **unit element**. Explicitly, for all $x, y, z \in A$, we have

1. $x \times (y + z) = x \times y + x \times z$ and $(x + y) \times z = x \times z + y \times z$,
2. $a \times 1_A = 1_A \times a$.

Definition 1.7.2 (Subring)**.** Let $(A, +, \times)$ be a ring and B be a part of A. We say that B is **subring** of A if

- $(B, +)$ is a subgroup of $(A, +)$.
- B is stable for multiplication, i.e. if $b_1, b_2 \in B$ then $b_1 \times b_2 \in B$.
- The **neutral multiplicative** of A belongs to B, i.e. $1_A \in B$.

Remark 1.7.3. Note that $(B, +, \times)$ itself a ring, with the same neutral multiplicative.

1.8 Body, Subbody

Let $\mathbb{K}$ be a non-empty set with two internal composition laws $+$ and $\times$.

Definition 1.8.1 (Body)**.** We say that $(\mathbb{K}; +; \times)$ is a **body** if

1. $(\mathbb{K}; +; \times)$ is a ring (or unit ring),
2. Any element of $\mathbb{K} - \{0_\mathbb{K}\}$ admits a symmetric for $\times$ in $\mathbb{K}$
If, moreover $\times$ is commutative, then $(\mathbb{K}; +; \times)$ is called an abelian body

Definition 1.8.2 (Subbody)**.** Let $(K; +; \times)$ be a body. A part $\mathbb{K}'$ of $\mathbb{K}$ is a **subbody** of $(\mathbb{K}; +; \times)$ if

1. $\mathbb{K}'$ is a subring of $\mathbb{K}$,
2. $\forall x \in \mathbb{K} - \{0_\mathbb{K}\} : x^{-1} \in \mathbb{K}'$.

Example 1.8.3. Prove that $S = \{x + y\sqrt[3]{3} + z\sqrt[3]{9}\}$ is subbody of $\mathbb{R}$.

Solution: It easy to prove that S is a subring of $\mathbb{R}$.
The multiplication is commutative in $\mathbb{R}$, so $1 = 1 + 0\sqrt[3]{3} + 0\sqrt[3]{9}$ is the neutral element for the multiplication, it is only necessary to verify that for $x + y\sqrt[3]{3} + z\sqrt[3]{9} \neq 0 \in S$, the inverse $\dfrac{x^2 - 3yz}{D} + \dfrac{3z^2 - xy}{D} + \dfrac{y^2 - xz}{D}$, where $D = x^3 + "y^3 + 9z^3 - 9xyz$ is in S.

1.9 Solved exercises

1.9.1 Exercises

1.1

Let A, B, C three parts of a set E. Prove the equalities

a) $A - (B \cup C) = (A - B) \cap (A - C)$

b) $A - (B \cap C) = (A - B) \cup (A - C)$

c) $A \cap (B - C) = (A \cap B) \cap (A - C)$

1.2 In the set $\mathcal{P}(E)$ of parts of a set E, we consider the equation $A \cap X = B$.

a) Indicate a necessary and sufficient condition such that this equation admits solutions.

b) Then solve this equation.

1.3 In the set $\mathcal{P}(E)$ of parts of a set E, we consider the equation $A \cup X = B$.

a) Indicate a necessary and sufficient condition such that this equation admits solutions.

b) Then solve this equation.

1.4

On $\mathbb{R}^2$ we define the relation $\mathcal{R}$ by

$$(x, y)\mathcal{R}(x', y') \Leftrightarrow x = x'$$

Prove that $\mathcal{R}$ is an equivalente relation, then determine the equivalence class of an element $(x_0, y_0) \in \mathbb{R}^2$.

1.5
On $\mathbb{R}^2$ we define the relation $\prec$ by

$$(x, y) \prec (x', y') \Leftrightarrow ((x < x') \text{ or } (x = x' \text{ and } y \leq y'))$$

Prove that $\prec$ is a relationship of order on $\mathbb{R}^2$.

1.6

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = \dfrac{2x}{1 + x^2}$
1. Is f surjective, injective ?
2. Prove that $f(\mathbb{R}) = [-1, +1]$
3. Prove that the restriction $g : [-1, +1] \longrightarrow [-1, +1]$, $g(x) = f(x)$ is bijective.

1.7

Let $\star$ be an internal law on $\mathbb{R}$

$$\forall (x, y) \in \mathbb{R}^2 \qquad x \star y = xy + (x^2 - 1)(y^2 - 1)$$

1. Is $\star$ associative on $\mathbb{R}$? commutative on $\mathbb{R}$? verify that $\mathbb{R}$ admits a neutral element for $\star$? Does this law give $\mathbb{R}$ a group structure?
2. Calculate the symetric of the real 2 for $\star$.
3. Solve the equations $2 \star x = 2$, and $2 \star x = 5$.

1.8

Velocity group in special relativity

Let $G =]-1, 1[$, we define on G tha law $\star$ by

$$\forall (x, y) \in G^2, \qquad x \star y = \frac{x + y}{1 + xy}$$

Show that $(G, \star)$ is a n abelian group.

1.9.2 Solutions

1.1

$$
\begin{aligned}
a)\ A-(B\cup C) &= A\cap(\overline{B\cup C}) = A\cap(\overline{B}\cap\overline{C})\\
&= (A\cap\overline{B})\cap(A\cap\overline{C}) = (A-B)\cap(A-C)\\
b)\ A-(B\cap C) &= A\cap(\overline{B\cap C}) = A\cap(\overline{B}\cup\overline{C})\\
&= (A\cap\overline{B})\cup(A\cap\overline{C}) = (A-B)\cup(A-C)\\
c)\ A\cap(B-C) &= A\cap(B\cap\overline{C}) = (A\cap B)\cap(A\cap\overline{C})\\
&= (A\cap B)\cap(A-C)
\end{aligned}
$$

1.2

a) If there exists a part X of E allowing us to write $A\cap X=B$, this implies $B\subset A$, which is therefore a necessary condition for the equation to have a solution.
If this condition is met, the equation admits the solution $X=B$ since $A\cap B=B$ results from the inclusion $B\subset A$.
The condition found is therefore necessary and sufficient for the equation to have solutions.
b) For a part X of E to be a solution, it is necessary and sufficient for X to be the union of B and a part of E contained in the complement of A ; in other words, it is necessary and sufficient for X to be of the form $B\cup(P\cap\overline{A})$, where P is in $\mathcal{P}(E)$.

1.3
This exercise is treated like the previous one.
a) $A\subset B$.
b) $X=B\cap(P\cup\overline{A})$; where $P\in\mathcal{P}(E)$.

1.4

The relation $\mathcal{R}$ is an equivalence relation, in fact, it is:
1) Reflexive: $(x,y)\mathcal{R}(x,y)$ because $x=x$.
2) Symetric: If $(x,y)\mathcal{R}(x',y')$ then $x=x'$ which is written $x'=x$ and which is equivalent to $(x',y')\mathcal{R}(x,y)$.
3) Transitive: If $(x,y)\mathcal{R}(x',y')$ and $(x',y')\mathcal{R}(x'',y'')$ then $x=x'$ and $x'=x''$ which gives $x=x''$ and which leads $(x,y)\mathcal{R}(x'',y'')$.
Let's find the equivalence class of (x_0,y_0), **i.e.**, determine all couples (x,y) such that $(x,y)\mathcal{R}(x_0,y_0)$ or $(x,y)\mathcal{R}(x_0,y_0)\Leftrightarrow x=x_0$, in another word $x=x_0$ and y is any real

number.
So $Cl((x_0, y_0)) = \{(x_0, y)/y \in \mathbb{R}\}$.

1.5

The order relation,
1) Reflexive: $x = x$ and $y \leq y$ which implies $(x, y) \prec (x, y)$.
2) Antisymetric: If $(x, y) \prec (x', y')$ and $(x', y') \prec (x, y)$ then from the first relation we have necessarily $x = x'$, and from the second relation we have $y \leq y'$ and $y' \leq y$, then $x = x'$ and $y \leq y'$.
3) Transitive: If $(x, y) \prec (x', y')$ and $(x', y') \prec (x'', y'')$
Either $x = x'$ and $x' = x''$ in this case we have $y \leq y'$ and $y' \leq y''$ then $y \leq y''$ and so $(x, y) \prec (x'', y'')$.
Or $x = x'$ and $x' < x''$ in this case, we have $x \leq x''$ and so $(x, y) \prec (x'', y'')$.
Or $x < x'$ and $x' = x''$ in this case we have $x < x''$ and so $(x, y) \prec (x'', y'')$.
1.6
1) f is not injective , because $f(2) = f(\frac{1}{2}) = \frac{4}{5}$.
f is not surjective because $y = 2$ has no antecedent,
in fact, the equation $f(x) = 2$ becomes $2x = 2(1 + x^2)$, then $x^2 - x + 1 = 0$, this equation has no solution in $\mathbb{R}$.
2) The equation $f(x) = y$ is equivalent to equation $yx^2 - 2x + y = 0....(1)$, we have $\triangle = 4 - 4y^2$ so the equation (1) has no solution except if $y \in [-1, +1]$, so we have exactly $f(\mathbb{R}) = [-1, +1]$.
3) Let $y \in]1, 1[-0$. The possible solutions of the equation $g(x) = y$ are $x = \frac{1 - \sqrt{1 - y^2}}{y}$ or $x = \frac{1 + \sqrt{1 - y^2}}{y}$. The second solution does not belong to $[1, 1]$ (It is strictly greater than 1 if $y > 0$, and strictly less than 1 if $y < 0$). On the other hand, $x = \frac{1 - \sqrt{1 - y^2}}{y} = \frac{y}{1 + \sqrt{1 - y^2}}$ is in $[1, 1]$. In fact,

$$1 \leq 1 + \sqrt{1 - y^2} \Rightarrow 0 < \frac{1}{1 + \sqrt{1 - y^2}} \leq 1$$

while $1 < y < 1$. We can deduce

$$-1 \leq \frac{-1}{1 + \sqrt{1 - y^2}} \leq x \leq \frac{1}{1 + \sqrt{1 - y^2}} \leq 1.$$

On the other hand, if $y = 1$, the equation $g(x) = 1$ has the only solution $x = 1$ while if $y = 1$, the equation $g(x) = 1$ has the only solution $x = 1$. Finally, if $y = 0$, the equation $g(x) = 0$ admits as the only solution $x = 0$.
In all cases, we have proved that for all $y \in [1, 1]$, the equation $g(x) = y$ admits a

unique solution with $x \in [1, 1]$. We have clearly proved that g is a bijection. Of course, a method of analysis, using continuity and strict monotonicity of g, would be easier.

1.7

Let us first notice that $\star$ clearly defines an internal composition law on $\mathbb{R}$ because the two usual operations $+$ and $\times$ are themselves internal composition laws on $\mathbb{R}$
1. The law $\star$ is not associative on $\mathbb{R}$ because $2 \star (3 \star 4) = 52533 \neq (2 \star 3) \star 4 = 13605$. It is on the other hand commutative on $\mathbb{R}$ Let's check it. Let x and y be two real numbers. The multiplication being commutative on $\mathbb{R}$, we have:

$$x \star y = xy + (x^2 - 1)(y2 - 1) = yx + (y^2 - 1)(x^2 - 1) = y \star x.$$

The commutativity property of the law $\star$ is deduced from that of the two usual laws $+$ and $\times$. We notice that $1 \star x = x = x \star 1$ for all real x. The neutral element is therefore the real 1. Of course, $\mathbb{R}$ does not have a group structure for the law $\star$ since the law is not associative on $\mathbb{R}$.
2. If s is a symmetric of element 2 for the law $\star$ in $\mathbb{R}$, it then verifies: $s \star 2 = 1 = 2 \star s$. Calculating the real s amounts to looking for solutions to the equation: $3s^2 + 2s - 4 = 0$. The real 2 has two symmetrical ones for the law $\star$ů These are the two reals: $\dfrac{-1 + \sqrt{13}}{3}$, $\dfrac{-1 - \sqrt{13}}{3}$.
3. The equation $2 \star x = 2$ (of unknown x) admits solutions 1 and $-5/3$; the equation $2 \star x = 5$ (of unknown x) admits for solutions $4/3$ and -2.
1.8

We have $\text{th}(u + v) = \dfrac{\text{th}u + \text{th}v}{1 + \text{th}u\text{th}v}$, (see chapter 4). So we put $x = \text{th}u$ and $y = \text{th}v$, and we have , $x \star y = \text{th}(u + v) = \text{th}(\text{argth}x + \text{argth}y)$
We deduce that:
- The law $\star$ is internal, since a hyperbolic tangent belongs to $]1, 1[$.
- $(x \star y) \star z = \text{th}(\text{argth}x + \text{argth}y) + \text{argth}z) = x \star (y \star z)$.
- 0 is a neutral element.
- The opposite of x is also its inverse for $\star$.

Direct method: We have

$$(\forall x \in \mathbb{R},\ x \in]-1, 1[) \Leftrightarrow (|x| < 1)$$

, from which we deduce

$$\forall (x, y) \in]-1, 1[^2,\ |xy| < 1,$$

consequently $1 + xy > 0$.
then $\dfrac{x + y}{1 + xy} < 1$ is equivalent to $x + y < 1 + xy$, that's to say $(1 - x)(1 - y) > 0$, this

inequality is verified for all x, y elemnts of $]-1,1[$.
We show, in an analogous manner, that

$$\forall (x,y) \in]-1,1[^2, \ \frac{x+y}{1+xy} > -1.$$

The law $\star$ is then internal in $]-1,1[$.
This law is obviously commutative and admits 0 as a neutral element. Any real element x admits $-x$ as symmetrical with respect to this law.
Moreover, we have $\forall (x,y,z) \in]-1,1[^3$,

$$x \star (y \star z) = \frac{x + y \star z}{1 + x(y \star z)} = \frac{x + \frac{y+z}{1+yz}}{1 + x.\frac{y+z}{1+yz}} = \frac{x+y+z+xyz}{1+xy+yz+zx} = (x \star y) \star z.$$

It follows that the law $\star$ is associative and that $(]-1,1[, \star)$ is a group.

Chapter 2

Numbers

2.1 Introduction

In the very beginning, human life was simple. An early ancient herdsman compared sheep of his herd with a pile of stones when the herd left for grazing and again on its return for missing animals In the earliest systems probaly the vertical strokes or bars such as I, II, III etc.. were used for numbers 1, 2, 3, etc. The symbol " IIII" was used by many people including the ancient egypcians for the number of fingers of one hand. Around 5000 B.C, the Egypcians has a number system based on 10. The Egyptians had very few signs (hieroglyphs) to count:

𓏺 : représents 1

𓎆 : représents 10

𓍢 : représents 100

𓆼 : représents 1000

𓂭 : représents 10000

𓆐 : représents 100000

𓁨 : représents 1000000

Their system is called "additive", like the Greeks and Romans: we "add the signs" to obtain the desired number. For example, what does :

Different people invented their own symbols for numbers. But these systems of otations proved to be inadequate with advencement of societies and were discarded. Ultimatly the set $\{1, 2, 3, 4, ...\}$ was adopted as the counting set (also called the set of natural numbers). The solution of the equation $x+2 = 2$ was not possible in the set of natural numbers. So the natural number system was extented to the set of whole numbers. No number in the set of whole numbers W could satisfy the equation $x + 4 = 2$ or $x+a = b$ if $a > b$, and $a, b \in W$. The negative integers $-1, -2, -3, ...$ were introduced to form the set of integers $\mathbb{Z} = \{0 \mp 1, \mp 2, ...\}$.
Again the equation of type $2x = 3$ or $bx = a$ where $a, b \in \mathbb{Z}$ and $b \neq 0$ had no solution in the set $\mathbb{Z}$., so the numbers of the form $\frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$, were invented to remove such difficulties. The set $\mathbb{Q} = \{\frac{a}{b}/a, b \in \mathbb{Z}$ with $b \neq 0\}$ was named as the set of rational numbers. Stoll the solution of equations such as $x^2 = 2$ or $x^2 = a$ (where a is not a perfect square) was not possible in the set $\mathbb{Q}$. So the irrational numbers of the type $\mp\sqrt{2}$ or $\mp\sqrt{a}$ where a is not a perfec square were introduced. This process enlargment of the number system ultimately led to the set of real numbers $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$ (the set $\mathbb{Q}'$ is the set of irrational numbers) which is used most frequently in everyday life.

2.2 Rational numbers and irrational numbers

We know that a rational number is a number which can be put in the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ with $q \neq 0$. The numbers $\sqrt{16}, 3.7, 4$ etc., are rational numbers. $\sqrt{16}$ can be reduced to the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ with $q \neq 0$, because $\sqrt{16} = 4 = \frac{4}{1}$.
Irrational numbers are those numbers which cannot be pu into the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ with $q \neq 0$. The numbers $\sqrt{2}, \sqrt{3}, \frac{7}{\sqrt{5}}, \sqrt{\frac{5}{16}}$ are irrational numbers.

2.2.1 Decimal representation of rational and irrational numbers

1) Terminating decimals: A decimal which has only a finite numbers of digits in its decimal part, is called a terminating decimal. Thus 202.04, 0.0000415, 100000.41237895 are examples of terminating decimals.

Since a terminating decimal can be converted into a common fraction, so every terminating decimal represents a **rational number.**
2) Recurring decimals: This is another type of **rational numbers.** In general, a recurring or periodic decimal is a decimal in which one or more digits repeat indefinitely.
It will be showed that a recurring decimal can be converte into a common fraction. So **every recurring decimal represents a rational number.**
A non-terminating decimal is a decimal which neither terminates nor it is recurring. It is not possible to convert such a decimal into a common fraction. Thus a **non-terminating , non-recurring decimal represents an irrational number.**

2.2.2 Example

1. $.25(=\frac{25}{100})$ is a rational number.

2. $.3333...(=\frac{1}{3})$ is a recurring decimal, it is a rational number.

3. $2.\overline{3}(=2.333...)$ is a rational number .

4. $0.142857142857....(=\frac{1}{7})$ is a rational number.

5. 0.01001000100001.... is a non-terminating number, non-periodic decimal, so it is an irrational number.

6. 214.121122111222111112222.... is also an irrational number.

7. 1.4142135.... is an irrational number.

8. 7.3205080.. is an irrational number.

9. 1.709975947... is an irrational numbber.

10. 3.141592654.... is an important irrational number called it π(pi) which denotes the constant ratio of the circumference of any circle to the length of its diameter. An approximate value of π is $\frac{22}{7}$, a better approximation is $\frac{355}{113}$ and a still better approximation is 3.14159. The value of π correct to 5 lac decimal places has been determined with the help of computer.

2.3 Real numbers and thier properties

We are already familiar with the set of real numbers and most of thier propreties. We now state them in a unified and systematic manner. Before stating them we give a preliminary definition.
Taken togather, the rational numbers (recurring decimals) and irrational numbers (non-recurring decimales) from the set of **real numbers**, denoted by $\mathbb{R}$.
We now state several propreties of $\mathbb{R}$, with which we will alrady familiar, although we may not have met thier names before. These properties are used frequently in Analysis, and we do not always refer to them explicitly by name.

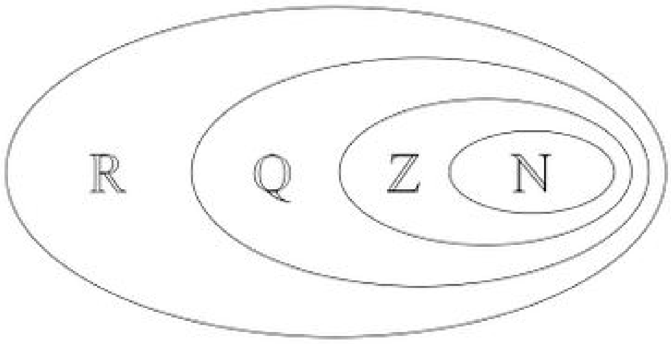


Figure 2.1: $\mathbb{R} \supset \mathbb{Q} \supset \mathbb{Z} \supset \mathbb{N}$

2.3.1 Properties

Property (R1): These are the properties that we have always practiced.
For $a, b, c \in \mathbb{R}$ we have

- $a + b = b + a$
- $0 + a = a$
- $a + b = 0 \Leftrightarrow a = -b$
- $a + (b + c) = (a + b) + c$
- $a \times (b + c) = a \times b + a \times c$
- $a \times b = b \times a$
- $a \times 1 = a$ si $a \neq 0$
- $a \times b = 1 \Leftrightarrow a = \dfrac{1}{b}$
- $a \times (b \times c) = (a \times b) \times c$
- $a \times b = 0 \Leftrightarrow a = 0$ ou $b = 0$

Property (R2): The relation $\leq$ on $\mathbb{R}$ is an order relation, and moreover, it is total.

We so we

- $\forall x \in \mathbb{R},\ x \leq x$ then xRx,
- $\forall x, y \in \mathbb{R}$; if xRy and yRx then $x = y$,
- $\forall x, y, z \in \mathbb{R}$; if xRy and yRz then xRz.

