Democratic and Popular Republic of Algeria
Ministry of Higher Education and Scientific Research
FERHAT ABBAS UNIVERSITY-SETIF 1-

FACULTY OF SCIENCES
DEPARTEMENT OF MATHEMATHICS

THESIS

Presented by:

MOUSSAOUI NOUHA

To obtain the title of Doctorate LMD in Sciences

Option

Optimization and control

THEME

Interior-point methods for convex quadratic optimization based on modified search directions.

Supported on:

In front of the jury composed of :

Chairman:	**Mrs. H. ROUMILI**	**Prof.**	**U. F. A. Sétif. 1**
Supervisor:	**Mr. M. ACHACHE**	**Prof.**	**U. F. A. Sétif. 1**
Examiners:	**Mr. M. ZERGUINE**	**Prof.**	**U. M. B. Batna. 2**
	Mrs. Z. KEBAILI	**M.C.A**	**U. F. A. Sétif. 1**

Academic year: 2022/2023

Acknowledgments

I am grateful to a number of people who have supported me during my studies and the development of this work and it is my pleasure to mention them here.
Firstly, I would like to express my sincere gratitude to my advisor Prof. **Mohamed Achache** for the continuous support of my Ph.D study and related research, for his patience, motivation, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis. I could not have imagined having a better advisor and mentor for my Ph.D study. I also thank the other members of my thesis committee: Professor **Hayet Roumili**, Professor **Zehira Kebaili** and Professor **Mohamed Zerguine**.
Also I would like to thank my friends in **Laboratoire de Mathématiques Fondamentales et Numériques**. Last but not the least, I would like to thank my family: my parents and to my brothers and sister, also to my husband for supporting me spiritually throughout writing this thesis and my life in general.

Dedication

There are a number of people without whom this thesis might not have been written, to whom I owe a lot.
Thanks most of all to my inspiring parents, without whom my success would not be possible.
To my husband who had been an inspiration with his patience, support and uninterrupted encouragement every single day throughout my journey And my friends who encourage and support me.
To my brothers and sister, to my big family and step family thank you.
My friends " **Rima, Souad, Ilhem, Aicha, Bouthaina, Bouchra, Rania**".
All the people in my life who touched my heart.

Contents

List of Figures

List of Tables

Abstract

In this thesis we are interested with theoretical and numerical study of convex quadratic optimization. For this purpose, we have introduced two methods of interior point. The first depends on the classical central-path with modified Newton search directions. Meanwhile, the second one is based on the weighted path and also new search directions. In the two cases, we have proved that the corresponding algorithms are defined and converge locally quadratically. In addition, this algorithm has the best known polynomial complexity. Finally, this study is followed by some numerical experiments for evaluation.

Keywords: Convex quadratic optimization,Interior-point methods, Short-step method, Polynomial complexity.

List of publications

Paper 1

M. Nouha, M. Achache. A Weighted-path Following Interior-point Algorithm for Convex Quadratic Optimization Based on Modified Search Directions. **STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING,** Vol. 10, June 2022, pp 873-889.

Glossary of Notation

Problem Classes

MP	:	Mathematical programming.
(CQO)	:	Convex quadratic optimization.
(LO)	:	Linear optimization.
(P)	:	Primal standard format.
(D)	:	Dual problem of (P).
(P_μ)	:	Primal barrier problem.
(D_μ)	:	Dual barrier problem.
AET	:	Algebraically equivalent transformation.
IP	:	Interior-Point.
IPA	:	Interior-Point Algorithm.
IPC	:	Interior-Point Condition.
IPMs	:	Interior-Point Methods.
K.K.T	:	Krush-Kuhn-Tucker.

Spaces

$\mathbb{R}$	:	The set of real numbers
$\mathbb{R}^n$	:	The real n-dimensional space.
$\mathbb{R}^n_+$	:	The nonnegative orthant of $\mathbb{R}^n$.
$\mathbb{R}^n_{++}$	:	The positive orthant of $\mathbb{R}^n$.
$\mathbb{R}^{n\times n}$	:	The set of all $n \times n$ squared matrices.

Vectors

x_i	:	The i-th component of x.
x^T	:	$(x_1, \ldots, x_n)$ the transpose of a vector x, with components x_i
$x \geq 0$	=	$x_i \geq 0$, $\forall i$.
$x > 0$	:	$x_i > 0,\ \forall i$.
e	=	$(1, \ldots, 1)^T \in \mathbb{R}^n$.
$\lvert x \rvert$	=	Absolute value of a vector $x \in \mathbb{R}^n$. $\lvert x\rvert = (\lvert x_i\rvert), i = 1, \ldots, n$.
$x^T y$	=	$\displaystyle\sum_{i=1}^{n} x_i y_i$ the standard inner product in $\mathbb{R}^n$.
xy	=	$(x_1 y_1, \ldots, x_n y_n)^T$ Hadamard product.
$\log(x)$	=	$(\log(x_1), \cdots, \log(x_n))^T,\ \ x > 0$.
$\frac{x}{y}$	=	$\left(\frac{x_1}{y_1}, \cdots, \frac{x_n}{y_n}\right)^T, y \neq 0$.
$\sqrt{x}$	=	$(\sqrt{x}_1, \cdots, \sqrt{x}_n)^T$ the square of a vector x, with components x_i.
x^{-1}	=	$(x_1^{-1}, \ldots, x_n^{-1})^T$
$\lVert x \rVert$	:	denote the Euclidean norm $(x^T x)^{1/2}$.
$\lVert x \rVert_\infty$	:	denote the Maximum norm $(\max_i \lvert x_i \rvert)$.

Matrices

$X \succeq 0$	:	Means X is positive semidefinite.
$X \succ 0$	:	Means X is positive definite.
I	:	Order identity matrix .
$0_{n \times n}$	:	The null matrix of type (n, n) .
A^T	:	The transposed matrix of A.
A^{-1}	:	The inverse of a regular matrix A.
$tr\,(A)$	=	$\displaystyle\sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i(A)$, (the trace of a matrix $A \in \mathbb{R}^{n \times n}$).
$\lVert A \rVert_2$	=	$\sqrt{\rho\,(A^T A)}$, (the spectral norm of A).
$D(x)$	=	diagonal matrix whose diagonal elements are the components of the vector x.
X	=	$diag(x)$, denote the diagonal matrix whose diagonal elements are the components of$x$.
Z	=	$diag(z)$, denote the diagonal matrix whose diagonal elements are the components of$z$.

Functions

$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \cdots, \frac{\partial f}{\partial x_n}(x)\right)^T$	:	the gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$.
$\nabla^2 f(x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right)_{1 \leq i,j \leq n}$	:	the Hessian matrix of $f : \mathbb{R}^n \to \mathbb{R}$.
$\psi(t)$	:	a kernel function.

Introduction

The CQO problems are an interesting class of nonlinear convex programming which have been proven to be useful in many domains of applied mathematics and engineering. Also the CQO includes the standard linear optimization (LO) [8]. Further, it can be seen as a special case of general symmetric conic optimization problems (see e.g.,[25, 29]). There are many direct and iterative methods for finding optimal solutions of CQOs. Beside the active set methods [43], a classical Wolfe like simplicial algorithm for solving CQOs has in the worst case an exponential complexity, the feasible path-following interior-point methods (IPMs) gained much more attention then others due to their highly numerical efficiency and their polynomial complexity (see e.g.,[40, 44, 45]). These methods used the central-path as a guideline to follow the central-path for reaching an optimal solution of CQO. However, their derived algorithms require that the starting point must be strictly feasible and on the central-path. Still, it is uneasy task to find a perfectly centered initial point. Primal-dual path-following algorithms are among the most effective methods for solving optimization problems.

In the last years, the determination of new search directions has a great influence to improving the complexity and the numerical efficiency of these methods. Among the proposals, the technique of algebraic equivalent transformations (AET) becomes a widely used tool for offering new search directions in interior-point methods. This technique was first introduced by Darvay's [22] for linear optimization (LO) where he offered a search direction induced by the univariate square root function. His full-Newton short-step IPA has the best well-known iteration bound, namely, $\mathcal{O}\left(\sqrt{n}\log\frac{n}{\epsilon}\right)$. Since then, this idea was extended successfully to convex quadratic and semidefinite optimization (SDO) problems, monotone and sufficient linear complementarity problems (LCP) (see e.g. [1, 4, 5, 8, 20, 31]). Therefore, it is worth to analyze other cases when starting points are not centered. It is well-known that with every algorithm which follows the central-path, we can associate a target sequence on the central-path. This idea leads to the concept of target-following methods introduced earlier by Jansen et al [31]. Weighted-path following IPMs can be viewed as a special case of them and which are used as an alternative to remedy this drawback [3, 7, 10, 23, 40]. In [18], Darvay proposed a weighted-path following IPMs for solving LO. Later, Achache [6], Mansouri et al. [34] and Wang et al. [38, 40] extended successfully Darvay's algorithm for solving monotone LCP, $P_*(\kappa)-$LCP over symmetric cones, monotone mixed and horizontal LCP, respectively. The relevance of the technique of AETs for the central-path equation has been treated in [19, 38] and subsequently in references [1, 5].

Later, different univariate functions are investigated for several optimization problems (see e.g. [20, 21, 38, 40, 46]). We also mention that the paradigm of so-called barrier kernel functions is a good way for obtaining new search directions (see e.g. [3, 4, 12, 14, 15, 24, 29, 47]).

Our main goal in this thesis is to investigate two proposed IPA for solving the CQOs. The first one is based on the classical full-Newton step based on AET introduced by the univariate function $\psi(t) = t^2$ applied to the centering equation which defines the central-path. However, the second one is based on a new weighted-path following IPA for solving the CQOs based also on an AET induced by the function $\psi(t) = t^{\frac{3}{2}}$. At each iteration, only full-Newton steps and the strategy of the central-path are used for getting an ϵ-approximated optimal solution for CQO. These algorithms achieves the best known polynomial complexity. Further, under some imported changes on the original versions of these algorithms, the new obtained numerical results are very encouraging.

Short Outline of the Thesis

The thesis contains three chapters, followed by a bibliography. This thesis is organized as follows

- Chapter 1: In this chapter, we present a mathematical background which will be useful throughout the thesis.

- Chapter 2: This chapter presents a new feasible full-Newton step path-following method for CQO based on a new modified Newton search direction obtained by the application of the AET technique introduced by the new univariate function $\psi(t) = t^2$. Under new appropriate defaults of τ and θ, the favorable iteration bound of the algorithm with short-step method is achieved, namely, $\mathcal{O}(\sqrt{n}\log\frac{n}{\epsilon})$ which is as good as the bound for CQOs. For evaluating the efficiency of our algorithm some numerical results are provided.

- Chapter 3: Getting a perfectly centered initial point for feasible path-following interior-point algorithms is a hard practical task. Therefore, it is worth to analyze other cases when the starting point is not necessarily centered. In this chapter, we propose a short-step weighted-path following interior-point algorithm (IPA) for solving convex quadratic optimization (CQO). The latter is based on a modified search direction which is obtained by using the technique of algebraically equivalent transformation (AET) introduced by a new univariate function to the Newton system which defines the weighted-path. At each iteration, the algorithm uses only full-Newton steps and the strategy of the central-path for tracing approximately the weighted-path. We show that the algorithm is well-defined and converges locally quadratically to an optimal solution of CQO. Moreover, we obtain the currently best known iteration bound, namely, $\mathcal{O}\left(\sqrt{n}\log\frac{n}{\epsilon}\right)$ which is as good as the bound for linear optimization analogue. Some numerical results are given to evaluate the efficiency of the algorithm.

Finally, we end the thesis by a conclusion and future work.

Chapter 1

Mathematical background

In this chapter, we present some mathematical background in matrix and convex analysis, differential calculus, quadratic optimization and some methods of resolution.

1.1 Matrix Analysis

We denote by $\mathbb{R}^n$, the finite-dimensional Euclidean vector space $n \in \mathbb{N}$ and by $\mathbb{R}^{n\times n}$, the set of real square matrices of order n.

- For all x and $y \in \mathbb{R}^n$, we denote by:

$$x^T y = \sum_{i=1}^{n} x_i y_i,$$

the usual scalar product of x and y. The vector x^T denotes the transpose of the vector column x of $\mathbb{R}^n$.

- Two vectors x and y are orthogonal only if $x^T y = 0$.

- The Euclidean norm associated with the usual scalar product, is given by:

$$\|x\|_2 = \sqrt{x^T x} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

Definition 1.1. A mapping $\|.\| : \mathbb{R}^n \to \mathbb{R}$ is said to be a vectorial norm if it satisfies the following conditions:

- $\|x\| \geq 0, \forall x \in \mathbb{R}^n, \|x\| = 0 \Leftrightarrow x = 0.$
- $\|\alpha x\| = |\alpha| \, \|x\| \, , \forall \alpha \in \mathbb{R}, \forall x \in \mathbb{R}^n.$
- $\|x + y\| \leq \|x\| + \|y\| \, , \forall x, y \in \mathbb{R}^n.$

- The Cauchy-Schwarz inequality:

$$\left|x^T y\right| \leq \|x\| \, \|y\| \ \ \forall x, y \in \mathbb{R}^n.$$

Definition 1.2. We say that the matrix A is positive semi-definite if:

$$x^T A x \geq 0 \ , \forall x \in \mathbb{R}^n,$$

and we say that A is positive definite if:

$$x^T A x > 0, \forall x \in \mathbb{R}^n (x \neq 0).$$

1.2 Differential calculus

In this paragraph, we introduce the notion of differentiability of a function. We start to give some basic topological notions.

Definition 1.3. Let $\mathcal{D}$ be a subset of $\mathbb{R}^n$ and $f : \mathcal{D} \to \mathbb{R}$. We say that f is continuous in $x \in \mathcal{D}$, if $f(x_k) \to f(x^\star)$ for any sequence $\{x_k\}$ of $\mathcal{D}$ such that $x_k \to x^\star$. So we say that f is continuous over all $\mathcal{D} \subset \mathbb{R}^n$, if it is continuous at any point of $\mathcal{D}$.

Suppose now that the set Ω, is an open set of $\mathbb{R}^n$ and f is a function $f : \Omega \to \mathbb{R}$.

Definition 1.4. For all $x \in \Omega$, and $h \in \mathbb{R}^n$, the directional derivative of f at x in the direction h, is given by

$$\frac{\partial f}{\partial h}(x) = \lim_{t \to 0} \frac{f(x + th) - f(x)}{t}.$$

The gradient of f at x, denoted by $\nabla f(x)$, is the column vector of $\mathbb{R}^n$ given by:

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x) \right)^T.$$

Recall the formula:

$$\frac{\partial f}{\partial h}(x) = h^T \nabla f(x), \ \ \forall x \in \Omega, \ \ \forall h \in \mathbb{R}^n.$$

- We say that $x^\star$ is a critical or stationary point of the function f if

$$\nabla f(x^\star) = 0.$$

- We denote for all $x \in \Omega$, $\nabla^2 f(x)$ the square symmetric matrix of order n, given by

$$(\nabla^2 f(x))_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}(x), \forall i, j = 1, \ldots, n.$$

$\nabla^2 f(x)$ is called the Hessian matrix H from f to x. We also have:

$$H = \nabla^2 f(x)h = \frac{\partial}{\partial x_i} h^T \nabla f(x).$$

- We say that f is of class $\mathcal{C}^2$ on Ω if the partial derivatives of order 2 of f, exist and are continuous.

