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Theoretical analysis of different problems in a stationary or dynamical regime in a three dimensional thin domain with various friction law

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I dedicate this work To my dear parents: **my mother** and **my father** who have always encouraged to go further in my life and in my studies.
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General introduction

Frictional contact problems usually arise in everyday life and play a very important role in many applications, technical and fluid systems such as brakes, machine tools, engines, turbines or wheel systems.

Fluid mechanics is to provide a well-structured mathematical theory of different fields, which shows its importance in several fields, especially in industry.

A fluid (liquid, gas or ionized gas) is considered as a continuous medium represented by density, pressure and velocity fields corresponding to the famous Navier-Stokes equation.

During the last decades, many works have been done on the mathematical theory of frictional contact. Well-known and common friction laws used in the mathematical literature are Tresca's law and Coulomb's law.

These mechanical problems which are the subject of our study in this thesis are very frequent in applications in nature and it is therefore important to be able to model these phenomena.

Dynamic and static contact problems with Tresca's law of friction which was applied to linearly elastic bodies and fluid rigid bodies were studied in 1972, by , Duvaut and Lions [22].

Under the assumptions of elasticity and viscoelasticity, for example the tire, the Lassa hemorrhagic fever model and the groundwater model due in a leaky aquifer [2, 3]. Other applications are associated with the mechanism of the balls.

A non-Newtonian fluid is a fluid whose flow characteristics are different from those of any Newtonian fluid. The first mathematical results noticed at the border with friction conditions were by Fujita et al [24,25] in the case of fluid

Results concerning the study of the phenomenon of lubrication by Newtonian fluids with sliding were obtained in [8], when the sliding is given by Coulomb's friction law, and by F. Saidi [12] when the temperature effect is also taken into account. The authors in [3], studied the asymptotic analysis of any problem of the fluid in a thin domain. The asymptotic behavior of the dynamical problem of non-isothermal elasticity with non linear friction of Tresca type was studied in [6]. in [4] the authors studied the asymptotic analysis of a dynamical problem of isothermal elasticity with non linear friction of Tresca type.

There are many phenomena in nature and industry that show the behavior of Herschel-Bulkley medium. For instance, the flow of metals, plastic solids and some polymers. There is a lot of literature on this subject, see eg. [23,42]. Recently, the authors of [32] have theoretically studied the dam failure flow of viscoplastic fluids using the Herschel-Bulkley constitutive law and the motion lubrication model.

The authors of [40] studied the asymptotic behavior of a rotationally coupled system of an incompressible Bingham fluid and the thermal energy equation, in a three-dimensional bounded domain with Tresca free boundary friction conditions. Asymptotic analysis of solutions of a thin film lubrication problem with Coulomb fluid-solid interface law study in [10]. In the case $\alpha^{\varepsilon} = 0$, Numerical solutions of Herschel-Bulkley fluid flow problems are studied, e.g. [28,30,35,36].

The object of the doctoral thesis is the study of the asymptotic behavior of some boundary problems modeling the behavior of fluids and different materials in the stationary case in three-dimensional bounded domains with the nonlinear friction conditions on the edge. One of the goals of asymptotic analysis is to obtain and describe a two dimensional problem from a three dimensional problem, passing to the limit on the thickness of the domain assumed to be already thin. That is, the physical domains are defined such that the height is much smaller than the length.

In this work, we are interested in the study of the asymptotic analysis of a stationary problem for the non-isothermal linear elasticity and the isothermal non-Newtonian fluid in a domain bounded in thin film Ω^{ε}. The boundary Ω^{ε} is denoted by Γ^{ε} and is composed on three parts: the bottom and fixed part ω, the later part Γ_L and the upper surface Γ_1.

The normal displacement and the normal velocity are equal to zero on the bottom. However, on the lateral surface the tangential displacement or velocity is unknown and

satisfies the Coulomb boundary conditions. We consider the boundary conditions of Dirichlet and Neumann on the upper surface.

The main parts of this work can be summarized as follows.

In the first chapter, we introduce some necessary notations and we are interested to recalling some basic definitions and theorems of functional analysis which allow us to better understand the content of this job. Including Sobolev's theorem, lower semicontinuity, convex-base definitions, and differential functions.The basic tools presented in this chapter are standard and can be found in many functional analysis books.

The second chapter, the Herschel-Bulkley fluid is a general model of a non-Newtonian fluid. The name is associated with Winslow Herschel and Ronald Bulkle [26] , the first in 1926, where the relationship between the stress tensor and the rate of symmetric strain. In this part we are devoted to the study of asymptotic behavior of the steady flow of the incompressible viscoplastic fluid of Herschel-Bulkley in a thin domain, the viscosity of which does not depend on the temperature.

The equation of conservation of momentum given by

$$\operatorname{div} \sigma^{\varepsilon} + f^{\ \varepsilon} = 0 \quad \text{in } \Omega^{\varepsilon}.$$

with the constitutive law of Herschel-Bulkley

$$\begin{cases} \sigma^{D,\varepsilon} = \alpha^{\varepsilon} \dfrac{D(u^{\varepsilon})}{|D(u^{\varepsilon})|} + 2\mu \left|D(u^{\varepsilon})\right|^{r-2} D(u^{\varepsilon}), & \text{when } D(u^{\varepsilon}) \neq 0, \quad r > 2, \\ \left|\sigma^{D,\varepsilon}\right| \leq \alpha^{\varepsilon} \ , & \text{when } D(u^{\varepsilon}) = 0, \end{cases}$$

where the parameter r, $\quad 1 < r < 2$, is the exponent of the power law of the material. When $r = 2$, we find the Bingham fluids, and when $r = 2$ and $\alpha^{\varepsilon} = 0$, we find the Navier-Stokes model (Newtonian fluid). The boundary conditions used are the homogeneous Dirichlet conditions on one part of the border and Coulomb's conditions on the other part. By looking for a priori estimates of speed and pressure and going to the limit for this by using the inequalities of Poincaré, Young, Hölder, Korn and Minty's lemma, we get the weak Reynolds equation.

Under the form :

$$\int_{\omega}\left(\frac{h^3}{12}\nabla\rho^*(\acute{x})+\tilde{F}(\acute{x},h)+\mu\int_0^h\int_0^y A^*(\acute{x},\xi)\tfrac{\partial u^*}{\partial\xi}(\acute{x},\xi)d\xi dy+\hat{\alpha}\int_0^h\int_0^y\frac{\partial u^*/\partial z}{|\partial u^*/\partial z|}(\acute{x},\xi)d\xi dy\right)\nabla\varphi(x)$$
$$-\int_{\omega}\left(\frac{\mu h}{2}\int_0^h A^*(\acute{x},\xi)\tfrac{\partial u^*}{\partial\xi}(\acute{x},\xi)d\xi+\frac{\hat{\alpha}h}{2}\int_0^h\tfrac{\partial u^*/\partial z}{|\partial u^*/\partial z|}(\acute{x},\xi)d\xi\right)\nabla\varphi(x)=0,\forall\varphi\in W^{1,r}(\omega).$$

where

$$\tilde{F}(\acute{x},z)=\int_0^h F(\acute{x},y)dy-\frac{h}{2}F(\acute{x},h)\text{ and }\ F(\acute{x},z)=\int_0^h\int_0^\xi \hat{f}_i^{\,\varepsilon}(\acute{x},\alpha)d\alpha d\xi$$

We provide an exact characterization of the limit form of the Coulomb boundary condition.

$$\begin{cases}\hat{\mu}(\varsigma^*)\hat{\xi}^*<\hat{k}\left|R\left(\hat{\sigma}_n^\varepsilon\left(-\rho^*\right)\right)\right|\Rightarrow s^*=s\\ \hat{\mu}(\varsigma^*)\hat{\xi}^*=\hat{k}\left|R\left(\hat{\sigma}_n^\varepsilon\left(-\rho^*\right)\right)\right|\Rightarrow\exists\beta\geq 0\text{ such that }s^*=s-\beta\hat{\xi}^*\end{cases}$$

where

$$\varsigma^*=T^*(\acute{x},0)\text{ and }\tau^*=\frac{\partial u^*}{\partial z}(\acute{x},0).$$

In the last chapter, we will consider the following problem:

$$\operatorname{div}\sigma^\varepsilon+f^{\,\varepsilon}=0\ \text{ in }\Omega^\varepsilon.$$

The constitutive law is given by

$$\sigma^\varepsilon_{i,j}(u^\varepsilon)=2\mu(T^\varepsilon)d_{i,j}(u^\varepsilon)+\lambda(T^\varepsilon)d_{kk}(u^\varepsilon)\delta_{ij}.$$

and as the strain in the non-isothermal case, by coupling the equation conservation of momentum with the equation conservation of energy deduced from the following Fourier law

$$\begin{cases}-\nabla(K^\varepsilon\nabla T^\varepsilon)=\sigma^\varepsilon:D(u^\varepsilon)+r^\varepsilon(T^\varepsilon),\\ \sigma^\varepsilon:D(u^\varepsilon)=\sum\limits_{i,j=1}^{3}\sigma^\varepsilon_{i,j}d_{i,j}(u^\varepsilon).\end{cases}$$

where, u^ε represents the field of displacements, T^ε represents the temperature. The boundary conditions are mixed, they are modeled by a condition of Dirichlet on part of the border and the friction conditions Coulomb type nonlinear on the other side for the equation of motion. As well as a homogeneous Neumann and Dirichlet condition for the energy conservation equation, of shape

$$\begin{cases} \operatorname{div}\sigma^{\varepsilon} + f^{\varepsilon} = 0 & \text{in } \Omega^{\varepsilon}, \\ \sigma^{\varepsilon}_{i,j}(u^{\varepsilon}) = 2\mu(T^{\varepsilon})d_{i,j}(u^{\varepsilon}) + \lambda(T^{\varepsilon})d_{kk}(u^{\varepsilon})\delta_{ij}. & \text{in } \Omega^{\varepsilon}, \\ u^{\varepsilon} = 0 & \text{on } \Gamma^{\varepsilon}_{1}, \\ u^{\varepsilon} = g & \text{on } \Gamma^{\varepsilon}_{L}, \\ u^{\varepsilon}.n = 0 & \text{on } \omega, \\ \left\{ \begin{array}{c} |\sigma^{\varepsilon}_{T}| < F^{\varepsilon}\,|\sigma^{\varepsilon}_{n}| \Rightarrow u^{\varepsilon}_{T} = s, \\ |\sigma^{\varepsilon}_{T}| = F^{\varepsilon}\,|\sigma^{\varepsilon}_{n}| \Rightarrow \exists\beta \geq 0 \text{ such that } u^{\varepsilon}_{T} = s - \beta\sigma^{\varepsilon}_{T}, \end{array} \right\} & \text{on } \omega, \\ T^{\varepsilon} = 0 & \text{on } \Gamma^{\varepsilon}_{L} \cup \Gamma^{\varepsilon}_{1}, \\ \dfrac{\partial T^{\varepsilon}}{\partial n} = 0 & \text{on } \omega. \end{cases}$$

We give the related weak formulation of the problem. Then, we discuss the existence and uniqueness theorem of the weak solution. Next, we study the asymptotic analysis according to the change of the variables $xz = \frac{x_3}{\varepsilon}$ to transform the initial problem posed in the domain Ω^{ε} which depends on a small parameter ε into a new problem posed on a fixed domain Ω which is independent of ε . Then, we find some estimates on the velocity and pressure. We obtain further the main results concerning the existence of a weak limit (u^*, p^*) of $(u^{\varepsilon}, p^{\varepsilon})$ such that (u^*, p^*) satisfies the weak form of the Reynolds equation

$$\int_{\omega}\left(\tilde{F} - \int_0^h \int_0^y \hat{\mu}(T^*(\acute{x},\xi)\frac{\partial u^*(\acute{x},\xi)}{\partial \xi}\partial\xi dy\right)\nabla\psi d\acute{x} = \int_{\omega}\frac{h}{2}\int_0^h \hat{\mu}(T^*(\acute{x},\xi)\frac{\partial u^*(\acute{x},\xi)}{\partial \xi}\partial\xi\nabla\psi d\acute{x}, \forall\psi \in H^1(\omega)$$

The study is based on the variational formulation, the Poincaré inequality, Cauchy-Schwartz, Young, Hölder, Korn. This study was carried out by H. Benseridi and M. Dilmi [4] in the isothermal case and by H. Benseridi and A. Saadallah [6].

Notations

Be Ω is a domain of $\mathbb{R}^d\,(d=2,3)$ and X a Banach space, we use the following notations:

$\overline{\Omega}$	the adhesion of Ω.
Γ	the total border of Ω.
Γ_1	the upper border of Ω.
Γ_L	the side border of Ω.
n	the outgoing unit normal on .
v_n, v_T	the normal and tangential components of a vector v.
$\lvert.\rvert$	the Euclidean norm of $\mathbb{R}^d$, with $\lvert.\rvert = \sqrt{\sum\limits_{i=1}^{d} x_i^2}$.
I_d	the second order identity tensor on $\mathbb{R}^d$.
δ_{ij}	the symbol of Krönecker.
$D'(\Omega)$	the distribution space .
$C^1(\omega)$	the space of real functions continuously differentiable on ω.
$H^1(\Omega)$	Sobolev space of order 1 on Ω.
$H_0^1(\Omega)$	the adhesion of $D(\Omega)$ in $H^1(\Omega)$.
$H^{-1}(\Omega)$	the dual space of $H_0^1(\Omega)$.
$L^2(\Omega)^d$	space $\left\{u=(u_i)\,/\,u_i\in L^2(\Omega)\,,\ i=\overline{1,d}\right\}$.
$H^1(\Omega)^d$	space $\left\{u=(u_i)\,/\,u_i\in H^1(\Omega)\,,\ i=\overline{1,d}\right\}$.
$\lVert\cdot\rVert_{0,\Omega}$	the norm of $L^2(\Omega)^d$.
$\lVert\cdot\rVert_{1,\Omega}$	the norm of $H^1(\Omega)^d$.
$\lVert\cdot\rVert_X$	the norm of X.
X^d	space $\left\{x=(x_i)\,/\,x_i\in X,\ i=\overline{1,d}\right\}$.
$x_n \rightharpoonup x$	the weak convergence of the sequence (x_n) to an element x in X.
$x_n \to x$	the strong convergence of the sequence (x_n) to an element x in X.
$\partial_i f$	the partial derivative of f with respect to the component x_i.
∇f	the gradient of f.
$\operatorname{div} f$	the divergence of f.

Chapter 1

Required and preliminary

To facilitate the reading of this thesis, this chapter will be divided into two sections to describe the necessary mathematical and mechanical tools. The results are standard and can be found in many references.

Recall functional spaces as well as Hilbert and Banach spaces, declare inequalities such as Korn and Poincaré...

The second section is based on the theory of continuum mechanics and the essential results of the constitutive law of linear elasticity, models the isothermal non-isothermal evolution of homogeneous isotropic elastic bodies and determines the nonlinear friction of the Coulomb type in the domain Ω of $\mathbb{R}^3$.

1.1 Functional spaces

A functional space is a space whose points are functions, the resolution of an analysis problem often consists in choosing a functional space provided with an associated norm.

Sobolev spaces play an important role in the study of elliptic and hyperbolic partial derivative equationes.

A Lipschitz domain or Lipschitz bounded domain is a domain in Euclidean space whose boundaries are "regular enough" in the sense that it can be considered local being the graph of a Lipschitz continuous function. Many of Sobolev embedding theorems require that the domain of study is a Lipschitz domain.

As a result, many partial differential equations and variational problems are defined in the Lipschitz domain.

1.1.1 Functional analysis reminders

We introduce in this subsection a summary of the functional analysis, and some results which help us in the study of the problems of this thesis.

Let Ω an open from $\mathbb{R}^d$. We must define some necessary spaces in Ω.

The space of $D(\Omega)$

Definition 1.1.1 *We call $D(\Omega)$ vector space on $\mathbb{R}$ or $\mathbb{C}$, which contains the set of class functions C^∞, with compact support in Ω, that is to say:*

$$C_c^\infty(\Omega) = D(\Omega) = \{u \in C^\infty(\Omega);\ \exists K \subset \Omega,\ K \text{ compact; } u = 0 \text{ in } K^c\}.$$

The distribution space

We denote by $D'(\Omega)$ the «dual» of $D(\Omega)$, that is to say the space of continuous linear forms on $D(\Omega)$. We notice $\langle T, \phi \rangle = T(\phi)$ the duality product between a distribution $T \in D'(\Omega)$ and a function $\phi \in D(\Omega)$: this duality product generalizes the usual integral $\int_\Omega T\phi dx$. Indeed, we check that if f is a locally integrable function in Ω, then we can define a distribution T_f through:

$$\langle T_f, \phi \rangle = \int_\Omega f\phi \; dx.$$

We now define **the derivation in the sense of the distributions :** if $T \in D'(\Omega)$, the derivative $\dfrac{\partial T}{\partial x_i} \in D'(\Omega)$ is defined by :

$$\langle \frac{\partial T}{\partial x_i}, \phi \rangle = -\langle T, \frac{\partial \phi}{\partial x_i} \rangle, \ \forall \phi \in D(\Omega). \tag{1.1.1}$$

We can also associate $D'(\Omega)$ by a notion of convergence: we say that a sequence $T_n \in D'(\Omega)$

Complete space

Definition 1.1.2 *We say that a metric space (X, d) is complete if any Cauchy sequence converges.*

Banach space

Definition 1.1.3 *That is E a normalized vector space, we say that it is a Banach space if it is complete.*

Hilbert space

Definition 1.1.4 *Let H a real vector space and $\langle .,. \rangle_H$ a dot product on H that is to say : $\langle .,. \rangle_H : H \times H \to \mathbb{R}$ is a symmetric and positive definite bilinear map.*

We denote by $|.|_H$ the application of $H \to \mathbb{R}_+$ defined by:

$$|\theta|_H = \langle \theta, \theta \rangle_H^{\frac{1}{2}},$$

and remember that $|.|_H$ is a standard on H. We say that H is a Hilbert space if H is complete for the standard defined previously.