For $(x, y) \in \mathbb{R} \times \mathbb{R}$ by definition we have

$$\begin{aligned} x &\leq y \iff y - x \in \mathbb{R}_+ = \{x \in \mathbb{R}; \quad x \geq 0\}, \\ x &< y \iff x \leq y \text{ and } x \neq y. \end{aligned}$$

The operations of $\mathbb{R}$ are compatible with the order relation $\leq$ in the following sense, for real numbers a, b, c, d

$$\begin{aligned} (a \leq b \text{ and } c \leq d) &\implies a + c \leq b + d \\ (a \leq b \text{ and } c \geq 0) &\implies a \times c \leq b \times c \\ (a \leq b \text{ and } c \leq 0) &\implies a \times c \geq b \times c \end{aligned}$$

We define the **maximum** and the **minimum** of two real values a and b by

$$\max(a, b) = \begin{cases} a & \text{if } a \geq b \\ b & \text{if } a < b \end{cases}$$

$$\min(a, b) = \begin{cases} b & \text{if } a \geq b \\ a & \text{if } a < b \end{cases}$$

Property (R3): Let $x \in \mathbb{R}$, there exists a unique relative integer, **the integer part** denoted $E(x)$, such that

$$E(x) \leq x \leq E(x) + 1$$

We also note $E(x) = [x]$.
Example 2.3.1. We have

- $E(2,853) = 2, \quad (2 \leq 2.853 \leq 3)$
- $E(\pi) = 3, \quad (3 \leq \pi \leq 4)$
- $E(-\pi) = -4, \quad (-4 \leq -\pi \leq -3)$

Property (R4):

1. Trichotomy Property: If $a, b \in \mathbb{R}$, then exactly one of thefollowing inequalities holds

$$a > b \quad \text{or} \quad a = b \quad \text{or} \quad a < b.$$

2. Transitive Property: If $a, b, c \in \mathbb{R}$, then

$$a < b \qquad \text{and} \qquad b < c \Longrightarrow a < c.$$

3. Archimedean Property: If $a \in \mathbb{R}$, then there is a positive integer n such that

$$n > a.$$

4. Density Property: $a, b \in \mathbb{R}$ and $a < b$, then there is a rational number x and an irrational number y such that

$$a < x < b \qquad \text{and} \qquad a < y < b.$$

2.3.2 Solving inequalities

Solving an inequality involving an unknown real number x means determining those values of x for which the given inequality holds; that is, finding the solution set of the inequality. We can often do this by rewriting the inequality in an equivalent, but simpler form, using the rules given in the last sub-section.

Example 2.3.2. Solve the inequality $\dfrac{x+2}{x+4} > \dfrac{x-3}{2x-1}$

Solution We rearrange this inequality to give a somewhat simpler inequality, using

$$\begin{aligned}\frac{x+2}{x+4} > \frac{x-3}{2x-1} &\Leftrightarrow \frac{x+2}{x+4} - \frac{x-3}{2x-1} > 0\\ &\Leftrightarrow \frac{x^2+2x+10}{(x+4)(2x-1)} > 0\\ &\Leftrightarrow \frac{(x+1)^2+9}{(x+4)(2x-1)} > 0.\end{aligned}$$

Now, the numerator is always positive. The denominator vanishes when $x = -4$ or $x = \frac{1}{2}$. By examining separately the sign of the denominator when $x < -4$, $-4 < x < \frac{1}{2}$ and $x > \frac{1}{2}$, we can deduce that the last fraction is positive precisely when $x < -4$ or $x > \frac{1}{2}$. Hence the solution set of the original inequality is $S =]-\infty, 4[\cup]\frac{1}{2}, +\infty[$.

Definition 2.3.3 (Absolute value)**.** For a real number x, we define **the absolute value** of x by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Thus, the representative curve of the absolute value function is the next

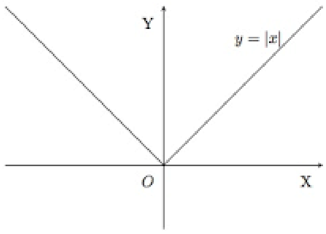


Figure 2.2: Absolute function

Proposition 2.3.4. $\forall x, y \in \mathbb{R}$; *we have*

1. $|x| \geq 0; \quad |-x| = |x|\,; \quad |x| > 0 \iff x \neq 0$

2. $\sqrt{x^2} = |x|$

3. $|x \times y| = |x| \times |y|$

4. $|x + y| \leq |x| + |y| \qquad$ (***Triangle inequality***)

5. $||x| - |y|| \leq |x - y| \qquad$ (***Second triangle inequality***)

Example 2.3.5. Solve the inequality $\mid x - 2 \mid < 1$
Solution We have

$$\begin{aligned} \mid x - 2 \mid < 1 &\Leftrightarrow -1 < x - 2 < 1 \\ &\Leftrightarrow 1 < x < 3. \end{aligned}$$

Example 2.3.6. Solve the inequality $\mid x - 2 \mid \leq \mid x + 1 \mid$
Solution We have

$$\begin{aligned} \mid x - 2 \mid \leq \mid x + 1 \mid &\Leftrightarrow (x-2)^2 \leq (x+1)^2 \\ &\Leftrightarrow x^2 - 4x + 4 \leq x^2 + 2x + 1. \\ &\Leftrightarrow 3 \leq 6x. \\ &\Leftrightarrow \frac{1}{2} \leq x. \end{aligned}$$

So the solution set of the original inequality is $S = [\frac{1}{2}, +\infty[$.

Example 2.3.7. Use the Triangle Inequality to prove that:
a) $\mid a \mid \leq 1 \Longrightarrow \mid 3 + a^3 \mid \leq 4$
b) $\mid b \mid \leq 1 \Longrightarrow \mid 3 - b \mid > 2$
Solution *a*)Suppose that $\mid a \mid \leq 1$. The triangle inequality then gives

$$\begin{aligned} \mid 3 + a^3 \mid &\leq \mid 3 \mid + \mid a^3 \mid \\ &= 3 + \mid a \mid^3 \\ &\leq 3 + 1 (\text{since} \mid a \mid < 1) \\ &\leq 4 \end{aligned}$$

b) Suppose that $\mid b \mid< 1$. The ' reverse form' of the triangle inequality then gives

$$\begin{aligned}|3 - b \mid &\geq \mid\mid 3 \mid - \mid b \mid \mid\\ &= |3 - \mid b \mid \mid\\ &\geq 3 - \mid b \mid\end{aligned}$$

Now $\mid b \mid< 1$, so that $- \mid b \mid> -1$. Thus $3- \mid b \mid> 3 - 1 = 2$.
And we can then deduce from the previous chain of inequalities that$\mid 3 - b \mid> 2$. So the solution set of the original inequality is $S = [\frac{1}{2}, +\infty[$.

Let $a, b \in \mathbb{R}$ be such that $a < b$, the intervals are parts of the totally ordered set $(\mathbb{R}; \leq)$.

Definition 2.3.8 (Intervals)**. 1.** We call **closed interval** of origin a and end b, the set defined as follows

$$[a, b] = \{x \in \mathbb{R}; \quad a \leq x \leq b\}.$$

2. We call **open interval** of origin a and end b, the set defined as follows

$$]a, b[= \{x \in \mathbb{R}; \quad a < x < b\}.$$

3. We call interval **semi-open on the right** of origin a and end b, the set defined as follows

$$[a, b[= \{x \in \mathbb{R}; \quad a \leq x < b\}.$$

4. We call interval **semi-open on the left** of origin a and end b, the set defined as follows

$$]a, b] = \{x \in \mathbb{R}; \quad a < x \leq b\}.$$

Definition 2.3.9. It is often practical to add the two ends to the number line

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$$

Definition 2.3.10. Let a be a real number, $V \subset \mathbb{R}$ a subset. We say that V is a **neighborhood** of a if there exists an open interval I such as $a \in I \subset V$.

Remark 2.3.11. The notion of neighborhood will be useful for limits.

2.4 Least upper bounds and greatest lower bounds

2.4.1 Upper and lower bounds

Definition 2.4.1 (Maximum, minimum)**.** Let A be a non-empty part of $\mathbb{R}$

1. A real M is a **maximum** or (**largest element**) of A if

$$M \in A \text{ and } \quad \forall x \in A; \quad x \leq M.$$

If it exists, the largest element is unique, we then note it $\max A$.

2. A real m is a **minimum** or (**smallest element**) of A if

$$m \in A \text{ and } \quad \forall x \in A; \quad x \geq m.$$

If it exists, the smallest element is unique, we then note it $\min A$.

Remark 2.4.2. It should be kept in mind that the largest element or the most small element do not always exist.

Example 2.4.3. • $\min\,]0,5]$ does not exist and $\max\,]0,5] = 5$,

- The interval $]a,b[$ has no largest element, nor smallest element.
- $\min(]1,2] \cup \{6\})$ does not exist and $\max(]1,2] \cup \{6\}) = 6$.

Definition 2.4.4 (Upper and lower bounds)**.** Let A be a non-empty part of $\mathbb{R}$

1. A real M is an **upper bound** of A if $\forall x \in A; \quad x \leq M$.

2. A real m is a **lower bound** of A if $\forall x \in A; \quad x \geq m$.

Example 2.4.5. • 3 is an upper bound of $A = \,]0,2[$, in addition the upper bounds of A are exactly the elements of $[2,+\infty[$.

- -3 is a lower bound of $A = \,]0,2[$, in addition the lower bounds of A are exactly the elements of $]-\infty,0]$.
- the lower bounds of $A = \,]-2,+\infty[$ are exactly the elements of $]-\infty,-2]$ . .

If an upper bound (resp. a lower bound) of A exists we say that A is **bounded above** (resp. **bounded below**).

there is not always an upper or lower bound, in addition we do not have uniqueness.

Definition 2.4.6 (Similar terminology applies to functions)**.** A function f defined on an interval $I \subset \mathbb{R}$ is said to:

- be **bounded above** by M if $f(x) \leq M$ for all $x \in I$; M is a an **upper bound** of f;
- be **bounded below** by m if $f(x) \geq m$ for all $x \in I$; m is a **lower bound** of f;
- have a **maximum** (or **maximum value**) M if M is an upper bound of f and

$f(x) = M$, for at least one $x \in I$;
• have a **minimum** (or **minimum value**) m if m is a Lower bound of f and $f(x) = m$, for at least one $x \in I$;

Example 2.4.7. Let f be the function defined by $f(x) = x^2, x \in [\frac{1}{2}, 3[$. Determine whether f is bounded above or below, and any maximum or minimum value of f.

Solution First, f is increasing on the interval $[\frac{1}{2}, 3[$, so that since $\frac{1}{2} < x < 3$ it follows that $\frac{1}{4} < f(x) < 9$. Hence f is bounded above and bounded below.
Next, since $f(\frac{1}{2}) = \frac{1}{4}$ and $\frac{1}{4}$ is a lower bound for f on the interval $[\frac{1}{2}, 3[$, it follows that f has a minimum value of $\frac{1}{4}$ on this interval.
Finally, 9 is an upper bound for f on the interval $[\frac{1}{2}, 3[$, but there is no point $x \in [\frac{1}{2}, 3[$ for which $f(x) = 9$. 9 cannot be a maximum for f on the interval.

2.4.2 Least upper bounds, greatest lower bounds

We have seen that the interval $[0, 2]$ has a maximum element 2, but $[0, 2[$ has no maximum element. However, the number 2 is rather like a maximum element of $[0, 2[$, because 2 is an upper bound of $[0, 2[$ and any number less than 2 is not an upper bound of $[0, 2[$. In other words, 2 is the least upper bound of $[0, 2[$.

Definition 2.4.8 (Least upper bound)**.** A real number M is the least upper bound, or supremum, of a set $E \subseteq \mathbb{R}$ if:
1. M is an upper bound of E;
2. if $M' < M$, then M' is not an upper bound of E. In this case, we write $M = \sup E$.

If E has a maximum element, $\max E$, then $\sup E = \max E$. For example, the closed interval $[0, 2]$ has least upper bound 2. We can think of the least upper bound of a set, when it exists, as a kind of generalised maximum element.
If a set does not have a maximum element, but is bounded above, then we may be able to guess the value of its least upper bound. As in the case $E = [0, 2[$, there may be an obvious missing point at the upper end of the set. However it is important to prove that your guess is correct. We now show you how to do this.

Example 2.4.9. Prove that the least upper bound of $[0, 2[$ is 2.

Solution We know that $M = 2$ is an upper bound of $[0, 2[$, because

$$x \leq 2 \text{ for all } x \in [0, 2[.$$

To show that 2 is the least upper bound, we must prove that each number $M' < 2$ is not an upper bound of $[0,2[$. To do this, we must find an element $x \in [0,2[$ which is greater than M'. But, if $M' < 2$, then there is a real number x such that $M' < x < 2$ and also $0 < x < 2$:
Since $x \in [0,2[$, the number M' cannot be an upper bound of $[0,2[$. Hence $M = 2$ is the least upper bound, or supremum, of $[0,2[$.

Similarly, we define the notion of a greatest lower bound.

Definition 2.4.10 (Greatest lower bound)**.** A real number m is the greatest lower bound, or infimum, of a set $E \subseteq \mathbb{R}$ if:
1. m is an lower bound of E;
2. if $m' > m$, then m' is not an upper bound of E. In this case, we write $m = \inf E$.

Example 2.4.11. • The upper bounds of $A =]0,1]$ are the elements of $[1,+\infty[$. Then the least upper bound is 1, the least bounds of $A =]0,1]$ are the elements of $]-\infty,0]$. Then the greatest lower bound is 0.

- $]0,+\infty[$ has not least upper bound, and $\inf]0,+\infty[= 0$.

Théorème 2.4.12. *Any non-empty part of $\mathbb{R}$ and bounded above admits an upper bound. In the same way, any non-empty part of $\mathbb{R}$ and bounded below admits a lower bound.*

An equivalent characterization of sup and inf by real sequences.

Théorème 2.4.13. ***1.*** *$M = \sup A$ if and only if M is an upper bound of A and it exists a sequence $(a_n)_{n\in\mathbb{N}}$ in A which converges to M.*

2. *$m = \inf A$ if and only if m is a lower bound of A and it exists a sequence $(a_n)_{n\in\mathbb{N}}$ in A which converges to m.*

Proposition 2.4.14. *Let A and B be two non-empty and bounded parts of $\mathbb{R}$; We have the following assertions*

1. $A \subset B \implies \sup A \le \sup B$ *and* $\inf B \le \inf A$,
2. $\sup A \cup B = \max\{\sup A, \ \sup B\}$,
3. $\inf A \cup B = \min\{\inf A, \ \inf B\}$.

Exercise 1. Study the existence of the minimum, maximum, lower bound and upper bound of the following parts of $\mathbb{R}$

$$A = [-1,2[\cap\mathbb{Q}, \quad B = \{-3n; \quad n \in \mathbb{N}\}.$$

Solution 2.4.15. **1.** A is the set of rationals contained in $[-1,2[$, we have $\inf A = \min A = -1$. We consider the sequence $a_n = 2 - \frac{1}{n} \in A$, we have $\lim\limits_{n\longrightarrow\infty} a_n = 2$ then $\sup A = 2$ but $\max A$ does not exist since $2 \notin A$.

2. $\sup B = \max B = 0$ (for $n = 0$) but $\inf B$ and $\min B$ do not exist ($\mathop{lim}\limits_{n\to+\infty} -3n = -\infty$).

2.5 Reasoning by recurrence

Proof by induction (Recursive reasoning): It is a way to prove that something is true for a sequence of numbers. This methode consists of proving the following points.

Let $P(n)$ be the property we want to demonstrate. To prove that $P(n)$ is true for any integer $n \geq k$, (k can be 0 or 1 or 2 or...), we proceed in three steps:

Step1 (Initialization): We show that $P(k)$ is true for $k = 0$ or, 1, or 2.
Step2 (Heredity): We assume that $P(n)$ is true and we show that $P(n+1)$ is still true.
Step3 (Conclusion): Once the two previous steps are established, we conclude that the property $P(n)$ is true for all $n \geq k$.
Example 2.5.1. Prove the following property by induction

$$\forall n \geq 1; \quad 1+2+3+\cdots+n = \frac{n\,(n+1)}{2}.$$

Consider $P\,(n) : 1+2+3+\cdots+n = \dfrac{n\,(n+1)}{2}$.
Step1 (Initialization): $P(1)$ is true since for $n = 1$, we have

$$1 = \frac{1\,(1+1)}{2},$$

Step2 (Heredity): We assume that $P(n)$ true, i.e.,

$$1+2+3+\cdots+n = \frac{n\,(n+1)}{2}$$

and we prove that $P(n+1)$ is true. Indeed

$$\begin{aligned}
1+2+3+\cdots+(n+1) &= 1+2+3+\cdots+n+(n+1) \\
&= \frac{n\,(n+1)}{2} + (n+1) \\
&= \frac{(n+1)\,(n+2)}{2}
\end{aligned}$$

So $P(n+1)$ is true.
Step3 (Conclusion): We conclude that $P(n)$ is true for all $n \geq 1$.

Théorème 2.5.2. ***(Bernoullis Inequality)*** *For any real number $x \geq -1$ and any natural number n, $(1+x)^n \geq 1+nx$.*

Proof. Let $P(n)$ be the statement

$$P(n) : (1+x)^n \geq 1 + nx, \text{ for all } x \geq -1$$

Step 1 First we show that $P(1)$ is true: $(1+x)^1 \geq 1 + 1x$. This is obviously true.
Step 2 We now assume that $P(k)$ holds for some $k \geq 1$, and prove that $P(k+1)$ is then true.
So, we are assuming that $(1+x)^k \geq 1+kx$, for all $x \geq -1$. Multiplying this inequality by $(1+x)$, we get

$$\begin{aligned}(1+x)^{k+1} &\geq (1+x)(1+kx)\\ &= 1+(k+1)x+kx^2\\ &\geq 1+(k+1)x\end{aligned}$$

Thus, we have $(1+x)^{k+1} \geq 1+(k+1)x$; in other words the statement : $P(k+1)$ holds.
So, $P(k)$ true for some $k \geq 1 \Longrightarrow P(k+1)$ is true.
It follows, by the Principle of Mathematical Induction, that $(1+x)^n \geq 1+nx$, for all$x \geq -1, n \geq 1$
Step 3 (Conclusion): We conclude that $P(n)$ is true for all $n \geq 1$. □

2.6 Solved exercises

2.6.1 Exercises

2.1
Arrange the following numbers in increasing order:

(a) $\frac{7}{36}, \frac{3}{20}, \frac{7}{6}, \frac{7}{45}, \frac{11}{60}$

(b) $0.\overline{465}, 0.4\overline{65}, 0.46\overline{5}, 0.4655, 0.4656$

2.2
Find the fraction whose decimal expansion are:

(a) $0.\overline{231}$, (b) $2.2\overline{81}$

2.3
Let $x = 0.\overline{21}$ and $y = 0.\overline{2}$ find $x+y$ and xy on decimal form.

2.4
Prove that:

$$\sqrt{a^2+b^2} \leq a+b, \quad \text{for} \quad a,b \geq 0$$

2.5
Prove the inequalities

$$(a) \quad 2^n \geq 1+n, \quad (b) \quad 2^{\frac{1}{n}} \leq 1+\frac{1}{n}$$

2.6
Let $E_1 = \{x : x \in \mathbb{Q}, 0 \leq x < 1\}$ and $E_2 = \{\left(1+\frac{1}{n}\right)^2 : n = 1, 2, ...\}$

1. Show that each of the sets E_1 and E_2 is bounded above. Which of them has a maximum element?
2. Chow that each of the sets E_1 and E_2 is bounded below. Which of them has a minimum element?
3. Determine the least upper bound of each of the sets E_1 and E_2.
4. Determine the greatest lower bound of each of the sets E_1 and E_2.

2.7
For each of the following functions, determine whether it has a maximum or a minimum, and determine its supremum and infimum:
(a) $f(x) = \dfrac{1}{1+x^2}$, $x \in [0,1[$, (b) $f(x) = 1-x+x^2$, $x \in [0,2[$.

2.6.2 Solutions

2.1
(a) We have $\frac{7}{36} = 0.19\overline{4}$, $\frac{3}{20} = 0.15$, $\frac{7}{6} = 1.1\overline{6}$, $\frac{7}{45} = 0.1\overline{5}$, $\frac{11}{60} = 1.18\overline{3}$

Then: $\frac{3}{20} < \frac{7}{45} < \frac{11}{60} < \frac{7}{36} < \frac{7}{6}$.

(b) We have $0.\overline{465} < 0.46\overline{5} < 0.4\overline{65} < 0.4655 < 0.46565$.