1.3 Convex analysis

In this paragraph, we cite some basic notions of convex analysis which will be useful afterwards.

Convex functions

Definition 1.5. A set $\mathcal{D} \subset \mathbb{R}^n$ is said to be convex if $\forall x, y \in \mathbb{R}^n$ the segment $[x, y] \subset \mathcal{D}$.

Definition 1.6. Let $\mathcal{D} \subset \mathbb{R}^n$ be a convex set and $f:\ \mathcal{D} \to \mathbb{R}$.
1- f is said to be convex on $\mathcal{D}$ if:

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y). \forall x, y \in \mathcal{D}, \forall t \in [0, 1].$$

2- f is said to be strictly convex if:

$$f((1-t)x + ty) < (1-t)f(x) + tf(y). \forall x, y \in \mathcal{D}, x \neq y, \forall t \in \,]0, 1[\, .$$

Definition 1.7. If f is of class $\mathcal{C}^2$ on $\mathcal{D}$, then f is convex over $\mathcal{D}$ if and only if the matrix Hessian $\nabla^2 f(x)$ is positive semi-definite. Likewise, f is said to be strictly convex on $\mathcal{D}$ if and only if the Hessian matrix is positive definite.

1.4 Unconstrained optimization problem

In this section, we only consider unconstrained optimization problem. We give the existence and the uniqueness of a minimum thus the characterization of the latter through its optimality conditions. A unconstrained optimization problem (P) is defined by

$$(P) \qquad \min_{x \in \mathbb{R}^n} f(x).$$

Definition 1.8. If $\mathcal{D} \subset \mathcal{R}^n, x^\star \in \mathcal{D}$ and $f : \mathcal{D} \to \mathbb{R}$.
1- We say that $x^\star$ is a global minimum of f on $\mathcal{D}$ if:

$$f(x) \geq f(x^\star), \forall x \in \mathcal{D}.$$

2- We say that $x^\star$ is a local minimum of f on $\mathcal{D}$ if there exists a neighborhood $\mathcal{V}$ of $x^\star$ such that :

$$f(x) \geq f(x^\star), \forall x \in \mathcal{D} \cap \mathcal{V}.$$

Remark 1.9. Any global minimum is a local minimum.

1.4.1 Results of existence and uniqueness of the solution

In order to be able to easily calculate or approach the solution of an optimization problem, it is interesting to know the hypotheses guaranteeing the existence and uniqueness of this solution.

Existence.

Theorem 1.10. *Let $\mathcal{D} \subset \mathbb{R}^n$. If $f : \mathcal{D} \to \mathbb{R}$, is continuous and if $\mathcal{D}$ is a compact set (closed and bounded), then f has a minimum on $\mathcal{D}$.*

Theorem 1.11. *We suppose that:*

1. *The set $\mathcal{D}$ is closed*
2. *f is lower semi-continuous on $\mathcal{D}$.*
3. $\lim\limits_{\|x\| \to +\infty} f(x) = +\infty$ *, (i.e. f is coercive).*

Then f has a minimum on $\mathcal{D}$.

Uniqueness.

The following result shows the impact of convexity in optimization problems.

Proposition 1.12. *Let f be a convex function defined on a convex set $\mathcal{D}$. Then any local minimum of f on $\mathcal{D}$ is a global minimum. If f is strictly convex, it there is at most a global minimum.*

1.4.2 Necessary and sufficient conditions for optimality

Necessary conditions

Theorem 1.13. *(First order necessary optimality conditions) Let $f : \mathbb{R}^n \to \mathbb{R}$, a differentiable functional. If $x^\star$ achieves a minimum (global or local) of f on $\mathbb{R}^n$ then:*

$$\nabla f(x^\star) = 0.$$

Theorem 1.14. *(Second-order necessary optimality conditions) Let $x^\star$ a minimum of f. If f is twice differentiable at the point $x^\star$, then:*

$$\left\langle \nabla^2 f(x^\star) y, y \right\rangle \geq 0, \quad \forall y \in \mathbb{R}^n.$$

Sufficient conditions

Theorem 1.15. *(First order sufficient condition) If f is convex and if:*

$$\nabla f(x^\star) = 0.$$

then $x^\star$ is a global minimum of f.

Theorem 1.16. *(Second order sufficient condition) If f is twice differentiable and if $\nabla f(x^\star) = 0$ and $\nabla^2 f(x^\star)$ positive definite then $x^\star$ is a global minimum of f.*

1.4.3 Optimality conditions

The constraint qualification

Definition 1.17. Let x be the feasible for convex optimization problem and put $I(x) = \{i \mid h_i(x) = 0\}$. We say that the linear independence constraint qualification hold at x if the gradients

$$\nabla g_i(x)\,(i = 1, \cdots, m),\ \nabla h_j(x)\,(j \in I(x)),$$

are linearly independent.

Theorem 1.18. ***(Karush-Kuhn-Tucker (K.K.T))****If x^* is a solution of (CP), if the gradients of f, g and h are finite at x^*, and if a constraint qualification is satisfied, then there exists an $y^* \in \mathbb{R}^m$ and $z^* \in \mathbb{R}^k$ such that:*

$$\nabla f(x^*) + \sum_{i=1}^{m} y_i^* \nabla g_i(x^*) + \sum_{j=1}^{k} z_j^* \nabla h_j(x^*) = 0,$$

$$z_j^* h_j(x^*) = 0, \quad j = 1, \cdots, k,$$

$$z_j^* \geq 0, \quad j = 1 \in I(x).$$

1.4.4 Order of convergence for a sequence of points

Let $x^\star$ be the limit of a sequence $\{x_k\}_{k\geq 0}$ produced by an iterative algorithm. We say that $x_k \to x^\star$ when $k \to +\infty \Leftrightarrow x_k - x^\star \to 0$ when $k \to +\infty \Leftrightarrow \|x_k - x^\star\| = 0$ when $k \to +\infty$. Now we try to characterize the speed of convergence of the quantity $x_k - x^\star \to 0$ when $k \to +\infty$. We say that:

- x_k converges linearly to $x^\star$, if there is a $c_k (0 < c_k < 1)$ such that:

$$\|x_{k+1} - x^\star\| \leq c_k \|x_k - x^\star\| \qquad \text{from a certain rank } k_0.$$

- x_k converges super linear to $x^\star$ if $\lim\limits_{k\to+\infty} c_k = 0$.
- x_k converges quadratically to $x^\star$, if there is a $c (0 < c < 1)$ such that:

$$\|x_{k+1} - x^\star\| \leq c \|x_k - x^\star\|^2 \quad \text{from a certain rank } k_0.$$

1.5 Convex quadratic optimization

Let us consider the convex quadratic optimization (CQO) problem in its standard format:

$$\min_x \left\{ \frac{1}{2} x^T Q x + c^T x \quad \textbf{s.t.} \;\; Ax = b, x \geq 0 \right\}, \tag{$\mathcal{P}$}$$

where $Q \in \mathbb{R}^{n\times n}$ is symmetric positive semidefinite, $A \in \mathbb{R}^{m\times n}$ with rank $(A) = m (m \leq n)$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ is the vector of variables.
If there exists a feasible solution to CQOs, then there exists an optimal solution.
We associate Lagrange multipliers $y \in \mathbb{R}^m$ and $z \in \mathbb{R}^n (z \geq 0)$ with the constraints $Ax = b$ and $x \geq 0$, and write **the Lagrangian**

$$L(x, y, z) = \frac{1}{2} x^T Q x + c^T x - y^T (Ax - b) - z^T x.$$

Dual quadratic format of $\mathcal{P}$

To determine the Lagrangian dual

$$L_D(y, z) = \min_x L(x, y, z)$$

we need stationarity with respect to x:

$$\nabla_x L(x, y, z) = Qx + c - A^T y - z = 0.$$

Hence

$$\begin{aligned} L_D(y, z) &= \frac{1}{2} x^T Q x + c^T x - y^T (Ax - b) - z^T x \\ &= b^T y + x^T (c + Qx - A^T y - z) - \frac{1}{2} x^T Q x \\ &= b^T y - \frac{1}{2} x^T Q x \end{aligned}$$

and **the dual problem** has the form:

$$\max_{x, y, z} \left\{ b^T y - \frac{1}{2} x^T Q x \quad \textbf{s.t.} A^T y + z - Qx = c, \; z \geq 0 \right\}, \tag{$\mathcal{D}$}$$

We define the following feasibility sets:
$\mathcal{F}_{\mathcal{P}} = \{x \in \mathbb{R}^n \mid Ax = b, \; x \geq 0\}$, the feasible solutions set of $(\mathcal{P})$.
$\mathcal{F}_{\mathcal{D}} = \{y \in \mathbb{R}^m \mid A^T y + z = Qx, \; z \geq 0\}$,the feasible solutions set of $(\mathcal{D})$.
$\mathcal{F}^0_{\mathcal{P}} = \{x \in \mathbb{R}^n \mid Ax = b, \; x > 0\}$,the strict feasible solutions setof $(\mathcal{P})$.
$\mathcal{F}^0_{\mathcal{D}} = \{y \in \mathbb{R}^m \mid A^T y + z = Qx, \; z > 0\}$, the strict feasible solutions set of $(\mathcal{D})$.

1.5.1 Resolution methods

Wolfe's algorithm for CQOs

It is one of the most widely used methods and was developed by P. Wolfe in 1959. Wolfe's algorithm can be directly applied to solve any quadratic programming problem. Wolfe's rule calls for the calculation of $\varphi'(t)$ in the linear search procedure.
We mention that this algorithm has an exponential complexity.

Step1(Input):
$\alpha_0 \in [0, 10^{99}]$, calculate $\varphi_k(0), \varphi'_k(0)$;
take $\rho = 0.1$, and $\sigma = 0.9$;
$\alpha_0 = 0, b_0 = 10^{99}, k = 0$;
begin
Step2: Calculate $\varphi_k(\alpha_k)$
• If $\varphi_k(\alpha_k) \leq \varphi_k(0) + \rho\alpha_k\varphi'_k(0)$; go to **Step3**,
• Else put $\alpha_{k+1} = \alpha_k, b_{k+1} = \alpha_k$; and go to **Step4**.
Step3 Calculate $\varphi'_k(\alpha_k)$
• If $\varphi'_k(\alpha_k) \geq \sigma\varphi'_k(0)(\varphi'_k(\alpha_k)) \leq -\sigma\varphi'_k(0)$ **Stop** and take $\hat{\alpha} = \alpha_k$;
• Else $\alpha_{k+1} = \alpha k, b_{k+1} = b_k$ and go to **Step4**;
• **Step4** Calculate α_{k+1}, $\alpha_{k+1} = \dfrac{\alpha_{k+1} + b_{k+1}}{2}$;
$k = k + 1$ and return to **Step2**.
end

Wolfe's Algorithm for CQOs

Interior Point Methods

Interior-point methods are a certain class of algorithms that solve linear and non-linear convex optimization problems. Arguably, interior-point methods were known as early as the 1960s. Interior point methods in mathematical programming have been the most important research area in optimization since the development of the simplex method for linear programming. Interior point methods have strongly influenced mathematical programming theory, practice and computation. For example linear programming is no longer synonymous with the simplex method, and linear programming is shown as a special case of nonlinear programming due to these developments. On the theoretical side, permanent research led to better computational complexity bounds for linear programming, quadratic programming, linear complementarity problems, semi-definite programming and some classes of convex programming problems. The interior point method attempts to overcome the potential weakness of the simplex method and it has been shown that the interior point methods can be solved in polynomial time. Interior-point methods solve the problem $(\mathcal{P})$ by applying Newton's method to a sequence of equality constrained problems. These methods is divided into three important classes such as:

- Projection's method for Karmarkar
- Potential method
- Path-following method.

In this work, we concentrate only on the third class which is path-following IPMs for solving CQOs. Here we give the principal for the basic ideas for the latter.

Perturbed problem (Log Barriers)

We replace the primal quadratic problem $(\mathcal{P})$ with primal Barrier problem:

$$\min_{x} \left\{ \frac{1}{2} x^T Q x + c^T x - \mu \sum_{i=1}^{n} \log x_i \quad \textbf{s.t.} \ \ Ax = b \right\}, \tag{$\mathcal{P}_\mu$}$$

and replace the dual quadratic problem $(\mathcal{D})$ with dual Barrier problem:

$$\max_{x,y,z} \left\{ b^T y - \frac{1}{2} x^T Q x + \mu \sum_{i=1}^{n} \log z_i \quad \textbf{s.t.} \quad A^T y + z - Qx = c, \right\}, \tag{$\mathcal{D}_\mu$}$$

where $\mu \geq 0$ is a barrier parameter.

First Order Optimality Conditions

Consider the primal barrier quadratic program $(\mathcal{P}_\mu)$:

$$\min_x \left\{ \frac{1}{2} x^T Q x + c^T x - \mu \sum_{i=1}^{n} \log x_i \quad \textbf{s.t.} \quad Ax = b \right\}. \qquad (\mathcal{P}_\mu)$$

We write out the Lagrangian

$$L(x, y, \mu) = \frac{1}{2} x^T Q x + c^T x - y^T (Ax - b) - \mu \sum_{i=1}^{n} \log x_i.$$

The conditions for a stationary point of the Lagrangian are:
$\nabla_x L(x, y, \mu) = c - A^T y - \mu X^{-1} e + Qx = 0,$
$\nabla_y L(x, y, \mu) = Ax - b = 0,$ where $X^{-1} = diag(x_1^{-1}, x_2^{-1}, \cdots, x_n^{-1})$.
Let us denote

$$z = \mu X^{-1} e, \quad \text{i.e.} \quad Xz = \mu e.$$

Thus we consider the system:

$$\begin{cases} Ax = b, x \geq 0, \\ A^T y + z - Qx = c, z \geq 0, \\ Xz = \mu e. \end{cases}$$

Newton Method

The first order optimality conditions for the barrier problem form a large system of nonlinear equations

$$F(x, y, z) = 0,$$

where $F : \mathbb{R}^{2n+m} \to \mathbb{R}^{2n+m}$ is an application defined as follows:

$$F(x, y, z) = \begin{bmatrix} Ax - b \\ A^T y + z - Qx - c \\ xz - \mu e. \end{bmatrix}.$$

Actually, the first two terms of it are linear, only the last one, corresponding to the complementarity condition, is nonlinear. Note that

$$\nabla F(x, y, z) = \begin{bmatrix} A & 0 & 0 \\ -Q & A^T & I \\ Z & 0 & X \end{bmatrix},$$

Thus, for a given point (x, y, z) we find the Newton direction $(\Delta x, \Delta y, \Delta z)$ by solving the system of linear equations:

$$\begin{bmatrix} A & 0 & 0 \\ -Q & A^T & I \\ Z & 0 & X \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} b - Ax \\ c - A^T y - z + Qx \\ \mu e - XZe \end{bmatrix}.$$

For more details see([26, 27])

Chapter 2

A feasible short-step IPA for convex quadratic optimization based on new search directions

2.1 A feasible full-Newton step IPA for CQO

Throughout the paper, we assume that both $(\mathcal{P})$ and $(\mathcal{D})$ satisfy the Interior-Point-Condition (IPC), i.e., there exists (x^0, y^0, z^0) such that:

$$Ax^0 = b,\; x^0 > 0,\; A^T y^0 + z^0 - Qx^0 = c,\; z^0 > 0.$$

Finding an optimal solution for $(\mathcal{P})$ and $(\mathcal{D})$ is equivalent to solving the following system, which represents the Karush-Kuhn-Tucker (KKT) optimality conditions

$$\begin{cases} Ax = b,\; x \geq 0, \\ A^T y + z - Qx = c,\; z \geq 0, \\ xz = 0. \end{cases} \tag{2.1}$$

2.1.1 The central-path of CQO

The primal-dual path-following IPA is usually replace the complementarity equation $xz = 0$, in (2.1) by the parameterized equation $xz = \mu e$ with $\mu \in \mathbb{R}^n_{++}$. Thus we consider the system:

$$\begin{cases} Ax = b, x \geq 0, \\ A^T y + z - Qx = c, z \geq 0, \\ xz = \mu e. \end{cases} \tag{2.2}$$

If the IPC holds, then system (2.2) has a unique solution for each fixed $\mu > 0$ denoted by $(x(\mu), y(\mu), z(\mu))$. The set $\{(x(\mu), y(\mu), z(\mu)) \mid \mu > 0\}$ is called the μ-center of CQO. The set of μ-centers is called the central-path of CQOs. If μ tends to zero then the limit of the central-path exists and since the limit point satisfies the complementarity condition, the limit yields an optimal solution for CQO. The relevance of the central-path for LO and CQO has been discussed in the following monographs, (see, e.g. [40, 44, 45]).

