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QUALITATIVE STUDY OF SOME CLASSES OF DIFFERENTIAL SYSTEMS VIA INTEGRABILITY

Presented by

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Qualitative Study of Some Classes of Differential Systems Via Integrability

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"I may not have gone where I intended to go, but I think I have ended up where I needed to be."

– Douglas Adams.

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Abstract

This thesis is concerned by the occurrence of non trivial periodic solutions called limit cycles for polynomial differential systems in the real plane. This work splits into two major parts. In the first one, we have studied the existence and the number of limit cycles for two classes of planar differential systems of degrees four and seven respectively. The second part is concerned by the study of crossing limit cycles for a class of piecewise discontinuous differential systems separated by a straight line and formed by arbitrary linear and cubic parts with an isochronous center at the origin.

Keywords: Polynomial differential system, first integral, periodic orbit, limit cycle, piecewise discontinuous differential systems, isochronous cubic center, linear system, crossing limit cycle.

Mathematics Subject Classification: 34A30 · 34C05 · 34C25 · 34C07 · 37G15.

Résumé

Cette thèse s'intéresse à l'occurrence de solutions périodiques non triviales appelées cycles limites pour les systèmes différentiels polynomiaux dans le plan réel. Ce travail se compose de deux grandes parties. Dans la première, nous avons étudié l'existence et le nombre de cycles limites pour deux classes de systèmes différentiels planaires de degrés quatre et sept respectivement. La deuxième partie concerne l'étude des cycles limites de croisement pour une classe de systèmes différentiels discontinus par morceaux séparés par une droite et formés par des parties arbitraires linéaires et cubiques avec un centre isochrone à l'origine.

Mots-clés: Systèmes différentiels polynomiaux, intégrale première, orbite périodique, cycle limite, systèmes différentiels discontinus par morceaux, centre cubique isochrone, système linéaire, cycle limite de croisement.

Classification des sujets de Mathématiques: 34A30 · 34C05 · 34C25 · 34C07 · 37G15.

ملخص

تتناول هذه الأطروحة بحدوث حلول دورية غير تقليدية تسمى دورات الحد للأنظمة التفاضلية المعرفة بكثيري حدود في المستوى الحقيقي. ينقسم هذا العمل إلى جزئين رئيسيين. في الأول ، درسنا وجود وعدد دورات الحد لفئتين من الأنظمة التفاضلية المستوية من الدرجتين الرابعة والسابعة على التوالي. يتعلق الجزء الثاني بدراسة دورات الحد التقاطعية لفئة من الأنظمة التفاضلية المتقطعة وغير المستمرة مفصولة بخط مستقيم وتتكون من جزء خطي وجزء مكعب ذات مركز متزامن.

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List of Publications

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General Introduction

1 Background

Differential equations are a fundamental component in the description of physical phenomena through mathematical models that cover almost every branch of human knowledge (biology, technology, electronics, mechanics, economics, etc). Historically, the notion of Ordinary Differential Equations appeared in the $17th$ century with the works of Jakob Bernoulli $(1654 - 1705)$, Isaac Newton $(1642 - 1727)$ and Gottfried W. von Leibniz $(1646 - 1716)$.

The qualitative theory of differential equation started at the end of the $19th$ century with the works of H. Poincaré $(1854 - 1912)$ in his series of publications "*Mémoire sur les courbes définies par une équation différentielle* "[62, 63], in which he gave a geometrical sense to differential equations. This innovative method involved the study of the topological structure of the solutions of a differential equation, which allowed to deduce properties about such solutions without explicitly knowing them. Also, in [64] (1878), Gaston Darboux studied the integrability of the first order nonlinear algebraic differential equations in the complex projective plane in his "*Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré* ". In the last years, these works has been intensively reconsidered and has taken more importance. For example, the mathematicians have reconsidered in a modern form the way he considered projective differential equations and the relationship he established between the existence of algebraic solutions, and the existence of first integrals. and how to

Poincaré proposed the concept of a limit cycle as a periodic orbit for which at least one trajectory of the vector field approaches in positive or negative time. An alternative definition is usually provided:

> "A limit cycle is a non trivial periodic orbit that is isolated in the set of all possible periodic solutions of a given differential equation".

He also defined other fundamental objects such as phase portrait, a name for the compilation of the minimal information which enables to determine the topological structure of the orbits of a differential system, or the notion of return map, which is also known as Poincaré map, which maps a radial initial condition ρ to the radial component $\Pi(\rho)$ after a 2π loop on the angular component. It also resulted from Poincaré's work with a contribution of I. Bendixon $(1861-1935)$ during the first years of the $20th$ century resulted in the well known Poincaré-Bendixson's Theorem, which states that under compacity conditions every solution tends to a singular solution which can be either a critical point, a periodic orbit or a connected set.
In the second ICM, celebrated in Paris (1900), D. Hilbert proposed a list of twenty-three problems to be solved during the $20th$ century. One of the most difficult problems suggested by Hilbert is the $16th$ Hilbert Problem and involves two subproblems: relative position and number of limit cycles. Smale [70], in Mathematical problems for the next century (1998), reformulates the second part of Hilbert problem as follows:

> Consider here two-dimensional polynomial differential systems of the form
>
> $$\frac{dx}{dt} = P\left(x, y\right), \quad \frac{dy}{dt} = Q\left(x, y\right), \tag{1}$$
>
> where P and Q are polynomials in $\mathbb{R}[x, y]$, which denotes the ring of the polynomials in the variables x and y with real coefficients. By definition, the degree of this system is $n = \max(\deg(P), \deg(Q))$. Is there a maximum bound of number the limit cycles?

The study of limit cycles is very interesting since they are involved in a large number of phenomena in nature or in social sciences where periodic behaviour can be observed. A classical example is the periodicity of the oscillations in RLC circuits and the existence of some isolated periodic orbits. In this line, the works of the engineer A. Liénard $(1869 - 1958)$ and the physicist B. Van der Pol $(1889 - 1959)$ are highly remarkable. Years later, A. A. Andronov $(1901-1952)$ and his collaborators tackled the problem of mechanical and electronical oscillators focusing on the limit cycles analysis. Actually, the Van der Pol equation $\frac{d^2x}{dt^2} + (1 - x^2)\frac{dx}{dt} + x = 0$, and Liénard equations $\frac{d^2x}{dt^2} + (f(x))\frac{dx}{dt} + g(x) = 0$, which can be written as a system as follows

$$
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= -f(x)\, y - g(y).
\end{aligned}
$$

where $\dot{x} = \frac{dx}{dt}$. These equations are actually subject to intense research thanks to their frequent apparitions in a wide class of models arising from different areas of sciences in which limit cycles can occur.

Among the main problems appearing in the qualitative theory of real planar differential systems of the form (1), are the determination of centres, limit cycles, and first integrals. It is very difficult to detect if a planar differential system is integrable or not and also to know if limit cycles for this system can occur. There is a huge literature about limit cycles, most of them deal essentially with their detection, their number and their stability. By contrast to that, there are just a few papers devoted to the determination of the explicit expressions of these limit cycles. Historically, the first known explicit limit cycles were algebraic (see, for example, [1, 11, 54] and references therein). On the other hand, it seems intuitively clear that "most" limit cycles of planar polynomial vector fields are non-algebraic, for instance, the limit cycle appearing in the Van der Pol equation is non-algebraic as it is proved by Odani [58]. The Van der

Pol equation can be written as a polynomial differential system (1) of degree 3, but its limit cycle is not known explicitly. In the chronological order, the first examples were explicit non-algebraic limit cycles appeared are those of A. Gasull and all [37] and by Al-Dosary, Khalil I. T. [2] for $n = 5$. In [14], R. Benterki and J. Llibre presented an example of an explicit non-algebraic limit cycle for $n = 3$. Later Bendjeddou and all published many papers giving explicit non-algebraic limit cycles for several polynomial differential systems of degree greater than or equal to 3 see [7, 8, 9, 10, 13, 19, 20].
Let us consider the system

$$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y), \tag{2}$$

where t is time f and g are tow arbitrary functions. There are systems of the form (2) with an infinite number of limit cycles just as there are systems without limit cycles as is the case for example of the linear ones. But if we assume the discontinuity hypothesis on the functions f and g, the situation changes drastically. It is partly for this raison that Andronov, Vitt and Khaikin have started in 1930 the study of the so called discontinuous piecewise differential systems in the plane. (see for instance their original monography [5]), where system is changed by two or more sub-systems acting each in an appropriate zone of the plan. From this date and till now, these new kind of systems have been a topic of great interest in the mathematical community due to their applications in various areas. They are particularly used for modeling real phenomena and different modern devices, see for instance the books [21, 68] and references quoted therein.
In the last few years, there has been an increasing interest in the study of the problem of bounding a number of limit cycles for planar piecewise differential systems, see [24, 29, 31, 34, 36, 40, 50, 71]. This interest has been mainly motivated by their wider range of applications in various fields of science (e.g., engineering, biology, control theory, design of electric circuits, mechanical systems, economics science, medicine,

chemistry, physics, etc.).

There are many papers studying planar piecewise linear differential systems with two zones (see, e.g., [15, 16, 17, 49, 51, 56, 57] and references therein). For the discontinuous planar nonlinear differential systems there are several papers studying the number of limit cycles (see [6, 24, 33, 44, 45, 50] and references therein). Note that for the piecewise cubic polynomial differential system, there are two recent papers see [41, 42] obtaining at least 24 and 18 small limit cycles, respectively.

2 Objective

This research project has a twofold objective. Firstly, we aim to undertake a qualitative study of some classes of planar polynomial differential systems of the form (1), with the help of the integrability property since the existence of a first integral for a given system determines its phase portrait. More precisely, we present two new results about this topic, we study the integrability and the existence of non trivial solutions that are limit cycles for same classes of planar polynomial differential system by given them explicitly thanks to the first integral. In a second part, we turn to the qualitative theory of discontinuous piecewise differential system of the form (2) formed by an arbitrary linear part and a cubic acting each in one of the two regions separated by the straight line $x = 0$. We prove the existence and the exact number of limit cycles of this case.

3 Outline of the thesis

The scientific contributions of this work are collected in Papers 1-3. In addition, there is a main body which includes a summary of the research performed and explanations of methods and theoretical tools used in the appended papers.

Chapter 1 describes briefly some of the basic concepts, definitions and results used through this work such as limit cycles, invariant curve, the integrability problem, poincaré compactification, and piecewise vector field. **Chapter 2** formed by two part.

The first part is devoted to the study of the integrability and the existence of limit cycle for a class of quartic differential system. We show that this class is integrable and we give the explicit expression for its first integral. After that, we prove under suitable conditions on the parameters, that this class admits a non-algebraic limit cycle surrounding an unstable node. At infinity, we use the Poincaré compactification technique to show that there is only one singular point at infinity. In the second part, we provide a new family of planar polynomial differential systems of degree seven with a non elementary point at the origin. We show the integrability of this family by transforming it into a Riccati equation. We determine sufficient conditions for the coexistence of algebraic and non-algebraic limit cycle surrounding such a point. Moreover, these limit cycles are explicitly given. Finally, **Chapter 3** is concerned specially by the study the existence and the number of crossing limit cycles that can arise in some class of planar piecewise differential systems formed by two regions and separated by a straight line, where in the left region we consider an arbitrary linear differential system, while in the right one the system is cubic with homogeneous nonlinearity and an isochronous center at the origin. More precisely, we show that this piecewise system may have at most zero, one or two explicit algebraic or non-algebraic limit cycles depending on the type of their linear parts, i.e. if those systems have foci, center, saddle, node with different eigenvalues, non-diagonalizable node with equal eigenvalues or linear system without equilibrium points.

Chapter I

Qualitative theory of dynamical systems in the plane

"The way to get started is to quit talking and begin doing."

– Walt Disney

1 Preface

This chapter aims to give to the reader an overview of the theoretical tools and concepts used in the appended papers. The chapter is outlined as follows. At first, we introduce the concept of phase portraits of a dynamical system in the plane and illustrate most of the characteristic behaviors that differentiate the nonlinear systems from the linear ones. The planar dynamical systems are then described. After that, we recall some basic notions about the main topics of this work, that are limit cycles, invariant curves, the integrability problem and the Poincaré compactification. The chapter concludes with a brief description of the particular case of the discontinous piecewise differential system.

2 Saddles, Nodes, Foci and Centers

In this section, we study the topology of the phase portrait for the linear system and nonlinear system near to the origin. We will successively study the trajectories of linear system, then the behavior of the trajectories of nonlinear system in the neighborhood of equilibrium points, and explain the differences between them.

2.1 Linear differential systems

Consider the linear system

$$\dot{X} = AX, \tag{I.1}$$

when $X \in \mathbb{R}^2$ and $A \in M_2(\mathbb{R})$. We begin by describing the portraits for the linear system

$$\dot{X} = BX, \tag{I.2}$$

where the matrix $B = P^{-1}AP; P \in GL_2(\mathbb{R})$, has one of the follows forms:

$$B = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

The phase portrait for the linear system (I.1) above is then obtained from the phase portrait for (I.2) under the linear transformation coordinates $x = Py$.

Case 1. $B = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ with $\lambda < 0 < \mu$. The system (I.2) is said to have a saddle at the origin. The phase portrait for the linear system (I.2) in this case is given in Figure (I.1).

Remark 1 *If $\mu < 0 < \lambda$, then the arrows in figure 1.1 are reversed.*

Case 2. $B = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ with $\lambda \leq \mu < 0$ or $B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ with $\lambda < 0$. The origin is referred to as a stable node in each of these cases. If $\lambda = \mu$ then the origin is a proper node. If $\lambda < \mu$ or $\lambda < 0$ then the origin is an improper node in the other two cases. The phase portrait for the linear system (I.2) in this case is given in Figure (I.1).

Remark 2 *If $\lambda \geq \mu > 0$ or if $\lambda > 0$ in case 2, then the arrows in figure. (I.2) are reversed and the origin is referred to as an unstable node. Thus, Then the stability of the node is determined by the sign of the eigenvalues:*

- *If $\lambda \leq \mu < 0$, then we have a stable node at the origin.*
- *If $\lambda \geq \mu > 0$, then we have a unstable node at the origin.*

Case 3. $B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $a < 0$. The origin is referred to as a stable focus in these cases, the phase portrait for the linear system (I.2) in this case is given in Figure (I.1).

Remark 3 *If $a > 0$, the trajectories spiral away from the origin with increasing t and the origin is called an unstable focus.*

Case 4. $B = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}$. The system (I.2) is said to have a center at the origin in this case. the phase portrait for the linear system (I.2) in this case is given in Figure (I.1).

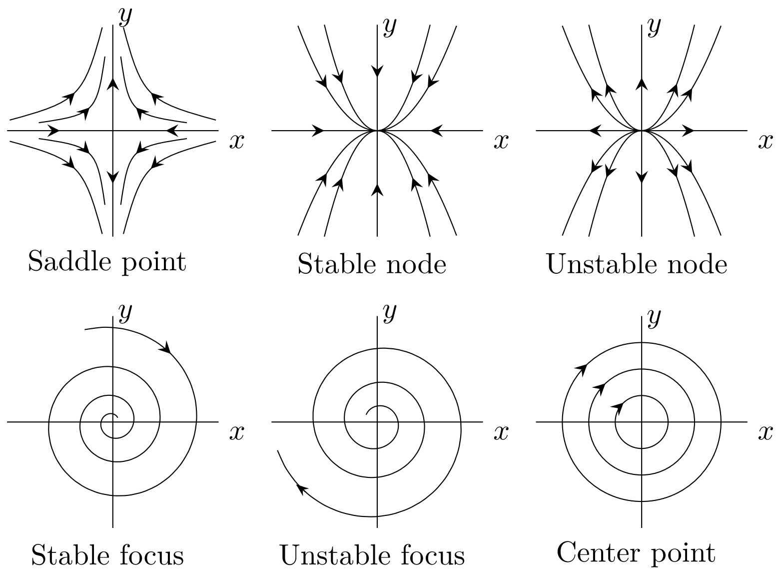


Figure I.1: Classification of phase portrait

Classification of equilibrium points by eigenvalues

Consider the differential system (I.1), let λ, μ be the eigenvalues of the matrix A, X_0 the equilibrium point. We distinguish the different cases according to the eigenvalues λ and μ of the matrix A.

1. If λ and μ are real nonzero and of different sign, then the equilibrium point X_0 is a saddle point, it is always unstable.
2. If λ and μ are real with the same sign, we have three cases:
   - If $\lambda < \mu < 0$, the equilibrium point X_0 is a stable node.
   - If $0 < \lambda < \mu$, the equilibrium point X_0 is an unstable node.
   - If $\lambda = \mu \neq 0$, the equilibrium point X_0 is a proper node, it is stable if $\lambda < 0$ and unstable if $\lambda > 0$.
3. If λ and μ are conjugate complexes and $Im\ (\lambda, \mu) \neq 0$, then the equilibrium point X_0 is a focus. It is stable if $Re(\lambda, \mu) < 0$ and unstable if $Re(\lambda, \mu) > 0$.
4. If λ and μ are pure imaginary, then the equilibrium point X_0 is a center.

Figure (I.1) summarizes the types of equilibrium.

2.2 Nonlinear differential systems

A phase portrait is a geometric representation of the trajectories of a dynamic system in the phase space, in with each set of initial conditions corresponds a curve or a point. To know the aspect of the trajectories of the dynamical systems at least locally, it is enough to know their behavior through the study of so-called singular points, as H.Poincaré did in [62].

We consider the nonlinear system of differential equations

$$\dot{X} = f\left(X\right), \tag{I.3}$$

where $X \in \mathbb{R}^2$ and $f : E \to \mathbb{R}^2$ is a continuous function and E is an open subset of $\mathbb{R}^2$. A point $X_0 \in \mathbb{R}^2$ is called a singular point (or an equilibrium point) of system (I.3) if $f(X_0) = 0$. If a singular point has a neighborhood that does not contain any other singular point, then that singular point is called an isolated singular point.

In what follows, any possible singular point is isolated and so let X_0 be a singular point of system (I.3). To study the behavior of the solutions near to X_0, we consider the associated linearized system (I.1) at this point. Here, the matrix A is given by $A = \left(\frac{\partial f_i}{\partial xj}\left(X_0\right)\right) = Df(X_0)$, where $i = 1, 2, j = 1, 2$.

Definition 4 X_0 *is said to be hyperbolic if any eigenvalue of the matrix of* $Df(X_0)$ *has a non-zero real part. Otherwise, it is said to be non-hyperbolic.*

Definition 5 X_0 *is called non-elementary if the eigenvalues of the linear part of the* $Df(X_0)$ *are both zero, and elementary otherwise. If both of the eigenvalues of the linear part of the* $Df(X_0)$ *are real. then the singular point* X_0 *is called hyperbolic.* X_0 *is said to be non-elementary is called degenerate if the linear part is identically zero, otherwise it is called nilpotent.*

Does the knowledge of the behavior of the linear approximation of a non-linear system in the neighborhood of an equilibrium point, allow us to deduce the behavior of the non-linear system? To clarify what we means by behavior, we introduce the following definition:

Definition 6 (Topological equivalence) *The trajectories (or the orbits) of two dynamical systems are topologically equivalent if there exists a homeomorphism (a bicontinuous bijection) that makes it possible to pass from a trajectory of the first system to a trajectory of the second.*

Theorem 7 (Hartman-Grobman) *If the equilibrium X_0 is hyperbolic, then the trajectories of the nonlinear system (I.3) in a neighborhood of the equilibrium point X_0 are topologically equivalent to those of the linear approximation (I.1) .*

Topologically equivalent trajectories have the same appearance. We can therefore speak of a node or a focus, or even a saddle, for the equilibrium points of nonlinear systems, by studying the eigenvalues of the matrix of the linear approximation, but not of the center.

Remark 8 *In the case of a non-hyperbolic equilibrium, it is the higher order terms, the very ones which have been neglected, which will locally determine the shape of the trajectories.*

Remark 9 *Let E be an open subset of $\mathbb{R}^2$ containing the origin and let $f \in C^1(E)$ with $f(X_0) = 0$. If the singular point X_0 is a center for the linear system (I.1) with $A = Df(X_0)$, then the singular point X_0 is either a center or a weak focus for the nonlinear system (I.3). The nature of X_0 requires further investigation: this is the "problem of the center".*

Stability of the equilibrium points

The behavior of the solutions in the neighbourhood towards or away from the equilibrium points as t goes to infinity is what stability of equilibrium points means.

Definition 10 (Stable equilibrium point) *An isolated equilibrium point X_0 is said to be stable if for given any neighbourhood V of X_0 every solution passes near X_0 remains in V as t increasing.*

Definition 11 (Asymptotically stable equilibrium point) *The equilibrium point X_0 is said to be asymptotically stable if every solution starts in the neighbourhood of X_0 approaches X_0 as t goes to infinity.*

Remark 12 *If the equilibrium point X_0 in not stable is said to be unstable.*

3 Dynamical systems and limit cycles

The mathematical formalism of differential equations has proven useful for describing dynamical systems, that is the evolution of a system with respect to an independent variable, usually time. The mathematical model of the behavior for such dynamical systems is described by a set of two first order autonomous nonlinear ordinary differential systems as follows

$$\begin{aligned} \dot{x} &= \tfrac{dx}{dt} = P\left(x\left(t\right), y\left(t\right)\right), \\ \dot{y} &= \tfrac{dy}{dt} = Q\left(x\left(t\right), y\left(t\right)\right), \end{aligned} \tag{I.4}$$

where P and Q are coprime polynomials in the ring $\mathbb{R}[x, y]$, of degrees n leas or equal n where $n \in \mathbb{N}^*$. The dot over a letter denotes the derivative with respect to time t. In the literature equivalent mathematical objects to refer to this planar differential systems appear: as a vector field

$$\mathcal{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y},$$

as a differntial form

$$\omega = Qdx - Pdy,$$

Definition 13 *A flow in $\mathbb{C}^2$ along a time $t \in \mathbb{R}$ is defined as*

$$\varphi : \mathbb{R} \times \mathbb{C}^2 \to \mathbb{C}^2$$
$$(t, \Omega) \to \varphi_t(\Omega)$$

such that

(i) $\varphi_0(\Omega) = \Omega$,

(ii) $\varphi_t(\varphi_s(\Omega)) = \varphi_{t+s}(\Omega)$,

for all Ω in $\mathbb{C}^2$ and $t, s \in \mathbb{R}$.

