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Par

CHEKAKTA Issam

THÈME

Contribution à l'observation d'état et au filtrage non linéaire pour les modèles Takagi-Sugeno

Soutenue le 26/06/2023 devant le Jury:

BADOUD Abdessalam	Professeur	Univ. Ferhat Abbas Sétif 1	Président
BELKHIAT Djamel Eddine Chouaib	Professeur	Univ. Ferhat Abbas Sétif 1	Directeur de thèse
GUELTON Kevin	Professeur	Univ. Reims Champagne Ardenne	Co-Directeur
LEITE Valter J.S.	Professeur	CEFET-MG, Divinopolis, Brésil	Examinateur
SARI Bilal	M.C.A.	Univ. Ferhat Abbas Sétif 1	Examinateur
MOTCHON Koffi.M.D	M.C.A.	Univ. Reims Champagne Ardenne	Invité

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By

Issam Chekakta

Contribution to the State Observation and Nonlinear Filtering of Takagi-Sugeno Models

Defended on 26/06/2023 in the presence of the committee:

Abdessalam Badoud	**Professor**	**Univ. Ferhat Abbas Sétif 1**	**President**
Djamel E.C. Belkhiat	**Professor**	**Univ. Ferhat Abbas Sétif 1**	**Thesis Supervisor**
Kevin Guelton	**Professor**	**Univ. Reims (URCA), France**	**Co-Supervisor**
Valter J.S. Leite	**Professor**	**CEFET-MG, Divinopolis, Brazil**	**Examiner**
Bilal Sari	**M.C.A.**	**Univ. Ferhat Abbas Sétif 1**	**Examiner**
Koffi.M.D Motchon	**M.C.A.**	**Univ. Reims (URCA), France**	**Invited**

Contribution to the state observation and nonlinear filtering of Takagi-Sugeno models

ISSAM CHEKAKTA

A THESIS SUBMITTED FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

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To my parents, my sisters and our little Touta.
To my beautiful unexpectation, my brilliant Ibtihel.
To me - a testament to my perseverance, determination, and unwavering belief in my abilities.

“If I have seen further it is by standing on the shoulders of Giants.”

— Isaac Newton

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IN the labyrinth of academia, where challenges lurk and uncertainties abound, I emerged as a traveler on a quest for knowledge. As I stand on the precipice of completing my journey as a PhD. student, I cannot help but reflect upon the multitude of individuals who have guided and supported me along the way. Their unwavering belief in my abilities has been the fuel that has propelled me forward.

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As I take my final steps towards defending my PhD thesis, I offer my profound gratitude to each and every person who has played a role in shaping my academic and personal growth.

– ***Arrivederci a tutti voi***

Abstract

The objective of this thesis is to develop methods for designing observers and H_∞ filters for continuous-time switched nonlinear systems with Takagi–Sugeno (T-S) fuzzy models representing the nonlinear system in each mode. These estimation techniques are investigated under state-dependent switching laws (dwell-time-independent), bounded disturbances, and asynchronous switching, where the observer's or filter's active switching mode does not necessarily match that of the switched T-S system. First, switched T-S observers are proposed along with Lipschitz constraints, which allow state estimations when the systems' premise variables are not necessarily measurable. However, due to the conservatism that might arise from the Lipschitz constraint, an interesting approach to dealing with unmeasured premise variables is investigated instead in the second part of this thesis in the context of H_∞ filtering. Thus, the switched T-S system is modelled as a switched T-S model with nonlinear consequent parts, in which the unmeasured nonlinearities are kept in the nonlinear consequent parts. Thanks to the incremental quadratic constraint employed to deal with unmeasured nonlinearities, the obtained conditions are less conservative compared to the Lipschitz constraint. Furthermore, acknowledging that T-S models only represent nonlinear ones on subsets of their state space, an optimization procedure to estimate the filtering error's domain of attraction is developed. To deal with the asynchronous switching modes, multiple Lyapunov function candidates are considered, along with a H_∞ criterion to minimize the transfer between the input/output disturbances and state estimation/filtering errors. Moreover, the proposed design conditions are formulated in terms of Linear Matrix Inequalities (LMI). Several illustrative examples are used throughout the manuscript to validate the proposed observers and filters, as well as the obtained results.

Résumé

L'objectif de cette thèse est de développer des méthodes de conception d'observateurs et de filtres H_∞ pour des systèmes non linéaires à commutations à temps continu avec des modèles flous Takagi-Sugeno (T-S) représentant le système non linéaire dans chaque mode. Ces techniques d'estimation sont étudiées dans le cadre de lois de commutation dépendant de l'état (indépendantes du temps de séjour), de perturbations limitées et de commutation asynchrone, où le mode de commutation actif de l'observateur ou du filtre ne correspond pas nécessairement à celui du système T-S à commutation. Tout d'abord, les observateurs T-S à commutations sont proposés avec des contraintes de Lipschitz, qui permettent des estimations d'état lorsque les variables de base des systèmes ne sont pas nécessairement mesurables. Cependant, en raison du conservatisme qui pourrait découler de la contrainte de Lipschitz, une approche intéressante pour traiter les variables de prémisse non mesurées est étudiée dans la deuxième partie de cette thèse dans le contexte du filtrage H_∞. Ainsi, le système T-S à commutation est modélisé comme un modèle T-S à commutation avec des parties conséquentes non linéaires, dans lequel les non-linéarités non mesurées sont conservées dans les parties conséquentes non linéaires. Grâce à la contrainte quadratique incrémentale utilisée pour traiter les non-linéarités non mesurées, les conditions obtenues sont moins conservatrices que la contrainte de Lipschitz. De plus, reconnaissant que les modèles T-S ne représentent les non-linéaires que sur des sous-ensembles de leur espace d'état, une procédure d'optimisation pour estimer le domaine d'attraction de l'erreur de filtrage est développée. Pour traiter les modes de commutation asynchrones, plusieurs fonctions de Lyapunov candidates sont considérées, ainsi qu'un critère H_∞ pour minimiser le transfert entre les perturbations d'entrée/sortie et les erreurs d'estimation d'état/filtrage. En outre, les conditions de conception proposées sont formulées en termes d'inégalités matricielles linéaires (LMI). Plusieurs exemples illustratifs sont utilisés tout au long du manuscrit pour valider les observateurs et les filtres proposés, ainsi que les résultats obtenus.

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Notations

Acronyms

HDS	Hybrid Dynamical Systems
SS	Switched Systems
T-S	Takagi-Sugeno
N-TS	T-S with nonlinear consequent parts
LMI	Linear Matrix Inequality
LPV	Linear Parameter Varying
UPVs	Unmeasured Premise variables
CQLF	Common Quadratic Lyapunov Function
MQLF	Multiple Quadratic Lyapunov Function

Sets, matrices and vectors

$\mathbb{R}$	Set of real numbers
$\mathcal{I}_m$	Set of integers $\{1, ..., m\}$
$\mathbb{R}^{n\times m}$	Set of real matrices of dimensions $n \times m$
M^T	The transpose of the M matrix
$\mathcal{H}_e(M)$	Stands for $M + M^T$
$M_{(v)}$	Denotes the v^{th} row of matrix M
$(*)$	Denotes a transpose quantity in an inequality
$M^\dagger$	Denotes the pseudo-inverse of M
$co\{X, Y\}$	Stands for a cone characterized by the sectors X and Y

CHAPTER 1

General Introduction

1.1 Introduction

Most physical systems are characterized by either a continuous model or a discrete event model, whereas, hybrid systems are a class of dynamic systems that combine both continuous and discrete dynamics. These systems are widely used in various fields such as power systems, traffic control, communications, networking and robotics, and cannot be described solely by a continuous model or a discrete event model. Indeed, the interplay between digital systems such as digital computers, software, logic components, etc, and continuous physical processes has resulted in the creation and formalization of a new class of systems known as Hybrid Dynamical Systems (HDS) in numerous industrial applications (see Figure 1.1).

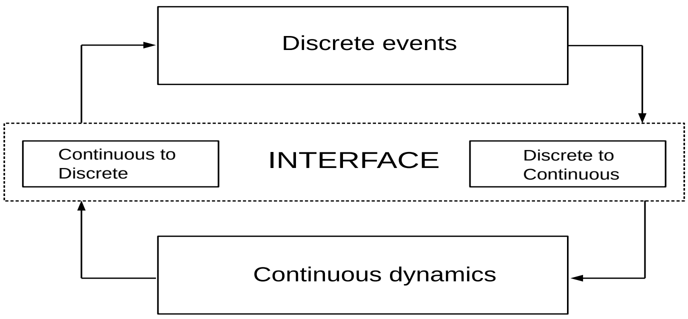

Figure 1.1: General structure of a hybrid dynamical system

This type of systems involve, in an explicit and simultaneous way, two types of dynamics, one continuous and the other event-driven (Dogruel and Ozguner, 1994; Lennartson et al., 1996; Branicky, 1998; Branicky et al., 1998). The continuous dynamics can be in continuous time or in discrete time. On the other hand, the discrete dynamics is related to a discrete event system whose state space is a finite discrete set. One of the main advantages of hybrid dynamical systems is that they can model a wide range of physical systems. They can handle smooth and continuous processes, such as the motion of robotic arm that can switch between different modes of operation, or the voltage in an electrical circuit, as well as discrete events, such as the switching of a relay or the actuation of a valve. This makes them very useful in modeling systems that exhibit both types of behavior.

Switched systems (SS) represent a popular class of hybrid dynamical systems, the study of switched systems has attracted the attention of both researchers interested in the study of continuous systems and those interested in the study of discrete event systems. The first research works on this type of system were carried out on modeling (Liberzon and Morse, 1999a). Indeed, the latter is a necessary step for control, estimation, filtering, stability analysis or fault-detection and control (Liberzon, 2003). It allows to represent in a homogeneous way the two types of dynamics. This generally allows, during the study of switched systems, not to use only classical methods from the automatic control of continuous systems or, but also to develop other methods that take into account the two types of dynamics and their interactions.

Despite the many advantages of hybrid systems, especially switched systems, there are also several challenges that must be addressed in the study of this class of systems. One of the main challenges is the design of estimation algorithms, that is to say, the design of observers and filters. Estimation for nonlinear systems is the primary focus on Academia and Industry for many decades. As a matter of fact, since Luenberger (1971) pioneering work, observers' design has caught the interest of a large number of control engineering researchers, and tremendous amounts of work have been accomplished in this field. In the case of switched systems, however, continuous state estimate is not the only concern; discrete state estimation must also be considered. In other words, the discrete state, often referred to as the set of active modes, must be frequently estimated online. The switched observer and the switched system may or may not have a synchronous or asynchronous relationship as a direct result of this. It is true that we refer to them as synchronous when the system and observer switch at the same time. This indicates that the active mode of the system must be known at each time instant, which is rarely realistic in practice. Therefore, a situation in which both the observer and the system exhibit asynchronous switching instants is a case that more closely resembles reality. When a switched system has more than one mode, there is the potential for asynchronicity to occur because it takes time to determine which mode is currently active. This could be the cause of the asynchronicity. In this regard, a look at the literature shows that a large number of studies related to switched observers have been performed (Juloski et al., 2002; Daafouz et al., 2002; Pettersson, 2005; Barbot et al., 2007) for continuous-time and discrete-time cases, and more recently (Belkhiat et al., 2011b, 2015; Zhao et al., 2015; Regaieg et al., 2019; Yin et al., 2017; Han et al., 2020),

The filtering problem is summarized in the estimation or filtering some of the system's outputs, which may not be measured or may be affected by noise disturbances, while minimizing the influence of external noisy disturbances on the filtering error. In general, the filter synthesis is formulated by specifying a performance criterion to be optimized. The first filter, dedicated to the estimation of the state of linear systems, was developed by Kalman in a stochastic framework (Kalman, 1960). In this case, the minimization criterion often used is the variance of the estimation error. Extensions of the Kalman filter have been proposed for particular classes of nonlinear systems, such as the Extended Kalman filter, Unscented Kalman filter (Julier and Uhlmann, 1997; Wan and Van Der Merwe, 2000). The Kalman filter has had a considerable impact in many practical problems and has been widely applied in target tracking, navigation, localization and fault detection in many other fields. However, this technique is sometimes difficult to apply in the real world, it is sometimes difficult to apply because of the constraints imposed. Indeed, the Kalman filter requires that the noises that affect the system are white noise. In practice, we do not always have at our disposal all the statistical properties of the external disturbances. To overcome these difficulties, an alternative approach, called H_∞ filtering has been introduced. Compared with other filtering approaches mentioned above, the main interest of H_∞ filtering technique is its robustness against unmodeled dynamics as well as it does not require any knowledge about the disturbances except that they are of bounded energy and it proceeds by minimizing the effect of the disturbances on the filtering error. There exist in the literature several approaches to solve the H_∞ filtering problem, such as the Riccati equation approach (Nagpal and Khargonekar, 1991), Lyapunov LMI approach (Nagpal and Khargonekar, 1991; Li and Fu, 1997), polynomial approaches (Deutscher, 2002). Linear matrix inequalities (LMI) have a certain interest because they are easily applicable. Their solution is based on convex optimization methods (Boyd et al., 1994), and they regularly benefit from many theoretical and computational advances. During the last few years, LMI constrained convex optimization has emerged as an essential tool in control systems and many complex problems are now solved in few seconds. In this context, several studies have been carried out to design H_∞ filters for linear systems, one can cite (Liu et al., 2008; Zhang and Boukas, 2009; de Souza et al., 2010; Li et al., 2016; El Hellani et al., 2017). Furthermore, in the framework of switched linear systems, H_∞ filters design have been the subject of several recent investigations (see, e.g., (Ding et al., 2011; Zhang et al., 2015, 2017; Ren et al., 2018; Qi et al., 2021)).

It is important to keep in mind that the aforementioned studies only addressed observers and H_∞ filters design for switched linear systems and linear systems in general. On the other hand, it is commonly acknowledged that the vast majority of real-world systems exhibit nonlinear dynamics. As a result, a substantial amount of research effort is also dedicated to the framework of nonlinear systems (Xiang and Xiang, 2008; Li and Jia, 2010; Xiang and Xiao, 2011; Yang et al., 2011; Abbaszadeh and Marquez, 2012; Xiang et al., 2012, 2014; Liu and Zhao, 2020). For instance, in (Xiang and Xiang, 2008; Xiang et al., 2012), switched observers have been designed for switched nonlinear systems through the use of Lipschitz conditions to handle the nonlinearities. Moreover, the asymptotic convergence of the designed switched observer was obtained via multiple Lyapunov function, leading to Linear Matrix Inequality (LMIs) conditions. More recently, with similar considerations of Lipschitz conditions to cope with the nonlinear terms, the design of synchronous switched adaptive observers for a class of uncertain switched nonlinear systems with average dwell time was studied in (Liu and Zhao, 2020). In Abbaszadeh and Marquez (2012), the authors investigated the design of H_∞ filters for Lipschitz nonlinear systems with time-varying uncertainties. Xiang et al. (2014) investigated the fault detection problem based on H_∞ filters for switched nonlinear systems using Lipschitz conditions. However, dealing with such constraints to handle nonlinear systems, i.e., Lipschitz approximations to handle nonlinear terms has a number of problems, the most significant being that they do not adequately reflect the global nonlinear system. This can lead to a loss of information regarding the systems under consideration as well as bad observers and filters design.

Takagi-Sugeno (T-S) fuzzy models (Takagi and Sugeno, 1985), when obtained by the sector nonlinearity approach (Tanaka and Wang, 2001), provide an intriguing framework for representing exactly nonlinear systems without loss of information. These models are commonly used to describe nonlinear systems as weighted combinations of linear systems, for which many tools designed specifically for linear systems can be employed with ease (see, e.g., (Benzaouia et al., 2011; Lendek et al., 2012; Zhao et al., 2016; Jabri et al., 2011; Yang et al., 2013; Lendek et al., 2014b,a; Wang et al., 2016b; Jabri et al., 2018; Su et al., 2020; Garbouj et al., 2020; Shi et al., 2020).

Concerning observers design for switched T-S systems, (Lendek et al., 2012) proposed the design of synchronous switched T-S observers for a class of periodic discrete-time T-S systems, where the switching sequences and switching instants were assumed priory known. Following this work, (Lendek et al., 2014b) proposed relaxed LMI conditions in which the switching sequences were no longer required to be known beforehand, but an admissible set of switches was required. The synchronous observer-based robust control problem for switched T-S systems with time-delay subject to uncertainties and external disturbances has been studied in (Yang et al., 2013). Another work dealing with synchronous switched observers for stabilization purpose has been proposed in (Yang and Tong, 2016), where the premise variables are supposed unmeasured. Synchronous switched robust interval observers for switched T-S systems have been proposed in (Ifqir et al., 2017), assuming that the premise variables are measurable, and in (Garbouj et al., 2019), in the case of unmeasured premise variables. Garbouj et al. (2020) proposed an optimal interval observers for continuous nonlinear switched systems. The nonlinear modes are described by T-S fuzzy systems where premise variables depending on the state vector that is unmeasurable and considered as bounded uncertainties. Robust observer-based controllers to stabilize switched T-S systems with state jumps and robust detection filters were proposed in (Ait Ladel et al., 2021) for simultaneous fault detection and control purposes with mode-dependent average dwell time approach.

Regarding H_∞ filters design for switched T-S systems, Zheng et al. (2018a) proposed a synchronous mixed H_∞ and passive Luenberger-like switched T-S filter for switched T-S systems. In (Hong et al., 2018), the H_∞ filtering problem for switched T-S fuzzy systems under constrained and asynchronous switching has been investigated. Using multiple Lyapunov functions approach and mode-dependent average dwell time technique, some sufficient conditions ensuring the convergence of the filtering error with a weighted H_∞ performance index have been formulated in terms of LMIs. Moreover, the premise variables are considered measurable. The design of a synchronous H_∞ filter for a class of switched T-S fuzzy systems with measurable premise variables and under constrained switching has been studied in (Shi et al., 2019). Using Lyapunov theory with persistent dwell time condition, a set of LMI conditions have been developed to ensure the convergence of the filtering error with a prescribed non-weighted H_∞ noise attenuation performance. In the same context, the authors in (Liu et al., 2020) investigated the design of a non-weighted asynchronous H_∞ filter for a class of continuous time switched T-S fuzzy systems. Using fuzzy multiple Lyapunov function, sufficient conditions have been formulated in terms of LMIs to guarantee the filtering error system is globally asymptotically stable with a non-weighted H_∞ performance. The filter was designed under minimum dwell time constraint and under the condition that the premise variables are measurable. In (Chekakta et al., 2022), dwell-time free LMI-based conditions for the design of an asynchronous H_∞ filters for switched T-S systems under arbitrary switching have been investigated. To reduce the conservatism of the design conditions, LMI-based relaxation techniques (Peaucelle et al., 2000; Tuan et al., 2001b) were considered together with a descriptor redundancy approach to ensure the convergence of the filtering error with H_∞ disturbance attenuation performances. However, the main weakness of these studies is that the premise variables in the considered T-S models are considered measurable, which is often unrealistic in practice.

Based on the above literature review, we can conclude that the following two major issues must be taken into account when designing observers and H_∞ filters for switched T-S systems:

- *Takagi-Sugeno modelling with UPVs:* The design of switched observers and H_∞ filters for switched T-S systems with Unmeasured Premise Variables (UPVs) has been rarely investigated. As a result, dealing with UPVs is the most important issue to consider when designing these observers and filters. Significant work has been done in the last decades to deal with UPVs outside the switched nonlinear systems context. These works can be broadly summarized as follows:
  - Using the Lipschitz condition, which can be easily implemented, is the most common method for dealing with the UPVs problem. This latter option is frequently considered as a way to circumvent the nonlinear additive term occurring in the dynamics of the state estimation error. Accordingly, there have been a number of research papers written about the Lipschitz condition. We may cite the studies conducted by (Ichalal et al., 2010; Moodi and Farrokhi, 2014; Moodi and Bustan, 2018; Xie et al., 2019; Chekakta et al., 2021). Further improvements in terms of conservatism are proposed in (Ichalal et al., 2011) to substitute the Lipschitz condition by applying the differential mean value theorem in order to deal with the additive term in the dynamic of the state error. Another approach to deal with UPVs is proposed in (López-Estrada et al., 2017), which consists in considering the error between the measured and unmeasured premise variables as model uncertainties, then applying robust control approaches. More recently, in (Ichalal et al., 2018), auxiliary dynamics and immersion techniques are considered to rewrite and augment the T-S model with UPVs as a new T-S model with weighting functions depending only on measured variables. However, this approach cannot be easily generalized and may fail to provide the required transformations for the initial model due to the possible infinite number of iterations and the nature of the nonlinear entries of the system.
  - To circumvent the occurrence of UPVs in standard T-S modelling, an elegant way is to consider T-S models with nonlinear consequent parts (N-TS) (see e.g., Bouarar et al. (2007); Yoneyama (2009); Dong et al. (2009); Moodi and Farrokhi (2015); Araújo et al. (2019); Moodi et al. (2019); Nagy et al. (2020, 2022)). In this case, the T-S modelling with the sector nonlinearity approach are modified. That is to say, the sector nonlinearity approach is only applied to the nonlinear terms depending on the measured state variables, leading to a N-TS model involving only measured premises variables, and where the unmeasured nonlinear terms are kept as nonlinear consequent parts. Furthermore, to cope with the nonlinear

consequent parts, previous studies usually consider Lipschitz conditions to provide LMI-based conditions for the design of N-TS filters or observers (see e.g., (Yoneyama, 2009; Nagy et al., 2020, 2022)). Moreover, (Pourasghar et al., 2022; Nguyen et al., 2021) investigated the design of observers for discrete-time T-S systems with nonlinear consequent parts, where the unmeasured nonlinearities have been dealt with by employing a zonotopic set-based approach and the differential mean value theorem, respectively. Similarly, Moodi and Farrokhi (2015) considered the design of observers-based controller for T-S systems with nonlinear consequent parts, where the incremental quadratic constraint (Açıkmeşe and Corless, 2011) is used to cope with the unmeasured nonlinearities, with significant conservative improvements regarding to the design of standard T-S filters or observers with UPVs.

- *Nature of the switching phenomena:* Another concern regarding the above literature review is related to the nature of the switching phenomena. There exists a vast literature dealing with such phenomena in the switched linear system's framework; see, e.g., (Liberzon, 2003; Daafouz et al., 2002; Belkhiat et al., 2011a; Zhang and Gao, 2010; Mao et al., 2014; Yuan et al., 2018; Ren et al., 2018; Han et al., 2020; Etienne et al., 2020; Fei et al., 2022; Alessandri and Sanfelice, 2022). The most easiest case remain when the switched control law, observer of filter, share the same switching law as the switched linear plant, i.e. the so-called switched synchronous case (Liberzon, 2003; Daafouz et al., 2002; Belkhiat et al., 2011a; Zhang et al., 2015; Tao et al., 2017; Qi et al., 2021). However, it is often unrealistic in practice to consider the synchronous case because it requires the availability of the plant's switching signals, which may be difficult to measure or estimate. As a result, several studies have been carried out in the asynchronous case, either in the switched linear case, see e.g., (Pettersson, 2005; Zhang and Gao, 2010; Yuan et al., 2018; Ren et al., 2018; Han et al., 2020; Etienne et al., 2020; Fei et al., 2022), or extended to the switched nonlinear system framework (Xiang et al., 2012; Wang and Tong, 2017; Hong et al., 2018; Shi et al., 2019; Zare et al., 2020; Liu and Zhao, 2020; Chekakta et al., 2021, 2022). Hence, the asynchronous case refers to a scenario in which the switching rules of the observer or filter do not match those of the plant. In this more realistic scenario, the mismatch can be a delay (for instance, the amount of time it takes to estimate the plant's switching rule) (Zhang and Gao, 2010; Yuan et al., 2018; Zhai et al., 2018; Etienne et al., 2020; Fei et al., 2022), or it can be due to modelling uncertainties in the context of state-dependant switching laws remaining on mismatching switching hyper-planes (Pettersson, 2005; Yang et al., 2019; Chekakta et al., 2021, 2022). So far, design conditions for switched systems can be classified into two classes:
  - Dwell-time dependent conditions (see e.g. (Zhang and Gao, 2010; Mao et al., 2014; Yuan et al., 2018; Fei et al., 2022)), which are useful to relax the conditions but require some knowledge on the switching rules. Especially, in this case, the considered switched systems are required to stay in each mode during at least a minimum positive dwell-time. However, such approach may be unsuitable for some switched systems, for instance when the switched systems involve state-dependent switching laws that are evolving regarding to uncertain

switching hyper-planes.

- Dwell-time free conditions (see e.g. (Pettersson, 2005; Yang et al., 2019; Chekakta et al., 2021, 2022)), which are useful for arbitrary or state-dependent switching laws evolving regarding to uncertain switching hyper-planes. Despite these conditions cannot benefit from the conservatism reduction brought by dwell-time dependent conditions, they allows to cope with a larger class of switched systems since they require less restrictive knowledge about the switching phenomena.

1.2 Organization

This thesis, composed of four chapters, is organized as follows:

Chapter 1 includes a general introduction, the structure of this thesis, the contributions made and the author's publications.

Chapter 2 introduces the fundamental concepts related to the classes of systems treated in this thesis. First, we propose a presentation of the structures and problems related to switched nonlinear systems, and switched nonlinear systems represented by Takagi-Sugeno models. Then, brief introduction to the main concepts of two theoretical topics is presented, namely state observation and robust nonlinear filtering. These topics constitute the problems addressed by the proposed methods in the next two chapters.

Chapter 3 addresses the asynchronous switched T-S observers design for continuous-time switched nonlinear systems under asynchronous switching and mismatching switching laws. The content of this chapter is divided into two parts. The first part is devoted to the observer's design for switched T-S systems where the premise variables are assumed to be measured, it starts with a presentation of the class of switched nonlinear systems and the considered assumptions. An approach is then proposed for the synthesis of asynchronous switched observers for switched nonlinear continuous-time systems, where Takagi-Sugeno fuzzy models represent the nonlinear modes, with mismatching switching laws and bounded output disturbances. The second part of this chapter was dedicated to the design of switched T-S observers for switched nonlinear systems with unmeasured premise variables. Lipschitz conditions were employed to deals with the premise variables. Several theorems were proposed in terms of LMIs, despite their computational cost, these theorems bring successive conservatism improvements, allowing the users to select the appropriate conditions regarding to the complexity of their applications. Several illustrative examples were given to compare the conservatism and to illustrate the effectiveness in time-simulation of the proposed asynchronous switched T–S observer design methodologies.

Chapter 4 addresses the nonlinear H_∞ robust filtering problem for switched nonlinear systems. The content of this chapter is divided into two parts. The first part is devoted to the filtering problem for switched T-S systems where the premise variables are assumed to be measured, using a descriptor redundancy-based approach to describe the filtering-error dynamics. The second part of this chapter is devoted to the filtering problem of switched T-S systems where the premise variables are not measured. To deal with this more general case, a switched Takagi-Sugeno (T-S) model with nonlinear consequent parts (taking into account the unmeasured variables) is used as an alternative representation to the usual techniques in the literature. Then, switched T-S filters with nonlinear consequent parts are proposed to estimate the unmeasured and/or disturbed outputs of the system. Compared to previous related work, these conditions have many advantages. Primarily, they are dwell time independent and less conservative, due to the use of incremental quadratic constraints to deal with the unmeasured premise variables. Furthermore, it is well known that T-S models represent nonlinear models only on subsets of their state space, a procedure for optimizing the estimation of the domain of attraction of the filtering error is developed. Several examples were given to show how the proposed switched T-S H_∞ filter design methodology improves conservatism over previous studies as well as its effectiveness in time-simulation.

The general conclusion summarizes the main results of this thesis and presents some research directions to complete this work.

1.3 Contributions

The research goals of this thesis focus on developing and improving the existing methods of state observers design and nonlinear filtering for a class of hybrid nonlinear systems represented by switched Takagi-Sugeno models. The main contributions of this thesis are briefly summarized:

- The first contribution is to design asynchronous observers for continuous-time switched T-S systems with measured/unmeasured premise variables using the Lipschitz assumption, subject to output disturbances (which may represent measurement bias or sensor noise.) with H_∞ performance specifications and under any sequences and arbitrary switching sets.

- The second contribution inherits from the ability to cope with arbitrary mismatches of switch sets and/or sequences between the switched systems and observers under consideration. Despite all the previous studies mentioned in the introduction, which assume that the considered switched systems and switched observers are initialized in the same switching modes, an important feature of this proposal is that they can now start from different initial modes (asynchronous initialization of the switched modes).

- The third contribution is to propose dwell-time free LMI-based conditions for the design of H_∞ filters for switched T-S systems, which assume that the premise variables are measured, and then apply this proposal to the design of asynchronous switched N-TS H_∞ filters with unmeasured premise variables by considering nonlinear consequent part approach, incremental quadratic constraints (Açıkmeşe and Corless, 2011) and extending the Lipschitz conditions to cope with the additive nonlinear term arising from the error between the measured and UPVs.

- The last contribution of this thesis is to propose an optimization procedure for the estimation of the filtering error attraction region based on the enlargement of Lyapunov level sets. The proposed results can only be locally valid, since T-S fuzzy models are only valid for a subset of the state space of a nonlinear system. Whereas, to the best of the authors' knowledge, this important feature has not been considered in any previous N-TS model based filters design studies from the literature.

1.4 Author's Publications

International Journal papers

- Tabbi, I., Jabri, D., **Chekakta, I**, & Belkhiat, DEC. (2023). Robust state and sensor fault estimation for switched nonlinear systems based on asynchronous switched fuzzy observers. International Journal of Adaptive Control and Signal Processing (Under review).
- **Chekakta, I**, Jabri D, Motchon KMD, Guelton K, Belkhiat DEC. Design of asynchronous switched Takagi–Sugeno model-based H_∞ filters with nonlinear consequent parts for switched nonlinear systems. Int J Adapt Control Signal Process. 2023;1-25. doi: 10.1002/acs.3588
- Menighed, K., Yamé, J. J., , & **Chekakta, I.** (2022). A non-cooperative distributed model predictive control using Laguerre functions for large-scale interconnected systems. *Journal Européen des Systèmes Automatisés (JESA)* 55.5: 555-572.
- **Chekakta, I.**, Belkhiat, D. E. C., Guelton, K., Jabri, D., & Manamanni, N. (2021). Asynchronous observer design for switched T–S systems with unmeasurable premises and switching mismatches. *Engineering Applications of Artificial Intelligence*, 104, 104371.

National Journal papers

- Menighed, K., & **Chekakta, I.** (2020). Level Control of Quadruple Tank Process using Laguerre Functions based Model Predictive Control Algorithm. Algerian Journal of Signals and Systems, 5(2), 130-137.

Papers published in conferences proceedings:

- **Chekakta, I.**, Jabri, D., Belkhiat, D.E.C, Motchon, K.M. and Guelton, K., (2022). Suivi d'attitude à base d'un modèle flou T-S d'un quadrirotor soumis aux saturations sur les entrées et perturbations externes. In Rencontres Francophones sur la Logique Floue et ses Applications (LFA2022) (pp. 1-8). Cépaduès.
- **Chekakta, I.**, Guelton, K., Jabri, D., Belkhiat, D. E., & Motchon, K. M. (2022). Regional T-S Model-based Attitude Tracking Control of a Quadrotor with Input Saturation and External Disturbances. IFAC-PapersOnLine, 55(15), 63-68.
- **Chekakta, I.**, Belkhiat, D. E. C., Guelton, K., Motchon, K. M., & Jabri, D. (2022). Asynchronous Switched Takagi-Sugeno H_∞ Filters Design For Switched Nonlinear Systems. IFAC-PapersOnLine, 55(1), 351-356.
- **Chekakta, I.**, Belkhiat, D. E. C., Motchon, K., Guelton, K., & Jabri, D. (2021). Synthèse de filtres H_∞ de type Takagi-Sugeno avec commutations asynchrones pour les systèmes non linéaires à commutations. In Rencontres Francophones sur la Logique Floue et ses Applications (LFA 2021) (pp. 103-110). Cépaduès-éditions.

- Belkhiat, D. E. C., Jabri, D., Guelton, K., Manamanni, N., & **Chekakta, I.** (2019). Asynchronous Switched Observers Design for Switched Takagi-Sugeno Systems Subject to Output Disturbances. IFAC-PapersOnLine, 52(11), 49-54.

Traduction en Français : Introduction générale

La plupart des systèmes physiques sont caractérisés par un modèle continu ou un modèle à événements discrets, tandis que les systèmes hybrides sont une classe de systèmes dynamiques qui combinent les dynamiques continues et discrètes. Ces systèmes sont largement utilisés dans divers domaines tels que les systèmes d'alimentation, le contrôle du trafic, les communications, les réseaux et la robotique, et ne peuvent être décrits uniquement par un modèle continu ou un modèle à événements discrets. En effet, l'interaction entre les systèmes numériques tels que les ordinateurs numériques, les logiciels, les composants logiques, etc., et les processus physiques continus a conduit à la création et à la formalisation d'une nouvelle classe de systèmes connus sous le nom de systèmes dynamiques hybrides (HDS) dans de nombreuses applications industrielles (voir Figure 1.1).

Ce type de systèmes implique, de manière explicite et simultanée, deux types de dynamiques, l'une continue et l'autre événementielle (Dogruel and Ozguner, 1994; Lennartson et al., 1996; Branicky, 1998; Branicky et al., 1998). La dynamique continue peut être en temps continu ou en temps discret. D'autre part, la dynamique discrète est liée à un système d'événements discrets dont l'espace d'état est un ensemble discret fini. L'un des principaux avantages des systèmes dynamiques hybrides est qu'ils peuvent modéliser un large éventail de systèmes physiques. Ils peuvent gérer des processus lisses et continus, comme le mouvement d'un bras robotique qui peut passer d'un mode de fonctionnement à un autre, ou la tension d'un circuit électrique, ainsi que des événements discrets, comme la commutation d'un relais ou l'actionnement d'une vanne. Cela les rend très utiles pour modéliser des systèmes qui présentent ces deux types de comportement.

Les systèmes à commutations (SS) représentent une classe populaire de systèmes dynamiques hybrides, l'étude des systèmes à commutations a attiré l'attention à la fois des chercheurs intéressés par l'étude des systèmes continus et de ceux intéressés par l'étude des systèmes à événements discrets. Les premiers travaux de recherche sur ce type de système ont été réalisés sur la modélisation (Liberzon and Morse, 1999a). En effet, cette dernière est une étape nécessaire pour la commande, l'estimation, le filtrage, l'analyse de stabilité ou la détection de défauts et le contrôle (Liberzon, 2003). Elle permet de représenter de manière homogène les deux types de dynamique. Cela permet généralement, lors de l'étude des systèmes à commutations, de ne pas utiliser uniquement les méthodes classiques de l'automatique des systèmes continus ou, mais de développer d'autres méthodes qui prennent en compte les deux types de dynamique et leurs interactions.

Malgré les nombreux avantages des systèmes hybrides, en particulier des systèmes à commutations, il existe également plusieurs défis à relever dans l'étude de cette classe de systèmes. L'un des principaux défis est la conception d'algorithmes d'estimation, c'est-à-dire la conception d'observateurs et de filtres. L'estimation pour les systèmes non linéaires est le principal centre d'intérêt du monde universitaire et de l'industrie depuis de nombreuses décennies. En fait, depuis les travaux pionniers de Luenberger (1971), la conception d'observateurs a suscité l'intérêt d'un grand nombre de chercheurs en ingénierie du contrôle, et d'énormes quantités de travail ont été accomplies dans ce domaine. Dans le cas des systèmes à commutations, cependant, l'estimation de l'état continu n'est pas la seule préoccupation ; l'estimation de l'état discret doit également être considérée. En d'autres termes, l'état discret, souvent désigné comme l'ensemble des modes actifs, doit être fréquemment estimé en ligne. L'observateur à commutation et le système à commutation peuvent ou non avoir une relation synchrone ou asynchrone en conséquence directe de ceci. Il est vrai que nous les qualifions de synchrones lorsque le système et l'observateur commutent en même temps. Cela indique que le mode actif du système doit être connu à chaque instant, ce qui est rarement réaliste en pratique. Par conséquent, une situation dans laquelle l'observateur et le système présentent tous deux des instants de commutation asynchrones est un cas qui ressemble davantage à la réalité. Lorsqu'un système à commutations possède plus d'un mode, il est possible que l'asynchronisme se produise car il faut du temps pour déterminer quel mode est actuellement actif. Cela pourrait être la cause de l'asynchronisme. À cet égard, un examen de la littérature montre qu'un grand nombre d'études relatives aux observateurs à commutations ont été réalisées. (Juloski et al., 2002; Daafouz et al., 2002; Pettersson, 2005; Barbot et al., 2007) pour les cas à temps continu et à temps discret, et plus récemment (Belkhiat et al., 2011b, 2015; Zhao et al., 2015; Regaieg et al., 2019; Yin et al., 2017; Han et al., 2020).

Le problème du filtrage se résume à l'estimation ou au filtrage de certaines des sorties du système, qui peuvent ne pas être mesurées ou être affectées par des perturbations sonores, tout en minimisant l'influence des perturbations sonores externes sur l'erreur de filtrage. En général, la synthèse du filtre est formulée en spécifiant un critère de performance à optimiser. Le premier filtre, dédié à l'estimation de l'état de systèmes linéaires, a été développé par Kalman dans un cadre stochastique (Kalman, 1960). Dans ce cas, le critère de minimisation souvent utilisé est la variance de l'erreur d'estimation. Des extensions du filtre de Kalman ont été proposées pour des classes particulières de systèmes non linéaires, comme le filtre de Kalman étendu, le filtre de Kalman noncentré (Julier and Uhlmann, 1997; Wan and Van Der Merwe, 2000). Le filtre de Kalman a eu un impact considérable dans de nombreux problèmes pratiques et a été largement appliqué dans la poursuite de cibles, la navigation, la localisation et la détection de défauts dans de nombreux autres domaines. Cependant, cette technique est parfois difficile à appliquer dans le monde réel, elle est parfois difficile à appliquer à cause des contraintes imposées. En effet, le filtre de Kalman nécessite que les bruits qui affectent le système soient des bruits blancs. En pratique, nous ne disposons pas toujours de toutes les propriétés statistiques des perturbations externes. Pour surmonter ces difficultés, une approche alternative, appelée filtrage H_∞ a été introduite. Par rapport aux autres approches de filtrage mentionnées ci-dessus, le principal intérêt de la technique de filtrage H_∞ est sa robustesse face aux perturbations externes et sa robustesse face à une dynamique non modélisée. De plus, elle ne nécessite aucune connaissance des perturbations, si ce n'est qu'elles sont d'énergie limitée, et elle procède en minimisant l'effet des perturbations sur l'erreur de filtrage. Il existe dans la littérature plusieurs approches pour résoudre le problème de filtrage de H_∞, telles que l'approche par les équations de Riccati (Nagpal and Khargonekar, 1991), l'approche LMI de Lyapunov (Nagpal and Khargonekar, 1991; Li and Fu, 1997), les approches polynomiales (Deutscher, 2002). Les inégalités matricielles linéaires (IML) présentent un certain intérêt car elles sont facilement applicables. Leur résolution est basée sur des méthodes d'optimisation convexe (Boyd et al., 1994), et elles bénéficient régulièrement de nombreuses avancées théoriques et informatiques. Au cours des dernières années, l'optimisation convexe sous contrainte LMI est apparue comme un outil essentiel dans les systèmes de contrôle et de nombreux problèmes complexes sont maintenant résolus en quelques secondes. Dans ce contexte, plusieurs études ont été menées pour concevoir des filtres H_∞ pour les systèmes linéaires, on peut citer (Liu et al., 2008; Zhang and Boukas, 2009; de Souza et al., 2010; Li et al., 2016; El Hellani et al., 2017). De plus, dans le cadre des systèmes linéaires à commutations, la conception des filtres H_∞ a fait l'objet de plusieurs recherches récentes (voir, par exemple, (Ding et al., 2011; Zhang et al., 2015, 2017; Ren et al., 2018; Qi et al., 2021)).

Il est important de garder à l'esprit que les études susmentionnées n'ont abordé que la conception d'observateurs et de filtres H_∞ pour les systèmes linéaires à commutations et les systèmes linéaires en général. D'autre part, il est communément admis que la grande majorité des systèmes du monde réel présentent une dynamique non linéaire. Par conséquent, une quantité importante d'efforts de recherche est également consacrée au cadre des systèmes non linéaires : (Xiang and Xiang, 2008; Li and Jia, 2010; Xiang and Xiao, 2011; Yang et al., 2011; Abbaszadeh and Marquez, 2012; Xiang et al., 2012, 2014; Liu and Zhao, 2020). Par exemple, dans la (Xiang and Xiang, 2008; Xiang et al., 2012), des observateurs à commutations ont été conçus pour les systèmes non linéaires à commutations grâce à l'utilisation de conditions de Lipschitz pour traiter les non-linéarités. En outre, la convergence asymptotique de l'observateur à commutation conçu a été obtenue via une fonction de Lyapunov multiple, conduisant à des conditions d'inégalités matricielles linéaires (LMI). Plus récemment, avec des considérations similaires de conditions de Lipschitz pour faire face aux termes non linéaires, la conception d'observateurs adaptatifs synchrones à commutations pour une classe de systèmes non linéaires incertains à commutations avec un temps de séjour moyen a été étudiée dans (Liu and Zhao, 2020). Dans Abbaszadeh and Marquez (2012), les auteurs ont étudié la conception de filtres H_∞ pour les systèmes non linéaires Lipschitz avec des incertitudes variant dans le temps. Xiang et al. (2014) ont étudié le problème de la détection des défauts basé sur les filtres H_∞ pour les systèmes non linéaires à commutations utilisant des conditions de Lipschitz. Cependant, l'utilisation de telles contraintes pour traiter les systèmes non linéaires, c'est-à-dire des approximations de Lipschitz pour traiter les termes non linéaires, présente un certain nombre de problèmes, le plus important étant qu'elles ne reflètent pas correctement le système non linéaire global. Cela peut conduire à une perte d'informations concernant les systèmes considérés ainsi qu'à une mauvaise conception des observateurs et des filtres.

Les modèles flous Takagi-Sugeno (T-S) (Takagi and Sugeno, 1985), lorsqu'ils sont obtenus par l'approche de la non-linéarité sectorielle (Tanaka and Wang, 2001), fournissent un cadre intriguant pour représenter les systèmes exactement non linéaires sans perte d'information. Ces modèles sont couramment utilisés pour décrire les systèmes non linéaires comme des combinaisons pondérées de systèmes linéaires, pour lesquels de nombreux outils conçus spécifiquement pour les systèmes linéaires peuvent être employés avec facilité (voir, par exemple, (Benzaouia et al., 2011; Lendek et al., 2012; Zhao et al., 2016; Jabri et al., 2011; Yang et al., 2013; Lendek et al., 2014b,a; Wang et al., 2016b; Jabri et al., 2018; Su et al., 2020; Garbouj et al., 2020; Shi et al., 2020).

Concernant la conception d'observateurs pour les systèmes T-S à commutations, (Lendek et al., 2012) a proposé la conception d'observateurs T-S à commutations synchrones pour une classe de systèmes T-S périodiques à temps discret, où les séquences de commutation et les instants de commutation étaient supposés connus à l'avance. Suite à ce travail, (Lendek et al., 2014b) a proposé des conditions LMI relaxées dans lesquelles les séquences de commutation n'avaient plus besoin d'être connues à l'avance, mais un ensemble admissible de commutations était requis. Le problème de contrôle robuste basé sur un observateur synchrone pour les systèmes T-S à commutations avec retard temporel soumis à des incertitudes et des perturbations externes a été étudié dans (Yang et al., 2013). Un autre travail portant sur les observateurs synchrones à commutations à des fins de stabilisation a été proposé dans (Yang and Tong, 2016), où les variables de base sont supposées non mesurées. Des observateurs d'intervalle robustes à commutations synchrone pour les systèmes T-S à commutations ont été proposés dans (Ifqir et al., 2017), en supposant que les variables de départ sont mesurables, et dans (Garbouj et al., 2019), dans le cas de variables de départ non mesurées. (Garbouj et al., 2020) a proposé un observateur d'intervalle optimal pour les systèmes continus non linéaires à commutation. Les modes non linéaires sont décrits par des systèmes flous T-S où les variables de prémisse dépendent du vecteur d'état qui n'est pas mesurable et considéré comme des incertitudes limitées. Des contrôleurs basés sur des observateurs robustes pour stabiliser les systèmes T-S à commutations avec des sauts d'état et des filtres de détection robustes ont été proposés dans (Ait Ladel et al., 2021) pour la détection simultanée des défauts et à des fins de contrôle avec une approche de temps de séjour moyen dépendant du mode.

Concernant la conception de filtres H_∞ pour les systèmes T-S à commutations, (Zheng et al., 2018a) a proposé un filtre T-S à commutation synchrone mixte H_∞ et passif de type Luenberger pour les systèmes T-S à commutations. Dans (Hong et al., 2018), le problème de filtrage H_∞ pour les systèmes flous T-S à commutations sous contrainte et asynchrones a été étudié. En utilisant l'approche des fonctions de Lyapunov multiples et la technique du temps de séjour moyen dépendant du mode, certaines conditions suffisantes assurant la convergence de l'erreur de filtrage avec un indice de performance H_∞ pondéré ont été formulées en termes de LMI. De plus, les variables de prémisse sont considérées comme mesurables. La conception d'un filtre H_∞ synchrone pour une classe de systèmes flous T-S à commutations avec des variables de prémisse mesurables et sous contrainte de commutation a été étudiée dans (Shi et al., 2019). En utilisant la théorie de Lyapunov avec une condition de temps de séjour persistant, un ensemble de conditions LMI ont été développées pour assurer la convergence de l'erreur de filtrage avec une performance prescrite d'atténuation du bruit non pondéré H_∞. Dans le même contexte, les auteurs de (Liu et al., 2020) ont étudié la conception d'un filtre H_∞ asynchrone non pondéré pour une classe de systèmes flous T-S à commutations à temps continu. En utilisant la fonction de Lyapunov multiple floue, des conditions suffisantes ont été formulées en termes de LMI pour garantir que le système d'erreur de filtrage est globalement asymptotiquement stable avec une performance H_∞ non pondérée. Le filtre a été conçu sous une contrainte de temps de séjour minimal et à la condition que les variables de prémisse soient mesurables. Dans (Chekakta et al., 2022), les conditions LMI sans temps de séjour pour la conception d'un filtre asynchrone H_∞ pour les systèmes T-S à commutations sous commutation arbitraire ont été étudiées. Pour réduire le conservatisme des conditions de conception, des techniques de relaxation basées sur LMI (Peaucelle et al., 2000; Tuan et al., 2001b) ont été considérées avec une approche de redondance des descripteurs pour assurer la convergence de l'erreur de filtrage avec des performances d'atténuation des perturbations H_∞. Cependant, la principale faiblesse de ces études est que les variables de prémisse dans les modèles T-S considérés sont considérées comme mesurables, ce qui est souvent irréaliste en pratique. Sur la base de la revue de la littérature ci-dessus, nous pouvons conclure que les deux problèmes majeurs suivants doivent être pris en compte lors de la conception d'observateurs et de filtres H_∞ pour les systèmes T-S à commutations :

- *Modélisation de Takagi-Sugeno avec UPVs:* La conception d'observateurs à commutations et de filtres H_∞ pour les systèmes T-S à commutations avec des variables de prémisse non mesurées (UPV) a été rarement étudiée. Par conséquent, le traitement des VPP est le problème le plus important à prendre en compte lors de la conception de ces observateurs et filtres. Des travaux importants ont été réalisés au cours des dernières décennies pour traiter les UPVs en dehors du contexte des systèmes non linéaires à commutations. Ces travaux peuvent être résumés de la manière suivante :

  – L'utilisation de la condition de Lipschitz, qui peut être facilement mise en œuvre, est la méthode la plus courante pour traiter le problème des UPVs. Cette dernière option est souvent considérée comme un moyen de contourner le terme additif non linéaire apparaissant dans la dynamique de l'erreur d'estimation d'état. En conséquence, de nombreux articles de recherche ont été écrits sur la condition de Lipschitz. On peut citer les études menées par (Ichalal et al., 2010; Moodi and Farrokhi, 2014; Moodi and Bustan, 2018; Xie et al., 2019; Chekakta et al., 2021). D'autres améliorations en termes de conservatisme sont proposées dans (Ichalal et al., 2011) pour substituer la condition de Lipschitz en appliquant le théorème de la valeur moyenne différentielle afin de traiter le terme additif dans la dynamique de l'erreur d'état. Une autre approche pour traiter les UPVsest proposée dans (López-Estrada et al., 2017), qui consiste à considérer l'erreur entre les variables de prémisse mesurées et non mesurées comme des incertitudes du modèle, puis à appliquer des approches de contrôle robuste. Plus récemment, dans (Ichalal et al., 2018), des dynamiques auxiliaires et des techniques d'immersion sont considérées pour réécrire et augmenter le modèle T-S avec UPVs comme un nouveau modèle T-S avec des fonctions de pondération dépendant uniquement des variables mesurées. Cependant, cette approche ne peut pas être facilement généralisée et peut échouer à fournir les transformations requises pour le modèle initial en raison du nombre infini possible d'itérations et de la nature des entrées non linéaires du système.

