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QUALITATIVE STUDY OF SOME CLASSES OF KOLMOGOROV DIFFERENTIAL SYSTEMS

Presented by

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In the name of **ALLAH** the Merciful, Praise to Allah, Lord of the Worlds, Praise be to the Lord of all worlds. Prayers and peace be upon our Prophet, Muhammad, his family and all of his companions.

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Dedication

This thesis is dedicated to:

My parents Chouader Rachid and Saliha Guerniche.

My big family Chouader.

My husband Hicham and our daughter Sondos.

To all those who encouraged me.

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1. **Rima Chouader, Salah Benyoucef, and Ahmed Bendjeddou** [34], Kolmogorov differential systems with prescribed algebraic limit cycles, Sib. Electron. Math. Rep. 18 (2021), no. 1, 1–8.

2. **Rima Chouader**, **Ahmed Bendjeddou, Salah Benyoucef,** A class of integrable Kolmogorov systems with non algebraic limit cycle. Transactions of A. Razmadze Mathematical Institute.(**Article submitted**)

Conferences

1. IEEE International conference on recent advances in mathematics and informatics (ICRAMI'2021) Tebessa, Algeria. On the non-existence of limit cycles for a familly of kolmogorov systems.

2. The First International Workshop on Applied Mathematics (1st-IWAM'2022), Constantine-Algeria. An integrable class of Kolmogorov systems transforms into a riccati equation.

3. The First Conference on Mathematics and Applications of Mathematics (1st CMAM 2021), Jijel,Algeria. On the non-existence of limit cycles in a class of cubic kolmogorov systems.

4. 2nd National Seminaire of Mathematics, Mentouri University Constantine 1, Algeria. Integrability of cubic Kolmogorov systems.

5. National Conference on Mathematics and Applications "NCMA 23", University Ferhat Abbas of Setif1, Algeria. Coexistence of limit cycle in a septic Kolmogorov system enclosing a non-elementary singular point.

6. The Second National Conference on Mathematics and its Applications (2nd SC-NMA 2022), Bordj Bou Arréridj, Algeria. Classes of polynomial differential systems with algebraic limit cycles.

General Introduction

Differential systems are an important branch of mathematics. They model phenomena resulting from practice, such as the interaction between two species in ecology and the balance between supply and demand in economy. Generally they are used in scientific fields such as: physics, biology, economics, mechanics, astronomy and in many problems of chemisty. The significant development of the theory of dynamical systems during this century has helped to develop methods of studying the properties of their solutions.The direct solution to the differential system is usually difficult or impossible. Since Andronov (1932), three different approaches have traditionally been used to study dynamic systems: qualitative methods, analytical methods and numerical methods the most important of them are qualitative methods. This study makes it possible to provide information on the behavior of the solutions of a differential system without the need to solve it explicitly, and it consists in examine the properties and characteristics of the solutions of this system, and to justify, among these solutions, the existence or non-existence of an isolated closed curve shape, more precisely the limit cycles, their number and their stability. A limit cycle is an isolated periodic solution in the set of all periodic solutions. Several books have dealt with the qualitative study of differential systems particularly ([5], [89], [65], [72], [88]).

In order to position our problem well, let us begin by giving a history. The concept of limit cycles appeared for the first time in the famous articles of H. Poincaré [73]. Through his book "On curves inscribed by a differential equation" ([73], [74], [75], [76]). Published between 1881 and 1886. **Henri Poincaré** (1854-1912) paved the way for the approach differential equations, where the effort is no longer focused on the direct solution, but on the technical study of the solutions. Look at the problem from the qualitative point of view, the method of study consists in studying the succession of points. The intersection of a path with a plane perpendicular to it. He said that if a the maximal solution of the equation

$$\frac{dy}{dx} = \frac{G(x,y)}{F(x,y)} \tag{1}$$

where F and G are polynomials of real variables (x, y) with real coefficients and any degrees, remains bounded, then either this solution converges to an equilibrium point, or its asymptotic behavior is a periodic function.

The Mathematician **Ivar Bendixson** (1861-1935) gave a rigorous proof of Poincaré's statement and extended it to the famous Poincaré-Bendixson theorem on the limit set of trajectories of dynamical systems in a bounded region [9]. The mathematician **David Hilbert** (1862-1943) presented, during the second congress. Mathematics International (1900), 23 famous problems considered as the col-election having had the most inuence in mathematics [53]. The second part of the sixteenth Hilbert's problem poses the question of the maximum number and arrangement of isolated periodic projections (limit cycles) for the system

$$\begin{cases} x' = \frac{dx}{dt} = F\left(x, y\right) \\ y' = \frac{dy}{dt} = G\left(x, y\right) \end{cases} \tag{2}$$

where F and G are polynomials of real variables (x, y) with real coefficients and any degrees.his problem until today, not completely solved, has been the subject of several works. We note H_n the maximum number of limit cycles of the system (2).

Dulac (1870-1955) proposed in 1923 a proof ensuring that H_n is fini for all n where $n = max(deg(F), deg(G))$, but his proof had an error. There Dulac's resolution to this problem was made independently by **Martinet & Mossy** (1987), **Ilyashenko** (1991) and **Ecalle** (1992). **Petrovsky & Landis** (1957) believed to find the value of H_2 but they noticed an error in their own demonstration before they were informed by a counter example of **Shi** (1982) [81] in which a quadratic system has 4 cycles boundaries. Thus, if H_n is a number ni for all n, we only know that $H_2 \geq 4$ [81], $H_3 \geq 13$ [58], $H_3 \geq 20$ [51], $H_5 \geq 28$ [87], $H_6 \geq 35$ [86], $H_7 \geq 50$ [59]. **Christopher & Lloyd** (1995) [36] gave a lower bound to the number H_n

$$H_n \geq n^2 \ln\left(n\right) \tag{3}$$

In mathematical biology, a model of a system of differential equations has been proposed, independently, by **Alfred James Lotka** (1880-1949) in 1925 and **Vito Volterra** (1860-1940) in 1926, hence the name Lotka-Volterra. He stated a simple system of ordinary differential equations. Volterra considered $x(t)$ and $y(t)$ the densities of prey and predators, and assuming linear behaviors and taking positive constants a, b, c and d, he stated the following model

$$\begin{cases} x' = x(a - by) \\ y' = y(-c + dx) \end{cases} \tag{4}$$

Later, Lotka–Volterra systems were generalized and considered in arbitrary dimension, i.e. Consequently, the applications of these systems started to multiply. New applications on population dynamics had been developed, but also these systems have been used for modeling many natural phenomena, such as chemical reactions, plasma physics or hydrodynamics just as other problems from social science and economics. In view of this, it is clear that population models, in particular competition or predator–prey models are important both in their original field and in their application to many other problems from other areas.

This model does not presents any limit cycles, but it remains the starting point of several models currently offered. Among these models is that of Kolmogorov (1903-1987).

We are interested in some systems of ordinary differential equations planars of the type

$$\begin{cases} x' = F(x,y) \\ y' = G(x,y) \end{cases} \tag{5}$$

and other systems modeling prey-predator problems of the Kolmogorov type

$$\begin{cases} x' = xF(x,y) \\ y' = yG(x,y) \end{cases} \tag{6}$$

where F, G are functions of class C^1 on an open set of $\mathbb{R}^2$ in $\mathbb{R}$.

Motivated by the work ([12], [13], [11], [15]) of Professor **Ahmed Bendjeddou and all**, as well as by his directives, we were able to make contributions by formulating a few theorems concerning the existence, the number and the explicit expressions of limit cycles of a few classes of differential systems of types (5) and (6).

There is an enormous amount of work focusing on the study of limit cycles for systems of Kolmogorov type, see for example ([45], [49], [50], [52]), but very few concern limit cycles expressed in an explicit way. The only known limit cycles were algebraic and explicitly given, see for example the work of Bendjeddou and all ([11], [15]). In this thesis we will focus on:

1) For a given algebraic curve, we exhibit the following Kolmogorov differential system

$$\begin{cases} x' = x\left(R(x,y)\,U - y\Phi(x,y)\,U_y\right) \\ y' = y\left(S(x,y)\,U + x\Phi(x,y)\,U_x\right) \end{cases} \tag{7}$$

Where $R(x,y), S(x,y), \Phi(x,y)$ and $U(x,y)$ are polynomial functions. Our contribution consist to show that the system (7) admits all the bounded components of $\Gamma = \{(x,y) \in \mathbb{R}^2, U(x,y) = 0\}$ as hyperbolic limit cycles if certain conditions on the parameters are satisfied.

2) We are interested on the Septic Kolmogorov system of the form:

$$\begin{cases} x' = (x+p)\,P_6(x,y) \\ y' = (y+q)\,Q_6(x,y) \end{cases}$$

where

$$P_6(x,y) = \begin{pmatrix} x(q+y)\left(ax^4+ay^4+2bxy^3+2bx^3y\right) \\ +cx(x+p)(q+y)^2+\left(x^2+y^2\right)\left(-4qy^3+x^4-3y^4\right) \end{pmatrix}$$
$$Q_6(x,y) = \begin{pmatrix} y(p+x)\left(ax^4+ay^4+2bxy^3+2bx^3y\right) \\ +cy(q+y)(x+p)^2+\left(x^2+y^2\right)\left(4px^3-y^4+3x^4\right) \end{pmatrix}$$

and p, q, a, b, c are real numbers. We show that under certain conditions on the parameters, this class has non-algebraic limit cycle. The tool we use here is the notion of integrability and the influence of the first integral on the solutions of systems.

This thesis is structured as follows, it is composed of:

- The first chapter, which is more of a glossary, brings together some basic concepts, introductory and necessary for understanding the entire thesis.
- The second chapter is devoted to construction three classes of Kolmogorov type systems of degree greater than n modeling the problems of prey-predator, where it suffices to check certain conditions on the parameters to directly conclude the existence and the number of limit cycles. In addition we have given the algebraic expression of these limit cycles. In the future, we hope to extend this work for non-algebraic limit cycles.
- The third chapter, we have studied the integrability and the existence of a non-algebraic limit cycle of a class Kolmogorov differential systems of degree septic, moreover, we characterized the conditions for the suggested system in order to find the implicit expression of limit cycle and we study the system at infinity by using the Poincaré compactification. We end our work with a conclusion and some perspectives.

Chapter 1

Preliminary on planar polynomial differential systems

1.1 Introduction

The main objective of the first chapter is to introduce some of the basic concepts needed for the qualitative study of dynamics systems and present some results, which we will use in the sequel. We start by definition of polynomial differential systems, moreover, we will discuss the solution of differential system, invariant curve, integrability, periodic solutions, limit cycles. We will recall the fundamentals theorems: existence and uniqueness of the solution of an initial value problem and of limit cycles. Finally, we will end this chapter with the concept of poincaré map and the method of Poincaré compactification.

1.1.1 Polynomial Differential System

We consider a planar polynomial differential system of the form:

$$\begin{cases} x' = \frac{dx}{dt} = P\left(x\left(t\right), y\left(t\right)\right) \\ y' = \frac{dy}{dt} = Q\left(x\left(t\right), y\left(t\right)\right) \end{cases} \tag{1.1}$$

where P and Q are polynomials in the variables x and y.The system (1.1) is of degree n where $n = max(deg(P), deg(Q))$, if P and Q do not explicitly depend on t, then the system (1.1) is said to be autonomous.

To know the aspect of the trajectories of the system (1.1) at least locally, we have to find your equilibrium points. A point $(x^*, y^*) \in \mathbb{R}^2$ is called an equilibrium point (singular point, critical point) of system (1.1) if

$$\begin{cases} P(x^*, y^*) = 0 \\ Q(x^*, y^*) = 0 \end{cases}$$

In the literature equivalent mathematical objects to refer to this planar differential systems appear: as vector field.

A vector field X is a region of the plane in which there exists at every point M of $\Omega \subseteq \mathbb{R}^2$ a vector $\vec{V}(M,t)$, i.e an application:

$$X : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$$
$$M\,(x,y) \to \vec{V}(M,t) = \begin{pmatrix} P\,(x,y) \\ Q\,(x,y) \end{pmatrix}$$

where P and Q are class C^1 on $\Omega \subseteq \mathbb{R}^2$. we can also write

$$\chi = P\,(x,y)\,\frac{\partial}{\partial x} + Q\,(x,y)\,\frac{\partial}{\partial y}$$

In this thesis, we assume that P and Q are C^1 functions, then the Cauchy Lipchitz conditions are satisfied at any point of system (1.1). So in each initial condition (x_0, y_0), the system (1.1) has a unique solution.

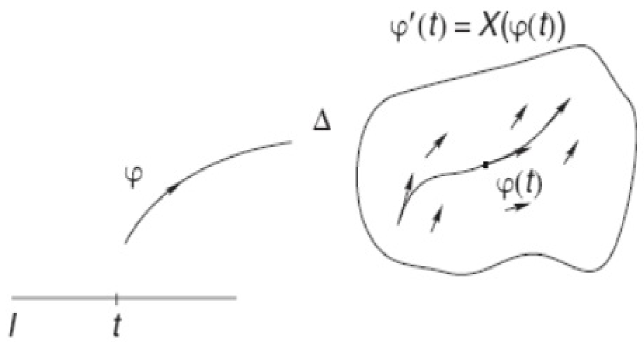

Figure 1.1 : Vector field.

1.1.2 Solutions of Differential System

A solution to differential system (1.1) is an application

$$\varphi : I \subseteq \mathbb{R} \to \mathbb{R}^2$$
$$t \to \varphi(t) = (x(t), y(t)),$$

where I is a non-empty interval such that, for all $t \in I$, the solution $\varphi(t) = (x(t), y(t))$ satisfies this system i.e: $\dot{\varphi}(t) = X(\varphi(t))$. If $\varphi_1(t) = (x_1(t), y_1(t))$ and $\varphi_2(t) = (x_2(t), y_2(t))$ are two solutions on I_1 and I_2 respectively. We say that $\varphi_2(t)$ is an extension of $\varphi_1(t)$ if $I_1 \subset I_2$ and $\forall t \in I$, $\varphi_1(t) = \varphi_2(t)$. The solution $(x(t), y(t))$ is called maximal solution on I if it does not admit any extension on I.

Definition 1 *The solution $\varphi(t) = (x(t), y(t))$ is a periodic solution of the system* (1.1) *if there exists a real number $T > 0$ such that $\forall t \in I, \varphi(t+T) = \varphi(t)$. The smallest number T is called the period of the solution φ.*

1.1.3 Existence and uniqueness of the solution

A solution to system (1.1) is a vector-valued function $X(t) = (x(t), y(t))$ with t in some interval $I \subseteq \mathbb{R}$, which satisfies

$$\dot{X} = \frac{dX}{dt} = F(x, y) \tag{1.2}$$

Or

$$\frac{dX}{dt} - \begin{pmatrix} P(x, y) \\ Q(x, y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

A solution is called maximal if no such extension exists on an interval $J \supset I$. The interval I is said to be the maximal interval on which a solution to (1.2) exists. If the solutions of differential system have the maximal interval equal to $\mathbb{R}$ then it is called complete [5].

