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MATHEMATICAL STUDY OF SOME MODELS APPLIED IN BIOLOGY AND MEDICINE

Presented by

Mr. **Houssem Eddine KADEM**

Supervisor : **Pr. Saida BENDAAS**
Co-supervisor : **Pr. Salim MESBAHI**

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Mr. Mohamed SELMANI	Prof	Ferhat Abbas University, Setif 1	President
Mme. Saida BENDAAS	Prof	Ferhat Abbas University, Setif 1	Supervisor
Mr. Salim MESBAHI	Prof	Ferhat Abbas University, Setif 1	Co- supervisor
Mr. Ahcene MERAD	Prof	Larbi Ben M’hidi University, Oum El-Bouaghi	Examiner
Mr. Yassine ADJABI	Prof	M'hamed Bougara University, Boumerdes	Examiner
Mr. Salah ZITOUNI	Prof	Mohammed-Cherif Messaadia University, Souk Ahras	Examiner

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بسم الله الرحمن الرحيم

If logic is the hygiene of the mathematician,
it is not logic that provides him with his food;
the daily bread on which he lives,
these are the great problems.

André Weil.

Dedication

To my dear parents,
To all my family,
To all my friends.

Acknowledgements

First of all, I would like to thank God the Almighty, for giving me enough courage and patience to accomplish this work.

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Finally, I address my most sincere thanks to my parents, all my family my friends, and all my relatives who have accompanied, helped, supported and encouraged me throughout the realization of this work.

Houssem Eddine Kadem

Setif, September 13th, 2022

العنوان : دراسة رياضية لبعض النماذج المطبقة في علم الأحياء والطب.

ملخص : العمل الذي يشكل هذه الأطروحة هو مساهمة في النمذجة والتحليل الرياضي لبعض النماذج المطبقة في علم الأحياء والطب. نحن مهتمون بدراسة وجود الحلول لبعض النماذج باستخدام تقنيات تعتمد على التحليل الوظيفي والتحليل غير القياسي. يتكون هذا العمل من خمسة فصول مستقلة، مسبوقة بمقدمة عامة تسلط الضوء على فن الموضوع والمشكلات التي تم تناولها.

كلمات مفتاحية : أنظمة تفاعل-انتشار، معادلة برجرس، نظرية النقطة الثابتة، نمذجة ظواهر الانتشار، تحليل غير قياسي.

Titre : ÉTUDE MATHÉMATIQUE DE QUELQUES MODÈLES APPLIQUÉS EN BIOLOGIE ET EN MÉDECINE.

Résumé :

Le travail constituant cette thèse est une contribution à la modélisation mathématique et à l'analyse de certains modèles appliqués en biologie et en médecine. Nous nous intéressons à l'étude de l'existence de solutions pour certains modèles en utilisant des techniques basées sur l'analyse fonctionnelle et l'analyse non standard. Ce travail se compose de cinq chapitres indépendants, précédés d'une introduction générale qui met en évidence l'art du sujet et les problèmes abordés.

Mots-clés : systèmes de réaction-diffusion, équation de Burgers, théorème de point fixe, modélisation des phénomènes de diffusion, analyse non standard.

Title : MATHEMATICAL STUDY OF SOME MODELS APPLIED IN BIOLOGY AND MEDICINE.

Abstract : The work constituting this thesis is a contribution to the mathematical modeling and analysis of certain models applied in biology and medicine. We are interested in studying the existence of solutions for certain models using techniques based on functional analysis and non-standard analysis. This work is then composed of five independent chapters, preceded by a general introduction which highlights the art of the subject and the problems addressed.

Keywords : reaction diffusion systems, Burgers equation, fixed point theorem, modeling of diffusion phenomena, non-standard analysis.

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General Introduction

In mathematics, more precisely in differential calculus, a partial differential equation (sometimes called partial differential equation and abbreviated to PDE) is a differential equation whose solutions are unknown functions depending on several variables verifying some conditions concerning their partial derivatives.

A PDE often has very many solutions, the conditions being less stringent than in the case of an ordinary single-variable differential equation; the problems often have boundary conditions that restrict the set of solutions. While the solution sets of an ordinary differential equation are parametrized by one or more parameters corresponding to the additional conditions, in the case of PDEs, the boundary conditions are instead in the form of a function; intuitively this means that the solution set is much larger, which is true in almost all problems.

They are ubiquitous in science provide mathematical models in various in the following provide mathematical models in various fields of application such as biology, medicine, ecology, quantum mechanics, image synthesis, weather forecasting, demography and finance, appearing as much in structural dynamics and fluid mechanics as in the theories of gravitation, electromagnetism (Maxwell's equations), or financial mathematics (Black-Scholes equation). They are essential in fields such as aeronautical simulation, image synthesis, or weather forecasting. Finally, the most important equations of general relativity and quantum mechanics are also PDEs.

Some names are given as examples: Euler's, Navier's and Stokes' in fluid mechanics, Fourier's for the heat equation mechanics, those of Fourier for the heat equation, of Maxwell for those of electromagnetism electromagnetism, of Schrodinger and Heisenberg for the equations of quantum mechanics and of course Einstein for the PDEs of the theory of relativity.

From the mathematical modeling of several phenomena in nature or in other fields in practice come certain systems of equations, called equations of reaction-diffusion.

In nature, the heat equation is one of the most interesting systems and physical phenomena that are most interesting and complex to study, and for this we go to the analytical resolution which brings us to exact solutions.

Burgers equation

One of the most important partial differential equations of the theory of nonlinear conservation laws, is the semi-linear diffusion equation. The study of this equation has a long history: In 1906, Forsyth, treated an equation that converts by some changes of variables into the Burgers equation. In 1915, Bateman in [21] introduced this one. He was interested in the case where $v \to 0$, and in studying the motion behavior of a viscous fluid when the viscosity tends to zero. In 1948, Burgers in [32] studied it (to which he owes his name),

$$u_t + uu_x - \nu u_{xx} = f$$

where u stands, generally, for a velocity, t the time variable, x the space variable and v the constant of viscosity (or the diffusion coefficient). Homogeneous Burgers equation (with $f = 0$), is one of the simplest models of nonlinear equations which have been studied.

Burgers continued his study of what he called “nonlinear diffusion equation”. This study treated mainly the static aspects of the equation. Among the most interesting applications of the one-dimensional Burgers equation, we mention traffic flow, growth of interfaces, and financial mathematics.

The mathematical structure of this equation includes a nonlinear convection term $u\partial_x u$ which makes the equation more interesting, and a viscosity term of higher order ∂_x^2 which regularizes the equation and produces a dissipation effect of the solution near a shock. As a particular case, When the viscosity coefficient vanishes $v = 0$, the Burgers equation is reduced to the transport equation, which represents the inviscid Burgers equation

$$u_t + uu_x = f$$

Reaction-Diffusion Systems

A reaction-diffusion system is a mathematical model that describes the evolution of the concentrations of one or more substances spatially distributed and subject to two processes: a process of local chemical reactions, in which the different substances are transformed, and a diffusion process that causes a distribution of these substances in space, it’s appear naturally in the mathematical modeling of a wide variety of phenomena, not only in the natural sciences, but also in engineering and economics, such as the dynamics of gases, fusion processes.

This description naturally implies that such systems are applied in chemistry. However, they can also describe dynamic phenomena of a different nature: biology, physics, geology or ecology are examples of fields where such systems appear. Mathematically, reaction-diffusion systems are represented by semi-linear parabolic partial differential equations, reaction-diffusion systems appear naturally in the mathematical modeling of a wide variety of phenomena, not only in the natural sciences, but also in engineering and economics, such as gas dynamics, fusion processes, some biological models, cellular processes, ecology, disease propagation, industrial processes, catalytic transport of contaminants in the environment, population dynamics, flame propagation and chemical reactions and others. biological models, cellular processes, ecology, disease propagation, industrial processes, catalytic transport of contaminants in the environment, population dynamics.

From the mathematical modelling of several phenomena in nature or in other fields in practice in practice come from certain systems of equations, called reaction-diffusion equations.

Most of these, at first glance, are phenomena that have a common denominator, the presence of diffusion (allowing the spread of an epidemic or a chemical substance), and reaction (which is the specific way in which the different chemical phases or components react, they are generically called reaction-diffusion systems. For the analysis of these types of problems, various methods and elaborate techniques have been proposed.

What is interesting is that reaction-diffusion systems appear naturally in the mathematical modeling of a wide variety of phenomena such as gas dynamics, fusion processes, some biological models such as cellular processes, disease propagation, catalytic transport of contaminants in the environment, population dynamics, flame propagation and chemical reactions, etc.

The objective of this work is the study of a system of quasi-linear reaction-diffusion equations modeling the propagation of an infectious disease in a population, it is to show the global existence of the solution and to give its asymptotic behavior.

for example, in nature, the heat equation is one of the most interesting physical systems and phenomena that are interesting systems and phenomena that are the most complex to study, and for this we go to the analytical resolution analytical resolution which brings us to exact solutions.

Non-standard analysis

In mathematics, and more precisely in analysis, non-standard analysis is a set of tools developed since 1960 in order to deal with the notion of infinitesimal in a rigorous way. For this purpose, a new notion is introduced, that of standard object (as opposed to non-standard object), or more generally of standard model or non-standard model. This allows to present the main results of the analysis in a more intuitive form than the one traditionally exposed since the 19th century. Non-standard analysis is better than standard analysis in that some proofs become simplified, and infinitessimal are somehow more intuitive to grasp than epsilon-delta arguments, The history of calculus is fraught with philosophical debates about the meaning and logical validity of fluxions or infinitesimal numbers. The standard way to resolve these debates is to define the operations of calculus using epsilon–delta procedures rather than infinitessimal. Nonstandard analysis [46], [73] and [89] instead reformulates the calculus using a logically rigorous notion of infinitesimal numbers.

Nonstandard analysis originated in the early 1960s by the mathematician Abraham Robinson, however the list of new applications in mathematics via non-standard analysis is still very small.

It has been established that a classical statement, having a proof in non-standard analysis, is true in classical mathematics. The situation is quite comparable to that of mathematicians before 1800, who were allowed to use imaginary numbers as long as the final result was real. Non-standard analysis allows us to give new (often simpler) proofs of classical theorems. Non-standard analysis also allows to manipulate the new concepts of infinitely small or infinitely large numbers which have caused so many problems to mathematicians and which had been banished from analysis. It is therefore more general than classical analysis, just as complex analysis is more general than real analysis, however, non-standard analysis has had little influence to date. Few new theorems have been developed by means of it, and for the moment it is essentially a rewriting of the whole analysis by means of new concepts. It should be noted that one cannot expect new results in elementary analysis; interesting applications must be sought, for example, in the study of differential systems.

In the nonstandard analysis community, there is a growing number of results that are not being translated into standard results, because the intuitive content of certain theorems is greater and/or clearer when left in nonstandard terminology. Examples include the use of nonstandard analysis in mathematical economics to describe the behavior of large economies and the use of nonstandard methods to give meaning to concepts that do not classically make sense, such as certain products of infinitely many independent, equally weighted random variables.

As a result, the use of non-standard analysis is much more restricted in the context of digital implementation, we will mention for example, but not exhaustively the work of Syed Ahmed Pasha [119] in their paper thy propose an alternative Non-Standard Finite Difference (NSFD) scheme that guarantees first-order accuracy in time and second-order accuracy in space whilst preserving positivity of the solution. Stability and consistency analyses of the proposed scheme are also presented.

In the literature, NSFD methods have been used to numerically solve ordinary and partial differential equations (PDEs) arising in a number of application domains. In [104], an NSFD scheme has been constructed for the Burgers PDE without diffusion which preserves the positivity and boundness properties of the continuous-time model. In [120], the NSFD method has been used to solve second-order, linear, singularly perturbed differential difference equations and the convergence properties have been studied. The authors in [90], have designed NSFD schemes for singularly perturbed two-point boundary value problems where the nonstandard denominator of the discrete derivative is related to some qualitative features of the governing differential equation. An NSFD-based approach to numerically solve a generalized nonlinear Black–Scholes model for illiquid markets has been presented in [17]. It has been shown that the proposed numerical scheme is stable and positivity of the solution. In [40], an NSFD scheme has been constructed for two-dimensional differential equations and the convergence properties studied.

For example, In [59] , an NSFD method has been presented to numerically solve a model for disease transmission. It has been shown that the NSFD method does not suffer from the shortcomings of standard finite difference methods. Moreover, using numerical simulations it was demonstrated that while standard finite difference methods fail to preserve the qualitative features such as positivity, the NSFD method does not suffer from this problem regardless of the choice of the step size in the numerical simulation. An NSFD scheme to numerically solve a virus dynamics model has been presented in [51]. This approach has been shown to preserve the positivity and boundedness of the solution.

An outline of this thesis is as follows: In the part I of this thesis we present one of the most important partial differential equations in the theory of non linear conservation laws is Burgers equation. she combining both nonlinear propagation effects and diffusive effect. It is known that this equation has a large variety of application in the modeling of statistics of flow problems mixing and turbulent diffusion cosmology and seismology, in medicine, e.g. dermatological ointment, medication for the treatment of a disease, or in agriculture, the watering of plants with chemicals. Burgers equation has different types and each of them has special application. This equation is parabolic when the viscous term is included. If the viscous term is null the remaining equation is hyperbolic.

The other part of this work, is devoted on one hand to study a reaction-diffusion systems in models of biological and chemical phenomena, and by the richness of the structure of their solution sets. Given the numerous and varied applications of these systems, we will give the approaches followed to model certain chemical problems such as oscillating chemical reactions (Brusselator), and other hand to study the link between reaction-diffusion systems and image processing indeed, a two-component reaction-diffusion system is a typical model of equations describing pattern recognition. They can be defined as a process that meaningfully describes the data and deeply understands the problem by manipulating a data set. They are proving very useful in image processing for pattern classification, where data preprocessing is useful for error correction, image enhancement, and recognition. Nonlinear reaction-diffusion models can describe many natural phenomena in a wide range of domains. Some surprising results have been observed in technical applications such as image processing. Among these applications, we mention the Fitzhugh-Nagumo model which allowed the detection of contours in noisy images. We also mention the anisotropic diffusion described by Perona and Malik which includes local information to reduce noise and improve contrast while preserving the edge. Hence the idea of Catté *et al*.

Situation of the thesis

This thesis is part of a multidisciplinary study and is divided into fifth chapters.

In the first chapter we will give some tools of functional analysis including definitions of theorems and some results in order to study and the resolution of diffusion reaction systems, with a few examples of reaction-diffusion systems found here and there in the literature as models for very different applications as Lotka-Volterra systems, Quadratic chemical reactions, Super quadratic reaction-diffusion systems, Diffusion of pollutants in atmosphere, after that we gave existence of global classical solutions with many boundary conditions. We will conclude this chapter by giving some mathematical model in physics, chemistry, medecin and in biology.

The second chapter is devoted to study the Burgers equation and its applications in the framework of nonstandard analysis. This equation has a large variety of application in the modeling of statistics of flow problems mixing and turbulent diffusion cosmology and seismology, in medicine, e.g. dermatological ointment, medication for the treatment of a disease, or in agriculture, the watering of plants with chemicals. Burgers equation has different types and each of them has special application. This equation is parabolic when the viscous term is included. If the viscous term is null the remaining equation is hyperbolic. We recall the Hopf-Cole transformation is a transformation that transforms the solution of the Burgers equation into the heat equation. We notice that this transformation, can be applied to the forced Burgers equation. It is simple to show that it leads to the parabolic differential equation.

Still within the scope of this chapter, we gave some models:

- The propagation of sound waves in soft biological tissues model
- Coronavirus model

- Malaria model
- Traffic flow
- inviscid and viscous Burgers equations
- Viscous Burgers' model
- Navier-Stokes model

Another application of Burger's equation is sonification technique is used to explore tissue images with details that are difficult to detect. By applying sonification, new images are obtained with a better visualization of the explored tissue.

In the third chapter, we propose a new model of nonlinear generic reaction diffusion system applied to edge detection and image restoration. We prove the existence of weak global solutions for a class of generic reaction diffusion systems for which two main properties hold: the quasi-positivity and a triangular structure condition on the nonlinearities.

The work constituting this chapter is the subject of an article published in (International Journal of Nonlinear Analysis and Applications), in collaboration with S. Mesbahi and S. Bendaas [68].

The aim of the fourth chapter is to examine the solutions of the boundary value problem of the nonlinear elliptic equation $\varepsilon^2 \triangle u = f(u)$. We describe the asymptotic behavior as ε tends to zero of the solutions on a spherical crown C of $\mathbb{R}^N$, $(N \geq 2)$ in a direct non-classical formulation which suggests easy proofs. We propose to look for interesting solutions in the case where the condition at the edge of the crown is a constant function. Our results are formulated in classical mathematics. Their proofs use the stroboscopic method which is a tool of the nonstandard asymptotic theory of differential equations. The work constituting this chapter is the subject of an article published in (Mathematica Slovaca), in collaboration with S. Bendaas [67].

The fifth chapter concerns the initial boundary value problem for the non linear dissipative Burgers equation.Our general purpose is to describe the asymptotic behavior of the solution in the cauchy problem with a small parametre ε for this equation and to discuss in particular the cases of the N wave shock and periodic wave shock. we show that the solution of cauchy problem of viscid equation approach the shock type solution for the cauchy problem of the inviscid equation for each case.The results are formulated in classical mathematics and proved with infinitesimal techniques of Non Standard Analysis. The work constituting this chapter is the subject of an article published in (International Journal of Analysis and Applications), in collaboration with Z. Nouri and S. Bendaas [116].

At the end of this thesis a conclusion and perspectives for both the diffusion reaction systems and the burger equation are given with a rich bibliography.

CHAPTER 1

Reaction-diffusion systems and its applications

In this chapter, we will give some tools of functional analysis including definitions, theorems and some results in order to study and the resolution of reaction-diffusion systems, with a few examples of reaction-diffusion systems found here and there in the literature as models for very different applications as Lotka-Volterra systems, Quadratic chemical reactions, Super quadratic reaction-diffusion systems, Diffusion of pollutants in atmosphere, after that we gave existence of global classical solutions with many boundary conditions. We will conclude this chapter by giving some mathematical model in physics, chemistry, medicine and in biology.

1.1 Preliminaries on functional analysis

In this section, we give some basic functional analysis tools.

1.1.1 Functional spaces

- We designate by $L^p(\Omega)$, $1 \leq p < \infty$ the space of functions (or more exactly of equivalence classes of functions, in the sense of equality almost everywhere) u measurable on Ω such as

$$\int_\Omega |u|^p \, dx < \infty$$

equipped with the norm

$$\|u\|^p_{L^p(\Omega)} = \int_\Omega |u|^p \, dx$$

- The spaces $L^p(\Omega)$ with this norm are Banach spaces. In particular $L^2(\Omega)$ is a Hilbert space with scalar product

$$(u,v) = \int_\Omega u(x)v(x)dx$$

- We denote by $L^\infty(\Omega)$ the space of measurable and essentially bounded functions u on Ω,

$$L^\infty(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable, } \exists c > 0 \text{ such that } |u| \le c \text{ almost everywhere on } \Omega\}$$

it is a complete vector space for the norm

$$\|u\|_{L^\infty(\Omega)} = \sup_{x\in\Omega} \mathbf{ess}\,|u(x)| = \inf\{c > 0, |u| \le c \text{ a.e. on } \Omega\}$$

- We define the spaces $L^p(0,T,X)$, $1 \le p < \infty$ and $L^p(0,T,X)$ as follows

$$L^p(0,T,X) = \left\{u : [0,T] \to X \text{ measurable, } \int_0^T \|u\|^p_X \, dt < \infty\right\}$$

equipped with the norm

$$\|u\|^p_{L^p(0,T,X)} = \int_0^T \|u\|^p_X \, dt$$

$$L^\infty(0,T,X) = \left\{u : [0,T] \to X \text{ measurable, } \sup_{t\in(0,T)} \mathbf{ess}\,\|u\|_X < \infty\right\}$$

equipped with the norm

$$\|u\|_{L^\infty(0,T,X)} = \sup_{t\in(0,T)} \mathbf{ess}\,\|u\|_X$$

Naturally we have

$$L^p(0,T,L^p(\Omega)) = L^p((0,T)\times\Omega), 1 \le p \le \infty.$$

- $C(\Omega)$ denotes the space of continuous functions with compact support in Ω, with the norm

$$\|u\|_{C(\Omega)} = \max_{x\in\Omega} |u(x)|$$

- $C^k(\Omega)$ (k positive integer), denotes the space of k times continuously differentiable on Ω, and we write

$$C^\infty(\Omega) = \underset{k\geq 0}{\cap} C^k(\Omega)$$

- $D(\Omega)$ it is the space of C^∞ functions with compact support.

- $H^1(\Omega)$ it is the Sobolev space defined by

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) : \frac{\partial u}{\partial x_i} \in L^2(\Omega),\ 1 \leq i \leq n \right\}$$

equipped with the norm

$$\begin{aligned} \|u\|^2_{H^1(\Omega)} &= \int_\Omega |u|^2\, dx + \int_\Omega \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2 dx \\ &= \int_\Omega |u|^2\, dx + \int_\Omega |\nabla u|^2\, dx \end{aligned}$$

- In general, for $m \in \mathbb{N}^*$ and $1 \leq p < \infty$, Sobolev spaces $H^m(\Omega)$ and $W^{m,p}(\Omega)$ are defined as follows

$$H^m(\Omega) = \left\{ u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega),\ \alpha \in \mathbb{N}^n, |\alpha| \leq m \right\}$$

equipped with the norm

$$\|u\|^2_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \int_\Omega \left| D^\alpha u \right|^2 dx = \sum_{|\alpha|\leq m} \left\| D^\alpha u \right\|^2_{L^2(\Omega)}$$

and

$$W^{m,p}(\Omega) = \left\{ u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega),\ |\alpha| \leq m \right\}$$

equipped with the norm

$$\|u\|^p_{m,p} = \sum_{|\alpha|\leq m} \left\| D^\alpha u \right\|^p_{L^p(\Omega)} dx$$

where

$$D^\alpha = \frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}},\ |\alpha| = \sum_{i=1}^n \alpha_i$$

is the derivative in the sense of distributions.

Naturally we have

$$W^{1,2}(\Omega) = H^1(\Omega) \quad , \quad W^{m,2}(\Omega) = H^m(\Omega)$$

Remark 1.1.

$$H_0^1(\Omega) = \{u \in H^1(\Omega),\ u\,|_\Gamma = 0\}$$

and for any $m \in \mathbb{N}$, we have the following inclusion

$$H^m(\Omega) \subset H^{m-1}(\Omega) \subset \cdots \subset H^1(\Omega) \subset L^2(\Omega)$$

1.1.2 Classical inequalities

- Cauchy's inequality

$$ab \le \frac{a^2}{2} + \frac{b^2}{2},\ \forall a, b \in \mathbb{R}$$

- Cauchy's inequality with ε

$$ab \le \varepsilon a^2 + \frac{b^2}{4\varepsilon},\ \forall \varepsilon \in \mathbb{R}_+^*$$

Theorem 1.1 (Young's inequality)**.** *Let $1 < p,\ q < +\infty$, where $\frac{1}{p} + \frac{1}{q} = 1$*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q},\ \forall a, b \in \mathbb{R}_+^*$$

Theorem 1.2 (Green's formula)**.** *Let Ω be an open with a boundary C^1 piecewise.*

Then, if u and v are functions of $H^1(\Omega)$, we have

$$\int_\Omega \frac{\partial u}{\partial x_i} v\,dx = -\int_\Omega u \frac{\partial v}{\partial x_i} dx + \int_\Gamma u v \eta_i\, d\sigma, \quad 1 \le i \le n$$

with η_i is the i-th director cosine of the normal η to Γ directed towards the exterior of Ω, we write $\eta_i = \left(\vec{\eta}.\vec{e_i}\right)$.$d\sigma$ is the surface measure on Ω.

Corollary 1.1. *For any function u of $H^2(\Omega)$ and any function v of $H^1(\Omega)$, we have the Green's formula*

$$\int_\Omega (\Delta u) v\,dx = \int_\Gamma \frac{\partial u}{\partial \eta} v\,d\sigma - \int_\Omega \nabla u.\nabla v\,dx$$

Remark 1.2. *We can write Green's formula as*

$$\int_\Omega \nabla u.v\,dx = \int_\Gamma u v\,d\sigma - \int_\Omega u.\nabla v\,dx$$

Theorem 1.3 (Hölder's inequality). *Let* $1 < p,\ q < +\infty$, *where* $\dfrac{1}{p} + \dfrac{1}{q} = 1$. *Let* u *be a function of* $L^p(\Omega)$ *and* v *a function of* $L^p(\Omega)$. *Then*

$$\int_\Omega |u.v|\, dx \leq \left(\int_\Omega |u|^p\, dx\right)^{\frac{1}{p}} . \left(\int_\Omega |v|^q\, dx\right)^{\frac{1}{q}}$$

Theorem 1.4 (Shauder fixed point). *Let* $(X, \|\ .\ \|)$ *be a Banach space over* $\mathbf{K}$ ($\mathbf{K} = \mathbb{R}$ *or* $\mathbf{K} = \mathbb{C}$) *and* $S \subset X$ *is closed, bounded, convex, and non empty. Any compact operator* $A : S \to S$ *has at least one fixed point.*

Lemma 1.1 (Gronwall's). *Let* φ, ψ *and* y *three continuous functions on a segment* $[a, b]$, *with positive values and verifying the inequality*

$$\forall t \in [a, b],\quad y(t) \leq \varphi(t) + \int_a^t \psi(s) y(s) ds$$

then

$$\forall t \in [a, b],\quad y(t) \leq \varphi(t) + \int_a^t \varphi(s)\psi(s) \exp\left(\int_s^t \psi(u) du\right) ds$$

Corollary 1.2. *Let* ψ *and* $y : [a, b] \to \mathbb{R}^+$ *two continues functions satisfies*

$$\exists c \geq 0 \ /\ \forall t \in [a, b],\quad y(t) \leq c + \int_a^t \psi(s) y(s) ds$$

then

$$\forall t \in [a, b],\quad y(t) \leq c \exp\left(\int_a^t \psi(s) ds\right)$$

Corollary 1.3. *Let* $y : [a, b] \to \mathbb{R}^n$ *a function of class* C^1 *satisfying*

$$\exists \alpha > 0, \exists \beta \geq 0, \forall t \in [a, b],\quad \left\| y'(t) \right\| \leq \beta + \alpha \left\| y(t) \right\|$$

then

$$\forall t \in [a, b],\quad \|y(t)\| \leq \|y(a)\|\, e^{\alpha(t-a)} + \frac{\beta}{\alpha}\left(e^{\alpha(t-a)} - 1\right)$$

1.2 Reaction-diffusion systems

During the recent years, reaction-diffusion systems have received a great deal of attention, motivated by their attention, motivated by their widespread occurrence in models of biological and chemical phenomena, and by the richness of the structure of their solution sets. Given the numerous and varied applications of these systems, we will give the approaches followed to model certain chemical problems such as oscillating chemical reactions (Brusselator). The individuals differ from one problem to another:

In chemistry, for example, they represent chemical substances. In biochemistry, they can represent molecules. In metallurgy, they represent atoms. In dynamics populations, they are humans. In population genetics, they represent characters. In biophysics, electrical charges or potential differences. In environment, they can represent animals or plants of a forest, a sea or a river... For most of these problems, we show that we end up with reaction-diffusion systems, see Alaa and Mesbahi et *al*. [2]-[9], [97]-[102], [134], [135], and also [76] and [128].

The edge conditions will be chosen according to the origin and the nature of the problem studied: if there is no immigration of individuals across the boundary of the domain on which the problem is posed, the homogeneous Neumann edge conditions are chosen. If there are no individuals on the boundary, we take the homogeneous Dirichlet edge conditions. The unknown (the solution we are looking for) is a vector whose components are generally positive functions: in chemistry, for example, it is a vector of chemical concentrations. In biochemistry or metallurgy, it is a vector of concentrations in numbers of molecules or atoms respectively. In population dynamics and in environment, it is a vector of densities of human, animal or vegetable populations. The initial conditions are generally positive; since they are concentrations, densities, electric charges.

Reaction-diffusion systems are coupled systems of partial differential equations. The general form of these systems is

$$\frac{\partial u}{\partial t} = \mathbf{div}(D(t,x,u,\nabla u).\nabla u) + f(t,x,u,\nabla u),\ x \in \Omega,\ t \geq 0 \tag{1.1}$$

where $u = u(t,x) = (u_1,...u_m) : \mathbb{R}^+ \times \Omega \to \mathbb{R}^m$ is a vector of variables. f is a linear or nonlinear vector function, which is called the reaction terms, it is a regular (at least locally lipschitzian) application. $D : \mathbb{R}^+ \times \Omega \times \mathbb{R}^m \times \mathbb{R}^{mN} \to \mathbb{R}^m$ is a regular function. When D is a square matrix it is called the diffusion matrix, in this case $\mathbf{div}(D(t,x,u,\nabla u).\nabla u) = D\Delta u$ are the diffusion terms.