- Pour contourner l'apparition des UPVsdans la modélisation T-S standard, une manière élégante est de considérer les modèles T-S avec des parties conséquentes non linéaires (N-TS) (voir par exemple, Bouarar et al. (2007); Yoneyama (2009); Dong et al. (2009); Moodi and Farrokhi (2015); Araújo et al. (2019); Moodi et al. (2019); Nagy et al. (2020, 2022)). Dans ce cas, la modélisation T-S avec l'approche de non-linéarité sectorielle est modifiée. En d'autres termes, l'approche de non-linéarité sectorielle n'est appliquée qu'aux termes non linéaires dépendant des variables d'état mesurées, ce qui conduit à un modèle N-TS n'impliquant que les variables locales mesurées, et où les termes non linéaires non mesurés sont conservés en tant que parties conséquentes non linéaires. En outre, pour faire face aux parties conséquentes non linéaires, les études précédentes considèrent généralement des conditions de Lipschitz pour fournir des conditions basées sur LMI pour la conception de filtres ou d'observateurs N-TS (voir par exemple, (Yoneyama, 2009; Nagy et al., 2020, 2022)). De plus, (Pourasghar et al., 2022; Nguyen et al., 2021) a étudié la conception d'observateurs pour les systèmes T-S à temps discret avec des parties conséquentes non linéaires, où les non-linéarités non mesurées ont été traitées en utilisant une approche basée sur les ensembles zonotopiques et le théorème de la valeur moyenne différentielle, respectivement. De même, Moodi and Farrokhi (2015) a considéré la conception d'un contrôleur basé sur des observateurs pour des systèmes T-S avec des parties conséquentes non linéaires, où la contrainte quadratique incrémentielle (Açıkmeşe and Corless, 2011) est utilisée pour faire face aux non-linéarités non mesurées, avec des améliorations conservatrices significatives par rapport à la conception de filtres T-S standard ou d'observateurs avec UPVs.

- *Nature des phénomènes de commutation:* Une autre préoccupation concernant l'analyse documentaire ci-dessus est liée à la nature des phénomènes de commutation. Il existe une vaste littérature traitant de tels phénomènes dans le cadre des systèmes linéaires à commutations ; voir, par exemple, (Liberzon, 2003; Daafouz et al., 2002; Belkhiat et al., 2011a; Zhang and Gao, 2010; Mao et al., 2014; Yuan et al., 2018; Ren et al., 2018; Han et al., 2020; Etienne et al., 2020; Fei et al., 2022; Alessandri and Sanfelice, 2022). Le cas le plus simple reste celui où la loi de commande à commutations, l'observateur du filtre, partagent la même loi de commutation que la système linéaire à commutations, c'est-à-dire le cas dit synchrone à commutation (Liberzon, 2003; Daafouz et al., 2002; Belkhiat et al., 2011a; Zhang et al., 2015; Tao et al., 2017; Qi et al., 2021). Cependant, il est souvent irréaliste en pratique de considérer le cas synchrone car il nécessite la disponibilité des signaux de commutation de système, qui peuvent être difficiles à mesurer ou à estimer. Par conséquent, plusieurs études ont été réalisées dans le cas asynchrone, soit dans le cas linéaire à commutation, voir par ex, (Pettersson, 2005; Zhang and Gao, 2010; Yuan et al., 2018; Ren et al., 2018; Han et al., 2020; Etienne et al., 2020; Fei et al., 2022), soit étendues au cadre des systèmes non linéaires à commutations (Xiang et al., 2012; Wang and Tong, 2017; Hong et al., 2018; Shi et al., 2019; Zare et al., 2020; Liu and Zhao, 2020; Chekakta et al., 2021, 2022). Par conséquent, le cas asynchrone fait référence à un scénario dans lequel les règles de commutation de l'observateur ou du filtre ne correspondent pas à celles de système. Dans ce scénario plus réaliste, l'inadéquation peut être un retard (par exemple, le temps nécessaire pour estimer la règle de commutation de l'usine) (Zhang and Gao, 2010; Yuan et al., 2018; Zhai et al., 2018; Etienne et al., 2020; Fei et al., 2022), ou elle peut être due à des incertitudes de modélisation dans le contexte de lois de commutation dépendant de l'état et demeurant sur des hyperplans de commutation mal adaptés (Pettersson, 2005; Yang et al., 2019; Chekakta et al., 2021, 2022). Jusqu'à présent, les conditions de conception des systèmes à commutations peuvent être classées en deux catégories :

- Conditions dépendantes du temps de séjour (voir par exemple (Zhang and Gao, 2010; Mao et al., 2014; Yuan et al., 2018; Fei et al., 2022)), qui sont utiles pour assouplir les conditions mais nécessitent une certaine connaissance des règles de commutation. En particulier, dans ce cas, les systèmes à commutations considérés doivent rester dans chaque mode pendant au moins un temps de séjour positif minimum. Cependant, cette approche peut être inadaptée à certains systèmes à commutations, par exemple lorsque les systèmes à commutations impliquent des lois de commutation dépendantes de l'état qui évoluent en fonction d'hyperplans de commutation incertains.
- Conditions sans temps de séjour (voir par exemple (Pettersson, 2005; Yang et al., 2019; Chekakta et al., 2021, 2022)), qui sont utiles pour les lois de commutation arbitraires ou dépendantes de l'état qui évoluent par rapport à des hyperplans de commutation incertains. Bien que ces conditions ne puissent pas bénéficier de la réduction de conservatisme apportée par les conditions dépendantes du temps de séjour, elles permettent de traiter une plus grande classe de systèmes à commutations puisqu'elles nécessitent une connaissance moins restrictive des phénomènes de commutation.

CHAPTER 2

Preliminaries on Switched Takagi-Sugeno Systems

Traduction en Français : Préliminaires sur les systèmes Takagi-Sugeno à commutations

L'objectif de ce chapitre est de présenter quelques préliminaires essentiels sur les systèmes dynamiques à commutations, la modélisation, la stabilité et l'introduction de la classe de systèmes utilisée dans cette thèse, ainsi que quelques préliminaires sur la conception des observateurs et le filtrage non linéaire des classes de systèmes étudiées dans cette thèse. Section 2.2 discute d'une classe populaire de systèmes dynamiques hybrides, celle qui reçoit une attention particulière au cours des dernières décennies, est la classe des systèmes à commutations. Ils ont été étudiés depuis longtemps dans la littérature (Liberzon, 2003; Alessandri and Coletta, 2001; Daafouz et al., 2002). Au cours des deux dernières décennies, cette classe de systèmes a suscité un intérêt particulier chez les chercheurs et les ingénieurs en raison de sa capacité à modéliser une grande classe de systèmes physiques. Dans un système à commutations, l'état continu évolue sur un intervalle de temps avec une dynamique (parmi un ensemble fini de dynamiques) et ensuite avec une autre sur l'intervalle de temps suivant, ce qui peut être vu comme des systèmes multi-modèles, où un sous-système (mode) est actif à un intervalle de temps selon une fonction de commutation. La commutation entre les différentes dynamiques est régie par une loi de commutation qui spécifie le mode de fonctionnement (la dynamique) actif à chaque instant. Par conséquent, le mécanisme de commutation utilisé dans cette thèse, qui est la commutation arbitraire (commutation dépendant de l'état ou commutation sans temps d'arrêt) est illustré dans cette section, de telles lois de commutation sont plus générales et surpassent la commutation contrainte (approche du temps d'arrêt).

Dans la Section 2.3 de ce chapitre, nous nous intéressons à la représentation multi-modèle des systèmes non linéaires, dans laquelle une représentation Takagi-Sugeno floue (Takagi and Sugeno, 1985) du système non linéaire ainsi qu'une transformation par secteurs non linéaires (Tanaka and Wang, 2001) qui permet d'obtenir un modèle T-S à partir d'un système non linéaire de manière systématique, car les approches quasi-LPV/T-S représentent avec précision (représentation exacte sans aucune perte d'information) les systèmes non linéaires comme des systèmes polytopiques convexes. L'idée principale des modèles flous de Takagi-Sugeno est basée sur l'utilisation d'un ensemble de modèles locaux de structures simples (fréquemment linéaires) où chaque modèle local contribue de manière variable dans le temps au comportement global du système. La contribution de chaque modèle local au modèle global (qui est une combinaison convexe des sous-systèmes), est définie par une fonction d'activation, en général non linéaire mais vérifiant la propriété de somme convexe. L'approche T-S permet donc de réécrire un système non linéaire d'une manière plus facile à étudier en le décomposant en unités plus simples. L'intérêt d'effectuer une décomposition du système non linéaire à l'aide de ce type de modèle est que, grâce à la propriété de somme convexe, les problèmes de stabilité, de conception de contrôleurs, d'observateurs et de filtres qui ont été largement étudiés dans le cas linéaire peuvent être étendus au cas non linéaire avec des outils similaires. Par exemple, on peut citer : (Tanaka and Wang, 2001; Tuan et al., 2001a; Guerra and Vermeiren, 2004; Guelton et al., 2009a). Les fonctions d'activation dépendent des variables dites de décision ou de prémisse. Ces variables peuvent être mesurables (entrées/sorties du système) ou non mesurables (état du système,..). Il est important de noter que la classe des modèles T-S avec des variables de prémisse non mesurables est plus grande que celle avec des variables de décision mesurables. Ceci est dû au fait que le modèle T-S est généralement obtenu par la transformation en secteurs non linéaires, qui dans la plupart des cas, fait apparaître une partie ou la totalité des variables d'état dans les fonctions d'activation. Ce point est également illustré dans les travaux de (Bergsten and Palm, 2000; Ichalal et al., 2010, 2011).

En utilisant le cadre T-S pour traiter les systèmes non linéaires, qui permet de représenter exactement les systèmes non linéaires, et par extension les systèmes non linéaires à commutations par des combinaisons pondérées de systèmes linéaires dans un ensemble compact de leurs espaces d'état, grâce à leur structure convexe polytopique, les modèles T-S permettent d'étendre certains des concepts de contrôle linéaire aux cas non linéaires. Par conséquent, les modèles Takagi-Sugeno à commutations seront le sujet principal de la Section 2.4. Ensuite, une discussion sur les observateurs et les filtres non linéaires pour les systèmes Takagi-Sugeno à commutations, les variables de prémisse mesurées et non mesurées seront données pour illustrer ce problème lors de la conception des observateurs et des filtres. Ce chapitre se termine par une conclusion, où les principales contributions de cette thèse, apportées dans les chapitres suivants, sont présentées.

2.1 Introduction

The purpose of this chapter is to present some essential preliminaries about switched dynamical systems, modeling, stability and introducing the class of systems used in this thesis, and some preliminaries about observers design and nonlinear filtering of the classes of systems investigated in this this. Section 2.2 discusses a popular class of hybrid dynamical systems, the one that receives particular attention over the last few decades, is the class of switched systems. They have been studied for a long time in the literature (Liberzon, 2003; Alessandri and Coletta, 2001; Daafouz et al., 2002). During the last two decades, this class of systems has been of particular interest to researchers and engineers because of its ability to model a large class of physical systems. In a switched system, the continuous state evolves on a time interval with one dynamic (among a finite set of dynamics) and then with another on the next time interval, which can be seen as multi-model systems, where one sub-system (mode) is active at a time interval according to a switching function. The switching between the different dynamics is governed by a switching law which specifies the mode of operation (the dynamics) active at each time. Therefore, the switching mechanism used in this thesis, which is the arbitrary switching (state dependent switching or dwell-time free switching) is illustrated in this section, such switching laws are more general and outperform the constrained switching (dwell-time approach).

In Section 2.3 of this chapter we are interested in the multi-model representation of nonlinear systems, in which a fuzzy Takagi-Sugeno representation (Takagi and Sugeno, 1985) of the nonlinear system along with a transformation by nonlinear sectors (Tanaka and Wang, 2001) which allows to obtain a T-S model from a nonlinear system in a systematic way, as quasi-LPV/T-S approaches accurately represent (exact representation without any loss of information) nonlinear systems as convex polytopic systems. The main idea of Takagi-Sugeno fuzzy models is based on the use of a set of local models of simple structures (frequently linear) where each local model contributes in a time-varying way to the global behavior of the system. The contribution of each local model to the global model (which is a convex combination of the sub-systems), is defined by an activation function, in general nonlinear but verifying the convex sum property. The T-S approach thus makes it possible to rewrite a nonlinear system in a way that is easier to study by decomposing it into simpler units. The interest of performing a decomposition of the nonlinear system using this type of model is that, thanks to the convex sum property, the problems of stability, controllers, observers and filters design have been widely studied in the linear case can be extended to the nonlinear case with similar tools. For example, we can cite (Tanaka and Wang, 2001; Tuan et al., 2001a; Guerra and Vermeiren, 2004; Guelton et al., 2009a). The activation functions depend on the so-called decision or premise variables. These variables can be measurable (inputs/outputs of the system) or non-measurable (state of the system,..). It is important to note that the T-S models with unmeasured premise variables may represent a larger class of nonlinear systems compared to the T-S model with measurable premise variables (Yoneyama, 2009). This is due to the fact that the T-S model is generally obtained by the transformation into nonlinear sectors, which in most cases, makes some or all of the state variables appear in the activation functions. This point is also illustrated in the work of (Bergsten and Palm, 2000; Ichalal et al., 2010, 2011).

By using the T-S framework to deal with nonlinear systems, which allows to represent exactly the nonlinear systems, and by extension switched nonlinear systems by weighted combinations of linear systems in a compact set of their state spaces, thanks to their polytopic convex structure, the T-S models allows extending some of the linear control concepts to the nonlinear cases. Therefore, switched Takagi-Sugeno models will be the main focus of Section 2.4, then a discussion about observers and nonlinear filters for switched Takagi-Sugeno systems, measured and unmeasured premise variables will be given to illustrate such issue when designing observers and filters. This chapter ends with a conclusion, where the main contributions of this thesis, brought in the next chapters, are introduced.

2.2 Switched dynamical systems

Among the various classes of hybrid systems, the one that receives particular attention in this work is the class of switched systems. They have been studied for a long time in the literature (Liberzon, 2003; Daafouz et al., 2002; Alessandri and Coletta, 2001). During the last three decades, this class of systems has been of particular interest to researchers because of its ability to model a large class of physical systems. In a switched system, the continuous state evolves on a time interval with one dynamic (among a finite set of dynamics) and then with another on the next time interval (See Figure 2.1). The switching between the different dynamics is governed by a switching law that specifies the mode of operation (the dynamics) active at each time.

The expression of the mathematical model of a switched dynamical systems in continuous-time state space is written in the following form:

$$\sum_{\sigma(t)} = \begin{cases} \dot{x}(t) = f_{\sigma(t)}(x(t), u(t)) \\ y(t) = g_{\sigma(t)}(x(t), u(t)) \end{cases} , \quad \sigma(t) \in \{1, \ldots, m\} \tag{2.1}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state vector; $y(t) \in \mathbb{R}^{n_y}$ is the measured output vector $u(t) \in \mathbb{R}^{n_w}$ is the input signal; $\sigma(t) \in \{1, \ldots, m\}$ is the switching law; $f_{\sigma(t)}(x(t), u(t)) \in \mathbb{R}^{n_x}$, $g_{\sigma(t)}(x(t), u(t)) \in \mathbb{R}^{n_y}$ are nonlinear vector or matrix-valued functions that describe the dynamics of the considered system.

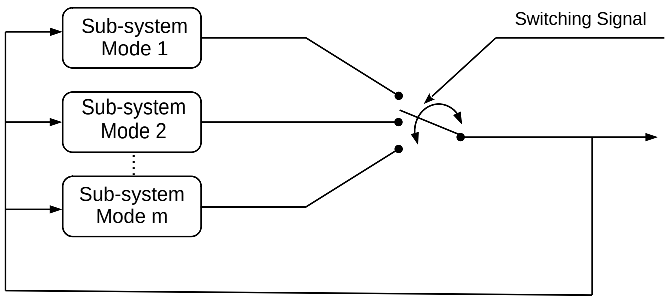

Figure 2.1: Switching mechanism of switched systems

Example 1: Temperature control system (Kurovszky, 2002)

Let's consider the example of the temperature control system in a room. The system has two distinct modes corresponding to the ON-OFF operation of the thermostat. The temperature dynamics change depending on the state of the room. The temperature increases when the heater is ON and decreases when it is OFF. Initially the heater is OFF and the room temperature is below a minimum temperature θ_{min}. When the temperature control process is started, the heater remains ON as long as the room temperature is below a maximum temperature. Once the sensor detects a room temperature equal to the maximum temperature θ_{max}, the heater turns OFF. The room temperature starts to decrease due to heat loss. When the sensor detects a room temperature equal to the minimum temperature, the heater turns ON again. The temperature of the room and the thermostat can be seen as a switched linear system whose continuous evolution is defined by the variation of the temperature x in the room and the discrete evolution by the passage of the system from the ON state to the OFF state and vice versa.

We consider that the evolution of the temperature can be modeled by the following differential equations (Kurovszky, 2002):

$$\dot{x}(t) = \begin{cases} f_1(x(t)) = -x(t) + au(t), & \text{If the heater is ON} \\ f_2(x(t)) = -x(t), & \text{If the heater is OFF} \end{cases} \tag{2.2}$$

Where $a \in \mathbb{R}^{+}$ is a positive real constant.

Then, the representation of switched dynamical system (2.2) is illustrated by the following figure:

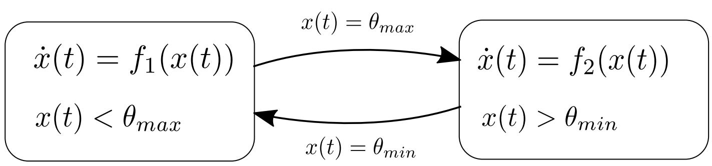

Figure 2.2: Temperature control switched system.

The nature of switching from one mode to another can be used to classify switched systems. In this context, one can distinguish, for example, the class of systems where the switching law is time dependent (dwell-time switching/constrained switching) (Xiang and Xiao, 2014), the class of systems where the switching law is state dependent (Arbitrary switching) (Yang et al., 2013), and the class of systems where the switching law is state and time dependent (Allerhand and Shaked, 2012). These two switching laws will be presented in the following sections.

2.2.1 Time-dependent switching law

In this case, the evolution of the discrete state is managed by time. Thus, the switched system evolves in a given time interval with one dynamic and then with another dynamic in the next interval. Each dynamic represents a mode. Figure shows the evolution of a switched system under a time-dependent switching law. Initially, the first mode operates with its own dynamics. At time $t = t_1$, the discrete state changes value, and the switched system evolves in the second mode until time t_2. The time between two switching instants is called the dwell time. It is noted as τ_1 when the switched system has evolved in mode 1 between time t_0 and time t_1 and τ_2 in the second mode between the two times t_1 and t_2 (Hespanha and Morse, 1999; Allerhand and Shaked, 2010; Zhang and Gao, 2010; Xiang and Xiao, 2014).

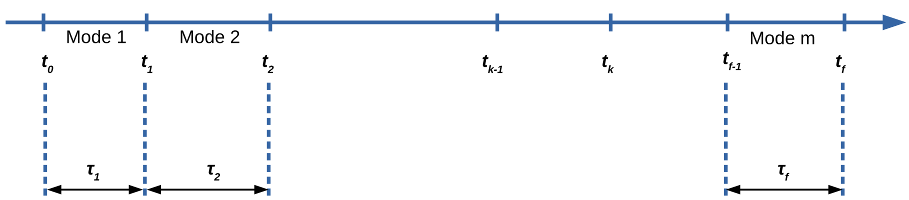

Figure 2.3: Time-dependent switching law (*Dwell-time*)

2.2.2 State-dependent switching law

In a switched system with an arbitrary switching (state-dependent switching law), switching from one mode to another depends only on the continuous state vector of the system $x(t)$. Indeed, the state space is divided into several regions of operation. In each region, a continuous-time dynamical system evolves according to appropriate differential equations. When the trajectory of the state reaches a surface of the state space called the switching surface or hyper-planes, a switching occurs instantaneously and the continuous state of the system changes dynamics (Pettersson, 2005; Yang et al., 2013; Kader et al., 2018; Yang et al., 2019). Then, two cases can be distinguished for the arbitrary switching according to linear hyper-planes.

Assumption 2.1. *In this thesis, we assume that there is no state jumps when switching occur in the switched nonlinear system, i.e. all the switched modes share the same state vector $x(t)$. Moreover, we assume that the output equation is common and linear for all the switched modes since it is usually dedicated to select the state variables (or their linear combinations) which are seen by the output $y(t)$, as well as to model the direct transfer from the exogenous disturbance input $w(t)$ to the output $y(t)$.*

Case 1 The system shares the same switching sets with the observer, filter or controller, i.e. $\hat{S}_{\hat{j}\hat{j}^+} = S_{jj^+}$ (see Figure 2.4), let us denote j and j^+ respectively the switched system's active mode at time t and its successor. Without loss of generality, we assume that the switches of the system occur within switching sets defined by linear hyper-planes defined by:

$$\mathscr{S}_{jj^+} = \{x \in \mathbb{R}^{n_x} : S_{jj^+}x = 0\}, \quad (jj^+) \in \mathcal{I}_s \tag{2.3}$$

where S_{jj^+} are real matrices with appropriate dimensions.

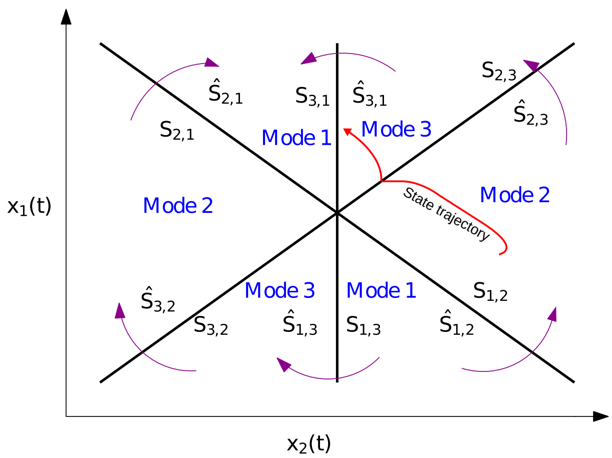

Figure 2.4: State-dependent switching law (*Case 1*)

Case 2 The system doesn't share the same switching sets with the observer, filter or controller, i.e. $\hat{S}_{\hat{j}\hat{j}^+} = S_{jj^+}$ (see Figure 2.4), where the mismatching linear hyper planes $\hat{S}_{\hat{j},\hat{j}^+}$ of the observer, filter or controller are defined as follows:

$$\mathscr{S}_{\hat{j}\hat{j}^+} = \left\{\hat{x} \in \mathbb{R}^{n_x} : \hat{S}_{\hat{j}\hat{j}^+}\hat{x} = 0\right\}, \quad \left(\hat{j}\hat{j}^+\right) \in \mathcal{I}_s \tag{2.4}$$

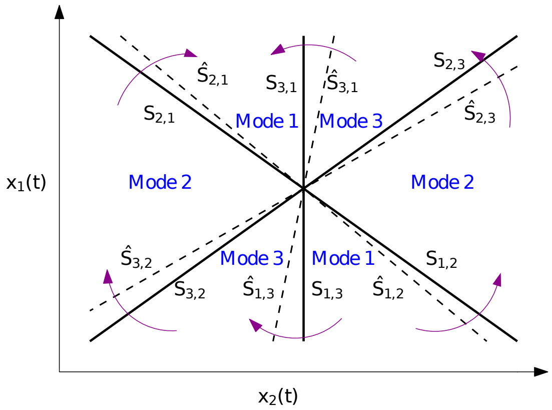

Figure 2.5: State-dependent switching law (*Case 2*)

In Figure 2.4 and 2.5, we have three modes $m = 3$, where the state space is divided using three linear hyper-planes $S_{12} = S_{21}$, $S_{23} = S_{32}$ and $S_{31} = S_{13}$. Each region defines the state space in which a nonlinear system (mode), $m = 1, 2, 3$ evolves.

These figures shows that when the state trajectory reaches the hyper-plane S_{23} between mode 2 and mode 3, the state changes dynamics and evolves in mode 3 with the appropriate differential equations for mode 3. In summary, state-dependent switching (arbitrary switching) refers to the switching according to hyper-planes related to the states in the phase plane, whereas time-dependent switching (dwell-time) occurs at a given time (Liberzon and Morse, 1999b).

Moroever, it can be observed from the second case of the state dependet switching law, that the system and the observer/filter do not share the same switching sets, which constitutes a larger class of state-dependent switching laws, where in practical applications, parametric identification of the system's switching law are not precise or biased. This latter unprecise identification can be seen as a source of anynchronism when designing observers/filter with different switching sets. Motivated by this practical issue of the mismatching switching sets, in this thesis, we are interested in this particular class of switched systems, namely, the class of nonlinear switched systems with mismatching switching sets, where each nonlinear mode is represented as Takagi-Sugeno system, and the switching mechanism is assumed to be arbitrary, i.e. state dependent switching.

2.2.3 Stability of switched dynamical systems

The stability analysis and observers/filters design of switched dynamical systems begins with a modeling phase that consists in finding, through the laws of physics and mathematical tools, a model that is close enough to the physical process considered, and it depends also on the process to be modeled and the assumptions made. A switched system is composed of a set of subsystems (modes) that switch with respect to each other according to a switching law, this class of systems can be described by the following equation Liberzon (2003); Sun and Ge (2011)

$$\dot{x}(t) = f_{\sigma(t)}(x(t), u(t)), \qquad \sigma(t) : \mathbb{R}^{+} \to \mathcal{I}_m = \{1, \ldots, m\} \tag{2.5}$$

and it can be written in a state space form as follows:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \qquad \sigma(t) : \mathbb{R}^{+} \rightarrow \mathcal{I}_m = \{1, \ldots, m\} \tag{2.6}$$

where $\sigma(t)$ is a piece-wise constant function called the switching law, $x(t) \in \mathbb{R}^{n_x}$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the control vector, $f_{\sigma(t)}(x(t))$, $A_{\sigma(t)}$ and $B_{\sigma(t)}$ are nonlinear functions and matrices, respectively, that describe the dynamics of each mode.

The objective of this section is not to make an in-depth study of the stability of switched dynamical systems. It is simply to make the reader aware of some tools that will be necessary to understand some developments that will be presented in the next two chapters.

Example 2: Switched linear system (Jabri, 2011)
Consider the following switched linear systems, composed of two subsystems:

$$\dot{x}(t) = A_i x(t), \text{ with } A_1 = \begin{bmatrix} -0.5 & -0.4 \\ 3 & -0.5 \end{bmatrix}, A_2 = \begin{bmatrix} -0.5 & -3 \\ 0.4 & -0.5 \end{bmatrix}, \tag{2.7}$$

Where the eigenvalues of the modes A_1, A_2 are respectively $\lambda_1^{1,2} = \lambda_2^{1,2} = -0.5 \pm 1.09i$.

Therefore, both modes are asymptotically stable as it is shown in Figure 2.6

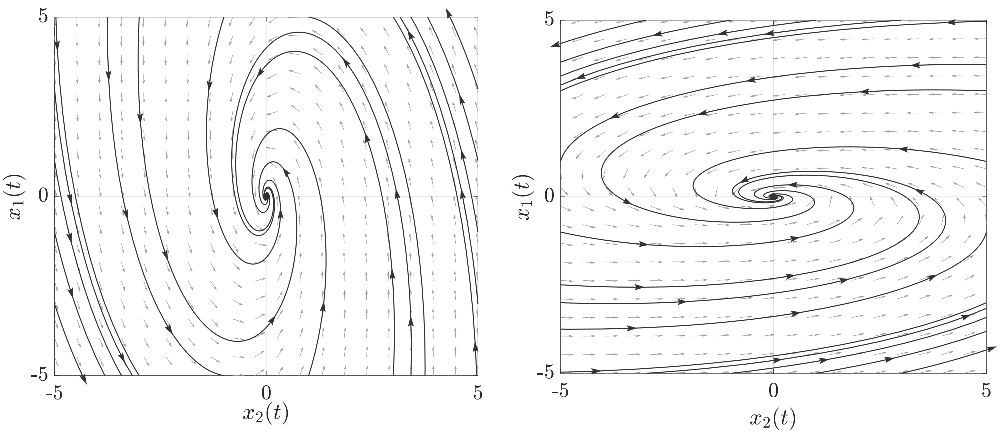

Figure 2.6: Phase trajectory of the two stable modes: Left) Mode 1. Right) Mode 2.

Now, let us consider two state-dependent switching law as follows for a particular initial conditions pair $(x_1(0) = 0.3, x_2(0) = -0.2)$:

$$\sigma_1(t) = \begin{cases} 1 & \text{If } x_1(t)x_2(t) \geq 0 \\ 2 & \text{If } x_1(t)x_2(t) < 0 \end{cases} \quad , \quad \sigma_2(t) = \begin{cases} 1 & \text{If } x_1(t)x_2(t) \leq 0 \\ 2 & \text{If } x_1(t)x_2(t) > 0 \end{cases} \tag{2.8}$$

As it is shown in the following figure, the stability of the switched system depends on the chosen switching law, the global stability of the switched system (2.7) under the switching law $\sigma_1(t)$ (unstable behavior) is depicted in the left side of Figure 2.7, where the switching law $\sigma_2(t)$ (stable behavior) is used to generate the trajectories for the global stability of the switched system (2.7) depicted in the right side of the figure. As a result, the choice of the switching law is crucial and has a considerable influence on the stability of switched systems.

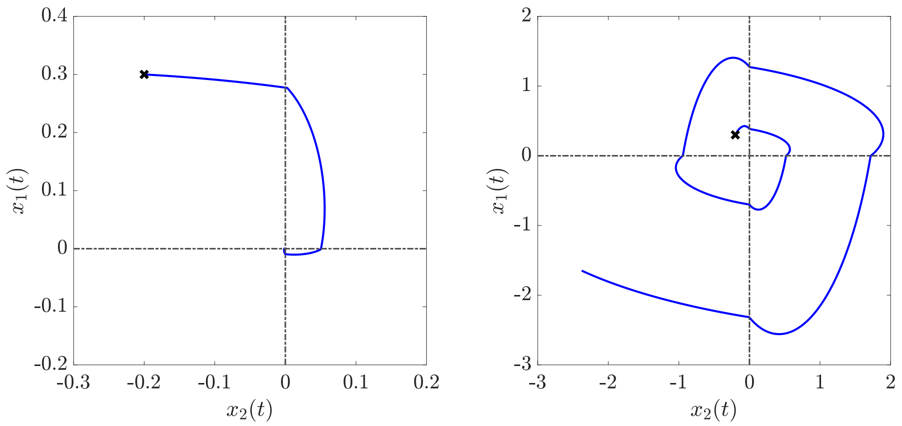

Figure 2.7: Phase planes of the two switching laws: Left) Stable behavior. Right) Unstable behavior

Two types of solutions have been proposed to verify the stability of a switched system. One way is to identify or design classes of stabilizing switching laws. Consequently, several works carried out to ensure the stability and the stabilization of switched systems exist in the literature (Yang et al., 2013; Yang and Tong, 2016; Liu et al., 2013; Allerhand and Shaked, 2012; Yang et al., 2018). Another way to verify the stability of a switched system is to establish stability conditions independent of this switching law. Hence, several works on the stability of switched systems carried out based on the second Lyapunov method (Lin and Antsaklis, 2009).

Let us now recall some concepts on the stability of switched dynamical systems in continuous-time based on Lyapunov theory. Stability in the Lyapunov sense is a general theory valid for any differential equation. This notion means that the solution of a differential equation initialized in the neighborhood of an equilibrium point remains sufficiently close to it.

Lyapunov's direct method

The direct method of Lyapunov makes it possible to analyze the stability of a system around its point equilibrium without solving it explicitly, in which the existence of a particular function provides system stability information. Consider the class of nonlinear systems described by the following equation :

$$\dot{x}(t) = f(x(t), t), \quad x(t_0) = x_0 \tag{2.9}$$

Definition 2.1. *Let $V(x,t) : \mathbb{R}^n \to \mathbb{R}$ be a continuous differentiable function. V is a candidate Lyapunov function if :*

1) $x = 0, \quad V(x,t) = 0$;

2) $\forall x \neq 0, \quad V(x,t) > 0$;

Then the nonlinear system (2.9) is locally asymptotically stable and V is a Lyapunov function if $\dot{V}(x,t) < 0, \forall x \neq 0$. Moreover, if V is radially unbounded, then the nonlinear system (2.9) is globally *asymptotically stable (GAS) if :*

$$\|x\| \to \infty \Rightarrow V(x) \to \infty \tag{2.10}$$

Therefore, if the system (2.9) admits a candidate Lyapunov function, then the origin is a point of equilibrium asymptotically or at least locally stable.

Remark 2.1. *Choosing the quadratic Lyapunov function $V(x,t) = x^T Px, P = P^T > 0$, a linear system $\dot{x}(t) = Ax(t)$ is globally asymptotically stable at the origin if and only if P is the solution of the equation $A^T P + PA < 0$.*

In the sequel, we will present some of the most used Lyapunov functions, namely, the Common Quadratic Lyapunov Function (CQLF), and the Multiple Quadratic Lyapunov Function (MQLF).

It is well known that if all subsystems of the switched system share a common quadratic Lyapunov function known as a CQLF, then the switched system is asymptotically stable for any switching signal (Dogruel and Ozguner, 1994; Lin and Antsaklis, 2009) and, according to Boyd et al. (1994), the existence conditions of a CQLF for a switched system can be expressed as LMIs.

Theorem 2.1: (Boyd et al., 1994)

Consider the following switched system:

$$\dot{x}(t) = A_\lambda x(t) \tag{2.11}$$

where $\lambda \in \mathcal{I}_m \subset \mathbb{N}^*$ is the discrete state of the system. Then, if there exists a symmetric positive matrix $P = P^T > 0 \in \mathbb{R}^{n\times n}$ such that the following inequalities are satisfied:

$$PA_i + A_i^T P < 0, \qquad i \in \mathcal{I}_m \tag{2.12}$$

Then $V(x) = x^T(t)Px(t)$ is a CQLF for the switched system given in (2.11) and it is asymptotically stable.

It should be noted that the above theorem is not sufficient, due to the fact that having a set of stable modes may not ensure the stability of the switched system. Additionally, the choice of the switching law, as mentioned earlier in this section, can affect the stability of the switched system as well. Furthermore, it is worth noting that a Common Quadratic Lyapunov Function (CQLF) is incapable of ensuring the stability of a switched system if one or more modes are unstable (i.e. having positive eigenvalues).

Due to the conservative nature of CQLFs, a different class of Lyapunov functions known as MQLFs, which is less conservative, has received a lot of attention in recent years (Peleties and DeCarlo, 1991; Branicky, 1998; Daafouz et al., 2002), the construction of a Multiple Quadratic Lyapunov Function (MQLF) involves the combination of several different quadratic Lyapunov functions. Each function is decreasing within its own region, but the global function may contain discontinuities and may not be decreasing along the state trajectories. In addition to this, it can be differentiated in a piece-wise fashion and it is continuous. These are the kinds of functions that can be utilised to make certain that a switched system is stable. In addition to this, one of the benefits of using these functions is that in comparison to CQLF, they are less conservative. The following is one way that the MQLFs can be defined when all of the switched system modes are stable:

$$V(t,x(t)) = x^T(t)P_\lambda x(t), \qquad P_\lambda = \sum_{j=1}^{m} \sigma_i(t)P_j, \quad \text{with } \sigma_j(t)\begin{cases} 1, & \text{if } j = l \\ 0, & \text{Otherwise} \end{cases} \quad \forall j \in \mathcal{I}_m. \tag{2.13}$$

And the matrix $P_j = P_j^T > 0$ verifies the asymptotic stability of the j^{th} mode.

2.3 Takagi-Sugeno fuzzy models

Takagi-Sugeno (T-S) models have been studied extensively since their introduction in 1985 (Takagi and Sugeno, 1985). These models belong to the class of convex polytopic systems and allow to extend some concepts of linear systems control and observation to the case of nonlinear systems. Historically based on the fuzzy formalism, the most recent methods of obtaining T-S models, such as the decomposition into nonlinear sectors (Tanaka and Wang, 2001), allow to exactly represent a nonlinear system on a compact space of its state variables. As a result, a T-S model is written, in an equivalent way to a Quasi-LPV (Linear Variable Parameter) model (Shamma and Cloutier, 1993), as a collection of linear dynamics (polytopes) interpolated by a set of nonlinear functions in which they verify the properties of convex sum.

In this section part of the second chapter we focus on the T-S representation of nonlinear systems as well as on the transformation by nonlinear sectors which allows us to systematically obtain a T-S model from a nonlinear system as follows:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} h_i(\xi(t))\,(A_i x(t) + B_i u(t)) \\ y(t) = \sum_{i=1}^{r} h_i(\xi(t)) C_i x(t) \end{cases} \tag{2.14}$$

Where $\xi(t)$ is the vector of premise variables, and the normalized membership functions $h_i(\xi(t))$ verify the convex sum properties:

$$h_i(\xi(t)) \geq 0, \qquad \sum_{i=1}^{r} h_i(\xi(t)) = 1 \tag{2.15}$$

2.3.1 Construction of T-S fuzzy Models

In the literature, there are mainly three approaches to obtain a nonlinear model of a system in T-S form. These different approaches are:

– **Identification**: By identification from experimental input-output data (Takagi and Sugeno, 1985; Babuška, 1998), this is historically the first method proposed and it does not require prior modeling.

– **Linearization**: By linearization of a pre-established nonlinear model around a set of operating points (Ma et al., 1998; Johansen et al., 2000; Tanaka and Wang, 2001), the degrees of freedom for the modeling is then based on the choice of these points as well as the membership functions.

– **Sector Nonlinearity Approach**: By decomposition into nonlinear sectors of a pre-established nonlinear model (Tanaka and Wang, 2001; Morère, 2001; Ohtake et al., 2003).

In the rest of this manuscript, the T-S models studied are all obtained from a decomposition into nonlinear sectors. The principle of this method is explained below.

2.3.2 Sector Nonlinearity approach

Let us explore here the well known approach to rewrite the nonlinearities in polytopic form and thus to transform a nonlinear systems into a T-S model.

Consider the following nonlinear system:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = g(x(t), u(t)) \end{cases} \tag{2.16}$$

With $x(t) \in \mathbb{R}^{n_x}$ is the system's state, $u(t) \in \mathbb{R}^{n_u}$ is control vector, and $y(t) \in \mathbb{R}^{n_y}$ is the output vector.

The nonlinear system (2.16) can be re-written in the following Quasi-LPV form:

$$\begin{cases} \dot{x}(t) = A(\xi(t))x(t) + B(\xi(t))u(t) \\ y(t) = C(\xi(t))x(t) + D(\xi(t))u(t) \end{cases} \tag{2.17}$$

Let p be the number of nonlinear functions in the nonlinear system (2.16). These functions appear in the state matrices $A(.) \in \mathbb{R}^{n_x \times n_x}$, $B(.) \in \mathbb{R}^{n_x \times n_u}$, $C(.) \in \mathbb{R}^{n_y \times n_x}$, and $D(.) \in \mathbb{R}^{n_y \times n_u}$; they generally depend on the state vector $x(t)$ and the command $u(t)$ and are denoted $\xi_i(t), i \in \mathcal{I}_p$.

Let us consider a continuous scalar nonlinear function $f : \mathbb{R} \to \mathbb{R}$, with $f(0) = 0$, there exists $\mathcal{D}_x \subseteq \mathbb{R}$ such that, $\forall x \in \mathcal{D}_x$, we have $f \in [s_1 s_2]x$ where s_1, s_2 are finite scalars. Then, we say the nonlinear sector is local if $\mathcal{D}_x \subset \mathbb{R}$ (e.g. $\mathcal{D}_x \in [-d, d]$, with $d > 0$), otherwise ($\mathcal{D}_x = \mathbb{R}$), the nonlinear sector is global (Tanaka and Wang, 2001) (see Figure 2.8).

Then, it should be noted that, an exact T-S model can be obtained from a nonlinear system by applying the sector nonlinearity approach (Tanaka and Wang, 2001), and it is valid only inside a defined domain of validity $\mathcal{D}_x$:

$$\mathcal{D}_x = \{x \in \mathbb{R}^n : \mathcal{L}_{(v)} x(t) \leq \mathcal{Q}_{(v)},\ v \in \mathcal{I}_\mathcal{V}\} \tag{2.18}$$

where $\mathcal{L}_{(v)}$, $\mathcal{Q}_{(v)}$ are given row vector and scalar respectively.

Suppose that there exists a compact $\mathcal{C}$ of $\xi(t)$ where the premise variables are bounded, in this case the ξ_j verify:

$$\xi_i(t) \in \left[\underline{\xi}_i, \bar{\xi}_i\right], \qquad \text{where} \begin{cases} \bar{\xi}_i = \max_{x,u} \{\xi_i(t)\} \\ \underline{\xi}_i = \min_{x,u} \{\xi_i(t)\}, \end{cases} \qquad i \in \mathcal{I}_p \tag{2.19}$$

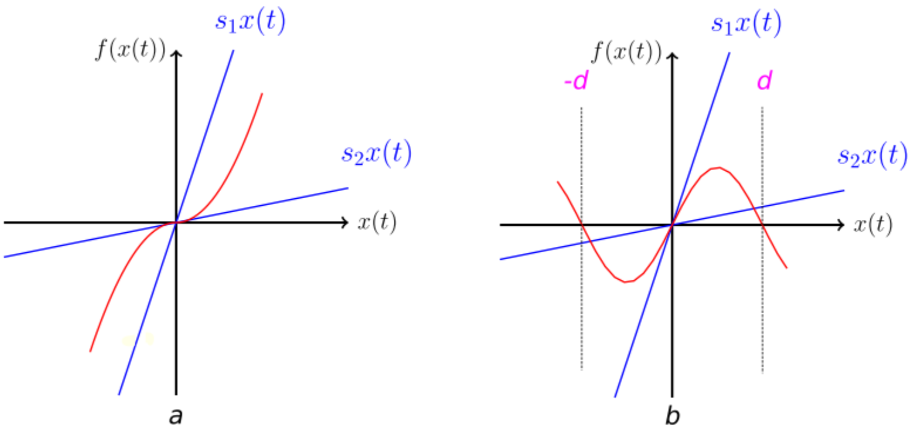

Figure 2.8: a) Global sector nonlinearity. b) Local sector nonlinearity.

The premise variables $\xi_i(t)$ can be re-written in the following form:

$$\xi_i(t) = F_i^1(\xi_i(t))\,\bar{\xi}_i + F_i^2(\xi_i(t))\,\underline{\xi}_i, \qquad \text{where} \begin{cases} F_i^1(\xi_i(t)) = \frac{\xi_i(t)-\underline{\xi}_i}{\bar{\xi}_i-\underline{\xi}_i} \\ F_i^2(\xi_i(t)) = \frac{\bar{\xi}_i-\xi_i(t)}{\bar{\xi}_i-\underline{\xi}_i} \end{cases}. \tag{2.20}$$

with $F_i^1(\xi_i(t)) + F_i^2(\xi_i(t)) = 1, F_i^1(\xi_i(t)) \geq 0, F_i^2(\xi_i(t)) \geq 0$.

The activation functions $h_i(\xi(t)), \in \mathcal{I}_r$ are obtained from the functions $F_{i,1}(\xi_i(t))$ and $F_{i,2}(\xi_i(t))$ as follows:

$$h_j(\xi(t)) = \prod_{i=1}^{r} F_i^{\delta}(\xi_i(t)), \quad \text{with} \begin{cases} h_1(\xi(t)) = F_1^1(\xi(t)) \times F_2^1(\xi(t)) \ldots F_p^1(\xi(t)) \\ h_2(\xi(t)) = F_1^1(\xi(t)) \times F_2^1(\xi(t)) \ldots F_p^2(\xi(t)) \\ h_{r-1}(\xi(t)) = F_1^2(\xi(t)) \times F_2^1(\xi(t)) \ldots F_p^1(\xi(t)) \\ h_r(\xi(t)) = F_1^2(\xi(t)) \times F_2^2(\xi(t)) \ldots F_p^2(\xi(t)) \end{cases}, \tag{2.21}$$

Remark 2.2. *The index δ is equal to 1 or 2 and it indicates which partition (F_i^1 or F_i^2) is used to define the sub-system j. Moreover, the T-S models obtained via the sector nonlinearity approach depend directly on the number of nonlinearities to be cut within the nonlinear model of the system to represent. Thus, for a system containing p nonlinear terms, the T-S model consists of $r = 2^p$ fuzzy rules.*

The matrices A_i, B_i, C_i and D_i are obtained by replacing $\xi_i(t)$ by $\bar{\xi}_i$ and $\underline{\xi}_i$ in the corresponding matrices $A(\xi(t)), B(\xi(t)), C(\xi(t))$ and $D(\xi(t))$ in (2.17). We obtain the following T-S model :

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} h_i(\xi(t))\,(A_i x(t) + B_i u(t)) \\ y(t) = \sum_{i=1}^{r} h_i(\xi(t))\,(C_i x(t) + D_i u(t)) \end{cases} \tag{2.22}$$

Where the membership functions $h_i(\xi(t))$ verify the convex sum properties:

$$h_i(\xi(t)) \geq 0, \quad \sum_{i=1}^{r} h_i(\xi(t)) = 1 \tag{2.23}$$

Example 3: T-S Modelling (Morère, 2001)
Let us consider the following nonlinear system:

$$\dot{x}(t) = x(t)\cos(x(t)) \tag{2.24}$$

the nonlinearity $f(x(t)) = \cos(x(t))$ is continuous and bounded by $\begin{bmatrix} -1 & 1 \end{bmatrix}$, it can be written in the following form:

$$\cos(x(t)) = \underbrace{\frac{\cos(x(t))+1}{2}}_{h_1(x(t))\geq 0} \times 1 + \underbrace{\frac{1-\cos(x(t))}{2}}_{h_2(x(t))\geq 0} \times (-1) \tag{2.25}$$

Therefore, the above nonlinear system can be rewritten as a T-S multi-model as follows:

$$\dot{x}(t) = \sum_{i=1}^{2} h_i(x(t)) a_i x(t) \tag{2.26}$$

where $a_1 = 1$ and $a_2 = -1$.

Example 4: T-S model of a quadrotor (Cherifi et al., 2018)
Let us consider the attitude dynamics of a quadrotor, described by:

$$\begin{cases} \ddot{\phi}(t) = \frac{J_r\dot{\theta}(t)}{I_{xx}} u_g(t) + I_{yzx}\dot{\theta}(t)\dot{\psi}(t) + \frac{u_1(t)}{I_{xx}} \\ \ddot{\theta}(t) = \frac{-J_r\dot{\phi}(t)}{I_{yy}} u_g(t) + I_{zxy}\dot{\phi}(t)\dot{\psi}(t) + \frac{u_2(t)}{I_{yy}} \\ \ddot{\psi}(t) = I_{xyz}\dot{\phi}(t)\dot{\theta}(t) + \frac{u_3(t)}{I_{zz}} \end{cases} \tag{2.27}$$

where $\phi(t)$, $\theta(t)$ and $\psi(t)$ denote respectively the roll, pitch and yaw angles, $\dot{\phi}(t)$, $\dot{\theta}(t)$ and $\dot{\psi}(t)$ represent their angular velocities, for $i \in \mathcal{I}_3$, $u_i(t)$, are the attitude torque control inputs, $u_g(t) = -w_1(t) + w_2(t) - w_3(t) + w_4(t)$ is the gyroscopic effect depending on the rotors' angular velocities w_j ($j \in \mathcal{I}_4$), we denotes $I_{yzx} = \frac{I_{yy}-I_{zz}}{I_{xx}}$, $I_{xyz} = \frac{I_{xx}-I_{yy}}{I_{zz}}$ and $I_{zxy} = \frac{I_{zz}-I_{xx}}{I_{yy}}$,

Let $x(t) = \left[\phi(t), \theta(t), \psi(t), \dot{\phi}(t), \dot{\theta}(t), \dot{\psi}(t)\right]^T$ and $u(t) = [u_1(t), u_2(t), u_3(t)]^T$ be respectively the state and input vectors. The nonlinear dynamics (2.27) can be written as the following affine-in-control nonlinear state space model:

$$\begin{cases} \dot{x}(t) = A(x(t))x(t) + B(u(t)) \\ y(t) = Cx(t) \end{cases} \tag{2.28}$$

with:

$$A(x(t)) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{J_r u_g(t)}{I_{xx}} & I_{yzx} x_5(t) \\ 0 & 0 & 0 & -\frac{J_r u_g(t)}{I_{yy}} & 0 & I_{zxy} x_4(t) \\ 0 & 0 & 0 & I_{xyz} x_5(t) & 0 & 0 \end{bmatrix},$$

$$B^T = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{I_{xx}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{I_{yy}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{I_{zz}} \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Using the sector nonlinearity approach (Tanaka and Wang, 2001), a T-S model matching exactly (2.28) can be obtained. Note that (2.28) contains a nonlinear entry $u_g(t)$, which depends on the input variables (and causes algebraic loop in practical implementation when designing controller). However, for modelling only purposes, such issue won't be investigated.

Now, assuming $|x_4(t)| \leq \bar{x}_4$ and $|x_5(t)| \leq \bar{x}_5$ and $|u_g(t)| \leq \bar{u}_g$, where $\bar{u}_g$ is a scalar defines the bound of $u_g(t)$.