2.2
(a) First we find the fraction x such that $x = 0.\overline{231}$ if we multiply both sides of this equation by 10^3 (because the recuring block has length 3)

$$1000x = 231.\overline{231} = 231 + x$$

Hens

$$999x = 231 \Longrightarrow x = \frac{231}{999} = \frac{7}{333}$$

$b)$ Let $x = 0.\overline{81}$
Multiplying bboth sides by 10^2 we obbtain

$$100 = 81x = 81 + x$$

Hence

$$99x = 81 \Longrightarrow x = \frac{81}{99} = \frac{9}{11}$$

Thus

$$2.2\overline{81} = 2 + \frac{2}{10} + \frac{9}{110} = \frac{259}{110}$$

2.3 First we write the numbers x and y in the fractional form, we obtain $x = \dfrac{43}{99}$ and $y = \dfrac{2}{9}$
then $x + y = \dfrac{43}{99} = 0.\overline{43}$ and $xy = \dfrac{42}{891} = 0\overline{047138}$.
2.4 We tackle this inequality using the various rearrangement rules and a chain of equivalent inequalities until we obtain an inequality that we know must be true

$$\begin{aligned}\sqrt{a^2+b^2} \leq a+b &\Leftrightarrow a^2+b^2 \leq (a+b)^2 \\ &\Leftrightarrow a^2+b^2 \leq a^2+2ab+b^2 \\ &\Leftrightarrow 0 \leq 2ab\end{aligned}$$

This final inequality is certainly true, since $a, b \geq 0$. It follows that the original inequality $\sqrt{a^2+b^2} \leq a+b$, is also true for $a, b \geq 0$.
2.5
(a) By the Binomial Theorem for $n \geq 1$

$$\begin{aligned}(1+x)^n &= 1 + nx + \frac{n(n-1)}{2!}x^2 + ... + x^n \\ &\geq 1 + nx\end{aligned}$$

Then, if we substitute $x = 1$ in this last inequality, we get

$$2^n \geq 1 + n \text{ for } n \geq 1.$$

(b) We start by rewriting the required result in an equivalent form:

$$2^{\frac{1}{n}} \leq 1 + \frac{1}{n} \Leftrightarrow 2 \leq \left(1 + \frac{1}{n}\right)^n \quad \text{by the power rule}$$

Now, if we substitute $x = \frac{1}{n}$ in the Binomial Theorem for $(1+x)^n$, we get

$$\begin{aligned}(1+\frac{1}{n})^n &= 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + ... + \left(\frac{1}{n}\right)^n \\ &\geq 1+1=2\end{aligned}$$

Since the inequality $2 \leq \left(1+\frac{1}{n}\right)^n$ for $n \geq 1$ is true, it follows that the original inequality $2^{\frac{1}{2}} \leq 1 + \frac{1}{n}$ for $n \geq 1$, is also true, as required.

2.6
For E_1
- E_1 is bounded above by the elements of the set $]1, +\infty[$
- E_1 is bounded below by the elements of $]-\infty, 0]$
- The least upper bound of E_1 is 1, the greatest lower bound is 0
- E_1 admits 0 as a minimum.
- E_1 has no maximum.

For E_2
- E_2 is bounded above by the elements of the set $]4, +\infty[$
- E_2 is bounded below by the elements of $]-\infty, 1]$
- The least upper bound of E_2 is 4, the greatest lower bound is 1
- E_2 admits 4 as a maximum.
- E_2 has no minimum.

2.7
For $f(x) = \frac{1}{1+x^2},\ x \in [0, 1[$
- f is bounded above by the elements of $[1, +\infty[$
- f has 1 as a maximum and supremum in the same time.
- f is bounded below by the elements of the set $]-\infty, 1[$
- f has no minimum.

For $f(x) = 1 - x - x^2,\ x \in [0, 2[$
- f is bounded above by the elements of the set $]3, +\infty[$
- f has no maximum and no supremum.
- f is bounded below by the elements of the set $]-\infty, \frac{3}{4}]$.
- f has $\frac{3}{4}$ as a minimum and infemum in the same time.

Chapter 3

Real functions with a real variable

3.1 Introduction

A technical definition of a what is a function in math is a relation from a set of inputs to a set of possible outputs where each input is related to exactly one output. Typical examples of function in math are from integers to integers or from the real numbers to real numbers.

In addition, it is a relation or a process which connects each element x of a set X to the domain of the function and to a single element y of another set Y (usually the same set), the codomain of the function.

Suppose if we call the function f, then we can denote this relation as $y = f(x)$ (read f of x), the element x is the argument or input of the function, and y denotes the value of the function, the output, or the image of x by f. Let us study what is a function in math in detail.

3.2 Notions of function

3.2.1 Definitions

Definition 3.2.1. A **function** of a real-valued real variable is a map f defined on $\mathbb{R}$ or a part D of $\mathbb{R}$ whth values in $\mathbb{R}$, we write $f : D \to \mathbb{R}$. In general, D is an interval or a interval meeting. We call D the **domain of definition** of function f.

Definition 3.2.2. The **graph** of function $f : D \to \mathbb{R}$ is a part of $\mathbb{R}^2$ defined by

$$\mathcal{G}_f = \{(x, f(x)); \quad x \in D\}.$$

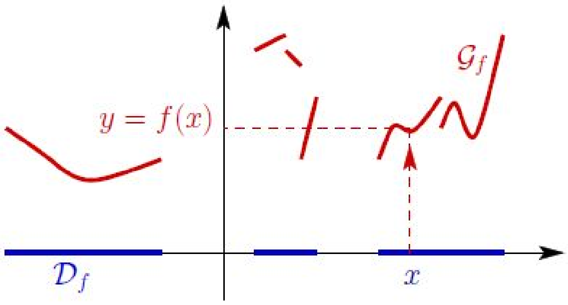


Figure 3.1: Graphic representation of a function

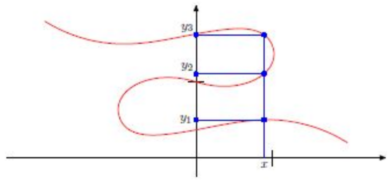


Figure 3.2: Set which is not the graph of a function

Definition 3.2.3 (Bounded functions)**.** Let $f : D \to \mathbb{R}$ be a function. We say that

- f is **bounded above** on D; if $\exists M \in \mathbb{R}$, $\forall x \in D$, $\quad f(x) \leq M$,
- f is **bounded below** on D; if $\exists m \in \mathbb{R}$, $\forall x \in D$, $\quad f(x) \geq m$,
- f is **bounded** on D; if f is bounded above and bounded below in the same time on D, i.e.,

$$\exists M, m \in \mathbb{R}, \forall x \in D; \quad m \leq f(x) \leq M.$$

Here is the graph of a bounded function below

Definition 3.2.4 (Increasing, decreasing functions)**.** Let $f : D \to \mathbb{R}$ be a function. We say that

- f is **increasing** on D if $\forall x, y \in D, \quad x < y \implies f(x) \leq f(y)$,
- f is **strictly increasing** on D if $\forall x, y \in D, \quad x < y \implies f(x) < f(y)$,
- f is **decreasing** on D if $\forall x, y \in D, \quad x < y \implies f(x) \geq f(y)$,
- f is **strictly decreasing** on D if $\forall x, y \in D, \quad x < y \implies f(x) > f(y)$,
- f is **monotonic** on D if f is increasing or decreasing on D,
- f is **strictly monotone** strictly increasing or strictly decreasing on D.

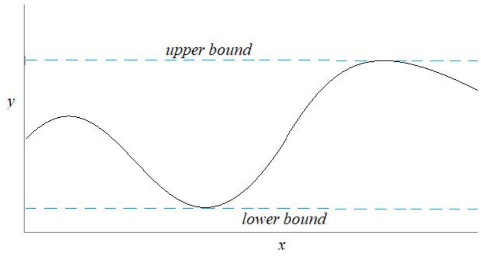


Figure 3.3: Graph of a bounded function

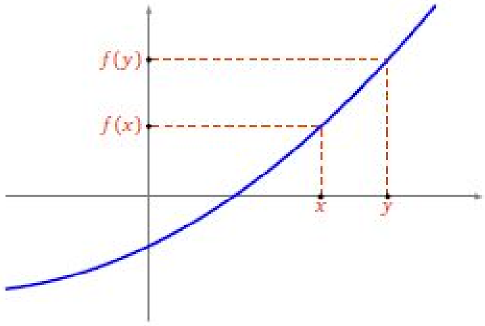


Figure 3.4: Increasing function

Definition 3.2.5 (Parity). Let I be an interval of $\mathbb{R}$ symmetric with respect to 0 (i.e. of the form $]-a,a[$ or $[-a,a]$ or $\mathbb{R}$). Let $f : D \to \mathbb{R}$ be a function defined on this interval. We say that

- f is **even** if $\forall x \in I, \quad f(-x) = f(x)$,
- f is **odd** if $\forall x \in I, \quad f(-x) = -f(x)$.

Remark 3.2.6. f is even if and only if its graph is symmetric with respect to on the y-axis and f is odd if and only if its graph is symmetric with respect to at the origin.

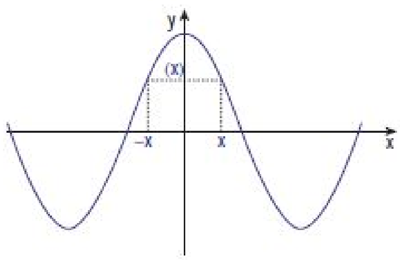


Figure 3.5: Even function

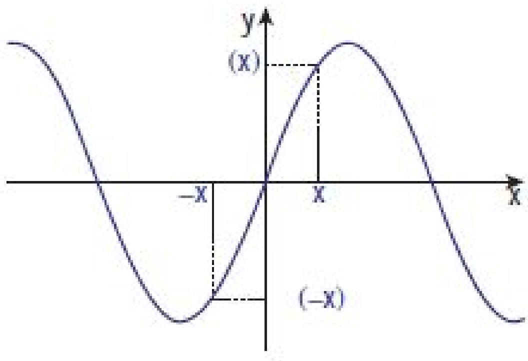


Figure 3.6: Odd function

Definition 3.2.7 (Periodicity)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function and T be a real number, $T > 0$. The function f is called **periodic** of period T if $\forall x \in \mathbb{R}, \quad f(x+T) = f(x)$.

Here is the graph of a periodic function.

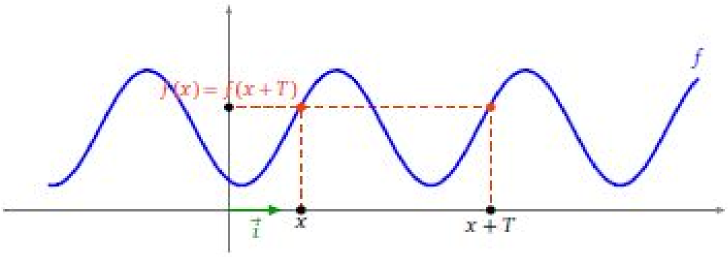


Figure 3.7: Periodic function

3.2.2 Function operations

Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be two functions defined on the same part D of $\mathbb{R}$. We can then define the following functions

- The **sum** of f and g is the function $f + g : D \to \mathbb{R}$ defined by $(f+g)(x) = f(x) + g(x)$ for all $x \in D$,
- The **product** of f and g is the function $f \cdot g : D \to \mathbb{R}$ defined by $(f \cdot g)(x) = f(x) \cdot g(x)$ for all $x \in D$,
- The **multiplication** by a scalar $\alpha \in \mathbb{R}$ of f is the function $\alpha \cdot f : D \to \mathbb{R}$ defined by $(\alpha \cdot f)(x) = f(x)$ for all $x \in D$.

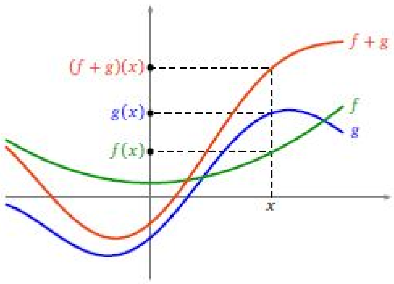


Figure 3.8: Sum of functions

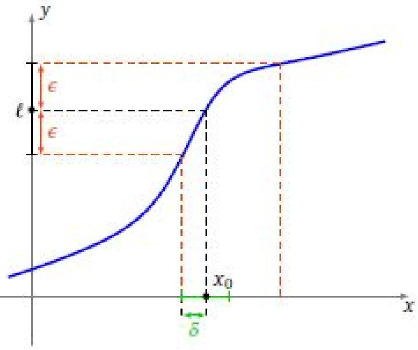
3.3 Limit of a function

3.3.1 Definitions

Let f be a function defined in the neighborhood of $x_0 \in \mathbb{R}$, except perhaps in x_0.
Definition 3.3.1 (Limit at a point)**.** We say that f admits a limit ℓ at the point x_0, if

$$\forall \varepsilon > 0, \quad \exists \delta > 0, \quad \forall x \in D_f : \qquad |x - x_0| < \delta \implies |f(x) - \ell| < \varepsilon,$$

We also say that $f(x)$ tends to ℓ when x tends to x_0. We then note $\lim\limits_{x \to x_0} f(x) = \ell$.



Remark 3.3.2. **1.** We can replace certain strict inequalities «<» by large inequalities «≤» in definition

$$\forall \varepsilon > 0, \quad \exists \delta > 0, \quad \forall x \in D_f : \qquad |x - x_0| \leq \delta \implies |f(x) - \ell| \leq \varepsilon,$$

2. δ generally depends on the ε. To mark this dependence we can write

$$\forall \varepsilon > 0, \quad \exists \delta(\varepsilon) > 0, \quad \forall x \in D_f : \qquad |x - x_0| \leq \delta \implies |f(x) - \ell| \leq \varepsilon,$$

3. The inequality $|x - x_0| < \delta$ is equivalent to $x \in]x_0 - \delta, x_0 + \delta[$. The inequality $|f(x) - \ell| \leq \varepsilon$ is equivalent to $f(x) \in]\ell - \varepsilon, \ell + \varepsilon[$.

Definition 3.3.3. • We say that f has limit $+\infty$ in x_0 if

$$\forall A > 0, \quad \exists \delta > 0, \quad \forall x \in D_f : \qquad |x - x_0| \leq \delta \implies f(x) > A.$$

We then note $\lim_{x \to x_0} f(x) = +\infty$.

• We say that f has limit $-\infty$ in x_0 if

$$\forall A > 0, \quad \exists \delta > 0, \quad \forall x \in D_f : \qquad |x - x_0| \leq \delta \implies f(x) < -A.$$

We then note $\lim_{x \to x_0} f(x) = -\infty$.

Definition 3.3.4 (Limit to infinity)**.** Let $f : I \to \mathbb{R}$ be a function defined on an interval of the form $I =]a, +\infty[$.

• We say that f admits a limit ℓ at $+\infty$ if

$$\forall \varepsilon > 0, \quad \exists B > 0, \quad \forall x \in I : \qquad x > B \implies |f(x) - \ell| < \varepsilon.$$

We then note $\lim_{x \to +\infty} f(x) = \ell$.

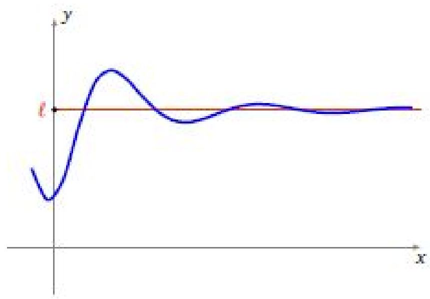


Figure 3.9: Limit at infinity

• We say that f admits a limit $+\infty$ at $+\infty$ if

$$\forall A > 0, \quad \exists B > 0, \quad \forall x \in I : \qquad x > B \implies f(x) > A.$$

We then note $\lim_{x \to +\infty} f(x) = +\infty$.

Remark 3.3.5. We would define the limit in $-\infty$ in the same way for functions defined on intervals of the type $]-\infty, a[$.

Definition 3.3.6 (Limit left and right)**.** • Say that f admits a limit $\ell \in \mathbb{R}$ to **right in** x_0 therefore means

$$\forall \varepsilon > 0, \quad \exists \delta > 0; \qquad x_0 < x < x_0 + \delta \implies |f(x) - \ell| \leq \varepsilon,$$

and we note $\lim_{x \to x_0^+} f(x) = \ell$.

- say that f admits a limit $\ell \in \mathbb{R}$ to **left in** x_0 therefore means

$$\forall \varepsilon > 0, \quad \exists \delta > 0; \qquad x_0 - \delta < x < x_0 \implies |f(x) - \ell| \leq \varepsilon,$$

and we note $\lim_{x \to x_0^-} f(x) = \ell$.

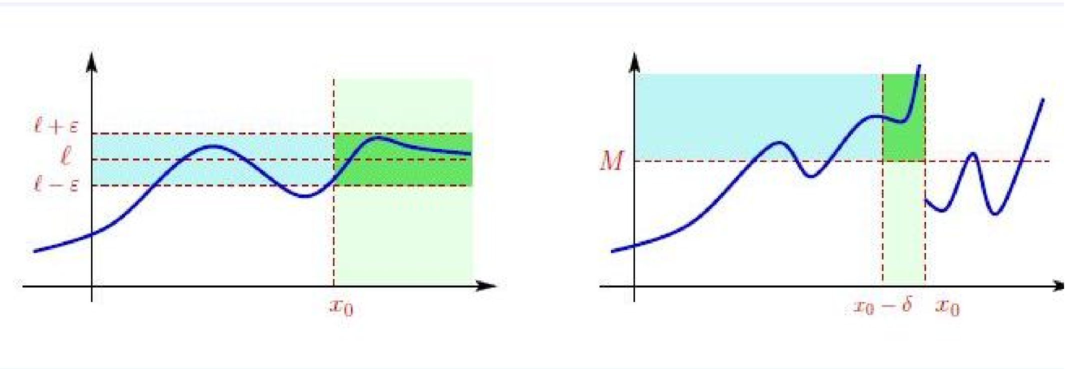


Figure 3.10: Limit left and right

Proposition 3.3.7.

$$\lim_{x \to x_0} f(x) = \ell \iff \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x)$$

3.3.2 Properties

Proposition 3.3.8 (Uniqueness of the limit)**.** *If a function admits a limit, then this limit is* ***unique****.*

Let there be two functions f and g. We assume that x_0 is a real, or that $x_0 = \pm\infty$.

Proposition 3.3.9 (Rules for limits)**.** *If* $\lim_{x\to x_0} f(x) = \ell \in \mathbb{R}$ *and* $\lim_{x\to x_0} g(x) = \ell' \in \mathbb{R}$, *then*

- $\lim_{x\to x_0} [f(x) + g(x)] = \ell + \ell'$
- $\lim_{x\to x_0} [f(x) \cdot g(x)] = \ell \cdot \ell'$
- $\lim_{x\to x_0} [\lambda \cdot f(x)] = \lambda \cdot \ell$, *fo all* $\lambda \in \mathbb{R}$
- $\lim_{x\to x_0} |f(x)| = |\ell|$
- *If* $\ell' \neq 0$, *then* $\lim_{x\to x_0} \left[\frac{f(x)}{g(x)}\right] = \frac{\ell}{\ell'}$

In addition, if $\lim_{x\to x_0} f(x) = +\infty$ *(or* $-\infty$*) then* $\lim_{x\to x_0} \frac{1}{f(x)} = 0$

Proposition 3.3.10 (Composition rule)**.**

$$\textit{If } \lim_{x\to x_0} f(x) = \ell \textit{ and } \lim_{x\to \ell} g(x) = \ell', \textit{ so } \lim_{x\to x_0} g \circ f = \ell'.$$

Example 3.3.11. Determine the following limit

$$\lim_{x\longrightarrow 0} \frac{\sin(\sin x)}{\sin x}.$$

Let $f(x) = \sin x,\ \ x \in \mathbb{R}$, and $g(x) = \frac{\sin x}{x}$, $x \neq 0$. Then $\lim_{x\longrightarrow 0} f(x) = \lim_{x\longrightarrow 0} \sin x$ and $\lim_{x\longrightarrow 0} g(x) = \lim_{x\longrightarrow 0} \frac{\sin x}{x} = 1$. Also, $f(x) = \sin x \neq 0$; in the punctured neighbouhood $]-\pi, 0[\cup]0, \pi[$ of 0 (for example).
It follows, by Coposition rule, that

$$\lim_{x\longrightarrow 0} g(f(x)) = \lim_{x\longrightarrow 0} \frac{\sin(\sin x)}{\sin x} = 1.$$

Proposition 3.3.12. *Let* f *and* g *be two functions defined in the neighborhood of* $x_0 \in \mathbb{R}$. *If* f *is bounded in the neighborhood of* x_0 *and if* $\lim_{x\to x_0} g(x) = 0$, *then* $\lim_{x\to \ell} (f(x).g(x)) = 0$.

Finally here is a very important proposition which means that we can pass to the limit in a large inequality.

Proposition 3.3.13.

- *If* $f \leq g$ *and if* $\lim_{x\to x_0} f(x) = \ell \in \mathbb{R}$ *and* $\lim_{x\to x_0} g(x) = \ell' \in \mathbb{R}$, *then* $\ell \leq \ell'$
- *If* $f \leq g$ *then*

$$\begin{aligned} \lim_{x\to x_0} f(x) &= +\infty \Longrightarrow \lim_{x\to x_0} g(x) = +\infty \\ \lim_{x\to x_0} g(x) &= -\infty \Longrightarrow \lim_{x\to x_0} f(x) = -\infty \end{aligned}$$

- *If* $f \geq g$ *and if* $\lim\limits_{x \to x_0} f(x) = +\infty$ *then* $\lim\limits_{x \to x_0} g(x) = +\infty$
- ***Squeeze Rule***

If $f \leq g \leq h$ *and if* $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} h(x) = \ell \in \mathbb{R}$ *, then* g *has a limit in* x_0 *and* $\lim\limits_{x \to x_0} g(x) = \ell$

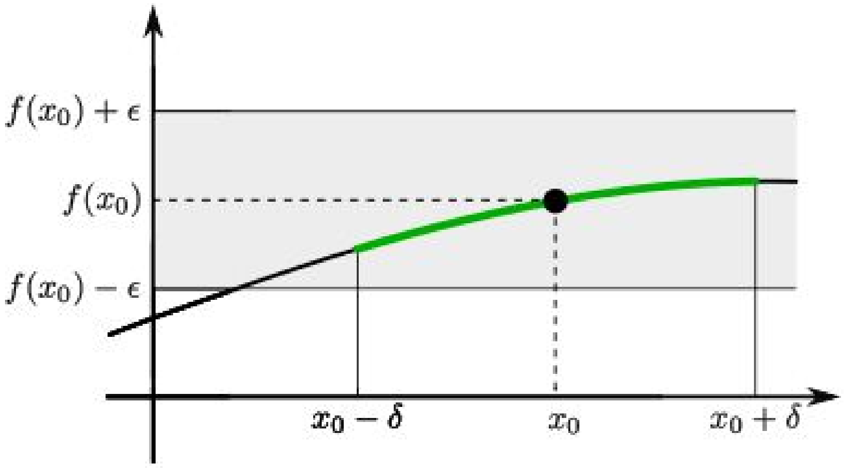
3.4 Continuity of a function

3.4.1 Continuity at a point

Let I be an interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ be function.
Definition 3.4.1. We say that f is **continuous** at a point $x_0 \in I$ if

$$\forall \varepsilon > 0, \quad \exists \delta > 0, \quad \forall x \in I : \qquad |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon,$$

that is to say if f admits a limit in x_0 (this limit is then worth necessarily $f(x_0)$).