Remark 2 ***(Existence and Uniqueness Theorem*** *[72]).*

All the planar differential systems considering in our work are polynomial, means that we do have always existence and uniqueness of solutions for all initial conditions; i.e.; the initial value problem (1.2) with $X(t_0) = (x(t_0), y(t_0))$, has a unique solution $X(t)$. Furthermore, $F(x, y)$ is locally Lipschitz function, then we have always a maximal solution for system (1.1). By using the existence and uniqueness theorem, a self intersection of a solution for an autonomous systems is impossible except the case when the solution is periodic. We emphasize that this is not the case if we consider a non-autonomous systems.

1.1.4 Phase Portrait and Flow

Although it is often impossible (or very difficult) to determine explicitly the solutions of a differential equation, it is still important to obtain information about the behavior of solutions, at least of qualitative nature.

We consider a planar polynomial differential system (1.1). We think of this as describing the motion of a point in the (x, y) plane **(which in this context is called the phase plane)**, with the independent variable as time. The path travelled by the point in a solution is called a trajectory of the system.

A phase portrait is a geometric representation tric of trajectories of a dynamical system in phases plane such that for a given initial conditions corresponds to a curve or a point.

1.1.5 Flow

A dynamical system can be defined by a function $\Phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ that makes X moving (forward) to X_t, which is defined for all $t \in \mathbb{R}$, gives the behaviour of X during the time t starting at $t = 0$ (or at any initial condition $t = t_0$) as Φ_0 such that $\Phi_0(X) = X$. Moving X_t backward gives naturally Φ_{-t} the inverse of Φ_t. In the same manner, moving X_t again by another elapsing time s gives $\Phi_{t+s}(X) = \Phi_t(\Phi_s(X))$. The formal definition of the function has these three properties is given as follows:

1. $\frac{d\Phi_t}{dt}(x, y) = (P(\Phi_t(x, y)), Q(\Phi_t(x, y)))$,
2. $\Phi_0(x, y) = (x, y)$,
3. $\Phi_{t+s}(x, y) = \Phi_t(\Phi_s(x, y))$,for all $(x, y) \in \mathbb{R}^2$ and $s, t \in \mathbb{R}$.

1.1.6 Jacobian Matrix and Linearization of polynomial system

The most natural approach to study the behavior of the trajectories of a nonlinear autonomous differential system, in the vicinity of a singular point, consists in reducing to the associated linear system, then in making the link between the trajectories of the two systems.

We consider the nonlinear system (1.1) near an equilibrium point (x^*, y^*), the linearization of the system (1.1) is given in matrix form by:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{\partial P}{\partial x}(x^*, y^*) & \frac{\partial P}{\partial y}(x^*, y^*) \\ \frac{\partial Q}{\partial x}(x^*, y^*) & \frac{\partial Q}{\partial y}(x^*, y^*) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The Jacobian matrix of the system (1.1) at a critical point (x^*, y^*) given by

$$A = J(x^*, y^*) = \begin{pmatrix} \frac{\partial P}{\partial x}(x^*, y^*) & \frac{\partial P}{\partial y}(x^*, y^*) \\ \frac{\partial Q}{\partial x}(x^*, y^*) & \frac{\partial Q}{\partial y}(x^*, y^*) \end{pmatrix}$$

Let λ_1 and λ_2 be the eigenvaluesof this matrix. We distinguish the different breaks according to the eigenvalues λ_1 and λ_2 of the matrix $J(x^*, y^*)$.

(i) λ_1 and λ_2 are real nonzero and have different designs, the critical point (x^*, y^*) is a saddle. It is still unstable.

(ii) λ_1 and λ_2 are real of the same sign

- If $\lambda_1 < \lambda_2 < 0$, the critical point (x^*, y^*) is a stable node.
- If $0 < \lambda_1 < \lambda_2$, the critical point (x^*, y^*) is an unstable node.
- If $\lambda_1 = \lambda_2 = \lambda$, the critical point (x^*, y^*) is a proper node, it is stable if $\lambda < 0$

  and unstable if $\lambda > 0$.

(iii) λ_1 and λ_2 are complex conjugates, ie $\forall j = 1, 2$, $\lambda_j = \alpha_j + i\beta_j$ and $Im(\lambda_j) \neq 0$, then the critical point (x^*, y^*) is a focus, it is stable if $Re(\lambda_j) < 0$ and unstable if $Re(\lambda_j) > 0$.

(iv) λ_1 and λ_2 are pure imaginary ie $Im(\lambda_j) \neq 0$ and $Re(\lambda_j) \neq 0$, $\forall j = 1, 2$, then the critical point (x^*, y^*) is a center, it is stable.

Stability of equilibrium point

Any non-linear system can have several equilibrium positions which can be stable or unstable, in some situations the stability of equilibrium required which is defined as follows:

Let (x^*, y^*) be a point of equilibrium and $X(t) = (x(t), y(t))$ the solution of the differential system (1.1) defined for all $t \in \mathbb{R}$.

- An equilibrium point $X^* = (x^*, y^*)$ of system (1.1) is stable if

$$\forall \epsilon > 0,\ \exists \sigma > 0,\ \ \|(x, y) - (x^*, y^*)\| < \sigma \Longrightarrow (\forall t > 0,\ \ \|X\,(t) - X^*\| < \epsilon)\,.$$

- An equilibrium point $X^* = (x^*, y^*)$ of system (1.1) is asymptotically stable if

  it is stable and there exist $\sigma > 0$ such that if $\|(x, y) - (x^*, y^*)\| < \sigma$ then

$$\lim_{t \to \infty} \|X\,(t) - X^*\| = 0.$$

- An equilibrium point $X^* = (x^*, y^*)$ of system (1.1) is unstable if it is not stable.

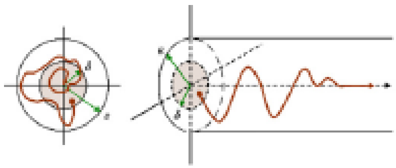

Figure 1.2: Asymptotic stability of equilibrium points.

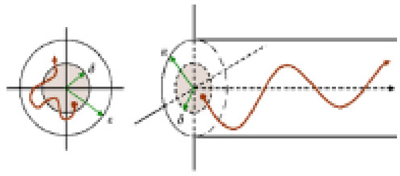

Figure 1.3: Stability of equilibrium points.

1.1.7 Invariant curve

In the study of differential equations the search for invariants curves is essential. He is indeed more interesting to consider the sets of the phase portrait which are not transformed by ow into other sets over time, see for example ([30], [32]), and also are used in the study of the existence and non-existence of limit cycles.

Definition 3 *We say that a curve defined by* $U(x,y) = 0$ *is an invariant curve for* (1.1) *if there exists a polynomial* $K(x,y)$ *satisfying*

$$P(x,y)\frac{\partial U}{\partial x}(x,y) + Q(x,y)\frac{\partial U}{\partial y}(x,y) = K(x,y)U(x,y) \tag{1.3}$$

The polynomial $K(x,y)$ *is called a cofactor of the curve* U*.*

- Equality (1.3) shows that on the invariant curve, the gradient $\left(\frac{\partial U}{\partial x}(x,y), \frac{\partial U}{\partial y}(x,y)\right)$ of U is orthogonal to the vector field $X = (P, Q)$, so at any point of the invariant curve the vector field is tangent to this curve.

- An invariant curve $U(x,y) = 0$ is said to be algebraic curve of degree n, if $U(x,y)$ is a polynomial of degree n, otherwise we say that the curve is a non algebraic curve.

- if the polynomial differential system (1.1) has an algebraic invariant curve

  $U(x,y) = 0$ of degree n, then any cofactor is of degree at most $n-1$.

Lemma 4 *Let* $f, g \in \mathbb{C}[x,y]$ *of class* C^1*. Then for the polynomial differential system* (1.1)*,* $fg = 0$ *is an algebraic invariant curve with cofactor* K_{fg} *if and only if* $f = 0$ *and* $g = 0$ *are algebraic curves invariant with* K_f *and* K_g *cofactors; respectively. Also,* $K_{fg} = K_f + K_g$*.*

Proposition 5 *Suppose* $U \in \mathbb{C}[x,y]$ *and let* $U = U_1^{n_1},...U_r^{n_r}r$ *be its factorization into irreducible factors over* $\mathbb{C}[x,y]$*.Then for a polynomial system* (1.1)*,* $U = 0$ *is an invariant algebraic curve with cofactor* k_u *if and only if* $U_i = 0$ *is an invariant algebraic curve for each* $i = 1,.....r$ *with cofactor* k_{ui}*. Moreover* $k_u = n_1k_{u1} + ... + n_rk_{ur}$ *.*

Theorem 6 *[48]We consider the system* (1.1) *and* $\Gamma(t)$ *a periodic orbit of period* $T > 0$*. We assume that* $U : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ *is an invariant curve,* $\Gamma(t) = \{(x,y) \in \Omega : U(x,y) = 0\}$*, and* $K(x,y) \in C^1$ *is the cofactor given in equation* (1.3)*, of the invariant curve* $U(x,y) = 0$*. Suppose that* $p \in \Omega$ *such that* $U(p) = 0$ *and* $\nabla U(p) \neq 0$*, then*

$$\int_0^T \operatorname{div}(\Gamma(t))dt = \int_0^T K(\Gamma(t))dt$$

The assumption $\nabla U(p) \neq 0$ means that U does not contain singular points.

1.1.8 Limit cycles

We have seen that the solution tend towards a singular point, another possible behavior for a trajectory is to tend towards a periodic movement in the case of a planar system, that means that the trajectories tend towards what is called a limits cycles $\Gamma = \{(x(t), y(t)), t \in [0,T]\}$ which is an isolated periodic solution in the set of all periodic solutions of system. Isolated means that neighborhood trajectories are note closed, they will either spiral away or towards the limit cycle.

- If a limit cycle is contained in a algebraic curve of the plan, then we say that it is algebraic, otherwise it is called non algebraic.
- The limit cycle appear only in non-linear differential systems.

Stability of limit cycle

In [56], d. Jordan B. Smith that adjacent tracks they are not closed and act like spirals approaching or moving away from them limit cycle when $t \to \infty$ is then a limit cycle: stable, unstable, or nearly stable depending on whether the near curves point to, away from, or two, respectively. These solutions are relatively of less importance in comparison with the solution periodic.

a) The limit cycle Γ is stable (or attractive), if the interior and exterior trajectories spirals tend towards the closed orbit when $t \to +\infty$

b) The limit cycle Γ is unstable (or repulsive), if the interior and exterior trajectories spirals tend towards the closed orbit when $t \to -\infty$

c) The limit cycle Γ is semi-stable, if the interior spiral trajectories tend towards the closed orbit when $t \to +\infty$, the others (external) tend towards when $t \to -\infty$, and viceversa.

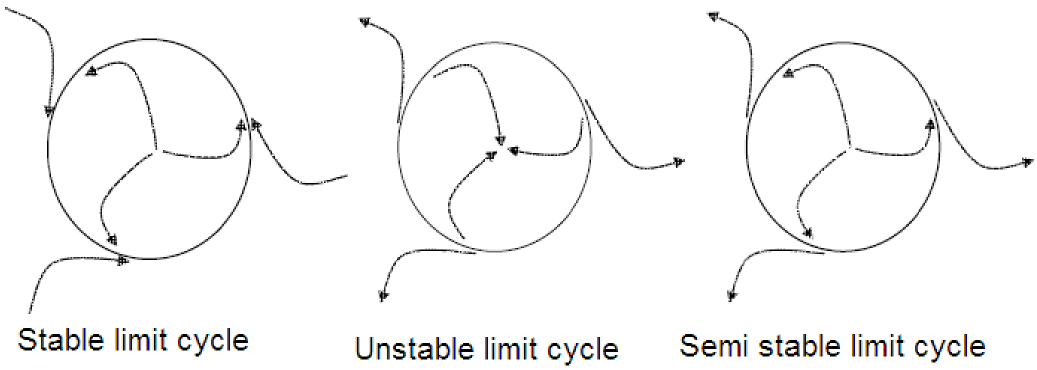

Figure 1.4: Classification of limit cycles.

Example 7 *We consider the following system:*

$$\begin{cases} x' = -y + ax\left(4 - x^2 - y^2\right) \\ y' = x + ay\left(4 - x^2 - y^2\right) \end{cases}$$

this system has a limit cycle Γ *represented by*

$$\Gamma(\theta) = (2\cos\theta, 2\sin\theta)$$

because we have

$$\operatorname{div}(P,Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = -4a\left(x^2 + y^2 - 2\right)$$

and

$$\begin{aligned} \int_0^T \operatorname{div}(\Gamma(t))\,dt &= \int_0^T \operatorname{div}(\cos\theta, \sin\theta)\,dt \\ &= a\int_0^{2\pi} -4\left((2\cos\theta)^2 + (2\sin\theta)^2 - 2\right)d\theta \\ &= -16\pi a. \end{aligned}$$

So the cycle $\Gamma(t) = (2\cos\theta, 2\sin\theta)$ *is a unstable limit cycle if* $a > 0$ *and is a stable* *limit cycle if* $a < 0$.

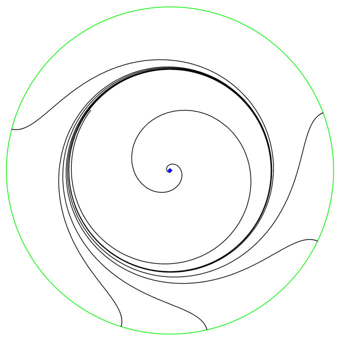

Figure 1.5 : Limit cyle for $a = -0.1$.

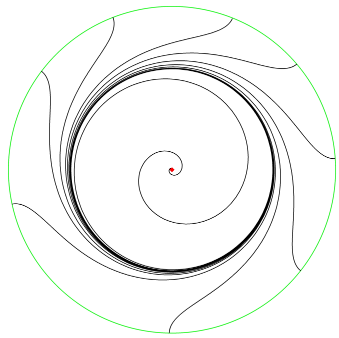

Figure 1.6 : Limit cyle for $a = 0.1$.

Criteria for the existence of limit cycles in the plane

In this subsection, we give some results which allow us it possible to prove the existence or the non existence of limit cycles for a polynomial differential system (1.1).