This equation is posed on an open domain $\Omega \subset \mathbb{R}^N$, and completed by conditions on the edge, for example, the homogeneous Dirichlet conditions ($u = 0$ on $\partial\Omega$) or the homogeneous Neumann conditions $\left(\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\right)$.

Reaction terms are the result of any interaction between the components of u; for example u can be a vector of chemical concentrations, and f is the effect of chemical reactions of those concentrations or the components of u can be plant or animal population densities, and f represents the effect of relationships (competitive or symbiotic) between predators and prey. Diffusion terms can represent molecular diffusions or some random movements of individuals in a population. For the mathematical analysis of reaction-diffusion systems, see Alaa and Mesbahi et *al*. [2]-[9], [97]-[102] and [122]-[124].

There are no general solutions for reaction-diffusion systems. However, we have qualitative information on the global existence of solutions and their expected behavior when the variable t tends to infinity.

The fact that these systems model real-world phenomena, the important mathematical questions that concern them are:

(i) Existence (and uniqueness) of weak and forts solutions.

(ii) Global character of the solution.

(iii) Positivity of the solution whenever the initial data are positive.

(iv) Asymptotic behavior of the global solution when time t tends to infinity.

(v) Continuous dependence of the solution on the initial data.

1.3 Dissipation or control of mass, Structure (P)+(M)

We will review most of the main results on global existence in time for the family of $m \times m$ reaction-diffusion systems satisfying the two main following properties:

- the nonnegativity of the solutions is preserved for all time

- the total mass of the components is a priori bounded on all finite intervals.

More precisely, let us introduce the general system

$$(1.2) \qquad \begin{cases} \forall i = 1, ..., m \\ \dfrac{\partial u_i}{\partial t} - d_i \Delta u_i = f_i(u_i, ..., u_m) \text{ on } (0,T) \times \Omega \\ \alpha_i \dfrac{\partial u_i}{\partial n} + (1-\alpha_i) u_i = \beta_i \text{ on } (0,T) \times \partial\Omega \\ u_i(0,.) = u_{i0} \end{cases}$$

where the $f_i : \mathbb{R}^m \to \mathbb{R}$ are C^1 functions of $u = (u_1, ..., u_m)$, and for all $i = 1, ..., m$, $d_i \in (0, \infty)$, $\alpha_i \in [0,1]$, $\beta_i \in C^2\left([0,T] \times \overline{\Omega}\right)$, $\beta_i \geq 0$. We denote $\Sigma_T = (0,T) \times \partial\Omega$.

By classical solution to (1.2) on $[0,T)$,we mean that

$$
(1.3) \qquad \begin{cases} u \in C\left([0,T); L^1(\Omega)^m\right) \cap L^\infty([0,T-\tau] \times \Omega)^m, \forall \tau \in (0,T) \\ \forall k,l = 1...N,\ \forall p \in [1,\infty),\ \partial_t u,\ \partial_{x_k} u,\ \partial_{x_k x_l} u \in L^p((\tau, T-\tau) \times \Omega)^m \\ \text{and equations in (1.2) are satisfied a.e.} \end{cases}
$$

Note that this regularity of u implies that $u; \partial_{x_k} u$ have traces in $L^p_{loc}(\Sigma_T)^m$ (see e.g. [80]). Most of the time, due to more regularity of f, the solutions will be regular enough so that derivatives may be understood in the usual sense (e.g. $u \in C^2\left((0,T) \times \overline{\Omega}\right)$ if f is C^2 itself).

Let us first recall the classical local existence result under the above assumptions (see e.g. [15], [62] and [132]).

Lemma 1.2. *Assume $u_0 \in L^\infty(\Omega)^m$. Then, there exist $T > 0$ and a unique classical solution of (1.2) on $[0,T)$. If T^* denotes the greatest of these T's, then*

$$
(1.4) \qquad \left[\sup_{t \in [0,T^*), 1 \leq i \leq m} \|u_i(t)\|_{L^\infty(\Omega)} < +\infty\right] \Longrightarrow \left[T^* = +\infty\right]
$$

If, moreover, the nonlinearity $(f_i)_{1 \leq i \leq m}$ is ***quasi-positive*** *(see (1.5) below), then*

$$
[\forall i = 1, \ldots, m,\ u_{i0} \geq 0] \Longrightarrow \left[\forall i = 1, \ldots, m,\ \forall t \in \left[0, T^*\right), u_i(t) \geq 0\right].
$$

Nonnegativity of the solutions is preserved if (and only if) the nonlinearity $f = (f_1, \ldots, f_m)$ is quasi-positive which means that

$$
(1.5) \qquad \textbf{(P)} \quad \forall r \in [0, +\infty)^m,\ \forall i = 1 \ldots m,\ f_i(r_1, \ldots, r_{i-1}, 0, r_{i+1}, \ldots, r_m) \geq 0,
$$

Where we denote $r = (r_1, ..., r_m)$.

Remark 1.3. *According to (1.4), in order to prove global existence of classical solutions for system (1.2), it is sufficient to prove that, if $T^* < +\infty$, then the solutions u_i are uniformly bounded on $[0,T^*)$. Thus, a priori L^∞-bounds imply global existence. As already noticed (see (1.2), (1.3) in [122]), it is the case if all the d_i's are equal, and global existence then holds for bounded initial data. The situation is quite more complicated if the diffusion coefficients are different from each other.*

Without any extra assumption on f, blows up generally occurs infinite time ($T^* < +\infty$). Here, we will assume that f satisfies a *"mass-control structure"*

$$\text{(1.6)} \qquad (\mathbf{M}) \quad \forall r \in [0,+\infty)^m, \sum_{1\le i\le m} f_i(r) \le C\left[1+\sum_{1\le i\le m} r_i\right]$$

Note that this was satisfied in example (1.1) [122] with $C=0$ and even with equality instead of inequality. With "correct" boundary conditions, (1.6) implies *that the total mass of the solution is bounded on each interval*. Indeed, let us set $W=\sum_{1\le i\le m} u_i$. Integrating the sum of the m equations of(1.2) leads to

$$\partial_t \int_\Omega W(t) - \int_{\partial\Omega} \nabla\left[\sum_i d_i u_i\right].n \le C\int_\Omega [1+W(t)].$$

Assume for instance that $\forall i, \alpha_i (0,1]$. Then, using boundary conditions

$$-\int_{\partial\Omega} \nabla\left[\sum_i d_i u_i\right].n = \int_{\partial\Omega} \sum_i d_i \frac{(1-\alpha_i)u_i - \beta_i}{\alpha_i} \ge -\int_{\partial\Omega}\sum_i \frac{d_i\beta_i}{\alpha_i} := -c$$

Thus, we have the Gronwall's inequality

$$\partial_t \int_\Omega W(t) \le c + C\int_\Omega 1+W(t)$$

which implies that, for each t in the interval of existence

$$\text{(1.7)} \qquad \int_\Omega W(t) \le e^{tC}\int_\Omega W(0) + k\left(e^{tC}-1\right),\ k=\frac{1}{C}\left(c+\int_\Omega C\right)$$

it follows that the total mass $\int_\Omega W(t)$ is bounded on any interval.

Now we will see how does this L^1-estimate help to provide global existence.

Instead of $(\mathbf{M})$, we could assume that, for some $a=(a_i)_{1\le i\le m}$ with $\forall i, a_i>0$

$$\text{(1.8)} \qquad (\mathbf{M}') \quad \forall r \in [0,+\infty)^m, \sum_{1\le i\le m} a_i f_i(r) \le C\left[1+\sum_{1\le i\le m} r_i\right]$$

Obviously, $(\mathbf{M}')$ may be reduced to $(\mathbf{M})$, after multiplying each i-th equation of (1.2) by a_i and upon replacing u_i by $a_i u_i$. For simplicity, we will mainly use $(\mathbf{M})$ in the following, although examples may arise with $(\mathbf{M}')$.

As shown above, i.e., by taking $\alpha_i > 0$ for all i the conditions $(\mathbf{P})+(\mathbf{M})$ give almost in all cases the controllability of the total mass. Nevertheless if $\alpha_i = 0$, $\beta_i \ne 0$ for some i and $\alpha_i > 0$ for others then L^1-estimates may be lacking, which suggests restricting the values of α_i, β_i in the section (see subsection (3.4) and (5.5) in [122]).

It is known in the literature that many systems naturally present the two properties (**P**) and (**M**) (or (**M**$'$)) in various applications, before giving some examples it would be necessary to ensure the global existence in time with these two properties alone. Moreover we will consider systems, where the total mass is bounded on any interval, but also the nonlinearities are bounded in $L^1(Q_T)$ for all $T < +\infty$ (as it is the case for the example (1.1) in [122]).

Using bootstrap arguments, we can show that solutions exist globally if the nonlinearities f_i are bounded in $L^1(Q_T)$ for all $T > 0$ and if their growth is smaller than $|u|^{\frac{N+2}{N}}$ as $|u| \to +\infty$ and therefore they are effectively bounded in $L^\infty(Q_T)$ (see for example [11]), what is interesting is to see other cases, i.e. not to be in the bootstrap state when the growth of the nonlinearities is not small, or, given a nonlinearity and this when the dimension is sufficiently high let us notice that the bootstrap argument is not valid for $N \geq 2$ even for quadratic nonlinearities.

1.4 Some examples of reaction-diffusion systems with properties (P) + (M)

Here, we give a few examples of reaction-diffusion systems found here and there in the literature as models for very different applications and for which the two properties $(P)+(M)$ hold.

(i) Lotka-Volterra systems. A general class of Lotka-Volterra Systems may be written (see for instance in [56] and [83])

$$\forall i = 1 \dots m,\ \partial_t u_i - d_i \Delta u_i = e_i u_i + u_i \sum_{1 \leq j \leq m} p_{ij} u_j,$$

with $e_i, p_{ij} \in \mathbb{R}$ and various boundary conditions. Condition (P) is always satisfied, and so is (M') -see (1.8)- if for instance for some $a_i > 0$ (see [83])

$$\forall w \in \mathbb{R}^m,\ \sum_{i,j=1}^{m} a_i p_{ij} w_i w_j \leq 0,$$

(ii) Quadratic chemical reactions. Many chemical reactions, when modeled through the mass action law, lead to reaction-diffusion systems with the above $(P)+(M)$ structure. Let us first take a typical example. We consider the reversible reaction

$$A + B \rightleftharpoons C + D$$

Then according to the mass action law, the evolution of the concentrations a,b,c,d of A,B,C,D is governed by the following reaction-diffusion system:

$$\begin{cases} \partial_t a - d_1\Delta a = -k_1 ab + k_2 cd \\ \partial_t b - d_2\Delta b = -k_1 ab + k_2 cd \\ \partial_t c - d_3\Delta c = k_1 ab - k_2 cd \\ \partial_t d - d_4\Delta d = k_1 ab - k_2 cd \end{cases}$$

with $k_1,k_2 > 0$. Our two conditions are obviously satisfied here. We may also exploit that the entropy is decreasing (this is actually the case in reversible reactions).

(iii) Superquadratic reaction-diffusion systems. We consider a general reversible chemical reaction of the form

$$p_1A_1 + p_2A_2 + ... + p_mA_m \rightleftharpoons q_1A_1 + q_2A_2 + ... + q_mA_m,$$

where p_i, q_i are nonnegative integers. According to the usual mass action kinetics and with classical diffusion operators, we model the evolution of the concentrations a_i of A_i by the following system of reaction-diffusion

$$\partial_t a_i - d_i\Delta a_i = (p_i - q_i)(k_2 \prod_{j=1}^{m} a_j^{q_j} - k_1 \prod_{j=1}^{m} a_j^{p_j}), \quad \forall i = 1...m,$$

where d_i are positive diffusion coefficients. A classical conservation property states that $\sum_i m_i p_i = \sum_i m_i q_i$ for some $m_i \in (0,\infty), i = 1...m$. Denoting by f_i the nonlinearity in the i-th equation, this implies $\sum_{i=1}^{m} m_i f_i = 0$, whence the condition (M'). The quasipositivity (P) is satisfied as well.

(iv) Diffusion of pollutants in atmosphere. Another interesting example comes from the modeling of pollutants transfer in atmosphere (here $N = 3$) this system of 20 equations is studied in [57] and, more recently in [123] (we refer to these two papers for more references):

$$\begin{cases} \partial_t \phi_i = d_i \partial^2_{zz}\phi_i + w.\nabla\phi + f_i(\phi) + g_i, \ \forall i = 1,\dots,20 \\ \text{+ Boundary and initial conditions.} \end{cases}$$

Here the nonlinearities f_i are given by

$$\begin{cases} f_1(\phi) = -k_1\phi_1 + k_{22}\phi_{19} + k_{25}\phi_{20} + k_{11}\phi_{13} + k_9\phi_{11}\phi_2 + k_3\phi_5\phi_2 \\ +k_2\phi_2\phi_4 - k_{23}\phi_1\phi_4 - k_{14}\phi_1\phi_6 + k_{12}\phi_{10}\phi_2 - k_{10}\phi_{11}\phi_1 - k_{24}\phi_{19}\phi_1, \\ f_2(\phi) = k_1\phi_1 + k_{21}\phi_{19} - k_9\phi_{11}\phi_2 - k_3\phi_5\phi_2 - k_2\phi_2\phi_4 - k_{12}\phi_{10}\phi_2, \\ f_3(\phi) = k_1\phi_1 + k_{17}\phi_4 + k_{19}\phi_{16} + k_{22}\phi_{19} - k_{15}\phi_3, \\ f_4(\phi) = -k_{17}\phi_4 + k_{15}\phi_3 - k_{16}\phi_4 - k_2\phi_2\phi_4 - k_{23}\phi_1\phi_4, \\ f_5(\phi) = 2k_4\phi_7 + k_7\phi_9 + k_{13}\phi_{14} + k_6\phi_7\phi_6 - k_3\phi_5\phi_2 + k_{20}\phi_{17}\phi_6, \\ f_6(\phi) = 2k_{18}\phi_{16} - k_8\phi_9\phi_6 - k_6\phi_7\phi_6 + k_3\phi_5\phi_2 - k_{20}\phi_{17}\phi_6 - k_{14}\phi_1\phi_6, \\ f_7(\phi) = -k_4\phi_7 - k_5\phi_7 + k_{13}\phi_{14} - k_6\phi_7\phi_6, \\ f_8(\phi) = k_4\phi_7 + k_5\phi_7 + k_7\phi_9 + k_6\phi_7\phi_6, \\ f_9(\phi) = -k_7\phi_9 - k_8\phi_9\phi_6, \\ f_{10}(\phi) = k_7\phi_9 + k_9\phi_{11}\phi_2 - k_{12}\phi_{10}\phi_2, \\ f_{11}(\phi) = k_{11}\phi_{13} - k_9\phi_{11}\phi_2 + k_8\phi_9\phi_6 - k_{10}\phi_{11}\phi_1, \\ f_{12}(\phi) = k_9\phi_{11}\phi_2, \\ f_{13}(\phi) = -k_{11}\phi_{13} + k_{10}\phi_{11}\phi_1, \\ f_{14}(\phi) = -k_{13}\phi_{14} + k_{12}\phi_{10}\phi_2, \\ f_{15}(\phi) = k_{14}\phi_1\phi_6, \\ f_{16}(\phi) = -k_{19}\phi_{16} - k_{18}\phi_{16} + k_{16}\phi_4, \\ f_{17}(\phi) = -k_{20}\phi_{17}\phi_6, \\ f_{18}(\phi) = k_{20}\phi_{17}\phi_6, \\ f_{19}(\phi) = -k_{21}\phi_{19} - k_{22}\phi_{19} + k_{25}\phi_{20} + k_{23}\phi_1\phi_4 - k_{24}\phi_{19}\phi_1, \\ f_{20}(\phi) = -k_{25}\phi_{20} + k_{24}\phi_{19}\phi_1. \end{cases}$$

where the k_i's are positive real numbers. These nonlinearities may seem complicated, but they are quadratic and, obviously satisfy $(P)+(M)$. The main new point in this system is that diffusion occurs only in the vertical direction. As a consequence, many of the tools, which are based on the regularizing effects of the diffusion, need to be revisited. Even the transport term may cause new difficulties due to the lack of diffusion in two directions. However, the general methods described in the next sections may be used to obtain some global existence results for this degenerate system.

We could go on with more and more examples arising in applications with $(P)+(M)$. We refer for instance to the books [52], [53], [62], [83], , [111], [118], and [132].

1.5 Existence of global classical solutions

1.5.1 A typical result on 2×2 systems

Let us consider the following 2×2 system

$$(1.9)\qquad \begin{cases} \partial_t u - d_1\Delta u = f(u,v) \\ \partial_t v - d_2\Delta v = g(u,v) \\ u(0,.) = u_0(.) \geq 0, v(0,.) = v_0(.) \geq 0, \\ \text{with either: } \dfrac{\partial u}{\partial n} = \beta_1, \dfrac{\partial v}{\partial n} = \beta_2 \text{ on } (0,+\infty)\times\partial\Omega, \\ \text{or : } u = \beta_1,\ v = \beta_2 \text{ on } (0,+\infty)\times\partial\Omega, \end{cases}$$

where $d_1, d_2 \in (0,+\infty), \beta_1, \beta_2 \in [0,+\infty)$ and $f, g : [0,+\infty)^2 \longrightarrow \mathbb{R}$ are C^1.

For $u_0, v_0 \in L^\infty(\Omega)$ with $u_0, v_0 \geq 0$, existence of classical nonnegative bounded solutions holds on some maximal interval $[0,T^*)$ (see Lemma 1.2). Then, we have the first following global existence result (see [64], [95])

Theorem 1.5. *Assume $(P)+(M)$ holds for (1.9) (see (1.5), (1.6)). Assume moreover that $u_0, v_0 \in L^\infty(\Omega)$, $u_0, v_0 \geq 0$,and, for some $U, C \geq 0$*

$$(1.10)\qquad \forall u \geq U, \forall v \geq 0,\ f(u,v) \leq C[1+u+v],$$

$$(1.11)\qquad \exists r \geq 1;\ \forall u,v \geq 0,\ |g(u,v)| \leq C\left[1+u^r+v^r\right]$$

Then, $T^* = +\infty$.

Comments: Condition (1.11) means that the growth of $g(u,v)$ as $u,v \longrightarrow +\infty$ is at most polynomial. The first condition (1.10) means that the first equation is "good".

A typical case is for instance when $f \leq 0$ in which case u is uniformly bounded on the time of existence by maximum principle.

It is more generally the case when $f \leq C(1+u)$. Actually, in the statement of Theorem 1.5, we may replace (1.10) by the a priori knowledge that u is uniformly bounded, no matter the reason of this bound (see [64]). In this case, the second condition (1.11) may be replaced by the weaker condition $g(u,v) \leq \varphi(u)(1+v^r)$ where $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is nondecreasing. The general idea of this theorem is that, for systems with the structure $(P)+(M)$, if moreover one of u or v is uniformly bounded on $[0;T^*)$, then, so is the other; whence global existence.

1.5.2 Extension to $m \times m$ systems

This L^p approach has been extended (see [56], [107], and [108]) to $m \times m$ systems with a so-called triangular structure, which means essentially that f_1, $f_1 + f_2$, $f_1 + f_2 + f_3, \ldots$ are all bounded above by a linear function of the u_i. we may for instance state the following

Theorem 1.6. *Let $f \in C^1([0,\infty)^m, \mathbb{R}^m)$ with at most polynomial growth and satisfying the quasipositivity* (P)*. Assume moreover that there exist $C \geq 0$, $b \in \mathbb{R}^m$ and a lower triangular invertible $m \times m$ matrix P with nonnegative entries such that*

$$\forall r \in [0,\infty)^m, Pf(r) \leq \left[1 + \sum_{1 \leq i \leq m} r_i\right] b \tag{1.12}$$

where the usual order in $\mathbb{R}^m$ is used. Then, the system

$$\begin{cases} \forall i = 1, \ldots, m \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, \ldots, u_m) \text{ on } (0,T) \times \Omega, \\ \forall i, \frac{\partial u_i}{\partial n} = \beta_i \geq 0 \text{ (or } \forall i, \partial u_i = \beta_i \geq 0) \text{ on } (0,T) \times \partial\Omega, \\ u_i(0,.) = u_{i0} \in L^\infty(\Omega), u_{i0} \geq 0, \end{cases} \tag{1.13}$$

has a global classical solution.

1.5.3 More remarks on global classical solutions

Using appropriate Lyapunov functions, the global existence of specific 2×2 systems with the above type of structure has been proved. A result has been given by Masuda [96] in the case

$$f(u,v) \leq 0, \ g(u,v) \leq -\varphi(u) f(u,v), \ g(u,v) \leq \varphi(u)(v + v^r), r \geq 1.$$

Haraux and Youkana [61] generalized this result by taking the growth of type $e^{\alpha v^\beta}$ with $\alpha > 0$ and $\beta < 1$ and in the case $g = -f$. If $\beta = 1$ can achieve exponential growth but, curiously, only with restrictions on the size of $\|u_0\|_\infty$ (see [20]). Recently this approach has been coupled with an interesting function change in [126]. What is interesting in their work is that they have demonstrated the global existence for new specific systems in this family.

If the growth is high (see problem 3 of section 7 in [122]) the problem does not admit any solution for example for the following formula

$$-f(u,v) = g(u,v) = ue^{\beta},\ \beta \geq 1. \tag{1.14}$$

(see below for the case where $\beta = 1$ in $\mathbb{R}^N$).

Exponential growth

we will see with the exponential growth

The exponential growth is a limit to most methods for system (1.14). In [63] in the case of the problem (1.14) with $\beta = 1$, although it is different with the other methods the global existence exists in the case of $\mathbb{R}^N$ with a careful analysis of the heat kernel. In this limit case other results are treated see references [69], [77] and [79], it is worth mentioning that in [63] the said method is optimal.

Nevertheless no example is provided. It is worth noting that the regularity results for L^p spaces could also be extended to Orlicz spaces they follow from the L^p duality method described above. This is an open problem, however the approximate analysis gives a suggestion to study the exponential growth see section 7 in [122] for more details.

More results

It is mentioned in [95] for the 2×2 system with $0 < g = -f$, and $\Omega = \mathbb{R}^N$ that, if the diffusion coefficients satisfy $d_1 \geq d_2$, then the global existence is assured, the demonstration is done by means of the properties of the heat kernel then it was generalized see [63]. On the other hand, always for the same system, this time via precise estimates of the Green's function on a bounded domain and with various boundary conditions, is also found in [70]. In [35], [70], [71], and [72] one finds global existence results obtained in any dimension for systems with $(\mathbf{P}) + (\mathbf{M})$ and with strictly sub-quadratic growth, note that the quadratic case is also treated for some systems in dimension $N \leq 2$ (see [44], [60] and [65].

Other boundary conditions

According to the results given in reference [22], the boundary conditions can play a role in that the explosion can take place in finite time for the following system

$$\begin{cases} \partial_t u - d_1 \Delta u = -uv^{\gamma} \\ \partial_t v - d_2 \Delta v = uv^{\gamma} \\ u = 1,\ \dfrac{\partial v}{\partial u} = 0\ on\ \Sigma_T, \end{cases}$$

with $\gamma > 2$, to be sure that this system does not really belong to this class, we must check $\int_\Omega u(t) + v(t)$ is not bounded. A question arises: what are the boundary conditions which answer the non membership of this system to this class? A general rule for 2×2 systems is that most techniques apply to cases where the boundary conditions are of the same type for both equations. When they are of different kind (like Dirichlet/Neumann).

Some other diffusion operators: Most of the results of this thesis, are stated with simple diffusion operators of the form $-d_i\Delta$ and this to facilitate to the reader the understanding of the different diffusion operators, which will allow us to focus on the main difficulty due to the fact that the diffusion operators intervening in the various equations are different from each other: this simple case proves to be very significant of this difficulty.

In several of the cited references, general diffusion operators are considered, among others, in the reference [95] the L^p technique is developed for general parabolic operators while in the reference [109] and its references nonlinear diffusion are considered.

1.6 Systems with nonlinearities bounded in L^1: weak global solutions

1.6.0.1 Introduction: an example

Recall the examples of the form

$$\begin{cases} \partial_t u - d_1\Delta u = \lambda u^p v^q - u^\alpha v^\beta = f(u,v) \\ \partial_t v - d_2\Delta v = -u^p v^q + u^\alpha v^\beta = g(u,v) \end{cases}$$

we noticed that

$$(M) \quad f(u,v) + g(u,v) = (\lambda - 1)u^p v^q \leq 0 \text{ if } \lambda \in [0,1]$$

but, we also have

$$(M_\lambda) \quad f(u,v) + \lambda g(u,v) = (\lambda - 1)u^\alpha v^\beta \leq 0,$$

which gives one more relation between f and g if $\lambda \leq 1$. It turns out that $(P)+(M)+(M_\lambda)$ with $\lambda \neq 1$ imply that the nonlinearities $f(u,v), g(u,v)$ are bounded in $L^1(Q_T)$ for all T.

Let us state this result for the more general system

$$(1.15) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - d_1\Delta u = f(u,v) \text{ on } Q_T \\ \dfrac{\partial v}{\partial t} - d_2\Delta v = g(u,v) \text{ on } Q_T \\ \alpha_0\dfrac{\partial u}{\partial n} + (1-\alpha_0)u = \alpha_0\dfrac{\partial u}{\partial n} + (1-\alpha_0)v = 0 \text{ on } \Sigma_T \\ u(0,.) = u_0(.) \geq 0, \ v(0,.) = v_0(.) \geq 0. \end{cases}$$

Proposition 1.1. *Assume* $(P), \alpha_0 \in [0,1]$*, and* $\exists C \geq 0$*,* $\exists \lambda \in [0,+\infty)$*,* $\lambda \neq 1$ *with*

$$f + g \leq C(1+u+v) \text{ and } f + \lambda g \leq C(1+u+v). \tag{1.16}$$

Then, if u, v *are solutions of* (1.15) *on* $(0,T)$,

$$\int_{Q_T} [|f(u,v)| + |g(u,v)|]\, dt\, dx \leq M = M(data) < +\infty.$$

Remark 1.4. *Note that the assumptions of Proposition 1.1 imply*

$$\forall \mu \in [\lambda, 1],\ f + \mu g \leq C(1+u+v).$$

Remark 1.5. *Many systems come naturally with an extra structure which makes the nonlinearities to be bounded in* $L^1(Q_T)$.

1.7 Relationship between reaction-diffusion systems and image processing

This part reviews one of the major applications of reaction-diffusion systems, namely the smoothing and restoration of images. The purpose of image restoration is to estimate the original image from the degraded data. This is mainly due to the mathematical formulation framing any Partial differential equations based approach that can give a good justification and explanation of the results obtained through these traditional and heuristic methods in image processing. In this context we prove the existence of weak global solutions for a class of generic reaction-diffusion systems for which two main properties hold: the quasi-positivity and a triangular structure condition on the nonlinearity.

Reaction-diffusion Systems are mathematical models which correspond to several physical phenomena and natural sciences often used to model how numbers of agents move and interact. They are naturally applied in chemistry explained by a local chemical reactions of substances in where they are transformed and a diffusion process that causes their distribution in space. However, they are also found in other field like biology, ecology, geology, and are characterized as a semi linear parabolic partial differential equations.

For example, in one dimension, the one component reaction-diffusion equation

$$\partial_t u = D\Delta u + f(u) \tag{1.17}$$

is referred to as the Kolmogorov-Petrovsky-Piskunov (KPP) equation in where for $f(u) = u(1-u)$ refers to Fisher's equation used at the base to describe the spreading of biological populations, while for $f(u) = u(1-u)(u-a)$ with $0 < a < 1$ describes the more general Zeldovich equation that appears in combustion theory.

The solutions of this kind of equation can present diverse behaviors among which the formation of progressive waves and wave phenomena or entropic patterns (bands, hexagons and other more complex patterns such as solitons).

Most studies of reaction-diffusion system was devoted to patterns formation phenomena like fingerprints, strips on zebra... In 1951, Alan Turing analyzed the possible cases of system (1.17) with two component and published his only paper on biological pattern formation of the idea that he had, which says that a state that is stable in the local system can become unstable in the presence of diffusion.