In this graphic, we may see that there is a region around x_0 , where function values differ by less than ϵ from $f(x_0)$. So in fact, there is a distance difference δ, such that all function values are inside the interval $]f(x_0) - \epsilon, f(x_) + \epsilon[$ highlighted in grey.

Example 3.4.2. The following functions are continuous:
- A constant function on an interval,
- The square root function $x \to \sqrt{x}$ on $[0, +\infty[$,
- The functions sin and cos on $\mathbb{R}$,

- The absolute value function $x \to |x|$ on $\mathbb{R}$,
- The exponential function exp on $\mathbb{R}$,
- The logarithmic function ln on $]0, +\infty[$.

Definition 3.4.3 (Continuity on the left and on the right)**.**

- We say that f is **continuous on the right** at a point $x_0 \in I$ if

$$\lim_{x \to x_0^+} f(x) = f(x_0),$$

that is to say

$$\forall \varepsilon > 0, \quad \exists \delta > 0, \quad \forall x \in I; \qquad x_0 < x < x_0 + \delta \implies |f(x) - f(x_0)| \leq \varepsilon,$$

- We say that f is **continuous on the letf** at a point $x_0 \in I$ if

$$\lim_{x \to x_0^-} f(x) = f(x_0),$$

that is to say

$$\forall \varepsilon > 0, \quad \exists \delta > 0, \quad \forall x \in I; \qquad x_0 - \delta < x < x_0 \implies |f(x) - f(x_0)| \leq \varepsilon$$

Théorème 3.4.4. *f is continuous at x_0 $\iff$ f is continuous on the right and continuous on the letf at x_0.*

Combination Rules for continuous functions

Proposition 3.4.5. *Let $f, g : I \to \mathbb{R}$ be two continuous functions at a point $x_0 \in I$. So*

- *$\lambda \cdot f$ is continuous at x_0 (for all $\lambda \in \mathbb{R}$),*
- *$f + g$ is continuous at x_0,*
- *$f \cdot g$ is continuous at x_0,*
- *if $f(x_0) \neq 0$, then $\dfrac{1}{f}$ is continuous at x_0.*

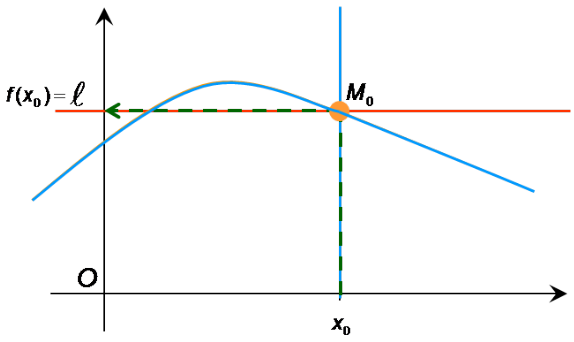
Proposition 3.4.6. *Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be two functions such that $f(I) \subset J$. If f is continuous at a point $x_0 \in I$ and if g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.*

Definition 3.4.7 (Continuity extension)**.** Let I be an interval, x_0 a point of I and $f : I - \{x_0\} \to \mathbb{R}$ a function.

- We say that f is **extendable by continuity** in x_0 if f admits a finite limit in x_0. Let us then write $\lim_{x \to x_0} f(x) = \ell$
- We then define the function $\widetilde{f} : I \to \mathbb{R}$ by setting for all $x \in I$

$$\widetilde{f}(x) = \begin{cases} f(x) & x \neq x_0 \\ \ell & x = x_0 \end{cases}$$

Then $\widetilde{f}$ is continuous at x_0 and we call it **the extension by continuity** of f in x_0.



3.4.2 Continuity over an interval

Let I be an interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a function x_0 a point of I.

Definition 3.4.8. We say that f is continuous on I if f is continuous at every point of I.

Notation 3.4.9. We denote by $\mathcal{C}(I;\mathbb{R})$ or $\mathcal{C}^0(I;\mathbb{R})$ the set of continuous functions on I whose values are in $\mathbb{R}$.

Proposition 3.4.10. *If f is continuous at x_0 and if $f(x_0) \neq 0$, then there exists $\delta > 0$ such that*

$$\forall x \in]x_0 - \delta, x_0 + \delta[\qquad f(x) \neq 0.$$

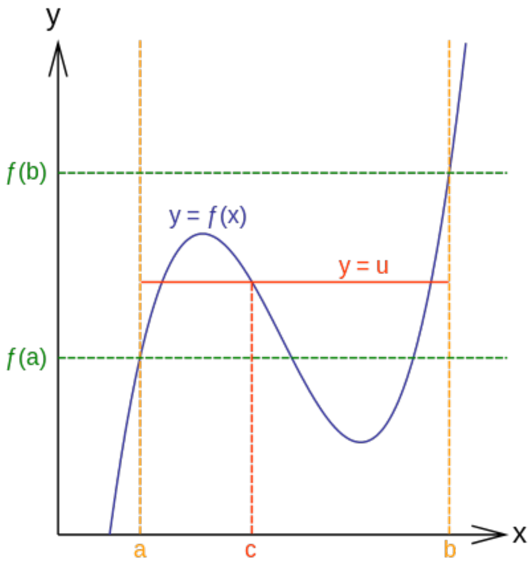
Théorème 3.4.11 (The Intermediate Value Theorem)**.** *Let $f : [a,b] \to \mathbb{R}$ be a continuous function on a segment. For any real y between $f(a)$ and $f(b)$, there exists $c \in [a,b]$ such that $f(c) = y$.*

Remark 3.4.12. An illustration of the intermediate value theorem, the real c is not necessarily unique. Moreover if the function is not continuous, the theorem is no longer true.

This is the most used version of the intermediate value theorem.

Proposition 3.4.13. *Let f be a continuous function on interval $[a,b]$, such that $f(a)f(b) < 0$, there exists $c \in]a,b[$ such that $f(c) = 0$.*

Example 3.4.14. Let f be a continuous function on $[0,1]$ such that for all x in this interval, $f(x) \in [0,1]$. Show that there exists an element $x \in [0,1]$ such that $f(x) = x$.
For all $x \in [0,1]$ we put $g(x) = f(x) - x$. The function g is continuous, because it is the sum of two continuous functions.
We have
- $g(0) = f(0) - 0 = f(0) > 0$.
- $g(1) = f(1) - 1 < 0$ because $f(1) < 1$.
So, according to the intermediate value theorem,there exists $x \in [0,1]$ such that $g(x) = 0$, that is $f(x) - x = 0$ or even $f(x) = x$.



3.4.3 Uniform continuity

Definition 3.4.15. Let f be a continuous function on an interval I.
f is uniformly continuous on I if, and only if,

$$\forall \epsilon > 0, \exists \alpha(\epsilon) > 0, \forall (x, y) \in I^2, \mid x - y \mid \Longrightarrow \mid f(x) - f(y) \mid < \epsilon.$$

Here the choice of α depends only on ϵ.

Remark 3.4.16. The notion of uniform continuity is a global notion, unlike the notion of continuity at a point, which is local.

Example 3.4.17. Consider the function $f : [0, 1] \longrightarrow \mathbb{R}$ defined by $f(x) = x^2$.
We have

$$\forall (x, y) \in \mathbb{R}^2, \qquad x^2 - y^2 = (x - y)(x + y).$$

It results

$$\forall (x, y) \in [0, 1]^2, \qquad |f(x) - f(y)| \leq 2|x - y|.$$

Let ϵ be a strictly positive real and α the real defined by $\alpha = \dfrac{\epsilon}{2}$.
we have the implication

$$\forall (x, y) \in [0, 1]^2, \qquad |x - y| < \alpha \Rightarrow |f(x) - f(y) < \epsilon.$$

So, the function f is uniformly continuous on $[0, 1]$.

Théorème 3.4.18. *If a function f is uniformly continuous on I, then it is continuous on I.*

Here is a theorem very used in practice to show that a function is bijective.

Théorème 3.4.19 (Bijection Theorem)**.** *Let $f : I \to \mathbb{R}$ be a function defined on an interval I of $\mathbb{R}$. If f is continuous and strictly monotonic on I, So*

1. *f establishes a bijection of the interval I in the image interval $J = f(I)$,*

2. *the inverse function $f^{-1} : J \to I$ is continuous and strictly monotonic on J and it has the same direction of variation as f.*

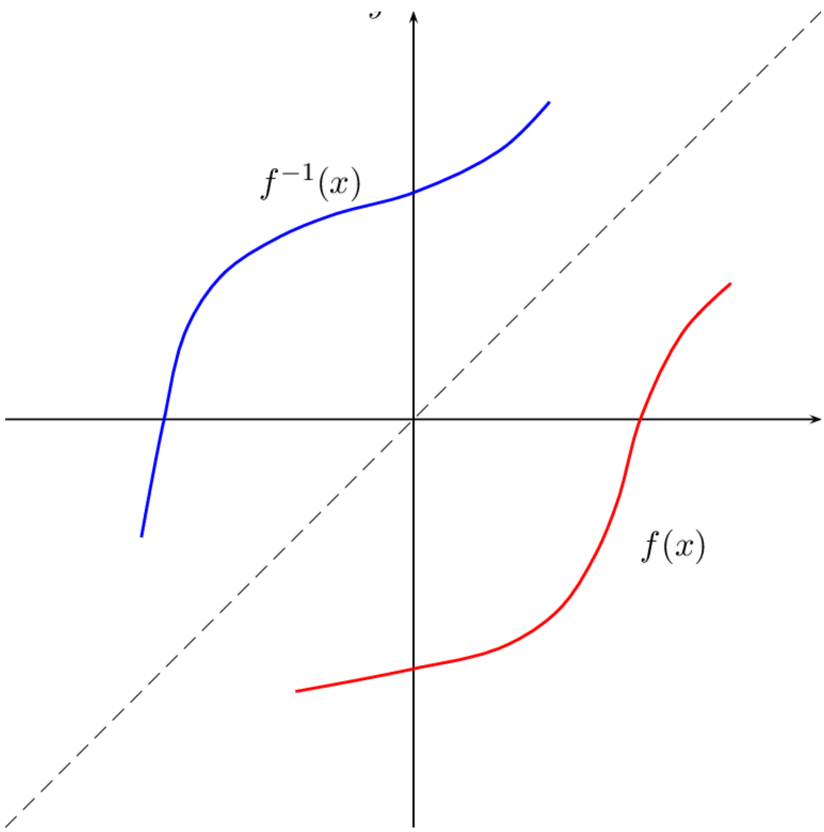


Figure 3.11: Representation of f and f^{-1}

3.5 Differentiation of a functions

3.5.1 Introduction

This section is an introduction to one of the most fabulous inventions of man, that of differential calculus, in the case of functions of a real variable with real values. The history of differential calculus begins largely with Galileo and Newton who needed new mathematical tools to develop the notions of speed and acceleration of a movement. But the possibility of calculating the slope of the tangent to a curve was essential in other problems such as those of extremum or for more applied questions. Newton and Leibniz were the first to attempt to formalize the notion of derivative. They disputed the paternity of this invention but it now seems certain that they discovered it independently. The notion of limit was only developed much later, in the 19th century by Cauchy and Weierstrass; the formalization of the derivation by Newton and Leibniz suffered from numerous shortcomings. Newton also refused to publish his work and Leibniz's writings were obscure and difficult to understand. Lagrange, a century later, introduced the term derivative as well as the notation f'. After having defined what a differentiable function is as well as its derivative, we will give the rules for calculating derivatives that you have known since high school. We will see in particular that the derivative allows us to approach a given function by an affine function. We will be interested in the global properties of differentiable functions. Rolle's theorem and that of finite increments will be in constant use in analysis. The inequality of finite increments which follows from the theorem of the same name is a real machine for manufacturing inequalities.

3.5.2 Definitions

Let I be an open interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ a function. Let $x_0 \in I$.

Definition 3.5.1. f is **differentiable** in x_0 if the rate of increase $\dfrac{f(x) - f(x_0)}{x - x_0}$ has a finite limit when x tends to x_0. The limit is then called **the derivative number** of f at x_0 and is denoted $f'(x_0)$. So

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

The function $x \to f'(x)$ is the **derivative function** of f, it is written f' or $\dfrac{df}{dx}$.

Example 3.5.2. The function defined by $f(x) = x^2$ is differentiable at every point $x_0 \in \mathbb{R}$. Indeed

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{x^2 - x_0^2}{x - x_0} = \frac{(x - x_0)(x + x_0)}{x - x_0} = (x + x_0) \underset{x \to x_0}{\longrightarrow} 2x_0.$$

We even showed that $f'(x) = 2x$.

Definition 3.5.3. f is differentiable on I if f is differentiable at every point $x_0 \in I$.

Proposition 3.5.4.

- *If f is differentiable at x_0 then f is continuous at x_0.*
- *If f is differentiable on I then f is continuous on I.*

Remark 3.5.5. The converse is false, for example, the absolute value function $f(x) = |x|$ is continues in 0 but is not differentiable in 0.

So, what do we do?

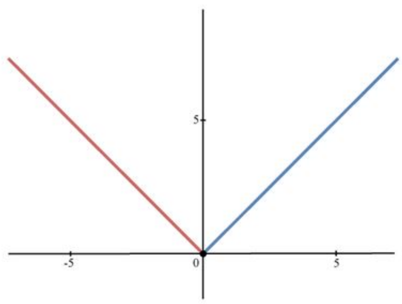
We use one-sided limits and our definition of derivative to determine whether or not the slope on the left and right sides are equal.

$$\lim_{h \longrightarrow 0^-} \frac{f(x+h) - f(x)}{h} = \lim_{h \longrightarrow 0^-} \frac{-(-x+h) - (-x)}{h} = \lim_{h \longrightarrow 0^-} \frac{x - h + x}{h} = \lim_{h \longrightarrow 0^-} \frac{-h}{h} = -1$$

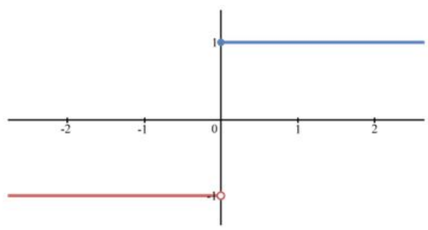
$$\lim_{h \longrightarrow 0^+} \frac{f(x+h) - f(x)}{h} = \lim_{h \longrightarrow 0^+} \frac{(x+h) - (x)}{h} = \lim_{h \longrightarrow 0^+} \frac{x + h - x}{h} = \lim_{h \longrightarrow 0^+} \frac{h}{h} = 1$$

And upon comparison, we find that the slope of the left-side equals -1 and the slope of the right-side equals $+1$, so they disagree.

Therefore, the function $f(x) = |x|$ is not differentiable at $x = 0$. While the function is continuous, it is not differentiable because the derivative is not continuous everywhere, as seen in the graphs below.



$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$



$$f'(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

Proposition 3.5.6. *Let $f, g : I \to \mathbb{R}$ be two differentiable functions on I. So for all $x \in I$, $f+g$, λf, $f.g$, and $\frac{f}{g}$, $g \neq 0$ are differentiable and we have:*

- $(f+g)'(x) = f'(x) + g'(x)$,
- $(\lambda \cdot f)'(x) = \lambda \cdot f'(x)$ *where λ is a fixed real,*
- $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$,
- $\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$, *(if $g(x) \neq 0$).*

Proposition 3.5.7. *If f is differentiable in x and g is differentiable in $f(x)$ then $g \circ f$ is differentiable in x of derivative*

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$$

Proposition 3.5.8. *Let I be an open interval. Let $f : I \to J$ be differentiable and bijective of which we denote $f^{-1} : J \to I$ the reciprocal bijection. If f' does not vanish on I then f^{-1} is differentiable and we have for all $x \in J$*

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

Proposition 3.5.9.
- $\forall x \in]a, b[, \quad f'(x) \geq 0 \iff f$ *is* ***increasing****,*
- $\forall x \in]a, b[, \quad f'(x) \leq 0 \iff f$ *is* ***decreasing****,*
- $\forall x \in]a, b[, \quad f'(x) = 0 \iff f$ *is* ***constant****,*
- $\forall x \in]a, b[, \quad f'(x) > 0 \iff f$ *is* ***strictly increasing****,*
- $\forall x \in]a, b[, \quad f'(x) < 0 \iff f$ *is* ***strictly decreasing****.*

Théorème 3.5.10 (Successive derivatives)**.** *Let $f : I \to \mathbb{R}$ be a differentiable function and let f' its derivative. If the function $f' : I \to \mathbb{R}$ is also differentiable we note $f'' = (f')'$ the* ***second derivative*** *of f. More generally we note*

$$f^{(0)} = f, \quad f^{(1)} = f', \quad f^{(2)} = f'' \text{ and } f^{(n+1)} = \left(f^{(n)}\right)'$$

If the ***derivative n-th*** *$f^{(n)}$ exists we say that f is* ***n-times differentiable****.*

- ***Leibniz formula***

$$(f \cdot g)^{(n)} = C_n^0 f^{(n)} \cdot g + C_n^1 f^{(n-1)} \cdot g^{(1)} + C_n^2 f^{(n-2)} \cdot g^{(2)} + ... + C_n^n f.g^{(n)}$$

In an other word

$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} C_n^k \cdot f^{(n-k)} \cdot g^{(k)},$$

such that

$$C_n^k = \frac{n!}{k!\,(n-k)!}.$$

Example 3.5.11. Let's calculate the $n-th$ derivatives of $e^x \cdot (x^2+1)$ for all $n \geq 0$.
Let us denote $f(x) = e^x$ then $f^{(k)} = e^x, \quad \forall k$

$$f^{(k)} = f^{(1)} = f^{(2)} = \ f^{(n+1)} = \left(f^{(n)}\right)'$$

Let us denote $g(x) = x^2 + 1$ then

$$g'(x) = 2x, \quad g''(x) = 2, \quad g^{(k)} = 0 \ \forall k \geq 3.$$

Let's apply Leibniz's formula

$$\begin{aligned}(f \cdot g)^{(n)} &= C_n^0 \cdot f^{(n)} \cdot g^{(0)} + C_n^1 \cdot f^{(n-1)} \cdot g^{(1)} + C_n^2 \cdot f^{(n-2)} \cdot g^{(2)} \\ (f \cdot g)^{(n)} &= e^x \left(x^2 + 2nx + n\,(n-1)\right).\end{aligned}$$

Théorème 3.5.12. *If f defined on an open interval $I =]a;b[$, is differentiable and admits a local maximum or minimum at $a \in I$, then $f'(a) = 0$.*

Théorème 3.5.13 (Rolle's Theorem)**.** *Let $f : [a,b] \to \mathbb{R}$ such that*

- *f is continuous on $[a,b]$,*
- *f is differentiable on $]a,b[$,*
- *$f(a) = f(b)$.*

Then there exists $c \in]a,b[$ such that $f'(c) = 0$.

Geometric interpretation:
There is at least one point of the graph of f where the tangent is horizontal.

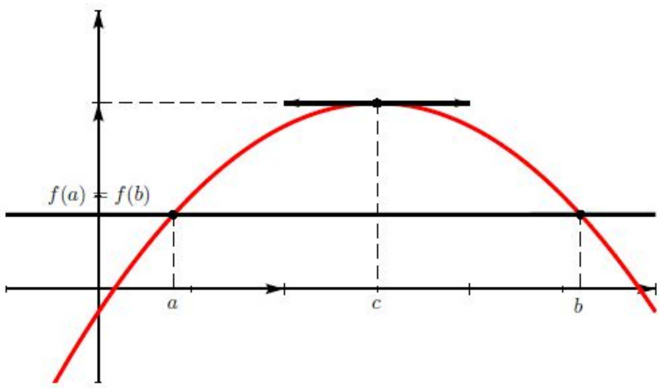


Figure 3.12: Rolle's theorem

Cinematic interpretation: From a moving point on an axis which returns to its starting position has seen its speed cancel out at a given moment.