2.1.2 The new search direction for CQO by using AETs

Similar to [1, 22], the AET technique for computing the new search direction for CQO is simply based on replacing the non linear equation $xz = \mu e$ by the new one:

$$\psi\left(\frac{xz}{\mu}\right) = \psi(e)$$

where $\psi(.) : (0, +\infty) \to \mathbb{R}$ is a continuously differentiable and invertible function. Using the function ψ, then (2.2) can be rewritten as the following system:

$$\begin{cases} Ax = b, x \geq 0, \\ A^T y + z - Qx = c, z \geq 0, \\ \psi(\dfrac{xz}{\mu}) = \psi(e), \end{cases} \tag{2.3}$$

where ψ is applied coordinate-wisely. Applying Newton's method to the modified system (2.3), we obtain the following system:

$$\begin{cases} A\Delta x = 0, \\ A^T \Delta y + \Delta z - Q\Delta x = 0, \\ z\Delta x + x\Delta z = \dfrac{\mu(\psi(e) - \psi(\frac{xz}{\mu}))}{\psi'(\frac{xz}{\mu})}, \end{cases} \tag{2.4}$$

where ψ' denotes the derivative of ψ. To simplify matters, we introduce the following notations

$$v := \sqrt{\frac{xz}{\mu}} \quad \text{and} \quad d := \sqrt{xz^{-1}}.$$

So the scaling directions are defined as

$$d_x = \frac{v\Delta x}{x} \text{ and } d_z = \frac{v\Delta z}{z}. \tag{2.5}$$

In addition, we have

$$x\Delta z + z\Delta x = \mu v(d_x + d_z). \tag{2.6}$$

Next, since Q is positive semidefinite matrix, therefore using (2.4), we deduce that

$$d_x^T d_z = (\Delta x)^T (\Delta z) = (\Delta x)^T Q\Delta x \geq 0. \tag{2.7}$$

Now, due to (2.5), (2.6) and (2.7), the scaled form of system (2.4) is given by

$$\begin{cases} \bar{A} d_x = 0, \\ \bar{A}^T \Delta y + d_z - \bar{Q} d_x = 0, \\ d_x + d_z = p_v \end{cases} \tag{2.8}$$

where

$$p_v = \frac{\psi(e) - \psi(v^2)}{v\psi'(v^2)} \tag{2.9}$$

with $\bar{A} = AD,\ \bar{Q} = DQD$ and $D :=$diag(d).

Next, by restricting ourselves to $\psi(t) = t^2$, then (2.9) and (2.4) are given by

$$p_v = \frac{1}{2}(v^{-3} - v) \tag{2.10}$$

and

$$\begin{cases} A\Delta x = 0, \\ A^T\Delta y + \Delta z - Q\Delta x = 0, \\ z\Delta x + x\Delta z = \frac{\mu}{2}((\frac{xz}{\mu})^{-1} - (\frac{xz}{\mu})). \end{cases} \tag{2.11}$$

Moreover, we address by:

$$x_+ := x + \Delta x; \quad y_+ := y + \Delta y; \quad z_+ := z + \Delta z$$

to the new iterate after a full-Newton step.

2.1.3 The proximity measure and some bounds

For the analysis of the algorithm, we use at each iterate a norm-based proximity measure $\delta(v;\mu)$ defined by

$$\delta(v) := \delta(xz;\mu) = 2\|p_v\| = \|v^{-3} - v\|. \tag{2.12}$$

It is clear that

$$\delta(v) = 0 \Leftrightarrow v = e \Leftrightarrow xz = \mu e.$$

So $\delta(v)$ is used to measure how far is a point (x, y, z) from the central-path.

Next, across system 2.8 and according to 2.12, we can easily prove the following inequalities.

Lemma 2.1. *Let $(d_x, \Delta y, d_z)$ be a solution of (2.8) and $\mu > 0$. If $\delta := \delta(v) > 0$, then*

$$0 \leq d_x^T d_z \leq \frac{1}{8}\delta^2 \tag{2.13}$$

and

$$\|d_x d_z\|_\infty \leq \frac{\delta^2}{16},\ \ \|d_x d_z\| \leq \frac{1}{8}\delta^2. \tag{2.14}$$

2.1.4 The algorithm

The full-Newton step IPA for solving the CQO works as follows. First, we use a suitable threshold (default) value $\tau > 0$, with $0 < \tau < 4$ and we suppose that a strictly feasible initial point (x^0, y^0, z^0) exists such that $\delta(x^0 z^0; \mu^0) \leq \tau$, for certain μ^0 is known. Using the obtained new search directions $(\Delta x, \Delta y, \Delta z)$ from (2.11) and taking a full-Newton step, the algorithm produces a new iterate $(x + \Delta x, y + \Delta y, z + \Delta z)$. Then it updates the barrier parameter μ to $(1-\theta)\mu$ with $0 < \theta < 1$ and solves the Newton system (2.11), and target a new μ-center and so on. This procedure is repeated until the stopping criterion $x^T z \leq \epsilon$ is satisfied for a given accuracy $\epsilon > 0$. The generic IPA for the CQO is stated in Figure 2.1 as follows.

Input:
A threshold parameter $0 < \tau < 4$ (default value $\tau = \frac{1}{4}$);
accuracy parameter $\epsilon > 0$;
a barrier update parameter θ, $0 < \theta < 1$ (default value $\theta = \frac{1}{12\sqrt{2n}}$);
a strictly feasible initial point (x^0, y^0, z^0) and μ^0 s.t. $\delta(x^0 z^0; \mu^0) \leq \frac{1}{4}$;
begin
Set $x := x^0; y := y^0; z := z^0; \mu := \mu^0$;
while $x^T z \geq \epsilon$ **do**
begin
- $\mu := (1-\theta)\mu$;
- calculate $(\Delta x, \Delta y, \Delta z)$ from (2.11);
- $(x, y, z) := (x + \Delta x, y + \Delta y, z + \Delta z)$;

endwhile
end.

Figure 2.1: Algorithm 1. The IPA for CQO

2.2 The analysis of the short-step algorithm

In this section, we will show across our new defaults that Algorithm 1, is well-defined, i.e., at each iteration, the full-Newton step is feasible and the condition $\delta(xz; \mu) \leq \tau$ are maintained through the algorithm.

In the next lemma, we investigate the feasibility of a full-Newton step under the condition $\delta < 4$.

Lemma 2.2. *Let $\delta < 4$, then the full-Newton step is strictly feasible.*

Proof. Let $\alpha \in [0,1]$, we define $x(\alpha) = x + \alpha\Delta x$ and $z(\alpha) = z + \alpha\Delta z$. Then

$$x(\alpha)z(\alpha) = xz + \alpha(x\Delta z + z\Delta x) + \alpha^2\Delta x\Delta z.$$

Using (2.5) and (2.6), we get

$$x(\alpha)z(\alpha) = \mu((1-\alpha)v^2 + \alpha(v^2 + vp_v + \alpha d_x d_z)). \tag{2.15}$$

Hence $x(\alpha)z(\alpha) > 0$ if $v^2 + vp_v + \alpha d_x d_z > 0$. By Lemma 2.1 (2.13), and from (2.10) substitution $p_v = \frac{1}{2}(v^{-3} - v)$ and let $\delta < 4$, it follows that

$$\begin{aligned} v^2 + vp_v + \alpha d_x d_z &\geq v^2 + vp_v - \alpha \left\|d_x d_z\right\|_\infty e \\ &\geq v^2 + vp_v - \alpha\frac{\delta^2}{16}e \\ &\geq v^2 + \frac{1}{2}v(v^{-3} - v) - \alpha\frac{\delta^2}{16}e \\ &\geq \frac{1}{2}v^2 + \frac{1}{2}v^{-2} - \alpha\frac{\delta^2}{16}e \\ &> \frac{1}{2}v^2 + \frac{1}{2}v^{-2} - e = \frac{1}{2v^2}\left(v^2 - e\right)^2 \geq 0, \forall v > 0. \end{aligned}$$

The functions $\alpha \to x(\alpha)$ and $\alpha \to z(\alpha) > 0$ are continuous, $x(0) > 0$ and $z(0) > 0$ and $x(\alpha)z(\alpha) > 0$ for each $\alpha \in [0,1]$. These imply that $x(1) = x_+ > 0$ and $z(1) = z_+ > 0$. This shows the strict feasibility of a full-Newton step. This completes the proof. □

Lemma 2.3. *Let $\delta < 4$, then*

$$\min v_+ \geq \frac{1}{4}\sqrt{16 - \delta^2}$$

where $v_+ = \sqrt{\frac{x_+ z_+}{\mu}}$.

Proof. Setting $\alpha = 1$ in (2.15), then from (2.10), we have

$$v_+^2 = v^2 + vp_v + d_x d_z = \frac{1}{2}v^2 + \frac{1}{2}v^{-2} + d_x d_z.$$

Lemma 2.2, implies that $\frac{1}{2}v^2 + \frac{1}{2}v^{-2} - e \geq 0$ if $\delta < 4$, so $\frac{1}{2}v^2 + \frac{1}{2}v^{-2} \geq e$. Hence, $v_+^2 \geq e + d_x d_z$. Next, due to (2.14), we obtain

$$v_+^2 \geq e + d_x d_z \geq (e - \|d_x d_z\|_\infty e) \geq \left(1 - \frac{\delta^2}{16}\right) e \geq \frac{1}{16}(16 - \delta^2)e.$$

This implies that

$$\min v_+ \geq \frac{1}{4}\sqrt{16 - \delta^2}.$$

This proves the lemma. □

Next, we prove that the iterate across the proximity measure is locally quadratically convergent through the algorithm 1.

Lemma 2.4. *Assume $\delta < 4$, then*

$$\delta_+ := \delta(v_+) := \delta(x_+z_+;\mu) \leq \delta_+ \leq \left(\frac{4}{\sqrt{16-\delta^2}} + \frac{64}{\sqrt{(16-\delta^2)^3}}\right)\frac{5}{8}\delta^2.$$

Moreover, if $\delta \leq \frac{1}{4}$ then $\delta_+ < \delta^2$ which means the local quadratic convergence of the full-Newton step.

Proof. Let $v_+ = \sqrt{\frac{x_+z_+}{\mu}}$ and $\delta_+ = \|v_+^{-3} - v_+\|$, then we have,

$$\delta_+ = \|v_+^{-3} - v_+\| = \left\|\frac{e - v_+^4}{v_+^3}\right\| = \left\|(e - v_+^2)\left(\frac{e + v_+^2}{v_+^3}\right)\right\|$$

For all $t \in \mathbb{R}^n_{++}$, we define the function g by

$$g(t) = \frac{1}{t} + \frac{1}{t^3}.$$

Using g, we deduce that

$$\delta_+ = \|g(v_+)(e - v_+^2)\| \leq \|g(v_+)\|_\infty \|e - v_+^2\|.$$

where $g(v_+) = (g_1(v_+)_1, \cdots, g_n(v_+)_n)$. The function g is continuous and monotonically decreasing and positive on $(0, +\infty)$. Hence

$$0 < |g((v_+)_i)| = g((v_+)_i) \leq g(\min v_+) \leq g\left(\frac{1}{4}\sqrt{16-\delta^2}\right).$$

Then

$$\|g(v_+)\|_\infty = \frac{4}{\sqrt{16-\delta^2}} + \frac{64}{\sqrt{(16-\delta^2)^3}}.$$

So

$$\delta_+ \leq \left(\frac{4}{\sqrt{16-\delta^2}} + \frac{64}{\sqrt{(16-\delta^2)^3}}\right)\|e - v_+^2\|.$$

Next, letting $\alpha = 1$ in (2.15) and due to (2.10), we have

$$\|e - v_+^2\| = \|e - (v^2 + vp_v + d_xd_z)\| = \|e - \frac{1}{2}v^2 - \frac{1}{2}v^{-2} - d_xd_z\|.$$

Then

$$\|e - v_+^2\| \leq \|e - \frac{1}{2}v^2 - \frac{1}{2}v^{-2}\| + \|d_xd_z\|.$$

Next, we may write

$$\left\|e - \frac{1}{2}v^2 - \frac{1}{2}v^{-2}\right\| = \left\|\phi(v).4p_v^2\right\|,$$

where

$$\phi(v) = \frac{v^4}{2(v^2+e)^2}.$$

Let us consider the function

$$\phi(t) = \frac{t^4}{2(t^2+1)^2}.$$

The function ϕ is continuous and monotonically increasing and positive for all $t \in (0, +\infty)$. Then we have

$$0 \leq \phi(t) < \frac{1}{2} = \lim_{t \mapsto \infty} \phi(t), \forall t > 0.$$

This yields

$$0 \leq \phi(v_i) < \frac{1}{2}, \forall\, i = 1, ..., n. \tag{2.16}$$

Then, as $\|p_v\|^2 = \frac{1}{4}\delta^2$, $\|\phi(v)\|_\infty = \max_i \phi(v_i) < \frac{1}{2}$ and $\|p_v^2\| \leq \|p_v\|^2$, this implies that

$$\left\| e - \frac{1}{2}v^2 - \frac{1}{2}v^{-2} \right\| \leq 4\|\phi(v)\|_\infty \|p_v\|^2 = \frac{1}{2}\delta^2.$$

By (2.14), we deduce that:

$$\|e - v_+^2\| \leq \frac{5}{8}\delta^2.$$

Next, for $\delta < 4$, we have

$$\delta_+ \leq \frac{5}{8}\left(\frac{4}{\sqrt{16-\delta^2}} + \frac{64}{\sqrt{(16-\delta^2)^3}} \right)\delta^2.$$

Next, let $\delta \leq \frac{1}{4}$, then $\delta_+ < \delta^2$. This completes the proof. □

In the next lemma, we obtain an upper bound for the duality gap after a full-Newton step.