Then applying the sector nonlinearity approach (Tanaka and Wang, 2001) on $A(x(t))$ with the vector of premises $\xi(t) = [x_4(t)\ x_5(t)\ u_g(t)]^T$, we obtain the following T-S model with $r = 2^3 = 8$ vertices, which exactly match (2.28) on a subset of its state space.

$$\dot{x}(t) = \sum_{i=1}^{8} h_i(\xi(t))(A_i x(t)) + Bu(t) \tag{2.29}$$

Then, the weighting functions are:

$$\begin{aligned} F_1^1(x_4(t)) &= \frac{x_4(t) - \bar{x}_4}{2\bar{x}_4}, \quad F_1^2(x_4(t)) = 1 - F_1^1(x_4(t)) \\ F_2^1(x_5(t)) &= \frac{x_5(t) - \bar{x}_5}{2\bar{x}_5}, \quad F_2^2(x_5(t)) = 1 - F_2^1(x_5(t)) \\ F_3^1\left(u_g(t)\right) &= \frac{u_g(t) - \bar{u}_g}{2\bar{u}_g}, \quad F_3^2\left(u_g(t)\right) = 1 - F_3^1\left(u_g(t)\right) \end{aligned} \tag{2.30}$$

The membership functions are computed as (2.21). For instance, the first membership function is:

$$h_1(\xi(t)) = F_1^1(x_4(t)) \times F_2^1(x_5(t)) \times F_3^1\left(u_g(t)\right) \tag{2.31}$$

Which satisfy the convex sum properties: $\sum_{i=1}^{8} h_i(\xi(t)) = 1.$ and $h_i(\xi(t)) \geq 0$. And the corresponding local linear models are obtained by substituting the corresponding values into the A matrix as follows:

$$\begin{aligned} A_1 &= A(-\bar{x}_4, -\bar{x}_5, -\bar{u}_g), \quad A_2 = A(-\bar{x}_4, -\bar{x}_5, \bar{u}_g) \\ A_3 &= A(-\bar{x}_4, \bar{x}_5, -\bar{u}_g), \quad A_4 = A(-\bar{x}_4, \bar{x}_5, \bar{u}_g) \\ A_5 &= A(\bar{x}_4, -\bar{x}_5, -\bar{u}_g), \quad A_6 = A(\bar{x}_4, -\bar{x}_5, \bar{u}_g) \\ A_7 &= A(\bar{x}_4, \bar{x}_5, -\bar{u}_g), \quad A_8 = A(\bar{x}_4, \bar{x}_5, \bar{u}_g) \end{aligned} \tag{2.32}$$

2.3.3 Preliminaries on Linear Matrix Inequalities

A large number of problems concerning dynamical systems can be put in the form of convex optimization problems of a particular type called semi-definite programming (SDPs). The main interest of SDPs is the possibility of computing the global minimum, these SDPs are also known as LMIs (Linear Matrix Inequalities). The book of Boyd et al. (1994) has popularized the use of these techniques based on convex optimization for the analysis and control of dynamical systems. Recall that a LMI is a condition expressing that a symmetric matrix whose entries are an-combinations of the decision variables is positive:

$$F(x) \triangleq F_0 + \sum_{i=1}^{m} x_i F_i > 0 \tag{2.33}$$

Where $x \in \mathbb{R}^m$ is a vector of the decision variables and $F_i \in \mathbb{R}^{n\times n}, i = 1, ..., m$ are symmetric matrices. Writing inequalities in LMI form often requires the use of special transformations. The results below present some of the most used ones.

Lemma 2.1. *(Congruence) Let be two square matrices P and Q of the same dimension. We suppose that the matrix Q is invertible, then the matrix P is positive if and only if the matrix QPQ^T is also positive.*

Lemma 2.2. *(Schur complement) Let be two symmetric matrices $P \in \mathbf{R}^{m\times m}$ and $Q \in \mathbf{R}^{n\times n}$ and let be a matrix $X \in \mathbf{R}^{m\times n}$. Then the following expressions are equivalent:*

$$\begin{bmatrix} Q & X^T \\ X & P \end{bmatrix} > 0, \quad \begin{cases} Q > 0 \\ P - XQ^{-1}X^T > 0 \end{cases}, \quad \begin{cases} P > 0 \\ Q - XP^{-1}X^T > 0 \end{cases} \tag{2.34}$$

Lemma 2.3. *(Zhou and Khargonekar, 1988) For any matrices X, Y with appropriate dimensions and $\lambda > 0$, the following inequality holds*

$$X^T Y + Y^T X \le \lambda X^T X + \lambda^{-1} Y^T Y \tag{2.35}$$

In T-S model-based approach, the problem of stability, observers, filters, and controllers design can be summarized in a positive semi-definite optimization problem involving one or more parameterized LMIs; the latter can be in the form of summations :

$$\Gamma_h = \sum_{i=1}^{r} h_i(\xi(t))\Gamma_i < 0 \tag{2.36}$$

$$\Gamma_{hh} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\xi(t))h_j(\xi(t))\Gamma_{ij} < 0 \tag{2.37}$$

A sufficient condition for the parameterized LMI (2.36) to be satisfied is that:

$$\Gamma_i < 0, \quad \forall i \in \{1,...,r\} \tag{2.38}$$

With no information available other than the convex sum properties (2.15) on the membership functions $h_i(\xi(t))$, these conditions are less conservative. On the other hand, these conditions turn out to be very restrictive in the case of parameterized LMIs with double (2.37) and triple summation. Several results are available in the literature allowing for its relaxation. In what follows, three of them ensuring different complexity-conservatism trade-offs are presented.

Lemma 2.4. *(Tanaka et al., 1998) The LMI condition (2.37) is verified for all $h_i, 1 \leq i \leq r$ satisfying the conditions (2.15) if:*

$$\begin{cases} \Gamma_{ii} < 0, & \forall i = 1,...,r \\ \Gamma_{ij} + \Gamma_{ji} < 0, & \forall (i,j) \in \{1,...,r\}^2 \text{ with } i < j \end{cases} \tag{2.39}$$

Lemma 2.5. *(Tuan et al., 2001a) The LMI condition (2.37) is verified for all $h_i, 1 \leq i \leq r$ satisfying the conditions (2.15) if:*

$$\begin{cases} \Gamma_{ii} < 0, & \forall i = 1,...,r \\ \frac{2}{r-1}\Gamma_{ii} + \Gamma_{ij} + \Gamma_{ji} < 0, & \forall (i,j) \in \{1,...,r\}^2 \text{ with } i \neq j \end{cases} \tag{2.40}$$

It is possible also to further reduce the conservatism and relax the LMI-based conditions, by introducing auxiliary variables or performing matrix transformations as in the following lemma.

Lemma 2.6. *(Peaucelle et al., 2000) For any matrices N, R, L, P and Q with appropriate dimensions, the following inequalities are equivalent.*

$$N^T P + P^T N + Q < 0 \iff \exists\, R, L : \begin{bmatrix} N^T L^T + LN + Q & (*) \\ P - L^T + R^T N & -R^T - R \end{bmatrix} < 0 \tag{2.41}$$

Moreover, to deal with the occurring zeros in the diagonals of the LMI conditions, the following lemma may be used.

Lemma 2.7. *S-Procedure (Derinkuyu and Pinar, 2006). Let $L \subset L^2$ be time invariant subspace and $\psi_k : L \to R, (k = 0, \ldots M)$, be continuous time invariant quadratic forms. Suppose that there exist $f_* \in L$ such that $\psi_1(f_*) > 0, \ldots, \psi_M(f_*) > 0$, then the following statement are equivalent:*

- *$\psi_0(f) \leq 0$ for all $f \in L$ such that $\psi_1(f) > 0, \ldots, \psi_M(f) > 0$.*
- *There exist $\tau_1 \geq 0, \ldots, \tau_M \geq 0$ such that $\psi_0(f) + \tau_1\psi_1(f) + \ldots + \tau_M\psi_M(f) \leq 0$ for all $f \in L$.*

2.3.4 Stability of Takagi-Sugeno models

Since the pioneering work of (Tanaka et al., 1998), the stability analysis and the synthesis of observers, filters, and control laws of a T-S model are mainly based on the direct Lyapunov method, whose principle was discussed in Section 2.2.3. Choosing appropriate Lyapunov functions is crucial for solving problems within the T-S framework. Different types of Lyapunov functions have been proposed in the literature, including quadratic (Tanaka and Wang, 2001), piece-wise quadratic (Johansson et al., 1999; Taniguchi and Sugeno, 2004), non-quadratic, fuzzy or multiple (Tanaka et al., 2003; Guerra and Vermeiren, 2004; Mozelli et al., 2009; Guerra et al., 2012), polynomial type (Sala, 2009; Guelton et al., 2013) or even fuzzy Lyapunov functions formulated as a line-integral (Rhee and Won, 2006; Guelton et al., 2009b). Exploiting all available information on the local behavior of the solutions is necessary to reduce conservatism. However, some approaches require additional assumptions such as knowledge of bounds on the state derivatives or membership functions. Others may result in LMI conditions of high complexity (in terms of number of constraints or variables) that cannot be solved with current LMI solvers (Nguyen et al., 2019) (in terms of the number of constraints or variables).

Let us consider the following autonomous T-S model:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(\xi(t))A_i x(t) \tag{2.42}$$

with the activation functions $h_i(\xi(t))$ verify the convex sum properties.

$$\begin{cases} \sum_{i=1}^{r} h_i(\xi(t)) = 1, & \forall t \\ 0 \leq h_i(\xi(t)) \leq 1, & i = 1, \dots, r \end{cases}$$

Moreover, let us consider a quadratic Lyapunov function of the following form:

$$V(x(t)) = x(t)^T P x(t) \tag{2.43}$$

where P is a symmetric, positive definite matrix. Then, the autonomous T-S model (2.42) is asymptotically stable (or at least locally) if, $\forall x \neq \mathcal{O}$:

$$\dot{V}(x(t)) = \dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t) = x^T(t) \sum_{i=1}^{r} h_i(\xi(t)) \left(A_i^T P + P A_i\right) x(t) \tag{2.44}$$

Because the autonomous T-S model (2.42) is a convex polytopic model, we always have $h_i(\xi(t)) \geq 0, (\forall i \in \mathcal{I}_r)$, which leads to quadratic stability conditions, given as a convex optimization procedure to be solved as LMIs and summarized by the following theorem:

Theorem 2.2: (Tanaka and Sugeno, 1992)

The autonomous T-S system (2.4) is asymptotically stable if there exists a symmetric positive matrix $P = P^T > 0 \in \mathbb{R}^{n_x \times n_x}$, such that the following LMI conditions are verified :

$$A_i^T P + P A_i < 0, \qquad i = 1, \dots, r \tag{2.45}$$

The proof of this lemma is based on the choice of a candidate Lyapunov function (2.6) and the convex sum properties (2.5), which allow us to obtain the above LMI conditions.

2.4 Switched Takagi-Sugeno Fuzzy Systems

Current physical systems are composed of many interacting components. Moreover, they have different dynamic behaviors depending on the conditions of use. These systems are represented by nonlinear dynamic models that are often complex to handle. In order to reduce the complexity of these models. In the rest of this manuscript, we use the combination of the previous two mentioned techniques in Section 2.2 and Section 2.3 based on a decomposition into simpler models, and they can be summarized as follows:

- Takagi-Sugeno (**multi-models**, Takagi and Sugeno (1985)): which are characterized by an aggregation of linear models to represent in an exact way the nonlinear behavior of the system, based on the sector nonlinearity approach (Tanaka and Wang, 2001).

- Switched systems (**multi-modes**, Liberzon and Morse (1999a)): which are characterized by a decomposition of the complex model of the system into a specific set of operation modes, with a switching mechanism between the various modes. Since we are interested in switched T-S systems under state-dependent switching (arbitrary switching) (Pettersson, 2005; Ding et al., 2011; Allerhand and Shaked, 2012; Liu et al., 2013; Yang et al., 2013, 2019). Consequently, the switching law $\sigma(t)$ is free to evolve in any way within the set $\mathcal{I}_m$. As a result of this, another representation of the switched system (2.6) can be obtained by decomposing the switching law $\sigma(t)$ into a set of m activation functions $\sigma_j(t) = 0, 1$. These activation functions will permit an indication of the one and only active mode at a given time instant. This representation is given by the following:

$$\dot{x}(t) = \sum_{j=1}^{m} \sigma_j(t)\Big(A_j x(t) + B_j u(t)\Big) \tag{2.46}$$

Where in this case the activation function $\sigma_j(t) = 1$ when the switched system is active in the l^{th} mode, otherwise $\sigma_j(t) = 0$. Generally, $\sigma_j(t)$ represents the system's switching laws defined such that the active system in the l^{th} mode leads to:

$$\begin{cases} \sigma_j(t) = 1 \text{ when } j = l. \\ \sigma_j(t) = 0 \text{ when } j \neq l. \end{cases} \tag{2.47}$$

The structural similarity between the T-S (multi-models) that were presented earlier (2.22) (Takagi and Sugeno, 1985) and switched systems (multi-modes) is made possible by the given representation of the switched system in (2.46) (Liberzon and Morse, 1999a; Hespanha and Morse, 1999; Liberzon, 2003; Daafouz et al., 2002). In point of fact, a switched system can be interpreted in this context as a convex polytopic system. This means that it is made up of a set of linear dynamics that are determined by activation functions $\sigma_j(t)$. However, the key distinction between T-S multi-models and switched systems is that in the former case, the activation functions are discrete (having values in {0,1}) and cannot be active at the same time. This is the fundamental difference between the two types of systems. On the other hand, activation functions in a T-S model are continuous (having values in the range [0,1]), and there is a combination of dynamics between the sub-models. This structural analogy will allow us to see switched systems as a part of multi-model approaches for modeling systems, which results in the formalization of a class of systems known as Switched Takagi-Sugeno systems **(multi-models multi-modes)**. Numerous works have been proposed for their stability analysis, stabilization, estimation and fault detection (Benzaouia et al., 2011; Lendek et al., 2011; Jabri, 2011; Lendek et al., 2012; Zouari, 2013; Lendek et al., 2014b; Yang and Tong, 2016; Hong et al., 2018; Jabri et al., 2018; Garbouj et al., 2020; Aravind and Balasubramaniam, 2022), the advantage of this representation is to extend the tools developed in the framework of linear systems to switched systems exhibiting nonlinear behavior at each mode (see Figure 2.9).

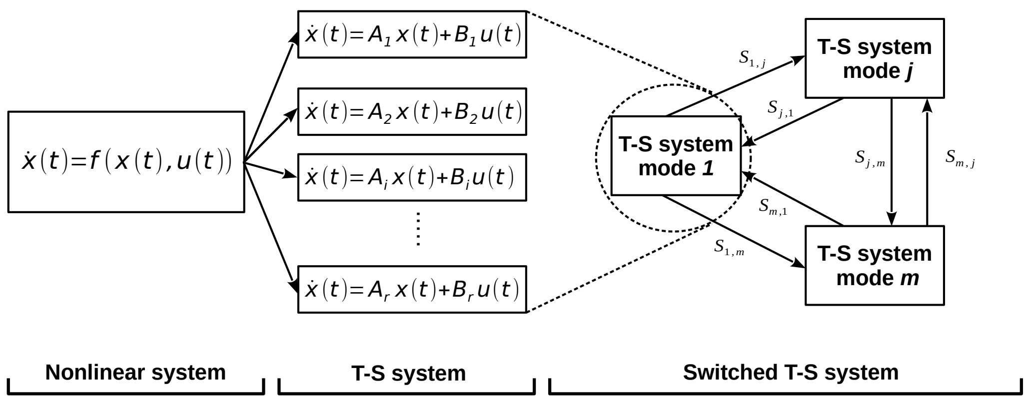

Figure 2.9: Graphical representation of switched T-S Systems.

Taking into account the two previous representations, a switched T-S system (Benzaouia et al., 2011; Lendek et al., 2014a; Jabri et al., 2018; Belkhiat et al., 2019) can be written in the following form:

$$\begin{cases} \dot{x}(t) = \sum_{j=1}^{m} \sum_{i_j=1}^{r_j} \sigma_j(t) h_{i_j}(\xi_j(t)) \left(A_{i_j} x(t) + B_{i_j} u(t) \right) \\ y(t) = Cx(t) + Ww(t) \end{cases} \tag{2.48}$$

where $x(t) \in \mathbb{R}^{n_x}, u(t) \in \mathbb{R}^{n_u}, y(t) \in \mathbb{R}^{n_y}$ and $w(t) \in \mathbb{R}^{n_w}$ are respectively the state vector, the input vector, the output vector and a time-varying L_2 norm bounded external disturbance vector. The number of switched modes is denoted by m and the number of fuzzy rules in the j^{th} mode by r_j $(j = 1, \dots, m)$, $\xi_j(t)$ are the vectors of premise variables and, $\forall i = 1, \dots, r_j$, $h_{i_j}(\xi_j(t)) \geq 0$ are fuzzy membership functions in each switched mode j, which satisfy the convex sum properties (2.15). $A_{i_j} \in \mathbb{R}^{n_x \times n_x}, B_{i_j} \in \mathbb{R}^{n_x \times n_u}, C \in \mathbb{R}^{n_y \times n_x}, W \in \mathbb{R}^{n_y \times n_u}$ are the associated matrices to each T-S subsystem and $\sigma_j(t)$ are switching functions (switching law) defined when the l^{th} mode is activated (see equation (2.47)).

Example 5: Switched nonlinear Chua's circuit (Tang and Zhao, 2019)
Let us consider the following switched nonlinear system shown in Figure 2.10, known as Chua's circuit (Zhang and Feng, 2007; Li and Zhao, 2016; Tang and Zhao, 2019; Shen et al., 2020) to illustrate the necessary steps required to get a switched T-S representation, the following set of equations describe its dynamics with $\sigma(t) \in \{1, 2\}$:

$$\begin{cases} \dot{v}_1(t) = -\frac{1}{C_{1\sigma(t)}R} v_1(t) + \frac{1}{C_{1\sigma(t)}R} v_2(t) - \frac{1}{C_{1\sigma(t)}} g_{\sigma(t)}(v_1(t)) - \frac{1}{C_{1\sigma(t)}} u(t) \\ \dot{v}_2(t) = \frac{1}{C_2 R} v_1(t) - \frac{1}{C_2 R} v_2(t) - \frac{1}{C_2} i_L(t) \\ \dot{i}_L(t) = \frac{1}{L} v_2(t) - \frac{1}{L} v_d(t). \end{cases} \tag{2.49}$$

where $v_1(t), v_2(t)$ represent the voltage across the capacitors $C_{1\sigma(t)}$ and C_2 with $i_L(t)$ as the current in the inductor L. $u(t)$ is the current from the generator (control action of the circuit). $v_d(t) = R_0 i_L(t)$ denotes the voltage loss or external disturbances. $g_{\sigma(t)}(v_1(t))$ is the current in resistor $R_{1\sigma(t)}$, which is a nonlinear function with $\alpha_{\sigma(t)} \in \{1, 0\}$:

$$g_{\sigma(t)}(v_1(t)) = \alpha_{\sigma(t)} v_1(t) + v_1^2(t) = \Big(\alpha_{\sigma(t)} + v_1(t)\Big) v_1(t) \tag{2.50}$$

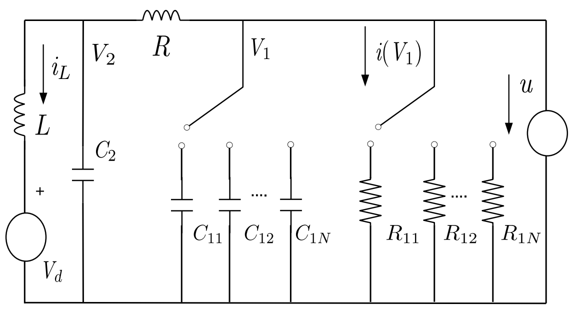

Figure 2.10: Switched Chua's Circuit.

Let $x_1(t) = v_1(t)$, $x_2(t) = v_2(t)$, $x_3(t) = i_L(t)$, then, the switched nonlinear Chua's circuit (2.49) can be written in the form of (2.6) as follows:

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} \frac{-1}{C_{1\sigma(t)}R}\Big(1 + (\alpha_{\sigma(t)} + \xi(t))\Big) & \frac{1}{C_{1\sigma(t)}R} & 0 \\ \frac{1}{C_2 R} & \frac{-1}{C_2 R} & \frac{-1}{C_2} \\ 0 & \frac{1}{L} & \frac{R_0}{L} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \frac{-1}{C_{1\sigma(t)}} \\ 0 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t) \end{cases} \tag{2.51}$$

Let us assume that $x_1(t) \in [-2, 2]$ with $\xi(t) = x_1(t)$, which can be decomposed, by applying the sector nonlinearity approach Tanaka and Wang (2001), as:

$$\xi(t) = \underbrace{\frac{x_1(t)+2}{4}}_{h_{1_j}(\xi(t))\geq 0} \times 2 + \underbrace{\frac{2-x_1(t)}{4}}_{h_{2_j}(\xi(t))\geq 0} \times (-2), \text{ where } h_{1_j}(\xi(t)) + h_{2_j}(\xi(t)) = 1, \forall j \in \{1,2\}. \tag{2.52}$$

Leading to a switched T-S model (2.48) with two switched modes ($j \in \{1,2\}$) and two T-S vertices ($i_j \in \{1,2\}$) specified by the following matrices:

$$A_{1_j} = \begin{bmatrix} \frac{-1}{C_{1j}R}\left(1+(\alpha_j+2)\right) & \frac{1}{C_{1j}R} & 0 \\ \frac{1}{C_2R} & \frac{-1}{C_2R} & \frac{-1}{C_2} \\ 0 & \frac{1}{L} & \frac{R_0}{L} \end{bmatrix}, A_{2_j} = \begin{bmatrix} \frac{-1}{C_{1j}R}\left(1+(\alpha_j-2)\right) & \frac{1}{C_{1j}R} & 0 \\ \frac{1}{C_2R} & \frac{-1}{C_2R} & \frac{-1}{C_2} \\ 0 & \frac{1}{L} & \frac{R_0}{L} \end{bmatrix}, B_{1_j} = B_{2_j} = \begin{bmatrix} \frac{-1}{C_{1j}} \\ 0 \\ 0 \end{bmatrix}. \tag{2.53}$$

Asynchronous switching: the problem of synchronicity can arise and is classified into two cases based on the evolution of the considered switched observer or H_∞ filter. Indeed, the first case, assuming a synchronous switching paradigm, corresponds to the case where the system and the observer/filter evolve in the same discrete mode at each instant, i.e., $\sigma_{\hat{j}}(t) = \hat{\sigma}_{\hat{j}}(t)$ (Shi et al., 2019; Zheng et al., 2018b; Liu and Zhao, 2020; Garbouj et al., 2020). The second case, known as the asynchronous switching paradigm, is when the observer/filter and the system operate in different modes, i.e., $\sigma_j(t) \neq \hat{\sigma}_{\hat{j}}(t)$ (Iqbal et al., 2014; Zhao et al., 2015; Zhai et al., 2018; Hong et al., 2018; Ren et al., 2018; Liu et al., 2020; Chekakta et al., 2021). The latter case (asynchronous switching), which is the focus of this thesis, is more general and practically realistic than the first (synchronous switching), but it is also more difficult to design using LMI. Furthermore, in the rest of this manuscript, the switched mechanisms of the system and the observer/filter are considered as arbitrary switching sequences (without dwell-time constraints (Mao et al., 2014; Zheng et al., 2018b; Yuan et al., 2018; Shi et al., 2019; Aravind and Balasubramaniam, 2022)) defined by linear hyperplanes (state-dependent switching) (Pettersson, 2005; Kader et al., 2018; Yang et al., 2013, 2019; Belkhiat et al., 2019; Chekakta et al., 2021, 2023)). There are two possibilities in this regard:

- The observer/filter and the system share the same switching sets, so they switch at the same instant.
- The observer/filter and the system switch according to different switching sets.

However, due to the imprecision in identifying the corresponding switching sets and the mismatch between the estimated states and the system states, especially during transients and when external disturbances occur, the first case is not practical (Pettersson, 2005; Belkhiat et al., 2019; Chekakta et al., 2021). These two cases will be considered in simulation in **Chapter 3** and **Chapter 4**, where the switched observer's/filter's mismatching hyperplanes $\bar{S}_{\hat{j},\hat{j}+}$ are defined in (2.4).

In order to present the general context of the approaches proposed in this thesis, we will review in the following section some fundamental concepts of state observation and nonlinear filtering of switched Takagi-Sugeno systems. A state of the art regarding the estimation and filtering of switched T-S systems will be presented. This allows us to highlight the advantages and limitations of the methods developed, and to define the directions of our thesis work.

2.4.1 Preliminaries on observation of switched T-S systems

In several systems, the real-time knowledge of certain variables is an important or even essential step for the synthesis of control laws, the detection, and diagnosis of faults, or supervision in a number of different types of systems (Figure 2.11) (Chen et al., 1996; Edwards et al., 2000). Utilizing physical sensors to directly measure certain quantities allows for the determination of these variables, which can be helpful in a number of contexts. However, due to various technical considerations, these are not always able to be installed, and even if they are, the cost may be prohibitive. Because of this, an additional viable option has been suggested. This one is constructed using the synthesis of state observers and robust filters, which makes it possible to estimate the current state of the system or the unmeasured/noisy outputs of certain systems(Kalman, 1960; Luenberger, 1971; Besançon, 2007). These observers are implemented as algorithms that are derived from a model of the system. They estimate the states of the system by using variables that are already known to them (measured output), specifically the command $u(t)$ and the measured output $y(t)$. In most cases, the feedback principle underpins an observer's capacity to synthesize information. In point of fact, if the initial state $x(0)$ is known and the discrete state is available, the observer is able to reconstruct from this initial state, the complete state $x(t)$ by integrating the system (2.46). However, if the initial state cannot be determined, the task at hand will be significantly more difficult. Therefore, the observer attempts to correct the estimate of the state $\hat{x}(t)$ based on the error that exists between the measured output $y(t)$ and the estimated output $\hat{y}(t)$, starting from some initial condition $\hat{x}(0)$.

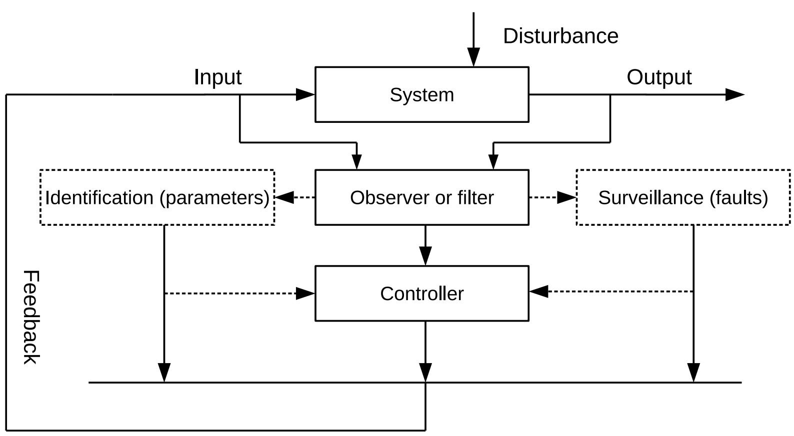

Figure 2.11: The role of an observer in a process

An extension of the Luenberger observer (Luenberger, 1971) to the class of switched T-S systems (Losero et al., 2016; Belkhiat et al., 2019; Chekakta et al., 2021; Krokavec and Filasová, 2023) is given as follows:

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{\hat{j}=1}^{m} \sum_{i_{\hat{j}}=1}^{r_{\hat{j}}} \hat{\sigma}_{\hat{j}}(t) h_{i_{\hat{j}}}(\hat{\xi}_{\hat{j}}(t)) \left(A_{i_{\hat{j}}} \hat{x}(t) + B_{i_{\hat{j}}} u(t) + K_{i_{\hat{j}}}(y - \hat{y}(t)) \right) \\ \hat{y}(t) = C\hat{x}(t) \end{cases} \tag{2.54}$$

where $\hat{x}(t) \in \mathbb{R}^{n_x}$ is the estimated state vector, $K_{i_{\hat{j}}} \in \mathbb{R}^{n_x \times n_y}$ are the observer gains to be designed, $\hat{j} = 1, \ldots, m$ denotes the observer's switching modes with $(\hat{j}, \hat{j}^+) \in \mathcal{I}_s$, $\hat{\xi}_{\hat{j}}(t)$ are vectors of the observer's premise variables, and $\hat{\sigma}_{\hat{j}}$ are the observer's switching functions, which are defined similarly to (2.47). Thus, we need to ensure the convergence of the estimation error $e(t) = x(t) - \hat{x}(t)$.

It should be noticed that the activation functions $h_{i_j}(t)$ of the switched T-S model (2.48) verify the convex sum property and depend on premise variables $\xi_j(t)$. These variables can be measurable (system inputs/outputs) or unmeasurable (system state). It is important to note that the class of switched T-S models with unmeasurable premise variables is larger than the class with measurable premise variables due to the fact that some or all of the state variables appear in the h_{i_j} functions (see (Patton et al., 1998; Bergsten and Palm, 2000; Ichalal et al., 2010, 2018), outside the switched systems framework). In **Chapter 3** and **Chapter 4**, both cases are presented for observers' and filter's design for switched T-S systems.

- *Case 1: Measured premise variables.* In the case of measurable premise variables, the observer synthesis relies on the availability of the premise variables. Therefore, the system and its observer share the same activation functions. This has the advantage that these can be factorized when studying the dynamics of the state estimation error $e(t)$ (Guelton, 2003; Lendek et al., 2011; Belkhiat et al., 2019).

- *Case 2: Unmeasured premise variables.* The study of switched T-S systems with unmeasurable premise variables is much more realistic and has been the subject of increasing interest from researchers in recent years outside the switched systems framework. This can be explained by the fact that this class of T-S models is broader than the one with measurable premise variables. Indeed, the T-S model obtained by the transformation into nonlinear sectors shows, in most cases that part or all of the state variables in the activation functions are not accessible. For example, the following work on the topic can be cited (Bergsten and Palm, 2000; Ichalal et al., 2011, 2012, 2018; Garbouj et al., 2019; Chekakta et al., 2021).

  The presence of unmeasurable premise variables prevents the dynamics of the state estimation error from being written in a simpler form. One of the proposed solutions is to consider Lipschitz conditions to deal with the mismatch that occurs from the UPVS (Bergsten and Palm, 2000; Chekakta et al., 2021). Another proposed approach is to use the differential mean value theorem instead of the Lipschitz condition in order to deal with the additive term in the dynamics of the estimation error (Ichalal et al., 2011). Another approach is to rewrite the T-S system with UPVs as an uncertain T-S system where the uncertainties are bounded (López-Estrada et al., 2017). The gains of the observer are then determined in such a way as to ensure the stability of the system generating the state estimation error while ensuring an L_2 attenuation of the transfer of the influence of the uncertainties to the state estimation error.

In (Ichalal et al., 2018), auxiliary dynamics and immersion techniques are considered to rewrite and augment the T-S model with UPVs as a new T-S model with weighting functions depending only on measured variables. However, this approach cannot be easily generalized and may fail to provide the required transformations for the initial model.

Another elegant approach to avoid UPVs is by introducing the nonlinear terms that depend on unmeasured states in the consequent parts of the T-S models, then using the incremental quadratic constraint (Açıkmeşe and Corless, 2011) to deal with nonlinear consequent parts brings a nice advantage since it allows to characterize the unknown nonlinearities satisfying incremental quadratic constraints as a set of multiplier matrices and includes as a special case Lipschitz conditions.

2.4.2 Preliminaries on H_∞ filtering of switched T-S systems

Filtering yet is another application of state estimation techniques that can be used. The filtering problem can be stated in the following manner: given a dynamic system with an exogenous input $w(t)$ and measured output $y(t)$, design a filter to estimate an output that has not been measured $z(t)$ (see Figure 2.12). Up to this point, a number of different approaches to the design of the filters have been developed. H_∞ filtering is one method that can be utilized to solve this issue. This method ensures that the mapping from the exogenous input to the filter error is minimized or is kept below a certain level that has been prescribed in terms of the H_∞ norm. This method has the advantage that the noise signals in the setting of H_∞ filtering are arbitrary signals with bounded energy, and it does not require the knowledge of exact statistics, making it more general than traditional Kalman filtering. In addition to this, it has been demonstrated that the H_∞ filter is significantly more robust against unmodeled dynamics than other systems, and many significant advancements have been achieved through various techniques (Chang, 2012; Yoneyama, 2009, 2013).

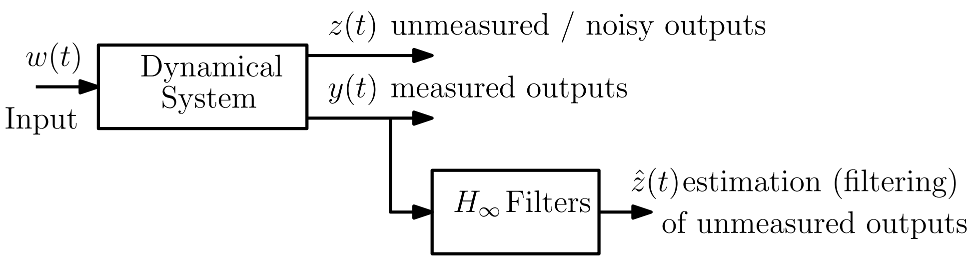

Figure 2.12: The placement of a filter in a process.

Considering the class of switched T-S systems mentioned in Section 2.4 in Equation 2.48. Then, in a similar way, a switched T-S H_∞ filter is proposed in **Chapter 4** to deal with the filtering problem under asynchronous switching as follows:

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{\hat{j}=1}^{m} \sum_{i_{\hat{j}}=1}^{r_{\hat{j}}} \hat{\sigma}_{\hat{j}}(t) h_{i_{\hat{j}}}(y(t)) \left(\hat{A}_{i_{\hat{j}}} \hat{x}(t) + \hat{B}_{i_{\hat{j}}} y(t) \right) \\ \hat{z}(t) = \sum_{\hat{j}=1}^{m} \sum_{i_{\hat{j}}=1}^{r_{\hat{j}}} \hat{\sigma}_{\hat{j}}(t) h_{i_{\hat{j}}}(y(t)) \hat{F}_{i_{\hat{j}}} \hat{x}(t) \end{cases} \tag{2.55}$$

where $\hat{x}(t) \in \mathbb{R}^{n_x}$ is the filter's state vector, $\hat{z}(t) \in \mathbb{R}^{n_z}$ is the estimate of $z(t)$, $\hat{A}_{i_{\hat{j}}} \in \mathbb{R}^{n_x \times n_x}$, $\hat{B}_{i_{\hat{j}}} \in \mathbb{R}^{n_x \times n_y}$, $\hat{F}_{i_{\hat{j}}} \in \mathbb{R}^{n_z \times n_x}$ are gain matrices to be synthesized and, for $\hat{j} = 1, \ldots, m$, $\hat{\sigma}_{\hat{j}}(t)$ is the switching law of the filter defined in (2.47), which is asynchronous regarding to the system's one ($\sigma_j(t)$).

The goal of **Chapter 4** is to propose dwell-time free LMI-based conditions for the design of H_∞ filters for switched T-S systems. To do so and to reduce the conservatism of the design conditions, LMI-based relaxation techniques (Peaucelle et al., 2000; Tuan et al., 2001a) are considered together with a descriptor redundancy approach (Schulte and Guelton, 2009; Guelton et al., 2009a; Bouarar et al., 2013; Jabri et al., 2020) to express the filtered error dynamics, such that the considered Lyapunov function candidate only depends state and output filtering errors. Two cases were investigated in **Chapter 4**, whether the premise variables are measured or unmeasured. Where the second case is handled using switched T-S systems with nonlinear consequent parts to circumvent the occurrence of the UPVs in the T-S models, and by using the incremental quadratic constraints (Açıkmeşe and Corless, 2011) to deal with the nonlinear consequent parts.

2.5 Conclusion

In this chapter, some preliminaries on state estimation techniques for switched T-S continuous systems have been presented. Since the main goal of this thesis is to investigate the state observation and robust nonlinear filtering of switched nonlinear systems. First, modelling of hybrid systems and particularly switched systems has been the focus of Section 2.2. Then, Takagi-Sugeno fuzzy models and their construction methods have been discussed in Section 2.3 . Combining these two previous classes of systems, namely, T-S fuzzy multi-models and switched systems (multi-modes), the class of switched Takagi-Sugeno systems was investigated in Section 2.4, in this section, observers design and H_∞ filtering for switched T-S systems under state-dependent switching (with mismatching switching sets) were introduced. Two main issues must be taken into consideration as it is explained in **General introduction**, namely, the occurrence of asynchronous switching and the unmeasured premise variables, which is realistic and very often the case in real world applications. These issues motivate the contributions presented in the sequel of this thesis in **Chapter 3** and **Chapter 4**. Several examples with or without physical meanings have been used to illustrate the effectiveness of the proposed theoretical work both in time-simulation and conservatism reduction.

Chapter 3

Observers design for Switched Takagi-Sugeno Systems

Traduction en Français : Synthèse des observateurs pour les systèmes Takagi-Sugeno à commutations

Initialement, l'estimation d'état se concentrait sur les systèmes linéaires, et pour ces systèmes, les filtres de Kalman et les observateurs de Luenberger donnaient des résultats satisfaisants : (Kalman, 1960; Luenberger, 1971). Dans le cas des systèmes stochastiques, le filtre de Kalman a été utilisé dans le but de minimiser la matrice de covariance de l'erreur d'estimation. D'autre part, l'observateur de Luenberger a été utilisé dans le cas de systèmes linéaires déterministes. Lorsqu'il s'agit de systèmes non linéaires, l'observation de l'état peut être un peu plus difficile, et il n'y a pas de méthode qui puisse être appliquée universellement à la synthèse des observateurs à l'heure actuelle. Les approches possibles sont soit une expansion des algorithmes linéaires existants, soit l'application d'algorithmes non linéaires particuliers. Les résultats des nombreuses études qui ont été menées sur le thème des algorithmes non linéaires spécifiques ont permis de développer un grand nombre d'algorithmes d'observation : (Phanomchoeng and Rajamani, 2010; Zemouche and Boutayeb, 2013). Les observateurs de Takagi-Sugeno sont un autre type d'observateurs basés sur les observateurs de Luenberger pour une classe de systèmes non linéaires à temps continu, où les non-linéarités ont été traitées en utilisant le cadre T-S et l'approche de non-linéarité sectorielle (Bergsten and Palm, 2000; Lendek et al., 2011; Guerra et al., 2015). Une extension des observateurs Takagi-Sugeno aux systèmes à commutations, a abouti à la formalisation des observateurs Takagi-Sugeno à commutations pour les systèmes Takagi-Sugeno à commutations. L'objectif de ce chapitre est d'aborder, d'une part, le problème de la synthèse des observateurs T-S à commutations pour les systèmes non linéaires à commutations décrits par des multi-modèles T-S avec des variables de prémisse mesurables et non mesurables. D'autre part, le problème de la commutation asynchrone dans le cadre d'une commutation dépendante de l'état (commutation arbitraire) sans considération de temps de séjour, où le système et l'observateur commutent selon des ensembles de commutation prédéfinis, donnés par des hyperplans linéaires. A ce sujet, deux cas sont considérés dans la suite de cette thèse, (*i*) le système et l'observateur partagent les mêmes ensembles de commutation ; (*ii*) l'observateur et le système commutent en fonction d'ensembles de commutation différents, ce qui est plus réaliste et pratique, en raison de l'imprécision dans l'identification des ensembles de commutation correspondants et du décalage entre les états estimés et les états du système.

Après la description du problème dans Section 3.2, nous procéderons à l'établissement de la conception de l'observateur pour les systèmes T-S à commutations sous commutation asynchrone dans le cas de variables prémisses mesurées dans Section 3.3. Section 3.4 sera consacré à la conception d'observateurs asynchrones à commutations pour les systèmes T-S à commutations avec des variables de prémisse non mesurables. Enfin, nous terminons ce chapitre par une conclusion.

3.1 Introduction

Initially, state estimation focused on linear systems, and for those systems, the Kalman filters and Luenberger observers produced satisfactory results (Kalman, 1960; Luenberger, 1971). In the case of stochastic systems, the Kalman filter was utilized for the purpose of minimizing the covariance matrix of the estimation error. On the other hand, the Luenberger observer has been utilized in the case of deterministic linear systems. When it comes to nonlinear systems, state observation can be a little more challenging, and there is no one method that can be applied universally to the synthesis of observers at this time. Either an expansion of existing linear algorithms or the application of particular nonlinear algorithms are the possible approaches. The results of the numerous studies that have been conducted on the topic of specific nonlinear algorithms have resulted in the development of a large number of observation algorithms (Phanomchoeng and Rajamani, 2010; Zemouche and Boutayeb, 2013). Takagi-Sugeno observers are another type of observers based on Luenberger observers for a class of continuous-time nonlinear systems, where the nonlinearities have been dealt with using the T-S framework and sector nonlinearity approach (Bergsten and Palm, 2000; Lendek et al., 2011; Guerra et al., 2015). An extension of Takagi-Sugeno observers to switched systems, has resulted in the formalization of switched Takagi-Sugeno observers for switched Takagi-Sugeno systems.

The aim of this chapter is to address on the one hand, the problem of synthesis of switched T-S observers for switched nonlinear systems described by T-S multi models with measurable and unmeasurable premise variables. On the other hand, the problem of asynchronous switching under state-dependent switching (arbitrary switching) without any dwell-time consideration, where the system and observer switch according to predefined switching sets, given by linear hyper-planes. In that matter, two cases are considered in the sequel of this thesis, (*i*) both the system and the observer share the same switching sets; (*ii*) the observer and the system switch accordingly to different switching sets, which more realistic and practical, due the imprecision in identification of the corresponding switching sets and the mismatch between the estimated states and the system states.

After the description of the problem in Section 3.2, we will proceed to the establishment of the observer design for switched T-S systems under asynchronous switching in the case of measured premise variables in Section 3.3. Section 3.4 will be devoted to the design of switched asynchronous observers for switched T-S systems with unmeasurable premise variables. Then, we end this chapter with a conclusion.

3.2 Problem statement

Let us recall the switched nonlinear systems described by switched T-S models given in (2.48) as follows:

$$\begin{cases} \dot{x}(t) = \sum_{j=1}^{m} \sum_{i_j=1}^{r_j} \sigma_j(t) h_{i_j}(\xi_j(t)) \Big(A_{i_j} x(t) + B_{i_j} u(t) \Big) \\ y(t) = Cx(t) + Ww(t) \end{cases} \tag{3.1}$$

where $x(t) \in \mathbb{R}^{n_x}, u(t) \in \mathbb{R}^{n_u}, y(t) \in \mathbb{R}^{n_y}$ are the state, the input, the output vectors respectively. $w(t) \in \mathbb{R}^{n_w}$ is a time-varying L2 norm bounded external disturbance. The number of switched modes is denoted by m and the number of fuzzy rules in the j^{th} mode by r_j $(j = 1, \ldots, m)$, $\xi_j(t)$ are the vectors of premise variables and $h_{i_j}(\xi_j(t)), i = 1, \ldots, r_j$ are normalized membership functions in each mode. $A_{i_j}^{n_x \times n_x}, B_{i_j}^{n_x \times n_u}, C^{n_y \times n_x}, W^{n_y \times n_w}$ are the associated matrices to each T-S subsystems, $\sigma_j(t)$ are switching functions that represent the switching law (2.47).

Our purpose in this section is the observer design for switched T-S systems. To do so, we will consider the well-known observer from the literature Luenberger (1971). which is extended to switched T-S systems and takes the same form as (3.1). However, state reconstruction under the T-S framework leads to an exact representation of the switched nonlinear systems by switched T-S systems with weighting functions depending on measurable premise variables ($\xi_j(t) \triangleq y(t)$) or (totally, partially) depend on unmeasurable premise variables, that is to say, the switched T-S system depend on unmeasured states. Furthermore, T-S systems with unmeasurable premise variables represent a larger class of systems, which is more realistic and more practical in engineering problems, despite the difficulty of the observer design and mathematical developments, this latter case will be investigated in Section 3.4

Let us consider a switched T-S observer for the first case, where the premise variables are supposed to be measurable as follows:

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{\hat{j}=1}^{m} \sum_{i_{\hat{j}}=1}^{r_{\hat{j}}} \hat{\sigma}_{\hat{j}}(t) h_{i_{\hat{j}}}(y(t)) \Big(A_{i_{\hat{j}}} \hat{x}(t) + B_{i_{\hat{j}}} u(t) + K_{i_{\hat{j}}}(y - \hat{y}(t)) \Big) \\ \hat{y}(t) = C\hat{x}(t) \end{cases} \tag{3.2}$$

where $\hat{x}(t) \in \mathbb{R}^{n_x}$ is the estimated state vector, $K_{i_{\hat{j}}} \in \mathbb{R}^{n_x \times n_y}$ are the observer gains to be designed, $\hat{j} = 1, \ldots, m$ denotes the observer's switching modes, $\xi_{\hat{j}}(t)$ are vectors of the observer's measured premise variables that depend on the measured states (outputs $y(t)$), hence, the system and observer share the same premise variables.

3.3 Observers design with measured premise variables

Several approaches have been proposed in the literature to deal with this class of systems and provide less conservative conditions (Guelton, 2003; Teixeira et al., 2003; Lendek et al., 2011; Guerra et al., 2015). In this section, we will investigate the design of robust observers for a class of switched T-S systems subject to external output disturbances, under the assumption that the premise vector depends on measured premise variables ($\xi_j(t) \triangleq y(t)$).

In order to study the convergence of the observer's states to the system's state, an estimation error is defined $e(t) = x(t) - \hat{x}(t)$, therefore, the observer's gains are determined by solving the differential equation of error dynamics $\dot{e}(t)$, and ensuring its convergence (Belkhiat et al., 2019). In this context, the dynamics of the estimation error can be expressed as follows:

$$\dot{e}(t) = \Big(A_{h_\sigma} - K_{h_{\hat{\sigma}}} C \Big) e(t) + \Big(A_{h_\sigma} - A_{h_\sigma} \Big) x(t) + \Big(B_{h_\sigma} - B_{h_\sigma} \Big) u(t) - K_{h_{\hat{\sigma}}} W w(t) \tag{3.3}$$

The following statements are defined as the observer design requirements (H_∞ Disturbance attenuation):

- *Convergence: the estimation error $e(t)$ converges to the origin without external disturbances ($w(t) = 0$).*
- *Robustness: for all non zero $w(t) \in L_2[0, \infty)$, the transfer between the external disturbances $w(t)$ and the estimation error $e(t)$ is minimized, which is represented by the following H_∞ criterion:*

$$\int_0^\infty e^T(t)e(t)dt \leq \gamma^2 \int_0^\infty w^T(t)w(t)dt \tag{3.4}$$

where $\gamma > 0$ is the disturbance attenuation level (to be minimized).

3.3.1 Main results

Now, the goal is to provide sufficient LMI-based conditions to design the gain matrices $K_{h_{\hat{\sigma}}}$ of the switched T-S observer (3.2) and ensuring the convergence of the error dynamics (3.3) (that is to say, the satisfaction of the robust H_∞ requirements defined above). The main result is summarized by the following theorem.

Theorem 3.1: (Belkhiat et al., 2019)

Let us consider the switched T-S system (3.1) and the asynchronous switched T-S observer (3.2). For given scalar $\varepsilon > 0$, if, for all combinations of $i = 1, \ldots, r_j$, $(j, \hat{j}) \in \{1, \ldots, m\}^2$ and $\left(\hat{j}, \hat{j}^+\right) \in \mathcal{I}_m$, there exist a scalar $\lambda > 0$ and the matrices $P_{\hat{j}} = P_{\hat{j}}^T > 0, G_{\hat{j}}, Y_{i_{\hat{j}}}$ such that the scalar $\gamma > 0$ is minimized and satisfying the following conditions:

i. The following equality and set of LMIs are satisfied:

$$P_{\hat{j}+} = P_{\hat{j}} + G_{\hat{j}}^T C + C^T G_{\hat{j}} \tag{3.5}$$

$$\begin{bmatrix} \mathcal{H}_e\left(P_{\hat{j}}A_{i_j} - Y_{i_{\hat{j}}}C\right) + (1+\lambda)I & (*) & (*) & (*) \\ A_{i_j}^T P_{\hat{j}} - A_{i_{\hat{j}}}^T P_{\hat{j}} & -\lambda\varepsilon & (*) & (*) \\ B_{i_j}^T P_{\hat{j}} - B_{i_{\hat{j}}}^T P_{\hat{j}} & 0 & -\lambda\varepsilon & (*) \\ -Y_{i_j} W & 0 & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{3.6}$$

ii. The state of the switched T-S observer is updated according to:

$$\hat{x}^+ = \left(I - R_{\hat{j}}^{-1}\left(CR_{\hat{j}}^{-1}\right)^\dagger C\right)\hat{x} + R_{\hat{j}}^{-1}\left(CR_{\hat{j}}^{-1}\right)^\dagger y \tag{3.7}$$

with: $R_{\hat{j}} = V_{\hat{j}}\sqrt{\Lambda_{\hat{j}}}V_{\hat{j}}^T$ and where $V_{\hat{j}} \in \mathbb{R}^{\eta\times\eta}$ is composed of the orthonormal eigenvectors of $P_{\hat{j}}$ and $\Lambda_{\hat{j}} \in \mathbb{R}^{\eta\times\eta}$ is the spectral matrix for $P_{\hat{j}}$, i.e. a Diagonal matrix composed with the eigenvalues of $P_{\hat{j}}$ and $\sqrt{\Lambda_{\hat{j}}}$ denotes a diagonal matrix containing the square root of these eigenvalues. Then the observer gains are given by $K_{i_{\hat{j}}} = P_{\hat{j}}^{-1} Y_{i_{\hat{j}}}$.

