Democratic and Populair Republic of Algeria

Ministry of Higher Eduction and Scientific Research

FERHAT ABBAS UNIVERSITY-SÉTIF 1-

FACULTY OF SCIENCES

DEPARTEMENT OF MATHEMATICS

THESIS

Presented by:

KHALDI MERZAKA

To obtain the title of Doctorate in Sciences

OPTION

APPLIED MATHEMATICS

THEME

The simplicial cone constrained convex quadratic optimization. Theoretical and Numerical study

Supported on: 19/06/2023

In front of the jury composed of:

Chairman:	**Mr. Bachir. MERIKHI**	**Prof.**	**U.F.A. Sétif-1**
Supervisor:	**Mr. Mohamed. ACHACHE**	**Prof.**	**U.F.A. Sétif-1**
Examiners:	**Mr. Mohamed. ZERGUINE**	**Prof.**	**U.M.B. Batna-2**
	Mr. Hamza. GUEBBAI	**Prof.**	**U.8 Mai 1945 Guelma**

Academic year 2022-2023

Acknowledgments

I want to thank all people who have contributed by some means or other to the success of this thesis.

First and foremost among these is my advisor Prof Mohamed Achache, who supported and guided me during the whole time of my studies. He gave me valuable insights in the optimization and always kept me on the right track in countless discussions. The subject of this thesis was proposed by him and without his ongoing encouragement and strong support, the completion of this work would not have been possible.
My sincere thanks go to him.

Let me express my best thanks to my colleagues, who enabled me to continue my scientific work study.
I also thank the members of my thesis committee : Professor Bachir Merikhi for making me the honor of being part of this jury and of being its president. Professor Hamza Guebbai and Professor Mohamed Zerguine for accepting to judge this work and to be its examiners.
Also I owe many thanks to my colleagues at Ferhat Abbas Sétif 1 University.

Last but not the least, I would like to thank my family : my parents and my husband and my children. for supporting me spiritually throughout writing this thesis and my life in general.

Dedicace

There are a number of people without whom this thesis might not have been written, to whom I owe a lot, I dedicate this work.

To my inspired parents, without whom my success would not be possible.

To my husband and my children Islam and Mahdi who shared with me the emotional moments during the realization of this work and who warmly supported and encouraged me throughout my journey..

To all my friends who have always encouraged me, and to whom I wish more success.

To all the people in my life who touched my heart.

Merzaka Khaldi 

Contents

List of Tables

List of Figures

Abstract

Because of its usefulness in production planning, financial and corporate planning, health care and hospital planning, quadratic programming (QP) problems have attracted considerable research and interest in recent years.

In this thesis, we present a theoretical analysis and numerical study for solving a special class of constrained convex quadratic programming problems, which is called the simplicial cone constrained convex quadratic optimization (abbreviated by SCQO). In the first part, and via its optimality K.K.T conditions, the SCQO is equivalent to finding the unique solution of an absolute value equations (AVE). For solving the latter, we applied across the (AVE), a new two-steps Picard's iterative fixed point iteration method. In particular, the sufficient conditions for the convergence of our algorithm are studied. The obtained numerical results illustrate that the algorithm is efficient and valid to solve SCQOs. In the second part, a feasible full-Newton step primal-dual path following interior-point algorithm (IPA) is proposed for solving SCQOs via its reformulation as a $\mathcal{P}$-LCP (i.e., a linear complementarity problem with a $\mathcal{P}$-matrix). Moreover, for the seek of benefit, the well-definiteness and the convergence of the proposed algorithm are proved. In addition, its complexity polynomial is computed. Some numerical results are presented to show the effectiveness of this algorithm. We followed it up with a comparison study between the numerical results obtained by these two algorithms through some SCQO examples.

Keywords: Quadratic programming, complementarity problem, absolute value equations, Picard's iterative methods, interior-point methods, primal-dual path algorithm, polynomial complexity.

List of publications

Paper 1

M. Khaldi, M. Achache. An interior-point algorithm for simplicial cone constrained convex quadratic optimization. **Advances in Mathematics: Scientific Journal**, Vol. 12 (2023), No. 1, 73-89.
https://doi.org/10.37418/amsj.12.1.5

Paper 2

M. Khaldi, M. Achache. A two-steps fixed-point method for the simplicial cone constrained convex quadratic optimization. To appear in **Stud. Univ. Babeş-Bolyai Math.**

Glossary of Notation

Problem Classes

QP	:	Quadratic Programming.
SCQO	:	Simplicial cone constrained convex quadratic optimization.
LCP	:	Linear complementarity problem.
HLCP	:	Horizontal linear complementarity problem .
MLCP	:	Monotone linear complementarity problem.
IP	:	Interior-Point.
IPC	:	Interior-Point-Condition.
IPMs	:	Interior-Point Methods.
K.K.T	:	Karush-Kuhn-Tucker.
CF	:	Cholesky Factorization.
AVE	:	Absolute value equations

Spaces

$\mathbb{R}^n$	:	The real n-dimensional space.
$\mathbb{R}^n_+$	:	The nonnegative orthant of $\mathbb{R}^n$.
$\mathbb{R}^n_{++}$	:	The positive orthant of $\mathbb{R}^n$.
$\mathbb{R}^{n\times n}$	:	The set of all $n \times n$ real matrices.

Vectors

x	:	The vevteur in $\mathbb{R}^n$.
x_i	:	The i-th component of vevteur x
x^T	:	$(x_1, \ldots, x_n)$ the transpose of a vector x, with components x_i
$x \geq 0$	=	$x_i \geq 0$, $\forall i$.
$x > 0$	:	$x_i > 0$, $\forall i$.
e	=	$(1, ..., 1)^T \in \mathbb{R}^n$.
$\|x\|$	=	Absolute value of a vector $x \in \mathbb{R}^n$. $\|x\| = (\|x_i\|), i = 1, \ldots, n$.
$x^T y$	=	$\sum_{i=1}^{n} x_i y_i$ the standard inner product in $\mathbb{R}^n$.
$sign(x)$	=	denotes a vector with the components equal to -1, 0 or 1.
$\|x\|$	:	denote the Euclidean norm $(x^T x)^{1/2}$.

Matrices

I	:	Identity matrix .
$0_{n\times n}$	:	The (n, n) null matrix .
A^T	:	The transposed matrix of A.
A^{-1}	:	The inverse of a regular matrix A.
A invertible	:	A nonsingular.
$\|A\|$	=	$(\|a_{ij}\|)$, the absolute value of a_{ij}.
$\lambda(A)$	:	The eigenvalue of $A \in \mathbb{R}^{n\times n}$.
$\lambda_{\max}(A)$	:	The largest eigenvalue of A.
$\lambda_{\min}(A)$	:	The smallest eigenvalue of A.
$\sigma_{\max}(A)$	:	The largest singular value of a matrix A.
$\sigma_{\min}(A)$	:	The smallest singular value of a matrix A.
$tr\,(A)$	=	$\sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i(A)$, (the trace of a matrix $A \in \mathbb{R}^{n\times n}$).
$\rho\,(A)$	=	$\max\|\lambda_i(A)\|$, (the spectral radius of A).
$\|A\|_2$	=	$\sqrt{\rho\,(A^T A)}$, (the spectral norm of A).
$D(x)$	=	diag(sign(x))

Introduction

Quadratic programming *(QP)* is the problem of optimizing a quadratic objective function and it is one of the simplest form of non-linear programming. (*QP*) is widely used in image and signal processing, computer vision, machine learning, to optimize financial portfolios, to perform the least-squares method of regression, to control scheduling in chemical plants, and in sequential quadratic programming, a technique for solving more complex non-linear programming problems. The problem was first explored in the early 1950s, most notably by Princeton University's Wolfe and Frank, who developed its theoretical background, and by Markowitz [48], who applied it to portfolio optimization, a subfield of finance. QPs are also extensively used in finance, as variance, which is used to measure risk, is a function containing squares. More specifically, Markowitz won the 1990. Nobel Prize in Economics for his widely-used model that employs quadratic programming to optimizes the amount of risk taken on based on variances. There are many approaches to solving a convex quadratic programming and The majority of these methods can be categorized into either interior point methods (which are discussed heavily in Chapter 3) or active-set methods (see [21, 23, 71]). Briefly, interior-point methods (*IPM*s) are among the most efficient methods for solving linear, and also wide classes of other convex optimization problems. The first known IP method is Frisch's (1955) logarithmic barrier method that was later extensively studied by Fiacco and McCormick. Since the path-breaking work of Karmarkar [32], much research was invested in IPMs. Many algorithmic variants were developed for Linear Optimization (*LO*). The new approach forced to reconsider all aspects of optimization problems. Not only the research on algorithms and complexity issues, but implementation strategies, duality theory and research on sensitivity analysis got also a new impulse. Several books were published that summarize and explore different aspects of IPMs. The seminal work of Nesterov and Nemirovski [51] provides the most general framework for polynomial time

IPMs for convex optimization. Den Hertog [19] gives a thorough survey of primal and dual path-following IPMs for linear and structured convex optimization problems. Many works discusses primal-dual target following algorithms for linear optimization and complementarity problems. The primal-dual path-following methods introduced by Kojima and al. in 1989 [36] and Monteiro and al. [50] are the most attractive and most of the corresponding algorithms are based on the logarithmic barrier function. In recent years, the majority of research in the field of IPs is for nonlinear optimisation problems, especially these methods are powerful tools to solve a wide large of mathematical problems such as *LO*,(see [1, 17, 20]), convex quadratic optimization (CQO) (see [2, 67, 74]), the linear complementarity problem (LCP) (see [7, 14, 18, 37, 46, 76]), the linear semidefinite optimization (SDO) and the semidefnite linear complementarity problem (SDLCP) (see [3, 6, 47, 66]). Recently, Achache [4] presented a short-step feasible IPMs for solving monotone standard LCP. He showed that the algorithm enjoys the iteration bound, namely, $\mathcal{O}\left(\sqrt{n}\log(\frac{n}{\varepsilon})\right)$. Furthermore, he reported some numerical results which confirmed the efficiency of this algorithm.

In this thesis, we are interested for solving an important special class of constrained convex quadratic programming problems, which is the simplicial cone constrained convex quadratic optimization SCQO of type:

$$\min_x \left\{ f(x) = \frac{1}{2}x^T Q x + x^T b + c \text{ subject to: } x \in \mathcal{S} \right\},$$

where $Q \in \mathbb{R}^{n\times n}$ is a given symmetric positive definite matrix, $b \in \mathbb{R}^n, c \in \mathbb{R}^n$ and

$$\mathcal{S} = \left\{ Ax \mid x \in \mathbb{R}^n_+ \right\}$$

is the simplicial cone associated with the nonsingular matrix $A \in \mathbb{R}^{n\times n}$. Simplicial cone constrained convex quadratic programming arises as a interesting problem in its own right, it has an subclass of positively constrained convex quadratic programming, or equivalently the problem of projecting the point onto a simplicial cone (see [13]). The interest in the subject of projection arises in several situations, having a wide range of applications in pure and applied mathematics such as Convex Analysis, Numerical Linear Algebra, Statistics, Computer Graphics. In [13] they particularized the Moreau's decomposition theorem for simplicial cones. This leads to an equivalence between the problem of projecting a point onto a simplicial cone and the one of finding the unique solution of a nonsmooth system of equations and they was proposed a semi-smooth Newton's method for solving the

obtained associated system.
The purpose of the work in this thesis is to solve SCQO via two different methods.

- A two-steps fixed-point method.
- An interior-point method.

Specifically, across the Karish-Khun-Tucker (K.K.T) optimality conditions of SCQO, we reformulate it as an equivalent LCPs (see [15]). First, from optimality conditions of SCQO and under suitable conditions, the convex quadratic programming under a simplicial cone constraints is equivalent to finding the unique solution of the following absolute value equation:

$$(A^TQA + I)x + (A^TQA - I)|x| = -A^Tb.$$

This equation is a special case of the general absolute value equations AVE of the type:

$$\bar{A}x - \bar{B}|x| = \bar{b}$$

where $\bar{A}, \bar{B}$ are given $(n \times n)$ real square matrices and $\bar{b} \in \mathbb{R}^n$. The AVE was first introduced by Rohn [60] and investigated in more general context in Mangasarian (see [44]). Other studies for the AVE can be found in [8, 9, 11, 26, 28, 30, 40, 42, 49, 52, 59, 69]. Besides some numerical methods are used to solve it. In particular, Mangasarian in [43] proposed a semi-smooth Newton's method for solving the AVE, and under suitable conditions he showed the finite and linear convergence to a solution of the AVE. However, other numerical approaches focus on reformulating the AVE as an horizontal linear complementarity problems (HLCP) (see [10]), where they introduce an infeasible path-following interior-point method for solving the AVE by using is equivalent reformulations as an HLCP. For solving the AVE, we propose a new two-steps fixed point iterative method which is introduced in [33], and under a new mild assumption we show that this method is always well-defined and the generated sequence converges globally and linearly to the unique solution of the AVE from any starting initial point. Finally, numerical results are provided to illustrate the efficiency of this algorithm to solving the SCQO. Secondly, we show that the corresponding LCP is a $\mathcal{P}$-LCP. Hence, due to R. W. Cottle, J. S. Pang and R. E. Stone [16], the $\mathcal{P}$-LCP has a unique solution and so is the SCQO. And across the $\mathcal{P}$-LCP, we introduce a simple feasible short-step interior-point algorithm for solving the SCQO. In fact

the latter uses at each iteration, only full-Newton steps with the advantage that no line search is required. For its well-definiteness and its local quadratic convergence to an optimal solution of SCQO, we suggest new appropriate defaults to ensure that the algorithm converges to the unique minimizer of SCQO. Moreover, the best known iteration bound, namely, $\mathcal{O}\left(\sqrt{n}\log\frac{n}{\epsilon}\right)$ is derived. Here, for its polynomial complexity, we have reconsidered the basic analysis used in [4] and other references and developed them to be suited for SCQOs. Finally, numerical tests were carried out to study the behavior of the algorithm with different proposed parameters.

Short Outline of the Thesis

The thesis contains three chapters, followed by a bibliography. This thesis is organized as follows

- **In the first chapter,** we present a background of matrices, the basic concepts of convex analysis, quadratic programming, differential calculus, equation in absolute values and some results of existence and uniqueness of the solution of the AVE, we present the definitions and terms that will be used throughout the thesis.

- **In the second chapter**, the reformulation of the SCQO as an absolute value equation AVE and the unique solvability of AVE is studied. Any solution of the AVE generates a solution of our convex quadratic programming problem SCQO. For its numerical solution, we propose an efficient two-steps fixed point iterative method for solving the AVE. In addition, we show that under suitable assumptions the proposed algorithm converges globally linear to the unique solution of the AVE and which is in turn an optimal solution of SCQO. Moreover, we also report some preliminary numerical results to show the practical performance and the efficiency of the proposed algorithm.

- **In the third chapter**, we deal with the study of the polynomial complexity and the numerical implementation for a full-Newton short-step primal-dual central-path interior-point method for solving the $\mathcal{P}$-LCP. We show that the corresponding algorithm enjoys the best known theoretical polynomial complexity, namely, $\mathcal{O}\left(\sqrt{n}\log\frac{n}{\epsilon}\right)$. A full theoretical study is done. The quadratic

convergence of full-Newton's iterations towards the central path is proved thus is calculated from the complexity polynomial of the algorithm which is of the polynomial order. Some numerical results are provided to illustrate its efficiency for solving the SCQOs. In the last of this chapter, a comparative study between the numerical results obtained by the these algorithms is presented where the performance of each method is deduced.
Finally, we end the thesis with a conclusion and future work.

Chapter 1

Mathematical background

In this chapter, we present some mathematical background in matrices and convex analysis, quadratic programming, differential calculus and the definition of the absolute value equations.

1.1 Matrix Analysis [27]

We denote by $\mathbb{R}^n$, the finite-dimensional Euclidean vector space $(n \in \mathbb{N})$ and by $\mathbb{R}^{n\times n}$ the set of$(n \times n)$ real square matrices.

- For all x and $y \in \mathbb{R}^n$ we denote by:

$$x^T y = \sum_{i=1}^{n} x_i y_i,$$

the usual scalar product of x and y. The vector x^T denotes the transpose of the vector column x of $\mathbb{R}^n$.

- Two vectors x and y are orthogonal if $x^T y = 0$.

- The Euclidean norm associated with the usual scalar product, is given by:

$$\|x\|_2 = \sqrt{x^T x} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

Definition 1.1. An application $\|.\| : \mathbb{R}^n \to \mathbb{R}$ is said to be a vectorial norm if it satisfies the following conditions:

- $\|x\| \geq 0, \forall x \in \mathbb{R}^n, \|x\| = 0 \Leftrightarrow x = 0.$
- $\|\alpha x\| = |\alpha| \|x\|, \forall \alpha \in \mathbb{R}, \forall x \in \mathbb{R}^n.$

- $\|x+y\| \leq \|x\| + \|y\|, \forall x, y \in \mathbb{R}^n$.

- The Cauchy-Schwarz inequality:

$$\left|x^T y\right| \leq \|x\| \|y\| \ \forall x, y \in \mathbb{R}^n.$$

Lemma 1.2. *For all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, we have:*

$$\| |x| - |y| \| \leq \|x - y\|.$$

Definition 1.3. Let $\mathbb{C}$ be the set of complex numbers and let A be a real square matrix of $\mathbb{R}^{n \times n}$. We recall that $\lambda \in \mathbb{C}$ is an eigenvalue of the matrix A if there exists a vector $x \in \mathbb{C}^n$ with $x \neq 0$ such that:

$$Ax = \lambda x,$$

we call x the eigenvector of A associated to the eigenvalue λ.

Definition 1.4. We say that the matrix A is positive semi-definite if:

$$x^T Ax \geq 0 \ , \forall x \in \mathbb{R}^n,$$

and we say that A is positive definite if:

$$x^T Ax > 0, \forall x \in \mathbb{R}^n (x \neq 0).$$

Definition 1.5. A matrix $A \in \mathbb{R}^{n \times n}$ is said to be symmetric if $A = A^T$. It is said to be skew symmetric if $A^T = -A$.