System $(I.4)$ defines a flow in $\mathbb{C}^2$, $\varphi(x, y)$. It is known that this flow is a smooth function defined for all (x, y) in some neighborhood of initial position and initial time. Also, it satisfies $(I.4)$ in the sense that

$$\frac{d}{dt}(\varphi_t(x, y))_{t=\tau} = \mathcal{X}(\varphi_\tau(x, y)).$$

Definition 14 *A solution of $(I.4)$ through a point $(x_0, y_0) \in \mathbb{C}^2$ is defined as $(x(t), y(t)) = \{\varphi_t(x_0, y_0), t \in \mathbb{R}\}$.*

3.1 Periodic solutions

Definition 15 *We call Periodic Solution of system $(I.4)$ any solution $X(t) = (x(t), y(t))$ for which there is a real number $T > 0$ such that*

$$\forall t \in \mathbb{R},\ X(t+T) = X(t).$$

The smallest number $T > 0$ that is suitable is then called period of this solution. Any periodic solution corresponds to a closed orbit in phase space.

3.2 Invariant curves

Invariant algebraic curves play an important role in the integrability of polynomial planar differential systems, see for example [23, 27, 33] and are also used in the study of existence and not-existence of periodic solutions and therefore the existence and non-existence of limit cycles.

Definition 16 *Let $f \in \mathbb{R}[x, y]$ a non-zero polynomial. The algebraic curve $f(x, y) = 0$ is an invariant algebraic curve of the polynomial system $(I.4)$ if there is $K \in \mathbb{R}[x, y]$ such that*

$$\mathcal{X} f = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf. \tag{I.5}$$

The polynomial K is called the cofactor of the invariant algebraic curve $f = 0$. Since the polynomial differential system has degree n, any cofactor has degree at most $n - 1$.

On the points of the algebraic curve $f = 0$ the gradient $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ of f is orthogonal to the vector field $\mathcal{X} = (P, Q)$. Hence at every point of $f = 0$ the vector field $\mathcal{X}$ is tangent to the curve $f = 0$. So the curve $f = 0$ is formed by trajectories of the vector field $\mathcal{X}$. This justifies the name "invariant algebraic curve" since it is invariant under the flow defined by $\mathcal{X}$.

Theorem 17 *[39] We consider the system $(I.4)$ and $\Gamma(t)$ a periodic orbit of period $T > 0$. We assume that $f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ is an invariant curve*

$$\Gamma(t) = \{(x, y) \in \Omega \ : f(x, y) = 0\},$$

and $K(x, y) \in C^1$ is the cofactor given in equation $(I.5)$, of the invariant curve $f(x, y) = 0$. Suppose that $p = (x_0, y_0) \in \Omega$ such that $f(p) = 0$ and $\nabla f(p) \neq 0$, then

$$\int_0^T div(\Gamma(t))\, dt = \int_0^T K(\Gamma(t))\, dt,$$

where div is the divergence of system $(I.4)$*, i.e.*

$$div\,(x,y) = \frac{\partial P}{\partial x}(x,y) + \frac{\partial Q}{\partial y}(x,y)\,.$$

Remark 18 *The assumption* $\nabla f(p) \neq 0$ *means that* f *does not contain singular points.*

3.3 Limit cycles

Consider system $(I.4)$, let $\gamma(t) = \{x, y : (x(t), y(t))\}$ a periodic orbit of period T .

Definition 19 *A limit cycle* γ *is an isolated periodic orbit, i.e.; for some arbitrarily smal neighborhood of the periodic orbit* γ *there are no other periodic orbits.*

Definition 20 *A limit cycle is said to be algebraic it's contained in an oval of an invariant algebraic curve of the system* $(I.4)$*, and non-algebraic otherwise.*

Definition 21 (Stability of limit cycles) *We say that the limit cycle* $\gamma = \{(x(t), y(t)), t \in [0,T]\}$ *of system* $(I.4)$ *is* ***stable*** *if all the inner and outer trajectories converging to* γ*, for* $t \to +\infty$*, and* ***unstable*** *that is if all trajectories wrap, spiral around* γ *for* $t \to -\infty$*.* ***Semi-stable (half-stable) limit cycle*** *will have trajectories such that, from one side spiral towards it, while from the other side spiral away from as* $t \to \infty$*.*

3.4 The Poincaré Map

The idea of the Poincaré map is the following: If Γ is a periodic orbit of the system $(I.4)$ throught the point $X_0 = (x_0, y_0)$ and Σ is a hyper-plane perpendicular to Γ at X_0, then for any point $X = (x, y) \in \Sigma$ sufficiently near X_0, the solution of $(I.4)$ throught X at $t = 0$, will cross Σ again at a point $P(X)$ near X_0; see figure $(I.2)$. The mapping $X \to P(X)$ is called the Poincaré map. The next theorem establishes the existence and continuity of the Poincaré map $P(X)$ and of it is first derivative $DP(X)$.

Theorem 22 *Let* E *be an open subset of* $\mathbb{R}^2$*. Suppose that* $\phi_t(x_0)$ *is a periodic solution* *of* $(I.4)$ *of period* T *and that the cycle*

$$\Gamma = \left\{ x \in \mathbb{R}^2 : X = \phi_t (X_0), 0 \leq t \leq T \right\}$$

is contained in E. *Let* Σ *is an hyperplane orthogonal to* Γ *at* X_0*; i.e., let*

$$\Sigma = \left\{ X \in \mathbb{R}^2 :< (X - X_0), (P(X_0), Q(X_0)) >= 0 \right\}.$$

Then there is a $\delta > 0$ *and a unique function* $\tau(X)$ *defined and continuously differentiable for* $X \in N_\delta (X_0)$, *such that* $\tau(X_0) = T$ *and*

$$\phi_{\tau(X)} (X) \in \Sigma, \forall X \in N_\delta (X_0).$$

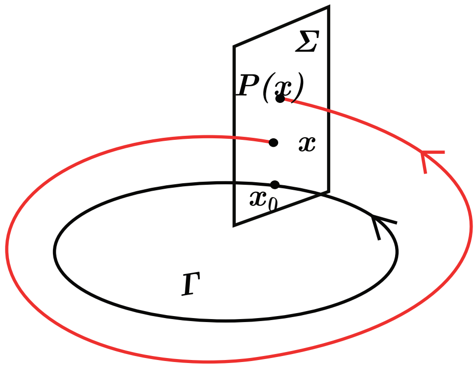


Figure I.2: The Poincaré map.

Definition 23 *Let* Γ, Σ, δ *and* $\tau(x)$ *be defined as in theorem* (22). *For* $x \in N_\delta (x_0) \cap \Sigma$, *the function* $P(x) = \phi_{\tau(x)} (x)$ *is called the Poincaré map for* Γ *at* x_0.

Remark 24 *It follows from theorem* (22), *that:* $P \in C^1 (U)$ *where* $U = N_\delta (x_0) \cap \Sigma$.

Theorem 25 *Let* E *be an open subset of* $\mathbb{R}^2$ *and suppose that* $f \in C^1 (E)$. *Let* $\gamma(t)$ *be a periodic solution of* $(I.4)$ *of period* T. *Then the derivative of the Poincaré map* $P(s)$ *along a straight line* Σ *normal to*

$$\Gamma = \left\{ x \in \mathbb{R}^2 : x = \gamma(t) - \gamma(0), 0 \leq t \leq T \right\}$$

at point $x = 0$ is given by

$$P(0) = \exp\left(\int_0^T \nabla.f(\gamma(t))\, dt\right).$$

Corollary 26 *Under the hypothese of theorem (25), the periodic solution $\gamma(t)$ is a stable limit cycle if*

$$\int_0^T \nabla.f(\gamma(t))\, dt < 0,$$

it is an unstable limit cycle if

$$\int_0^T \nabla.f(\gamma(t))\, dt > 0,$$

and it may be semi-stable limit cycle if

$$\int_0^T \nabla.f(\gamma(t))\, dt = 0.$$

4 The integrability problem

Definition 27 (First integral) *A C^1 function $H : \Omega \to \mathbb{R}$ such that it is constant on each trajectory of polynomial differential system $(I.4)$ and it is not locally constant is called a first integral of system $(I.4)$ in $\Omega \sqsubseteq \mathbb{R}^2$. The equation $H(x(t), y(t)) =$ constant for all values of t for which the solution $(x(t), y(t))$ is defined and contained in Ω, these conditions are equivalent*

$$\frac{\partial H}{\partial t} = P(x,y)\frac{\partial H}{\partial x}(x,y) + Q(x,y)\frac{\partial H}{\partial y}(x,y) = 0,$$

and $H(x,y) =$ constant are formed by the orbits or trajectories of the system $(I.4)$.

The problem of finding such a first integral and determining the functional class to which it must belong is what we refer to as the integrability problem. ”The importance of the existence of first integral for a differential system is that it determines its phase portrait”.

To find an integrating factor or an inverse integrating factor for system (I.4) is closely related to finding its first integral. When considering the integrability problem, we are also asked to study whether an (inverse) integrating factor belongs to a certain given class of functions.

Definition 28 (Integrating factor) *Let U an open set of $\mathbb{R}^2$. A function $\mu : U \to \mathbb{R}$ of class $C^i(U), i > 1$, is called an integrating factor of system $(I.4)$ if that satisfies the following linear partial differential equation*

$$P(x,y)\frac{\partial \mu}{\partial x}(x,y) + Q(x,y)\frac{\partial \mu}{\partial y}(x,y) = -\mu(x,y)\, div\,(P,Q), \tag{I.6}$$

where $div\,(P,Q)$ is called the divergence of a vector field $V = (P,Q)$ defined as

$$div\,(V) = div\,(P,Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

The first integral H associated to the integrating factor μ is given by

$$H(x,y) = \int \mu(x,y)\, P(x,y)\, dy + h(x),$$

satisfying $\frac{\partial H}{\partial x} = -\mu Q$.

When a polynomial differential system has an integrating factor μ we can make a time rescaling and the associated 1-form $\omega = \mu Q dx + \mu P dy$ because closed.

Definition 29 (Inverse integrating factor) *A function $V(x,y)$ is an inverse of integrating factor of system (I.4) in an open subset $U \subseteq \mathbb{R}^2$ if $V \in C^1(U)$, $V \neq 0$ in U and*

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + Q\frac{\partial Q}{\partial y}\right)V.$$

Clearly, from the definition, $V = 0$ is an invariant curve of system $(I.4)$, not algebraic at first. Moreover, it is easy to check that the function $\mu = 1/V$ defines an integrating factor in $U \setminus \{V = 0\}$ of system $(I.4)$.

Invariant algebraic curves are the main objects used in the Darboux theory of integrability. In [32], G. Darboux gives a method for finding an explicit first integral for a system $(I.4)$ in case that $d\,(d+1)\,/2+1$ different irreducible invariant algebraic curve are known, where d is the degree of the system. In this case, a first integral of the form

$$H\,(x,y) = f_1^{\lambda_1} f_2^{\lambda_2} ... f_n^{\lambda_s},$$

where $f_i\,(x,y) = 0$ is an invariant algebraic curve for system $(I.4)$ and $\lambda_i \in \mathbb{C}$ not all of them null, for $i = 1, 2, ..., s; s \in \mathbb{N}$ can be constructed. The functions of this type are called *Darboux functions.*

Some generalizations of the classical Darboux theory of integrability may be found in the literature. For instance, independent singular points can be taken into account to reduce the number of invariant algebraic curves necessary to ensure the Darbouxian integrability of the system, see [26]. A good summary of many of these generalizations can be found in [59] and a survey on the integrability of two-dimensional systems can be found in [25]. One of the most important definitions in this sense is the notion of exponential factor which is given by C. Christopher in [30], when he studies the multiplicity of an invariant algebraic curve. The notion of exponential factor is a particular case of invariant curve for system $(I.4)$.

Definition 30 (Exponential factor) *Let two coprime polynomial $h, g \in \mathbb{R}[x,y]$, the function $e^{h/g}$ is called an exponential factor for system $(I.4)$ if there exists a polynomial k of degree at most $d-1$ called cofactor, where d is the degree of the system, the following relation is satisfied*

$$P\left(\frac{\partial e^{h/g}}{\partial x}\right) + Q\left(\frac{\partial e^{h/g}}{\partial y}\right) = K\,(x,y)\,e^{h/g}.$$

Proposition 31 *If $F = e^{h/g}$ is an exponential factor and g is not a constant, then $g = 0$ is an invariant algebraic curve, and h satisfies the equation $P\left(\frac{\partial h}{\partial x}\right) + Q\left(\frac{\partial h}{\partial y}\right) = hk_g + gk_F$ where k_g and k_F are the cofactoers fo g and F, respectively.*

This proposition is proved in [30]. Since the exponential factor is the most current generlization in the Darboux theory of integrability, any function of the form

$$f_1^{\lambda_1}...f_p^{\lambda_p}\left[\exp\left(\frac{h_1}{g_1^{n_1}}\right)\right]^{\mu_1}...\left[\exp\left(\frac{h_q}{g_q^{n_q}}\right)\right]^{\mu_q},$$

where $p, q \in \mathbb{N}$, $f_i(x,y) = 0\,(1 \leq i \leq p)$ and $g_j(x,y) = 0\,(1 \leq j \leq q)$ are invariant algebraic curve of system $(I.4)$, $h_j(x,y) \in \mathbb{C}[x,y]$ and $\lambda_i, \mu_j \in \mathbb{C}$ and n_j $(1 \leq j \leq q)$ are non-negative integers, is called a Darbouxian function. For more information on the Darboux theory of integrability see for instance [33], [43], [55] and the references therein. Note that a polynomial or a rational first integral is a particular case of a Darboux first integral.

We recall that the integrability problem consists in finding the class of functions a first integral of a given system $(I.4)$ must belong to. We have the system $(I.4)$ defined in a certain class of functions, in this case, the polynomials with real cofficients $\mathbb{R}[x,y]$ and we consider the problem whether there is a first integral in another, possibly larger, class. For instence in [65], H. Poincaré stated the problem of determining when a system $(I.4)$ has a rattional first integral. The works of M.J. Prelle and M.F. Singer [66] and M.F. Singer [69] go on this direction since they give a characterization of when a polynomial system $(I.4)$ has an elementary or a Liouvillian first integral.

A planar differential system can have equilibrium points at infinity and therefore these points can influence the topology of the phase plane. To detect them and recognize their nature, we use the so-called Poincaré compactification technique.

5 Poincaré compactification

Let S^2 be the set of points $(s_1, s_2, s_3) \in \mathbb{R}^3$ such that $s_1^2 + s_2^2 + s_3^2 = 1$. We will call this set the Poincaré sphere. Given a polynomial vector field

$$\mathcal{X}(x,y) = (\dot{x}, \dot{y}) = (P(x,y), Q(x,y))$$

in $\mathbb{R}^2$ of degree d. It can be extended analytically to the Poincaré sphere by projecting each point $(x_1, x_2) \in \mathbb{R}^2$ identified by $(x_1, x_2, 1) \in \mathbb{R}^3$ into the Poincaré sphere using the straight line through x and the origin of $\mathbb{R}^2$. In this way we obtain two copies of X : one on the northen hemisphere $\{(s_1, s_2, s_3) \in S^2 : s_3 > 0\}$ and another on the southern hemisphere $\{(s_1, s_2, s_3) \in S^2 : s_3 < 0\}$. The equator $S^1 = \{(s_1, s_2, s_3) \in S^2 : s_3 = 0\}$ corresponds to the infinity of $\mathbb{R}^2$. The local charts needed for doing the calculations on the Poincaré sphere are

$$U_i = \left\{s \in S^2 : s_i > 0\right\}, \quad V_i = \left\{s \in S^2 : s_i < 0\right\},$$

where $s = (s_1, s_2, s_3)$, with the corresponding local maps

$$\varphi_i\left(s\right) : U_i \to \mathbb{R}^2, \quad \psi_i\left(s\right) : V_i \to \mathbb{R}^2,$$

such that $\varphi_i\left(s\right) = -\psi_i\left(s\right) = \left(s_m/s_i, s_n/s_i\right)$ for $m < n$ and $m, n \neq i$, for $i = 1, 2, 3$. The expression for the corresponding vector field on S^2 in the local chart U_1 is given by

$$\dot{u} = v^d \left[-uP\left(\tfrac{1}{v}, \tfrac{u}{v}\right) + Q\left(\tfrac{1}{v}, \tfrac{u}{v}\right)\right], \quad \dot{v} = -v^{d+1} P\left(\tfrac{1}{v}, \tfrac{u}{v}\right);$$

the expression for U_2 is

$$\dot{u} = v^d \left[P\left(\tfrac{u}{v}, \tfrac{1}{v}\right) - uQ\left(\tfrac{u}{v}, \tfrac{1}{v}\right)\right], \quad \dot{v} = -v^{d+1} Q\left(\tfrac{u}{v}, \tfrac{1}{v}\right);$$

and the expression for U_3 is just

$$\dot{u} = P\left(u, v\right), \quad \dot{v} = Q\left(u, v\right),$$

where d is degree of the vector field X. The expression for the charts V_i are those for the carts U_i multiplied by $\left(-1\right)^{d-1}$, for $i = 1, 2, 3$. Hence, to study the vector field X, it is enough to study its Poincaré compactification restricted to the northern hemisphere plus S^1, which we denote by D call the Poincaré disk. To draw the phase portraits we will consider the projection $\pi\left(s_1, s_2, s_3\right) = \left(s_1, s_2\right)$ of the Poincaré disk into the unit disk centered at the origin.

Finite singular points of X are the singular points of its compactification which are in $D^2\backslash S^1$, and they can be studied using U_3. *Infinite singular points* of X are the singular points of the corresponding vector field on the Poincaré disk D lying on S^1. Clearly a point $s \in S^1$ is an infinite singular point if and only if $-s \in S^1$, and the local behavior of one is the same as the other multiplied by $(-1)^{d-1}$. Hence to study the infinite singular points it suffices to look only at $U_1|_{v=0}$ and at the origin of U_2. For more details, see [33].

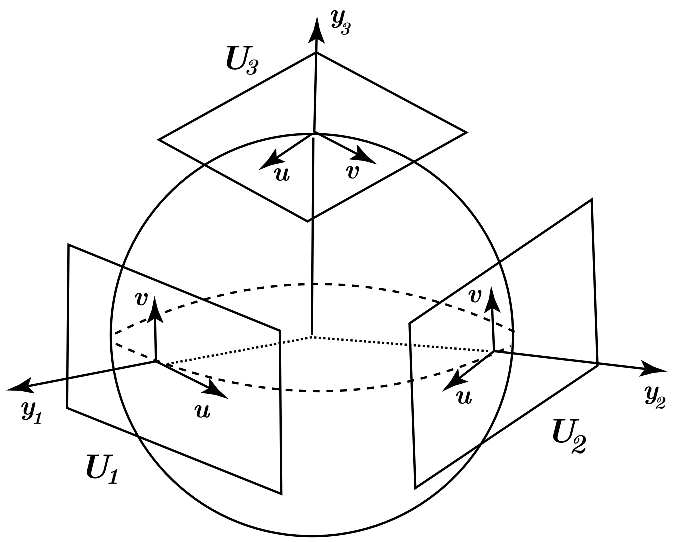


Figure I.3: The local charts (U_k, ϕ_k) for $k = 1, 2, 3$ of the Poincaré sphere.

6 Piecewise vector field

A discontinuous piecewise differential system on $\mathbb{R}^2$ is a pair of C^r (with $r \geq 1$) differential systems in $\mathbb{R}^2$ separated by a codimension one manifold Σ. The line of discontinuity Σ of the discontinuous piecewise differential system is defined by $\Sigma = h^{-1}(0)$, where $h : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function having 0 as a regular value. Note that Σ is the separating boundary of the regions or zones $\Sigma^+ = \{(x, y) \in \mathbb{R}^2 | \quad h(x, y) > 0\}$ and $\Sigma^- = \{(x, y) \in \mathbb{R}^2 | \quad h(x, y) < 0\}$. So C^r vector field associated to a piecewise differential system with line of discontinuity Σ is

$$Z(x,y) = \begin{cases} Z^{+}(x,y), & \text{if } (x,y) \in \Sigma^{+}, \\ Z^{-}(x,y). & \text{if } (x,y) \in \Sigma^{-}, \end{cases} \tag{I.7}$$

As usual, system (I.7) is denoted by $Z = (Z^{+}, Z^{-}, \Sigma)$ or simply by $Z = (Z^{+}, Z^{-})$, when separation line Σ is well understood. In order to establish a definition for the trajectories of $\varphi_Z(p,t)$ and investigate its behavior, we need a criterion for the transition of the orbits between Σ^{+} and Σ^{-} across Σ. The contact between the vector field Z^{+} (resp. Z^{-}) and the line of discontinuity Σ is characterized by the derivative of h in the direction of the vector field Z^{+} i.e.

$$Zh(p) = \langle \nabla h(p), Z(p) \rangle,$$

where $\langle ., . \rangle$ is the usual inner product in $\mathbb{R}^2$. The basic results of the discontinuous piecewise differential systems in this context were stated by Filippov [35], [47]. We can divide the line of discontinuity Σ in the following sets:

(a) Crossing set: $\Sigma^c : \{p \in \Sigma : Z^{+}h(p).Z^{-}h(p) > 0\}$.

(b) Escaping set: $\Sigma^e : \{p \in \Sigma : Z^{+}h(p) > 0 \text{ and } Z^{-}h(p) < 0\}$.

(c) Sliding set: $\Sigma^s : \{p \in \Sigma : Z^{+}h(p) < 0 \text{ and } Z^{-}h(p) > 0\}$.

The escaping set Σ_e or Sliding set Σ^s are respectively defined on points of Σ where both vector fields Z^{+} (resp. Z^{-}) simultaneously point outwards or inwards from Σ while the interior of its complement in Σ defines the crossing region Σ^c (see Figure $(I.4)$). The complementary of the union of these regions is the set formed by the tangency points between Z^{+} (resp. Z^{-}) with Σ.

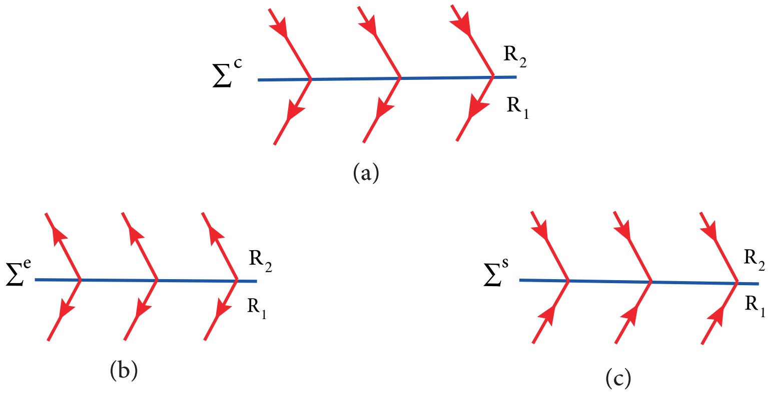


Figure I.4: The regions Σ^c in (a), Σ^e in (b) and Σ^s in (c).