Proof. Let us consider the multiple Lyapunov function candidate:

$$V(t) = e^T(t)P_{\hat{\sigma}}e(t) \tag{3.8}$$

where $P_{\hat{\sigma}} = \sum_{\hat{j}=1}^{m} \sigma_{\hat{j}}(t)P_{\hat{j}}$, $P_{\hat{j}} = P_{\hat{j}}^T > 0$.
From (3.3), the time derivative of (3.8) can be written as:

$$\begin{aligned}\dot{V}(t) &= e(t)^T P_{\hat{\sigma}}\Big((A_{h_\sigma} - K_{h_{\hat{\sigma}}}C)e(t) + (A_{h_\sigma} - A_{h_\sigma})x(t) + (B_{h_\sigma} - B_{h_\sigma})u(t) - K_{h_{\hat{\sigma}}}Ww(t)\Big) \\ &\quad + \Big((A_{h_\sigma} - K_{h_{\hat{\sigma}}}C)e(t) + (A_{h_\sigma} - A_{h_\sigma})x(t) + (B_{h_\sigma} - B_{h_\sigma})u(t) - K_{h_{\hat{\sigma}}}Ww(t)\Big)^T P_{\hat{\sigma}}e(t) \\ &= 2\Big(e(t)^T P_{\hat{\sigma}}(A_{h_\sigma} - K_{h_{\hat{\sigma}}}C)e(t) + e(t)^T P_{\hat{\sigma}}(A_{h_\sigma} - A_{h_{\hat{\sigma}}})x(t) \\ &\quad + e(t)^T P_{\hat{\sigma}}(B_{h_\sigma} - B_{h_{\hat{\sigma}}})u(t) - e(t)^T P_{\hat{\sigma}}K_{h_{\hat{\sigma}}}Ww(t)\Big)\end{aligned} \tag{3.9}$$

Moreover, the H_∞ criterion (3.4) is satisfied if:

$$\dot{v}(t) + e^T(t)e(t) - \gamma^2 w^T(t)w(t) < 0 \tag{3.10}$$

Or equivalently with the augmented vector $\psi(t) = [e^T(t)\ x^T(t)\ u^T(t)\ w^T(t)]^T$ if:

$$\psi(t)^T\bar{\Omega}(t)\psi(t) < 0 \tag{3.11}$$

$$\text{Where } \bar{\Omega}(t) = \begin{bmatrix} \mathcal{H}_e\Big(P_{\hat{\sigma}}A_{h_\sigma} - P_{\hat{\sigma}}K_{h_{\hat{\sigma}}}C\Big) + I & (*) & (*) & (*) \\ A_{h_\sigma}^T P_{\hat{\sigma}} - A_{h_{\hat{\sigma}}}^T P_{\hat{\sigma}} & 0 & 0 & 0 \\ B_{h_\sigma}^T P_{\hat{\sigma}} - B_{h_{\hat{\sigma}}}^T P_{\hat{\sigma}} & 0 & 0 & 0 \\ -P_{\hat{\sigma}}K_{h_{\hat{\sigma}}}W & 0 & 0 & -\gamma^2 I \end{bmatrix} \tag{3.12}$$

To deal with the inequality (3.12), with a scalar $\varepsilon > 0$, let us consider the following constraint:

$$\|e(t)\|_2^2 - \varepsilon\Big(\|x(t)\|_2^2 + \|u(t)\|_2^2\Big) > 0 \tag{3.13}$$

From (3.13) and applying Lemma 2.7 , the inequality (3.12) is satisfied if there exists a scalar $\lambda > 0$ such that:

$$\begin{bmatrix} \mathcal{H}_e\left(P_{\hat{\sigma}}A_{h_\sigma} - Y_{h_j}C\right) + (1+\lambda)I & (*) & (*) & (*) \\ A^T_{h_\sigma}P_{\hat{\sigma}} - A^T_{h_{\hat{\gamma}}}P_{\hat{\sigma}} & -\lambda\varepsilon & 0 & 0 \\ B^T_{h_\sigma}P_{\hat{\sigma}} - B^T_{h_{\hat{\sigma}}}P_{\hat{\sigma}} & 0 & -\lambda\varepsilon & (*) \\ -P_{\hat{\sigma}}K_{h_{\hat{\sigma}}}W & 0 & 0 & -\gamma^2 I \end{bmatrix} < 0. \tag{3.14}$$

Then, the observer gains are determined by $Y_{h_{\hat{\sigma}}} = P_{\hat{\sigma}}K_{h_{\hat{\sigma}}}$ and the observer's state $\hat{x}(t)$ converge asymptotically to the system's state $x(t)$ at each mode. However, there is no guarantee that the global estimation error $e(t)$ will converge to zero ($e(t) \longrightarrow 0$ when $t \longrightarrow \infty$) even if the estimation error converges for each mode.

Moreover, what we need to prove is the decreasing of the Lyapunov at the observer's switching instants (when $w(t) = 0$) to avoid any dwell time conditions, by following the pioneering work of Pettersson (2005), and it is verified if:

$$(x - \hat{x}^+)^T P_{\hat{j}^+}(x - \hat{x}^+) \leq (x - \hat{x})^T P_{\hat{j}}(x - \hat{x}) \tag{3.15}$$

where $\hat{j}$ and $\hat{j}^+$ denotes respectively the observer mode and its successor, $\hat{x}^+$ is the updated value of the observer's state vector.

Assuming that $\hat{x}^+$ satisfies $y = C\hat{x}^+$, we have $C(x - \hat{x}^+) = 0$, and so, for any matrix T of appropriate dimension Pettersson (2005):

$$(x - \hat{x}^+)^T \left(T^T C + C^T T\right)(x - \hat{x}^+) = 0 \tag{3.16}$$

Therefore, from (3.16), if there exist $G_{\hat{j}}$ such that the equality (3.5) is satisfied, then the inequality (3.15) yields and we can write:

$$(x - \hat{x}^+)^T P_{\hat{j}}(x - \hat{x}^+) \leq (x - \hat{x})^T P_{\hat{j}}(x - \hat{x}) \tag{3.17}$$

Furthermore, the updated value of the observer's/filter's state $\hat{x}^+(t) \in \bar{S}_{\hat{j},\hat{j}^+}$ have now to be determined such that the previous inequality is satisfied. To this end, consider the spectral decomposition $P_{\hat{j}} = Q^T_{\hat{j}}Q_{\hat{j}}$, with $Q_{\hat{j}} = V_{\hat{j}}\sqrt{\Lambda}_{\hat{j}}V^T_{\hat{j}} \in \Re e^{n\times n}$ Derinkuyu and Pinar (2006), (3.17) is satisfied if:

$$\|Q_{\hat{j}}(x - \hat{x}^+)\| \leq \|Q_{\hat{j}}(x - \hat{x})\| \tag{3.18}$$

To find the updated value $\hat{x}^+$, lying on the hyper plane $y = C\hat{x}^+$, such that the distance $\|Q_{\hat{j}}(x - \hat{x}^+)\|$ is minimized, the optimization problem is defined as follows:

$$\begin{aligned} &\min_{\hat{x}^+} \|Q_{\hat{j}}(\hat{x}^+ - \hat{x})\| \\ &\text{subject to } : C\hat{x}^+ = y \end{aligned} \tag{3.19}$$

By introducing a scalar $\alpha_{\hat{j}} = Q_{\hat{j}}(\hat{x}^+ - \hat{x})$, we have $Q_{\hat{j}}\hat{x}^+ = \alpha_{\hat{j}} + Q_{\hat{j}}\hat{x}$, then the above stated optimization problem can be reformulated as:

$$\begin{aligned} &\min_{\hat{x}^+} \|\alpha_{\hat{j}}\| \\ &\text{subject to } : CQ_{\hat{j}}^{-1}\alpha_{\hat{j}} = y - C\hat{x} \end{aligned} \tag{3.20}$$

which admits for solution the minimum least square length to $y - C\hat{x}$, i.e.:

$$\alpha_{\hat{j}} = \left(CQ_{\hat{j}}^{-1}\right)^{\dagger}(y - C\hat{x}) \tag{3.21}$$

and so:

$$Q_{\hat{j}}\hat{x}^+ = Q_{\hat{j}}\hat{x} + \left(CQ_{\hat{j}}^{-1}\right)^{\dagger}(y - C\hat{x}) \tag{3.22}$$

Finally, left multiplying (3.22) by $Q_{\hat{j}}^{-1}$, the updated value $\hat{x}^+$ can be computed as (3.7). This ends the proof. □

3.3.2 Simulation results and discussion

The following section is dedicated to illustrate the effectiveness of the proposed switched T-S observer design approach. To this end, Let us consider a switched T-S system (3.1), with $r_j = 2$ fuzzy rules in each $m = 2$ switched mode ($j = 1, 2$), specified by the following matrices:

$$\begin{aligned} &A_{1_1} = \begin{bmatrix} -0.2 & 10 \\ -2 & -1 \end{bmatrix}, A_{2_1} = \begin{bmatrix} 1.5 & 2 \\ -6 & -0.5 \end{bmatrix}, B_{1_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_{2_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ &A_{1_2} = \begin{bmatrix} -2 & 12 \\ -2 & -1 \end{bmatrix}, A_{2_2} = \begin{bmatrix} 1.5 & 2 \\ -2.5 & 1.9 \end{bmatrix}, B_{1_2} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, B_{2_2} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, C^T = \begin{bmatrix} -1 \\ -1.5 \end{bmatrix}, W^T = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}. \end{aligned} \tag{3.23}$$

With the membership functions:

$$\text{Mode 1: } \begin{cases} h_{1_1}(y(t)) = 1 - \sin^2(y(t)) \\ h_{2_1}(y(t)) = 1 - h_{1_1}(y(t)) \end{cases}, \quad \text{Mode 2: } \begin{cases} h_{1_2}(y(t)) = 1 - (\sin(3y(t)))^2 \\ h_{2_2}(y(t)) = 1 - h_{1_2}(y(t)) \end{cases} \tag{3.24}$$

Solving the LMI-based conditions given in Theorem 3.1, we obtain the following parameters:

$$K_{1_1} = \begin{bmatrix} 53.5 \\ -62.8 \end{bmatrix}, K_{2_1} = \begin{bmatrix} 31.2 \\ -40.3 \end{bmatrix}, K_{1_2} = \begin{bmatrix} 20.3 \\ -21.8 \end{bmatrix}, K_{2_2} = \begin{bmatrix} 17.4 \\ -20.5 \end{bmatrix}, P_1 = \begin{bmatrix} 17.9 & 15.5 \\ 15.5 & 14.4 \end{bmatrix}, P_2 = \begin{bmatrix} 10.8 & 10.1 \\ 10.1 & 14.3 \end{bmatrix}, \tag{3.25}$$

with $\gamma = 5.5$, $\varepsilon = 20$, $G_1 = \begin{bmatrix} 3.6 & 0.04 \end{bmatrix}$ and $\lambda = 8.6$.

Moreover, the following external disturbance and the input signal are applied to the system:

$$w(t) = \begin{cases} \sin(0.8\pi t + 10), \forall t \in [2.5, 4], \\ 0 \text{ otherwise.} \end{cases} \quad u(t) = \begin{cases} 1, \forall t \in [0, 2], \\ 10\sin(\pi t + 2), \forall t \in [2, 5], . \end{cases} \tag{3.26}$$

The system and the observer are initialized respectively as: $x^T(0) = \begin{bmatrix} 3 & 16.5 \end{bmatrix}, \hat{x}^T(0) = \begin{bmatrix} -2 & -10 \end{bmatrix}$.

For the rest of this subsection, let us consider two cases in simulations regarding the switching hyper-planes of the switched T-S observer.

- *Case 1:* The designed observers (3.2) and the switched T-S system (3.1) share the same switching sets (2.4) ($\hat{S}_{\hat{j}\hat{j}^+} = S_{jj^+}$), which hyper-planes are defined by:

$$S_{12} = \begin{bmatrix} 1 & 0.25 \end{bmatrix}, S_{21} = \begin{bmatrix} 1 & 1 \end{bmatrix} \tag{3.27}$$

  The simulation results for this scenario are depicted in Figures 3.1 to 3.3. The trajectories of both state and estimated vectors are depicted in Figure 3.1. Figure 3.3 illustrates the estimation error evolution and the evolution of both the system's and observer's modes. In addition, the phase planes of the observer and the system are depicted in Figure 3.3. This simulation demonstrates the observer's ability to accurately estimate the system's state when they share the same switching set. Furthermore, let us emphasize that even though the system and observer share the same switching sets, they remain asynchronous (i.e.$\hat{\sigma}(t) \neq \sigma(t)$) since they do not cross the hyper-planes boundaries at precisely the same time, and the fact that the system and the observer initialized in different modes (1^{st} and 2^{nd} respectively).

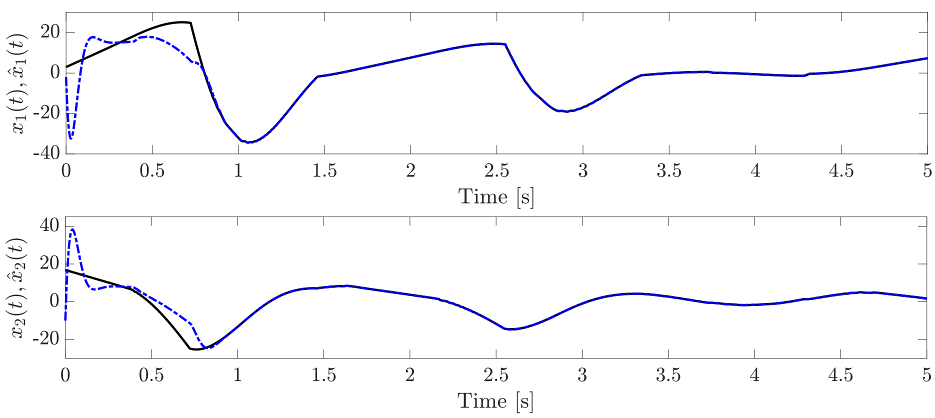

Figure 3.1: States of the switched T-S system (3.1) their estimates (*Same switching sets Case*).

- *Case 2:* This second case demonstrates what happens when only a relatively basic knowledge of the system's switching law is available. In practice, this is frequently the case when parametric identification of the system's switching law is biased or imprecise. The only difference between this case and *Case 1* is that the observer's switching sets differ from those of the switched system

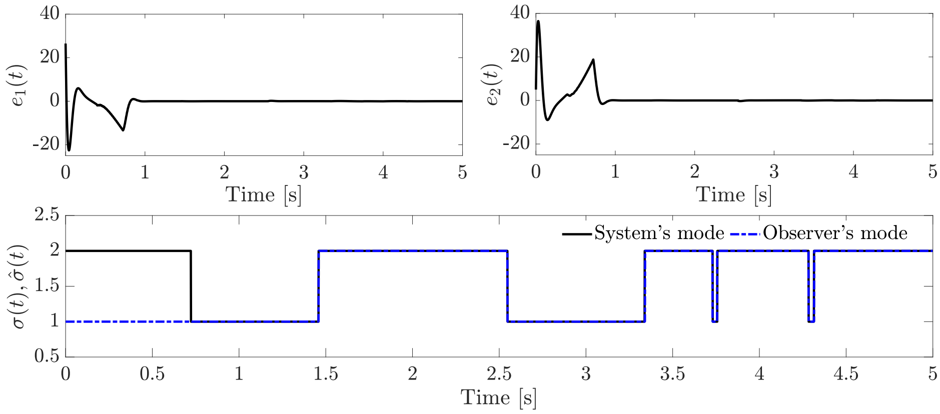

Figure 3.2: Top) Evolution of the estimation errors, Bottom) Evolution of the switched T-S system (3.1) and the switched T-S observer (3.2) modes.

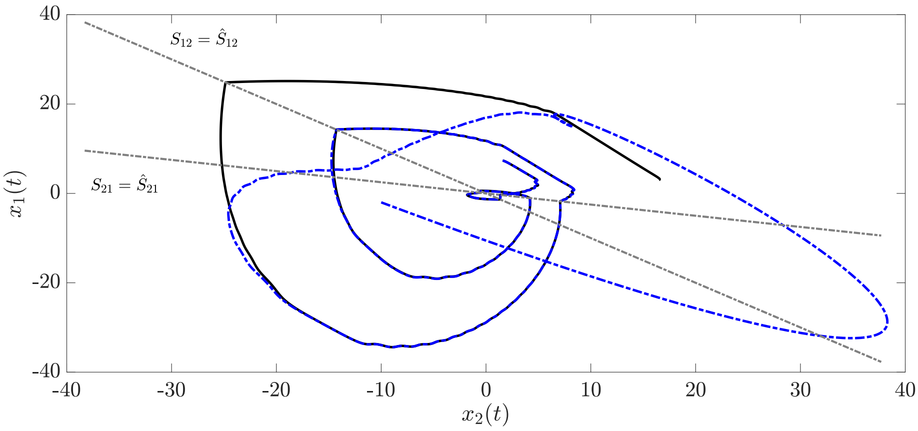

Figure 3.3: Phase planes of the switched T-S System (3.1) and the switched T-S observer (3.2).

$(\hat{S}_{\hat{j}\hat{j}^+} \neq S_{jj^+})$. To perform the simulation in this case, we define the observer's switching sets as:

$$\hat{S}_{12} = \begin{bmatrix} 0.97 & 0.22 \end{bmatrix}, S_{21} = \begin{bmatrix} 0.97 & 0.90 \end{bmatrix} \tag{3.28}$$

Figures 3.4 to 3.6 illustrate the simulation results. In contrast to the first case, Figure 3.4 demonstrates that the system and the observer switch at different time instants due to the fact that they do not share the same switching sets (see Figure 3.6). Indeed, over some intervals, the switched system (3.1) and observer (3.2) operate asynchronously ($\hat{\sigma}(t) \neq \sigma(t)$). This is clearly demonstrated in Figure 3.4 for $t \in [1.5s, 2.6s]$, where the system is in mode 2 and the observer is in mode 1, resulting in poorer estimation (see Figure 3.4 and Figure 3.5). It is also worth noting that the external output disturbance is active during this interval, making it difficult for the observer to quickly converge. However, after a while (for $t > 2.7s$), the observer better captures the system's state, and we can see that their respective switching laws tend to synchronize (see Figure 3.5).

This demonstrates that after a finite time interval, the state estimation is produced with high accuracy, confirming the validity of the proposed asynchronous switched T-S observer design methodology.

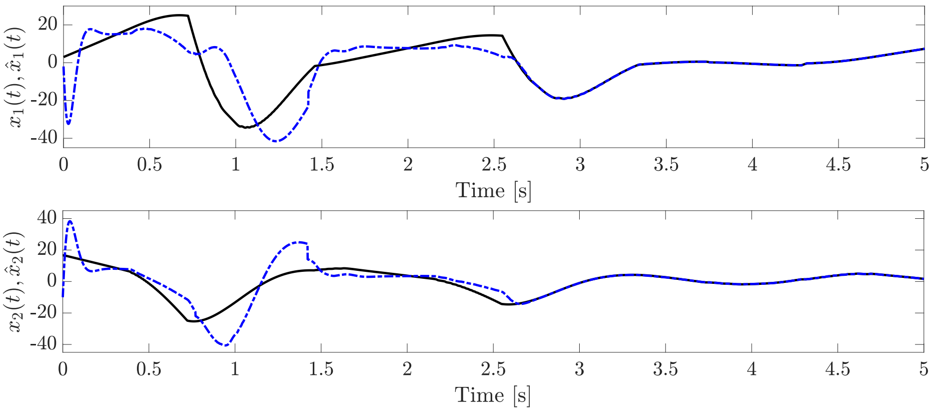

Figure 3.4: States of the switched T-S system (3.1) and their estimates (*Mismatching switching sets Case*).

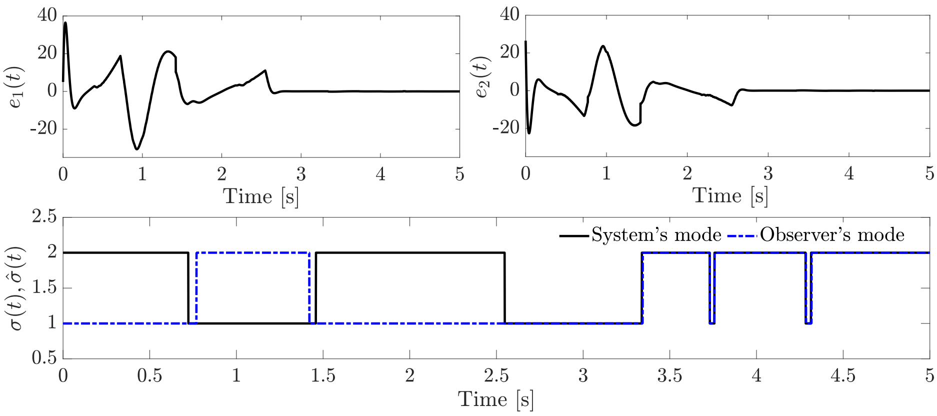

Figure 3.5: Top) Evolution of the estimation errors, Bottom) Evolution of the switched T-S System (3.1) and the switched T-S observer (3.2) modes.

The following section will be dedicated to the proposed LMI-based conditions for the design of asynchronous switched T-S observers with unmeasured premise variables and switching mismatches.

3.4 Observers design with unmeasured premise variables

The objective of the next section is to design switched T-S observers for switched T-S systems with unmeasured premise variables (the case of measured premise variables was examined in the previous section) and switching mismatches, which typically occur when implementing switched T-S observers for switched nonlinear systems. To this end, we might argue that the proposed method is useful when the system's switching sets are not precisely identified or, more broadly, when the system's switching signals are unavailable online. In addition, for T-S models, dealing with UPVs is crucial for estimating

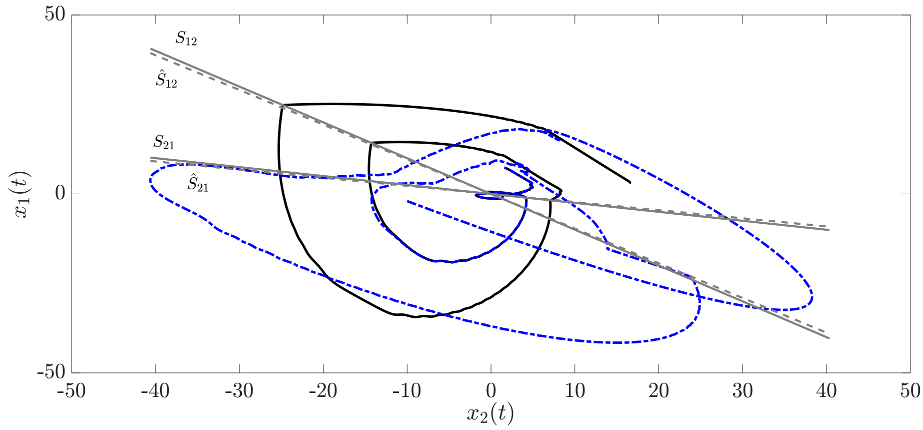

Figure 3.6: Phase planes of the switched T-S System (3.1) and the switched T-S observer (3.2).

their state (Ichalal et al., 2011, 2018). In fact, in T-S fuzzy modeling, premise variables are typically state-dependent and hence, by definition, cannot be measured online when the state estimation is necessary. All of these characteristics give the switched T-S observer design methodology proposed in the following some robustness properties against some modeling imprecision or unavailability of the considered switched nonlinear system, particularly in terms of its nonlinear entries (handled as premise variables in T-S modeling) and switching phenomena.

Let us now consider a switched T-S observer for the second case, (i.e., the premise variables are not measurable):

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{\hat{j}=1}^{m} \sum_{i_{\hat{j}}=1}^{r_{\hat{j}}} \hat{\sigma}_{\hat{j}}(t) h_{i_{\hat{j}}}(\hat{\xi}_{\hat{j}}(t)) \left(A_{i_{\hat{j}}} \hat{x}(t) + B_{i_{\hat{j}}} u(t) + K_{i_{\hat{j}}}(y - \hat{y}(t)) \right) \\ \hat{y}(t) = C\hat{x}(t) \end{cases} \tag{3.29}$$

where $\hat{\xi}_{\hat{j}}(t)$ denote the vector of unmeasurable premise variables that depend on the unmeasured states ($x(t)$). In this section, sufficient LMI-based conditions for the design of robust asynchronous switched T-S observers for a large class of switched T–S systems with UPVs by using the Lipschitz assumption are considered. In addition, H_∞ performance specifications are considered to cope with bounded output disturbances, which may represent measurement bias, sensors noise, or even faults.

Defining the estimation error $e(t) = x(t) - \hat{x}(t)$, then, its dynamics can be expressed as:

$$\begin{aligned} \dot{e}(t) =& A_{h_\sigma} x(t) + B_{h_\sigma} u(t) - A_{\hat{h}_{\hat{\sigma}}} \hat{x}(t) - B_{\hat{h}_{\hat{\sigma}}} u(t) - K_{\hat{h}_{\hat{\sigma}}}(y - \hat{y}(t)) \\ =& (A_{\hat{h}_{\hat{\sigma}}} - K_{\hat{h}_{\hat{\sigma}}} C) e(t) + (A_{h_\sigma} - A_{h_{\hat{\sigma}}}) x(t) + (B_{h_\sigma} - B_{h_{\hat{\sigma}}}) u(t) \\ &+ f_{h_{\hat{\sigma}}}(x, u) - f_{\hat{h}_{\hat{\sigma}}}(\hat{x}, x, u) - K_{\hat{h}_{\hat{\sigma}}} W w(t), \end{aligned} \tag{3.30}$$

where $f_{h_{\hat{\sigma}}}(x, u) = A_{h_{\hat{\sigma}}} x(t) + B_{h\hat{\sigma}} u(t)$ and $f_{\hat{h}_{\hat{\sigma}}}(\hat{x}, x, u) = A_{\hat{h}_{\hat{\sigma}}} x(t) + B_{\hat{h}_{\hat{\sigma}}} u(t)$.

In the sequel, we assume that f is Lipschitz with respect to x, then there exist positive scalars $\mu_{\hat{\sigma}}$

such that the following inequality holds see (Xiang and Xiang, 2008):

$$\|f_{h_{\hat{\sigma}}}(x,u) - f_{\hat{h}_{\hat{\sigma}}}(x,\hat{x},u)\| \leq \mu_{\hat{\sigma}}\|x(t) - \hat{x}(t)\| \tag{3.31}$$

Moreover, to deal with the occurring zeros in the diagonal of the proposed LMI conditions (see Equation 3.12), the S-procedure lemma (Lemma 2.7) was used in the previous section. To relax even further the LMI-based conditions for the design of the asynchronous switched T-S observer, an augmented H_∞ criterion is used and can be written as follows:

$$\int_0^\infty e^T(t)\mathcal{S}e(t)dt \leq \gamma^2 \int_0^\infty w^T(t)\mathcal{R}w(t)dt \tag{3.32}$$

where $\gamma > 0$ is the disturbance attenuation level (to be minimized), $\mathcal{S} > 0$, $\mathcal{R} = \text{diag}(\epsilon_1, \epsilon_2, I) > 0$ are given weighting diagonal matrices, and $\tilde{w}(t) = [x(t)^T\ u(t)^T\ w(t)^T]^T$.

3.4.1 Main results

In this section, sufficient LMI-based conditions are presented for the design of switched T-S observers (3.29) with UPVs to satisfy the robust H_∞ requirements described in Chapter 2 (Section 3.3) with the augmented criterion given in (3.32). These conditions are divided into four theorems with successive conservatism improvements. The cost of such enhancements is an increase in the computational burden required to solve the required convex optimization problems. Therefore, the following theorems are provided so that users can select the most applicable one based on the complexity of the applications under consideration. The following theorem summarizes the design conditions and serves as a foundation for the subsequent theorems.

Theorem 3.2: (Chekakta et al., 2021)

Consider the switched T-S system (3.1) and the asynchronous observer with unmeasured premise variables (3.29). For all combinations of $k_j \in \{1,\dots,r_j\}$, $(i_{\hat{j}}, q_{\hat{j}}) \in \{1,\dots,r_{\hat{j}}\}^2$, $(j,\hat{j}) \in \{1,\dots,m\}^2$ and $(\hat{j},\hat{j}^+) \in \mathcal{I}_m$, if there exist the scalars $\beta_{\hat{j}} > 0$, $\lambda_{\hat{j}} > 0$ and the matrices $Y_{i_{\hat{j}}} \in \mathbb{R}^{n_x \times n_y}$, $G_{\hat{j}} \in \mathbb{R}^{n_x \times n_y}$, $0 < P_{\hat{j}} = P_{\hat{j}}^T \in \mathbb{R}^{n_x \times n_x}$ such that the positive scalar γ^2 is minimized and satisfies:

$$P_{\hat{j}^+} = P_{\hat{j}} + G_{\hat{j}}^T C + C^T G_{\hat{j}} \tag{3.33}$$

$$\begin{bmatrix} A_{i_{\hat{j}}}^T P_{\hat{j}} + P_{\hat{j}} A_{i_{\hat{j}}} - C^T Y_{i_{\hat{j}}}^T - Y_{i_{\hat{j}}} C + (\beta_{\hat{j}} + \mathcal{S})I & (*) & (*) & (*) & (*) \\ A_{k_j}^T P_{\hat{j}} - A_{q_{\hat{j}}}^T P_{\hat{j}} & -\epsilon_1 \gamma^2 I & 0 & 0 & 0 \\ B_{k_j}^T P_{\hat{j}} - B_{q_{\hat{j}}}^T P_{\hat{j}} & 0 & -\epsilon_2 \gamma^2 I & 0 & 0 \\ -Y_{i_{\hat{j}}} W & 0 & 0 & -\gamma^2 I & 0 \\ P_{\hat{j}} & 0 & 0 & 0 & -\lambda_{\hat{j}} I \end{bmatrix} < 0 \tag{3.34}$$

then, with the gains $K_{i_{\hat{j}}} = P_{\hat{j}}^{-1} Y_{i_{\hat{j}}} \in \mathbb{R}^{n_x \times n_y}$, the switched T-S observer is asymptotically convergent (without external disturbances) and satisfies the H_∞ criterion (3.32) with the disturbance attenuation level γ. providing that, at the switching instants, the updated switched T-S observer states are computed as:

$$\hat{x}^+ = \left(I - Q_j^{-1}\left(CQ_j^{-1}\right)^\dagger C\right)\hat{x} + Q_j^{-1}\left(CQ_j^{-1}\right)^\dagger y, \, \forall \hat{x} \in \bar{S}_{j,j^+} \tag{3.35}$$

with $Q_{\hat{j}} = V_{\hat{j}} \sqrt{\Lambda}_{\hat{j}} V_{\hat{j}}^T \in \mathbb{R}^{n_x \times n_x}$, such that $V_{\hat{j}} \in \mathbb{R}^{n_x \times n_x}$ is a matrix composed of the orthonormal eigenvectors of $P_{\hat{j}}$ and $\Lambda_{\hat{j}} \in \mathbb{R}^{n_x \times n_x}$ is the spectral matrix for $P_{\hat{j}}$, i.e. a diagonal matrix composed with the eigenvalues of $P_{\hat{j}}$, and $\sqrt{\Lambda}_{\hat{j}}$ a diagonal matrix composed with the square root of these eigenvalues.

Proof. Let us consider the multiple Lyapunov function candidate:

$$V(t) = e^T(t) P_{\hat{\sigma}} e(t) \tag{3.36}$$

where $P_{\hat{\sigma}} = \sum_{\hat{j}=1}^{m} \sigma_{\hat{j}}(t) P_{\hat{j}}$, $P_{\hat{j}} = P_{\hat{j}}^T > 0$.
From (3.30), the time derivative of (3.36) can be written as:

$$\begin{aligned}\dot{V}(t) =& 2e^T(t)P_{\hat{\sigma}}\dot{e}(t)\\ =& 2e^T(t)P_{\hat{\sigma}}(A_{\hat{h}_{\hat{\sigma}}} - K_{\hat{h}_{\hat{\sigma}}}C)e(t) + 2e^T(t)P_{\hat{\sigma}}(A_{h_\sigma} - A_{h_{\hat{\sigma}}})x(t) + 2e^T(t)P_{\hat{\sigma}}(B_{h_\sigma} - B_{h_{\hat{\sigma}}})u(t)\\ &+ 2e^T(t)P_{\hat{\sigma}}(f_{h_{\hat{\sigma}}}(x,u) - f_{\hat{h}_{\hat{\sigma}}}(x,\hat{x},u)) - 2e^T(t)P_{\hat{\sigma}}K_{\hat{h}_{\hat{\sigma}}}Ww(t)\end{aligned} \tag{3.37}$$

Moreover, the H_∞ criterion (3.32) is satisfied if:

$$\begin{aligned}&\dot{V}(t) + e^T(t)\mathcal{S}e(t) - \gamma^2 w^T(t)\mathcal{R}w(t)\\ &= 2e^T(t)P_{\hat{\sigma}}(A_{\hat{h}_{\hat{\sigma}}} - K_{\hat{h}_{\hat{\sigma}}}C)e(t) + 2e^T(t)P_{\hat{\sigma}}(A_{h_\sigma} - A_{h_{\hat{\sigma}}})x(t)\\ &\quad + 2e^T(t)P_{\hat{\sigma}}(B_{h_\sigma} - B_{h_{\hat{\sigma}}})u(t) + 2e^T(t)P_{\hat{\sigma}}(f_{h_{\hat{\sigma}}}(x,u) - f_{\hat{h}_{\hat{\sigma}}}(x,\hat{x},u))\\ &\quad - 2e^T(t)P_{\hat{\sigma}}K_{\hat{h}_{\hat{\sigma}}}Ww(t) + e^T(t)e(t) - \gamma^2 w^T(t)w(t) < 0\end{aligned} \tag{3.38}$$

From Lemma 2.3, for any positive scalars $\lambda_{\hat{\sigma}}$, we have:

$$\begin{aligned}&2e^T(t)P_{\hat{\sigma}}(f_{h_{\hat{\sigma}}}(x,u) - f_{\hat{h}_{\hat{\sigma}}}(x,\hat{x},u))\\ &\leq \lambda_{\hat{\sigma}}(f_{h_{\hat{\sigma}}}(x,u) - f_{\hat{h}_{\hat{\sigma}}}(x,\hat{x},u))^T(f_{h_{\hat{\sigma}}}(x,u) - f_{\hat{h}_{\hat{\sigma}}}(x,\hat{x},u)) + \lambda_{\hat{\sigma}}^{-1}e^T(t)P_{\hat{\sigma}}P_{\hat{\sigma}}e(t)\end{aligned} \tag{3.39}$$

Moreover, assuming the Lipschitz condition (3.31) (Bergsten and Palm, 2000; Xiang and Xiang, 2008; Ichalal et al., 2012), it follows:

$$(f_{h_{\hat{\sigma}}}(x,u) - f_{\hat{h}_{\hat{\sigma}}}(x,\hat{x},u))^T(f_{h_{\hat{\sigma}}}(x,u) - f_{\hat{h}_{\hat{\sigma}}}(x,\hat{x},u)) \leq \mu_{\hat{\sigma}}^2 e^T(t)e(t) \tag{3.40}$$

Therefore, from (3.39) and (3.40), the inequality (3.38) is satisfied if:

$$\begin{aligned}&2e^T(t)P_{\hat{\sigma}}(A_{\hat{h}_{\hat{\sigma}}} - K_{\hat{h}_{\hat{\sigma}}}C)e(t) + 2e^T(t)P_{\hat{\sigma}}(A_{h_\sigma} - A_{h_{\hat{\sigma}}})x(t) + 2e^T(t)P_{\hat{\sigma}}(B_{h_\sigma} - B_{h_{\hat{\sigma}}})u(t)\\ &- 2e^T(t)P_{\hat{\sigma}}K_{\hat{h}_{\hat{\sigma}}}Ww(t) + \lambda_{\hat{\sigma}}\mu_{\hat{\sigma}}^2 e^T(t)e(t) + \lambda_{\hat{\sigma}}^{-1}e^T(t)P_{\hat{\sigma}}P_{\hat{\sigma}}e(t) + e^T(t)\mathcal{S}e(t) - \gamma^2 w^T(t)\mathcal{R}w(t) < 0\end{aligned} \tag{3.41}$$

Or equivalently with the augmented vector $\psi(t) = [e^T(t)\ x^T(t)\ u^T(t)\ w^T(t)]^T$ if:

$$\psi(t)^T\bar{\Omega}(t)\psi(t) < 0 \tag{3.42}$$

where $\bar{\Omega}(t) = \begin{bmatrix} A_{\hat{h}_{\hat{\sigma}}}^T P_{\hat{\sigma}} + P_{\hat{\sigma}}A_{\hat{h}_{\hat{\sigma}}} - C^T Y_{\hat{h}_{\hat{\sigma}}}^T - Y_{\hat{h}_{\hat{\sigma}}}C + (\lambda_{\hat{\sigma}}\mu_{\hat{\sigma}}^2 + \mathcal{S})I + \lambda_{\hat{\sigma}}^{-1}P_{\hat{\sigma}}P_{\hat{\sigma}} & (*) & (*) & (*)\\ A_{h_\sigma}^T P_{\hat{\sigma}} - A_{h_{\hat{\sigma}}}^T P_{\hat{\sigma}} & \epsilon_1\gamma^2 I & (*) & (*)\\ B_{h_\sigma}^T P_{\hat{\sigma}} - B_{h_{\hat{\sigma}}}^T P_{\hat{\sigma}} & 0 & \epsilon_2\gamma^2 I & (*)\\ -Y_{\hat{h}_{\hat{\sigma}}}W & 0 & 0 & -\gamma^2 I\end{bmatrix}$.

That is to say, by applying the Schur Complement:

$$\begin{bmatrix} A_{\hat{h}_{\hat{\sigma}}}^T P_{\hat{\sigma}} + P_{\hat{\sigma}}A_{\hat{h}_{\hat{\sigma}}} - C^T Y_{\hat{h}_{\hat{\sigma}}}^T - Y_{\hat{h}_{\hat{\sigma}}}C + (\beta_{\hat{\sigma}} + \mathcal{S})I & (*) & (*) & (*) & (*)\\ A_{h_\sigma}^T P_{\hat{\sigma}} - A_{h_{\hat{\sigma}}}^T P_{\hat{\sigma}} & -\epsilon_1\gamma^2 I & (*) & (*) & (*)\\ B_{h_\sigma}^T P_{\hat{\sigma}} - B_{h_{\hat{\sigma}}}^T P_{\hat{\sigma}} & (*) & -\epsilon_2\gamma^2 I & (*) & (*)\\ -Y_{\hat{h}_{\hat{\sigma}}}W & 0 & 0 & -\gamma^2 I & (*)\\ P_{\hat{\sigma}} & 0 & 0 & 0 & -\lambda_{\hat{\sigma}}I\end{bmatrix} < 0 \tag{3.43}$$

with the bijective change of variable $\beta_{\hat{\sigma}} = \lambda_{\hat{\sigma}}\mu_{\hat{\sigma}}^2$ and $Y_{\hat{h}_{\hat{\sigma}}} = P_{\hat{\sigma}}K_{\hat{h}_{\hat{\sigma}}}$.
Moreover, for the proof of the Lyapunov decreasing at the switching instant and the update of the observer states, the reader may refer to the proof of Theorem 3.1. □

Now that the LMI-based conditions given in Theorem 3.2 have been established, it is important to note that they were derived without any relaxation scheme and are, by definition, merely sufficient. Consequently, there is still a possibility for improvements. The remaining theorems provide further relaxed LMI-based conditions by way of extensions, i.e. reducing conservatism by introducing slack decision matrices through the use of Lemma 2.6 in a variety of ways. The subsequent theorem summarizes the first.

Theorem 3.3: (Chekakta et al., 2021)

Consider the switched T-S system (3.1) and the asynchronous observer with UPVs (3.29). For all combinations of $k_j \in \{1,\dots,r_j\}$, $(i_{\hat{j}}, q_{\hat{j}}) \in \{1,\dots,r_{\hat{j}}\}^2$, $(j,\hat{j}) \in \{1,\dots,m\}^2$ and $(\hat{j},\hat{j}^+) \in \mathcal{I}_m$, if there exist the scalars $\beta_{\hat{j}} > 0$, $\lambda_{\hat{j}} > 0$ and the matrices $Y_{i_{\hat{j}}} \in \mathbb{R}^{n_x \times n_y}$, $L_{\hat{j}} \in \mathbb{R}^{n_x \times n_x}$, $R_{\hat{j}} \in \mathbb{R}^{n_x \times n_x}$,

$G_{\hat{j}} \in \mathbb{R}^{n_x \times n_y}$, $0 < P_{\hat{j}} = P_{\hat{j}}^T \in \mathbb{R}^{n_x \times n_x}$ such that the positive scalar γ^2 is minimized and satisfies the conditions expressed in Theorem 3.2 with, instead of (3.34), the following inequalities:

$$\begin{bmatrix} A_{i_{\hat{j}}}^T L_{\hat{j}}^T + L_{\hat{j}} A_{i_{\hat{j}}} - Y_{i_{\hat{j}}} C - C^T Y_{i_{\hat{j}}}^T + (\beta_{\hat{j}} + \mathcal{S})I & (*) & (*) & (*) & (*) & (*) \\ P_{\hat{j}} - L_{\hat{j}}^T + R_{\hat{j}}^T A_{i_{\hat{j}}} & -R_{\hat{j}}^T - R_{\hat{j}} & (*) & (*) & (*) & (*) \\ A_{k_j}^T P_{\hat{j}} - A_{q_{\hat{j}}}^T P_{\hat{j}} & 0 & -\epsilon_1 \gamma^2 I & (*) & (*) & (*) \\ B_{k_j}^T P_{\hat{j}} - B_{q_{\hat{j}}}^T P_{\hat{j}} & 0 & 0 & -\epsilon_2 \gamma^2 I & (*) & (*) \\ -Y_{i_{\hat{j}}} W & 0 & 0 & 0 & -\gamma^2 I & (*) \\ P_{\hat{j}} & 0 & 0 & 0 & 0 & -\lambda_{\hat{j}} \end{bmatrix} < 0, \tag{3.44}$$

Then, with the gains are given by $K_{i_{\hat{j}}} = P_{\hat{j}}^{-1} Y_{i_{\hat{j}}} \in \mathbb{R}^{n_x \times n_y}$, the observer is asymptotically convergent (without external disturbances) and the H_∞ criterion (3.4) is satisfied with the external disturbance attenuation level γ.

Proof. Straightforward from the conditions expressed in Theorem 3.2, by the application of Lemma 2.6 on the first diagonal bloc of (3.34) in Theorem 3.2. □

By introducing the slack decision variables $L_{\hat{j}}$ and $R_{\hat{j}}$ from the application of Lemma 2.6 on the first diagonal bloc of (3.34) in Theorem 3.3. This latter provides the simplest proposed way to relax the conditions of Theorem 3.2. Also, given Lemma 2.6, it is clear that Theorem 3.3 includes Theorem 3.2 as a special case. To further relax the proposed LMI-based criteria, we will apply Lemma 2.6 in a more generalized way, i.e. on the entire matrix inequality (3.34). As a summary, the objective of the following two theorems is to relax further the conditions provided in Theorem 3.3. Thus, a different approach to apply Peaucelle's transformation (see Lemma 2.6) is given in Theorems 3.4 and 3.5 to reduce even more the conservatism.

Theorem 3.4: (Chekakta et al., 2021)

Consider the switched T-S system (3.1) and the asynchronous observer with UPVs (3.29). For all combinations of $k_j \in \{1, \ldots, r_j\}$, $(i_{\hat{j}}, \tilde{i}_{\hat{j}}, q_{\hat{j}}) \in \{1, \ldots, r_{\hat{j}}\}^3$, $(j, \hat{j}) \in \{1, \ldots, m\}^2$ and $(\hat{j}, \hat{j}^+) \in \mathcal{I}_m$, if there exist the scalars $\beta_{\hat{j}} > 0$, $\lambda_{\hat{j}} > 0$ and the matrices $Y_{i_{\hat{j}}} \in \mathbb{R}^{n_x \times n_y}$, $G_{\hat{j}} \in \mathbb{R}^{n_x \times n_y}$, $L_{\tilde{i}_{\hat{j}}} \in \mathbb{R}^{(2n_x+n_u)\times(2n_x+n_u)}$, $R_{\tilde{i}_{\hat{j}}} \in \mathbb{R}^{(2n_x+n_u)\times(2n_x+n_u)}$, $Z_{\hat{j}}^1 \in \mathbb{R}^{n_x \times n_x}$, $Z_{\hat{j}}^2 \in \mathbb{R}^{n_x \times n_x}$, $Z_{\hat{j}}^3 \in \mathbb{R}^{n_x \times n_u}$, $Z_{\hat{j}}^4 \in \mathbb{R}^{n_u \times n_x}$, $Z_{\hat{j}}^5 \in \mathbb{R}^{n_u \times n_x}$, $Z_{\hat{j}}^6 \in \mathbb{R}^{n_u \times n_u}$ and $P_{\hat{j}} = P_{\hat{j}}^T > 0 \in \mathbb{R}^{n_x \times n_x}$ such that the positive scalar γ^2 is minimized and satisfies the conditions expressed in Theorem 3.2 with, instead of (3.34), the following inequality:

$$\left[\begin{array}{c|c|c} \tilde{A}^T_{i_{\hat{j}}k_jq_{\hat{j}}}L^T_{\tilde{i}_{\hat{j}}} + L_{\tilde{i}_{\hat{j}}}\tilde{A}_{i_{\hat{j}}k_jq_{\hat{j}}} + \tilde{H}_{i_{\hat{j}}} & (*) & (*) \\ \hline \tilde{P}_{\hat{j}} - L^T_{\tilde{i}_{\hat{j}}} + R^T_{\tilde{i}_{\hat{j}}}\tilde{A}_{i_{\hat{j}}k_jq_{\hat{j}}} & -R^T_{\tilde{i}_{\hat{j}}} - R_{\tilde{i}_{\hat{j}}} & (*) \\ \hline \begin{matrix} -Y_{i_{\hat{j}}}W & 0 & 0 \\ P_{\hat{j}} & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} -\gamma^2 I & (*) \\ 0 & -\lambda_{\hat{j}} I \end{matrix} \end{array}\right] < 0 \tag{3.45}$$

with

$$\tilde{A}^T_{i_{\hat{j}}k_jq_{\hat{j}}} = \begin{bmatrix} A^T_{i_{\hat{j}}} & 0 & 0 \\ A^T_{k_j} - A^T_{q_{\hat{j}}} & 0 & 0 \\ B^T_{k_j} - B^T_{q_{\hat{j}}} & 0 & 0 \end{bmatrix}, \tilde{P}_{\hat{j}} = \begin{bmatrix} P_{\hat{j}} & 0 & 0 \\ Z^1_{\hat{j}} & Z^2_{\hat{j}} & Z^3_{\hat{j}} \\ Z^4_{\hat{j}} & Z^5_{\hat{j}} & Z^6_{\hat{j}} \end{bmatrix}, \tilde{H}_{i_{\hat{j}}} = \begin{bmatrix} \mathcal{H}e(-Y_{i_{\hat{j}}}C) + (\beta_{\hat{j}} + \mathcal{S})I & 0 & 0 \\ 0 & -\epsilon_1\gamma^2 I & 0 \\ 0 & 0 & -\epsilon_2\gamma^2 I \end{bmatrix}$$

Then, with the gains given by $K_{i_{\hat{j}}} = P^{-1}_{\hat{j}} Y_{i_{\hat{j}}} \in \mathbb{R}^{n_x \times n_y}$, the observer is asymptotically convergent (without external disturbances) and the H_∞ criterion (3.32) is satisfied with the external disturbance attenuation level γ.

Proof. From Lemma 2.6, if the conditions of Theorem 3.4 holds, then the following inequalities are satisfied:

$$\left[\begin{array}{c|c} \tilde{A}^T_{i_{\hat{j}}k_jq_{\hat{j}}}\tilde{P}_{\hat{j}} + \tilde{P}_{\hat{j}}\tilde{A}_{i_{\hat{j}}k_jq_{\hat{j}}} + \tilde{H}_{i_{\hat{j}}} & (*) \\ \hline \begin{matrix} -Y_{i_{\hat{j}}}W & 0 & 0 \\ P_{\hat{j}} & 0 & 0 \end{matrix} & \begin{matrix} -\gamma^2 I & 0 \\ 0 & -\lambda_{\hat{j}} I \end{matrix} \end{array}\right] < 0 \tag{3.46}$$

which are equivalent, by opening the matrices $\tilde{A}_{i_{\hat{j}}k_jq_{\hat{j}}}$, $\tilde{P}_{\hat{j}}$ and $\tilde{H}_{i_{\hat{j}}}$, to the inequalities (3.34) of Theorem 3.2. □

Once again, Theorem 3.4 being obtained from the application of Lemma 2.6, it obviously includes Theorem 3.2 as a special case. Moreover, it also includes Theorem 3.3 as a special case. Indeed, from (3.45), the inequality (3.44) can be recovered by setting the slack decision matrices as the particular case:

$$Z^1_{\hat{j}} = Z^2_{\hat{j}} = \cdots = Z^6_{\hat{j}} = 0, \; L_{\tilde{i}_{\hat{j}}} = \begin{bmatrix} L_{\hat{j}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } R_{\tilde{i}_{\hat{j}}} = \begin{bmatrix} R_{\hat{j}} & 0 & 0 \\ 0 & -\delta I & 0 \\ 0 & 0 & -\delta I \end{bmatrix}, \tag{3.47}$$

Then, by applying the Schur complement to cope with the last two diagonal blocs of $R_{\tilde{i}_{\hat{j}}}$ and taking the scalar $\delta > 0$ as small as possible ($\delta \to 0$).