Théorème 3.5.14 (Hospital Rule). *Let $f, g : I \to \mathbb{R}$ be two differentiable functions and let $x_0 \in I$. We suppose that*

- $f(x_0) = g(x_0) = 0$,
- $\forall x \in I - \{0\}\,, \quad g'(x) \neq 0$. *If* $\lim\limits_{x \to x_0} \dfrac{f'(x)}{g'(x)} = \ell \quad (\ell \in \mathbb{R})\,,$ *then* $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = \ell.$

Théorème 3.5.15 (Mean value theorem). *Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$ and differentiable on $]a, b[$. There exists $c \in]a, b[$ such that*

$$f(b) - f(a) = (b - a)\, f'(c)$$

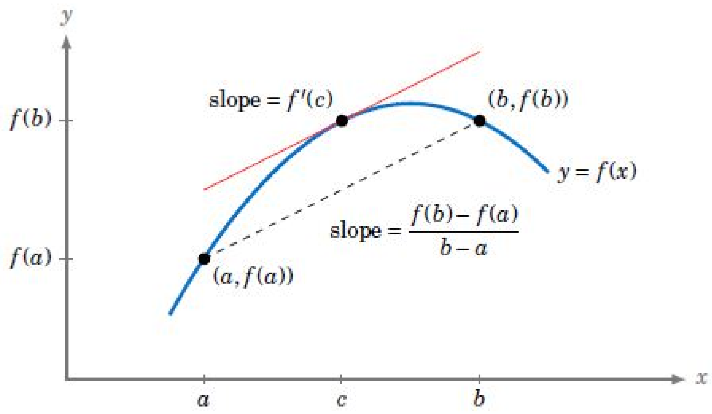


Figure 3.13: Mean value theorem

3.6 Convex functions

3.6.1 Definitions

Let $f : I \longrightarrow \mathbb{R}$ be a function defined on an interval $I \subset \mathbb{R}$. We say that f is convex if:

$$\forall (x, y) \in I, \forall \lambda \in [0, 1];\ f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

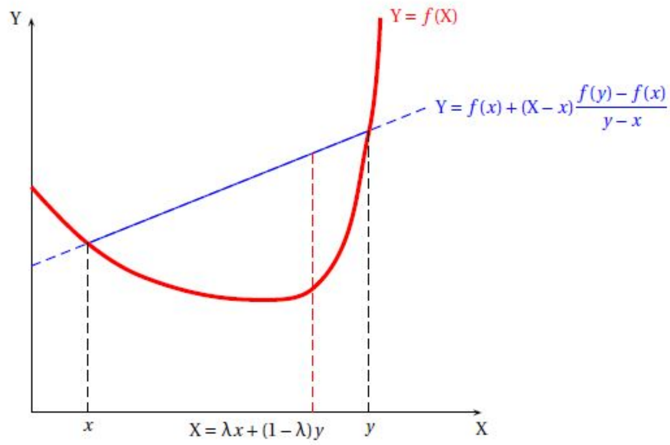


Figure 3.14: Convex function. A function which is convex cannot have a line segment drawn between two points that is below the curve.

A function on a graph is convex if a line segment drawn through any two points on the line of the function never lies below the curved line segment. I.e., basically, a convex function has its curve opening upward like a cup. Whereas a concave function has its curve opening downward like a hat or cap.

3.6.2 Graph of convex function

Théorème 3.6.1 (**The graph of a convex function is located above all its tangents**). *Let $f : I \longrightarrow \mathbb{R}$ a convex function and differentiable then:*

$$\forall x_0 \in I, \forall x \in I; \quad f(x) \geq f(x_0) + f'(x_0)(x - x_0)$$

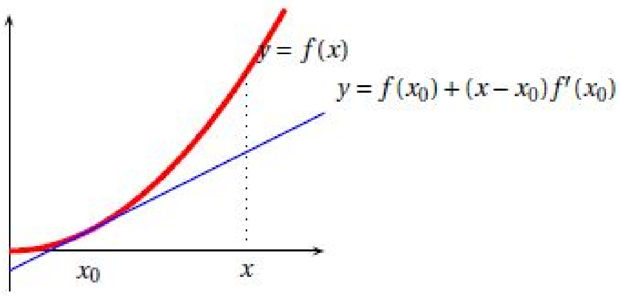


Figure 3.15: The graph of a convex function is located above its tangents

Example 3.6.2. The application $f : \mathbb{R} \longrightarrow \mathbb{R}, \quad f(x) = x^2$ is convex on $\mathbb{R}$
In fact, $\forall (x, y) \in \mathbb{R}^2, \ \forall \lambda \in [0, 1]$, we have

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &- (\lambda f(x) + (1-\lambda)f(y)) \\ &= (\lambda x + (1-\lambda)y)^2 - (\lambda x^2 + (1-\lambda)(y)^2) \\ &= \lambda(\lambda-1)(x^2 - 2xy + y^2) \\ &= \lambda(\lambda-1)(x-y)^2 \end{aligned}$$

This quantity is negative for $\lambda \in [0, 1]$, then

$$\forall (x, y) \in \mathbb{R}^2, \ \forall \lambda \in [0, 1], f(\lambda x + (1-\lambda)y) \leq (\lambda f(x) + (1-\lambda)f(y))$$

3.7 Derivative of usual functions

The following table is a summary of the main formulas to know, x is a variable.

Function	Derivative	Function	Derivative
x^n	$nx^{n-1} \quad (n \in \mathbb{Z})$	$\arccos(x)$	$\frac{-1}{\sqrt{1-x^2}} \quad \forall x \in]-1, 1[$
$\frac{1}{x}$	$\frac{-1}{x^2}$	$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}} \quad \forall x \in]-1, 1[$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$\arctan(x)$	$\frac{1}{1+x^2} \quad \forall x \in \mathbb{R}$
e^x	e^x	$\mathrm{ch}x$	$\mathrm{sh}x$
$\ln x$	$\frac{1}{x}$	$\mathrm{sh}x$	$\mathrm{ch}x$
$\cos x$	$-\sin x$	$\mathrm{th}x$	$1 - \mathrm{th}^2 x = \frac{1}{\mathrm{ch}^2 x}$
$\sin x$	$\cos x$	$\arg\mathrm{ch}(x)$	$\frac{1}{\sqrt{x^2-1}}$
$\tan x$	$(1 + \tan^2 x) = \frac{1}{\cos^2 x}$	$\arg\mathrm{sh}(x)$	$\frac{1}{\sqrt{x^2+1}}$
x^α	$\alpha x^{\alpha-1} \quad (\alpha \in \mathbb{R})$	$\arg\mathrm{th}(x)$	$\frac{1}{x^2-1} \quad \forall x \in]-1, 1[$

3.8 Solved exercises

3.8.1 Exercises

3.1
Using the definition of the notion of limit at a point, show that:

1. $\lim\limits_{x\to\infty} \frac{1}{x} = 0$
2. $\lim\limits_{x\to 0^+} \frac{1}{x} = +\infty$
3. $\lim\limits_{x\to 1^-} \frac{1}{1-x^2} = +\infty$
4. $\lim\limits_{x\to a} \sqrt{x} = \sqrt{a}$ with $a \in \mathbb{R}_+^\star$

3.2
Determine the following limits:

1. $\lim\limits_{x\to +\infty} \frac{x^4+2x^2+1}{x^2-1}$
2. $\lim\limits_{x\to\infty} x - \sqrt{x^2-2x}$
3. $\lim\limits_{x\to 0} \frac{\sin x}{x}$
4. $\lim\limits_{x\to 0^+} x^x$
5. $\lim\limits_{x\to 1} \frac{x^3+3x^2-3x-1}{x^2+x-2}$
6. $\lim\limits_{x\to +\infty} \frac{\cos x^2}{x}$

3.3
Determine the following limits:

1. $\lim\limits_{x\to +\infty} \frac{x\cos(e^x)}{x^2+1}$
2. $\lim\limits_{x\to 1^+\infty} \ln x(\ln(\ln x))$
3. $\lim\limits_{x\to 0} 2x - 1 + \frac{\sqrt{x^2}}{x}$
4. $\lim\limits_{x\to +\infty} \ln(1+x^2e^{-x})$
5. $\lim\limits_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2-x}}{x}$
6. $\lim\limits_{x\to 0} \frac{\ln(1+x)}{x}$

3.4
Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function defined by:

$$f(x) = \begin{cases} x + \frac{\sqrt{x^2}}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Determine the set on which f is continuous.

3.5
Let a, b be two real numbers.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function defined by:

$$f(x) = \begin{cases} \frac{\sin(ax)}{x} & \text{if } x < 0 \\ a & \text{if } x = 0 \\ e^{bx} - x & \text{if } x > 0 \end{cases}$$

1. Using the Hospital's rule Determine the limit

$$\lim_{x\longrightarrow 0} \frac{x\cos x - \sin x}{x^2}$$

2. Determine a, b so that f is continuous on $\mathbb{R}$.
3. Determine a, b so that f is differentiable on $\mathbb{R}$.

3.6
Calculate, when they exist, the derivatives of the following functions:

1. $f_1(x) = \ln(3 + \sin x)$
2. $f_2(x) = \ln(\sqrt{1 + x^2})$
3. $f_3(x) = \ln\left(\dfrac{2x \cos x}{1 - \cos x}\right)$
4. $f_4(x) = x^x$
5. $f_5(x) = \sin((e^x)^2)$
6. $f_6(x) = x^{\frac{\sin x}{x}}$

3.7
Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function defined by:

$$f(x) = \begin{cases} \dfrac{3 - x^2}{2} & \text{if } x \leq 1 \\ \dfrac{1}{x} & \text{if } x > 1 \end{cases}$$

Show that there exists $c \in]0, 2[$ such that $f(2) - f(0) = (2 - 0)f'(c)$
Determie the possible values of c.

3.8
A tractor starting from a point A located on a straight road must reach a point B located in a field. The tractor goes twice as fast on the road as in the field. It is assumed that the tractor moves on the road and in the field at constant speed. The distance AC is designated by L and the distance CB by d. Determine point D where the tractor must leave the road so that the travel time from A to B is minimal. We will discuss the solution according to the values of L and d.

3.8.2 Solutions

3.1
1. Let $\epsilon > 0$ We look for $m \in \mathbb{R}$ such that if $x \in]m, +\infty[$ then we have $\left|\frac{1}{x} - 0\right| = \frac{1}{|\, x \,|} < \epsilon$ this inequality is equivalent to $|\, x \,| > \frac{1}{\epsilon}$. posing $m = \frac{1}{\epsilon}$ we have for all $x \in]m, +\infty[$, $\left|\frac{1}{x} - 0\right| < \epsilon$. This proves that $\lim\limits_{x\to\infty} \frac{1}{x} = 0$.
2. Let $M \in \mathbb{R}$, we can suppose that $M \geq 1$. We lokk for $\delta \geq$, such that , for all $x \in \mathbb{R}_+^\star$, if $|\, x - 0 \,| = |\, x \,| = x < \delta$.
Then $\frac{1}{n} > M$. This inequality is equivalent to $\frac{1}{M} > x$. Then it is sufficient to choose $\delta = \frac{1}{M}$. So $\lim\limits_{x\to 0^+} \frac{1}{x} = +\infty$
3. Let $M > 0$, we search $\delta > 0$ such that, for all $x > 0$, if $1 - x < \delta$ then $\frac{1}{1 - x^2}$; we

have

$$\frac{1}{1-x^2} > M \Leftrightarrow x > \sqrt{1-\frac{1}{M}} \Leftrightarrow 1-x \leq 1-\sqrt{1-\frac{1}{M}}.$$

We take $\delta = 1-\sqrt{1-\frac{1}{M}}$.
4. Let $\epsilon > 0$, we search $\delta \in \mathbb{R}$ such that, for all $x \in \mathbb{R}_+$, if $\mid x-a \mid < \delta$ then $\mid \sqrt{x}-\sqrt{a} \mid < \epsilon$.
For $x \in \mathbb{R}_+$ such that $x > a$ we have

$$\sqrt{x}-\sqrt{a} < \epsilon \Leftrightarrow x < (\epsilon+\sqrt{a})^2 \Leftrightarrow x-a < (\epsilon+\sqrt{a})^2 - a = \epsilon^2 + 2\epsilon\sqrt{a}.$$

If $x < a$, we show that

$$\sqrt{a}-\sqrt{x} < \epsilon \Leftrightarrow a-x < 2\epsilon\sqrt{a}-\epsilon^2.$$

We take $\delta = \min(2\epsilon\sqrt{a}+\epsilon^2, 2\epsilon\sqrt{a}-\epsilon^2)$

3.2

1. $\dfrac{x^4+2x^2+1}{x^2-1} = \dfrac{x^4}{x^2}\dfrac{1+\frac{2}{x^2}+\frac{1}{x^4}}{1-\frac{1}{x^2}} \underset{x\to+\infty}{\longrightarrow} +\infty$
2. $\forall x \in \mathbb{R}^\star,\ \dfrac{\sin x}{x} = \dfrac{\sin x - \sin 0}{x-0} \underset{x\to 0}{\longrightarrow} \sin' 0 = 1$
3. $x-\sqrt{x^2-2x} = \dfrac{(x-\sqrt{x^2-2x})(x+\sqrt{x^2-2x})}{(x+\sqrt{x^2-2x})} = \dfrac{x^2-(x^2-2x)}{1+\sqrt{x^2-2x}} = \dfrac{x}{x}\dfrac{2}{1+\sqrt{1-\frac{2}{x}}} \underset{x\to+\infty}{\longrightarrow}$
1
4. $x^x = e^{x\ln x}$, $\left(X = x\ln x \underset{x\to 0^+}{\longrightarrow} 0\right)$, then $x^x \underset{x\to 0}{\longrightarrow} 1$
5. $\dfrac{x^3+3x^2-3x-1}{x^2+x-2} = \dfrac{(x-1)(x^2+4x+1)}{(x-1)(x+2)} \underset{x\to+1}{\longrightarrow} 2.$
6. For all $x \in \mathbb{R}_+^\star$, $\dfrac{-1}{x} \leq \dfrac{\cos x^2}{x} \leq \dfrac{1}{x}$ then the limit is 0.

3.3

1. For $x \in \mathbb{R}_+, 0 \leq \left|\dfrac{x\cos e^x}{x^2+1}\right| \leq \dfrac{1}{x}$, then the limit requested is 0.
2. $\ln x \ln(\ln x) = X\ln X$ with $X = \ln x \underset{x\to 1^+}{\longrightarrow} 0$, then the limit is 0.
3.

$$2x-1+\frac{x^2}{x} = 2x+\frac{\mid x \mid}{x} = \begin{cases} 2x-1+1 & \text{if } x \geq 0 \\ 2x-1-1 & \text{if } x \leq 0 \end{cases}$$

And $\lim\limits_{x\longrightarrow 0^+} 2x-1+\dfrac{x^2}{x} = 0$, $\lim\limits_{x\longrightarrow 0^-} 2x-1+\dfrac{x^2}{x} = -2$.
4. $x^2e^{-x} \underset{x\to+\infty}{\longrightarrow} 0$ then $\lim\limits_{x\longrightarrow+\infty} \ln(1+x^2e^{-x}) = 0.$

5. $\dfrac{\sqrt{2+x-\sqrt{2-x}}}{x} = \dfrac{2x}{x(\sqrt{2+x+\sqrt{2-x})}} \underset{x\to 0}{\longrightarrow} \dfrac{\sqrt{2}}{2}$.

6. $\dfrac{\ln(x+1)}{x} = \dfrac{\ln(x+1)-\ln 1}{x-0} \underset{x\to 0}{\longrightarrow} 1$.

3.4

Let us already note that this function is defined on $\mathbb{R}$ and continues on $\mathbb{R}^\star$ it remains to study the continuity at 0.
On the other hand $\sqrt{x^2} =\mid x\mid$, we will therefore distinguish two cases $x<0$ and $x>0$.
If $x<0$ so $f(x) = x + \dfrac{-x}{x} = x-1$, then $\lim\limits_{0\longrightarrow 0^-} f(x) = -1$
If $x<0$ so $f(x) = x + \dfrac{x}{x} = x+1$, then $\lim\limits_{0\longrightarrow 0^+} f(x) = 1$
Therefore $\lim\limits_{0\longrightarrow 0^-} f(x) \neq \lim\limits_{0\longrightarrow 0^+} f(x)$
Which shows that is not continuous at 0.

3.5

1.$\dfrac{(x\cos x-\sin x)'}{(x^2)} = \dfrac{-\sin x}{x} \underset{x\to 0}{\longrightarrow} 0$ then
$\lim\limits_{x\longrightarrow 0} \dfrac{x\cos x-\sin x}{x^2} = 0$
2. For $x\neq 0$, f is defined, continuous and differentable, we study the continuity and differientiability at $x=0$.
If $a\neq 0$ $\dfrac{\sin(ax)}{x} = a\dfrac{\sin(ax)}{ax} \underset{x\to 0}{\longrightarrow} a$
If $a=0$ $\dfrac{\sin(ax)}{x} = 0 \underset{x\to 0}{\longrightarrow} 0 = a$, and $e^{bx}-x \underset{x\to 0}{\longrightarrow} 1$
f is continuous at 0 if and only if $\begin{cases} \lim\limits_{x\to 0^-} = f(1) \\ \lim\limits_{x\to 0^+} = f(1) \end{cases}$

$\Leftrightarrow \begin{cases} a = f(1) \\ 1 = 1 \end{cases} \Leftrightarrow a = 1$
3. f must be continuous, then $a=1$.
if $x<0$ so that $f(x) = \dfrac{\sin x}{x}$, $f'(x) = \dfrac{x\cos x-\sin x}{x^2} \underset{x\to 0^-}{\longrightarrow} 0$ from question 1.
if $x>0$, so that $f(x) = e^{bx}-x$, $f'(x) = be^{bx}-1 \underset{x\to^+}{\longrightarrow} -1$.
For $a=1$ and $b=1$ f' admits one limite at 0. f continuous, f is C^1 at 0 then f is differentiable.

3.6

1.
$$\forall x \in \mathbb{R} \quad f_1'(x) = \frac{\cos x}{3 + \sin x}$$

2.
$$\forall x \in \mathbb{R} \quad f_2'(x) = \frac{x}{1 + x^2}$$

3.
$$\forall x \in \mathbb{R} \quad f_3'(x) = \frac{-4 \sin x}{4 - \cos^2 x}$$

4.
$$\forall x \in \mathbb{R}_+^\star \quad f_4'(x) = \frac{x \ln x + x + 1}{x} e^{(x+1)\ln x}$$

5.
$$\forall x \in \mathbb{R} \quad f_5'(x) = 2e^{2x} \cos(e^{2x})$$

6.
$$\forall x \in \mathbb{R}_+^\star \quad f_6'(x) = \frac{x \cos x \ln x - \sin x \ln x + \sin x}{x^2} e^{\frac{\sin x}{x} \ln x}$$

3.7

To use the mean value theorem, we must first show that f is differentiable on $\mathbb{R}$.
If $x \neq 0$, f is differentiable. Lets study the function at $x = 1$.
$\lim\limits_{x \longrightarrow 1^-} f(x) = \lim\limits_{x \longrightarrow 1^-} \frac{3 - x^2}{2} = 1 = f(1)$
and $\lim\limits_{x \longrightarrow 1^+} f(x) = \lim\limits_{x \longrightarrow 1^-} \frac{1}{x} = 1 = f(1)$
Which shows that the function is continuous at $x = 1$.
For $x < 1$,

$$\lim_{x \longrightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \longrightarrow 1^-} \frac{\frac{3 - x^2}{2} - 1}{x - 1} = \lim_{x \longrightarrow 1^-} \frac{1 - x^2}{x - 1} = \lim_{x \longrightarrow 1^-} -\frac{1 + x}{2} = -1$$

For $x > 1$,

$$\lim_{x \longrightarrow 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \longrightarrow 1^+} \frac{\frac{1}{x} - 1}{x - 1} = \lim_{x \longrightarrow 1^+} \frac{1 - x}{x(x - 1)} = \lim_{x \longrightarrow 1^+} -\frac{1}{x} = -1$$

Then f is differentiable at $x = 1$.
f is differentiable on $\mathbb{R}$, in particular f is continuous on $[0, 2]$, we can apply the mean value theorem $[0, 2]$ so there exists $c \in]0, 2[$ such that $f(2) - f(0) = (2 - 0)f(c)$.

$$f(\frac{1}{2}) = \frac{1}{2} \quad \text{and} \quad f(0) = \frac{3 - 0^2}{2} = \frac{3}{2}$$

Therefore,

$$f(2) - f(0) = (2-0)f(c) \Leftrightarrow \frac{1}{2} - \frac{3}{2} = 2f'(c) \Leftrightarrow f'(c) = -\frac{1}{2}$$

Suppose that $0 \leq c \leq 1$ then

$$f'(c) = -\frac{1}{2} \Leftrightarrow -c = -\frac{1}{2} \Leftrightarrow c = \frac{1}{2}$$

We verify that $0 \leq \frac{1}{2} \leq 1$ then $c = \frac{1}{2}$ is a solution.
Suppose $1 < c \leq 2$, so,

$$f'(c) = -\frac{1}{2} \Leftrightarrow -\frac{1}{x^2} = -\frac{1}{2} \Leftrightarrow x^2 = 2 \Leftrightarrow x = \pm\sqrt{2}$$

we have $-\sqrt{2} \notin]1,2[$ and $\sqrt{2} \in]1,2[$.
So, $\sqrt{2}$ is solution, there are two solutions $c = \frac{1}{2}$ and $c = \sqrt{2}$.
3.8

We note x the distance AD, see the figure, and v the speed of the tractor on the road. The travel time on the road is $\frac{x}{v}$. The distance traveled in the field is

$$BD = \sqrt{d^2 + (L-x)^2}$$

and the travel time in the field is $\frac{BD}{\left(\frac{v}{2}\right)}$. The total travel time to go from A to B depending on the distance x traveled on the road is therefore

$$T(x) = \frac{x}{v} + \frac{2\sqrt{d^2 + (L-x)^2}}{v} = \frac{1}{v}\left(x + 2\sqrt{d^2 + (L-x)^2}\right)$$

We are therefore led to determine the minimum of the function T on $[0, L]$.
Let's start by determining the possible values for the extrema of T.
We verify that for all $x \in [0, L]$

$$T'(x) = \frac{1}{v}\left(1 - \frac{2(L-x)}{\sqrt{d^2 + (L-x)^2}}\right)$$

We then have

$$T'(x) = 0 \Leftrightarrow \sqrt{d^2 + (L-x)^2} = 2(L-x) \Leftrightarrow d^2 + (L-x)^2 = 4(L-x)^2 \Leftrightarrow 3(L-x)^2 = d^2 \Leftrightarrow L - x = \frac{d}{\sqrt{3}}$$

either

$$x_0 = L - \frac{d}{\sqrt{3}}, \qquad x_0 \in [0, L] \Leftrightarrow L > \frac{d}{\sqrt{3}}.$$

We have tow cases:
1) If $L \leq \frac{d}{\sqrt{3}}$, so T' has no zeros on $]0, L[$.
The sign of T' is that of $T'(L) = \frac{1}{v}$. It is positive. The function T then is strictly increasing on $[0, L]$, and its minimum is reached at 0 and take the value $T(0) = \frac{2}{v}\sqrt{L^2 + d^2}$
To minimize travel time, the tractor must enter the field at A. 2) If $L > \frac{d}{\sqrt{3}}$, so T' equal to zero at $x_0 \in]0, L[$, we have $T'(L) = \frac{1}{v} > 0$ and

$$T'(0) = \frac{1}{v\sqrt{d^2 + L^2}}\left(\sqrt{d^2 + L^2} - 2L\right) = \frac{1}{v\sqrt{d^2 + L^2}}\frac{d^2 - 3L^2}{\sqrt{d^2 + L^2} + 2L} < 0,$$

Because $d^2 - 3L^2 < 0$ under the hypothesis $L > \frac{d}{\sqrt{3}}$. T is then decreasing on $[0, x_0]$ and increasing on $[x_0, L]$.
We deduce that T admits a minimum at x_0. To minimize travel time, the tractor must leave the road at a distance $x_0 = L - \frac{d}{\sqrt{3}}$ from point A. The travel time is then $T(x_0) = \frac{L + d\sqrt{3}}{v}$.