A two component reaction-diffusion system is a typical model of equations describing the pattern recognition. They can be defined as a process that describe significantly the data and understand deeply the problem through the manipulation of a data set. They turn out to be very useful in image processing for pattern classification where preprocessing data is useful for error corrections, image enhancement, and recognition. Nonlinear reaction-diffusion models can describe many natural phenomena in a wide range of disciplines. Over the last few years, some amazing results were observed in engineering applications such as image processing. Among these applications, we cite Fitzhugh-Nagumo et *al*. [50] model which allowed the detection of noisy image contours. We also cite the anisotropic diffusion described by Perona and Malik which includes local information to reduce noise and enhance contrast while preserving the edge. From where the idea of Catté et *al*. [37] to integrate directly the regularization into the equation by convolving the image with the Gaussian filter on the gradient of the noisy image to smooth the image first in order to avoid the dependence of the numerical scheme between the solution and the regularization procedure, this makes the problem well posed and the existence and uniqueness of the problem was proven by Catté et *al*. [37]. Other generalization of this work were made by Whitaker and Pizer, Li and Chen [84] and Weickert and Benhamouda [157]. In 2006, Morfu [106] proposed a model performing noise filtering and contrast enhancement where he combined the nonlinear diffusion process ruled by Fischer equation that was originally used to describe the spreading process of biological population without establishing any existence or consistency result Until the work of Alaa et *al.* [7] combining the regularization procedure in Catté with Morfu model, the authors were able to demonstrate the existence and consistency of the their proposed model. We build up on their works by providing a generalization to the case of reaction-diffusion systems.

1.8 Physical origin of the contrast

See Boccara et al. [30].

1.8.1 Tomography of biological tissues

Biological tissues, in the spectral zones where they are not too absorbent (red and near infrared), scatter the light strongly, so direct observations (in depth) are not feasible. However, the optical contrast as it appears for example to the anatomopathologist on the surgical specimens is often a relevant source of information. How can virtual (optical) biopsies be performed without physically penetrating the tissue?

For shallow observation ($< 1mm$), ballistic photons even very strongly attenuated (~ a billion times) are still detectable provided that they are isolated from the background of much more intense scattered light (~ a few million times).

Confocal microscopy, by spatial filtering, allows to realize ”sections” at a hundred μm depth.

Non-linear microscopy (fluorescence after a bi or tri photonic absorption or harmonic generation) uses a source in the near infrared, less scattered than blue or ultraviolet light and which is only active in the vicinity of the focal zone. Indeed, only the high power density obtained at the focal points of microscopes with lasers whose pulses last a few tens to a few hundreds of femtosecond allows the observation of these signals. It is important to underline the considerable impact that this approach has had in biology in a few years.

Coherence tomography (optical coherence tomography (**OCT**) or optical coherence microscopy (**OCM**)) presents without doubt the greatest dynamic range (~ 120 dB), it uses the interference between the back scattered light from a ”slice” of sample whose thickness is close to the coherence length of the source (~ 1 to 10μm) and a reference beam to make cuts in the depth of the tissue.

OCT has undergone spectacular developments in ophthalmology and is also valuable for monitoring embryonic development.

At greater depths the ballistic photons are completely attenuated and it is necessary to work with *scattered photons*.

Most often the use of fast *time gates* (electronic or based on non-linear optics) allows, by selecting the photons that have remained close to the ballistic trajectory, to partially escape the negative effects of scattering in terms of contrast and resolution: these photons, known as serpentiles, exist all the more so as the cells that make up the biological tissues scatter very strongly towards the front.

Despite certain advances in experimental approaches and in the resolution of the inverse problem, progress remains to be made in order to achieve the performance desired by radiologists ($\sim 1mm$ instead of 1cm today for tumors located at a few cm depth).

A simpler situation, for the resolution of the inverse problem, consists in highlighting a temporal variation of the spatial distribution of the optical properties without trying to reconstitute the optical properties of the whole structure. This situation occurs if there is injection of a contrast agent or activation (cerebral for example) by a outdoor stimulus. This field, which has been developed on the east coast of the USA for a decade, is beginning to be explored in France.

We can finally take advantage of acoustics to reveal optical contrasts with a resolution that is that of acoustic ultrasound ($\sim 1mm$).

Two physical effects can be used:

- *the acousto-optic effect* consists in .locally marking the photons by the ultrasounds. Provided that a laser of sufficient coherence length is used so that the emerging photons create an observable speckle, the ultrasound will periodically change the position of the scatterers and modulate the intensity of each speckle grain. We can thus create a virtual source of marked photons of millimeter size (in the area where ultrasound and photons are simultaneously present) capable of revealing the local optical properties of tissues. For example, this source will see its luminance decrease if it is placed in an absorbing area (vascularized tumor).

- *the opto-acoustic effect* is linked to the absorption of a short laser pulse by a zone of the sample. This absorption is followed by a rapid thermalization and a local variation of the density at constant volume which is the source of ultrasonic waves. The intensity of the acoustic signal is related to the distribution of local absorption coefficients (e.g. by brain activation). The inverse problem consists in reconstructing the distribution of these acoustic sources, here again the resolution is that of acoustics and the contrast is related to the absorption.

1.8.2 Extreme spectral ranges and large instruments

The very high luminance sources that operate over a very wide spectral range have been the source of new instrumental developments for imaging and especially spectro-imaging at very short (e.g. European Synchrotron Radiation Facility (**ESRF**) for X-rays or long (e.g. CLIO for infrared) wavelengths. With **ESRF**, the European scientific community can benefit from an operational platform for nano-tomography.

In the same way, platforms using intense lasers have made it possible to create laser beams in the X-ray domain, opening the way to direct imaging or holography at very high resolution.

The spectral domains of vacuum UV and X-rays have largely benefited from the progress of diffracted optics, which is essential in a domain where no material is transparent. These components, manufactured by lithographic techniques, deflect the beams as diffraction gratings do and not as refractive lenses.

1.8.3 Nuclear imagery

Nuclear imaging consists of measuring over time the specific binding of a radiolabeled tracer, by an β^+ isotope in Positron Emission Tomography (**PET**) or a γ emitter in Single Photon Emission Tomography (**SPECT**). It thus gives access to spatial information (where is the tracer fixed?) and to temporal information (what are the kinetics of tracer fixation?). The construction of the specific molecule of a biological target and its labelling, generally achieved by substitution or addition of a radioactive atom to the initial molecule, require the intervention of chemists and radiochemists. As for most biomedical imaging modalities, the implementation of nuclear imaging requires a highly multidisciplinary participation including chemists, instrumentalists, computer scientists, biologists and physicians. Nuclear imaging has 2 major assets:

- That of having a high sensitivity since it gives access to the quantification of molecular concentrations up to 10^{-12} mole/1.

- The ability to benefit from a very wide variety of radio-labeled molecules. Among the applications, we will particularly mention the measurement of energy metabolism, which is widely used today in oncology for its sensitivity, the pharmacokinetics of molecules of interest or even gene expression (a new and very promising field of investigation in imaging).

In contrast, nuclear imaging has several limitations: it does not provide anatomical data and has a relatively low spatial resolution (a few millimeters in humans). Finally, we should mention among the limitations of nuclear imaging, its cost and its difficulty of implementation, in particular for **PET** which requires dedicated facilities to produce radioisotopes *in situ* given their low radioactive half-life.

Among the recent progress made in nuclear imaging, a large part was motivated by the need to associate this technique with animal imaging. The stakes were high, since it was necessary to reduce the spatial resolution from $5mm$ for **PET** and $1cm$ for **SPECT** adapted to humans to a resolution of the order of one mm to meet the dimensions of the mouse. It is in the field of **PET** that these advances have been most significant. Although the value of $1mm$ has not yet been reached, systems with resolutions of less than $2mm$ are now operational.

1.8.4 Digital image restoration

Another interesting application, is the digital image restoration, it's it can be done by using a partial differential equations (PDE) in image processing and computer vision has as its ultimate goal the analysis and processing of digital images, in order to extract quantitative information and to develop programs to allow a computer to recognize in digital images the objects that are present in the visual scene, the spatial relationships between them, and any information useful for a certain task that we want to accomplish in the physical space represented by this scene. Research in this area has also influenced mathematical morphology, which has benefited from work on the application of PDEs. mathematical morphology which has benefited from the work on the application of PDEs, in this sense it is important to point out that the majority of computer vision problems are ill-posed. In other words, there is not always a single solution to a vision problem, but often an acceptable solution is sought by assuming true assumptions that are in most cases technical. This field of research has aroused the interest of a very large community of researchers for a long time and has become major and unavoidable thanks to the very clear increase in the number of journals and conferences in the last few years which testifies to this determination and enthusiasm. Nevertheless, this observation also highlights the difficulties encountered. Several failures, but also some successes, have marked this period where computer science, signal processing and information theory, applied mathematics, statistics, probability theory, geometry and pattern recognition... have played a driving role at one time or another during this period which started in the $70s$.

PDE-based approaches have the advantage of obtaining existence and uniqueness results for classical problems in many cases. They also bring numerical schemes already proven in other fields (numerical analysis, mechanics). Thus the mathematical development is among the well known successes of this research framework and extends to a wide variety of problems in computer vision such as selective smoothing, segmentation, enhancement, curve evolution, motion detection, contrast enhancement, edge detection or extraction and localization of objects in an image and especially restoration. Image restoration has been the subject of much research in image processing. The main difficulty comes from the fact that the contours of the objects and the discontinuities of the image must be preserved. This makes it necessary to introduce non-linear, variational [9] or stochastic methods [146], which allow smoothing of homogeneous regions except for the discontinuity areas of the image. It is interesting to note that these methods based on partial differential equations (PDEs) and anisotropic filtering techniques [38], which are now well established, meet these requirements and have been particularly studied in recent years.

1.8.5 Very high resolution and local probe microscopies

Electron microscopy remains the preferred field for the observation of materials and structures at a scale ranging from the atom to a few tens of nm. With the development of nanotechnologies, it has become an essential visualization tool, but also a tool for manufacturing and chemical analysis, which has taken a decisive step forward with the introduction of field effect sources.

However, in many cases, local probe microscopies prove to be more suitable and less cumbersome: going beyond the limits that were thought to be linked to the physics itself in terms of resolution has certainly been the most spectacular challenge of these methods. We will come back to this aspect later, but let us emphasize here the very wide range of physical quantities or phenomena at the origin of the signals that are revealed in the images.

The tunneling current associated with the density of electronic states has been through the Scaning Tunneling Microscopy (**STM**) at the origin of a very wide variety of methods capable of providing local information in the form of images.

Atomic force microscopes (**AFM**) reveal the topography of nano-objects or nano-structures but also allow access to the local mechanical properties of heterogeneous materials, composites or biological structures.

Today, the images obtained reflect magnetic (nano bits), electrical, thermal, electro-chemical, optical etc. properties at scales between $0.1nm$ and $100nm$.

The images obtained in optical near-field (at distances much smaller than the wave-length associated with the measurement) can highlight physical phenomena that cannot be observed in the far field, such as thermal emission or violation of the selection rules related to the size of the wave vector.

This field of imaging is typical of what can be expected from a field strongly linked to both physics and medicine.

Ultrasound images probe the elastic properties of matter, the technical evolutions have been spectacular and the quality of the 3-D images makes one dream.

Ultrasound does not only propagate in soft tissues (which have an acoustic impedance close to that of water) but through complex, multi-scale, absorbing or scattering hetero-geneous structures: physical models are then necessary to interpret and quantify the results.

Besides the spectacular applications to osteoporosis, high frequency ultrasound reveals structural anomalies at the cellular level related to changes in the walls or dimensions of the nuclei.

In the low frequency domain the absorption of the waves is weak and the propagation equations are, in the linear regime, invariant by time reversal. It is therefore possible to record the distribution of the acoustic field very widely multi-diffused and to re-emit a spatially focused and temporally compressed field like the initial pulse.

This research has opened the way to a multitude of fundamental and applied physical studies, from the self-focusing of ultrasound for the destruction of kidney stones to the realization of focal areas far below the wavelength. New concepts for telecommunica-tions in the presence of scatterers (such as city buildings) have also been successfully tested: they are applicable to other types of waves, such as electromagnetic waves in the microwave domain.

The non-linear domain has also made great progress, with new forms of contrast and better resolution being achieved with harmonics, but also the possibility of inducing shear waves capable of palpating tumors deep in the tissue.

1.8.6 Magnetic imagery : polarized gases with very low fields

Magnetic Resonance Imaging (**MRI**) is undoubtedly the most efficient field of medical imaging today, both in terms of morphological and functional information. Contrast is linked to variations in longitudinal and transverse relaxation times with the environment, and the presence of contrast products further enhances the performance of this imaging mode. The precision achieved for the observation of the human body is of the order of a millimeter and applies equally well to the observation of pathologies such as cerebral lesions, the anarchic development of the microcirculation in the vicinity of a tumor and soon the anomalies of the coronary circulation. Functional **MRI** is also a tool that has made it possible to observe brain activation by monitoring the metabolism accompanying the performance of cognitive tasks or effects related to the unconscious.

Polarized gases (He, Xe) are very efficient to study hollow volumes. like lungs for example. The low density of matter likely to provide a signal when using a gas is compensated by the strong decrease of the spin temperature obtained by optical pumping of the nuclear spins (method invented by Alfred Kastler). The images of lungs of heavy smokers are, in this respect, very revealing of the evils of tobacco.

Multimodal approaches with coupled **MRI**-acoustic contrasts (visualization of shear waves) or **MRI**-optical (colored probes to guide the surgeon) are particularly promising.

Finally, it is possible to make images in very low field thanks to ultra-sensitive detectors (SQUID) operating at very low temperature.

1.9 Application

An interesting example of application is the Modified Fitz-Hugh-Nagumo Model for image restoration, see [4], where the source terms have the following form

$$\begin{aligned} f(u,v) &= \frac{1}{\tau}u(u-a)(1-u)+\mu v \\ g(u,v) &= u-bv \end{aligned}$$

Remark 1.6. *When $\mu \geq 0$, the nonlinearities satisfy quasipostivity, mass control, and the triangular structure and therefore by direct application of the main result we can deduce global existence. It is also worth noting that there is no restriction on the growth of f, g. Consequently other types of non-polynomial nonlinearities can be handled.*

If $\mu < 0$, the expression above do not satisfy the quasipostivity. However we can use the fact that

$$uf(u,v) \leq L_1(1+u^2+v^2)$$
$$uf(u,v)+vg(u,v) \leq L_2(1+u^2+v^2)$$

multiplying each equation by its respective unknown in the truncated problem and summing up we directly obtain the following estimations

$$\sup_{t\in[0,T]} \int_\Omega u_n^2+v_n^2 \leq C$$
$$\int_{\Omega_T} |\nabla u_n|^2+|\nabla v_n|^2 \leq C$$
$$\int_{\Omega_T} |u_n|^4 \leq C$$

where C depends only on T, $|\Omega|$ and initial conditions on $L^2(\Omega)$, which are sufficient to pass to the limit and obtain the result of the main theorem. In both cases, the modified Fitz-Hugh-Nagumo model admits a weak solution for initial conditions u_0, v_0 in $L^2(\Omega)$.

To illustrate the performance of the studied model we present in this paragraph some numerical results. The modified Fitz-Hugh-Nagumo can be approximate by the explicit scheme bellow

$$\frac{u_{i,j}^{n+1}-u_{i,j}^n}{dt} - \mathbf{div}(A^n\nabla u_{i,j}^n) = \frac{1}{\tau}(u_{i,j}^n-a)(1-u_{i,j}^n)+\mu v_{i,j}^n$$
$$\frac{v_{i,j}^{n+1}-v_{i,j}^n}{dt} - d_v\Delta v_{i,j}^n = u_{i,j}^n+bv_{i,j}^n$$
$$A^n = A\left(\left|\nabla(G_\sigma * u_{i,j}^n)\right|, \lambda^n\right)$$
$$\lambda^n = 1.4826\; median(|\nabla u^n| - median(|\nabla u^n|))/\sqrt{2}$$

where *median* represents the median of an image over all its pixels and A is a function that lowers the diffusion rate d_u over regions of high gradients. An example of such function is given by $A(s,\lambda) = \dfrac{d_u}{\sqrt{1+\left(\frac{s}{\lambda}\right)^2}}$. Simulations done on a standard noisy image using the parameters $a = 0.5$, $\tau = 10^{-3}$, $du = 150$, $d_v = 250$, $dt = 1e^{-2}$ are represented in Figure 1.1. In order to quantitatively measure the performance of the model we illustrate in Table.1 to indicators

- The measure of enhancement (EME) measure the quality improvement of the image. It is defined by: Let an image $u(N,M)$ be split into k_1k_2 blocks $\omega_{k,l}$ of sizes l_1l_2 then we define

$$EME = \frac{1}{k_1k_2}\sum_{l=1}^{k_1}\sum_{k=1}^{k_2} 20\log\left(\frac{u^\omega_{\max;k,l}}{u^\omega_{\min;k,l}}\right)$$

  where $u^\omega_{\max;k,l}$ and $u^\omega_{\min;k,l}$ are respectively maximum and minimum values of the image $u(N,M)$ inside the block $\omega_{k,l}$.

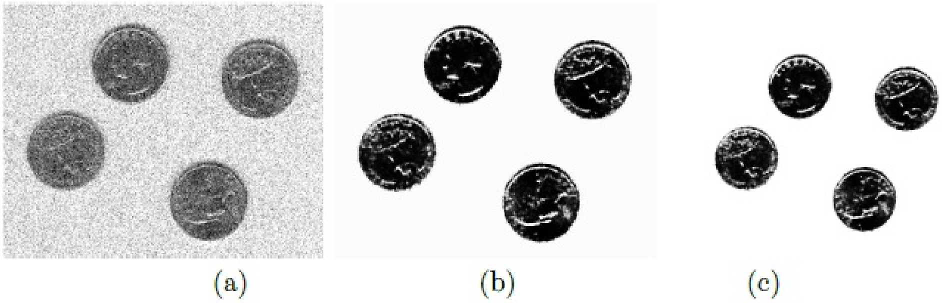

FIGURE 1.1. Restoration of a noisy image using the modified Fitz-Hugh-Nagumo: (a) noisy image, (b) $b = -1$ and $\mu = -1$, (c) $b = 1$ and $\mu = 1$.

The peak signal-to-noise ratio (PSNR) evaluates the performance of noise filtering. It is obtained by

$$PSNR = 10\log_{10}\left(\frac{255^2}{SNR}\right)$$

with

$$SNR = \frac{1}{MN}\sum_{i=1}^{M}\sum_{j=1}^{N}\left[u_{i,j} - u_{i,j}^{ref}\right]^2$$

A higher value of EME and PSNR indicates that the image is well filtered and well enhanced.

Parameters	PSNR	EME
$b = 1, \mu = 1$	25.0153	12.7673
$b = 1, \mu = -1$	25.0482	14.1897

Table.1: EME and PSNR values for the noisy image eight.tif for two different set of parameters.

Summing up, we demonstrated the existence of a global weak solution of the considered model. Also, we proved that the truncated problem admits a weak solution according to Schauder fixed point theorem. For unbounded nonlinearities satisfying suitable conditions, we established equi-integrablity and we derived a compactness results to be able to pass to the limit to get the desired result. To showcase the importance of the obtained result, a new application in the field of image restoration was given however its usefulness is not limited to this application and can be extended to resolve a range of problems in other fields.

CHAPTER 2

BURGERS EQUATION AND ITS APPLICATIONS

In this chapter, we review the non-standard analysis with some applications using the Burgers equation, this equation is transformed into a heat equation using the Hopf-Cole transformation it should be noted that the use of this transformation can be applied in several cases, among others to the multidimensional Burgers equation, to the forced Burgers equation which leads to a parabolic differential equation. Then always within the framework of this equation we will give some models, model of sound wave propagation in soft biological tissues, several applications of the Burgers equation are sonification which is a technique used to explore images of tissues whose details are difficult to detect or else Coronavirus model, Malaria model, Traffic flow model, inviscid and viscous Burgers equations model, viscous Burgers' model. We will finish with the Navier-Stokes model.

2.1 Introduction

The purpose of this chapter, is to apply these Non-Standard Analysis techniques to a boundary value problem of PDE namely Burgers's equation. The infinitesimal techniques of Non Standard Analysis have made it possible to evaluate qualitatively the problems of singular perturbations concerning the differential equations

One of the most important partial differential equations in the theory of non linear conservation laws is Burgers equation. She combining both nonlinear propagation effects and diffusive effect. This equation has a large variety of application in the modeling of statistics of flow problems mixing and turbulent diffusion cosmology and seismology, in medicine, e.g. dermatological ointment, medication for the treatment of a disease, or in agriculture, the watering of plants with chemicals. Burgers equation has different types and each of them has special application. This equation is parabolic when the viscous term is included. If the viscous term is null the remaining equation is hyperbolic. In this case the Burgers equation has the form

$$u_t + uu_x = \varepsilon u_{xx} \tag{2.1}$$

where ε is a positive parameter small enough.

In particular case When ε is null, this equation approach to the Euler's equations in one dimension who govern the flows of perfect fluids. If ε is not null, we approach to the Navier Stokes equations in one dimension. If the viscous term is removed from the Burgers equation, discontinuities can appear in finite time, even if the initial condition is smooth. They give rise to the phenomenon of shock waves, which have important applications in physics and biology.

In order to study the initial value problem of Burgers equation, the classical methods are based on the search of solutions of the reduced problem to deduce the existence and the asymptotic behavior of the solutions which ε tend to 0, the passage of the limit is very complicated, but in general the limit exists and it is a solution for the reduced problem (when $\varepsilon = 0$).

For small ε, the solution $u(x,t)$ is approximated by this limit [93]. Different methods are based on a weak formulation of the burgers equation seen as a conservation law satisfied on each of the computational domains called cells or finite volumes. The stochastic particle method is thus used for.

One of the major challenges in the field of complex systems is the thorough understanding of the turbulence phenomenon. Direct numerical simulations have contributed greatly to our understanding of the disordered flow phenomena that inevitably occur at high Reynolds numbers. However, an effective theory of the that would predict the characteristics of technologically important phenomena such as turbulent mixing is still lacking, turbulent convection and turbulent combustion on the basis of the fundamental equations of fluid dynamics.

This is due to the fact that already the evolution equation for the simplest fluids, which are the incompressible fluids called Newtonian, must take into account the non-linearity. The incompressible fluids, called Newtonian, must take into account the nonlinear evolution equations as well as non-local properties

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t}u(x,t)+u(x,t).\nabla u(x,t)=-\nabla p(x,t)+\nu\Delta u(x,t) \\ \nabla u(x,t)=0 \end{array} \right. \tag{2.2}$$

The nonlinearity comes from the convection term and the pressure term, while the none locality comes into play because of the pressure term. Due to the incompressibility, the pressure is defined by a Poisson equation

$$\nabla p(x,t)=-\nabla.u(x,t).\nabla u(x,t) \tag{2.3}$$

In 1939, the Dutch scientist J.M. Burgers simplified the Navier-Stokes equation (2.2) by simply deleting the pressure term. Unlike equation (2.2), this equation can be studied in one spatial dimension (physicists like to refer to this problem as a 1+1 dimensional problem in order to emphasize that there is a spatial dimension and a temporal dimension coordinate)

$$\frac{\partial}{\partial t}u(x,t)+u(x,t)\frac{\partial}{\partial x}u(x,t)=\nu\frac{\partial^2}{\partial t^2}u(x,t)+F(x,t) \tag{2.4}$$

Note that usually the Burgers equation is considered without the external force $F(x,t)$. However, we will include this external force field.

Burgers equation (2.4) is nonlinear and one expects to find phenomena similar to turbulence. However, as Hopf [66] and Cole [39] have shown, the homogeneous Burgers equation does not have the most important property attributed to turbulence. The solutions do not exhibit chaotic characteristics such as sensitivity to the initial conditions. This can be demonstrated explicitly by using the Hopf-Cole transformation which transforms the Burgers equation into a linear parabolic equation, see [87].

However, from a numerical point of view, this is important because it allows us to compare the numerically obtained solutions of the nonlinear equation with the exact solutions.

Numerically obtained solutions of the nonlinear equation with the exact solution. This comparison is important to study the quality of the applied numerical schemes applied. Moreover, the equation still has interesting applications in physics and astrophysics. We will briefly mention some of them.

2.2 Interface growth: deposition models

The Burgers equation (2.4) is equivalent to the so-called Kardar-Parisi-Zhang which is a model for a solid surface growing by vapor deposition, or, in the opposite case, the erosion of material from a solid surface. The location of the surface is described in terms of a height function $h(x,t)$. This height evolves in time according to Kardar-Parisi-Zhang equation

$$\frac{\partial}{\partial t}h(x,t) - \frac{1}{2}(\nabla h(x,t))^2 = \nu \frac{\partial^2}{\partial x^2}h(x,t) + F(x,t) \tag{2.5}$$

This equation is obtained from the simple advection equation for a surface at $z = h(x,t)$ moving with a velocity $U(x,t)$

$$\frac{\partial}{\partial t}h(x,t) + U.\nabla h(x,t) = 0$$

The velocity is assumed to be proportional to the gradient of $h(x,t)$, i.e., the surface evolves in the direction of its gradient. The diffusion of the surface is described by the diffusion term.

The Burgers equation (2.4) is obtained from the Kardar-Parisi-Zhang equation simply by forming the gradient of $h(\mathbf{x},t)$

$$u(x,t) = -\nabla h(x,t)$$

2.3 Hopf-Cole Transformation

The Hopf-Cole transformation is a transformation that transforms the solution of the Burgers equation (2.4) into the heat equation

$$\frac{\partial}{\partial t}\Psi(x,t) = \nu \nabla \Psi(x,t)$$

By doing some algebraic calculation we obtain

$$\frac{\partial}{\partial t}h - \frac{1}{2}(\nabla h)^2 = \nu \Delta h$$

However, this is exactly the Kardar-Parisi-Zhang equation (2.5). The complete transformation is then obtained by combining

$$u(x,t) = -\frac{1}{2\nu}(\nabla \log \Psi(x,t))$$

We see explicitly that the Hopf-Cole transformation transforms the nonlinear Burgers equation into a linear heat conduction equation. Since the heat conduction equation is explicitly solvable in terms of the so-called heat kernel, we obtain a general solution of the

Burgers equation. Before constructing this general solution, we want to emphasize that the Hopf-Cole transformation applied to the point out that the Hopf-Cole transformation applied to the multidimensional Burgers equation leads to the general solution only if the initial conditions are satisfied.

The Hopf-Cole transformation applied to the multidimensional Burger's equation leads to the general solution only if the initial condition $u(x,0)$ is a gradient field. For general initial conditions, in particular for initial fields with $\nabla \times (x,t)$, the solution cannot be constructed using the Hopf-Cole transformation and, therefore, is not known in analytical terms. In one space dimension it is not necessary to distinguish between these two cases.

2.4 Forced Burgers Equation

The Hopf-Cole transformation can be applied to the forced Burgers equation. It is simple to show that it leads to the parabolic differential equation

$$\frac{\partial}{\partial t}\Psi(x,t) = \nu\Delta\Psi(x,t) - U(x,t)\Psi(x,t)$$

in which the potential is related to the force

$$F(x,t) = -\frac{1}{2\nu}\nabla U(x,t)$$

The relation to the Schrodinger equation for a particle moving in the potential $U(x,t)$ is obvious. The Burgers equation with a fluctuating force has been studied. It is interesting to note that the Burgers equation with a linear force, i.e., a quadratic potential

$$U(x,t) = a(t)x^2$$

for an arbitrary time-dependent coefficient $a(t)$ could be solved analytically.

2.5 Models

2.5.1 Propagation of sound waves in soft biological tissues model

In [133], Rugina et *al.* studied the propagation of sound waves in soft biological tissues (blood, veins, kidney, liver, lung,.. etc.) by using the Burgers' equation. The propagation depends on the properties of the tissue at the ultrasonic range of frequencies. The propagation of waves in soft tissues is used for diagnostic and tissue therapy. Utility of the Burgers' equation to sonification technique is highlighted next to a medical image used to surgical operation. The Burgers' equation is written as

$$u_t = u_{xx} + uu_x \tag{2.6}$$

where u is the acoustic velocity, and the subscript means the differentiation with specified variable.

This equation reminds of the Riemann equation

$$p_t + pp_x = 0$$

as a particular case of (2.6), where p is the pressure deviation of a medium (air, for example). The Riemann equation is used in nonlinear acoustic waves propagation for which the viscosity of the medium is not taken into account.

Another application of the Burgers' equation is the sonification which is a technique used to explore images of tissues whose details are difficult to detect. By applying sonification, new images are obtained with a better visualization of the explored tissue.

The approach was exercised on fictive images of fibrotic rat liver samples inspired from a study of effects of Ginkgo Biloba leaf extract against hepatic toxicity induced by methotrexate in rats.