Definition 1.6. A matrix A is nonsingular (invertible) matrix if there exists a matrix denoted by A^{-1} such that $AA^{-1} = I$ where I denotes the identity matrix in $\mathbb{R}^{n \times n}$.

Lemma 1.7. *A is nonsingular matrix if its determinant is nonzero.*

Lemma 1.8. *Let $A, B \in \mathbb{R}^{n \times n}$. Then*

$$(AB)^T = B^T A^T,$$

and if α and β are scalars

$$(\alpha A + \beta B)^T = \alpha A^T + \beta B^T,$$

if A and B are non-singular matrices, we have

$$(AB)^{-1} = B^{-1} A^{-1}.$$

Lemma 1.9. *The square real matrix A is symmetric if and only if there exists an orthogonal matrix O such that $OO^T = I$ and $A = ODO^T$ where D is a diagonal matrix, i.e.,*

$$D := diag(\lambda_i(A)), \lambda_i(A) \in Sp(A),$$

where $Sp(A)$ denotes spectrum of the matrix A.

Definition 1.10. We call a singular value of a matrix $A \in \mathbb{R}^{m\times n}$ any square root of an eigenvalue of the semi-definite symmetric matrix A^TA, that is to say:

$$\sigma(A) = \sqrt{\lambda(A^TA)},$$

such that $\lambda(\ A^TA) \geq 0$, is the eigenvalue of A^TA.
- The minimal singular value of A is given through :

$$\sigma_{\min}(A) = \min_{\|x\|_2=1} \sqrt{x^TA^TAx},$$

likewise the maximal singular value of A, is given by:

$$\sigma_{\max}(A) = \max_{\|x\|_2=1} \sqrt{x^TA^TAx}.$$

We note if the matrix A is symmetric i.e., $A = A^T$, then the minimal singular value of A coincides with the minimal eigenvalue of A and we have:

$$\lambda_{\min}(A) = \min_{\|x\|_2=1} \|Ax\|,$$

likewise the maximal singular value of A coincides with the maximal eigenvalue of A, and we have:

$$\lambda_{\max}(A) = \max_{\|x\|_2=1} \|Ax\|.$$

Lemma 1.11. *Any positive definite matrix A is nonsingular.*

Lemma 1.12. *Any real symmetric matrix A, has real eigenvalues and there exist n eigenvectors of A forming a basis of $\mathbb{R}^n$.*

Lemma 1.13. *If A is a real symmetric matrix. Then*

$$x^TAx \geq \lambda_{\min}(A)\,\|x\|_2^2\,, \forall x \in \mathbb{R}^n,$$

where $\lambda_{\min}(A)$ denotes the smallest eigenvalue of A.

Lemma 1.14. *If the matrix $A \in \mathbb{R}^{n\times n}$ is symmetric, then A is positive semi-definite if and only if $\lambda_{\min}(A) \geq 0$, and A is positive definite if and only if $\lambda_{\min}(A) > 0$.*

Definition 1.15. Let $A, B \in \mathbb{R}^{n\times n}$, the following application:

$$\|.\| : \mathbb{R}^{n\times n} \to \mathbb{R},$$

is said to be a matrix norm if it satisfies the following properties:

- $\|A\| \geq 0, \forall A \in \mathbb{R}^{n\times n}, \|A\| = 0 \Leftrightarrow A = 0.$
- $\|\alpha A\| = |\alpha| \|A\|, \forall \alpha \in \mathbb{R}, \forall A \in \mathbb{R}^{n\times n}.$
- $\|A + B\| \leq \|A\| + \|B\|, \forall A, B \in \mathbb{R}^{n\times n}.$
- $\|AB\| \leq \|A\| \|B\|, \forall A, B \in \mathbb{R}^{n\times n}.$

Definition 1.16. - Recall that a subordinate matrix norm induced by the vector norm is defined as follows:

$$\|A\| = \sup_{x\neq 0,\, x\in\mathbb{R}^n} \frac{\|Ax\|}{\|x\|}.$$

- If the vector norm, is the Euclidean norm, then:

$$\|A\|_2 = \sqrt{|\lambda_{\max}(A^T A)|},$$

where $\lambda_{\max}(A^T A)$ denotes the largest eigenvalue of the matrix $A^T A$.
- If A is real symmetric and positive definite, then

$$\|A\|_2 = \lambda_{\max}(A).$$

Definition 1.17. The matrix $A \in \mathbb{R}^{n\times n}$ is said to be $\mathcal{P}$-matrix if and only if its principal minors are positive. Consequently any positive definite matrix is a $\mathcal{P}$-matrix.

Definition 1.18. The absolute value of a matrix A is defined by

$$|A| = (|a_{ij}|),$$

where $A = (a_{ij})$.

Matrix factorization

In this section, we focus on the topic of a matrix factorization the theoretical foundation of the finite elimination methods for solving linear equations. There are many Kinds of matrix factorizations, we review only the LU and Cholesky factorization .

Theorem 1.19. *Let $A \in \mathbb{R}^{n\times n}$. Suppose the first $n-1$ leading principal minors of A are non zero. Then there exist a lower triangular matrix L with unit diagonal entries an upper triangular matrix U such that*

$$A = LU. \tag{1.1}$$

The representation (1.1) is called the LU (triangular) factorization .

Proposition 1.20. *Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive definite. There exists a unique lower triangular real matrix L, such that all its elements diagonals are strictly positive and which verifies:*

$$A = LL^T.$$

The Cholesky decomposition or Cholesky factorization is a decomposition of a symmetric positive-definite matrix A into the product of a lowertriangular matrix and its transpose. The Cholesky decomposition is roughly twice as efficient as the LU decomposition for solving systems of linear equations.

The Cholesky decomposition of a symmetric positive-definite matrix A is a decomposition of the form $A = LL^T$, where L is a lower triangular matrix with real and positive diagonal entries, and L^T denotes the conjugate transpose of L. Every symmetric positive-definite matrix has a unique Cholesky decomposition.

The Cholesky factorization is not difficult to compute, it can be computed by a form of Gaussian elimination that takes advantage of the symmetry and definiteness. Equating (i, j) elements in the equation $A = LL^T$ gives:

$$\begin{aligned} l_{jj} &= (a_{jj} - \sum_{k-1}^{j-1} l_{kj}^2) \text{ for } i = j \\ l_{ij} &= \frac{1}{l_{ii}}(a_{ij} - \sum l_{ki} l_{kj}) \text{ for } i > j \end{aligned}$$

The algorithm of Cholesky requires $n^3/3 + O(n^2)$ flops and n square roots, where a flop is any of the four elementary scalar arithmetic operations +, -, *, and /.

The Cholesky decomposition is mainly used for the numerical solution of linear equations $Ax = b$. If A is symmetric and positive definite, then we can solve $Ax = b$ by first computing the Cholesky decomposition $A = LL^T$, then solving $Ly = b$ for y by forward substitution, and finally solving $L^T x = y$ for x by back substitution.

1.2 Convex analysis

In this paragraph, we cite some basic notions of convex analysis which will be useful afterwards.

Convex sets and convex functions

Definition 1.21. A subset $\mathcal{D} \subset \mathbb{R}^n$ is said to be convex if $\forall x, y \in \mathcal{D}$ implies that

$$(1-\lambda)x + \lambda y \in \mathcal{D},\ \forall \lambda \in [0,1]\,.$$

Thus a set $\mathcal{D} \subset \mathbb{R}^n$ is said to be convex if $\forall x, y \in \mathcal{D}$ the segment $[x, y] \subset \mathcal{D}$.

Definition 1.22. (**Convex Combinations**). Let $\{x_1, x_2, \ldots, x_n\}$ be any set of points in $\mathbb{R}^n$. A convex combination of this set is a point of the form

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_r x_r;\ \sum_{i=1}^{r} \alpha_i = 1, \alpha_1, ...\alpha_r \geq 0$$

Proposition 1.23. *For a set $\mathcal{D} \subset \mathbb{R}^n$, it holds that $conv\mathcal{D}$ is the set of all convex combinations of elements of $\mathcal{D}$, i.e.,*

$$conv\mathcal{D} = \left\{ \sum_{i=1}^{r} \alpha_i x_i; x_i \in \mathcal{D},\ \alpha_i \geq 0,\ \sum_{i=1}^{r} \alpha_i = 1 \right\}$$

.

Definition 1.24. Let $\mathcal{D} \subset \mathbb{R}^n$ be a convex set and $f :\ \mathcal{D} \to \mathbb{R}$.
1- f is said to be convex on $\mathcal{D}$ if:

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y). \forall x, y \in \mathcal{D}, \forall t \in [0,1].$$

2- f is said to be strictly convex if:

$$f((1-t)x + ty) < (1-t)f(x) + tf(y). \forall x, y \in \mathcal{D}, x \neq y, \forall t \in\,]0,1[\,.$$

Convex cones

Cones are fundamental geometric objects associated with sets. They play a key role in several aspects of mathematics.

Definition 1.25. A non-empty subset $\mathbb{K} \subset \mathbb{R}^n$ is called a cone if it satisfies:

$$\lambda x \in \mathbb{K}\ \forall x \in \mathbb{K}, \forall \lambda \geq 0.$$

We call a closed set $\mathbb{K} \subset \mathbb{R}^n$ with nonempty interior a closed convex cone if the following conditions hold:

$$\lambda x + \mu y \in \mathbb{K} \text{ for any } \lambda, \mu \geq 0 \text{ and } x, y \in \mathbb{K}.$$

Definition 1.26. Let $\mathbb{K} \subset \mathbb{R}^n$ be a closed convex cone. The dual cone of $\mathbb{K}$ is the following set

$$\mathbb{K}^* := \{x \in \mathbb{R}^n | \langle x, y\rangle \geq 0, \forall y \in \mathbb{K}\}.$$

Examples

1) The orthant $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n / x_i \geq 0, i = 1, ..., n\}$ and its interior

$$Int(\mathbb{R}^n_+) = \{x \in \mathbb{R}^n \mid x_i > 0, i = 1, ..., n\}$$

are two convex cones widely used in optimization theory.

2) The simplicial cone associated to a nonsingular matrix $A \in \mathbb{R}^{n\times n}$, is defined by

$$\mathcal{S} = A\mathbb{R}^n_+ = \left\{Ax : x \in \mathbb{R}^n_+\right\},$$

is a convex cone.

Coercive functions

Definition 1.27. Let $\mathcal{D}$ be an unbounded set of $\mathbb{R}^n$ and $f : \mathcal{D} \to \mathbb{R}$. f is said to be coercive on $\mathcal{D}$ if:

$$\lim_{\|x\|\to+\infty} f(x) = +\infty.$$

Proper functions

Definition 1.28. We say that f, is an eigenfunction (proper function) on $domf$ if $f(x) < +\infty$. The set $domf \subset \mathbb{R}^n$, denotes the effective domain of the function f.

Strongly convex functions

Let $\mathcal{D}$ be an unbounded set of $\mathbb{R}^n$ and $f : \mathcal{D} \to \mathbb{R}$, f is said to be strongly convex if and only if there exists $m > 0$ such that:

$$\langle f(x) - f(y), x - y\rangle \geq m \|x - y\|^2, \forall x, y \in \mathcal{D}.$$

1.3 Differential calculus

In this paragraph, we introduce the notion of differentiability of a function. We start to give some basic topological notions.
- For all $x \in \mathbb{R}^n$ and $r > 0$, the open ball of $\mathbb{R}^n$, denoted by $\mathcal{B}(x, r)$, is given through:

$$\mathcal{B}(x, r) = \{y \in \mathbb{R}^n : \|x - y\| < r\}.$$

- If $\{x_k\}$ is a sequence of $\mathbb{R}^n$ and $x^\star$ is an element of $\mathbb{R}^n$. We say that $\{x_k\}$ converges to $x^\star$ (denoted $x_k \to x^\star$) if $\|x_k - x^\star\| \to 0$ when $k \to \infty$.
- Let $\mathcal{D} \subset \mathbb{R}^n$.

1- We define the interior of $\mathcal{D}$ as the set of elements $x \in \mathcal{D}$ for which there exists an $r > 0$ such that $\mathcal{B}(x, r) \subset \mathcal{D}$.

2- $\mathcal{D}$ is said to be open if $\forall x \in \mathcal{D}$ exists $\mathcal{B}(x, r) \subset \mathcal{D}$.

3- $\mathcal{D}$ is said to be closed if for any sequence $\{x_k\}$ from $\mathcal{D}$ such that $x_k \to x^\star$ we have: $x^\star \in \mathcal{D}$.

- Let a and b, the set noted $[a, b]$ of $\mathbb{R}^n$, given by:

$$[a, b] = \{a + t(b - a) = (t - 1)a + tb : t \in [0, 1]\}.$$

The set $[a, b]$ is called the segment connecting a and b.

Definition 1.29. Let $\mathcal{D}$ be a subset of $\mathbb{R}^n$ and $f : \mathcal{D} \to \mathbb{R}$. We say that f is continuous at $x \in \mathcal{D}$, if $f(x_k) \to f(x^\star)$ for any sequence $\{x_k\}$ of $\mathcal{D}$ such that $x_k \to x^\star$. So we say that f is continuous over all $\mathcal{D} \subset \mathbb{R}^n$, if it is continuous at any point of $\mathcal{D}$.

Definition 1.30. The function $f : \mathbb{R}^n \to \mathbb{R}$ is lower semi-continuous at x if

$$f(x) = \lim_{y \to x} inf f(y),$$

and lower semi-continuous on $\mathbb{R}^n$ if this holds for every $x \in \mathbb{R}^n$.

Suppose now that the set Ω is an open set of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ is a function.

Definition 1.31. For all $x \in \Omega$, and $h \in \mathbb{R}^n$, the directional derivative of f at x in the direction h, is given by:

$$\frac{\partial f}{\partial h}(x) = \lim_{t \to 0} \frac{f(x + th) - f(x)}{t}.$$

The gradient of f at x, denoted by $\nabla f(x)$, is the column vector of $\mathbb{R}^n$ given by:

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x) \right)^T.$$

Recall the formula:

$$\frac{\partial f}{\partial h}(x) = h^T \nabla f(x), \quad \forall x \in \Omega, \quad \forall h \in \mathbb{R}^n.$$

- We say that $x^\star$ is a critical or stationary point of the function f if

$$\nabla f(x^\star) = 0.$$

- We denote for all $x \in \Omega$, $\nabla^2 f(x)$ the symmetric square matrix of order n, given by

$$(\nabla^2 f(x))_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}(x), \forall i, j = 1, \ldots, n.$$

$\nabla^2 f(x)$ is called the Hessian matrix H from f to x. We also have:

$$H = \nabla^2 f(x) h = \frac{\partial}{\partial x_i} h^T \nabla f(x).$$

- We say that f is of class $\mathcal{C}^2$ on Ω if the partial derivatives of order 2 of f exist and are continuous.

Definition 1.32. If f is of class $\mathcal{C}^2$ on $\mathcal{D}$, then f is convex over $\mathcal{D}$ if and only if the matrix hessian $\nabla^2 f(x)$ is semi-definite positive. Likewise, f is said to be strictly convex on $\mathcal{D}$ if and only if the hessian matrix is positive definite.

Some important examples:
- Consider the following quadratic function:

$$f(x) = x^T A x.$$

Then

$$\nabla f(x) = (A + A^T)x \text{ and } \nabla^2 f(x) = A + A^T.$$

In particular, if A is real symmetric, then

$$\nabla f(x) = 2Ax, \; \nabla^2 f(x) = 2A.$$

Let $f(x) = b^T x$, then:

$$\nabla f(x) = b, \quad \nabla^2 f(x) = 0.$$

Generalized Jacobian in the sense of Clarke

In this paragraph, one gives the definition of the generalized Jacobian matrix within the meaning of Clarke [56]. Indeed, we define the generalized Jacobian for non-differentiable functions. For this purpose, we begin to give the definition of a Lipschitizian function.

Definition 1.33. Let $f : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function, f is said to be Lipschitzian on $\mathcal{D}$ if there exists $L > 0$, such as:

$$\|f(x) - f(y)\| \leq L \|x - y\|, \forall x, y \in \mathcal{D}.$$

Definition 1.34. Let $f : \mathcal{D} \to \mathbb{R}^m$ be a Lipschitzian function. So the set

$$\partial f_B(x) = \left\{V \in \mathbb{R}^{m\times n} : \exists \{x_k\} \subset \mathcal{D} \text{ with } x_k \to x, \nabla f(x_k) \to V\right\}$$

is called the $B-$sub-differential of f at x.

Definition 1.35. The generalized Jacobian matrix ∂f is defined as follows:

$$\partial f(x) = conv(\partial f_B(x)),$$

where $conv$ denotes the convex hull.

Definition 1.36. Let the function $f : \mathbb{R}^n \to \mathbb{R}^n$.

$$f(x) = |x|,$$

then the Jacobian generalized is of f given by:

$$D(x) = diag(sign(x)),$$

where D is the diagonal matrix whose elements are equal to $1, 0$ or -1, with

$$sign\ (x) = \begin{cases} -1 & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \\ 1 & \text{if} \quad x > 0. \end{cases}$$

1.4 Unconstrained optimization problem

In this section, we only consider unconstrained optimization problem. We give the existence and the uniqueness of a minimum thus the characterization of the latter through its optimality conditions.

An unconstrained optimization problem is defined by

$$\min_{x\in\mathbb{R}^n} f(x).$$

Definition 1.37. If $\mathcal{D} \subset \mathbb{R}^n$, $x^\star \in \mathcal{D}$ and $f : \mathcal{D} \to \mathbb{R}$.
1- We say that $x^\star$ is a global minimum of f on $\mathcal{D}$ if:

$$f(x) \geq f(x^\star), \forall x \in \mathcal{D}.$$

2- We say that x is a local minimum of f on $\mathcal{D}$ if there exists a neighborhood $\mathcal{V}$ of $x^\star$ such than:

$$f(x) \geq f(x^\star), \forall x \in \mathcal{D} \cap \mathcal{V}.$$

Remark 1.38. Any global minimum is a local minimum.