Lemma 2.5. *After a full-Newton step it holds*

$$x_+^T z_+ \leq 2\mu n. \tag{2.17}$$

Proof. As $v_+^2 = \frac{x_+ z_+}{\mu}$, we have

$$\begin{aligned}
(x_+)^T z_+ &= e^T(x_+z_+) = \mu e^T v_+^2 = \mu e^T(v^2 + vp_v + d_x^T d_z) \\
&= \mu e^T(e + v^2 + vp_v - e) + \mu d_x^T d_z \\
&= \mu e^T e + e^T(v^2 + vp_v - e) + \mu d_x^T d_z \\
&= \mu e^T e + \mu d_x^T d_z + \mu e^T \left(\frac{\frac{1}{2}v^2 + \frac{1}{2}v^{-2} - e}{(v^{-3}-v)^2} .4p_v^2 \right).
\end{aligned}$$

Then, after some simplifications, we get

$$(x_+)^T z_+ = \mu e^T e + \mu d_x^T d_z + 4\mu e^T \phi(v) p_v^2,$$

where $\phi(v) = (\phi(v_1), \phi(v_2), \cdots, \phi(v_n))$ with

$$\phi(v_i) = \frac{v_i^4}{2(v_i^2+e)^2}, \text{ for } i = 1, \cdots, n.$$

Due to (2.16), we have

$$0 < |\phi(v_i)| = \phi(v_i) < \frac{1}{2}, \forall i.$$

Therefore

$$\begin{aligned} e^T v_+^2 &\leq \mu e^T e + \mu d_x^T d_z + 4\mu e^T \max_i |\phi(v_i)| p_v^2 \\ &\leq \mu e^T e + \mu d_x^T d_z + 2\|p_v\|^2 \\ &\leq n\mu + \frac{1}{8}\delta^2 \mu + \frac{1}{2}\delta^2 \mu \\ &\leq \mu\left(n + \frac{5}{8}\delta^2\right) \leq (n+\delta^2)\mu. \end{aligned}$$

Let $\delta \leq \frac{1}{4}$, then $e^T v_+^2 \leq (n+1)\mu$, but since $(n+1) \leq 2n, \forall n \geq 1$, it follows that $e^T v_+^2 \leq 2\mu n$. This completes the proof. □

In the following lemma, we investigate the effect of a full Newton-step on the proximity measure followed by updating the parameter μ by a factor $(1-\theta)$, where $0 < \theta < 1$.

Theorem 2.6. *Let $\mu_+ = (1-\theta)\mu$ and let $x_+ > 0, z_+ > 0$, then we have*

$$\delta(v_+; \mu_+) \leq \delta_+ + \frac{2\sqrt{2n}\theta}{1-\theta}.$$

In addition, let $\delta \leq \frac{1}{4}$, and $\theta = \frac{1}{12\sqrt{2n}}$, $n \geq 2$, then $\delta(v_+; \mu_+) \leq \frac{1}{4}$.

Proof. As $\sqrt{\frac{x_+ z_+}{\mu_+}} = \frac{1}{\sqrt{1-\theta}} v_+$ and $\delta_+ = \left\|v_+^{-3} - v_+\right\|$, we then have,

$$\begin{aligned} \delta(x_+ z_+; \mu_+) &= \left\| \left(\frac{x_+ z_+}{\sqrt{1-\theta}\mu}\right)^{-3} - \sqrt{\frac{x_+ z_+}{(1-\theta)\mu}} \right\| = \left\| (1-\theta)^{\frac{3}{2}} v_+^{-3} - \frac{1}{\sqrt{1-\theta}} v_+ \right\| \\ &= \left\| (1-\theta)^{\frac{3}{2}} v_+^{-3} - (1-\theta)^{\frac{3}{2}} v_+ + (1-\theta)^{\frac{3}{2}} v_+ - \frac{1}{\sqrt{1-\theta}} v_+ \right\| \\ &\leq \left\| (1-\theta)^{\frac{3}{2}} v_+^{-3} - (1-\theta)^{\frac{3}{2}} v_+ \right\| + \left\| (1-\theta)^{\frac{3}{2}} v_+ - \frac{1}{\sqrt{1-\theta}} v_+ \right\| \\ &= (1-\theta)^{\frac{3}{2}} \delta_+ + \left| (1-\theta)^{\frac{3}{2}} - \frac{1}{\sqrt{1-\theta}} \right| \|v_+\| \\ &\leq \delta_+ + \left| (1-\theta)^{\frac{3}{2}} - \frac{1}{\sqrt{1-\theta}} \right| \|v_+\|, \text{ for } \theta \in (0,1). \end{aligned}$$

By Lemma 2.5, $\|v_+\| \leq \sqrt{2n}$ and since $\left|(1-\theta)^{\frac{3}{2}} - \frac{1}{\sqrt{1-\theta}}\right| = \frac{\theta|2-\theta|}{\sqrt{1-\theta}}, \forall\theta \in (0,1)$, then

$$\delta(x_+z_+;\mu_+) \leq \delta_+ + \frac{\theta|2-\theta|}{\sqrt{1-\theta}}\sqrt{2n}.$$

As $\frac{\theta|2-\theta|}{\sqrt{1-\theta}} \leq \frac{2\theta}{\sqrt{1-\theta}}, \forall\theta \in (0,1)$, we obtain

$$\delta(x_+z_+;\mu_+) \leq \delta_+ + \frac{2\sqrt{2n}\,\theta}{\sqrt{1-\theta}}.$$

As $\delta \leq \frac{1}{4}$ so $\delta_+ \leq 1.254904\,\delta^2$, then we get

$$\delta(x_+z_+;\mu_+) \leq 0.078431 + \frac{2\sqrt{2n}\,\theta}{\sqrt{1-\theta}}.$$

Let $\theta = \frac{1}{12\sqrt{2n}}$, $n \geq 2$ hence $\theta \in \left[0, \frac{1}{24}\right]$ from which we deduce that $\delta(v_+;\mu_+) \leq \xi(\theta)$ where

$$\xi(\theta) = 0.078431 + \frac{1}{6\sqrt{1-\theta}}, \forall\theta \in (0,1).$$

Since $\xi'(\theta) > 0$, then $\xi(\theta)$ is strictly increasing on $\left[0, \frac{1}{24}\right]$ and so $\xi(\theta) \leq \xi\left(\frac{1}{24}\right) = 0.248682$. This implies that $\delta(x_+z_+;\mu_+) \leq \frac{1}{4}$. This completes the proof. □

Theorem 2.6, shows that Algorithm 1 is well-defined since the conditions $x > 0, z > 0$, and $\delta(xz;\mu) \leq \frac{1}{4}$ are maintained through the algorithm.

In the next lemma, we derive an upper bound for the total number of iterations produced by Algorithm 1.

Lemma 2.7. *Assume that x^0 and z^0 are strictly feasible such that $\delta(x^0z^0;\mu^0) \leq \frac{1}{4}$, with $\mu^0 > 0$. Moreover, let x^k and z^k denote the vectors obtained by Algorithm 1, after k-iterations with $\mu^0 := \mu^k$. Then $(x^k)^T z^k \leq \epsilon$ holds if*

$$k \geq \left\lceil \frac{1}{\theta}\log\frac{2\mu^0 n}{\epsilon} \right\rceil.$$

Proof. From Lemma 2.5 (2.17), we deduce that $(x^k)^T z^k \leq 2(1-\theta)^k\mu^0 n$. So the inequality $(x^k)^T z^k \leq \epsilon$ holds if $2(1-\theta)^k\mu^0 n \leq \epsilon$. Taking logarithms, we get

$$k\log(1-\theta) \leq \log\epsilon - \log 2n\mu^0 \leq \log\epsilon - \log 2\mu^0 n.$$

Next, as $-\log(1-\theta) \geq \theta, 0 < \theta < 1$, then the above inequality holds if

$$k\theta \geq \log 2n\mu^0 - \log\epsilon = \log\frac{2n\mu^0}{\epsilon}.$$

This completes the proof. □

Theorem 2.8. *Let* $\theta = \frac{1}{12\sqrt{2n}}$ *and* $\mu^0 = \frac{1}{2}$*, then Algorithm 1, requires at most*

$$O\left(\sqrt{n}\log\frac{n}{\epsilon}\right)$$

iterations for getting an approximated primal-dual solution of CQO with the output $x^T z \leq \epsilon$.

Proof. Let $\theta = \frac{1}{12\sqrt{2n}}$ and $\mu^0 = \frac{1}{2}$, the results follows directly from Lemma 2.7. □

2.3 Numerical results

In this section, we implemented Algorithm 1 by using **Mat lab R2010a** and run on a **PC** with **CPU 2.00 GHz** and **4.00 G RAM** memory and double precision format on some examples of CQOs with different size. Here our accuracy is $\epsilon = 10^{-4}$. The strictly feasible initial point and the optimal solution for CQO problems are denoted by $(x^0 > 0, y^0, z^0 > 0)$ and $(x^\star, y^\star, z^\star)$, respectively. Here, we display the following notations:"Iter" and "CPU" to denote the number of iterations and the elapsed time produced by our algorithm. Also to improve our numerical results we have relaxed the barrier parameter $\mu^0 \in \{1, 0.5, 0.05\}$, with $\theta = \frac{1}{12\sqrt{2n}}, n \geq 2$. We also display for each example, two tables, one is for different appropriate initial points taken in Algorithm 1 according to each $\mu^0 > 0$ such that $\delta(x^0 z^0, \mu^0) \leq \frac{1}{4}$ and the other is for the number of iterations and the elapsed time produced by our algorithm for each example.

Example 1. We consider the convex quadratic optimization, where

$$A = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

$c = (6.8565, -3.5720, -5.6797, 0.6479)^T$, $b = (0.5, 3)^T$. The different strictly feasible starting points of Algorithm 1 are stated in the table below:

Table 2.1: Initial points for Example 1.

$\mu^0 \downarrow$	$x^0 > 0$	$z^0 > 0$	$\delta(x^0 z^0; \mu^0)$
1	$(1.6243, 1.0033, 4.5688, 2.3291)^T$	$(0.6064, 1.0293, 0.2162, 0.4211)^T$	0.0842
0.5	$(1.0298, 0.9873, 1.0092, 0.9628)^T$	$(0.5, 0.5, 0.5, 0.5)^T$	0.1016
0.05	$(0.05, 0.04, 0.05, 0.05)^T$	$(0.9, 1.195, 1.001, 1.01)^T$	0.2416

Next, the number of iterations and CPU times for this example are summarized in the table below.

Table 2.2: Numerical results for Example 1.

$\mu^0 \rightarrow$					
1		0.5		0.05	
Iter	CPU	Iter	CPU	Iter	CPU
355	0.207200	332	0.205742	332	0.175473

An optimal solution of the COQs in Example 1 is:

$$x^\star = (0.3431, 0.7002, 0.1428, 0.2132)^T,$$
$$y^\star = (-5.3941, 1.0742)^T,$$
$$z^\star = (0.0001, 0.0000, 0.0000, 0.0001)^T.$$

Example 2. Let the matrices A and Q be given as

$$A = \begin{pmatrix} 1 & 1.2 & 1 & 1.8 & 0 \\ 3 & -1 & 1.5 & -2 & 1 \\ -1 & 2 & -3 & 4 & 2 \end{pmatrix},$$

$$Q = \begin{pmatrix} 20 & 1.2 & 1 & 1.8 & 0 \\ 1.2 & 32 & 1 & 1 & 1 \\ 0.5 & 1 & 14 & 1 & 1 \\ 0.5 & 1 & 1 & 15 & 1 \\ -1 & 1 & 1 & 1 & 16 \end{pmatrix},$$

$c = (-21, -37, -16, -22, -19)^T$, $b = (5, 2.5, 4)^T$. The initial points and the obtained numerical results are stated in the following tables.

Table 2.3: Initial points for Example 2.

$\mu^0 \downarrow$	$x^0 > 0$	$z^0 > 0$	$\delta(x^0z^0;\mu^0)$
1	e	$(1.0298, 0.9873, 0.9690, 1.0092, 0.9628)^T$	0.1201
0.5	$(1.0298, 0.9873, 0.9690, 1.0092, 0.9628)^T$	$(0.5000, 0.5000, 0.5000, 0.5000, 0.5000)^T$	0.1201
0.05	$(0.5182, 0.4920, 0.4811, 0.5057, 0.4773)^T$	$e/10$	0.1474

Table 2.4: numerical results of Example 2.

$\mu^0 \rightarrow$					
1		0.5		0.05	
Iter	CPU	Iter	CPU	Iter	CPU
406	0.264340	380	0.243439	293	0.165237

An optimal solution for this example is:

$$x^\star = (1.0182, 0.9920, 0.9811, 1.0057, 0.9773)^T,$$
$$y^\star = (-0.3172, 0.0913, -0.6271)^T,$$
$$z^\star = 10^{-4}(0.4573, 0.1949, 0.1971, 0.1923, 0.0489)^T.$$

Problem 3. We consider the CQO, where $n = 2m$ and

$$A = (a_{ij}) = \begin{cases} 0 & \text{if} \quad i \neq j \text{ or } (i+1) \neq j \\ 1 & \text{if} \quad j = i + m. \end{cases}$$

$$Q[i,j] = \begin{cases} 2j-1 & \text{if} \quad i > j \quad \text{or } i = j = n \\ 2i-1 & \text{if} \quad i < j \\ i(i+1) & \text{if} \quad i = j. \end{cases} \qquad c = \begin{cases} 0 & \text{if} \qquad i = i + m \\ 1 & \text{else if,} \end{cases},$$

$$b[i] = 0.5.$$

In this example, we take $x^0 = 0.5e$, $y^0 = 0_{\mathbb{R}^m}$ and $z^0 = e$ as the strictly feasible initial point. The obtained numerical results are showed in the table below.

Table 2.5: Numerical results for Example 3 with different size.

Size $(m,n)\downarrow$	$\tau = 1$	
	$\mu^0 = 0.5$	
	Iter	CPU
$(m,n) = (10,20)$	2892	4.345861
$(m,n) = (50,100)$	6761	77.349749
$(m,n) = (100,200)$	9736	150.012162
$(m,n) = (1000,2000)$	15753	2480.426429

An optimal solution for Example 3 is:

$$x^\star = (0.0000, \cdots, 0.4185)^T,$$
$$y^\star = (0.0003, \cdots, 0.0003)^T,$$
$$z^\star = (0.3876, \cdots, 0.0000)^T.$$

Comments. Across the obtained numerical results stated in the previous tables, we see that the number of iterations produced by Algorithm , is too large. This is due to the fact that the default θ provided by our analysis is very poor when the size n of the problem becomes very large. Consequently, the rate of decrease in the sequence of barrier parameters $\{\mu^k\}$ approaches to one. This leads to a slow convergence of Algorithm 1, to an optimal solution for large size CQO problems.