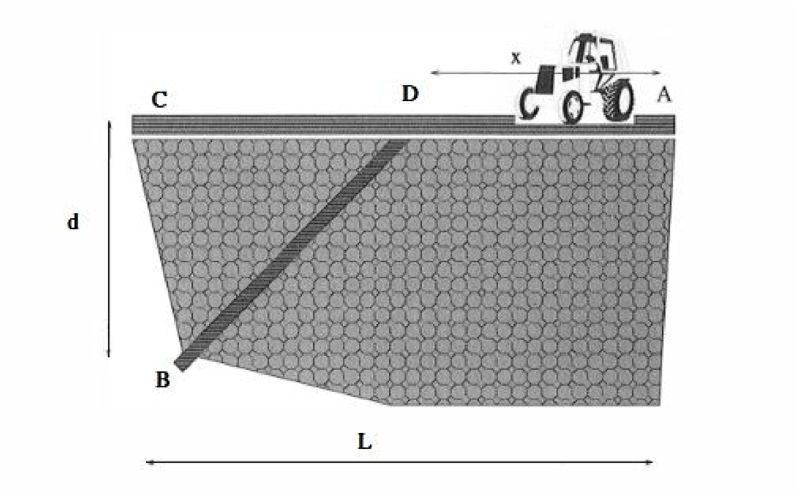


Figure 3.16: Situation considered

Chapter 4

Some elementary functions

4.1 Introduction

The term elementary function is very often mentioned in many math classes and in books. In fact, the vast majority of the functions that students and scientists come across are elementary functions of a real variable. However, there is a lack of a precise mathematical definition of elementary functions. Only a few authors in their textbooks, e.g. Stewart in his Calculus books try to give a description of elementary functions. Unfortunately, these descriptions are not given properly.
Thus, this note is written to introduce a precise mathematical definition of some elementary functions of a real variable. First we traited the trigonometric functions and their inverse function, second the hyperbolic functions and their inverses are studied. After the definition is introduced, it is easy to see that the elementary functions of a real variable posses properties that could greatly simplify the mathematical analysis needed to be done on them. Also, many problems in mathematics deal with elementary functions or even if the functions are non-elementary, very often the studying of these non-elementary functions leads to elementary functions.
The properties of elementary functions given in this note allow for problems of continuity of functions, which often arise in calculus, to be reduced to finding the set of admissible values for a given elementary function.
Then the properties of the fundamental elementary functions can be applied to finding the set of admissible values for any given elementary function, which becomes the set of points for which the elementary function is continuous.

4.2 Reciprocal circular functions

4.2.1 Brief reminders of trigonometric functions

Let's review trigonometric functions.

Proposition 4.2.1 (Sine function). *The sine function, denoted sin, is:*
- *Defined on* $\mathbb{R}$.
- *With values in* $[-1, +1]$.
- *Odd.*
- 2π*-periodic.*
- *Continuous on* $\mathbb{R}$.
- *Differentiable on* $\mathbb{R}$. *and* $\forall x \in \mathbb{R}$ $\sin' x = \cos x$.
- *Class* $\mathcal{C}^\infty$ *on* $\mathbb{R}$.
- *Forthermore, the restriction of the sine function to* $[-\frac{\pi}{2}, +\frac{\pi}{2}]$ *is strictly increasing.*

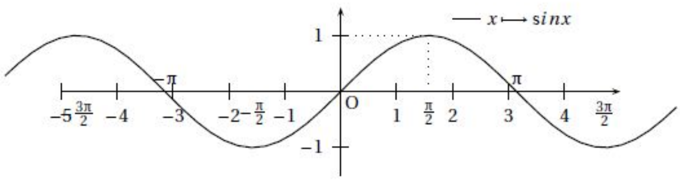


Figure 4.1: Sine function

Proposition 4.2.2 (Cosine function). *The cosine function, denoted cos, is:*
- *Defined on* $\mathbb{R}$.
- *With values in* $[-1, +1]$.
- *Even.*
- 2π*-periodic.*
- *Continuous on* $\mathbb{R}$.
- *Differentiable on* $\mathbb{R}$. *and* $\forall x \in \mathbb{R}$ $\cos' x = -\sin x$.
- *Class* $\mathcal{C}^\infty$ *on* $\mathbb{R}$.
- *Forthermore, the restriction of the cosine function to* $[0, \pi]$ *is strictly decreasing.*

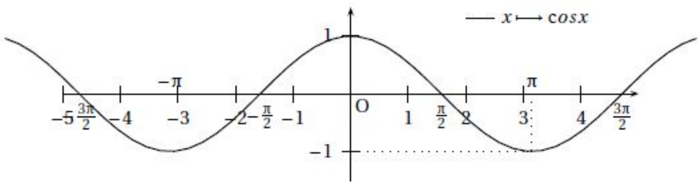


Figure 4.2: Cosine function

Proposition 4.2.3 (Tangent function)**.** *The tangent function, denoted tan, it is given by:*

$$\forall x \in \mathbb{R} - \left\{\frac{\pi}{2} + k\pi / k \in \mathbb{Z}\right\}, \tan x = \frac{\sin x}{\cos x}.$$

And it is:

- *Defined on* $\mathbb{R} - \left\{\frac{\pi}{2} + k\pi / k \in \mathbb{Z}\right\}$.
- *With values in* $\mathbb{R}$.
- *Even.*
- π*-periodic.*
- *Continuous on* $\mathbb{R} - \left\{\frac{\pi}{2} + k\pi / k \in \mathbb{Z}\right\}$.
- *Differentiable on* $\mathbb{R} - \left\{\frac{\pi}{2} + k\pi / k \in \mathbb{Z}\right\}$. *and* $\forall x \in \mathbb{R} - \left\{\frac{\pi}{2} + k\pi / k \in \mathbb{Z}\right\}$ $\tan' x = 1 + \tan^2 x = \dfrac{1}{\cos^2 x}$.
- *Class* $\mathcal{C}^{\infty}$ *on* $\mathbb{R} - \left\{\frac{\pi}{2} + k\pi / k \in \mathbb{Z}\right\}$.
- *Forthermore, the restriction of the tangent function to* $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ *is strictly increasing.*

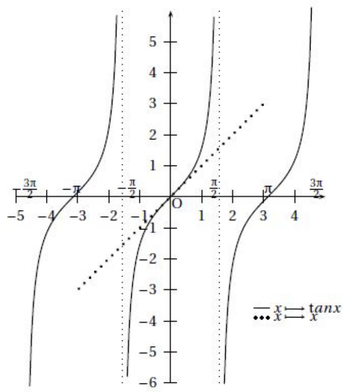


Figure 4.3: Tangent function

4.2.2 Arcsine function

Proposition 4.2.4. *The sine function is a bijection from* $[-\frac{\pi}{2}, +\frac{\pi}{2}]$ *on* $[-1, +1]$*. The reciprocal bijection is called the arcsine function and is noted arcsin*

$$\arcsin : \begin{cases} [-1, +1] & \longrightarrow & [-\frac{\pi}{2}, +\frac{\pi}{2}] \\ y & \longmapsto & \arcsin y \end{cases}$$

$$\forall y \in [-1, +1] \ , \ \sin(\arcsin y) = y$$

$$\forall x \in [-\frac{\pi}{2}, +\frac{\pi}{2}] \ , \ \arcsin(\sin x) = x$$

Forthermore, the arcsin function is:
- Strictly increasing on $[-1, +1]$
- Odd.
- Continuous on $[-1, +1]$*.*
- Differentiable on $]-1, +1[$ *and*

$$\forall y \in]-1, +1[, \arcsin' y = \frac{1}{\sqrt{1 - y^2}}$$

- Class $\mathcal{C}^\infty$ *on* $]-1, +1[$
- Realise a bijection from $]-1, +1[$ *on* $[-\frac{\pi}{2}, +\frac{\pi}{2}]$

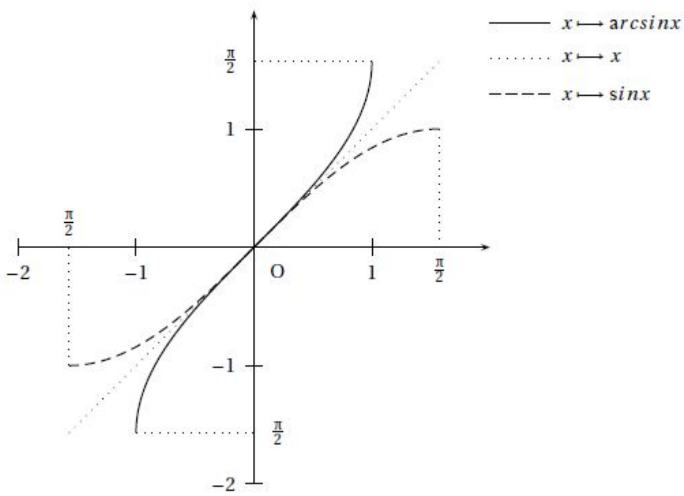


Figure 4.4: Arcsine function

4.2.3 Arccosine function

Proposition 4.2.5. *The cosine function is a bijection from* $[0, \pi]$ *on* $[-1, +1]$*. The reciprocal bijection is called the arccosine function and is noted arccos*

$$\arccos : \begin{cases} [-1, +1] & \longrightarrow & [0, \pi] \\ y & \longmapsto & \arccos y \end{cases}$$

$$\forall y \in [-1, +1] \ , \ \cos(\arccos y) = y$$

$$\forall x \in [0, \pi] \ , \ \arccos(\cos x) = x$$

Forthermore, the arcsin function is:
- Strictly decreasing on $[-1, +1]$*.*
- Continuous on $[-1, +1]$*.*
- Differentiable on $]-1, +1[$ *and*

$$\forall y \in]-1, +1[, \arccos' y = \frac{-1}{\sqrt{1 - y^2}}$$

- Class $\mathcal{C}^\infty$ *on* $]-1, +1[$
- Realise a bijection from $]-1, +1[$ *on* $[0, \pi]$

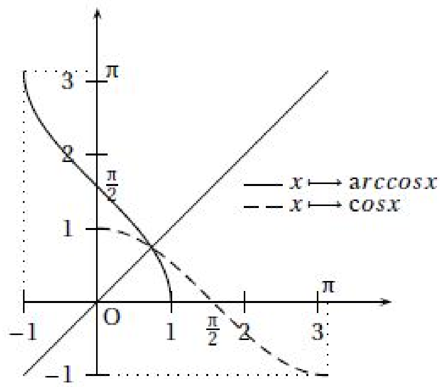


Figure 4.5: Arccosine function

4.2.4 Arctangent function

Proposition 4.2.6. *The tangent function is a bijection from* $]-\frac{\pi}{2}, \frac{\pi}{2}[$ *with values in* $\mathbb{R}$*. The reciprocal bijection is called the arctangent function denoted arctan*

$$\arctan : \begin{cases} \mathbb{R} & \longrightarrow]-\frac{\pi}{2}, \frac{\pi}{2}[\\ y & \longmapsto \arctan y \end{cases}$$

$$\forall y \in \mathbb{R} \ , \ \tan(\arctan y) = y$$

$$\forall x \in]-\frac{\pi}{2}, \frac{\pi}{2}[\ , \ \arctan(\tan x) = x$$

Forthermore, the arctan function is:
- Strictly increasing on $\mathbb{R}$.
- Continuous on $\mathbb{R}$.
- Odd - Differentiable on $\mathbb{R}$ and

$$\forall y \in \mathbb{R}, \arctan' y = \frac{1}{1+y^2}$$

- Class $\mathcal{C}^\infty$ on $\mathbb{R}$
- Realise a bijection from $\mathbb{R}$ on $]-\frac{\pi}{2}, \frac{\pi}{2}[$

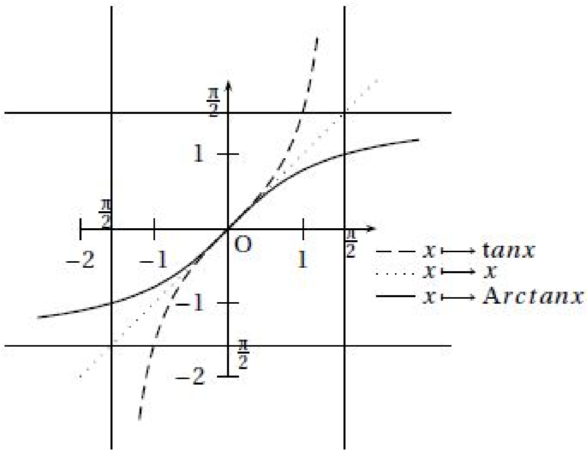


Figure 4.6: Arctangent function

4.3 Hyperbolic functions

4.3.1 Definitions and first properties

Hyperbolic sine and cosine

Definition 4.3.1. The functions hyperbolic sine sh and hyperbolic cosine ch are defined on $\mathbb{R}$ by

$$\text{ch} : \begin{cases} \mathbb{R} & \longrightarrow \mathbb{R} \\ x & \longmapsto \dfrac{e^x + e^{-x}}{2} \end{cases}, \qquad \text{sh} : \begin{cases} \mathbb{R} & \longrightarrow \mathbb{R} \\ x & \longmapsto \dfrac{e^x - e^{-x}}{2} \end{cases}$$

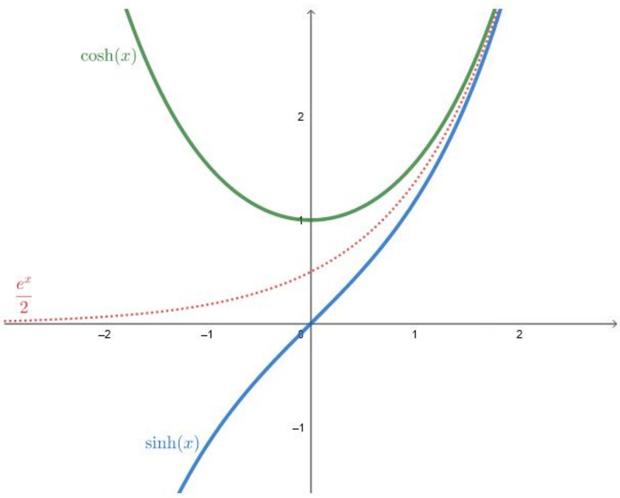
Remark 4.3.2. Any function $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ decomposes uniquely into the sum of an even function and of an odd function

$$\forall x \in I, f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

Indeed, $\dfrac{f(x) + f(-x)}{2}$ is even and $\dfrac{f(x) - f(-x)}{2}$ is odd. The hyperbolic cosine and hyperbolic sine functions are respectively the even part and the odd part of the exponential function in this decomposition.

Proposition 4.3.3. *For all* $x \in \mathbb{R}$
1. $\text{ch}x + \text{sh}x = e^x$
2. $\text{ch}x - \text{sh}x = e^{-x}$
3. $\text{ch}^2 x - \text{sh}^2 x = 1$

Proposition 4.3.4. *The functions* ch *and* sh *are differentiable on* $\mathbb{R}$*, for all* $x \in \mathbb{R}$
$\text{ch}'(x) = \text{sh}(x), \qquad \text{sh}'(x) = \text{ch}(x)$



Proposition 4.3.5. - *The function* sh *is odd, strictly increasing on* $\mathbb{R}$*, strictly negative on* $\mathbb{R}_-^\star$ *and strictly positive on* $\mathbb{R}_+^\star$ *and vanishes at* 0*.*
- *The function* ch *is even, strictly positive on* $\mathbb{R}$*, strictly decreasing on* $\mathbb{R}_-^\star$ *and strictly increasing on* $\mathbb{R}_+^\star$ *and* $\forall x \in \mathbb{R}, \mathrm{ch}x \geq 1$*.*

Hyperbolic tangent

Definition 4.3.6. The function hyperbolic tangent th is defined on $\mathbb{R}$ by

$$\mathrm{th} : \begin{cases} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{\mathrm{sh}x}{\mathrm{ch}x} \end{cases}$$

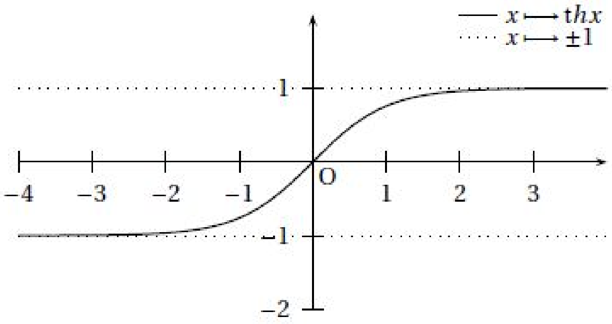


Figure 4.7: Hyperbolic tangent

Proposition 4.3.7. *The function* th *is odd, differentiable on* $\mathbb{R}$*, and for all* $x \in \mathbb{R}$

$$\mathrm{th}'x = 1 - \mathrm{th}^2 x = \frac{1}{\mathrm{ch}^2 x}.$$

Consequently, th is strictly increasing on $\mathbb{R}$ *and vanishes at* 0*. It admits at* $-\infty$ *a horizontal asymptote of equation* $y = -1$ *and at* $+\infty$ *a horizontal asymptote of equation* $y = -1$

4.3.2 Addition formulas for hyperbolic functions

Proposition 4.3.8. *For all $x, y \in \mathbb{R}$*

$$\begin{aligned} \mathrm{ch}(x+y) &= \mathrm{ch}x\mathrm{ch}y + \mathrm{sh}x\mathrm{sh}y \\ \mathrm{ch}(x-y) &= \mathrm{ch}x\mathrm{ch}y - \mathrm{sh}x\mathrm{sh}y \\ \mathrm{sh}(x+y) &= \mathrm{sh}x\mathrm{ch}y + \mathrm{ch}x\mathrm{sh}y \\ \mathrm{ch}(x-y) &= \mathrm{ch}x\mathrm{ch}y - \mathrm{sh}x\mathrm{sh}y \end{aligned}$$

$$\mathrm{th}(x+y) = \frac{\mathrm{th}x + \mathrm{th}y}{1 + \mathrm{th}x\mathrm{th}y} \qquad \mathrm{th}(x-y) = \frac{\mathrm{th}x - \mathrm{th}y}{1 - \mathrm{th}x\mathrm{th}y}$$

4.3.3 Inverse hyperbolic functions

Hyperbolic sine argument function argsh

Proposition 4.3.9. *The hyperbolic sine function defines a bijection of $\mathbb{R}$ on its image $\mathbb{R}$. The reciprocal application is called a function hyperbolic sine argument and denoted* argsh*:*

$$\mathrm{argsh} : \begin{cases} \mathbb{R} & \longrightarrow \mathbb{R} \\ y & \longmapsto \mathrm{argsh}y \end{cases}$$

$$\forall y \in \mathbb{R}, \qquad \mathrm{sh}(\mathrm{argsh}y) = y$$

$$\forall x \in \mathbb{R}, \qquad \mathrm{argsh}(\mathrm{sh}x) = x$$

The function argsh *is:*
- *Odd.*
- *Continuous on $\mathbb{R}$.*
- *Diffrentiable on $\mathbb{R}$ and*

$$\forall y \in \mathbb{R}, \qquad \mathrm{argsh}'y = \frac{1}{\sqrt{1+y^2}}$$

- *Strictly increasing on $\mathbb{R}$.*
- *Realise a bijection from $\mathbb{R}$ to $\mathbb{R}$.*
- *$\mathcal{C}^\infty$ on $\mathbb{R}$.*

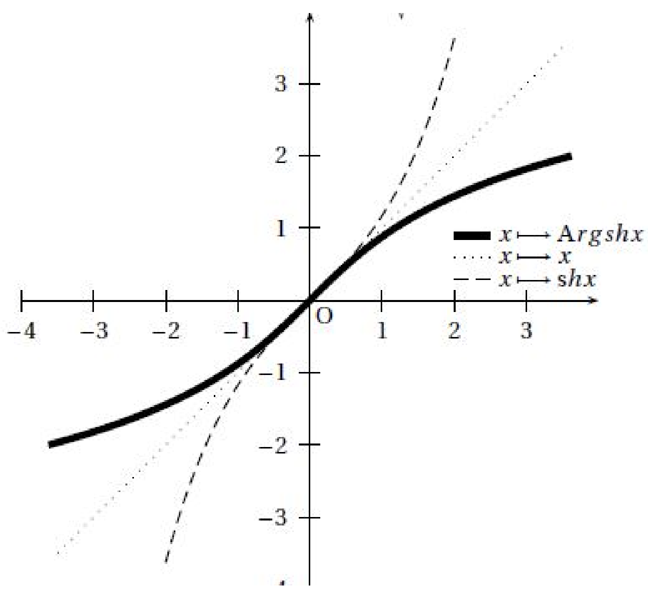


Figure 4.8: Hyperbolic sine functions and sine argument

Hyperbolic cosine argument function argch

Proposition 4.3.10. *The hyperbolic cosine function restricted on* $\mathbb{R}_+$ *define a bijection from* $\mathbb{R}_+^\star$ *on its image* $[1,+\infty[$. *The reciprocal application is called a function hyperbolic cosine argument and denoted* argch*:*

$$\text{argch} : \begin{cases} [1,+\infty[& \longrightarrow & \mathbb{R} \\ y & \longmapsto & \text{argch}y \end{cases}$$

$$\forall y \in [1,+\infty[, \qquad \text{ch}(\text{argch}y) = y$$

$$\forall x \in \mathbb{R}_+, \qquad \text{argch}(\text{ch}x) = x$$

The function argch *is:*
- *Continuous on* $[1,+\infty[$.
- *Diffrentiable on* $]1,+\infty[$ *and*

$$\forall y \in]1,+\infty[, \qquad \text{argch}'y = \frac{1}{\sqrt{1-y^2}}$$

- *Strictly increasing on* $[1,+\infty[$.
- *Realise a bijection from* $[1,+\infty[$ *to* $\mathbb{R}$.
- $\mathcal{C}^\infty$ *on* $]1,+\infty[$.