Theorem 8 *[46] Consider two closed curves* C *and* C'*, one surrounding the other. If in each point of* C*, the velocity vector field* (P, Q) *of the trajectory passing through it is directed towards outside, and if at each point of* C' *it is directed inwards, then there exists at least a limit cycle between* C *and* C'*. See Figure 1.7.*

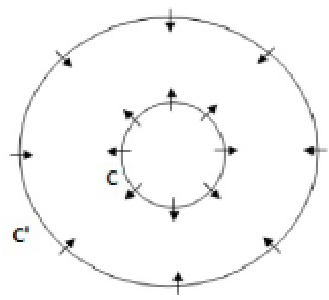

Figure 1.7 : Existence of limit cycle located between C *and* C'.

Giacomini, Llibre and Viano in [45], presented a method to study the existence and non-existence of limit cycles of a planar vector field, based on the following two criteria:

Theorem 9 (Criterion1) *[45] Let (P, Q) be a vector field of class C^1 defined on a non-empty open set D of $\mathbb{R}^2$, $(x(t), y(t))$ a periodic solution of period T of the system* (1.1) *and $K : D \to \mathbb{R}$ a C^1 map such that*

$$\int_0^T K(x(t), y(t))dt \neq 0$$

and $U = U(x, y)$ a C^1 solution of the linear partial differential equation

$$P(x,y)\frac{\partial U}{\partial x}(x,y) + Q(x,y)\frac{\partial U}{\partial y}(x,y) = K(x,y)U(x,y)$$

then the closed trajectory $\Gamma = \{(x(t), y(t)) \in D : t \in [0, T]\}$ is contained in $\Sigma = \{(x, y) \in \Omega : U(x, y) = 0\}$ and Γ is not contained in a period annulus of (P, Q). Moreover, if the vector field (P, Q) and the functions K and U are analytic, then Γ is a limit cycle.

Theorem 10 (Criterion2) *Consider the system* (1.1) *and assume that:*

1. P, Q of class C^1.
2. $U = U(x, y)$ of class C^1 (Ω) a solution of the partial differential equation:

$$P(x,y)\frac{\partial U}{\partial x}(x,y) + Q(x,y)\frac{\partial U}{\partial y}(x,y) = \left(\frac{\partial P}{\partial x}(x,y) + \frac{\partial P}{\partial y}(x,y)\right)U(x,y)$$

If γ is a limit cycle, then is contained in

$$\Sigma = \{(x, y) \in \Omega : U(x, y) = 0\}$$

Criteria for a non existence of limit cycles in the plane

One of the most important criterion in dynamical systems that permits us to confirm the absence of limit cycles is the Poincar e-Bendixon Theorem known as Bendixon's Criterion [38],[66].

Theorem 11 (Bendixon's criterion) *[38] Let D be a connected domain of $\mathbb{R}^2$.*

If the divergence $\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)$ is nonzero and of constant sign on D, then the differential system

$$\begin{cases} x' = P(x, y), \\ y' = Q(x, y) \end{cases}$$

does not admit a periodic solution entirely contained in D.

Theorem 12 *[33] If the system has no singular point, then it has no limit cycles.*

A generalized form of Theorem 11 was introduced by Dulac (see [56]) which can be given as follows:

Theorem 13 ***(Dulac's criterion)*** *[56] Let D be a simply connected domain of $\mathbb{R}^2$, and let ψ be a class function C^1 on D.*

If the quantity $\left(\frac{\partial(\psi P)}{\partial x}+\frac{\partial(\psi Q)}{\partial y}\right)$ is nonzero and of constant sign on D, then the differential system

$$\begin{cases} x' = P\left(x,y\right), \\ y' = Q\left(x,y\right) \end{cases}$$

does not admit a periodic solution entirely contained in D.

1.1.9 Notion of integrability

The aim of this section is to introduce the terminology of the notions of invariant curve and the first integral, Integrating factors, Inverse integrating factor and Exponential factor for real planar polynomial differential systems. For a detailed discussion of this theory see [38].

First integral

Definition 14 *We say that a non-locally constant C^1 function $H:\Omega\to\mathbb{R}$ is a first integral of the differential system* (1.1) *in Ω if H is constant on the trajectories of the system* (1.1) *contained in Ω , i.e, if*

$$\frac{dH}{dt}(x,y)=P(x,y)\frac{\partial H}{\partial x}(x,y)+Q(x,y)\frac{\partial H}{\partial y}(x,y)\equiv 0$$

Moreover, $H=k$ is the general solution of this equation, where k is an arbitrary constant.And a system (1.1) *is integrable in Ω if it has a first integral H in Ω.*

Integrating factors

It is possible to deduce the expression for the first integral by means of two concepts of factor of integration and factor of inverse integration.

Definition 15 *Let Ω be an open subset of $\mathbb{R}^2$ and $R:\Omega\to\mathbb{R}$ be an analytic function which is not identically zero on The function R is an integrating factor of the* *differential system* (1.1), *or of on* Ω *if one of the following three equivalent conditions:*

$$\begin{aligned} \frac{\partial (RP)}{\partial x} &= -\frac{\partial (RQ)}{\partial y} \\ div(RP, RQ) &= 0 \\ XR &= -Rdiv(X) \end{aligned}$$

on Ω. *As usual the divergence of the vector field* X *is defined by*

$$\operatorname{div}(X) = div(P, Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

It is clear that the function H *satisfying*

$$\begin{cases} x' = RP = \frac{\partial H}{\partial y} \\ y' = RQ = -\frac{\partial H}{\partial x} \end{cases}$$

The first integral H *associated to the integrating factor* R *is given by*

$$H(x, y) = -\int R(x, y)P(x, y)dy + h(x)$$

Or

$$H(x, y) = \int R(x, y)Q(x, y)dx + h(y)$$

Inverse integrating factor

The inverse integrating factor is among the tools that are used in the study of the existence and non-existence of limit cycles.

Definition 16 *A non-zero function* $V : \Omega \to \mathbb{R}$ *is said to be an inverse integrating factor of system* (1.1) *if of class* $C^1(\Omega)$, *not locally null and satisfies the following linear partial differential equation*

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) V$$

It is easy to verify that the function $R = \frac{1}{V}$ *defines an integrating factor in* $\Omega \backslash \{V = 0\}$ *of the system* (1.1).

The following theorem, proved in [45], gives an important relation between a limit cycle and an inverse integrating factor.

Theorem 17 *[45] Let* $V : \Omega \to \mathbb{R}$ *be an inverse integrating factor of system* (1.1). *If* Γ *is a limit cycle of* (1.1), *then* Γ *is contained in the set*

$$\Gamma = \{(x, y) \in \Omega : V(x, y) = 0\}$$

Proposition 18 *[45] If system* (1.1) *has two different integrating factors* R_1 *and* R_2 *on the open subset* $U \subseteq \mathbb{R}^2$ *and have no constant common factors other than 1, then the function* $R = R_1/R_2$ *is also an integration factor on* $U \setminus \{R_2 = 0\}$.

Exponential factor

There is another object, which is called exponential factor, which plays the same role as the invariant algebraic curves to obtain a first integral of a polynomial differential system (1.1).

Definition 19 *Let $h, g \in \mathbb{R}[x, y]$ either be coprime polynomials in the ring $\mathbb{R}[x, y]$, or $h \equiv 1$. Then, for the polynomial differential system* (1.1)*, the function $exp(g/h)$ is an exponential factor if there is a polynomial $K \in \mathbb{R}[x, y]$ of degree at most $m-1$, such that:*

$$P\frac{\partial\left(exp(\frac{g}{h})\right)}{\partial x}+Q\frac{\partial\left(exp(\frac{g}{h})\right)}{\partial y}=K\left(exp(\frac{g}{h})\right)$$

Then, for the exponential factor $exp(g/h)$ we stated that K was its cofactor. Since the exponential factor cannot vanish, it does not define invariant curves of the polynomial system (1.1) *.*

1.1.10 Return map (Poincaré Map)

In 1881[74], Henri Poincaré defined the Poincaré map or first return map which is one of the most basic tool for studying the stability of periodic orbits. The idea of the Poincaré map is the following: If Γ is a periodic orbit of the system (1.1) through the point $X_0 = (x_0, y_0)$ and Σ is a hyperplane perpendicular to Γ at X_0, then for any $X = (x, y) \in \Sigma$ sufficiently near X_0, the solution of (1.1) through X at $t = 0$, will cross Σ again at a point $\Pi(X)$ near X_0, (Figure 1.8). The mapping $X \to \Pi(X)$ is called the Poincaré map. The next theorem establishes the existence and continuity of the Poincaré map $\Pi(X)$ and of its first derivative $D\Pi(X)$.

Theorem 20 *[72] Let Ω be an open subset of $\mathbb{R}^2$. Suppose that $\Phi_t(X_0)$ is a periodic solution of* (1.1) *of period T and that the cycle*

$$\Gamma = \{X \in \mathbb{R}^2 / X = (X_0), 0 \leq t \leq T\}$$

is contained in Ω. Let Σ be the hyperplane orthogonal to Γ at X_0, i.e., let

$$\Sigma = \{X \in \mathbb{R}^n / (X - X_0).(P(X_0), Q(X_0)) = 0\}$$

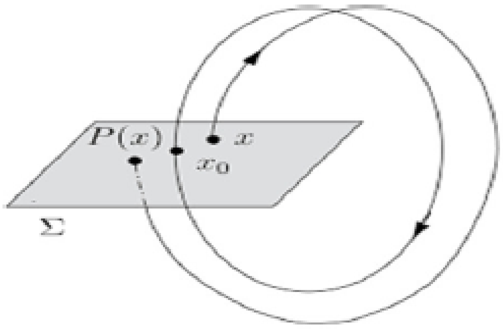

Figure 1.8 : The Poincare map.

Then there is $\delta > 0$ and a unique function $\tau(x, y)$, defined and continuously differentiable such that $\tau(X) = T$ and $\Phi_{\tau(X)}(X) \in \Sigma$ for all $(X) \in N_\delta(X_0)$.

Definition 21 *Let Γ, Σ, δ and $\tau(X)$ be defined as in Theorem* 20. *Then for $X_0 \in N_\delta(X_0) \cap \Sigma$ the function*

$$\Pi(X) = \Phi_{\tau(X)}(X)$$

is called the Poincaré map for Γ at X_0.

Theorem 22 *[72] Let $\Gamma(t)$ be a periodic solution of* (1.1) *of period T. Then the derivative of the Poincaré map $\Pi(s)$ along a straight line Σ normal to*

$$\Gamma = \{X \in \mathbb{R}^2 / \ X = \Gamma(t) - \Gamma(0), 0 \leq t \leq T\}$$

at $X = (0, 0)$ is given by

$$\Pi'(0, 0) = \exp \int_0^T \operatorname{div}(P\,(\Gamma(t))\,, Q\,(\Gamma(t)))dt$$

Stability of Poincaré map

The following corollary characterizes the stability of a limit cycle.

Corollary 23 *[72] Under the hypotheses of Theorem* 22. *Consider $\Gamma(t)$ is a periodic orbit of system* (1.1) *of periodic T.*

1 If $\int_0^T \text{div}(P\left(\Gamma(t)\right), Q\left(\Gamma(t)\right))dt < 0$, then Γ is a stable limit cycle.

2 If $\int_0^T \text{div}(P\left(\Gamma(t)\right), Q\left(\Gamma(t)\right))dt > 0$, then Γ is a unstable limit cycle.

3 If $\int_0^T \text{div}(P\left(\Gamma(t)\right), Q\left(\Gamma(t)\right))dt = 0$, then Γ may be a stable, unstable or semi-stable limit cycle or it may belong to a continuous band of cycles.

4 If $\int_0^T \text{div}(P\left(\Gamma(t)\right), Q\left(\Gamma(t)\right))dt \neq 0$, then Γ is a hyperbolic limit cycle.

1.2 Kolmogorov systems

Models are always a simplification of reality and will help to see a process from a global perspective. If we would like to develop methods to analyze the dynamics of a system with 2 differential equations, for example a prey-predator model. A prey-predator model is a model where we consider two species, one of which is eaten by the other. To model a population of foxes and rabbits that are interacting, it will be necessary to introduce a system with 2 differential equations.

1.2.1 Prey-predator systems

Predator-prey models are arguably the building blocks of the bio- and ecosystems as biomasses are grown out of their resource masses. Species compete, evolve and disperse simply for the purpose of seeking resources to sustain their struggle for their very existence. Depending on their specific settings of applications, they can take the forms of resource-consumer, plant-herbivore, parasite-host, tumor cells (virus)-immune system, susceptible-infectious interactions, etc. They deal with the general loss-win interactions and hence may have applications outside of ecosystems. When seemingly competitive interactions are carefully examined, they are often in fact some forms of predator-prey interaction in disguise.

Consider two populations whose sizes at a reference time t are denoted by $x(t)$,$y(t)$, respectively. The functions x and y might denote population numbers or concentrations (number per area) or some other scaled measure of the populations sizes, but are taken to be continuous functions. Changes in population size with time are described by the time derivatives $x' = dx/dt$ and $y' = dy/dt$, respectively, and a general model of interacting populations is written in terms of two autonomous differential equations

$$\begin{cases} x' = xf(x,y), \\ y' = yg(x,y). \end{cases}$$

(i.e., the time t does not appear explicitly in the functions $xf(x,y)$ and $yg(x,y)$). The functions f and g denote the respective per capita growth rates of the two species.

1.2.2 Lotka-Volterra prey-predator model

Alfred Lotka and Vito Volterra independently proposed in 1925-26 a simple system of population dynamics. The Lotka-Volterra model describes the interactions between a population of prey and a population of predators: the state variables are x the number of prey and y the number of predators. It is assumed that in the absence of predators, the prey population grows exponentially with an exponent $a > 0$ and that in the absence of prey, the predator population decreases exponentially with an exponent $c < 0$. When the two populations coexist, it is assumed that the number of prey decreases and the number of predators increases proportionally to the product xy with factors $b < 0$ and $d > 0$ respectively. We therefore obtain the following system:

$$\begin{cases} x' = x\,(a - by) \\ y' = y\,(-c + dx) \end{cases}$$

Note that the disadvantage of this model is that it does not admit a limit cycle.This is why the russian A.Kolmogorov generalized this model so that the new model admits limit cycles.