Cauchy sequence

Definition 1.1.5 *Let (u_n) a real sequence, we say that (u_n) is a Cauchy sequence or satisfies the Cauchy criterion if*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall (p,n) \in N^2, p \geq N \text{ and } n \geq N \Rightarrow |u_p - u_n| < \varepsilon.$$

$L^p(\Omega)$ **Spaces**

We consider the Lebesgue space $L^p(\Omega)$

Definition 1.1.6 *[14] Let $p \in \mathbb{R}$, with $1 \leq p < \infty$, the Lebesgue space $L^p(\Omega)$ is defined to be*

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^p \, dx < \infty. \right\}$$

The space $L^p(\Omega)$ is equipped with the norm

$$\|u\|_{L^{,p}(\Omega)} = \left(\int_\Omega |u(x)|^p \, dx \right)^p.$$

If $p = \infty$ and $u : \Omega \to \mathbb{R}$ is measurable then we define the $L^\infty(\Omega)$

$$L^\infty(\Omega) = \{ u : \Omega \to \mathbb{R} \text{ is measurable and there exist a constant } C \text{ such that } |u(x)| < C, \text{ in } \Omega. \}$$

its norm is

$$\|u\|_{L^{,\infty}(\Omega)} = \inf \{ C ; |u(x)| < C, \text{ in } \Omega. \}.$$

Theorem 1.1.1 ***(Separability).****[14] $L^p(\Omega)$ is separable space for any $p \in [1, \infty[$.*

Theorem 1.1.2 ***(Reflexivity).****[14] $L^p(\Omega)$ is reflexive space for any $p \in]1, \infty[$.*

Definition 1.1.7 *[14] We denote by C_c the space of continuous function in Ω with compact support in Ω.*

$$C_c(\Omega) = \{ f \in C(\Omega);\ f(x) = 0, \forall x \in \Omega \diagdown K \subset \text{ where } K \subset \Omega \text{ is compact} \}.$$

$C^k(\Omega)$ is the space of function k times continuously differentiable in Ω.

$$\begin{aligned} C^\infty(\Omega) &= \cap C^k(\Omega) \\ C_c^k(\Omega) &= C^k(\Omega) \cap C_c(\Omega) \\ C_c^\infty(\Omega) &= C^\infty(\Omega) \cap C_c(\Omega) \end{aligned}$$

convergence

Strong convergence and Weak convergence

Let X is a Banach space and X' the dual space of X and denote $\langle .,. \rangle$ the duality product between X and its topological dual space X'.

Definition 1.1.8 ***(Strong convergence) [14].*** *A sequence u_n is said to be strongly converge into u if*

$$u_n, u \in X \ and \lim_{n\to\infty} \|u_n - u\|_X = 0$$

noted this convergence by $u_n \to u$ in X.

Definition 1.1.9 ***(Weak convergence) [14].*** *A sequence u_n is said to be weakly converge into u if*

$$u_n, u \in X \ and \lim_{n\to\infty} \langle u_n, v\rangle_{X\times X'} = \langle u, v\rangle_{X\times X'}, \forall v \in X'$$

noted this convergence by $u_n \rightharpoonup u$ in X'.

Proposition 1.1.1 *u_n a sequence in X. Then*

1. *If $u_n \to u$ strongly in X, then $u_n \rightharpoonup u$ weakly in X.*
2. *If $u_n \rightharpoonup u$ weakly in X, then $\| u_n\|_X$ is bounded and $\| u\|_X \leq \lim_{n\to\infty} \inf \| u_n\|_X$.*
3. *If $u_n \rightharpoonup u$ weakly in X and if $v_n \longrightarrow v$ strongly in X', then $\langle u_n, v_n\rangle_{X\times X'} = \langle u, v\rangle_{X\times X'}$.*

Convergence on $D(\Omega)$

Let $\varkappa_n$ a series of $D(\Omega)$ and $\varkappa \in D(\Omega)$ we say that $\varkappa_n \to \varkappa$ for the topology of $D(\Omega)$ if supp $\varkappa_n \subset K$, and supp $\varkappa \subset K$, K is a compact of Ω.

$$\varkappa_n \to \varkappa \text{ evenly on } K \text{ and } D^\alpha \varkappa_n \to D^\alpha \varkappa \text{ evenly on } K.$$

converges in the sense of the distributions

towards $T \in D'(\Omega)$ if, for everything $\phi \in D(\Omega)$,

$$\lim_{n\to+\infty} \langle T_n, \phi\rangle = \langle T, \phi\rangle.$$

Weak convergence in a Banach space.

Definition 1.1.10 *Let E a Banach space. Be $(u_n)_{n\in\mathbb{N}} \subset E$ and $u \in E$, we say that u_n weakly converges to u in E when $n \to \infty$, and we note $u_n \rightharpoonup u$, if :*

$$T(u_n) \to T(u), \ \forall T \in E', \ or \ E' \text{ is the dual of } E.$$

Weak convergence in a Hilbert space.

Let H a real Hilbert space and $\langle .,. \rangle_H$ a dot product of H. That is $(x_n)_{n\in N}$ a squence in H and x also an element of H.

We say that the sequence $(x_n)_{n\in N}$ weakly converges to x (x is the weak limit of the squence (x_n)) if :

$$x_n \rightharpoonup x \iff \forall y \in H, \lim_{n\longrightarrow\infty} \langle x_n, y\rangle = \langle x, y\rangle, \text{ (weak convergence)}.$$

We will denote this convergence by the symbol $\rightharpoonup$ to distinguish it from strong convergence (that is to say for the Hilbert norm) :

$$\begin{aligned} u_n &\rightarrow u \iff \lim_{n\longrightarrow\infty} |u_n - u|_H = 0, \text{ (strong convergence)}. \\ u_n &\rightharpoonup u \iff \forall v \in H, \lim_{n\longrightarrow\infty} \langle u_n, v\rangle = \langle u, v\rangle, \text{ (weak convergence)}. \end{aligned}$$

Theorem 1.1.3 ***(Weak compactness theorem of the closed unit ball of Hilbert spaces).***

If H a Hilbert space, then any bounded sequence in H admits a weakly convergent subsequence.

1.1.2 Reminders on Sobolev's spaces

We define the Sobolev spaces which are the spaces of functions allowing to solve the variational formulations of partial differential equations.

Be Ω an open from $\mathbb{R}^d$, $L^2(\Omega)$ the space of measurable functions of squared summable in Ω, provided with a scalar product:

$$(u, v)_{L^2(\Omega)} = \int_\Omega u(x)v(x)\, dx,$$

$L^2(\Omega)$ is a Hilbert space. We notice :

$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega |u(x)|^2\, dx \right)^{\frac{1}{2}}$$

the corresponding standard.

Definition 1.1.11 *Let Ω an open from $\mathbb{R}^d$. The Sobolev space $H^1(\Omega)$ is defined by :*

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) such\ as\ \frac{\partial u}{\partial x_i} \in L^2(\Omega),\ \forall i \in \{1,\ ..., d\} \right\},$$

where $\frac{\partial u}{\partial x_i}$ is the partial derivative of u in the meaning of distributions (1.1.1).

Proposition 1.1.2 *The Sobolev $H^1(\Omega)$ endowed with the scalar product*

$$(u, v)_{H^1(\Omega)} = \int_\Omega (u(x)v(x)\ + \nabla u(x).\nabla v(x))\ dx$$

and the standard

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega (|u(x)|^2\ + |\nabla u(x)|^2)\ dx \right)^{\frac{1}{2}}.$$

$H^1(\Omega)$ is a **Hilbert** space.

$H_0^1(\Omega)$ **Space**

Definition 1.1.12 $H_0^1(\Omega)$ *denote the vector subspace of the functions of $H^1(\Omega)$ zero on Γ.*

$$H_0^1(\Omega) = \left\{ u \in H^1(\Omega),\ u = 0\ on\ \Gamma \right\}.$$

The norm of $H_0^1(\Omega)$ is defined by :

$$\|u\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla u(x)|^2 \right)^{\frac{1}{2}} = \|\nabla u\|_{L^2(\Omega)}.$$

$W^{1,p}(\Omega)$ **Space**

Definition 1.1.13 *[14] Let $p \in \mathbb{R}$, with $1 \leq p \leq \infty$, the sobolev space $W^{1,p}(\Omega)$ is defined to be*

$$W^{1,p}(\Omega) = \left\{ u \in L^p, \exists g_i \in L^p\ such\ that \int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_i \varphi, \forall \varphi D(\Omega), i = 1, 2, 3. \right\}$$

for $u \in W^{1,p}$ we have

$$\frac{\partial u}{\partial x_i} = g_i\ and\ \nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3} \right).$$

The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{i=1}^{3} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)}.$$

Proposition 1.1.3 *[14] The space $W^{1,p}(\Omega)$ is Banach space for $1 \leq p \leq \infty$. It is reflexive for $1 < p < \infty$, and separable for $1 \leq p < \infty$.*

Definition 1.1.14 *[14] For $1 \leq p < \infty$, the space $W_0^{1,p}(\Omega)$ desing the closure of $C_c^1(\Omega)$ in $W^{1,p}(\Omega)$.*

The space $W_0^{1,p}(\Omega)$ equipped the same norm of $W^{1,p}(\Omega)$ and is separable, Banach space and it's reflexive for $1 \leq p < \infty$.

Definition 1.1.15 *[14] The dual space of $W_0^{1,p}(\Omega)$ is denoted by $W^{-1,q}(\Omega)$.*

Duality

Let V is a Hilbert space we denote V' its dual, which represents the set of continuous linear forms on V.

The dual of $L^2(\Omega)$ is identified with $L^2(\Omega)$. We can also define the dual of a Sobolev space.

We are interested in the dual of $H_0^1(\Omega)$ which plays a particular role in the sequence.

Definition 1.1.16 *The dual of Sobolev's space $H_0^1(\Omega)$ is called $H^{-1}(\Omega)$.*

We notice : $\langle L, \phi \rangle_{H^{-1},H_0^1(\Omega)} = L(\phi)$ the product of duality between $H_0^1(\Omega)$ and its dual for any continuous linear form $L \in H^{-1}(\Omega)$ and any function $\phi \in H_0^1(\Omega)$.

Proposition 1.1.4 ***(properties of spaces** $H^{-1}(\Omega)$ **)***

(1) *Space $H^{-1}(\Omega)$ is characterized by :*

$$H^{-1}(\Omega) = \left\{ f = v_0 + \sum \frac{\partial v_i}{\partial x_i} \quad \text{with } v_0, v_1, ..., v_d \in L^2(\Omega) \right\}.$$

(2) *$H^{-1}(\Omega)$ is a Banach space when provided with the norm :*

$$\|L\|_{H^{-1}(\Omega)} = \sup_{\|\phi\|_{H_0^1(\Omega)} \leq 1} \langle L, \phi \rangle_{H^{-1},H_0^1(\Omega)}.$$

Remark 1.1.1 *$H^{-1}(\Omega)$ is a Hilbert space for the previous norm.*

Remark 1.1.2 *For everything* Ω *an open from* $\mathbb{R}^d$*, we have :*

$$H_0^1(\Omega) \subset L^2(\Omega) \equiv \left(L^2(\Omega)\right)' \subset H^{-1}(\Omega).$$

Theorem 1.1.4 ***(Density).****[14] The space* $D(\Omega)$ *is dense in* $L^p(\Omega)$ *for any* $1 \leq p < \infty$.

Theorem 1.1.5 ***(Dual of*** $L^p(\Omega)$ ***).*** *[1] Let* p *be a real number such that* $1 < p < \infty$*. The topological dual of* $L^p(\Omega)$ *is* $(L^p(\Omega))' = L^q(\Omega)$.

Proposition 1.1.5 *[1] The dual of* $L^1(\Omega)$ *is* $L^\infty(\Omega)$.

Proposition 1.1.6 ***(Poincare inequality).*** *Let* Ω *a bounded open of* $\mathbb{R}^d$*, then there is a constant* C_Ω *only depends on* Ω *such as for any function* $u \in H_0^1(\Omega)$*,*

$$\|u\|_{L^2(\Omega)} \leq C_\Omega \|\nabla u\|_{L^2(\Omega)}.$$

Theorem 1.1.6 ***(Green's formula).*** *Let* Ω *a regular class bounded open* C^1*. If* u *and* v *are functions of* $H^1(\Omega)$*, they check :*

$$\int_\Omega u(x)\frac{\partial v}{\partial x_i}dx = -\int_\Omega v(x)\frac{\partial u}{\partial x_i}dx + \int_{\partial\Omega} u(x)v(x)\eta_i(x)dx,$$

where $\eta = (\eta_i)_{1\leq i\leq d}$ *is the unit normal outside* $\partial\Omega$.

Theorem 1.1.7 ***(Cauchy-Schwarz inequality).*** *Let* H *a prehilbertian space.*

Then :

$$\forall (x,y) \in H^2, |\langle x,y\rangle| \leq \|x\| . \|y\| .$$

Lemma 1.1.1 ***(Korn inequality)****. [37] For everything* $\omega \in V$*, we have :*

$$\|\nabla\omega\|_{L^2(\Omega)} \leq C \|D(\omega)\|_{L^2(\Omega)}, \text{ for } \omega \in V.$$

With V *is a* ***Hilbert*** *space.*

Theorem 1.1.8 ***(Hölder inequality).[14]*** *If $p \in]1,+\infty[$, the conjugate exponent of p is the only one $q \in]1,+\infty[$*

such as $\dfrac{1}{p}+\dfrac{1}{q}=1$. For everything $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, we have :

$$uv \in L^1(\Omega) \text{ and } \|uv\|_{L^1(\Omega)} \leq \|u\|_{L^p(\Omega)} \cdot \|v\|_{L^q(\Omega)} .$$

This is a generalization of the Cauchy-Schwarz inequality.

Theorem 1.1.9 ***(Young inequality).***

$$ab \leq \frac{a^p}{p}+\frac{b^q}{q} \quad \forall (a,b) \in \mathbb{R}_+^2, \qquad .$$

Lax-Milgram theorem

We introduce in what follows some useful results which are valid in Hilbert spaces. This concerns the Riesz representation theorem and Lax-Milgram Theorem.

Definition 1.1.17 ***(Bilinear form).*** *Let X be a vector space. A scalar product (u,v) is a bilinear from $X \times X$ with values in $\mathbb{R}$ such that*

$$\begin{aligned}
(u,v) &= (v,u), \quad \forall u,v \in X, \qquad \textit{(symetry)},\\
(u,v) &\geq 0, \quad \forall u \in X, \qquad \textit{(positive)},\\
(u,v) &\neq 0, \quad \forall u \neq 0, \qquad \textit{(definite)}.
\end{aligned}$$

Theorem 1.1.10 ***(Riesz representation theorem).*** *Let X be a Hilbert space and $\varphi \in X'$ there exist unique $f \in X$ such that*

$$\langle \varphi, u \rangle = \langle f, u \rangle, \forall u \in X.$$

Moreover

$$\|\varphi\|_{X'} = \|f\|_X .$$

Definition 1.1.18 *Let X be a Hilbert space. A bilinear form $a : X \times X \to \mathbb{R}$*

1. *Continuous, if there is a constant C such that*

$$|a(u;v)| \leq C\,|u|\,|v|\,, \forall u;v \in X :$$

2. *Coercive, if there exists a constant $\gamma > 0$ such that*

$$|a(u;v)| \geq \gamma\,|u|^2\,, u \in X.$$

Theorem 1.1.11 ***(Lax-Milgram).*** *Assume that $a(u,v)$ is a continuous coercive bilinear form on X.*

Then, given any $\varphi \in X'$, there exist a unique element $u \in X$ such that

$$a(u,v) = \langle u,v\rangle\,, \forall v \in X.$$

A **semi-continuous,monotone**

Soit X un espace de Banach et X' son dual.

Definition 1.1.19 *Let A be the operator $X \to X'$*

A is called semi-continuous if for all sequences $(\lambda_n)_n$ converging to λ :

$$(A(u+\lambda_n v), w) \to (A(u+\lambda v), w), \forall (u,v,w) \in X^3$$

A is called monotone only if:

$$(A(u) - A(v), u-v) \geq 0, \forall (u,v) \in X^2$$

1.2 Reminders on the mechanics of continuous media

This section consists in establishing the mathematical model describing the evolution of a deformable body having a linear elastic law under the action of the external forces in the presence of conditions of friction on a part at the edge of the field. This results mathematically by the establishment of a system of partial differential equations in a domain of $\mathbb{R}^d(d = 2, 3)$. This system includes the constitutive law of the material, the equation of conservation of the momentum of the body as well as the boundary conditions to which it is subjected.

Consider a continuous medium which occupies a bounded domain Ω of $\mathbb{R}^3$.

1.2.1 Conservation equation of momentum

Let $u(x)$ the field of **displacement** of the elastic body at the point $x = (x_1, x_2, x_3) \in \Omega$ of the continuous medium in motion with respect **to the reference** Ox.

An elastic body: is a body that spontaneously returns to its original shape when external forces are removed.

Stress field *:* is a representation used in the mechanics of continuous media to characterize the state of stress, that is to say all the internal cohesion forces exerted on a part of the solid under the effect of external loads. The term was introduced by Cauchy around 1822.

As the internal forces are defined for each surface intersecting the medium, the tensor is defined locally, at each point of the solid.

There are several formulations of the fundamental law of mechanics. According to the chosen statement. All of these formulations are equivalent.