Results of existence and uniqueness of the optimal solution

In order to be able to easily calculate or to approximate the solution of an optimization problem, it is interesting to know the hypotheses guaranteeing the existence and uniqueness of this solution.
Existence: Existence of minimizers is given by the classical Weierstrass Theorem.

Theorem 1.39. *Let $\mathcal{D} \subset \mathbb{R}^n$. If $f : \mathcal{D} \to \mathbb{R}$, is continuous and if $\mathcal{D}$ is a compact set (closed and bounded), then f has a minimum on $\mathcal{D}$.*

Theorem 1.40. *We suppose that:*

1. *The set $\mathcal{D}$ is closed*
2. *f is lower semi-continuous on $\mathcal{D}$.*
3. $\lim_{\|x\| \to +\infty} f(x) = +\infty$ *, (i.e. f is coercive).*

Then f has a minimum on $\mathcal{D}$.

Uniqueness. The following result shows the impact of convexity in optimization problems.

Proposition 1.41. *Let f be a convex function defined on a convex set $\mathcal{D}$. Then any local minimum of f on $\mathcal{D}$ is a global minimum. If f is strictly convex, it there is at most a global minimum.*

Necessary and sufficient conditions for optimality

Necessary conditions

Theorem 1.42. *(Necessary condition of first order optimality) Let $f : \mathbb{R}^n \to \mathbb{R}$, a differentiable functional. If $x^\star$ achieves a minimum (global or local) of f on $\mathbb{R}^n$ then:*

$$\nabla f(x^\star) = 0.$$

Theorem 1.43. *(Necessary condition of second-order optimality) Let $x^\star$ un minimum of f. If f is twice differentiable at the point $x^\star$, then:*

$$\left\langle \nabla^2 f(x^\star) y, y \right\rangle \geq 0, \quad \forall y \in \mathbb{R}^n.$$

Sufficient conditions

Theorem 1.44. *(First order sufficient condition) If f is convex and if:*

$$\nabla f(x^\star) = 0,$$

then $x^\star$ is a global minimum of f.

Theorem 1.45. *(Second order sufficient condition) If f is twice differentiable and if $\nabla f(x) = 0$ and $\nabla^2 f(x)$ positive definite then $x^\star$ is a global minimum of f.*

Convergence of a sequence of points (Order of convergence)

Let $x^\star$ be the limit of a sequence $\{x_k\}_{k\geq 0}$ produced by an iterative algorithm. We say that $x_k \to x^\star$ when $k \to +\infty \Leftrightarrow x_k - x^\star \to 0$ when $k \to +\infty \Leftrightarrow \|x_k - x^\star\| = 0$ when $k \to +\infty$. Now we try to characterize the speed of convergence of the quantity $x_k - x^\star \to 0$ when $k \to +\infty$. We say that:

- x_k converges linearly to $x^\star$, if there is a $c_k (0 < c_k < 1)$ such that:

$$\|x_{k+1} - x^\star\| \leq c_k \|x_k - x^\star\| \qquad \text{from a certain rank } k_0.$$

- x_k converges superlinear to $x^\star$ if $\lim_{k\to+\infty} c_k = 0$.
- x_k quadratically converges to $x^\star$, if there is a $c (0 < c < 1)$ such that:

$$\|x_{k+1} - x^\star\| \leq c \|x_k - x^\star\|^2 \quad \text{from a certain rank } k_0.$$

1.5 Quadratic programming

Quadratic problems form an important class of nonlinear optimization problem.They involve the minimization of a quadratic objective function subject to linear constraints on optimization variables. Its importance lies in its theoretical properties, its applications in different scientific fields and several disciplines such as: finance, medicine, economics and telecommunications. In fact, several real and academic problems can be modeled under form of quadratic program for example: portfolio optimization in finance, transmission of multi-layer beams, and geometric problems.

The most general quadratic program is given by:

$$(QP) \quad \begin{cases} \min\limits_{x} \dfrac{1}{2} x^T Q x + c^T x \\ \text{s.t. } Ax = a, \\ \qquad Bx \leq b, \\ \qquad x \geq 0 \end{cases} \tag{1.2}$$

where Q is a symmetric matrix in $\mathbb{R}^{n\times n}$ (not necessarily positive semidefinite), $A \in \mathbb{R}^{m_1 \times n}$, rank$(A) = m_1, (m_1 \leq n)$, $B \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $a \in \mathbb{R}^{m_1}$.

There are two cases:

(a) Q is symmetric and positive semi-definite) QP is convex.

(b) Q is symmetric but not positive semi-definite) QP is non-convex.

Any quadratic program can be reduced to one of the following forms:

Equality-constrained QP's

If only equality constraints are imposed, the (QP) reduces to:

$$(EQP) \quad \begin{cases} \min\limits_{x} \dfrac{1}{2} x^T Q x + c^T x \\ \text{s.t. } Ax = a; \quad x \in \mathbb{R}^n. \end{cases}$$

where Q is a symmetric matrix in $\mathbb{R}^{n\times n}$ (not necessarily positive definite), $A \in \mathbb{R}^{m_1 \times n}$, $m_1 \leq n$, $c \in \mathbb{R}^n$, $a \in \mathbb{R}^{m_1}$. For the time being we assume that A has full row rank m_1.

The KKT conditions for the solution $x^\star \in \mathbb{R}^n$ of the EQP give rise to the following linear system

$$\begin{cases} Qx^\star + A^T \lambda^\star = -c \\ Ax^\star = a \end{cases}$$

or

$$\begin{pmatrix} Q & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x^\star \\ \lambda^\star \end{pmatrix} = \begin{pmatrix} -c \\ b \end{pmatrix}$$

where $\lambda^\star \in \mathbb{R}^n$ is the associated Lagrange multiplier.

The matrix

$$K = \begin{pmatrix} Q & A^T \\ A & 0 \end{pmatrix}$$

is called the K.K.T matrix.

Optimality for equality-constrained QP's

Assumptions:

- A has linearly independent rows, i.e. A has full row rank.
- Q is positive definite in the null space of A.

Theorem 1.46. *The matrix K is nonsingular.*

Theorem 1.47. *The solution $x^\star$ of the K.K.T system is the unique global solution of the equality-constrained EQP.*

The equality-constrained QP is a convex problem under the above assumptions.

Inequality-constrained QP's

$$(IQP) \begin{cases} \min\limits_x \dfrac{1}{2} x^T Q x + c^T x \\ \text{s.t. } Bx \leq b, \;\; x \geq 0, \end{cases}$$

where Q is a symmetric positive semi-definite matrix in $\mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$.

Optimality conditions for inequality-constrained QP's

The KKT conditions simplify to x is an optimal solution of (IQP) if there exist $y \in \mathbb{R}^m_+$, $\lambda \in \mathbb{R}^n_+$ such that:

$$\begin{cases} c + Qx + B^T y - \lambda = 0 \\ y^T(b - Bx) = 0 \\ (b - Bx) \leq 0 \\ \lambda^T x = 0 \\ y \geq 0, \lambda \geq 0 \end{cases}$$

Methods for solving a convex quadratic program

In recent years many algorithms have been developed to solve convex quadratic problems, among these algorithms, let us mention

Interior-Point Methods

The interior point method gets its name from the fact that the solution points move through the interior of the feasible region towards the optimum rather than along the boundary. The interior point method attempts to overcome the potential weakness of the simplex method and it has been shown that the interior point methods can be solved in polynomial time Interior methods IPMs solve the QP by converting it into a parameterized sequence of unconstrained problems whose solutions draw on the power of Newton's method to solve systems of nonlinear equations by solving a series of linearized equation systems. The resulting algorithms run in polynomial-time [70]. In the simplest case, the path is parameterized by a positive scalar parameter μ, such that as $\mu \to 0$, a point $x(\mu)$ on the path converges to the solution of the QP, and thus are believed to be effective for large-scale problems, which are efficient for solving convex problems (linear or not) of big size.
The interior point methods are divided into several families.
- The projective method (Optimization on ellipsoids) introduced by Karmarker [32].
- The potential reduction method (concept of barrier, central trajectory, relaxation).
- The central trajectory method is the most important representative of this family.
- The path-following methods (also known as trajectory-following) offer a good alternative to the earlier active-set methods. The path following method falls under the category of the so called barrier methods, which incorporate the constraints into the objective function as logarithmic terms to remove the inequalities from the problem. The resulting function becomes a function of a so called barrier parameter. This parameter is decreased at each iteration. The solution points form a so called central path through the interior of the region towards the optimum point. Although path-following methods may be applied to the general problem QP.

Active Set Methods

The Active-set method (ASM) is a classic method, developed in the early seventies for solving linear and quadratic programming problems. It applies to optimization problems with linear constraints of the inequality or mixed type (equalities and inequalities). Active Set methods keeps track of those constraints which are active, that is, constraints for which strict equality holds at a local minimum point. The idea of the method is to add active constraints to an active set provided certain conditions hold. If the conditions are not satisfied the corresponding constraint is dropped form the set. Active set methods start by finding a feasible point during an initial phase and then search for a solution along the edges and faces of the feasible set by solving a sequence of equality-constrained QPs. Active set methods differ from the simplex method for linear programming in that neither the iterates nor the solution need to be vertices's of the feasible set. When the quadratic programming problem is nonconvex, these methods usually find a local minimizer. Finding a global minimizer is a more difficult task.

Active-set methods for quadratic programming problems of the form (1,2) solve a sequence of linear equations that involve the y-variables and a subset of the x-variables. Each set of equations constitutes the optimality conditions associated with an equality constrained quadratic subproblem. The goal is to predict the optimal active set, i.e., the set of constraints that are satisfied with equality, at the solution of the problem. A conventional active-set method has two phases. In the first phase, a feasible point is found while ignoring the objective function; in the second phase, the objective is minimized while feasibility is maintained. A useful feature of active-set methods is that they are well-suited for "warm starts", where a good estimate of the optimal active set is used to start the algorithm. This is particularly useful in applications where a sequence of quadratic programs is solved. Other applications of active-set methods for quadratic programming include mixed integer nonlinear programming, portfolio analysis, structural analysis and optimal control.

Wolfe's method

Although it is not the first method that was developed for solving quadratic programming problems, it is one of the most widely used methods and was developed by P. Wolfe in 1959 (see [70]). This method solves the problems of convex

quadratic programming at linear constraints by algorithms related to that of the Simplex. It is based on the conditions of (K.K.T), which are then necessary and sufficient. To find a point satisfying the Kuhn-Tucker conditions Wolfe's algorithm proposes a modified version of Phase One of the Two-phase simplex method, applied to the linear equations and the non-negativity constraints of (K.K.T) . Essentially the method adds artificial variables to all the constraints and then attempts to minimize the sum of the artificial variables.

Projected gradient method

This method is inspired by the usual gradient methods known for unconstrained optimization. Suppose, in general, that we want to minimize a function f on a set of constraints C. Then, we build a sequence iterates of the form $x_{k+1} = x_k + kd_k$, where d_k is a descent direction. For ensure that all iterates belong to C, we call the projection on this last.

1.6 Linear complementarity problems

LCP is not an optimization problem, but it has robust relationship with both linear programming (LP) and convex quadratic programming (CQP) problems. This strong relationship is based on the fact that Karush-Kuhn-Tucker (KKT) optimality conditions for LP and QP can be converted into LCP. Many optimization problems from engineering, finance, transportation, etc. can be directly written as LCP. Therefore, solving LCPs has been very important topic for many years.

The standard Linear complementarity Problem

The standard linear complementarity problem (abbreviated by SLCP) consists to find a couple of vectors $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ such that:

$$x \geq 0,\ y \geq 0, y = Mx + q,\ x^T y = 0, \tag{1.3}$$

where $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$ are given. We will denote by:

$$\mathcal{F} = \left\{(x, y) \in \mathbb{R}^{2n} : y = Mx + q, x \geq 0, y \geq 0\right\},$$

the feasible set of the (SLCP) (1.3). The solution set of the (SLCP) is given by:

$$Sol(M, q) = \left\{(x, y) \in \mathcal{F} : x^T y = 0\right\}.$$

Monotone linear complementarity problem

The (*MLCP*) requires the composition of a victor pair $(y,z) \in \mathbb{R}^{2n}$ satisfying:

$$y = Mx + q,\ (x, y) \geq 0,\ x^T y = 0, \tag{1.4}$$

where $q \in \mathbb{R}^n$ and M is $n \times n$ matrix supposed positive semidefinite.
- If M is $\mathcal{P}$-matrix then the linear complementarity problem (1.4) is called $\mathcal{P}$-LCP. The following result was proved by Cottle, Pang and Stone [16], where any $\mathcal{P}$-LCP has a unique solution for every $q \in \mathbb{R}^n$.

Theorem 1.48. [*16, Theorem 3.3.7*] *A matrix $M \in \mathbb{R}^{n\times n}$ is a $\mathcal{P}$-matrix if and only if the LCP has a unique solution for $q \in \mathbb{R}^n$. In this case the LCP is denoted by $\mathcal{P}$-LCP.*

The horizontal linear complementarity problem

Given two square matrices $M, N \in \mathbb{R}^{n\times n}$ and a vector $q \in \mathbb{R}^n$, the horizontal linear complementarity problem (HLCP) consists in finding a pair $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ such that:

$$Ny - Mx = q, x \geq 0, y \geq 0,\ x^T y = 0. \tag{1.5}$$

We will denote

$$\mathcal{F}^\star = \left\{(x, y) \in \mathbb{R}^{2n} : Ny = Mx + q, x \geq 0, y \geq 0\right\}.$$

The feasible set of points of the HLCP as defined in (1.5). The solution set of the HLCP is given by:

$$Sol(M, N, q) = \left\{(x, y) \in \mathcal{F}^\star : x^T y = 0\right\}.$$

Remark 1.49. If $N = I$ or N^{-1} exists, the HLCP reduced to the standard LCP.

Transformation of a convex quadratic program into a linear complementary problem

$$(QP) \begin{cases} \min\limits_x \dfrac{1}{2} x^T Q x + c^T x \\ \text{Subject to: } Ax \leq b, x \geq 0 \end{cases} \tag{1.6}$$

where Q is a symmetric semidefinite matrix in $\mathbb{R}^{n\times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$ with $rank(A) = m$. Recall that the set of constraints is

$$C = (x \in \mathbb{R}^n : Ax = b, x \geq 0\},$$

Since the problem (QP) is convex and the constraints are linear then the KKT conditions are necessary and sufficient and are written as follows:

$x \in \mathbb{R}^n_+$ is an optimal solution of (QP) if and only if there exist $y \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}^n$ such that:

$$\begin{cases} c + Qx + A^T y - \lambda = 0 \\ y^T(Ax - b) = 0 \\ \lambda^T x = 0 \\ y \geq 0, \lambda \geq 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \lambda = c + A^T y + Qx \\ v = b - Ax \\ \lambda^T x = 0,\ y^T v = 0 \\ (x, \lambda, v, y) \geq 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \begin{pmatrix} \lambda \\ v \end{pmatrix} = \begin{pmatrix} Q & A^T \\ -A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c \\ b \end{pmatrix} \\ \langle \begin{pmatrix} \lambda \\ v \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \rangle = 0 \\ \begin{pmatrix} \lambda \\ v \end{pmatrix} \geq 0, \begin{pmatrix} x \\ y \end{pmatrix} \geq 0. \end{cases}$$

By taking

$$\begin{aligned} w &= \begin{pmatrix} \lambda \\ v \end{pmatrix}, z = \begin{pmatrix} x \\ y \end{pmatrix}, q = \begin{pmatrix} c \\ b \end{pmatrix} \\ M &= \begin{bmatrix} Q & A^T \\ -A & 0 \end{bmatrix} \end{aligned}$$

(QP) can be written in the following form :

$$(LCP) \begin{cases} \text{Find } y, z \in \mathbb{R}^n \text{ such that} \\ w = Mz + q \geq 0 \\ z \geq 0 \\ z^T y = 0 \end{cases}$$

1.7 Absolute value equations

In this section, we give the definition of absolute value equations, and some results of existence and uniqueness of the solution.
Absolute value equations (abbreviated as AVE) are defined as follows:

$$Ax - B|x| = b, \tag{1.7}$$

where $A, B \in \mathbb{R}^{n\times n}$ are given, $b \in \mathbb{R}^n$, and $|x|$ is a vector whose i-th entry is the absolute value of the i-th entry of x. If $B = I$, the identity matrix, then the AVE (1.7) can be reduced to the type:

$$Ax - |x| = b. \tag{1.8}$$

Reformulation of SLCP and HLCP as an AVE

Let us make the following variable change:

$$x = |z| + z,\ y = |z| - z,$$

it is easy to verify that:

$$x = |z| + z \geq 0, y = |z| - z \geq 0,$$

and

$$x^T y = (|z| + z)^T(|z| - z) = 0, \text{where } z \in \mathbb{R}^n.$$

With this change of variables, the (SLCP) turns into the following absolute value equations:

$$(I + M)z - (I - M)|z| = -q.$$

Using the previous results for the AVE, we deduce that:

Lemma 1.50. *If the matrix $(I - M)$ is nonsingular, then the SLCP becomes:*

$$(I - M)^{-1}(I + M)z - |z| = -(I - M)^{-1}q, \tag{1.9}$$

where

$$A = (I - M)^{-1}(I + M), b = -(I - M)^{-1}q.$$

Remark 1.51. The solution of (SLCP) is therefore computed via the formula:

$$x = |z| + z,$$

with z is the solution of the Absolute value equations (1.9).

With the same change of variable for SLCP, the HLCP is transformed into an Absolute value equations as follows.

$$(N+M)z-(N-M)\,|z|=-q,$$

with $A=N+M, B=N-M$ and $b=-q$.