Now, to further relax the condition (3.45), let us point out that it refers to a four sum parameterized LMI (with the index $i_{\hat{j}}$, $\tilde{i}_{\hat{j}}$, k_j and $q_{\hat{j}}$). However, the structure of the slack decision variables ($L_{\tilde{i}_{\hat{j}}}$, $R_{\tilde{i}_{\hat{j}}}$, $Z^1_{\hat{j}}$, $Z^2_{\hat{j}}$,...) can be arbitrarily extended to provide full index compensations of the whole inequalities, but ineluctably, with an increase of the computational cost. This last theoretical result is summarized by the following theorem.

Theorem 3.5: (Chekakta et al., 2021)

Consider the switched T-S system (3.1) and the asynchronous observer with UPVs (3.29). For all combinations of $(i_{\hat{j}}, \tilde{i}_{\hat{j}}, k_j, \tilde{k}_j, q_{\hat{j}}, \tilde{q}_{\hat{j}}) \in \{1, \dots, r_{\hat{j}}\}^6$, $(j, \hat{j}) \in \{1, \dots, m\}^2$ and $(\hat{j}, \hat{j}^+) \in \mathcal{I}_m$, if there exist the scalars $\beta_{\hat{j}} > 0$, $\lambda_{\hat{j}} > 0$ and the matrices $Y_{i_{\hat{j}}} \in \mathbb{R}^{n_x \times n_y}$, $G_{\hat{j}} \in \mathbb{R}^{n_x \times n_y}$, $L_{\tilde{i}_{\hat{j}}\tilde{k}_j\tilde{q}_{\hat{j}}} \in \mathbb{R}^{(2n_x+n_u)\times(2n_x+n_u)}$, $R_{\tilde{i}_{\hat{j}}\tilde{k}_j\tilde{q}_{\hat{j}}} \in \mathbb{R}^{(2n_x+n_u)\times(2n_x+n_u)}$, $Z^1_{i_{\hat{j}}\tilde{i}_{\hat{j}}k_j\tilde{k}_jq_{\hat{j}}\tilde{q}_{\hat{j}}} \in \mathbb{R}^{n_x \times n_x}$, $Z^2_{i_{\hat{j}}\tilde{i}_{\hat{j}}k_j\tilde{k}_jq_{\hat{j}}\tilde{q}_{\hat{j}}} \in \mathbb{R}^{n_x \times n_x}$, $Z^3_{i_{\hat{j}}\tilde{i}_{\hat{j}}k_j\tilde{k}_jq_{\hat{j}}\tilde{q}_{\hat{j}}} \in \mathbb{R}^{n_x \times n_u}$, $Z^4_{i_{\hat{j}}\tilde{i}_{\hat{j}}k_j\tilde{k}_jq_{\hat{j}}\tilde{q}_{\hat{j}}} \in \mathbb{R}^{n_u \times n_x}$, $Z^5_{i_{\hat{j}}\tilde{i}_{\hat{j}}k_j\tilde{k}_jq_{\hat{j}}\tilde{q}_{\hat{j}}} \in \mathbb{R}^{n_u \times n_u}$, $Z^6_{i_{\hat{j}}\tilde{i}_{\hat{j}}k_j\tilde{k}_jq_{\hat{j}}\tilde{q}_{\hat{j}}} \in \mathbb{R}^{n_u \times n_u}$ and $P_{\hat{j}} = P_{\hat{j}}^T > 0 \in \mathbb{R}^{n_x \times n_x}$, such that the positive scalar γ^2 is minimized and satisfies the conditions expressed in Theorem 3.2 with, instead of (3.34), the following inequality:

$$\left[\begin{array}{c|c|c} \tilde{A}^T_{i_{\hat{j}}k_jq_{\hat{j}}} L^T_{\tilde{i}_{\hat{j}}\tilde{k}_j\tilde{q}_{\hat{j}}} + L_{\tilde{i}_{\hat{j}}\tilde{k}_j\tilde{q}_{\hat{j}}} \tilde{A}_{i_{\hat{j}}k_jq_{\hat{j}}} + \tilde{H}_{i_{\hat{j}}} & (*) & (*) \\ \hline \tilde{P}_{\hat{j}} - L^T_{\tilde{i}_{\hat{j}}\tilde{k}_j\tilde{q}_{\hat{j}}} + R^T_{\tilde{i}_{\hat{j}}\tilde{k}_j\tilde{q}_{\hat{j}}} \tilde{A}_{i_{\hat{j}}k_jq_{\hat{j}}} & -R^T_{\tilde{i}_{\hat{j}}\tilde{k}_j\tilde{q}_{\hat{j}}} - R_{\tilde{i}_{\hat{j}}\tilde{k}_j\tilde{q}_{\hat{j}}} & (*) \\ \hline \begin{matrix} -Y_{i_{\hat{j}}}W & 0 & 0 \\ P_{\hat{j}} & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} -\gamma^2 I & (*) \\ 0 & -\lambda_{\hat{j}} I \end{matrix} \end{array}\right] < 0 \tag{3.48}$$

with:

$$\tilde{A}^T_{i_{\hat{j}}k_jq_{\hat{j}}} = \begin{bmatrix} A^T_{i_{\hat{j}}} & 0 & 0 \\ A^T_{k_j} - A^T_{q_{\hat{j}}} & 0 & 0 \\ B^T_{k_j} - B^T_{q_{\hat{j}}} & 0 & 0 \end{bmatrix}, \tilde{H}_{i_{\hat{j}}} = \begin{bmatrix} \mathcal{H}e(-Y_{i_{\hat{j}}}C) + (\beta_{\hat{j}} + \mathcal{S})I & 0 & 0 \\ 0 & -\epsilon_1\gamma^2 I & 0 \\ 0 & 0 & -\epsilon_1\gamma^2 I \end{bmatrix}$$

$$\tilde{P}_{\hat{j}} = \begin{bmatrix} P_{\hat{j}} & 0 & 0 \\ Z^1_{i_{\hat{j}}\tilde{i}_{\hat{j}}k_j\tilde{k}_jq_{\hat{j}}\tilde{q}_{\hat{j}}} & Z^2_{i_{\hat{j}}\tilde{i}_{\hat{j}}k_j\tilde{k}_jq_{\hat{j}}\tilde{q}_{\hat{j}}} & Z^3_{i_{\hat{j}}\tilde{i}_{\hat{j}}k_j\tilde{k}_jq_{\hat{j}}\tilde{q}_{\hat{j}}} \\ Z^4_{i_{\hat{j}}\tilde{i}_{\hat{j}}k_j\tilde{k}_jq_{\hat{j}}\tilde{q}_{\hat{j}}} & Z^5_{i_{\hat{j}}\tilde{i}_{\hat{j}}k_j\tilde{k}_jq_{\hat{j}}\tilde{q}_{\hat{j}}} & Z^6_{i_{\hat{j}}\tilde{i}_{\hat{j}}k_j\tilde{k}_jq_{\hat{j}}\tilde{q}_{\hat{j}}} \end{bmatrix},$$

Then, with the gains given by $K_{i_{\hat{j}}} = P^{-1}_{\hat{j}} Y_{i_{\hat{j}}} \in \mathbb{R}^{n_x \times n_y}$, the observer is asymptotically convergent (without external disturbances) and the H_∞ criterion (3.32) is satisfied with the external disturbance attenuation level γ.

Proof. Straightforward from the proof of Theorem 3.4 by choosing the structure of the slack decision variables as $L_{\tilde{i}_{\hat{j}}\tilde{k}_j\tilde{q}_{\hat{j}}}$, $R_{\tilde{i}_{\hat{j}}\tilde{k}_j\tilde{q}_{\hat{j}}}$, $Z^1_{i_{\hat{j}}\tilde{i}_{\hat{j}}k_j\tilde{k}_jq_{\hat{j}}\tilde{q}_{\hat{j}}}$, $Z^2_{i_{\hat{j}}\tilde{i}_{\hat{j}}k_j\tilde{k}_jq_{\hat{j}}\tilde{q}_{\hat{j}}}$, and so on. □

From the proof of Theorem 3.5, it is clear that it includes Theorem 3.4, which in turn includes Theorem 3.3, Theorem 3.2 and Theorem 3.1 in Section 3.3 as well. Moreover, as pointed out in the following remark, if they are suitable for the design of asynchronous switched T-S observers (3.29), they also include the synchronous case as a particular one.

Remark 3.1. *Let us highlight that the synchronous observer design is a particular case of the proposed approach in Theorem 3.1, Theorem 3.2, Theorem 3.3, Theorem 3.4 and Theorem 3.5. Indeed, the LMI-based conditions* (3.5), (3.6) *(or* (3.34) *or* (3.44)*, or* (3.45)*, or* (3.48)*) can be readily adapted to provide sufficient LMI-based synchronous observer design. To do so, just replace the index $\hat{j}$ by j and the index $q_{\hat{j}}$ by k_j. This leads to eliminate the terms $A_{k_j}^T P_{\hat{j}} - A_{q_j}^T P_{\hat{j}}$ and $B_{k_j}^T P_{\hat{j}} - B_{q_j}^T P_{\hat{j}}$ from LMIs* (3.6), (3.34), (3.44), (3.45) *and* (3.48)*. Moreover, it is also worth mentioning that the conditions of Theorem 3.2 for asynchronous T-S switched T-S observer design with UPVs include, as a special case, the results obtained in Theorem 3.1, where the premises were assumed to be measurable.*

Remark 3.2. *It should be noted that the presented theorems above are applicable to the design of asynchronous switched T-S observers with arbitrary switching sequences. Furthermore, as previously stated, the cost of successively provided conservative enhancements is a drastic increase in computational cost. When the set of admissible switches $\mathcal{I}_s$ is known, it is unnecessary to solve the LMI conditions for all combinations of the m switching modes. As a result, the algorithm provided in the following remark may help to lower the computational cost of proper implementation of the LMI conditions in Theorems 3.1 to 3.5.*

Remark 3.3. *Let j_0 and $\hat{j}_0$ be respectively the initial modes of the considered switched T-S system* (3.1) *and of the asynchronous switched T-S observers* (3.2) *and* (3.29) *(j and $\hat{j}$ are their actual modes in the set of admissible switches $\mathcal{I}_s$). The equality and LMI conditions of Theorems 3.1 to 3.5 can be implemented according to Algorithm 1 in order to reduce their computational cost.*

Algorithm 1 Implementation procedure for the LMI-based conditions provided in Theorems 3.1 to 3.5

Initialization
 $j \leftarrow j_0$, $\hat{j} \leftarrow \hat{j}_0$
Main Loop
 Solve (3.6) (or(3.34) or (3.44), or (3.45), or (3.48), depending on the chosen theorem),
 Compute the Lyapunov matrix $P_{\hat{j}^+}$ according to (3.5),
 Set $j_0 \leftarrow j^+$ as the forthcoming mode of the system in $\mathcal{I}_s$,
 Set $\hat{j}_0 \leftarrow \hat{j}^+$ as the forthcoming mode of the observer in $\mathcal{I}_s$
 Repeat for all the admissible switches in $\mathcal{I}_s$.

The theoretical part of this chapter is now established, we conclude the section on asynchronous switched T-S observers with unmeasured premise variables. In the following section, simulation tests, discussion, and feasibility comparisons are proposed to illustrate the effectiveness of the above-proposed asynchronous switched T-S observer design methodologies.

3.4.2 Simulation Results and discussion

In this section, two simulation examples are proposed to illustrate the effectiveness of the proposed robust asynchronous observer design methodology for switched T-S systems with UPVs. The first example is an academic one, dedicated to compare the conservatism and the effectiveness of the proposed conditions with regard to several previous related studies Garbouj et al. (2019); Zheng et al. (2018b); Hong et al. (2018); Belkhiat et al. (2019). Then, the second example shows the effectiveness of the proposed methodology for the design of a robust asynchronous observer with UPVs for a switched nonlinear system having a physical meaning, i.e., a switched tunnel diode circuit. Note that these simulation examples have been implemented in Matlab (using the ode23 solver) and the LMI conditions of the above proposed theorems have been solved using the YALMIP Toolbox Lofberg (2004) with the semidefinite programming solver SeDuMi Labit et al. (2002).

3.4.2.1 Academic example for conservatism comparison

The goal of this academic example is to discuss the conservatism of the LMI-based conditions proposed in Theorem 3.1 in (Belkhiat et al., 2019) and Theorems 3.2 to 3.5, with respect to previous results Garbouj et al. (2019); Zheng et al. (2018b); Hong et al. (2018). Note that, from the previous literature, we failed to find suitable LMI-based conditions for switched T-S observers that exhibit both UPVs and asynchronous switched modes. Therefore, for the conservatism comparison purpose, we consider the following recent and closely related studies:

- Theorem 1 in Garbouj et al. (2019), which considers the design of interval observers for switched T-S systems with UPVs in the synchronous case,
- Theorem 1 in Hong et al. (2018), which proposes the design of H_∞ filters for switched T-S systems with asynchronous switched modes but without UPVs,
- Theorem 1 in Zheng et al. (2018b), where an average dwell-time approach is proposed for the design of switched T-S Luenberger-like filters, assuming that the premises variables are fully measurable and with synchronous switched modes,
- Theorem 3.1 in Belkhiat et al. (2019), which constitutes a special case of the present study where the premises variables are assumed fully measurable, and without the consideration of relaxation techniques such like the use of Lemma 2.6 (see Section 2.2).

Let us consider a switched T-S system (3.1), with $r_j = 2$ fuzzy rules in each $m = 4$ switched modes ($j = 1, \ldots, 4$), specified by the following matrices:

$$A_{1_1} = \begin{bmatrix} -3.6 & 10 \\ -2 & -1 \end{bmatrix}, A_{2_1} = \begin{bmatrix} -2 & 12 \\ -2 & -1 \end{bmatrix}, A_{1_2} = \begin{bmatrix} 4b+2.5+a & -2+b \\ -6a & -0.5 \end{bmatrix}, A_{2_2} = \begin{bmatrix} b+2.5+a & 2a \\ -2.5 & 1.9+4b \end{bmatrix}$$

$$A_{1_3} = \begin{bmatrix} -1.2 & 1 \\ -1.1 & -3.2 \end{bmatrix}, A_{2_3} = \begin{bmatrix} -1.5 & 0 \\ -1 & -3.2 \end{bmatrix}, A_{1_4} = \begin{bmatrix} -2.2 & 0 \\ -1.1 & -3.2 \end{bmatrix}, A_{2_4} = \begin{bmatrix} -2.3 & 1 \\ -1+b & -3.2 \end{bmatrix}$$

,

$$B_{i_1} = B_{i_3} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_{i_2} = B_{i_4} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix}, W = 0.5,$$

where a and b are two scalar parameters dedicated to check the feasibility fields of the proposed LMI-based conditions, and with the membership functions given by:

$$\text{Mode 1:}\begin{cases} h_{1_1}(\xi_1(t)) = \frac{1}{2}\left(1 - sin(\xi_1(t))\right) \\ h_{2_1}(\xi_1(t)) = 1 - h_{1_1}(\xi_1(t)) \end{cases} \quad \text{Mode 2:}\begin{cases} h_{1_2}(\xi_2(t)) = \frac{1}{2}\left(1 - sin(\xi_2(t))\right) \\ h_{2_2} = 1 - h_{1_2}(\xi_2(t)) \end{cases}$$
$$\text{Mode 3:}\begin{cases} h_{1_3}(\xi_3(t)) = cos^2(\xi_3(t)) \\ h_{2_3}(\xi_3(t)) = 1 - h_{1_3}(\xi_3(t)) \end{cases} \quad \text{Mode 4:}\begin{cases} h_{1_4}(\xi_4(t)) = sin^2(\xi_4(t)) \\ h_{2_4}(\xi_4(t)) = 1 - h_{1_3}(\xi_4(t)) \end{cases} \tag{3.49}$$

In this example, we consider the switching sequence $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ depicted in Figure 3.7, where $\mathcal{V} = \{1, 2, 3, 4\}$ denotes the sets of switched modes and $\mathcal{E} = \{(1, 2), (2, 3), (3, 4), (4, 1), (1, 4), (3, 2)\}$ the set of the admissible switches $(\mathcal{I}_s)$ between modes (Lendek et al., 2014a).

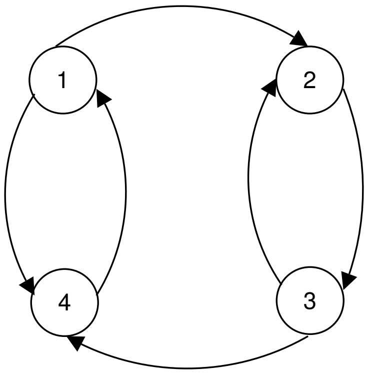

Figure 3.7: Graphical illustration of the considered switched system.

For several values of $a = [-15, 15]$ and $b = [-10, 25]$, with a step of 1, the feasibility of the conditions proposed in Theorems 3.1 to 3.5, as well as the ones proposed in Garbouj et al. (2019); Zheng et al. (2018b); Hong et al. (2018), has been checked with the YALMIP toolbox and the semi-definite programming solver SeDumi in Matlab (Lofberg, 2004). This results to the feasibility fields plotted in Figure 3.8.

As illustrated in Figure 3.8, over the 1116 points (a, b) that have been tested for each considered LMI conditions, the solutions obtained from Theorem 1 in Garbouj et al. (2019) provide 208 feasible solutions (18.6%), which are mostly included in those obtained by solving Theorem 3.3 (581 feasible solutions, i.e. 52.1%), except for 3 points when $a = 0$. Moreover, all the feasible solutions obtained from Theorem 1 in Zheng et al. (2018b) (119 feasible solutions, 10.7%) and from Theorem 1 in Hong et al. (2018) (86 feasible solutions, 7.7%) are included in those obtained from Theorem 3.3.

Remark 3.4. *Let us recall that the LMI-based conditions provided as Theorem 1 in Garbouj et al. (2019) are only valid for the design of synchronous switched interval observers, while the conditions of Theorem 3.3 are valid for the more general case of asynchronous switched T-S observers design. This explains why Theorem 1 Garbouj et al. (2019) succeeded to find a feasible solution for the three points* $(a, b) = \{(0, 0), (0, 1), (0, 2)\}$, *whereas Theorem 3.3 failed. Nevertheless, excepted for these three particular points, Theorem 3.3 provides an overall larger feasibility fields than the one obtained from Garbouj et al. (2019), which comfort the*

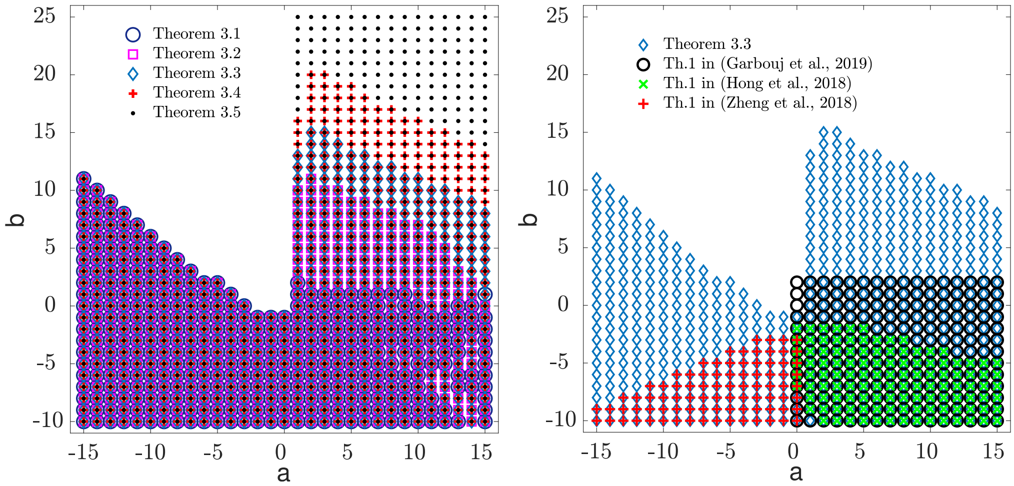

Figure 3.8: Left) Feasibility fields obtained from Theorems 3.1 to 3.5. Right) Feasibility fields obtained from Theorem 3.2 with respect to previous related works.

conservatism improvements brought by the proposed LMI-based conditions. Moreover, note that the Luenberger-like observer designed in Zheng et al. (2018b) is a switched synchronous filter, which means that the LMI-based conditions in Theorem 1 of this latter verifies one mode at a time, hence, there is no asynchronicity and no interactions between modes in the LMIs.

Figure 3.8 shows also the comparison of the feasibility fields obtained from Theorem 3.1 in Belkhiat et al. (2019) and Theorems 3.1, 3.2, 3.4 and 3.5. The feasibility field obtained from Theorem 3.5 (781 solutions, 70.0%) includes the one from Theorem 3.4 (655 solutions, 58.7%), which in turn includes the one of Theorem 3.3 (581 solutions, 52.1%), then Theorem 3.2 (511 solutions, 45.9%). Moreover, note that the feasibility field obtained from Theorem 3.1 in Belkhiat et al. (2019) (396 solutions, 35.5%) is always included and outperformed by the ones obtained from Theorems Theorems 3.2 to 3.5.

These comparisons of the feasibility fields clearly indicate that, over the whole tested area, the LMI conditions proposed in this paper provide significant improvements in terms of conservatism reduction regarding to the considered previous related results (Garbouj et al., 2019; Zheng et al., 2018b; Hong et al., 2018; Belkhiat et al., 2019).

Let us now discuss the computational complexity of the proposed LMI-based conditions, compared with the complexity of the LMI-based results proposed in Belkhiat et al. (2019); Garbouj et al. (2019); Zheng et al. (2018b); Hong et al. (2018). This comparison is detailed in Table 3.1, for this numerical example ($r = 2$ and $m = 4$), with regards to the conservatism achievements (Feasibility in % of the whole tested area $(a, b) \in [-15, 15] \times [-10, 25]$) for each results. Three criteria have been considered to evaluate the computational complexity of each LMI-based conditions: the number of decision variables (v) and LMI constraints (c) to be optimized, and finally, the ratio $\eta = v/c$. The latter constitutes the overall computational performance index we chose to evaluate, that is to say, the higher η is, the more is the computational complexity. Therefore, we observe that the price to pay for less conservative results is the increase of the computational complexity. Moreover, the computational complexity of

Theorems 3.2 and 3.3 are somewhat comparable, but with significant conservatism improvements, to the one of the previous considered results (Theorem 3.1 in Belkhiat et al. (2019); Garbouj et al. (2019); Zheng et al. (2018b); Hong et al. (2018)). Theorem 3.4, and even more 3.5, suffer from a significant increase of their computational complexities. However, if this can be seen as a drawback, especially for systems with large orders, let us recall that LMI computations are done offline, and as far as the computational capacity grows for daily computers, such complexity should be alleviated.

Table 3.1: Comparison of the computational complexity (Section 3.4.2.1)

Method	Feasibility (%)	Nb of dec. var. (v)	Nb of LMIs (c)	$\eta = v/c$
Theorem 3.2	**45.9%**	26	62	**0.42**
Zheng et al. (2018b)	7.7%	16	32	0.5
Theorem 3.1	**35.5%**	16	29	**0.55**
Theorem 3.3	**52.1%**	34	62	**0.55**
Hong et al. (2018)	10.7%	23	26	0.88
Garbouj et al. (2019)	16.6%	21	21	1
Theorem 3.4	**58.7%**	193	110	**1.75**
Theorem 3.5	**70.0%**	3193	398	**8.02**

For the rest of this subsection and for simulation purposes, let us now consider this numerical example at the particular point $(a, b) = (0, -1)$. The switched T-S observer gain matrices and parameters, listed in Table 3.2, have been obtained from Theorem 3.3 and Theorem 3.1. Moreover, we assume that the premise $x_1(t)$ is not measured and the designed observers and the switched system share the same switching sets (2.4), which hyper planes are defined by:

$$\begin{aligned} S_{12} &= \begin{bmatrix} -\frac{1}{2} & 1 \end{bmatrix}, \quad S_{23} = \begin{bmatrix} 1 & -20 \end{bmatrix} \\ S_{32} &= \begin{bmatrix} 1 & 10 \end{bmatrix}, \quad S_{34} = \begin{bmatrix} 1 & 6 \end{bmatrix} \\ S_{14} &= \begin{bmatrix} -1 & 1 \end{bmatrix}, \quad S_{41} = \begin{bmatrix} 4 & 1 \end{bmatrix} \end{aligned} \tag{3.50}$$

Figure 3.9 shows the simulation results for both observers designed from Theorem 3.2 and Theorem 3.1 (Belkhiat et al., 2019). The observer and the system have been respectively initialized in different modes (2^{nd} and 3^{rd}) with the initial conditions $x^T(0) = \begin{bmatrix} 2 & 1 \end{bmatrix}$ and $\hat{x}^T(0) = \begin{bmatrix} 5 & 5 \end{bmatrix}$, as shown in Figure 3.9 to Figure 3.11. Moreover, the following external disturbance and the input signal are applied to the system:

$$w(t) = \begin{cases} \sin(1.6\pi t + 0.5), \forall t \in [2.5, 3.5], \\ 0 \text{ otherwise.} \end{cases} \qquad u(t) = \begin{cases} 1, \forall t \in [0, 2], \\ 10 \sin(\pi t + 0.25), \forall t \in [2, 5],. \end{cases} \tag{3.51}$$

Table 3.2: Observer gains and parameters obtained from Theorem 3.3 and Theorem 3.1.

Method	Observer gains	Scalar parameters
Theorem 3.3	$K_{1_1} = \begin{bmatrix} 1.08 \\ 3.27 \end{bmatrix}, K_{2_1} = \begin{bmatrix} 3.87 \\ 3.31 \end{bmatrix}$	$\beta_1 = 0.76, \lambda_1 = 4.0$
	$K_{1_2} = \begin{bmatrix} -136.60 \\ 105.87 \end{bmatrix}, K_{2_2} = \begin{bmatrix} -88.85 \\ 73.26 \end{bmatrix}$	$\beta_2 = 0.29, \lambda_2 = 18.0$
	$K_{1_3} = \begin{bmatrix} -34.94 \\ 15.21 \end{bmatrix}, K_{2_3} = \begin{bmatrix} -37.58 \\ 16.04 \end{bmatrix}$	$\beta_3 = 0.98, \lambda_3 = 10.0$
	$K_{1_4} = \begin{bmatrix} -17.91 \\ 7.52 \end{bmatrix}, K_{2_4} = \begin{bmatrix} -16.40 \\ 7.15 \end{bmatrix}$	$\beta_4 = 1.32, \lambda_4 = 8.0$
		$\epsilon_1 = 20, \epsilon_2 = 20, \gamma = 1.81$
Theorem 3.1 in Belkhiat et al. (2019)	$K_{1_1} = \begin{bmatrix} -87.97 \\ 52.37 \end{bmatrix}, K_{2_1} = \begin{bmatrix} -49.69 \\ 32.84 \end{bmatrix}$	$\lambda = 10.0$
	$K_{1_2} = \begin{bmatrix} -61.69 \\ 8.30 \end{bmatrix}, K_{2_2} = \begin{bmatrix} -59.19 \\ 8.0 \end{bmatrix}$	$\epsilon = 7.3e^3$
	$K_{1_3} = \begin{bmatrix} -98.12 \\ 59.26 \end{bmatrix}, K_{2_3} = \begin{bmatrix} -52.25 \\ 35.16 \end{bmatrix}$	$\tau = 10^{-3}$
	$K_{1_4} = \begin{bmatrix} -12.97 \\ 2.59 \end{bmatrix}, K_{2_4} = \begin{bmatrix} -11.67 \\ 2.53 \end{bmatrix}$	$\gamma = 1.14$

As we can see on Figure 3.9, the switched T-S observer with UPVs designed from Theorem 3.2 is properly estimating the states, while with the observer designed from Theorem 3.1 in Belkhiat et al. (2019) provides poor estimations. This is confirmed by the state errors and switched modes estimations plotted in Figure 3.11, where the observer with the gains design from Theorem 3.1 inBelkhiat et al. (2019) shows poor results. Of course, this was expected since Theorem 3.1 in Belkhiat et al. (2019) doesn't cope with UPVs, which emphasizes the importance of this observers' class, .i.e the design of switched T-S observers with UPVs.

To conclude this first example, let us now post-verify that the gamma-level attenuation (defined by the H_∞ criterion (3.32)) is achieved by the proposed design. To do so, from the the simulation shown in Figure 3.9, where the final simulation time is $t_f = 5s$, we can compute an approximation of the effective disturbance attenuation level as:

$$\sqrt{\frac{\int_0^{t_f} e^T(t)\mathcal{S}e(t)dt}{\int_0^{t_f} w^T(t)\mathcal{R}w(t)dt}} = 0.029$$

which is lower than $\gamma = 1.81$ minimized from Theorem 3.3. This confirm the effectiveness of the proposed robust switched T-S H_∞ filters design.

3.4.2.2 Case study of a Switched Tunnel Diode Circuit

This example is devoted to illustrate the effectiveness of the proposed switched T-S observer design methodology (Theorem 3.3) on a system having a physical meaning. To do so, let us consider the modified tunnel diode circuit system, depicted in Figure 3.12, which state space realization is given by Shen et al. (2020):

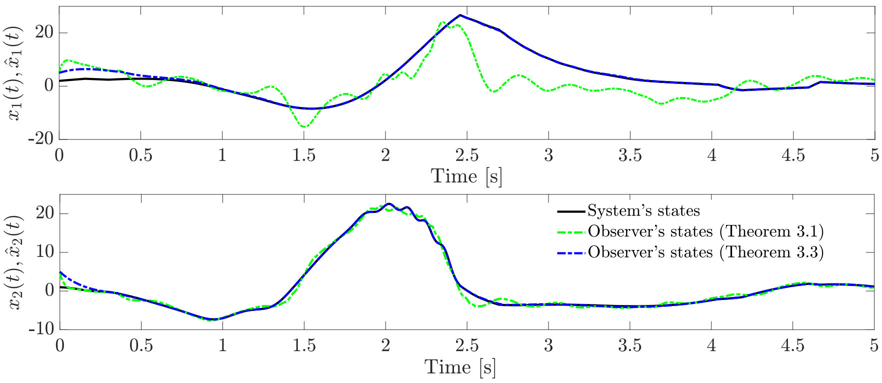

Figure 3.9: States of the switched T-S system (3.1) and their estimates.

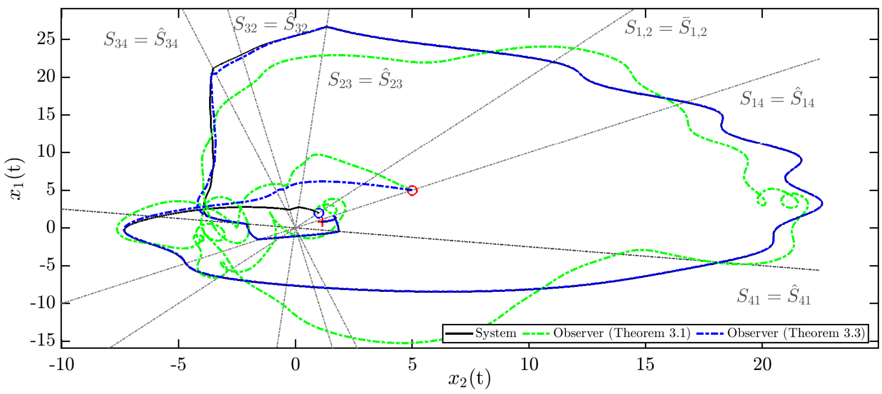

Figure 3.10: Phase planes of the switched T-S System (3.1) and the switched T-S observer (3.29).

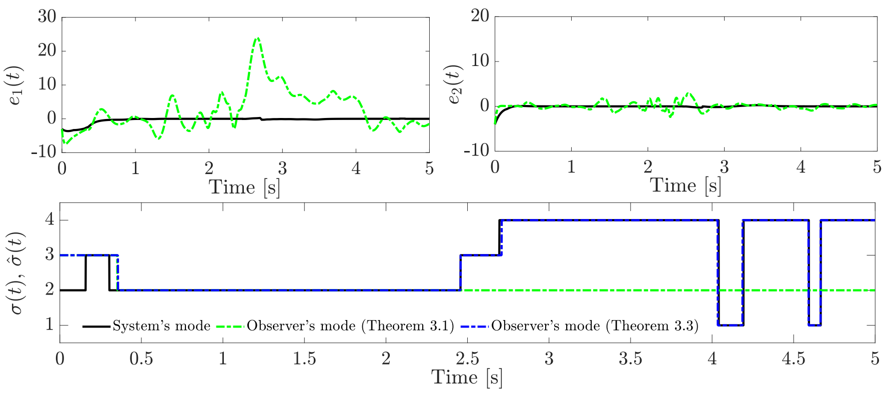

Figure 3.11: Top) Evolution of the estimation errors. Bottom) Evolution of switched T-S System (3.1) and the switched T-S observer (3.29) modes.

$$\begin{cases} \dot{x}_1(t) = \frac{0.2}{C} x_1(t) + \frac{0.01}{C} x_1^3(t) + \frac{1}{C} x_2(t) \\ \dot{x}_2(t) = -\frac{1}{L} x_1(t) - \frac{R_{\sigma(t)}}{L} x_2(t) + \frac{1}{L} u(t) \end{cases} \tag{3.52}$$

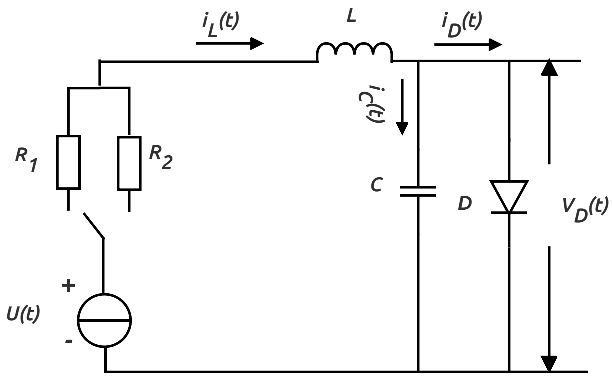

Figure 3.12: Switched Tunnel diode circuit

where $x_1(t) = v_D$ and $x_2(t) = i_D$ are respectively the voltage and current of the tunnel diode (state variables); $\sigma(t) \in \{1, 2\}$ denotes the switching modes, whereas the resistances $R_{\sigma(t)}$ switches between two distinct values ($R_1 = 1\,\Omega$ and $R_1 = 2\,\Omega$); $C = 0.1\,F$ is the circuit capacitance; $L = 1\,H$ is the circuit inductance. In the sequel, we assume that only $x_2(t)$ is measured and the output signal is affected by a disturbance $w(t)$ such that $y(t) = Cx(t) + Ww(t)$ with $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and $W = 1$. Moreover, assuming $x_1(t) \in [-3, 3]$, the state dependent premise variables $\xi_1(t) = \xi_2(t) = x_1^2(t) \in [0, 9]$ and $x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T$, the switched nonlinear system (3.52) can be exactly rewritten as a switched T-S system (3.1), by applying the sector nonlinearity approach Tanaka and Wang (2001), with $m = 2$, $r_1 = r_2 = 2$,

$$A_{1_1} = \begin{bmatrix} 2 & 10 \\ -1 & -1 \end{bmatrix}, A_{2_1} = \begin{bmatrix} 2.9 & 10 \\ -1 & -1 \end{bmatrix}, A_{1_2} = \begin{bmatrix} 2 & 10 \\ -1 & -2 \end{bmatrix}, A_{2_2} = \begin{bmatrix} 2.9 & 10 \\ -1 & -2 \end{bmatrix}, B_{1_1} = B_{1_2} = B_{2_1} = B_{2_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and the membership functions:

$$\begin{cases} h_{1_1}(\xi_1(t)) = h_{1_2}(\xi_2(t)) = 1 - \frac{\xi_1(t)}{9}. \\ h_{2_1}(\xi_1(t)) = h_{2_2}(\xi_2(t)) = 1 - h_{1,1}(\xi_1(t)) \end{cases} \tag{3.53}$$

The conditions of Theorem 3.3 have been solved using MATLAB and YALMIP Lofberg (2004) (with parameters $\epsilon_1 = \epsilon_2 = 480$). With the attenuation level $\gamma = 2.97$, we obtain the following switched T-S observer gain matrices:

$$K_{1_1} = \begin{bmatrix} -342.23 \\ 29.32 \end{bmatrix}, K_{2_1} = \begin{bmatrix} -314.03 \\ 26.97 \end{bmatrix}, K_{1_2} = \begin{bmatrix} -930.75 \\ 57.48 \end{bmatrix}, K_{2_2} = \begin{bmatrix} -820.09 \\ 50.67 \end{bmatrix},$$

the Lyapunov matrices:

$$P_1 = \begin{bmatrix} 0.10 & 0.91 \\ 0.91 & 10.90 \end{bmatrix}, P_2 = \begin{bmatrix} 0.10 & 1.33 \\ 1.33 & 21.81 \end{bmatrix},$$

and the decision variables $G_1 = \begin{bmatrix} 0.43 & 5.45 \end{bmatrix}$, $\beta_1 = 0.055$, $\beta_2 = 0.25$, $\lambda_1 = 179.34$ and $\lambda_2 = 162.04$.

To check the effectiveness of the designed switched T-S observer, the following input signal is set in simulation to the tunnel diode circuit:

$$\begin{cases} u(t) = \sin(1.6\pi t + 0.5), \forall t \in [1.4, 5] \\ u(t) = 0, \text{ otherwise} \end{cases}$$

Also, a noisy output disturbance signal $w(t)$, and the initial conditions of the system and the observer are set as:

$$\begin{cases} w(t) = 0.02\sin(1.65\pi t + 0.4) + r(t), \forall t \in [1.6s, 2.6s] \\ w(t) = r(t), \text{ otherwise} \\ x(0) = \begin{bmatrix} 0.3 \\ -0.06 \end{bmatrix}, \hat{x}(0) = \begin{bmatrix} -0.3 \\ -0.7 \end{bmatrix} \end{cases}$$

where $r(t)$ is an additive white Gaussian noise with a signal to noise ratio equal to $20dB$.

For the rest of this subsection, let us consider two cases in simulations regarding the switching hyper-planes of the switched T-S observer.

- *Case 1:* The designed observers (3.29) and the switched T-S system (3.1) share the same switching sets (2.4) ($\hat{S}_{\hat{j}\hat{j}^+} = S_{jj^+}$), which hyper-planes are defined by:

$$S_{12} = \begin{bmatrix} -0.01 & 2 \end{bmatrix}, S_{21} = \begin{bmatrix} 2 & 0 \end{bmatrix} \tag{3.54}$$

  In this first case, the system and the observer are respectively initialized in their second and first modes. The simulation results are shown in Figures 3.13 to 3.15. Figure 3.13 shows the trajectories of both states of the system and their estimates. Figure 3.14 exhibits the estimation errors and the evolution of the switched modes of the system and the observer. Figure 3.15 shows the output of the system subject to noisy disturbances, the same without noise (for indication), and the estimated output from the observer. It is worth to point-out that, due to the employed H_∞ criterion (3.32), the transfer between the external disturbances to the state estimation error is successfully attenuated, so that the estimated output provides a filtered estimation of the system's one (despite the presence of noise). Moreover, even if the system and the observer share the same switching sets, their switching modes evolve asynchronously because they are not initialized in the same mode and the system output is affected by external disturbances. However, in this case, when the state estimation error converges, the observer modes provide fine estimates of the systems ones (see Figure 3.14). To conclude this first simulation case, from Figure 3.13 and

Figure 3.14, we observe that the switched system's states are accurately estimated by the designed asynchronous switched T-S observer, despite the presence of output disturbances, as illustrated by Figure 3.15 where the system noisy output and the observer filtered one are plotted.

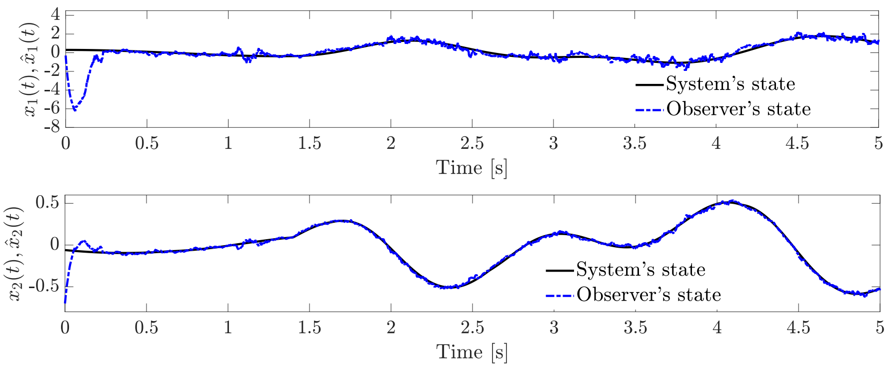

Figure 3.13: States of the switched tunnel diode (3.52) and their estimates in the presence of noise (Same switching sets case).

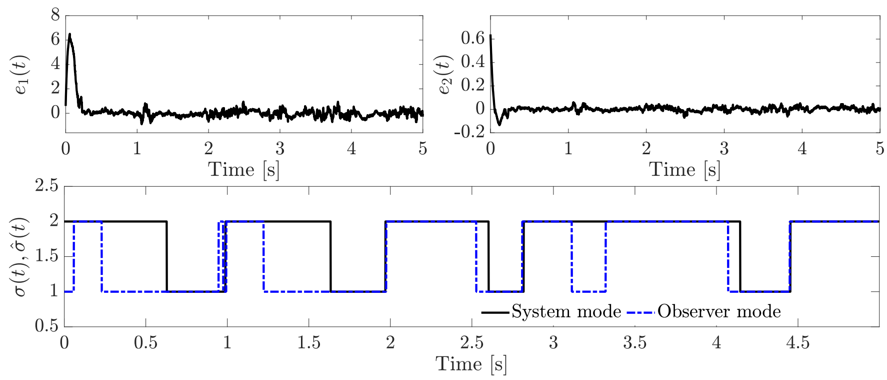

Figure 3.14: Estimation errors and switched modes of the switched tunnel diode (3.52) and the switched T-S observer (3.29)(*Same switching sets case*).

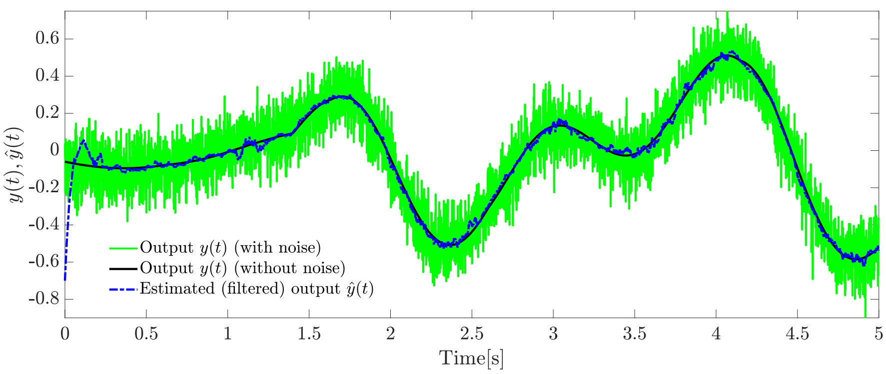

Figure 3.15: Switched tunnel diode (3.52) and the switched T-S observer's (3.29) output with and without noise (*Same switching sets case*).

— *Case 2:* Let us now we assume that the switching sets of the system and the observer are mismatched That is to say $\bar{S}_{\hat{j},\hat{j}^+} \neq S_{j,j^+}$ defined by the hyper planes:

$$\begin{aligned} S_{12} &= \begin{bmatrix} -0.01 & 2 \end{bmatrix}, S_{21} = \begin{bmatrix} 2 & 0 \end{bmatrix} \\ \hat{S}_{12} &= \begin{bmatrix} -0.1 & 3 \end{bmatrix}, \hat{S}_{21} = \begin{bmatrix} 3 & 1 \end{bmatrix} \end{aligned} \tag{3.55}$$

The system and the observer are respectively initialized in their second and first modes, with the same initial conditions as in *case 1.* The simulation results are shown in Figures 3.16 to 3.18. The trajectories of both state and their estimates are depicted in Figure 3.16. Figure 3.17 shows the estimation errors as well as the evolution the switched modes of the system and observer. In addition, Figure 3.18 depicts the observer's output, which provide a filtered estimation of the system's one, affected by the external disturbance with noise (the system's output without noise is given for indication).

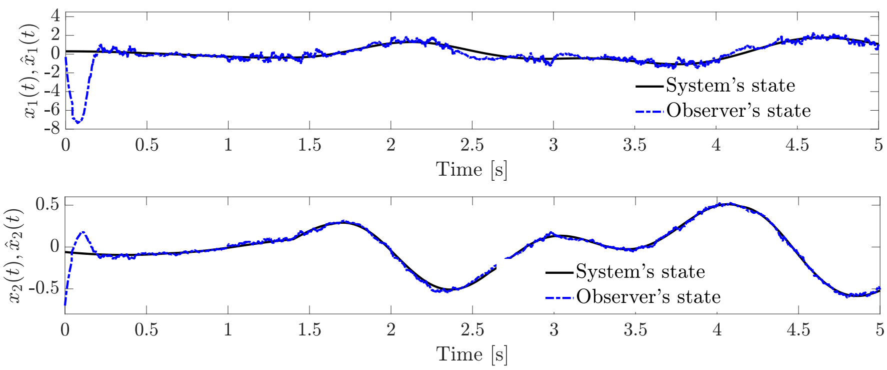

Figure 3.16: States of the switched tunnel diode (3.52) and their estimates in the presence of noise (Mismatching switching sets case).

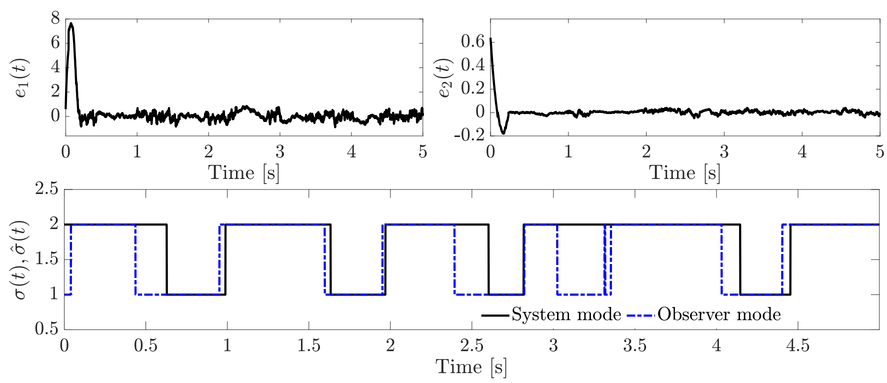

Figure 3.17: Estimation errors and switched modes of the switched tunnel diode (3.52) and the switched T-S observer (3.29)(*Mismatching switching sets case*).

From these figures, we can conclude that the observer successfully estimates the states of the

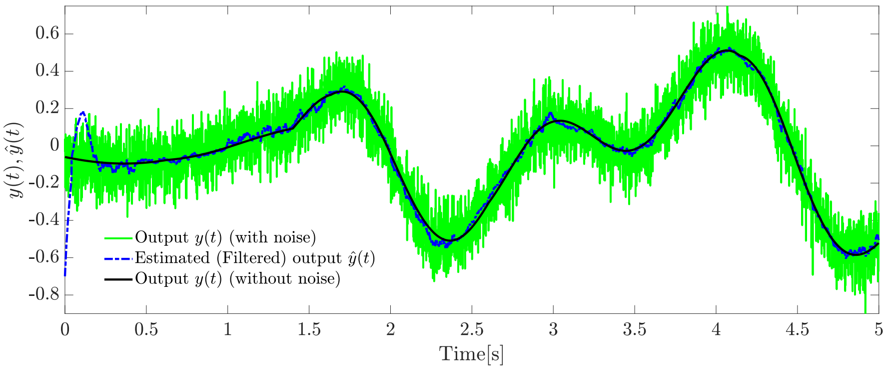

Figure 3.18: Switched tunnel diode (3.52) and the switched T-S observer's (3.29) output with and without noise (*Mismatching switching sets case*).

system despite the mismatch of the switching sets, the presence of the noisy modes disturbance, and the fact that the system and the observer are initialized in different modes.

Remark 3.5. *A systematic way to obtain a switched T-S system from a switched nonlinear one is to apply the well known sector nonliterary approach Tanaka and Wang (2001) on each nonlinear mode. In this case, the resulting T-S model is valid globally for global nonlinear sectors (like in the first example given in Section 3.4.2.1) or locally in a compact subset of the state space for local nonlinear sectors, which is the case of the second example (presented in Section 3.4.2.2) where we assume $x_1 \in [-3, 3]$. Of course, in this second case, if the state variables exit the validity domain of the switched T-S model, the convergence of the state estimation error cannot be guaranteed, which is one of the main limitation of T-S model-based observers obtained from local sector nonlinearity approaches. In this context, the estimation of the domain of attraction of the designed state estimation error dynamics should be taken into consideration, for instance by computing the maximum Lyapunov level set included in the validity domain of the switched T-S model, which will be investigated in **Chapter 4**.*

3.5 Conclusion

In this chapter, asynchronous switched Takagi-Sugeno observers design methodology has been presented for continuous-time switched Takagi-Sugeno systems subject to output disturbances and switching mismatches. Several theorems have been proposed in terms of LMI conditions based on switched quadratic Lyapunov Functions with significant improvements regarding conservatism with respect to previous related studies. The designed observers are investigated using the poly-topic modeling framework using sector nonlinearity approach for both cases, where the premise variables are measured or unmeasured. For both cases, the switching signals are assumed to be asynchronous and arbitrary (state-dependent switching), where the switching sets are different, which is usually the case in real world applications. Several examples have been presented to illustrate the effectiveness of the proposed design methodology both in simulation and in conservatism reduction.