1.2.3 Kolmogorov systems

Andrei Kolmogorov generalized the formalism introduced by Lotka and Volterra by introducing the following type of systems:

$$\begin{cases} x' = xf\,(x,y) \\ y' = yg\,(x,y) \end{cases}$$

with the following conditions on the functions f and g assumed to be of class C^1:

$$\frac{\partial f}{\partial y} < 0, \frac{\partial g}{\partial y} < 0, \frac{\partial f}{\partial x} < 0 \text{ for large } x \ \frac{\partial g}{\partial x} > 0$$

These conditions emerge when natural constraints are imposed on the dynamic interaction between populations. Indeed, the first two conditions result from the hypothesis that if the number of predators increases, then the growth rate of the two populations decreases. The last two conditions are sufficient to guarantee the existence and uniqueness of a state of equilibrium where the two populations coexist, i.e. a singular point of the dynamics in $\{x > 0, y > 0\}$

Since the paper of May (1972), determining conditions which guarantee uniqueness of limit cycles in predator-prey models has become an outstanding problem in mathematical ecology. Recently, Kuang and Freedman (1988). Huang (1988), Huang and Merrill (1989) provided some criteria for uniqueness of limit cycles in predator-prey models. The idea of their criteria is based on a uniqueness theorem of limit cycles for a general Lienard equation by Zhang (1958, 1986). In this paper, we propose a general Kolmogorov-type model which consists of the above two and many other models (Cheng, 1981, for example) as special cases. We have proved the existence and uniqueness theorems of limit cycles in the model. The method we used in the proof of uniqueness of limit cycles is not dependent on Zhang's theorem (see Huang. 1989).

The general model is

$$\begin{cases} x' = \phi(x)(F(x) - \pi(y)) \\ y' = \varphi(y)(\psi(x) + \xi(y)) \end{cases}$$

where x is the prey density, y is the predator density, $\phi(x)$,$\psi(x)$ are predator response functions, $\pi(y)$, $\varphi(y)$, $\xi(y)$ are predator $\phi(x) F(x)$ density functions, is the "relative" or "per capita" growth function which governs the growth of the prey in the absence of predators, $\phi(x)\pi(y)$ the predator,is the death rate of the prey due to $\varphi(y)(\psi(0) + \xi(y))$ is the death rate of predator in the absence of prey.

One of the most classic examples of a Kolmogoroff type model is the model introduced by Robert May in 1972. It is obtained from the Lotka-Volterra model by replacing the exponential growth of the prey in the absence of the predator by growth called logistic (implying asymptotic saturation) and by introducing a saturation effect to the predator's efficiency.

Many researchers have been interested in the study of Kolmogorov systems,in particularment, integrability and the existence of limit cycles.The search for the explicit expression of the limit cycles of differential systems is a difficult task,especially that of Kolmogorov.To ourknowledge,all explicit expressions of limit cycles now a days were only algebraic(see the works from Bendjeddou and all [11],[21],[22]).

Example 24 *A general cubic kolmogorov system has the form*

$$\begin{cases} x' = (x+p)\left(y(mx+ny+l) + ax^2 + bx\right) \\ y' = (y+q)\left(-x(mx+ny+l) + ay^2 + by\right) \end{cases} \tag{1.4}$$

This system can have one or more limit cycles,if we choose $p = q = 4$, $a = n = 1, b = \frac{1}{4}, m = \frac{1}{2}$ *and* $l = 2$ *then the system* (1.4) *is reduced to the following system:*

$$\begin{cases} x' = (x+4)\left(y\left(\frac{1}{2}x+y+2\right)+x^2+\frac{1}{4}x\right) \\ y' = (y+4)\left(-x\left(\frac{1}{2}x-y-2\right)+y^2+\frac{1}{4}y\right) \end{cases}$$

By using the translation : $u = x+4, v = y+4$, we obtain

$$\begin{cases} u' = u\left(u^2+\frac{1}{2}uv-\frac{39}{4}u+v^2-8v+31\right) \\ v' = v\left(6u-\frac{15}{4}v-uy-\frac{1}{2}u^2+v^2-1\right) \end{cases}$$

and back to the notation (x, y) the previous system become

$$\begin{cases} x' = x\left(x^2+\frac{1}{2}xy-\frac{39}{4}x+y^2-8y+31\right) \\ y' = y\left(-\frac{1}{2}x^2-xy+6x+y^2-\frac{15}{4}y-1\right) \end{cases}$$

This system have a limit cycle represented by the figure 1.9

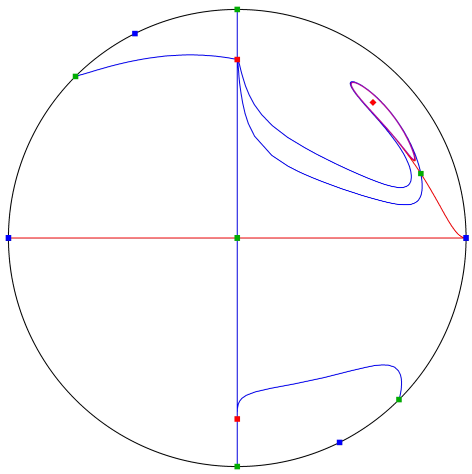

Figure 1.9 : The phase portrait in the Poincare disk of system (1.4)

1.3 Poincaré Compactification

In the definition of the Poincaré compactification we base ourselves on the reference of [38]. In order to study the behavior of the trajectories of a planar differential system near infinity it is possible to use a compactification. One of the possible constructions relies on stereographic projection of the sphere onto the plane, in which case a single “point at infinity” is adjoined to the plane, see Bendixson [9]. A better approach for studying the behavior of trajectories near infinity is to use the so called Poincaré sphere, introduced by Poincaré [76]. It has the advantage that the singular points at infinity are spread out along the equator of the sphere and are therefore of a simpler nature than the singular points of the Bendixson sphere. However, some of the singular points at infinity on the Poincaré sphere may still be very complicated.

In order to improve the construction, we introduce the so called Poincaré–Lyapunov sphere, which is however based on a construction of a more abstract nature than the previous ones. The singularities are also spread along the equator but are in general simpler than for the Poincaré sphere. In the Poincaré–Lyapunov compactification we prefer to work on a hemisphere, calling it as the Poincaré–Lyapunov disk, and similarly talk about a Poincaré disk, if we restrict the Poincaré sphere to one of the hemispheres separated by the equator that represents the points at infinity.

1.3.1 Infinite Singular Points

In order to draw the phase portrait of a vector field, we would have to work over the complete real plane $\mathbb{R}^2$, which is not very practical. If the functions defining the vector field are polynomials, we can apply Poincaré compactification, which will tell us how to draw it in a finite region. Even more, it controls the orbits which tend to or come from infinity.

In this part, we study the behavior of the trajectories of a planar differential systems near infinity. Let $X(x,y) = (P(x,y), Q(x,y))$ represent a vector field to each system which we are going to study its phase portraits, then for doing this we use the so called a Poincaré compactification. We consider the Poincaré sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3$ $(x^*, y^*) : x^2 + y^2 + z^2 = 1\}$, and we define the central projection $f : T_{(0,0,1)}S^2 \to S^2$ (with $T_{(0,0,1)}S^2$ the tangent space of S^2 at the point $(0,0,1)$), such that for each point $m \in T_{(0,0,1)}S^2$, $T_{(0,0,1)}S^2(m)$ associates the two intersection points of the straight line which connects the point m and $(0,0)$). The equator $S^2 = \{(x,y,z) \in S^2 : z = 0\}$ represent the infinity points of $\mathbb{R}^2$. In summary we get a vector field X' defined in S^2 $\setminus S^1$, which is formed by to symmetric copies of X, and we prolong it to a vector field $n(X)$ on S^2. By studying the dynamics of $n(X)$ near S^1 we get the dynamics of X at infinity. We need to do the calculations on the Poincaré sphere near the local charts $U_i = \{Y \in S^2 : y_i > 0\}$, and $V_i = \{Y \in S^2 : y_i < 0\}$ for $i = 1, 2, 3$. With the associated diffeomorphisms $L_i : U_i \to \mathbb{R}^2$ and $G_i : V_i \to R^2$ for $i = 1,\ 2, 3$. After a rescaling in the independent variable in the local chart (U_1, L_1) the expression for $n(X)$ is

$$u' = v^n \left[Q\left(\frac{1}{v}, \frac{u}{v}\right) - uP\left(\frac{1}{v}, \frac{u}{v}\right) \right], \qquad v' = -v^{n+1} P\left(\frac{1}{v}, \frac{u}{v}\right), \tag{1.5}$$

in the local chart (U_2, L_2) the expression for $n(X)$ is

$$u' = v^n \left[\left(P\left(\frac{u}{v}, \frac{1}{v}\right)\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right)\right], \qquad v' = -v^{n+1} Q\left(\frac{u}{v}, \frac{1}{v}\right), \tag{1.6}$$

in the local chart (U_3, L_3) the expression for $n(X)$ is

$$u' = P(u, v), \qquad v' = Q(u, v). \tag{1.7}$$

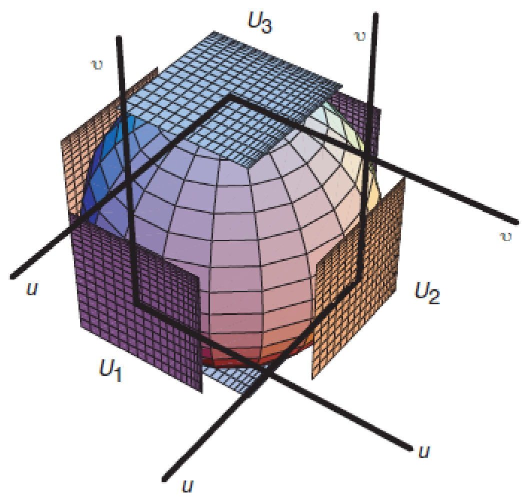

Figure 1.10 : The local charts (U_k, ϕ_k) for $k = 1, 2, 3$ of the Poincare sphere.

Note that for studying the singular points at infinity we only need to study the infinite singular points of the chart U_1 and the origin of the chart U_2, because the singular points at infinity appear in pairs diametrally opposite.

For more details on the Poincaré compactification see Chapter 5 of [38].

1.3.2 Application

Example 25 *we study the phase portrait on the Poincaré disk of the system*

$$\begin{cases} x' = -x - y^2 \\ y' = y + x^2 \end{cases} \tag{1.8}$$

This is a Hamiltonian system because it can be written as

$$\begin{cases} x' = -\frac{\partial H}{\partial y} \\ y' = \frac{\partial H}{\partial x} \end{cases}$$

with Hamiltonian $H(x, y) = \frac{1}{3}(x^3 + y^3) + xy$.

Therefore H is a first integral of system (1.8); i.e., H is constant on the solutions of (1.8), because on any solution $(x(t), y(t))$ of (1.8) we have

$$\frac{dH}{dt}(x(t), y(t)) = \frac{\partial H}{\partial x}\left(-x-y^2\right) + \frac{\partial H}{\partial y}\left(y+x^2\right)\Big|_{(x,y)=(x(t),y(t))} = 0$$

System (1.8) has two finite singular points, a saddle at $(0, 0)$ and a linear center at $(-1, -1)$ with eigenvalues $\pm\sqrt{3}i$. Since the first integral H is well defined at $(-1, -1)$, this singular point is a center.

Now we shall compute the infinite singular points. Let X be the vector field associated to (1.8). Then the expression for $p(X)$ in the local chart U_1 is

$$\begin{cases} u' = 1 + 2uv + u^3 \\ v' = v^2 + vu^2 \end{cases}$$

Therefore on U_1 there is a unique singular point, $(-1, 0)$ which is an unstable node at infinity. Since the degree of X is 2, the diametrically opposite point is a stable node in V_1.

The expression for $p(X)$ in the local chart U_2 is

$$\begin{cases} u' = -1 - 2uv - u^3 \\ v' = -v^2 - vu^2 \end{cases}$$

Since the origin of U_2 is not a singular point there do not exist additional infinite singular points.

The unique separatrices of system (1.8) are the separatrices of the saddle $(0, 0)$. Since $H(0, 0) = 0$, in order to locate such separatrices it is sufficient to draw the curve $H(x, y) = 0$. Hence the phase portrait of system (1.8) on the Poincaré disk as is given in Figure 1.11.

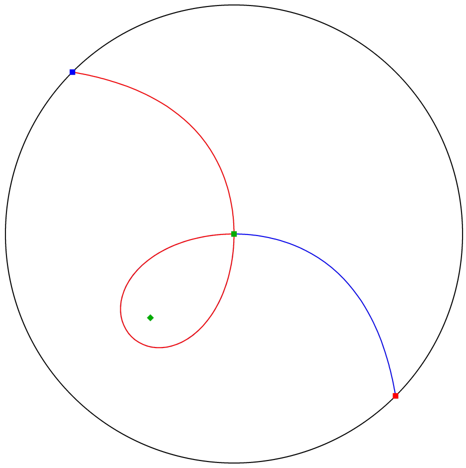

Figure 1.11 : The phase portrait in the Poincar disk of system (1.8) .

Chapter 2

Kolmogorov differential systems with prescribed algebraic limit cycles

2.1 Introduction

Many mathematical models in biology and population dynamics are modeled by systems of ordinary differential equations having the form:

$$\begin{cases} x' = \frac{dx}{dt} = xF(x,y) \\ y' = \frac{dy}{dt} = yG(x,y) \end{cases}$$

Where $x(t)$ and $y(t)$ represent the population density of two species at time t, and $F(x,y)$, $G(x,y)$ are the capita growth rate of each specie, usually, such systems are called Kolmogorov systems.

Generally, Kolmogorov system is introduced as the structure of many natural phenomena models. Their applications can be appear in several fields such as, physics, biology, chemical reactions, hydrodynamics, fluid dynamics,economics, etc. for more detail see ([8], [27], [78], [83]).

We consider the following polynomial differential system

$$\begin{cases} x' = x(ax + by + c) \\ y' = y(mx + ny + l) \end{cases}$$

This system known as Lotka–Volterra predator–prey model, if $a = n = 0$ the system becomes

$$\begin{cases} x' = x(by + c) \\ y' = y(mx + l) \end{cases}$$

where

- The variable x is the population density of prey (for example, the number of rabbits per square kilometre).
- The variable y is the population density of some predator (for example, the number of foxes per square kilometre).
- $\frac{dx}{dt}$ and $\frac{dy}{dt}$ represent the instantaneous growth rates of the two populations.
- t represents time
- The prey's parameters, a and b, describe, respectively, the maximum prey per capita growth rate, and the effect of the presence of predators on the prey growth rate.
- The predator's parameters, m, n, respectively describe the predator's per capita death rate, and the effect of the presence of prey on the predator's growth rate.
- All parameters are positive and real.

The Lotka-Volterra system of equations is a phenomenological model of competing population dynamics, it can model the dynamics of ecological systems with predator-prey interactions, competition, disease and mutualism.It is known that the model of Lotka-volterra have no limit cycle, but it remains the starting point of several models currently in applied science.