Lemma 1.52. *If the matrix $(N-M)$ is nonsingular, then the (HLCP) becomes:*

$$(N-M)^{-1}(N+M)z-|z|=-(N-M)^{-1}q. \tag{1.10}$$

So we set

$$A=(N-M)^{-1}(N+M), b=-(N-M)^{-1}q.$$

and z is the solution of the Absolute value equations (1.10) such that

$$x=|z|+z.$$

Some results of existence and uniqueness of the solution of the AVE

For unique solvability of AVE, we cite the most well-known established results until today. In [44], Mangasarian and Meyer presented a sufficient condition, namely, $1<\sigma_{\min}(A)$ for AVE (1.18). In [58], Rohn generalized this result to unique solvability of AVE (1.7) where he imposed the following sufficient condition

$$\sigma_{\max}(|B|)<\sigma_{\min}(A), \tag{1.11}$$

where $\sigma_{\max}(|B|)$ denotes the maximal singular value of matrix $|B|=(|b_{ij}|)$ and the $\sigma_{\min}(A)$ denotes the smallest singular values of matrix A.

Theorem 1.53 (Theorem 1 [58])**.** *Under Assumption 1.11, the AVE (1.7) is uniquely solvable for every $b\in\mathbb{R}^n$.*

In [10], Achache and Hazzam relaxed this result, where they assumed that the AVE (1.7) satisfies the following condition.

• **Assumption 1.** The pair of the matrices $[A,B]$ satisfies

$$\sigma_{\max}(B)<\sigma_{\min}(A),$$

From the theory of the HLCP and Assumption 1, we will show, based on the following theorem, that the AVE (1.7) is uniquely solvable for every $b\in\mathbb{R}^n$.

Theorem 1.54 (Theorem 2 [10]). *Under Assumption 1, the AVE (1.7) is uniquely solvable for every* $b \in \mathbb{R}^n$.

On the other hand, in [9] Achache and Anane give some weaker conditions and a simple proof for unique solvability of AVE. Furthermore, they demonstrate with an example that these results are reliable to detect unique solvability of AVE. For numerical applications they suggest a Picard iterative method to compute an approximated solutions of some uniquely solvable AVE problems where its globally linear convergence is guaranteed via one of ther weaker sufficient condition. The following result concerning the unique solvability of AVE (1.7).

Lemma 1.55 (Iemma 2 [9]). *Each of three conditions below implies the non singularity of the matrix* $(A - BD)$. *for all diagonal matrix* D *whose diagonal elements are* ± 1 *or* 0.

1. $\sigma_{\min}(A) > \sigma_{\max}(B)$,
2. $\left\|A^{-1}B\right\| < 1$, *provided* A *is nonsingular,*
3. *the matrix* $A^TA - \|B\|^2 I$ *is positive definite.*

Then, it is clear that the AVE (1.7) is uniquely solvable for any b if the matrix of coefficients $(A - BD)$ is nonsingular for all diagonal matrix D whose diagonal elements are ± 1 or 0.

Theorem 1.56 (Theorem 2 [9]). *If matrices* A *and* B *satisfy*

1. $\sigma_{\min}(A) > \sigma_{\max}(B)$,
2. $\left\|A^{-1}B\right\| < 1$, *provided* A *is non singular,*
3. *the matrix* $A^TA - \|B\|^2 I$ *is positive definite,*

then the AVE (1.7) is uniquely solvable for any b.

Furthermore, Lotfi and Veiseh [42], imposed other sufficient condition that if the following matrix:

$$A^TA - \||B|\|^2 I, \tag{1.12}$$

is positive definite, then AVE (1.7) is uniquely solvable for any $b \in \mathbb{R}^n$.
Their proof is based on the regularity of interval matrix. To simplify this concept, we give a short overview on this topic. We will give first the notion of an interval matrix.

Definition 1.57. Given matrices A and B in $\mathbb{R}^{n\times n}$, the set of matrices

$$\mathbf{I} = [A - |B|, A + |B|] := \{S : |S - A| \leq |B|\} = \{S : A - |B| \leq S \leq A + |B|\}$$

, is called an interval matrix (with midpoint A and radius $|B|$).

Now, we have this definition introducing the an important distinction:

Definition 1.58. A square interval matrix A is called regular if each $S \in \mathbf{I}$ is regular and singular otherwise, i.e. $\mathbf{I}$ contains a singular matrix.

The following result concerning the unique solvability of AVE (1.7).

Lemma 1.59 (Lemma 2.3 in [42]). *If the interval matrix $\mathbf{I}$ is regular, then the AVE (1.7) for any b has a unique solution.*

1.8 Fixed point method

Let F be a function from to $\mathbb{R}^n$ to $\mathbb{R}^n$, and consider the equation $F(x) = 0$. It is clear that x^* is a solution of $F(x) = 0$ if and only if x^* is a root of the equation:

$$F(x^*) = 0.$$

The idea of the fixed point iteration methods is to first reformulate the equation $F(x) = 0$ as an equivalent fixed point principle:

$$F(x) = 0 \Leftrightarrow \Phi(x) = x,$$

where $\Phi(x)$ is a given function. So finding the root of $F(x) = 0$, is equivalent to finding the unique fixed point of $\Phi(x) = x$. Then, the fixed point method for getting the root x^* of the equation $F(x) = 0$ is based on the following recurrent iterative schema:

$$\begin{cases} x_0 \in \mathbb{R}^n \text{ given} \\ x_{k+1} = \Phi(x_k), k \geq 0. \end{cases}$$

Theorem 1.60. *(Banach's fixed point theorem). Let (X, d) be a non-empty complete metric space, $0 \leq \alpha \leq 1$ and $T : X \longrightarrow X$ a mapping satisfying:*

$$d(T(x), T(y)) \leq \alpha d(x, y) \text{ for all } x, y \in X.$$

Then there exists a unique $x \in X$ such that $T(x) = x$. Furthermore, x can be found as follows: start with an arbitrary element $x_0 \in X$ and define a sequence $\{x_k\}$ by:

$$x_{k+1} = T(x_k),$$

then

$$\lim_{k\longrightarrow\infty} x_k = x,$$

and the following inequalities hold:

$$d(x, x_{k+1}) \leq \frac{\alpha}{1-\alpha} d(x_{k+1}, x_k),\ d(x, x_{k+1}) \leq \alpha d(x, x_k).$$

1.9 Newton-Raphson's Method for nonlinear system of equations

Now we interested to the Newton-Raphson method for solving nonlinear system of equations. This latter plays an important role in the devlopment of interoir-point methods. We have the following definition.

Definition 1.61. A system of nonlinear equations is a set of equations which given as following :

$$\left\{ \begin{array}{c} F_1(x_1, x_2, \ldots, x_n) = 0 \\ F_2(x_1, x_2, \ldots, x_n) = 0 \\ \vdots \\ F_n(x_1, x_2, \ldots, x_n) = 0 \end{array} \right\},$$

where $x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$ and F_i is given a nonlinear real function, $i = 1, ..., n$. To simplify the notation even further, define the vector valued function :

$$F(x) = \left[\begin{array}{c} F_1(x) \\ F_2(x) \\ \vdots \\ F_n(x) \end{array} \right] = \left[\begin{array}{c} F_1(x_1, x_2, \ldots, x_n) \\ F_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ F_n(x_1, x_2, \ldots, x_n) \end{array} \right].$$

The $n \times n$ jacobian matrix $J(x)$ of $F(x)$ is :

$$J(x) = \left[\begin{array}{cccc} \frac{\partial F_1}{\partial x_1}(x) & \frac{\partial F_1}{\partial x_2}(x) & \cdots & \frac{\partial F_1}{\partial x_n}(x) \\ \frac{\partial F_2}{\partial x_1}(x) & \frac{\partial F_2}{\partial x_2}(x) & \cdots & \frac{\partial F_2}{\partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1}(x) & \frac{\partial F_n}{\partial x_2}(x) & \cdots & \frac{\partial F_n}{\partial x_n}(x) \end{array} \right],$$

where $\frac{\partial F_i}{\partial x_j}(x)$ is called the partial derivative of F_i at x_j. The Newton-Raphson Method gives a successive approximations to the solution in the following form:

$$F(x_k) + J(x_k)(x_{k+1} - x_k) = 0,\ k = 0, 1, ...$$

Then, the fixed point method for getting the root x^* of the equation $F(x) = 0$ is based on the following recurrent iterative schema:

$$\begin{cases} x_0 \in \mathbb{R}^n \text{ a given initial vector} \\ x_{k+1} = x_k - J^{-1}F(x_k), k = 0, 1, ... \end{cases}$$

where J^{-1} denotes the inverse of J.

Chapter 2

A two-steps fixed-point method for SCQO

In this chapter, we have introduced the reformulation of problem SCQO (2.1) as absolute value equations AVE and the unique solvability of AVE is studied. Any solution of the AVE generates a solution of our convex quadratic programming problem SCQO. Numerically, we propose a new two-steps fixed point iterative method for solving the AVE (2.4), and under a new mild assumption we show that this method is always well-defined and the generated sequence converges globally and linearly to the unique solution of the AVE from any starting initial point. Finally, numerical results are provided to illustrate the efficiency of this algorithm to solving the SCQO.

2.1 The SCQO as absolute value equations

Recall that the SCQO problem is given by:

$$\min_x \left[f(x) = \frac{1}{2}x^T Q x + x^T b + c \right] \text{ subject to: } x \in \mathcal{S}. \tag{2.1}$$

Starting from the definition of simplicial cone $\mathcal{S}$ associated with the nonsingular matrix A, the problem (2.1) can be formulated as a quadratic programming problem under positive constraints:

$$\min_y \left[f(y) = \frac{1}{2}y^T A^T Q A y + y^T A^T b + c \right] \text{ subjet to: } y \in \mathbb{R}^n_+. \tag{2.2}$$

As the problem (2.2) is convex and the constraints are positive then the optimality conditions of K.K.T are necessary and sufficient and we have, $y \in \mathbb{R}^n_+$ is an optimal

solution of problem (2.2) if and only if there exists $z \in \mathbb{R}^n_+$ such that:

$$z - A^T Q A y = A^T b,\ z^T y = 0,\ y \geq 0,\ z \geq 0. \tag{2.3}$$

The K.K.T optimality conditions in (2.3) are only a standard LCP (see [16]). Next, letting $z = |s| - s$ and $y = |s| + s$ with $s \in \mathbb{R}^n$, then the LCP (2.3) is reformulated as the following absolute value equations (AVE) of type

$$\bar{A}s + \bar{B}\,|s| = \bar{b}, \tag{2.4}$$

where

$$\bar{A} = A^T Q A + I,\ \bar{B} = A^T Q A - I,\ \bar{b} = -A^T b.$$

Hence, solving problem (2.1) is equivalent to solving AVE (2.4). The following result is needed to guarantee the unique solvability of the AVE.

Theorem 2.1 (Theorem 8 [8]). *Assume that $\bar{A}$ is nonsingular and the matrices $\bar{A}, \bar{B}$ satisfy the following condition $\left\|\bar{A}^{-1}\bar{B}\right\| < 1$, then the AVE (2.4) has a unique solution for any $\bar{b} \in \mathbb{R}^n$.*

For our case since $\bar{A} = A^T Q A + I$ and $\bar{B} = A^T Q A - I$ where Q is symmetric positive definite and A is invertible, the condition $\left\|\bar{A}^{-1}\bar{B}\right\| < 1$ of Theorem 2.1, is satisfied. We check this result through the following lemma.

Lemma 2.2. *Let $\bar{A} = A^T Q A + I$ and $\bar{B} = A^T Q A - I$ such that A is nonsingular matrix and Q is symmetric positive definite. Then the matrix $\bar{A}$ is nonsingular and $\left\|\bar{A}^{-1}\bar{B}\right\| < 1$.*

Proof. Because Q is symmetric positive definite and A is nonsingular, then the matrix $A^T Q A$ is symmetric positive definite, hence $\bar{A}$ is symmetric positive definite too, which implies that $\bar{A}$ is nonsingular. Next, since $A^T Q A$ is symmetric positive definite, then $A^T Q A$ has positive real eigenvalues denoted by $\lambda_i(A^T Q A) := \lambda_i > 0, \forall i = 1, ..., n$. In addition, it is known that the eigenvalues of $\bar{A}$ and $\bar{B}$, are given by $\lambda_i + 1 > 0$ and $\lambda_i - 1$, respectively. Because, $\bar{A}$ and $\bar{B}$ are real symmetric matrices, we then have,

$$\begin{aligned} \left\|\bar{A}^{-1}\bar{B}\right\| &\leq \left\|\bar{A}^{-1}\right\| \left\|\bar{B}\right\| = \rho(\bar{A}^{-1})\rho(\bar{B}) \\ &= \max_i \left(\left| \frac{\lambda_i - 1}{\lambda_i + 1} \right| \right). \end{aligned}$$

As $\lambda_i > 0$, then

$$\left|\frac{\lambda_i - 1}{\lambda_i + 1}\right| < 1.$$

So $\left\|\bar{A}^{-1}\bar{B}\right\| < 1$. This gives the required result. □

Proposition 2.3. *If s^* is the solution of the AVE (2.3) then $(y^*, z^*) = (|s^*|+s^*, |s^*|-s^*)$ is the solution of the LCP (2.3). Consequently, Ay^* is the optimal solution of problem (2.1).*

Proof. Let s^* be the unique solution of the AVE, then

$$\bar{A}s^* + \bar{B}\,|s^*| = \bar{b}.$$

So

$$\begin{aligned}\left(A^TQA + I\right)s^* + \left(A^TQA - I\right)|s^*| &= -A^Tb \\ \Leftrightarrow A^TQA(|s^*| + s^*) + |s^*| - s^* &= -A^Tb \\ \Leftrightarrow z^* - A^TQAy^* &= A^Tb.\end{aligned}$$

Next, since $y^* = |s^*| + s^* = 2(s^*)^+, z^* = |s^*| - s^* = 2(s^*)^-$, then we have $y^* \geq 0$, $z^* \geq 0$ and $z^{*T}y^* = 0$, hence, the pair (y^*, z^*) is a solution of LCP (2.3). Finally, we deduce that Ay^* is an optimal solution of the SCQO problem. This completes the proof. □

2.2 Two-steps Picard's fixed point iterative method for SCQO

In this section, we derive a new fixed-point iterative approach for solving the equation (2.4). Let $t = |s|$ then, the AVE (2.4) is equivalent to the following system:

$$\begin{cases} \bar{A}s + \bar{B}t = \bar{b} \\ -|s| + t = 0. \end{cases} \tag{2.5}$$

The latter can be expressed as follows:

$$\begin{pmatrix} \bar{A} & \bar{B} \\ -D(s) & I \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} \bar{b} \\ 0 \end{pmatrix}, \tag{2.6}$$

where $D(s)$:=diag (sign (s)), $s \in \mathbb{R}^n$. Note that the system (2.6) is nonlinear, it is generally impossible to obtain an exact solution. We will therefore be satisfied

with an approximated solution. Since the matrix $\bar{A}$ is nonsingular hence from (2.5) we can obtain the following fixed point equation:

$$\begin{cases} s^* = \bar{A}^{-1}\left(-\bar{B}t^* + \bar{b}\right) \\ t^* = (1-r)\,t^* + r\,|s^*| \end{cases} \tag{2.7}$$

where $r > 0$, is a suitable parameter that we shall specified it later. According to the fixed-point equation, we generate a sequence $(s^{(k)}, t^{(k)})$ converging to the solution of AVE. So the new fixed-point iteration is given by:

$$\begin{cases} s^{(k+1)} = \bar{A}^{-1}\left(-\bar{B}t^{(k)} + \bar{b}\right) \\ t^{(k+1)} = (1-r)\,t^{(k)} + r\left|s^{(k+1)}\right|, k = 0, 1, \ldots \end{cases} \tag{2.8}$$

The details of our algorithm for solving the AVE (2.4) is described in Figure 2.1.