For our part, we state the law which expresses the equivalence between external forces and the acceleration tensor for any system, leads to the equation of motion.

$$\rho \frac{\partial^2 u}{\partial t^2} = \operatorname{div} \sigma + f \ , \ \text{ in } \Omega \tag{1.2.1}$$

where

$$Div\ \sigma = \sum_{i=1}^{d} \frac{\partial \sigma_i}{\partial x_i},\ (d = 2, 3).$$

In this equation, ρ denotes the mass density, and $\dfrac{\partial^2 u}{\partial t^2}$ the field of accelerations, f represent the external forces applied on the body and which are data of the problem, $(\operatorname{div} \sigma)$ is the divergence of the stress field.

The process modeled by $(1.2.1)$ is called dynamic process. In some situations the equation $(1.2.1)$ can be simplified, for example in the case where $\dfrac{\partial u}{\partial t} = 0$, it is a balancing process, or in the case where the speed field $\dfrac{\partial u}{\partial t}$ varies very slowly with respect to time, that is to say the term $\rho\dfrac{\partial^2 u}{\partial t^2}$ can be over looked. In these two cases the equation $(1.2.1)$ becomes:

$$\operatorname{div}\sigma + f = 0, \quad \text{in } \Omega.$$

The laws of conservation of the momentum are insufficient to describe the motions of continuous media, because we have several unknowns with respect to the numbers of the equations in the mathematical point of view, so we are obliged to treat other relations which characterize the behavior of the material called the constitutive laws of linear elasticity.

1.2.2 Linear constitutive laws of elastic materials

The constitutive laws express the relations which exist between the stress tensor σ and D the strain tensor, thier relations depend on the nature of the material. One will generally impose that the stress is linearly related to the strain.

The linearized strain field is defined as follows:

$$D(u) = d_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), 1 \leq i, j \leq 3.$$

$d_{ij}(u)$ indicate the displacement of the strain field d compared to the displacement field u.

Generalized Hooke's Law

Hooke's law was generalized by Cauchy (1789-1857), who proposed to express each component of the stress tensor as a linear function of the components of the strain tensor. Hooke's law is therefore often written today in the form:

$$\sigma_{ij}(u) = C_{ijkl}\, d_{kl}(u), \tag{1.2.2}$$

with C is the symmetric tensor of the fourth order called stiffness tensor or elasticity tensor.

Components C_{ijkl} are called elasticity coefficients.

In the case of a homogeneous material

The coefficients C_{ijkl} are constants (independent to the point x of Ω).

In the case of an isotropic material

In practice, this law is often too general and it is possible to simplify it. If we consider an isotropic material, then the elasticity tensor C is defined by :

$$C_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\delta_{ij}\delta_{kl}. \tag{1.2.3}$$

By using (1.2.2) and (1.2.3), we can then prove that the stress tensor σ of a homogeneous elastic and isotropic body is completely described by the two parameters:

$$\sigma(u) = \lambda TrD(u)\, I + 2\mu D(u),$$

where :

$$TrD(u) = \sum_{k=1}^{d} d_{kk}(u).$$

δ_{ij} : denotes the Krönecker symbol.

$\lambda,\ \mu$: are the independent coefficients and are called the Lamé constants. They are homogeneous at pressures.

Rheological tests make it possible to determine these constants, which are specific to a given material. These tests actually directly measure two other values: Young's modulus E and Poisson's ratio v. These last values are related to the Lamé coefficients by the equations :

$$\lambda = \frac{Ev}{(1-2v)(1+v)} \quad , \quad \mu = \frac{E}{2(1+v)}.$$

with

E : intuitively corresponds to the stiffness of the material.

v : measure its incompressibility.

According to the establishment of all these laws we are now looking for the partial differential equations modeling the evolution of a linear homogeneous body in a three dimensional bounded open domain Ω.

Then, it is assumed that there is a tangential force on part of the border noted by ω, we say that we have a contact with friction. One is brought to introduce a friction law which connects this tangential component to the other variables of the system. In this thesis, we will consider the nonlinear friction of the Coulomb type.

1.2.3 Coulomb type friction law

This contact force comprises a normal component σ_n^ε, perpendicular to the contact plane between the two solids, and a tangential component σ_T^ε, belonging to the contact plane. Coulomb shows, from experiments, that the sliding between the two solids occurs if σ_n^ε and σ_T^ε satisfy the relation of proportionality: $\sigma_T^\varepsilon = \mathcal{F}^\varepsilon \, |\sigma_n^\varepsilon|$, where $\mathcal{F}^\varepsilon$ is the coefficient of friction which characterizes the state of the two surfaces in contact. As long as the force σ_T^ε is less than $\mathcal{F}^\varepsilon \, |\sigma_n^\varepsilon|$(we say that the contact force remains inside the Coulomb cone), the slip does not occur, and we speak of adhesion between the two solids.

$$\begin{cases} |\sigma_T^\varepsilon| < \mathcal{F}^\varepsilon \, |\sigma_n^\varepsilon| \Rightarrow u_T^\varepsilon = s, \\ |\sigma_T^\varepsilon| = \mathcal{F}^\varepsilon \, |\sigma_n^\varepsilon| \Rightarrow \exists \beta \geq 0 \text{ such that } u_T^\varepsilon = s - \beta \sigma_T^\varepsilon, \end{cases}$$

Chapter 2

Study of a generalized non-Newtonian fluid in a thin film with Coulomb's law

Abstract. The objective of this chapter is the study of the isothermal flow of an incompressible Herschel-Bulkley fluid in the stationary regime in a thin domain with the nonlinear friction conditions of Coulomb type on part of the boundary and the Dirichelet conditions on the other part.

2.1 Description of the problem and basic equations

We consider a nonlinear model which describes the behavior of a Herschel-Bulkley fluid in the isothermal case in thin domain Ω^ε, Where ε is a positive real belonging to $]0,1[$ and which tends towards zero.

The border of Ω^ε will be noted $\Gamma^\varepsilon = \overline{\omega} \cup \overline{\Gamma_1^\varepsilon} \cup \overline{\Gamma_L^\varepsilon}$.

Γ_1^ε the upper border of Ω^ε with equation $x_3 = \varepsilon h(x_1, x_2)$.

Γ_L^ε the side border of Ω^ε.

ω is a bounded domain of $\mathbb{R}^3$, with equation $x_3 = 0$,
and it's the bottom of the domain and it's supposed that has a Lipschitz boundary.

We notice $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $\acute{x} = (x_1, x_2) \in \mathbb{R}^2$.

The domain Ω^ε is given by

$$\Omega^\varepsilon = \{(\acute{x}, x_3) \in \mathbb{R}^3,\ (\acute{x}, 0) \in \omega,\ 0 \leq x_3 \leq \varepsilon h(\acute{x})\}.$$

Where h is a function of class C^1 defined on ω with

$$0 \leq h_* \leq h(\acute{x}) \leq h^*,\ \forall (\acute{x}, 0) \in \omega.$$

Let σ^ε denotes the total Cauchy stress tensor

$$\sigma^\varepsilon = -\rho^\varepsilon I + \sigma^{D,\varepsilon}, \tag{2.1.1}$$

where ρ^ε the pressure and $\sigma^{D,\varepsilon}$ is the deviatoric part. This type of fluids is supposed to be viscoplastic.

We give the Herschel-Bulkley model which gives the relation between $\sigma^{D,\varepsilon}$ and $D(u^\varepsilon)$

$$\left.\begin{array}{ll} \sigma^{D,\varepsilon} = \alpha^\varepsilon \dfrac{D(u^\varepsilon)}{|D(u^\varepsilon)|} + \mu \left|D(u^\varepsilon)\right|^{r-2} D(u^\varepsilon), & \text{when } D(u^\varepsilon) \neq 0, \\ |\sigma^{D,\varepsilon}| \leq \alpha^\varepsilon\ , & \text{when } D(u^\varepsilon) = 0, \end{array}\right\} \text{in } \Omega^\varepsilon, \tag{2.1.2}$$

where

$\alpha^\varepsilon \geq 0$ is the yield stress,

$\mu > 0$ is the constant viscosity,

the notation $|D|$ represents the matrix norm,
u^ε is the velocity field and $D(u^\varepsilon) = \dfrac{1}{2}(\nabla u^\varepsilon + (\nabla u^\varepsilon)^T)$, and $1 < r < \infty$.
The law of conservation of momentum is given

$$-\mathrm{div}(\sigma^\varepsilon) = f^\varepsilon \quad \text{in } \Omega^\varepsilon, \tag{2.1.3}$$

where $f^\varepsilon = (f_i^\varepsilon)_{1\leq i\leq 3}$, denotes the body forces.
The incompressibility equation

$$\mathrm{div}(u^\varepsilon) = 0 \;\; \text{in } \Omega^\varepsilon, \tag{2.1.4}$$

we first introduce the vector function $g = (g_1, g_2, g_3)$ such that:

$$\int_{\Gamma^\varepsilon} g.nds = 0.$$

There exists a function G^ε ([22]) such that:

$$G^\varepsilon \in \left(W^{1,r}(\Omega^\varepsilon)\right)^3 \text{ with } div(G^\varepsilon) = 0 \text{ in } \Omega^\varepsilon, G^\varepsilon = g \text{ on } \Gamma^\varepsilon$$

The boundary conditions are described as

∘ On Γ_1^ε, the upper surface is assumed to be fixed, therefore

$$u^\varepsilon = 0. \tag{2.1.5}$$

∘ On Γ_L^ε, the velocity is known and is parallel to the $\omega-$plane

$$u^\varepsilon = g \text{ with } g_3 = 0. \tag{2.1.6}$$

∘ On ω, there is no-flux condition across ω so that

$$u^\varepsilon.n = 0. \tag{2.1.7}$$

the tangential velocity on ω is unknown and satisfies **Coulomb** boundary conditions:

$$\left.\begin{array}{c} |\sigma_T^\varepsilon| < k^\varepsilon\, |\sigma_n^\varepsilon| \Longrightarrow u_T^\varepsilon = s, \\ |\sigma_T^\varepsilon| = k^\varepsilon\, |\sigma_n^\varepsilon| \Longrightarrow \exists\lambda > 0 \text{ such that } u_T^\varepsilon = s - \lambda\sigma_T^\varepsilon, \end{array}\right\} \text{on } \omega. \tag{2.1.8}$$

Where $k^{\varepsilon} \geq 0$ is the coefficient of friction (see [22]) , $|.|$ is the Euclidean norm, $n = (n_1, n_2, n_3)$ is the unit outward normal to Γ^{ε}.

One defines the normal and tangential components of velocity by

$$u_n^{\varepsilon} = u^{\varepsilon}.n$$
$$u_{T_i}^{\varepsilon} = u_i^{\varepsilon} - \ u_n^{\varepsilon}.n_i$$

In the same way, the normal and tangential components of the tensor of the stresses are defined by

$$\sigma_n^{\varepsilon} = (\sigma^{\varepsilon}.n_i)\,.n_j$$
$$\sigma_{T_i}^{\varepsilon} = \sigma_{ij}^{\varepsilon}.n_j - \ \sigma_n^{\varepsilon}.n_i.$$

The complete problem (P^{ε}), consists to find a velocity field

$u^{\varepsilon} = (u_i^{\varepsilon})_{1\leq i\leq 3} : \Omega^{\varepsilon} \to \mathbb{R}^3$, which checks the following equations and boundary conditions:

$$\left\{\begin{array}{lr} -\mathrm{div}(\sigma^{\varepsilon}) = f^{\varepsilon} & \text{in } \Omega^{\varepsilon}, \\ \sigma^{\varepsilon} = -\rho^{\varepsilon} I + \sigma^{D,\varepsilon}, & \text{in } \Omega^{\varepsilon}, \\ \mathrm{div}(u^{\varepsilon}) = 0 & \text{in } \Omega^{\varepsilon}, \\ u^{\varepsilon} = 0 & \text{on } \Gamma_1^{\varepsilon}, \\ u^{\varepsilon} = g \text{ with } g_3 = 0. & \text{on } \Gamma_L^{\varepsilon}, \\ u^{\varepsilon}.n = 0 & \text{on } \omega, \\ \left.\begin{array}{r} |\sigma_T^{\varepsilon}| < k^{\varepsilon}\,|\sigma_n^{\varepsilon}| \Longrightarrow u_T^{\varepsilon} = s, \\ |\sigma_T^{\varepsilon}| = k^{\varepsilon}\,|\sigma_n^{\varepsilon}| \Longrightarrow \exists\lambda > 0 \text{ such that } u_T^{\varepsilon} = s - \lambda\sigma_T^{\varepsilon}, \end{array}\right\} & \text{on } \omega. \end{array}\right. \qquad (P^{\varepsilon})$$

Before giving the variational formulation of the strong problem (P^{ε}) we set up the following lemma.

Lemma 2.1.1 *The condition* (2.1.8) *is equivalent to the relation:*

$$(u_T^{\varepsilon} - s)\,\sigma_T^{\varepsilon} + k^{\varepsilon}\,|\sigma_n^{\varepsilon}|\,|u_T^{\varepsilon} - s| = 0, \ \textit{on } \omega \tag{2.1.9}$$

Proof. ·Assume that u^{ε} meet the boundary conditions of **Coulomb** (2.1.8).
→ If $|\sigma_T^{\varepsilon}| < k^{\varepsilon}\,|\sigma_n^{\varepsilon}|$ then $u_T^{\varepsilon} = s$.

$$(u_T^\varepsilon - s)\,\sigma_T^\varepsilon + k^\varepsilon\,|\sigma_n^\varepsilon|\,|u_T^\varepsilon - s| = (s-s)\sigma_T^\varepsilon + (s-s)k^\varepsilon\,|\sigma_n^\varepsilon| = 0$$

$\rightarrow$ If $|\sigma_T^\varepsilon| = k^\varepsilon\,|\sigma_n^\varepsilon|$, there is $\lambda \geq 0$ such as

$$u_T^\varepsilon = s - \lambda\sigma_T^\varepsilon,$$

from where

$$(u_T^\varepsilon - s)\,\sigma_T^\varepsilon + k^\varepsilon\,|\sigma_n^\varepsilon|\,|u_T^\varepsilon - s| = -\lambda\,|\sigma_T^\varepsilon|^2 + \lambda\,|\sigma_T^\varepsilon|^2 = 0$$

$\cdot$Conversely, suppose that

$$(u_T^\varepsilon - s)\,\sigma_T^\varepsilon + k^\varepsilon\,|\sigma_n^\varepsilon|\,|u_T^\varepsilon - s| = 0$$

$\rightarrow$ If $|\sigma_T^\varepsilon| = k^\varepsilon\,|\sigma_n^\varepsilon|$ so

$$(u_T^\varepsilon - s)\,\sigma_T^\varepsilon = -\,|\sigma_T^\varepsilon|\,|u_T^\varepsilon - s|\,,$$

hence the existence of a $\lambda \geq 0$ such as

$$\begin{aligned} u_T^\varepsilon - s &= \lambda\sigma_\tau^\varepsilon, \\ u_T^\varepsilon &= s - \lambda\sigma_T^\varepsilon. \end{aligned}$$

$\rightarrow$ If $|\sigma_T^\varepsilon| < k^\varepsilon\,|\sigma_n^\varepsilon|$ so

$$\begin{aligned} (u_T^\varepsilon - s)\,\sigma_T^\varepsilon + k^\varepsilon\,|\sigma_n^\varepsilon|\,|u_T^\varepsilon - s| &= 0 \geq -\,|u_T^\varepsilon - s|\,|\sigma_T^\varepsilon| + k^\varepsilon\,|\sigma_n^\varepsilon|\,|u_T^\varepsilon - s| \\ &\geq |u_T^\varepsilon - s|\,(k^\varepsilon\,|\sigma_n^\varepsilon| - |\sigma_T^\varepsilon|). \end{aligned}$$

■

2.2 Weak formulation in the domain Ω^ε

In this section we define the functional framework in which we are going to work, and we obtain the weak formulation of the problem (P^ε).