In [57] Kolmogorov developed a more general system than that of Lotka–Volterra predator–prey model, this is the system

$$\begin{cases} x' = xF(x, y) \\ y' = yG(x, y) \end{cases}$$

This model was then the subject of several extensions by Rascigno and Richardson [77], May [68] and Albrecht et al [2]. The functions F and G are respectively the growth rates of the two populations x and y, they are of class $C^1([0, +\infty[)$ and satisfy the following conditions :

1. For a fixed prey density, the population growth rate of prey x decreases with the growt of the population of predators y. This leads to the condition

$$\frac{\partial F}{\partial y}(x, y) < 0$$

2. For a fixed density of predators, the growth rate of the predator population y increases by the growth of the prey population x. This leads to the condition

$$\frac{\partial G}{\partial x}(x, y) > 0$$

3. For low densities of both populations, the prey population grows and therefore we have
$$F(0,0) > 0$$

4. There is a sufficient number of predators for which, a small number of prey cannot grow any longer. This leads to the condition
$$\exists A > 0, \text{ such that } F(0, A) = 0$$

5. The medium admits a limited carrying capacity. This leads to the condition
$$\exists B > 0, \text{ such that } F(B, 0) = 0$$

6. If there is a sufficient number of prey, the number of predators increases, otherwise it decreases. This leads to the condition
$$\exists C > 0, \text{ such that } G(C, 0) = 0$$

7. According to Kolmogorov [57] and Rascigno and Richardson [77], if $B \leq C$, the predators are in extinction and the prey saturates the environment (this is also true for the model of Lotka–Volterra predator–prey). However, to have a coexistence between the populations of prey and predators, it is necessary to ensure that $C < B$.

To our knowledge, the first result which gives an system differential of kolmogorov type with an algebraic limit cycle whose exact expression is known is found in the article [11], by A. Bendjeddou et al. And then we find the work of A. Bendjeddou, S. Benyoucef ([21],[22]). Who showed the existence of a finite class cycle of the Kolmogorov system, and gives its explicit form.

This chapter is a contribution to the study of a class of differential systems of the kolmogorov type. It is interesting to see if the differential system recognizes a limit cycle or not and if we know the exact expression for that limit cycle it is even better. Motivated by some papers presenting planar polynomial systems with one or more algebraic term cycles given analytically (see for instance work of Bendjeddou and Cheurfa [12],[13],[17]), and mainly based on the papers of C. Christopher [35],S. Benyoucef [20], we will extend the same concept to Kolmogorov systems, where once we have selected components of the Kolmogorov system that satisfy certain conditions, we can directly infer the number and explicit form of limit cycles.

More precisely, in the following we introduce three different classes of differential systems of order n of the Kolmogorov type, in which we will study hyperbolic algebraic limit cycles.

$$\begin{cases} x' = x\left(R\left(x,y\right)U\left(x,y\right) - y\Phi\left(x,y\right)U_y\right) \\ y' = y\left(S\left(x,y\right)U\left(x,y\right) + x\Phi\left(x,y\right)U_x\right) \end{cases} \tag{2.1}$$

Where $R\left(x,y\right), S\left(x,y\right), U\left(x,y\right)$ and $\Phi\left(x,y\right)$ are polynomial functions.Our contribution consist to show that the system (2.1) admits all the bounded components of Γ as hyperbolic limit cycles if certain conditions on the parameters are satisfied.

This result was the subject of a publication entitled "Kolmogorov differential systems with prescribed algebraic limit cycles" in the journal Siberian Electronic Mathematical Reports (2021), 1-8 see [34].

2.2 The main results

In this section we introduce the following theorems that are of particular importance in the study of some class differential kolmogorov system .

Theorem 26 *Let U is C^1 function in open subset $V = \{(x,y) \in \mathbb{R}^2, x > 0, y > 0\}$, if $U = 0$ is non-singular of polynomial differential system*

$$\begin{cases} x' = x\left(\left(P\left(y\right) + axy + b\right)U\left(x,y\right) - \alpha yU_y\right) \\ y' = y\left(Q\left(x\right) + cxy + d\right)U\left(x,y\right) + \alpha xU_x\right) \end{cases} \tag{2.2}$$

Where $P\left(y\right)$ and $Q\left(x\right)$ are polynomial of any degree, α, a, b, c, d are real numbers satisfying $\alpha \neq 0$, $a + c \neq 0$, then the system (2.2) *admit all the bounded components of $U = 0$ as hyperbolic limit cycles.*

Proof of theorem 26 Let $\Gamma = \{(x,y) \in \mathbb{R}^2,\ U\left(x,y\right) = 0\}$ is a trajectory of the system (2.2).

To show that all the bounded components of Γ are hyperbolic limit cycles of system (2.2), we will prove that Γ is an invariant curve of the system (2.2), and $\int_0^T \operatorname{div}\left(\Gamma\right) dt \neq 0$ (See for instance Perko [72] [17, Pages 216-217]).

1) Calculate the differential of the function $U(x,y)$ with respect to the differential system (2.2):

$$\begin{aligned} \frac{dU}{dt} &= U_x x' + U_y y' \\ &= U_x\left(x\left(\left(P\left(y\right) + axy + b\right)U - \alpha yU_y\right)\right) + U_y\left(y\left(Q\left(x\right) + cxy + d\right)U + \alpha xU_x\right)\right) \\ &= U\left(xU_x\left(P\left(y\right) + axy + b\right) + yU_y(Q\left(x\right) + cxy + d)\right). \end{aligned}$$

which shows that $U(x,y) = 0$ an invariant curve of the system (2.2) with a cofactor $K\left(x,y\right) = \left(xU_x\left(P\left(y\right) + axy + b\right) + yU_y(Q\left(x\right) + cxy + d)\right).$

2) We show that $\int_0^T \operatorname{div}(\Gamma)\, dt$ is nonzero. Since we know that

$$\int_0^T \operatorname{div}(\Gamma)\, dt = \int_0^T K(x(t), y(t))\, dt$$

We have

$$\begin{aligned}\int_0^T K(x(t), y(t))\, dt &= \int_0^T (xU_x(P(y) + axy + b) + yU_y(Q(x) + cxy + d))\, dt \\ &= \int_0^T (xU_x(P(y) + axy + b))\, dt + \int_0^T (yU_y(Q(x) + cxy + d)\, dt.\end{aligned}$$

Since

$$dt = \frac{dx}{x((P(y) + axy + b)U - \alpha yU_y)} = \frac{dy}{y(Q(x) + cxy + d)U + \alpha xU_x)}$$

then

$$\int_0^T \operatorname{div}(\Gamma)\, dt = \oint_\Gamma \frac{xU_x(P(y) + axy + b)}{axyU_x} dy + \oint_\Gamma \frac{yU_y(Q(x) + cxy + d)}{-\alpha xyU_y} dx$$

Since $U(x,y) = 0$ is algebraic curve, then we have $U_x(x,y)dx + U_y(x,y)dy = 0$, thus $U_x(x,y)dx = -U_y(x,y)dy$, and we get

$$\begin{aligned}\int_0^T \operatorname{div}(\Gamma)\, dt &= \oint_\Gamma \tfrac{(P(y)+axy+b)}{ay} dy - \oint_\Gamma \tfrac{(Q(x)+cxy+d)}{\alpha x} dx \\ &= \tfrac{1}{\alpha}\left(\oint_\Gamma \left(ax + \tfrac{P(y)}{y} + \tfrac{b}{y}\right) dy - \oint_\Gamma \left(cy + \tfrac{Q(x)}{x} + \tfrac{d}{x}\right) dx\right).\end{aligned}$$

By applying the GREEN formula we obtain

$$\begin{aligned}&\tfrac{1}{\alpha}\left(\oint_\Gamma \left(ax + \tfrac{P(y)}{y} + \tfrac{b}{y}\right) dy - \oint_\Gamma cy + \tfrac{Q(x)}{x} + \tfrac{d}{x} dx\right) \\ &= \tfrac{1}{\alpha}\iint_{int(\Gamma)} \left(\frac{\partial\left(ax + \frac{P(y)}{y} + \frac{b}{y}\right)}{\partial x} + \frac{\partial\left(cy + \frac{Q(x)}{x} + \frac{d}{x}\right)}{\partial y}\right) dxdy \\ &= \tfrac{1}{\alpha}\iint_{int(\Gamma)} (a + c)\, dxdy,\end{aligned}$$

where $int(\Gamma)$ denotes the interior of Γ.

As $a + c \neq 0$, and $\alpha \neq 0$, then $\int_0^T K(x(t), y(t))\, dt$, is nonzero.This complete the proof of Theorem 26.

Theorem 27 *Let U is C^1 function in open subset* $V = \{(x, y) \in \mathbb{R}^2, x > 0, y > 0\}$

We consider a polynomial differential system

$$\begin{cases} x' = x\left(aU(x,y) - \alpha xy^2 U_y\right) \\ y' = y\left(bU(x,y) + \alpha yx^2 U_x\right) \end{cases} \tag{2.3}$$

Where $\alpha \neq 0$, $a + b \neq 0$, then the system (2.3) admits all the bounded components of $U = 0$ as hyperbolic limit cycles.

Proof of theorem 27 Let $\Gamma = \{(x, y) \in \mathbb{R}^2, U(x, y) = 0\}$ is a trajectory of the system (2.3).

To show that all the bounded components of Γ are hyperbolic limit cycles of system (2.3), we will prove that Γ is an invariant curve of the system (2.3), and $\int_0^T \operatorname{div}(\Gamma)\, dt \neq 0$ (See for instance Perko [72] [17, Pages 216-217]).

1) Calculate the differential of the function $U(x, y)$ with respect to the differential system (2.3):

$$\begin{aligned} \tfrac{dU}{dt} &= U_x x' + U_y y' \\ &= U_x\left(x\left(aU - \alpha xy^2 U_y\right)\right) + U_y\left(y\left(bU + \alpha yx^2 U_x\right)\right) \\ &= U\left(axU_x + byU_y\right). \end{aligned}$$

which shows that $U(x, y) = 0$ an invariant curve of the system (2.3) with a cofactor $K(x, y) = (axU_x + byU_y)$.

2) We show that $\int_0^T \operatorname{div}(\Gamma)\, dt$ is nonzero. We know that

$$\int_0^T \operatorname{div}(\Gamma)\, dt = \int_0^T K(x(t), y(t))\, dt$$

We have

$$\begin{aligned} \int_0^T K(x(t), y(t))\, dt &= \int_0^T (axU_x + byU_y)\, dt \\ &= \int_0^T (axU_x)\, dt + \int_0^T (byU_y)\, dt. \end{aligned}$$

Since

$$dt = \frac{dx}{x\left(aU - \alpha xy^2 U_y\right)} = \frac{dy}{y\left(bU + \alpha yx^2 U_x\right)}$$

then

$$\int_0^T \operatorname{div}(\Gamma)\, dt = \oint_\Gamma \frac{axU_x}{-\alpha x^2 y^2 U_x} dy + \oint_\Gamma \frac{byU_y}{-\alpha y^2 x^2 U_y} dx$$

Since $U(x, y) = 0$ is algebraic curve, then we have $U_x(x, y)dx + U_y(x, y)dy = 0$, thus $U_x(x, y)dx = -U_y(x, y)dy$, and we get

$$\int_0^T \operatorname{div}(\Gamma)\,dt = \oint_\Gamma \frac{a}{\alpha x y^2} dy - \oint_\Gamma \frac{b}{\alpha y x^2} dx$$
$$= \frac{1}{\alpha}\left(\oint_\Gamma \left(\frac{a}{xy^2}\right) dy - \oint_\Gamma \left(\frac{b}{x^2 y}\right) dx\right).$$

By applying the GREEN formula we obtain

$$\left(\oint_\Gamma \left(\frac{a}{xy^2}\right) dy - \oint_\Gamma \left(\frac{b}{x^2 y}\right) dx\right)$$
$$= \iint_{int(\Gamma)} \left(\frac{\partial\left(\frac{a}{xy^2}\right)}{\partial x} + \frac{\partial\left(\frac{b}{x^2y}\right)}{\partial y}\right) dxdy$$
$$= -\frac{a+b}{\alpha}\iint_{int(\Gamma)} \frac{1}{x^2y^2}\, dxdy,$$

where $int(\Gamma)$ denotes the interior of Γ.

As $\alpha \neq 0$,and $a+b \neq 0$, then $\int_0^T \operatorname{div}(\Gamma)\,dt$ is nonzero.This complete the proof of Theorem 27

Theorem 28 *Let U and Φ are C^1 functions in open subset*

$V = \{(x,y) \in \mathbb{R}^2, x > 0, y > 0\}$ such that $\Phi(x,y) = \frac{c}{a+b}xy + \frac{y^a}{x^b}$,where a,b,c are nonzero reals and $a+b \neq 0$.If the curve $\Phi(x,y) = 0$ lies outside all bounded components of the non-singular curve $U = 0$, then the differentiel system

$$\begin{cases} x' = x\left(aU(x,y) - y\Phi(x,y)\,U_y\right) \\ y' = y\left(bU(x,y) + x\Phi(x,y)\,U_x\right) \end{cases} \tag{2.4}$$

admits all the bounded components of $U = 0$ as hyperbolic limit cycles.

Proof of theorem 28 Let $\Gamma = \{(x,y) \in \mathbb{R}^2, U(x,y) = 0\}$ is a trajectory of the system (2.4).

To show that all the bounded components of Γ are hyperbolic limit cycles of system (2.4), we will prove that Γ is an invariant curve of the system (2.4) ,and $\int_0^T \operatorname{div}(\Gamma)\,dt \neq 0$ (See for instance Perko [72] [17, Pages 216-217]).

1) Calculate the differential of the function $U(x,y)$ with respect to the differential system (2.4):

$$\frac{dU}{dt} = U_x x' + U_y y'$$
$$= U_x\left(x\left(aU - y\Phi(x,y)\,U_y\right)\right) + U_y\left(y\left(bU + x\Phi(x,y)\,U_x\right)\right)$$
$$= \left(axU_x + byU_y\right)U.$$

which shows that $U(x,y) = 0$ an invariant curve of the system (2.4)with a cofactor $K(x,y) = \left(axU_x + byU_y\right).$

2) We show that $\int_0^T \operatorname{div}(\Gamma)\, dt$ is nonzero.