Algorithm

Input
An accuracy parameter $\epsilon > 0$;
a parameter r such that $0 < r < \dfrac{2}{\left\|\bar{A}^{-1}\bar{B}\right\| + 1}$;
an initial starting point $t^0 \in \mathbb{R}^n$;
compute $t^1 = (1-r)t^0 + r|s^1|$, $s^1 = \bar{A}^{-1}\left(-\bar{B}t^{(0)} + \bar{b}\right)$; $k := 0$;
While $\dfrac{\left\|t^{k+1} - t^k\right\|}{\left\|\bar{b}\right\|} \geq \epsilon$ **do**
begin
compute : $\begin{cases} t^{(k+1)} = (1-r)\,t^{(k)} + r\left|s^{(k+1)}\right|, \\ s^{(k+1)} = \bar{A}^{-1}\left(-\bar{B}t^{(k)} + \bar{b}\right) \end{cases}$
$k := k + 1$;
end
end

Figure 2.1: Algorithm. 2.2.1

In this section, we give detailed proof for the convergence of Algorithm 2.2.1

Theorem 2.4. *Let $Q \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix, $b \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ is a nonsingular matrix then the sequence $(s^{(k)}, t^{(k)})$ generated by the iterative schema (2.8) for solving problem (2.5) is well-defined for any starting point $t^0 \in \mathbb{R}^n$. In addition, if*

$$0 < r < \frac{2}{\left\|\bar{A}^{-1}\bar{B}\right\| + 1},$$

then the sequence $(s^{(k)}, t^{(k)})$ *converges linearly to the solution* (s^*, t^*) *of the nonlinear equation (2.5). Consequently,* $A(|s^*| + s^*)$ *is the solution of the problem (2.1).*

Proof. First, we check that the sequence $(s^{(k)}, t^{(k)})$ is well-defined, it suffices to show that the matrix $\bar{A} = A^T Q A + I$ is nonsingular. This claim was proven by Lemma 2.2. Next, using formula (2.8) and Lemma 1.2, we have, on one hand that

$$\begin{aligned} \left\|t^{k+1} - t^*\right\| &= \left\|(1-r)\, t^k + r\left|s^{(k+1)}\right| - (1-r)\, t^* + r\,|s^*|\right\| \\ &= \left\|(1-r)\,(t^k - t^*) + r(\left|s^{(k+1)}\right| - |s^*|)\right\| \\ &\leq |1-r|\left\|(t^k - t^*)\right\| + r\left\|s^{k+1} - s^*\right\|. \end{aligned}$$

So

$$\begin{aligned} \left\|s^{k+1} - s^*\right\| &= \left\|\bar{A}^{-1}\left(-\bar{B}t^k + \bar{b}\right) - \bar{A}^{-1}\left(-\bar{B}t^* + \bar{b}\right)\right\| \\ &= \left\|-\bar{A}^{-1}\bar{B}(t^k - t^*)\right\| \leq \left\|\bar{A}^{-1}\bar{B}\right\|\left\|(t^k - t^*)\right\|. \end{aligned}$$

Therefore

$$\left\|t^{k+1} - t^*\right\| \leq (|1-r| + r\left\|\bar{A}^{-1}\bar{B}\right\|)\left\|(t^k - t^*)\right\|.$$

On the other hand,

$$\begin{aligned} \left\|s^{k+1} - s^*\right\| &\leq \left\|-\bar{A}^{-1}\bar{B}(t^k - t^*)\right\| \\ &\leq \left\|-\bar{A}^{-1}\bar{B}(t^k - (1-r)t^{k-1} + (1-r)t^{k-1} - t^*)\right\| \\ &\leq \left\|-\bar{A}^{-1}\bar{B}(r\left|s^k\right| + (1-r)t^{k-1} - t^*)\right\|. \end{aligned}$$

As $|s^*| = t^*$, we find

$$\begin{aligned} \left\|s^{k+1} - s^*\right\| &\leq \left\|-\bar{A}^{-1}\bar{B}(r\left|s^k\right| - r\,|s^*|) - (1-r)\bar{A}^{-1}\bar{B}(t^{k-1} - t^*)\right\| \\ &\leq r\left\|\bar{A}^{-1}\bar{B}\right\|\left\|s^k - s^*\right\| + |1-r|\left\|s^k - s^*\right\| \\ &\leq (|1-r| + r\left\|\bar{A}^{-1}\bar{B}\right\|)\left\|s^k - s^*\right\|. \end{aligned}$$

The sequence $(s^{(k)}, t^{(k)})$ is convergent if the following condition

$$|1-r| + r\left\|\bar{A}^{-1}\bar{B}\right\| < 1$$

holds. For that we distinguish two cases.
Case 1. If $0 < r \leq 1$, then

$$\begin{aligned} |1-r| + r\left\|\bar{A}^{-1}\bar{B}\right\| < 1 &\Leftrightarrow 1 - r + r\left\|\bar{A}^{-1}\bar{B}\right\| < 1 \\ &\Leftrightarrow r(\left\|\bar{A}^{-1}\bar{B}\right\| - 1) < 0. \end{aligned}$$

Since $\left\|\bar{A}^{-1}\bar{B}\right\| < 1$ then,

$$r(\left\|\bar{A}^{-1}\bar{B}\right\| - 1) < 0, \forall\, 0 < r \leq 1.$$

Case 2. If $r \geq 1$, then

$$\begin{aligned} |1-r| + r\left\|\bar{A}^{-1}\bar{B}\right\| < 1 \quad &\Leftrightarrow \quad -1 + r + r\left\|\bar{A}^{-1}\bar{B}\right\| < 1 \\ &\Leftrightarrow \quad r < \frac{2}{\left\|\bar{A}^{-1}\bar{B}\right\| + 1}. \end{aligned}$$

Finally, regrouping the two cases, this gives the required result. □

2.3 Numerical results

In this section, we present numerical results for Algorithm 2.2.1 by using $\epsilon = 10^{-6}$ and $r = 0.9$. The algorithm has been applied on three examples of SCQO problem. The iterations have been carry out by **MATLAB** R2016a and run on a personal pc with 1.40 GHZ AMD E1-2500 APU Radeon(TM) HD Graphic, 8 GB memory and Windows 10 operating system. The starting point and the unique solution by t^0 and s^*, respectively. The stopping criterion used in our algorithm is the relative residue, i.e.,

$$RES := \frac{\left\|t^{k+1} - t^k\right\|}{\left\|\bar{b}\right\|} \leq 10^{-6}.$$

In view of the influence of the initial point on the convergence of our algorithm, different values are used. For each problem, the hypotheses of Theorem 2.4. are checked. In the tables below, the symbols "It" and "CPU" denote the number of iterations produced by the algorithm and the elapsed times, respectively.

Example 2.3.1. *Consider the SCQO problem where Q, A and b are given by :*

$$Q = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}, A = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0.5 & 3 & 0 & 0 & 0 \\ -1 & 0.5 & 3 & 0 & 0 \\ -1 & -1 & 0.5 & 3 & 0 \\ -1 & -1 & -1 & 0.5 & 3 \end{bmatrix}$$

and $b = [-3,\ 1,\ -10,\ -12,\ -2]^T$.

The starting point in this example is taken as:

$$t^0 = [0,\, -1,\, -1,\, 2,\, 1]^T.$$

After 21 iterations, the unique solution s^* of AVE is:

$$s^* = [0.2071,\, -7.6143,\, 0.5262,\, 0.7886,\, -2.2308]^T,$$

and

$$y^* = |s^*| + s^* = [0.4142,\, 0,\, 1.0525,\, 1.5771,\, 0]^T.$$

Therefore, the unique solution of Problem SCQO(2.1), is given by:

$$x^* = Ay^* = [1.2426,\, 0.2071,\, 2.7433,\, 4.8435,\, -0.6781]^T.$$

Example 2.3.2. *Let the matrices Q, A and the vector b of this example are given by:*

$$Q = (q_{ij}) = \begin{cases} 4, & \textit{for} \quad i = j, \\ \dfrac{1}{2}, & \textit{for} \quad |i-j| = 1, i = 1, 2, \ldots n-2, \\ 1, & \textit{for} \quad \begin{cases} j = i-2, i = 1, 2, \ldots n, \\ i = j-2, j = 1, 2, \ldots n, \end{cases} \\ 0, & \textit{otherwise} \end{cases}$$

$$A = (a_{ij}) = \begin{cases} -2, & \textit{for} \quad i = j, \\ 4, & \textit{for} \quad j = i-1, i = 2, \ldots n, \\ -1, & \textit{for} \quad i = j-1, j = 2, \ldots n, \\ \dfrac{1}{2}, & \textit{for} \quad j > i+2, i = 1, 2, \ldots n, \\ \dfrac{1}{5}, & \textit{for} \quad i > j+1, j = 1, 2, \ldots n, \end{cases}$$

and

$$b = -2\bar{A}^{-1}(\bar{A} + \bar{B})e.$$

An exact solution of AVE is given by:

$$s^* = [2,\, 2,\, \ldots,\, 2]^T,$$

and

$$y^* = [4,\, 4,\, \ldots,\, 4]^T.$$

An exact solution of problem (2.1) is given by: $x^* = Ay^*$.
The computational results with different size of n are shown in Table 2.1. For the initialization of Problem 2, we take different values of t^0.

n		$t^0 = [0, \ldots, 0]^T$	$t^0 = [5, \ldots, 5]^T$	$t^0 = [-10, \ldots, -10]^T$
10	CPU	$0.04966s$	$0.04552s$	$0.10820s$
	It	17	11	65
	RES	$9.923e-07$	$8.542e-07$	$9.606e-07$
50	CPU	$0.03704s$	$0.03391s$	$0.36887s$
	It	4	3	42
	RES	$9.136e-07$	$9.025e-07$	$9.576e-07$
100	CPU	$0.07339s$	$0.05023s$	$0.26949s$
	It	3	3	22
	RES	$6.032e-07$	$1.508e-07$	$9.365e-07$
1000	CPU	$5.34351s$	$4.42180s$	$5.36816s$
	It	2	1	2
	RES	$5.992e-07$	$3.371e-07$	$7.501e-07$
2000	CPU	$34.17236s$	$33.23488s$	$39.15840s$
	It	1	1	2
	RES	$3.370e-07$	$8.427e-07$	$1.873e-07$

Table 2.1: Computational results with $r = 0.9$ of Example 2.3.2.

Example 2.3.3. *The bloc matrices Q, A and the vector b of this example are given by:*

$$Q = \begin{bmatrix} Q_{11} & I_n \\ I_n & Q_{22} \end{bmatrix}, \; A = \begin{bmatrix} A_{11} & I_n \\ B & A_{11} \end{bmatrix}$$

where

$$Q_{11} = (q_{11})_{ij} = \begin{cases} 6, & \textit{for} \quad i = j, \\ -1, & \textit{for} \quad |i-j| = 1, i = 1, 2, \ldots n, \\ 0, & \textit{otherwise.} \end{cases}$$

$$Q_{22} = (q_{22})_{ij} = \begin{cases} 5, & \textit{for} \quad i = j, \\ -2, & \textit{for} \quad |i-j| = 1, i = 1, 2, \ldots n, \\ \frac{1}{4}, & \textit{for} \quad \begin{cases} j = i+1, i = 1, 2, \ldots n-1, \\ j = i-1, j = 1, 2, \ldots n-1, \end{cases} \\ 0, & \textit{otherwise.} \end{cases}$$

$$A_{11} = (a_{11})_{ij} = \begin{cases} -2, & \textit{for} \quad i = j, \\ -1, & \textit{for} \quad j = i-1, i = 3, \dots n, \\ 3, & \textit{for} \quad j > i, \;\; i = 3, \dots n, \\ 0.5, & \textit{otherwise.} \end{cases}$$

$$B = (b_{ij}) = \begin{cases} -1, & \textit{for} \quad i = j, \\ 0, & \textit{for} \quad \begin{cases} j = i+1, i = 1, 2, \dots n-1, \\ j = i-1, j = 1, 2, \dots n, \end{cases} \\ \frac{1}{n}, & \textit{for} \quad \begin{cases} j = i+2, i = 1, 2, \dots n-2, \\ j = i-2, j = 4, \dots n, \end{cases} \\ 0, & \textit{otherwise,} \end{cases}$$

and

$$b = [-8, \dots, -8]^T.$$

For the initialization, we take:

$$t^0 = [0, \dots, 0, -1, \dots, -1]^T.$$

The numerical results with different size of n are summarized in Table 2.2.

n	10	50	100	1000	2000
CPU	$0.14101s$	$1.0919s$	$1.11202s$	$8.37809s$	$35.76440s$
It	122	137	92	6	3
RES	$9.81e-07$	$9.98e-07$	$9.95e-07$	$7.436e-07$	$7.35e-07$

Table 2.2: Computational results with $r = 1 - \frac{1}{n}$ of Example 2.3.3.

For example, if $n = 10$ then,

$$s^* = [0.0984,\ 0.0042, 0.409,\ -1.782,\ 1.6329,\ 0.3763,\ 0.4448, 0.881, 0.7929, 2.7841]^T,$$

and

$$y^* = |s^*| + s^* = [0.1967, 0.0082, 0.8179, 0, 3.2658, 0.7525, 0.8896, 1.7619, 1.586, 5.568]^T.$$

THe solution of SCQO (2.1) is given by:

$$x^* = Ay^* = [2.3928, 2.475, 2.3819, 2.206, 2.1049, 2.0303, 3.038, 3.3649, 3.2356, 2.3168]^T.$$

Chapter 3

An interior-point algorithm for SCQO

In this chapter, we present a feasible full-Newton step primal-dual interior-point algorithm for solving SCQOs via the $\mathcal{P}$-LCP, we study first the central-path of $\mathcal{P}$-LCP and the search directions, then we state the generic full-Newton step feasible interior-point algorithm for $\mathcal{P}$-LCP in Section 3.2 the complexity analysis and the currently best known iteration bound for short-step methods are established in Section 3.3. Finally, some numerical results are provided to show the efficiency of the proposed algorithm.

3.1 Reformulation of SCQO as a $\mathcal{P}$-LCP

Next task is to reformulate the SCQO (2.1) as a standard LCP. Starting from the definition of the simplicial cone $\mathcal{S}$ associated with the nonsingular matrix A, letting $x = Ay$, then the problem (2.1) can be reformulated as the following convex quadratic optimization problem under positive constraints:

$$\min_{y} \left[f(y) = \frac{1}{2} y^T M y + y^T q + c \right] \text{ s.t. } y \in \mathbb{R}^n_+, \tag{3.1}$$

where

$$M = A^T Q A,\ q = A^T b.$$

The optimality conditions of K.K.T are necessary and sufficient. Then, $y \in \mathbb{R}^n_+$ is an optimal solution of problem (3.1) if and only if there exists $z \in \mathbb{R}^n_+$ such that:

$$z = My + q,\ z^T y = 0,\ y \geq 0,\ z \geq 0. \tag{3.2}$$

The system (3.2) is only a standard LCP with $M = A^T Q A$ and $q = A^T b$.

Theorem 3.1. *Let $Q \in \mathbb{R}^{n \times n}$ be symmetric positive definite and A is nonsingular then $M = A^T Q A$ is a $\mathcal{P}$-matrix. Hence, the LCP (3.2) is a $\mathcal{P}$-LCP.*

Proof. Since $Q \in \mathbb{R}^{n \times n}$ is assumed to be symmetric positive definite and A is nonsingular then for all nonzero $v \in \mathbb{R}^n$, $v^T M v = v^T A^T Q A v = \|Qs\|^2 > 0$ where $s = Av \neq 0$, M is positive definite, therefore M is a $\mathcal{P}$-matrix and so the LCP (3.2) is a $\mathcal{P}$-LCP. By Theorem 1.48, the $\mathcal{P}$-LCP has a unique solution and so is the SCQO (2.1). □

Corollary 3.2. *The vector $x^\star = Ay^\star$ is the unique minimizer of SCQO if and only if the pair of vectors $(y^\star, z^\star)$ is the unique solution of $\mathcal{P}$-LCP (3.2).*

3.2 A feasible full-Newton step interior-point algorithm for SCQO

In this section, we solve the SCQO (2.1) by the application of a feasible full-Newton step interior-point algorithm to the equivalent $\mathcal{P}$-LCP (3.2). To do so, we discuss first the notion of central-path of $\mathcal{P}$-LCP and the Newton search direction.

Then the generic feasible interior-poin algorithm for SCQO (2.1) is presented. In the sequel, we assume that $\mathcal{P}$-LCP (2.3) satisfies the interior-point condition (IPC), i.e., there exists $y^0 > 0$ and $z^0 > 0$ such that $z^0 = My^0 + q$.

The central-path for $\mathcal{P}$-LCP

The basic idea of the path-following interior-point algorithm is to replace the second equation in (2.3), the so-called complementarity condition by the perturbed equation $zy = \mu e$ where $\mu > 0$. Hence, we obtain the following system of equations:

$$z = A^T QAy + A^T b,\ zy = \mu e,\ y \geq 0,\ z \geq 0. \tag{3.3}$$

By IPC assumption it is shown that for any $\mu > 0$ the parameterized system (3.3) has an unique solution denoted by $(y(\mu), z(\mu))$, which is called the $\mu-$center of $\mathcal{P}$-LCP. The set of μ -centers constructs the so-called central-path. Moreover, if μ tends to zero then the limit of central-path exists and converges to a solution of $\mathcal{P}$-LCP ([72]).

The search direction for $\mathcal{P}$-LCP

Applying Newton's method to system (3.3) for a given strictly feasible point (y, z) and the Newton search direction $(\Delta y, \Delta z)$ at this point is the unique solution of the system:

$$\begin{pmatrix} -A^T QA & I \\ Z & Y \end{pmatrix} \begin{pmatrix} \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} 0 \\ \mu e - yz \end{pmatrix}, \tag{3.4}$$

where Y:=diag(y), Z:=diag(z). By simple calculations, the system (3.4) can be written as follows:

$$\begin{cases} (A^T QA + Y^{-1}Z)\Delta y = Y^{-1}(\mu e - yz) \\ \Delta z = A^T QA\Delta y. \end{cases} \tag{3.5}$$

The unique solution $(\Delta y, \Delta z)$ of the system (3.5) is guaranteed by our assumptions since the matrix

$$(A^T QA + Y^{-1}Z), \tag{3.6}$$

is nonsingular.

Lemma 3.3. *Let $M = (A^T QA + Y^{-1}Z)$ such that A is nonsingular matrix and Q is symmetric positive definite, and if (y, z) are strictly feasible. Then the matrix M is nonsingular.*

Proof. For any nonzero $v \in \mathbb{R}^n$, we have $v^T(A^TQA + Y^{-1}Z)v = v^TA^TQAv + v^TY^{-1}Zv > 0$ because $v^T(A^TQA)v > 0$ and $v^TY^{-1}Zv > 0$ for any nonzero $v \in \mathbb{R}^n$ and for all $y > 0$ and $z > 0$ (Y:=diag(x), Z:=diag(z) are positive definite matrices), then the matrix $(A^TQA + Y^{-1}Z)$ is positive definite too and so it is nonsingular. □

Since $(A^TQA + Y^{-1}Z)$ is a symmetric positive definite matrix, the method of Cholesky and the conjugate gradient method are the most convenient for solving the system (3.5).
The new iterates are then given as follows:

$$y_+ := y + \Delta y,\ z_+ := z + \Delta z, \tag{3.7}$$

which indicates that a full-Newton step is taken along the Newton search direction.
Now, to simplify matters, we introduce the following notations:

$$v = \sqrt{\frac{yz}{\mu}},\ d = \sqrt{\frac{y}{z}}. \tag{3.8}$$

So the scaled directions are given by

$$d_y = \frac{v\Delta_y}{y},\ d_z = \frac{v\Delta_z}{z}, \tag{3.9}$$

where $y > 0,\ z > 0$ and $\mu > 0$. Then we have

$$\mu d_y d_z = \Delta y \Delta z \text{ and } y\Delta z + z\Delta y = \mu v\left(d_y + d_z\right). \tag{3.10}$$

Hence the system defining Newton search directions can be written as:

$$\begin{cases} -\overline{M}d_y + d_z = 0 \\ d_y + d_z = p_v, \end{cases} \tag{3.11}$$

where $\overline{M} = DMD^{-1} = DA^TQAD^{-1}$, D:=diag(d), and

$$p_v = v^{-1} - v. \tag{3.12}$$

For the analysis of the algorithm and according to (3.12), we use a norm-based proximity measure $\delta\left(v\right)$ defined by:

$$\delta := \delta(v) = \delta\left(yz;\mu\right) = \frac{1}{2}\|p_v\|. \tag{3.13}$$

Clearly, the value of $\delta\left(v\right)$ can be considered as a measure for the distance between any given pair (y, z) to $\left(y\left(\mu\right), z\left(\mu\right)\right)$. Note that for any strictly feasible pair (y, z), we have

$$\delta\left(v\right) = 0 \Leftrightarrow v = e \Leftrightarrow yz = \mu e.$$

The interior-point algorithm for SCQO

Let $\epsilon > 0$ be a given tolerance and $\theta \in]0,1[$ the update parameter (default $\theta = \frac{1}{\sqrt{3n}}$), the algorithm starts with a strictly feasible initial point (y^0, z^0) such that $\delta(y^0 z^0; \mu_0) \leq \tau$ where $0 < \tau < 1$. Determining the search directions $(\Delta y, \Delta z)$, the algorithm produces a new iterate $(y_+, z_+) = (y + \Delta y, z + \Delta z)$. Then, it updates the barrier parameter μ to $(1-\theta)\,\mu$ and solves the Newton system. This procedure is repeated until the stopping criterion $y_+^T z_+ \leq \epsilon$ is satisfied. The generic feasible full-Newton step interior-point algorithm for SCQO is stated in Figure 3.1.