2.3.1 A numerical modification of Algorithm 1

In this subsection, according to the above comments and in order to accelerate the convergence of Algorithm 1, we import some modifications on it, where instead of using the default of θ stated in our analysis, we take it as a constant belongs to the set $\{0.1, ..., 0.9\}$. In addition, to ensure that iterates remain interior, we introduce a primal and dual step-size $\alpha_{\max} > 0$ such that $x + \rho\alpha_{\max}\Delta x > 0$ and $z + \rho\alpha_{\max}\Delta z > 0$ where $\alpha_{\max} = \min(\rho\alpha, 1)$ with $\rho \in (0, 1)$ and $\alpha = \min(\alpha_x, \alpha_z)$ such that α_x and α_z are computed as follows:

$$\alpha_x = \min\{(-x_i)/(\Delta x_i) : \Delta x_i < 0\} \quad \alpha_z = \min\{(-z_i)/(\Delta z_i) : \Delta z_i < 0\}.$$

Based, on our modifications the new obtained numerical results are stated in tables below.

Table 2.6: New obtained numerical results for Example 1.

$\theta \downarrow$	$\mu^0 \rightarrow$					
	1		0.5		0.05	
	Iter	CPU	Iter	CPU	Iter	CPU
0.1	101	0.062610	94	0.062887	73	0.043659
0.2	48	0.039662	45	0.033867	35	0.033420
0.3	30	0.028877	28	0.024989	22	0.024648
0.4	21	0.030101	20	0.023720	15	0.023575
0.5	16	0.023684	15	0.025608	11	0.026514
0.6	12	0.023404	11	0.024542	9	0.021171
0.7	9	0.023096	9	0.019677	7	0.019430
0.8	7	0.020431	7	0.020159	5	0.021679
0.9	5	0.017440	5	0.016987	4	0.017399

Table 2.7: New obtained numerical results for Example 2.

$\theta \downarrow$	$\mu^0 \rightarrow$					
	1		0.5		0.05	
	Iter	CPU	Iter	CPU	Iter	CPU
0.1	103	0.062552	97	0.054235	75	0.045948
0.2	49	0.038470	46	0.036995	36	0.031934
0.3	31	0.028234	29	0.026437	22	0.027367
0.4	22	0.028094	20	0.027767	16	0.024999
0.5	16	0.023248	15	0.023424	12	0.022347
0.6	12	0.024802	12	0.021535	9	0.021052
0.7	9	0.018651	9	0.024192	7	0.021086
0.8	7	0.020868	7	0.022089	5	0.020154
0.9	5	0.017233	5	0.016867	4	0.017240

Table 2.8: New obtained numerical results for Example 3.

Size $(m,n)\downarrow$	$\theta\rightarrow$							
	0.1		0.5		0.65		0.95	
	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
$(10,20)$	364	0.574110	56	0.094392	37	0.023738	13	0.019624
$(50,100)$	380	4.114823	58	0.528836	39	0.141632	14	0.055535
$(100,200)$	386	13.905783	59	2.078463	39	0.664945	14	0.256223
$(1000,2000)$	403	1320.457701	62	542.518584	41	41.610999	15	107.677837

Chapter 3

A weighted-path following IPA for convex quadratic optimization based on modified search directions

3.1 A weighted-path following IPA for CQO

In this section, we study first the existence and uniqueness of the weighted-path of CQO and the new modified search directions. Finally, we state the generic weighted-path following full-Newton step IPA for CQO.

3.1.1 The weighted-path of CQO

Throughout the paper, we assume that both problems $(\mathcal{P})$ and $(\mathcal{D})$ satisfy the following conditions.

- Interior-Point-Condition (IPC). There exists a triplet of vectors (x^0, y^0, z^0) such that:

$$Ax^0 = b, x^0 > 0, A^T y^0 + z^0 - Qx^0 = c, z^0 > 0.$$

- Symmetric and positive semi-definiteness. The matrix Q is symmetric positive semidefinite, i.e. $Q = Q^T$ and $v^T Q v \geq 0$, for all $v \in \mathbb{R}^n$.

Getting an optimal solution for both problems $(\mathcal{P})$ and $(\mathcal{D})$ is equivalent to solving the following system of optimality conditions:

$$\begin{cases} Ax = b, x \geq 0, \\ A^T y + z - Qx = c, z \geq 0, \\ xz = 0. \end{cases} \tag{3.1}$$

Similar to the standard central-path methods, the basic idea behind weighted-path following IPMs is to replace the third equation (complementarity condition) in (3.1) by the parametrized equation $xz = \omega^2$; where $\omega \in \mathbb{R}^n_{++}$. Thus we consider the following parametrized system:

$$\begin{cases} Ax = b, x \geq 0, \\ A^T y + z - Qx = c, z \geq 0, \\ xz = \omega^2. \end{cases} \tag{3.2}$$

Under our assumptions, system (3.2) has a unique solution denoted by $(x(\omega), y(\omega), z(\omega))$ for all fixed $\omega \in \mathbb{R}^n_{++}$. The set

$$\{(x(\omega), y(\omega), z(\omega)) \ : \omega > 0\}$$

is called the weighted-path of both problems $(\mathcal{P})$ and $(\mathcal{D})$. If ω tends to zero then the limit of the weighted-path exists and since the limit point satisfies the complementarity condition, the limit yields an optimal solution for CQO. Similar to the monotone LCP [6], the existence and the uniqueness of the weighted-path can be derived in the following way. Consider the following log-barrier problem

$$\min_{(x,y,z)} x^T z - \sum_{i=1}^{n} \omega_i^2 \ln x_i z_i \ \text{ s.t. } \ Ax = b, \ A^T y + z - Qx = c, \ x > 0, \ z > 0.$$

Therefore, the necessary and sufficient optimality conditions of the log-barrier problem are characterized by the solutions of system (3.2). In other word, the existence and the uniqueness of the weighted-path is equivalent to the existence of unique minimizers for the log-barrier problem for each weight $\omega > 0$. It is easy to verify by the IPC, that the objective function of the log-barrier problem is strictly convex for each $\omega > 0$. Now, by the application of Newton's method for system (3.2), we get the classical Newton search directions [34]. Note that if $\omega = \sqrt{\mu} e$ with μ is a positive scalar, then the weighted-path reduces to the classical central-path. The relevance of the central-path has been discussed in the monographs, (see, e.g.,[44, 45]).

3.1.2 The new modified search direction

Following [6, 18], we turn now to describe the new modified Newton search direction for CQO. The AET based directions for CQO is simply based in replacing the weighted equation $xz = \omega^2$ by the new equation

$$\psi(xz) = \psi(\omega^2)$$

where $\psi(.) : (0, +\infty) \to \mathbb{R}$ is continuously differentiable and invertible function. Then, system (3.2) is converted to the following system

$$\begin{cases} Ax = b, x \geq 0, \\ A^T y + z - Qx = c, z \geq 0 \\ \psi(xz) = \psi(\omega^2), \end{cases} \tag{3.3}$$

where ψ is applied coordinate-wisely. As system (3.2) has a unique solution so is the system (3.3). Applying Newton's method to system (3.3) for a given strictly feasible point (x, y, z), i.e. the IPC holds, we obtain the following system:

$$\begin{cases} A\Delta x = 0, \\ A^T \Delta y + \Delta z - Q\Delta x = 0, \\ z\Delta x + x\Delta z = \dfrac{\psi(\omega^2) - \psi(xz)}{\psi'(xz)}, \end{cases} \tag{3.4}$$

where ψ' denotes the derivative of ψ.
To simplify matters, we define the vectors

$$v := \sqrt{xz} \text{ and } d := \sqrt{xz^{-1}}.$$

The vector d is used to scale the vectors x and z to the same vector v as:

$$d^{-1}x = dz = v. \tag{3.5}$$

Due to (3.5), the scaling directions are given by

$$d_x = d^{-1}\Delta x \text{ and } d_z = d\Delta z. \tag{3.6}$$

In addition, we have

$$x\Delta z + z\Delta x = v(d_x + d_z). \tag{3.7}$$

Now, since Q is positive semidefinite matrix, it follows that

$$d_x^T d_z = (\Delta x)^T(\Delta z) = (\Delta x)^T Q\Delta x \geq 0. \tag{3.8}$$

Hence, from (3.5), (3.7) and (3.8), system (3.4) can be written as

$$\begin{cases} \bar{A}d_x = 0, \\ \bar{A}^T \Delta y + d_z - \bar{Q}d_x = 0, \\ d_x + d_z = p_v \end{cases} \tag{3.9}$$

where

$$p_v = \frac{\psi(\omega^2) - \psi(v^2)}{v\psi'(v^2)} \tag{3.10}$$

with $\bar{A} = AD$, $\bar{Q} = DQD$ and $D :=\text{diag}(d)$.

Next, substituting $\psi(t) = t^{\frac{3}{2}}$ in (3.10) and in (3.4), yields

$$p_v = \frac{2}{3} v^{-2}(\omega^3 - v^3) \tag{3.11}$$

and

$$\begin{cases} A\Delta x = 0, \\ A^T\Delta y + \Delta z - Q\Delta x = 0, \\ z\Delta x + x\Delta z = \frac{2}{3}\frac{(\omega^3 - \sqrt{(xz)^3})}{\sqrt{xz}}. \end{cases} \tag{3.12}$$

Therefore, the new unique modified search directions $(\Delta x, \Delta y, \Delta z)$ are obtained by solving system (3.12). Moreover, the new iterate is computed by taking a full-Newton step as follows:

$$x_+ := x + \Delta x; \quad y_+ := y + \Delta y; \quad z_+ := z + \Delta z.$$

We end this subsection with this remark. By choosing function $\psi(t)$ appropriately, the system (3.12) can be used to define a class of new search directions. For example:
• $\psi(t) = t$ yields $p_v = v^{-1}(\omega^2 - v^2)$, we recuperate the standard weighted search directions (see [7, 8, 40]).
• $\psi(t) = \sqrt{t}$ yields $p_v = 2(\omega - v)$, we get Darvay's weighted search directions [18].

3.1.3 The proximity measure

For any positive vector v and according to (3.11), we define the norm-based proximity measure $\delta(v;\omega)$ as follows:

$$\delta(v;\omega) = \frac{3\|p_v\|}{2\omega_i} = \frac{\|v^{-2}(\omega^3 - v^3)\|}{\omega_i}, \ \forall i = 1, \cdots, n. \tag{3.13}$$

It is clear that

$$\delta(v;\omega) = 0 \Leftrightarrow \omega^3 = v^3 \Leftrightarrow xz = \omega^2.$$

So $\delta(v;\omega)$ is to measure the distance of a point (x, y, z) to the weighted-path $(x(\omega), y(\omega), z(\omega))$. Let us define another measure $\sigma_C(\omega)$ as follows:

$$\sigma_C(\omega) = \frac{\max(\omega)}{\min(\omega)} \geq 1. \tag{3.14}$$

The quantity $\sigma_C(\omega)$ is to measure the closeness of ω to the central-path. Here,

$$\min(\omega) = \min_i(\omega_i)$$

and likewise

$$\max(\omega) = \max_i(\omega_i).$$

Note that in (3.14), $\sigma_C(\omega) = 1$ if ω is on the central-path.

3.1.4 The generic weighted-path full-Newton step IPA for CQO

The weighted-path following IPA for CQO works as follows. First, we use a suitable threshold (default) value $\tau > 0$ with $0 < \tau < 3$ and we suppose that a strictly feasible initial point $(x^0 > 0, y^0, z^0 > 0)$ such that $\delta(x^0 z^0; \omega_0) \leq \tau$ for some known vector ω_0. Using the obtained search directions from (3.12) and taking a full Newton-step the algorithm produces a new iterate $(x + \Delta x, y + \Delta y, z + \Delta z)$. Then, the vector ω is reduced by the factor $(1 - \theta)$ with $0 < \theta < 1$ and solves system (3.12), and so target a new iterate and so on. This procedure is repeated until the stopping criterion $n \max(\omega^2) \leq \epsilon$ is satisfied for a given accuracy parameter $\epsilon > 0$. The generic IPA is stated in Algorithm 2 as follows.

Input:
A threshold parameter $\tau \leq 3$ (default $\tau = 1$);
an accuracy parameter $\epsilon > 0$;
a barrier update parameter θ, $0 < \theta < 1$ (default $\theta = \dfrac{1}{36\sqrt{2n}\sigma_c(\omega_0)}$);
a starting point (x^0, y^0, z^0) and ω_0 s.t. $\delta(x^0 z^0; \omega_0) \leq 1$;
begin
Set $x := x^0; y := y^0; z := z^0; \omega := \omega_0$;
while $n \max(\omega^2) \geq \epsilon$ **do**
begin
• $\omega := (1 - \theta)\omega$;
• Solve system (3.12) to obtain the direction $(\Delta x, \Delta y, \Delta z)$;
• Update $x := x + \Delta x, y := y + \Delta y, z := z + \Delta z$;
endwhile
end

Figure 3.1: Algorithm 2. The weighted-path full Newton step IPA for CQO

3.2 Convergence analysis

In this section, we will show across our new defaults that Algorithm 2 is well-defined and solves the CQO in polynomial complexity.

We first quote the following technical results which will be used later in the analysis of Algorithm 2.

Lemma 3.1. *Let* $(d_x, \Delta y, d_z)$ *be a solution of (3.9) and* $\omega > 0$. *If* $\delta := \delta(v; \omega) > 0$, *then* $\forall i = 1, \cdots, n$, *one has*

$$0 \leq d_x^T d_z \leq \frac{2}{9}\delta^2 \omega_i^2, \ \forall i \tag{3.15}$$

and

$$\|d_x d_z\|_\infty \leq \frac{\delta^2}{9}\omega_i^2, \;\|d_x d_z\| \leq \frac{2\delta^2}{9}\omega_i^2, \,\forall i. \tag{3.16}$$

Proof. For the first claim, we have

$$0 \leq \|d_x\|^2 + 2d_x^T d_z + \|d_z\|^2 = \|d_x + d_z\|^2 = \|p_v\|^2.$$

But since $d_x^T d_z \geq 0$ by (3.8), it follows that

$$d_x^T d_z \leq \frac{1}{2}\|p_v\|^2 = \frac{2}{9}\delta^2\omega_i^2, \,\forall i.$$

For the second claim, as

$$d_x d_z = \frac{1}{4}((d_x + d_z)^2 - (d_x - d_z)^2)$$

then, we have

$$\begin{aligned}\|d_x d_z\|_\infty &= \frac{1}{4}(\|(d_x + d_z)^2 - (d_x - d_z)^2\|_\infty) \\ &\leq \frac{1}{4}\max(\|d_x + d_z\|_\infty^2, \|d_x - d_z\|_\infty^2) \\ &\leq \frac{1}{4}\max(\|d_x + d_z\|^2, \|d_x - d_z\|^2).\end{aligned}$$

Since $d_x^T d_z \geq 0$, we have $\|d_x + d_z\|^2 \geq \|d_x - d_z\|^2$, then we obtain

$$\|d_x d_z\|_\infty \leq \frac{1}{4}\|d_x + d_z\|^2 = \frac{1}{4}\|p_v\|^2 = \frac{\delta^2}{9}\omega_i^2, \,\forall i.$$

For the last claim, we have

$$\begin{aligned}\|d_x d_z\|^2 = e^T(d_x d_z)^2 &= \frac{1}{16}e^T((d_x + d_z)^2 - (d_x - d_z)^2)^2 \\ &= \frac{1}{16}(\|(d_x + d_z)^2 - (d_x - d_z)^2)\|^2) \leq \frac{1}{16}(\|d_x + d_z\|^2 + \|d_x - d_z\|^2)^2 \\ &\leq \frac{1}{8}(\|d_x + d_z\|^4 + \|d_x - d_z\|^4) = \frac{1}{4}\|d_x + d_z\|^4 = \frac{1}{4}\|p_v\|^4 = \frac{4}{81}\delta^4\omega_i^4.\end{aligned}$$

Hence,

$$\|d_x d_z\| \leq \frac{2}{9}\delta^2\omega_i^2, \,\forall i.$$

This completes the proof. □

The next lemma investigates the feasibility of a full-Newton step.