To get a weak formulation, we introduce

Let $L^r(\Omega^\varepsilon)$ be the usual **Lebesgue** space with the norm denoted by $\|.\|_{L^r(\Omega^\varepsilon)}$

$$\left(W^{1,r}(\Omega^\varepsilon)\right)^3=\left\{v\in(L^r(\Omega^\varepsilon))^3;\qquad \frac{\partial v_i}{\partial x_j}\in L^r(\Omega^\varepsilon);\quad \forall i,j=1,2,3\right\},$$

$W^{1,r}(\Omega^\varepsilon)$ is a **Sobolev** space with the following norm

$$\|v\|_{L^r(\Omega^\varepsilon)}=\left(\sum_{i=1}^{3}\int\limits_{\Omega^\varepsilon}|v|^r\,d\acute{x}dx_3+\sum_{i=1}^{3}\int\limits_{\Omega^\varepsilon}\left|\frac{\partial v}{\partial x_j}\right|^r d\acute{x}dx_3\right)^{\frac{1}{r}}$$

$W_0^{1,r}(\Omega^\varepsilon)$ is a vector subspace of the functions $W^{1,r}(\Omega^\varepsilon)$ nulls on Γ^ε, and it's the closured of $D(\Omega^\varepsilon)$ in $W^{1,r}(\Omega^\varepsilon)$

we denote by $W^{-1,q}(\Omega^\varepsilon)$ its topological dual where, $\dfrac{1}{r}+\dfrac{1}{q}=1$

Moreover, we need

$$K^\varepsilon=\left\{\varphi\in\left(W^{1,r}(\Omega^\varepsilon)\right)^3:\varphi=G^\varepsilon \text{ on } \Gamma_L^\varepsilon,\ \varphi=0 \text{ on } \Gamma_1^\varepsilon \text{ and } \varphi.n=0 \text{ on } \omega\right\},$$

$$K_{\text{div}}^\varepsilon=\{\varphi\in K^\varepsilon;\quad \text{div}(\varphi)=0\ \},$$

$$L_0^q(\Omega^\varepsilon)=\left\{q\in L^q(\Omega^\varepsilon);\int\limits_{\Omega^\varepsilon}q\,d\acute{x}dx_3=0\right\},$$

The variational formulation of the problem (P^ε) we noted by (pb_1) and is written

$$\left\{\begin{array}{l}\text{Find } u^\varepsilon\in K_{\text{div}}^\varepsilon \text{ and } \rho^\varepsilon\in L_0^r(\Omega^\varepsilon \text{ such that}\\ \quad a(u^{\ \varepsilon},\varphi-u^\varepsilon)-(\rho^\varepsilon,\text{div}(\varphi))+\tilde{j}(u^\varepsilon,\varphi)\\ \quad -\tilde{j}(u^\varepsilon,u^\varepsilon)\geq(f^{\ \varepsilon},\varphi-u^{\ \varepsilon}),\quad \forall\varphi\in K^{\ \varepsilon}.\end{array}\right. \qquad (pb_1)$$

where

$$a(u^\varepsilon,\varphi)=\sum_{i,j=1}^{3}\int_{\Omega^\varepsilon}\mu^\varepsilon\left|D(u^\varepsilon)\right|^{r-2}D(u^\varepsilon)D(\varphi)d\acute{x}dx_3, \qquad (2.2.1)$$

$$(\rho^\varepsilon,\text{div}(\varphi))=\int_{\Omega^\varepsilon}\rho^\varepsilon\,\text{div}(\varphi)d\acute{x}dx_3, \qquad (2.2.2)$$

$$(f^\varepsilon, \varphi) = \int_{\Omega^\varepsilon} f^\varepsilon \varphi d\acute{x} dx_3 = \sum_{i=1}^{3} \int_{\Omega^\varepsilon} f_i^\varepsilon \varphi_i d\acute{x} dx_3, \tag{2.2.3}$$

$$\tilde{j}(u^\varepsilon, \varphi) = \int_\omega k^\varepsilon |\sigma_n| |\varphi - s| d\acute{x} + \sqrt{2}\alpha^\varepsilon \int_{\Omega^\varepsilon} |D(\varphi)| d\acute{x} dx_3, \tag{2.2.4}$$

The integral $\tilde{j}(u^\varepsilon, v)$ has no meaning for $u^\varepsilon \in K^\varepsilon$. Indeed, σ_n^ε is defined by duality as an element of $W^{-\frac{1}{2},r}(\omega)$ and $|\sigma_n^\varepsilon|$ is not well defined on ω. So following ([22]) we replace σ_n^ε by some regularization $R(\sigma_n^\varepsilon)$, where R is a regularization operator from $W^{-\frac{1}{2},r}(\omega)$ into $L_+^r(\omega)$ can be obtained by convolution with a positive regular function and defined by

$$\forall \tau \in W^{-\frac{1}{2},r}(\omega), R(\tau) \in L_+^r(\omega), R(\tau)(x) = \left| \langle \tau, \varphi(x - \tau) \rangle_{W^{-\frac{1}{2},r}(\omega), W_{00}^{\frac{1}{2},r}(\omega)} \right|,$$

φ is a given positive function of class C^∞ with compact support in ω and $W^{-\frac{1}{2},r}(\omega)$ is the dual space to

$$W_{00}^{\frac{1}{2},r}(\omega) = \left\{ \varphi_{|\omega} : \varphi \in W^{1,r}(\Omega^\varepsilon);\ \varphi = 0 \text{ on } \Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon \right\}.$$

After the regularization, we get the new problem:

Problem 2.2.1 **($P_\kappa^{\varepsilon,\phi}$)** *.Find* $(u^\varepsilon, \rho^\varepsilon) \in K_{\text{div}}^\varepsilon \times L_0^q(\Omega^\varepsilon)$, *such that*

$$a(u^\varepsilon, \varphi - u^\varepsilon) - (\rho^\varepsilon, \text{div}(\varphi)) + j(u^\varepsilon, \varphi) - j(u^\varepsilon, u^\varepsilon) \geq (f^\varepsilon, \varphi - u^\varepsilon), \quad \forall \varphi \in K^\varepsilon(\Omega^\varepsilon) \quad (pb2)$$

where

$$j(u^\varepsilon, \varphi) = \int_\omega k^\varepsilon |R(\sigma_n^\varepsilon)| |\varphi - s| d\acute{x} + \sqrt{2}\alpha^\varepsilon \int_{\Omega^\varepsilon} |D(\varphi)| d\acute{x} dx_3.$$

Lemma 2.2.1 *if* $(u^\varepsilon, \rho^\varepsilon) \in K_{\text{div}}^\varepsilon \times L_0^q(\Omega^\varepsilon)$, *are solutions of the problem* (P^ε) *then they check* $(pb2)$.

Proof. By multiplying (2.3.1), by $\varphi - u^\varepsilon$, where $\varphi \in K^\varepsilon$ and using the **Green**'s formula, we get

$$\int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon \frac{\partial}{\partial x_i}(\varphi_i - u_i^\varepsilon) d\acute{x} dx_3 - \int_{\Gamma^\varepsilon} \sigma_{ij}^\varepsilon n_j (\varphi_i - u_i^\varepsilon) ds = \int_{\Omega^\varepsilon} f_i{}^\varepsilon (\varphi_i - u_i^\varepsilon) d\acute{x} dx_3.$$

According to the boundary conditions $(2.1.5)-(2.1.6)$ we find

$$\int\limits_{\Gamma^\varepsilon} \sigma_{ij}^\varepsilon n_j(\varphi_i - u_i^\varepsilon)ds = \int\limits_{\omega} \sigma_{ij}^\varepsilon n_j(\varphi_i - u_i^\varepsilon)d\acute{x}.$$

We know that: $\sigma_{ij}^\varepsilon n_j = \sigma_{T_i}^\varepsilon + \sigma_n^\varepsilon n_i$ then we get

$$\int_{\Gamma^\varepsilon} \sigma_{ij}^\varepsilon n_j(\varphi_i - u_i^\varepsilon)ds = \int_{\omega} \sigma_{T_i}^\varepsilon(\varphi_i - u_i^\varepsilon)d\acute{x} + \int_{\omega} \sigma_n^\varepsilon n_i(\varphi_i - u_i^\varepsilon)d\acute{x}$$

as $n_i(\varphi_i - u_i^\varepsilon) = 0,$ we have

$$\int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon \frac{\partial}{\partial x_i}(\varphi_i - u_i^\varepsilon)d\acute{x}dx_3 - \int_{\omega} \sigma_{T_i}^\varepsilon(\varphi_i - u_i^\varepsilon)d\acute{x} = \int_{\Omega^\varepsilon} f_i{}^\varepsilon(\varphi_i - u_i^\varepsilon)d\acute{x}dx_3, \tag{2.2.5}$$

In $(2.2.5),$ we add and subtract the term $\int_\omega k^\varepsilon\,|\sigma_n^\varepsilon|\,(|\varphi_T - s| - |u_T^\varepsilon - s|)d\acute{x},$ we obtain

$$\begin{aligned}
&\int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon \frac{\partial}{\partial x_i}(\varphi_i - u_i^\varepsilon)d\acute{x}dx_3 - \int_{w} \sigma_{T_i}^\varepsilon(\varphi_i - u_i^\varepsilon)d\acute{x} + \\
&\int_{\omega} k^\varepsilon\,|\sigma_n^\varepsilon|\,(|\varphi_T - s| - |u_T^\varepsilon - s|)d\acute{x} - \int_{\omega} k^\varepsilon\,|\sigma_n^\varepsilon|\,(|\varphi_T - s| - |u_T^\varepsilon - s|)d\acute{x}, \\
= &\int\limits_{\Omega^\varepsilon} f_i{}^\varepsilon(\varphi_i - u_i^\varepsilon)d\acute{x}dx_3.
\end{aligned}$$

set

$$T = \int\limits_{\omega} \sigma_{T_i}^\varepsilon((\varphi_T - s) - (u_T^\varepsilon - s))d\acute{x} + \int_{\omega} k^\varepsilon\,|\sigma_n^\varepsilon|\,(|\varphi_T - s| - |u_T^\varepsilon - s|)d\acute{x},,$$

then

$$T = \int_{\omega} \sigma_{T_i}^\varepsilon(\varphi_T - s)d\acute{x} + \int_{\omega} k^\varepsilon\,|\sigma_n^\varepsilon|\,|\varphi_T - s|\,d\acute{x},$$

thus

$$\sigma_{T_i}^\varepsilon(\varphi_T - s) \geq -\left|\sigma_{T_i}^\varepsilon\right|\,|\varphi_T - s| \geq -k^\varepsilon\,|\sigma_n^\varepsilon|\,|\varphi_T - s| > 0, \text{ on } \omega,$$

and

$$T = \int_{\omega} \sigma_{T_i}^\varepsilon(\varphi_T - s)d\acute{x} + \int_{\omega} k^\varepsilon\,|\sigma_n^\varepsilon|\,|\varphi_T - s|\,d\acute{x} \geq 0,$$

which gives

$$\int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon \frac{\partial}{\partial x_i}(\varphi_i - u_i^\varepsilon)d\acute{x}dx_3 + \int_{\omega} k^\varepsilon\,|\sigma_n^\varepsilon|\,|\varphi_T - s|\,d\acute{x} - \int_{\omega} k^\varepsilon\,|\sigma_n^\varepsilon|\,|u_T^\varepsilon - s|\,d\acute{x} \geq \int_{\Omega^\varepsilon} f_i{}^\varepsilon(\varphi_i - u_i^\varepsilon)d\acute{x}dx_3.$$

Replacing $\sigma_{ij}^{\varepsilon}$ by the expression which given in $(2.1.2)$

$$
\begin{aligned}
& a(u^{\varepsilon},\varphi-u^{\varepsilon})+\alpha^{\varepsilon}\int\limits_{\Omega^{\varepsilon}}\frac{dij(u^{\varepsilon})}{|D(u^{\varepsilon})|}d_{ij}(\varphi-u^{\varepsilon})d\acute{x}dx_3- \\
& \int_{\Omega^{\varepsilon}}\rho^{\varepsilon}\operatorname{div}(\varphi-u^{\varepsilon})d\acute{x}dx_3+\int_{\omega}k^{\varepsilon}R\left|\sigma_n^{\varepsilon}\right|\left|\varphi_T-s\right|d\acute{x}- \\
& -\int_{\omega}k^{\varepsilon}\left|\sigma_n^{\varepsilon}\right|\left|u_T^{\varepsilon}-s\right|d\acute{x} \\
\geq\ & \int_{\Omega^{\varepsilon}}f^{\ \varepsilon}\varphi d\acute{x}dx_3,\ \forall\varphi\in K^{\varepsilon}
\end{aligned}
$$

according to the inequality of **Cauchy Swartz**, we deduce

$$d_{ij}(u^{\varepsilon})d_{ij}(\varphi)\leq|D(u^{\varepsilon})|\,|D(\varphi)|$$

we get

$$
\begin{aligned}
& a(u^{\varepsilon},\varphi-u^{\varepsilon})+\alpha^{\varepsilon}\int\limits_{\Omega^{\varepsilon}}\frac{D(u^{\varepsilon})}{|D(u^{\varepsilon})|}d_{ij}(\varphi-u^{\varepsilon})d\acute{x}dx_3- \\
& \int_{\Omega^{\varepsilon}}\rho^{\varepsilon}\operatorname{div}(\varphi-u^{\varepsilon})d\acute{x}dx_3+\int_{\omega}k^{\varepsilon}\left|\sigma_n^{\varepsilon}\right|\left|\varphi_T-s\right|d\acute{x}- \\
& -\int_{\omega}k^{\varepsilon}\left|\sigma_n^{\varepsilon}\right|\left|u_T^{\varepsilon}-s\right|d\acute{x} \\
\geq\ & \int_{\Omega^{\varepsilon}}f^{\ \varepsilon}\varphi d\acute{x}dx_3,\ \forall\varphi\in K^{\varepsilon}
\end{aligned}
$$

then (pb_1) ■

Remark 2.2.1 *If $\varphi\in K_{\operatorname{div}}^{\varepsilon}$, the problem $(pb1)$ becomes:*

$$
\begin{cases}
\text{Find } \ u^{\varepsilon}\in K_{\operatorname{div}}^{\varepsilon} \ \text{ such that} \\
a(u^{\varepsilon},\varphi-u^{\varepsilon})+j(u^{\varepsilon},\varphi)-j(u^{\varepsilon},u^{\varepsilon})\geq(f^{\ \varepsilon},\varphi-u^{\ \varepsilon}),\quad\forall\varphi\in K_{\operatorname{div}}^{\ \varepsilon}(\Omega^{\varepsilon}),
\end{cases}
\tag{pb_3}
$$

Theorem 2.2.1 *If $f^{\varepsilon}\in(L^q(\Omega^{\varepsilon}))^3$ and the friction coefficient k^{ε} is a non-negative function in $L^{\infty}(\omega)$ then there exists $(u^{\varepsilon},\rho^{\varepsilon})\in k_{\operatorname{div}}^{\varepsilon}\times L_0^q(\Omega^{\varepsilon})$ solution to the problem $\left(P_{\kappa}^{\varepsilon,\phi}\right)$.*

Moreover, for small k^{ε} the solution is unique.

Proof. To proof the existence and uniqueness result of $(pb1)$, we define the following intermediate problem

$$\begin{aligned} a(u^\varepsilon, \varphi - u^\varepsilon) + \int_\omega Y(|\varphi - s| - |u^\varepsilon - s|)d\acute{x} + \delta_{K^\varepsilon_{\text{div}}}(\varphi) - \delta_{K^\varepsilon_{\text{div}}}(u^\varepsilon) \\ \geq (f^{\,\varepsilon}, \varphi - u^{\,\varepsilon}), \quad \forall \varphi \in W^{1,r}_{\text{div}}(\Omega^\varepsilon), \end{aligned} \tag{2.2.7}$$

where, Y is the mapping defined from $L^r(\omega)$ into $L^r(\omega)$ by $Y \to k^\varepsilon R(\sigma^\varepsilon_n)$ and

$$W^{1,r}_{\text{div}}(\Omega^\varepsilon) = \left\{ v \in (W^{1,r}(\Omega^\varepsilon))^3;\ \text{div}(v) = 0 \right\},$$

$$\delta_{K^\varepsilon_{\text{div}}} \begin{cases} 0 \text{ if } \varphi \in K^\varepsilon_{\text{div}} \\ +\infty \text{ if } \varphi \notin K^\varepsilon_{\text{div}} \end{cases}$$

· $a(u^\varepsilon, \varphi - u^\varepsilon)$ is bounded, coercive, hemicontinuous and strictly monotone ([30]) .

· $Y + \delta_{K^\varepsilon_{\text{div}}}$ is a proper, convex and continuous function in $L^r(\omega)$, then we apply **Tichovo**'s fixed point theorem (as in [15]) .

·We show the existence and the uniqueness of $u^\varepsilon \in K^\varepsilon_{\text{div}}$ satisfy $(2.2.7)$. The proof of the existence of $\rho^\varepsilon \in L^q_0(\Omega^\varepsilon)$ such that $(u^\varepsilon, \rho^\varepsilon)$ satisfy $(pb1)$ is given in Reference ([40]). ■

2.3 Asymptotic analysis of problem in fixed domain

The object of this section is the asymptotic analysis of the previous problem by changing the scale, to bring the study back to a domain Ω independent of ε, on which we set new unknowns.

We obtain the variational formulation in the fixed domain, we prove a priori estimates on the independent solution of ε, by different inequalities and we deduce the results of convergence.

2.3.1 problem in transpose form

For the asymptotic analysis of the problem (P^ε), we use the approach which consists in transposing the problem initially posed in the field Ω^ε which depends on a small parameter ε to an equivalent problem whose fixed domain Ω independent of ε.

To do this, we introduce a small change of variable $z = \dfrac{x_3}{\varepsilon}$, so $(\acute{x}, x_3)$ in Ω^ε we have $(\acute{x}, z)$ in Ω with

$$\Omega = \{(\acute{x}, z) \in \mathbb{R}^3,\ (\acute{x}, 0) \in \omega,\ 0 < z < h(\acute{x})\}.$$

We designate by $\Gamma = \overline{\omega} \cup \overline{\Gamma_L} \cup \overline{\Gamma_1}$, the fixed domain boundary.