Input:
An accuracy parameter $\epsilon > 0$;
A threshold parameter $0 < \tau < 1$ (default $\tau = \sqrt{\frac{3}{7}}$);
A barrier update parameter $0 < \theta < 1$ (default $\theta = \frac{1}{\sqrt{3n}}$);
A strictly feasible initial point (y^0, z^0) and $\mu_0 = \frac{1}{2}$ s.t. $\delta(y^0 z^0, \mu_0) \leq \tau$;
begin
$y := y^0$, $z := z^0$, $\mu := \mu_0$;
While $n\mu \geq \epsilon$ **do**
Solve system (3.5) to obtain $(\Delta y, \Delta z)$;
Update $y := y + \Delta y$; $z := z + \Delta z$;
$\mu := (1-\theta)\mu$;
end while
end.

Figure 3.1: Algorithm 3.2.3

3.3 Complexity analysis

In this section, we will show under our new defaults $\tau = \sqrt{\frac{3}{7}}$ and $\theta = \frac{1}{\sqrt{3n}}$ that Algorithm 3.2.3 solves the SCQOs in polynomial and ensures the locally quadratic convergence of the Newton process through the algorithm. Our analysis is straightforward to monotone LCPs ([4]).
We first quote the following technical lemma which will be used later.

Lemma 3.4. *[7, Lemma 2.2.1] Let $\delta > 0$ and (d_y, d_z) be a solution of system (3.11). Then, we have*

$$0 \leq d_y^T d_z \leq 2\delta^2, \tag{3.14}$$

and

$$\|d_y d_z\|_\infty \leq \delta^2, \|d_y d_z\| \leq \sqrt{2}\,\delta^2. \tag{3.15}$$

Lemma 3.5. *Let (y, z) be a strictly feasible primal-dual, then $y_+ = y + \Delta y$ and $z_+ = z + \Delta z$ are positive if and only if $e + dydz > 0$.*

Proof. we have,

$$y_+ z_+ = (y + \Delta y)(z + \Delta z) = yz + y\Delta z + z\Delta y + \Delta y \Delta z.$$

Due to (3.9) and (3.10), we have $\mu d_y d_z = \Delta_y \Delta_z$, and $y\Delta_z + z\Delta_y = \mu v\,(d_y + d_z)$, then we get

$$y_+ z_+ = \mu\,(e + d_y d_z)\,.$$

If $y_+ > 0$ and $z_+ > 0$ then $y_+ z_+ > 0$ and so $e + d_y d_z > 0$. conversely, for each $0 \leq \alpha \leq 1$ denote $y(\alpha) := y + \alpha\Delta y, z(\alpha) := z + \alpha\Delta z$. Therfore,

$$y(\alpha)\, z(\alpha) = yz + \alpha(\mu e - yz) + \alpha^2 \Delta y \Delta z.$$

If $e + dydz > 0$ then $\mu e + \Delta y \Delta z > 0$ and $\Delta y \Delta z > -\mu e$. Hence

$$y(\alpha)\, z(\alpha) \geq (1 - \alpha) yz + (\alpha - \alpha^2)\mu e \geq 0.$$

So $y(\alpha)\, z(\alpha) \geq 0$ for each $0 \leq \alpha \leq 1$. Since $y(\alpha)$ and $z(\alpha)$ are linear functions of α and $y(0) = y > 0$ and $z(0) = z > 0$, then $y(1) = y_+ > 0$ and $z(1) = z_+ > 0$.
this completes the proof. □

In the following lemma, we show that the feasibility of the full-Newton step when the proximity $\delta(yz, \mu) < 1$.

Lemma 3.6. *Let $\delta = \delta(yz, \mu) < 1$. Then $y_+ > 0$ and $z_+ > 0$, which means that the full-Newton step is strictly feasible.*

Proof. Due to Lemma 3.5, $y_+ > 0, z_+ > 0$ if and only if $e + d_y d_z > 0$, since

$$1 + (dyd_z)_i \geq 1 - \mid (d_y d_z)_i \mid \geq 1 - \parallel d_y d_z \|_\infty, \forall i.$$

Then by (3.15), it follows that

$$1 + (d_y d_z)_i \geq 1 - \delta^2.$$

Thus $e + d_y d_z > 0$ if $\delta < 1$ and then y_+ and z_+ are strictly feasible. This completes the proof. □

For convenience, we may write

$$v_+ = \sqrt{\frac{y_+ z_+}{\mu}}.$$

It is easy to deduce that

$$v_+^2 = e + d_y d_z \Leftrightarrow y_+ z_+ = \mu(e + d_y d_z). \tag{3.16}$$

The next lemma shows the influence of the full-Newton step on the proximity measure.

Lemma 3.7. *If $\delta < 1$. Then*

$$\delta_+ := \delta(v_+; \mu) \leq \frac{\delta^2}{\sqrt{2(1 - \delta^2)}}.$$

In addition, if $\delta \leq \sqrt{\frac{3}{7}}$, thus $\delta_+ \leq \delta^2$ which means the full-Newton step converges locally quadratically through the algorithm.

Proof. By definition

$$2\delta_+ = \|v_+ - v_+^{-1}\|.$$

Due to (3.16), we have

$$v_+ = \sqrt{e + d_y d_z} \text{ and } v_+^{-1} = \frac{e}{\sqrt{e + d_y d_z}},$$

Then

$$2\delta_+ = \|\frac{d_y d_z}{\sqrt{e + d_y d_z}}\| \leq \frac{\|d_y d_z\|}{\sqrt{1 - \|d_y d_z\|_\infty}}.$$

Hence from (3.15),

$$\delta_+ \leq \frac{\delta^2}{\sqrt{2(1 - \delta^2)}}.$$

This completes the proof . □

In the following lemma, we obtain the upper bound of a duality gap after a full-Newton step.

Lemma 3.8. *Let* $\delta \leq \sqrt{\frac{3}{7}}$ *and suppose that the vectors* y_+ *and* z_+ *are obtained by using a full-Newton step, thus* $y_+ = y + \Delta y$ *and* $z_+ = z + \Delta z$ *we have*

$$y_+^T z_+ \leq 2\mu n. \tag{3.17}$$

Proof. Using (3.8) and (3.9) we have

$$\begin{aligned} y_+ z_+ &= (y + \Delta y)(z + \Delta z) \\ &= yz + y\Delta y + z\Delta z + \Delta y \Delta z \\ &= \mu \left(e + d_y d_z\right), \end{aligned}$$

then

$$y_+^T z_+ = \mu e^T \left(e + d_y^T d_z\right) = \mu \left(n + d_y^T d_z\right).$$

Next, let $\delta \leq \sqrt{\frac{3}{7}}$ then $\delta^2 \leq \frac{3}{7}$, using (3.13) we deduce that

$$y_+^T z_+ \leq \mu \left(n + \frac{6}{7}\right).$$

But as $n + \frac{6}{7} \leq 2n, \forall n \geq 1$, this gives the required result. □

In the following theorem, we investigate the influence on the proximity measure of Newton process followed by a step along the central-path.

Theorem 3.9. *Let* $\delta \leq \sqrt{\frac{3}{7}}$ *and* $\mu_+ = (1 - \theta)\mu$*, where* $0 < \theta < 1$*. Then*

$$\delta^2 \left(y_+ z_+; \mu_+\right) \leq \frac{9}{56} + \frac{\theta^2 \left(n + \frac{6}{7}\right)}{4\left(1 - \theta\right)} + \frac{15}{56}\theta.$$

In addition, if $\theta = \frac{1}{\sqrt{3n}}$ *and* $n \geq 3$*, then* $\delta \left(y_+ z_+, \mu_+\right) \leq \sqrt{\frac{3}{7}}$.

Proof. we have

$$\begin{aligned}
4\delta^2\left(y_+z_+;\mu_+\right) &= \| \sqrt{1-\theta}v_+^{-1} - \frac{1}{\sqrt{1-\theta}}v_+\|^2 \\
&= \|\sqrt{1-\theta}\left(v_+^{-1}-v_+\right) - \frac{\theta}{\sqrt{1-\theta}}v_+\|^2 \\
&= (1-\theta)\|v_+^{-1}-v_+\|^2 + \frac{\theta^2}{1-\theta}\|v_+\|^2 - 2\theta\left(v_+^{-1}-v_+\right)^T v_+ \\
&= (1-\theta)\| v_+^{-1}-v_+\|^2 + \frac{\theta^2}{1-\theta}\|v_+\|^2 - 2\theta\left(v_+^{-1}\right)^T v_+ + 2\theta v_+^T v_+ \\
&= 4\delta_+^2(1-\theta) + \frac{\theta^2}{1-\theta}\|v_+\|^2 - 2\theta n + 2\theta\|v_+\|^2.
\end{aligned}$$

Because $\left(v_+^{-1}\right)^T v_+ = n$ and $v_+^T v_+ = \| v_+\|^2$, and according Lemma 3.8 we get

$$\| v_+\|^2 = \frac{1}{\mu}y_+^T z_+ \leq \left(n+\frac{6}{7}\right),$$

which implies that

$$\delta^2\left(y_+z_+;\mu_+\right) \leq (1-\theta)\,\delta_+^2 + \frac{\theta^2\left(n+\frac{6}{7}\right)}{4(1-\theta)} + \frac{3\theta}{7}.$$

As $\delta \leq \sqrt{\frac{3}{7}}$, Lemma 3.7 implies that $\delta_+^2 \leq \frac{\delta^2}{\sqrt{2\left(1-\delta^2\right)}} = \frac{9}{56}$. Therefore, after some simplifications, we obtain

$$\delta^2\left(y_+z_+;\mu_+\right) \leq \frac{9}{56} + \frac{\theta^2\left(n+\frac{6}{7}\right)}{4(1-\theta)} + \frac{15}{56}\theta.$$

Let $\theta = \frac{1}{\sqrt{3n}}$ then $\theta^2 = \frac{1}{3n}$, this imply that

$$\delta^2\left(y_+z_+;\mu_+\right) \leq \frac{9}{56} + \frac{\frac{n+\frac{6}{7}}{3n}}{4(1-\theta)} + \frac{15}{56}\theta.$$

And since $\frac{n+\frac{6}{7}}{3n} \leq \frac{3}{7}$ for $n \geq 3$ then

$$\delta^2\left(y_+z_+;\mu_+\right) \leq \frac{9}{56} + \frac{3}{28(1-\theta)} + \frac{15}{56}\theta.$$

For $n \geq 3, \theta \in \left[0, \frac{1}{3}\right]$, we consider the following function:

$$f(\theta) = \frac{9}{56} + \frac{15}{56}\theta + \frac{3}{28(1-\theta)}.$$

As

$$f'(\theta) = \frac{1}{28(1-\theta)^2} + \frac{15}{56} > 0,$$

so f is continuous and monotone increasing on $\left[0, \frac{1}{3}\right]$. Consequently

$$f(\theta) \leq f\left(\frac{1}{3}\right) \simeq 0.4107 \leq \frac{3}{7}, \text{ for all } \theta \in \left[0, \frac{1}{3}\right].$$

Then, after the barrier parameter is update to $\mu_+ = (1-\theta)\mu$ with $\theta = \frac{1}{\sqrt{3n}}$ and if $\delta \leq \sqrt{\frac{3}{7}}$, we get $\delta(y_+ z_+; \mu_+) \leq \sqrt{\frac{3}{7}}$. This completes the proof . □

Theorem 3.9 indicates that the algorithm is well-defined since the conditions $y_+ > 0, z_+ > 0$ and $\delta(y_+ z_+; \mu_+) \leq \sqrt{\frac{3}{7}}$ hold through the algorithm.

Iteration bound

In the following lemma, we derive the upper bound for the total number of iterations produced by the algorithm.

Lemma 3.10. *Suppose that y^0 and z^0 are strictly feasible starting point such that $\delta\left(y^0 z^0; \mu_+\right) \leq \sqrt{\frac{3}{7}}$ for each $\mu_0 > 0$. Moreover, let y^k and z^k be the vectors obtained after k iterations. Then the inequality $\left(y^k\right)^T z^k \leq \epsilon$ is satisfied if*

$$k \geq \frac{1}{\theta} \log\left(\frac{2n\mu_0}{\epsilon}\right).$$

Proof. From (3.13), it follows that:

$$\left(y^k\right)^T z^k \leq 2n\mu_k = 2n(1-\theta)^k \mu_0.$$

Then the inequality $\left(y^k\right)^T z^k \leq \epsilon$ holds if $2n(1-\theta)^k \mu_0 \leq \epsilon$. We take logarithms, so we may write

$$k \log(1-\theta) \leq \log\epsilon - \log(2n\mu_0).$$

We know that $-\log(1-\theta) \geq \theta$ for $0 \leq \theta \leq 1$. So the inequality holds only if

$$k\theta \geq \log\epsilon - \log(2n\mu_0) = \log\left(\frac{2n\mu_0}{\epsilon}\right).$$

This completes the proof. □

We end this subsection with a theorem that gives the iteration bound of the algorithm .

Theorem 3.11. *Using the defaults* $\theta = \frac{1}{\sqrt{3n}}$ *and* $\mu_0 = \frac{1}{2}$, *we obtain that the algorithm given in Fig 3.1 at most requires at* $\mathcal{O}\left(\sqrt{n}\log\frac{n}{\epsilon}\right)$ *iterations for getting an* $\epsilon-$ *approximated solution of* $\mathcal{P}$*-LCP.*

Proof. Let $\theta = \frac{1}{\sqrt{3n}}$ and $\mu_0 = \frac{1}{2}$, by using Lemma 3.8, the result follows. □

3.4 Numerical results

In this section, we present some numerical problems of different sizes for testing the effectiveness of Algorithm 3.2.3, each example is followed by a table containing the computational results obtained by the algorithm. All programs were implemented in **MATLAB** R2016a on a personal PC with 1.40 GHZ AMD E1-2500 APU Radeon(TM) HD Graphic, 8 GB memory and Windows 10 operating system. In the implementation, we set $\epsilon = 10^{-6}$ and different values of the barrier parameter μ_0, θ are used in order to improve the performances of our algorithm. Here, the starting point and the unique solution of $\mathcal{P}$-LCP are denoted by (y^0, z^0) and $(y^\star, z^\star)$, respectively. The unique solution of SCQO is then computed via $x^\star = Ay^\star$. The number of iterations required and the elapsed time by the algorithm are denoted by "Iter" and "CPU", respectively.

Example 3.4.1. *Let us consider the problem of SCQO where*

$$Q = \begin{bmatrix} 6 & 0.5 & 6 & 1 & 3 & 2 & -2 & 0 & 0 & 4 \\ 0.5 & 8.25 & -3.5 & 1 & -3.5 & 2 & 1.5 & -2.5 & -6 & -4.5 \\ 6 & -3.5 & 38 & -1.5 & 7 & -6 & -1 & 2.5 & 16 & 3 \\ 1 & 1 & -1.5 & 8.25 & -2 & 2 & -1.5 & 0. & 0 & -6 \\ 3 & -3.5 & 7 & -2 & 11 & -4 & -1 & -0.5 & 0 & -5 \\ 2 & 2 & -6 & 2 & -4 & 8 & -4 & 0 & -2.5 & 8 \\ -2 & 1.5 & -1 & -1.5 & -1 & -4 & 7 & -4 & 1 & -4 \\ 0 & -2.5 & 2.5 & 0 & -0.5 & 0 & -4 & 7.25 & -0.5 & 4 \\ 0 & -6 & 16 & 0 & 0 & -2.5 & 1 & -0.5 & 16.25 & 9.5 \\ 4 & -4.5 & 3 & -6 & -5 & 8 & -4 & 4 & 9.5 & 41 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ -1 & -2 & 0 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ -1 & -1 & -2 & 0 & 3 & 3 & 3 & 3 & 3 & 3 \\ -1 & -1 & -1 & -2 & 0 & 3 & 3 & 3 & 3 & 3 \\ -1 & -1 & -1 & -1 & -2 & 0 & 3 & 3 & 3 & 3 \\ -1 & -1 & -1 & -1 & -1 & 2 & 0 & 3 & 3 & 3 \\ -1 & -1 & -1 & -1 & -1 & -1 & -2 & 0 & 3 & 3 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -2 & 0 & 3 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -2 & 0 \end{bmatrix},$$

$b = (-1, -4, 4, -2, 1, 10, 4, 0, 5, -11)^T$.
The unique solution of the $\mathcal{P}$-LCP is given by

$$y^\star = (0, 0.09, 0, 0, 0.0549, 0, 0, 0, 0, 0,)^T,$$

$$z^\star = (4.3634, 0, 1.5624, 5.5552, 0, 19.9944, 9.3423, 69.6119, 86.007, 48.1573)^T.$$

Then the unique minimizer of this problem is:

$$x^\star = Ay^\star = (0.27, 0.1646, -0.0154, 0.0746, -0.09, -0.1998, -0.1449, ..., -0.1449)^T.$$

The obtained number of iterations and the elapsed time via specified different values of μ_0 and θ are summarized in the Table 3.1:

$\mu_0 \longrightarrow$	0.5		0.05		0.005		0.0005	
$\theta \downarrow$	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
0.7	13	0.05229	11	0.04637	9	0.04747	8	0.04486
0.5	23	0.05135	19	0.04978	16	0.04944	13	0.04537
$\frac{1}{\sqrt{3n}}$	77	0.06364	66	0.05992	54	0.05507	43	0.05120

Table 3.1: The numerical results with $\theta = 0.7, \theta = 0.5$ and $\theta = \frac{1}{\sqrt{3n}}$, after the algorithm reaches $n\mu \leq 10^{-6}$.