When $p \in \Sigma_c^{\pm}$ it is natural to consider that the trajectories of $Z(p) = Z^-(p)$ are given by concatenations of trajectories of Z^+ (resp. Z^-). So, for to determine all possible trajectories of such a vector field it is necessary to define the dynamics in regions $\Sigma_e^{\pm}$ and $\Sigma_s^{\pm}$. In $\Sigma_e^{\pm} \cup \Sigma_s^{\pm}$ we define a vector field Z^+ (resp. Z^-) given by the unique convex conbination of Z^+ (resp. Z^-) wich is tangent to $\Sigma^{\pm}$ at p, see Figure $(I.5)$. This is if $p \in \Sigma_e^{\pm} \cup \Sigma_s^{\pm}$, then

$$Z_s^{\pm}(p) = \frac{1}{Z^+H(p) - Z^-H(p)} \left(Z^+H(p)\, Z^-(p) - Z^-H(p)\, Z^+(p) \right).$$

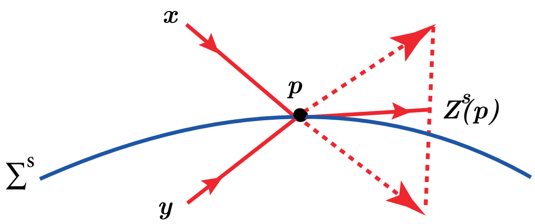


Figure I.5: Filippov vector field $Z_s^{\pm}(p)$ when $p \in Z_s^{\pm}(p)$.

We have that $Z(p) = Z^+$ (resp. $Z(p) = Z^-$), then given a trajectory $\varphi_Z(q,t) \in \Sigma^+ \cup \Sigma^-$ and $\Sigma^{\pm}$, q is said to be a departing point (resp. arriving point) of $\varphi_Z(p,t)$ if there exist $t_0 < 0$ (resp. $t_0 > 0$) such that $\lim_{t \to t_0^+} \varphi_Z(q,t) = p$ (resp. $\lim_{t \to t_0^-} \varphi_Z(q,t) = p$). With these definitions if $p \in \Sigma_c^{\pm}$, then it is departing point (resp. arriving point) of

$\varphi_Z(q,t)$ for any $q \in \gamma^1(p)$ (resp. $q \in \gamma^2(p)$), where

$$\gamma^1(p) = \{\varphi_Z(p,t) : t \in I \cap t \geq 0\} \quad \text{and} \quad \gamma^2(p) = \{\varphi_Z(p,t) : t \in I \cap t \leq 0\},$$

are the trajectories of differential system Z^+ (resp. Z^-) in Σ^+ and Σ^- through p, respectively. these definitions are analogous if the piecewise differential system $(I.7)$ is defined in n connected and open regions.

Definition 32 *A periodic orbit Γ of the discontinuous piecewise differential system $(I.7)$ is a smooth piecewise curve which is formed by pieces of orbits of each part of this system, so Γ writes $\Gamma = \gamma^1 \cup \gamma^2$, where $\gamma^1 \subset \Sigma^+$ and $\gamma^2 \subset \Sigma^-$, and it is such that $\varphi_Z(p,t+T) = \varphi_Z(p,t)$, for some $T > 0$, where T is called of period of the periodic orbit Γ.*

If $\gamma^i \cap \Sigma \subset \Sigma_c$ for all $i = 1, 2$, then the periodic orbit Γ is called the crossing periodic orbit, otherwise is called sliding periodic orbit.

Definition 33 *If a periodic orbit Γ is isolated in the set of all periodic orbit of Z, then it is called limit cycle of piecewise differential system $(I.7)$.*

Definition 34 *Consider two discontinous vector field Z and $\hat{Z}$ defined in open sets, U and $\hat{U}$, of $\mathbb{R}^2$ with discontinutiy curve Σ and $\hat{\Sigma}$, respectively. Then,*

- *Z and $\hat{Z}$ are Σ-equivalent if there exists an orientation preserving homeomorphism $h : U \to \hat{U}$ that sends Σ to $\hat{\Sigma}$ and sends orbits of Z to orbit of $\hat{Z}$;*
- *Z and $\hat{Z}$ are topologically aquivalent if ther exists an orientation preserving homeomorphism $h : U \to \hat{U}$ sends orbits of Z to orbits of $\hat{Z}$.*

Definition 35 *We say that*

(a) *a point $p \in \Sigma^- \cup \Sigma^+$ is a regular equilibrium point associated to the vector field Z^+ (resp. Z^-) if $Z^+(p) = 0$ (resp. $Z^-(p) = 0$). Besides then, p is real (admissible) if $p \in \Sigma^\pm$ or virtual (non admissible) if $p \in \Sigma^\mp$.*

(b) *a point* $p \in \Sigma$ *is a boundary equilibrium point associated to the vector field* Z^+ *(resp.* Z^-*) if* $Z^+(p) = 0$ *(resp.* $Z^-(p) = 0$*).*

(c) *a point* $p \in \Sigma$ *is a pseudo-equilibrium point if there is* $\alpha \in (0,1)$ *such that* $(1-\alpha) Z^-(p) + \alpha Z^+(p) = 0$*, i.e.,* Z^+ *(resp.* Z^-*) are anti-collinear to* Σ *at* p*. Obviously,* p *is an equilibrium of* Z^s*. If* p *behaves as a node, a focus or a saddle for the dynamics of* Z^s*, we say that* p *is a pseudo-node, pseudo-focus or pseudo-saddle, respectively.*

One of the most important tools in the study of flows in the neighborhood of periodic orbits is obviously the Poincaré map, which for this case, if defined as follows:
When the trajectory of Z^+ (resp. Z^-) through $p \in \Sigma_c^+$ returns to Σ_c^- (by the first time) after a positive time $t_+(p)$, we define the half returns map of Z^+ by $\Pi_+(p) = \varphi_{Z^+}(t_+(p), p) = q \in \Sigma_c^-$. When the trajectory of Z^- through $q \in \Sigma_c^-$ returns to Σ_c^+ (by the first time) after a positive time $t_-(q)$, we define the half return map of Z^- by $\Pi_-(q) = \varphi_{Z^-}(t_-(q), q) = \mu \in \Sigma$. The first return map or Poincaré return map of $Z = (Z^+, Z^-)$ is defined by the composition of these two half return maps, that is,

$$\Pi(p) = \Pi_- \circ \Pi_+(p) = \varphi_{Z^-}(t_-(\varphi_{Z^+}(t_+(p), p)), \varphi_{Z^+}(t_+(p), p))$$

or in the reverse order, applying first the flow of Z^- and after the flow of Z^+, more details about this construction are provided in [33], [46].

Chapter II

Limit cycles of some classes of planar polynomial differential systems

"Errors, like straws, upon the surface flow; He who would search for pearls, must dive below."

– John Dryden

1 Preface

As we have said in the introduction, we are interested by the problem which consists in finding explicitly the limit cycles, if exist, for a given planar polynomial differential system. In that direction, we have obtained via the integrability approach two results. In fact, this chapter is composed of two parts: the first one is concerned by the explicit limit cycle for a class of quartic polynomial differential system surrounding a node. The second one is concerned by the coexistence of two limit cycles for a planar differential system of degree seven. In these results, the limit cycles are obviously explicitly obtained.

2 Limit cycle for a class of quartic differential systems with an unstable node

In this section, we are interested by the integrability and the existence of limit cycles for the class of quartic differential systems, given by

$$\begin{aligned} \dot{x} &= \left(hlx - 2ly + (1+h)x^2 - (2-h)xy - y^2\right)\left(x^2 + xy + y^2\right) + x(l + x + y)^2, \\ \dot{y} &= \left(hly + x^2 + (2+h)xy - (1-h)y^2\right)\left(x^2 + xy + y^2\right) + y(l + x + y)^2, \end{aligned} \tag{II.1}$$

where h and l are non-zero real constants. Moreover, under some suitable conditions, we will show that system (II.1) exhibits one non-algebraic limit cycle explicitly given surrounding an unstable node at the origin. The study of the existence and nature of singular points at infinity is done via the Poincaré compactification method. We show that this system admits one singular point at infinity.

2.1 The main result

As a main result, we prove the following theorem.

Theorem 36 *Consider the class of quartic differential system* (II.1)*, the following statements hold*

1 The origin $(0,0)$ *is unstable node.*

2 System (II.1) *is integrable and its first integral is*

$$I(x,y) = \frac{(x^2+y^2)\exp(-h\arctan\frac{y}{x})}{l+x+y} - 2\int_0^{\arctan\frac{y}{x}} \frac{\exp(-hs)}{2+\sin 2s} ds.$$

3 If $h<0$ *and* $l>0$*, then system* (II.1) *has an explicit non-algebraic limit cycle, in polar coordinates* $(r;\theta)$ *given by*

$$r(\theta, r_*) = \frac{1}{2}\left((\cos\theta + \sin\theta)\rho(\theta, r_*) + \sqrt{4l\rho(\theta, r_*) + (\sin 2\theta + 1)\rho^2(\theta, r_*)} \right),$$

where

$$\rho(\theta, r_*) = \exp(h\theta)\left(\frac{r_*^2}{l+r_*} + 2\int_0^{\theta} \frac{e^{-hs}}{2+\sin 2s} ds \right) > 0, \text{ for all } \theta \in \mathbb{R},$$

and

$$r_* = \frac{e^{2\pi h}}{e^{2\pi h}-1}\left(\sqrt{\left(\int_0^{2\pi} \frac{e^{-hs}}{\sin 2s+2} ds\right)^2 + 2l\left(e^{-2\pi h}-1\right)\int_0^{2\pi} \frac{e^{-hs}}{\sin 2s+2} ds} - \int_0^{2\pi} \frac{e^{-hs}}{\sin 2s+2} ds \right),$$

4 The class of quartic system (II.1) *has one singular point at infinity.*

Proof

For the study of singular points at finite distance, we know that if (x,y) is singular point of the system (II.1), then this point is included in the set $\{x\dot{y} - y\dot{x} = 0\}$ such that $(x,y) \in \mathbb{R}^2$, So we find that the origin $(0,0)$ is a singular point and other singular points if they exist, are included in the line given by $\{2l + x + y = 0\}$.

Statement (1) The Jacobian matrix of the differential system (II.1) at the origin is

$$Dj(0,0) = \begin{pmatrix} l^2 & 0 \\ 0 & l^2 \end{pmatrix}$$

$tr(Dj(0,0)) = 2l^2$ and $det(Dj(0,0)) = l^4$, then $\Delta = (tr)^2 - 4det = 0$, then the origin $(0,0)$ is unstable node.

Statement (2) The differential system (II.1) in polar coordinates becomes

$$\begin{aligned} \dot{r} &= F(\theta)r^3 + G(\theta)r^2 + H(\theta)r + 4l^2, \\ \dot{\theta} &= M(\theta)r^2 + N(\theta)r, \end{aligned} \tag{II.2}$$

where

$$\begin{aligned} F(\theta) &= (5h+3)\cos\theta + (5h-3)\sin\theta + (1-h)\cos 3\theta + (1+h)\sin 3\theta, \\ G(\theta) &= 2(hl+2)\sin 2\theta + 4(hl+1), \\ H(\theta) &= 8l(\cos\theta + \sin\theta), \\ M(\theta) &= 5h(\cos\theta + \sin\theta) + \sin 3\theta - \cos 3\theta, \\ N(\theta) &= 8l + 4l\sin 2\theta. \end{aligned}$$

Taking θ as an independent variable, system (II.2) is equivalent to the nonlinear differential equation

$$\frac{dr}{d\theta} = \frac{F(\theta)r^3 + G(\theta)r^2 + H(\theta)r + 4l^2}{M(\theta)r^2 + N(\theta)r}. \tag{II.3}$$

To solve this equation, we use the following change of variable

$$\rho = \frac{r^2}{l + r(\cos\theta + \sin\theta)}. \tag{II.4}$$

The differential equation (II.3) leads to the linear differential equation

$$\dot{\rho} = \frac{d\rho}{d\theta} = h\rho + \frac{2}{2 + \sin 2\theta}. \tag{II.5}$$

Hence the general solution of (II.5) is readily obtained by

$$\rho(\theta) = \exp(h\theta)(k + \int_0^\theta \frac{2\exp(-hs)}{2 + \sin 2\theta} ds), \tag{II.6}$$

where $k \in \mathbb{R}$. By replacing the expression of $\rho(\theta)$ of (II.6), we get

$$k = \frac{r^2 \exp(-h\theta)}{l + r\cos\theta + r\sin\theta} - 2\int_0^\theta \frac{\exp(-hs)}{2+\sin 2\theta} ds.$$

We return to Cartesian coordinates to obtain the first integral of system (II.1) by

$$I(x,y) = \frac{(x^2+y^2)\exp(-h\arctan\frac{y}{x})}{l+x+y} - 2\int_0^{\arctan\frac{y}{x}} \frac{\exp(-hs)}{2+\sin 2s} ds. \tag{II.7}$$

Statement (3) From the (II.4) we can obtain the two solutions of the nonlinear differential equation (II.3)

$$r_1(\theta,k) = \frac{1}{2}\left((\cos\theta+\sin\theta)\,\rho(\theta,k) - \sqrt{4l\rho(\theta,k) + (\sin 2\theta + 1)\,\rho^2(\theta,k)}\right),$$
$$r_2(\theta,k) = \frac{1}{2}\left((\cos\theta+\sin\theta)\,\rho(\theta,k) + \sqrt{4l\rho(\theta,k) + (\sin 2\theta + 1)\,\rho^2(\theta,k)}\right).$$

We note that the $r_1(\theta,k)\, r_2(\theta,k) = -l\rho < 0$ for any θ from the which condition. Therefore, we choose the greater and positive solution

$$r(\theta,k) = r_2(\theta,k) = \frac{1}{2}\left((\cos\theta+\sin\theta)\,\rho(\theta,k) + \sqrt{4l\rho(\theta,k) + (\sin 2\theta + 1)\,\rho^2(\theta,k)}\right), \tag{II.8}$$

where

$$\rho(\theta,k) = \exp(h\theta)\left(k + 2\int_0^\theta \frac{\exp(-hs)}{2+\sin 2s} ds\right).$$

Now, we prove that system (II.1) possesses a non-algebraic limit cycle. Among the solutions of (II.3), we consider the solutions such that $r(0,k) = r_0 > 0$. This requires that

$$k = \frac{r_0^2}{l + r_0}.$$

We note by $r(0, r_0) = r_0$. The equation (II.8) becomes

$$r(\theta,r_0) = \frac{1}{2}\left((\cos\theta+\sin\theta)\,\rho(\theta,r_0) + \sqrt{4l\rho(\theta,r_0) + (\sin 2\theta + 1)\,(\rho(\theta,r_0))^2}\right), \tag{II.9}$$

where

$$\rho(\theta, r_0) = \exp(h\theta)\left(\frac{r_0^2}{l + r_0} + 2\int_0^\theta \frac{\exp(-hs)}{2 + \sin 2s} ds\right). \tag{II.10}$$

A periodic solution must verify the following condition $r(2\pi, r_0) = r(0, r_0) = r_0$. Assuming $\varphi(\theta) = \cos\theta + \sin\theta$, we note that $\varphi(2\pi) = \varphi(0) = 1$, then $r(2\pi, r_0) = r(0, r_0)$ if and only if $\rho(2\pi, r_0) = \rho(0, r_0)$. Hence the condition $r(2\pi, r_0) = r_0$ gives two different values of r_0

$$r_* = \frac{e^{2\pi h}}{1 - e^{2\pi h}} \left(\int_0^{2\pi} \frac{e^{-hs}}{\sin 2s + 2} ds - \sqrt{\left(\int_0^{2\pi} \frac{e^{-hs}}{\sin 2s + 2} ds\right)^2 + 2l\left(e^{-2\pi h} - 1\right)\int_0^{2\pi} \frac{e^{-hs}}{\sin 2s + 2} ds}\right),$$

$$r_* = \frac{e^{2\pi h}}{1 - e^{2\pi h}} \left(\int_0^{2\pi} \frac{e^{-hs}}{\sin 2s + 2} ds + \sqrt{\left(\int_0^{2\pi} \frac{e^{-hs}}{\sin 2s + 2} ds\right)^2 + 2l\left(e^{-2\pi h} - 1\right)\int_0^{2\pi} \frac{e^{-hs}}{\sin 2s + 2} ds}\right).$$

We consider only values of r_* such that $r(2\pi, r_*) = r_* > 0$. Since $\frac{e^{-hs}}{\sin 2s+2} > 0$, for all $\theta \in \mathbb{R}$, thus $\int_0^{2\pi} \frac{e^{-hs}}{\sin 2s+2} ds > 0$ for all $\theta \in \mathbb{R}$.

As $h < 0$ and $l > 0$, it follows that $\frac{e^{\pi h}}{1-e^{2\pi h}} > 0$ and $2l\left(e^{-2\pi h} - 1\right)\int_0^{2\pi} \frac{e^{-hs}}{\sin 2s+2} ds > 0$ then

$$\sqrt{\left(\int_0^{2\pi} \frac{e^{-hs}}{\sin 2s + 2} ds\right)^2 + 2l\left(e^{-2\pi h} - 1\right)\int_0^{2\pi} \frac{e^{-hs}}{\sin 2s + 2} ds} > \int_0^{2\pi} \frac{e^{-hs}}{\sin 2s + 2} ds.$$

Then, we have only one positive choice of r_0:

$$r_* = \frac{e^{\pi h}}{1-e^{2\pi h}} \left(\int_0^{2\pi} \frac{e^{-hs}}{\sin 2s+2} ds + \sqrt{\left(\int_0^{2\pi} \frac{e^{-hs}}{\sin 2s+2} ds\right)^2 + 2l\left(e^{-2\pi h} - 1\right)\int_0^{2\pi} \frac{e^{-hs}}{\sin 2s+2} ds}\right), \tag{II.11}$$

After the substitution of this value r_* into $r(\theta, r_0)$ we obtain

$$r(\theta, r_*) = \frac{1}{2}\left((\cos\theta + \sin\theta)\,\rho(\theta, r_*) + \sqrt{4l\rho(\theta, r_*) + (\sin 2\theta + 1)\,(\rho(\theta, r_*))^2}\right),$$

where $\rho(\theta, r_*)$ are the functions defined in the statement of Theorem (36). Since $r_* > 0$, $l > 0$ and $\int_0^\theta \frac{e^{-hs}}{\sin 2s+2}\,ds > 0$ for all $\theta \in \mathbb{R}$, then

$$\rho(\theta, r_*) = \exp(h\theta)\left(\frac{r_*^2}{l + r_*} + 2\int_0^\theta \frac{e^{-hs}}{2 + \sin 2s} ds\right) > 0, \text{ for all } \theta \in \mathbb{R},$$

and $4l\rho(\theta, r_*) > 0$, thus

$$r(\theta, r_*) = \frac{1}{2}\left((\cos\theta + \sin\theta)\,\rho(\theta, r_*) + \sqrt{4l\rho(\theta, r_*) + (\sin 2\theta + 1)\,(\rho(\theta, r_*))^2}\right) > 0, \tag{II.12}$$

for all $\theta \in \mathbb{R}$. Now, $r(\theta, r_*)$ is a periodic solution. In order to prove that the periodic orbit is a hyperbolic limit cycle, we consider (II.9) and introduce the Poincaré return map $r_0 \longmapsto \Pi(2\pi, r_0) = r(2\pi, r_0)$. Therefore, a limit cycles of system (II.1) is hyperbolic if, and only if

$$\left.\frac{dr(2\pi, r_0)}{dr_0}\right|_{r_0 = r_*} \neq 1,$$

where

$$r_* = \frac{e^{2\pi h}}{1 - e^{2\pi h}}\left(\int_0^{2\pi} \frac{e^{-hs}}{\sin 2s+2}\,ds + \sqrt{\left(\int_0^{2\pi} \frac{e^{-hs}}{\sin 2s+2}\,ds\right)^2 + 2l\left(e^{-2\pi h} - 1\right)\int_0^{2\pi} \frac{e^{-hs}}{\sin 2s+2}\,ds}\right),$$

after some calculations we obtain that

$$\left.\frac{dr(2\pi, r_0)}{dr_0}\right|_{r_0 = r_*} = e^{2\pi h} < 1.$$

Consequently, the limit cycle of differential system (II.1) is hyperbolic and stable

Statement (4) In the local chart U_1, where $x = \frac{1}{v}$ and $y = \frac{u}{v}$, system (II.1) becomes

$$\dot{u} = (u + 2lv + 1)(u^4 + u^3 + 2u^2 + u + 1),$$
$$\dot{v} = -v(h + 1 + (2h - 1)u + (1 + hl)v + 2(h - 1)u^2 + l(h - 2 + 2)uv + 2lv^2 + (h - 3)u^3 + (1 - 2l + hl)u^2v + 2luv^2 + l^2v^3 - u^4 - 2lu^3v).$$

We have $\dot{u}|_{v=0} = u^5 + 2u^4 + 3u^3 + 3u^2 + 2u + 1$, the system (II.1) has one infinite singular point that is $(-1, 0)$.

At this singular point, the Jacobian matrix of system (II.1) in the chart U_1 is

$$Dj(-1, 0) = \begin{pmatrix} 2 & 4l \\ 0 & -2 \end{pmatrix}$$

$tr(Dj(-1, 0)) = 0$ and $det(Dj(-1, 0)) = -4$, then $\Delta = (tr)^2 - 4det = 16$. Therefore the point $(-1, 0)$ is saddle.

In the local chart U_2, where $x = \frac{u}{v}$ and $y = \frac{1}{v}$, the differential system (II.1) writtes as

$$\dot{u} = -(u + 2lv + 1)(u^4 + u^3 + 2u^2 + u + 1),$$
$$\dot{v} = -v(h - 1 + (1 + 2h)u + (1 + hl)v + 2(h + 1)u^2 + (2 + hl + 2l)uv + 2lv^2 + (h + 3)u^3 + (4l + hl + 1)u^2v + 2luv^2 + l^2v^3 + u^4).$$

thus, the origin is not a singular point in the chart U_2.