Notice that

$$\int_0^T \operatorname{div}(\Gamma)\, dt = \int_0^T K\left(x\left(t\right), y\left(t\right)\right) dt$$

We have

$$\int_0^T K\left(x\left(t\right), y\left(t\right)\right) dt = \int_0^T \left(axU_x + byU_y\right) dt$$

Since

$$dt = \frac{dx}{x\left(aU - y\Phi\left(x,y\right)U_y\right)} = \frac{dy}{y\left(bU + x\Phi\left(x,y\right)U_x\right)}$$

then

$$\int_0^T \operatorname{div}(\Gamma)\, dt = \oint_\Gamma \frac{axU_x}{yx\Phi\left(x,y\right)U_x} dy + \oint_\Gamma \frac{byU_y}{-yx\Phi\left(x,y\right)U_y} dx$$

Since $U(x,y) = 0$ is algebraic curve, then we have $U_x(x,y)dx + U_y(x,y)dy = 0$, thus $U_x(x,y)dx = -U_y(x,y)dy$, and we get

$$\int_0^T \operatorname{div}(\Gamma)\, dt = \oint_\Gamma \frac{a}{y\Phi\left(x,y\right)} dy - \oint_\Gamma \frac{b}{x\Phi\left(x,y\right)} dx.$$

By applying the GREEN formula we obtain

$$\left(\oint_\Gamma \frac{a}{y\Phi(x,y)} dy - \oint_\Gamma \frac{b}{x\Phi(x,y)} dx\right)$$

$$= \iint_{int(\Gamma)} \left(\frac{a}{y}\frac{\partial\left(\frac{1}{\Phi(x,y)}\right)}{\partial x} + \frac{b}{x}\frac{\partial\left(\frac{1}{\Phi(x,y)}\right)}{\partial y}\right) dxdy$$

$$= -\iint_{int(\Gamma)} \frac{\left(\frac{a}{y}\frac{\partial(\Phi(x,y))}{\partial x} + \frac{b}{x}\frac{\partial(\Phi(x,y))}{\partial y}\right)}{\left(\Phi(x,y)\right)^2} dxdy$$

$$\Phi\left(x,y\right) = \frac{c}{a+b}xy + \frac{y^a}{x^b} \Rightarrow \frac{\partial\left(\frac{c}{a+b}xy + \frac{y^a}{x^b}\right)}{\partial y} = \frac{1}{x^b y(a+b)}\left(a^2y^a + aby^a + cxx^by\right)$$

$$\frac{a}{y}\frac{\partial(\Phi(x,y))}{\partial x} + \frac{b}{x}\frac{\partial(\Phi(x,y))}{\partial y} = \frac{a}{y}\frac{\partial\left(\frac{c}{a+b}xy + \frac{y^a}{x^b}\right)}{\partial x} + \frac{b}{x}\frac{\partial\left(\frac{c}{a+b}xy + \frac{y^a}{x^b}\right)}{\partial y}$$

$$\frac{a}{y}\left(-\frac{1}{x^{b+1}(a+b)}\left(b^2y^a + aby^a - cxx^by\right)\right) + \frac{b}{x}\left(\frac{1}{x^by(a+b)}\left(a^2y^a + aby^a + cxx^by\right)\right) = c.$$

So $\int_0^T \operatorname{div}(\Gamma)\, dt = -\iint_{int(\Gamma)} \frac{c}{\left(\Phi(x,y)\right)^2} dxdy$,

where $int\left(\Gamma\right)$ denotes the interior of Γ.

As $c \neq 0$, then $\int_0^T \operatorname{div}(\Gamma)\, dt$ is nonzero. This complete the proof of Theorem 28.

2.3 Applications

As applications of our theorems, we present three systems of kind of Kolmogorov systems as follow :

Example 29 *In the model* (2.2), *we take* $a = b = c = d = \alpha = 1$,

$P(y) = y^2, Q(x) = x^2$ and
$U(x,y) = \left(\left((x-2)^2 + (y-3)^2 - 1\right)^2 - (x-2)^2(y-3) + 4(x-2)(y-3)^2\right)$, then we obtain

$$\begin{cases} x' = x\left(\begin{array}{l} \left(y^2+xy+1\right)\left(\left((x-2)^2+(y-3)^2-1\right)^2-(x-2)^2(y-3)+4(x-2)(y-3)^2\right) \\ -y\left(4x^2y-13x^2-16xy+56x+4y^3-36y^2+120y-156\right)) \end{array} \right) \\ y' = y\left(\begin{array}{l} \left(x^2+xy+1\right)\left(\left((x-2)^2+(y-3)^2-1\right)^2-(x-2)^2(y-3)+4(x-2)(y-3)^2\right) \\ +x\left(4x^3-24x^2+4xy^2-26xy+86x-8y^2+56y-120\right)) \end{array} \right) \end{cases} \tag{2.5}$$

We remark that this system satisfy the conditions of Theorem 26, hence the system (2.5) possess two limit cycles represented by the curve
$U(x,y) = \left(\left((x-2)^2 + (y-3)^2 - 1\right)^2 - (x-2)^2(y-3) + 4(x-2)(y-3)^2\right) = 0$. See Figure 2.1

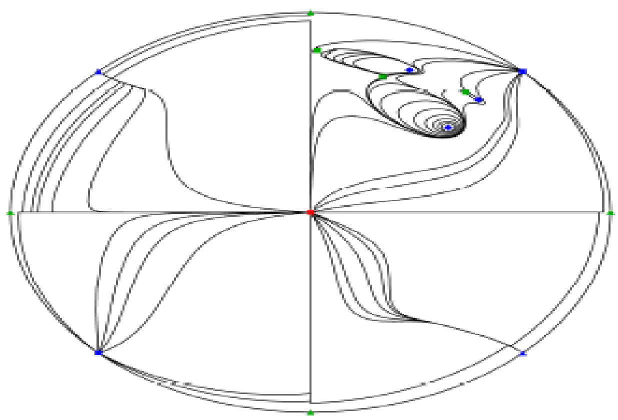

Figure 2.1 : The phase portrait in the Poincare disc of the polynomial differential system (2.5) which present two limit cycles.

Example 30 *Let $a = b = 1$, and*

$U\left(x, y\right) = ((x-2)^2 + (y-2)^2 - 1)^2 - (x-2)^2(y-2) - (x-2)(y-2)^2$,
then the differentiel system (2.3) becomes

$$\begin{cases} x' = x(((x-2)^2 + (y-2)^2 - 1)^2 - (x-2)^2(y-2) - (x-2)(y-2)^2 \\ \qquad -xy^2(4x^2y - 9x^2 - 18xy + 40x + 4y^3 - 24y^2 + 64y - 68)) \\ y' = y(((x-2)^2 + (y-2)^2 - 1)^2 - (x-2)^2(y-2) - (x-2)(y-2)^2 \\ \qquad +yx^2(4x^3 - 24x^2 + 4xy^2 - 18xy + 64x - 9y^2 + 40y - 68)) \end{cases} \tag{2.6}$$

It is easy to verify that all conditions of Theorem 27 are satisfied. Then system (2.6) has three limit cycles represented by the curve
$U\left(x, y\right) = ((x-2)^2 + (y-2)^2 - 1)^2 - (x-2)^2(y-2) - (x-2)(y-2)^2 = 0$. See figure 2.2

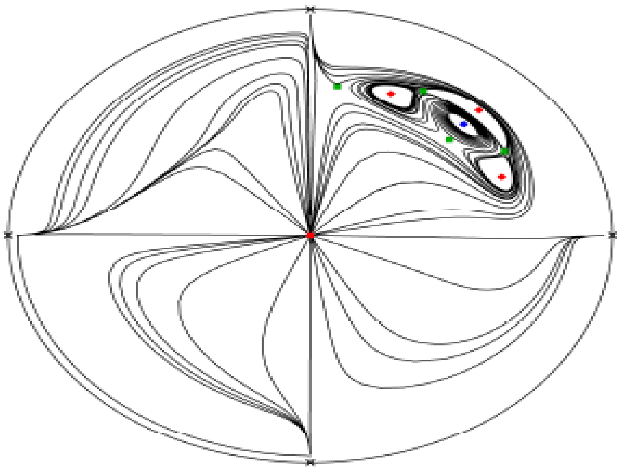

Figure 2.2 : The phase portrait in the Poincare disc of the polynomial differential system (2.6) which present three cycle limit.

Example 31 *If we take $a = b = 1, c = 2$ and $U(x, y) = 2x^2 - 10x + 3y^2 - 12y + 20$,*

then the differential system (2.4) reads

$$\begin{cases} x' = x\left(\left(2x^2 - 10x + 3y^2 - 12y + 20\right) - \frac{1}{x}y^2\left(x^2+1\right)(6y-12)\right) \\ y' = y\left(\left(2x^2 - 10x + 3y^2 - 12y + 20\right) + y\left(x^2+1\right)(4x-10)\right) \end{cases} \tag{2.7}$$

It is easy to verify that all conditions of Theorem 28 are satisfied. We conclude that the system (2.7) has one limit cycle represented by the curve

$2x^2 - 10x + 3y^2 - 12y + 20 = 0$. See Figure 2.3

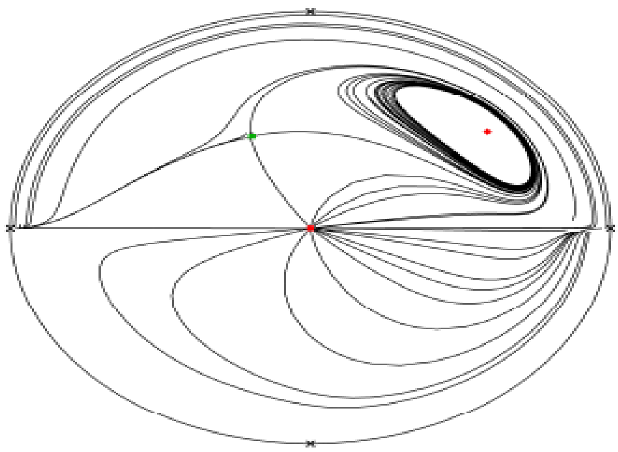

Figure 2.3 :The phase portrait in the Poincare disc of the polynomial differential system (2.7) which present one cycle limit.

Chapter 3

A class of integrable Kolmogorov systems with non-algebraic limit cycle

3.1 Introduction

In biology, the first model with a limit cycle is the one introduced in 1936 by the Russian mathematician Andrei Kolmogorov (1903-1987) which bears his name and has the following form:

$$\begin{cases} x' = xf(x,y) \\ y' = yg(x,y) \end{cases} \tag{3.1}$$

where f and g are functions of class C^1 on an open set of $\mathbb{R}^2$ in $\mathbb{R}$.

More specifically, in the following we introduce a Kolmogorov models whish widely used in ecology to describe the interaction between two populations, and a limit cycle of system (3.1) is an isolated periodic orbit and it is said to be algebraic if it is contained in the zero set of an algebraic curve, otherwise it is called non-algebraic, and it also corresponds to an equilibrium state of the system.There are works which are interested in the study of limit cycles for Kolmogorov type systems, see for example ([50], [52], [61], [63], [64]), but very few concern limit cycles expressed in an explicit way. The only known limit cycles were algebraic, see for example the work of A. Bendjeddou, A. Berbache and R. Cheurfa [11],S. Benyoucef and A. Bendjeddou ([21], [22]),and R. Boukoucha, A. Bendjeddou [26].With respect to algebraic limit cycles.

One of the most important topics in qualitative theory of planar dynamical systems is related to the second part of the unsolved Hilbert 16th problem which consisted to study the maximum number of limit cycles and their relative distributions of the real planar polynomial system of degree n: There is an extensive literature on that subject, most of it deals essentially with detection, number and stability of limit cycles, see for instance ([38], [41]).

We say that a polynomial differential system is integrable if it admits a first integral. We note that the determination of a first integral for a given differential system is not an easy task. The importance of the existence of first integral for a differential system is that it completely determines its phase portrait see ([29], [30], [43]).

Many different methods have been used for proving the existence and nonexistence of limit cycles in simply connected region, for instance see ([12], [45]).

In recent years, existence and nonexistence of limit cycle for some class of Kolmogorov system has been studied, see for instance ([11], [20], [21], [34], [61], [62]).

In the literature, we can find also another interesting but even more difficult problem is to give an explicit expression of a limit cycle.

In 1995, Odani [71] showed that the limit cycle appearing in Van der Pol's equation is not algebraic but without expressing it explicitly. Since 2006, other mathematical researchers have published articles where they give the explicit expression of the non-algebraic limit cycles whose existence they have proven for polynomial systems ([3], [17], [19], [42], [47]).

In this chapter, we are interested in studying the integrability and the existence of limit cycles for a septic Kolmogorov systems of the form:

$$\begin{cases} x' = (x+p)\, P_6\,(x,y) \\ y' = (y+q)\, Q_6\,(x,y) \end{cases} \tag{3.2}$$

where

$$\begin{aligned} P_6\,(x,y) &= \left(\begin{array}{c} x\,(q+y)\left(ax^4+ay^4+2bxy^3+2bx^3y\right) \\ +cx(x+p)(q+y)^2+\left(x^2+y^2\right)\left(-4qy^3+x^4-3y^4\right) \end{array} \right) \\ Q_6\,(x,y) &= \left(\begin{array}{c} y\,(p+x)\left(ax^4+ay^4+2bxy^3+2bx^3y\right) \\ +cy(q+y)(x+p)^2+\left(x^2+y^2\right)\left(4px^3-y^4+3x^4\right) \end{array} \right) \end{aligned}$$

and p, q, a, b, c are real numbers. We show that our system exhibiting an explicit expression of first integral. Moreover, according to certain conditions on the parameters the system admits a non algebraic limit cycle which can be explicitly given. We give an examples to illustrate this proved result.

3.2 The main result

As a main result, we shall prove the following theorem.

Theorem 32 *Consider Kolmogorov differential system* (3.2). *The following statements are holds.*

Statement 1 : System (3.2) has the first integral given by the expression

$$I(x,y)=\frac{\left(x^4+y^4\right)\exp\left(-\int\limits_0^{\arctan\frac{y}{x}}F(s)\,ds\right)}{(x+p)(y+q)}-\int\limits_0^{\arctan\frac{y}{x}}\frac{G(s)}{\exp\left(\int\limits_0^{s}F(w)\,dw\right)}ds \qquad (3.3)$$

Where

$$\begin{aligned}F(s)&=\frac{(3a+a\cos 4s+4b\sin 2s)}{3+\cos 4s}\\ G(s)&=c\end{aligned}$$

Statement 2 *:* if $r_0q+pq>0$ and $a<0$, $c>0$, where ρ_0 should satisfy $\rho_0=\frac{r_0^4}{(r_0q+pq)}$ the system (3.2) has a *non-algebraic limit cycle* implicitly *given*

$$\rho(\theta,\rho_0^*)=\exp\left(a(\theta)+b\arctan(\tan(\theta))^2\right)\left(\rho_0^*+\int_0^\theta c\exp\left(-as-b\arctan(\tan s)^2\right)ds\right)$$

where

$$\rho_0^*=ce^{2\pi a}\frac{\int_0^{2\pi}\exp\left(-b\arctan\left(\tan^2 s\right)-as\right)\,ds}{1-e^{2\pi a}}$$

Moreover, this limit cycle is stable and hyperbolic.