Some generalization of (2.6) are the following :

- Burgers-Huxley equation

$$u_t + \alpha uu_x = \mu u_{xx} + \mu u + \eta u^2 - \delta u^3 \tag{2.7}$$

- Kolmogorov-Petrovsky-Piskunov equation (Fisher equation)

$$u_t = \mu u_{xx} + \mu u + \eta u^2 \tag{2.8}$$

- Korteweg-de Vries-Burgers equation

$$u_t + \alpha uu_x + u_{xxx} = \mu u_{xx} \tag{2.9}$$

- Kuramoto-Sivashinsky equation

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0 \tag{2.10}$$

The Burgers' equation can take the form of the modified Westervelt equation

$$w_x - \frac{\beta}{c_0^2} w w_\tau = \frac{b}{2\rho_0 c_0^3} w w_{\tau\tau} \tag{2.11}$$

with $\tau = t - x/c_0$ the delayed time, c_0 the speed of sound propagation in the linear approximation, β the Burgers coefficient that quantifies nonlinear effects, ρ_0 the density of the medium, and b the shear viscosity coefficient. Equation (2.11) is used to describe the propagation of waves in soft tissue

The exact solutions of the nonlinear equations (2.7)-(2.11) are obtained by different methods such as the tanh-method [153], the pseudo-spectral method [41], the inverse scattering method [1], the Cnoidal method [110], the Bakland transformation [151], and variational methods [88]. There are also iterative numerical methods that converge rapidly.

Application to sonification

An interesting application of the Burgers equation is sonification. The sonification technique is used to explore images of tissues whose details are difficult to detect. By applying sonification, new images are obtained with better visualization of the explored tissue.

The approach was exercised on mock images of fibrotic rat liver samples inspired by a study of the effects of Ginkgo Biloba leaf extract against methotrexate-induced liver toxicity in rats in the papers [136] and [147].

The sonification procedure is as follows: a digital image B, seen as a collection of N pixels, is subjected to the force $f(t)$ expressed as the sum of the harmonic excitation force $F_p(t)$ and the generation sound force $F_s(t)$.

The last force is introduced to construct the sonification operator. The behavior of the digital image is described by the Burgers equation (2.11). The forces F_s are determined from the minimum of the acoustic power radiated by B

$$\frac{\partial W}{\partial F_s} = 0, \quad W = \frac{\mathscr{A}}{2} v^T p$$

where v is the speed of sound and p is the sound pressure vector, $\mathscr{A}$ is the area of the rectangular image, and the index T is the transposition.

The unknown parameters $P = \{m, k, \omega, \tilde{\varphi}, \alpha, \gamma, \lambda\}$ are found by a genetic algorithm that minimizes the objective function $\Upsilon(P_j)$ given by

$$\Upsilon(P_j) = 3^{-1} \sum_{j=1}^{3} \delta_{1j}^2 - \delta_2^2$$

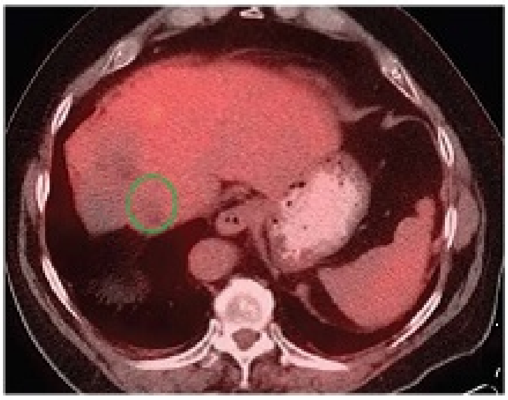

FIGURE 2.1. The MR image of a liver.

where δ_{1j} and δ_2 and are residuals which must tend to zero

$$\begin{aligned}
\delta_{1j} &= \frac{\partial v_j}{\partial x_j} - \frac{\beta_j}{c_0^2} v_j \frac{\partial v_j}{\partial \tau} - \frac{b_j}{2\rho_0 c_0^3} v_j \frac{\partial^2 v_j}{\partial \tau^2}, \\
\delta_2 &= \frac{\partial W}{\partial F_s}.
\end{aligned}$$

After sonification, the mapped data is completed and filled with colors and geometric lines, so that the final image may contain details that do not appear in the original image.

We now consider a fictitious image of an area of coagulation necrosis in the liver, and intentionally hide an area in this image (shown in green in Figure 2.1). Figure 2.2 is the image used for sonification. Sonification of this image was successful in the sense that the initially hidden area is found (Figure 2.3).

The propagation of sound waves in soft biological tissues (blood, veins, kidneys, liver, lungs, etc.) is described in [133] using the Burgers equation in the frequency range $1-10$ MHz. The solutions of the Burgers equation are determined by the Cnoidal method. The solutions are localized waves that retain their properties when interacting with other waves. The symmetry properties of the Burgers equation, i.e., the symmetry of spatial and temporal translations, the odd reflection symmetry, and the Galilean invariance are the main symmetries of the equation are analyzed. The utility of the Burgers equation for the sonification technique is then demonstrated on a medical image used for surgical operations. For a better application of the Burgers equation, the acoustic properties of the tissue at ultrasonic frequencies (acoustic velocity, ultrasonic attenuation, and factors affecting acoustic velocity) must be taken into account and ultrasonic attenuation - temperature, frequency, anisotropy) and the mechanical properties of the fabric

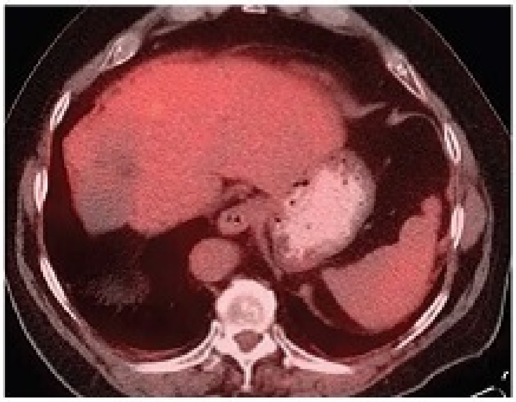

FIGURE 2.2. An image with a hidden area used for sonification.

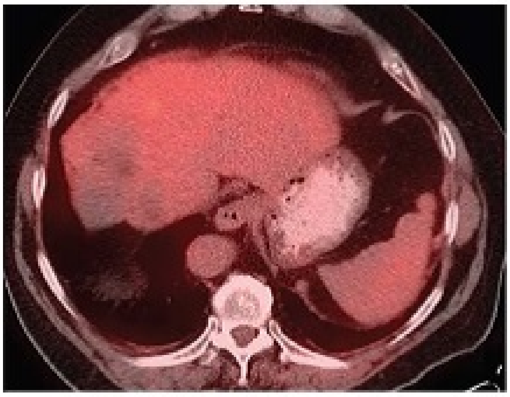

FIGURE 2.3. The hidden area is found by sonification.

(mass density, elastic moduli, viscoelastic moduli and factors affecting elasticity and viscoelasticity) must be well known.

Ultrasonic velocity and attenuation are functions of frequency, it is observed in the literature that the in vitro acoustic velocity is different between tumors and normal human liver. Compared to normal liver, ultrasound propagates about 1.5%(±1%) slower, is attenuated about 20%(±30%) less at 3 MHz in the tumors that were measured [18].

It is observed that the ultrasonic velocity decreases with increasing water and fat contents. An increase in water content is related to the decrease in attenuation, and there are positive dependencies between acoustic characteristics and fat content [19] , [49] and [158].

Knowledge of the acoustic properties of the tissue at ultrasound frequencies and the mechanical properties of the tissue is necessary in surgical procedures where the surgeon and the robot work with the same tool, with the aim of minimizing vascular damage and bleeding [125] and [148]. The inverse problem of extracting tissue acoustic properties at ultrasound frequencies and tissue mechanical properties from experimental data represents an interested work.

2.5.2 Coronavirus model

The model of Noor [115] treat the dynamics of a Coronavirus in humans with the case of the mathematical model in which the human population is divided into three components of subpopulations. They consider population function, which is presented by $N(t), t \geq 0$ as defined $N : [0,\infty) \to \mathbb{R}$.Also, the non-negative differential function of each component of the population as $S,\ I,\ R : [0,\infty) \to \mathbb{R}$. The representation of the components of the human population is described as $S(t)$ (denotes the susceptible humans who shall have the maximum probability to catch the fatal virus), $I(t)$ (denotes the infected humans who have convicted with the virus), and $R(t)$ (denotes the recovered humans who have got the vaccination or delay strategies and due to their internal level of immunity).

The model of the differential equations as follows

$$\begin{cases} S'(t) = a - cS(t)I(t)(1+\gamma I(t)) - \mu S(t) + \alpha R(t) \\ I'(t) = cS(t)I(t)(1+\gamma I(t)) - (\beta+\mu+\delta-b)I(t) \\ R'(t) = \beta I(t) - (\alpha+\mu)R(t) \end{cases}$$

where

c represented the convex incidence rate under the law of mass action.

δ denotes the rate at which infected humans die due to Coronavirus.

α represented the rate of humans after recovery may become susceptible.

μ the rate at which humans die from other diseases and natural incidences.

β the rate of humans may recover from the infection due to vaccination, hospitalization, social distancing, travel restrictions, quarantine, etc.

b the rate of infected human's immigrant to one location to another.

In this model, it is analyzed the computational dynamical analysis of the stochastic susceptible-infected-recovered pandemic model of the novel Coronavirus. It is adopted two ways for stochastic modelling like as transition probabilities and parametric perturbation techniques.

Different and well-known computational methods like Euler Maruyama, stochastic Euler, and stochastic Runge Kutta are applied to study the dynamics of the model.

Unfortunately, these computational methods do not restore the dynamical properties of the model like positivity, boundedness, consistency, and stability in the sense of biological reasoning, as desired. Then, for the given stochastic model, it is developed a *stochastic non-standard finite difference method*, which is shown to satisfy all of the model's dynamical properties. To that end, several simulations are presented to compare the proposed method's efficiency to that of existing stochastic methods.

2.5.3 Malaria model

Malaria is caused by a protozoa of the genus plasmodium and is transmitted by the female anopheles mosquito (vector). In contrast to Macdonald (1957), Chitnis et *al*. (2008) introduced the model where the number of bites of mosquitoes per human depends on the population sizes of both mosquitoes and humans. The authors of this work used a nonstandard finite difference scheme, is a general set of methods in numerical analysis that gives numerical solutions to differential equations by constructing a discrete model. The general rules for such schemes are not precisely known

For numerical simulations they use the nonstandard scheme due to the established consistency with the following model. Standard numerical methods fail to preserve the dynamics of continuous models. From Chitnis et *al*., see [16], the dynamics of malaria is

$$
\begin{cases}
S_h'(t) = \Lambda_h + \psi_h N_h(t) + \rho_h R_h(t) - c(N_h(t), N_v(t))\beta_{hv} I_v(t) S_h(t) \\
-f_h(N_h(t)) S_h(t) \\
E_h'(t) = c(N_h(t), N_v(t))\beta_{hv} I_v(t) S_h(t) - \upsilon_h E_h(t) - f_h(N_h(t)) E_h(t) \\
I_h'(t) = \upsilon_h E_h(t) - (\gamma_h + f_h(N_h(t)) + \delta_h) I_h(t) \\
R_h'(t) = \gamma_h I_h(t) - \rho_h R_h(t) - f_h(N_h(t)) E_h(t) \\
S_v'(t) = \psi_v N_v(t) - c(N_h(t), N_v(t))(\beta_{hv} I_h(t) + \tilde{\beta}_{vh} R_h(t)) S_v(t) \\
-f_v(N_v(t)) S_v(t) \\
E_v'(t) = c(N_h(t), N_v(t))(\beta_{hv} I_h(t) + \tilde{\beta}_{vh} R_h) S_v(t) - \nu_v E_v(t) \\
-f_v(N_v(t)) E_v(t) \\
I_v'(t) = \nu_v E_v(t) - f_v(N_v(t)) I_v(t)
\end{cases}
\tag{2.12}
$$

with

$$
\begin{aligned}
f_h &= \mu_{1h} + \mu_{2h} N_h(t) \\
f_v &= \mu_{1v} + \mu_{2v} N_v(t) \\
N_h(t) &= N_h(t) + E_h(t) + I_h(t) + R_h(t) \\
N_v(t) &= N_v(t) + E_v(t) + I_v(t) + R_v(t) \\
c(N_h(t), N_v(t)) &= \frac{\sigma_v \sigma_h}{\sigma_v N_v(t) + \sigma_h N_h(t)}
\end{aligned}
$$

S_h : Number of susceptible humans,

E_h : Number of exposed humans,

I_h : Number of infective humans,

R_h : Number of recovered (immune and asymptomatic, but slightly infectious) humans,

S_v : Number of susceptible mosquitoes,

E_v : Number of exposed mosquitoes,

I_v : Number of infective mosquitoes,

Λ_h : immigration rate,

ψ_h : relative birth rate,

μ_{1h} : density-independent force of mortality/out-migration rate,

μ_{2h} : density-dependent force of mortality/out-migration rate,

σ_h : bites tolerated by a human per unit time,

β_{hv} : probability of transmission of infection from infective mosquito,

$\frac{1}{\upsilon_h}$: average duration of the latent period,

γ_h : recovery rate,

ρ_h : loss of immunity rate,

δ_h : disease-induced death rate,

ψ_h : relative birth rate,

μ_{1h} : density-independent force of mortality,

μ_{2h} : density-dependent force of mortality,

σ_h : bites required by a mosquito per unit time,

β_{hv} : probability of transmission of infection from infective human,

$\widetilde{\beta}_{hv}$: probability of transmission of infection from recovered human,

$\frac{1}{\upsilon_h}$: average duration of the latent period.

A Nonstandard finite difference scheme

The authors design a nonstandard finite difference (NSFD) scheme, which is consistent with the dynamics of the continuous malaria model of equation (2.12). For the numerical approximation of the model of equation (2.12), they replace the continuous time variable $t \in [0,\infty)$ by discrete nodes $t_n = n\Delta t, n \in \mathbb{Z}$ where Δt is the step size. In order to find approximate solutions $S_h^n, E_h^n, I_h^n, R_h^n, S_v^n, E_v^n, I_v^n, N_h^n$ and N_ν^n of $S_h, E_h, I_h, R_h, S_v, E_v, I_v, N_v$ and N_h at the time $t = t_n$. The NSFD scheme reads as

$$\begin{cases} \dfrac{S_h^{n+1}-S_h^n}{\phi(\Delta t)} = \Lambda_h + \psi_h N_h^n + \rho_h R_h^{n+1} - c(N_h^n, N_v^n)\beta_{hv} I_v^n S_h^{n+1} \\ -f_h(N_h^n) S_h^{n+1} \\ \dfrac{E_h^{n+1}-E_h^n}{\phi(\Delta t)} = c(N_h^n, N_v^n)\beta_{hv} I_v^n S_h^{n+1} - \upsilon_h E_h^{n+1} - f_h(N_h^n) E_h^{n+1} \\ \dfrac{I_h^{n+1}-I_h^n}{\phi(\Delta t)} = \upsilon_h E_h^{n+1} - (\gamma_h + f_h(N_h^n) + \delta_h) I_h^{n+1} \\ \dfrac{R_h^{n+1}-R_h^n}{\phi(\Delta t)} = \gamma_h I_h^{n+1} - \rho_h R_h^{n+1} - f_h(N_h^n) R_h^{n+1} \\ \frac{S_h^{n+1}-S_h^n}{\phi(\Delta t)} = \psi_\nu N_\nu^n - c(N_h^n, N_\nu^n)(\beta_{\upsilon h} I_h^n + \tilde{\beta}_{\upsilon h} R_h^n) S_\nu^{n+1} \\ -f_\nu(N_\nu^n) S_\nu^{n+1} \\ \frac{E_\nu^{n+1}-E_\nu^n}{\phi(\Delta t)} = c(N_h^n, N_\upsilon^n)(\beta_{h\upsilon} I_\upsilon^n + \tilde{\beta}_{\upsilon h} R_h^n) S_\nu^{n+1} - \nu_\nu E_\nu^{n+1} \\ -f_\nu(N_\nu^n) E_\nu^{n+1} \\ \frac{I_h^{n+1}-I_h^n}{\phi(\Delta t)} = \upsilon_\upsilon E_\upsilon^{n+1} - f_\nu(N_\nu^n) I_\nu^{n+1} \end{cases}$$

The numerical scheme of equation (2.12) is called a nonstandard finite difference method (Mickens, 1994; Anguelov and Lubuma, 2001, 2003) because nonlinear terms are approximated in a non local way by using more than one mesh point: for instance $c(N_h, N_\upsilon)\beta_{h\upsilon} I_\upsilon S_h$ is approximated by $c(N_h^n, N_\upsilon^n)\beta_{h\upsilon} I_\upsilon^n S_h^{n+1}$ instead of $c(N_h^n, N_\upsilon^n)\beta_{h\upsilon} I_\upsilon^n S_h^n$ and because the standard denominator Δt of the discrete derivatives is replaced by a more complex function positive $\phi(\Delta t)$ which satisfies the condition:

$$\phi \equiv \phi(\Delta t) = \Delta t + O((\Delta t)^2).$$

2.5.4 Traffic flow model

The model of Landajuela [82] consider the flow of cars on a highway, let $\rho(x,t)$ designates the density of cars and $f(x,t)$ the traffic flow. Landajuela also consider ρ^* as the restriction of ρ to some interval, $0 \le \rho^* \le \rho_{\max}$, where $\rho_{\max}$ is the value at which the cars are bumper to bumper.

Since the cars are conserved, the car density and the flow must be related by the continuity equation

$$\frac{\partial \rho^*}{\partial t^*} + \frac{\partial f}{\partial x^*} = 0 \tag{2.13}$$

Obviously, the first expression in which one thinks of for the flow is $f = v\rho^*$ where v is the velocity.

However, it turns out that to reflect the fact that drivers will reduce their speed to account for a forward increasing density, we should assume that f is a function of the density gradient as well. A simple assumption is to take

$$f(\rho^*) = \rho^* v(\rho^*) - D\frac{\partial \rho^*}{\partial x^*} \tag{2.14}$$

where D is constant.

We also assume that the speed v is a given function of ρ^*: On a highway, we would like in optimally way to drive at a certain speed $v_{\max}$ (the speed limit perhaps) but with a heavy traffic, we slow down and the speed decreases when the density increases.

The simplest relationship that takes this into account is the following

$$v(\rho^*) = \frac{v_{\max}}{\rho_{\max}}(\rho_{\max} - \rho^*) \tag{2.15}$$

Putting (2.14) and (2.15) into (2.13) leads to

$$\frac{\partial \rho^*}{\partial t^*} + \frac{d}{dx^*}\left[(\frac{v_{\max}}{\rho_{\max}}(\rho_{\max} - \rho^*)\rho^* \right] = D\frac{\partial {\rho^*}^2}{\partial x^{*2}}$$

Scaling through $v_{\max} = \frac{x_0}{t_0}, \rho = \rho_{\max}\rho^*, x = x_0 x^*$ and $t = t_0 t^*$ result in

$$\rho_t + \left[(1-\rho)\rho\right]_x = \varepsilon \rho_{xx}$$

with $\varepsilon = \dfrac{D}{v_{\max} x_0}$ and $0 \le \rho \le 1$.

The transformation $u = 2\rho - 1$ leads to the viscous Burgers equation (2.1) with the conditions $-1 \le u \le 1$.

2.5.5 inviscid and viscous Burgers equations model

In [26], Bendaas studied the inviscid and viscous Burgers equations with initial conditions in the half-plane $x \in \mathbb{R}; t > 0$. She first considers the Burgers equations with initial conditions admitting two and three shocks and uses the Hopf-Cole transformation to linearize the problems and solve them explicitly.

Then, she studied the Burgers equation and solved the problem with an initial value. Then she studied the asymptotic behavior of the solutions and showed that the exact solution of the boundary value problem of the viscous Burgers equation, when the viscosity parameter is sufficiently small, approaches the shock-type solution of the boundary value problem of the inviscid Burgers equation.

A new method has been developed to find the exact solutions. The results are formulated in classical mathematics and proved by non-standard infinitesimal analysis techniques. The model deals with the initial boundary value problem for inviscid Burgers' equation. She consider the inviscid Burgers equation

$$u_t + uu_x = 0 \tag{2.16}$$

in $x \in \mathbb{R}; t > 0$ with the initial condition

$$u(\xi, 0) = f(\xi) \tag{2.17}$$

where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

This problem not admit the regular solutions but some weak solutions with certain regularity exist. The Burgers equation is known to possess traveling waves solutions. The solution of (2.16) and (2.17) may be given in a parametric form and shocks must be fitted in such that

$$U = \frac{1}{2}(u_1 + u_2) = \frac{1}{2}(f(\xi_1) + f(\xi_2))$$

where ξ_1 and ξ_2 are the value of on the two sides of the shock.

2.5.6 Viscous Burgers' model

In [45], Diaz and Gonzales gave some applications to the linear heat equation with boundary conditions of Robin type are given. So they consider the viscous Burgers' problem

$$\begin{cases} u_t - u_{xx} + uu_x = 0 & , x \in (0, \infty), t > 0 \\ u(0, t) = 0 & , \liminf_{x \to \infty} u(t, x) \geq 0, t > 0 \\ u(x, 0) = u_0(x) & , \text{on } (0, +\infty) \end{cases}$$

They consider the Burgers equation and prove a property that seems to have been observed until now: there is no limitation on the growth of the initial nonnegative data $u_0(x)$ to infinity when the problem is formulated on unbounded intervals, such as $(0 + \infty)$, and the solution is unique without prescribing its behavior at infinity.

2.5.7 Navier-Stokes model

Consider the Navier-Stokes equations

$$\nabla . v = 0 \tag{2.18}$$

$$(\rho v)_t + \nabla.(\rho v v) + \nabla p - \nabla^2 v = 0 \tag{2.19a}$$

It is well known that when we consider ρ the density, p the pressure, v the velocity and μ the viscosity of a fluid, equations (2.18) and (2.19a) describe the dynamics of a fluid without divergence (2.18) incompressible ($\rho_t = 0$) flow where gravitational effects are negligible.

The simplification in (2.19a) of the x component of the velocity vector, which we will call v^x, gives

$$\rho \frac{\partial v^x}{\partial t} + \rho v^x \frac{\partial v^x}{\partial x} + \rho v^y \frac{\partial v^x}{\partial y} + \rho v^z \frac{\partial v^x}{\partial z} + \frac{\partial p}{\partial x} \\ - \mu \left(\frac{\partial^2 v^x}{\partial x^2} + \frac{\partial^2 v^x}{\partial y^2} + \frac{\partial^2 v^x}{\partial z^2} \right) = 0$$

If we consider a 1D problem with no pressure gradient, the above equation reduces to

$$\rho \frac{\partial v^x}{\partial t} + \rho v^x \frac{\partial v^x}{\partial x} - \mu \frac{\partial^2 v^x}{\partial x^2} = 0 \tag{2.20}$$

If we now use the traditional variable u instead of v^x and take ε the kinematic viscosity, i.e., $\varepsilon = \dfrac{\mu}{\rho}$ then the last equation simply becomes the viscous Burgers equation as presented in (2.18, 2.19a).

When the viscosity μ of the fluid is almost zero, one could think, as an idealization, to simply delete the term of the second derivative in (2.20). This would lead to

$$\rho \frac{\partial v^x}{\partial t} + \rho v^x \frac{\partial v^x}{\partial x} = 0 \tag{2.21}$$

which, after making $u = v^x$ and dividing by ρ, becomes the inviscid Burgers equation (2.1). It turns out that in order to use (2.21) as a model of the dynamics of an inviscid fluid, it is necessary to use the Burgers equation inviscous fluids, it must be completed by other physical conditions which will prevent the equation (2.21) from developing meaningless physical solutions.

This extension is interesting because working with (2.21) is much easier than working with (2.20), see [82].

CHAPTER 3

EXISTENCE RESULT OF GLOBAL SOLUTIONS FOR A CLASS OF GENERIC REACTION-DIFFUSION SYSTEMS

In this chapter we prove the existence of weak global solutions for a class of generic reaction diffusion systems for which two main properties hold: the quasi-positivity and a triangular structure condition on the nonlinearities. The work constituting this chapter is the subject of an article published in (International Journal of Nonlinear Analysis and Applications), in collaboration with S. Mesbahi and S. Bendaas [68].

3.1 Introduction

Reaction diffusion systems are widely used in biology, ecology, physics and chemistry. What we observe in modern scientific studies is the great interest of scientists in studying this type of system, which confirms once again its importance in developing sciences in all fields. Many models and real examples in various scientific fields, as well as course notes can be found in Lions [86], Murray [112], [113], Pazy [121], Pierre [122] and the references therein.

This paper reviews one of the major applications of reaction diffusion systems, namely the smoothing and restoration of images. The purpose of image restoration is to estimate the original image from the degraded data. Applications range from medical imaging, astronomical imaging, to forensic science, etc. In recent years, this field has attracted the attention of many researchers in computer vision. This is mainly due to the mathematical formulation framing any PDEs-based approach that can give a good justification and explanation of the results obtained through these traditional and heuristic methods in image processing.

There are many important studies and models that have been studied in recent decades on image processing and its applications. The reader can see some of them and some similar models of the problem that we will study in this paper in Alaa et *al*. [3]-[7] and [97], Alvarez et *al*. [12] and [13], Catté et *al*. [37], Weickert et *al*. [58], [154]-[156], Morfu [105] and the references therein. He will also find some of the methods and techniques used to study these questions.

In this paper, we propose a new model of nonlinear generic reaction diffusion system applied to edge detection and image restoration. We tackle the problem of the global existence of solutions for the following system

$$\begin{cases} \dfrac{\partial u}{\partial t} - \textbf{div}\ (A(|\nabla u_\sigma|)\nabla u) = f(t,x,u,v,w) & \text{, in } Q_T \\ \dfrac{\partial v}{\partial t} - \textbf{div}\ (B(|\nabla v_\sigma|)\nabla v) = g(t,x,u,v,w) & \text{, in } Q_T \\ \dfrac{\partial w}{\partial t} - d_w \Delta w = h(t,x,u,v,w) & \text{, in } Q_T \end{cases} \tag{3.1}$$

with the boundary conditions

$$\frac{\partial u}{\partial v} = \frac{\partial v}{\partial v} = \frac{\partial w}{\partial v} = 0 \ \ , \ \ \text{on } \Sigma_T \tag{3.2}$$

and the initial conditions

$$u(0,x) = u_0(x) \ \ , \ \ v(0,x) = v_0(x) \ \ , \ \ w(0,x) = w_0(x) \quad \text{in } \Omega \tag{3.3}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ and $T \in (0,\infty[$, $Q_T =]0,T[\times \Omega$ and $\Sigma_T =]0,T[\times \partial\Omega$ where $\partial\Omega$ denotes the boundary of Ω. v is the outward normal to the domain and $\frac{\partial}{\partial v}$ is the normal derivative.

Let $\sigma > 0$, ∇u_σ, ∇v_σ are the regularizations by convolution of ∇u and ∇v respectively. We define

$$\nabla u_\sigma = \nabla(G_\sigma * u) \quad \text{and} \quad \nabla v_\sigma = \nabla(G_\sigma * v)$$

where G_σ is the Gaussian function. The anisotropic diffusivities A and B are smooth nonincreasing functions, such that

$$A(0) = B(0) = 1 \quad \text{and} \quad \lim_{s\to\infty} A(s) = \lim_{s\to\infty} B(s) = 0$$

Note that the function $s \mapsto \dfrac{1}{1+s^2}$ satisfies these conditions.

We have found a good idea to present our work as follows : In the next section, we present our main result. In the third section, we provide some preliminary results on our problem in the case where the nonlinearities are bound which we need later. In the last section, we truncate the problem and show that the approximate problem admits weak solutions using the Schauder fixed point theorem. Afterward, we will provide some essential compactness and equi-integrability results in order to pass to the limit and rigorously demonstrate the existence of global weak solution to the considered model.

3.2 Statement of the main result

3.2.1 Assumptions

Throughout this note we will assume that : The nonlinear functions $f,\ g,\ h : Q_T \times \mathbb{R}^2 \to \mathbb{R}$ are measurable and $f(t,x,.),\ g(t,x,.),\ h(t,x,.) : \mathbb{R}^3 \to \mathbb{R}$ are continuous. In addition the nonlinearities satisfy the quasi-positivity property

$$f(t,x,0,s,q) \geq 0 \quad , \quad g(t,x,r,0,q) \geq 0 \quad , \quad h(t,x,r,s,0) \geq 0 \quad , \quad \forall r,s,q \geq 0 \tag{3.4}$$

and the triangular structure condition

$$\begin{cases} (f+g+h)(t,x,r,s,q) \leq L_1(1+r+s+q) \\ (g+h)(t,x,r,s,q) \leq L_2(1+r+s+q) \\ h(t,x,r,s,q) \leq L_3(1+r+s+q) \end{cases} \tag{3.5}$$

where L_1, L_2 and L_3 are positive constants. Furthermore,

$$\sup_{|r|+|s|+|q| \leq R} (|f(t,x,r,s,q)| + |g(t,x,r,s,q)| + |h(t,x,r,s,q)|) \in L^1(Q_T) \tag{3.6}$$

for $R > 0$. The initial conditions u_0, v_0, w_0 are only assumed to be square integrable.