2.2 Application

The following example illustrate our result. Let $h = -2$ and $l = 1.5$, system (II.1) becomes

$$\begin{aligned} \dot{x} &= -(3x + 3y + x^2 + 0.5xy + y^2)(x^2 + xy + y^2) + x(1.5 + x + y)^2, \\ \dot{y} &= (3x - 3y + x^2 + 3.5xy - 3y^2)(x^2 + xy + y^2) + y(1.5 + x + y)^2. \end{aligned} \tag{II.13}$$

The system (II.1) has a non-algebraic limit cycle whose expression in polar coordinates (r, θ) is

$$r(\theta, r_*) = \frac{1}{2}\left((\cos\theta + \sin\theta)\,\rho(\theta, r_*) + \sqrt{2\rho(\theta, r_*) + (\sin 2\theta + 1)(\rho(\theta, r_*))^2}\right) > 0, \tag{II.14}$$

where

$$\rho(\theta, r_*) = \exp(-2\theta)\left(\frac{r_*^2}{l + r_*} + 2\int_0^\theta \frac{e^{2s}}{2 + \sin 2s}\,ds\right),$$

and $r_\star = 0.33849$. The phase portrait in the Poincaré disc of differential system (II.13) is plotted in figure (II.1)

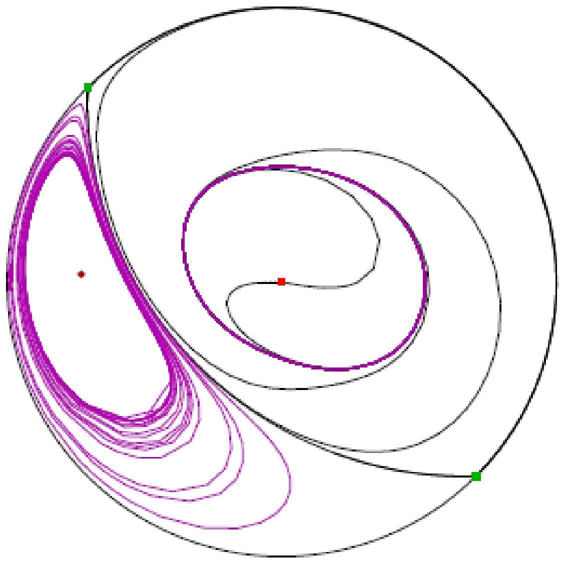


Figure II.1: Phase portrait of differential system (II.13).

3 Coexistence of limit cycles in a septic planar differential system enclosing a non-elementary singular point, using riccati equation

In this section, we consider the family of the polynomial differential systems of the form

$$\begin{aligned} \dot{x} &= x\left(x^2+y^2\right)^2\left(x^2+y^2-h\right)+\left(\left(a+b\right)x^2+\left(a-b\right)y^2\right) \\ &\quad \left(h^2cx-2hcx^3+2hx^2y-2hcxy^2+2hy^3+cx^5+2cx^3y^2+cxy^4\right), \\ \dot{y} &= y\left(x^2+y^2\right)^2\left(x^2+y^2-h\right)+\left(\left(a+b\right)x^2+\left(a-b\right)y^2\right) \\ &\quad \left(h^2cy-2hx^3-2hcx^2y-2hxy^2-2hcy^3+cx^4y+2cx^2y^3+cy^5\right), \end{aligned} \tag{II.15}$$

where a, b, h and $c \in \mathbb{R}_*^+$. We prove that these systems are integrable. Moreover, we determine sufficient conditions for the polynomial differential system $(II.15)$ to possess exactly two limit cycles, one of them is algebraic and the other non-algebraic, and both are explicitly given.

Let's recall that the first result for the coexistence of algebraic and non-algebraic limit cycles goes back to J. Giné and M. Grau [40] for $n = 9$, Bendjeddou and Cheurfa [12] for $n = 5$. B. Ghermoul et all [38], we find a result concerning a class for $n = 5$, for which a unique non-algebraic limit cycle around a non-elementary critical point is given. The non-algebraic limit cycle is also constructed explicitly, by passage to a Bernoulli equation. Here we give a result for a class of septic planar differential systems. By transforming the differential system to a Riccati type equation, we show the coexistence of two limit cycles one is algebraic, while the other is non-algebraic, surrounding both a non-elementary singular point, namely the origin. Let us recall that the Riccati differential equation appear in many applications such as in control theory, flow, econometric models and diffusion problems (see for instance [48], [67]). On the other hand a Riccati equation has at most two limit cycles. In the work [40], J. Giné and M. Grau. proved their result by transforming their system to a Riccati equation and expressing their solution in terms of the solutions of the corresponding second order homogeneous linear differential equation. Our method is different since we have a system with an invariant algebraic curve which constitute a particular solution of the Riccati equation obtained after a suitable change of variable in our system. Using Polyanin et al. [72], pp 31, we obtain the general solution of this equation and this allow us to obtain the first integral of the system.

3.1 The main result

Our main result is the following one.

Theorem 37 *The multi-parameter polynomial differential system (II.15) possess exactly two limit cycles: the circle* $(\gamma_1) : x^2 + y^2 - h = 0$ *surrounding a transcendental limit cycle* (γ_2) *explicitly given in polar coordinates* (r, θ) *by the equation*

$$r(\theta, r_*) = \left(\frac{h \exp(\varphi(\theta)) \left(\frac{r_*^2}{r_*^2 - h} + c \int_0^\theta \exp(-\varphi(s))\, ds \right)}{-1 + \exp(\varphi(\theta)) \left(\frac{r_*^2}{r_*^2 - h} + c \int_0^\theta \exp(-\varphi(s))\, ds \right)} \right)^{\frac{1}{2}},$$

with $\varphi(\theta) = \int_0^\theta \frac{ds}{a + b\cos 2s}$ *and*

$$r_* = \left(\frac{h \exp(\varphi(2\pi)) \left(c \int_0^{2\pi} \exp(-\varphi(s))\, ds \right)}{-1 + \exp(\varphi(2\pi)) \left(1 + c \int_0^{2\pi} \exp(-\varphi(s))\, ds \right)} \right)^{\frac{1}{2}},$$

if the following conditions are assumed

$$b^2 - a^2 < 0, \tag{II.16}$$

and

$$\exp(\varphi(2\pi)) < c \int_0^{2\pi} \exp(-\varphi(s))\, ds. \tag{II.17}$$

Proof

Firstly, we have

$$y\dot{x} - x\dot{y} = 2h \left(x^2 + y^2 \right)^2 \left((a + b)\, x^2 + (a - b)\, y^2 \right).$$

We see from (II.16), that the origin $O(0;0)$ is the unique critical point at finite distance. Also, it is obvoious, by considering the jacobian matrix of the differential system $(II.15)$, that the origin is a non elementary point.

We prove that $(\gamma_1) : x^2 + y^2 - h = 0$ is an invariant algebraic curve of the differential system $(II.15)$. Indeed, if we put

$$\begin{aligned} P &= x\left(x^2+y^2\right)^2\left(x^2+y^2-h\right)+\left((a+b)\,x^2+(a-b)\,y^2\right) \\ &\quad \left(ch^2x-2chx^3+2hx^2y-2chxy^2+2hy^3+cx^5+2cx^3y^2+cxy^4\right), \\ Q &= y\left(x^2+y^2\right)^2\left(x^2+y^2-h\right)+\left((a+b)\,x^2+(a-b)\,y^2\right) \\ &\quad \left(ch^2y-2hx^3-2chx^2y-2hxy^2-2chy^3+cx^4y+2cx^2y^3+cy^5\right), \\ U\left(x,y\right) &= x^2+y^2-h \end{aligned}$$

Immediately we have

$$P(x,y)\frac{\partial U}{\partial x}(x,y)+Q(x,y)\frac{\partial U}{\partial y}(x,y)=K\left(x,y\right)U\left(x,y\right),$$

where

$$K\left(x,y\right)=2\left(x^2+y^2\right)\left(\left(x^2+y^2\right)^2+c\left(\left(a+b\right)x^2+\left(a-b\right)y^2\right)\left(x^2+y^2-h\right)\right).$$

Therefore, the circle $(\gamma_1) : x^2 + y^2 - h = 0$ is an invariant curve of system $(II.15)$. Of course (γ_1) defines a periodic solution $\gamma_1(t) = (x\left(t\right), y\left(t\right))$ of system $(II.15)$, since it do not pass through the origin. To see that (γ_1) is in fact a limit cycle, we recall a classic result characterizing limit cycles among other periodic orbits (see for instance [54] for more details), which means that (γ_1) is a limit cycle when $\int_0^T div(\gamma_1\left(t\right))dt \neq 0$ stable if $\int_0^T div\left(\gamma_1\left(t\right)\right)dt < 0$, and instable otherwise $\int_0^T div(\gamma_1(t))dt > 0$, where T is the period of the periodic solution $\gamma_1\left(t\right)$, we use also a practical result of H. Giacomini and al. [52], which asserts that

$$\int_0^T div(\gamma_1\left(t\right))dt = \int_0^T K\left(x\left(t\right), y\left(t\right)\right)dt.$$

Since $b^2 - a^2 < 0$, so the curve $K(x, y) = 0$ do not cross cercle γ_1, thus $K(x, y) \neq 0$ inside $\gamma_1 \backslash \{(0,0)\}$ and $K(0,0) = 0$, hence $\int_0^T K(x, y)\, dt \neq 0$. Consequently (γ_1) defines an algebraic limit cycle for system $(II.15)$.

The search for the non-algebraic limit cycle, requires the integration of our system. To prove our results we write the polynomial differential system $(II.15)$ in polar coordinates (r, θ), defined by $x = r\cos\theta$ and $y = r\sin\theta$. Then the system $(II.15)$ become

$$\begin{aligned} \dot{r} &= r^3 \left(h - r^2\right)\left(-r^2 - acr^2 + ach - bcr^2\cos 2\theta + bch\cos 2\theta\right), \\ \dot{\theta} &= -2hr^4\left(a + b\cos 2\theta\right). \end{aligned} \tag{II.18}$$

Taking as independent variable the coordinate θ, this differential system writes

$$\frac{dr}{d\theta} = \frac{1}{2r}\left(h - r^2\right)\frac{r^2 + acr^2 - ach + bcr^2\cos 2\theta - bch\cos 2\theta}{h\left(a + b\cos 2\theta\right)}. \tag{II.19}$$

Note that since $h > 0$, $a > 0$ and $b > 0$, therefore, $\dot{\theta} < 0$ and the orbits $r(\theta)$ of the differential equation $(II.19)$ has reversed their orientation with respect to the orbits $(r(t), \theta(t))$ or $(x(t), y(t))$ of the differential systems $(II.18)$ and $(II.15)$, respectively.

Via the change of variables $\rho = r^2$, this equation is transformed into the Riccati equation

$$\frac{d\rho}{d\theta} = (h - \rho)\frac{\rho + ac\rho - ach + bc\rho\cos 2\theta - bch\cos 2\theta}{h\left(a + b\cos 2\theta\right)} \tag{II.20}$$

$$= -\frac{ac + bc\cos 2\theta + 1}{h\left(a + b\cos 2\theta\right)}\rho^2 + \frac{2ac + 2bc\cos 2\theta + 1}{a + b\cos 2\theta}\rho - ch. \tag{II.21}$$

Since $\rho = h$ is evidently a particular solution, so from the last lines of the introduction, the general solution can be written as

$$\rho(\theta, k) = h + \Phi(\theta)\left(k - \int_0^\theta \Phi(s)\left(-\frac{ac + bc\cos 2s + 1}{h\left(a + b\cos 2s\right)}\right) ds\right),$$

where

$$\Phi(\theta) = \exp\left(\int_0^\theta \exp\left(-\int_0^s \frac{dw}{a + b\cos 2w}\right) ds\right),$$

and then

$$\rho(\theta, k) = \frac{h\exp(\varphi(\theta))\left(k + c\int_0^\theta \exp(-\varphi(s))\, ds\right)}{-1 + \exp(\varphi(\theta))\left(k + c\int_0^\theta \exp(-\varphi(s))\, ds\right)}, \tag{II.22}$$

with $\varphi(\theta) = \int_0^\theta \frac{ds}{a+b\cos 2s}$ and $k \in \mathbb{R}$.

Consequently, the general solution of $(II.19)$ is

$$r(\theta, k) = \left(\frac{h\exp(\varphi(\theta))\left(k + c\int_0^\theta \exp(-\varphi(s))\, ds\right)}{-1 + \exp(\varphi(\theta))\left(k + c\int_0^\theta \exp(-\varphi(s))\, ds\right)}\right)^{\frac{1}{2}}.$$

By passing to Cartesian coordinates, we deduce the first integral

$$H(x, y) = \frac{(x^2 + y^2)}{(x^2 + y^2 - h)\exp\left(\varphi(\arctan\frac{y}{x})\right)} - c\int_0^{\arctan\frac{y}{x}} \exp(-\varphi(s))\, ds.$$

The trajectories of system $(II.15)$ are the level curves $H(x, y) = k$, $k \in \mathbb{R}$ and since these curves are obviously non-algebraic except to the curve (γ_1) which corresponds to $k \to +\infty$, thus any other limit cycle, if exists, should also be non-algebraic.

To go a step further, if we put $\theta = 0$ in the solution, we get $r(0, k) = \left(\frac{hk}{-1+k}\right)^{\frac{1}{2}}$. Let $r_0 = \left(\frac{hk}{-1+k}\right)^{\frac{1}{2}}$, so $r(0, r_0) = r_0 > 0$, corresponds to the value $k = \frac{r_0^2}{r_0^2 - h}$ providing a rewriting of the general solution of $(II.18)$ as

$$r(\theta, r_0) = \left(\frac{h\exp(\varphi(\theta))\left(\frac{r_0^2}{r_0^2-h} + c\int_0^\theta \exp(-\varphi(s))\, ds\right)}{-1 + \exp(\varphi(\theta))\left(\frac{r_0^2}{r_0^2-h} + c\int_0^\theta \exp(-\varphi(s))\, ds\right)}\right)^{\frac{1}{2}}, \tag{II.23}$$

where $r_0 = r(0)$ and $\varphi(\theta) = \int_0^\theta \frac{ds}{a+b\cos 2s}$. A periodic solution of system $(II.15)$ must satisfy the condition

$$r(2\pi, r_0) = r(0, r_0) \tag{II.24}$$

The condition (II.24) equivalent to

$$r_0^2 = \frac{h\exp(\varphi(2\pi))\left(c\int_0^{2\pi} \exp(-\varphi(s))\, ds\right)}{-1 + \exp(\varphi(2\pi))\left(1 + c\int_0^{2\pi} \exp(-\varphi(s))\, ds\right)}, \tag{II.25}$$

provided two distinct values of $r_0 = h$ and thanks to (II.17), the well defined second value is

$$r_0 = r_* = \left(\frac{h \exp\left(\varphi(2\pi)\right) \left(c \int_0^{2\pi} \exp\left(-\varphi(s)\right) ds \right)}{-1 + \exp\left(\varphi(2\pi)\right) \left(1 + c \int_0^{2\pi} \exp\left(-\varphi(s)\right) ds \right)} \right)^{\frac{1}{2}}. \tag{II.26}$$

r_* is the intersection of the periodic orbit with the positive x-semi.axis, providing the value $r_* > 0$. Indeed, since $h, a, b, c \in \mathbb{R}_*^+$ and $b^2 - a^2 < 0$, then $\varphi(\theta) > 0$ for all $\theta \in \mathbb{R}$, so

$$\begin{aligned} r_* &= \left(\frac{h \exp\left(\varphi(2\pi)\right) \left(c \int_0^{2\pi} \exp\left(-\varphi(s)\right) ds \right)}{-1 + \exp\left(\varphi(2\pi)\right) \left(1 + c \int_0^{2\pi} \exp\left(-\varphi(s)\right) ds \right)} \right)^{\frac{1}{2}} \\ &> \left(\frac{h \exp\left(\varphi(2\pi)\right) \left(c \int_0^{2\pi} \exp\left(-\varphi(s)\right) ds \right)}{\exp\left(\varphi(2\pi)\right) \left(1 + c \int_0^{2\pi} \exp\left(-\varphi(s)\right) ds \right)} \right)^{\frac{1}{2}} > 0 \text{ for all } s \in [0, 2\pi]. \end{aligned}$$

Injecting this value of r_* in (II.23), we get the candidate solution

$$r(\theta, r_*) = r(\theta, r_0) = \left(\frac{h \exp\left(\varphi(\theta)\right) \left(\frac{r_*^2}{r_*^2 - h} + c \int_0^{\theta} \exp\left(-\varphi(s)\right) ds \right)}{-1 + \exp\left(\varphi(\theta)\right) \left(\frac{r_*^2}{r_*^2 - h} + c \int_0^{\theta} \exp\left(-\varphi(s)\right) ds \right)} \right)^{\frac{1}{2}}. \tag{II.27}$$

Next we prove that $r(\theta, r_*) > 0$. Indeed

$$\begin{aligned} r(\theta, r_*) &= \left(\frac{h \exp\left(\varphi(\theta)\right) \left(\frac{r_*^2}{r_*^2 - h} + c \int_0^{\theta} \exp\left(-\varphi(s)\right) ds \right)}{-1 + \exp\left(\varphi(\theta)\right) \left(\frac{r_*^2}{r_*^2 - h} + c \int_0^{\theta} \exp\left(-\varphi(s)\right) ds \right)} \right)^{\frac{1}{2}}. \\ &> \left(\frac{h \exp\left(\varphi(\theta)\right) \left(\frac{r_*^2}{r_*^2 - h} + c \int_0^{\theta} \exp\left(-\varphi(s)\right) ds \right)}{\exp\left(\varphi(\theta)\right) \left(\frac{r_*^2}{r_*^2 - h} + c \int_0^{\theta} \exp\left(-\varphi(s)\right) ds \right)} \right)^{\frac{1}{2}} \\ &= \sqrt{h} > 0, \end{aligned}$$

because $h > 0$ and $\exp\left(\varphi(\theta)\right) \left(\frac{r_*^2}{r_*^2 - h} + c \int_0^{\theta} \exp\left(-\varphi(s)\right) ds \right) > 0$ for all $\theta \in [0, 2\pi]$. In order to prove that the periodic orbit is an hyperbolic limit cycle, we consider (II.23), and introduce the Poincaré return map $r_0 \mapsto P(r_0) = r(2\pi, r_0)$, see [33] and [60], We compute $\dfrac{dr(2\pi, r_0)}{dr_0}$ at the value $r_0 = r_*$. We find that

$$\left. \frac{dr(2\pi, r_0)}{dr_0} \right|_{r_0 = r_*} = \frac{h r_*}{\left(r_*^2 - h\right)^2} \frac{\left(h \exp\left(\varphi(2\pi)\right)\right)^{\frac{1}{2}} \left(\frac{r_*^2}{r_*^2 - h} + c \int_0^{2\pi} \exp\left(-\varphi(s)\right) ds \right)^{\frac{-1}{2}}}{\left(-1 + \exp\left(\varphi(2\pi)\right) \left(\frac{r_*^2}{r_*^2 - h} + c \int_0^{2\pi} \exp\left(-\varphi(s)\right) ds \right) \right)^{\frac{3}{2}}}.$$

Taking into account (II.17), we deduce that $\left.\dfrac{dr(2\pi, r_0)}{dr_0}\right|_{r_0=r_*} \neq 1$, and finally (γ_2) is the expected non-algebraic limit cycle. Obviously (γ_2) lies inside (γ_1) when $r_* < h$. Since the Poincaré return map do not possess other fixed points, the system (II.15) admits exactly two limit cycles.

3.2 Application

For $a = 2$, $b = c = \frac{1}{4}$ and $h = 1$ the system $(II.15)$ becomes

$$\begin{aligned}
\dot{x} &= x\left(x^2+y^2\right)^2\left(x^2+y^2-1\right) + \left(\tfrac{9}{4}x^2 + \tfrac{7}{4}y^2\right) \\
&\quad \left(\tfrac{1}{4}x - \tfrac{1}{2}x^3 + 2x^2y - \tfrac{1}{2}xy^2 + 2y^3 + \tfrac{1}{4}x^5 + \tfrac{1}{2}x^3y^2 + \tfrac{1}{4}xy^4\right), \\
\dot{y} &= y\left(x^2+y^2\right)^2\left(x^2+y^2-1\right) + \left(\tfrac{9}{4}x^2 + \tfrac{7}{4}y^2\right) \\
&\quad \left(\tfrac{1}{4}y - 2x^3 - \tfrac{1}{2}x^2y - 2xy^2 - \tfrac{1}{2}y^3 + \tfrac{1}{4}x^4y + \tfrac{1}{2}x^2y^3 + \tfrac{1}{4}y^5\right).
\end{aligned} \tag{II.28}$$

This system possesses two limit cycles : the circle $(\Gamma_1) : x^2 + y^2 - 1 = 0$ surrounding a transcendental limit cycle (Γ_2) explicitly given in polar coordinates (r, θ) by the equation

$$r(\theta, r_*) = \left(\frac{\exp(\varphi(\theta))\left(\frac{r_*^2}{r_*^2-1} + \frac{1}{4}\int_0^\theta \exp(-\varphi(s))\,ds\right)}{-1 + \exp(\varphi(\theta))\left(\frac{r_*^2}{r_*^2-1} + \frac{1}{4}\int_0^\theta \exp(-\varphi(s))\,ds\right)}\right)^{\frac{1}{2}}$$

with $\varphi(\theta) = \int_0^\theta \frac{ds}{2+\frac{1}{4}\cos 2s}$ and $r_* = \left(\frac{he^{\varphi(2\pi)}\left(c\int_0^{2\pi}\exp(-\varphi(s))ds\right)}{-1+e^{\varphi(2\pi)}\left(1+c\int_0^{2\pi}\exp(-\varphi(s))ds\right)}\right)^{\frac{1}{2}} \simeq 0.80204$, see figure (II.2).

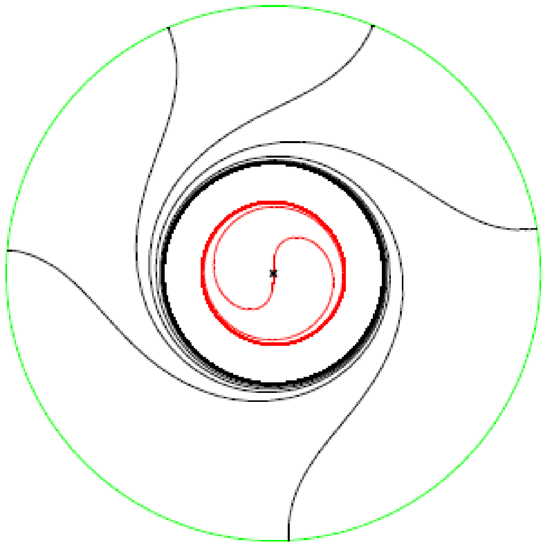


Figure II.2: Two limit cycles for system (II.28).

4 Discussions and conclusions

Although the integrability of the differential system is not an easy task. We have succeeded in constructing some classes of integrable differential systems with explicit non-algebraic limit cycle. First, we have a quartic polynomial differential systems with unstable node, we give the exact expression of the first integral. On the other hand, we show the existence of a limit cycle which is not algebraic. Secondly we have studied a new family of polynomial planar differential systems of degree seven with a non-elementary point at the origin. We show that under certain conditions this family is transformed into a Riccati type equation. This allowed us to determine the first integral, and to show that this family admits two limit cycles one is algebraic and the other is not algebraic.