Statement 3 *: if $p=q=0$* the system (3.2) has no limit cycle.

3.3 Proof of the main result

For the demonstration of Theorem 32 , we need the following lemma

Lemma 33 *Consider a planar diffrential system which in polar coordinates has the form:*

$$\begin{cases} r'=F(\theta)H(r,\theta)-\dfrac{\partial H}{\partial\theta}(r,\theta)+G(\theta)\\ \theta'=\dfrac{\partial H}{\partial r}(r,\theta)\end{cases}$$

then this system possess a first integral expressed as

$$W(r,\theta)=H(r,\theta)\exp\left(-\int\limits_0^\theta F(s)\,ds\right)-\int\limits_0^\theta G(s)\exp\left(-\int\limits_0^s F(w)\,dw\right)ds$$

Proof of Lemma 33**:** Let us set

$$U(\theta) = H(r,\theta)\exp\left(-\int_0^\theta F(s)\,ds\right) andV(\theta) = \int_0^\theta G(s)\exp\left(-\int_0^s F(w)\,dw\right)ds$$

The derivatives of U and V with respect to θ are

$$\begin{aligned}\frac{dU}{d\theta} &= \left(\frac{\partial H}{\partial\theta}(r,\theta) - F(\theta)H(r,\theta)\right)\exp\left(-\int_0^\theta F(s)\,ds\right),\\ \frac{dV}{d\theta} &= G(\theta)\exp\left(-\int_0^\theta F(s)\,ds\right),\end{aligned}$$

By replacing the expression of derivatives of U and V with respect to θ in the expression of W, it follows that

$$\frac{\partial W}{\partial\theta}(r,\theta) = \left(\frac{\partial H}{\partial\theta}(r,\theta) - F(\theta)H(r,\theta) - G(\theta)\right)\exp\left(-\int_0^\theta F(s)\,ds\right)$$

By the chain rule, the derivative of W with respect to t is given by following expression,

$$\begin{aligned}\frac{dW}{dt}(r(t),\theta(t)) &= \frac{dW}{dr}(r,\theta)\frac{dr}{dt} + \frac{dW}{d\theta}(r,\theta)\frac{d\theta}{dt}\\ &= \left(\frac{\partial H}{\partial r}(r,\theta)\exp\left(-\int_0^\theta F(s)\,ds\right)\right)\left(F(\theta)H(r,\theta) - \frac{\partial H}{\partial\theta}(r,\theta) + G(\theta)\right)\\ &\quad + \left(\left(\frac{\partial H}{\partial\theta}(r,\theta) - F(\theta)H(r,\theta) - G(\theta)\right)\exp\left(-\int_0^\theta F(s)\,ds\right)\right)\frac{\partial H}{\partial r}(r,\theta)\\ &= 0.\end{aligned}$$

So $W(r,\theta)$ is a first integral of system. This complete the proof of Lemma 33.

Beginning by discussing the singular points of system (3.2).We have

$$xy' - yx' = \left(x^2+y^2\right)\left(x^4+y^4\right)(4pq + 3py + 3qx + 2xy)$$

and any singular point must satisfy

$$\begin{cases}(x+p)P_6(x,y) = 0\\ (y+q)Q_6(x,y) = 0\end{cases}$$

so the singular points are $(0,0)$, $(-p,-q)$ and any point of the form $\left(x, -q\frac{4p+3x}{3p+2x}\right)$ where x is solution of equation $P_6\left(x, -q\frac{4p+3x}{3p+2x}\right) = 0$.

- If $\left(-81cp^2+1024q^4+768apq^3\right)=0$ then $\left(0,-\frac{4}{3}q\right)$ is a singular point.
- If $\left(-81cq^2-1024p^4+768ap^3q\right)=0$ then $\left(-\frac{4}{3}p,0\right)$ is a singular point.
- If $p=q=0$, the unique singular point is $(0,0)$.

Proof of statement (1) of Theorem 32 :

In order to prove the statement(1) of Theorem 32, it must to transform the system to polar coordinates (r,θ) where $x=r\cos\theta$, $y=r\ \sin\theta$, then the system (3.2) becomes

$$\begin{cases} r'=f_1(\theta)r^7+f_2(\theta)r^6+f_3(\theta)r^5+f_4(\theta)r^4+f_5(\theta)r^3+f_6(\theta)r^2+f_7r \\ \theta'=f_8(\theta)r^6+f_9(\theta)r^5+f_{10}(\theta)r^4 \end{cases} \tag{3.5}$$

Where

$$\begin{aligned}
&f_1(\theta)=4b+18\cos 2\theta-2\cos 6\theta-4b\cos 4\theta+5a\sin 2\theta+a\sin 6\theta,\\
&f_2(\theta)=(6p+6aq+4bp)\cos\theta+(6ap-6q+4bq)\sin\theta+(5p+aq-4bp)\cos 3\theta\\
&+(5q-ap+4bq)\sin 3\theta+(3q+ap)\sin 5\theta+(aq-3p)\cos 5\theta,\\
&f_3(\theta)=\begin{pmatrix} c\cos^2\theta\sin^2\theta+apq\cos^4\theta+apq\sin^4\theta \\ +(-4pq+2bpq)\cos\theta\sin^3\theta+(4pq+2bpq)\cos^3\theta\sin\theta \end{pmatrix},\\
&f_4(\theta)=cp\cos\theta+cq\sin\theta-cp\cos 3\theta+cq\sin 3\theta,\\
&f_5(\theta)=cp^2+cq^2+\left(-cp^2+cq^2\right)\cos 2\theta+4cpq\sin 2\theta,\\
&f_6(\theta)=2cp^2q\sin\theta+2cpq^2\cos\theta,\\
&f_7(\theta)=cp^2q^2,\\
&f_8(\theta)=5\sin 2\theta+\sin 6\theta,\\
&f_9(\theta)=18q\cos\theta+18p\sin\theta+3q\cos 3\theta+3q\cos 5\theta-3p\sin 3\theta+3p\sin 5\theta,\\
&f_{10}(\theta)=pq(3+\cos 4\theta).
\end{aligned}$$

The differential system (3.5) where

$$r^4\left(f_8(\theta)r^2+f_9(\theta)r+f_{10}(\theta)\right)\neq 0,$$

can be written as the equivalent differential equation

$$\frac{dr}{d\theta}=\frac{f_1(\theta)r^6+f_2(\theta)r^5+f_3(\theta)r^4+f_4(\theta)r^3+f_5(\theta)r^2+f_6(\theta)r+f_7(\theta)}{r^3\left(f_8(\theta)r^2+f_9(\theta)r+f_{10}(\theta)\right)}. \tag{3.6}$$

We use the transformation of variable $\rho=\frac{\left((r\cos\theta)^4+(r\sin\theta)^4\right)}{((r\cos\theta)+p)((r\sin\theta)+q)}$ the equation (3.6) becomes a linear differential equation

$$\frac{d\rho}{d\theta}=F(\theta)\rho+G(\theta), \tag{3.7}$$

where $F(\theta)=\frac{(3a+a\cos 4\theta+4b\sin 2\theta)}{\cos 4\theta+3}$ and $G(\theta)=c$.

The general solution of this linear equation is

$$\rho(\theta) = \exp\left(\int_0^\theta F(s)\,ds\right)\left(h + \int_0^\theta G(s)\exp\left(-\int_0^s F(w)\,dw\right)ds\right). \tag{3.8}$$

Where $h \in \mathbb{R}$, and the functions $F(s)$, $G(s)$ are defined in theorem.

Consequently, the implicit solution of equation (3.8) is given by

$$H(r,\theta) - \exp\left(\int_0^\theta F(s)\,ds\right)\left(h + \int_0^\theta G(s)\exp\left(-\int_0^s F(w)\,dw\right)ds\right) = 0 \tag{3.9}$$

Where

$$H(r,\theta) = \frac{(r\cos\theta)^4 + (r\sin\theta)^4}{((r\cos\theta)+p)((r\sin\theta)+q)}$$

By passing to Cartesian coordinates, we deduce the first integral is

$$I(x,y) = \frac{\left(x^4+y^4\right)}{(x+p)(y+q)\exp\left(a\arctan\frac{y}{x} + b\arctan\left(\frac{y}{x}\right)^2\right)} - \int_0^{\arctan\frac{y}{x}} c\exp\left(-as - b\arctan(\tan s)^2\right)ds.$$

Hence the statement(1) of Theorem is proved.

Proof of statement (2) of Theorem 32**:**

Notice that system (3.2) has a periodic orbit if and only if the equation (3.8) has a strictly positive 2π-periodic solution. This, moreover, is equivalent to the existence of a solution of equation (3.8) that satisfies $r(2\pi, r_0) = r(0, r_0)$ and $r(0, r_0) > 0$, with $r_0 > 0$, for this, it suffices to verify that $\rho(2\pi, \rho_0) = \rho(0, \rho_0)$ and $\rho_0 > 0$ for any θ in $[0; 2\pi]$.

For $\theta = 0$,we remark that the solution $\rho(\theta, \rho_0)$ of the differential equation ,such as $\rho(0, \rho_0) = \rho_0$, corresponds to the value

$$k = \rho_0 = \frac{r_0^4}{(r_0 q + pq)}$$

and as we have $r_0 q + pq > 0$, then $\rho_0 > 0$.

We replace the value of k in the general solution of equation (3.8) we obtain

$$\rho(\theta,\rho_0) = \exp\left(a(\theta) + b\arctan(\tan(\theta))^2\right)\left(\rho_0 + \int_0^{(\theta)} c\exp\left(-as - b\arctan(\tan s)^2\right)ds\right)$$

The condition of the periodic solution with 2π-periodic starting at $\rho(0, \rho_0) = \rho_0 > 0$ is $\rho(2\pi, \rho_0) = \rho(0, \rho_0)$.

For $\theta = 2\pi$, we have

$$\rho(2\pi,\rho_0) = e^{2\pi a}\left(\rho_0 + c\int_0^{2\pi}\exp\left(-b\arctan\left(\tan^2 s\right) - as\right)ds\right),$$

and if $ac < 0$, we obtain

$$\rho_0^* = ce^{2\pi a} \frac{\int_0^{2\pi} \exp\left(-b \arctan\left(\tan^2 s\right) - as\right) ds}{1 - e^{2\pi a}} > 0.$$

We substituted the value of ρ_0^* in the general solution of equation (3.8) we obtain

$$\rho(\theta, \rho_0^*) = \exp\left(a(\theta) + b\arctan(\tan(\theta))^2\right)\left(\rho_0^* + \int_0^\theta c\exp\left(-as - b\arctan(\tan s)^2\right) ds\right) \tag{3.10}$$

Strict positivity of $\rho(\theta, \rho_0^*)$ for all θ in $[0; 2\pi]$

$$\rho(\theta, \rho_0^*) = \exp\left(a(\theta) + b\arctan(\tan(\theta))^2\right)\left(\rho_0^* + \int_0^\theta c\exp\left(-as - b\arctan(\tan s)^2\right) ds\right).$$

If ($a < 0, c > 0$), $\rho_0^* > 0$ and also $\rho(\theta, \rho_0^*) > 0$.Therefore $\rho(\theta, \rho_0^*)$ is strictly positive.

In order to determine the hyperbolicity of these periodic orbit, see for instance [12], we consider the equation (3.8), and we introduce the Poincaré return map $\lambda \to \Pi(2\pi, \lambda) = \rho(2\pi, \lambda)$ Therefore, a limit cycles of system (3.2) are hyperbolic if and only if

$$\rho(2\pi, \rho_0) = \exp\left(a(2\pi) + b\arctan(\tan(2\pi))^2\right)\left(\rho_0 + \int_0^{(2\pi)} c\exp\left(-as - b\arctan(\tan s)^2\right) ds\right)$$

$$\left.\frac{d\Pi(2\pi,\lambda)}{d\lambda}\right|_{\lambda=\rho_0} = \frac{d\left(\exp\left(a(2\pi)+b\arctan(\tan(2\pi))^2\right)\left(\rho_0+\int_0^{(2\pi)} c\exp\left(-as-b\arctan(\tan s)^2\right)ds\right)\right)}{d\rho_0}$$
$$= \exp\left(\int_0^{2\pi} F(s)\, ds\right) = e^{2\pi a} \neq 1,$$

the limit cycle is hyperbolic and it is stable since $a < 0$.

Proof of statement (3) of Theorem 32**:** If $p = q = 0$ then

$$\rho(\theta) = \frac{(r\cos\theta)^4 + (r\sin\theta)^4}{(r\cos\theta)(r\sin\theta)}. \tag{3.11}$$

So the solution $r(\theta)$ given by solving equation (3.11) we obtain

$$-\frac{1}{4}r^2\left(-r^2\cos 4\theta - 3r^2 + 4\rho\cos\theta\sin\theta\right) = 0,$$

is

$$r_1(\theta) = 0,$$
$$r_2(\theta) = -\tfrac{2}{\cos 4\theta + 3}\sqrt{\rho(\cos\theta\sin\theta)(\cos 4\theta + 3)},$$
$$r_3(\theta) = \tfrac{2}{\cos 4\theta + 3}\sqrt{\rho(\cos\theta\sin\theta)(\cos 4\theta + 3)},$$

if $p = q = 0$ then the origin is the unique singular point , on the other hand $r_1(0) = r_2(0) = r_3(0) = 0$. We notice that the two axes $x = 0$ and $y = 0$ are invariant curves, so if there is a limit cycle surrounding the origin which intersects the two axes, therefore there is no limit cycle. This complete the proof of theorem 32.

3.4 Applications

The following examples are given to illustrate our result.