Lemma 3.2. *Let (x, y, z) be a strictly feasible point and assume $\delta := \delta(v; \omega) < 3$, then $x_+ = x + \Delta x > 0$ and $z_+ = z + \Delta z > 0$, i.e., x_+ and z_+ are strictly feasible.*

Proof. Let $\alpha \in [0, 1]$, we define $x(\alpha) = x + \alpha\Delta x$ and $z(\alpha) = z + \alpha\Delta z$. Then, we have

$$x(\alpha)z(\alpha) = xz + \alpha(x\Delta z + z\Delta x) + \alpha^2\Delta x\Delta z.$$

Using (3.7) and (3.8), we get

$$x(\alpha)z(\alpha) = (1-\alpha)v^2 + \alpha(v^2 + vp_v + \alpha d_x d_z). \tag{3.17}$$

Hence $x(\alpha)z(\alpha) > 0$ if $v^2 + vp_v + \alpha d_x d_z > 0$. By Lemma 3.1 (3.16), and from (3.11) and let $\delta < 3$, it follows that

$$\begin{aligned} v^2 + vp_v + \alpha d_x d_z &\geq v^2 + vp_v - \alpha \|d_x d_z\|_\infty e \\ &\geq v^2 + vp_v - \alpha \frac{\delta^2}{9}\omega_i^2 \\ &> \frac{1}{3}v^2 + \frac{2}{3}\omega_i^3 v^{-1} - \omega_i^2, \ \forall i. \end{aligned}$$

Clearly, $x(\alpha)z(\alpha) > 0$ if

$$\frac{1}{3}v^2 + \frac{2}{3}\omega_i^3 v^{-1} - \omega_i^2 \geq 0, \ \forall i.$$

Letting

$$g(t) = \frac{1}{3}t^2 + \frac{2\omega_i^3}{3}t^{-1} - \omega_i^2, t > 0, \ \forall i.$$

g is a strictly convex function and has a minimum at $t = \omega_i$, and so $g(t) \geq g(\omega_i) = 0$. Hence,

$$\frac{1}{3}v^2 + \frac{2\omega_i^3}{3}v^{-1} - \omega_i^2 \geq 0, \ \forall i.$$

Therefore, $\forall \alpha \in [0,1]$, $x(\alpha)z(\alpha) > 0$. Since x and z are positive which implies that $x(\alpha) > 0$ and $z(\alpha) > 0$ for all $\alpha \in [0,1]$. So by continuity the vectors $x(1) = x_+$ and $z(1) = z_+ > 0$. This implies the lemma. □

For the new iterates x_+ and z_+, we define the vector $v_+ = \sqrt{x_+ z_+}$.

Lemma 3.3. *Assume $\delta < 3$, then*

$$(v_+)_i \geq \frac{\omega_i}{3}\sqrt{9-\delta^2}, \ \ \forall i.$$

Proof. In (3.17), setting $\alpha = 1$, then from (3.11), we have

$$(v_+^2)_i = v^2 + vp_v + d_x d_z = \frac{1}{3}v^2 + \frac{2}{3}v^{-1}\omega_i^3 + d_x d_z, \ \forall i.$$

From the proof of Lemma 3.2, we have $\frac{1}{3}v^2 + \frac{2}{3}v^{-1}\omega_i^3 - \omega_i^2 \geq 0$ if $\delta < 3$, $\forall i$. From which we deduce that $\frac{1}{3}v^2 + \frac{2}{3}v^{-1}\omega_i^3 \geq \omega_i^2$, $\forall i$. Consequently,

$$(v_+^2)_i \geq \omega_i^2 + d_x d_z, \ \forall i.$$

Now, due to (3.16), we deduce that

$$\omega_i^2 + d_x d_z \geq \left(\omega_i^2 - \|d_x d_z\|_\infty e\right) \geq \frac{\omega_i^2}{9}(9-\delta^2), \ \forall i.$$

Hence,

$$(v_+)_i \geq \frac{\omega_i}{3}\sqrt{9-\delta^2},\ \forall i.$$

This proves the lemma. □

Next, we prove that the iterate across the proximity measure is locally quadratically convergent during the Newton process.

Lemma 3.4. *Assume $\delta < 3$, then*

$$\delta_+ := \delta(v_+) := \delta(x_+ z_+;\omega) \leq \frac{5}{9}\left(\frac{9}{9-\delta^2} + \frac{3}{3+\sqrt{9-\delta^2}}\right)\delta^2.$$

In addition, if $\delta \leq 1$ then

$$\delta_+ \leq \left(\frac{5}{8} + \frac{5}{9+6\sqrt{2}}\right)\delta^2,$$

which means the local quadratic convergence of the full-Newton step.

Proof. We have

$$\begin{aligned}
\delta(v_+;\omega) &= \frac{1}{\omega_i}\left\|v_+^{-2}(\omega^3 - v_+^3)\right\| \\
&= \frac{1}{\omega_i}\left\|\frac{\omega^3 - v_+^3}{v_+^2}\right\| \\
&= \frac{1}{\omega_i}\left\|\frac{(\omega - v_+)(\omega^2 + \omega v_+ + v_+^2)}{v_+^2}\right\| \\
&= \frac{1}{\omega_i}\left\|\frac{(\omega^2 - v_+^2)(\omega^2 + \omega v_+ + v_+^2)}{v_+^2(\omega + v_+)}\right\|.
\end{aligned}$$

For all fixed $\omega \in \mathbb{R}^n_{++}$, i.e. $\omega_i > 0,\ \forall i$, we define the function g by

$$g(t) = \frac{\omega_i^2 + \omega_i t + t^2}{t^2(\omega_i + t)} = \frac{\omega_i}{t^2} + \frac{1}{\omega_i + t},\ \forall i.$$

Using g, we deduce that

$$\delta_+ = \frac{1}{\omega_i}\|g(v_+)(\omega^2 - v_+^2)\| \leq \frac{1}{\omega_i}\|g(v_+)\|_\infty\|\omega^2 - v_+^2\|,\ \forall i,$$

where $g(v_+) = (g_1(v_+)_1, \cdots, g_n(v_+)_n)$. The function g is continuous and monotonically decreasing and positive on $(0, +\infty)$. Hence, by Lemma 3.3

$$0 < |g_i((v_+)_i)| = g_i((v_+)_i) \leq g(v_+) \leq g\left(\frac{\omega_i}{3}\sqrt{9-\delta^2}\right),\ \forall i.$$

Then

$$\|g(v_+)\|_\infty \leq \frac{9\omega_i}{9\omega_i^2 - \delta^2\omega_i^2} + \frac{3}{\omega_i(3+\sqrt{9-\delta^2})},\forall i.$$

This implies that

$$\delta^+ \leq \frac{1}{\omega_i^2}\left(\frac{9}{9-\delta^2} + \frac{3}{3+\sqrt{9-\delta^2}}\right)\|\omega^2 - v_+^2\|,\ \forall i.$$

Next, in (3.17), setting $\alpha = 1$ and from (3.11), we have

$$\begin{aligned}\|\omega^2 - v_+^2\| =& \|\omega^2 - (v^2 + vp_v + d_x d_z)\| \\ =& \|\omega^2 - \frac{1}{3}v^2 - \frac{2}{3}v^{-1}\omega^3 - d_x d_z\|.\end{aligned}$$

Then

$$\|\omega^2 - v_+^2\| \leq \|\omega^2 - \frac{1}{3}v^2 - \frac{2}{3}v^{-1}\omega^3\| + \|d_x d_z\|.$$

Next, we may write

$$\left\|\omega^2 - \frac{1}{3}v^2 - \frac{2}{3}v^{-1}\omega^3\right\| = \left\|\frac{\omega^2 - \frac{1}{3}v^2 - \frac{2}{3}v^{-1}\omega^3}{v^{-4}(\omega^3 - v^3)^2} . \frac{9p_v^2}{4}\right\|.$$

And after elementary reductions, we get

$$\left\|\omega^2 - \frac{1}{3}v^2 - \frac{2}{3}v^{-1}\omega^3\right\| = \left\|\varphi(v) . \frac{9p_v^2}{4}\right\|$$

where

$$\varphi(v) = \frac{v^3(v + 2\omega)}{3(v^2 + v\omega + \omega^2)^2}.$$

Let us consider for all i, the following function

$$\varphi(t) = \frac{t^3(t + 2\omega_i)}{3(t^2 + \omega_i t + \omega_i^2)^2}.$$

$\varphi(t)$ is continuous and monotonically increasing and positive for all $t \in (0, +\infty)$. Then we have

$$0 \leq \varphi(t) < \frac{1}{3} = \lim_{t \mapsto \infty} \varphi(t), \forall t > 0.$$

This yields

$$0 \leq \varphi(v_i) < \frac{1}{3}, \forall i = 1, ..., n.$$

Consequently,

$$0 < |\varphi(v_i)| = \varphi(v_i) \leq \frac{1}{3}. \tag{3.18}$$

Then, as $\|p_v\|^2 = \frac{4}{9}\omega_i^2\delta^2$, $\|\varphi(v)\|_\infty = \max_i \varphi(v_i) \leq \frac{1}{3}$ and $\|p_v^2\| \leq \|p_v\|^2$, these imply that

$$\left\|\omega^2 - \frac{1}{3}v^2 - \frac{2}{3}v^{-1}\omega^3\right\| \leq \|\varphi(v)\|_\infty \frac{9}{4}\|p_v\|^2 = \frac{1}{3}\omega_i^2\delta^2, \ \forall i.$$

Due to (3.16), it follows

$$\|\omega^2 - v_+^2\| \leq \frac{5}{9}\delta^2\omega_i^2, \ \forall i.$$

Next, for $\delta < 3$, we have

$$\delta_+ \leq \frac{5}{9}\left(\frac{9}{9 - \delta^2} + \frac{3}{3 + \sqrt{9 - \delta^2}}\right)\delta^2.$$

Now, let $\delta \leq 1$, then $\frac{9}{9-\delta^2} \leq \frac{9}{8}$ and $\frac{3}{3+\sqrt{9-\delta^2}} \leq \frac{3}{3+2\sqrt{2}}$. Hence, after some simplifications, we obtain

$$\delta_+ \leq \left(\frac{5}{8} + \frac{5}{9+6\sqrt{2}}\right)\delta^2 < \delta^2.$$

This completes the proof. □

The next lemma gives an upper bound for the duality gap after a full-Newton step.

Lemma 3.5. *After a full-Newton step it holds*

$$x_+^T z_+ \leq 2n \max(\omega^2).$$

Proof. As $v_+^2 = x_+ z_+$, we have

$$\begin{aligned}
(x_+)^T z_+ &= e^T v_+^2 = e^T(v^2 + vp_v + d_x d_z) \\
&= e^T(\omega^2 + v^2 + vp_v - \omega^2) + d_x^T d_z \\
&= e^T\omega^2 + e^T(v^2 + vp_v - \omega^2) + d_x^T d_z \\
&= e^T\omega^2 + d_x^T d_z + e^T(v^2 + \frac{2}{3}v^{-1}\omega^3 - \frac{2}{3}v^2 - \omega^2) \\
&= e^T\omega^2 + d_x^T d_z + e^T\left(\frac{\frac{1}{3}v^2 + \frac{2}{3}v^{-1}\omega^3 - \omega^2}{v^{-4}(\omega^3 - v^3)^2}\frac{9p_v^2}{4}\right).
\end{aligned}$$

Then after some reductions, we get

$$(x_+)^T z_+ = e^T\omega^2 + d_x d_z + \frac{9}{4}e^T\varphi(v)p_v^2$$

where $\varphi(v) = (\varphi(v_1), \varphi(v_2), \cdots, \varphi(v_n))$ with

$$\varphi(v_i) = \frac{v_i^3(v_i + 2\omega_i)}{3(v_i^2 + v_i\omega_i + \omega_i^2)^2}, \text{ for } i = 1, \cdots, n.$$

Due to (3.18), we have

$$0 < |\varphi(v_i)| = \varphi(v_i) \leq \frac{1}{3}, \forall i.$$

Therefore, by Lemma 3.1 and from (3.13), we have

$$\begin{aligned}
e^T v_+^2 &\leq e^T\omega^2 + d_x^T d_z + \frac{9}{4}\max_i|\varphi(v_i)| e^T p_v^2 \\
&\leq e^T\omega^2 + d_x^T d_z + \frac{3}{4}\|p_v\|^2 \\
&\leq n\max(\omega^2) + \frac{2}{9}\delta^2\omega_i^2 + \frac{1}{3}\delta^2\omega_i^2, \forall i. \\
&\leq n\max(\omega^2) + \frac{2}{9}\delta^2\max(\omega^2) + \frac{1}{3}\delta^2\max(\omega^2) \\
&\leq \left(n + \frac{5}{9}\delta^2\right)\max(\omega^2) \leq (n + \delta^2)\max(\omega^2).
\end{aligned}$$

Let $\delta \leq 1$, then $e^T v_+^2 \leq (n+1)\max(\omega^2)$, but since $(n+1) \leq 2n, \forall n \geq 1$, it follows that $e^T v_+^2 \leq 2n \max(\omega^2)$. This gives the required result. $\square$

Next lemma investigates the effect of a full Newton-step on the proximity measure followed by updating the weighted vector ω by a factor $(1-\theta)$, where $0 < \theta < 1$.

Theorem 3.6. *Let $\omega_+ = (1-\theta)\omega$ and let $x_+ > 0, z_+ > 0$, then we have*

$$\delta(v_+;\omega_+) \leq \delta_+ + \frac{3\sqrt{2n}\theta}{1-\theta}\sigma_C(\omega).$$

In addition, let $\delta \leq 1$, $\sigma_C(\omega) \geq 1$, and $\theta = \dfrac{1}{36\sqrt{2n}\sigma_C(\omega)}$, $n \geq 2$, then $\delta(v_+;\omega_+) \leq 1$.