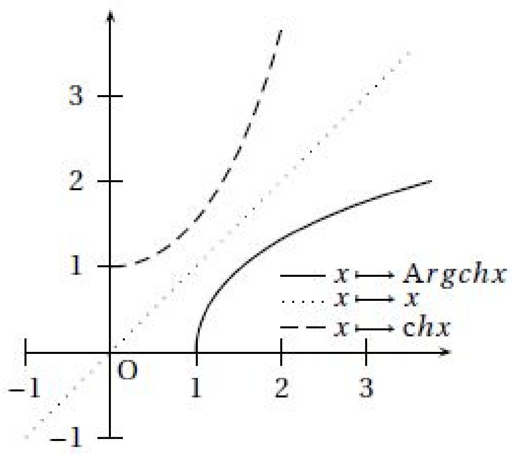


Figure 4.9: Hyperbolic cosine functions and cosine argument

Hyperbolic tangent argument function argth

Proposition 4.3.11. *The hyperbolic tangent function defines a bijection of* $\mathbb{R}$ *on its image* $]1,1[$. *The reciprocal application is called Hyperbolic tangent argument and is denoted argth.*

$$\text{argth} : \begin{cases}]-1,+1[\longrightarrow \mathbb{R} \\ y \longmapsto \text{argth}y \end{cases}$$

$$\forall y \in]-1,+1[, \qquad \text{th}(\text{argth}y) = y$$

$$\forall x \in \mathbb{R}, \qquad \text{argth}(\text{th}x) = x$$

The function argth *is:*
- *Odd.*
- *Continuous on* $]-1,+1[$.
- *Diffrentiable on* $]-1,+1[$ *and*

$$\forall y \in]1,+\infty[, \qquad \text{argth}'y = \frac{1}{1-y^2}$$

- *Strictly increasing on* $]-1,+1[$.
- *Sealise a bijection from* $]-1,+1[$ *to* $\mathbb{R}$.
- $\mathcal{C}^\infty$ *on* $]-1,+1[$.

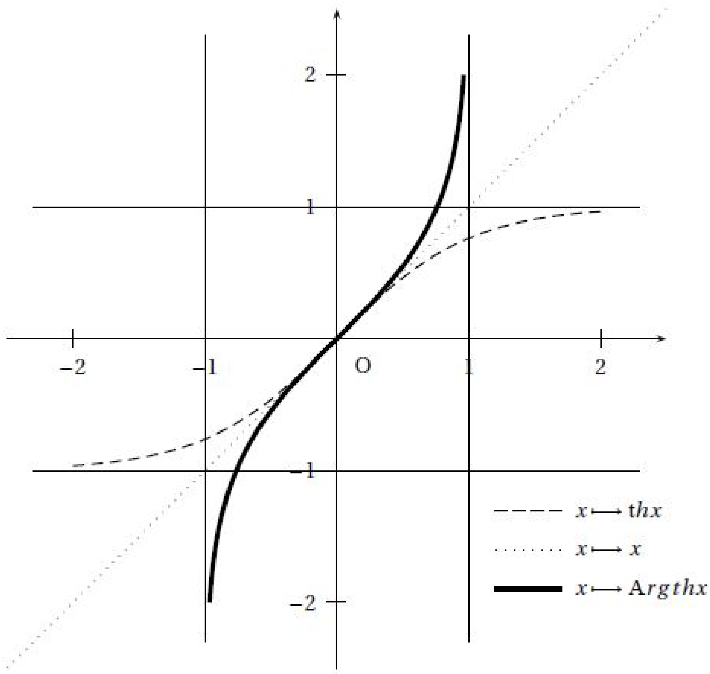


Figure 4.10: Hyperbolic tangent and tangent argument functions

4.4 Solved exercises

4.4.1 Exercises

4.1

We traced the trigonometric circle in a direct orthonormal coordinate system (see figure below). The angle α is measured in radians. The triangles OAH and OBC are rectangular respectively in H and C. We recall that the area of the angular sector OAC is $\frac{\alpha}{2}$.

1. Calculate the area of triangle OAH. Deduce that : $\forall\alpha \in]0, \frac{\pi}{2}],\ 0 < \sin\alpha < \alpha$.

2. Prove that for $\alpha \in]0, \frac{\pi}{2}]$, we have $1 > \cos^2\alpha > 1 - \alpha^2$. Deduce that $\lim_{\alpha\to 0} \cos\alpha = 1$.

3. Calculate the area of triangle OBC. Deduce the inequalities:
$\forall\alpha \in]0, \frac{\pi}{2}],\ \sin\alpha < \alpha < \tan\alpha$.

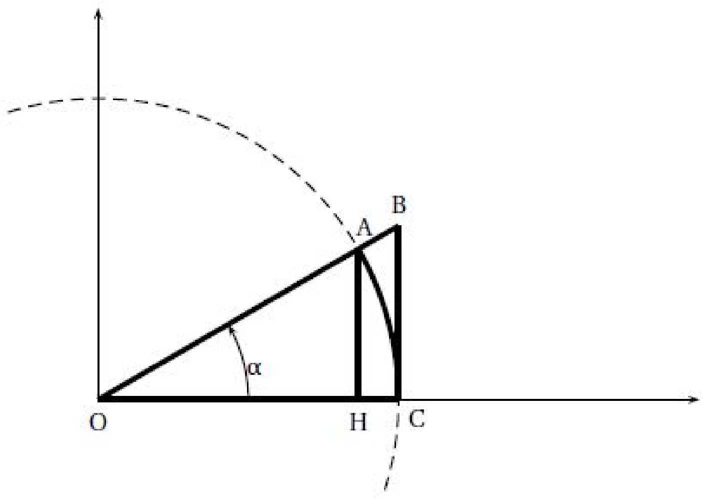
4. Deduce from the previous questions that $\lim_{\alpha\to 0} \frac{\sin\alpha}{\alpha} = 1$ and $\lim_{\alpha\to 0} \frac{\tan\alpha}{\alpha} = 1$. We thus prove that sin and tan are differentiable at 0. Explain why.

5. For $\alpha \in]0, \dfrac{\pi}{2}]$. Establish inequalities:

$$\alpha \leq \cos^2\alpha \leq \cos\alpha,\ 0 \leq 1-\cos\alpha \leq \sin^2\alpha,\ 0 \leq 1-\cos\alpha \leq \alpha^2$$

6. Deduce then limit $\lim\limits_{\alpha\to 0} \dfrac{\cos\alpha}{\alpha}$

7. Deduce $\lim\limits_{h\to 0} \dfrac{\cos(\alpha+h)-\cos\alpha}{h} = 1$ and $\lim\limits_{h\to 0} \dfrac{\sin(\alpha+h)-\cos\alpha}{h}$. What important property of cos and sin do we get to prove ?



4.2

Calculate: $\arcsin(\sin(\dfrac{3\pi}{4}))$, $\arccos(\cos(\dfrac{2009\pi}{3}))$

4.3

1. Let $x \in [-1,+1]$ sipmlify
(a) $\cos(\text{arcsin}x)$.
(b) $\sin(\text{arccos}x)$.

2. Let $x \in \mathbb{R}$ simplify
(a) $\cos(3\text{arctan}x)$.
(b) $\cos^2(\dfrac{1}{2}\text{arctan}x)$.

4.4

Solve the equation: $\text{arcsin}x = 2\text{arctan}x$

4.5

- Show that : $\text{arctan}x + \text{arctan}\frac{1}{x} = \begin{cases} \frac{\pi}{2} & \text{if } x > 0 \\ -\frac{\pi}{2} & \text{if } x < 0 \end{cases}$

- Show that $\forall x \in [-1, 1]$: $\text{arcsin}x + \text{arccos}x = \frac{\pi}{2}$

4.6

Prove that : $\forall x \in \mathbb{R},\ \forall n \in \mathbb{N}, (\text{ch}x + \text{ch}x)^n = \text{ch}nx + \text{ch}nx.$

4.7

Simplify, where they are defined, the following expressions:

1. $\text{ch}(\text{argsh}x)$
2. $\text{th}(\text{argsh}x)$
3. $\text{sh}(2\text{argsh}x)$
4. $\text{sh}(\text{argch}x)$
5. $\text{th}(\text{argch}x)$
6. $\text{ch}(\text{argth}x)$

4.4.2 Solutions

4.1

1. The area of triangle OAH is given by $\frac{OC \times HA}{2} = \frac{\sin\alpha}{2}$. If $\alpha \in]0, \frac{\pi}{2}]$ then the triangle OAH is not flat and its area is positve therefore $\sin\alpha > 0$. Furthermore, the triangle OAH is included in the angular sector OAC and therefore $\frac{\sin\alpha}{2} < \frac{\alpha}{2}$ which proves the second inequality.
2. If $\alpha \in]0, \frac{\pi}{2}]$. We use the previous question. From $0 < \sin\alpha < \alpha$ we have $0 < \sin^2\alpha < \alpha^2$ because the function $f(x) = x^2$ is increasing on $\mathbb{R}_+$. then $1 > 1 - \sin^2\alpha > 1 - \alpha^2$ which gives $1 \geq \cos^2\alpha > 1 - \alpha^2$, if $\alpha \longrightarrow 1$ then $\lim_{\alpha \longrightarrow 0} \cos\alpha = 1$.
3. The area of triangle OBC is $\frac{OC \times BC}{2} = \frac{\tan\alpha}{2}$. As the AHC triangle is strictly included in the OAC sector and that this sector is strictly included in the triangle OBC, we deduce that $\sin\alpha < \alpha < \tan\alpha$.

4.Let $\alpha \in]0, \frac{\pi}{2}]$. We deduce from the inequality of question 1. that $0 < \frac{\sin\alpha}{\alpha} < 1$. Likewise, from $\tan\alpha < \alpha$ we deduce $\sin\alpha > \alpha\cos\alpha$ and therfore, $\frac{\sin\alpha}{\alpha > \cos\alpha} \underset{\alpha \to 0^+}{\longrightarrow} 1$, then we deduce $\lim_{\alpha \longrightarrow 0^+} \frac{\sin\alpha}{\alpha} = 1$. At 0^- we find same limit by parity. Secod limit is obvious. We recognize that $\frac{\sin\alpha}{\alpha}$ is the rate of increase of sin at 0. It then admits a limit when $\alpha \longrightarrow 0$ and sin is differentiable at 0 with a derivative equal to 1. Same for tan.

5. We know that $0 \leq \cos\alpha \leq 1$ so by multiplying by $\cos\alpha$ which is positive, we obtain the first inequality. We deduces that $1 \geq 1\cos^2\alpha \geq 1\cos\alpha$ and therefore the second inequality. The last follows by using that $\sin\alpha < \alpha$.
6. We divide the last inequality by $\alpha \in]0, \frac{\pi}{2}]$:

$$0 \leq \frac{1-\cos^2\alpha}{\alpha} \leq \frac{\alpha^2}{\alpha} = \alpha \underset{\alpha\to 0^+}{\longrightarrow} 0$$

So $\lim\limits_{\alpha\longrightarrow 0^+} \frac{1-\cos\alpha}{\alpha} = 0$, By parity, it follows that $\lim\limits_{\alpha\longrightarrow 0} \frac{1-\cos\alpha}{\alpha} = 0$.
7. Using the addition formulas and previous questions:

$$\frac{\cos(\alpha+h)-\cos\alpha}{h} = \cos\alpha\frac{\cos h - 1}{h} - \sin\alpha\frac{\sin h}{h} \underset{h\to 0}{\longrightarrow} -\sin\alpha$$

We recognize in the first term of the previous line the rate of increase of cos in α. It has been proven that it tends to $\sin\alpha$ when $h \longrightarrow 0$. Therefore cos is differentiable in α and its derivative is $\sin\alpha$. We proceed in the same way for sin.

4.2

1. we know that $\arcsin : [-1,1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$, we must determine the real $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\sin x = \sin(\frac{3\pi}{4}$. So, $\sin(\frac{3\pi}{4}) = \sin(\frac{\pi}{2}+\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} = \sin\frac{\pi}{4}$, then $\arcsin(\sin(\frac{3\pi}{4})) = \frac{\pi}{4}$.
2 We have $\arccos : [-1,1] \longrightarrow [0,\pi]$ so we must determine the real $x \in [0,\pi]$ such that $\cos x = \cos(\frac{2009\pi}{3})$. But $2009 = 3 \times 670 - 1$, so $2009\pi = -\frac{\pi}{3}[2\pi]$. But $\cos(-\frac{\pi}{3}) = \cos\frac{\pi}{3}$. So $\arccos(\cos(\frac{2009\pi}{3})) = \frac{\pi}{3}$.

4.3

1.Let $x \in (-1,1]$.
(a) $\arcsin x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, so $\cos(\arcsin x) = \sqrt{1-\sin^2(\arcsin x)} = \sqrt{1-x^2}$.
(b) $\arccos x \in [0,\pi]$, so $\sin(\arccos x) = \sqrt{1-\cos^2(\arccos x)} = \sqrt{1-x^2}$.
2. Let $x \in \mathbb{R}$. Remark that, for $X \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, as $1+\tan^2 X = \frac{1}{\cos^2 X}$. It comes $\cos X = \sqrt{1+\tan^2 x}$, so $\cos\arctan x = \frac{1}{\sqrt{1+x^2}}$ because $\arctan x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$

(a) We know that $\cos(3X) = 4\cos^3 X - 3\cos X$ it comes,

$$\begin{aligned}\cos(3\text{arctan}x &= 4\cos^3 \text{arctan}x - 3\cos\text{arctan}x \\ &= \frac{4}{(1+x^2)^{\frac{3}{2}}} - \frac{3}{(1+x^2)^{\frac{1}{2}}} \\ &= \frac{1-3x^2}{(1+x^2)^{\frac{3}{2}}}\end{aligned}$$

(b) As $\cos^2 X = \dfrac{1+\cos 2X}{2}$:

$$\cos^2\left(\frac{1}{2}\text{arctan}x\right) = \frac{1}{2}(\cos\text{arctan}x + 1) = \frac{1}{2\sqrt{1+x^2}} + \frac{1}{2}.$$

4.4

For all $X \in]-\dfrac{\pi}{2}, \dfrac{\pi}{2}[$, as $1+\tan^2 X = \dfrac{1}{\cos^2 X}$, it comes that $X = \sqrt{1+\tan^2 X}$, so $\cos\text{arctan}x = \dfrac{1}{\sqrt{1+x^2}}$, because $\arctan x \in]-\dfrac{\pi}{2}, \dfrac{\pi}{2}[$, and as $\sin X = \mp\sqrt{1-\cos^2 X}$, also we have $\sin\text{arctan}x = \dfrac{x}{\sqrt{1+x^2}}$.

We have then:

$$\begin{aligned}&\text{arcsin}x = 2\text{arctan}x \\ \Longrightarrow\quad & x = \sin(2\text{arctan}x) \\ \Longrightarrow\quad & x = 2\sin(\text{arctan}x)\cos(\text{arctan}x) \\ \Longrightarrow\quad & x = \frac{2x}{1+x^2} \\ \Longrightarrow\quad & x^3 - x = 0 \Longrightarrow x = -1,\ x = 0 \text{ or } x = 1.\end{aligned}$$

4.5

The same method applies in both questions.

1. Let $\theta_1 : \begin{cases} \mathbb{R}^\star \longrightarrow \mathbb{R} \\ x \longmapsto \text{arctan}x + \text{arctan}\dfrac{1}{x} \end{cases}$, θ_1 is differentiable on $\mathbb{R}^\star$ and $\theta_1' = 0$.

Therefore, there exist real c_1 and c_2 such that $\theta_{1|\mathbb{R}^\star_-} = c_1$ and $\theta_{1|\mathbb{R}^\star_+} = c_2$. By taking the limit of θ_1 when x tends towards $-\infty$ and $+\infty$ we show that $c_1 = -\dfrac{\pi}{2}$ and $c_2 = \dfrac{\pi}{2}$.

2. Let $\theta_2 : \begin{cases} [-1,+1] \longrightarrow \mathbb{R} \\ x \longmapsto \text{arcsin}x + \text{arccos}x \end{cases}$, θ_2 is differentiable on $[-1,+1]$ and $\theta_2' = 0$.

θ_2 is therefore constant on $[-1,1]$ and evaluating the expression at$x = 0$, we show that this constant is equal to $\dfrac{\pi}{2}$.

4.7

1. Let $x \in \mathbb{R}$, as $\text{ch}x > 0$ on $\mathbb{R}$, then $\text{ch}x = \sqrt{1+\text{sh}^2 x}$

$$\text{ch}(\text{argsh}x) = \sqrt{1+\text{sh}^2(\text{argsh}x)} = \sqrt{1+x^2}$$

2 Let $x \in \mathbb{R}$, as $\text{th}x = \dfrac{\text{sh}x}{\text{ch}x}$:

$$\text{th}(\text{argsh}x) = \frac{\text{sh}(\text{argsh}x)}{\text{ch}(\text{argsh}x)} = \frac{x}{\sqrt{1+x^2}}$$

3. Let $x \in \mathbb{R}$, Using the addition formulas,

$$\text{sh}(2\text{argsh}x) = 2\text{ch}(\text{argsh}x)\text{sh}(\text{argsh}x) = 2x\sqrt{1+x^2}$$

4. Let $x \in [1, +\infty[$, as $\text{sh}x > 0$, then $\text{sh}x = \sqrt{\text{ch}^2 x - 1}$

$$\text{sh}(\text{argch}x) = \sqrt{\text{ch}^2(\text{argch}x) - 1} = \sqrt{x^2-1}$$

5. Let $x \in [1, +\infty[$,

$$\text{th}(\text{argch}x) = \frac{\text{sh}(\text{argch}x)}{\text{ch}(\text{argch}x)} = \frac{\sqrt{x^2-1}}{x}$$

6. Let $x \in \mathbb{R}$, from, $1 - \text{th}^2 x = \dfrac{1}{\text{ch}^2 x}$

$$\text{ch}(\text{argth}x) = \frac{1}{\sqrt{1-\text{th}^2(\text{argth}x)}} = \frac{1}{\sqrt{1-x^2}}.$$

Chapter 5

Linear algebra

5.1 Introduction

Linear algebra is a universal language used to describe many phenomena in mechanics, electronics, and economics, for example. It is therefore not surprising to find this subject taught at the beginning of many university courses because it is necessary to be able to express more advanced concepts in subsequent years. So it is crucial for a student to master their vocabulary and grammar as early as possible. However, even if it is a field of mathematics, it is not necessary to be a sophisticated mathematician to learn it, Fortunately. This course aims to try to learn this beautiful language that is linear algebra to first year students of matter sciences.

5.2 Vector spaces , vector subspaces

In this chapter, $\mathbb{K}$ denotes a body. In most examples, this will be the field of reals $\mathbb{R}$.