We now set in Ω new functions **independent of** ε

$$\begin{cases} \hat{u}_i^\varepsilon(\acute{x}, z) = u_i^\varepsilon(\acute{x}, x_3),\ i = 1, 2, \\ \hat{u}_3^\varepsilon(\acute{x}, z) = \varepsilon^{-1} u_3^\varepsilon(\acute{x}, x_3). \\ \quad \hat{\rho}^\varepsilon(\acute{x}, x_3) = \ \varepsilon^r\ \hat{\rho}^\varepsilon(\acute{x}, z). \end{cases} \tag{2.3.1}$$

$$\begin{cases} \hat{f}\ (\acute{x}, z) = \varepsilon^r\ f^\varepsilon(\acute{x}, x_3), \\ \hat{\alpha} = \varepsilon^{r-1} \alpha^\varepsilon, \\ \hat{k} = \varepsilon^{r-1} k^\varepsilon. \end{cases} \tag{2.3.2}$$

Let $\hat{G}(\acute{x}, z)$ such as

$$\operatorname{div} \hat{G} = 0 \text{ and } \hat{G} = \hat{g} \text{ on } \Gamma.$$

The vector $\hat{G}$ introduced previously is defined as follows

$$\begin{aligned} \hat{G}_i(\acute{x}, z) &= G_i^\varepsilon(\acute{x}, x_3),\ i = 1, 2, \\ \hat{G}_3(\acute{x}, z) &= \varepsilon^{-1} G_3^\varepsilon(\acute{x}, x_3). \end{aligned} \tag{2.3.3}$$

Let

$$K = \left\{ \hat{\varphi} \in \left(W^{1,r}(\Omega)\right)^3;\quad \hat{\varphi} = \hat{G} \text{ on } \Gamma_L\ \cup \Gamma_1 \text{ and } \hat{\varphi}.n = 0 \text{ on } \omega \right\},$$

$$K_{\text{div}} = \{\hat{\varphi} \in K;\quad \operatorname{div}(\hat{\varphi}) = 0\ \},$$

$$\Pi(K) = \left\{ \hat{\varphi} \in \left(W^{1,r}(\Omega)\right)^3;\ \hat{\varphi}_i = \hat{G} \text{ on } \Gamma_L \cup \Gamma_1\ , i = 1, 2 \right\},$$

$$\tilde{\Pi}(K) = \left\{ \hat{\varphi} \in \Pi(K); \quad \bar{\varphi} \text{ satisfies } (\acute{D}) \right\}.$$

The condition $(\acute{D})$ given by

$$\int_{\Omega} \left(\hat{\varphi}_1 \frac{\partial \theta}{\partial x_1} + \hat{\varphi}_2 \frac{\partial \theta}{\partial x_2} \right) d\acute{x} dz = 0, \ \forall \hat{\varphi} \in \left(L^r(\Omega) \right)^2 \text{ and } \theta \in C_0^{\infty}(\Omega).$$

We denote by

$$V_z = \left\{ \hat{\varphi} = (\hat{\varphi}_1, \hat{\varphi}_2) \in \left(L^r(\Omega) \right)^2; \quad \frac{\partial \hat{\varphi}_i}{\partial z} \in L^r(\Omega), i = 1, 2; \hat{\varphi} = 0 \text{ on } \Gamma_1 \right\}.$$

It is clear that V_z is a Banach space with the norm

$$\|\hat{\varphi}\|_{V_z} = \left(\sum_{i=1}^{2} \|\hat{\varphi}_i\|_{L^r(\Omega)}^r + \left\| \frac{\partial \hat{\varphi}_i}{\partial z} \right\|_{L^r(\Omega)}^r \right)^{\frac{1}{r}}.$$

and define its linear subspace

$$\tilde{V}_z = \left\{ \hat{\varphi} \in \tilde{\Pi}(K); \quad \hat{\varphi} \text{ satisfies the condition } (\acute{D}) \right\}$$

By injecting the new data and unknown factors in $(pb1)$, and after multiplication by ε^{r-1} the variational problem $\left(P_{\kappa}^{\varepsilon,\phi}\right)$ is equivalent to the following

Problem 2.3.1 (P_{κ}) *.Find* $(\hat{u}, \hat{\rho}) \in K_{\text{div}} \times L_0^q(\Omega)$, *such that*

$$\left\{ \begin{array}{c} a_0(\hat{u}^{\varepsilon}, \hat{\varphi} - \hat{u}^{\varepsilon}) - (\hat{\rho}^{\varepsilon}, \operatorname{div}(\hat{\varphi} - \hat{u})) \\ +\hat{j}(\hat{u}^{\varepsilon}, \hat{\varphi}) - \hat{j}(\hat{u}^{\varepsilon}, \hat{u}^{\varepsilon}) \geq (\hat{f}, \hat{\varphi} - \hat{u}^{\varepsilon}), \quad \forall \hat{\varphi} \in K^{\varepsilon}(\Omega), \end{array} \right. \tag{2.3.4}$$

Where

$$\begin{aligned} a_0(\hat{u}^\varepsilon, \hat{\varphi} - \hat{u}^\varepsilon) &= \mu\varepsilon^2 \sum_{i,j=1}^{2} \int_\Omega \left|\tilde{D}(\hat{u}^\varepsilon)\right|^{r-2} \left(\frac{1}{2}\left(\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i}\right)\right) \frac{\partial}{\partial x_j}(\hat{\varphi}_i - \hat{u}_i^\varepsilon) d\acute{x} dz \\ &\quad + \mu \sum_{i,j=1}^{2} \int_\Omega \left|\tilde{D}(\hat{u}^\varepsilon)\right|^{r-2} \left(\frac{1}{2}\left(\frac{\partial \hat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i}\right)\right) \frac{\partial}{\partial z}(\hat{\varphi}_i - \hat{u}_i^\varepsilon) \\ &\quad + \int_\Omega \left|\tilde{D}(\hat{u}^\varepsilon)\right|^{r-2} \varepsilon^2 \frac{\partial}{\partial x_j}(\hat{\varphi}_3 - \hat{v}_3^\varepsilon) d\acute{x} dz + \mu\varepsilon^2 \int\limits_\Omega \left|\tilde{D}(\hat{u}^\varepsilon)\right|^{r-2} \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \frac{\partial}{\partial z}(\hat{\varphi}_3 - \hat{u}_3^\varepsilon) d\acute{x} dz \\ &\quad + \mu \sum_{,j=1}^{2} \int_\Omega \left|\tilde{D}(\hat{u}^\varepsilon)\right|^{r-2} \left(\frac{1}{2}\left(\varepsilon^2 \frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial z}\right)\right) \frac{\partial}{\partial x_j}(\hat{\varphi}_i - \hat{u}_i^\varepsilon) d\acute{x} dz \end{aligned}$$

$$(\hat{\rho}^\varepsilon, \mathrm{div}(\hat{\varphi} - \hat{u}^\varepsilon)) = \int_\Omega \hat{\rho}^\varepsilon \,\mathrm{div}((\hat{\varphi} - \hat{u}^\varepsilon)) d\acute{x} dz, \tag{2.3.5}$$

$$(\hat{f}, \hat{\varphi} - \hat{u}^\varepsilon) = \sum_{i=1}^{2} \int_\Omega \hat{f}_i (\hat{\varphi}_i - \hat{u}_i^\varepsilon) d\acute{x} dz + \int_\Omega \varepsilon \hat{f}_3 (\hat{\varphi}_3 - \hat{u}_3^\varepsilon) d\acute{x} dz, \tag{2.3.6}$$

$$\hat{\jmath}(\hat{u}^\varepsilon, \hat{\varphi}) = \int_\omega \hat{k} \left| R(\hat{\sigma}_n) \right| \left| \hat{\varphi} - s \right| d\acute{x} + \sqrt{2}\hat{\alpha} \int\limits_\Omega \left| \tilde{D}(\hat{\varphi}) \right| d\acute{x} dz, \tag{2.3.7}$$

$$\left|\tilde{D}\ (\hat{u}^\varepsilon)\right| = \left[\frac{1}{4} \sum_{i,j=1}^{2} \varepsilon^2 \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i}\right)^2 + \frac{1}{2} \sum_{i=1}^{2} \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i}\right)^2 + \varepsilon^2 \left(\frac{\partial \hat{u}_3^\varepsilon}{\partial z}\right)^2\right]^{\frac{1}{2}}.$$

We introduce some inequalities which will be used in the next. Can found in ([2]) the detail description

Korn inequality

$$\|\nabla u^\varepsilon\|_{L^r(\Omega^\varepsilon)} \le C \|Du^\varepsilon\|_{L^r(\Omega^\varepsilon)} \tag{2.3.8}$$

Poincaré inequality

$$\|u^\varepsilon\|_{L^r(\Omega^\varepsilon)} \le \varepsilon h^* \left\|\frac{\partial u^\varepsilon}{\partial z}\right\|_{L^r(\Omega^\varepsilon)} \tag{2.3.9}$$

Young inequality

$$ab \le \frac{a^r}{r} + \frac{b^{\acute{r}}}{\acute{r}}, \forall (a,b) \in \mathbb{R}^2 \tag{2.3.10}$$

2.4 Main convergence results and uniqueness of solution

In this section, we prove the following convergence results of $(\hat{u}^\varepsilon, \hat{\rho}^\varepsilon)$ to (u^*, ρ^*) and give the limit problem in the fixed domain Ω.

Finally, we prove the uniqueness of the solution.

2.4.1 Convergence results

Theorem 2.4.1 *Under the assuptions as in Theorem* $(2.2.1)$, *then there exist* $u^* = (u_1^*, u_2^*) \in \tilde{V}_z$ *and* $\rho^* \in L_0^r(\Omega)$ *such that*

$$\hat{u}_i^\varepsilon \rightharpoonup \hat{u}_i^* \quad (1 \le i \le 2) \text{ weakly in } \tilde{V}_z \tag{2.4.1}$$

$$\varepsilon \frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} \rightharpoonup 0, \quad (1 \le i \le 2) \text{ weakly in } L^r(\Omega) \tag{2.4.2}$$

$$\varepsilon \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \rightharpoonup 0 \text{ weakly in } L^r(\Omega) \tag{2.4.3}$$

$$\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \rightharpoonup 0, \quad (1 \le i \le 2) \text{ weakly in } L^r(\Omega) \tag{2.4.4}$$

$$\varepsilon \hat{u}_3^\varepsilon \rightharpoonup 0 \text{ weakly in } L^r(\Omega) \tag{2.4.5}$$

$$\hat{\rho}^\varepsilon \rightharpoonup \rho^* \qquad \text{weakly in } L_0^q(\Omega) \tag{2.4.6}$$

Proof. The proof of $(2.4.1) - (2.4.6)$ is based on the following lemma ■

Lemma 2.4.1 *Let* $(\hat{u}^\varepsilon, \hat{\rho}^\varepsilon) \in K_{\mathrm{div}}^\varepsilon(\Omega) \times L_0^q(\Omega)$ *be the solution of variational problem* $(2.3.4)$, *then there are constants* C, C' *independent of* ε *such that*

$$\sum_{i,j=1}^{2} \left\| \varepsilon \frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} \right\|_{L^r}^r + \left\| \varepsilon \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \right\|_{L^r}^r + \sum_{i=1}^{2} \left(\left\| \frac{\partial \hat{u}_i^\varepsilon}{\partial z} \right\|_{L^r}^r + \left\| \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \right\|_{L^r}^r \right) \le C \tag{2.4.7}$$

$$\left\| \frac{\partial \hat{\rho}^\varepsilon}{\partial x_i} \right\|_{W^{-1,q}} \le C' \text{ for } i = 1, 2 \tag{2.4.8}$$

$$\left\| \frac{\partial \hat{\rho}^\varepsilon}{\partial z}(t) \right\|_{W^{-1,q}} \le \varepsilon C' \tag{2.4.9}$$

Proof. choosing $\varphi = G^\varepsilon$ in $(pb3)$ and using the fact that $G^\varepsilon = s$ on $\omega,$ we find

$$a(u^\varepsilon, u^\varepsilon) \leq a(u^\varepsilon, G^\varepsilon) + (f^\varepsilon, u^\varepsilon) - (f^\varepsilon, G^\varepsilon) \tag{2.4.10}$$

From **Korn**'s inequality, we have $C_K > 0$ independent of ε ,such that

$$a(u^\varepsilon, u^\varepsilon) \geq 2\mu C_K \left\| \nabla u^\varepsilon \right\|^r_{L^r(\Omega^\varepsilon)} \tag{2.4.11}$$

Using **Hölder** and **Young** inequalities, we get

$$a(u^\varepsilon, G^\varepsilon) \leq \frac{\mu C_K}{2} \int_{\Omega^\varepsilon} \mu \left| D(u^\varepsilon) \right|^{q(r-1)} d\acute{x} dx_3 + \frac{2^{(r-1)}\mu}{r\,(qC_K)^{\frac{r}{q}}} \int_{\Omega^\varepsilon} \mu \left| D(G^\varepsilon) \right|^r d\acute{x} dx_3 \tag{2.4.12}$$

By $(2.3.8)$, the inequality $(2.4.12)$ becomes

$$a(u^\varepsilon, \varphi) \leq \left\| \nabla u^\varepsilon \right\|^r_{L^r(\Omega^\varepsilon)} + \frac{2^{(r-1)}\mu}{r\,(qC_K)^{\frac{r}{q}}} \left\| \nabla \varphi^\varepsilon \right\|^r_{L^r(\Omega^\varepsilon)} \tag{2.4.13}$$

Applying $(2.3.9) - (2.3.10)$, gives the analogue of $(2.4.13)$

$$\left| (f^\varepsilon, u^\varepsilon) \right| \leq \frac{\mu C_K}{2} \left\| \nabla u^\varepsilon \right\|^r_{L^r(\Omega^\varepsilon)} + \frac{(\varepsilon h^*)^q}{q\left(\frac{1}{2}\mu r C_K\right)^{\frac{q}{r}}} \left\| f^\varepsilon \right\|^q_{L^q(\Omega^\varepsilon)}, \tag{2.4.14}$$

$$\left| (f^\varepsilon, G^\varepsilon) \right| \leq \frac{\mu C_K}{2} \left\| \nabla G^\varepsilon \right\|^r_{L^r(\Omega^\varepsilon)} + \frac{(\varepsilon h^*)^q}{q\left(\frac{1}{2}\mu r C_K\right)^{\frac{q}{r}}} \left\| f^\varepsilon \right\|^q_{L^q(\Omega^\varepsilon)} \tag{2.4.15}$$

From $(2.4.10) - (2.4.15)$, we get

$$\mu C_K \left\| \nabla u^\varepsilon \right\|^r_{L^r(\Omega^\varepsilon)} \leq \left(\frac{2^{(r-1)}\mu}{r\,(qC_K)^{\frac{r}{q}}} + \frac{\mu C_K}{2} \right) \left\| \nabla G^\varepsilon \right\|^r_{L^r(\Omega^\varepsilon)} + \frac{2\,(\varepsilon h^*)^q}{q\left(\frac{1}{2}\mu r C_K\right)^{\frac{q}{r}}} \left\| f^\varepsilon \right\|^q_{L^q(\Omega^\varepsilon)} \tag{2.4.16}$$

Multiplying $(2.4.16)$ by ε^{r-1} hen using the fact that

as $\varepsilon^q \left\| f^\varepsilon \right\|^q_{L^q(\Omega)} = \varepsilon^{1-r} \left\| \hat{f} \right\|^q_{L^q(\Omega)}$ and $\left\| \dfrac{\partial u_i^\varepsilon}{\partial x_3} \right\|^r_{L^r(\Omega^\varepsilon)} = \varepsilon^{1-r} \left\| \dfrac{\partial \hat{u}_i^\varepsilon}{\partial z} \right\|^r_{L^r(\Omega)}$, for $i = 1, 2,$

we deduce $(2.4.1)$ with

$$C = \frac{1}{\mu C_k} \left[\left(\frac{2^{(r-1)}\mu}{r\,(qC_K)^{\frac{r}{q}}} + \frac{\mu C_K}{2} \right) \left\| \nabla \hat{G}^\varepsilon \right\|^r_{L^r(\Omega^\varepsilon)} + \frac{2\,(\varepsilon h^*)^q}{q\left(\frac{1}{2}\mu r C_K\right)^{\frac{q}{r}}} \left\| \hat{f}^\varepsilon \right\|^q_{L^q(\Omega^\varepsilon)} \right]$$

For get the estimate (2.4.8), we choose in (2.3.4); $\hat{\varphi} = \hat{u}^\varepsilon + \psi, \psi \in W_0^{1;r}(\Omega)$, we obtain

$$a(\hat{u}^\varepsilon, \psi) - (\hat{\rho}^\varepsilon, \operatorname{div} \psi) + \hat{\alpha} \int_\Omega \left| \tilde{D}(\hat{u}^\varepsilon + \psi) \right| d\acute{x} dz - \hat{\alpha} \int_\Omega \left| \tilde{D}(\hat{u}^\varepsilon) \right| d\acute{x} dz \geq \left(\hat{f}^\varepsilon, \psi \right)$$

then

$$(\hat{\rho}^\varepsilon, \operatorname{div} \psi) \leq a(\hat{u}^\varepsilon, \psi) + \sqrt{2}\hat{\alpha} \int_\Omega \left| \tilde{D}(\hat{u}^\varepsilon + \psi) \right| \ d\acute{x} dz - \sqrt{2}\hat{\alpha} \int_\Omega \left| \tilde{D}(\hat{u}^\varepsilon) \right| d\acute{x} dz - \left(\hat{f}^\varepsilon, \psi \right),$$

as

$$\left| \tilde{D}(\hat{u}^\varepsilon + \psi) \right| \leq \sqrt{2} \left| \tilde{D}(\hat{u}^\varepsilon) \right| + \sqrt{2} \left| \tilde{D}(\psi) \right|$$

we get

$$(\hat{\rho}^\varepsilon, \operatorname{div} \psi) \leq a(\hat{u}^\varepsilon, \psi) + 2\hat{\alpha} \int_\Omega \left| \tilde{D}(\psi) \right| d\acute{x} dz + \left(2 - \sqrt{2} \right) \hat{\alpha} \int_\Omega \left| \tilde{D}(\hat{u}^\varepsilon) \right| d\acute{x} dz - \int_\Omega \hat{f}^\varepsilon \psi d\acute{x} dz.$$

As

$$\left\| \tilde{D}(\psi) \right\|_{L^r(\Omega)} \leq \|\psi\|_{W^{1,r}(\Omega)}$$

Using **Hölder** formula

$$\begin{aligned} (\hat{\rho}^\varepsilon, \operatorname{div} \psi) \ &\leq \ \mu \left\| \tilde{D}(\hat{u}^\varepsilon) \right\|_{L^r(\Omega)}^{\frac{r}{q}} \ \|\psi\|_{W^{1,r}(\Omega)} + 2\hat{\alpha} \left| \Omega \right|^{\frac{1}{q}} \|\psi\|_{W^{1,r}(\Omega)} \\ &+ \left(2 - \sqrt{2} \right) \hat{\alpha} \left| \Omega \right|^{\frac{1}{q}} \|\hat{u}^\varepsilon\|_{W^{1,r}(\Omega)} + \left\| \hat{f} \right\|_{L^q(\Omega^\varepsilon)} \|\psi\|_{W^{1,r}(\Omega)}. \end{aligned} \tag{2.4.17}$$