Proof. Let $\delta(x_+z_+;\omega_+)$ and $\omega_+ = (1-\theta)\omega$ where $\theta \in (0,1)$. We have

$$\begin{aligned}
\delta(v_+;\omega_+) &= \frac{1}{(\omega_+)_i}\|v_+^{-2}(\omega_+^3 - v_+^3)\| \\
&= \frac{1}{(1-\theta)\omega_i}\|\frac{((1-\theta)^3\omega^3 - (v_+)^3)}{v_+^2}\| \\
&= \frac{1}{(1-\theta)\omega_i}\|v_+^{-2}\left((1-\theta)^3\omega^3 + (1-\theta)^3 v_+^3 - (1-\theta)^3 v_+^3 - v_+^3\right)\| \\
&\leq \frac{1}{(1-\theta)\omega_i}\left(\|(1-\theta)^3 v_+^{-2}(\omega^3 - v_+^3)\| + \|v_+((1-\theta)^3 - 1)\|\right) \\
&= (1-\theta)^2\delta_+ + \frac{|(1-\theta)^3 - 1|}{1-\theta}\frac{\|v_+\|}{\omega_i} \\
&\leq \delta_+ + \frac{|(1-\theta)^3 - 1|}{1-\theta}\frac{\|v_+\|}{\omega_i} \\
&\leq \delta_+ + \frac{|\theta(\theta^2 - 3\theta + 3)|}{1-\theta}\frac{\|v_+\|}{\min(\omega)}.
\end{aligned}$$

As $0 < \theta^2 - 3\theta + 3 < 3, \forall\theta \in (0,1)$, we obtain

$$\delta(v_+;\omega_+) \leq \delta_+ + \frac{3\theta}{1-\theta}\frac{\|v_+\|}{\min(\omega)}.$$

By Lemma 3.5, we have

$$\|v_+\| \leq \sqrt{2n}\max(\omega).$$

Next, Lemma 3.4 implies that

$$\delta_+ \leq \left(\frac{5}{8} + \frac{5}{9+6\sqrt{2}}\right)\delta^2.$$

Now, let $\delta \leq 1$, then we get

$$\delta(v_+;\omega_+) \leq \left(\frac{5}{8} + \frac{5}{9+6\sqrt{2}}\right) + \frac{3\sqrt{2n}\sigma_C(\omega)\theta}{1-\theta}.$$

Let $\theta = \dfrac{1}{36\sqrt{2n}\sigma_c(\omega)}$, $n \geq 2$ and $\sigma_c(\omega) \geq 1$ so $\theta \in \left[0, \dfrac{1}{72}\right]$ from which we deduce that

$$\delta(v_+; \omega_+) \leq \xi(\theta)$$

where

$$\xi(\theta) = \left(\frac{5}{8} + \frac{5}{9 + 6\sqrt{2}}\right) + \frac{1}{12(1-\theta)}.$$

As $\xi'(\theta) = \dfrac{1}{12\,(\theta-1)^2} > 0$, then $\xi(\theta)$ is strictly increasing on the interval $\left[0, \dfrac{1}{72}\right]$. Hence $\xi(\theta) \leq \xi\left(\dfrac{1}{72}\right) = 0.9966 < 1$. This proves the theorem. □

Theorem 3.6, shows that Algorithm 2 is well-defined since the conditions $x > 0, z > 0$, and $\delta(xz;\omega) \leq 1$ are maintained throughout the algorithm. Also observe that $\sigma_C(\omega) = \sigma_C(\omega_0)$ for all iterates produced by Algorithm 2.

The next lemma derives an upper bound for the total number of iterations produced by Algorithm 2.

Lemma 3.7. *Let x^k and z^k be the k-th iteration produced by Algorithm 2. Then*

$$(x^k)^T z^k \leq \epsilon$$

if

$$k \geq \left\lceil \frac{1}{\theta} \log \frac{2n \max(\omega_0)^2}{\epsilon} \right\rceil.$$

Proof. After k iterations, we have $\omega_k = (1-\theta)^k \omega_0$. By Lemma 3.7, we get that

$$(x^k)^T z^k \leq 2n(1-\theta)^{2k} \max(\omega_0)^2.$$

Thus the inequality $(x^k)^T z^k \leq \epsilon$ holds if

$$2n(1-\theta)^{2k} \max(\omega_0)^2 \leq \epsilon.$$

Taking logarithms, we find

$$2k \log(1-\theta) \leq \log \epsilon - \log 2n \max(\omega_0)^2.$$

Using the inequality $-\log(1-\theta) \geq \theta$ where $0 < \theta < 1$, so the above inequality holds if

$$k\theta \geq \frac{1}{2} \log \frac{2n \max(\omega_0)^2}{\epsilon}.$$

This completes the proof. □

Theorem 3.8. *Suppose that (x^0, y^0, z^0) is a strictly feasible starting point, $\omega_0 = \dfrac{x^0 z^0}{\sqrt{2}\max(x^0 z^0)}$ and $\delta(x^0 z^0; \omega_0) \leq 1$. Let $\theta = \dfrac{1}{36\sqrt{2n}\sigma_c(\omega_0)}$, then Algorithm 2, requires at most*

$$O\left(\sqrt{n}\sigma_C(\omega_0)\log\frac{n}{\epsilon}\right)$$

iterations for getting an ϵ-approximate solution of CQO.

Proof. Let $\theta = \dfrac{1}{36\sqrt{2n}\sigma_C(\omega_0)}$, Theorem 3.8 follows directly from Lemma 3.7. □

Corollary 3.9. *If we take $\omega_0 = \dfrac{1}{\sqrt{3}}e$, then Algorithm 2, requires at most $\mathcal{O}\left(\sqrt{n}\log\dfrac{n}{\epsilon}\right)$ iterations which is the currently best known iteration bound for short-update method.*

Proof. The proof is an immediate consequence of Theorem 3.8. □

3.3 Numerical results

In this section, we implement Algorithm 2 on some examples of CQO with different size by using **Mat lab R2010a** and run on a **PC** with **CPU 2.00 GHz** and **4.00 G RAM** memory and double precision format. Here our accuracy is set to $\epsilon = 10^{-4}$. The strictly feasible initial point $(x^0 > 0, y^0, z^0 > 0)$ is taken such that $\delta(x^0 z^0, \omega_0) \leq 1$. The optimal primal-dual solution is denoted by $(x^\star, y^\star, z^\star)$. Here, we display the following notations: the "Iter" denotes the number of iterations produced by the algorithm to obtain an approximated optimal solution. The "CPU" denotes the time (in second) required to obtain an approximate optimal solution for CQO. Also to improve our numerical results we have relaxed the barrier vector $\omega_0 = \{\dfrac{1}{\sqrt{2}}e, \sqrt{x^0 z^0}, \dfrac{x^0 z^0}{\sqrt{2}\max(x^0 z^0)}\}$, with the update barrier $\theta = \dfrac{1}{36\sqrt{2n}\sigma_C(\omega_0)}$, $n \geq 2$. We also display a table for the number of iterations and the elapsed time for each example.

Example 4. We consider the convex quadratic optimization, where

$$A = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{pmatrix}, \; Q = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

$c = (1.3333, 1.3333, 1.3333, 1.3333)^T$, $b = (0.33, 2)^T$. The strictly feasible starting point x^0 and z^0 are chosen according to each value of ω_0 and such that $\delta(x^0 z^0; \omega_0) \leq 1$. For this example, we take

$$x^0 = (0.3333, 0.3333, 0.3333, 0.3333)^T, \; y^0 = (-2, -2)^T, \; z^0 = (2, 2, 2, 2)^T.$$

Table 3.1: Numerical results for Example 1.

$\omega_0 \to$					
$\frac{1}{\sqrt{2}}e$		$\sqrt{x^0 z^0}$		$\frac{x^0 z^0}{\sqrt{2}\max(x^0 z^0)}$	
Iter	CPU	Iter	CPU	Iter	CPU
1862	0.53163	1876	0.137857	1862	0.139970

A primal-dual optimal solution of Example 1 is:

$$
\begin{aligned}
x^\star &= (0.2000, 0.5333, 0.0000, 0.0000)^T, \\
y^\star &= (-2.0800, -1.1733)^T, \\
z^\star &= (0.0000, 0.0000, 1.4133, 0.5067)^T.
\end{aligned}
$$

Example 5. The data of the following convex quadratic problem is given by

$$
A = \begin{pmatrix}
1 & -1 & 1.9 & 1.25 & 1.2 & 0.4 & -0.7 & 1.06 & 1.5 & 1.05 \\
1.3 & 1.2 & 0.15 & 2.15 & 1.25 & 1.5 & 0.4 & 1.52 & 1.3 & 1 \\
1.5 & -1.1 & 3.5 & 1.25 & 1.8 & 2 & 1.95 & 1.2 & 1 & -1
\end{pmatrix}
$$

$$
Q = \begin{pmatrix}
30 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 21 & 0 & 1 & -1 & 1 & 0 & 1 & 0.5 & 1 \\
1 & 0 & 15 & -0.5 & -2 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & -0.5 & 30 & 3 & -1 & 1 & -1 & 0.5 & 1 \\
1 & -1 & -2 & 3 & 27 & 1 & 0.5 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 & 16 & -0.5 & 0.5 & 0 & 1 \\
1 & 0 & 0 & 1 & 0.5 & -0.5 & 8 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 & 0.5 & 1 & 24 & 1 & 1 \\
1 & 0.5 & 1 & 0.5 & 1 & 0 & 1 & 1 & 39 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 11
\end{pmatrix},
$$

$c = (2.1, 3.45, 4.25, 2.5, 2.75, 4, 4.7, 2.95, 1.4, 4)^T$, $b = (0.7660, 1.1770, 1.21)^T$. The initial point is taken as:

$$
\begin{aligned}
x^0 &= (0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1)^T, \\
y^0 &= (-1, -1, -1)^T, \\
z^0 &= (6, 6, 6, 6, 6, 6, 6, 6, 6, 6)^T.
\end{aligned}
$$

Table 3.2: Numerical results for Example 2.

$\omega_0 \rightarrow$					
$\frac{1}{\sqrt{2}}e$		$\sqrt{x^0z^0}$		$\frac{x^0z^0}{\sqrt{2}\max(x^0z^0)}$	
Iter	CPU	Iter	CPU	Iter	CPU
3022	0.293683	3037	0.293173	3022	0.294073

A primal-dual optimal solution for this example is:

$$x^\star = (0.0973, 0.0000, 0.0236, 0.1725, 0.0810, 0.2061, 0.0000, 0.1124, 0.0730, 0.0000)^T,$$
$$y^\star = (-1.5354, 2.0035, 0.5461)^T,$$
$$z^\star = (0.0000, 1.5550, 0.0000, 0.0000, 0.0000, 0.0000, 0.5015, 0.0000, 0.0000, 3.8707)^T.$$

Example 3. We consider the CQO, where $n = 2m$ and

$$A[i,j] = \begin{cases} 0 & \text{if} \quad i \neq j \text{ or } (i+1) \neq j \\ 1 & \text{if} \quad j = i+m. \end{cases}$$

$$Q[i,j] = \begin{cases} 2j-1 & \text{if} \quad i > j \quad \text{or } i = j = n \\ 2i-1 & \text{if} \quad i < j \\ i(i+1) & \text{if} \quad i = j. \end{cases} \qquad c = \begin{cases} 0 & \text{if} \qquad i = i+m \\ 1 & \text{else if,} \end{cases},$$
$$b[i] = 0.5.$$

For this example, we take $x^0 = 0.5e$, $y^0 = 0_{\mathbb{R}^m}$ and $z^0 = e$ as the strictly feasible initial point. The obtained numerical results with $\theta = \frac{1}{36\sqrt{2n}\sigma_C(\omega_0)}$ are showed in the table below.

Table 3.3: Numerical results for Example 3 with different sizes.

Size $(m,n)\downarrow$	$\omega_0 \rightarrow$					
	$\frac{1}{\sqrt{2}}e$		$\sqrt{x^0z^0}$		$\frac{x^0z^0}{\sqrt{2}\max(x^0z^0)}$	
	Iter	CPU	Iter	CPU	Iter	CPU
$(10,20)$	4356	0.819771	4356	0.797447	4356	0.806954
$(50,100)$	10162	98.984386	10163	134.182516	10163	148.585150
$(100,200)$	14624	577.192434	14625	694.052203	14625	708.916735
$(1000,2000)$	51120	1481.75235	51120	1482.24890	51120	1481.06895

A primal-dual optimal solution of Example 3 is:

$$x^\star = (0.0000, \cdots, 0.0000, 0.4185)^T,$$
$$y^\star = (0.0388, \cdots, 0.0388)^T,$$
$$z^\star = (0.3876, \cdots, 0.3876, 0.0000)^T.$$

Comment. Across the obtained numerical results, we see that Algorithm 2 computes a primal-dual optimal solution for CQOs but in a large number of iterations and with a significant elapsed time. This means that the algorithm converges slowly to an optimal solution while using θ stated in our analysis. The cause is due to the fact that θ becomes very small for problems with a large size n. Consequently, the rate of decrease $(1-\theta)$ in the sequence of barrier vectors $\{\omega_k\}$ approaches to one.

3.3.1 A numerical amelioration of Algorithm 2

In this subsection, based on our comment and in order to improve our numerical results, we import some changes on the original version of Algorithm 2, where instead of using the updating θ provided by our analysis, we take it as a constant belongs to the set $\{0.1, ..., 0.9\}$. Moreover, to guarantee that the iterates remain interior, we introduce a step-size $\alpha_{\max} > 0$ such that $x + \rho\alpha_{\max}\Delta x > 0$ and $z + \rho\alpha_{\max}\Delta z > 0$ with $\alpha_{\max} = \min\{\alpha_{\mathcal{P}}, \alpha_{\mathcal{D}}\}$ and $\rho \in (0,1)$ where $\alpha_{\mathcal{P}}$ and $\alpha_{\mathcal{D}}$ are given by

$$\alpha_{\mathcal{P}} = \begin{cases} \min\limits_i \left(-\dfrac{x_i}{\Delta x_i}\right) & \text{if} \quad \Delta x_i < 0 \\ 1 & \text{if} \quad \Delta x_i \geq 0, \end{cases} \qquad \alpha_{\mathcal{D}} = \begin{cases} \min\limits_i \left(-\dfrac{z_i}{\Delta z_i}\right) & \text{if} \quad \Delta z_i < 0 \\ 1 & \text{if} \quad \Delta z_i \geq 0. \end{cases}$$

Based, on the imported changes our new obtained numerical results for the same examples are stated in tables below.

Table 3.4: Numerical results for Example 1 with different value of θ and ω_0 .

$\theta \downarrow$	$\omega_0 \rightarrow$					
	$\frac{1}{\sqrt{2}}e$		$\sqrt{x^0 z^0}$		$\frac{x^0 z^0}{\sqrt{2}\max(x^0 z^0)}$	
	Iter	CPU	Iter	CPU	Iter	CPU
0.1	175	0.139169	176	0.193436	175	0.117749
0.2	84	0.083488	83	0.081854	83	0.062218
0.3	52	0.067883	52	0.068502	52	0.046201
0.4	36	0.063592	36	0.060969	36	0.039543
0.5	27	0.054806	27	0.051760	27	0.039456
0.6	20	0.054244	20	0.054322	20	0.032333
0.7	19	0.048853	19	0.055084	19	0.031448
0.8	19	0.051142	19	0.048136	19	0.036626
0.9	18	0.051417	18	0.047807	18	0.035654

Table 3.5: Numerical results for Example 2 with different value of θ and ω_0 .