In Pierre [122], we find some examples of reaction diffusion systems as models for very different applications and for which the two properties (3.4) and (3.5) hold. We also refer to Murray's books [112] and [113], in which we find many important models in multiple fields. An interesting example where the result of this paper can be applied is the Modified Fitz-Hugh-Nagumo Model for image restoration. To learn more about this model, we refer to Alaa and Zirhem [4].

We introduce the notion of solution to the problem (3.1) − (3.3), (3.4) − (3.6) used here :

Definition 3.1. We say that (u,v,w) is a weak solution of the system (3.1) – (3.3) under the assumptions (3.4) – (3.6), if

(i) $u,\ v,\ w \in L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$,

(ii) $\forall \phi,\ \psi,\ \eta \in C^1(Q_T)$, such that $\phi(\cdot,T)=0$, $\psi(\cdot,T)=0$ and $\eta(\cdot,T)=0$, we have

$$\begin{aligned}
\int_{Q_T} -u\frac{\partial \phi}{\partial t} + A(|\nabla u_\sigma|)\nabla u \nabla \phi &= \int_{Q_T} f(t,x,u,v,w)\phi + \int_\Omega u_0\phi(0,\cdot)\\
\int_{Q_T} -v\frac{\partial \psi}{\partial t} + B(|\nabla v_\sigma|)\nabla v \nabla \psi &= \int_{Q_T} g(t,x,u,v,w)\psi + \int_\Omega v_0\psi(0,\cdot)\\
\int_{Q_T} -w\frac{\partial \eta}{\partial t} + d_w \nabla w \nabla \eta &= \int_{Q_T} h(t,x,u,v,w)\eta + \int_\Omega w_0\eta(0,\cdot)
\end{aligned}$$

where $f,\ g,\ h \in L^1(Q_T)$.

3.2.2 The main result

Now, we can state the main result of this work :

Theorem 3.1. *Under the assumptions* (3.4) – (3.6) *and for a continuous functions f, g and h as described above. The reaction diffusion system* (3.1) – (3.3) *admits a global weak solution (u,v,w) in the sense defined in Definition 3.1 for all $u_0,v_0,w_0 \in L^2(\Omega)$ such that u_0,v_0,w_0 are positive.*

3.3 Preliminary results for bounded nonlinearities

Before treating the nonlinear case, we will prove an existence result for bounded nonlinearities. In what follows, we denote $\mathcal{V} = H^1(\Omega)$ and $\mathcal{H} = L^2(\Omega)$.

Theorem 3.2. *Under the above assumptions on the nonlinearities, if there exist M_1, M_2, $M_3 \geq 0$, such that for almost every $(t,x) \in Q_T$,*

$$|f(t,x,r,s,q)| \leq M_1 \quad , \quad |g(t,x,r,s,q)| \leq M_2 \quad , \quad |h(t,x,r,s,q)| \leq M_3 \quad , \quad \forall r,s,q \in \mathbb{R}$$

then for every $u_0,\ v_0,\ w_0 \in L^2(\Omega)$, there exists a weak solution (u,v,w) to the considered system (3.1) – (3.3)*. Moreover there exists a constant C depends on M_1, M_2, M_3, σ, T,*

$\|u_0\|_{L^2(\Omega)}$, $\|v_0\|_{L^2(\Omega)}$ *and* $\|w_0\|_{L^2(\Omega)}$, *such that*

$$\|(u,v,w)\|_{L^\infty(0,T;\mathscr{H})^2} + \|(u,v,w)\|_{L^2(0,T;\mathscr{V})^2} \le C$$

Furthermore, if u_0, v_0, w_0 *are positive and* f, g, h *are quasi-positive then* $u(t,x) \ge 0$, $v(t,x) \ge 0$ *and* $w(t,x) \ge 0$ *for a.e.* $(t,x) \in Q_T$.

Proof. We introduce the space

$$\mathscr{W}(0,T) = \left\{ u,v,w \in L^2(0,T;\mathscr{V}) \cap L^\infty(0,T;\mathscr{H}) \mid \frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t} \in L^2(0,T;\mathscr{V}') \right\}$$

Let $z = (z_1, z_2, z_3) \in \mathscr{W}(0,T)$ and let (u,v,w) be the solution of a linearization of problem (3.1) − (3.3), (3.4) − (3.6) given by

$$(3.7) \qquad \begin{cases} (u,v,w) \in L^2(0,T;\mathscr{V}) \cap C(0,T;\mathscr{H}) \\ \forall \phi, \psi, \eta \in C^1(Q_T) \text{ such that } \phi(\cdot,T) = 0, \psi(\cdot,T) = 0, \eta(\cdot,T) = 0 \\ \int_{Q_T} -u\dfrac{\partial \phi}{\partial t} + A(|\nabla (z_1)_\sigma|)\nabla u \nabla \phi = \int_{Q_T} f(t,x,z_1,z_2,z_3)\phi + \int_\Omega u_0 \phi(0,\cdot) \\ \int_{Q_T} -v\dfrac{\partial \psi}{\partial t} + B(|\nabla (z_2)_\sigma|)\nabla v \nabla \psi = \int_{Q_T} g(t,x,z_1,z_2,z_3)\psi + \int_\Omega v_0 \psi(0,\cdot) \\ \int_{Q_T} -w\dfrac{\partial \eta}{\partial t} + d_w \nabla w \nabla \eta = \int_{Q_T} h(t,x,z_1,z_2,z_3)\eta + \int_\Omega w_0 \eta(0,\cdot) \end{cases}$$

The application $z \in \mathscr{W}(0,T) \mapsto (u,v,w) \in \mathscr{W}(0,T)$ is clearly well defined. In fact, z_1, z_2, z_3 are in $L^\infty(0,T;\mathscr{H})$, G_σ is $C^\infty(Q_T)$ therefore $A(|\nabla (z_1)_\sigma|)$ and $B(|\nabla (z_2)_\sigma|)$ are in $C^\infty(Q_T)$ and since A and B are nonincreasing, it results

$$(3.8) \qquad a \le A(|\nabla z_\sigma|) \le b \quad \text{and} \quad c \le B(|\nabla z_\sigma|) \le d$$

where $b, d > 0$ and a, c are a positive constants which only depend on A and B respectively. The property (3.8) with the fact that nonlinearities are bounded implies that the differential operators in (3.7) are continuous and coercive thus by application of the standard theory of Partial Differential Equations, we obtain (u,v,w) the solution of the linearized problem (3.7). To learn more about this existence, we refer to Amann [14], Benilan and Brezis [29] and Lions [85].

Now, we establish some important estimates to reformulate the problem in the form of a fixed point problem. The following result holds for $t \in [0,T]$.

$$
\begin{cases}
\frac{1}{2}\int_\Omega u^2(t) + \int_{Q_T} A(|\nabla (z_1)_\sigma|)|\nabla u|^2 = \frac{1}{2}\int_\Omega u_0^2 + \int_{Q_T} u\ f(t,x,z_1,z_2,z_3) \\
\frac{1}{2}\int_\Omega v^2(t) + \int_{Q_T} B(|\nabla (z_2)_\sigma|)|\nabla v|^2 = \frac{1}{2}\int_\Omega v_0^2 + \int_{Q_T} v\ g(t,x,z_1,z_2,z_3) \\
\frac{1}{2}\int_\Omega w^2(t) + d_w\int_{Q_T} |\nabla w|^2 = \frac{1}{2}\int_\Omega w_0^2 + \int_{Q_T} w\ h(t,x,z_1,z_2,z_3)
\end{cases} \tag{3.9}
$$

Consequently,

$$
\begin{cases}
\int_\Omega u^2(t) \le M_1 + \int_{Q_T} u^2 + \int_\Omega u_0^2 \\
\int_\Omega v^2(t) \le M_2 + \int_{Q_T} v^2 + \int_\Omega v_0^2 \\
\int_\Omega w^2(t) \le M_3 + \int_{Q_T} w^2 + \int_\Omega w_0^2
\end{cases} \tag{3.10}
$$

Using Gronwall's inequality, we get

$$
\begin{cases}
\int_{Q_T} u^2 \le \left(e^T - 1\right)\left(M_1 + \int_\Omega u_0^2\right) \\
\int_{Q_T} v^2 \le \left(e^T - 1\right)\left(M_2 + \int_\Omega v_0^2\right) \\
\int_{Q_T} w^2 \le \left(e^T - 1\right)\left(M_3 + \int_\Omega w_0^2\right)
\end{cases}
$$

By substituting the above expression in (3.9), we obtain

$$
\begin{cases}
\sup\limits_{0\le t\le T} \int_\Omega u^2(t) \le M_1 + \left(e^T - 1\right)\left(M_1 + \int_\Omega u_0^2\right) + \int_\Omega u_0^2 := C_u \\
\sup\limits_{0\le t\le T} \int_\Omega v^2(t) \le M_2 + \left(e^T - 1\right)\left(M_2 + \int_\Omega v_0^2\right) + \int_\Omega v_0^2 := C_v \\
\sup\limits_{0\le t\le T} \int_\Omega w^2(t) \le M_3 + \left(e^T - 1\right)\left(M_3 + \int_\Omega w_0^2\right) + \int_\Omega w_0^2 := C_w
\end{cases}
$$

Therefore by setting $C_1 = \max\{C_u, C_v, C_w\}$, we obtain

$$
\|(u,v,w)\|_{L^\infty(0,T;\mathcal{H})^3} \le C_1
$$

Using (3.9) and (3.10), we deduce

$$
\begin{cases}
\int_{Q_T} u^2 + |\nabla u|^2 \le \dfrac{M_1 + \int_{Q_T} u^2 + \int_\Omega u_0^2}{\min\{\frac{1}{2}, a\}} \le C'_u \\
\int_{Q_T} v^2 + |\nabla v|^2 \le \dfrac{M_2 + \int_{Q_T} v^2 + \int_\Omega v_0^2}{\min\{\frac{1}{2}, b\}} \le C'_v \\
\int_{Q_T} w^2 + |\nabla w|^2 \le \dfrac{M_3 + \int_{Q_T} w^2 + \int_\Omega w_0^2}{\min\{\frac{1}{2}, d_w\}} \le C'_w
\end{cases}
$$

Setting $C_2 = \max\{C'_u, C'_v, C'_w\}$, we conclude that

$$\|(u,v,w)\|_{L^2(0,T;\mathcal{V})^3} \leq C_2$$

Next we estimate $\frac{\partial u}{\partial t}$, $\frac{\partial v}{\partial t}$ and $\frac{\partial w}{\partial t}$ in $L^2(0,T;\mathcal{V}')$. We know that

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathbf{div}\ (A(|\nabla u_\sigma|)\nabla u) + f(t,x,u,v,w) \\ \dfrac{\partial v}{\partial t} = \mathbf{div}\ (B(|\nabla v_\sigma|)\nabla v) + g(t,x,u,v,w) \\ \dfrac{\partial w}{\partial t} = d_w \Delta w + h(t,x,u,v,w) \end{cases}$$

It follows that

$$\begin{cases} \left\| \dfrac{\partial u}{\partial t} \right\|_{L^2(0,T;\mathcal{V}')} \leq C\, \|\nabla u\|_{L^2(Q_T)} + M_1 T \\ \left\| \dfrac{\partial v}{\partial t} \right\|_{L^2(0,T;\mathcal{V}')} \leq C'\, \|\nabla v\|_{L^2(Q_T)} + M_2 T \\ \left\| \dfrac{\partial w}{\partial t} \right\|_{L^2(0,T;\mathcal{V}')} \leq d_w\, \|\nabla w\|_{L^2(Q_T)} + M_3 T \end{cases}$$

and

$$\begin{cases} \left\| \dfrac{\partial u}{\partial t} \right\|_{L^2(0,T;\mathcal{V}')} \leq C C_1 + M_1 T \\ \left\| \dfrac{\partial v}{\partial t} \right\|_{L^2(0,T;\mathcal{V}')} \leq C' C_1 + M_2 T \\ \left\| \dfrac{\partial w}{\partial t} \right\|_{L^2(0,T;\mathcal{V}')} \leq d_w C_1 + M_3 T \end{cases}$$

Eventually,

$$\left\| \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t} \right) \right\|_{L^2(0,T;\mathcal{V}')} \leq \max\{CC_1 + M_1 T, C'C_1 + M_2 T, d_w C_1 + M_3 T\} := C_3$$

Now, we can apply the Schauder fixed point theorem in the functional space

$$\begin{aligned} \mathcal{W}_0(0,T) = \quad & \{u,v,w \in L^2(0,T;\mathcal{V}) \cap L^\infty(0,T;\mathcal{H}) : \|(u,v,w)\|_{L^\infty(0,T;\mathcal{H})^3} \leq C_1; \\ & \|(u,v,w)\|_{L^2(0,T;\mathcal{V})^3} \leq C_2; \quad \left\| \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t} \right) \right\|_{L^2(0,T;\mathcal{V}')} \leq C_3, \\ & u(\cdot,0) = u_0,\ \ v(\cdot,0) = v_0,\ \ w(\cdot,0) = w_0\} \end{aligned}$$

We can easily verify that $\mathcal{W}_0(0,T)$ is a nonempty closed convex in $\mathcal{W}(0,T)$. To use Schauder's theorem, we will show that the application

$$F : z \in \mathcal{W}_0(0,T) \longrightarrow F(z) = (u,v,w) \in \mathcal{W}_0(0,T)$$

is weakly continuous.

Let us consider a sequence $z_n \in \mathscr{W}_0(0,T)$ such that z_n converges weakly in $W_0(0,T)$ toward z, and let $F(z_n) = (u_n, v_n, w_n)$. Thus,

$$
\left\{
\begin{aligned}
\frac{\partial u_n}{\partial t} &= \mathbf{div}\ \left(A\left(\left|\nabla z_{1n_\sigma}\right|\right)\nabla u_n\right) + f(t,x,u_n,v_n,w_n)\\
\frac{\partial v_n}{\partial t} &= \mathbf{div}\ \left(B\left(\left|\nabla z_{2n_\sigma}\right|\right)\nabla v_n\right) + g(t,x,u_n,v_n,w_n)\\
\frac{\partial w_n}{\partial t} &= d_w \Delta w_n + h(t,x,u_n,v_n,w_n)
\end{aligned}
\right.
\tag{3.11}
$$

Based on the previous estimation, (u_n, v_n, w_n) is bounded in $L^2(0,T;\mathscr{V})^3$ and $\left(\frac{\partial u_n}{\partial t}, \frac{\partial v_n}{\partial t}, \frac{\partial w_n}{\partial t}\right)$ is bounded in $L^2(0,T;\mathscr{V}')^3$. Then, by using Aubin-Simon compactness in Simon [144], we have that (u_n, v_n, w_n) is relatively compact on $\left(L^2(Q_T)\right)^3$; which allows us to extract a subsequence denoted $z_n = (u_n, v_n, w_n)$, such that:

- $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ and $w_n \rightharpoonup w$ in $L^2(0,T;\mathscr{V})$,
- $f(t,x,z_n) \to f(t,x,z)$, $g(t,x,z_n) \to g(t,x,z)$ and $h(t,x,z_n) \to h(t,x,z)$ in $\left(L^2(Q_T)\right)$,
- $u_n \to u$, $v_n \to v$ and $w_n \to w$ in $L^2(0,T;\mathscr{H})$ and a.e. in Q_T,
- $\nabla u_n \rightharpoonup \nabla u$, $\nabla v_n \rightharpoonup \nabla v$ and $\nabla w_n \rightharpoonup \nabla w$ in $L^2(0,T;\mathscr{H})$,
- $z_n \longrightarrow z$ in $L^2(0,T;\mathscr{H})$ and a.e. in Q_T,
- $A(|\nabla z_{1n\sigma}|) \to A(|\nabla z_{1\sigma}|)$ and $B(|\nabla z_{2n\sigma}|) \to B(|\nabla z_{2\sigma}|)$ in $L^2(0,T;\mathscr{V})$,
- $\frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u}{\partial t}$, $\frac{\partial v_n}{\partial t} \rightharpoonup \frac{\partial v}{\partial t}$ and $\frac{\partial w_n}{\partial t} \rightharpoonup \frac{\partial w}{\partial t}$ in $L^2(0,T;\mathscr{V}')$.

Using this convergences, we can pass to the limit in (3.11) and show that the limit u, v and w are solutions of the problem

$$
\left\{
\begin{aligned}
\frac{\partial u}{\partial t} &= \mathbf{div}\ \left(A(|\nabla z_{1\sigma}|)\nabla u\right) + f(t,x,z_n)\\
\frac{\partial v}{\partial t} &= \mathbf{div}\ \left(B(|\nabla z_{2\sigma}|)\nabla v\right) + g(t,x,z_n)\\
\frac{\partial w}{\partial t} &= d_w \Delta w + h(t,x,z_n)
\end{aligned}
\right.
$$

Thus $F(z) = (u, v, w)$, then F is weakly continuous which proves the desired results.

Note that the condition of quasi-positivity (3.4) leads to the positivity of u, v and w. For more details, we refer to Alaa et *al*. [7] and [97]. ■

3.4 Existence result for unbounded nonlinearities

3.4.1 Approximating scheme

First, we truncate f, g and h using truncation function $\Psi_n \in C_c^\infty(\mathbb{R})$, such that $0 \le \Psi_n \le 1$ and

$$\Psi_n(r) = \begin{cases} 0 & \text{if } |r| \le n \\ 1 & \text{if } |r| \ge n+1 \end{cases}$$

We can say that the approximate problem

$$\begin{cases} \dfrac{\partial u_n}{\partial t} = \mathbf{div}\ (A(|\nabla u_{n\sigma}|)\nabla u_n) + f_n(t,x,u_n,v_n,w_n) \\ \dfrac{\partial v_n}{\partial t} = \mathbf{div}\ (B(|\nabla v_{n\sigma}|)\nabla v_n) + g_n(t,x,u_n,v_n,w_n) \\ \dfrac{\partial w_n}{\partial t} = d_w \Delta w_n + h_n(t,x,u_n,v_n,w_n) \end{cases} \tag{3.12}$$

where

$$\begin{aligned} f_n(t,x,u_n,v_n,w_n) &= \Psi_n(|u_n|+|v_n|+|w_n|)\cdot f(t,x,u_n,v_n,w_n) \\ g_n(t,x,u_n,v_n,w_n) &= \Psi_n(|u_n|+|v_n|+|w_n|)\cdot g(t,x,u_n,v_n,w_n) \\ h_n(t,x,u_n,v_n,w_n) &= \Psi_n(|u_n|+|v_n|+|w_n|)\cdot h(t,x,u_n,v_n,w_n) \end{aligned}$$

admits a weak solution by means of Theorem 3.2.

3.4.2 A priori estimates

In what follows, In what follows, C denotes a constant independent of n. Now we show that up to a subsequences (u_n, v_n, w_n) converges to the weak solution (u,v,w) of problem (3.1) − (3.3), (3.4) − (3.6). For this, we will prove some important results.

Lemma 3.1. *Under the assumptions of the main result and for (u_n, v_n, w_n) a weak solution of the truncated problem, there exists $C > 0$, such that*

$$\|u_n + v_n + w_n\|_{L^2(Q_T)} \le C\left[1 + \|v_n\|_{L^2(Q_T)} + \|w_n\|_{L^2(Q_T)}\right]$$

Proof. This estimate is based on the duality method, for more details, see Pierre [122]. Let $\theta \in \mathscr{C}_c^\infty(Q_T)$ be such that $\theta \geq 0$ and let ϕ be a solution of

$$
\begin{cases}
\dfrac{-\partial \phi}{\partial t} - \mathbf{div}\,\left(A(|\nabla u_{n\sigma}|)u_n \nabla \phi\right) = \theta \\
\dfrac{\partial \phi}{\partial n} = 0 \\
\phi(T,0)
\end{cases}
\tag{3.13}
$$

We know that there exists $C > 0$ such that $\|\phi\|_{H^2(Q_T)} \leq C\|\theta\|_{L^2(Q_T)}$. Further details can be found in Ladyzhenskaya et *al.* [80] and Schmitt [141]. We set $W = \exp(-L_1 t)(u_n + v_n + w_n)$, by the mass control the following inequality holds,

$$
\begin{aligned}
&\int_{Q_T} \partial_t W \phi + \int_{Q_T} \exp(-L_1 t)\left[\mathbf{div}\,(A(|\nabla u_{n\sigma}|)u_n) + \mathbf{div}\,(B(|\nabla v_{n\sigma}|)v_n) + d_w \Delta w_n\right]\phi \\
\leq\ & \int_{Q_T} L_1 \exp(-L_1 t)\phi
\end{aligned}
$$

Integrating by parts and using (3.13), we get

$$
\begin{aligned}
\int_{Q_T} W\theta \leq \int_{Q_T} & \exp(-L_1 t)[d_w \Delta\phi - A(|\nabla u_{n\sigma}|)\Delta\phi - \nabla A(|\nabla u_{n\sigma}|)\nabla\phi - \\
& B(|\nabla v_{n\sigma}|)\Delta\phi - \nabla B(|\nabla v_{n\sigma}|)\nabla\phi]w_n \\
& + \int_{Q_T} L_1 \exp(-L_1 t)\phi + \int_\Omega (u_0 + v_0 + w_0)\phi(0,\cdot)
\end{aligned}
$$

where $A(|\nabla u_{n\sigma}|)$, $B(|\nabla v_{n\sigma}|)$, $\nabla A(|\nabla u_{n\sigma}|)$ and $\nabla B(|\nabla v_{n\sigma}|)$ are bounded independently of n in $L^\infty(Q_T)$; hence

$$
\begin{aligned}
\int_{Q_T} W\theta &\leq C\left[1 + \|u_0 + v_0 + w_0\|_{L^2(\Omega)} + \|v_n\|_{L^2(Q_T)} + \|w_n\|_{L^2(Q_T)}\right]\|\phi\|_{H^2(Q_T)} \\
&\leq C\left[1 + \|v_n\|_{L^2(Q_T)} + \|w_n\|_{L^2(Q_T)}\right]\|\theta\|_{L^2(Q_T)}
\end{aligned}
$$

which by duality completes the proof. ■

Lemma 3.2. *Let (u_n, v_n, w_n) be the solution of the approximate problem (3.12). Then*

(i) There exists a constant M depending only on $\int_\Omega u_0$, $\int_\Omega v_0$, $\int_\Omega w_0$, L_1, T and $|\Omega|$, such that

$$
\int_{Q_T} (u_n + v_n + w_n) \leq M \quad , \quad \forall t \in [0,T]
$$

(ii) There exists $C_1 > 0$, such that

$$\int_{Q_T} \left(|\nabla u_n|^2 + |\nabla v_n|^2 + |\nabla w_n|^2\right) \leq C_1$$

(iii) There exists $C_2 > 0$, such that

$$\int_{Q_T} (|f_n| + |g_n| + |h_n|) \leq C_2$$

Proof. (i) The triangular structure of problem (3.1) − (3.3), (3.4) − (3.6) implies that

$$\frac{\partial u_n}{\partial t} + \frac{\partial v_n}{\partial t} + \frac{\partial w_n}{\partial t} - \mathbf{div}\ (A(|\nabla u_{n\sigma}|)\nabla u_n)$$

$$-\mathbf{div}\ (B(|\nabla v_{n\sigma}|)\nabla v_n) - d_w \Delta w_n \leq L_1(1 + u_n + v_n + w_n)$$

The integration over Q_T leads to

$$\int_\Omega (u_n + v_n + w_n)(t) \leq \int_\Omega (u_0 + v_0 + w_0) + L_1 \int_{Q_T} (1 + u_n + v_n + w_n)$$

According to Gronwall's lemma, we get

$$\int_\Omega (u_n + v_n + w_n)(t) \leq \left[\int_\Omega (u_0 + v_0 + w_0) + L_1 |Q_T|\right] \exp(L_1 T)$$

It is what we want to prove.

(ii) We have

$$\frac{\partial w_n}{\partial t} - d_w \Delta w_n = h_n \leq L_3(1 + u_n + v_n + w_n)$$

The integration over Q_T leads to

$$\frac{1}{2}\int_{Q_T} \left(w_n^2\right)_t + d_w \int_{Q_T} |\nabla w_n|^2 \leq L_3 \int_{Q_T} (1 + u_n + v_n + w_n) w_n$$

According to Young's inequality and lemma 3.1, we get

$$\begin{aligned} \frac{1}{2}\int_\Omega w_n^2 + d_w \int_{Q_T} |\nabla w_n|^2 &\leq \frac{1}{2}\int_\Omega w_0^2 + L_3\left[\int_{Q_T} (1 + u_n + v_n + w_n)^2 + \int_{Q_T} w_n^2\right] \\ &\leq \frac{1}{2}\int_\Omega w_0^2 + C\int_{Q_T} w_n^2 \end{aligned}$$

and by Gronwall's lemma, we deduce that

$$\int_{Q_T} w_n^2 \leq C$$

which ensures that $\int_{Q_T} |\nabla w_n|^2$ and $\int_{Q_T} w_n^2$ are bounded. Now let us show that $\int_{Q_T} |\nabla v_n|^2$ are bounded. We have $v_n + w_n$ satisfies

$$\partial_t(v_n + w_n) - \mathbf{div}\,\left(B\left(\left|\nabla v_{n_\sigma}\right|\right)\nabla v_n\right) - d_w \Delta w_n = g_n + h_n \le L_2(1 + u_n + v_n + w_n)$$

Letting $\gamma = \exp(-L_2 t)(v_n + w_n)$, he comes

$$\int_{Q_T} \frac{\partial \gamma}{\partial t}\,\gamma + I + \int_{Q_T} \exp(-L_2 t) d_w \nabla w_n \nabla(v_n + w_n) \le \int_{Q_T} \exp(-L_2 t) L_2 \gamma \tag{3.14}$$

where

$$\begin{aligned} I &= \int_{Q_T} \exp(-L_2 t) B\left(\left|\nabla v_{n_\sigma}\right|\right)\nabla v_n \nabla(v_n + w_n) \\ &= \int_{Q_T} \exp(-L_2 t) B\left(\left|\nabla v_{n_\sigma}\right|\right)|\nabla(v_n + w_n)|^2 \\ &\quad - \int_{Q_T} \exp(-L_2 t) B\left(\left|\nabla v_{n_\sigma}\right|\right)\nabla w_n \nabla(v_n + w_n) \end{aligned}$$

Since $B\left(\left|\nabla v_{n_\sigma}\right|\right) \ge c$, we have

$$I \ge c\int_{Q_T} |\nabla(v_n + w_n)|^2 - \int_{Q_T} \exp(-L_2 t) B\left(\left|\nabla v_{n_\sigma}\right|\right)\nabla w_n \nabla(v_n + w_n)$$

Substituting in (3.14), he comes

$$\begin{aligned} &\frac{1}{2}\int_\Omega \gamma^2(T) + c\int_{Q_T} |\nabla(v_n + w_n)|^2 \\ \le\ & C + \int_{Q_T} \exp(-L_2 t)\left(d_w - B\left(\left|\nabla v_{n_\sigma}\right|\right)\right)\nabla w_n \nabla(v_n + w_n) \end{aligned}$$

According to Young's inequality on $|\nabla v_n \nabla(v_n + w_n)|$, we have

$$\begin{aligned} c\int_{Q_T} |\nabla(v_n + w_n)|^2 &\le C + \int_{Q_T} \exp(-L_2 t)(d_w - d)\left[\frac{|\nabla v_n|^2}{2\varepsilon} + \frac{\varepsilon|\nabla(v_n + w_n)|^2}{2}\right] \\ &\le C\left(1 + \frac{\exp(-L_2 t)(d_w - d)}{2\varepsilon C}\left[\int_{Q_T} |\nabla v_n|^2 + \varepsilon^2 \int_{Q_T} |\nabla(v_n + w_n)|^2\right]\right) \\ &\le C\left(1 + C(\varepsilon)\left[\int_{Q_T} |\nabla v_n|^2 + \varepsilon^2 \int_{Q_T} |\nabla(v_n + w_n)|^2\right]\right) \end{aligned}$$

Hence by choosing a suitable ε we deduce that $\int_{Q_T} |\nabla(v_n + w_n)|^2$ is bounded and because $\int_{Q_T} |\nabla w_n|^2$ is bounded, $\int_{Q_T} |\nabla v_n|^2$ is bounded as well.