According to (2.4.7), there exists a constant C does not depend on ε such that:

$$\int_\Omega \frac{\partial \hat{\rho}^\varepsilon}{\partial x_i} \psi d\acute{x} dz \leq \mu C \|\psi\|_{W^{1,r}(\Omega)} + 2\hat{\alpha} \left| \Omega \right|^{\frac{1}{q}} \|\psi\|_{W^{1,r}(\Omega)} + \left(2 - \sqrt{2} \right) \hat{\alpha} \left| \Omega \right|^{\frac{1}{q}} C + \left\| \hat{f} \right\|_{L^q(\Omega^\varepsilon)} \|\psi\|_{W^{1,r}(\Omega)}. \tag{2.4.18}$$

In a similar manner, we choose in (2.3.4), $\hat{\varphi} = \hat{u}^\varepsilon - \psi, \psi \in W_0^{1;r}(\Omega)$, we obtain

$$\begin{aligned} -\int_\Omega \frac{\partial \hat{\rho}^\varepsilon}{\partial x_i} \psi d\acute{x} dz \ &\leq \ \mu \left\| \tilde{D}(\hat{u}^\varepsilon) \right\|_{L^r(\Omega)}^{\frac{r}{q}} \ \|\psi\|_{W^{1,r}(\Omega)} + 2\hat{\alpha} \left| \Omega \right|^{\frac{1}{q}} \|\psi\|_{W^{1,r}(\Omega)} \\ &+ \left(2 - \sqrt{2} \right) \hat{\alpha} \left| \Omega \right|^{\frac{1}{q}} C + \left\| \hat{f} \right\|_{L^q(\Omega^\varepsilon)} \|\psi\|_{W^{1,r}(\Omega)}, \end{aligned} \tag{2.4.19}$$

From $(2.4.18) - (2.4.19)$, we deduce

$$\left|\int_{\Omega} \frac{\partial \hat{\rho}^{\varepsilon}}{\partial x_i} \psi d\acute{x} dz\right| \leq \mu \left\|\tilde{D}(\hat{u}^{\varepsilon})\right\|_{L^r(\Omega)}^{\frac{r}{q}} \|\psi\|_{W^{1,r}(\Omega)} \tag{2.4.20}$$
$$+2\hat{\alpha} |\Omega|^{\frac{1}{q}} \|\psi\|_{W^{1,r}(\Omega)} + \left(2-\sqrt{2}\right) \hat{\alpha} |\Omega|^{\frac{1}{q}} C + \left\|\hat{f}\right\|_{L^q(\Omega^{\varepsilon})} \|\psi\|_{W^{1,r}(\Omega)}.$$

When $i = 1; 2$ we choose $\psi = (\psi_1, 0, 0)$ then $\psi = (0, \psi_2, 0)$ in the inequality $(2.4.20)$, then

$$\left|\int_{\Omega} \frac{\partial \hat{\rho}^{\varepsilon}}{\partial x_i} \psi d\acute{x} dz\right| \leq \left(\mu C + 2\hat{\alpha} |\Omega|^{\frac{1}{q}} + \left\|\hat{f}_i\right\|_{L^q(\Omega)}\right) \|\psi\|_{W^{1,r}(\Omega)} + \left(2-\sqrt{2}\right) \hat{\alpha} |\Omega|^{\frac{1}{q}} C,$$

gettin $(2.4.8)$ follows for $i = 1, 2$.

For $(2.4.9)$, we take in the inequality $(2.4.20)$, $\psi = (0; 0; \psi_3)$, we find

$$\frac{1}{\varepsilon}\left|\int_{\Omega} \frac{\partial \hat{\rho}^{\varepsilon}}{\partial z} \psi d\acute{x} dz\right| \leq \left(\mu C + 2\hat{\alpha} |\Omega|^{\frac{1}{q}} + \left\|\hat{f}_3\right\|_{L^q(\Omega^{\varepsilon})}\right) \|\psi\|_{W^{1,r}(\Omega)} + \left(\sqrt{2}-1\right) \hat{\alpha} |\Omega|^{\frac{1}{q}} C,$$

This completes the proof of **Lemma 2.4.1**

Now, the convergence $(2.4, 1) - (2.4, 6)$ of **Theorem 2.4.1,**

are a direct result of inequalities $(3.4.7) - (3.4.9)$.

Indeed, from $(2.4.7)$ there exists a fixed constant C which does not depend on ε such that

$$\left\|\frac{\partial \hat{u}_i^{\varepsilon}}{\partial z}\right\|_{L^r(\Omega)} \leq C, i = 1, 2 \tag{2.4.21}$$

It is clear that $(2.4.1)$ deduces directly from $(2.4.21)$ and the using of the **Poincaré's** inequality in the fixed domain Ω .

Also $(2.4.2) - (2.4.4)$ follows from $(2.4.7)$.

The obtaining of $(2.4.5)$ is done as in $([12])$.

Finally, it is easy $(2.4.6)$ follows from $(2.4.8) - (2.4.9)$. ∎

To be able to pass to the limit in the problem (P_k), we must prove the strong convergence of the integral term defined on ω .

Lemma 2.4.2 *There exists a subsequence of* $R(\hat{\sigma}_n^\varepsilon(\hat{u}^\varepsilon,\hat{\rho}^\varepsilon))$ *converging strongly towards* $R(-\rho^*)$ *in* $L^r(\omega)$.

Proof. From the equilibrium equation (2.1.3), we have

$$-\operatorname{div}(\sigma^\varepsilon) = f^\varepsilon \quad \text{in } \Omega^\varepsilon,$$

with $f^\varepsilon \in L^q(\Omega)$.

By the convergence **Theorem 2.4.1** we deduce that $(\hat{u}^\varepsilon,\hat{\rho}^\varepsilon)$ are bounded in $\tilde{V}_z \times L_0^q(\Omega)$, then $\hat{\sigma}^\varepsilon$ is bounded in

$$H_{\text{div}} = \left\{ v \in L^r(\Omega) : \operatorname{div}(v) \in L^q(\Omega), \forall i,j = 1,...,3 \right\},$$

which shows that there exists a subsequence converging weakly towards σ^*.

Now, we show that $\hat{\sigma}_n^\varepsilon(\hat{u}^\varepsilon,\hat{\rho}^\varepsilon)$ converges weakly to $(-\rho^*)$ in $W^{\frac{-1}{2},r}(\omega)$

Indeed, as $\sigma_n^\varepsilon = \sigma_n^\varepsilon n_{in_j}, 1 \le i,j \le 3,$ we have

$$\begin{aligned}\hat{\sigma}_n(\hat{u}^\varepsilon,\hat{\rho}^\varepsilon) = &\sum_{i=1}^{2}\left(\varepsilon^2\mu\left|\tilde{D}(\hat{u}^\varepsilon)\right|^{r-2}\frac{\partial \hat{u}_i^\varepsilon}{\partial x_i} + \varepsilon\hat{\alpha}\left(\left|\tilde{D}(\hat{u}^\varepsilon)\right|\right)^{-1}\frac{\partial \hat{u}_i^\varepsilon}{\partial x_i} - \hat{\rho}^\varepsilon\right)\\ &+\left(\varepsilon^2\mu\left|\tilde{D}(\hat{u}^\varepsilon)\right|^{r-2}\frac{\partial \hat{u}_3^\varepsilon}{\partial z} + \varepsilon\hat{\alpha}\left(\left|\tilde{D}(\hat{u}^\varepsilon)\right|\right)^{-1}\frac{\partial \hat{u}_3^\varepsilon}{\partial z} - \hat{\rho}^\varepsilon\right)\end{aligned}$$

Since $\hat{\sigma}^\varepsilon$ is bounded in $H_{\text{div}}(\Omega)$, then there exists a subsequence converging weakly towards σ^* in $H_{\text{div}}(\Omega)$.

The continuity of the trace operator from $H_{\text{div}}(\Omega)$ into $W^{\frac{-1}{2},r}(\omega)$

give $\hat{\sigma}_n^\varepsilon(\hat{u}^\varepsilon,\hat{\rho}^\varepsilon)$ weakly converges towards $\hat{\sigma}_n^\varepsilon(u^*,\rho^*)$ in $W^{\frac{-1}{2},r}(\omega)$.

Using the convergence results of **Theorem 2.4.1** in the expression of $\hat{\sigma}_n^\varepsilon(\hat{u}^\varepsilon,\hat{\rho}^\varepsilon)$, we get the desired result.

For the rest of proof, using the same techniques as in ([3], [26],) we get the result. ■

Theorem 2.4.2 *Under the same assumptions as in* ***Theorem 2.4.1,*** *the solution* (u^*,ρ^*) *satisfies relations*

$$\hat{u}_i^\varepsilon \to \hat{u}_i^* \quad (1 \le i \le 2) \text{ strongly in } \tilde{V}_z, \forall 1 < r \le 2. \tag{2.4.22}$$

$$\mu\sum_{i=1}^{2}\int_{\Omega}2\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial u_i^*}{\partial z}\right)^2\right)^{\frac{r-2}{2}}\frac{\partial u^*}{\partial z}\frac{\partial\left(\hat{\varphi}_i-u_i^*\right)}{\partial z}d\acute{x}dz$$
$$-\sum_{i=1}^{2}\int_{\Omega}\rho^*(\acute{x})\frac{\partial\left(\hat{\varphi}_i-u_i^*\right)}{\partial z}d\acute{x}dz+\hat{\alpha}\frac{\sqrt{2}}{2}\int_{\Omega}\left(\left|\frac{\partial\hat{\varphi}}{\partial z}\right|-\left|\frac{\partial u^*}{\partial z}\right|\right)d\acute{x}dz \tag{2.4.23}$$
$$+\int_{\omega}\hat{k}\left|R\left(-\rho^*\right)\right|\left(\left|\hat{\varphi}-s\right|-\left|u^*-s\right|\right)d\acute{x}\geq\sum_{i=1}^{2}\int_{\Omega}\left(\hat{f}_i,\hat{\varphi}_i-u_i^*\right)d\acute{x}dz,\forall\hat{\varphi}\in\Pi(K).$$

Proof. For u^{ε} the solution on $(pb3)$, we obtain for $\varphi\in K_{\text{div}}^{\varepsilon}$

$$a(u^{\varepsilon},u^{\varepsilon}-\varphi)-a(\varphi,u^{\varepsilon}-\varphi)-j(u^{\varepsilon},\varphi)+j(u^{\varepsilon},u^{\varepsilon})\leq(f^{\varepsilon},\varphi-u^{\varepsilon})+a(\varphi,u^{\varepsilon}-\varphi). \tag{2.4.24}$$

Using the inequality $([37])$

$$\left(|x|^{r-2}x-|y|^{r-2}y,x-y\right)\geq(r-1)(|x|+|y|)^{r-2}|x-y|^2,\forall 1<r<2,\text{ for }x,y\in\mathbb{R}^n$$

and the **Korn's** inequality, we deduce

$$(r-1)\,\mu C_k\sum_{i,j=1}^{3}\int_{\Omega^{\varepsilon}}\left(\left|\frac{\partial u_i^{\varepsilon}}{\partial x_j}\right|^{r-2}+\left|\frac{\partial\varphi_i}{\partial x_j}\right|^{r-2}\right)\left(\left|\frac{\partial}{\partial x_j}\left(u_i^{\varepsilon}-\varphi_i\right)\right|^2\right)d\acute{x}dx_3$$
$$-j(u^{\varepsilon},\varphi)+j(u^{\varepsilon},u^{\varepsilon})\leq(f^{\varepsilon},\varphi-u^{\varepsilon})+a(\varphi,u^{\varepsilon}-\varphi).$$

Multiplying the last inequality by ε^{r-1}, then by the convergence of **Theorem 2.4.1** we get in the fixed domain Ω

$$(r-1)\,\mu C_k\sum_{i=1}^{2}\left\|\frac{\partial}{\partial z}\left(\hat{u}_i^{\varepsilon}-\hat{\varphi}_i\right)\right\|_{L^r(\Omega)}^r-\hat{j}(\hat{u}^{\varepsilon},\hat{\varphi})+\hat{j}(\hat{u}^{\varepsilon},\hat{u}^{\varepsilon})$$
$$\leq\sum_{i=1}^{2}\int_{\Omega}(\hat{f}_i,\hat{u}_i^{\varepsilon}-\hat{\varphi}_i)d\acute{x}dz+a(\hat{\varphi},\hat{u}^{\varepsilon}-\hat{\varphi}).$$

We put, $\bar{u}^{\varepsilon}=(\hat{u}_1^{\varepsilon},\hat{u}_2^{\varepsilon})$, $u^*=(u_1^*,u_2^*)$, $\bar{\varphi}=(\hat{\varphi}_1,\hat{\varphi}_2)$,so $\bar{\varphi}\in\tilde{\Pi}(K)$ and

$$\limsup_{\varepsilon\to 0}\left[(r-1)\,\mu C_k\left\|\frac{\partial}{\partial z}\left(\bar{u}^{\varepsilon}-\hat{\varphi}_i\right)\right\|_{L^r(\Omega)}^r-\hat{j}(\bar{u}^{\varepsilon},\bar{\varphi})+\hat{j}(\bar{u}^{\varepsilon},\bar{u}^{\varepsilon})\right]$$
$$\leq\mu\int_{\Omega}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial\hat{\varphi}_i}{\partial z}\right)^2\right)^{\frac{r-2}{2}}\frac{\partial\bar{\varphi}}{\partial z}\frac{\partial}{\partial z}\left(\bar{\varphi}-u^*\right)d\acute{x}dz+\sum_{i=1}^{2}\int_{\Omega}(\hat{f}_i,u_i^*-\hat{\varphi}_i)d\acute{x}dz.$$

Consequently

$$(r-1)\,\mu C_k \left\| \frac{\partial}{\partial z}\left(\bar{u}^\varepsilon - \hat{\varphi}_i\right) \right\|^r_{L^r(\Omega)} - \hat{\jmath}(\bar{u}^\varepsilon, \bar{\varphi}) + \hat{\jmath}(\bar{u}^\varepsilon, \bar{u}^\varepsilon)$$
$$\leq \mu \int_\Omega \left(\frac{1}{2} \sum_{i=1}^{2} \left(\frac{\partial \hat{\varphi}_i}{\partial z} \right)^2 \right)^{\frac{r-2}{2}} \frac{\partial \bar{\varphi}}{\partial z} \frac{\partial}{\partial z}\left(\bar{\varphi} - u^*\right) d\acute{x} dz +$$
$$+ \sum_{i=1}^{2} \int_\Omega (\hat{f}_i, u_i^* - \hat{\varphi}_i) d\acute{x} dz + \delta \text{ for } \varepsilon < \varepsilon(\delta)$$

where $\delta > 0$ is arbitrary.

So there exists a sequence of functions $\bar{\varphi} \in \tilde{\Pi}(K)$ which has u^* is a limit in $\tilde{V}_z$, which gives

$$(r-1)\,\mu C_k \left\| \frac{\partial}{\partial z}\left(\bar{u}^\varepsilon - u^*\right) \right\|^r_{L^r(\Omega)} - \hat{\jmath}(\bar{u}^\varepsilon, \bar{\varphi}) + \hat{\jmath}(\bar{u}^\varepsilon, u^*)$$
$$\leq \delta, \forall\ \varepsilon < \varepsilon(\delta)\,.$$

Using the fact that $\hat{\jmath}$ is convex and lower semi-continuous, $(\liminf\ \hat{\jmath}(\bar{u}^\varepsilon) \to \hat{\jmath}(u^*))$, we deduce the strong convergence of $\bar{u}^\varepsilon$ to u^* in $\tilde{V}_z$ as well as $\hat{\jmath}(\bar{u}^\varepsilon, \bar{u}^\varepsilon) \to \hat{\jmath}(\bar{u}^\varepsilon, u^*)$ for $\varepsilon \to 0$, which gives the convergence (2.4.22).

If $r = 2$, we follow the same techniques but (2.4.24) we will be replaced by

$$\left(|x|^{r-2}\,x - |y|^{r-2}\,y, x-y\right) \geq (\frac{1}{2})^{r-1}\,|x-y|^r\,, \text{ for } x, y \in \mathbb{R}^n \tag{2.4.25}$$

The proof of the inequality (2.4.23) needs the following lemma. ■

Lemma 2.4.3 *(Minty). [23] Let E be a reflexive* ***Banach space,***

A be a nonempty closed convex subset of E, and E' be the dual of E.

Let $T : E \to E'$ be a monotone operator which is continuous on finite dimensional subspaces (or at least hemicontinuous).