Definition 5.2.1 (Vector spaces)**.** Let E be a non-empty set. We say that $(E;+;\times)$ is a $\mathbb{K}$ **-vector space** (or **vector space on** $\mathbb{K}$) if and only if

1. $(E;+)$ is an abelian group,

2. The $\times$ law is external on E, i.e.

$$\begin{aligned} \mathbb{K}\times E &\to E \\ (\lambda, u) &\to \lambda \cdot u \end{aligned}$$

which verify the following properties

- $\forall (u,v) \in E^2; \quad \forall \lambda \in \mathbb{K} : \lambda \cdot (u+v) = \lambda \cdot u + \lambda \cdot v,$

- $\forall u \in E; \quad \forall(\lambda, \mu) \in \mathbb{K}^2 : (\lambda + \mu) \cdot u = \lambda \cdot u + \mu \cdot u,$
- $\forall u \in E; \quad \forall(\lambda, \mu) \in \mathbb{K}^2 : (\lambda + \mu) \cdot u = \lambda \cdot (\mu \cdot u) = \lambda \cdot (\mu \cdot u),$
- $\forall u \in E; \quad 1_K \cdot u = u.$

Remark 5.2.2. The elements of E are called **vectors** and those of K **scalars**.

Definition 5.2.3 (Vector subspace)**.** Let E be a $\mathbb{K}$-vector space. A part F of E is called a **vector subspace** if

- $0_E \in F,$
- $\forall(u, v) \in F^2; \quad u + v \in F,$
- $\forall u \in F; \quad \forall \lambda \in \mathbb{K} : \lambda \cdot u \in F.$

Example 5.2.4. The set $F = \left\{(x, y) \in \mathbb{R}^2 / \ x + y = 0\right\}$ is a vector subspace of $\mathbb{R}^2$. Indeed

(a) $(0, 0) \in F,$

(b) If $u = (x_1, y_1)$ and $v = (x_2, y_2)$ belong to F, then

$$u = x_1 + y_1 = 0 \text{ and } v = x_2 + y_2 = 0$$

So

$$(x_1 + x_2) + (y_1 + y_2) = 0$$

and so

$$u + v = (x_1 + x_2, y_1 + y_2) \in F,$$

(c) if $u = (x, y) \in F$ and $\lambda \in \mathbb{R}$, then $x + y = 0$ So

$$\lambda \cdot x + \lambda \cdot y = 0,$$

hence $\lambda \cdot u \in F$.

Definition 5.2.5 (Linear combinations)**.** Let $n \geq 1$ be an integer, let $v_1, v_2, ..., v_n$, n vectors of a vector space E. Any vector of the form

$$u = \lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n v_n,$$

where $\lambda_1, \lambda_2, ..., \lambda_n$ are elements of $\mathbb{K}$ is called **linear combination** of vectors $v_1, v_2, ..., v_n$.

Remark 5.2.6. The scalars $\lambda_1, \lambda_2, ..., \lambda_n$ are called **coefficients** of the linear combination.

Example 5.2.7. In the $\mathbb{R}$-vector space $\mathbb{R}^3$, $(3, 3, 1)$ is linear combination of the vectors $(1, 1, 0)$ and $(1, 1, 1)$ because we have equality

$$(3, 3, 1) = 2(1, 1, 0) + (1, 1, 1).$$

Proposition 5.2.8 (Intersection of two subspaces)**.** *Let F, G be two vector subspaces of a $\mathbb{K}$-space vector E.* ***The intersection*** *$F \cap G$ is a subspace vector of E.*

Proof. As an exercise left to the students. □

Definition 5.2.9 (Sum of two vector subspaces)**.** Let F and G be two vector subspaces of a $\mathbb{K}$-space vector E. All

$$F + G = \{u + v \;/\; u \in F \text{ and } v \in G\},$$

is called **sum** of the vector subspaces F and G.

Definition 5.2.10 (Direct sum of two subspaces)**.** Let F and G be two vector subspaces of E. F and G are **direct sum** in E if

- $F \cap G = \{0_E\}$,
- $F + G = E$,

We then denote $F \oplus G = E$.

Definition 5.2.11 (Subspaces vector additional)**.** If F and G are a direct sum, we say that F and G are **additional** vector subspaces in E.

Definition 5.2.12 (Generated subspace)**.** Let $V = \{v_1, v_2, ..., v_n\}$ be a finite set of vectors of a $\mathbb{K}$-vector space E. We call **vector subspace generated** by V the set of linear combinations of the vectors of V and is denoted $Vect\,(V)$. So we have

$$u \in Vect(V) \text{ there exists} \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K} \text{ such that } u = \lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n v_n.$$

Example 5.2.13.

1. The vector subspace $F = \{(x; y) \in \mathbb{R}; \; x = y\}$ is generated by $\{(1; 1)\}$, indeed

$$\begin{aligned} F &= \{(x, y) \in \mathbb{R}; \; x = y\} \\ F &= \{(x, x); \; x \in \mathbb{R}\} \\ F &= \{x(1, 1); \; x \in \mathbb{R}\} \\ F &= Vect\,\{(1, 1)\}. \end{aligned}$$

2. The vector subspace $F = \left\{(x, y, z) \in \mathbb{R}^3; \; x + y + z = 0\right\}$ is generated by $\{(-1, 1, 0), (-1, 0, 1)\}$, indeed

$$\begin{aligned} F &= \left\{(x, y, z) \in \mathbb{R}^3; \; x + y + z = 0\right\}, \\ F &= \left\{(x, y, z) \in \mathbb{R}^3; \; x = -y - z\right\}, \\ F &= \{(-y - z, y, z); \; y, z \in \mathbb{R}\}, \\ F &= \{(-y, y, 0), (-z, 0, z); \; y, z \in \mathbb{R}\}, \\ F &= \{y(-1, 1, 0), z(-1, 0, 1); \; y, z \in \mathbb{R}\}, \\ F &= Vect\,\{(-1, 1, 0), (-1, 0, 1)\}. \end{aligned}$$

Definition 5.2.14 (Generating family of a vector space)**.** Let E be a $\mathbb{K}$-vector space and $\mathcal{F} = \{v_1, v_2, ..., v_n\}$ a family of vectors of E. We say that $\mathcal{F}$ is **generator** of E if and only if $E = \mathcal{V}ect(\mathcal{F})$, i.e.

$$\forall u \in E \text{ there exists } \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K} \text{ such that } u = \sum_{i=0}^{n} \lambda_i u_i$$

Definition 5.2.15 (Free family)**.** Let $(E; +; \times)$ be a $\mathbb{K}$-vector space and $V = \{v_1, v_2, ..., v_n\}$ a family of vectors of E. V is said **free** in E if and only if none of the vectors v_i can be written as a linear combination of the other vectors.

- In other words, if they exist $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K}$ such that

$$\sum_{i=0}^{n} \lambda_i v_i = 0_E \implies \lambda_i = 0, \quad \forall i = 1, ..., n$$

We say in this case that the vectors $v_1, v_2, ..., v_n$ are **linearly independent**.

Definition 5.2.16 (Linked family)**.** In the case where V is not free, we say that it is **bound** or that the vectors $v_1, v_2, ..., v_n$ are **linearly dependent**.

- We say in this case that the vectors $v_1, v_2, ..., v_n$ are linearly dependent.

Example 5.2.17. $\mathcal{F} = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$ is a generating family of $\mathbb{R}^4$, because $\forall x, y, z, t \in \mathbb{R}$

$$(x, y, z, t) = x\,(1, 0, 0, 0) + y(0, 1, 0, 0) + z(0, 0, 1, 0) + t(0, 0, 0, 1).$$

Definition 5.2.18. We say that a vector space E is of **finite dimension**, if we can find a generating family in E. Otherwise, E is of infinite dimension.

In all that follows, the vector spaces considered are finite dimensions.

Definition 5.2.19 (Basis of a vector space)**.** Let E be a $\mathbb{K}$-vector space and $\mathcal{B}$ be a vector family of E. $\mathcal{B}$ is said to be **base** of E if and only if

1. $\mathcal{B}$ is generator of E, i.e. $\mathcal{V}ect(\mathcal{B}) = E$.
2. $\mathcal{B}$ is free in E.

Example 5.2.20.

- $\mathcal{B}_1 = \{(1, 0), (0.1)\}$ is the canonical basis of $\mathbb{R}^2$,
- $\mathcal{B}_2 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is the canonical base of $\mathbb{R}^3$,
- $\mathcal{B}_3 = \{(1, 0, ..., 0), (0, 1, 0, ..., 0), ..., (0, ..., 0, 1)\}$ is the canonical basis of $\mathbb{R}^n$.

Definition 5.2.21 (Dimension of a vector space)**.** Let E be a $\mathbb{K}$-vector space and $\mathcal{B}$ be any basis of E. The **dimension** of E is equal to the number of vectors of $\mathcal{B}$ and we write

$$\dim E = Card(\mathcal{B}).$$

Proposition 5.2.22. *If E is a finite-dimensional vector space, then all bases of E have the same number of vectors.*

Remark 5.2.23. If $E = \{0_E\}$ then $\dim E = 0$.

Proposition 5.2.24. *If E is a finite-dimensional $\mathbb{K}$-vector space and F is a vector subspace of E, then*

$$\begin{aligned} \dim F &\leq \dim E, \\ \textit{if } \dim F &= \dim E, \textit{ then } E = F. \end{aligned}$$

5.3 Linear application

Let E and F be two $\mathbb{K}$-vector spaces.

Definition 5.3.1. A map f from E to F is a linear map if it satisfies the following two conditions

1. $f(u+v) = f(u) + f(v)$, for all $u, v \in E$,

2. $\lambda \cdot f(u) = f(\lambda \cdot u)$, for all $u \in E$ and all $\lambda \in \mathbb{K}$.

Notation 5.3.2. The set of linear maps of E in F is denoted $\mathcal{L}(E, F)$.

Example 5.3.3. The map f defined by

$$f : \mathbb{R}^3 \to \mathbb{R}^2$$
$$(x, y, z) \to f(x, y, z) = (-2x, y + 3z)$$

is a linear map. Indeed, let $u = (x, y, z)$ and $v = (x', y', z')$ two elements of $\mathbb{R}^3$ and λ a real.

$$\begin{aligned} f(u+v) &= f(x+x', y+y', z+z') \\ &= (-2(x+x'), y+y'+3(z+z')) \\ &= (-2x, y+3z) + (-2x', y'+3z') \\ &= f(x,y,z) + f(x',y',z') \\ &= f(u) + f(v), \end{aligned}$$

and

$$\begin{aligned} f(\lambda \cdot u) &= f(\lambda x, \lambda y, \lambda z) = (-2\lambda x, \lambda y + 3\lambda z) \\ &= \lambda \cdot (-2x, y+3z) = \lambda \cdot f(x,y,z) \\ &= \lambda \cdot f(u). \end{aligned}$$

Proposition 5.3.4. *If f is a linear map of E into F, then*

- $f(0_E) = 0_F$,
- $f(-u) = -f(u)$, *for everything* $u \in E$,
- f *is linear if and only if*

$$\forall u, v \in E; \quad \forall \lambda, \mu \in \mathbb{K} \quad f(\lambda \cdot u + \mu \cdot v) = \lambda \cdot f(u) + \mu \cdot f(v).$$

Definition 5.3.5 (Image of a linear map)**. The image** of f, denoted $\operatorname{Im}(f)$ is the part of F defined by

$$\operatorname{Im}(f) = \{f(u) \in F; \quad u \in E\}.$$

Definition 5.3.6 (Kernal of a linear application)**.** The **kernel** of f, denoted $\operatorname{Ker}(f)$, is the set of elements of E whose image is 0_F

$$\operatorname{Ker}(f) = \{u \in E; \quad f(u) = 0_F\}.$$

Proposition 5.3.7.
- *The image of f is a vector subspace of F,*
- *The kernel of f is a vector subspace of E,*
- *f is injective if and only if* $\operatorname{Ker}(f) = \{0_E\}$.
- *f is surjective if and only if* $\operatorname{Im}(f) = F$.

Example 5.3.8. In example (5.3.3), we have

$$\begin{aligned}
\operatorname{Ker}(f) &= \left\{(x,y,z) \in \mathbb{R}^3; \quad f(x,y,z) = 0_{\mathbb{R}^2}\right\} \\
&= \left\{(x,y,z) \in \mathbb{R}^3; \quad (-2x, y+3z) = (0,0)\right\} \\
&= \left\{(x,y,z) \in \mathbb{R}^3; \quad (-2x, y+3z) = (0,0)\right\} \\
&= \left\{(x,y,z) \in \mathbb{R}^3; \quad x = 0, y = -3z\right\} \\
&= \{(0,-3z,z); \quad z \in \mathbb{R}\} \\
&= \mathcal{V}ect\,\{(0,-3,1)\}.
\end{aligned}$$

et

$$\begin{aligned}
\operatorname{Im}(f) &= \left\{f(x,y,z) \in \mathbb{R}^2; \quad (x,y,z) \in \mathbb{R}^3\right\} \\
\operatorname{Im}(f) &= \left\{(-2x, y+3z); \quad (x,y,z) \in \mathbb{R}^3\right\} \\
\operatorname{Im}(f) &= \left\{(-2x,0) + (0,y) + (0,3z); \quad (x,y,z) \in \mathbb{R}^3\right\} \\
\operatorname{Im}(f) &= \mathcal{V}ect\,\{(-2,0),(0,1),(0,3)\}.
\end{aligned}$$

Definition 5.3.9 (Rank of a linear map)**.** Let $f \in \mathcal{L}(E;F)$ with E of finite dimension. The **rank** of f is the dimension of the image of f

$$rank\,(f) = \dim \operatorname{Im}(f).$$

Example 5.3.10. In example (5.3.3), we have

$$\mathrm{Im}(f) = \mathcal{V}ect\,\{(-2,0),(0,1),(0,3)\}\,,$$

but the vectors $(-2,0),(0,1),(0,3)$ are related because

$$\begin{aligned}\lambda_1\,(-2,0)+\lambda_2(0,1)+\lambda_3(0,3) &= (0,0) \qquad \forall\lambda_1,\lambda_2,\lambda_3\in\mathbb{R},\\ (-2\lambda_1,\lambda_2+3\lambda_3) &= (0,0)\,,\\ \lambda_1 &= 0, \qquad \lambda_2=-3\lambda_3,\end{aligned}$$

and as the vectors $(-2,0),(0,1)$, are free since

$$\begin{aligned}(-2\lambda_1,0)+\lambda_2(0,1) &= (0,0) \qquad \forall\lambda_1,\lambda_2\in\mathbb{R},\\ (-2\lambda_1,\lambda_2) &= (0,0) \Longrightarrow \lambda_1=\lambda_2=0,\end{aligned}$$

then the family $\mathcal{B}=\{(-2,0),(0,1)\}$ is a basis of $\mathrm{Im}(f)$, therefore $\dim\mathrm{Im}(f)=2=rank\,(f)\,.$

Théorème 5.3.11 (Rank theorem)**.** *Let $f\in\mathcal{L}(E;F)$ with E of finite dimension. So*

$$\dim\mathrm{Ker}(f)+\dim\mathrm{Im}(f)=\dim E.$$

Example 5.3.12. The rank theorem is verified for the linear application of example (5.3.3). Indeed, we have shown that $\mathrm{Ker}(f)=\mathcal{V}ect\,\{(0,-3,1)\}\,$, so $\dim\mathrm{Ker}(f)=1$ therefore

$$\dim\mathrm{Ker}(f)+\dim\mathrm{Im}(f)=1+2=3=\dim\mathbb{R}^3.$$

5.4 Solved exercises

5.4.1 Exercises

5.1

Are the following parts vector subspaces of $\mathbb{R}^2$?

1. $F_1=\{(x,y)\in\mathbb{R}^2/2x+y\geq 0\}$
2. $F_2=\{(x,y)\in\mathbb{R}^2/x^2+y^2=1\}$
3. $F_3=\{(x,y)\in\mathbb{R}^2/y=x\}$
4. $F_4=\{(x,y)\in\mathbb{R}^2/x-2y=3\}$

5.2

Let $F = \{(x, y, z) \in \mathbb{R}^3 / x + y + z = 0\}$ and $G = \{(s - t, s + t, t) \in \mathbb{R}^3 / s, t \in \mathbb{R}\}$
1. Show that F and G are a vector subspaces of $\mathbb{R}^3$
2. Determine $F \cap G$.

5.3

Show that the following sets are vector subspaces by describing them in the form $Vect(\mathcal{F})$
1. $F_1 = \{(x, y) \in \mathbb{R}^2 / x - y = 0\}$
2. $F_2 = \{(x, y) \in \mathbb{R}^2 / 2x - y = 0\}$
3. $F_3 = \{(t, -2t) \in \mathbb{R}^2 / t \in \mathbb{R}\}$

5.4

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that:

$$f(x, y) = (2x - y, x + y)$$

1. Prove that f is linear.
2. Determine Ket$f$ and Imf.

5.5

Determine the kernel and the image of the linear map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ where $f(x, y, z) = (x + y - z, x - y + 2z)$. Is f injective? surjective?

5.4.2 Solutions

5.1

Recall that a part of $\mathbb{R}^2$ is a vector subspace of $\mathbb{R}^2$ if and only if it is the singleton 0, a vector line or whole $\mathbb{R}^2$.
1.The couple $(0, 1)$ is an element of F_1 but this is not the case for the couple $(0, 1)$ which is nevertheless collinear with it. F_1 is therefore not stable by linear combination and it cannot be a vector subspace of $\mathbb{R}^2$.
2. The zero pair $(0, 0)$ is not an element of F_2 and therefore F_2 cannot be a vector subspace of $\mathbb{R}^2$.
3. We easily verify that F_3 is a non-empty part of $\mathbb{R}^2$. If $(x, y), (x', y') \in F_3$ and if $\alpha, \beta \in \mathbb{R}$ then we check easily that $\alpha x + \beta x' = \alpha y + \beta y'$, and therefore that

$\alpha(x,y)+\beta(x',y')\in F_3$, F_3 is therefore a vector subspace of $\mathbb{R}^2$.
4. The zero pair $(0,0)$ is not an element of F_4 and therefore F_4 is not a vector subspace of $\mathbb{R}^2$.

5.2

Recall that a part of $\mathbb{R}^3$ is a vector subspace of $\mathbb{R}^3$ if and only if it is the singleton 0, a vector line, a vector plane or all $\mathbb{R}^3$.
1. F is a non-empty part of $\mathbb{R}^3$. If $(x,y,z),(x',y',z')\in F$ and if $\alpha,\beta\in\mathbb{R}$, then we easily verify that the triplet $\alpha(x,y,z)+\beta(x',y',z')$ verify the equation $x+y+z=0$. F is therefore stable by linear combination and forms a subspace vector of $\mathbb{R}^3$ (we will have recognized that F is a vector plane of space). We also check that G is a non-empty subset of $\mathbb{R}^3$ and if $(s-t,s+t,t)$ an if $(s'-t',s'+t',t')$ are two elements of G with $(s,s',t,t'\in\mathbb{R})$ and if then $\alpha,\beta\in\mathbb{R}$ then:

$$\alpha(s-t,s+t,t)+\beta(s'-t',s'+t',t')=(S-T,S+T,T)$$

with $S=\alpha s+\beta s'$ and $T=\alpha t+\beta t'$, and therefore G is also stable by linear combination (We will again have noticed that G is a vector plane of space).
2. To determine $F\cap G$ it is enough to solve the system $\begin{cases} x+y+z=0\\ x=s-t\\ y=s+t\\ z=t\end{cases}$ and we obtain as a solution set that parameterized by $\begin{cases} x=-\dfrac{3}{2}t\\ y=\dfrac{1}{2}t\\ z=t\end{cases}$ (we recognize the parameterized equation of a vector line).

5.3

1. $F_1=\Big\{(x,y)\in\mathbb{R}^2/x-y=0\Big\}=\Big\{(x,x)\in\mathbb{R}^2/x\in\mathbb{R}\Big\}=Vect((1,1))$
2. $F_2=\Big\{(x,y)\in\mathbb{R}^2/2x-y=0\Big\}=\Big\{(x,2x)\in\mathbb{R}^2/x\in\mathbb{R}\Big\}=Vect((1,2))$
3. $F_3=\{(t,-2t)/t\in\in\mathbb{R}\}=Vect((1,-2))$

5.4

1. We easily verify that f is linear.
2. Let us show that f is bijective. We will deduce that $\mathrm{Ker}f=\{0\}$ and $\mathrm{Im}(f)=\mathbb{R}^2$. It is enough to show that there exists one and only couple $(x,y)\in\mathbb{R}$ such that $f(x,y)=(X,Y)$. To do this, let's solve:

$\begin{cases} 2x - y = X \\ x + y = Y \end{cases}$ The unique solution is $\left(x = \dfrac{X+Y}{3}, y = \dfrac{2X-Y}{3}\right)$, and f is therefore bijective.

5.5

We have $(x, y, z) \in \text{Ker} f \Leftrightarrow \begin{cases} x + y - z = 0 \\ x - y + 2z = 0 \end{cases} \Leftrightarrow \begin{cases} z = -2x \\ y = -3x \end{cases}$ So $\text{Ker} f = Vect((1, -3, -2))$, then f is not injective.
Moreover,
$\text{Im} f = \{(x+y-z, x-y+2z)/(x, y, z) \in \mathbb{R}^3\} = \{x(1,1)+y(1,-1)+z(-1,2)/(x, y, z) \in \mathbb{R}^3\} = Vect((1,1),(1-1),(-1,2)) = \mathbb{R}^2$ because $(1,1),(1,-1)$ are not independent, so f in then surjective.

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