In the same way, taking $u_n + v_n + w_n$, we deduce that $\int_{Q_T} |\nabla(v_n + w_n + w_n)|^2$ is bounded and because $\int_{Q_T} |\nabla w_n|^2$ and $\int_{Q_T} |\nabla v_n|^2$ are bounded, we conclude that $\int_{Q_T} |\nabla u_n|^2$ is bounded as well.

(iii) For w_n solution of

$$\frac{\partial w_n}{\partial t} - d_w \Delta w_n = h_n \leq L_3(1 + u_n + v_n + w_n)$$

We can write

$$\frac{\partial w_n}{\partial t} - d_w \Delta w_n + L_3(1 + u_n + v_n + w_n) - h_n = L_3(1 + u_n + v_n + w_n)$$

which implies

$$\int_{Q_T} \frac{\partial w_n}{\partial t} + \int_{Q_T} [L_3(1 + u_n + v_n + w_n)] - h_n = \int_{Q_T} L_3(1 + u_n + v_n + w_n)$$

Then

$$\int_{\Omega} w_n(T) - \int_{\Omega} w_n(0) + \int_{Q_T} [L_3(1 + u_n + v_n + w_n) - h_n] = \int_{Q_T} L_3(1 + u_n + v_n + w_n)$$

We know that $\int_{Q_T} L_3(1 + u_n + v_n + w_n)$ is bounded, which follows that

$$\|L_3(1 + u_n + v_n + w_n) - h_n\|_{L^1(Q_T)} \leq C$$

Therefore

$$\|h_n\|_{L^1(Q_T)} \leq C_h$$

Since $L_2(1 + u_n + v_n + w_n) - g_n - h_n \geq 0$, we obtain the same for $g_n + h_n$, hence

$$\|g_n\|_{L^1(Q_T)} \leq C_g$$

and since $L_1(1 + u_n + v_n + w_n) - f_n - g_n - h_n \geq 0$, we obtain the same for $f_n + g_n + h_n$, hence

$$\|f_n\|_{L^1(Q_T)} \leq C_f$$

■

3.4.3 Convergence

Our objective is to show that (u_n, v_n, w_n) converges to some (u,v,w) solution of our problem. According to Lemma 3.2, (u_n, v_n, w_n) is bounded in $\left(L^2(0,T;\mathcal{V})\right)^3$ and $\left(\dfrac{\partial u_n}{\partial t}, \dfrac{\partial v_n}{\partial t}, \dfrac{\partial w_n}{\partial t}\right)$ is bounded in $\left(L^2(0,T;\mathcal{V}') + L^1(Q_T)\right)^3$. Therefore, by Aubin-Simon, (u_n, v_n, w_n) is relatively compact in $\left(L^2(Q_T)\right)^3$, see Simon [144], then we can extract a subsequence also noted (u_n, v_n, w_n) in $\left(L^2(Q_T)\right)^3$, such that :

- $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ and $w_n \rightharpoonup w$ in $L^2(Q_T)$ and a.e. in Q_T,
- $\nabla G_\sigma * u_n \rightharpoonup \nabla G_\sigma * u$ and $\nabla G_\sigma * v_n \rightharpoonup \nabla G_\sigma * v$ in $L^2(Q_T)$ and a.e. in Q_T,
- $A(|\nabla u_{n\sigma}|) \longrightarrow A(|\nabla u_\sigma|)$ and $B(|\nabla v_{n\sigma}|) \to B(|\nabla v_\sigma|)$ in $L^2(Q_T)$,
- $f_n(t,x,u_n,v_n,w_n) \to f(t,x,u,v,w)$ for a.e. in Q_T,
- $g_n(t,x,u_n,v_n,w_n) \to g(t,x,u,v,w)$ for a.e. in Q_T,
- $h_n(t,x,u_n,v_n,w_n) \to h(t,x,u,v,w)$ for a.e. in Q_T.

This is not sufficient to ensure that (u_n, v_n, w_n) is a weak solution of our problem. In fact, we have to prove that the previous convergences are in $L^1(Q_T)$. In view of the Vitali theorem, to show that $f_n \to f$, $g_n \to g$ and $h_n \to h$ strongly in $L^1(Q_T)$, is equivalent to proving that f_n, g_n and h_n are equi-integrable in $L^1(Q_T)$. This is confirmed by the following lemma.

Lemma 3.3. *Under the additional assumption that, for $R > 0$,*

$$\sup_{|r|+|s|+|q|\le R} \left(|f(t,x,r,s,q)| + |g(t,x,r,s,q)| + |h(t,x,r,s,q)|\right) \in L^1(Q_T)$$

(i) There exists $C > 0$, such that

$$\int_{Q_T} (u_n + 2v_n + 3w_n)(|f_n| + |g_n| + |h_n|) \le C$$

(ii) f_n, g_n and h_n are equi-integrable in $L^1(Q_T)$.

Proof. (i) Let

$$\begin{aligned} R_n &= L_1(1+u_n+v_n+w_n) - f_n - g_n - h_n \\ S_n &= L_1(1+u_n+v_n+w_n) - g_n - h_n \\ P_n &= L_1(1+u_n+v_n+w_n) - h_n \end{aligned}$$

and

$$\theta_n = u_n + 2v_n + 3w_n \quad \text{and} \quad E_n = u_n + v_n + w_n$$

we have by hypothesis (3.5)

$$R_n \geq 0 \ , \ S_n \geq 0 \ , \ P_n \geq 0$$

Combining the equations of system (4.1), we have

$$\begin{aligned}
\frac{\partial \theta_n}{\partial t} - \xi_n &= f_n + 2g_n + 3h_n \\
&= -R_n + L_1(1 + u_n + v_n + w_n) \\
&\quad -S_n + L_2(1 + u_n + v_n + w_n) \\
&\quad -P_n + L_3(1 + u_n + v_n + w_n)
\end{aligned}$$

where

$$\xi_n = \mathbf{div}\ (A(|\nabla u_{n\sigma}|)\nabla u_n) + 2\mathbf{div}\ (B(|\nabla v_{n\sigma}|)\nabla v_n) + 3d_w \Delta w_n$$

Multiplying by $u_n + 2v_n + 3w_n$ and integrating over Q_T, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \theta_n^2(T) + \int_{Q_T} \nabla\xi_n.\nabla\theta_n + \int_{Q_T} (R_n + S_n + P_n)\theta_n \\
= &\frac{1}{2}\int_\Omega \theta_n^2(0) + (L_1 + L_2 + L_3)\int_{Q_T} (1 + u_n + v_n + w_n)\theta_n
\end{aligned}$$

which implies

$$\begin{aligned}
\int_{Q_T} (R_n + S_n + P_n)\theta_n &\leq \int_{Q_T} |\nabla\xi_n|.|\nabla\theta_n| + \frac{1}{2}\int_\Omega \theta_n^2(0) \\
&\quad + (L_1 + L_2 + L_3)\int_{Q_T} (1 + u_n + v_n + w_n)\theta_n
\end{aligned}$$

Using Young's inequality, we conclude that

$$\begin{aligned}
\int_{Q_T} (R_n + S_n + P_n)\theta_n &\leq \frac{1}{2}\int_{Q_T} \left[|\nabla\xi_n|^2 + |\nabla\theta_n|^2\right] + \frac{1}{2}\int_\Omega \theta_n^2(0) \\
&\quad + (L_1 + L_2 + L_3)\int_{Q_T} (1 + u_n + v_n + w_n)\theta_n \\
&\leq C
\end{aligned}$$

By the previous lemmas, we obtain the desired result

(ii) We know that f_n, g_n and h_n converge almost everywhere toward f, g and h. We will show that f_n, g_n and h_n are equi-integrable in $L^1(Q_T)$. The proof will be given for f_n, however the same result holds for g_n and h_n. For this, we let $\varepsilon > 0$ and prove that there exists $\delta > 0$ such that $|Y| < \delta$ implies that $\int_Y f_n < \varepsilon$. We have

$$\begin{aligned}
\int_A |f_n(t,x,u_n,v_n,w_n)| &= \int\limits_{A\cap[E_n>k]} |f_n| + \int\limits_{A\cap[E_n\leq k]} |f_n| \\
&\leq \int\limits_{A\cap[\theta_n>k]} |f_n| + \int\limits_{A\cap[E_n\leq k]} |f_n| \\
&\leq \frac{1}{k}\int_A (u_n+2v_n+3w_n).|f_n| + |A| \sup_{|u_n|+|v_n|+|w_n|\leq k} |f(t,x,u_n,v_n,w_n)|
\end{aligned}$$

We can choose δ small enough and a large k such that $\int_Y f_n < \varepsilon$.

In the same way, we treat g_n and h_n. ■

CHAPTER

4

On Rapidly Oscillating Solutions of a Nonlinear Elliptic Equation

The aim of this chapter is to examine the solutions of the boundary value problem of the nonlinear elliptic equation $\varepsilon^2 \triangle u = f(u)$. We describe the asymptotic behavior as ε tends to zero of the solutions on a spherical crown C of $\mathbb{R}^N$, $(N \geq 2)$ in a direct non-classical formulation which suggests easy proofs. We propose to look for interesting solutions in the case where the condition at the edge of the crown is a constant function. Our results are formulated in classical mathematics. Their proofs use the stroboscopic method which is a tool of the nonstandard asymptotic theory of differential equations.

The work constituting this chapter is the subject of an article published in (Mathematica Slovaca), in collaboration with S. Bendaas [67].

4.1 Introduction

The following boundary value problem of ordinary differential equations

$$(O_\varepsilon) \begin{cases} \varepsilon^2 x'' = f(x) & , 0 < t < 1 \\ x(0) = a & , x(1) = b, \text{ prescribed} \end{cases}$$

inspired a large number of Authors in the field of Non Standard Analysis. When ε is infinitely small, they described the asymptotic behavior of the solutions, when ε tends to zero. They proved in a simple way by Non Standard Analysis techniques the existence of rapidly oscillating solutions for certain values of a and b in a suitable observability plane see: [43], [81], [91] and [127]. This is a typical case where classical development methods are inapplicable. Indeed, G. F. Carrier and C.E. Pearson [36] and P.C. Fife and W.M. Greenlee [55] have constructed formal solutions of (O_ε) which present transition layers at arbitrary points of $]0,1[$ and thus cannot all approach true solutions when ε approaches zero.

In [94], R.E. O'Malley attempted a geometric approach based on the prime integral $\varepsilon^2 x^2 - F(x)$, where $F(x)$ is a primitive of $2f$, the intuitive arguments he uses reveal many limit solutions admitting transition layers whose location depends only on derivatives of f where it vanishes. However, these arguments give incorrect results when the zeros of f (what it means the solutions of the reduced problem) are of order greater than 1 (if they are all of order 1, the locations are coincidentally correct). He proposed to completely describe the asymptotic behavior of the solutions in a direct non-classical formulation which suggests easy proofs.

The infinitesimal techniques of Non Standard Analysis have made it possible to evaluate qualitatively the problems of singular perturbations concerning the differential equations. Little work has so far been devoted to the application of these techniques to EDPs, we can cite [23], [25], [26], [27], [28], [43] and [116].

The purpose of this work is to apply these Non-Standard Analysis techniques to a boundary value problem of P.D.E. The paper is devoted to the problem

$$(P_\varepsilon)\begin{cases} \varepsilon^2 \Delta u = f(u) & \text{in } C \\ u = cste & \text{on } \partial C \end{cases}$$

where ε is an infinitely small fixed positive real and f is a standard function continuous in u.

We will describe the asymptotic behavior of the solutions of the problem (P_ε) using new infinitesimal techniques which allow to approach the problem in a geometric way by avoiding the formal developments and the usual approximation techniques as in ([25], [26], [27], [47] and [116]). For $\varepsilon = 0$, the reduced problem

$$(P_0)\begin{cases} f(u) = 0, & \text{in } C \\ u = cste & \text{on } \partial C \end{cases}$$

In the problem (P_0), $f(u) = 0$ is an ordinary equation whose solutions in general cannot verify the boundary condition. We can expect that the solutions of problem (P_ε)

are infinitely close to one of the solutions of the reduced problem (P_0) outside of various areas that we would like to determine [23], [25], [26] and [116].

The only results known in this sense report a weak limiting behavior, which does not specify the behavior point by point [81], [91]. When the boundary condition is a functiont, the difficulty is to determine the hyper transition surfaces where u jumps from one root to another.

In this work we are based on the Stroboscopy method of Non Standard Analysis, to describe the asymptotic behavior of solutions in a boundary value problem of PDE. This technique allows us to describe the shape of the trajectories of the induced field without going through often complex asymptotic expansion calculations very common in classical Mathematics. This asymptotic behavior of the solutions was determined by the shadows of the integral curves of the equivalent differential equation [92], [129].

Historically, the Non Standard perturbation theory of differential equations, which is today a well-established tool in asymptotic theory, has its roots in the seventies, when the Reebian school (see [47], [91], [92]) introduced the use of Non Standard Analysis (NSA) into the field of perturbed differential equations. For more information on the subject, the interested reader is referred to texts such as [48] (Chapters 4 and 10) and to papers such as [47], [81], [139] among many others.

We propose to look for interesting solutions for the problem P_ε. In this case it is reasonable to seek solutions which only depend on the distance t at the center of the crown, since the Laplacian is invariant by isometry around the origin. For such a supposed solution of class C^2, we have

$$\Delta u = u''(t) + \frac{(N-1)}{t} u'(t).$$

If the radius of the crown (C) centered at the origin are: $r_1 = t_1$ and $r_2 = t_2$, we obtain the boundary problem of ordinary differential equations

$$\left\{ \begin{array}{ll} \varepsilon^2 \left(u'' + \dfrac{(N-1)}{t} u' \right) = f(u) & (E) \\ u(r_1) = a, \; u(r_2) = b & \end{array} \right. \tag{4.1}$$

where r_1 and r_2 are both standard and $0 < r_1 < r_2$.

It is therefore a non-autonomous variant of the boundary value problem of ordinary differential equations (O_ε).

If $\varepsilon = 0$,we obtain the corresponding reduced problem

$$\left\{ \begin{array}{ll} f(u) = 0 & (E)_0 \\ u(r_1) = a, \; u(r_2) = b & \end{array} \right. \tag{4.2}$$

In the following we restrict ourselves to the case $N = 2$, which contains the essential aspects of this problem.

The idea to use NSA in perturbation theory of differential equations goes back to the 1970s with the Reebian School. Relative to this use, among many works we refer the interested reader for instance, to [47], (see the five-digits classification 34E18 of the 2000 Mathematical Subject Classification). It has become today a well established tool in asymptotic theory of differential equations. Among the famous discoveries of the Non Standard asymptotic theory of differential equations we can cite the canards which appear in slow-fast vector fields and are closely related to the stability loss delay phenomenon in dynamical bifurcations.

Our goal in this paper is to apply theses Non Standard Analysis techniques to the Problem (P_ε). The structure of the paper is as follows: Section 2 concerns the preliminaries, we study the slow-fast vector field for the problem (4.1) in a suitable observability space. In Section 3 we start with a short tutorial to NSA and then state our main (nonstandard) tool, the so-called stroboscopic method. Section 4 concerns the oscillatory movement, we place ourselves in the case of an oscillatory movement and we propose to describe this movement. In space $(t, u', v = \varepsilon u')$, the differential equation admits an almost prime integral $v^2 - -2\int_0^u f(\xi)d\xi = k(t)$, where k evolves slowly over time. Based on the Stroboscopic method described by J.L.Callot and T. Sari, we obtain this evolution which ensures the existence of rapidly oscillating solutions having nothing to do with the solutions of the reduced equation $f(u) = 0$ obtained when $\varepsilon = 0$, and deals with our Main Results and its proof. The last section deals with some special cases of the boundary problem (4.1).

4.2 Preliminaries

To equation (E) in problem (4.1) is associated in the phase space $(t, u, v = u')$ the system

$$\left\{\begin{array}{l} u' = v \\ v' = \dfrac{f(u)}{\varepsilon^2} - \dfrac{v}{t} \\ t' = 1 \end{array}\right. \tag{4.3}$$

The corresponding slow-fast vector field is almost vertical, of infinitely large modulus at any limited point (t, u, v) located outside the halo of the "slow variety" $f(u) = 0$.The rapid movement highlighted is studied in the "Space of observability" $(t, u', v = \varepsilon u')$ in which we obtain the system

$$\begin{cases} u' = \dfrac{v}{\varepsilon} \\ v' = \dfrac{f(u)}{\varepsilon} - \dfrac{v}{t} \\ t' = 1 \end{cases} \tag{4.4}$$

for which the slow Sub-Variety is defined by $v = 0 = f(u)$.Outside the Galaxy of this subset defined by $\frac{v}{\varepsilon}$ limited and $\frac{1}{\varepsilon}f(u)$ limited. The vector field corresponding to the system (4.4) is almost contained in a vertical plane $t = \bar{t}$ and has an infinitely large module; to determine the direction, we carry out the time change

$$T = \frac{1}{\varepsilon}(t - \bar{t})$$

we then get

$$\begin{cases} u^{'} = v \\ v' = f(u) - \dfrac{\varepsilon v}{\varepsilon T + \bar{t}} \\ T^{'} = 1 \end{cases} \tag{4.5}$$

For T limited, the integral curves are infinitely close to the solutions of the autonomous system

$$\begin{cases} u' = v \\ v' = f(u) \end{cases} \tag{4.6}$$

who has the first integral $v^2 - F(u)$, where

$$F(u) = 2\int_0^u f(\xi)d\xi$$

Let $\gamma(t) = (t, u(t), v(t))$ be the integral curve of the system (4.4) passing through the limited point $\left(\bar{t}, \bar{u}, \bar{v}\right)$, and let (C) be the related component of the equation curve

$$\begin{cases} v^2 - F(u) = \overline{v}^2 - F(\bar{u}) \\ t = \bar{t} \end{cases}$$

containing the point $\left(\bar{t}, \bar{u}, \bar{v}\right)$. Then $\gamma(t)$ is infinitely close to (C) as long as t differs from $\bar{t}$ only by a quantity of order ε.The solution $\gamma(t)$ quickly runs along the equation surface $v^2 - F(u) = k(t)$,where $k(t)$ varies slowly.

We propose to describe the oscillatory movements by establishing the equation $v^2 - F(u) = k(t)$ in the halo of which is the integral curve $\gamma(t) = (t, u(t), v(t))$ solution of (4.4). The movement is fast infinitely large outside the Galaxy of the subvariety: $v = 0 = f(u)$. The shadows of the integral curves of (4.4) are infinitely close to the equation curves: $v^2 - F(u)$ constant as long as the variation of t is of the order of ε i.e. as long as these curves are described a standard number of times.

However, how do these curves evolve when time varies appreciably. In other words, what happens when the variation of t is no longer infinitely small. We suppose that the function $f(u)$ cancels out in a finite number of points.

4.3 Some Notions on Non Standad Analysis

4.3.1 Internal Set Theory

In this section we give a short tutorial of NSA. Additional information can be found in [92], [127], [139] and [140]. *Internal Set Theory* (IST) [43] is an extension of ordinary mathematics, that is, Zermelo-Fraenkel set theory plus axiom of choice (ZFC). The theory IST gives an axiomatic approach of Robinson's Nonstandard Analysis [114], [129]. We adjoin to ZFC a new undefined unary predicate standard (**st**) and add to the usual axioms of ZFC three others for governing the use of the new predicate. All theorems of ZFC remain valid in IST. What is new in IST is an addition, not a change. We call a formula of IST *internal* in the case where it does not involve the new predicate (**st**); otherwise, we call it *external*. The theory IST is a *conservative extension* of ZFC, that is, every internal theorem of IST is a theorem of ZFC. Some of the theorems which are proved in IST are external and can be reformulated so that they become internal. Indeed, there is a *reduction algorithm* due to Nelson which reduces any external formula $F(x_1,\dots,x_n)$ of IST without other free variables than $x_1,\dots,x_n$ to an internal formula $F'(x_1,\dots,x_n)$ with the same free variables, such that $F \equiv F'$, that is, $F \Longleftrightarrow F'$ for all standard values of the free variables. We will need the following reduction formula which occurs frequently

$$\forall x \left(\forall^{st} y A \Longrightarrow \forall^{st} z B\right) \equiv \forall z \ \ \exists^{fin} y' \forall x \left(\forall y \in y' \ \ A \Longrightarrow B\right)$$

where A (respectively B) is an internal formula with free variable y (respectively z) and standard parameters. The notation $\forall^{st}$ means "for all standard" and $\exists^{fin}$ means "there is a finite".

A real number x is said to be infinitesimal if $|x| < a$ for all standard positive real numbers a and limited if $|x| \leq a$ for some standard positive real number a. A limited real number which is not infinitesimal is said to be appreciable. A real number which is not limited is said to be *unlimited*. The notations $x \simeq 0$ and $x \simeq \infty$ are used to denote, respectively, x is infinitesimal and x is *unlimited* positive.

Let D be a standard subset of some standard normed space E. A vector $x \in D$ is infinitesimal (resp. limited, unlimited) if its norm $\|x\|$ is infinitesimal (resp. limited, unlimited). Two elements $x, y \in D$ are said to be infinitely close, in symbols $x \simeq y$, if $\|x - y\| \simeq 0$. An element $x \in D$ is said to be near-standard (resp. near-standard in D) if $x \simeq x_0$ for some standard $x_0 \in E$ (resp. for some standard $x_0 \in D$). The element x_0 is called the standard part or shadow of x. It is unique and is usually denoted by ${}^o x$. Note that when $E = \mathbb{R}$ each limited vector $x \in D$ is near-standard (but not necessary near-standard in D).

The shadow of a subset A of E, denoted by ${}^o A$, is the unique standard set whose standard elements are precisely those standard elements $x \in E$ for which there exists $y \in A$ such that $y \simeq x$. Note that ${}^o A$ is a closed subset of E and if $A \subset B$ then ${}^o A \subset {}^o B$. When A is standard, ${}^o A = A$.

Let $I \subset \mathbb{R}$ be some interval and $f : I \to \mathbb{R}^d$ be a function, with d standard. We say that f is $S-$*continuous* at a standard point $x \in I$ if, for all $y \in I$, $y \simeq x$ implies $f(y) \simeq f(x)$, f is $S-$*continuous* on I if f is $S-$*continuous* at each standard point of I, and f is $S-$*uniformly-continuous* on I if, for all $x, y \in I$, $x \simeq y$ implies $f(x) \simeq f(y)$. If I is standard and compact, S-*continuity* on I and S-*uniform-continuity* on I are the same. When f (and then I) is standard, the first definition is the same as saying that f is continuous at a standard point x, the second definition corresponds to the continuity of f on I and the last one to the uniform continuity of f on I.

We need the following result on S-*uniformly-continuous* functions on compact intervals of $\mathbb{R}$. This result is a particular case of the so-called "Continuous Shadow Theorem"
.

Theorem 4.1. *Let I be a standard and compact interval of $\mathbb{R}$, D be a standard subset of $\mathbb{R}^d$ (with d standard) and $x : I \to D$ be a function. If x is S-uniformly continuous on I and for each $t \in I$, $x(t)$ is near-standard in D, then there exists a standard and continuous function $y : I \to D$ such that, for all $t \in I$, $x(t) \simeq y(t)$.*

The function y in Theorem 4.1 is unique. It is defined as the unique standard function y which, for t standard in I, is given by $y(t) = {}^\circ(x(t))$.

4.3.2 The Stroboscopic Method for ODEs

We present in this subsection a fundamental lemma of the nonstandard perturbation theory of differential equations. The stroboscopic method was proposed by J. L. Callot and G. Reeb and improved by R. Lutzand T. Sari (see [34], [140] and [149]). Let O be a standard open subset of $\mathbb{R}^n$, $F : O \to \mathbb{R}^n$ a standard continuous function. Let J be an interval of $\mathbb{R}$ containing 0 and $\phi : J \to \mathbb{R}^n$ a function such that $\phi(0)$ is nearstandard in O, that is, there exists a standard $x_0 \in O$ such that $\phi(0) \simeq x_0$. Let I be a connected subset of J, eventually external, such that $0 \in I$.

Definition 4.1 (Stroboscopic property)**.** Let t and t' be in I. The function ϕ is said to satisfy the stroboscopic property $S(t,t')$, if $t' \simeq t$, and $\phi(s) \simeq \phi(t)$ for all s in $[t,t']$ and

$$\frac{\phi(t)-\phi(t')}{t-t'} \simeq F\left(\phi(t)\right)$$

Under suitable conditions, the Stroboscopy lemma asserts that the function ϕ is approximated by the solution of the initial value problem

$$\frac{dx}{dt} = F(x), \quad x(0) = x_0 \tag{4.7}$$

Theorem 4.2 (Stroboscopic Lemma)**.** *Suppose that*

(i) There exist $\mu > 0$ such that, whenever $t \in I$ is limited and $\phi(t)$ is nearstandard in O, there is $t' \in I$ such that $t' - t \geq \mu$ and the function ϕ,satisfies the stroboscopic property $s(tt')$.

(ii) The initial value problem (4.7) has a unique solution $x(t)$.

Then, for any standard L in the maximal positive interval of definition of $x(t)$, we have $[0,L] \subset I$ and $\phi(t) \simeq x(t)$ for all $t \in [0,L]$.

Proof. The proof of Stroboscopic lemma for ODEs and additional information can be found in [48] (Chapters 4 and 10). ■

4.4 Oscillatory Movement

In the following, we place ourselves in the case of an oscillatory movement and we propose to describe this movement by establishing the equation $w(t) = v^2(t) - F(u(t))$ in the halo of which is the integral curve $\gamma(t) = (t,u(t),v(t))$, solution of the system (4.4) supposed to rotate around an almost center in projection on the plane (u,v) in a time interval I with an appreciable amplitude. Let I be an open interval, such that for all t in I there is

(i) An interval $K_t =]w_-(t),w_+(t)[$, such that for all w in K_t the equation $w+F(u) = v^2$ defines closed curves.

(ii) An interval $J_t(w) =]u_-(t,w),u_+(t,w)[$ such that for each value w, $u_+(t,w)$ and $u_-(t,w)$ are the only roots of the equation $-F(u) = w$ between which u oscillates.

(iii) Let $D = \{(t,w); t \in I, w \in K_t\}$ be a domain, and let $u_-(t,w)$ and $u_+(t,w)$, be two defined and continuous functions on D, with valuesin J_t such that $u_-(t,w) < u_+(t,w)$.

In D we then define the function

$$\varphi(t,w) = -\frac{2}{t}\frac{\int_{u_-(t,w)}^{u_+(t,w)}\sqrt{w+F(u)}\,du}{\int_{u_-(t,w)}^{u_+(t,w)}\frac{du}{\sqrt{w+F(u)}}}$$

this function is well defined in D. Indeed, if $(t,w) \in D$, the function $h(u) = \sqrt{w+F(u)}$ defined on the closed interval $]u_-(t,w),u_+(t,w)[$ is positive and is only canceled in u_- and u_+. It is differentiable over the open interval $]u_-(t,w),u_+(t,w)[$. Its derivative

$$h'(u) = \frac{f(u)}{\sqrt{w+F(u)}}$$

tends to (∞) when u tends to $u_-(w)$ because $f(u_\pm) \neq 0$.Thus, its inverse is integrable over the interval $[u_-(t,w),u_+(t,w)]$, because near u_- and u_+ we have

$$\sqrt{w+F(u)} \sim \sqrt{|u-u_\pm|w+f(u_\pm)}, \quad f(u_\pm) \neq 0$$

Our main result for the problem (4.1), is the following

Theorem 4.3. *Let $\gamma(t) = (t,u(t),v(t))$, be the integral curve of the system (4.4) from point (t_0,u_0,v_0) with $t_0 > 0$. Let $w_0 = v_0^2 - F(u_0)$. Let us assume that $u_-(t_0,w_0) < u_0 < u_+(t_0,w_0)$. So the shadow of the curve $\gamma(t)$ is the surface of equation*

$$v^2 - F(u) = w(t)$$

where $w(t)$ is solution of the differential equation

$$\begin{cases} w'(t) = \varphi(t,w) \\ w(t_0) = w_0 \end{cases}$$

and this as long as t remains limited and that $(t,w(t))$ belongs to D.

Proof. Let $\gamma(t) = (t, u(t), v(t))$ be an integral curve of the system (4.4). Denote $k(t) = v^2(t) - F(u(t))$, so

$$\frac{dk}{dt} = -2\frac{v^2}{t}$$

and the system (4.4) is equivalent to the two systems

$$\left\{\begin{array}{l} \dfrac{dk}{dt} = -2\dfrac{v^2(t)}{t} \\ \dfrac{du}{dt} = \pm\dfrac{\sqrt{k+F(u)}}{\varepsilon} \end{array}\right. \tag{4.8}$$

defined for $\sqrt{k+F(u(t))} > 0$.