Journal and Conference Papers Related to the Chapter

Cn) Qualitative Study of a Class of Quartic Differential System with an Unstable Node

Rebeiha ALLAOUA, Rachid CHEURFA and Ahmed BENDJEDDOU. "Qualitative Study of a Class of Quartic Differential System with an Unstable Node". International Conference on Recent Advances in Mathematics and Informatics (ICRAMI), 2021. Tebessa, Algeria. pp. 1-3, see [3].

Jr) Coeexistence of Limit Cycles in a Septic Planar Differential System Enclosing a Non-Elementary Singular Point, Using Riccati Equation

Rebeiha ALLAOUA, Rachid CHEURFA and Ahmed BENDJEDDOU. "Coeexistence of Limit Cycles in a Septic Planar Differential System Enclosing a Non-Elementary Singular Point, Using Riccati Equation". São Paulo Journal of Mathematical Sciences 16 (2022), 997–1006, see [4].

Chapter III

Limit cycles for discontinuous piecewise differential systems formed by an arbitrary linear system and a cubic isochronous center

"It is during our darkest moments that we must focus to see the light."

– Aristotle

1 Preface

The study of the discontinuous piecewise differential systems, more recently also called Filippov systems, has attracted the attention of the mathematicians during these past decades due to their applications. These piecewise differential systems in the plane are formed by different differential systems defined in distinct regions separated by a straight line. A pioneering work on this subject was due to Andronov, Vitt and Khaikin in 1920's, and later on Filippov in 1988 provided the theoretical bases for this kind of differential systems. Nowadays a vast literature on these differential systems is available.

There are many papers studying planar piecewise linear differential systems with two zones (see, e.g., [15, 16, 17, 51, 56, 57] and references therein). For the discontinuous planar nonlinear differential systems there are several papers studying the number of limit cycles (see [12, 24, 33, 40, 50] and references therein). Note that for the piecewise cubic polynomial differential system, there are two recent papers see [23, 42] obtaining at least 18 and 24 small limit cycles, respectively.

Another interesting and natural problem is to express analytically the limit cycles. Nevertheless, in most of these papers explicit limit cycles do not appear. The present paper is a contribution in that direction, motivated by the recent publication of some research papers exhibiting planar polynomial systems with algebraic or non-algebraic limit cycles given analytically (see, e.g., [15, 16, 22, 53]).

In this chapter, we analyse the version of Hilbert's 16Th problem for a piecewise differential system in the plane for a particular case. We estimate the maximum number of crossing limit cycles for a planar piecewise differential system separated by straight line Σ and formed by two regions. More precisely, we study a class of planar discontinuous piecewise differential systems with two linearity regions separated by a straight line $\Sigma = \{(x, y) \in \mathbb{R}^2 : x = 0\}$. We assume that the two linearity regions in the phase plane are the left and right half-planes

$$\Sigma_L = \{(x, y) \in \mathbb{R}^2 : x < 0\}, \;\; \Sigma_R = \{(x, y) \in \mathbb{R}^2 : x > 0\},$$

formed by an arbitrary linear differential system and by cubic systems with homogeneous nonlinearity with an isochronous center at the origin. We can write such systems as

$$\begin{aligned} &\dot{x} = -y + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ &\dot{y} = x + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3, \end{aligned} \quad \text{in } \Sigma_R,$$
$$\begin{aligned} &\dot{x} = \alpha x + \beta y + \gamma, \\ &\dot{y} = \eta x + \delta y + \xi, \end{aligned} \quad \text{in } \Sigma_L, \tag{III.1}$$

where $\alpha, \beta, \gamma, \eta, \delta, \xi$ and a_{ij}, b_{ij} for $i, j \in \{0, 1, 2, 3\}, i + j = 3$ are the real constants. Consider the piecewise differential systems $(III.1)$. In order to state precisely our results, we introduce first some notations and definitions. In accordance with Filippov [35], we distinguish the following open regions in the discontinuity set Σ.

1. Crossing region:

$$\Sigma_c = \{(0, y) \in \Sigma : \left(a_{03}y^3 - y\right)(\beta y + \gamma) > 0\}, \tag{III.2}$$

2. Sliding region:

$$\Sigma_s = \{(0, y) \in \Sigma : \left(a_{03}y^3 - y\right)(\beta y + \gamma) \leq 0\}. \tag{III.3}$$

As usual, isolated periodic orbits are called *limit cycles*. There are two types of limit cycles ″ crossing and sliding ones ″ in the planar discontinuous piecewise differential systems. The first type of the limit cycles contains in some arc of discontinuity lines that separate the different differential systems (for more detail see [61]), and the second type contains only isolated points of the lines of discontinuity. But we shall work only with crossing limit cycles. An equilibrium point is called a real (resp. virtual) singular point of the right system of $(III.1)$ if this point locates in the region Σ_R (resp. Σ_L). A similar definition can be done for the left system of $(III.1)$.

The linear differential system that we consider in the second half-plane Σ_L is either a focus (we include in this class the centers), or a saddle, or a node with different eigenvalues, or node with equal eigenvalues whose linear part does not diagonalize, or linear without equilibrium points. Note that if a piecewise differential system with two pieces separated by a straight line has a star node (node with equal eigenvalues whose linear part diagonalize), this prevents the existence of periodic orbits.

2 Canonical forms

We assume that the discontinuous piecewise differential systems are formed by an arbitrary linear differential system and a cubic system with homogeneous nonlinearity with an isochronous center at the origin.
A center of a planar polynomial differential system is said isochronous if there exists a neighborhood of this point such that all periodic orbits in this neighborhood have the same period. Due to Theorem 11.1 of [28], a cubic polynomial differential system with homogeneous nonlinearity and with an isochronous center at the origin has one of the

following canonical forms

$$(S_1) \quad : \quad \begin{cases} \dot{x} = -y - 3xy^2 + x^3, \\ \dot{y} = x + 3x^2y - y^3. \end{cases}$$

$$(S_2) \quad : \quad \begin{cases} \dot{x} = -y + x^2y, \\ \dot{y} = x + 3xy^2. \end{cases}$$

$$(S_3) \quad : \quad \begin{cases} \dot{x} = -y\left(1 + 3x^2\right), \\ \dot{y} = x\left(1 + 2x^2 - 9y^2\right). \end{cases}$$

$$(S_4) \quad : \quad \begin{cases} \dot{x} = -y\left(1 - 3x^2\right), \\ \dot{y} = x\left(1 - 2x^2 + 9y^2\right). \end{cases}$$

It is known that system (S_1) has the first integral

$$H_1(x, y) = \frac{\left(x^2 + y^2\right)^2}{1 + 4xy}. \tag{III.4}$$

The period annulus of this system is given by:

$$\left\{(x, y) \in \mathbb{R}^2 : H_1(x, y) = h_1, \;\; h_1 \in (0, +\infty)\right\}.$$

System (S_2) has the first integral

$$H_2(x, y) = \frac{x^2 + y^2}{1 + x^2}. \tag{III.5}$$

The period annulus of this system is given by:

$$\left\{(x, y) \in \mathbb{R}^2 : H_2(x, y) = h_2, \;\; h_2 \in (0, +\infty)\right\}.$$

System (S_3) has the first integral

$$H_3(x, y) = \frac{(x + 2x^3)^2 + y^2}{(1 + 3x^2)^3}. \tag{III.6}$$

The period annulus of this system is given by:

$$\left\{(x,y)\in\mathbb{R}^2 : H_3(x,y)=h_3, h_3\in\left(0,\frac{4}{27}\right)\right\},$$

System (S_4) has the first integral

$$H_4(x,y)=\frac{(x-2x^3)^2+y^2}{(1-3x^2)^3}. \tag{III.7}$$

The period annulus of this system is given by:

$$\left\{(x,y)\in\mathbb{R}^2 : H_4(x,y)=h_4,\ h_4\in(0,+\infty)\right\}.$$

The following lemma provides a normal form for an arbitrary linear differential system having a real focuss (resp. a center), saddle, node with different eigenvalues and non-diagonalizable node with equal eigenvalues, respectively

Lemma 38 . *i) A linear differential system having a focus (resp. a center) can be written as*

$$\dot{x}=\alpha x+\beta y+\gamma,\ \dot{y}=-\frac{1}{\beta}\left((\alpha-\lambda)^2+\omega^2\right)x+(2\lambda-\alpha)y+\xi, \tag{III.8}$$

with $\omega>0,\ \beta\neq 0,\ \lambda\neq 0$ *(resp.* $\omega>0,\ \beta\neq 0$ *and* $\lambda=0$*). Moreover, when* $\lambda\neq 0$ *this system has the first integral*

$$\begin{aligned}H_f(x,y)=&\Big(\left(\omega^2+(\alpha-\lambda)^2\right)x^2+2\beta(\alpha-\lambda)xy+\beta^2y^2+2\tfrac{\lambda\gamma\left((\alpha-\lambda)^2+\omega^2\right)+\beta\xi\left(\alpha\lambda-\lambda^2-\omega^2\right)}{\lambda^2+\omega^2}x\\&+2\beta\left(\gamma+\tfrac{\lambda(\beta\xi-\gamma(2\lambda-\alpha))}{\lambda^2+\omega^2}\right)y+\tfrac{\gamma^2\left((\alpha-\lambda)^2+\omega^2\right)+\beta\xi(2\gamma(\alpha-\lambda)+\beta\xi)}{\lambda^2+\omega^2}\Big)\\&\times\exp\left(-\tfrac{2\lambda}{\omega}\arctan\left(\tfrac{\omega\left(\left(\lambda^2+\omega^2\right)x+2\lambda\gamma-\alpha\gamma-\beta\xi\right)}{(\lambda^2+\omega^2)(\alpha-\lambda)x+\beta(\lambda^2+\omega^2)y+(\omega^2-\lambda^2+\alpha\lambda)\gamma+\beta\lambda\xi}\right)\right).\end{aligned} \tag{III.9}$$

and if $\lambda=0$*, the first integral of (III.8) is*

$$H_c(x,y)=\left(\omega^2+\alpha^2\right)x^2+2\beta\alpha xy+\beta^2y^2-2\beta\xi x+2\beta\gamma y.$$

ii) A linear differential system having a saddle (resp. a node with different eigenvalues) *(resp. a non-diagonalizable node with equal eigenvalues) can be written as*

$$\dot{x} = \alpha x + \beta y + \gamma, \quad \dot{y} = \frac{1}{\beta}\left(\rho^2 - (\alpha - r)^2\right) x + (2r - \alpha)\, y + \xi, \tag{III.10}$$

with$\beta \neq 0$ *and* $\rho^2 > r^2 > 0$ *(resp.* $\beta \neq 0$ *and* $r^2 > \rho^2 > 0$*) (resp.* $\beta \neq 0$, $r \neq 0$ *and* $\rho = 0$*). Moreover, when* $\rho \neq 0$ *this system has the first integral*

$$H_{s,n}(x,y) = \left((\alpha - r - \rho)x + \beta y + \gamma + \tfrac{\beta\xi - \gamma(2r-\alpha)}{r-\rho}\right)\left((\alpha - r + \rho)x + \beta y + \gamma + \tfrac{\beta\xi - \gamma(2r-\alpha)}{r+\rho}\right)^{\frac{\rho - r}{r+\rho}}, \tag{III.11}$$

when $\rho = 0$*, the first integral of (III.10) is given by*

$$H_{n'}(x,y) = \frac{1}{r\,(r-\delta)\,x + \beta r y + r\gamma + \beta\xi - \gamma\delta} \exp\left(\frac{r^2 x - \beta\xi + \gamma\delta}{r\,(r-\delta)\,x + \beta r y + r\gamma + \beta\xi - \gamma\delta}\right). \tag{III.12}$$

Proof. Consider the general linear differential system

$$\dot{x} = \alpha x + \beta y + \gamma, \quad \dot{y} = \eta x + \delta y + \xi. \tag{III.13}$$

Its eigenvalues are given by $\lambda_{1,2} = \frac{1}{2}\left(\alpha + \delta \pm \sqrt{(\alpha - \delta)^2 + 4\beta\eta}\right)$.

i) We know that system (III.13) has a focus if $\frac{1}{2}(\alpha + \delta) = \lambda$ and $(\alpha - \delta)^2 + 4\beta\eta = -4\omega^2$ for some $\omega > 0$, $\beta\eta < 0$ and $\lambda \in \mathbb{R}$, then $\delta = 2\lambda - \alpha$ and $\eta = -\frac{1}{\beta}((\alpha - \lambda)^2 + \omega^2)$. Therefore, we obtain system $(III.8)$.

ii) The linear differential system $(III.13)$ has a saddle if $\frac{1}{2}(\alpha + \delta) = r$ and $(\alpha - \delta)^2 + 4\beta\eta = 4\rho^2$ for some $r^2 < \rho^2$, then $\alpha = 2r - \delta$ and $\eta = -\frac{1}{\beta}\left((r - \delta)^2 - \rho^2\right)$. Therefore, we obtain system $(III.10)$.

Analogously to the previous case, the linear differential system $(III.13)$ has a node with different eigenvalues if $\frac{1}{2}(\alpha + \delta) = r$ and $(\alpha - \delta)^2 + 4\beta\eta = 4\rho^2$ for some $r^2 > \rho^2$, then $\alpha = 2r - \delta$, $\eta = -\frac{1}{\beta}\left((r - \delta)^2 - \rho^2\right)$.

We know that system $(III.13)$ has a non-diagonalizable node with equal eigenvalues if $(\alpha - \delta)^2 + 4\beta\eta = 0$ and $\frac{1}{2}(\alpha + \delta) = r \neq 0$, then $\delta = 2r - \alpha$ and $\eta = -\frac{1}{\beta}(r - \delta)^2$.

Therefore, we obtain system $(III.10)$ with $\rho = 0$.

It is clear that H_f, H_c, $H_{s,n}$, and $H_{n'}$ are first integrals of systems $(III.8)$, $(III.8)$ with $\lambda = 0$, $(III.10)$ with $\rho \neq 0$ and $(III.10)$ with $\rho = 0$, respectively. In fact, all the following equations are satisfied:

$$\frac{dH_i}{dt} = \dot{x}\frac{\partial H_i}{\partial x} + \dot{y}\frac{\partial H_i}{\partial y} \equiv 0, \quad i = f, c, s, n, n'.$$

■

The following Lemma provides a first integral for a linear differential system without equilibrium points.

Lemma 39 *[17]. A linear differential system without equilibrium points can be written as*

$$\dot{x} = ax + by + c, \qquad \dot{y} = \mu ax + \mu by + d, \tag{III.14}$$

where a, b, c, μ and d are real constants such that $d \neq \mu c$ and $\mu \neq 0$. Moreover, this system has the first integral

$$H_w(x, y) = \begin{cases} b\mu^2x^2 - 2b\mu xy - 2dx + by^2 + 2cy & \text{if } a + b\mu = 0, \\ ((a + b\mu)(ax + by) + ac + bd)\, e^{\frac{a+b\mu}{d-c\mu}(\mu x - y)} & \text{if } a + b\mu \neq 0. \end{cases} \tag{III.15}$$

Remark 40 . *According to Lemma 39 and Lemma 38 , it seems clear that limit cycles (if exist) of systems $(III.1)$ are algebraic, when the left subsystem of $(III.1)$ is one of the following types:*

-a linear center;

-a linear saddle;

-a linear node with different eigenvalues;

-a linear system without equilibria $(III.14)$ with $a + b\mu = 0$.

While that limit cycles (if exist) are non-algebraic when the left subsystem of $(III.1)$

is one of the following types:

-a linear focus;

-a non-diagonalizable node with equal eigenvalues;

-a linear system without equilibria $(III.14)$ *with* $a + b\mu \neq 0$.

3 Statement of the main results

The main result is the following:

Theorem 41 . *The following statements hold for the discontinuous piecewise differential systems* $(III.1)$

(1) *if* $(III.1)$ *is of the type linear focus and cubic isochronous center at the origin, then the piecewise differential systems* $(III.1)$ *have at most two crossing limit cycles. Moreover, these limit cycles, if exist are non-algebraic and there are system of this type with one or two limit cycles.*

(2) *if* $(III.1)$ *is of the type linear center and cubic isochronous center at the origin, then the piecewise differential systems* $(III.1)$ *have no crossing limit cycles.*

(3) *if* $(III.1)$ *is of the type linear saddle and cubic isochronous center at the origin, then the piecewise differential systems* $(III.1)$ *have at most one crossing limit cycle. Moreover, this limit cycle, if exists, is algebraic and there are systems of this type with one limit cycle.*

(4) *if (III.1) is of the type linear node with different eigenvalues and cubic isochronous center at the origin, then the piecewise differential systems (III.1) have at most one crossing limit cycle. Moreover, this limit cycle, if exists, is algebraic and there are systems of this type with one limit cycle.*

(5) *if (III.1) is of the type non-diagonalizable linear node with equal eigenvalues and cubic isochronous center at the origin, then the piecewise differential systems*

(III.1) have at most one crossing limit cycle. Moreover, this limit cycle, if exists, is non-algebraic and there are systems of this type with one limit cycle.

(6) *if (III.1) of the type linear without equilibrium point and cubic isochronous center at the origin, then the piecewise differential systems (III.1) have at most one crossing limit cycle. Moreover, this limit cycle, if there exists, is non-algebraic and there are systems of this type with one limit cycle.*

Theorem 41 will be proved in section 2.

In the next proposition, we show that there are discontinuous piecewise differential systems (III.1) of the type linear real focus and cubic isochronous center at the origin, with two non-algebraic crossing limit cycles.

Proposition 42 . *The discontinuous piecewise differential systems (III.1) formed by one of the four cubic isochronous centers* (S_1), *or* (S_2), *or* (S_3), *or* (S_4), *and a family with one parameter of linear differential system of the form*

$$\dot{x} = -\tfrac{2}{5}x + \beta y - \tfrac{1}{8}, \qquad \dot{y} = -\tfrac{1}{50\beta}\left(52x + 5\right), \tag{III.16}$$

with $\beta \in (-\infty, -0.563\,86)$, *have exactly two nested non-algebraic crossing limit cycles. Moreover, these limit cycles are given by*

$$\begin{aligned} \Gamma_1 &= \{(x,y) \in \Sigma_R : H_j(x,y) = h_j\} \cup \left\{(x,y) \in \Sigma_L : H_{f1}(x,y) = 5.\,974\,1 \times 10^{-2}\right\}, \\ \Gamma_2 &= \left\{(x,y) \in \Sigma_R : H_j(x,y) = h_j'\right\} \cup \{(x,y) \in \Sigma_L : H_{f1}(x,y) = 0.101\,04\}, \end{aligned}$$

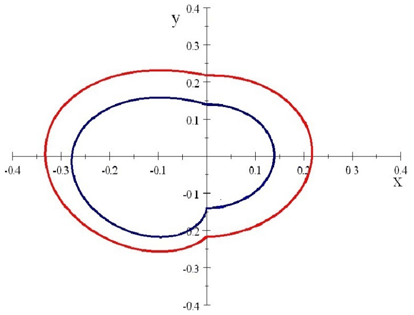
where $j \in \{1,2,3,4\}$,

$$\begin{aligned} h_1 &= \left(\tfrac{0.139\,44}{\beta}\right)^4, \quad h_2 = h_3 = h_4 = \left(\tfrac{0.139\,44}{\beta}\right)^2, \\ h_1' &= \left(\tfrac{0.217\,03}{\beta}\right)^4, \quad h_2' = h_3' = h_4' = \left(\tfrac{0.217\,03}{\beta}\right)^2 \end{aligned}$$

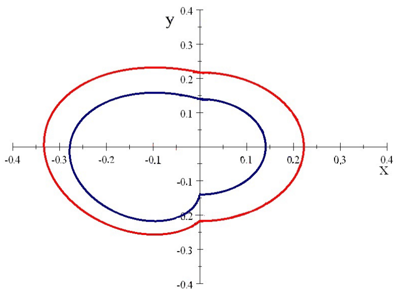
and

$$H_{f1}(x,y) = \left(\frac{26}{25}x^2 - \frac{2}{5}\beta xy + \frac{61}{260}x + \beta^2 y^2 - \frac{11}{52}\beta y + \frac{17}{832}\right) e^{-\frac{2}{5}\arctan\frac{520x+50}{104x-520\beta y+55}}.$$

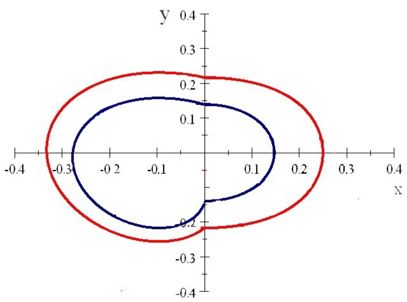
See Figure III.1.



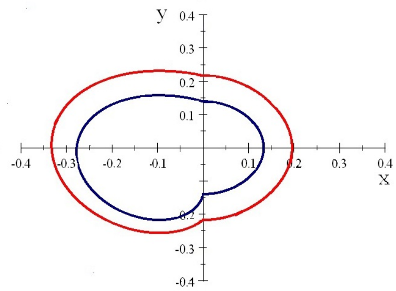
(a) Case when $S_j = S_1$.



(b) Case when $S_j = S_2$.



(c) Case when $S_j = S_3$.



(d) Case when $S_j = S_4$.

Figure III.1: The two nested crossing limit cycles of systems $(III.16) + (S_j)$ with $\beta = -1$.

In the next proposition, we show that there are discontinuous piecewise differential systems (III.1) of the type linear virtual focus and cubic isochronous center at the origin, with one non-algebraic crossing limit cycle.