Chapter 4

H_∞ filtering for switched Takagi-Sugeno Systems

Traduction en Français : Synthèse de filtres H_∞ pour les systèmes Takagi-Sugeno à commutations

Au cours des dernières décennies, l'estimation d'états a attiré l'attention des chercheurs et des ingénieurs, puisque les variables d'état ne sont pas toujours disponibles, une application de ces techniques d'estimation est le filtrage H_∞ (robuste). Le problème du filtrage est l'une des questions les plus importantes dans le domaine de l'automatique, il peut être appliqué à de nombreux problèmes, notamment la commande par retour de sortie, la modélisation de systèmes, la détection de défauts, l'estimation et la surveillance, pour estimer certains états du système ou filtrer certaines sorties du système, qui peuvent ne pas être mesurées ou être affectées par des perturbations dues au bruit. Le filtrage H_∞ peut être résumé comme la minimisation du transfert de l'entrée de la perturbation exogène à la sortie, ou pas plus grand qu'un certain niveau prescrit en termes de norme H_∞. L'un des avantages de l'utilisation des filtres H_∞ est qu'il n'est pas nécessaire de disposer d'informations sur les signaux de bruit, à l'exception de leur énergie limitée, ce qui rend cette technique de filtrage plus générale et plus intéressante que le filtrage de Kalman classique (Tuan et al. (2001b)). De plus, par rapport aux approches de filtrage classiques, les filtres H_∞ sont connus pour leur robustesse face aux dynamiques non modélisées et aux incertitudes (Chang, 2012; Zhang and Boukas, 2009).

Plusieurs études ont été réalisées sur la question du filtrage des systèmes linéaires, des systèmes non linéaires, et de nombreux résultats prometteurs ont été publiés dans la littérature. Les lecteurs peuvent se référer aux études suivantes (Yoneyama (2009, 2013); Chang (2012, 2014)), aux thèses en (Zerrougui (2011); Li (2013)) et aux références qui y figurent. En comparaison avec le problème de filtrage des systèmes linéaires et non linéaires, le filtrage des systèmes non linéaires à commutations est encore plus difficile en raison du phénomène de commutation et nécessite des investigations plus approfondies. Le filtrage de H_∞ des systèmes Takagi-Sugeno à commutations a fait l'objet d'études récentes utilisant la commutation contrainte en utilisant l'approche de temps de séjour (Shi et al., 2019; Hong et al., 2018; Liu and Zhao, 2020). Les travaux précités n'étaient valables que pour le filtrage H_∞ des systèmes Takagi-Sugeno à commutations sous des considérations de dwell-time, ce qui peut être inutile lorsque la séquence à commutatione est incontrôlée, arbitraire ou inconnue.

Dans ce chapitre, la conception de filtres Takagi-Sugeno H_∞ pour une classe de systèmes à commutations non linéaires à temps continu est étudiée, où des conditions LMI sans temps d'arrêt ont été dérivées dans le cadre d'une commutation asynchrone, pour estimer les sorties non mesurées et/ou perturbées du système, même lorsque la loi de commutation du filtre ne correspond pas à celle du système T-S à commutation, qui peut être en pratique inconnu ou mesuré de manière imprécise. Après la description du problème du filtrage H_∞ dans Section 4.2. Après la description du problème du filtrage H_∞ dans la Section 4.2, la Section 4.3 sera consacrée à la conception de filtres H_∞ à commutations basés sur une approche de redondance du descripteur (Schulte and Guelton, 2009; Guelton et al., 2009a; Bouarar et al., 2013; Jabri et al., 2020) pour des systèmes T-S à commutations sous commutation asynchrone et loi de commutation dépendante de l'état (sans temps de séjour), où les variables de prémisse sont supposées être mesurables. Dans le document Section 4.4, une extension des filtres H_∞ conçus dans le document Section 4.3 à une plus grande classe de systèmes T-S à commutations avec des variables de prémisse non mesurées est proposée. Dans ce contexte, le système T-S à commutation est modélisé comme un système T-S à commutation avec des parties conséquentes non linéaires, c'est-à-dire où les termes non linéaires non mesurés sont conservés dans les parties conséquentes non linéaires afin de contourner l'apparition de variables de prémisse non mesurées, ce qui est généralement le cas dans la modélisation T-S conventionnelle sans parties conséquentes non linéaires. En raison du conservatisme qui découle des conditions de Lipschitz lorsqu'on traite des variables de prémisse non mesurées, la conception de filtres H_∞ pour les systèmes T-S à commutations avec des parties conséquentes non linéaires, ainsi que les contraintes quadratiques incrémentielles pour traiter la partie conséquente non linéaire non mesurée, seront au centre de cette section. De plus, il est bien connu que les modèles flous T-S représentent des systèmes non linéaires sur un sous-ensemble de leur espace d'état, par conséquent, une procédure d'optimisation pour élargir le domaine d'attraction de l'erreur de filtrage est proposée. Enfin, nous terminons ce chapitre par une conclusion.

4.1 Introduction

Over the past few decades, state estimations have attracted widespread attention from researchers and engineers, since the state variables are not always available, an application of such estimation techniques is the H_∞ (robust) filtering. Filtering problem is one of the most important issues in the area of systems and control, it can be applied in many problems including output feedback control, system modeling, fault detection, estimation and monitoring, to estimate some of the system's states or filter some of the system's outputs, which can be not measured or affected by noises disturbances. H_∞ filtering can be summarized as the minimization of the transfer from the exogenous disturbance input to the output, or no larger than some prescribed level in term of the H_∞ norm. One of the advantages of using H_∞ filters is the non-necessity of any information about the noise signals except their energy boundedness, which makes such filtering technique more general and attractive than the classical Kalman filtering (Tuan et al., 2001b). Moreover, compared to the classical filtering approaches, H_∞ filters are known for their robustness against unmodeled dynamics and uncertainties (Zhang and Boukas, 2009; Chang, 2012). Several studies have been done on the filtering issue of linear systems, nonlinear systems, and there have been a lot of promising results published in the literature. The readers are referred to the following studies (Yoneyama, 2009, 2013; Chang, 2012, 2014), the theses in (Zerrougui, 2011; Li, 2013) and references therein. In comparison with the filtering problem of linear and nonlinear systems, filtering of switched nonlinear systems is even more challenging due to the switching phenomenon and requires deeper investigations. H_∞ filtering of switched Takagi-Sugeno systems has been the subject of recent investigations using constrained switching using dwell-time approaches (Shi et al., 2019; Hong et al., 2018; Liu and Zhao, 2020). The aforementioned works were only valid for the H_∞ filtering of switched Takagi-Sugeno systems under dwell-time considerations, which may be impractical when the switched sequence is uncontrolled, arbitrary or unknown.

In this chapter, the design of Takagi-Sugeno H_∞ filters for a class of continuous-time nonlinear switched systems is investigated, where dwell-time free LMI-based conditions were derived under asynchronous switching, to estimate unmeasured and/or disturbed system's outputs, even when the filter's switching law mismatches the switched T-S system's one, which can be in practice unknown or imprecisely measured. After the description of the problem of H_∞ filtering in Section 4.2. Section 4.3 will be devoted to the design of switched H_∞ filters based on a desciptor redundancy approach (Schulte and Guelton, 2009; Guelton et al., 2009a; Bouarar et al., 2013; Jabri et al., 2020) for switched T-S systems under asynchronous switching and state-dependent switching law (dwell-time free), where the premise variables are assumed to be measurable. In Section 4.4, an extension of the H_∞ filters designed in Section 4.3 to a larger class of switched T-S systems with unmeasured premise variables is proposed. In this context, the switched T-S system is modelled as a switched T-S system with nonlinear consequent parts, i.e. where unmeasured nonlinear terms are kept in the nonlinear consequent parts in order to circumvent the occurrence of unmeasured premise variables, which is usually faced in conventional T-S modelling without nonlinear consequent parts. Due to the conservatism that arises from the Lipschitz conditions when dealing with unmeasured premise variables, H_∞ filters design for switched T-S systems with nonlinear consequent parts along with the incremental quadratic constraints to deal with the unmeasured nonlinear consequent part will be the main focus of this section. Moreover, it is well known that T-S fuzzy models represent nonlinear systems on a subset of their state space, therefore, an optimization procedure to enlarge the domain of attraction of the filtering error is proposed. Then, we end this chapter with a conclusion.

4.2 Problem Statement

In this chapter, we consider a class of switched nonlinear systems given by the following set of equations (4.1) :

$$\begin{cases} \dot{x}(t) & = f_{x,\sigma(t)}(x(t)) + f^{w}_{x,\sigma(t)}(x(t))w(t) \\ z(t) & = f_{z,\sigma(t)}(x(t)) \\ y(t) & = Cx(t) + Dw(t) \end{cases} \tag{4.1}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state vector; $y(t) \in \mathbb{R}^{n_y}$ is the measured output vector and $z(t) \in \mathbb{R}^{n_z}$ is the unmeasured output to be estimated; $w(t) \in \mathbb{R}^{n_w}$ is a time-varying L_2 norm bounded exogenous disturbance input vector; $\sigma(t) \in \{1, ..., m\}$ is the switching law; $f_{x,\sigma(t)}(x(t)) \in \mathbb{R}^{n_x}$, $f_{z,\sigma(t)}(x(t)) \in \mathbb{R}^{n_z}$ and $f^{w}_{x,\sigma(t)}(x(t)) \in \mathbb{R}^{n_x \times n_w}$ are nonlinear vector or matrix-valued functions that describe the dynamics of the considered system, $C \in \mathbb{R}^{n_y \times n_x}$ and $D \in \mathbb{R}^{n_y \times n_w}$ are respectively the state selection and direct transfer matrices of the measured output equation.

Now, using the definition of the switching law given in (2.47), the switched nonlinear system (4.1) can be equivalently rewritten as:

$$\begin{cases} \dot{x}(t) = \sum_{j=1}^{m} \sigma_j(t)\left(f_{x,j}(x(t)) + f_{x,j}^{w}(x(t))w(t)\right) \\ z(t) = \sum_{j=1}^{m} \sigma_j(t) f_{z,j}(x(t)) \\ y(t) = Cx(t) + Dw(t) \end{cases} \tag{4.2}$$

Assumption 4.1. *$\forall j \in \{1,\dots,m\}$, the nonlinear vector-valued functions $f_{x,j}(x(t)) \in \mathbb{R}^{n_x}$ and $f_{z,j}(x(t)) \in \mathbb{R}^{n_z}$ are smooth sector bounded nonlinear functions (where $f_{x,j}(0) = 0$ and $f_{z,j}(0) = 0$). Also, together with the matrix-valued function $f_{x,j}^{w}(x(t)) \in \mathbb{R}^{n_x \times n_w}$, we assume that their entries contain p scalar bounded state-dependant nonlinear functions $\xi_i(x(t)) \in \left[\underline{\xi}_i, \bar{\xi}_i\right], i = 1,\dots,p$, with $\bar{\xi}_i = \max_{x,u}\{\xi_i(t)\}$ and $\underline{\xi}_i = \min_{x,u}\{\xi_i(t)\}$.*

From Assumption 4.1, an exact T-S model Takagi and Sugeno (1985) of each mode j of the switched nonlinear system (4.2) can be obtained by applying the well-known sector nonlinearity approach Tanaka and Wang (2001) on each nonlinearity $\xi_i(x(t)) \in \left[\underline{\xi}_i, \bar{\xi}_i\right]$ (see Section 2.3).

4.3 H_∞ filtering with measured premise variables

The aim of this section is to design asynchronous H_∞ filters for switched nonlinear systems described by T-S fuzzy models, to estimate the unmeasured output vector $z(t)$, where the premise variables are assumed to be measured, i.e. the premise variables $\xi_j(t)$ depend on measured state variables $y(t)$.

The switched nonlinear system in (4.1) can be represented by Takagi-Sugeno fuzzy models as follows :

$$\begin{cases} \dot{x}(t) = \sum_{j=1}^{m} \sum_{i_j=1}^{r_j} \sigma_j(t) h_{i_j}(\xi_j(t))\left(A_{i_j} x(t) + B_{i_j} w(t)\right) \\ z(t) = \sum_{j=1}^{m} \sum_{i_j=1}^{r_j} \sigma_j(t) h_{i_j}(\xi_j(t)) F_{i_j} x(t) \\ y(t) = Cx(t) + Dw(t) \end{cases} \tag{4.3}$$

where the number of fuzzy rules in the j^{th} mode is given by r_j ($j = 1,\dots,m$), $\xi_j(t)$ are the vectors of premise variables in each modes and, $\forall i_j = 1,\dots,r_j$, $h_{i_j}(\xi_j(t))$ are normalized membership functions satisfying the convex sum properties: $h_{i_j}(\xi_j(t)) > 0$ and $\sum_{i=1}^{r_j} h_{i_j}(\xi_j(t)) = 1$. The matrices $A_{i_j} \in \mathbb{R}^{n_x \times n_x}$, $B_{i_j} \in \mathbb{R}^{n_x \times n_w}$, $F_{i_j} \in \mathbb{R}^{n_z \times n_x}$, C and D define the vertices of T-S subsystems in each switched mode, $\sigma_j(t)$ is the switching law such that $\sigma_j(t) = 1$ when the active system j is in the l^{th} mode and $\sigma_j(t) = 0$ when $j \neq l$.

Let us consider the following asynchronous mode-dependent T-S fuzzy filter:

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{\hat{j}=1}^{m} \sum_{i_{\hat{j}}=1}^{r_{\hat{j}}} \hat{\sigma}_{\hat{j}}(t) h_{i_{\hat{j}}}(y(t)) \left(\hat{A}_{i_{\hat{j}}} \hat{x}(t) + \hat{B}_{i_{\hat{j}}} y(t) \right) \\ \hat{z}(t) = \sum_{\hat{j}=1}^{m} \sum_{i_{\hat{j}}=1}^{r_{\hat{j}}} \hat{\sigma}_{\hat{j}}(t) h_{i_{\hat{j}}}(y(t)) \hat{F}_{i_{\hat{j}}} \hat{x}(t) \end{cases} \tag{4.4}$$

where $\hat{x}(t) \in \mathbb{R}^{n_x}$ is the filter's state vector, $\hat{z}(t) \in \mathbb{R}^{n_z}$ is the estimate of $z(t)$, $\hat{A}_{i_{\hat{j}}} \in \mathbb{R}^{n_x \times n_x}$, $\hat{B}_{i_{\hat{j}}} \in \mathbb{R}^{n_x \times n_y}$, $\hat{F}_{i_{\hat{j}}} \in \mathbb{R}^{n_z \times n_x}$ are gain matrices to be synthesized and, for $\hat{j} = 1, \dots, m$, $\hat{\sigma}_{\hat{j}}(t)$ is the switching law of the filter, which may be asynchronous regarding to the system's one ($\sigma_j(t)$).

In the sequel, for all the matrices with fuzzy summation structures, we will adopt the following notations:

$$\mathcal{M}_{h_\sigma} = \sum_{j=1}^{m} \sum_{i_j=1}^{r_j} \sigma_j(t) h_{i_j}(\xi_j(t)) \mathcal{M}_{i_j},\ \mathcal{M}_{h_{\hat{\sigma}}} = \sum_{\hat{j}=1}^{m} \sum_{i_{\hat{j}}=1}^{r_{\hat{j}}} \hat{\sigma}_{\hat{j}}(t) h_{i_{\hat{j}}}(\xi_{\hat{j}}(t)) \mathcal{M}_{i_{\hat{j}}},$$

$$\mathcal{M}_{h_\sigma h_{\hat{\sigma}}} = \sum_{j=1}^{m} \sum_{\hat{j}=1}^{m} \sum_{i_j=1}^{r_j} \sum_{i_{\hat{j}}=1}^{r_{\hat{j}}} \sigma_j(t) \hat{\sigma}_{\hat{j}}(t) h_{i_j}(\xi_j(t)) h_{i_{\hat{j}}}(\xi_{\hat{j}}(t)) \mathcal{M}_{i_j i_{\hat{j}}}.$$

Moreover, the time t as functions argument will be omitted when no ambiguity arises. $\mathcal{M}^\dagger$ stands for the pseudo-inverse of non square matrices $\mathcal{M}$. For real square matrices $\mathcal{M}$, we denote $\mathcal{H}e(\mathcal{M}) = \mathcal{M} + \mathcal{M}^T$. Finally, $\mathcal{M}_{(\nu)}$ denotes the ν^{th} row of a matrix $\mathcal{M}$.

Let us define the filtering errors $e_z(t) = z(t) - \hat{z}(t)$ and $e_x(t) = x(t) - \hat{x}(t)$. Then, consider the augmented vectors $e(t) = \begin{bmatrix} e_x(t)^T & e_z(t)^T \end{bmatrix}^T$ and $\tilde{w}(t) = \begin{bmatrix} x(t)^T & \hat{x}(t)^T & w(t)^T \end{bmatrix}^T$, the dynamics of the filtering error can be written as the following descriptor:

$$E\dot{e}(t) = \tilde{A}_{h_\sigma} e(t) + \tilde{B}_{h_\sigma h_{\hat{\sigma}}} \tilde{w}(t) \tag{4.5}$$

with:

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \tilde{A}_{h\sigma} = \begin{bmatrix} A_{h_\sigma} & 0 \\ 0 & -I \end{bmatrix}, \tilde{B}_{h_\sigma h_{\hat{\sigma}}} = \begin{bmatrix} -\hat{B}_{h_{\hat{\sigma}}} C & A_{h_\sigma} - \hat{A}_{h_{\hat{\sigma}}} & B_{h_\sigma} - \hat{B}_{h_{\hat{\sigma}}} D \\ F_{h_\sigma} & -\hat{F}_{h_{\hat{\sigma}}} & 0 \end{bmatrix}.$$

Therefore, the aim of this section is to provide new LMI-based conditions for the design of asynchronous switched T-S H_∞ filters (4.4), to estimate the unmeasured output $z(t)$ for the switched T-S system (4.3) with exogenous input $w(t)$ and measured output $y(t)$, which uses arbitrary signals with bounded energy specified levels in terms of the H_∞ norm, so that the mapping from the exogenous input to the filtering error is minimal, or no bigger than a prescribed attenuation level. Which can be summarized as follows:

i. When $\tilde{w}(t) = 0$, the filtering error dynamics (4.5) is asymptotically converging to the origin.

ii. $\forall \tilde{w}(t) \neq 0$, the transfer between $\tilde{w}(t)$ and the output estimation error $e_z(t)$ is minimized with an attenuation level γ defined such that:

$$\frac{\|e_z\|_2^2}{\|\tilde{w}\|_2^2} \leq \gamma^2 \tag{4.6}$$

that is to say, regarding to the following H_∞ criterion:

$$\int_0^\infty e_z^T(t)e_z(t)dt \leq \gamma^2 \int_0^\infty \tilde{w}^T(t)\tilde{w}(t)dt \tag{4.7}$$

4.3.1 Main results

Based on the problem statement given above, the goal now is to provide the LMI-based conditions for the H_∞ filtering problem for the first case, i.e. with measured premise variables. The following theorem summarizes the main results.

Theorem 4.1: Chekakta et al. (2022)

$\forall (j,\hat{j}) \in \{1,...,m\}^2$, $\forall (i_j,k_j) = \{1,...,r_j\}^2$ and $\forall q_{\hat{j}} \in \{1,...,r_{\hat{j}}\}$, the conditions i. and ii. defined above of the H_∞ filters (4.27) design are satisfied if there exist a real scalar γ and real matrices $\hat{A}_{k_j}$, $\hat{B}_{k_j}$, $\hat{F}_{k_j}$, $G_{\hat{j}}$, $L_{k_jq_{\hat{j}}}$, $R_{k_jq_{\hat{j}}}$, $X_{\hat{j}}^1 = X_{\hat{j}}^{1^T}$, $X_{\hat{j}}^2$ and $X_{\hat{j}}^3$, which verify the following optimization problem:

$$\begin{cases} \min \gamma^2, \\ \text{subject to (4.9), (4.10) and } X_{\hat{j}}^1 > 0 \end{cases} \tag{4.8}$$

$$X_{\hat{j}^+}^1 = X_{\hat{j}}^1 + G_{\hat{j}}^T C + C^T G_{\hat{j}} \tag{4.9}$$

$$\frac{1}{r_j - 1}\Gamma_{i_j i_j q_{\hat{j}}} + \frac{1}{2}(\Gamma_{i_j k_j q_{\hat{j}}} + \Gamma_{k_j i_j q_{\hat{j}}}) < 0 \tag{4.10}$$

with :

$$\Gamma_{i_jk_jq_{\hat{j}}} = \begin{bmatrix} \Phi^1_{i_jk_jq_{\hat{j}}} & (*) & (*) & (*) \\ \Phi^2_{i_jk_jq_{\hat{j}}} & -R^T_{i_jq_{\hat{j}}} - R_{i_jq_{\hat{j}}} & (*) & (*) \\ \tilde{B}^T_{i_jq_{\hat{j}}} & 0 & -\gamma^2 I & (*) \\ X^3_{\hat{j}}\phi & 0 & 0 & -I \end{bmatrix}, \begin{cases} \Phi^1_{i_jk_jq_{\hat{j}}} = \tilde{A}_{i_j} L^T_{k_jq_{\hat{j}}} + L_{k_jq_{\hat{j}}} \tilde{A}^T_{i_j} \\ \Phi^2_{i_jk_jq_{\hat{j}}} = X_{\hat{j}} - L^T_{k_jq_{\hat{j}}} + R^T_{k_jq_{\hat{j}}} \tilde{A}^T_{i_j} \end{cases}$$

and $\tilde{A}_{i_j} = \begin{bmatrix} A_{i_j} & 0 \\ 0 & -I \end{bmatrix}$ and $\tilde{B}_{i_j q_{\hat{j}}} = \begin{bmatrix} -\hat{\mathcal{B}}_{q_{\hat{j}}} C & A_{i_j} - \hat{A}_{q_{\hat{j}}} & B_{i_j} - \hat{B}_{q_{\hat{j}}} D \\ F_{i_j} & -\hat{F}_{q_{\hat{j}}} & 0 \end{bmatrix}$, $\phi = \begin{bmatrix} 0 & I \end{bmatrix}$.

In that case, the filter's state vector is updated at the switching instants according to:

$$\hat{x}^+ = \left(I - Q_{\hat{j}}^{-1}\left(CQ_{\hat{j}}^{-1}\right)^{\dagger} C\right)\hat{x} + Q_{\hat{j}}^{-1}\left(CQ_{\hat{j}}^{-1}\right)^{\dagger} y \tag{4.11}$$

with $Q_{\hat{j}} = \mathbf{V}_{\hat{j}}\sqrt{\Lambda_{\hat{j}}}\mathbf{V}_{\hat{j}}^T$, where $\mathbf{V}_{\hat{j}} \in \mathbb{R}^{n\times n}$ are composed of the orthonormal eigenvectors of $P_{\hat{j}}^1 = (X_{\hat{j}}^1)^{-1}$, and where $\Lambda_{\hat{j}} \in \mathbb{R}^{n\times n}$ are diagonal matrices, which entries are the eigenvalues of $P_{\hat{j}}^1 = (X_{\hat{j}}^1)^{-1}$.

Proof. Let us consider a multiple Lyapunov candidate function given by:

$$V(t, e(t)) = e(t)^T E^T P_{\hat{\sigma}} e(t) \tag{4.12}$$

where $EP_{\hat{\sigma}} = P_{\hat{\sigma}}^T E > 0$, $P_{\hat{\sigma}} = \sum_{\hat{j}=1}^{m} \hat{\sigma}_{\hat{j}}(t) P_{\hat{j}}$ with $P_{\hat{j}} = \begin{bmatrix} P_{\hat{j}}^1 & 0 \\ P_{\hat{j}}^2 & P_{\hat{j}}^3 \end{bmatrix}$, $E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ and $P_{\hat{j}}^1 = P_{\hat{j}}^{1^T} > 0$.

Taking the time derivative of (4.12), yields:

$$\begin{aligned} \dot{V}(t, e(t)) &= \dot{e}^T(t) E^T P_{\hat{\sigma}} e(t) + e^T(t) P_{\hat{\sigma}}^T E \dot{e}(t) \\ &= 2e^T(t) P_{\hat{\sigma}}^T \left(\tilde{A}_{h_\sigma} e(t) + \tilde{B}_{h_\sigma h_{\hat{\sigma}}} \tilde{w}(t)\right) \end{aligned} \tag{4.13}$$

Hence, the items i. and ii. of the H_∞ filtering problem defined above are satisfied if there exists γ such that the following inequality is satisfied:

$$\begin{aligned} &\dot{V}(t, e(t)) + e_z(t)^T e_z(t) - \gamma^2 \tilde{w}(t)^T \tilde{w}(t) \\ &= \begin{bmatrix} e \\ \tilde{w} \end{bmatrix}^T \begin{bmatrix} \tilde{A}_{h_\sigma}^T P_{\hat{\sigma}} + P_{\hat{\sigma}}^T \tilde{A}_{h_\sigma} + \psi & (*) \\ \tilde{B}_{h_\sigma h_{\hat{\sigma}}}^T P_{\hat{\sigma}} & -\gamma^2 I \end{bmatrix} \begin{bmatrix} e \\ \tilde{w} \end{bmatrix} < 0, \quad \text{with } \psi = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}. \end{aligned} \tag{4.14}$$

For all $\begin{bmatrix} e \\ \tilde{w} \end{bmatrix} \neq 0$, taking the congruence by $\begin{bmatrix} X_{\hat{\sigma}} & 0 \\ 0 & I \end{bmatrix}$, where $X_{\hat{\sigma}} = P_{\hat{\sigma}}^{-1}$, then, the inequality (4.14) is satisfied if:

$$\begin{bmatrix} X_{\hat{\sigma}}^T \tilde{A}_{h_\sigma}^T + \tilde{A}_{h_\sigma} X_{\hat{\sigma}} + X_{\hat{\sigma}}^T \psi X_{\hat{\sigma}} & (*) \\ \tilde{B}_{h_\sigma h_{\hat{\sigma}}}^T & -\gamma^2 I \end{bmatrix} < 0 \tag{4.15}$$

Applying Lemma 2.6, Equation 4.15 holds if $\exists L_{h_\sigma h_{\hat\sigma}}$ and $\exists R_{h_\sigma h_{\hat\sigma}}$ such that:

$$\begin{bmatrix} \Phi^1_{h_\sigma h_\sigma h_{\hat\sigma}} + X^T_{\hat\sigma}\psi X_{\hat\sigma} & (*) & (*) \\ \Phi^2_{h_\sigma h_\sigma h_{\hat\sigma}} & -R^T_{h_\sigma h_{\hat\sigma}} - R_{h_\sigma h_{\hat\sigma}} & (*) \\ \tilde{B}^T_{h_\sigma h_{\hat\sigma}} & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{4.16}$$

with $\Phi^1_{h_\sigma h_\sigma h_{\hat\sigma}} = \tilde{A}_{h_\sigma} L^T_{h_\sigma h_{\hat\sigma}} + L_{h_\sigma h_{\hat\sigma}} \tilde{A}^T_{h_\sigma}$, $\Phi^2_{h_\sigma h_\sigma h_{\hat\sigma}} = X_{\hat\sigma} - L^T_{h_\sigma h_{\hat\sigma}} + R^T_{h_\sigma h_{\hat\sigma}} \tilde{A}^T_{h_\sigma}$
Then, applying the Schur complement (Lemma 2.2), Equation 4.16 becomes:

$$\begin{bmatrix} \Phi^1_{h_\sigma h_\sigma h_{\hat\sigma}} & (*) & (*) & (*) \\ \Phi^2_{h_\sigma h_\sigma h_{\hat\sigma}} & \Phi^3_{h_\sigma h_{\hat\sigma}} & (*) & (*) \\ \tilde{B}^T_{h_\sigma h_{\hat\sigma}} & 0 & -\gamma^2 I & (*) \\ X^3_{\hat\sigma}\phi & 0 & 0 & -I \end{bmatrix} < 0, \text{ with } \phi = \begin{bmatrix} 0 & I \end{bmatrix} \tag{4.17}$$

Now, applying Lemma 2.5, Equation 4.17 holds if the LMIs (4.10) are satisfied.

Moreover, for the proof of the Lyapunov decreasing at the switching instant and the update of the filter states, the reader may refer to the proof of Theorem 3.1.

□

Remark 4.1. *Note that the LMI-based conditions proposed in Theorem 1 are generic and holds for arbitrary switching sequences. Nevertheless, when the switching sequence is priory known, a straightforward simplification arises by reducing the number of LMI constraints with the proper switching set S_{jj^+}, containing all admissible switches $\mathcal{I}_s$ for the switched T-S system, and $\hat{S}_{\hat{j}\hat{j}^+}$ for the switched T-S filter. Moreover, the synchronous switched filter design constitutes a special case of Theorem 4.1 where $\hat{j} = j \in [1, m]$.*

4.3.2 Simulation results and discussion

To illustrated the effectiveness of the proposed switched T-S filter design methodology, let us consider an example of a switched T-S model (4.3), inspired by (Wang et al., 2016a; Liu and Zhao, 2020; Shi et al., 2019). This model, defined by the following matrices and membership functions, switches between $m = 3$ modes ($j \in \{1, 2, 3\}$), and have two fuzzy rules in each mode ($r_j = 2$).

$$A_{1_1} = \begin{bmatrix} 5a & 1 \\ -4.9 & 3(b+a) \end{bmatrix}, A_{2_1} = \begin{bmatrix} a & -b \\ -4.2 & -3 \end{bmatrix}, A_{1_2} = \begin{bmatrix} 0 & 1 \\ -4.3 & -3.4 \end{bmatrix}, A_{2_2} = \begin{bmatrix} 0 & 1 \\ 2b-0.5 & -3.4 \end{bmatrix}$$

$$A_{1_3} = \begin{bmatrix} 0 & 1 \\ -5.5 & -3.2 \end{bmatrix}, A_{2_3} = \begin{bmatrix} 0 & 1 \\ -2.9 & b-2.2 \end{bmatrix}, \forall i_j, B_{i_j} = \begin{bmatrix} 0 & 0.1 \end{bmatrix}^T, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0.2, F_{i_j} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

And the membership functions:

$$\forall j \in [1,m] \begin{cases} h_{1_j}(x_1(t)) = \sin^2(x_1(t)) \\ h_{2_j}(x_1(t)) = \cos^2(x_1(t)) \end{cases}$$

where a and b are two extra parameters dedicated to investigate the feasibility fields of the LMI conditions ($a = 0$ and $b = -1$ recover the vertices considered in (Wang et al., 2016a; Liu and Zhao, 2020; Shi et al., 2019)). To compare the feasibility fields of the proposed LMI conditions , Matlab and the YALMIP toolbox (Lofberg, 2004) with SeDuMi solvers are used. Then, for several values of $a = [-2, 1]$ and $b = [-5, 1]$ with a step of 0.5. Figure 4.1 compares the feasibility fields obtained from Theorem 4.1, Theorem 1 in (Hong et al., 2018), Theorem 3 in (Shi et al., 2019) and Theorem 1 in (Liu and Zhao, 2020).

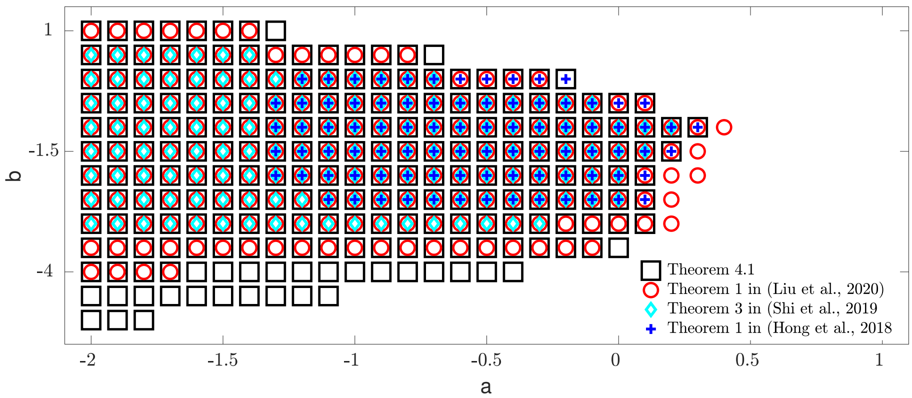

Figure 4.1: Feasibility areas of Theorem 4.1 with respect to previous works.

We notice that over the 403 points tested to draw the feasibility fields of each theorem, the solutions obtained from the proposed LMI-based conditions in Theorem 4.1 ($56.6\%, 227$ solutions) are close to the one obtained from Theorem 1 in (Liu and Zhao, 2020) ($50.4\%, 203$ solutions). Nevertheless, it is worth to point-out that the conditions of Theorem 1 in (Liu and Zhao, 2020) requires dwell-time prescriptions, which is not the case of the conditions proposed in Theorem 4.1. This constitutes the main advantage of our proposal compared to (Liu and Zhao, 2020). Moreover, the feasibility fields obtained from Theorem 4.1 includes the ones obtained from Theorem 1 in (Hong et al., 2018) and Theorem 3 in (Shi et al., 2019), which confirms the conservatism improvement raised by the present proposal. Finally, even if the computational complexity of the LMI-based conditions proposed in Theorem 4.1 is higher than the ones proposed in the considered previous related studies (Hong et al., 2018; Shi et al., 2019; Liu and Zhao, 2020), it doesn't constitute a big drawback since these conditions are usually solved offline.

For the rest of this subsection and for simulation purposes, let us now consider this numerical example at the particular point ($a = 0$, $b = -1$) to illustrate the effectiveness of the proposed LMI conditions for the design of the asynchronous switched T-S H_∞ filters. The switched T-S H_∞ filter gain matrices have been obtained using YALMIP toolbox and SeDuMi solver in Matlab by solving the LMI-based conditions of Theorem 4.1, which provides a minimized H_∞ performance index of $\gamma = 1.41$ and the filter gains given by:

$$\hat{A}_{1_1}=\begin{bmatrix}0.111 & 1.0\\ -4.243 & -3.220\end{bmatrix}, \hat{A}_{1_2}=\begin{bmatrix}0.112 & 0.996\\ -4.244 & -3.220\end{bmatrix}, \hat{A}_{2_1}=\begin{bmatrix}0.112 & 0.994\\ -4.245 & -3.225\end{bmatrix}, \hat{A}_{2_2}=\begin{bmatrix}0.112 & 0.995\\ -4.242 & -3.225\end{bmatrix},$$

$$\hat{A}_{3_1}=\begin{bmatrix}0.113 & 0.995\\ -4.244 & -3.225\end{bmatrix}, \hat{A}_{3_2}=\begin{bmatrix}0.112 & 0.994\\ -4.243 & -3.225\end{bmatrix}, \hat{F}_{i_j}=\begin{bmatrix}0 & 1\end{bmatrix}, \hat{B}_{i_j}^T=\begin{bmatrix}0.0 & 0.5\end{bmatrix}, (\forall i_j).$$

Moreover, the exogenous disturbance input signal and the initial conditions of the switched T-S system and the switched T-S H_∞ filter are given by:

$$\begin{cases} w(t) = 20\cos(6t)e^{-0.4t} + v(t) \\ x(0) = \begin{bmatrix}0.03\\ 0.1\end{bmatrix}, \quad \hat{x}(0) = \begin{bmatrix}0.01\\ -0.2\end{bmatrix}\end{cases} \tag{4.18}$$

Where $v(t)$ is a noise set as normally distributed random numbers updated with a sampling period of $0.0005s$.

For the rest of the simulations, we will consider only the case where the switched T-S system (4.3) and the switched T-S H_∞ filters (4.4) do not share the same switching sets ($\hat{S}_{\hat{j}\hat{j}^+} \neq S_{jj^+}$).

$$S_{jj^+}: \begin{cases} S_{12}=\begin{bmatrix}-\frac{1}{2} & 1\end{bmatrix}, \quad S_{23}=\begin{bmatrix}1 & -40\end{bmatrix}, \quad S_{31}=\begin{bmatrix}1 & 6\end{bmatrix} \\ S_{13}=\begin{bmatrix}-1 & 1\end{bmatrix}, \quad S_{32}=\begin{bmatrix}1 & 10\end{bmatrix}, \quad S_{21}=\begin{bmatrix}4 & 1\end{bmatrix}\end{cases} \tag{4.19}$$

$$\hat{S}_{\hat{j}\hat{j}^+}: \begin{cases} \hat{S}_{12}=\begin{bmatrix}-0.8 & 1\end{bmatrix}, \quad \hat{S}_{23}=\begin{bmatrix}1 & -50\end{bmatrix}, \quad \hat{S}_{31}=\begin{bmatrix}3 & 8\end{bmatrix} \\ \hat{S}_{13}=\begin{bmatrix}-1.5 & 1\end{bmatrix}, \quad \hat{S}_{32}=\begin{bmatrix}2 & 15\end{bmatrix}, \quad \hat{S}_{21}=\begin{bmatrix}4 & 2\end{bmatrix}\end{cases} \tag{4.20}$$

The simulation results are shown in Figures 4.2 to 4.4. The measured output of the system and the exogenous disturbance input signal are plotted in Figure 4.2. Figure 4.3 exhibits the switching signals for both the system and the filter. The unmeasured output $z(t)$ and its estimate $\hat{z}(t)$ are shown in Figure 4.4 along with the estimation error $e_z(t)$.

We see from the simulation results depicted in these figures that the designed switched T-S H_∞ filters provide a good estimation of the unmeasured output, despite the switching sets mismatches and the fact that the asynchronous switching mechanism are arbitrarily chosen. This confirm the effectiveness of the proposed asynchronous filter design methodology.

The following section will be devoted to the design of asynchronous H_∞ filters for switched nonlinear systems described by T-S fuzzy models, to estimate the unmeasured output vector $z(t)$, where the premise variables are not measured, i.e. the premise variables $\xi_j(t)$ depend completely or partially on the state variables $x(t)$. To do so, a reformulation of the switched nonlinear system (4.1) is done to cope with the unmeasured nonlinearities by using nonlinear consequent parts, where the nominal

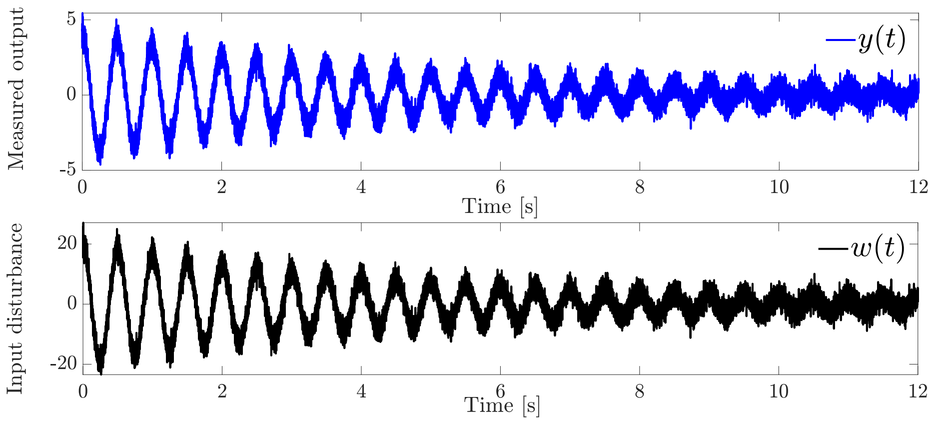

Figure 4.2: Measured output $y(t)$ and input disturbance $w(t)$ of the switched T-S system (4.3).

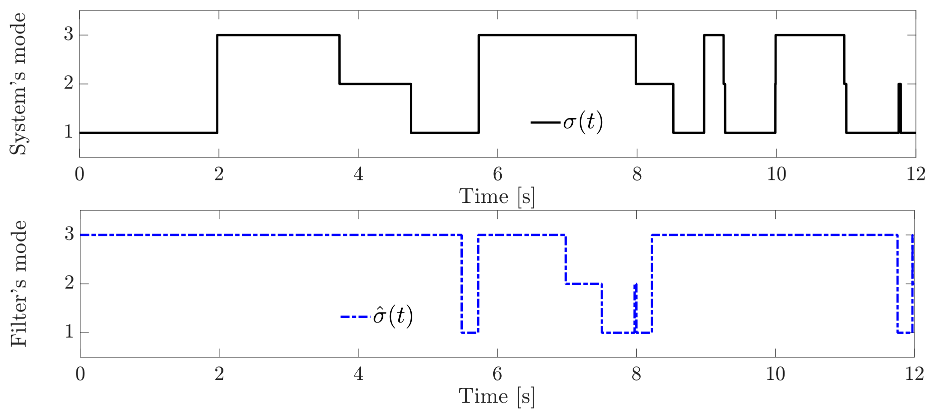

Figure 4.3: Switched modes evolution of the switched T-S system (4.3) and the designed switched T-S H_∞ filter (4.4).

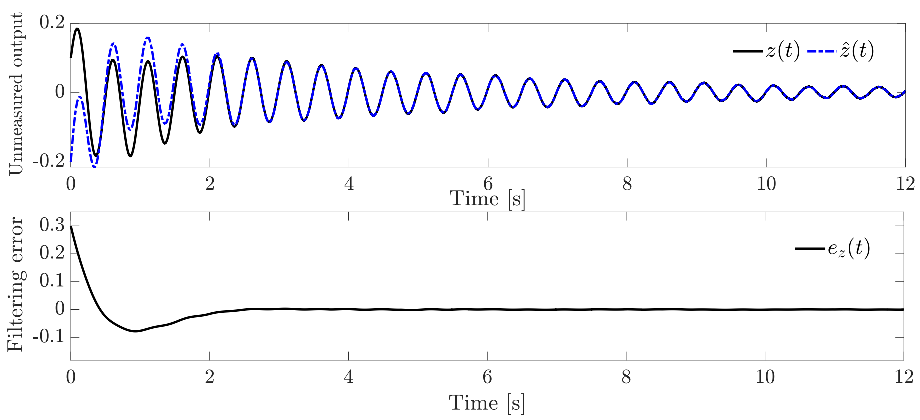

Figure 4.4: Unmeasured output $z(t)$ of the switched T-S system (4.3), its estimate $\hat{z}(t)$ and filtering error $e_z(t)$.

switched T-S model will contain only the measured premise variables and the unmeasured nonlinearities will be handled using sector bounded conditions and incremental quadratic constraints (Açıkmeşe and Corless, 2011; Moodi and Farrokhi, 2014).

4.4 H_∞ filtering with unmeasured premise variables

Let us recall that, by applying the well known sector nonlinearity approach to each mode of the switched nonlinear system (4.1), will lead to a switched Takagi-Sugeno system with unmeasured premise variables as it is explained in **Chapter 3**. The reader may refer to these references (Ichalal et al., 2010; Moodi and Farrokhi, 2014; Moodi and Bustan, 2018; Xie et al., 2019; Chekakta et al., 2021). To circumvent the occurrence of unmeasured premise variables and reduce the conservatism brought by the previously mentioned approaches in the introduction and the references within, i.e., Lipschitz approach (Ichalal et al., 2010; Chekakta et al., 2021), differential mean-value theorem (Ichalal et al., 2011), immersion techniques and auxiliary dynamics (Ichalal et al., 2018). An elegant approach based on the decomposition of the switched nonlinear system into measured nonlinearities and unmeasured nonlinearities, where the measured ones will be used to construct a nominal switched T-S system and the unmeasured ones will kept in the nonlinear consequent parts. Hence, the H_∞ filters design with unmeasured premise variables for the class of switched T-S systems with nonlinear consequent parts (switched N-TS systems) will be the focus of this section.

let us rewrite the sector-bounded nonlinear vector-valued functions (4.1), under Assumption 4.1, as:

$$f_{x,j}(x(t)) = \bar{f}_{x,j}(Mx(t))x(t) + g_{x,j}(Mx(t))\bar{\phi}(Nx(t)) \tag{4.21}$$

$$f_{z,j}(x(t)) = \bar{f}_{z,j}(Mx(t))x(t) + g_{z,j}(Mx(t))\bar{\phi}(Nx(t)) \tag{4.22}$$

where $M \in \mathbb{R}^{n_m \times n_x}$ and $N \in \mathbb{R}^{(n_x - n_m)\times n_x}$ are known matrices selecting respectively the measured and unmeasured state variables, so the nonlinear functions $\bar{f}_{x,j}(Mx(t)) \in \mathbb{R}^{n_x \times n_x}$, $\bar{f}_{z,j}(Mx(t)) \in \mathbb{R}^{n_z \times n_x}$, $g_{x,j}(Mx(t)) \in \mathbb{R}^{n_x \times n_s}$ and $g_{z,j}(Mx(t)) \in \mathbb{R}^{n_z \times n_s}$ contain only measurable nonlinear terms while the unmeasured terms are reported in the vector valued sector-bounded nonlinear functions $\bar{\phi}(Nx(t)) \in \mathrm{co}\{\underline{U}x(t), \bar{U}x(t)\} \subseteq \mathbb{R}^{n_s}$, where $\underline{U} \in \mathbb{R}^{n_s \times n_x}$ and $\bar{U} \in \mathbb{R}^{n_s \times n_x}$.

Remark 4.2. *Let us notice that, for the sake of convenience and to lighten further mathematical developments, we consider in the present study that $f^w_{x,j}(x(t))$ do not depends on unmeasured state variable, i.e. $f^w_{x,j}(x(t)) \equiv f^w_{x,j}(Mx(t))$. Obviously, this choice can be considered as slightly restrictive w.r.t. this class of systems but it doesn't affect the proof of concept in our proposal since this term only relates to the external disturbances $\omega(t)$, which in practice usually affect the system's dynamics without any nonlinear transformation (i.e. $f^w_{x,j}(x(t)) = B$ constant).*

Now, applying the well-known sector nonlinearity approach Tanaka and Wang (2001), only on the nonlinear terms involving measured state variables, we can write:

$$\bar{f}_{x,j}(Mx(t)) = \sum_{i_j=1}^{r_j} h_{i_j}(Mx(t))\mathcal{A}_{i_j},\ \bar{f}_{z,j}(Mx(t)) = \sum_{i_j=1}^{r_j} h_{i_j}(Mx(t))\mathcal{F}_{i_j},\ f^w_{x,j}(Mx(t)) = \sum_{i_j=1}^{r_j} h_{i_j}(Mx(t))B_{i_j}$$
$$g_{x,j}(Mx(t)) = \sum_{i_j=1}^{r_j} h_{i_j}(Mx(t))H^x_{i_j},\ g_{z,j}(Mx(t)) = \sum_{i_j=1}^{r_j} h_{i_j}(Mx(t))H^z_{i_j} \tag{4.23}$$

where r_j is the number of vertices (or fuzzy rules) in the j^{th} mode; $\forall i_j \in \{1,\dots,r_j\}$, $h_{i_j}(Mx(t)) \geq 0$ are normalized membership functions, which depend only on measured state variables and satisfy the convex sum property $\sum_{i=1}^{r_j} h_{i_j}(Mx(t)) = 1$; the matrices $\mathcal{A}_{i_j} \in \mathbb{R}^{n_x \times n_x}$, $\mathcal{F}_{i_j} \in \mathbb{R}^{n_z \times n_x}$, $B_{i_j} \in \mathbb{R}^{n_x \times n_w}$, $H^x_{i_j}$ and $H^z_{i_j}$ define the vertices in each mode j.

From the above developments, the switched nonlinear system (4.1) can be exactly represented on the compact subset $\mathcal{D}_x$ (2.18) of its state space by a switched T-S model with nonlinear consequent parts (switched N-TS) as follows:

$$\begin{cases} \dot{x}(t) = \sum_{j=1}^{m} \sum_{i_j=1}^{r_j} \sigma_j(t) h_{i_j}(Mx(t))\Big(\mathcal{A}_{i_j}x(t) + H^x_{i_j}\bar{\phi}(Nx(t)) + B_{i_j}w(t)\Big) \\ z(t) = \sum_{j=1}^{m} \sum_{i_j=1}^{r_j} \sigma_j(t) h_{i_j}(Mx(t))(\mathcal{F}_{i_j}x(t) + H^z_{i_j}\bar{\phi}(Nx(t))) \\ y(t) = Cx(t) + Dw(t) \end{cases} \tag{4.24}$$

Furthermore, a change of origin for the nonlinear consequent term is performed such that $\phi(Nx(t)) = \bar{\phi}(x(t)) - \underline{U}x(t)$. Thus we have $\phi(Nx(t)) \in \text{co}\{0, Ux(t)\} \subseteq \mathbb{R}^{n_s}$, with $U = \bar{U} - \underline{U}$, and the N-TS (4.24) can be rewritten as:

$$\begin{cases} \dot{x}(t) = \sum_{j=1}^{m} \sum_{i_j=1}^{r_j} \sigma_j(t) h_{i_j}(Mx(t))\Big(A_{i_j}x(t) + H^x_{i_j}\phi(Nx(t)) + B_{i_j}w(t)\Big) \\ z(t) = \sum_{j=1}^{m} \sum_{i_j=1}^{r_j} \sigma_j(t) h_{i_j}(Mx(t))(F_{i_j}x(t) + H^z_{i_j}\phi(Nx(t))) \\ y(t) = Cx(t) + Dw(t) \end{cases} \tag{4.25}$$

with $A_{i_j} = \mathcal{A}_{i_j} + H^x_{i_j}\underline{U}$ and $F_{i_j} = \mathcal{F}_{i_j} + H^z_{i_j}\underline{U}$.