Example 3.4.2. *Consider the following SCQO problem where Q and A are given by:*

$$Q = \begin{bmatrix} 3 & 1 & \dots & 0 \\ 1 & 3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 3 \end{bmatrix}, A = \begin{bmatrix} 1 & 2 & \dots & n \\ 0 & 1 & \ddots & n-1 \\ \vdots & \ddots & \ddots & 2 \\ 0 & \dots & 0 & 1 \end{bmatrix},$$

and

$$b = (A^T)^{-1}(e - A^T Q A e).$$

The strictly feasible initial point taken in the algorithm is :

$$y^0 = \frac{1}{2}(1, 1, \dots, 1)^T > 0, \; z^0 = My^0 + q > 0.$$

A solution of $\mathcal{P}$-LCP is given by:

$$y^\star = (0, 1.2154, 1.0615, 1, \dots, 1)^T, \; z^\star = (0.1846, 0, 0, \dots, 0)^T.$$

For example if $n = 10$, then the unique minimizer of this problem is

$$x^\star = Ay^\star = (54.6154, 45.3385, 36.0615, 28, 21, 15, 10, 6, 3, 1)^T.$$

The details of obtained numerical results with different values of μ_0 are in the table 3.2.
The details of obtained numerical results with different sizes of n, relaxed values of μ_0 and $\theta = \frac{1}{\sqrt{3n}}$ are in the Table 3.3.

$\mu_0 \longrightarrow$	0.5		0.05		0.005		0.0005	
$\theta \downarrow$	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
$\frac{1}{2\sqrt{n}}$	90	0.0345	77	0.0271	63	0.0229	50	0.0177
$\frac{1}{\sqrt{3n}}$	77	0.0227	66	0.0213	54	0.0169	43	0.0128

Table 3.2: The numerical results, after the algorithm reaches $n\mu \leq 10^{-6}$.

$\mu_0 \longrightarrow$	0.5		0.05		0.005		0.0005	
size $n \downarrow$	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
20	117	0.0559	100	0.0483	84	0.0446	67	0.0255
50	200	0.2569	173	0.1967	146	0.1554	119	0.1336
100	299	2.0932	260	1.8765	221	1.4960	182	1.3049
200	442	13.1413	387	11.4398	332	10.0121	277	8.3507
1000	624	99.1524	552	95.9942	481	93.3013	409	90.7823

Table 3.3: The numerical results with $\theta = \frac{1}{\sqrt{3n}}$, after the algorithm reaches $n\mu \leq 10^{-6}$

Example 3.4.3. *Let the matrixes of problem SCQO:*

$$Q = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & \ddots & 0 \\ 0 & \ddots & \ddots & 1 \\ 0 & \dots & 1 & 2 \end{pmatrix}; A = \begin{pmatrix} 3 & 0 & \dots & 0 \\ 0.5 & 3 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ -1 & \dots & 0.5 & 3 \end{pmatrix}$$

$$b = -2(A^{-1})^T 2Me, \text{where } M = A^T QA$$

The strictly feasible starting point taken in the algorithm is :

$$y^0 = (3, 3, \dots 3)^T.$$

A solution of $\mathcal{P}$-LCP is given by:

$$y^* = (4, 4, \dots\dots\dots, 4)^T; z^* = (0, 0, \dots\dots\dots 0)^T.$$

The solution of SCQO is $x^* = Ay^* = (12, 14, 10, 6, 2, -6, -10, \dots, -78)$
Numerical results for the problem SCQO with some sizes of n, and with $\theta = \frac{1}{\sqrt{3n}}$ are in the Table 3.4.

$\mu_0 \longrightarrow$	0.5		0.05		0.005		0.0005	
n↓	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
5	39	0.071465	33	0.005415	25	0.003894	21	0.002997
10	61	0.02272	52	0.01419	43	0.009145	34	0.011438
20	94	0.03094	52	0.01419	43	0.009145	34	0.011438
100	120	0.05828	103	0.04782	87	0.0416249	84	0.036102

Table 3.4: The numerical results, after the algorithm reaches $n\mu \leq 10^{-6}$.

Example 3.4.4. *Let the matrices:*

$$Q = \begin{bmatrix} 4 & -1 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 4 & \ddots & \ddots & \vdots & 0 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 4 & -1 & \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & -1 & 4 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 5 & -2 & 0.25 & \cdots & 0.5 \\ 0 & 1 & \ddots & \ddots & \vdots & -2 & 5 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 & 0.25 & \ddots & \ddots & \ddots & 0.25 \\ \vdots & \ddots & \ddots & 1 & 0 & \vdots & \ddots & \ddots & 5 & -2 \\ 0 & \cdots & 0 & 0 & 1 & 0.25 & \cdots & 0.25 & -2 & 5 \end{bmatrix},$$

and

$$A = \begin{bmatrix} -2 & -1 & 0.5 & \cdots & 0.5 & 1 & 0 & 0 & \cdots & 0 \\ 4 & -2 & \ddots & \ddots & \vdots & 0 & 1 & \ddots & \ddots & \vdots \\ 3 & \ddots & \ddots & \ddots & 0.5 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -2 & -1 & \vdots & \ddots & \ddots & 1 & 0 \\ 3 & \cdots & 3 & 4 & -2 & 0 & \cdots & 0 & 0 & 1 \\ -1 & 0 & 0.1 & \cdots & 0 & -2 & -1 & 0.5 & \cdots & 0.5 \\ 0 & -1 & \ddots & \ddots & \vdots & 4 & -2 & \ddots & \ddots & \vdots \\ 0.1 & \ddots & \ddots & \ddots & 0.1 & 3 & \ddots & \ddots & \ddots & 0.5 \\ \vdots & \ddots & \ddots & -1 & 0 & \vdots & \ddots & \ddots & -2 & -1 \\ 0 & \cdots & 3 & 0.1 & -1 & 3 & \cdots & 3 & 4 & -2 \end{bmatrix},$$

and

$$b = -2QAe.$$

The strictly feasible initial point taken in the algorithm is:

$$y^0 = (5, 5, \ldots, 5)^T,\ z^0 = My^0 + q.$$

The solution of $\mathcal{P}$-LCP is

$$y^\star = (2, 2, \ldots, 2)^T,\ z^\star = (0, 0, \ldots, 0)^T.$$

Then the unique minimizer of this problem is computed via $x^\star = Ay^\star$.
The numerical results with different size of n are summarized in next Table.

$\mu_0 \longrightarrow$	0.5		0.05		0.005		0.0005	
n↓	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
10	13	0.0347	11	0.0342	9	0.0335	8	0.0315
50	15	0.0517	13	0.0492	11	0.0465	9	0.0449
100	15	0.1255	13	0.10829	11	0.1011	9	0.0944
200	16	0.5258	14	0.3967	12	0.3443	10	0.2927
1000	17	35.2648	15	27.3913	13	26.9554	11	23.1565
1600	18	176.1260	16	158.1731	14	98.6910	12	80.8734

Table 3.5: The numerical results, after the algorithm reaches $n\mu \leq 10^{-6}$.

3.5 Comparative study

In this section we give a comparative study between the numerical results obtained by the algorithms (a two-steps fixed-point algorithm and The interior-point algorithm). This comparison takes place through examples. It should be mentioned here that depending solely on the CPU time in measuring the performance of an algorithm might be misleading. especially, if the operating system is multiprogrammed the CPU time becomes longer and less reliable. Consequently, the number of iterations should be used together with the CPU time to get a better insight into the performances of the different algorithms.

Example 3.5.1. *Consider the SCQO problem , where $Q, A \in \mathbb{R}^{10\times 10}$ and b are given by:*

$$Q = \begin{bmatrix} 100 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 100 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 100 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 100 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 100 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 100 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 & 100 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 10 & 100 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 100 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 100 \end{bmatrix}$$

$$A = \begin{bmatrix} 0.2 & -1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ -1 & 0.2 & -1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ 0.05 & -1 & 0.2 & -1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ 0.05 & 0.05 & -1 & 0.2 & -1 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\ 0.05 & 0.05 & 0.05 & -1 & 0.2 & -1 & 0.05 & 0.05 & 0.05 & 0.05 \\ 0.05 & 0.05 & 0.05 & 0.05 & -1 & 0.2 & -1 & 0.05 & 0.05 & 0.05 \\ 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & -1 & 0.2 & -1 & 0.05 & 0.05 \\ 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & -1 & 0.2 & -1 & 0.05 \\ 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & -1 & 0.2 & -1 \\ 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & -1 & 0.2 \end{bmatrix}$$

$$b = [1, -2, -3, 0, 2, 0, -1, 0, 1, 10]^T .$$

The starting point is taken as:

$$x_0 = [1, 1, 1, 1, 1, 1, 1, 1, 1, 1]^T .$$

The unique solution of this problem is given by:

$$x^\star = [0.0032, 0.0032, 0.0032, 0.005, -0.0029, 0.0014, -0.0033, -0.0509, 0.0062, -0.0502]^T .$$

The obtained numerical results are stated in Table 3.6.

Algorithms→	**Algorithm. 2.2.1**	**IP Algorithm**
Iter	505	11
CPU(s)	1.163234	0.114184

Table 3.6: The numerical results for Example 3.5.1

Example 3.5.2. *Consider the SCQO problem , where $Q, A \in \mathbb{R}^{n\times n}$ and b are given by:*

$$Q = \begin{bmatrix} 3 & -2 & 0 & \cdots & 0 & 0 \\ -2 & 3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 3 & \cdots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 3 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 3 \end{bmatrix}, A = \begin{bmatrix} 1 & 2 & 0 & \cdots & 0 & 1/n \\ 0 & 1 & 2 & \cdots & 0 & 1/(n-1) \\ 0 & 0 & 1 & \cdots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 2 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 2 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

$$b = (A^T)^{-1}(e - A^T Q A e).$$

The starting point is taken as: $x_0 = (1/2)e$.
The numerical results with different size of n are summarized in Table 3.7

Algorithms→	**Algorithm. 2.2.1**		**IP Algorithm**	
Size n	Iter	CPU(s)	Iter	CPU(s)
10	93	0.029143	19	0.031565
20	111	0.35727	20	0.049872
50	112	0.055592	22	0.054265
100	112	0.515329	23	0.233119
500	112	14.174202	25	8.284128
1000	112	102.07466	26	49.446352

Table 3.7: The numerical results for Example 3.5.2.

Example 3.5.3. *The matrices $Q, A \in \mathbb{R}^{n\times n}$ and the vector $b \in \mathbb{R}^n$ of the SCQO*

problem are given by:

$$Q = \begin{bmatrix} 2n & 1/n & 0.125 & \cdots & 0.125 & 0.125 \\ 1/n & 2n & 1/n & \cdots & 0.125 & 0.125 \\ 0.125 & 1/n & 2n & \cdots & 0.125 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1/n & 0.125 \\ 0.125 & 0.125 & 0.125 & \cdots & 2n & 1/n \\ 0.125 & 0.125 & \cdots & 0.125 & 1/n & 2n \end{bmatrix}.$$

$$A = \begin{bmatrix} 4 & -1 & 0.3 & \cdots & 0.3 & 0.3 \\ -1 & 4 & -1 & \cdots & 0.3 & 0.3 \\ 0 & -1 & 4 & \cdots & 0.3 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 & 0.3 \\ 0 & 0 & 0 & \cdots & 4 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 4 \end{bmatrix}$$

$$b = [5, 0, \ldots, 5, 0]^T$$

The starting point is taken as: $x_0 = (1/2)e$.
The numerical results with different size of n *are summarized in Table 3.8.*

Algorithms$\rightarrow$	**Algorithm. 2.2.1**		**IP Algorithm**	
Size n	Iter	CPU(s)	Iter	CPU(s)
10	570	0.091436	11	0.032565
20	1248	0.235727	12	0.035872
50	991	0.671777	13	0.047478
100	5520	0.185329	13	0.180890
500	11411	36.174202	15	6.884128
1000	$---$	$---$	16	31.287818
1500	$---$	$---$	16	112.885273

Table 3.8: The numerical results for Example 3.5.3

Example 3.5.4. *Consider the following SCQO problem where Q and A are given by:*

$$
\begin{aligned}
Q &= \begin{bmatrix}
4 & -1 & 0 & \cdots & 0 & 0.2 & 0 & 0 & \cdots & 0 \\
-1 & 4 & \ddots & \ddots & \vdots & 0 & 0.2 & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 & 0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & 4 & -1 & \vdots & \ddots & \ddots & 0.2 & 0 \\
0 & \cdots & 0 & -1 & 4 & 0 & \cdots & 0 & 0 & 0.2 \\
0.2 & 0 & 0 & \cdots & 0 & 5 & -2 & 1/4 & \cdots & 0 \\
0 & 0.2 & \ddots & \ddots & \vdots & -2 & 5 & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 & 1/4 & \ddots & \ddots & \ddots & 1/4 \\
\vdots & \ddots & \ddots & 0.2 & 0 & \vdots & \ddots & \ddots & 5 & -2 \\
0 & \cdots & 0 & 0 & 0.2 & 0 & \cdots & 1/4 & -2 & 5
\end{bmatrix} \\
A &= \begin{bmatrix}
-2 & -1 & 1/5 & \cdots & 1/5 & 1 & 0 & 0 & \cdots & 0 \\
0.4 & -2 & \ddots & \ddots & \vdots & 0 & 1 & \ddots & \ddots & \vdots \\
3 & \ddots & \ddots & \ddots & 1/5 & 0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & -2 & -1 & \vdots & \ddots & \ddots & 1 & 0 \\
3 & \cdots & 3 & 0.4 & -2 & 0 & \cdots & 0 & 0 & 1 \\
-6 & 0 & 2.5 & \cdots & 0 & -4 & -2 & 0.4 & \cdots & 0.4 \\
0 & -6 & \ddots & \ddots & \vdots & 0.8 & -4 & \ddots & \ddots & \vdots \\
0.1 & \ddots & \ddots & \ddots & 2.5 & 6 & \ddots & \ddots & \ddots & 0.4 \\
\vdots & \ddots & \ddots & -6 & 0 & \vdots & \ddots & \ddots & -4 & -2 \\
0 & \cdots & 0.1 & 0 & -6 & 6 & \cdots & 6 & 0.8 & -4
\end{bmatrix} \\
b &= (A^T)^{-1}(e - A^T Q A e).
\end{aligned}
$$

The strictly feasible initial point taken in the algorithm is:

$$x^0 = (5, 5, \ldots, 5)^T.$$

The numerical results with different size of n are summarized in Table 3.9.

3.6 Contributions and some remarks

Through the numerical tests that we carried out on examples of different dimensions, we notice that:

Algorithms→	**Algorithm. 2.2.1**		**IP Algorithm**	
Size n	Iter	CPU(s)	Iter	CPU(s)
10	8	0.037128	7	0.032268
20	9	0.045672	7	0.033391
50	11	0.157435	8	0.039197
200	19	0.552646	8	0.236422
500	7	3.453799	9	4.335142
1000	4	9.991263	9	20.115544
2000	3	54.420731	9	138.211867

Table 3.9: The numerical results for Example 3.5.4

1. In terms of robustness, there appears to be no significant difference between the two algorithms tested, These algorithms are simple and easy to implement and give the solution of the considered problems. although IP Algorithm is slightly more reliable.

2. Among advantages of interior point methods are Theoretical efficiency: These methods have polynomial convergence and good numerical behavior. Unlike the Algorithm. 2.2.1 converges globally and linearly.

3. In terms of function evaluations (or iterations), we observe some significant differences between the two algorithms. Algorithm. 2.2.1 requires more iterations overall than the interior-point method. As expected, the last one typically perform between 5 and 20 iterations to reach convergence. Since the geometry of bound constraints is simple, only nonlinearity in the objective function cause interior-point methods to perform a large number of iterations. Algorithm. 2.2.1 sometimes converges in a very small number of iterations (e.g. 3.5.4), but on other problems it requires significantly more iterations than the interior-point algorithms.

4. Interior point methods required number of iterations grows very slowly with the size of the problem, which makes it so attractive for large-scale optimization problems. these methods make it possible to solve problems of very large size that no other known algorithm could process in an acceptable time.

5. It is mentioned that interior point methods suffer to obtain a strictly feasible starting point and close to central-path to initiate their associate algorithm.

This because the used neighborhood of the central-path is narrow since the threshold ($0 < \tau < 1$).

6. The update barrier parameter θ depends on the size n of the problem. If n is large enough this leads to a slow convergence of the IP algorithm.

7. CPU Time, the computation times of these methods are competitive. It is clear from the Tables that Algorithm. 2.2.1 requires the largest amount of computing time among all the solvers. This test set contains a significant number of problems with ill-conditioned Hessians and the step computation of Algorithm. 2.2.1 is dominated by the large number of steps performed.