Proposition 43 . *The discontinuous piecewise differential systems (III.1) formed by one of the four cubic isochronous centers* (S_1) *or* (S_2) *or* (S_3) *or* (S_4) *and a class with one parameter of linear differential system of the form*

$$\dot{x} = -x + \beta y + 1, \qquad \dot{y} = \tfrac{1}{\beta}\left(3\beta y - 5x + 5\right), \tag{III.17}$$

with $\beta \in (-\infty, -5.466\,4)$, *have exactly one non-algebraic crossing limit cycle. Moreover, this limit cycle is given by*

$$\Gamma_1 = \{(x,y) \in \Sigma_R : H_j(x,y) = h_j\} \cup \left\{(x,y) \in \Sigma_L : H_{f2}(x,y) = 5.374\,3 \times 10^{-2}\right\},$$

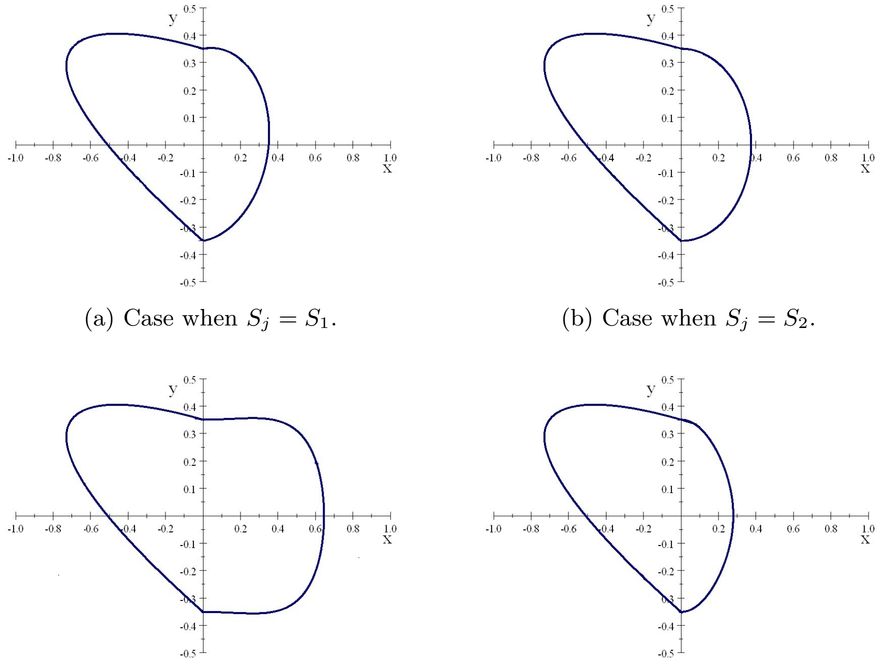
where $j \in \{1, 2, 3, 4\}$,

$$h_1 = \left(\frac{2.1040}{\beta}\right)^4, h_2 = h_3 = h_4 = \left(\frac{2.1040}{\beta}\right)^2$$

and

$$H_{f2}(x, y) = \left(5x^2 - 4\beta xy - 10x + \beta^2 y^2 + 4\beta y + 5\right) e^{-2\arctan \frac{x-1}{\beta y - 2x + 2}}.$$

See Figure III.2.



(a) Case when $S_j = S_1$.

(b) Case when $S_j = S_2$.

(c) Case when $S_j = S_3$.

(d) Case when $S_j = S_4$..

Figure III.2: The unique crossing limit cycle of systems $(III.17) + (S_j)$ with $\beta = -6$.

In the next proposition, we show that there are discontinuous piecewise differential systems (III.1) of the type linear saddle and cubic isochronous center at the origin, with one algebraic crossing limit cycle.

Proposition 44 . *The discontinuous piecewise differential systems (III.1) formed by one of the four cubic isochronous centers* (S_1) *or* (S_2) *or* (S_3) *or* (S_4) *and a class with one parameter of linear differential system of the form*

$$\dot{x} = x + \beta y + \tfrac{1}{10}, \qquad \dot{y} = \tfrac{1}{4\beta}(3x + 4), \tag{III.18}$$

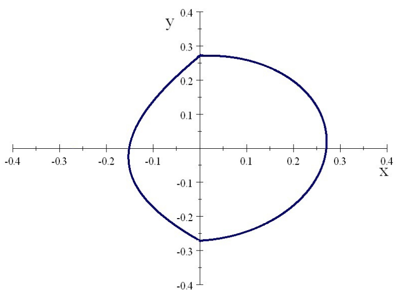
with $\beta \in (-\infty, -1.4056)$, *have exactly one algebraic crossing limit cycle. Moreover, this limit cycle is given by*

$$\Gamma = \{(x,y) \in \Sigma_R : H_j(x,y) = h_j\} \cup \{(x,y) \in \Sigma_L : H_s(x,y) = -3.2818\},$$

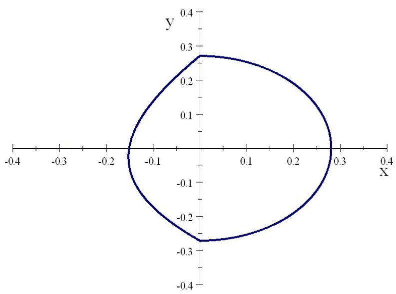
where $j \in \{1,2,3,4\}$, $h_1 = \left(\frac{0.54103}{\beta}\right)^4$ *and* $h_2 = h_3 = h_4 = \left(\frac{0.54103}{\beta}\right)^2$

$$H_s(x,y) = \left(-\tfrac{1}{2}x + \beta y - \tfrac{19}{10}\right)^3 \left(\tfrac{3}{2}x + \beta y + \tfrac{23}{30}\right).$$

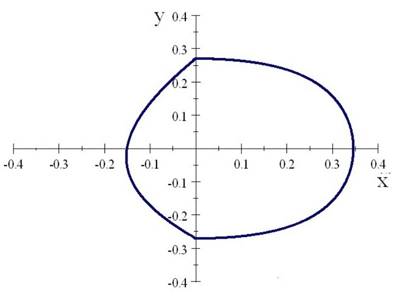
See Figure III.3.



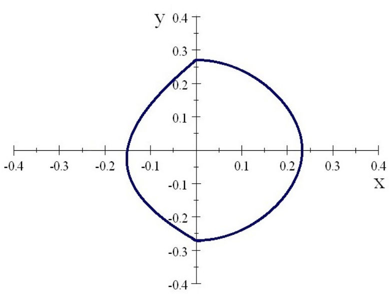
(a) Case when $S_j = S_1$.



(b) Case when $S_j = S_2$.



(c) Case when $S_j = S_3$.



(d) Case when $S_j = S_4$.

Figure III.3: The unique crossing limit cycle of systems $(III.18) + (S_j)$ with $\beta = -2$.

In the next proposition, we show that there are discontinuous piecewise differential systems (III.1) of the type linear node with different eigenvalues and cubic isochronous center at the origin; with one algebraic crossing limit cycle.

Proposition 45 . *The discontinuous piecewise differential systems (III.1) formed by one of the four cubic isochronous centers* (S_1) *or* (S_2) *or* (S_3) *or* (S_4) *and a class with* *one parameter of linear differential system of the form*

$$\dot{x} = 6x + \beta y + 1, \qquad \dot{y} = \tfrac{8}{\beta}x + \tfrac{4}{\beta}, \tag{III.19}$$

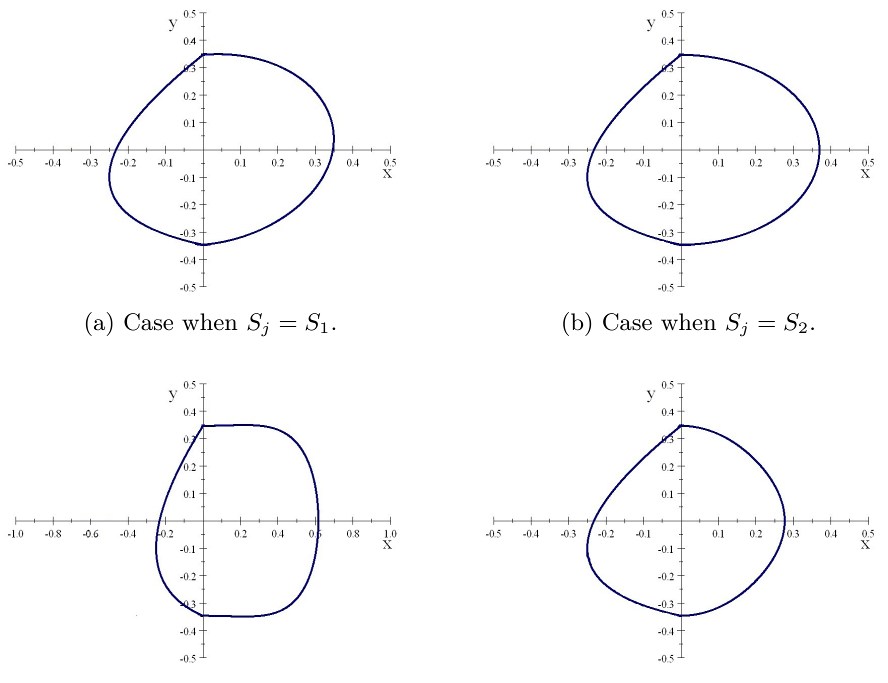
with $\beta \in (-\infty, -4.500\,1)$, *have exactly one algebraic crossing limit cycle. Moreover, this limit cycle is given by*

$$\Gamma = \{(x,y) \in \Sigma_R : H_j\,(x,y) = h_j\} \cup \{(x,y) \in \Sigma_L : H_n\,(x,y) = 36.\,008\}\,,$$

where $j \in \{1,2,3,4\}\,,\ h_1 = \left(-\frac{1.\,732\,1}{\beta}\right)^4$ *and* $h_2 = h_3 = h_4 = \left(-\frac{1.\,732\,1}{\beta}\right)^2$

$$H_n\,(x,y) = \frac{(2x+\beta y+3)^{\,4}}{(4x+\beta y+2)^2}\,.$$

See Figure III.4.



(a) Case when $S_j = S_1$. (b) Case when $S_j = S_2$.

(c) Case when $S_j = S_3$. (d) Case when $S_j = S_4$.

Figure III.4: The unique crossing limit cycle of systems $(III.19) + (S_j)$ with $\beta = -5$.

In the next proposition, we show that there are discontinuous piecewise differential systems (III.1) of the type linear non-diagonalizable node with equal eigenvalues and cubic isochronous center at the origin, with one non-algebraic crossing limit cycle.

Proposition 46 . *The discontinuous piecewise differential systems (III.1) formed by one of the four cubic isochronous centers* (S_1) *or* (S_2) *or* (S_3) *or* (S_4) *and a family with one parameter of linear differential system of the form*

$$\dot{x} = x + \beta y - 1, \qquad \dot{y} = -\tfrac{1}{\beta}\left(4x + 3\beta y - 4\right), \tag{III.20}$$

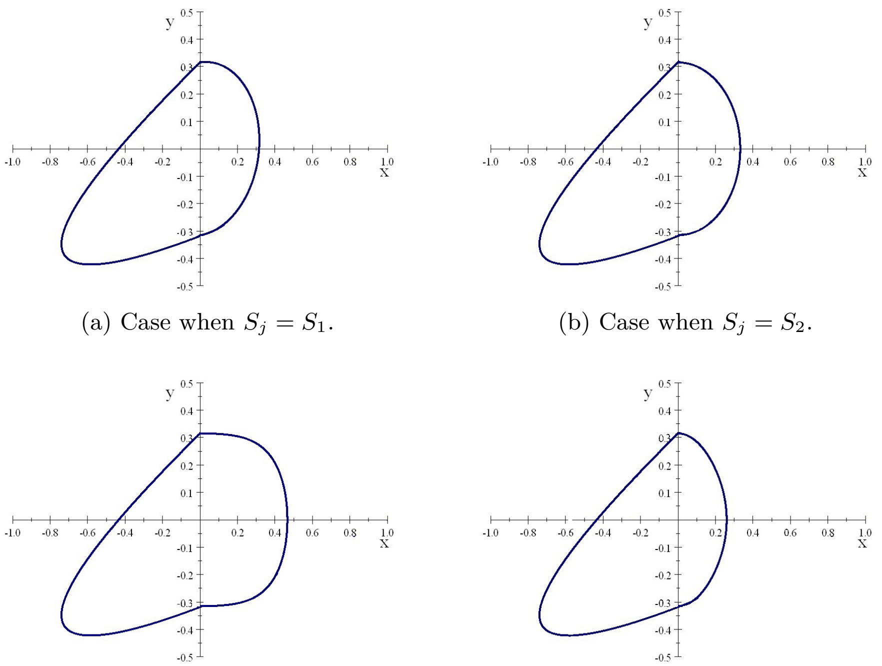
with $\beta \in (-\infty, -4.\,092\,5)$*, have exactly one non-algebraic crossing limit cycle. Moreover, this limit cycle is given explicitly by*

$$\Gamma = \left\{(x,y) \in \Sigma_R : H_j\left(x,y\right) = h_j\right\} \cup \left\{(x,y) \in \Sigma_L : H_{n'}\left(x,y\right) = 0.211\,46\right\},$$

where $j \in \{1,2,3,4\}$ *,* $h_1 = \left(\frac{1.\,575\,2}{\beta}\right)^4$ *and* $h_2 = h_3 = h_4 = \left(\frac{1.\,575\,2}{\beta}\right)^2$

$$H_{n'}\left(x,y\right) = \frac{1}{2 - \beta y - 2x} e^{\frac{x-1}{2-\beta y-2x}}.$$

See Figure III.5.



(a) Case when $S_j = S_1$.

(b) Case when $S_j = S_2$.

(c) Case when $S_j = S_3$.

(d) Case when $S_j = S_4$.

Figure III.5: The unique crossing limit cycle of systems $(III.20) + (S_j)$ with $\beta = -5$.

Remark 47 *The assumption on the parameter* β *in Propositions 42–46 is a necessary*

condition such that the cubic polynomial differential system with homogeneous nonlinearity (S_3) *has an unbounded period annulus surrounding the origin (a necessary condition is* $h_3 < \frac{4}{27}$ *which is also a necessary condition for the existence of crossing limit cycles of systems* $(S_3) + (III.16) - (S_3) + (III.20)$.
For $(S_j) + (III.16) - (S_j) + (III.20)$, $j = 1$ *or* 2 *or* 4, *the assumption* $\beta < 0$ *is a sufficient condition for the existence of crossing limit cycles because if* $\beta < 0$, *the two intersection points* $(0, y_0)$ *and* $(0, y_1)$ *of the orbit arc in* Σ_R *and the orbit arc in* Σ_L*; satisfy* $(-y)(\beta y + \gamma) \leq 0$. *This implies that the two intersection points* $(0, y_0)$ *and* $(0, y_1)$ *are sliding points and this prevents the existence of crossing limit cycle.*

In the next proposition, we show that there are discontinuous piecewise differential systems (III.1) of the type linear without equilibria and cubic isochronous center at the origin, with one non-algebraic crossing limit cycle.

Proposition 48 . *The discontinuous piecewise differential systems (III.1) formed by one of the four cubic isochronous centers* (S_1) *or* (S_2) *or* (S_3) *or* (S_4) *and a class with one parameter of linear differential system of the form*

$$\dot{x} = (\mu - 1)\, x - y - \tfrac{1}{100}, \qquad \dot{y} = \mu\, (\mu - 1)\, x - \mu y - \tfrac{\mu+100}{100}, \tag{III.21}$$

when $\mu \neq 0$, *have one explicit non-algebraic crossing limit cycle given by*

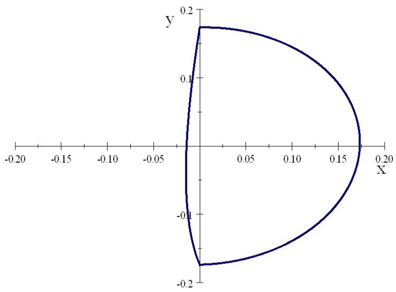
$$\Gamma = \{(x, y) \in \Sigma_R : H_j\,(x, y) = h_j\} \cup \{(x, y) \in \Sigma_L : H_w\,(x, y) = 0.995\,01\},$$

where $j = j \in \{1, 2, 3, 4\}$, $h_1 = (0.173\,38)^4$ *and* $h_2 = h_3 = h_4 = (0.173\,38)^2$ *and*

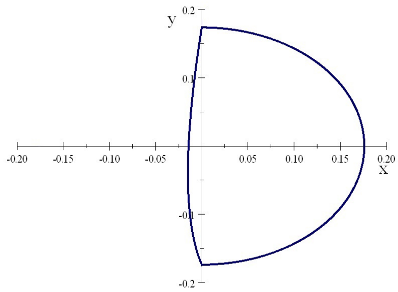
$$H_w\,(x, y) = \left(\frac{101}{100} + (1 - \mu)\, x + y\right) e^{\mu x - y}.$$

See Figure III.6.

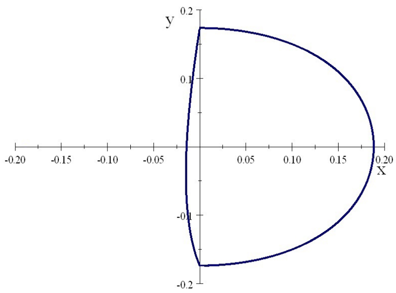
Remark 49 *The assumption* $b < 0$ *in Proposition 48 is a necessary condition for the existence of crossing limit cycle of the system because the crossing region of*



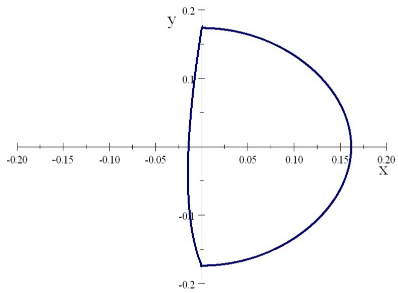
(a) Case when $S_j = S_1$.



(b) Case when $S_j = S_2$.



(c) Case when $S_j = S_3$.



(d) Case when $S_j = S_4$.

Figure III.6: The unique crossing limit cycle of systems $(III.21) + (S_j)$ with $\mu = 3$.

these systems is given by $-by\left(y + \frac{1}{100}\right) > 0$*, hence this last inequality implies that the crossing region is an open interval* $\left(-\frac{1}{100}, 0\right)$ *of the line* Σ *if* $b > 0$ *and is an open interval* $(0, +\infty) \cup \left(-\infty, -\frac{1}{100}\right)$ *of the line* Σ *if* $b < 0$*. Since the intersection points* $(0, y_1)$ *and* $(0, y_2)$*, where* $y_1 = -0.173\,38$ *and* $y_2 = 0.173\,38$*, are located in* $\left(-\infty, -\frac{1}{100}\right) \cup (0, +\infty)$*, we have to choose* $b < 0$*.*

4 Proof of the Theorem

We consider the discontinuous piecewise differential systems (III.1). If there exists a limit cycle of the discontinuous piecewise differential systems (III.1), it must intersect the discontinuity line Σ at two different points $(0, y_0)$ and $(0, y_1)$. In order to investigate the limit cycles of these systems, we use the Poincaré map of $(III.1)$.

We can define a right return map P_R as $y_1 = P_R(y_0)$ and a left return map P_L as $y_2 = P_L(y_1)$. Composing the right return map P_R with the left return map P_L, the

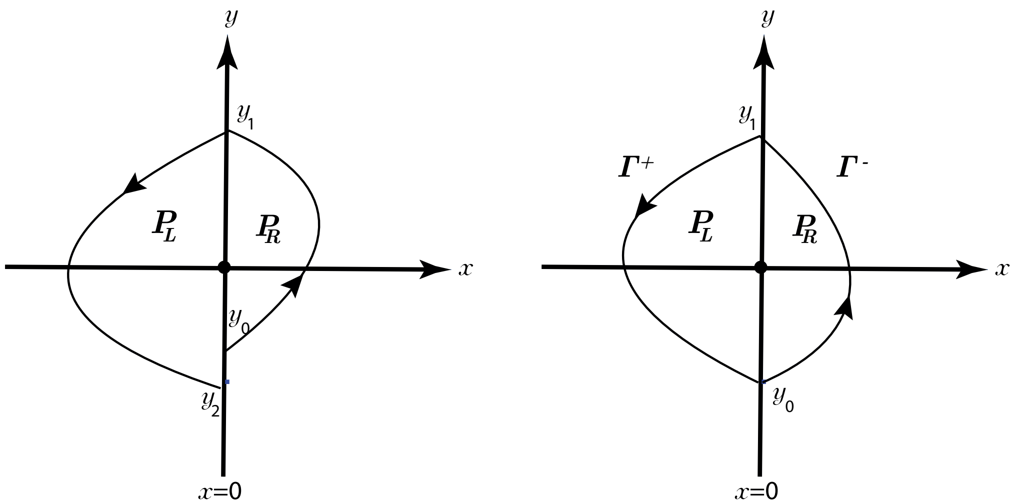
Poincaré map P of $(III.1)$ can be constructed by P_L and P_R as follows:

$$y_2 = P(y_0) = P_L \circ P_R(y_0).$$

It is obvious that the zeros of equation

$$F(y_0) = y_0 - P(y_0),$$

correspond to the periodic solution of the discontinuous piecewise differential systems (see figure III.7).