Such a change of variable is useful for design purpose so that the following property holds.

Property 1. *(Dong et al., 2009) The vector of nonlinearities $\phi(Nx(t)) \in \text{co}\{0, Ux(t)\}$ satisfies the following sector-boundedness condition:*

$$\phi(Nx(t))^T \Upsilon(\phi(Nx(t)) - Ux(t)) \leq 0 \tag{4.26}$$

where $\Upsilon \in \mathbb{R}^{n_s \times n_s}$ is any positive-definite diagonal matrix.

Remark 4.3. *Recall that an interesting feature of switched N-TS systems is that they can be used to overcome the occurrence of UPVs efficiently, see e.g. Bouarar et al. (2007); Yoneyama (2009); Moodi and*

Farrokhi (2015); Dong et al. (2009); Moodi et al. (2019); Araújo et al. (2019); Nagy et al. (2020). Indeed, as highlighted from the above developments, a switched N-TS model consists in transforming the original switched nonlinear system (4.1) into a nominal switched T-S model with additive nonlinear consequent parts. In this case, the premise variables of the N-TS will only depend on the measured states, while the nonlinear terms depending on unmeasured variables are reported into the nonlinear consequent part. Moreover, another interesting feature of N-TS approaches is that the resulting number of vertices involved in the design conditions can be significantly reduced to help relaxing the conservatism or the computational complexity v.s. classical T-S modelling approaches.

Given switched N-TS systems (4.25), we propose the design of asynchronous switched N-TS filters given by:

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{\hat{j}=1}^{m} \sum_{i_{\hat{j}=1}}^{r_{\hat{j}}} \hat{\sigma}_{\hat{j}}(t) h_{i_{\hat{j}}}(M\hat{x}(t)) \Big(\hat{A}_{i_{\hat{j}}} \hat{x}(t) + H^x_{i_{\hat{j}}} \phi(N\hat{x}(t)) + \hat{B}_{i_{\hat{j}}} y(t) \Big) \\ \hat{z}(t) = \sum_{\hat{j}=1}^{m} \sum_{i_{\hat{j}=1}}^{r_{\hat{j}}} \hat{\sigma}_{\hat{j}}(t) h_{i_{\hat{j}}}(M\hat{x}(t)) \Big(\hat{F}_{i_{\hat{j}}} \hat{x}(t) + H^z_{i_{\hat{j}}} \phi(N\hat{x}(t)) \Big) \end{cases} \tag{4.27}$$

where $\hat{x}(t)$ is the filter's state vector, $\hat{A}_{i_{\hat{j}}}$, $\hat{B}_{i_{\hat{j}}}$, $\hat{F}_{i_{\hat{j}}}$ are gain matrices to be synthesized; $\forall \hat{j} \in \{1, \dots, m\}$, $\hat{\sigma}_{\hat{j}}(t) \in \{0, 1\}$, with $\sum_{i_{\hat{j}}}^{m} \hat{\sigma}_{\hat{j}}(t) = 1$, define the switching law of the filter (2.47), which is assumed to evolve according to the filter's own switching sets $\hat{\mathscr{S}}_{\hat{j}\hat{j}^+}$ defined by the hyper-planes (2.5).

Let us define the filtering errors $e_x(t) = x(t) - \hat{x}(t)$ and $e_z(t) = z(t) - \hat{z}(t)$, we can write:

$$\begin{aligned} \dot{e}_x(t) =& \Big(A_{h_\sigma} x(t) + H^x_{h_\sigma} \phi(Nx(t)) + B_{h_\sigma} w(t) \Big) - \Big(\hat{A}_{h_{\hat{\sigma}}} \hat{x}(t) + H^x_{h_{\hat{\sigma}}} \phi(N\hat{x}(t)) + \hat{B}_{h_{\hat{\sigma}}} y(t) \Big) \\ =& A_{h_\sigma} e_x(t) + (A_{h_\sigma} - \hat{A}_{h_{\hat{\sigma}}}) \hat{x}(t) + H^x_{h_\sigma} \phi_e(t) + (H^x_{h_\sigma} - H^x_{h_{\hat{\sigma}}}) \phi(N\hat{x}(t)) + B_{h_\sigma} w(t) - \hat{B}_{h_{\hat{\sigma}}} (Cx(t) + Dw(t)) \end{aligned} \tag{4.28}$$

and:

$$\begin{aligned} e_z(t) &= \Big(F_{h_\sigma} x(t) + H^z_{h_\sigma} \phi(Nx(t)) \Big) - \Big(\hat{F}_{h_{\hat{\sigma}}} \hat{x}(t) + H^z_{h_{\hat{\sigma}}} \phi(N\hat{x}(t)) \Big) \\ &= F_{h_\sigma} e_x(t) + (F_{h_\sigma} - \hat{F}_{h_{\hat{\sigma}}}) \hat{x}(t) + H^z_{h_\sigma} \phi_e(t) + (H^z_{h_\sigma} - H^z_{h_{\hat{\sigma}}}) \phi(N\hat{x}(t)) \end{aligned} \tag{4.29}$$

where $\phi_e(t) = \phi(Nx(t)) - \phi(N\hat{x}(t))$.

Then, considering the augmented state error vector $e(t) = \begin{bmatrix} e_x(t)^T & e_z(t)^T \end{bmatrix}^T$ and introducing $\phi_a(t) = \begin{bmatrix} \phi^T(N\hat{x}(t)) & \phi_e^T(t) \end{bmatrix}^T$ and $\tilde{w}(t) = \begin{bmatrix} x^T(t) & \hat{x}^T(t) & w^T(t) \end{bmatrix}^T$, a compact form of the filtering-error dynamics can be written as the following descriptor:

$$E\dot{e}(t) = \tilde{A}_{h_\sigma} e(t) + \Gamma_{h_\sigma h_{\hat{\sigma}}} \phi_a(t) + \tilde{B}_{h_\sigma h_{\hat{\sigma}}} \tilde{w}(t) \tag{4.30}$$

with

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \tilde{A}_{h_\sigma} = \begin{bmatrix} A_{h_\sigma} & 0 \\ F_{h_\sigma} & -I \end{bmatrix}, \Gamma_{h_\sigma h_{\hat{\sigma}}} = \begin{bmatrix} H^x_{h_\sigma} - H^x_{h_{\hat{\sigma}}} & H^x_{h_\sigma} \\ H^z_{h_\sigma} - H^z_{h_{\hat{\sigma}}} & H^z_{h_\sigma} \end{bmatrix}, \tilde{B}_{h_\sigma h_{\hat{\sigma}}} = \begin{bmatrix} -\hat{B}_{h_{\hat{\sigma}}} C & A_{h_\sigma} - \hat{A}_{h_{\hat{\sigma}}} & B_{h_\sigma} - \hat{B}_{h_{\hat{\sigma}}} D \\ 0 & F_{h_\sigma} - \hat{F}_{h_{\hat{\sigma}}} & 0 \end{bmatrix} \tag{4.31}$$

Assumption 4.2. *Let us assume that the characterisation of the nonlinear term $\phi(Nx(t))$ can be made based on a set $\mathcal{W}$ of symmetric matrices $W_{h_\sigma} = \text{diag}\left(W_{h_\sigma}^{11}, W_{h_\sigma}^{22}\right)$, with $= W_{h_\sigma}^{11^T} > 0$ and $W_{h_\sigma}^{22} = W_{h_\sigma}^{22^T} < 0$. Hence, Following the work of Açıkmeşe and Corless Açıkmeşe and Corless (2011), all $W_{h_\sigma} \in \mathcal{W}$ satisfies the Incremental Quadratic Constraint (δQC) given by:*

$$\varphi(t, q_1, q_2) = \begin{bmatrix} q_1 - q_2 \\ \phi(q_1, t) - \phi(q_2, t) \end{bmatrix}^T W_{h_\sigma} \begin{bmatrix} q_1 - q_2 \\ \phi(q_1, t) - \phi(q_2, t) \end{bmatrix} \geq 0 \tag{4.32}$$

where $q_1 = Nx(t)$ and $q_2 = N\hat{x}(t)$.

Remark 4.4. *In order to cope with the term $\phi_e(t)$ in the filtering error dynamics (4.30), an alternative is to consider Lipschitz conditions, however this may lead to conservatism. Instead, the δQC (4.32) is more general since it includes as a special case the Lipschitz condition. Indeed, note that the Lipschitz condition $\|\phi(q_1, t) - \phi(q_2, t)\| \leq \mu \|q_1 - q_2\|$, with $\mu \geq 0$, can be rewritten as:*

$$\begin{bmatrix} q_1 - q_2 \\ \phi(q_1, t) - \phi(q_2, t) \end{bmatrix}^T W_{h_\sigma} \begin{bmatrix} q_1 - q_2 \\ \phi(q_1, t) - \phi(q_2, t) \end{bmatrix} \geq 0,$$

with the incremental multiplier matrix $W_{h_\sigma} = \begin{bmatrix} \mu^2 I & 0 \\ 0 & -I \end{bmatrix}$ satisfying the δQC (4.32).

Now, let us recall that the switched N-TS model (4.25) is valid and guarantee an exact representation of the switched nonlinear system (4.1) inside a domain of validity $\mathcal{D}_x$ defined in (2.18). Hence, to design a switched N-TS filter that has the same structure as the considered switched N-TS system, it is necessary to consider a domain of validity $\mathcal{D}_{\hat{x}}$, similarly to the one of the switched N-TS system (4.25) (i.e. using the same nonlinear sectors). These allow to define the domain of validity of the estimation error $\mathcal{D}_{e_x}$ as follows:

$$\mathcal{D}_{e_x} = \{e_x \in \mathbb{R}^{n_x} : \mathfrak{L} e_x(t) \leq 2\mathcal{Q}\} \tag{4.33}$$

Remark 4.5. *Note that the initialization of the filter's state is arbitrary by nature. Indeed, for practical reasons, it is well-known that the whole system's state is not supposed to be available (else, filtering or any other estimation techniques would be irrelevant). Therefore, it is not realistic to assume that the system's initial state $x(0)$ is known and the filter can be initialized in any other initial state $\hat{x}(0) \in \mathcal{D}_{\hat{x}}$. For the sake of convenience, in this study, we choose to set the filter's initial state as $\hat{x}(0) = 0$ so that the estimation of the domain of attraction will be easier since, in this case we have $e_x(0) = x(0)$.*

The objective of this work is to synthesize the gain matrices of the asynchronous switched N-TS filters (4.27) such that the following requirements are satisfied, together with maximizing the estimate $\mathcal{D}_a \subseteq \mathcal{D}_{e_x}$ of the domain of attraction, i.e. the guaranteed domain of convergence of the filtering errors.

i. $\forall e_x \in \mathcal{D}_a \subseteq \mathcal{D}_{e_x}$, the unmeasured output filtering error $e_z(t)$ is converging to the origin, i.e. $\lim_{t\to+\infty} e_z(t)=0$, when $w(t)=0$.

ii. For all non-zero $\tilde{w}(t) \in L_2[0,\infty)$, the filtering error dynamics (4.30) has a prescribed disturbance attenuation level γ, such that:

$$\frac{\|e_z(t)\|_S^2}{\|\tilde{w}(t)\|_R^2} \leq \gamma^2, \tag{4.34}$$

that is to say, if the following H_∞ criterion is satisfied:

$$\int_0^\infty e_z^T(t)\mathcal{S}e_z(t)dt \leq \gamma^2 \int_0^\infty \tilde{w}^T(t)\mathcal{R}\tilde{w}(t)dt \tag{4.35}$$

where $\mathcal{R}$ and $\mathcal{S}$ are known weighting positive diagonal matrices with appropriate dimension.

4.4.1 Main results

Based on the problem defined above, sufficient LMI-based conditions for the design of asynchronous switched N-TS filters (4.27) with nonlinear consequent parts, incremental quadratic constraints and arbitrary switching (without dwell-time conditions) are provided. These are summarized in the following theorem.

Theorem 4.2: (Chekakta et al., 2023)

Consider the switched N-TS system (4.25), and the switched N-TS H_∞ filters (4.27). $\forall (j,\hat{j}) \in \{1,...,m\}^2$, $\forall (i_j,k_j) \in \{1,...,r_j\}^2$ and $\forall q_{\hat{j}} \in \{1,...,r_{\hat{j}}\}$, both requirements of the above problem statement are satisfied if there exist a scalar $\gamma > 0$ and real matrices $\hat{A}_{i_{\hat{j}}}$, $\hat{B}_{i_{\hat{j}}}$, $\hat{F}_{i_{\hat{j}}}$, $G_{\hat{j}}$, L_{k_j}, R_{k_j}, $X_{\hat{j}}^1 = X_{\hat{j}}^{1^T}$, $X_{\hat{j}}^2$, $X_{\hat{j}}^3$ and diagonal matrices $\Upsilon_{i_j} \geq 0$, which verify the following optimization problem:

$$\begin{cases} \min \gamma^2, \max trace(X_{\hat{j}}^1) \\ \text{s.t.} (4.37), (4.38), (4.39) \text{ and } X_{\hat{j}}^1 > 0 \end{cases} \tag{4.36}$$

where:

$$\frac{1}{r_j-1}\Xi_{i_j i_j q_{\hat{j}}} + \frac{1}{2}\left(\Xi_{i_j k_j q_{\hat{j}}} + \Xi_{k_j i_j q_{\hat{j}}}\right) < 0,\ \forall (i_j,k_j) \in \{1,\dots,r_j\}^2, \tag{4.37}$$

$$X_{\hat{j}^+}^1 = X_{\hat{j}}^1 + G_{\hat{j}}^T C + C^T G_{\hat{j}},\ \forall (\hat{j},\hat{j}^+) \in \{1,\dots,m\}^2 \tag{4.38}$$

$$\begin{bmatrix} X_{\hat{j}}^1 & (*) \\ \mathfrak{L}_{(\nu)} X_{\hat{j}}^1 & 4\mathcal{Q}_{(\nu)}^2 \end{bmatrix} \geq 0,\ \forall \nu \in \{1,\dots,\upsilon\},\ \ \forall \hat{j} \in \{1,\dots,m\}, \tag{4.39}$$

with:

$$\Xi_{i_j k_j, q_{\hat{j}}} = \begin{bmatrix} \mathcal{H}e(L_{k_j}\tilde{A}_{i_j}^T) & (*) & (*) & (*) & (*) & (*) & (*) \\ X_{\hat{j}} - L_{k_i}^T + R_{k_j}^T \tilde{A}_{i_j}^T & -R_{k_j} - R_{k_j}^T & (*) & (*) & (*) & (*) & (*) \\ B_{i_j q_{\hat{j}}}^T & 0 & -\mathcal{R}\gamma^2 & (*) & (*) & (*) & (*) \\ \Gamma_{i_j q_{\hat{j}}}^{1^T} & 0 & \Upsilon_{i_j} U \bar{\Psi} & -2\Upsilon_{i_j} & (*) & (*) & (*) \\ \Gamma_{i_j k_{\hat{j}}}^{2^T} & 0 & 0 & 0 & W_{i_j}^{22} & (*) & (*) \\ \psi X_{\hat{j}} & 0 & 0 & 0 & 0 & -I & (*) \\ \widetilde{N} X_{\hat{j}} & 0 & 0 & 0 & 0 & 0 & -\overline{W}_{i_j}^{11} \end{bmatrix} \tag{4.40}$$

where:

$$\tilde{A}_{i_j} = \begin{bmatrix} A_{i_j} & 0 \\ F_{i_j} & -I \end{bmatrix}, \tilde{B}_{i_j q_{\hat{j}}} = \begin{bmatrix} -\hat{B}_{q_{\hat{j}}} C & A_{i_j} - \hat{A}_{q_{\hat{j}}} & B_{i_j} - \hat{B}_{q_{\hat{j}}} D \\ 0 & F_{i_j} - \hat{F}_{q_{\hat{j}}} & 0 \end{bmatrix}, \Gamma_{i_j q_{\hat{j}}}^1 = \begin{bmatrix} H_{i_j}^x - H_{q_{\hat{j}}}^x \\ H_{i_j}^z - H_{q_{\hat{j}}}^z \end{bmatrix}, \Gamma_{i_j q_{\hat{j}}}^2 = \begin{bmatrix} H_{i_j}^x \\ H_{i_j}^z \end{bmatrix} \tag{4.41}$$

and

$$\psi = \mathrm{diag}\left(0, \mathcal{S}^{1/2}\right), \widetilde{N} = \begin{bmatrix} N & 0 \end{bmatrix}, \bar{\Psi} = \begin{bmatrix} 0 & I & 0 \end{bmatrix}, \overline{W}_{i_j}^{11} = \left(W_{i_j}^{11}\right)^{-1}. \tag{4.42}$$

In this case, the filter's state vector must be updated at the switching times according to:

$$\hat{x}^+(t) = \left(I - \mathcal{L}_{\hat{j}}^{-1}\left(C\mathcal{L}_{\hat{j}}^{-1}\right)^{\dagger} C\right)\hat{x}(t) + \mathcal{L}_{\hat{j}}^{-1}\left(C\mathcal{L}_{\hat{j}}^{-1}\right)^{\dagger} y(t) \tag{4.43}$$

with $\mathcal{L}_{\hat{j}} = \mathbf{V}_{\hat{j}}\sqrt{\Lambda_{\hat{j}}}\mathbf{V}_{\hat{j}}^T$, where $\mathbf{V}_{\hat{j}} \in \mathbb{R}^{n\times n}$ are composed of the orthonormal eigenvectors of $P_{\hat{j}}^1 = \left(X_{\hat{j}}^1\right)^{-1}$, and where $\Lambda_{\hat{j}} \in \mathbb{R}^{n\times n}$ are diagonal matrices, which entries are the eigenvalues of $P_{\hat{j}}^1$.

Furthermore, the intersection of Lyapunov level sets $\mathscr{L}(1)$ (Jungers and Castelan, 2011), defined by (4.44), provides an estimate of the domain of attraction $\mathcal{D}_a$ of e_x.

$$\mathscr{L}(1) = \bigcap_{\hat{j}\in\mathcal{I}_m} \left\{ e_x(0) \in \mathbb{R}^{n_x} : e_x^T(0) P_{\hat{j}}^1 e_x(0) \leq 1 \right\} \tag{4.44}$$

Proof 4.1. *Let us consider a multiple quadratic Lyapunov candidate function given by:*

$$V(t, e(t)) = e(t)^T E P_{\hat{\sigma}} e(t) \tag{4.45}$$

where $EP_{\hat\sigma} = P_{\hat\sigma}^T E > 0$, *with* $P_{\hat\sigma} = \begin{bmatrix} P_{\hat\sigma}^1 & 0 \\ P_{\hat\sigma}^2 & P_{\hat\sigma}^3 \end{bmatrix}$ *and* $P_{\hat\sigma}^1 = P_{\hat\sigma}^{1^T} > 0$.
Taking the time derivative of (4.45), *yields:*

$$\dot V(t,e(t)) = 2e^T(t)P_{\hat\sigma}^T E\dot e(t) = 2e^T(t)P_{\hat\sigma}^T\Big(\tilde A_{h_\sigma} e(t) + \Gamma_{h_\sigma h_{\hat\sigma}}\phi_a(t) + \tilde B_{h_\sigma h_{\hat\sigma}}\tilde w(t)\Big) \tag{4.46}$$

Hence, the items i. and ii. of the problem statement defined above are satisfied if there exists γ *such that the following inequality holds:*

$$\begin{aligned}&\dot V(t,e(t)) + e_z(t)^T\mathcal{S}e_z(t) - \gamma^2\tilde w(t)^T\mathcal{R}\tilde w(t)\\ &= \begin{bmatrix} e(t) \\ \tilde w(t) \\ \phi(N\hat x(t)) \\ \phi_e(t)\end{bmatrix}^T \begin{bmatrix} \mathcal{H}e(\tilde A_{h_\sigma}^T P_{\hat\sigma}) + \psi^\top\psi & (*) & (*) & (*) \\ \tilde B_{h_\sigma h_{\hat\sigma}}^T P_{\hat\sigma} & -\mathcal{R}\gamma^2 & (*) & (*) \\ \Gamma_{h_\sigma h_{\hat\sigma}}^{1^T} P_{\hat\sigma} & 0 & 0 & (*) \\ \Gamma_{h_\sigma h_{\hat\sigma}}^{2^T} P_{\hat\sigma} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ \tilde w(t) \\ \phi(N\hat x(t)) \\ \phi_e(t)\end{bmatrix} < 0\end{aligned} \tag{4.47}$$

On the other hand, the δQC (4.32) *is equivalent to:*

$$\begin{bmatrix} e(t) \\ \phi_e(t)\end{bmatrix}^T \begin{bmatrix} \widetilde N^T W_{h_\sigma}^{11}\widetilde N & 0 \\ 0 & W_{h_\sigma}^{22}\end{bmatrix}\begin{bmatrix} e(t) \\ \phi_e(t)\end{bmatrix} \geq 0 \tag{4.48}$$

From (4.26) *and* (4.48), *the inequality* (4.47) *holds,* $\forall \begin{bmatrix} e^T(t) & \tilde w^T(t) & \phi^T(N\hat x(t)) & \phi_e^T(t)\end{bmatrix} \neq 0$, *if:*

$$\begin{bmatrix} \mathcal{H}e(\tilde A_{h_\sigma}^T P_{\hat\sigma}) + \widetilde N^T W_{h_\sigma}^{11}\widetilde N + \psi^T\psi & (*) & (*) & (*) \\ B_{h_\sigma h_{\hat\sigma}}^T P_{\hat\sigma} & -\mathcal{R}\gamma^2 & (*) & (*) \\ \Gamma_{h_\sigma h_{\hat\sigma}}^{1^T} P_{\hat\sigma} & \Upsilon_{h_\sigma} U\bar\Psi & -2\Upsilon_{h_\sigma} & (*) \\ \Gamma_{h_\sigma h_{\hat\sigma}}^{2^T} P_{\hat\sigma} & 0 & 0 & W_{h_\sigma}^{22}\end{bmatrix} < 0 \tag{4.49}$$

Performing a congruence transformation to (4.49) *with the diagonal matrix* $diag(X_{\hat\sigma}, I, I, I)$ *where* $X_{\hat\sigma} =$

$P_{\hat{\sigma}}^{-1}$, *the following inequality is obtained:*

$$\begin{bmatrix} \mathcal{H}e(X_{\hat{\sigma}}^T \tilde{A}_{h_\sigma}^T) + X_{\hat{\sigma}}^T \widetilde{N}^T W_{h_\sigma}^{11} \widetilde{N} X_{\hat{\sigma}} + X_{\hat{\sigma}}^T \psi^\top \psi X_{\hat{\sigma}} & (*) & (*) & (*) \\ B_{h_\sigma h_{\hat{\sigma}}}^T & -\mathcal{R}\gamma^2 & (*) & (*) \\ \Gamma_{h_\sigma h_{\hat{\sigma}}}^{1^T} & \Upsilon_{h_\sigma} U \bar{\Psi} & -2\Upsilon_{h_\sigma} & (*) \\ \Gamma_{h_\sigma h_{\hat{\sigma}}}^{2^T} & 0 & 0 & W_{h_\sigma}^{22} \end{bmatrix} < 0 \tag{4.50}$$

By applying the Schur complement, the inequality (4.50) *is equivalent to:*

$$\begin{bmatrix} \mathcal{H}e(X_{\hat{\sigma}}^T \tilde{A}_{h_\sigma}^T) & (*) & (*) & (*) & (*) & (*) \\ B_{h_\sigma h_{\hat{\sigma}}}^T & -\mathcal{R}\gamma^2 & (*) & (*) & (*) & (*) \\ \Gamma_{h_\sigma h_{\hat{\sigma}}}^{1^T} & \Upsilon_{h_\sigma} U \bar{\Psi} & -2\Upsilon_{h_\sigma} & (*) & (*) & (*) \\ \Gamma_{h_\sigma h_{\hat{\sigma}}}^{2^T} & 0 & 0 & W_{h_\sigma}^{22} & (*) & (*) \\ \psi X_{\hat{\sigma}} & 0 & 0 & 0 & -I & (*) \\ \widetilde{N} X_{\hat{\sigma}} & 0 & 0 & 0 & 0 & -\overline{W}_{h_\sigma}^{11} \end{bmatrix} < 0 \tag{4.51}$$

Then, by applying Lemma 2.6, the inequality (4.51) *holds if* $\exists (L_{h_\sigma}, R_{h_\sigma})$ *such that:*

$$\begin{bmatrix} \mathcal{H}e(L_{h_\sigma} \tilde{A}_{h_\sigma}^T) & (*) & (*) & (*) & (*) & (*) & (*) \\ X_{\hat{\sigma}} - L_{h_\sigma}^T + R_{h_\sigma}^T \tilde{A}_{h_\sigma}^T & -R_{h_\sigma} - R_{h_\sigma}^T & (*) & (*) & (*) & (*) & (*) \\ B_{h_\sigma h_{\hat{\sigma}}}^T & 0 & -\mathcal{R}\gamma^2 & (*) & (*) & (*) & (*) \\ \Gamma_{h_\sigma h_{\hat{\sigma}}}^{1^T} & 0 & \Upsilon_{h_\sigma} U \bar{\Psi} & -2\Upsilon_{h_\sigma} & (*) & (*) & (*) \\ \Gamma_{h_\sigma h_{\hat{\sigma}}}^{2^T} & 0 & 0 & 0 & W_{h_\sigma}^{22} & (*) & (*) \\ \psi X_{\hat{\sigma}} & 0 & 0 & 0 & 0 & -I & (*) \\ \widetilde{N} X_{\hat{\sigma}} & 0 & 0 & 0 & 0 & 0 & -\overline{W}_{h_\sigma}^{11} \end{bmatrix} < 0 \tag{4.52}$$

which lead, after the application of Lemma 2.5, to the conditions expressed as LMIs in (4.37).

Moreover, for the proof of the Lyapunov decreasing at the switching instant and the update of the filter states, the reader may refer to the proof of Theorem 3.1.

To conclude this proof, it remains to provide an estimate of the domain of attraction $\mathcal{D}_a \subseteq \mathcal{D}_{e_x}$ *(see requirement i. in the problem statement defined above). Notice that the Lyapunov function* (4.45) *can be rewritten as* $V(t, e(t)) = e_x^T(0) P_{\hat{j}}^1 e_x(0)$. *Hence, let us consider the Lyapunov level set* $\mathscr{L}(1)$ *defined, at* $t = 0$, *by* (4.44). *Performing a congruence transformation on* (4.39) *by* $\mathrm{diag}\left(P_{\hat{j}}^1, 1\right)$, *then applying the*

Schur Complement, we get:

$$P^1_{\hat{j}} - \frac{\mathfrak{L}^T_{(\nu)}\mathfrak{L}_{(\nu)}}{4Q^2_{(\nu)}} \geq 0,\ \forall \hat{j} \in \{1,\dots,m\}. \tag{4.53}$$

Pre and post multiplying (4.53) by $e_x^T(0)$ and its transpose provides:

$$e_x^T(0)P^1_{\hat{j}}e_x^T(0) - \frac{e_x^T(0)\mathfrak{L}^T_{(\nu)}\mathfrak{L}_{(\nu)}e_x^T(0)}{4Q^2_{(\nu)}} \geq 0,\ \forall \hat{j} \in \{1,\dots,m\}. \tag{4.54}$$

Consequently, for any $e_x(0) \in \mathscr{L}(1)$, the inequality $e_x^T(0)\mathfrak{L}^T_{(\nu)}\mathfrak{L}_{(\nu)}e_x^T(0) \leq 4Q^2_{(\nu)}$ holds and, from the definition (4.33) of the error domain $\mathcal{D}_{e_x}$, one gets that $\mathscr{L}(1) \subseteq \mathcal{D}_{e_x}$. Finally, a simple procedure to enlarge $\mathscr{L}(1)$ is to maximize the trace of $X^1_{\hat{j}}$ with $\hat{j} \in \{1,\dots,m\}$, as proposed in (4.36). □

Remark 4.6. *The asynchronous switched N-TS H_∞ filters design conditions includes as special case the design of synchronous switched Filter by replacing the index $\hat{j}$ with j in the LMI-based conditions of Theorem 4.2. Moreover, another special case includes Theorem 4.1 for the design of asynchronous switched T-S filter without UPVs presented in Chekakta et al. (2022), which can be recovered by eliminating the 4^{th}, the 5^{th} and the 7^{th} rows, and by setting $\mathcal{R} = I$, $\mathcal{S} = I$. Finally, the special case which considers Lipchitz conditions to deal with the nonlinear consequent part (see remark 4.4) can be retrieved by setting $W^{11}_{i_j} = \mu^2 I$ and $W^{22}_{i_j} = -I$.*

The following section will be dedicated to illustrate its effectiveness over numerical simulations, compared with related recent results from the literature.

4.4.2 Simulation results and Discussion

In order to illustrate the effectiveness of the proposed asynchronous switched N-TS H_∞ filters design methodology, two simulation examples are presented in this section. The first one is an academic example devoted to compare the conservatism of the LMI conditions provided in Theorem 4.2, Theorem 4.1 in Chekakta et al. (2022) with regards to related recent results proposed in Hong et al. (2018); Shi et al. (2019); Liu et al. (2020). The second example illustrates the effectiveness of the present proposal in simulation by considering a switched model of a mass-spring system drawn from Zhang et al. (2015); Ren et al. (2018), extended to the switched nonlinear framework by considering nonlinear stiffness, as proposed in Khalil (2002), subject to external disturbances and considering arbitrary mismatching switching laws.

4.4.2.1 Academic example for conservatism comparison

Let us consider a switched nonlinear system with three modes ($m = 3$, i.e. $\sigma(t) \in \{1, 2, 3\}$) defined by:

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & \Delta_{\sigma(t)}\xi(x_1(t)) + \theta_{\sigma(t)} \\ \beta_{\sigma(t)}(3 + \xi(x_1(t))) & -2\sigma(t) \end{bmatrix}}_{\bar{f}_{x,\sigma(t)}(x_1(t))} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{g_{x,\sigma(t)} = H^x} \underbrace{sin(x_2(t))}_{\bar{\phi}(x_2(t))} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{5} + \sigma(t)e^{-2} \end{bmatrix}}_{f^w_{x,\sigma(t)} = B_{\sigma(t)}} w(t) \\ z(t) = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{f_{z,\sigma(t)}} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \frac{1}{2}w(t) \end{cases} \tag{4.55}$$

where $\xi(x_1(t)) = x_1(t)$ and with the switching parameters $\Delta_1 = \frac{3}{2}$, $\Delta_2 = 2 + \frac{1+2a-b}{6}$, $\Delta_3 = 3 + \frac{3+ab}{6}$, $\theta_1 = 2$, $\theta_2 = 1 + a - \frac{b}{2}$, $\theta_3 = 2 + \frac{ab}{2}$, $\beta_1 = \frac{b}{6}$ and $\beta_2 = \beta_3 = 0$. Moreover, a and b are two scalar parameters dedicated to further check the feasibility fields of the design conditions proposed in Theorem 4.2, then to compare their conservatism with the following previous related results from the literature:

- Theorem 1 in Hong et al. (2018), which proposes the design of H_∞ filters for switched T-S systems with asynchronous switching and measured premises,
- Theorem 3 in Shi et al. (2019), which considers synchronous filtering for switched T-S systems with persistent dwell-time with measured premises.
- Theorem 1 in Liu et al. (2020), which considers non-weighted asynchronous H_∞ filtering design for switched T-S systems with minimum dwell time switching,
- Theorem 4.1 in Chekakta et al. (2022), which constitutes a special case (see Section 4.3) of this chapter, where the premise variables were assumed measurable.

To obtain a classical switched T-S representation of the above nonlinear switched system, we should consider applying the sector nonlinearity approachTanaka and Wang (2001) on both the terms $\xi_1(x_1(t)) = x_1(t)$ and $\xi_2(x_2(t)) = sinc(x_2(t))$, leading to 4 vertices in each switched mode for standard T-S modelling without nonlinear consequent parts. However, assuming that the second state variable $x_2(t)$ is unmeasured, this would make $\xi_2(x_2(t))$ unavailable. Hence, to circumvent the problem of UPVs in conventional T-S modelling, let us consider the N-TS modelling approach, which allows to reduce the number of vertices to 2 in each modes. To do so, let us assume that $x_2(t) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is unmeasured. Then, let $\bar{\phi}(x_2(t)) = sin(x_2(t)) \in \text{co}\left\{\frac{2}{\pi}x_2(t), x_2(t)\right\}$ and apply a change of origin such that $\phi(x_2(t)) = \bar{\phi}(x_2(t)) - \frac{2}{\pi}x_2(t) \in \text{co}\left\{0, \frac{\pi-2}{\pi}x_2(t)\right\}$, leading to $U = \begin{bmatrix} 0 & (\pi-2)/\pi \end{bmatrix}$ (see Property 1 and Figure 4.5). So, applying the sector nonlinearity approach Tanaka and Wang (2001) for $x_1(t) \in [-3, 3]$, we can write for each switched mode $j = \{1,2,3\}$:

$$\xi(x_1(t)) = h_{1_j}(x_1(t)) \times (-3) + h_{2_j}(x_1(t)) \times 3 \tag{4.56}$$

with the membership functions:

$$h_{1_j}(x_1(t)) = \frac{3 - x_1(t)}{6},\ h_{2_j}(x_1(t)) = \frac{x_1(t)+3}{6} \text{ and } h_{1_j}(x_1(t)) + h_{2_j}(x_1(t)) = 1, \tag{4.57}$$

Hence, the switched nonlinear system (4.55) can be rewritten as a switched N-TS system (4.25) with

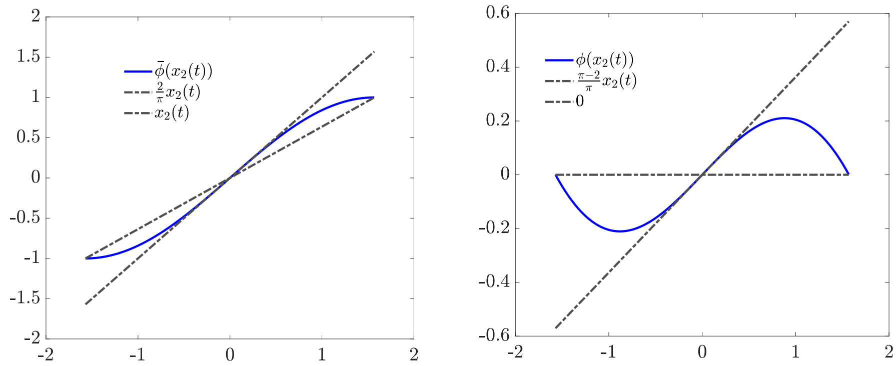

Figure 4.5: Nonlinearity in (4.55) before and after changing the origin (Property 1).

$r_j = 2$ vertices in each of its $m = 3$ modes ($j = \{1,2,3\}$) with the matrices:

$$A_{1_1} = \begin{bmatrix} -1 & -3\Delta_1 + \theta_1 \\ 0 & -2+\frac{2}{\pi} \end{bmatrix}, A_{2_1} = \begin{bmatrix} -1 & 3\Delta_1 + \theta_1 \\ b & -2+\frac{2}{\pi} \end{bmatrix}, A_{1_2} = \begin{bmatrix} -1 & -3\Delta_2 + \theta_2 \\ 0 & -4+\frac{2}{\pi} \end{bmatrix}, A_{2_2} = \begin{bmatrix} -1 & 3\Delta_2 + \theta_2 \\ 0 & -4+\frac{2}{\pi} \end{bmatrix},$$

$$A_{1_3} = \begin{bmatrix} -1 & -3\Delta_3 + \theta_3 \\ 0 & -6+\frac{2}{\pi} \end{bmatrix}, A_{2_3} = \begin{bmatrix} -1 & 3\Delta_3 + \theta_3 \\ 0 & -6+\frac{2}{\pi} \end{bmatrix},$$

$$B_{1_1} = B_{2_1} = \begin{bmatrix} 0 \\ \frac{1}{5}+1e^{-2} \end{bmatrix}, B_{1_1} = B_{2_2} = \begin{bmatrix} 0 \\ \frac{1}{5}+2e^{-2} \end{bmatrix}, B_{1_3} = B_{2_3} = \begin{bmatrix} 0 \\ \frac{1}{5}+3e^{-2} \end{bmatrix}, H^x_{i_j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$F_{i_j} = \begin{bmatrix} 0 & 1 \end{bmatrix}, H^z_{i_j} = 0\,(\forall j \in \{1,2,3\} \text{ and } \forall i_j \in \{1,2\}),\ M = \begin{bmatrix} 1 & 0 \end{bmatrix}, N = \begin{bmatrix} 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = \frac{1}{2}. \tag{4.58}$$

Note that, since we consider here that $x_1(t) \in [-3,3]$ and $x_2(t) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the obtained N-TS represents exactly the switched nonlinear system (4.55) on a validity domain $\mathcal{D}_x$ defined in (2.18) with:

$$\mathfrak{L} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \text{ and } \mathcal{Q} = \begin{bmatrix} 3 \\ 3 \\ \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}. \tag{4.59}$$

For different values of $a \in [-1, 3]$ and $b \in [-2, 0.5]$ with a step of 0.1, the feasibility regions obtained from Theorem 4.2 using YALMIP and SeduMi in MATLAB Lofberg (2004) is compared with the ones obtained from Theorem 4.1 in Chekakta et al. (2022) and from the previous related studies Hong et al. (2018); Shi et al. (2019); Liu et al. (2020). These results are shown in Figure 4.6. For these tests, each considered LMI-based conditions are checked over 1066 points (a, b). The feasibility field provided by Theorem 1 in Hong et al. (2018)(65 solutions, 6.1%) contains the one provided by Theorem 3 in Shi et al. (2019) (91 solutions, 8.54%), which in turn includes the one obtained with Theorem 1 in Liu et al. (2020) (179 solutions, (16.8%). Theorem 4.1 in Chekakta et al. (2022), which is a particular result of Theorem 4.2 provides 210 feasible solutions (19.7%) and it is logically included in the feasibility field we have obtained with the proposed dwell-time free LMI conditions. This clearly confirms that our proposal, using N-TS modelling to circumvent the occurrence of UPVs, provides better results and thus a significant conservatism improvement.

Moreover, to emphasize the computational complexity of the LMI-based conditions proposed in Theorem 4.2, compared with the ones proposed in Theorem 4.1 and in Hong et al. (2018); Shi et al. (2019); Liu et al. (2020), Table 4.1 provides the size S_{lr} of the LMI rows, the number N_c of conditions, and the number N_d of decision variables. This shows that the conditions presented in Theorem 4.2 require fewer decision variables and LMI conditions than the ones proposed in Theorem 4.1 and Liu et al. (2020). Then, compared to Hong et al. (2018) and Shi et al. (2019), the difference of computational complexity remains reasonable and above all, the dwell-time free LMI conditions proposed in this section achieve the lowest H_∞ performance γ, listed in Table Table 4.1 as the minimal values obtained from each results over all the test points (a, b). Furthermore, for the particular point $(a, b) = (2, 0)$, the computational time achieved to solve the different conditions are also given in Table Table 4.1 (we use for this test a 2013 HP notebook-2000 laptop having a 2.0GHz Intel Core I7 2^{nd} generation processor and 8GB of memory). This emphasizes that the conditions of Theorem 4.2, which always provide the best results, are less computationally costly than the other tested results from the literature.

Now, for simulation purposes, let us consider this first example at the particular point ($a = 1$, $b = -1$), where no solution can be found from previous results Hong et al. (2018); Shi et al. (2019); Liu et al. (2020); Chekakta et al. (2022). Let the weighting parameters of the H_∞ criterion (4.35) be set as $\mathcal{R} = \text{diag}(150I_2, 150I_2, 1)$, $\mathcal{S} = 1$ and the δQC constraints (see Assumption 4.2) be defined by $W_{i_j}^{11} = 0.2I$, $W_{i_j}^{22} = -1$. Using YALMIP and SeDuMi in Matlab Lofberg (2004); Labit et al. (2002) to solve the conditions of Theorem 4.2, we obtain a minimized H_∞ performance index of $\gamma = 0.0735$ and

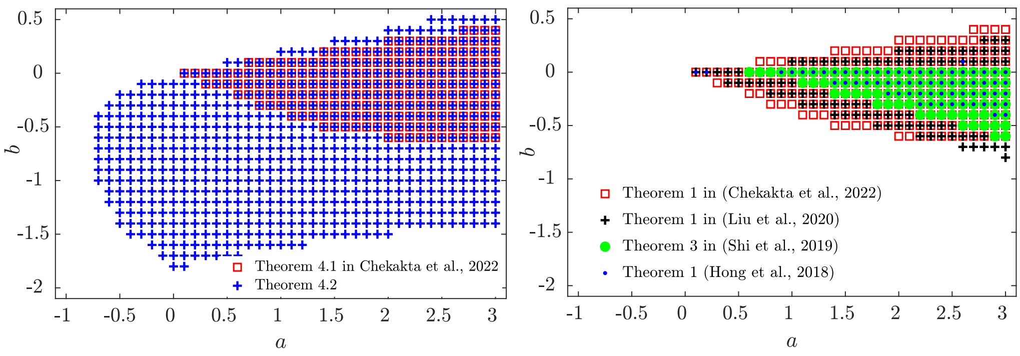

Figure 4.6: Feasibility regions provided by Theorem 4.2 and the considered studies.

Table 4.1: Computational complexity of the considered studies

Method	Feasibility	$\mathcal{S}_{lr}$	$\mathcal{N}_c$	$\mathcal{N}_d$	γ	Sol time (s)
Th. 1 in Hong et al. (2018)	6.1%	$5n_x + n_z + n_w$	136	181	0.1936	4.77
Th. 3 in Shi et al. (2019)	8.54%	$3n_x + n_w + n_y$	52	148	0.4725	1.21
Th. 1 in Liu et al. (2020)	16.8%	$12n_x + n_w + n_z$	652	1097	0.1530	24.79
Theorem 4.1	19.7%	$4n_x + 3n_z + n_w$	580	2693	1.0	18.80
Theorem 4.2	63.79%	$5n_x + 3n_z + 2n_\phi + n_w$	90	218	0.0245	0.84

the switched N-TS filter gain matrices given by:

$$\hat{A}_{1_1} = \begin{bmatrix} -1.0395 & -5.3603 \\ 0.0023 & -2.7921 \end{bmatrix}, \hat{A}_{2_1} = \begin{bmatrix} -0.4579 & 8.0586 \\ -0.5248 & -2.1121 \end{bmatrix}, \hat{A}_{1_2} = \begin{bmatrix} -1.0393 & -5.4133 \\ 0.0028 & -2.7823 \end{bmatrix},$$

$$\hat{A}_{2_2} = \begin{bmatrix} -0.4271 & 8.1495 \\ -0.5397 & -2.1560 \end{bmatrix}, \hat{A}_{1_3} = \begin{bmatrix} -1.0392 & -5.4596 \\ 0.0037 & -2.7776 \end{bmatrix}, \hat{A}_{2_3} = \begin{bmatrix} -0.4209 & 8.2376 \\ -0.5528 & -2.1814 \end{bmatrix},$$

$$\hat{B}_{1_1} = \begin{bmatrix} -0.0077 \\ 0.4232 \end{bmatrix}, \hat{B}_{2_1} = \begin{bmatrix} 0.0123 \\ 0.4206 \end{bmatrix}, \hat{B}_{1_2} = \begin{bmatrix} -0.0074 \\ 0.4236 \end{bmatrix}, \hat{B}_{2_2} = \begin{bmatrix} 0.0115 \\ 0.420 \end{bmatrix},$$

$$\hat{B}_{1_3} = \begin{bmatrix} -0.0076 \\ 0.4240 \end{bmatrix}, \hat{B}_{2_3} = \begin{bmatrix} 0.0104 \\ 0.4188 \end{bmatrix}, \hat{F}^T_{1_1} = \begin{bmatrix} -0.0001 \\ 1.0092 \end{bmatrix}, \hat{F}^T_{2_1} = \begin{bmatrix} -0.0073 \\ 1.0136 \end{bmatrix},$$

$$\hat{F}^T_{1_2} = \begin{bmatrix} 0.0003 \\ 1.00092 \end{bmatrix}, \hat{F}^T_{2_2} = \begin{bmatrix} -0.0087 \\ 1.0146 \end{bmatrix}, \hat{F}^T_{1_3} = \begin{bmatrix} 0.0005 \\ 1.0094 \end{bmatrix}, \hat{F}^T_{2_3} = \begin{bmatrix} -0.0107 \\ 1.0148 \end{bmatrix},$$

as well as the Lyapunov matrices:

$$P_1^1 = \begin{bmatrix} 35.8408 & -0.3675 \\ -0.3675 & 3.4015 \end{bmatrix}^{-1}, P_2^1 = \begin{bmatrix} 34.6381 & -0.4356 \\ -0.4356 & 3.4015 \end{bmatrix}^{-1}, P_3^1 = \begin{bmatrix} 33.6029 & -0.5111 \\ -0.5111 & 3.4015 \end{bmatrix}^{-1}.$$

We assume that the system is switching according to hyper-planes S_{jj^+} which depend on the system's states, while the filter is switching according to the estimated hyper-planes $\hat{S}_{\hat{j}\hat{j}^+}$ depending on the estimated states. This makes the system's and the filter's switching laws asynchronous, despite the fact that they are evolving according to the same hyper-planes. For simulation purpose, these linear hyper-planes (2.3) are defined by:

$$\begin{gathered} S_{12} = \begin{bmatrix} 3 & -1 \end{bmatrix}, S_{13} = \begin{bmatrix} 4 & -5 \end{bmatrix}, \\ S_{21} = \begin{bmatrix} 2 & -9 \end{bmatrix}, S_{23} = \begin{bmatrix} 1 & 2 \end{bmatrix}, \\ S_{31} = \begin{bmatrix} 3 & 2 \end{bmatrix}, S_{32} = \begin{bmatrix} -1 & 0.2 \end{bmatrix} \end{gathered}$$

Moreover, to better highlight such an asynchronous switching behavior, the system and the filter have been initialized in different switching modes ($j_{t=0} = 3, \hat{j}_{t=0} = 1$). Finally, to perform the simulation, we also assume a disturbance input as well as the initial conditions for the filter and the system (see Remark 4.5), set as:

$$\begin{cases} w(t) = 20\cos(3.2\pi t)e^{-0.04t} + v(t) \\ x(0) = \begin{bmatrix} 0 & 0.9 \end{bmatrix} \\ \hat{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix} \end{cases}$$

where $v(t)$ is a normally distributed random signal (with a sampling period of $0.00025\,s$, a zero mean value, a maximal amplitude of 4 and a standard deviation equal to 1). Figure 4.7 shows the evolution of the switching signals, where the asynchronous switching behavior of the designed switched N-TS filter with regard to the switched nonlinear system can be clearly observed. Figure 4.8 shows the measured output $y(t)$ and the input disturbance signal $w(t)$. Then, the unmeasured output $z(t)$ and its estimate $\hat{z}(t)$, as well as the unmeasured output filtering error $e_z(t)$, are plotted in Figure 4.9.

Moreover, let us notice that from these simulations, for $t \in [0, t_f]$ with $t_f = 9s$, we can provide an estimate of the achieved disturbance attenuation level as:

$$\sqrt{\frac{\int_0^{t_f} e_z^T(t)e_z(t)dt}{\int_0^{t_f} \tilde{w}^T(t)R\tilde{w}(t)dt}} = 0.0079 \tag{4.60}$$

It shows that the γ-level disturbance attenuation criterion (4.35) is satisfied since it is lower than $\gamma = 0.0735$ obtained from the application of Theorem 4.2. Furthermore, the evolution of the δQC function $\varphi(t, q_1, q_2) = \begin{bmatrix} (q_1 - q_2)^\top & (\phi(q_1, t) - \phi(q_2, t))^\top \end{bmatrix} W_{h_\sigma} \begin{bmatrix} (q_1 - q_2)^\top & (\phi(q_1, t) - \phi(q_2, t))^\top \end{bmatrix}^\top$ is

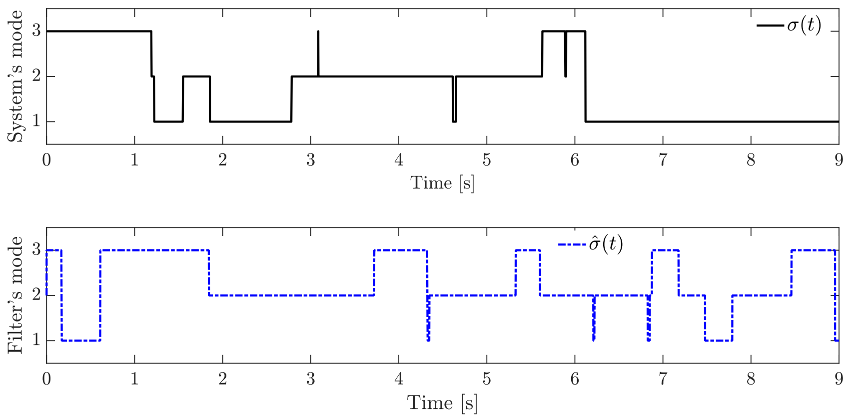

Figure 4.7: Switched modes evolution of the switched N-TS system (4.55) and the designed switched N-TS H_∞ filter (4.27).