An oscillatory movement is performed by alternately borrowing the integral curves of the two vector fields. Thus, $k(t)$ remains almost constant during a standard number of oscillations. We evaluate the variations of k by observing the solution using a stroboscope [33]. Let t_n of I be an observation instant and $k_n = k(t_n)$, under the magnifying glass

$$\begin{array}{r} T = \dfrac{t-t_n}{\varepsilon} \\ K(T) = \dfrac{k(t_n+\varepsilon T) - k(t_n)}{\varepsilon} \end{array} \tag{4.9}$$

The systems (4.8) become

$$\left\{\begin{array}{l} \dfrac{dK}{dT} = -2\dfrac{v^2(t_n+\varepsilon T)}{t_n+\varepsilon T} \\ \dfrac{du}{dT} = \pm\sqrt{k_n+\varepsilon k(T)+F(u(t_n+\varepsilon T))} \end{array}\right. \tag{4.10}$$

whose integral curves are infinitely close as long as T is limited to solutions of the systems

$$\left\{\begin{array}{l} \dfrac{dK}{dT} = -2\dfrac{v^2(t_n)}{t_n} \\ \dfrac{du}{dT} = \pm\sqrt{k_n+F(u(t_n))} \end{array}\right. \tag{4.11}$$

The systems (4.10) are defined for u lying between $u_-(t_n, k_n)$ and $u_+(t_n, k_n)$. Thus the movement which is established when one alternately follows their trajectories is periodical of period

$$P = 2\left[\int_{u_-(k(t_n))}^{u_+(k(t_n))} \frac{du}{\sqrt{k_n+F(u(t_n))}}\right] \tag{4.12}$$

and, we have

$$K(P) = -2\int_0^P \frac{v^2(t_n)}{t_n} dT \sim -4\left[\int_{u_-(k(t_n))}^{u_+(k(t_n))} \frac{\sqrt{k_n + F(u(t_n))}}{t_n} du\right]$$

We define the following observation instant of the Stroboscopy Method through

$$t_{n+1} = t_n + \varepsilon P$$

So

$$\frac{k_{n+1} - k_n}{t_{n+1} - t_n} = \frac{K(P)}{P} \approx \varphi(t_n, k_n)$$

It follows that $k(t)$ is infinitely close to the solution $w(t)$ of the differential equation

$$\begin{cases} \dfrac{dw}{dt} = \varphi(t,w) \\ w(t_0) = w_0 \end{cases}$$

As long as t is limited and that $(t, w(t))$ remains in D. which proves the theorem. ■

4.5 Special Cases for the Boundary Problem (4.1)

Various cases can arise for the boundary problem (4.1). Among these cases we can cite.

4.5.1 The First Case

If we start from a certain value a at the time $r_1 = t_1$, the solution may not reach the value b at the time $r_2 = t_2$ while remaining in the halo of the surface equation $v^2 - F(u) = k(t)$, Figure 4.1.

4.5.2 The Second Case

We will explain the discussion for the function $f(u) = u(u^2 - 1)$. The equipotentials of $F(u) = \dfrac{u^4}{2} - u^2$, admit a center at 0 between two passes in -1 and 1. (Figure 4.1).

We are interested in solutions around the center and we propose to describe the slow evolution of $k(t)$ given by Theorem 4.3. So we get

$$\frac{dw}{dt} = \varphi(t,w) \tag{4.13}$$

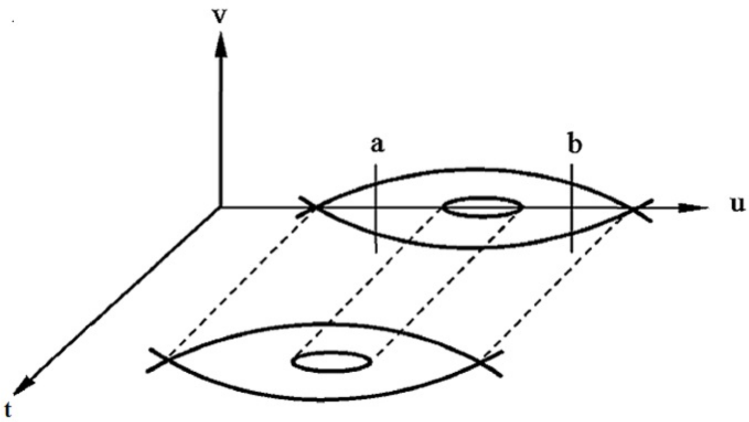

FIGURE 4.1. Grapchic representation of the evolution $v^2 - F(u) = k(t)$.

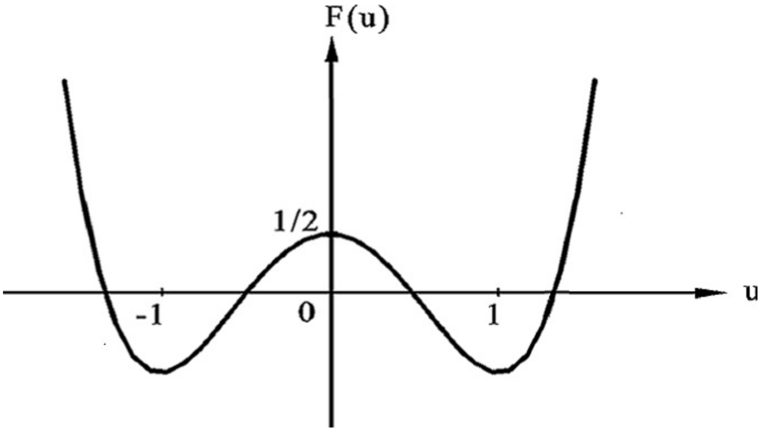

FIGURE 4.2. The equipotentials of $F(u) = \frac{u^4}{4} - u^2$.

with

$$\varphi(t,w) = -\frac{2}{t}\frac{\int_{\alpha_-(w)}^{\alpha_+(w)} \sqrt{(u^2-\alpha^2)(u^2-\beta^2)}\,du}{\int_{\alpha_-(w)}^{\alpha_+(w)} \frac{du}{\sqrt{(u^2-\alpha^2)(u^2-\beta^2)}\,du}} \quad \text{for all } t \in [t_n, t_{n+1}]$$

The function $\varphi(t,w)$ is negative so that the solutions $w(t)$ are decreasing. The two positive roots of the equation $w + F(u) = 0$, are

$$\alpha^2 = 1 - \sqrt{1-2w} \qquad (4.14)$$
$$\beta^2 = 1 + \sqrt{1-2w}$$

For $0 < w < \dfrac{1}{2}$, we get oscillating solutions. Therefore, we have the following lemma

Lemma 4.1. *The standard function*

$$\begin{aligned} I(\alpha) &= \int_{\alpha_-(w)}^{\alpha_+(w)} \frac{du}{\sqrt{k+F(u)}} \\ &= \int_{\alpha_-(w)}^{\alpha_+(w)} \frac{du}{\sqrt{[(u^2-\alpha^2)(u^2-2+\alpha^2)]}} \end{aligned}$$

is increasing with respect to k.

Proof. The time required for aride around is such that

$$\begin{aligned} I(\alpha) &= \int_{\alpha_-(w)}^{\alpha_+(w)} \frac{du}{\sqrt{k+F(u)}} \\ &= 2\int_0^{\alpha} \frac{du}{\sqrt{[(u^2-\alpha^2)(u^2-2+\alpha^2)]}} \end{aligned}$$

Let λ and α be two solutions with $\lambda > \alpha$, corresponding to k_1 and k_2 where $k_1 > k_2$. Let us show that $I(\lambda) > I(\alpha)$, what it means let us show that the function I is increasing. Indeed

$$I(\lambda) = 2\int_0^{\lambda} \frac{dv}{\sqrt{(\lambda^2-v^2)(2-\lambda^2-v^2)}}$$

Let $u = \dfrac{\alpha v}{\lambda}$, we obtain

$$I(\lambda) = 2\int_0^{\alpha} \frac{\lambda}{\alpha} \frac{du}{\sqrt{(\lambda^2-(\lambda^2u^2/\alpha^2))(2-\lambda^2-(\lambda^2u^2/\alpha^2))}}.$$

From where

$$I(\lambda)-I(\alpha) = 2\int_0^{\alpha} \frac{du}{\sqrt{\alpha^2-u^2}} \frac{1}{\sqrt{(2-\lambda^2-(\lambda^2u^2/\alpha^2)}} - \frac{1}{\sqrt{(2-\alpha^2-u^2)}}$$

But, u is between 0 and α. Then a simple calculation [28] shows that

$$\frac{1}{\sqrt{(2-\lambda^2-(\lambda^2u^2/\alpha^2)}} - \frac{1}{\sqrt{(2-\alpha^2-u^2)}} > 0$$

and therefore $I(\lambda) > I(\alpha)$ for $\lambda > \alpha$ and the function $I(\alpha)$ is then increasing. Which proves the lemma. ∎

This allows us to establish the following corollary.

Corollary 4.1. *According to the evolution of time given by the function I that a trajectory takes to make a turn, we deduce that for two solutions λ and α corresponding to k_1 and k_2, the time $\frac{t}{\varepsilon} = I$ set by the trajectory λ is greater than that of the trajectory α if and only if $k_1 \gg k_2$. In addition, for $\alpha \simeq 0$, we will have $I(\alpha) \approx 2\sqrt{2}$. And for $\alpha = 1$, we have $I(\alpha) = \infty$.*

Hence the following remark.

Remark 4.1. *Let α and λ be two trajectories corresponding to k_1 and k_2 with $k_1 \gg k_2$. When the first trajectory turns, the second one wins an appreciable lap during the same time.*

Let's discuss the following standard differential system

$$\left\{ \begin{array}{l} \dfrac{dw}{dt} = \varphi(t,w) \\ \dfrac{dt}{dt} = 1 \end{array} \right. \tag{4.15}$$

$\varphi(\ t,\ w)$ is a decreasing function with respect to t which gives the evolution of $w(t)$. However in our case we have

$$\frac{dw}{dt} = \frac{1}{2}\left[\beta^2 \frac{d\alpha^2}{dt} + \alpha^2 \frac{d\beta^2}{dt}\right] = (1-\alpha^2)\frac{d\alpha^2}{dt}$$

Therefore, the differential equation (4.13) is equivalent to the equation

$$\frac{d(\alpha^2)}{dt} = \frac{-1}{t(1-\alpha^2)} \frac{\int_{-\alpha}^{\alpha} \sqrt{(u^2-\alpha^2)(u^2-2+\alpha^2)}\,du}{\int_{-\alpha}^{\alpha} \dfrac{du}{\sqrt{(u^2-\alpha^2)(u^2-2+\alpha^2)}}} \tag{4.16}$$

To estimate the integrals in the differential equation (4.16), we put

$$\begin{aligned} g(\alpha) &= \int_{-\alpha}^{\alpha} \sqrt{(u^2-\alpha^2)(u^2-2+\alpha^2)}\,du \\ &= 2\int_{0}^{\alpha} \sqrt{(u^2-\alpha^2)(u^2-2+\alpha^2)}\,du \end{aligned}$$

For $0 \le \xi \le \alpha$

$$g(\alpha) = 2\alpha \int_{0}^{\alpha} \sqrt{(u^2-\alpha^2)(u^2-2+\alpha^2)}\,du$$

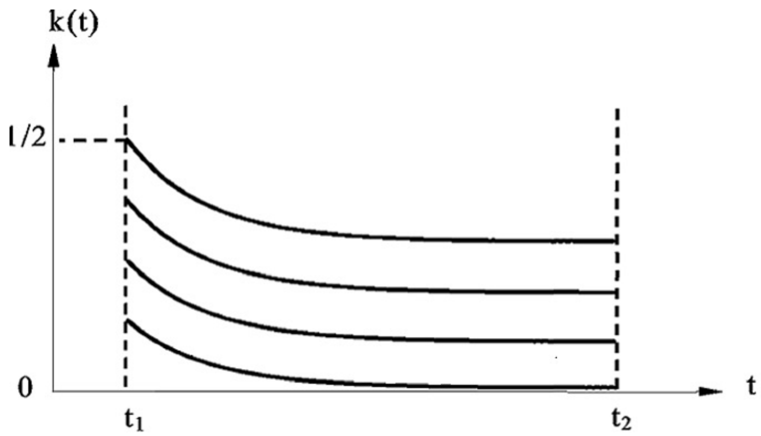

FIGURE 4.3. The Evolution of function $k(t)$ between t_1 and t_2.

which is a consequence of the Mean theorem And

$$h(\alpha) = \int_{-\alpha}^{\alpha} \frac{du}{\sqrt{(u^2-\alpha^2)(u^2-2+\alpha^2)}}$$

For ε infinitely small we then find that

$$h(\alpha) \simeq \int_{-\alpha+\varepsilon}^{\alpha+\varepsilon} \frac{du}{\sqrt{(u^2-\alpha^2)(u^2-2+\alpha^2)}}$$

For $0 \le \eta(\varepsilon,\alpha) \le \alpha$, we have

$$h(\alpha) = \frac{2(\alpha-\varepsilon)}{\sqrt{(\eta^2-\alpha^2)(\eta^2-2+\alpha^2)}} \sim \frac{2\alpha}{\sqrt{(\eta^2-\alpha^2)(\eta^2-2+\alpha^2)}}$$

By substitution in equation (4.16), we then obtain

$$\frac{d(\alpha^2)}{dt} \sim \frac{\sqrt{(\xi^2-\alpha^2)(\xi^2-2+\alpha^2)(\eta^2-\alpha^2)(\eta^2-2+\alpha^2)}}{t(1-\alpha^2)} \le \frac{\alpha^4}{(t-(2-\alpha))}$$

The function $k(t)$ given by Theorem (4.3) is represented by Figure 4.3.

From Figure d3 and formula (4.14), we get the following discussion

(i) Near $\alpha = 1$, corresponding to $k = 1/2$, the function $\alpha^2(t)$ is decreasing.

(ii) Near $\alpha = 0$, corresponding to $k = 0$, the function $\alpha^2(t)$ has a limit which tends to 0. In addition, the differential system (4.15) extends in a Lipchitzian way to zero, because of the unique solution, the solution $\alpha(t) = 0$, cannot be crossed by a solution coming from a point $\alpha(t_1) > 0$. As illustrated in Figure d3, the initial condition curve $k(t_1) = 1/2$, is decreasing. It gives a value in t_2 which delimits the interval in which it is necessary locate $b = u(t)$ so that there can exist a solution, and in the opposite case for any value b

in this interval and whatever the value a between -1 and 1, we will have a solution. The values a and b are then said to be admissible under these conditions. And as a result, we get the following proposition

Proposition 4.1. ***(i)*** *There is an appreciable interval I for the initial condition $w(t)$ which is suitable for the value a.*

(ii) *In all subinterval i of I, there is an initial condition $w(t_1)$ which gives a solution. So there are an infinitely large number of solutions.*

Proof. Let two trajectories γ_1 and γ_2 corresponding to two initial conditions k_1 and k_2. If $\gamma_1(t_1) \ll \gamma_2(t_1)$, we have $\forall t > t_1$, $\gamma_1(t) \ll \gamma_2(t)$. When the first trajectory makes a turn, the other gains an appreciable end of turn a_1 during the same time (according to the previous remark). The cumulation of these differences over an infinitely large number of laps between times t_1 and t_2 ($t_1 \ll t_2$).

By continuity with respect to the initial condition, it exists for any admissible couple (a,b) and in all subinterval $i \subset I$ an initial speed such as $k(t_1) \in i$ for which $u(t_1) = a$, because by varying $k(t_1)$ appreciably in I, the end of the trajectory in t_2 to the vertical of b passes an infinitely large number of times. So we can even say that there are an infinitely large number of solutions.

As the appreciable interval I has an infinitely large number of appreciable subintervals i_j, there exists an infinitely large number of solutions for the boundary problem (4.1). ■

4.6 Conclusion

In this work we rely on the Stroboscopy method of Non Standard Analysis, to describe the asymptotic behavior of solutions in a boundary value problem of PDE . This technique allowed us to describe the shape of the trajectories of the induced field without going through often complex calculations of asymptotic development very common in classical mathematics. This asymptotic behavior of the solutions was determined by the shadows of the integral curves of an equivalent differential equation. In the observability space, this equation admits an almost first integral $v^2 - 2\int_0^u F(u) = k(t)$, where k evolves slowly as a function of time t. The method of infintesimal stroboscopy described by J.L. Callot and T. Sari in [33] and [48](Chapters 4 and 10) gave us this evolution which ensures the existence of solutions for the problem (4.1) which have nothing to do with those of the reduced problem (4.2). Regarding the particular boundary problem

$$\begin{cases} \varepsilon^2\left(u'' + \dfrac{(n-1)}{t}u'\right) = u(u^2-1) \\ u(t_1) = a, \quad u(t_2) = b \end{cases}$$

where t_1 and t_2 are standard and $t_1 < t_2$, we have shown that

(i) There are boundary conditions for which there are an infinitely large number of solutions.

(ii) On each appreciable size interval, none of these solutions is infinitely close to a solution of the reduced problem $f(u) = 0$, given by $u = 0$, $u = 1$ and $u = -1$.

CHAPTER 5

N Wave and Periodic Wave Solutions for Burgers Equations

This chapter concerns the initial boundary value problem for the non linear dissipative Burgers equation.Our general purpose is to describe the asymptotic behavior of the solution in the cauchy problem with a small parametre ε for this equation and to discuss in particular the cases of the N wave shock and periodic wave shock. we show that the solution of cauchy problem of viscid equation approach the shock type solution for the cauchy problem of the inviscid equation for each case.The results are formulated in classical mathematics and proved with infinitesimal techniques of Non Standard Analysis.

The work constituting this chapter is the subject of an article published in (International Journal of Analysis and Applications), in collaboration with Z. Nouri and S. Bendaas [116]..

5.1 Introduction

Burgers equation is the scalar partial differential equation

(5.1) $$u_t + uu_x = \varepsilon u_{xx}$$

where $x \in X \subseteq \mathbb{R}$, $t \geqslant 0$, and $u : X \times \mathbb{R}^+ \to \mathbb{R}$. The parameter ε is typically referred to as the viscosity due to the connection between this equation and the stydy of fluid dynamics. When $\varepsilon > 0$ it is often reffered to the viscous Burgers equation , and when $\varepsilon = 0$, it is often reffered to the inviscid Burgers equation. Burgers equation was proposed as a model of turbulent fluid motion by J.M. Burgers in a series of sevral articles. It is one of the most important PDEs in the theory of non linear consevation laws. she combining both nonlinear propagation effects and diffusive effect. This equation is the approximation for the one-dimensional propagation of weak shock waves in a fluid. It can also be used in the description of the variation in vehicle density in highway traffic. Burgers introduced the equation to describe the behavior of shock waves, traffic flow and acoustic transmission. This equation plays a relevant role in many different areas of the mathematical physics, specially in Fluid Mechanics. Moreover the simplicity of its formulation, in contrast with the Navier-Stokes system, makes of the Burgers equation a suitable model equation to test different numerical algorithms and results of a varied nature [24], [25]. If the viscous term is null, the remaining equation is hyperbolic this is the inviscid Burgers equation

$$u_t + u u_x = 0 \tag{5.2}$$

If the viscous term is dropped from the Burgers equation, discontinuities may appaer in finite time, even if the initial condition is smooth they give rise to the phenomen of shock waves with important application in physics [75]. This properties make Burgers equation a proper model for testing numerical algorithms in flows where severe gradients or shocks are anticipated [137], [145], [152]. Recently, Kunjan and Twinkle [142] used mixture of new integral transform and Homotopy Perturbation Method to find the solution of Bugers equation arising in the longitudinal dispersion phenomenon in fluid flow through porous media. Discretization methods are well-knoun techniques for sollving Burgers equation. Ascher and Mclachlan established many methods as multisymplectic box sheme. Olayiwola et *al*. [149] also presented the modified variational iteration method for the numerical solution of generalized Burgers-Huxley equation. For the boundary value problem, Sinai [145] is intersted to the initial condition case: null on $\mathbb{R}_{-,}$and Brownian on $\mathbb{R}_+$. She, Aurell and Frich [143] with a numerical calculs particularly examine the initial conditions of Brownian fractionnair type. To study the initial value problem of Burgers equation, classical methods are based on search of solutions of the reduiced problem to deduce existence and asymptotic behavoiur of the solutions as ε tends to 0, the passasge of the limit is very complicated, but in general the limit exist and it's a solution for the reduced problem (when $\varepsilon = 0$). For ε small the solution $u(x,t)$ is approximated by this limit [75]. Other methods are based on a weak formulation of burgers equation seen as a conservation law satisfied on each of the computational domain called cell or finite volume. Stochastic particle method is so used for with different initial conditions. In particular we will take the same initial conditions that we took for the inviscid.

This paper completes recent works on the study of boundary value problems of Burgers equations for different initial conditions [24], [26]. In the presented work, our general purpose is to describe the asymptotic behavior of solutions in boundary value problems with a small parameter ε and to discuss in particular the cases of N wave and periodic wave shocks with new techniques infinitesimal of Non-Standard Analysis. We can conclude that the solution of the cauchy problem of inviscid equation in each case is infinitely close to the solution of he cauchy problem of viscid equation as ε is a parameter positif sufficiently small. We introduce the infintesimal techniques to give a simple formulation for the asymptotic behaviour. It is worth noting that our contribution is an elegant combination of infinitesimal techniques of Non-Standard Analysis and the Van Den Berg method [24], [25], [149].

Historically the subject non standard was developed by Robinson, Reeb, Lutz and Gose [92]. The Nonstandard perturbation theory of differential equations, which is today a well-established tool in asymptotic theory, has its roots in the seventies, when the Reebian school (see [92], [149]) introduced the use of Non-Standard Analysis into the field of perturbed differential equations. Our goal in this paper is to generalize these techniques on PDE.

The paper is organised as follows: Section 2 concerns the boundary value problems of inviscid Burgers equation, we start with the Fitting discontinuous shock then we describe the asymptotic behavoir of solutions for this problem in the N wave and periodic wave cases. Section 3 concerns the boundary value problems of viscid Burgers equation, it contains basic preliminaries results and deals with our main results about N wave and periodic wave shock cases and its proof, we present it in a non standard form.

5.2 Inviscid Burgers Equation

We will focus first on equation (5.2). Specifically, we will deal with the initial value problem

$$\begin{cases} u_t + u u_x = 0 & , \forall x \in \mathbb{R},\ t > 0 \\ u(\xi, 0) = f(\xi) & , t = 0 \end{cases} \tag{5.3}$$

As it as has been suggested previously, although (5.3) seems to be a very innocent problem a priori it hides many unexpected phenomena.This problem does not admit the regular solutions but some weak solutions with certain regularity exist.The Burgers equation on the whole line is known to possess traveling waves solutions. Using the characteristic method, the solution of the problem (5.3) may be given in a parametric form

$$\begin{cases} u = f(\xi) \\ x = \xi + f(\xi)t \end{cases} \tag{5.4}$$

and shocks must be fitted in such that

$$U = \frac{1}{2}(u_1 + u_2) = \frac{1}{2}(f(\xi_1) + f(\xi_2)$$

where $f : R \to R$. is a standard continuous function. ξ_1 and ξ_2 are the values of ξ on the two sides of the shock [28].

According to (5.4), the solution at time t is obtained from the initial profile $u = f(\xi)$ by translating each point a distance $f(\xi)t$ to the right. The shock cuts out the part corresponding to $\xi_2 \geq \xi \geq \xi_1$. If the discontinuity line it is a straight line chord betewen the points $\xi = \xi_1$ and $\xi = \xi_2$ on the curve $f(\xi)$. Moreover since areas are preserved under the mapping, the equal area property still holds. The chord on the f curve cuts off lobes of equal area. The shock determination can then be describe entirely on the fixe $f(\xi)$ curve by drawing all the chords with the equal area propperty can be written analytically as

$$\frac{1}{2}\{(f(\xi_1) + f(\xi_2)\}(\xi_1 - \xi_2) = \int_{\xi_2}^{\xi_1} f(\xi)d\xi \tag{5.5}$$

This is the differential equation for the shock line chord wich verifies the entropic condition such as [28]. Since the left hand side is the area under the chord and the right hand side is the area under the f curve. If the shock is at $x = s(t)$ at time t, we also have

$$s(t) = \xi_1 + f(\xi_1)t \tag{5.6}$$

$$s(t) = \xi_2 + f(\xi_2)t \tag{5.7}$$

From (5.6) and (5.7), we have

$$t = \frac{\xi_1 - \xi_2}{f(\xi_1) - f(\xi_2)} \tag{5.8}$$

5.2.1 Single Hump

To describe the solutions of the problem (5.3), we assume that the initial condition f verify the following assumptions

(H_1) f is equal to a constant u_0 outside the range $0 < \xi < L$.

(H_2) $f(\xi) > u_0$ in the range.

The theorem bellow gives the assymptotic behavior to the solution behind the shock and at the shock.

Theorem 5.1. *Suppose that* (H_1) *and* (H_2) *are satisfied, the solution of the problem* (5.3) *is given by* (5.4) *with* $0 < \xi < \xi_2$ *and the asymptotic form is*

$$u \sim \frac{x}{t}, \text{ for } u_0 t < x < u_0 t + \sqrt{2At}$$

Proof. We consider the problem (5.3) and we suppose that (H_1) and (H_2) are satisfied. Equation (5.5) may be written as

$$\frac{1}{2}\{(f(\xi_1) + f(\xi_2) - 2u_0\}(\xi_1 - \xi_2) = \int_{\xi_1}^{L} (f(\xi) - u_0)d\xi$$

As time goes on ξ_1 increase and eventually exceed L. At this stage $f(\xi_1) = u_0$ and the shock is moving into the constant region $u = u_0$. The function $\xi_1(t)$ can then be eliminated for we have

$$\frac{1}{2}(f(\xi_2) - u_0)(\xi_1 - \xi_2) = \int_{\xi_2}^{L} (f(\xi) - u_0)d\xi \ , \ t = \frac{\xi_1 - \xi_2}{f(\xi_2) - u_0}$$

There for

$$\frac{1}{2}(f(\xi_2) - u_0)^2 t = \int_{\xi_2}^{L} (f(\xi) - u_0)d\xi \tag{5.9}$$

At this stage the shock position and the value of u just behind the shock are given by

$$\begin{cases} u = f(\xi_2) \\ s(t) = \xi_2 + f(\xi_2)t \end{cases} \tag{5.10}$$

where ξ_2 satisfie (5.9). As t is infinitely large we have ξ_2 infinitesimal and $f(\xi_2)$ approach u_0, hence the equation for $\xi_2(t)$ takes the limiting form

$$\frac{1}{2}(f(\xi_2) - u_0)^2 t \sim A$$

where

$$A = \int_0^L (f(\xi) - u_0) d\xi$$

is the area of the hump above the undisturbed value u_0.We have ξ_2 infinitesimal and

$$f(\xi_2) \sim u_0 + \sqrt{2A/t}$$

Therefore the asymptotic formul as for $s(t)$ and u in (5.10) are

$$s(t) \sim u_0 t + \sqrt{2At}$$

$$u - u_0 \sim \sqrt{2A/t}$$

at the shock. The shock curve is asymptotically parabolic. The solution behind the shock is given by (5.4) with $0 < \xi < \xi_2$. Since ξ_2 is small enouhg as t is small enouhg, all the relevant values of ξ also small enouhg and the asymptotic form is

$$u \sim \frac{x}{t}, \text{ for } u_0 t < x < u_0 t + \sqrt{2At}$$

The asymptotic solution and the corresponding (x,t) diagram are shown in Figure 5.1.

■

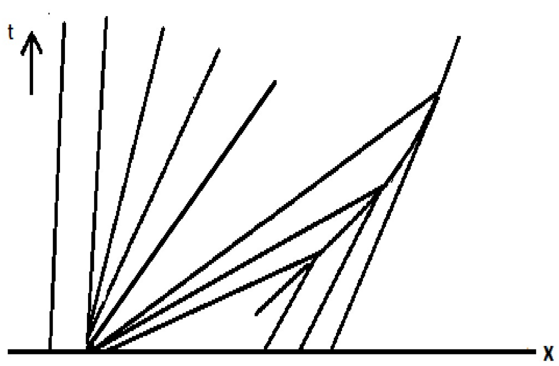

FIGURE 5.1. The Asymptotique triangular wave.

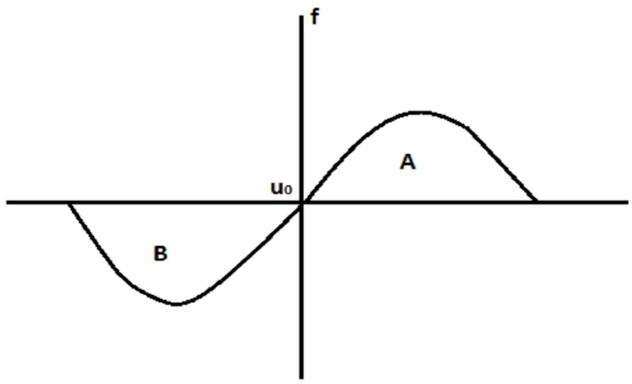

FIGURE 5.2. The profile of the initial condition in the N wave case.