Then, the followings are equivalent:

$$u \in A, \langle Tu; v-u \rangle_{E' \times E} + j(v) - j(u) \geq \langle f; v-u \rangle_{E' \times E}\ , \forall v \in E$$
$$u \in A, \langle Tv; v-u \rangle_{E' \times E} + j(v) - j(u) \geq \langle f; v-u \rangle_{E' \times E}\ , \forall v \in E$$

*By using **Minty's** Lemma and the fact that* $div(\hat{u}^{\varepsilon}) = 0$ *in* $\Omega,$ *then* (2.3.4) *is equivalent to*

$$a(\hat{\varphi}, \hat{\varphi} - \hat{u}^{\varepsilon}) - \sum_{i=1}^{2} \left(\hat{\rho}^{\varepsilon}, \frac{\partial \hat{\varphi}_i}{\partial x_i}\right) - \left(\hat{\rho}^{\varepsilon}, \frac{\partial \hat{\varphi}_3}{\partial z}\right) + \hat{\jmath}(\hat{u}^{\varepsilon}, \hat{\varphi}) - \hat{\jmath}(\hat{u}^{\varepsilon}, \hat{u}^{\varepsilon})$$
$$\geq \sum_{i=1}^{2} \int_{\Omega} \hat{f}_i(\hat{\varphi}_i - \hat{u}_i^{\varepsilon}) d\acute{x} dz + \int_{\Omega} \varepsilon \hat{f}_3(\hat{\varphi}_3 - \hat{u}_3^{\varepsilon}) d\acute{x} dz, \forall \hat{\varphi} \in K$$

*We apply the convergence of **Theorem 2.4.1, Lemma 2.4.2** and the fact* $\hat{\jmath}$ *is convex and lower semi-continuous, we obtain*

$$\mu \sum_{i=1}^{2} \int_{\Omega} \frac{1}{2} \left(\frac{1}{2} \sum_{i=1}^{2} \left(\frac{\partial \hat{\varphi}_i}{\partial z} \right)^2 \right)^{\frac{r-2}{2}} \frac{\partial \hat{\varphi}_i}{\partial z} \frac{\partial \left(\hat{\varphi}_i - u_i^*\right)}{\partial z} d\acute{x} dz$$
$$- \int_{\Omega} \rho^*(\acute{x}) \left(\frac{\partial \hat{\varphi}_1}{\partial x_1} + \frac{\partial \hat{\varphi}_2}{\partial x_2} \right) d\acute{x} dz + \hat{\alpha} \frac{\sqrt{2}}{2} \int_{\Omega} \left(\left| \frac{\partial \hat{\varphi}}{\partial z} \right| - \left| \frac{\partial u^*}{\partial z} \right| \right) d\acute{x} dz$$
$$+ \int_{\omega} \hat{k} \left| R\left(-\rho^*\right) \right| \left(\left| \hat{\varphi} - s \right| - \left| u^* - s \right| \right) d\acute{x} \geq \sum_{i=1}^{2} \int_{\Omega} \hat{f}_i(\hat{\varphi}_i - u_i^*).$$

ρ^* *independent of* z*, then using again Minty's Lemma for the second time, we deduce* (2.4.23)*, like* (***lemma*** 5.1 *in* [*8*]).

Theorem 2.4.3 *With the same assumptions as in **Theorem 2.4.1**, and if* $\left|\dfrac{\partial u^*}{\partial z}\right| \neq 0$ *then* (u^*, ρ^*) *satisfy*

$$\rho^* \in W^{1,q}(\omega) \tag{2.4.26}$$

$$-\frac{\partial}{\partial z} \left[\frac{1}{2} \mu \left(\frac{1}{2} \sum_{i=1}^{2} \left(\frac{\partial u_i^*}{\partial z} \right)^2 \right)^{\frac{r-2}{2}} \frac{\partial u^*}{\partial z} + \hat{\alpha} \frac{\sqrt{2}}{2} \frac{\partial u^* / \partial z}{\left| \partial u^* / \partial z \right|} \right] = \hat{f} - \nabla \rho^*, \ \text{in } L^q(\Omega) \tag{2.4.27}$$

Proof. we can choose $\hat{\varphi}$ in (2.4.23) as in ([10], **lemma**5.3) such that $\hat{\varphi}_i = u_i^* + \psi_i, i = 1, 2,$ with $\psi_i \in W_0^{1;r}(\Omega)$ we find

$$\mu \sum_{i=1}^{2} \int_{\Omega} \frac{1}{2} \left(\frac{1}{2} \sum_{i=1}^{2} \left(\frac{\partial u_i^*}{\partial z} \right)^2 \right)^{\frac{r-2}{2}} \frac{\partial u_i^*}{\partial z} \frac{\partial \psi_i}{\partial z} d\acute{x} dz$$
$$- \sum_{i=1}^{2} \int_{\Omega} \rho^*(\acute{x}) \left(\frac{\partial \psi_i}{\partial z} \right) d\acute{x} dz = \sum_{i=1}^{2} \int_{\Omega} \hat{f}_i, \psi_i d\acute{x} dz.$$

using now the **Green**'s formula, and choosing $\psi_1 = 0$ and $\psi_2 \in W_0^{1;r}(\Omega)$, then $\psi_1 \in W_0^{1;r}(\Omega)$ and $\psi_2 = 0$, we get (2.4.26).

Now, for the prove of (2.4.27).

For this, we use the following techniques.

Firstly, we choose $\hat{\varphi}$ in (2.4.23) by $\hat{\varphi} = u^* + \lambda\psi$ then $\hat{\varphi} = u^* - \lambda\psi$, $\psi \in W_0^{1;r}(\Omega)$, we obtain

$$\begin{aligned}&\mu\sum_{i=1}^{2}\int_{\Omega}\frac{1}{2}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial u_i^*}{\partial z}\right)^2\right)^{\frac{r-2}{2}}\frac{\partial u_i^*}{\partial z}\frac{\partial(\lambda\psi_i)}{\partial z}d\acute{x}dz-\sum_{i=1}^{2}\int_{\Omega}\rho^*(\acute{x})\frac{\partial(\lambda\psi_i)}{\partial z}d\acute{x}dz\\&+\hat{\alpha}\frac{\sqrt{2}}{2}\int_{\Omega}\left(\left|\frac{\partial(u^*+\lambda\psi)}{\partial z}\right|-\left|\frac{\partial u^*}{\partial z}\right|\right)d\acute{x}dz\geq\sum_{i=1}^{2}\int_{\Omega}\hat{f}_i(\lambda\psi_i)\,d\acute{x}dz,\forall\psi\in W_0^{1;r}(\Omega)\end{aligned}\tag{2.4.28}$$

$$\begin{aligned}&\mu\sum_{i=1}^{2}\int_{\Omega}\frac{1}{2}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial u_i^*}{\partial z}\right)^2\right)^{\frac{r-2}{2}}\frac{\partial u_i^*}{\partial z}\frac{\partial(\lambda\psi_i)}{\partial z}d\acute{x}dz-\sum_{i=1}^{2}\int_{\Omega}\rho^*(\acute{x})\frac{\partial(\lambda\psi_i)}{\partial z}d\acute{x}dz\\&-\hat{\alpha}\frac{\sqrt{2}}{2}\int_{\Omega}\left(\left|\frac{\partial(u^*+\lambda\psi)}{\partial z}\right|-\left|\frac{\partial u^*}{\partial z}\right|\right)d\acute{x}dz\leq\sum_{i=1}^{2}\int_{\Omega}\hat{f}_i(\lambda\psi_i)\,d\acute{x}dz,\forall\psi\in W_0^{1;r}(\Omega)\end{aligned}\tag{2.4.29}$$

Secondly, dividing (2.4.28) and by the passage to the limit when λ tends to zero, we find

$$\begin{aligned}&\mu\sum_{i=1}^{2}\int_{\Omega}\frac{1}{2}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial u_i^*}{\partial z}\right)^2\right)^{\frac{r-2}{2}}\frac{\partial u_i^*}{\partial z}\frac{\partial\psi_i}{\partial z}d\acute{x}dz-\sum_{i=1}^{2}\int_{\Omega}\rho^*(\acute{x})\frac{\partial\psi_i}{\partial z}d\acute{x}dz\\&+\hat{\alpha}\frac{\sqrt{2}}{2}\int_{\Omega}\left(\left|\frac{\partial u^*}{\partial z}\right|\right)^{-1}\frac{\partial u^*}{\partial z}\frac{\partial\psi}{\partial z}d\acute{x}dz\geq\sum_{i=1}^{2}\int_{\Omega}\hat{f}_i\psi_i d\acute{x}dz,\forall\psi\in W_0^{1;r}(\Omega)\end{aligned}\tag{2.4.30}$$

$$\begin{aligned}&\mu\sum_{i=1}^{2}\int_{\Omega}\frac{1}{2}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial u_i^*}{\partial z}\right)^2\right)^{\frac{r-2}{2}}\frac{\partial u_i^*}{\partial z}\frac{\partial\psi_i}{\partial z}d\acute{x}dz-\sum_{i=1}^{2}\int_{\Omega}\rho^*(\acute{x})\frac{\partial\psi_i}{\partial z}d\acute{x}dz\\&+\hat{\alpha}\frac{\sqrt{2}}{2}\int_{\Omega}\left(\left|\frac{\partial u^*}{\partial z}\right|\right)^{-1}\frac{\partial u^*}{\partial z}\frac{\partial\psi}{\partial z}d\acute{x}dz\leq\sum_{i=1}^{2}\int_{\Omega}\hat{f}_i\psi_i d\acute{x}dz,\forall\psi\in W_0^{1;r}(\Omega)\end{aligned}\tag{2.4.31}$$

So the last two formulas, we give:

$$\begin{aligned}&\mu\sum_{i=1}^{2}\int_{\Omega}\frac{1}{2}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial u_i^*}{\partial z}\right)^2\right)^{\frac{r-2}{2}}\frac{\partial u_i^*}{\partial z}\frac{\partial\psi_i}{\partial z}d\acute{x}dz-\sum_{i=1}^{2}\int_{\Omega}\rho^*(\acute{x})\frac{\partial\psi_i}{\partial z}d\acute{x}dz\\&+\hat{\alpha}\frac{\sqrt{2}}{2}\int_{\Omega}\left(\left|\frac{\partial u^*}{\partial z}\right|\right)^{-1}\frac{\partial u^*}{\partial z}\frac{\partial\psi}{\partial z}d\acute{x}dz=\sum_{i=1}^{2}\int_{\Omega}\hat{f}_i\psi_i d\acute{x}dz,\forall\psi\in W_0^{1;r}(\Omega)\end{aligned}\tag{2.4.32}$$

By the Green's formula, we get (2.4.27). ■

Theorem 2.4.4 *Under the same assumptions of the previous Theorems, the traces* τ^*, s^* *satisfy the following inequality*

$$\sum_{i=1}^{2}\int_{\omega}\hat{k}\left|R\left(\hat{\sigma}_n\left(-\rho^*\right)\right)\right|\varphi_i\left(s_i^*-s_i\right)d\acute{x}-\int_{\omega}\hat{\mu}\tau^*\varphi\left(s^*-s\right)d\acute{x}\geq 0,\forall\varphi\in L^r(\omega) \tag{2.4.33}$$

and the following limit form of the ***Coulomb*** *boundary conditions*

$$\begin{cases} \hat{\mu}\left|\tau^*\right|<\hat{k}\left|R\left(\hat{\sigma}_n^{\varepsilon}\left(-\rho^*\right)\right)\right|\Rightarrow s^*=s, \\ \hat{\mu}\left|\tau^*\right|=\hat{k}\left|R\left(\hat{\sigma}_n^{\varepsilon}\left(-\rho^*\right)\right)\right|\Rightarrow\exists\beta\geq 0 \text{ such that } s^*=s-\beta\tau^*. \end{cases} \text{on } \omega. \tag{2.4.34}$$

With

$$s^*(\acute{x})=\frac{\partial u^*(\acute{x},0)}{\partial z},A^*(\acute{x},\xi)=\frac{1}{2}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial u_i^*}{\partial z}(\acute{x},\xi)\right)^2\right)^{\frac{r-2}{2}},\tau^*=A^*(\acute{x},0)\frac{\partial u^*}{\partial z}(\acute{x},0)$$

$$F(\acute{x},y)=\int_0^y\int_0^{\xi}\hat{f}^{\,\varepsilon}(\acute{x},t)dtd\xi,\quad \tilde{F}(\acute{x},y)=\int_0^h F(\acute{x},y)dy-\frac{h}{2}\ F(\acute{x},h).$$

Another u^* *and* ρ^* *satisfies the following weak form of the* ***Reynolds*** *equation*

$$\int_{\omega}\left(\frac{h^3}{12}\nabla\rho^*(\acute{x})+\tilde{F}(\acute{x},h)+\mu\int_0^h\int_0^y A^*(\acute{x},\xi)\frac{\partial u^*}{\partial\xi}(\acute{x},\xi)d\xi dy+\hat{\alpha}\int_0^h\int_0^y\frac{\partial u^*/\partial z}{|\partial u^*/\partial z|}(\acute{x},\xi)d\xi dy\right)\nabla\varphi(x)$$
$$-\int_{\omega}\left(\frac{\mu h}{2}\int_0^h A^*(\acute{x},\xi)\frac{\partial u^*}{\partial\xi}(\acute{x},\xi)d\xi-\frac{\hat{\alpha}h}{2}\int_0^h\frac{\partial u^*/\partial z}{|\partial u^*/\partial z|}(\acute{x},\xi)d\xi\right)\nabla\varphi(x)=0,\forall\varphi\in W^{1,r}(\omega), \tag{2.4.35}$$

Proof. We take in (2.4.23), $\hat{\varphi}_i=u_i^*+\lambda\Psi_i, i=1,2,$ for all $\Psi_i\in W^{1,r}_{\Gamma_1\cup\Gamma_L}(\Omega)$ with

$$W^{1,r}_{\Gamma_1\cup\Gamma_L}(\Omega)=\left\{\Psi\in W^{1,r}(\Omega),\Psi=\left(\Psi_1,\Psi_2\right),\Psi=0 \text{ on } \Gamma_1\cup\Gamma_L\ \right\}$$

,

then

$$\mu\sum_{i=1}^{2}\int_{\Omega}\frac{1}{2}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial u_i^*}{\partial z}\right)^2\right)^{\frac{r-2}{2}}\frac{\partial u_i^*}{\partial z}\frac{\partial(\lambda\Psi_i)}{\partial z}d\acute{x}dz$$
$$-\sum_{i=1}^{2}\int_{\Omega}\left(\rho^*(\acute{x}),\frac{\partial(\lambda\Psi_i)}{\partial x_i}\right)d\acute{x}dz+\hat{\alpha}\frac{\sqrt{2}}{2}\sum_{i=1}^{2}\int_{\Omega}\left(\left|\frac{\partial(\lambda\Psi_i+u_i^*)}{\partial z}\right|-\left|\frac{\partial u_i^*}{\partial z}\right|\right)d\acute{x}dz$$
$$+\int_{\omega}\hat{k}\left|R(-\rho^*)\right|(|\lambda\Psi+s^*-s|-|s^*-s|)\,d\acute{x}$$
$$\geq\sum_{i=1}^{2}\left(\hat{f}_i,\lambda\Psi_i\right).$$

Dividing the last inequality by λ and the passage to the limit when λ tends to zero,

$$\mu\sum_{i=1}^{2}\int_{\Omega}\frac{1}{2}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial u_i^*}{\partial z}\right)^2\right)^{\frac{r-2}{2}}\frac{\partial u_i^*}{\partial z}\frac{\partial\Psi_i}{\partial z}d\acute{x}dz$$
$$-\sum_{i=1}^{2}\int_{\Omega}\rho^*(\acute{x})\frac{\partial\Psi_i}{\partial x_i}d\acute{x}dz+\hat{\alpha}\frac{\sqrt{2}}{2}\sum_{i=1}^{2}\int_{\Omega}\left(\left|\frac{\partial u_i^*}{\partial z}\right|^{-1}\frac{\partial u_i^*}{\partial z}\frac{\partial\Psi_i}{\partial z}\right)d\acute{x}dz$$
$$+\quad\int_{\omega}\hat{k}\left|R(-\rho^*)\right|\frac{\Psi_i(s^*-s)}{|s^*-s|}d\acute{x}\geq\sum_{i=1}^{2}\left(\hat{f}_i,\Psi_i\right). \tag{2.4.36}$$

Finallly, using the Green formula in $(2.4.36)$ and from $(2.4.27)$, we find

$$\sum_{i=1}^{2}\int_{\omega}\hat{k}\left|R\left(\hat{\sigma}_n(-\rho^*)\right)\right|\varphi_i\left(s_i^*-s_i\right)d\acute{x}-\int_{\omega}\hat{\mu}\tau^*\varphi\left(s^*-s\right)d\acute{x}\geq 0,\forall\varphi\in W^{1,r}_{\Gamma_1\cup\Gamma_L}(\Omega)$$

This inequality remains valid for any $\Psi\in D(\omega)^2$,

and by the density of $D(\omega)$ in $L^r(\omega)$ we deduce $(3.4.33)$.