$\theta \downarrow$	$\omega_0 \rightarrow$					
	$\frac{1}{\sqrt{2}}e$		$\sqrt{x^0 z^0}$		$\frac{x^0 z^0}{\sqrt{2}\max(x^0 z^0)}$	
	Iter	CPU	Iter	CPU	Iter	CPU
0.1	179	0.187614	180	0.201249	179	0.190970
0.2	85	0.146417	86	0.115407	85	0.262326
0.3	53	0.092770	53	0.081417	53	0.069546
0.4	37	0.077989	37	0.067789	37	0.052182
0.5	27	0.066829	27	0.064044	27	0.045523
0.6	21	0.057692	21	0.057452	21	0.042347
0.7	21	0.057708	21	0.054687	21	0.042074
0.8	21	0.063730	21	0.058077	21	0.039714
0.9	21	0.059037	21	0.059576	21	0.036237

Table 3.6: Numerical results for Example 3 with different value of θ and ω_0 .

$\omega_0 \downarrow$	Size $(m, n) \downarrow$	$\theta \rightarrow$						
		0.1		0.3		0.5		0.9
		Iter	CPU	Iter	CPU	Iter	CPU	Iter
$\frac{1}{\sqrt{2}}e$	$(10, 20)$	182	0.428436	53	0.087912	28	0.060326	18
	$(50, 100)$	190	2.366860	55	0.648054	29	0.340302	18
	$(100, 200)$	193	8.695862	57	2.523648	29	1.421619	18
	$(1000, 2000)$	205	5240.8705	59	2237.279743	31	835.91287	19
$\sqrt{x^0 z^0}$	$(10, 20)$	182	0.255420	53	0.090728	28	0.061706	18
	$(50, 100)$	190	2.404953	55	0.694129	29	0.400725	18
	$(100, 200)$	193	8.743116	57	2.542191	29	1.338784	18
	$(1000, 2000)$	204	5239.9845	59	2236.4251	31	803.36214	19
$\frac{x^0 z^0}{\sqrt{2}\max(x^0 z^0)}$	$(10, 20)$	182	0.274023	53	0.086902	28	0.059837	18
	$(50, 100)$	190	2.003090	55	0.640520	29	0.365868	18
	$(100, 200)$	193	8.822089	57	2.344394	29	1.343666	18
	$(1000, 2000)$	204	5239.5095	59	2237.26837	31	784.092613	19

General conclusion and perspectives

In this dissertation, we have developed a feasible full-Newton step path-following method for CQO based on a modified Newton search direction obtained by the application of an AET to the central equations introduced by the univariate function $\psi(t) = t^2$. Under new appropriate selection of defaults of τ and θ, the algorithm is well-defined and the favorable iteration bound of the algorithm with short-step method is achieved, namely, $\mathcal{O}(\sqrt{n}\log\frac{n}{\epsilon})$. This iteration bound is as good as for CQOs. Meanwhile, the obtained numerical results by Algorithm 1 in its original version are not good for solving CQO problems with large size. But, with the imported modifications, the obtained numerical results are significantly ameliorated.
We have also presented a new weighted full-Newton step path-following interior-point method for CQO based on a new modified Newton search direction obtained by the application of the AET technique introduced by the new univariate function $\psi(t) = t^{\frac{3}{2}}$ for the Newton system which defines the weighted-path. New appropriate choices of the defaults of τ and θ are proposed where the favorable iteration bound of the algorithm with short-step method is achieved, namely, $\mathcal{O}(\sqrt{n}\log\frac{n}{\epsilon})$. This iteration bound is as good as for LO analogue. Meanwhile, for the obtained numerical results by Algorithm 2 for its first version are not good for CQOs problems with a large size. But, with the imported changes on Algorithm 2, the obtained numerical results are significantly improved.
Finally some perspectives are given.
1- A good topic of research is the extension of this two algorithms for other type of optimization problems.
2- The ameliorated algorithm good topic of study the convergence and so on.

Appendix

LU decomposition

The system $Ax = b$ of linear equations can be solved quickly if A can be factored as $A = LU$ where L and U are of a particularly nice form. Certain matrices are easier to work with than others. In this section, we will see how to write any square matrix A as the product of two matrices that are easier to work with. We will write $A = LU$, where:

- L is lower triangular. This means that all entries above the main diagonal are zero. In notation, $L = (l^i_j)$ with $l^i_j = 0$ for all $j > i$.

$$L = \begin{pmatrix} l^1_1 & 0 & 0 & 0 \\ l^2_1 & l^2_2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ l^n_1 & l^n_2 & \cdots & l^n_n \end{pmatrix},$$

- U is upper triangular. This means that all entries below the main diagonal are zero. In notation, $U = (u^i_j)$ with $u^i_j = 0$ for all $j < i$.

$$U = \begin{pmatrix} u^1_1 & u^1_2 & \cdots & u^1_n \\ 0 & u^2_2 & \cdots & u^2_n \\ \vdots & 0 & \cdots & \vdots \\ 0 & 0 & \cdots & u^n_n \end{pmatrix},$$

$A = LU$ is called an LU decomposition of A.
This is useful trick for many computational reasons. It is much easier to compute the inverse of un upper or lower triangular matrix. Since inverses are useful for solving linear systems, this makes solving any linear system associated to the matrix much faster as well.

Bibliography

[1] M. Achache. A new primal-dual path-following method for convex quadratic programming. Comput. Appl. Math, 25 : 97-110 (2006).

[2] M. Achache. Complexity analysis of a weighted-full-Newton step interior-point algorithm for $P_*(\kappa)$-LCP, RAIRO-Oper. Res. 50 : 131-143 (2016).

[3] M. Achache. A new parameterized kernel function for LO yielding the best known iteration bound for a large-update interior point algorithm. Afrika Matematika, 27(3): 591-601 (2016).

[4] M. Achache. Complexity analysis of an interior point algorithm for the semidefinite optimization based on a kernel function with a double barrier term. Acta Mathematica Sinica, English Series, 31(3): 543-556 (2015).

[5] M. Achache. Complexity analysis and numerical implementation of a short-step primal-dual algorithm for linear complementarity problems, Appl. Math. Comput., 216(7), 1889–1895 (2010).

[6] M. Achache, A weighted path-following method for the linear complementarity problem, Studia Univ. Babes-Bolyai. Ser. Inform. 49(1): 61-73 (2004).

[7] M. Achache, A weighted full-Newton step primal-dual interior point algorithm for convex quadratic optimization, Statistics, Optimization & Information Computing, (2): 21-32 (2014).

[8] M. Achache, M. Goutali. A primal-dual interior-point algorithm for convex quadratic programs, Studia Univ. Babes-Bolyai. Ser. Inform. LVII(1): 48-58 (2012).

[9] M. Achache, M. Goutali. Complexity analysis and numerical implementation of a full-Newton step interior-point algorithm for LCCO. Numer. Algorithms. 70: 393-405 (2015).

[10] M. Achache, R. Khebchache. A full-Newton step feasible weighted primal-dual interior point algorithm for monotone LCP. Afrika Matematika. 26 : 139-151 (2015).

[11] M. Achache, H. Roumili and A. Keraghel. A numerical study of an infeasible primal-dual path following algorithm for linear programming, Appl Math Comput, 186 (2), 1472-1479 (2007).

[12] M. Achache, N. Tabchouche. Complexity analysis of an interior point algorithm for the semidefinite optimization based on a kernel function with a double barrier term. Optimization, 67(8): 1211-1230 (2018).

[13] B. Alzalg. A logarithmic barrier interior-point method based on majorant functions for second-order cone programming, Optimization Letters, 14, 729–746 (2020).

[14] Y.Q. Bai, M. El Ghami and C. Roos. A comparative study of kernel functions for primal-dual interior point algorithms in linear optimization. SIAM J. Optim. 15(1), 101-128 (2004).

[15] Y.Q. Bai, G.Q. Wang and C. Roos. A new kernel function yielding the best known iteration bounds for primal-dual interior point method. Acta Math. Sinica, 25(12), 2169-2178 (2009).

[16] L. B. Cherif and B. Merikhi. A variant of a logarithmic method for non linear convex programming, RAIRO-Operations Research, 53 (1), 29-38 (2019).

[17] J-P. Crouzeix and B. Merikhi. A logarithm barrier method for semi-definite programming, RAIRO - Operations Research - Recherche Oprationnelle, 42 (2), 123-139 (2008).

[18] Zs. Darvay, A weighted-path-following method for linear optimization, Studia Universitatis Babes-Bolyai Series Informatica, 47(2): 3-12 (2002).

[19] Zs. Darvay. New interior-point algorithms for linear optimization. Advanced Modeling Optimization. 5 (1): 51-92 (2003).

[20] Zs. Darvay, I.M. Papp and P.R. Takacs. Complexity analysis of a full-Newton step interior-point method for linear optimization, Period. Math. Hung. 73 (1)27-42 (2016).

[21] Zs. Darvay, P.R. Takacs. New method for determining search directions for interior-point algorithms in linear optimization, Optim. Let. 12 1099-1116 (2018).

[22] Zs. Darvay. New interior-point algorithms for linear optimization, Advanced Modeling Optimization, 5 (1), 51-92 (2003).

[23] J. Ding, and T.Y. Li, An algorithm based on weighted logarithmic barrier functions for linear complementarity problems, Arabian Journal for Science and Engineering, 15(4): 679-685 (1990).

[24] S. Fathi-Hafshejani, Z. Moaberfard. An interior-point algorithm for linearly constrained convex optimization based on kernel function and application in non-negative matrix factorization. Optim Eng, 21(3): 1019-1051 (2020).

[25] L. Goran, G. Wang, and A. Oganian, A Full Nesterov-Todd Step Infeasible Interior-point Method for Symmetric Optimization in the Wider Neighborhood of the Central Path,Statistics, Optimization & Information Computing, 9(2), 250-267 (2021).

[26] J. Gondzio. Interior Point Methods for Convex Quadratic Programming, NAT-COR, Edinburgh (2018).

[27] M. Goutali. Complexité et implémentation numérique d'une méthode de points intérieurs pour la programmation convexe, Universite Ferhat Abbas Sétif 1 (2018).

[28] L.M. Grana Drummond and BF. Svaiter. On well definiteness of the central path, J. Optim. Theory Appl, 102 (2), 223-237 (1999).

[29] L. Guerra, M. Achache. A parametric Kernel Function Generating the best Known Iteration Bound for Large-Update Methods for CQSDO. Statistics, Optimization & Information Computing, 8: 876-889 (2020).

[30] M. Haddou, J. Omer and T. Migot. A Generalized Direction in Interior-Point Method for Monotone Linear Complementarity Problems, Optim. Lett. 13, 35-53 (2019).

[31] B. Jansen, C. Roos, T. Terlaky and J.P. Vial. Primal-dual target-following algorithms for linear programming. Annals of Operations Research. 62 : 197-231 (1996).

[32] A. Leulmi. Logarithmic Barrier Interior Point Method for Linearly Constrained Convex Programming, Universite Ferhat Abbas Sétif 1 (2021).

[33] A. Leulmi, B. Merikhi and D. Benterki. Study of a Logarithmic Barrier Approach for Linear Semidefinite Programming, Journal of Siberian Federal University. Mathematics and Physics, 11 (3), 300-312 (2018).

[34] H. Mansouri, M. Pirhaji, and M. Zanglabadi, A weighted-path-following interior-point algorithm for cartesian $P_*(\kappa)$-LCP over symmetric cone, Korean Math. Soc.32, (3): 765-778 (2017).

[35] L. Menniche and D. Benterki. A logarithmic barrier approach for linear programming, J. Comput. App. Math, 312, 267-275 (2017).

[36] Y.E. Nesterov and A. Nemirovski. Interior- Point Polynomial Algorithms in Convex Programming, SIAM. (1994).

[37] J. Peypouquet. Convex Analysis in Normed Space. Theory, Methods and Examples, Springer. (2015).

[38] P.R. Rigó. New trends in algebraic equivalent transformation of the central-path and its applications, PhD thesis, Budapest University of Technology and Economics, Institute of Mathematics, Hungary. (2020).

[39] R.T. Rockafellar. Convex Analysis, Princeton University Press, New Jersey. (1970).

[40] C. Roos , T. Terlaky and J.Ph. Vial. Theory and Algorithms for Linear Optimization. An interior Point Approach. John-Wiley and Sons, Chichester, UK. (1997).

[41] G.Q. Wang, Y.J. Yue, and X.Z. Cai, A weighted path-following method for monotone horizontal linear complementarity problem. Fuzzy Information and Engineering. 54: 479-487 (2009).

[42] G.Q. Wang, Y.J. Yue, and X.Z. Cai, Weighted path-following method for monotone mixed linear complementarity problem, Fuzzy Information and Engineering. 4: 435-445 (2009).

[43] E. Wong. Active-Set Methods for Quadratic Programming, PhD thesis, Department of Mathematics, University of California San Diego, La Jolla, CA. (2011).

[44] S.J. Wright. Primal-dual Interior-Point Methods. Copyright by SIAM. (1997).

[45] Y. Ye. Interior-Point Algorithm. Theory and Analysis. New-York: John Wiley. (1997).

[46] M. Zhang, YQ. Bai and GQ. Wang. A new primal-dual path-following interior-point algorithm for linearly constrained convex optimization. Journal of Shanghai University, 12(6): 475-480 (2008).

[47] D. Zhao, M. Zhang. A primal-dual large-update interior-point algorithm for semi-definite optimization based on a new kernel function. Statistics, Optimization & Information Computing, 1(1), 41-61 (2013).

ملخص

في هذه الأطروحة نهتم بالدراسة النظرية والعددية في حل مسائل الأمثلة التربيعية المحدبة. لهذا الغرض، قدمنا طريقتين ذواتي النقطة الداخلية. الأولى تعتمد على المسار المركزي واتجاهات نيوتن المعدلة والثانية تعتمد على المسار ذي الثقل. في كلتا الحالتين ، أثبتنا أن الخوارزميتين الملحقتين معرفة وتتقارب تربيعيا نحو الحل الأمثل. زيادة أثبتنا أن التقارب ذات تكلفة حدودية.

أخيرا هده الدراسة ألحقت بالتجارب العددية للتقييم.

الكلمات المفتاحية : الأمثلة التربيعية المحدبة ، طرق النقطة الداخلية ، طرق الخطوة القصيرة التكلفة الحدودية.

Abstract

In this thesis we are interested with theorical and numerical study of convex quadratic optimization. For this purpose, we have introduced two methods of interior point. The first depends on the classical central-path with modified Newton search directions. Meanwhile, the second one is based on the weighted path and also new search directions. In the two cases, we have proved that the corresponding algorithms are defined and converge locally quadratically. In addition, those algorithms have the best known polynomial complexity.

Finally, this study is followed by some numerical experiments for evaluation.

Keywords: Convex quadratic optimization, Interior-point methods, Short-step method, Polynomial complexity.

Résumé

Dans cette thèse, on a intéressé par l’étude théorique et numérique de la programmation quadratique convexe. Pour ce but, on a introduit deux méthodes de point intérieur. La première méthode dépend de trajectoire centrale classique avec des directions de Newton modifiée, la deuxième est basée sur la trajectoire avec poids et aussi avec des nouvelles directions. Dans les deux cas, on a montré que les algorithmes sont bien définis et convergent localement quadratique. De plus, ces algorithmes ont la meilleur complexité polynomiale.

Finalement, cette étude est suivie par quelque résultats numériques pour évaluation.

Mots clés : Optimisation quadratique convexe, Méthodes de point-intérieur, Méthode à petit pas, Complexité polynomiale.