(a) The poincaré map of $(III.1)$ (b) The periodic solution of $(III.1)$

Figure III.7: The poincaré map and the periodic solution of $(III.1)$

In what follows, we give the detailed calculations for right and left return maps. For determine the right return map P_R, we use the first integrals for the right side systems of (III.1). Assume that the orbits starting at the point $(0, y_0)$ go into the right zone Σ_R under the flow of the right differential systems. If these orbits can reach Σ again at some point $(0, y_1)$, then $(0, y_0)$ and $(0, y_1)$ must satisfy the following equation

$$e_j = H_j(0, y_0) - H_j(0, y_1) = 0, \tag{III.22}$$

where $j \in \{1, 2, 3, 4\}$ and H_j are given by $(III.4), (III.5), (III.6)$ and (III.7) respectively. The equations III.22 for $j \in \{1,2,3,4\}$ are equivalent to $(y_0 - y_1)(y_1 + y_0) = 0$. From this equation, the unique solution satisfying $y_0 < y_1$ is $y_1 = -y_0$. Then, we can define a right Poincaré map as

$$P_R(y_0) = -y_0. \tag{III.23}$$

Proof of statement (1) of Theorem 41. First, we consider the case where the left subsystem of (III.1) is a linear focus type satisfying (III.8) with $\lambda \neq 0$. To determine the left return map P_L, we use the parametric representation of the solution of the linear differential system (III.8) in Σ_L. Thus the solution of this system with $\lambda \neq 0$ starting at the point $(0, y_1)$ is given by

$$\begin{aligned} x_L(t) &= \frac{e^{\lambda t}\left(\lambda\left(\beta\xi - \gamma\left(2\lambda - \alpha\right)\right) + \left(\lambda^2 + \omega^2\right)\left(\gamma + \beta y_1\right)\right)\sin\omega t}{\omega\left(\lambda^2 + \omega^2\right)} \\ &\quad - \frac{\left(\beta\xi - \gamma\left(2\lambda - \alpha\right)\right)\left(e^{t\lambda}\cos\omega t - 1\right)}{\lambda^2 + \omega^2}, \\ y_L(t) &= \frac{\left(\gamma\left(\left(\alpha - \lambda\right)^2 + \omega^2\right) + \beta\xi\alpha\right)\left(e^{t\lambda}\cos\omega t - 1\right)}{\beta\left(\lambda^2 + \omega^2\right)} + e^{\lambda t}\left(\cos\omega t - \frac{\alpha - \lambda}{\omega}\sin\omega t\right) y_1 \\ &\quad - \frac{\left(\lambda\left(\alpha - \lambda\right)\left(\gamma\left(\alpha - \lambda\right) + \beta\xi\right) + \left(\lambda\gamma - \beta\xi\right)\omega^2\right)e^{\lambda t}\sin\omega t}{\omega\beta\left(\lambda^2 + \omega^2\right)}. \end{aligned}$$

Then from the equation $x_L(t) = 0$, we obtain

$$y_1(t) = -\frac{\omega\left(\beta\xi - \gamma\left(2\lambda - \alpha\right)\right)\left(e^{-t\lambda} - \cos\omega t\right) + \left(\gamma\left(\lambda^2 + \omega^2\right) + \lambda\left(\beta\xi - \gamma\left(2\lambda - \alpha\right)\right)\right)\sin\omega t}{\beta\left(\lambda^2 + \omega^2\right)\sin\omega t}. \tag{III.24}$$

For this case, the parametric representation of the left return map P_L is

$$\begin{aligned} P_L(y_1) &= \frac{\left(\gamma\left(\left(\alpha - \lambda\right)^2 + \omega^2\right) + \beta\xi\alpha\right)\left(e^{t\lambda}\cos\omega t - 1\right)}{\beta\left(\lambda^2 + \omega^2\right)} + e^{\lambda t}\left(\cos\omega t - \frac{\left(\alpha - \lambda\right)}{\omega}\sin\omega t\right) y_1 \\ &\quad - \frac{\left(\lambda\left(\alpha - \lambda\right)\left(\gamma\left(\alpha - \lambda\right) + \beta\xi\right) + \left(\lambda\gamma - \beta\xi\right)\omega^2\right)e^{\lambda t}\sin\omega t}{\omega\beta\left(\lambda^2 + \omega^2\right)}. \end{aligned}$$

Since $y_1 = -y_0$, the zeros of function F are the zeros of the function G given by

$$G(t) = -y_1(t) - P_L(y_1(t)).$$

When substituting the previous expressions of $y_1(t)$ and $P_L(y_1(t))$ into the equation $G(t) = 0$, we obtain the equation

$$\frac{1}{\beta(\lambda^2+\omega^2-)}\left(\left(\gamma\left(\omega^2-\lambda^2\right)+\lambda(\beta\xi+\gamma\alpha)\right)-\omega(\beta\xi-\gamma(2\lambda-\alpha))\frac{\sinh\lambda t}{\sin\omega t}\right)=0. \tag{III.25}$$

Now, it is easy to see that the existence of a crossing limit cycle is equivalent to the existence of a positive t satisfying (III.25). For convenience, we use the notation

$$f_1(t)=\left(\gamma\left(\omega^2-\lambda^2\right)+\lambda(\beta\xi+\gamma\alpha)\right)-\omega(\beta\xi-\gamma(2\lambda-\alpha))\frac{\sinh\lambda t}{\sin\omega t}. \tag{III.26}$$

So, the way of solving equation (III.25 is the same as that of the equation $f_1(t) = 0$. In order to investigate a number of solutions of $f_1(t) = 0$, and since f_1 is a C^1- function in $\mathbb{R}\setminus\{0\}$, we use the first derivative of the function f_1 with respect to the variable t. Simple calculations yield

$$f_1'(t)=-\frac{\omega(\alpha\gamma-2\lambda\gamma+\beta\xi)}{\sin^2\omega t}(\lambda\sin\omega t\cosh\lambda t-\omega\cos\omega t\sinh\lambda t).$$

Note that the zeros of $f_1'(t)$ are the zeros of $K_1(t)$, where

$$K_1(t)=\lambda\sin\omega t\cosh\lambda t-\omega\cos\omega t\sinh\lambda t.$$

Note that the left linear differential system (III.8) has the eigenvalues $\lambda \pm i\sqrt{\omega}$, $\omega > 0$, at its singularity

$$(x_0,y_0)=\left(\frac{\beta\xi-\gamma(2\lambda-\alpha)}{\lambda^2+\omega^2},-\frac{\gamma\left((\alpha-\lambda)^2+\omega^2\right)+\beta\xi\alpha}{\beta(\lambda^2+\omega^2)}\right).$$

So, it follows that the frequency is ω, and consequently if (x_0, y_0) is a virtual focus, we have $t \in \left(0, \frac{\pi}{\omega}\right)$ and $t \in \left(\frac{\pi}{\omega}, \frac{2\pi}{\omega}\right)$ for (x_0, y_0) is a real focus.

i) If (x_0, y_0) is a virtual focus, i.e., if $\beta\xi - \gamma(2\lambda - \alpha) > 0$, we have $K_1(0) = 0$ and $K_1(t) \neq 0$ for $t \neq 0$, since $K_1'(t) = (\lambda^2 + \omega^2)\sin\omega t \sinh\lambda t$ cannot vanish in $\left(0, \frac{\pi}{\omega}\right)$, so, $f_1'(t) \neq 0$ for $t \in \left(0, \frac{\pi}{\omega}\right)$. Therefore, the equation $f_1(t) = 0$ with $\beta\xi - \gamma(2\lambda - \alpha) > 0$

may have at most one solution in $\left(0, \frac{\pi}{\omega}\right)$. Hence systems (III.1) has at most one crossing limit cycle.

ii) If (x_0, y_0) is a real focus, i.e., if $\beta\xi - \gamma(2\lambda - \alpha) < 0$, and since $K_1(\frac{\pi}{\omega}) = \omega \sinh\left(\lambda\frac{\pi}{\omega}\right)$ and $K_1(\frac{2\pi}{\omega}) = -\omega \sinh\left(\lambda\frac{2\pi}{\omega}\right)$ and $sign\left(K_1(\frac{\pi}{\omega})K_1(\frac{2\pi}{\omega})\right) < 0$, while $K_1'(t) \neq 0$ in $\left(\frac{\pi}{\omega}, \frac{2\pi}{\omega}\right)$, then K_1 is a strictly monotone function in $\left(\frac{\pi}{\omega}, \frac{2\pi}{\omega}\right)$. Thus $f_1'(t) = 0$ has exactly one solution in $\left(\frac{\pi}{\omega}, \frac{2\pi}{\omega}\right)$ and consequently, the equation (III.26) has at most two zeros in $\left(\frac{\pi}{\omega}, \frac{2\pi}{\omega}\right)$. From the above analysis, we conclude that systems (III.1) have at most two crossing limit cycles when $\beta\xi - \gamma(2\lambda - \alpha) < 0$. Using the first integrals of both differential systems of (III.1) and knowing that the non-algebraic crossing periodic orbits passes through the points $(0, y_{1i})$ and through the point $(0, y_{0i})$, $i = 1, 2$ where y_{1i} is defined by (III.24) and $y_{0i} = -y_{1i}$. we obtain the following expressions:

$$\Gamma_i = \{(x, y) \in \Sigma_R : H_j(0, y) = H_j(0, y_{0i})\} \cup \{(x, y) \in \Sigma_L : H(x, y) = H(0, y_{0i})\}, i = 1, 2,$$

where $j \in \{1, 2, 3, 4\}$ and H_j are given by $(III.4), (III.5), (III.6), (III.7)$ respectively.

Proof of statement (2) of Theorem 41. Using the notation introduced in the proof of statements (1), we consider that the left subsystem of (III.1) is a linear center type satisfying (III.8) with $\lambda = 0$. The solution of system (III.8) with $\lambda = 0$ starting at the point $(0, y_1)$ is

$$\begin{aligned} x_L(t) &= \frac{1}{\omega^2}\left((\alpha\gamma + \beta\xi) + \omega(\gamma + \beta y_1)\sin\omega t - \alpha\gamma\cos\omega t - \beta\xi\cos\omega t\right), \\ y_L(t) &= \frac{(\gamma(\alpha^2 + \omega^2) + \beta\xi\alpha)(\cos\omega t - 1)}{\beta\omega^2} + \left(\cos\omega t - \frac{\alpha}{\omega}\sin\omega t\right) y_1 + \frac{\xi\sin\omega t}{\omega}. \end{aligned}$$

Then, from equation $x_L(t) = 0$, we obtain

$$y_1(t) = -\frac{((\beta\xi + \gamma\alpha) + \gamma\omega\sin\omega t + (-\gamma\alpha - \beta\xi)\cos\omega t)}{\beta\omega\sin\omega t}.$$

For this case, the parametric representation of the left return map P_L is

$$P_L(y_1) = \frac{(\gamma(\alpha^2+\omega^2)+\beta\xi\alpha)(\cos\omega t-1)}{\beta\omega^2} + \left(\cos\omega t - \frac{\alpha}{\omega}\sin\omega t\right)y_1 + \frac{\xi\sin\omega t}{\omega}.$$

Since $y_1 = -y_0$, substituting the previous two expressions into the equation $G(t) = -y_1(t) - P_L(y_1(t)) = 0$, we obtain $\frac{2}{\beta}\gamma = 0$. Hence, if $\gamma \neq 0$, this last equality does not hold, the equation $F(y_0) = 0$ has no solutions and consequently, the discontinuous piecewise differential systems (III.1) have no periodic solutions. If $\gamma = 0$, then $G(t) = 0$ for all $t > 0$, i.e., $F(y_0) = 0$ has a continuum of solutions. So, the discontinuous piecewise differential systems (III.1) either does not have periodic solutions, or it has a continuum of periodic orbits, and consequently, these differential systems have no limit cycles.

Proof of statements (3) and (4) of Theorem 41. Now, we assume that the left subsystem of (III.1) is a linear system satisfying (III.10) with $\rho \neq 0$. We recall that if $r^2 > \rho^2 > 0$, then system (III.10) has a real or a virtual node with two different eigenvalues, while if $r^2 > \rho^2 > 0$ the system has a real or a virtual saddle. We have to study these cases simultaneously. To determine the left return map P_L of (III.1), we use the parametric representation of the solution of the linear differential system (III.10) with $\rho \neq 0$ in Σ_L starting at the point $(0, y_1)$, this solution is

$$\begin{aligned} x_L(t) &= \frac{e^{(r+\rho)t}(\gamma(\alpha-r+\rho)+\beta\xi+\beta(r+\rho)y_1)}{2\rho(\rho+r)} - \frac{\beta\xi-\gamma(2r-\alpha)}{\rho^2-r^2} \\ &\quad + \frac{e^{(r-\rho)t}(\gamma(\alpha-r-\rho)+\beta\xi+\beta(r-\rho)y_1)}{2\rho(\rho-r)}, \\ y_L(t) &= \frac{e^{(r+\rho)t}\left(\gamma\rho^2-\gamma(\alpha-r)^2+(\beta\xi+\beta(r+\rho)y_1)(\rho+r-\alpha)\right)}{2\beta\rho(r+\rho)} \\ &\quad - \frac{\left(\gamma(\alpha-r)^2+2\beta\xi r-\beta\xi(2r-\alpha)-\gamma\rho^2\right)}{\beta(r^2-\rho^2)} \\ &\quad + \frac{e^{(r-\rho)t}\left(\gamma(\alpha-r)^2-\gamma\rho^2+(\beta\xi+\beta(r-\rho)y_1)(\alpha-r+\rho)\right)}{2\beta\rho(r-\rho)}. \end{aligned}$$

Hence, the left Poincaré map is written as follows:

$$\begin{aligned} P_L(y_1) &= \frac{\left(\gamma\rho^2 - \gamma(\alpha - r)^2 + (\beta\xi + \beta(r+\rho)y_1)(\rho + r - \alpha)\right)}{2\beta\rho(r+\rho)} e^{(r+\rho)t} \\ &+ \frac{\left(\gamma(\alpha - r)^2 - \gamma\rho^2 + (\beta\xi + \beta(r-\rho)y_1)(\alpha - r + \rho)\right)}{2\beta\rho(r-\rho)} e^{(r-\rho)t} \\ &- \frac{\gamma(\alpha - r)^2 - \gamma\rho^2 + \beta\xi(\alpha - r)}{\beta(r^2 - \rho^2)}, \end{aligned}$$

and from the equation $x_L(t) = 0$, we obtain

$$\begin{aligned} y_1(t) &= \frac{e^{(\rho+r)t}(\rho - r)(\gamma(\alpha - r + \rho) + \beta\xi) - 2\rho(\beta\xi - \gamma(2r - \alpha))}{\beta(\rho^2 - r^2)(e^{(r-\rho)t} - e^{(r+\rho)t})} \\ &+ \frac{e^{(r-\rho)t}(r+\rho)(\gamma(\alpha - r - \rho) + \beta\xi)}{\beta(\rho^2 - r^2)(e^{(r-\rho)t} - e^{(r+\rho)t})}. \end{aligned} \tag{III.27}$$

But $y_1 = -y_0$, so this reduces the equation $y_0 - P(y_0) = 0$ to the form

$$\frac{1}{\beta(\rho^2 - r^2)}\left(\left(\rho^2 - r^2\right)(-2\gamma) + 2r(\beta\xi - \gamma(2r - \alpha))\right)\frac{\sinh\rho t}{\sinh rt} - 2\rho(\beta\xi - \gamma(2r - \alpha)) = 0. \tag{III.28}$$

For convenience, we use the notation

$$f_2(t) = \left(2r(\beta\xi - \gamma(2r - \alpha)) - 2\gamma\left(\rho^2 - r^2\right)\right)\frac{\sinh\rho t}{\sinh rt} - 2\rho(\beta\xi - \gamma(2r - \alpha)). \tag{III.29}$$

Now, the way of solving (III.28) is equivalent to that of finding the solutions t of the equation $f_2(t) = 0$. In order to investigate the number of the equation of $f_2(t) = 0$, and since f_2 is a C^1- function in $\mathbb{R}\setminus\{0\}$, we consider its first derivative. Simple calculations yield

$$f_2'(t) = -\frac{2r(\beta\xi - \gamma(2r - \alpha)) - 2\gamma(\rho^2 - r^2)(r\cosh rt\sinh\rho t - \rho\sinh rt\cosh\rho t)}{\sinh^2 rt}.$$

Note that any zero of f_2' is a zero of K_2, where

$$K_2(t) = r\cosh rt\sinh t\rho - \rho\sinh rt\cosh t\rho.$$

Since $K_2(0) = 0$ and $K_2'(t) = (r^2 - \rho^2)\sinh rt\sinh\rho t \neq 0$ for any $t > 0$ (because $r \neq 0$

and $\rho \neq 0$), we can conclude that equation (III.28) has at most one real solution, and there are values of $r, \gamma, \rho, \beta, \xi$ and α for which this solution exists. Hence systems (III.1) have at most one crossing limit cycle. Using the first integrals of both differential systems of (III.1) and knowing that the algebraic crossing periodic orbit passes through the points $(0, y_0)$ and $(0, y_1)$, where y_1 is defined by (III.27) and $y_0 = -y_1$, we get the following expressions of the limit cycle when exist:

$$\Gamma = \{(x, y) \in \Sigma_R : H_j(0, y) = H_j(0, y_0)\} \cup \{(x, y) \in \Sigma_L : H(x, y) = H(0, y_0)\},$$

where $j \in \{1, 2, 3, 4\}$ and H_j are given by (III.4), (III.5), (III.6), and (III.7) respectively. This completes the proof of statements (3) and (4) of Theorem 41.

Proof of statement (5) of Theorem 41. Now, we consider the case where the left subsystem of (III.1) is a linear system satisfying (III.10) with $\rho = 0$. By an analogous analysis of previous statements, the solution $(x_L(t), y_L(t))$ of system (III.10) with $\rho = 0$ which pass through the point $(0, y_1)$ is

$$\begin{aligned} x_L(t) &= \frac{e^{rt}\left(\gamma(2r-\alpha)-\beta\xi\right)-\left(\gamma(2r-\alpha)-\beta\xi\right)}{r^2}+\frac{te^{rt}\left(\gamma(\alpha-r)+\beta\xi+\beta r y_1\right)}{(r)}, \\ y_L(t) &= \frac{e^{rt}\left(\gamma(\alpha-r)^2+\beta\left(\alpha\xi+r^2y_1\right)\right)-\left(\gamma(\alpha-r)^2+\beta\xi(\alpha-r)\right)}{\beta r^2} \\ &\quad -\frac{te^{rt}\left(\gamma(\alpha-r)^2+\beta(\alpha-r)(\xi+ry_1)\right)}{\beta r}, \end{aligned}$$

whence we get the following parametric representation for the left Poincaré map:

$$\begin{aligned} P_L(y_1) &= \frac{e^{rt}\left(\gamma(\alpha-r)^2+\beta\left(\alpha\xi+r^2y_1\right)\right)-\left(\gamma(\alpha-r)^2+\beta\xi(\alpha-r)\right)}{\beta r^2} \\ &\quad -\frac{te^{rt}\left(\gamma(\alpha-r)^2+\beta(\alpha-r)(\xi+ry_1)\right)}{\beta r}. \end{aligned} \quad \text{(III.30)}$$

From the equation $x_L(t) = 0$, we obtain

$$y_1(t) = -\frac{1}{t\beta e^{rt}}\left(-\frac{1}{r^2}\left(\gamma\left(2r-\alpha\right)-\beta\xi\right)+\frac{1}{r^2}e^{rt}\left(\gamma\left(2r-\alpha\right)-\beta\xi\right)+\frac{t}{r}e^{rt}\left(\beta\xi-\gamma\left(r-\alpha\right)\right)\right). \tag{III.31}$$

Using (III.30), (III.31), and taking into account that $y_0 = -y_1(t)$ and $P(y_0) = P_L(y_1(t))$, the equation $y_0 - P(y_0) = 0$ become

$$\frac{1}{\beta r}\left(\frac{\left(\gamma\left(2r-\alpha\right)-\beta\xi\right)\sinh rt}{rt}+\gamma\left(\alpha-r\right)+\beta\xi\right)=0, \tag{III.32}$$

the previous equation is equivalent to $f_3(t) = 0$, where

$$f_3(t) = \frac{\sinh rt}{rt}+\frac{\beta\xi-\gamma\left(r-\alpha\right)}{\gamma\left(2r-\alpha\right)-\beta\xi}. \tag{III.33}$$

Now, the way of solving (III.32) is equivalent to that of finding the solutions t of the equation $f_3(t) = 0$. The study of the maximum number of zeros of $f_3(t) = 0$ is equivalent to finding of the maximum number of intersection points z_i of the curve $\mathcal{F} : y = \frac{\sinh z}{z}$ with the horizontal line $\mathcal{L} : y = -\frac{\beta\xi-\gamma(r-\alpha)}{\gamma(2r-\alpha)-\beta\xi}$.

It is easy to check that $K_3(z) = \frac{\sinh z}{z}$ is an even function and $K_3(z)$ is strictly increasing for $z > 0$ and strictly decreasing for $z < 0$, and that $\lim\limits_{z\to 0} K_3(z) = 1$.

Clearly, we can choose the values of the parameters of system (III.10) with $\rho = 0$ such that the straight line $\mathcal{L}$ intersects the curve $\mathcal{F}$ at either zero or one or at two points.

If $\mathcal{L}$ does not intersect $\mathcal{F}$, then $f_3(t) = 0$ has no solution, and systems (III.1) has no limit cycles.

If $\mathcal{L}$ intersects $\mathcal{F}$ in a unique point, then the intersection point is multiple to two, this point should be $y = 1$ and $z = 0$. This implies that $t = 0$; again, systems (III.1) have no limit cycles.

If there are two intersection points, says (z_1, y_1') and (z_2, y_2'). Taking into account the evenness of the function $K_3(z) = \frac{\sinh z}{z}$, it follows that $z_1 = -z_2$ and $y_1' = y_2'$. So, the equation $f_3(t) = 0$ has at most one solution in $t \in (0, +\infty)$ for $z = rt$, and consequently, a unique solution for y_1 and y_0. Hence, systems (III.1) has at most one crossing limit cycle. Using the first integrals of both differential systems of (III.1) and knowing that the algebraic crossing periodic orbit passes through the points $(0, y_0)$ and $(0, y_1)$ where y_1 is defined by (III.31) and $y_{0i} = -y_{1i}$, we obtain the following expressions for limit cycle:

$$\Gamma = \{(x,y) \in \Sigma_R : H_j(0,y) = H_j(0,y_0)\} \cup \{(x,y) \in \Sigma_L : H(x,y) = H(0,y_0)\},$$

where $j \in \{1,2,3,4\}$ and H_j are given by (III.4), (III.5), (III.6) and (III.7) respectively. This completes the proof of the statement (5) of theorem 41.

Proof of statement (6) of Theorem 41. Finally, we consider the case when the left subsystem of (III.1) is a linear system satisfying (III.14) having without equilibria, neither real nor virtual. In a similar way as in the previous cases, the solution of system (III.14) with $a + \mu b \neq 0$ starting at the point $(0, y_1)$, is

$$\begin{aligned} x_L(t) &= \frac{b(c\mu - d)(a + b\mu)t + \left(e^{at+bt\mu} - 1\right)(ac + bd + b(a + b\mu)y_1)}{(a + b\mu)^2}, \\ y_L(t) &= \frac{a(d - c\mu)t + ay_1}{(a + b\mu)} - \frac{\mu(ac + bd)}{(a + b\mu)^2} + \frac{e^{at+bt\mu}\mu(ac + bd + b(a + b\mu)y_1)}{(a + b\mu)^2}. \end{aligned}$$

Then the left Poincaré map is

$$P_L(y_1) = \frac{a(d - c\mu)t + ay_1}{(a + b\mu)} - \frac{\mu(ac + bd)}{(a + b\mu)^2} + \frac{e^{at+bt\mu}\mu(ac + bd + b(a + b\mu)y_1)}{(a + b\mu)^2},$$

and from equation $x_L(t) = 0$, we obtain

$$y_1(t) = -\frac{\left(e^{at+bt\mu} - 1\right)(ac + bd) - b(a + b\mu)(d - c\mu)t}{b\left(e^{at+bt\mu} - 1\right)(b\mu + a)}, \tag{III.34}$$

Since $y_1 = -y_0$, the equation $y_0 - P(y_0) = 0$ is equivalent to $-y_1(t) - P_L(y_1(t)) = 0$.

Substituting the previous two expressions into $-y_1(t) - P_L(y_1(t)) = 0$, we obtain

$$-(d-c\mu)\, t \coth\left(\frac{1}{2}(a+b\mu)\, t\right) - \frac{2}{b(a+b\mu)}(ac+bd) = 0, \tag{III.35}$$

or, equivalently $f_4(t) = 0$, where

$$f_4(t) = \frac{1}{2}(a+b\mu)\, t \coth\left(\frac{1}{2}(a+b\mu)\, t\right) + \frac{ac+bd}{b(d-c\mu)}. \tag{III.36}$$

In order to investigate the number of solutions of $f_4(t) = 0$, we find by an analogous analysis of the previous case, the number of intersection points z_i of the curve $\mathcal{F}' : y = z\coth(z)$ with the straight line $\mathcal{L}' : y = \frac{ac+bd}{b(c\mu-d)}$.