5.2.2 N Wave

Other problem can be worked out in similar way, one important case is when $f(\xi)$ has a positive and a negative phase about an unidisturbed value u_0 as in Figure 5.2.

There are now two shocks, corresponding to the two compression phases at the front and at the back where $f'(\xi) < 0$. The families of chords for each are shown in the figure 5.2. As t is infinitely large, the pair (ξ_2, ξ_1) for the front shock approach $(0, \infty)$,where as for the rear shock (ξ_2, ξ_1) approach $(-\infty, 0)$. Asymptotically the front shock is

$$s \simeq u_0 t + \sqrt{2At}$$

and the jump of u is

$$u - u_0 \sim \sqrt{2A/t}$$

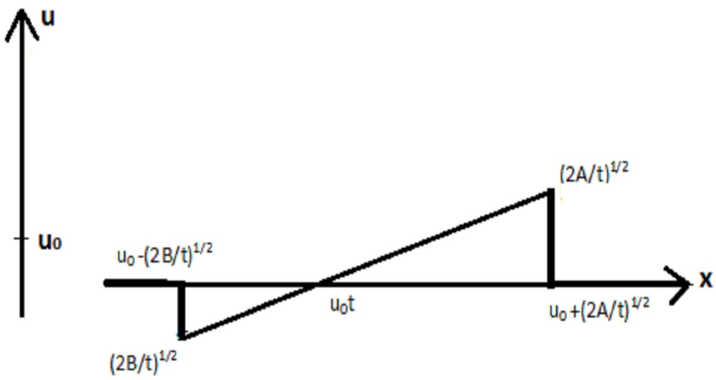

FIGURE 5.3. Shock construction for N wave.

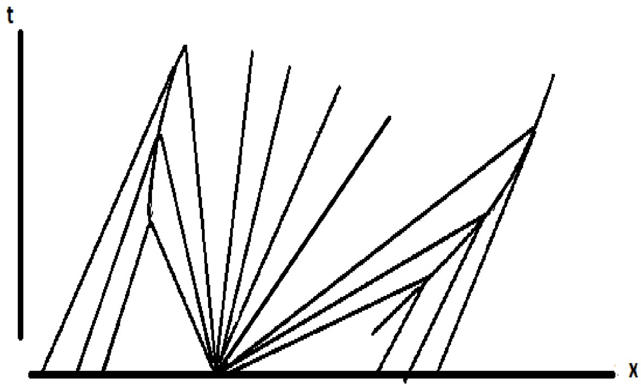

FIGURE 5.4. The asymptotic N wave.

where A is the area of the f curve above $u = u_0$.The rear shock has

$$x \sim u_0 t - \sqrt{2Bt} \tag{5.11}$$

$$u - u_0 \sim -\sqrt{2B/t}$$

where B is the area below $u = u_0$. The solution between the shocks is again asymptotically

$$u \sim \frac{x}{t},\ u_0 t - \sqrt{2Bt} < x < u_0 t + \sqrt{2At}$$

The asymptotic form and the (x,t) diagramm are shown in Figure 5.3 and Figure 5.4.

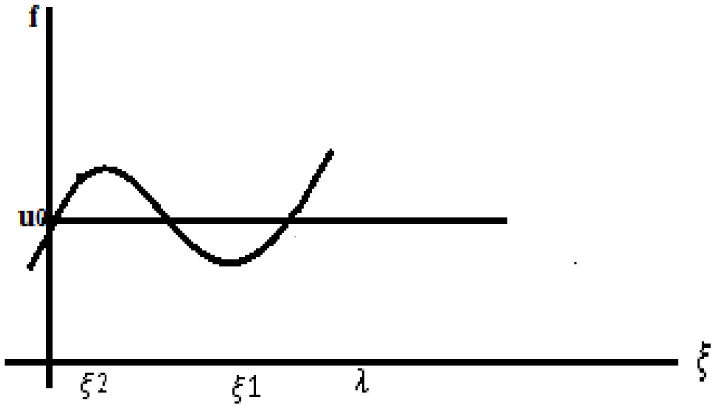

FIGURE 5.5. Shock construction for a periodic wave.

5.2.3 Periodic Wave

Another intersting problem is that of an initial distribution

$$f(\xi) = u_0 + a \sin \frac{2\pi\xi}{\lambda}$$

In this case, the shock equations (5.5) simplify considerably for all times t. Consider one period $0 < \xi < \lambda$ as in Figure 5.5. Relations (5.5) becomes

$$(\xi_1 - \xi_2) \sin \frac{\pi}{\lambda}(\xi_1 + \xi_2) \cos \frac{\pi}{\lambda}(\xi_1 - \xi_2) = \frac{\lambda}{\pi} \sin \frac{\pi}{\lambda}(\xi_1 - \xi_2) \sin \frac{\pi}{\lambda}(\xi_1 + \xi_2)$$

and the relevant choice is the trivial one

$$\sin \frac{\pi}{\lambda}(\xi_1 + \xi_2) = 0 \text{ that is } \xi_1 + \xi_2 = \lambda$$

From the difference and sum of (5.6) and (5.7) we have

$$t = \frac{\xi_1 - \xi_2}{2a \sin \frac{\pi}{\lambda}(\xi_1 - \xi_2)}$$

$$s = u_0 t + \frac{\lambda}{2}$$

Respectively, the discontinuity in u at the shock is

$$\begin{aligned} u_2 - u_1 &= a \sin \frac{2\pi\xi_1}{\lambda} - a \sin \frac{2\pi\xi_2}{\lambda} \\ &= 2a \sin \frac{\pi}{\lambda}(\xi_1 - \xi_2) \end{aligned}$$

If we introduce

$$\xi_1 - \xi_2 = \frac{\lambda\theta}{\pi}, \; \xi_1 + \xi_2 = \lambda$$

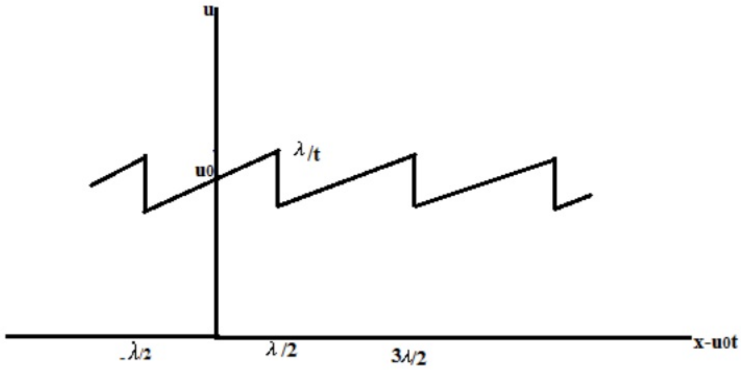

FIGURE 5.6. Asymptotic form of a periodic wave.

we have

$$t = \frac{\lambda}{2\pi a} \cdot \frac{\theta}{\sin\theta}$$

$$s = u_0 t + \frac{\lambda}{2}$$

$$\frac{u_2 - u_1}{u_0} = \frac{2a}{u_0} \sin\theta$$

The shock has constant velocity u_0 and this result could have been deduced in advance from the symmetry of the problem. The shock starts with zero strength corresponding to $\theta = 0$ at time $t = \lambda/2\pi a$. It reaches a maximum strength of $2a/u_0$ for $\theta = \pi/2$, $t = \lambda/4a$ and decays ultimately with θ approach π, when t is infinitly large

$$\frac{u_2 - u_1}{u_0} \sim \frac{\lambda}{u_0 t}$$

It is intersting that the final decay formula does not even depend explicitly on the amplitude a, however the condition for its application is $t \gg \lambda/a$. For any periodic sinusoidal $f(\xi)$ or not $\xi_1 - \xi_2 \to \lambda$ as t infinitely large, thence from (5.8)

$$\frac{u_2 - u_1}{u_0} = \frac{f(\xi_2) - f(\xi_1)}{u_0} \sim \frac{\lambda}{u_0 t}$$

Between successive shocks, the solution for u is linear in x with slope $\frac{1}{t}$ as before, and the asymptotic form of the entire profile is the Sawtooth shown in Figure 5.6.

5.3 Viscid Burgers Equation

In this section we shall present and prove our main results, we discuss the N wave and periodic wave cases in the boundary value problem of viscid Burgers equation

$$\begin{cases} u_t + uu_x = \varepsilon u_{xx} & , \forall x \in \mathbb{R},\ t > 0 \\ u(\xi, 0) = f(\xi) & , t = 0 \end{cases} \tag{5.12}$$

Before going further in this cases we need the following proposition and lemma

5.3.1 The Cole-Hopf Transformation

Cole and Hopf noted the remarkable result [66] that the viscid Burgers equation (5.1) may be reduced to the linear Heat equation

$$\varphi_t = \varepsilon \varphi_{xx} \tag{5.13}$$

by the non linear transformation

$$u = -2\varepsilon[\log \varphi]_x \tag{5.14}$$

It is again convenient to do the trasformation in two steps. Firstly are introduced

$$u = \psi_x$$

so that (5.1) may be integrated to

$$\psi_t + \frac{1}{2}\psi_x^2 = \varepsilon \psi_{xx}$$

then we introduce

$$\psi = -2\varepsilon[\log \varphi]$$

to obtain (5.13).

The non linear transformation just eliminates the nonlinear term. The general solution of the Heat equation (5.13) is well khown and can be handled by a variety of methods. The basic problem considered in section 2 is the initial value problem

$$u = f(\xi), \text{ at } t = 0$$

this is transformed by (5.14) to the initial value problem

$$\varphi = \phi(x) = \exp\left\{-\frac{1}{2\varepsilon}\int_0^x f(\eta)d\eta\right\},\ t = 0$$

For the Heat equation, the solution for φ is

$$\varphi = \frac{1}{\sqrt{4\pi\varepsilon t}} \int_{-\infty}^{+\infty} \phi(\eta) \exp\left\{ -\frac{(x-\eta)^2}{4\varepsilon t} \right\} d\eta$$

Through (5.14), the solution for u is

$$u(x,t) = \frac{\int_{-\infty}^{+\infty} \frac{x-\eta}{t} \exp\left(\frac{-G}{2\varepsilon}\right) d\eta}{\int_{-\infty}^{+\infty} \exp\left(\frac{-G}{2\varepsilon}\right) d\eta} \tag{5.15}$$

where

$$G(\eta, x, t) = \int_0^{\eta} f(\nu) d\nu + \frac{(x-\eta)^2}{2t}$$

5.3.2 The Behavior of Solutions as ε Small Enough

The behavior of the exact solution (5.15) is now considered as ε is small enough. For x, t and $f(x)$ are held fixed as ε is small enough, the dominant contributions to the integrals in (5.15) come from the neighborhood of the stationnary points of G. A stationary point is where

$$\frac{\partial G}{\partial \eta} = f(\eta) - \frac{x-\eta}{t} = 0$$

Let $\eta = \xi(x,t)$ be such a point that is $\xi(x,t)$ is difined as a solution of

$$f(\xi) - \frac{x-\xi}{t} = 0 \tag{5.16}$$

The contribution from the neighborhood of a stationary point $\eta = \xi$ in an integral is given with the lemma 5.2.

Lemma 5.1 (The Van. Den .Berg lemma 12)**.** *let G be a standard function definied and increasing on $[0, +\infty[$ such that $G(v) = av^r\ (1+\delta)$ for $v \simeq 0$ and $G(v) > m(v)^q$. Let φ be an intern function definied on $]0, +\infty[$ such that $\varphi(v) = bv^s(1+\delta)$ for $v \approx 0$ and such that $\forall d > 0$, $\exists$ standard k and c such that $|\varphi(v)| < k\exp(\cosh(v))$ for $v > d$. Then*

$$\int_0^{\infty} \varphi(v) \exp\left(-\frac{G(v)}{2\varepsilon}\right) dv = \frac{b\Gamma\left(\frac{(s+1)}{r}\right)}{ra^{\frac{(s+1)}{r}}} \frac{1}{\left(\frac{1}{2\varepsilon}\right)^{\frac{(s+1)}{r}}}$$

where a and r are positifs standard, m and q are the both positifs. δ is a positif real small enough. b and s are standard, $b \neq 0$ and $s > -1$.

Lemma 5.2 (The Nonstandard formula of the method of steepest descents)**.** *Let ε be a positif real small enough and let φ and G be two standard functions such that G is a C^2 class function verifie the lemma 5.1, and admits on the ξ point an unique absolute minimum $\left(G'(\xi)=0 \text{ et } G''(\xi)>0\right)$. $\varphi(\xi)\neq 0$, it is S- continuous on ξ and satisfie the conditions of the lemma 5.2 in the two sens. Then*

$$\int_{-\infty}^{+\infty}\varphi(\eta).\exp\left(-\frac{G}{2\varepsilon}\right)d\eta=\varphi(\xi)\frac{\sqrt{4\pi\varepsilon}}{\sqrt{G''(\xi)}}.\exp\left(-\frac{G}{2\varepsilon}\right)(1+\delta)$$

δ is a positif real small enouhg.

Proof. Suppose first that there is only one stationary point $\xi(x,t)$ wich satisfies (5.16) then

$$\int_{-\infty}^{+\infty}\frac{x-\eta}{t}\exp\left(-\frac{G}{2\varepsilon}\right)d\eta=\frac{x-\eta}{t}\frac{\sqrt{4\pi\varepsilon}}{\sqrt{G''(\xi)}}\exp\left(-\frac{G}{2\varepsilon}\right)(1+\delta)$$

$$\int_{-\infty}^{+\infty}\exp\left(-\frac{G}{2\varepsilon}\right)d\eta=\frac{\sqrt{4\pi\varepsilon}}{\sqrt{G''(\xi)}}\exp\left(-\frac{G}{2\varepsilon}\right)(1+\delta)$$

and in (5.16) we have

$$u\simeq\frac{x-\xi}{t} \tag{5.17}$$

where $\xi(x,t)$ is difinied by (5.16).This asymptotic solution may be rewriten as

$$\begin{cases} u=f(\xi) \\ x=\xi+f(\xi)t \end{cases}$$

It is exactly the solution of (5.3) witch was discussed in section 2. The stationary point $\xi(x,t)$ becomes the characteristic variable. ■

5.3.3 The Main Results

5.3.3.1 N Wave

Another example, we consider is more easly derived by choosing appropriate solutions for φ to satisfy the Heat equation and then substituting in (5.4) to obtain u as a rough qualitative guide to the appropriate choice. The profile of u will be some thing like φ_x. To obtain an N wave of u. We choose the source solution of the Heat equation for φ

$$\varphi = 1 + \sqrt{\frac{a}{t}}.\exp\left(\frac{-x^2}{4\varepsilon t}\right) \tag{5.18}$$

Since φ has a δ function behavior as t is infinitesimal, this is a little hard to interpret as an initial value problem on u. However, for any $t > 0$,it has the form shown in Figure 5.2 with a positive and a negative phase and we may take the profile at any $t = t_0$ to be the initial profile. It should typical of all N wave solutions.

Theorem 5.2. *(i). Assume that the initial data f has a profile shown in Figure 5.2, the problem* (5.3) *admit an unique solution for $t > 0$ given by*

$$u \sim \begin{cases} \dfrac{x}{t}, & -\sqrt{2At} < x < \sqrt{2At} \\ 0, & |x| > \sqrt{2At} \end{cases}$$

where $A = \int_{-\infty}^{+\infty}(f(x) - u_0)dx$.

(ii). Such solution present N wave chocks, and for ε small enough, this solution is infinitely close to the solution of the inviscid problem (5.4) *given in section* 2.

Proof. To obtain an N wave for u, we choose the source solution of the Heat equation for φ given by (5.17). Then the corresponding solution for u is

$$u = -2\varepsilon\frac{\varphi_x}{\varphi} = \frac{x}{t}.\frac{\sqrt{\frac{a}{t}}.\exp\left(-\frac{x^2}{4\varepsilon t}\right)}{1 + \sqrt{\frac{a}{t}}.\exp\left(-\frac{x^2}{4\varepsilon t}\right)} \tag{5.19}$$

Since φ has a δ function behavior as t is infinitely small, this is a little hard to interpret as an initial value problem on u.

However for any $t > 0$ it has the form shown in Figure 5.6 with a positive and negative phase and we may take the profile at any $t = t_0$ to be the initial profile. It should be typical of all N Wave solution. The area under the positive phase of the profile is

$$\int_0^{+\infty} u\,dx = -2\varepsilon[\log\varphi]_0^{\infty} = 2\varepsilon\log[1 + \sqrt{\frac{a}{t}}]$$

The positive phase is infinitely small when t is infinitely large. If the value of (5.18) at the initial time t_0 is denoted by A we may introduce a Reynolds number

$$R_0 = \frac{A}{2\varepsilon} = \log(1+\sqrt{\frac{a}{t_0}})$$

but as time goes on the effective, Reynolds number will be

$$R(t) = \frac{1}{2\varepsilon}\int_0^{+\infty} u dx = \log(1+\sqrt{\frac{a}{t}}) \tag{5.20}$$

and this is infinitely small as t is infinitely large.

If $R_0 \gg 1$, We may expect the "Inviscid Theory" of (5.3) and (5.4) to be a good approximation for some time but as t is infinitely large, $R(t)$ is eventually become dominant. In terms of R_0 and t_0, $a = t_0(\exp(R_0)-1)$. Hence (5.18) may be written

$$u = \frac{x}{t}.\left\{1+\sqrt{\frac{t}{t_0}}\frac{\exp(\frac{x^2}{4\varepsilon t})}{\exp(R_0)-1}\right\}^{-1}$$

and For $R_0 \gg 1$ (corresponding to $t_0 \ll 0$),we have

$$\exp(R_0)-1 \sim \exp(R_0)$$

and

$$\frac{\exp(\frac{x^2}{4\varepsilon t})}{\exp(R_0)-1} \sim \exp(\frac{x^2}{4\varepsilon t} - R_0) \sim \exp R_0(\frac{x^2}{2At}-1).$$

Or u in (5.20) may be approximated by

$$u = \frac{x}{t}.\left\{1+\sqrt{\frac{t}{t_0}\exp[R_0(\frac{x^2}{2At}-1)]}\right\}^{-1} \tag{5.21}$$

For x and t. Now for fixed t limited and R_0 infinitely large we have

$$u \sim \begin{cases} \frac{x}{t}, & \text{if } \frac{x^2}{2At}-1<0,\ -\sqrt{2At}<x<\sqrt{2At} \\ 0, & \text{if } \frac{x^2}{2At}-1<0,\ |x|>\sqrt{2At} \end{cases}$$

this is exactly the inviscid solution.

However, for any fixed a and ε we see directely from (5.19) [and it may be verified also from (5.21)] that

$$u \sim \frac{x}{t}.\sqrt{\frac{a}{t}}\exp(-\frac{x^2}{4\varepsilon t})$$

as t is infinitely large. This is the dipole solution of the Heat equation. The diffusion dominates the nonlinear term in the final decay. It should be remembered though,that this final period of decay is for extremely large times; the Inviscid theory is adequate for most of the interesting range. ■

5.3.3.2 Periodic Wave

A periodic solution may be obtained by taking for φ a distribution of Heat sources spaced a distance λ apart. Then

$$\varphi = \frac{1}{\sqrt{4\pi\varepsilon t}}\sum_{n=-\infty}^{\infty}\exp\left\{-\frac{(x-n\lambda)^2}{4\varepsilon t}\right\}$$

Theorem 5.3. *(i) Assume that the initial data f has a profile shown in Figure 5.5. When $\lambda^2/4\varepsilon t \gg 1$, the problem (5.12) admit an unique solution for $t>0$ given by*

$$u \sim \frac{x-m\lambda}{t},\ (m-1/2)\lambda < x < (m+1/2)\lambda$$

(ii) Such solution is the periodic wave chock, and for ε small enough, this solution is infinitely close to the solution of the inviscid problem (5.3) given in section 2.

Proof. To obtain a periodic wave for u, we choose for φ a distribution of Heat sources spaced a distance λ given by (5.18).

Then the corresponding solution for u is

$$u = -2\varepsilon\frac{\varphi_x}{\varphi} = \frac{\sum_{n=-\infty}^{\infty}\left(\frac{x-n\lambda}{t}\right)\exp\left\{-\frac{(x-n\lambda)^2}{4\varepsilon t}\right\}}{\sum_{n=-\infty}^{\infty}\exp\left\{-\frac{(x-n\lambda)^2}{4\varepsilon t}\right\}}$$

For $\lambda^2/4\varepsilon t \gg, 1$ this implies that $\sqrt{\varepsilon t} \ll \lambda/2$, and $\sum_{n=-\infty}^{\infty} = 2\sum_{n=0}^{\infty}$, then for $n=m$ we have

$$\frac{(x-n\lambda)^2}{4\varepsilon t} = \frac{(x-m\lambda)^2}{\varepsilon t} \ll 1$$

wich gives $|x - m\lambda| \ll \sqrt{\varepsilon t}$ and the exponential with the minimum value of $(x - n\lambda)^2/4\varepsilon t$ will dominate over all the others.

Therefore the term witch will dominate for

$$(m - 1/2)\lambda < x < (m + 1/2)\lambda$$

and (5.19) is approximately

$$u \sim \frac{x - m\lambda}{t}, \text{ for } (m - 1/2)\lambda < x < (m + 1/2)\lambda$$

This is a sawtooth wave with a periodic set of shocks a distance λ apart, and u jumps from $-\lambda/2t$ to $\lambda/2t$ at each shock. The result agrees with he inviscid solution given by (5.11). ■

Corollary 5.1. *If* $\dfrac{\lambda^2}{4\varepsilon t} \ll 1$, *the initial data can be expanded in a Fourier series as*

$$\varphi = \frac{1}{\lambda}\left\{1 + 2\sum_{n=1}^{\infty} \exp\left(-\frac{4\pi^2 n^2}{\lambda^2}\varepsilon t\right)\cos\frac{2\pi n x}{\lambda}\right\}$$

and for $t > 0$, *we have*

$$u \sim \frac{8\pi\varepsilon}{\lambda}\exp\left\{-\frac{4\pi^2\varepsilon t}{\lambda^2}\right\}\sin\frac{2\pi x}{\lambda}$$

this is a solution of the Heat equation.

Proof. To study the Final decay $(\lambda^2/4\varepsilon t) \ll 1$, we may use an altrnative form of the solution. The expression (5.18) is periodic in x, and in the interval $\dfrac{-\lambda}{2} < x < \dfrac{\lambda}{2}$, $\varphi \to \delta(x)$, as t is infinitly small. The initial condition can be expanded in a Fourier series as

$$\phi(x) = \frac{1}{\lambda}\left\{1 + 2\sum_{n=1}^{\infty}\cos\frac{2\pi n x}{\lambda}\right\}$$

and the corresponding solution of the Heat equation for φ is

$$\varphi = \frac{1}{\lambda}\left\{1 + 2\sum_{n=1}^{\infty} \exp\left(-\frac{4\pi^2 n^2}{\lambda^2}\varepsilon t\right)\cos\frac{2\pi n x}{\lambda}\right\}$$

It may be verified directly that this is the Fourier series of .(3.14). In this form

$$u = -2\varepsilon\frac{\varphi_x}{\varphi} = \frac{\frac{8\pi\varepsilon}{\lambda}\sum_{n=1}^{\infty} n\exp\left(-\frac{4\pi^2 n^2}{\lambda^2}\varepsilon t\right)\sin\frac{2\pi n x}{\lambda}}{1+2\sum_{n=1}^{\infty}\exp\left(-\frac{4\pi^2 n^2}{\lambda^2}\varepsilon t\right)\cos\frac{2\pi n x}{\lambda}}$$

when $\frac{\varepsilon t}{\lambda^2} \gg 1$,the term with $n = 1$ dominate the series and we have

$$u \sim \frac{8\pi\varepsilon}{\lambda}\exp\left\{-\frac{4\pi^2\varepsilon t}{\lambda^2}\right\}\sin\frac{2\pi x}{\lambda}$$

this is a solution of

$$u_t = \varepsilon u_{xx}$$

and the diffusion dominate in the ultimate decay. ■

5.4 Conclusion

This paper completes recent works on the study of boundary value problems of Burgers equations for different initial conditions [24], [26]. Our general purpose is to describe the asymptotic behavior of solutions in boundary value problem with a small parameter ε and to discuss in particular the N wave shocks and Periodic wave shocks cases. The originality of this work consists in introducing new infinitesimal techniques of Non-Standard Analysis. We can conclude that the solution of the cauchy problem of inviscid equation in each case is infinitely close to the solution of he cauchy problem of viscid equation as ε is a parameter positif sufficiently small. We introduce the infintesimal techniques to give a simple formulation for the asymptotic behavior. It is worth noting that our contribution is an elegant combination of infinitesimal techniques of non standard analysis and the Van Den Berg method [24], [25] and [149].

APPENDIX A

CONCLUSION AND PERSPECTIVES

This study focuses on the analysis and mathematical modeling of models frequently used in various natural sciences, particularly in biology, ecology and medicine.

- **In the first part of this thesis, we presented a major application of reaction diffusion systems, namely the smoothing and restoration of images. The purpose of image restoration is to estimate the original image from the degraded data. Applications range from medical imaging, astronomical imaging, to forensic science, etc. In recent years, this field has attracted the attention of many researchers in computer vision. This is mainly due to the mathematical formulation framing any PDEs-based approach that can give a good justification and explanation of the results obtained through these traditional and heuristic methods in image processing.**

- **In the second part of this work, we have used a method of Non Standard Analysis, to describe the asymptotic behavior of solutions in a boundary value problem of PDE This technique allowed us to describe the shape of the trajectories of the induced field without going through often complex calculations of asymptotic development very common in classical mathematics. As application, we have introduced the study of the Burg-**

ers equation by means of non-standard analysis applied to the study of some problems in biology and medicine. If the complexity and the importance of these problems related to the unavailability via non-standard analysis.

- **Many important results have been obtained with additional assumptions that can be applied to several models in biology, ecology, physics and others as appropriate.**

- **We have developed original methods to overcome certain difficulties, and despite the complexity of the models studied, we have managed to obtain several important results, original and solve very difficult news problems.**

We can also address the following interesting questions in addition to this work :

(i) The mathematical analysis of anisotropic system, which consists in adding diffusion coefficients to the studied system depending on (t,x) or more generally depending on $(t,x,u,\nabla u)$.

(ii) Studying the asymptotic behavior of solutions.

(iii) It would be helpful to implement some numerical simulations in order to gain a better understanding of the solutions at large times.

(iv) It is important to study the same models under different conditions.

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العنوان : دراسة رياضية لبعض النماذج المطبقة في علم الأحياء والطب.

ملخص : العمل الذي يشكل هذه الأطروحة هو مساهمة في النمذجة والتحليل الرياضي لبعض النماذج المطبقة في علم الأحياء والطب. نحن مهتمون بدراسة وجود الحلول لبعض النماذج باستخدام تقنيات تعتمد على التحليل الوظيفي والتحليل غير القياسي. يتكون هذا العمل من خمسة فصول مستقلة، مسبوقة بمقدمة عامة تسلط الضوء على فن الموضوع والمشكلات التي تم تناولها.

كلمات مفتاحية : أنظمة تفاعل-انتشار، معادلة برجرس، نظرية النقطة الثابتة، نمذجة ظواهر الانتشار، تحليل غير قياسي.

Titre : ÉTUDE MATHÉMATIQUE DE QUELQUES MODÈLES APPLIQUÉS EN BIOLOGIE ET EN MÉDECINE.

Résumé :

Le travail constituant cette thèse est une contribution à la modélisation mathématique et à l'analyse de certains modèles appliqués en biologie et en médecine. Nous nous intéressons à l'étude de l'existence de solutions pour certains modèles en utilisant des techniques basées sur l'analyse fonctionnelle et l'analyse non standard. Ce travail se compose de cinq chapitres indépendants, précédés d'une introduction générale qui met en évidence l'art du sujet et les problèmes abordés.

Mots-clés : systèmes de réaction-diffusion, équation de Burgers, théorème de point fixe, modélisation des phénomènes de diffusion, analyse non standard.

Title : MATHEMATICAL STUDY OF SOME MODELS APPLIED IN BIOLOGY AND MEDICINE.

Abstract : The work constituting this thesis is a contribution to the mathematical modeling and analysis of certain models applied in biology and medicine. We are interested in studying the existence of solutions for certain models using techniques based on functional analysis and non-standard analysis. This work is then composed of five independent chapters, preceded by a general introduction which highlights the art of the subject and the problems addressed.

Keywords : reaction diffusion systems, Burgers equation, fixed point theorem, modeling of diffusion phenomena, non-standard analysis.