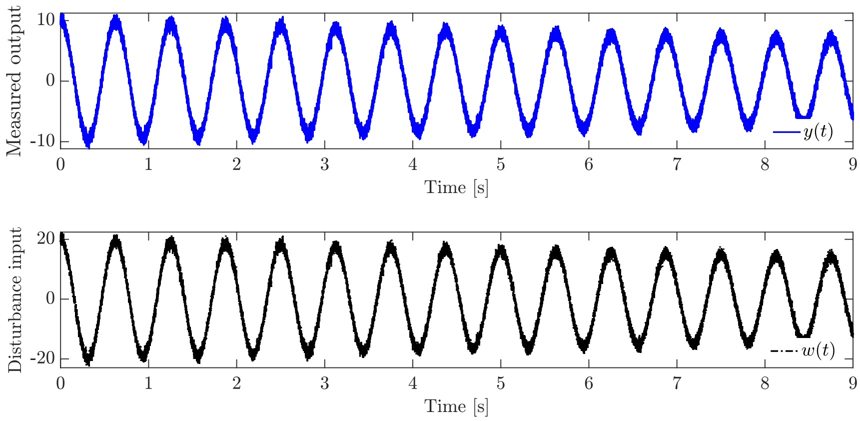

Figure 4.8: Measured output $y(t)$ and disturbance input $w(t)$ of the switched N-TS system (4.55).

depicted in Figure 4.10, which is always positive. Hence, it verifies the Incremental Quadratic Constraint (4.32).

Finally, 4.11 shows the estimate of the filtering error domain of attraction $\mathcal{D}_a$, projected on the planes (e_{x_1}, e_{x_2}) and (x_1, x_2). We can observe that the state and error trajectories remain in $\mathcal{D}_a$ for initial conditions taken on its edge.

From this first numerical example, we can conclude that the presented simulations show the efficiency of the proposed switched N-TS H_∞ filter design methodology in providing good estimation $\hat{z}(t)$ of the unmeasured output $z(t)$, with significant conservatism improvements compared to previous related results (Theorem 4.1, Hong et al. (2018); Shi et al. (2019); Liu et al. (2020)), despite the occurrence of asynchronous switching and the application of a noisy input disturbance signal.

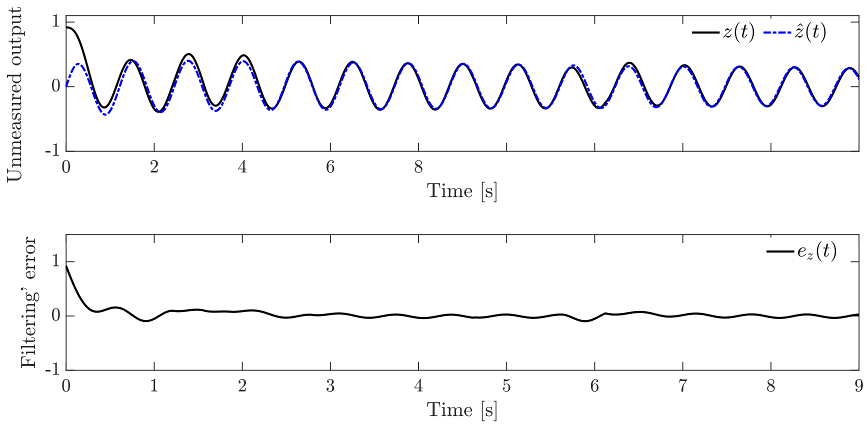

Figure 4.9: Unmeasured output $z(t)$ of the switched T-S system (4.3), its estimate $\hat{z}(t)$ and filtering error $e_z(t)$.

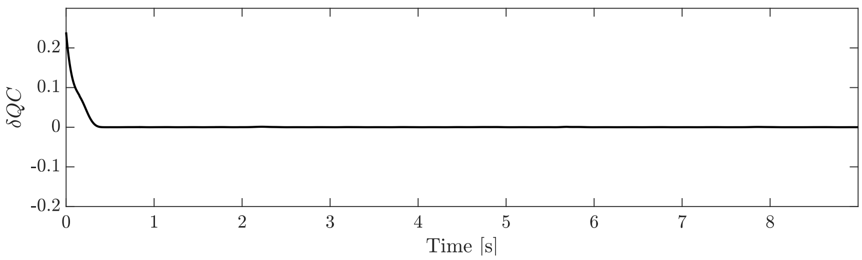

Figure 4.10: Satisfaction of the incremental quadratic constraint (4.32).

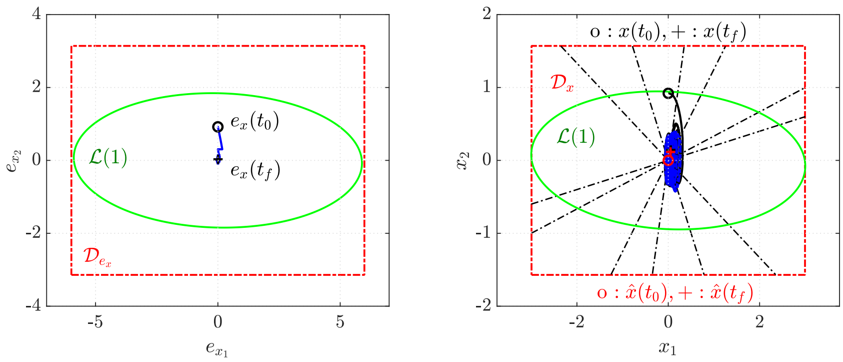

Figure 4.11: Projection of $\mathcal{D}_a = \mathscr{L}(1)$ (green lines), $\mathcal{D}_{e_x}$, $\mathcal{D}_x$ (red dashed-lines) on planes of interest, states and output error trajectories (blue lines for the filter (4.27), black lines for the system (4.3)), switching hyper-plane frontiers (dashed-dotted black lines).

4.4.2.2 Case study of a switched mass-spring system

Let us consider the switched mass-spring system depicted in Figure 4.12, drawn from Zhang et al. (2015); Ren et al. (2018), and governed by the following switched state space model with $\sigma(t) \in \{1, 2\}$:

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_c + k_{\sigma(t)}}{m_1} & \frac{k_{\sigma(t)}}{m_1} & -\frac{c}{m_1} & 0 \\ \frac{k_{\sigma(t)}}{m_2} & -\frac{k_{\sigma(t)}}{m_2} & 0 & -\frac{c}{m_2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix} w(t) \\ z(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} x(t) \\ y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x(t) + 0.8 w(t) \end{cases} \tag{4.61}$$

where $x_1(t)$ and $x_2(t)$ denotes the displacements of respectively the masses $m_1 = 6\,kg$ and $m_2 = 1\,kg$; k_c is the stiffness of the left spring while k_1 and k_2 are the stiffness of both the right springs which are assumed to switch according to further defined switching hyper-planes.

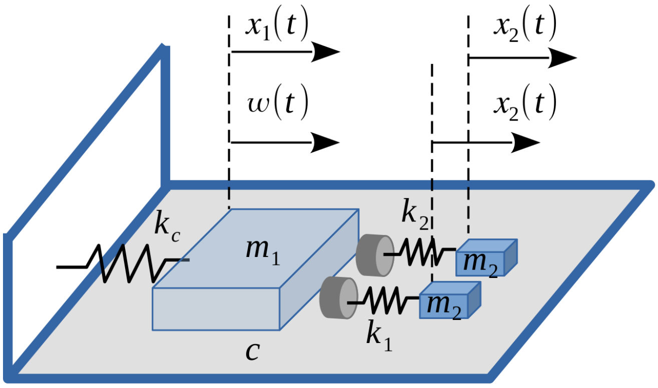

Figure 4.12: Switched mass-spring system.

Now, to extend the mass-spring system (4.61) in the nonlinear framework, let us assume hardening springs, as modelled in Khalil (2002), where:

$$k_c = \kappa_c\left(1 + a_c^2 x_1^2(t)\right) \text{ and } k_{\sigma(t)} = \kappa_{\sigma(t)}\left(1 + a_{\sigma(t)}^2 x_2^2(t)\right), \text{ for } \sigma(t) \in \{1, 2\}. \tag{4.62}$$

where $\kappa_c = 10\,N/m$, $\kappa_1 = 10\,N/m$ and $\kappa_2 = 20\,N/m$ are the nominal springs' stiffness; $a_c = 0.4$, $a_1 = 0.1$ and $a_2 = 0.2$ are the spring's hardening coefficients.

Hence, by substituting (4.62) in (4.61), and assuming that x_2 is unmeasured, it yields the following nonlinear switched model of the considered nonlinear mass-spring system:

$$\begin{cases}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \ddot{x}_1(t)\\ \ddot{x}_2(t)\end{bmatrix}=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\frac{\kappa_c}{m_1}\left(1+a_c^2x_1^2(t)\right)-\frac{\kappa_{\sigma(t)}}{m_1} & \frac{\kappa_{\sigma(t)}}{m_1} & -\frac{c}{m_1} & 0\\ \frac{\kappa_{\sigma(t)}}{m_2} & -\frac{\kappa_{\sigma(t)}}{m_2} & 0 & -\frac{c}{m_2}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ \dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}+\begin{bmatrix}0\\ 0\\ \frac{\kappa_{\sigma(t)}}{m_1}a_{\sigma(t)}^2\\ -\frac{\kappa_{\sigma(t)}}{m_2}a_{\sigma(t)}^2\end{bmatrix}\bar{\phi}(x(t))+\begin{bmatrix}0\\ 0\\ \frac{1}{m_1}\\ 0\end{bmatrix}w(t)\\ z(t)=\begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}x(t)\\ y(t)=\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}x(t)+0.8w(t)\end{cases} \tag{4.63}$$

With $\bar{\phi}(x(t))=\left(x_2^3(t)-x_2^2(t)x_1(t)\right)$. Let us assume that $x_1(t)\in[-2,2]$ and $x_2(t)\in[-1,1]$. Thus, we can define the validity domain $\mathcal{D}_x$ in (2.18) with:

$$\mathfrak{L}=\begin{bmatrix}1 & 0 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & -1 & 0 & 0\end{bmatrix}\text{ and }\mathcal{Q}=\begin{bmatrix}2\\ 2\\ 1\\ 1\end{bmatrix}. \tag{4.64}$$

Also, we can write:

$$\bar{\phi}(x(t))=x_2^3(t)-x_2^2(t)x_1(t)=\begin{bmatrix}-x_2^2(t) & x_2^2(t)\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}\in\text{co}\{-Ux(t),Ux(t)\} \tag{4.65}$$

with $U=\begin{bmatrix}-1 & 1 & 0 & 0\end{bmatrix}$.

Hence, we can apply the change of variable $\phi(x(t)=\bar{\phi}(x(t))+Ux(t)\in\text{co}\{0,2Ux(t)\}$, so that the switched nonlinear system (4.63) can be rewritten as:

$$\begin{cases}\dot{x}(t)=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\frac{\kappa_c}{m_1}\left(1+a_c^2\xi(x_1(t))\right)+\frac{\kappa_{\sigma(t)}}{m_1}\left(a_{\sigma(t)}^2-1\right) & \frac{\kappa_{\sigma(t)}}{m_1}\left(1-a_{\sigma(t)}^2\right) & -\frac{c}{m_1} & 0\\ \frac{\kappa_{\sigma(t)}}{m_2}\left(1-a_{\sigma(t)}^2\right) & \frac{\kappa_{\sigma(t)}}{m_2}\left(a_{\sigma(t)}^2-1\right) & 0 & -\frac{c}{m_2}\end{bmatrix}x(t)+\begin{bmatrix}0\\ 0\\ \frac{\kappa_{\sigma(t)}}{m_1}a_{\sigma(t)}^2\\ -\frac{\kappa_{\sigma(t)}}{m_2}a_{\sigma(t)}^2\end{bmatrix}\phi(x(t))+\begin{bmatrix}0\\ 0\\ \frac{1}{m_1}\\ 0\end{bmatrix}w(t)\\ z(t)=\begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}x(t)\\ y(t)=\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}x(t)+0.8w(t)\end{cases} \tag{4.66}$$

with $\xi(x_1(t))=x_1^2(t)$, which can be decomposed, by apply the sector nonlinearity approach Tanaka and Wang (2001), as:

$$\xi(x_1(t)) = \underbrace{\frac{x_1^2(t)}{4}}_{h_{1_j}(x_1(t))\geq 0} \times 4 + \underbrace{\frac{4-x_1^2(t)}{4}}_{h_{2_j}(x_1(t))\geq 0} \times 0, \text{ where } h_{1_j}(x_1(t)) + h_{2_j}(x_1(t)) = 1, \forall j \in \{1,2\}, \tag{4.67}$$

leading to a N-TS model (4.25) with two switched modes ($j \in \{1,2\}$) and two T-S vertices ($i_j \in \{1,2\}$) specified by the matrices:

$$A_{1_j} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\kappa_c}{m_1}\left(1+4a_c^2\right)+\frac{\kappa_j}{m_1}\left(a_j^2-1\right) & \frac{\kappa_j}{m_1}\left(1-a_j^2\right) & -\frac{c}{m_1} & 0 \\ \frac{\kappa_j}{m_2}\left(1-a_j^2\right) & \frac{\kappa_j}{m_2}\left(a_j^2-1\right) & 0 & -\frac{c}{m_2} \end{bmatrix}, A_{2_j} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\kappa_c}{m_1}+\frac{\kappa_j}{m_1}\left(a_j^2-1\right) & \frac{\kappa_j}{m_1}\left(1-a_j^2\right) & -\frac{c}{m_1} & 0 \\ \frac{\kappa_j}{m_2}\left(1-a_j^2\right) & \frac{\kappa_j}{m_2}\left(a_j^2-1\right) & 0 & -\frac{c}{m_2} \end{bmatrix},$$
$$B_{1_j} = B_{2_j} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix}, H_{1_j}^x = H_{2_j}^x = \begin{bmatrix} 0 \\ 0 \\ \frac{\kappa_j}{m_1}a_j^2 \\ -\frac{\kappa_j}{m_2}a_j^2 \end{bmatrix}, F_{1_j} = F_{2_j} = N = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}^T, H_{1_j}^z = H_{2_j}^z = 0, C = M = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T, D = \frac{4}{5}. \tag{4.68}$$

To illustrate the effectiveness of the proposed switched nonlinear design procedure, let the weighting parameters of the H_∞ criterion (4.35) be set as $\mathcal{R} = \text{diag}(250I_4, 250I_4, 1)$ and $\mathcal{S} = 1$ and the δQC constraints (see Assumption 4.2) be defined by $W_{i_j}^{11} = 0.26I$, $W_{i_j}^{22} = -0.1$. Solving the LMI conditions of Theorem 4.2 in Matlab with the use of YALMIP and SeDuMi Lofberg (2004), one obtains a minimized H_∞ performance index $\gamma = 0.0775$ and the switched N-TS filter (4.27) gain matrices given by:

$$\hat{A}_{1_1} = \begin{bmatrix} 0.0753 & 0.1450 & 1.0000 & 0.0000 \\ 0.0444 & 0.0673 & -0.0000 & 1.0000 \\ -4.1267 & 2.1224 & -0.8333 & -0.0000 \\ 10.4952 & -10.2954 & 0.0000 & -5.0000 \end{bmatrix}, \hat{A}_{2_1} = \begin{bmatrix} 0.0833 & 0.1442 & 1.0000 & -0.0000 \\ 0.0619 & 0.0922 & -0.0000 & 1.0000 \\ -3.4674 & 1.3555 & -0.8333 & 0.0000 \\ 9.8202 & -9.4648 & -0.0000 & -5.0000 \end{bmatrix}, \hat{B}_{1_1} = \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.2071 \\ -0.0003 \end{bmatrix}, \hat{B}_{2_1} = \begin{bmatrix} 0.0000 \\ 0.0001 \\ 0.2070 \\ -0.0000 \end{bmatrix}$$
$$\hat{A}_{1_2} = \begin{bmatrix} 0.1238 & 0.0954 & 1.0000 & -0.0000 \\ 0.0836 & 0.0693 & -0.0000 & 1.0000 \\ -5.7447 & 3.7354 & -0.8333 & 0.0000 \\ 19.7566 & -19.9103 & 0.0000 & -5.0000 \end{bmatrix}, \hat{A}_{2_2} = \begin{bmatrix} 0.1015 & 0.1080 & 1.0000 & -0.0000 \\ 0.0971 & 0.0910 & 0.0000 & 1.0000 \\ -5.0383 & 2.9238 & -0.8333 & -0.0000 \\ 19.0457 & -18.9358 & -0.0000 & -5.0000 \end{bmatrix}, \hat{B}_{1_2} = \begin{bmatrix} 0.0000 \\ 0.0001 \\ 0.2071 \\ 0.0000 \end{bmatrix}, \hat{B}_{2_2} = \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.2071 \\ 0.0001 \end{bmatrix},$$
$$\hat{F}_{1_1}^T = \begin{bmatrix} 0.0029 \\ 1.0014 \\ 0.0000 \\ 0.0000 \end{bmatrix}, \hat{F}_{2_1}^T = \begin{bmatrix} 0.0015 \\ 1.0029 \\ -0.0000 \\ -0.0000 \end{bmatrix}, \hat{F}_{1_2}^T = \begin{bmatrix} 0.0044 \\ 1.0046 \\ 0.0000 \\ -0.0000 \end{bmatrix}, \hat{F}_{2_2}^T = \begin{bmatrix} 0.0039 \\ 1.0037 \\ -0.0000 \\ 0.0000 \end{bmatrix}, \tag{4.69}$$

and the Lyapunov matrices:

$$P_1^1 = \begin{bmatrix} 1.9411 & 1.7887 & -1.1547 & 0.1795 \\ 1.7887 & 3.9090 & -3.0718 & -4.2428 \\ -1.1547 & -3.0718 & 5.2046 & 0.8520 \\ 0.1795 & -4.2428 & 0.8520 & 26.4345 \end{bmatrix}^{-1}, P_2^1 = \begin{bmatrix} 1.8585 & 1.6720 & -1.1351 & 0.2570 \\ 1.6720 & 3.9090 & -3.0718 & -4.2428 \\ -1.1351 & -3.0718 & 5.2046 & 0.8520 \\ 0.2570 & -4.2428 & 0.8520 & 26.4345 \end{bmatrix}^{-1}. \tag{4.70}$$

For simulation purposes, we assume that the switched nonlinear system (4.63) switches according to the hyper-planes (2.4):

$$S_{12} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}$$
$$S_{21} = \begin{bmatrix} 1 & -2 & 0 & 0 \end{bmatrix}$$

Moreover, to better highlight the asynchronous behavior of the switched mass-spring system and the designed switched N-TS filter, it is assumed that the latter switches according to different hyper-planes (2.5):

$$\hat{S}_{12} = \begin{bmatrix} 2 & -1 & 0 & 0 \end{bmatrix}$$
$$\hat{S}_{21} = \begin{bmatrix} 1 & -4 & 0 & 0 \end{bmatrix}$$

We set the noise disturbance input, and the initial conditions, so that the system and the filter are respectively initialized in different modes, i.e. $j_{t=0} = 2$ and $\hat{j}_{t=0} = 1$. as:

$$\begin{cases} w(t) = 8\sin(0.8\pi t)e^{-0.01t} + v(t) \\ x(0) = \begin{bmatrix} 0.5 \\ 0.4 \\ 0.01 \\ 0.2 \end{bmatrix}, \hat{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{cases}$$

Where $v(t)$ is a normally distributed random signal (with a sampling period of $0.001\,s$, a zero mean value, a maximal amplitude of 2 and a standard deviation equal to 1).

Figure 4.13 shows the evolution of the measured output $y(t)$ and the noisy disturbance $w(t)$. Figure 4.14 shows the evolution of the asynchronous switching signals. The unmeasured output $z(t)$ and its estimate $\hat{z}(t)$ are depicted in Figure 4.15, together with the filtering error $e_z(t)$. Figure 4.16 confirms that the δQC constraints (4.32) is always verified since it remains positive. Finally, the estimation of the domain of attraction $\mathcal{D}_a$ are plotted in Figure 4.17, projected on the planes of interest (e_{x_1}, e_{x_2}) and (x_1, x_2), together with the system's and filter's state trajectories. As expected, for initial conditions taken inside $\mathcal{D}_a$, the state estimation error $e_x(t)$, the state trajectory $x(t)$ and the filter's state $\hat{x}(t)$ remains in $\mathcal{D}_a$. Furthermore, from the simulation results depicted in Figures 4.13 and 4.14, for $t \in [0, t_f]$

with $t_f = 30s$, we obtain an estimate of the achieved disturbance attenuation level as:

$$\sqrt{\frac{\int_0^{t_f} e_z^T(t)e_z(t)dt}{\int_0^{t_f} \tilde{w}^T(t)R\tilde{w}(t)dt}} = 0.0039, \tag{4.71}$$

which satisfies the γ-level disturbance attenuation criterion (4.35) since it is lower than the obtained $\gamma = 0.0775$ from the solution of Theorem 4.2.

From these simulations, we can conclude that the designed switched N-TS filter successfully estimates the unmeasured displacement $z(t) = x_2(t)$ despite the switched asynchronous phenomena as well as the noisy input disturbances affecting the dynamic behaviour of the considered switched nonlinear mass-spring system.

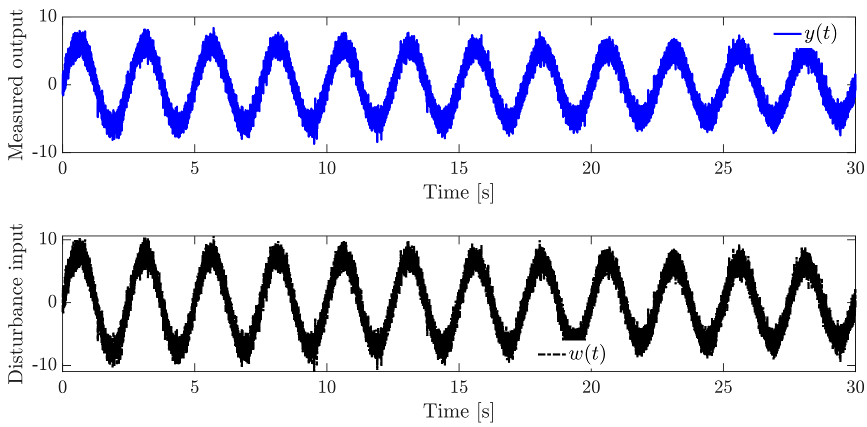

Figure 4.13: Measured output $y(t)$ and disturbance input $w(t)$ of the switched mass-spring system (4.63).

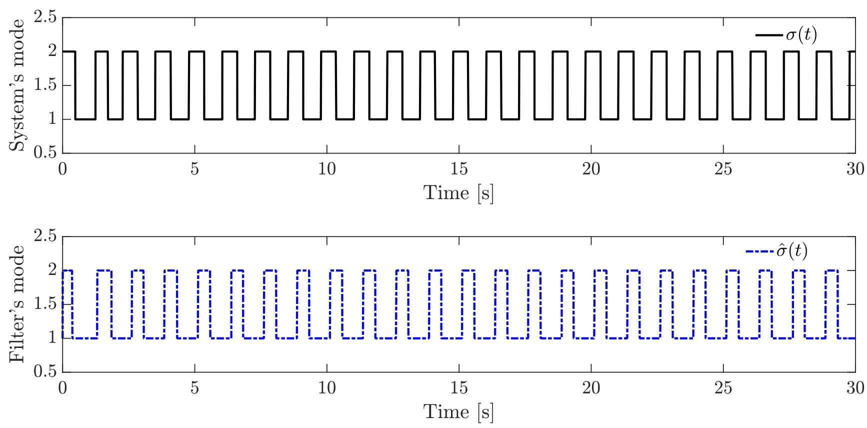

Figure 4.14: Switched modes evolution of the switched mass-spring system (4.63) and the designed switched N-TS H_∞ filter (4.27).

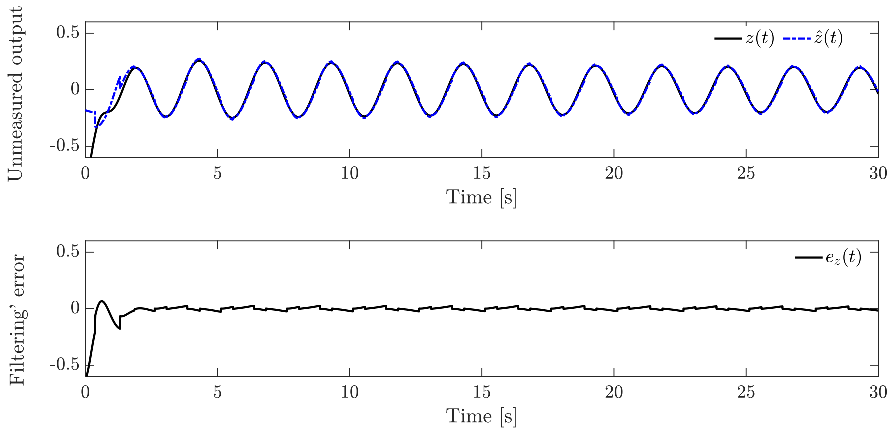

Figure 4.15: Unmeasured output $z(t)$ the switched mass-spring system (4.63), its estimate $\hat{z}(t)$ and filtering error $e_z(t)$.

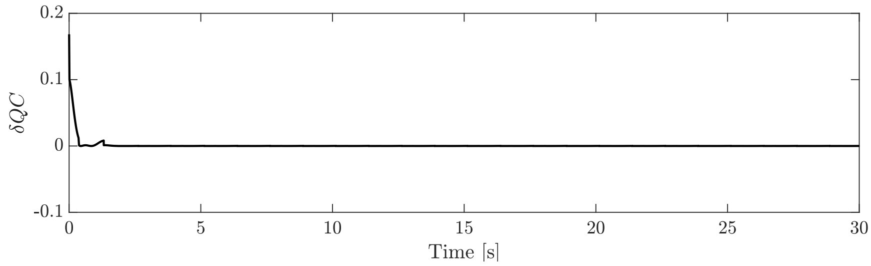

Figure 4.16: Satisfaction of the incremental quadratic constraint (4.32).

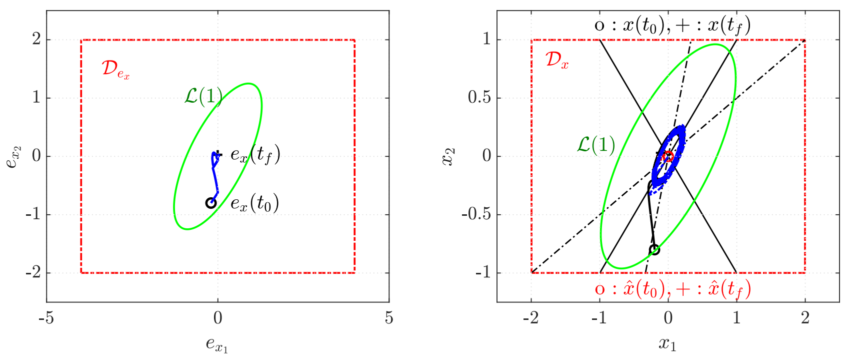

Figure 4.17: Projection of $\mathcal{D}_a = \mathscr{L}(1)$ (green lines), $\mathcal{D}_{e_x}$, $\mathcal{D}_x$ (red dashed-lines) on planes of interest, states and output error trajectories (blue lines for the filter (4.27), black lines for the system (4.3)), system's switching hyper-plane frontiers (black lines), filter's switching hyper-plane frontiers (dashed-dotted black lines)

4.5 Conclusion

In this chapter, asynchronous switched Takagi-Sugeno H_∞ filters have been proposed to estimate/filter the unmeasured outputs of continuous-time switched Takagi-Sugeno systems under state-dependent switching with switching sets mismatching (asynchronous switching). This chapter was mainly divided into two parts, where the first part was dedicated to the design of H_∞ filters for switched T-S systems with measured premise variables. And in the second part, unmeasured premise variables were the main focus, by transforming the switched T-S system into a nominal switched T-S system and the unmeasured nonlinearities were kept in the nonlinear consequent parts. Then, for conservatism reduction purposes, the incremental quadratic constraint was used as an alternative to Lipschitz conditions used in **Chapter 3**. The proposed conditions were derived using multiple switched Lyapunov functions with the descriptor redundancy approach, and expressed in terms of LMIs, with noticeable improvements regarding conservatism with respect to previous related studies. Several examples have been presented to illustrate the effectiveness of the proposed design methodology both in time-simulation and conservatism reduction.

General Conclusion

The work addressed in this thesis was devoted to the study of state estimation using different techniques for a class of continuous-time switched nonlinear systems, in the presence of bounded disturbances and asynchronous switching. The contributions were made according to two possible situations regarding the availability of the T-S premise variables (measured/unmeasured), and according to two possible situations regarding the switching phenomena (same/different switching sets).

- The representation of a nonlinear system with a T–S model often leads to consider state dependent premise variables that are not always measurable, which leads to the occurrence of two classes of T-S systems, T-S modelling with measured premise variables and T-S modelling with unmeasured premise variables.
    - T-S modelling with measured premise variables: The majority of the work done on the synthesis of observers and filters for systems described by the T-S representation, assumes that the premise variables are measured, this implies, that the observer/filter shares the same premise variables as the system (Yoneyama, 2009; Ichalal et al., 2010).
    - T-S modelling with unmeasured premise variables: dealing with unmeasured premise variables is the most important issue to consider when designing observers and filters, since it represents a larger class of systems than the ones with measured premise variables (Bergsten and Palm, 2000; Ichalal et al., 2011, 2018). Significant work has been done in the last decades to deal with unmeasured premise variables outside the switched nonlinear systems context (see **Chapter 1** and references therein). The design of switched observers and H_∞ filters for switched T-S systems with unmeasured premise variables has been rarely investigated in the literature. therefore, this issue has been taken into consideration in our proposals in **Chapter 3** and **Chapter 4**.
- The switched mechanisms of the system and the observer/filter investigated in this work are regarded as arbitrary switching sequences (state-dependent switching) defined by the linear hyper planes. In that regard, two cases can be distinguished.
    - The observer/filter and the system use the same switching sets, i.e. they switch at the same time. However, this case is not practical due to the resulting difficulties in predicting/identifying the exact switching sets of the switched system.
    - The observer/filter and the system use different switching sets (switching mismatches). Due to the precision in identifying the corresponding switching sets and the mismatch between the estimated/filtered states and the system states, especially during transients

and when external disturbances occur, this case is more challenging and more likely to be encountered in real-world applications.

Before examining the estimation problem, we have given a general introduction in **Chapter 1**, which presents the state of the art of state estimation for switched nonlinear systems, the overall organization of the manuscript, and the main contributions. Furthermore, in line with the main objectives of this thesis, preliminaries on continuous-time switched T-S systems have been presented in **Chapter 2**. In this context, the nonlinear systems are represented by Takagi-Sugeno models and driven by switching laws that describe the discrete dynamics of the switched nonlinear system. Thus, we are particularly interested in studying state estimation of switched nonlinear systems based on T-S and switched multi-models. Note that these two classes of systems are represented by similar structures. Indeed, their concept is based on a decomposition of the global system into a set of local models, valid in defined regions of the state space and linked together by interpolation or switching mechanisms. Thus, if the interconnection is defined by discrete functions, we speak of a switched system. On the other hand, if the relations between local models are defined by weighting functions, we speak of fuzzy representations in the form of multi-models T-S. First, we have introduced the formal definition of switched systems, which represent a particular class of hybrid dynamical systems (HDS). Because of their ability to model a wide variety of physical systems, this category of systems has attracted considerable interest from researchers and engineers, as they can be viewed as multi-model systems in which a subsystem (mode) is active at a time interval according to a switching function, where the switching between the different dynamics is governed by a discrete switching law that specifies the mode (dynamics) that is active at each time interval. Then, to deal with all the nonlinearities arising from modeling switched nonlinear systems, we presented the well-known framework for modelling nonlinear systems known as Takagi-Sugeno (T-S) fuzzy models (Takagi and Sugeno, 1985), which, when obtained by the sector nonlinearity approach (Tanaka and Wang, 2001), provide an intriguing framework for representing precisely nonlinear systems without information loss. Frequently, these models are employed to characterize nonlinear systems as weighted combinations of linear systems, for which many tools designed specifically for linear systems can be applied easily. The objective was then to present switched Takagi-Sugeno systems by combining T-S multi models and switched systems so as to have the elementary notions allowing us to approach **Chapter 3** and **Chapter 4** as well as to consider their state estimation and filtering. To sum up, the state of the art and the preliminaries concerning observation, filtering, the classes considered, the different switching laws and the illustrative examples encountered in this chapter have allowed the formulation of the problem addressed in this thesis, which can be divided into two well-balanced approaches, namely observation and filtering, which are presented in the third and fourth chapters.

In **Chapter 3**, our contributions to the design of asynchronous switched T-S observers for continuous-time switched T-S systems subject to output disturbances and switching mismatches have been presented, considering both cases where the premise variables are measured/unmeasured. Most of the available studies in the switched T-S framework (observers design) consider that the observer and the system to be estimated share the same switched sets and that both switch at the same instants (synchronous switching), whether the switching signal is *priori* known or the switching instants are known (Pettersson, 2005; Hamdi et al., 2009; Lendek et al., 2014a; Ethabet et al., 2018; Garbouj et al., 2020). The proposed asynchronous switched T-S observers offer a significant robustness against switching mismatches by enabling the handling of arbitrary mismatches in switch sets and/or sequences between the observer and switched system. Despite most of the previous studies mentioned along the manuscript, which assume that the considered switched systems and switched observers are initialized in the same switching modes, an important feature of our proposals is that they can start from different initial modes (asynchronous initialization of the switched modes). It should be noted that the proposed LMI based conditions along the manuscript are dwell-time free conditions (see e.g. (Pettersson, 2005; Kader et al., 2018; Yang et al., 2019; Chekakta et al., 2021, 2023)), which are useful for arbitrary or state-dependent switching laws that evolve with respect to uncertain switching hyper planes. Although these conditions cannot benefit from the reduction in conservatism brought by dwell-time-dependent conditions, they allow to deal with a larger class of switched systems, since they require less restrictive knowledge about the switching phenomena. This chapter was divided into two main parts, in the first part, the design of asynchronous observers for switched T-S systems under arbitrary switching (state-dependent switching) and with measured premise variables was investigated (Belkhiat et al., 2019), the derived LMI conditions were obtained by using a multiple switched Lyapunov function, together with an H_∞ criterion to minimize the transfer between the output disturbance (e.g. representing measurement bias, noise or fault) and the state estimation errors. We have assumed that the exact switching sequence is unknown (or not precisely known), but that only the set of all admissible switches is known. The proposed method ensures that, in the absence of external disturbances, the estimation error asymptotically converges to zero regardless of the system and observer switches. As mentioned above, T-S fuzzy systems with unmeasured premise variables represent a larger class of systems than the ones with measured premise variables (Bergsten and Palm, 2000; Ichalal et al., 2011, 2018). Therefore, by extending the results proposed in the first part of **Chapter 3** (Belkhiat et al., 2019) to the design of robust asynchronous switched observers for a large class of switched T–S systems using the Lipschitz assumption (Chekakta et al., 2021) to cope with the unmeasured premise variables. Thus, in the second part of this chapter, based on a multiple switched Lyapunov function together with an H_∞ criterion to attenuate the output disturbances on the state estimation error, the derived LMI based conditions have been declined into four theorems with successive conservatism improvements, naturally at the expense of the computational cost. Several examples have been presented to illustrate the effectiveness of our proposal regarding switched T-S observers with measured/unmeasured premise variables and under state-dependent switching with same/different switching sets in time-simulation. Moreover, it has been shown that the feasibility fields comparisons clearly indicated that the proposed LMI conditions provide significant improvements in terms of conservatism reduction regarding to previous related works.

Another estimation technique have been investigated deeply, which is robust H_∞ filtering in **Chapter 4**. LMI based conditions for the design of asynchronous switched T-S H_∞ filters with nonlinear consequent parts for a class of continuous-time nonlinear switched rewritten as Takagi-Sugeno fuzzy systems with nonlinear consequent parts (N-TS), subject to bounded exogenous disturbances and mismatching switching law were derided. The proposed asynchronous switched T-S H_∞ filter has been proposed to estimate unmeasured outputs under arbitrary switching, without any consideration of dwell-time (dwell-time independent), and by using the descriptor redundancy approach. The design conditions were derived based on a multiple Lyapunov functional and an H_∞ criterion to attenuate the disturbances for both cases (measured and unmeasured cases). Similar to **Chapter 3**, this chapter is divided into two parts, the first part dealt with the design of asynchronous switched T-S H_∞ filters for switched T-S systems where the premise variables are assumed to be measurable (Chekakta et al., 2022). Then, by extending the results obtained in the first part of this chapter to the design of asynchronous switched T-S H_∞ filters for switched T-S systems where the premise variables are assumed to be unmeasurable (Chekakta et al., 2023). To do so, the nonlinear consequent parts approach were employed to separate the unmeasured nonlinear terms from the nominal T-S model. Then, the Incremental Quadratic Constraints were used to cope with the unmeasured premise variable instead of Lipschitz conditions, since it includes this latter as a special case. An interesting feature of N-TS approaches in the switched systems framework is that the resulting number of vertices involved in the design conditions can be significantly reduced when the number of modes and nonlinearities increase to help relaxing the conservatism or the computational complexity v.s. classical T-S modelling approaches. Recalling the limitation of T-S models locality mentioned above, for that matter, an optimization procedure was performed to estimate the closed-loop domains of attraction of the filtering-error to guarantee its convergence. Several examples have been presented to illustrate the effectiveness of our proposal regarding switched T-S H_∞ filters with measured/unmeasured premise variables and under state-dependent switching (dwell-time free) with same/different switching sets in time-simulation. Moreover, it has been shown that the feasibility fields comparisons clearly indicated that the proposed LMI conditions provide significant improvements in terms of conservatism reduction regarding to previous related works as well as the significant decrease of the computational complexity and computational time to solve the LMIs compared to recent studies.

To conclude, let us highlight our contributions regarding asynchronous estimation and filtering of switched T-S systems. To the best of the author's knowledge, no prior results were found addressing the following issues:

- The design of asynchronous observers for continuous-time switched T-S systems with measured/unmeasured premise variables using the Lipschitz assumption, subject to output disturbances (which may represent measurement bias or sensor noise.) with $H\chi\infty$ performance specifications, under any sequences, arbitrary and mismatched switching sets.

- The second proposal consists in the design of asynchronous switched T-S H_∞ filters for switched T-S systems, for both cases where the premise variables are assumed to be measured/unmeasured, by considering the nonlinear consequent parts approach along with the incremental quadratic constraints, which enlarges the Lipschitz conditions (used in **Chapter 3**). Then, an optimization procedure was performed to estimate the closed-loop domains of attraction of the filtering-error.

However, there are still some limitations of our proposals, and these limitations can be discussed in order to provide some points of view on this work.

- Although we made significant improvements regarding the feasibility fields, the design conditions suffer from a significant increase of their computational complexity. Moreover, our proposal has another limitation due to the T-S models representation, such as locality of the results. Applying the well-known sector nonlinear approach (Tanaka and Wang, 2001) to each nonlinear mode is a systematic way to transform a switched nonlinear system into a switched T-S one. The resulting T-S model is either globally valid for all nonlinear sectors (as in the first example given in Section 3.4.2.1) or locally valid in a subset of the state space (as in the second example given in Section Section 3.4.2.2). One of the main limitations of T-S model-based observers derived from local sector nonlinearity nonlinearity approaches is that the convergence of the state estimation convergence of the state estimation error cannot be guaranteed if the state variables leave the validity domain of the switched T-S model. Specifically, the maximum Lyapunov level set contained in the validity domain of the switched T-S model should be computed in order to investigate the bounds of the designed state estimation error dynamics. In **Chapter 4**, we've dived deeper into this key element.

- Another limitation of our proposal is related to the T-S modelling with unmeasured premised variables of the switched T-S systems, using the Lipschitz conditions, which can be easily implemented, are the most common method for dealing with the unmeasured premise variables. These conditions frequently considered as a way to circumvent the nonlinear additive term occurring in the dynamics of the state estimation error. However, the Lipschitz conditions may imply very conservative conditions and the choice of the Lipschitz constant can lead to infeasible solutions. Several approaches were proposed recently to substitute the Lipschitz condition and to deal with the additive term in the dynamic of the state error (Ichalal et al., 2011; López-Estrada et al., 2017; Ichalal et al., 2018; Nguyen et al., 2021). However, the majority of the proposed approaches cannot be easily generalized and may fail to provide less conservative conditions. T-S models with nonlinear consequent parts (N-TS) are an effective way (Bouarar et al., 2007; Moodi and Farrokhi, 2014; Araújo et al., 2019; Nagy et al., 2022). In this case, the sector nonlinearity approach was applied only to the nonlinear terms that depend on the measured state variables, resulting in an N-TS model with only measured premise variables, and we retain the unmeasured nonlinear terms as nonlinear consequent parts. The design of asynchronous T-S H_∞ filters for switched T-S systems with nonlinear consequent parts was the main contributions in **Chapter 4**.

- Another interesting limitation of our proposals which requires a deeper investigation in future work, is related to the use of Pettersson approach (Pettersson, 2005) to ensure the decreasing of the Lyapunov functions of the switched T-S observer/switched T-S H_∞ filter at the switching instant (In the presence of disturbance), especially in the filtering case, which may not be an efficient approach to deal with such issue in switched systems subject to higher amplitude disturbances.

In perspective of this work, our future research will be focused on relying to the state estimation of a wider class of nonlinear switched systems, including network-induced phenomena, which is a key point in many daily applications. Also, recall that the present study assumed arbitrary switching sequences driven by given switching sets. If mismatches are now allowed between the switching laws of the observer and the system, there is still open issues to be dealt with or improved, such like the online system's switched modes estimation or identification. Moreover, an interesting direction to improve our proposals it to explore other possibilities related to the switching law (state-dependent switching is not always the best switching mechanism to use) by using alternative approaches to Pettersson (2005).

Furthermore, another interesting way to follow could be the co-design of the observer switching mechanism to further improve the transient of the estimation error dynamics. In the context of H_∞ filtering, our further works will focus on the adaptation of such filtering techniques in the context of non-fragile design or fault tolerant control.

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المساهمة في مراقبة الحالة والتصفية غير الخطية لنماذج Takagi-Sugeno

خلاصة : الهدف من هذه الأطروحة هو تطوير طرق لتصميم المراقبين ومرشحات H_∞ للأنظمة غير الخطية بتبديل الوقت المستمر مع نماذج Takagi-Sugeno (T-S) الضبابية التي تمثل النظام غير الخطي في كل وضع. يتم التحقق في تقنيات التقدير هذه بموجب قوانين التبديل المعتمدة على الحالة (المسكن المستقل عن الوقت) ، والاضطرابات المحددة ، والتبديل غير المتزامن ، حيث لا يتطابق وضع التبديل النشط للمراقب أو المرشح بالضرورة مع نظام التبديل T-S . أولاً ، يتم اقتراح مراقبي T-S المبدلين جنبًا إلى جنب مع قيود Lipschitz ، والتي تسمح بتقديرات الحالة عندما لا تكون متغيرات فرضية الأنظمة قابلة للقياس بالضرورة. ومع ذلك ، نظرًا للتحفظ الذي قد ينشأ من قيود Lipschitz ، تم البحث عن طريقة مثيرة للاهتمام للتعامل مع متغيرات الفرضية غير المقاسة بدلاً من ذلك في الجزء الثاني من هذه الأطروحة في سياق تصفية H_∞ . وبالتالي ، تم تصميم نظام T-S المحول كنموذج T-S بتبديل مع الأجزاء اللاحقة غير الخطية ، حيث يتم الاحتفاظ بالأجزاء غير الخطية غير المقاسة في الأجزاء اللاحقة غير الخطية. بفضل القيد التدريجي التربيعي المستخدم للتعامل مع غير الخطية غير المقاسة ، فإن الشروط التي تم الحصول عليها أقل تحفظًا مقارنة بقيد Lipschitz . علاوة على ذلك ، مع الاعتراف بأن نماذج T-S لا تمثل سوى النماذج غير الخطية على مجموعات فرعية من مساحة الدولة الخاصة بهم ، تم تطوير إجراء تحسين لتقدير مجال الجذب لخطأ التصفية. للتعامل مع أوضاع التبديل غير المتزامنة ، يتم النظر في العديد من وظائف Lyapunov المرشحة ، جنبًا إلى جنب مع معيار H_∞ لتقليل النقل بين اضطرابات الإدخال / الإخراج وتقدير الحالة / أخطاء التصفية. علاوة على ذلك ، تمت صياغة شروط التصميم المقترحة من حيث متباينات المصفوفة الخطية (LMI) . تم استخدام العديد من الأمثلة التوضيحية في جميع أنحاء المخطوطة للتحقق من صحة المراقبين والمرشحات المقترحة ، وكذلك النتائج التي تم الحصول عليها.

كلمات مفتاحية: أنظمة Takagi-Sugeno المحولة ، تبديل المراقبين غير المتزامنين ،تبديل فلاتر H_∞ ، متغيرات فرضية غير قابلة للقياس ، عدم تطابق قوانين التحويل ، الأجزاء اللاحقة غير الخطية ، الشروط المستندة إلى LMI .

Contribution to the State Observation and Nonlinear Filtering of Takagi-Sugeno Models

Abstract: The objective of this thesis is to develop methods for designing observers and H_∞ filters for continuous-time switched nonlinear systems with Takagi–Sugeno (T-S) fuzzy models representing the nonlinear system in each mode. These estimation techniques are investigated under state-dependent switching laws (dwell-time-independent), bounded disturbances, and asynchronous switching, where the observer's or filter's active switching mode does not necessarily match that of the switched T-S system. First, switched T-S observers are proposed along with Lipschitz constraints, which allow state estimations when the systems' premise variables are not necessarily measurable. However, due to the conservatism that might arise from the Lipschitz constraint, an interesting approach to dealing with unmeasured premise variables is investigated instead in the second part of this thesis in the context of H_∞ filtering. Thus, the switched T-S system is modelled as a switched T-S model with nonlinear consequent parts, in which the unmeasured nonlinearities are kept in the nonlinear consequent parts. Thanks to the incremental quadratic constraint employed to deal with unmeasured nonlinearities, the obtained conditions are less conservative compared to the Lipschitz constraint. Furthermore, acknowledging that T-S models only represent nonlinear ones on subsets of their state space, an optimization procedure to estimate the filtering error's domain of attraction is developed. To deal with the asynchronous switching modes, multiple Lyapunov function candidates are considered, along with a H_∞ criterion to minimize the transfer between the input/output disturbances and state estimation/filtering errors. Moreover, the proposed design conditions are formulated in terms of Linear Matrix Inequalities (LMI). Several illustrative examples are used throughout the manuscript to validate the proposed observers and filters, as well as the obtained results.
Keywords: Switched Takagi–Sugeno systems, Switched asynchronous observers, Switched H_∞ filters, Unmeasurable premise variables, Mismatching switching laws, Nonlinear consequent parts, LMI-based conditions.

Contribution à l'observation d'état et au filtrage non linéaire pour les modèles Takagi-Sugeno

Résumé: L'objectif de cette thèse est de développer des méthodes de conception d'observateurs et de filtres H_∞ pour des systèmes non linéaires à commutations à temps continu avec des modèles flous Takagi-Sugeno (T-S) représentant le système non linéaire dans chaque mode. Ces techniques d'estimation sont étudiées dans le cadre de lois de commutation dépendant de l'état (indépendantes du temps de séjour), de perturbations limitées et de commutation asynchrone, où le mode de commutation actif de l'observateur ou du filtre ne correspond pas nécessairement à celui du système T-S à commutation. Tout d'abord, les observateurs T-S à commutations sont proposés avec des contraintes de Lipschitz, qui permettent des estimations d'état lorsque les variables de base des systèmes ne sont pas nécessairement mesurables. Cependant, en raison du conservatisme qui pourrait découler de la contrainte de Lipschitz, une approche intéressante pour traiter les variables de prémisse non mesurées est étudiée dans la deuxième partie de cette thèse dans le contexte du filtrage H_∞. Ainsi, le système T-S à commutation est modélisé comme un modèle T-S à commutation avec des parties conséquentes non linéaires, dans lequel les non-linéarités non mesurées sont conservées dans les parties conséquentes non linéaires. Grâce à la contrainte quadratique incrémentielle utilisée pour traiter les non-linéarités non mesurées, les conditions obtenues sont moins conservatrices que la contrainte de Lipschitz. En outre, reconnaissant que les modèles T-S ne représentent les non-linéaires que sur des sous-ensembles de leur espace d'état, une procédure d'optimisation pour estimer le domaine d'attraction de l'erreur de filtrage est développée. Pour traiter les modes de commutation asynchrones, plusieurs candidats de fonction de Lyapunov sont considérés, ainsi qu'un critère H_∞ pour minimiser le transfert entre les perturbations d'entrée/sortie et les erreurs d'estimation d'état/filtrage. En outre, les conditions de conception proposées sont formulées en termes d'inégalités matricielles linéaires (LMI). Plusieurs exemples illustratifs sont utilisés tout au long du manuscrit pour valider les observateurs et les filtres proposés, ainsi que les résultats obtenus.
Mots clés: Systèmes Takagi-Sugeno à commutations, Observateurs asynchrones à commutations, Filtres H_∞ à commutations, Variables de prémisses non mesurables, Lois de commutation non concordantes, Parties conséquentes non linéaires, Conditions basées sur LMI