The function $K(z) = z\coth z$ is even, strictly increasing for $z > 0$ and strictly decreasing for $z < 0$, and $K(0) = 0$.

Clearly, the straight line $\mathcal{L}'$ may intersect the curve $\mathcal{F}'$ at either zero, one or two points.

If $\mathcal{L}'$ does not intersect $\mathcal{F}'$, then the equation $f_4(t) = 0$ has no solution, and systems $(III.1)$ have no limit cycles.

If $\mathcal{L}'$ intersects $\mathcal{F}'$ at a unique point, the intersection point is multiple to two, this point should be $y = 0$ and $z = 0$. This implies that $t = 0$ (because $a + b\mu \neq 0$); again, systems (III.1) have no limit cycles.

If the intersection points are two, we denote them by (z_1, y_1') and (z_2, y_2'). Since $K_3(z) = z\coth z$ is an even function and the the straight line $\mathcal{L}'$ is horizontal, it follows that $z_1 = -z_2$ and $y_1' = y_2'$. So, equation (III.35) has at most one solution $t > 0$ for $z = \frac{1}{2}t(a+b\mu)$, and consequently, a unique solution for y_1 and y_0 follows from (III.34) and (III.23), respectively. To obtain in this way at most one limit cycle for the discontinuous piecewise differential systems, we use the first integrals of both differential systems knowing that the non-algebraic periodic orbit passes through the points $(0, y_0)$ and $(0, -y_0)$, where y_1 is defined by (III.34) and $y_{0i} = -y_{1i}$, we get the following expression:

$$\Gamma = \{(x,y) \in \Sigma_R : H_j(0,y) = H_j(0,y_0)\} \cup \{(x,y) \in \Sigma_L : H(x,y) = H(0,y_0)\},$$

where $j \in \{1,2,3,4\}$ and H_j are given by (III.4), (III.5), (III.6) and (III.7) respectively.

Now, we consider system (III.14) with $a + \mu b = 0$. In this case, the solution of system (III.14), starting at the point $(0, y_1)$, is

$$\begin{aligned} x_L(t) &= (c + by_1)t + \frac{1}{2}(bd - bc\mu)t^2, \\ y_L(t) &= y_1 + \frac{1}{2}\left(bd\mu - bc\mu^2\right)t^2 + (by\mu + d)t. \end{aligned}$$

If $x_L(t) = 0$, we get

$$y_1(t) = -\frac{1}{2b}(2c + b(d - c\mu)t),$$

and the parametric representation of the left Poincaré map is

$$P_L(y_1) = y_1 + \frac{1}{2}\left(bd\mu - bc\mu^2\right)t^2 + (by_1\mu + d)t.$$

Hence $y_1 = -y_0$, the equation $F(y_0) = y_0 - P(y_0) = 0$ is equivalent to $G(t) = -y_1(t) - P_L(y_1(t)) = 0$. Substituting the previous two expressions of $y_1(t)$ and $P_L(y_1(t))$ in $G(t) = 0$, we obtain $\frac{2}{b}c = 0$. Hence, if $c \neq 0$, this last equality does not hold, and $F(y_0) = 0$ has no solutions and consequently, the discontinuous piecewise differential systems (III.1) have no periodic solutions. If $c = 0$, then $G(t) = 0$ for all $t > 0$, i.e., $F(y_0) = 0$ has a continuum of solutions. So, in this case, the discontinuous piecewise differential systems (III.1) either do not have periodic solutions, or have a continuum of periodic orbits, and consequently, these differential systems have no limit cycles. So, statement (6) of theorem 41 is proved.

5 Proof of propositions

Proof of proposition 42. We consider the piecewise differential systems $(S_j)+(III.16)$ with $j=1$ or 2 or 3 or 4. We remark that the equilibrium point $\left(-\frac{5}{52},\frac{9}{104\beta}\right)$ of system (III.16) has as eigenvalues $-\frac{1}{5}\pm i$. So, this equilibrium point a real focus. The systems $(S_j)+(III.16)$ have the first integral

$$H_{f1}(x,y)=\left(\frac{26}{25}x^2-\frac{2}{5}\beta xy+\beta^2y^2\frac{61}{260}x+-\frac{11}{52}\beta y+\frac{17}{832}\right)e^{-\frac{2}{5}\arctan\frac{520x+50}{104x-520\beta y+55}},$$

when $x\in\Sigma_L$, and the first integral H_j are given by (III.4),(III.5), (III.6) and (III.7) respectively when $x\in\Sigma_R$.

For the piecewise differential systems $(S_j)+(III.16)$ with $j\in\{1,2,3,4\}$, the function (III.26) becomes

$$f_1(t)=\frac{10\sinh\frac{1}{5}t+11\sin t}{100\sinh\frac{1}{5}t}$$

The equation $f_1(t)=0$ has exactly two positive zeros $t_1\simeq 4.1438$ and $t_2\simeq 4.7492$, which gives using (III.24), two values of y_1: $y_{11}=-\frac{0.13944}{\beta}$ and $y_{12}=-\frac{0.21703}{\beta}$. Since, we get $y_0=-y_1$, we have $y_{01}=\frac{0.13944}{\beta}$ and $y_{02}=\frac{0.21703}{\beta}$. Thus, these two solutions will correspond to the isolated periodic orbits Γ_1 and Γ_1 of systems $(S_j)+(III.16)$, i.e., to two limit cycles of those systems. The smallest one Γ_1 intersects the switching line Σ at the points $(0,y_{01})$ and $(0,y_{11})$ and the biggest limit cycle Γ_2 intersects the switching line Σ at the points $(0,y_{02})$ and $(0,y_{12})$. Straightforward computations show that the solution of $(S_j)+(III.16)$ with $j=1$ or 2 or 3, or 4, passing through the crossing points $(0,y_{01})$ and $(0,y_{11})$, correspond to

$$\Gamma_1=\{(x,y)\in\Sigma_R: H_j(x,y)=h_j\}\cup\left\{(x,y)\in\Sigma_L: H_{f1}(x,y)=5.9741\times10^{-2}\right\},$$

where $h_1=\left(\frac{0.13944}{\beta}\right)^4$, $h_2=h_3=h_4=\left(\frac{0.13944}{\beta}\right)^2$, and the solution of $(S_j)+(III.16)$, passing through the crossing points $(0, y_{02})$ and $(0, y_{12})$, correspond to

$$\Gamma_2 = \left\{(x, y) \in \Sigma_R : H_j(x, y) = h'_j\right\} \cup \{(x, y) \in \Sigma_L : H_{f1}(x, y) = 0.101\,04\},$$

where $h'_1 = \left(\frac{0.217\,03}{\beta}\right)^4, h'_2 = h'_3 = h'_4 = \left(\frac{0.217\,03}{\beta}\right)^2$. Moreover, Γ_1 and Γ_2 are non-algebraic and travel in a counterclockwise sense around the sliding segment
$\Sigma_s = \left\{(0, y) \in \Sigma : 0 \leq y \leq -\frac{1}{8\beta}\right\}$.
See Figure (III.1).

Proof of proposition 43. We consider the piecewise differential systems $(S_j) + (III.17)$ with $j = 1$ or 2 or 3 or 4. The equilibrium point $(1, 0)$ of system (III.17) has the eigenvalues $1 \pm i$, so it is a virtual focus, those piecewise differential systems have the first integral

$$H_{f2}(x, y) = \left(5x^2 - 4\beta xy + \beta^2 y^2 - 10x + 4\beta y + 5\right) e^{-2 \arctan \frac{x-1}{\beta y - 2x + 2}},$$

if $x \in \Sigma_L$, and the first integral H_j with $j = 1$ or 2 or 3 or 4, where H_j are given by $(III.4), (III.5)$, $(III.6)$ and $(III.7)$, respectively, if $x \in \Sigma_R$.

For the piecewise differential systems $(S_j) + (III.17)$ with $j \in \{1, 2, 3, 4\}$, the function $(III.26)$ becomes

$$f_1(t) = -\frac{2 \sinh t - 4 \sin t}{\sinh t}.$$

From the equation $f_1(t) = 0$, we obtain the unique solution $t = 1.435\,4$. From this value of t and using $(III.24)$, we get the values $y_1 = -\frac{2.104\,0}{\beta}$, because $y_1 = -y_0$, then $y_0 = \frac{2.104\,0}{\beta}$. So, the discontinuous piecewise differential systems $(S_j) + (III.17)$ have exactly one crossing limit cycle. Straightforward computations show that the solution of $(S_j) + (III.17)$ passing through the crossing points $(0, y_0)$ and $(0, y_1)$ correspond to

$$\Gamma_1 = \{(x, y) \in \Sigma_R : H_j(x, y) = h_j\} \cup \left\{(x, y) \in \Sigma_L : H_{f2}(x, y) = 5.374\,3 \times 10^{-2}\right\},$$

where $j \in \{1, 2, 3, 4\}$, $h_1 = \left(\frac{2.104\,0}{\beta}\right)^4$ and $h_2 = h_3 = h_4 = \left(\frac{2.104\,0}{\beta}\right)^2$. We note that this limit cycle is non-algebraic and travels in a counterclockwise sense, around the sliding segment $\Sigma_s = \left\{(0, y) \in \Sigma : 0 \leq y \leq -\frac{1}{\beta}\right\}$.

Proof of proposition 44. We consider the piecewise differential systems $(S_j) + (III.18)$ with $j \in \{1, 2, 3, 4\}$. Since the eigenvalues of the matrices of the linear differential system $(III.18)$ are $\frac{3}{2}, -\frac{1}{2}$, this system has a real saddle at the equilibrium point $\left(-\frac{4}{3}, \frac{37}{30\beta}\right)$. The piecewise differential systems $(S_j) + (III.18)$ with $j = 1$ or 2 or 3, or 4, have the first integral

$$H_s(x, y) = \left(-\frac{1}{2}x + \beta y - \frac{19}{10}\right)^3 \left(\frac{3}{2}x + \beta y + \frac{23}{30}\right),$$

if $x \in \Sigma_L$, and the first integral H_j with $j = 1$ or 2 or 3, or 4, where H_j are given by $(III.4), (III.5)$, $(III.6)$ and $(III.7)$, respectively, if $x \in \Sigma_R$.

For the piecewise differential systems $(S_j) + (III.18)$ with $j \in \{1, 2, 3, 4\}$, the function $(III.29)$ becomes

$$f_2(t) = \frac{17}{20} \frac{\sinh t}{\sinh \frac{1}{2}t} - 2.$$

The unique solution of $f_1(t) = 0$ is $t = 1.1714$. From this value of t and using $(III.24)$, we get the values of $y_1 = -\frac{0.54103}{\beta}$. Since $y_1 = -y_0$, we have $y_0 = \frac{0.54103}{\beta}$. So, the discontinuous piecewise differential systems $(S_j) + (III.18)$ have exactly one crossing limit cycle. Straightforward computations show that the crossing limit cycle passing through the crossing points $(0, y_1)$ and $(0, y_2)$ correspond to

$$\Gamma = \{(x, y) \in \Sigma_R : H_j(x, y) = h_j\} \cup \{(x, y) \in \Sigma_L : H_s(x, y) = -3.2818\},$$

where $j = 1$ or 2 or 3, or 4, $h_1 = \left(\frac{0.54103}{\beta}\right)^4$ and $h_2 = h_3 = h_4 = \left(\frac{0.54103}{\beta}\right)^2$. Moreover, this limit cycle is algebraic and the sliding region of systems $(S_j) + (III.19)$ is defined by $\Sigma_s = \left\{(0, y) \in \Sigma : 0 \leq y \leq -\frac{1}{10\beta}\right\}$, which is inside the periodic orbit. Drawing the orbit Γ we obtain the limit cycle in figure III.3, which travels in a counterclockwise sense.

Proof of proposition 45. We consider the piecewise differential systems $(S_j)+(III.19)$ with $j = 1$ or 2 or 3 or 4. The equilibrium point $\left(\frac{1}{2}, -\frac{4}{\beta}\right)$ of system $(III.19)$ has eigenvalues $4, 2$, so, it is a virtual node. On the other hand, the piecewise differential systems $(S_j)+(III.19)$ with $j \in \{1,2,3,4\}$ have the first integral

$$H_n(x,y) = \frac{(2x+\beta y+3)^4}{(4x+\beta y+2)^2},$$

if $x \in \Sigma_L$, and the first integral H_j with $j = 1$ or 2 or 3, or 4, where H_j are given by $(III.4), (III.5), (III.6)$ and $(III.7)$, respectively, if $x \in \Sigma_R$.

For the piecewise differential systems $(S_j)+(III.19)$ with $j \in \{1,2,3,4\}$, the function $(III.29)$ becomes

$$f_2(t) = 20\frac{\sinh t}{\sinh 3t} - 4.$$

Now, solving the equation $f_2(t) = 0$ with respect to the variable t, we get $t = 0.658\,48$. Using the expression of y_1 given by $(III.27)$ and taking into account that $y_1 = -y_0$, we get $y_1 = -\frac{1.7321}{\beta}$, and $y_0 = \frac{1.7321}{\beta}$. So, the discontinuous piecewise differential systems $(S_j)+(III.19)$ have exactly one crossing limit cycle, see Figure III.4. Straightforward computations show that the crossing limit cycle passing through the crossing points $(0,y_1)$ and $(0,y_2)$ correspond to

$$\Gamma = \{(x,y) \in \Sigma_R : H_j(x,y) = h_j\} \cup \{(x,y) \in \Sigma_L : H_n(x,y) = 36.008\},$$

where $j \in \{1,2,3,4\}$, $h_1 = \left(-\frac{1.7321}{\beta}\right)^4$ and $h_2 = h_3 = h_4 = \left(-\frac{1.7321}{\beta}\right)^2$. Moreover, Γ is non-algebraic and travels in a counterclockwise sense, around the sliding set $\Sigma_s = \left\{(0,y) \in \Sigma : 0 \le y \le -\frac{1}{\beta}\right\}$.

Proof of proposition 46. We consider the piecewise differential systems $(S_j)+(III.20)$ with $j = 1$ or 2 or 3 or 4. Since the eigenvalues of the matrices of the linear differential system $(III.20)$ is -1, this system has a virtual node with eigenvalue of multiplicity 2 whose linear part does not diagonalize at the equilibrium point $(1,0)$. The piecewise differential systems $(S_j)+(III.20)$ with $j=1$ or 2 or 3, or 4, have the first integral

$$H_{n'}(x,y)=\frac{1}{2-\beta y-2x}e^{\frac{x-1}{2-\beta y-2x}},$$

if $x\in\Sigma_L$, and the first integral H_j with $j=1$ or 2 or 3 or 4, where H_j are given by $(III.4),(III.5)$, $(III.6)$ and $(III.7)$, respectively, if $x\in\Sigma_R$.

For the piecewise differential systems $(S_j)+(III.20)$ with $j\in\{1,2,3,4\}$, the function $(III.33)$ becomes

$$f_3(t)=\frac{1}{t}(\sinh t-2t).$$

Now, solving $f_3(t)=0$, we get $t=2.1773$, substituting this value of t in the expression of y_1 given by $(III.31)$ and taking into account that $y_1=-y_0$, we get $y_1=-\frac{1.5752}{\beta}$, and $y_0=\frac{1.5752}{\beta}$. So, the discontinuous piecewise differential systems $(S_j)+(III.20)$ have exactly one crossing limit cycle, see Figure III.5. Straightforward computations show that the crossing limit cycle passing through the crossing points $(0,y_1)$ and $(0,y_2)$ correspond to

$$\Gamma=\{(x,y)\in\Sigma_R:H_j(x,y)=h_j\}\cup\{(x,y)\in\Sigma_L:H_{n'}(x,y)=0.21146\},$$

where $j\in\{1,2,3,4\}$, $h_1=\left(\frac{1.5752}{\beta}\right)^4$ and $h_2=h_3=h_4=\left(\frac{1.5752}{\beta}\right)^2$. Moreover, this limit cycle is non-algebraic and surrounds the sliding segment $\Sigma_s=\left\{(0,y)\in\Sigma:\frac{1}{\beta}\leq y\leq 0\right\}$ counterclockwise.

Proof of proposition 48. We consider the piecewise differential systems $(S_j)+(III.21)$ with $j\in\{1,2,3,4\}$. The planar linear differential system $(III.21)$ has the first integral

$$H_w(x,y)=\left(\frac{101}{100}+(1-\mu)x+y\right)e^{\mu x-y},$$

in Σ_L and the cubic polynomial differential systems (S_j) with $j\in\{1,2,3,4\}$ have the first integral H_j, where H_j are given by $(III.4),(III.5)$, $(III.6)$ and $(III.7)$ with $j=1$

or 2 or 3 or 4, respectively, in Σ_R. It is easy to see that $(III.21)$ has no equilibria, neither real nor virtual.

Then for the discontinuous piecewise differential systems $(S_j) + (III.21)$, the function $(III.36)$ becomes

$$f_4(t) = t \coth \frac{1}{2} t - \frac{101}{50}.$$

This function $f_4(t)$ has exactly a unique positive root $t = 0.346\,76$. From this value of t and using $(III.34)$, we get the values of $y_1 = 0.173\,38$. Since $y_1 = -y_0$, we have $y_0 = -0.173\,38$. So, the discontinuous piecewise differential systems $(S_j) + (III.21)$ have exactly one non-algebraic crossing limit cycle, see Figure III.6. Straightforward computations show that the crossing limit cycle passing through the crossing points $(0, y_1)$ and $(0, y_2)$ correspond to

$$\Gamma = \{(x, y) \in \Sigma_R : H_j = h_j\} \cup \{(x, y) \in \Sigma_L : H_w(x, y) = 0.995\,01\},$$

where $j \in \{1, 2, 3, 4\}$, $h_1 = (0.173\,38)^4$ and $h_2 = h_3 = h_4 = (0.173\,38)^2$. This limit cycle surrounds the sliding set $\Sigma_s = \left\{(0, y) \in \Sigma : \frac{-1}{100} \le y \le 0\right\}$.

6 Discussions and conclusions

In this chapter, we studied the maximum numbers of crossing limit cycles for the piecewise differential system separated by straight line and formed by an arbitrary linear differential system and by cubic systems with homogeneous nonlinearity with an isochronous center at the origin. We show that these systems may have at most zero or one or two explicit algebraic or non-algebraic limit cycles depending on the type of their linear part.

Journal and Conference Papers Related to the Chapter

Jr) Limit Cycles for Piecewise Differential Systems Formed by an Arbitrary Linear System and a Cubic Isochronous Center

Aziza BERBACHE, Rebeiha ALLAOUA, Rachid CHEURFA and Ahmed BENDJEDDOU. "Limit Cycles for Piecewise Differential Systems Formed by an Arbitrary Linear System and a Cubic Isochronous Center". Memoirs on Differential Equations and Mathematical Physics, Volume 89, (2023), 17–38, see [18].

Conclusion and Perspectives

The central theme of this thesis turns around an important aspect of the qualitative theory of planar polynomial differential systems, that is the occurrence of non trivial periodic solutions called limit cycles for planar polynomial differential systems in the real plane. They remained the most sought solutions when modelling physical systems in the plane. This work splits into two major parts. In the first one, we have studied the existence and the number of limit cycles for two classes of planar differential systems of degrees four and seven respectively. When the limit cycles exist, they are explicitly obtained via the integrability approach. The second part is concerned by the study of crossing periodic orbits for some families of piecewise vector fields using two important tools that are first integrals of each vector field that compose these families and the other one is the Poincaré return map, which are separated by a straight line $x = 0$ and formed by an arbitrary linear differential system and a cubic polynomial differential system with homogeneous nonlinearity and an isochronous center at the origin. We have shown that these systems may have at most two explicit algebraic or non-algebraic limit cycles depending on the type of their linear part.

As a continuation of this work, it would be interesting to undertake a similar task for systems of lower degrees and to see weather our results still hold. As a second outcome, on can study the existence of periodic orbits and estimate the number of limit cycles for discontinuous piecewise differential systems in the case where these systems are separated by one or more continuous curves homomorphic to a straight line and are composed both of an arbitrary linear system and of a given quadratic one. Exploring a comparable investigation for differential systems arising from various scientific disciplines would be also of significant interest.

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ملخص

تتعلق هذه الأطروحة بحدوث حلول دورية غير تقليدية تسمى دورات الحد للأنظمة التفاضلية المعرفة بكثيري حدود في المستوى الحقيقي. ينقسم هذا العمل إلى جزئين رئيسيين. في الأول ، درسنا وجود وعدد دورات الحد لفئتين من الأنظمة التفاضلية المستوية من الدرجتين الرابعة والسابعة على التوالي. يتعلق الجزء الثاني بدراسة دورات الحد التقاطعية لفئة من الأنظمة التفاضلية المتقطعة وغير المستمرة مفصولة بخط مستقيم وتتكون من جزء خطي وجزء مكعب ذات مركز متزامن.

الكلمات الرئيسية: أنظمة تفاضلية المعرفة بكثيري حدود ، تكامل اولي ، مدار دوري، دورة حد ، أنظمة تفاضلية متعددة الأجزاء متقطعة ، مركز تكعيبي متساوي الزمان ،نظامخطي، دورة حد عبور تقاطعية.

Abstract

This thesis is concerned by the occurrence of non trivial periodic solutions called limit cycles for polynomial differential systems in the real plane. This work splits into two major parts. In the first one, we have studied the existence and the number of limit cycles for two classes of planar differential systems of degrees four and seven respectively. The second part is concerned by the study of crossing limit cycles for a class of piecewise discontinuous differential systems separated by a straight line and formed by arbitrary linear and cubic parts with an isochronous center at the origin.

Keywords : Polynomial differential system, first integral, periodic orbit, limit cycle, piecewise discontinuous differential systems, isochronous cubic center, linear system, crossing limit cycle.

Résumé

Cette thèse s'intéresse à l'occurrence de solutions périodiques non triviales appelées cycles limites pour les systèmes différentiels polynomiaux dans le plan réel. Ce travail se compose de deux grandes parties. Dans la première, nous avons étudié l'existence et le nombre de cycles limites pour deux classes de systèmes différentiels planaires de degrés quatre et sept respectivement. La deuxième partie concerne l'étude des cycles limites de croisement pour une classe de systèmes différentiels discontinus par morceaux séparés par une droite et formés par des parties arbitraires linéaires et cubiques avec un centre isochrone à l'origine.

Mot-clé : Systèmes différentiels polynomiaux, intégrale première, orbite périodique, cycle limite, systèmes différentiels discontinus par morceaux, centre cubique isochrone, système linéaire, cycle limite de croisement.