3.4.1 Septic Kolmogorov system with non algebraic stable limit cycle

Example 34 *Let $a = -1$ and $b = c = p = q = 1$, then the system* (3.2) *becomes*

$$\begin{cases} x' = (x+1)\begin{pmatrix} x(1+y)\left(-x^4 - y^4 + 2xy^3 + 2x^3y\right) \\ +x(x+1)(1+y)^2 + \left(x^2+y^2\right)\left(-4y^3 + x^4 - 3y^4\right) \end{pmatrix} \\ y' = (y+1)\begin{pmatrix} y(1+x)\left(-x^4 - y^4 + 2xy^3 + 2x^3y\right) \\ +y(1+y)(x+1)^2 + \left(x^2+y^2\right)\left(4x^3 - y^4 + 3x^4\right) \end{pmatrix} \end{cases} \tag{3.12}$$

The system (3.12) has the first integral

$$I(x,y) = \frac{\left(x^4+y^4\right)\exp\left(-\int_0^{\arctan\frac{y}{x}} F(s)\,ds\right)}{(x+1)(y+1)} - \int_0^{\arctan\frac{y}{x}} \frac{G(s)}{\exp\left(\int_0^s F(w)\,dw\right)} ds.$$

Where

$$\begin{aligned} F(s) &= \frac{(-3 - \cos 4s + 4\sin 2s)}{3+\cos 4s}. \\ G(s) &= 1. \end{aligned}$$

And it has five singular points: $(0,0)$ is an unstable node, $(0.0848055, -1.34225)$ is an unstable focus, $(-1.41584, 1.47036)$ is an unstable focus, $(-1.90897, -2.11129)$ is a stable focus, $(-1,-1)$ is a saddle point.

We study points (u_*, v_*) at infinity of system (3.12). Using (1.4), we get the Poincaré compactification in the local chart (u_1, v_1).

Let $x = \frac{1}{v}$, $y = \frac{u}{v}$, $v > 0$, then the system (3.12) becomes

$$\begin{cases} u' = \left(2u + 3v + 3uv + 4v^2\right)\left(u^6 + u^4 + u^2 + 1\right) \\ v' = -v(v+1)\begin{pmatrix} -3u^6 - 4u^5v - u^5 - u^4v - u^4 - 2u^3v + u^2v^3 + u^2v^2 \\ +3u^2 + 2uv^4 + 2uv^3 + 2uv - u + v^5 + v^4 - v + 1 \end{pmatrix} \end{cases} \tag{3.13}$$

Let $v = 0$, the infinite singular points of the system (3.13) in chart (u_1, v_1) should satisfy $(2u)\left(u^6 + u^4 + u^2 + 1\right) = 0$.

$(u', v')|_{(0,0)} = 0$, the origin is the singular point of system (3.13)

$$Df(0,0) = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix}$$

Which is saddle, with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$.

In the local chart (U_2, V_2) the expression for $n(X)$ is

$$\begin{cases} u' = -\left(2u + 3v + 3uv + 4v^2\right)\left(u^6 + u^4 + u^2 + 1\right) \\ v' = -v\,(v+1)\begin{pmatrix} 3u^6 + 4u^5v - u^5 - u^4v + 5u^4 + 6u^3v + u^2v^3 + u^2v^2 \\ +u^2 + 2uv^4 + 2uv^3 + 2uv - u + v^5 + v^4 - v - 1 \end{pmatrix} \end{cases} \tag{3.14}$$

Let $v = 0$, the infinite singular points of the system (3.14) in chart (u_1, v_1) should satisfy $-(2u)\left(u^6 + u^4 + u^2 + 1\right) = 0$.

$(u', v')|_{(0,0)} = 0$, the origin is the singular point of system (3.14)

$$Df\,(0,0) = \begin{pmatrix} -2 & -3 \\ 0 & 1 \end{pmatrix}$$

which is saddle, with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -2$.

The expression of limit cycle is implicitly given by

$$\frac{(r\cos\theta)^4 + (r\sin\theta)^4}{(1 + r\cos\theta)(1 + r\sin\theta)} = \exp\left(-\theta + \arctan\left(\tan\left(\theta\right)\right)^2\right)\left(\rho_0^* + \int_0^\theta \exp\left(s - \arctan\left(\tan^2 s\right)\right)\,ds\right)$$

where

$$\rho_0^* = e^{-2\pi}\frac{\int_0^{2\pi}\exp\left(-\arctan\left(\tan^2 s\right) + s\right)\,ds}{1 - e^{-2\pi}} = 0.623\,29$$

Clearly, the system has one limit cycle as it is shown on the phase portrait of Poincaré disc in figure 3.1.

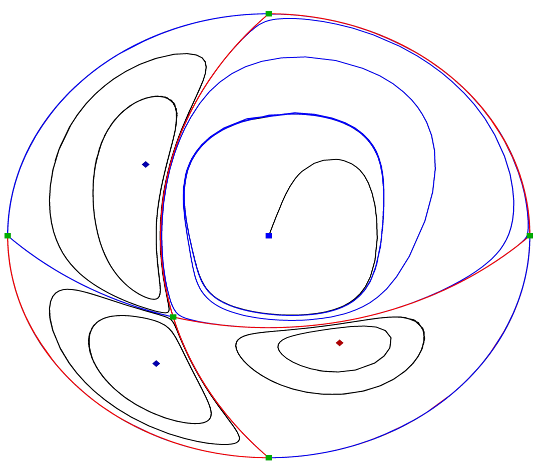

Figure 3.1: Phase potrait of the system (3.12) on the Poincare disc.

3.4.2 Septic Kolmogorov system with non algebraic unstable limit cycle

Example 35 *Let $a = b = c = 1, p = 2, q = -1$, the system* (3.2) *becomes*

$$\begin{cases} x' = (x+2)\begin{pmatrix} x(-1+y)\left(x^4+y^4+2xy^3+2x^3y\right) \\ +x(x+2)(-1+y)^2+\left(x^2+y^2\right)\left(4y^3+x^4-3y^4\right)\end{pmatrix} \\ y' = (y-1)\begin{pmatrix} y(2+x)\left(x^4+y^4+2xy^3+2x^3y\right) \\ +y(-1+y)(x+2)^2+\left(x^2+y^2\right)\left(8x^3-y^4+3x^4\right)\end{pmatrix}\end{cases} \tag{3.15}$$

The system (3.15) *has the first integral*

$$I(x,y) = \frac{\left(x^4+y^4\right)\exp\left(-\int_0^{\arctan\frac{y}{x}} F(s)\,ds\right)}{(x+2)(y-1)} - \int_0^{\arctan\frac{y}{x}} \frac{G(s)}{\exp\left(\int_0^s F(w)\,dw\right)}ds$$

Where

$$\begin{aligned} F(s) &= \frac{(3+\cos 4s+4\sin 2s)}{3+\cos 4s}. \\ G(s) &= 1. \end{aligned}$$

And it has five singular points: $(0,0)$ is an unstable node, $(-2.862983, -3.049953)$ is an stable focus, $(-3.405015, 3.405015)$ is an stable focus, $(-0.182396, 1.323842)$ is a unstable focus, $(-2,1)$ is a saddle point.

The expression of limit cycle is implicitly given by

$$\frac{(r\cos\theta)^4+(r\sin\theta)^4}{(2+r\cos\theta)(r\sin\theta-1)} = \exp\left(-\theta+\arctan\left(\tan(\theta)\right)^2\right)\left(\rho_0^*+\int_0^\theta \exp\left(s-\arctan\left(\tan^2 s\right)\right)\,ds\right),$$

where

$$\rho_0^* = e^{2\pi}\frac{\int_0^{2\pi}\exp\left(-\arctan\left(\tan^2 s\right)-s\right)\,ds}{1-e^{2\pi}} = 0.623\,29$$

Clearly, the system (3.15) has one limit cycle as it is shown on the phase portrait of Poincaré disc in figure 3.2.

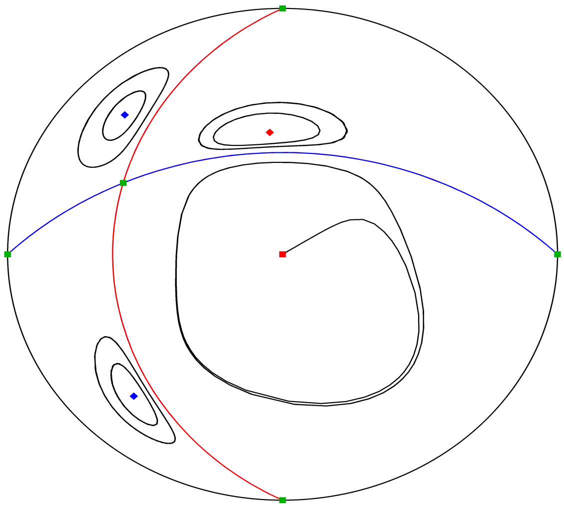

Figure 3.2 : Phase potrait of the system (3.15) on the Poincare disc.

3.4.3 Non existence of limit cycle of Septic Kolmogorov system

Example 36 *Let* $a = b = 1, c = -1, p = q = 0$, *the system* (3.2) *becomes*

$$\begin{cases} x' = x\left(xy\left(x^4 + y^4 + 2xy^3 + 2x^3y\right) - x^2y^2 + \left(x^2 + y^2\right)\left(x^4 - 3y^4\right)\right) \\ y' = y\left(yx\left(x^4 + y^4 + 2xy^3 + 2x^3y\right) - x^2y^2 + \left(x^2 + y^2\right)\left(-y^4 + 3x^4\right)\right) \end{cases} \quad (3.16)$$

For the system (3.16), we notice that the two axes $x = 0$ and $y = 0$ are invariant curves, so if there is a limit cycle surrounding the origin which intersects the two axes, therefore there is no limit cycle. See figure 3.3.

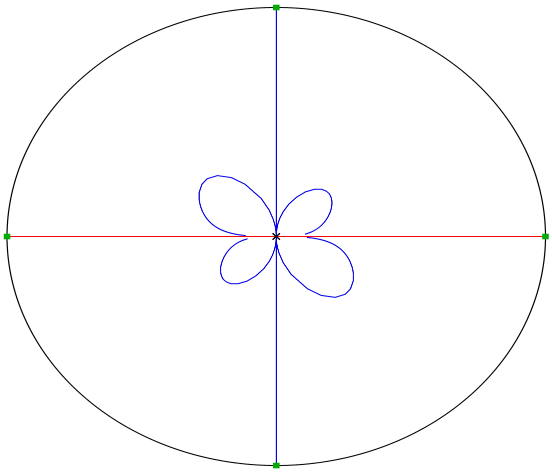

Figure 3.3 : Phase potrait of the system (3.16)
on the Poincar disc.

Conclusion and persepectives

One of the most important topics in qualitative theory of planar dynamical systems is related to the second part of the unsolved Hilbert 16th problem which consisted to study the existence, the non-existence and the maximum number of limit cycles which shown to model many real-world phenomena.

In this work we were interested in the qualitative study of Kolmogorov differential systems. Firstly, we presented some basics concerning the qualitative theori differential systems, especially planar differential systems. Secondly, we studied some classes of differential system of order n of the next Kolmogorov type

$$\begin{cases} x' = x\left(R\left(x,y\right)U - y\Phi\left(x,y\right)U_y\right) \\ y' = y\left(S\left(x,y\right)U + x\Phi\left(x,y\right)U_x\right) \end{cases}$$

Where $R\left(x,y\right), S\left(x,y\right)$ and $\Phi\left(x,y\right)$ are polynomial functions, and show that these systems admit exactly the limits Curve components as hyperbolic term cycles if certain conditions on the system parameters are saturated.

Thirdly, we have studied the integrability and the existence of a non-algebraic limit cycle of a following class of Kolmogorov differential systems of degree septic:

$$\begin{cases} x' = (x+p)\begin{pmatrix} x\left(q+y\right)\left(ax^4+ay^4+2bxy^3+2bx^3y\right) \\ +cx(x+p)(q+y)^2+\left(x^2+y^2\right)\left(-4qy^3+x^4-3y^4\right) \end{pmatrix} \\ y' = (y+q)\begin{pmatrix} y\left(p+x\right)\left(ax^4+ay^4+2bxy^3+2bx^3y\right) \\ +cy(q+y)(x+p)^2+\left(x^2+y^2\right)\left(4px^3-y^4+3x^4\right) \end{pmatrix} \end{cases}$$

where p, q, a, b, c are real numbers, moreover, we have determined their explicit expression.

In perspective, it is convenient to apply our method of study on Kolmogorov type systems where the degree less than five. Knowing that there are results of existence of limit cycles for classes of quartic and cubic system differential planaire, but to our knowledge none of these works gives the limit cycle in the case of existence in an explicit way, as we did it in this thesis.

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ملخص

الهدف من هذه الأطروحة هو الدراسة النوعية لبعض أصناف أنظمة كولموغوروف التفاضلية. أولاً ، قدمنا بعض المفاهيم الأساسية اللازمة للدراسة النوعية للأنظمة الديناميكية. في الجزء الثاني ، استطعنا تقديم فئات من أنظمة تفاضلية من نوع كولموغوروف لنمذجة ظاهرة حيوية في الطبيعة ، حيث درسنا وجود وعدد دورات الحد، كما استطعنا تقديم العبارة الجبرية الصريحة لدورات الحد . و في الفصل الثالث درسنا قابلية التكامل، ووجود دورة الحد غير جبرية لفئة من أنظمة تفاضلية لكولموغوروف من الدرجة السابعة، وقدمنا العبارة الغير جبرية لدورة الحد. و في آخر كل فصل عرضنا بعض الأمثلة التوضيحية لنتائج المتحصل عليها.

الكلمات المفتاحية : نظام كولموغوروف التفاضلي ، دورة الحد ، دورة الحد غير الجبرية ، الحل الدوري ، التكامل ، التكامل الأول.

RESUME

L'objectif de cette thèse est l'étude qualitative de certaines classes de systèmes différentiels de type Kolmogorov. Dans le premier chapitre, nous avons introduit des concepts de base nécessaires à l'étude qualitative des systèmes dynamiques. Dans le deuxième chapitre, nous avons présenté notre premier résultat sur la construction d'une classe de systèmes différentiels de type Kolmogorov, avec cycles limites algébriques de plus on donne l'expression explicit de ces cycles limites. Dans le chapitre trois, nous avons présenté notre deuxième résultat sur l'intégrabilité d'une classe de systèmes différentiels de degré sept de type Kolmogorov avec cycle limite non algébrique et on donne son expression explicite. A la fin de chaque chapitre, on donne des exemples d'illustration .

Mots clés : Système différentiel de Kolmogorov, Cycle limite, cycle limite non algébrique, solution périodique, intégrabilité, intégrale première.

ABSTRACT

The objective of this thesis is the qualitative study of some classes of Kolmogorov differential systems. In the first chapter, we introduced some of the basic concepts needed for the qualitative study of dynamics systems. In the second chapter, we presented our first result on the construction of a class of differential systems of Kolmogorov type, with algebraic limit cycles and we give the explicit expression of these limit cycles. In chapter three,we presented our second result on the integrability of a class of Kolmogorov type differential systems of degree seven with a non-algebraic limit cycle and we give its explicit expression. At the end of each chapter, examples of illustrations are given.

Keywords : Kolmogorov differential system, Limit cycle, non algebraic limit cycle, periodic solution, integrability, first integral.