General conclusion and future work

In this thesis, a convex quadratic programming problem under simplicial cone constraints were studied, and via its K.K.T optimality conditions is transformed into a $\mathcal{P}$-LCP. For its numerical solution, we have proposed two different methods. In the first part, the $\mathcal{P}$-LCP is reduced to finding the unique solution of an absolute value equation AVE. For solving this AVE we applied a new two-steps Picard's iterative fixed point iteration. In particular, the sufficient conditions for the convergence of our algorithm are studied. The obtained numerical results deduced from the testing examples illustrate that the suggested algorithm is efficient and valid to solve the SCQO problems. In the second part, for solving $\mathcal{P}$-LCP we have implemented a feasible full-step primal-dual path following interior-point algorithm. The advantage of this algorithm is to use only full-Newton steps to obtain a solution, i.e., the step-size $\alpha = 1$. The algorithm produces a sequence of strictly feasible iterations and near to the central path. Here, we suggested new defaults such as $\delta \leq \sqrt{\frac{3}{7}}$ and $\theta = \frac{1}{\sqrt{3n}}$ for its well-definiteness and its convergence to the unique minimizer of SCQOs. Further, its best iteration bound is derived. The obtained numerical results illustrate that the algorithm is efficient and valid to solve the SCQO problems. We had end this Second part of the thesis, with a comparison of our obtained numerical results.

future works

We conclude by pointing out a few research topics that one may consider for extending the work in this thesis.

- The generalization of our convergence guarantees for two-steps Picard's method in chapter 2 with A is singular matrix or Q is symmetric positive semi-definite .
- There are many more methods that have not been discussed in this thesis and therefore it could be extended to incorporate other algorithms available for AVE.
- One topic of interest would be extending these techniques for solving general convex QPs.

- Another interesting topic of research in the future is to solve the SCQOs by introducing the active set methods and to compare our full-step primal-dual path-following interior point algorithm with active set methods.
- The development of infeasible interior-point algorithm based on the analysis (feasible) given in the second part seems to be an interesting.
- Introduces a new primal-dual modified barrier method for solving SCQOs in starting from any given feasible point.

Bibliography

[1] M. Achache. A new parameterized kernel function for LO yielding the best known iteration bound for a large-update interior point algorithm. Afrika Matematika. 27(3)(2016), 591-601.

[2] M. Achache. A new primal-dual path-following method for convex quadratic programming. Computational and Applied Mathematics. 25(1)(2006), 97-110.

[3] M. Achache. Complexity analysis of an interior point algorithm for the semidefinite optimization based on a Kernel function with a double barrier term. Acta Mathematica Sinica. Vol. 31, No. 3.(2015), 543-556.

[4] M. Achache. Complexity analysis and numerical implementation of a short-step primal-dual algorithm for linear complementarity problems. Applied Mathematics and Computation. 216(7)(2010), 1889-1895.

[5] M. Achache and M. Goutali. Complexity analysis and numerical implementation of a full-Newton step interior-point algorithm for LCCO. Numerical Algorithms. 70:(2015), 393-405.

[6] M. Achache, N. Boudiaf. Complexity analysis of primal-dual algorithms for semidefinite linear complementarity problems. Revue d'Analyse Numérique et de Théorie de l'Approximation. 40(2)(2011), 95-106.

[7] M. Achache, N.Tabchouche. A full-Newton step feasible interior-point algorithm for monotone horizontal linear complementarity problems, Optimization Letters, (2018), 1-19.

[8] M. Achache, On the unique solvability and numerical study of absolute value equations. J. Numer. Anal. Approx. Theory, vol. 48 (2019) no. 2, 112-121.

[9] M. Achache, N. Anane. On unique solvability and Picard's iterative method for absolute value equations. Bulletin of the Transilvania University of Brasov. Vol 13(62), No. 1 - 2020.

[10] M. Achache and N. Hazzam. Solving absolute value equations via linear complementarity and interior-point methods. Journal of Nonlinear Functional Analysis. Article. ID 39: (2018), 1-10.

[11] N. Anane, M. Achache. Preconditioned conjugate gradient methods for absolute value equations J. Numer. Anal. Approx. Theory, vol. 48 (2020) no. 1, 3-14.

[12] Y. Q. Bai, M. El Ghami and C. Roos. A Comparative study of kernel functions for primal-dual interior point algorithms in linear optimization. SIAM J. OPTI, Vol. 15, (2004), No. 1. 101-128.

[13] J.G. Barrios, O.P. Ferreira and S.Z. Németh. A semi-smooth Newton method for solving convex quadratic programming problem under simplicial cone constraints. arXiv: 1503.0275 v1 [math. OC] (2015).

[14] J.E. Bonnans and C.C. Gonzaga. Convergence of interior-point algorithms for monotone linear complementarity problem. Mathematics of Operations Research. 21(1)(1996), 1-15.

[15] R. W. Cottle, G. Dantzig. Complementary pivot theory of mathematical programming. Lin. Alg. Appl., 1,(1968), 103-125.

[16] R. W. Cottle, J. S. Pang and R. E. Stone. The Linear Complementarity Problem. Academic Press. New-York (1992).

[17] Zs. Darvay, New interior-point algorithms in linear programming, Adv. Model. Optim. 5 (2003), 51-92.

[18] Zs. Darvay, T. Illes, and Cs. Majoros. Interior-point algorithm for sufficient LCPs based on the technique of algebraically equivalent transformation. Optim. Lett., 2020.

[19] D. Den Hertog. Interior Point Approach to Linear, Quadratic and Convex Programming, volume 277 of Mathematics and its Applications. Kluwer Academic Publishers, Dordrecht, The Netherlands, 1994.

[20] M. El Ghami, I. D. Ivanov, C. Roos and T. Steihaug. A polynomial-time algorithm for LO based on generalized logarithmic barrier functions. International Journal of Applied Mathematics. 21 (1)(2008), 99-115.

[21] P. E. Gill and E. Wong. Methods for convex and general quadratic programming. Math. Prog. Comp., 7(1): (2015), 71-112,

[22] R. Fletcher, Practical Methods of Optimization, John Wiley and Sons, New-York, 1987.

[23] R. Fletcher. A general quadratic programming algorithm. J. Inst. Math. Applics., (1971), 7:76-91.

[24] O.P. Ferreira and S.Z. Németh. Projection onto simplicial cones by a semi-smooth Newton method. Optimization Letters. 9(4) (2015), 731-741.

[25] Haddou, M., Migot, T., Omer, J.: A generalized direction in interior point method for monotone linear complementarity problems. Optim. Lett. 13(1), (2019), 35-53

[26] M. Hladick. Bounds for the solution of absolute value equations. Computational Optimization and Applications. 69(1): (2018),243-266.

[27] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, 1995.

[28] S.L. Hu, Z.H. Huang, Q. Zhang. A generalized Newton method for absolute value equations associated with second order cones. J. Comput. Appl. Math. **235**(2011), 1490-1501.

[29] Z.H. Huang. Polynomiality of high-order feasible interior-point method for solving the horizontal linear complementarity problems. Journal of System Science and Mathematical Sciences. 20(4): (2000), 432-438

[30] J. Iqbal, M.A. Noor, K.I. Noor, On iterative method for solving absolute value equations, Optimiz. Lett., 6 (2012), pp. 1027-1033.

[31] P. K. Kaluarachchi. Full-Newton-step interior-point method for the linear Complemontarity Problems Master of Science. Georgia Southern University, (2011).

[32] N. K. Karmarkar. A new Polynomial time Algorithm for Linear Programming. Combinatorica 4 (4)(1984), 373-395.

[33] Y. Ke. The new iteration algorithm for absolute value equation. Applied Mathematics Letters. 99 (2020). ISSN 0893-9659.

[34] S. Ketabchi, H. Moosaei, An efficient method for optimal correcting of absolute value equations by minimal changes in the right hand side, Comput. Math. Appl., 64 (2012), pp. 1882-1885.

[35] M. Khaldi, M. Achache. An interior-point algorithm for simplicial cone constrained convex quadratic optimization. **Advances in Mathematics: Scientific Journal,** Vol. 12 (2023), No. 1, 73-89.

[36] M. Kojima, N. Megiddo, T. Noma, and A. Yoshise, A primal-dual interior point algorithm for linear programming, In : N. Megiddo (Ed.), Progress in Mathematical Programming ; Interior Point Related Methods, 10 Springer Verlag, New York,(1989), 29-47.

[37] M. Kojima, N. Megiddo, T. Noma and A. Yoshise, A unified Approach to Interior Point Algorithms for Linear Complementarity Problems, Volume 538 of Lecture Notes in Computer Science, Springer Verlag, Berlin, (1991).

[38] Kreyszig, E., Introductory functional analysis with applications, John Wiley Sons, New-York, London, Sydney. (1978).

[39] L. Lehmann, M. Radons, M. Rump and CH. Strom. Sign controlled solvers for the absolute value equation with an application to support vector machines. (2010).

[40] C.X. Li. A Modified generalized Newton method for absolute value equations. J. Optim. Theory Appl. (2016), 1055-1059.

[41] Z. Liu, W Sun and F Tian. A full-Newton step infeasible interior-poit algorithm for linear programming based on a Kernel function. Appl Math Optim (2009).

[42] T. Lotfi and H. Veiseh. A note on unique solvability of the absolute value equations. Journal of Linear and Topological Algebra. Vol 2(2): (2013), 77-81.

[43] O.L. Mangasarian. A generalized Newton method for absolute value equations. Optimization Letters. **3**(2009),101-108.

[44] O.L. Mangasarian, and R.R. Meyer. Absolute value Equations. Linear Algebra and Its Applications, **419**(2006), 359-367.

[45] O.L. Mangasarian. Linear complementarity as absolute value equation solution. Optimization Letters. **8**(2014),1529-1534.

[46] H. Mansouri, M. PirhajiA Polynomial Interior-Point Algorithm for Monotone Linear Complementarity Problems Published online: 11 October 2012.

[47] H. Mansouri, M. Zangiabadi and M. Pirhaji. A polynomial-time interior-point algorithm for semidefinite linear complementarity problems. Mathematical Analysis and Optimization. (1): (2016), 37-45.

[48] Markowitz, H.M. (1959). Portfolio Selection: Efficient Diversification of Investments. New York: John Wiley & Sons. (reprinted by Yale University Press, (1970).

[49] T. Migot, L. Abdellah, M. Haddou, Solving absolute value equation using complementarity and smoothing functions, Comput.Appl.Math.,327(2018), pp. 196-207.

[50] Monteiro, R.D.C. and Adler, I., Interior path-following primal-dual algorithms. Part I: Linear programming, Mathematical Programming, 44(1989), 27-41.

[51] Y. E. Nesterov and A. S. Nemirovskii, Interior Point Polynomial Algorithms in Convex Programming : Theory and Applications, SIAM, Philadelphia, PA, 1994.

[52] M. A. Noor, J. Iqbal, S. Khattri and E. Al-Said. A new iterative method for solving absolute value equations. Int. J. Phy. Sci.6(7), (2011), 1793-1797.

[53] M. A. Noor, J. Iqbal, K. I. Noor, and E. Al-Said. On an iterative method for solving absolute value equations. Optim. Lett., (2011).

[54] J. Peng, C. Roos, T. Terlaky. New complexity analysis of the primal-dual Newton method for linear optimization. Ann. Oper. Res. 99: (2000), 23–39

[55] J. Peng, C. Roos and T. Terlaky. Self-regular functions and new search directions for linear and semidefinite optimization. Mathematical Programming. 93(1): (2002),129-171.

[56] B.T. Polyak. Introduction to optimization. Optimization Software Inc. Publication Division. New-York. (1987).

[57] O. Prokopyev. On equivalent reformulations for absolute value equations.Comput. Optim. Appl., 44, (2009), 363–372.

[58] J. Rohn, On unique solvability of the absolute value equations, Optimiz. Lett., 3 (2009), pp. 603-606

[59] J. Rohn, An algorithm for computing all solutions of an absolute value equation, Optimiz. Lett., 6 (2012), pp. 851-856.

[60] J. Rohn. A theorem of the alternatives for the equations $Ax - B|x| = b$. Linear and Multilinear Algebra. 52 (6): (2004), 421-426.

[61] J. Rohn. A theorem of the alternatives for the equations $Ax - |B||x| = b$. Optimization Letters (6): (2012), 585-591.

[62] Y. Shi, Modified quasi-Newton methods for solving systems of linear equations, Int. J. Contemp. Math. Sci., 15 (2007), pp. 737-744.

[63] I. Touil, Dj. Benterki and A. Yassine. A feasible primal-dual interior point method for linear semidefinite programming. Computation and Applied Mathematics. 312: (2017), 216-230.

[64] N. Ujevic, A new iterative method for solving linear systems, Appl. Math. Comput.,179 (2006), pp. 725-730.

[65] S.L. Wang and C.X. Li. A note on unique solvability of the absolute value equation. Optimization Letters. https://doi.org. 10.1007/s11590-019-01478-x. (2019).

[66] G.Q. Wang and Y.Q. Bai. A new primal-dual path-following interior point algorithm for semidefinte optimization. Journal of Mathematical Analysis and Applications. 353: (2009), 339-349.

[67] G. Wang and D. Zhu. A unified kernel function approach to primal-dual interiorpoint algorithms for convex quadratic SDO. Numerical Algorithms. 57: (2011), 537-558.

[68] S.J. Wright. Primal-Dual Interior-Point Methods. Copyright by SIAM. (1997).

[69] S. L. Wu, C. X. Li, The unique solution of the absolute value equations, Appl. Math. Lett., 76 (2018), 195–200.

[70] P. Wolfe, The Simplex Method for Quadratic Programming, Econometrica, Vol.27, (1959), pp. 382-398.

[71] E. Wong. Active-Set Methods for Quadratic Programming. PhD thesis, Department of Mathematics, University of California San Diego, La Jolla, CA, 2011.

[72] Y. Ye. Interior-point algorithms. Theory and Analysis. Jhon-Wiley. Sons, Chichester, UK (1997)

[73] L. Yong. Iterative method for a class of linear complementarity problems. International Conference on Information Computing and Applications. 1: (2010), 390-398.

[74] M.W. Zhang. A large-update interior-point algorithm for convex quadratic semidefinite optimization based on a new kernel function. Acta Mathematica Sinica. Vol. 28, No. 11. 2313-2328.

[75] F. Zhang. Matrix theory: Basic results and techniques, Springer-Verlag New York Inc., (1999).

[76] L. Zhang and Y. Xu. A full-Newton step primal-dual interior-point algorithm for linear complementarity problem. Journal of Information and Computing Science. 5(4): (2010), 305-313.

[77] G. Zoutendijk, Nonlinear Programming, Computational Methods, in: Integer and Nonlinear Programming, J. Abadie, ed., North- Holland, Amsterdam, 1970, pp. 37-86.

ملخص :

نقدم في هذه الأطروحة دراسة نظرية و عددية لحل مسألة تربيعية محدبة مقيدة بمخروط بسيط (SCQO). في الجزء الأول ، ومن خلال شروط الأمثلية KKT فان حل SCQO يكافئ إيجاد الحل الوحيد لمعادلات ذات القيمة المطلقة (AVE). لحل هذه الاخيرة طبقنا طريقة جديدة للنقطة الثابتة التكرارية لـ Picard مكونة من خطوتين. على وجه الخصوص ، يتم دراسة الشروط الكافية لتقارب الخوارزمية حيث توضح النتائج العددية التي تم الحصول عليها أن الخوارزمية فعالة وصالحة لحل مشاكل SCQO. في الجزء الثاني ، تم استغلال طريقة النقطة الداخلية لحل SCQOs عبر حل P-LCP. علاوة على ذلك ، يتم حساب التكلفة الحدودية لها ويتم إعطاء النتائج العددية لإظهار فعالية هذه الأخيرة. ثم نقدم دراسة مقارنة بين النتائج العددية للخوارزميتان من خلال بعض الأمثلة.

الكلمات المفتاح : البرمجة التربيعية ، مسائل التكامل الخطي ، معادلات القيمة المطلقة ، طرق بيكارد التكرارية ، طرق النقطة الداخلية ، التكلفة الحدودية.

Résumé :

Dans cette thèse, nous présentons une analyse théorique et une étude numérique pour la résolution d'un problème d'optimisation quadratique convexe sous le cône simplicial (SCQO). Dans la première partie, et à travers ses conditions d'optimalité de K.K.T, La résolution de SCQO est équivalente à trouver l'unique solution d'une équation en valeur absolue AVE. Pour la résoudre nous avons appliqué une nouvelle itération itérative en point fixe de Picard en deux étapes. En particulier, les conditions suffisantes pour la convergence de notre algorithme sont étudiées. Les résultats numériques obtenus montrent que l'algorithme est efficace et valide pour résoudre les problèmes SCQO. Dans la deuxième partie, un algorithme de trajectoire centrale de type primal-dual à petit pas est proposé pour résoudre les SCQOs via un P-LCP. De plus, son complexité polynomiale est calculé et des résultats numériques sont donnés pour montrer l'efficacité de ce dernier. Suivi par une étude comparative entre les résultats numériques obtenus par ces deux algorithmes à travers quelques exemples.

Mots clés : Programmation quadratique, problème de complémentarité, equations aux valeurs absolues, méthodes itératives de Picard, méthodes des points intérieurs, complexité polynomiale.

Abstract :

In this thesis we present a theoretical analysis and numerical study for solving a simplicial cone constrained convex quadratic optimization problems (SCQO). In the first part, and across its optimality K.K.T conditions, solving SCQO is equivalent to finding the unique solution of an absolute value equation AVE. For solving it we applied a new two-steps Picard's iterative fixed point iteration. In particular, the sufficient conditionsfor the convergence of our algorithm are studied. The obtained numerical results illustrate that the algorithm is efficient and valid to solve the SCQO problems. In the second part, a feasible a short-step primal-dual interior-point algorithm is proposed for solving SCQOs via a P-LCP. Moreover, its complexity polynomial is computed and somenumerical results are given to show the effectiveness of this latter. Followed by with a comparison study between the numerical results obtained by these two algorithms through some examples.

Keywords: Quadratic programming, complementarity problem, absolute value equations, Picard's iterative methods, interior-point methods, polynomial complexity.