To prove $(2.4.35)$, we integrat $(2.4.27)$ from 0 to z we obtain:

$$\begin{aligned}&-\mu\int_0^z A^*(\acute{x},\xi)\frac{\partial u^*}{\partial\xi}(\acute{x},\xi)d\xi-\hat{\alpha}\frac{\sqrt{2}}{2}\int_0^z\frac{\partial u^*/\partial z}{|\partial u^*/\partial z|}(\acute{x},\xi)d\xi\\&+\mu\tau^*(\acute{x})z+\hat{\alpha}\frac{\sqrt{2}}{2}\frac{\rho^*(\acute{x})}{|\rho^*(\acute{x})|}z=\int_0^z\int_0^\xi\hat{f}(\acute{x},t)dtd\xi-\frac{z^2}{2}\nabla\rho^*,\end{aligned}\tag{2.4.36}$$

for $z = h$ we have

$$\begin{aligned}
&-\mu\int_0^h A^*(\acute{x},\xi)\frac{\partial u^*}{\partial \xi}(\acute{x},\xi)d\xi - \hat{\alpha}\frac{\sqrt{2}}{2}\int_0^h \frac{\partial u^*/\partial z}{|\partial u^*/\partial z|}(\acute{x},\xi)d\xi \\
&+\mu\tau^*(\acute{x})h + \hat{\alpha}\frac{\sqrt{2}}{2}\frac{\rho^*(\acute{x})}{|\rho^*(\acute{x})|}h = \int_0^z\int_0^\xi \hat{f}_i(\acute{x},t)dtd\xi - \frac{h^2}{2}\nabla\rho^*.
\end{aligned} \tag{2.4.37}$$

By integrating $(2.4.36)$ from 0 to h we obtain:

$$\begin{aligned}
&-\mu\int_0^h\int_0^y A^*(\acute{x},\xi)\frac{\partial u^*}{\partial \xi}(\acute{x},\xi)d\xi - \hat{\alpha}\frac{\sqrt{2}}{2}\int_0^h\int_0^y \frac{\partial u^*/\partial z}{|\partial u^*/\partial z|}(\acute{x},\xi)d\xi dy + \mu\tau^*(\acute{x})\frac{h^2}{2} \\
&+\hat{\alpha}\frac{\sqrt{2}}{2}\frac{\rho^*(\acute{x})}{|\rho^*(\acute{x})|}\frac{h^2}{2} = \int_0^h\int_0^y\int_0^\xi \hat{f}\ (\acute{x},t)dtd\xi dy - \frac{h^3}{6}\nabla\rho^*.
\end{aligned} \tag{2.4.38}$$

From $(2.4.37)$, we deduce

$$\begin{aligned}
&\left(\mu\tau^*(\acute{x}) + \hat{\alpha}\frac{\sqrt{2}}{2}\frac{\rho^*(\acute{x})}{|\rho^*(\acute{x})|}\right)\frac{h^2}{2} = \mu\frac{h}{2}\int_0^h A^*(\acute{x},\xi)\frac{\partial u^*}{\partial \xi}(\acute{x},\xi)d\xi + \\
&\hat{\alpha}\frac{h}{2}\frac{\sqrt{2}}{2}\int_0^h \frac{\partial u^*/\partial z}{|\partial u^*/\partial z|}(\acute{x},\xi)d\xi + \frac{h}{2}\int_0^h\int_0^y \hat{f}\ (\acute{x},\xi)d\xi dy - \frac{h^3}{4}\nabla\rho^*
\end{aligned} \tag{2.4.39}$$

By $(2.4.38)-(2.4.39)$, we deduce $(2.4.35)$. ■

2.4.2 Uniqueness of the solutions

Theorem 2.4.5 *There exists a positive constant sufficiently small* k^* *such that* $\left\|\hat{k}\right\|_{L^\infty(\omega)} \leq k^*$ *the solution*

(u^*, ρ^*) *in* $\tilde{V}_z \times (L_0^q(\omega) \cap W^{1,q}(\omega))^2$ *of limite problem* $(2.4.27)$, *is unique.*

Proof. suppose there are two solutions $(u^{*,1}, \rho^{*,1})$ and $(u^{*,2}, \rho^{*,2})$ of the problem (2.4.27)

$$\begin{aligned}\mu \sum_{i=1}^{2} \int_{\Omega} \left(\sum_{i=1}^{2} \left(\frac{\partial u_i^{*,1}}{\partial z} \right)^2 \right)^{\frac{r-2}{2}} \frac{\partial u_i^{*,1}}{\partial z} \frac{\partial(\hat{\varphi}_i - u_i^{*,1})}{\partial z} d\acute{x} dz - \int_{\Omega} \left(\rho^*, \frac{\partial \hat{\varphi}_1}{\partial x_1} + \frac{\partial \hat{\varphi}_2}{\partial x_2} \right) d\acute{x} dz \\ + \hat{\alpha} \tfrac{\sqrt{2}}{2} \int_{\Omega} \left(\left| \frac{\partial \hat{\varphi}}{\partial z} \right| + \left| \frac{\partial u^{*,1}}{\partial z} \right| \right) d\acute{x} dz + \int_{\omega} \hat{k} \left| R(-\rho^{*,1}) \right| \left(|\hat{\varphi} - s| - |u^{*,1} - s| \right) d\acute{x} \\ \geq \sum_{i=1}^{2} \left(\hat{f}_i, \hat{\varphi}_i - u_i^{*,1} \right) . \forall \hat{\varphi} \in K(\Omega)\end{aligned} \tag{2.4.40}$$

$$\begin{aligned}\mu \sum_{i=1}^{2} \int_{\Omega} \left(\frac{1}{2} \sum_{i=1}^{2} \left(\frac{\partial u_i^{*,2}}{\partial z} \right)^2 \right)^{\frac{r-2}{2}} \frac{\partial u_i^{*,2}}{\partial z} \frac{\partial(\hat{\varphi}_i - u_i^{*,2})}{\partial z} d\acute{x} dz - \int_{\Omega} \left(\rho^*, \frac{\partial \hat{\varphi}_1}{\partial x_1} + \frac{\partial \hat{\varphi}_2}{\partial x_2} \right) d\acute{x} dz \\ + \hat{\alpha} \tfrac{\sqrt{2}}{2} \int_{\Omega} \left(\left| \frac{\partial \hat{\varphi}}{\partial z} \right| + \left| \frac{\partial u^{*,2}}{\partial z} \right| \right) d\acute{x} dz + \int_{\omega} \hat{k} \left| R(-\rho^{*,2}) \right| \left(|\hat{\varphi} - s| - |u^{*,2} - s| \right) d\acute{x} \\ \geq \sum_{i=1}^{2} \left(\hat{f}_i, \hat{\varphi}_i - u_i^{*,2} \right) . \forall \hat{\varphi} \in K(\Omega).\end{aligned} \tag{2.4.41}$$

We take $\hat{\varphi} = u^{*,2}$ in (2.4.40) and $\hat{\varphi} = u^{*,1}$ in (2.4.41),

$$\begin{aligned}\mu \sum_{i=1}^{2} \int_{\Omega} \left(\sum_{i=1}^{2} \left(\frac{\partial u_i^{*,1}}{\partial z} \right)^2 \right)^{\frac{r-2}{2}} \frac{\partial u_i^{*,1}}{\partial z} \frac{\partial(u_i^{*,2} - u_i^{*,1})}{\partial z} d\acute{x} dz - \int_{\Omega} \left(\rho^*, \frac{u_1^{*,2}}{\partial x_1} + \frac{u_2^{*,2}}{\partial x_2} \right) d\acute{x} dz \\ + \hat{\alpha} \tfrac{\sqrt{2}}{2} \int_{\Omega} \left(\left| \frac{u^{*,2}}{\partial z} \right| + \left| \frac{\partial u^{*,1}}{\partial z} \right| \right) d\acute{x} dz + \int_{\omega} \hat{k} \left| R(-\rho^{*,1}) \right| \left(|u^{*,2} - s| - |u^{*,1} - s| \right) d\acute{x} \\ \geq \sum_{i=1}^{2} \left(\hat{f}_i, u_i^{*,2} - u_i^{*,1} \right) . \forall \hat{\varphi} \in K(\Omega)\end{aligned} \tag{2.4.42}$$

$$\begin{aligned}\mu \sum_{i=1}^{2} \int_{\Omega} \left(\frac{1}{2} \sum_{i=1}^{2} \left(\frac{\partial u_i^{*,2}}{\partial z} \right)^2 \right)^{\frac{r-2}{2}} \frac{\partial u_i^{*,2}}{\partial z} \frac{\partial(u_i^{*,1} - u_i^{*,2})}{\partial z} d\acute{x} dz - \int_{\Omega} \left(\rho^*, \frac{\partial u_1^{*,1}}{\partial x_1} + \frac{\partial u_2^{*,1}}{\partial x_2} \right) d\acute{x} dz \\ + \hat{\alpha} \tfrac{\sqrt{2}}{2} \int_{\Omega} \left(\left| \frac{\partial u^{*,1}}{\partial z} \right| + \left| \frac{\partial u^{*,2}}{\partial z} \right| \right) d\acute{x} dz + \int_{\omega} \hat{k} \left| R(-\rho^{*,2}) \right| \left(|\partial u^{*,1} - s| - |u^{*,2} - s| \right) d\acute{x} \\ \geq \sum_{i=1}^{2} \left(\hat{f}_i, \partial u_i^{*,1} - u_i^{*,2} \right) . \forall \hat{\varphi} \in K(\Omega).\end{aligned} \tag{2.4.43}$$

then by summing the two inequalities, we find

$$\mu\sum_{i=1}^{2}\int_{\Omega}\left[\left(\left(\frac{1}{2}\right)^{\frac{r}{2}}\sum_{i=1}^{2}\left(\frac{\partial u_i^{*,1}}{\partial z}\right)^2\right)^{\frac{r-2}{2}}\frac{\partial u_i^{*,1}}{\partial z}-\left(\left(\frac{1}{2}\right)^{\frac{r}{2}}\sum_{i=1}^{2}\left(\frac{\partial u_i^{*,2}}{\partial z}\right)^2\right)^{\frac{r-2}{2}}\frac{\partial u_i^{*,2}}{\partial z}\right]\frac{\partial}{\partial z}\left(u_i^{*,1}-u_i^{*,2}\right)d\acute{x}dz$$
$$-\int_{\omega}\hat{k}\left|R(-\rho^{*,1})-R(-\rho^{*,2})\right|\left|u^{*,1}-u^{*,2}\right|d\acute{x}\leq 0, \tag{2.4.44}$$

we use the following inequality

$$\left(|x|^{r-2}x-|y|^{r-2}y,x-y\right)\geq(r-1)(|x|+|y|)^{r-2}\left|x-y\right|^2,x,y\in\mathbb{R}^n,\forall 1<r\leq 2,$$

we get

$$\mu\left\|\frac{\partial}{\partial z}\left(u^{*,1}-u^{*,2}\right)\right\|_{L^r(\Omega)}^r\leq\left\|\hat{k}\right\|_{L^\infty}\int_{\omega}\left|R(-\rho^{*,1})-R(-\rho^{*,2})\right|\left|u^{*,1}-u^{*,2}\right|d\acute{x}. \tag{2.4.45}$$

So that

$$\mu\left\|u^{*,1}-u^{*,2}\right\|_{L^r(\Omega)}^r\leq\mu h^*\left\|\frac{\partial}{\partial z}\left(u_i^{*,1}-u_i^{*,2}\right)\right\|_{L^r(\Omega)}^r$$

Now, we apply the Hölder inequality on the second term of (2.4.45), we have

$$\mu\left\|u^{*,1}-u^{*,2}\right\|_{L^r(\Omega)}^r\leq\mu h^*\left\|\frac{\partial}{\partial z}\left(u^{*,1}-u^{*,2}\right)\right\|_{L^r(\omega)}^r$$
$$\leq h^*\left\|\hat{k}\right\|_{L^\infty(\omega)}C_0\int_{\omega}\left(\left|R(-\rho^{*,1})-R(-\rho^{*,2})\right|^q\right)^{\frac{1}{q}}d\acute{x}\left\|u^{*,1}-u^{*,2}\right\|_{L^r(\omega)}$$

whence

$$\left\|u^{*,1}-u^{*,2}\right\|_{L^r(\omega)}^{r-1}\leq\frac{h^*\left\|\hat{k}\right\|_{L^\infty(\omega)}C_0}{\mu}\left\|R(-\rho^{*,1})-R(-\rho^{*,2})\right\|_{L^q(\omega)} \tag{2.4.46}$$

Using the fact that R is a linear continuous operator $W^{-\frac{1}{2},r}(\omega)$ into $L^r(\omega)$, there exists a constant $C1$ depending on R, such that

$$\left\|R(-\rho^{*,1})-R(-\rho^{*,2})\right\|_{L^q(\omega)}\leq C_1\left\|\rho^{*,1}-\rho^{*,2}\right\|_{L^q(\omega)}. \tag{2.4.47}$$

Combining (2.4.46) and (2.4.47) we deduce that if $\left\|\hat{k}\right\|_{L^\infty(\omega)}\leq k^*$ for sufficiently small k^* then we have

$$\left\|\rho^{*,1}-\rho^{*,2}\right\|_{L^q(\omega)}=0 \text{ and } \left\|u^{*,1}-u^{*,2}\right\|_{V_z}.$$

This ends the proof of the Theorem ■

Chapter 3

Study of a Non-Isothermal Hooke Operator in Thin Domain with Friction on the Bottom Surface

Abstract. This chapter is devoted to the achievement of the mathematical model in a thin domain Ω^ε of $\mathbb{R}^3$, such that the height is smaller compared to the other two lengths. This problem related with deformations of a non isothermal elastic homogeneous body " depends on a temperature coefficient", with nonlinear friction conditions of the Coulomb type.

It's made up of five sections. Firstly, we give ratings, boundary conditions, and then we formulate the variational formulation of the strong problem. Finally, we concern ourselves with the study of the existence and uniqueness of the solution of the weak problem.

3.1 Stating of the problem

Consider the problem of uniform elasticity in the thin domain Ω^ε, and steady-state deformation of an isotropic body (a material with the same behavior in all directions).Where ε positive real numbers belonging to $]0,1[$ and it tends towards zero.

Denote the boundary of Ω^ε by $\Gamma^\varepsilon = \overline{\omega} \cup \overline{\Gamma_1^\varepsilon} \cup \overline{\Gamma_L^\varepsilon}$.

Γ_1^ε is the upper bound of Ω^ε with equation $x_3 = \varepsilon h(x_1, x_2)$.
Γ_L^ε is the side border of Ω^ε.
ω is a bounded domain of $\mathbb{R}^3$, with equation $x_3 = 0$,
and is the bottom of the domain.

We notice $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $\acute{x} = (x_1, x_2) \in \mathbb{R}^2$.

The domain Ω^ε is given by :

$$\Omega^\varepsilon = \{(\acute{x}, x_3) \in \mathbb{R}^3,\ (\acute{x}, 0) \in \omega,\ 0 < x_3 < \varepsilon h(\acute{x})\}.$$

Where h is a function of class C^1 defined on ω with :

$$0 \leq h_* \leq h(\acute{x}) \leq h^*,\ \forall (\acute{x}, 0) \in \omega.$$

$f^{\,\varepsilon}$, represents the mass density of the external forces in the field Ω^ε.

The momentum conservation equation in domain Ω^ε, is determined by non-isothermal steady flow:

$$\operatorname{div} \sigma^\varepsilon + f^\varepsilon = 0 \ \text{ in } \Omega^\varepsilon. \tag{3.1.1}$$

We represent the stress tensor by $\sigma^\varepsilon = (\sigma_{ij})^\varepsilon_{1\leq i,j\leq 3}$, and the deformation tensor by $D = (d_{ij})_{1\leq i,j\leq 3}$.

$$d_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),\ 1 \leq i,j \leq 3.$$

The law of behavior is believed to follow **Hooke's** law.

$$\sigma^\varepsilon_{i,j}(u^\varepsilon) = 2\mu(T^\varepsilon) d_{i,j}(u^\varepsilon) + \lambda(T^\varepsilon) d_{kk}(u^\varepsilon)\delta_{ij}. \tag{3.1.2}$$

Define the normal and tangential components of the displacement as follows:

$$u_n^\varepsilon = u^\varepsilon.n$$
$$u_\tau^\varepsilon = u^\varepsilon - u_n^\varepsilon.n$$

Similarly, the normal and tangential components of the stress tensor are defined as:

$$\sigma_n^\varepsilon = (\sigma^\varepsilon.n).n$$
$$\sigma_\tau^\varepsilon = \sigma^\varepsilon.n - \sigma_n^\varepsilon.n$$

The law of conservation of energy is:

$$\begin{cases} -\nabla(K^\varepsilon \nabla T^\varepsilon) = \sigma^\varepsilon : D(u^\varepsilon) + r^\varepsilon(T^\varepsilon), \\ \sigma^\varepsilon : D(u^\varepsilon) = \sum_{i,j=1}^{3} \sigma_{i,j}^\varepsilon d_{i,j}(u^\varepsilon). \end{cases} \tag{3.1.3}$$

Where K^ε is the thermal conductivity and $r^\varepsilon(T^\varepsilon)$ is the heat source.

Before describing the limiting conditions, first let us introduce the vectorial function $g = (g_1, g_2, g_3)$ like:

$$\int_{\Gamma^\varepsilon} g.n.ds = 0.$$

There is a function G^ε such that:

$$G^\varepsilon \in H^1(\Omega^\varepsilon)^3 \text{ with } G^\varepsilon = g \text{ on } \Gamma^\varepsilon.$$

Assume that the displacement on the upper surface Γ_1^ε, is given by:

$$u^\varepsilon = g = 0 \text{ on } \Gamma_1^\varepsilon. \tag{3.1.4}$$

On Γ_L^ε, the displacement is known and parallel to the ω-plane

$$u_i^\varepsilon = g_i \text{ with } g_3 = 0,\ i = 1, 2, \tag{3.1.5}$$

Now let's talk about the condition on ω.

Bilateral contact is given by :

$$u^\varepsilon.n = 0 \text{ on } \omega, \tag{3.1.6}$$

Also assume the friction on the common plane ω,

this friction is unknown and is modeled by a nonlinear **Coulomb**'s law:

$$\begin{cases} |\sigma_\tau^\varepsilon| < \digamma^\varepsilon |\sigma_n^\varepsilon| \Rightarrow u_\tau^\varepsilon = s, \\ |\sigma_\tau^\varepsilon| = \digamma^\varepsilon |\sigma_n^\varepsilon| \Rightarrow \exists \beta \geq 0 \text{ such that } u_\tau^\varepsilon = s - \beta\sigma_\tau^\varepsilon, \end{cases} \tag{3.1.7}$$

with $\digamma^\varepsilon \geq 0$ is the coefficient of friction.

For temperature, we make the following assumptions

homogeneous **Dirichlet**

$$\begin{cases} T^\varepsilon = 0 \quad \text{ on } \Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon, \text{ (homogenous } \textbf{Dirichlet}) \\ \dfrac{\partial T^\varepsilon}{\partial n} = 0 \text{ on } \omega \text{ , (homogenous } \textbf{Neumann}) \end{cases} \tag{3.1.8}$$

The complete problem $(3.1.9)$, is to find the displacement field

$u^\varepsilon = (u_i^\varepsilon)_{1\leq i\leq 3} : \Omega^\varepsilon \to \mathbb{R}^3$, check the following equations and boundary conditions:

$$\begin{cases} \operatorname{div} \sigma^\varepsilon + f^{\,\varepsilon} = 0 & \text{in } \Omega^\varepsilon, \\ \sigma_{i,j}^\varepsilon(u^\varepsilon) = 2\mu(T^\varepsilon) d_{i,j}(u^\varepsilon) + \lambda(T^\varepsilon) d_{kk}(u^\varepsilon)\delta_{ij}. & \text{in } \Omega^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \Gamma_1^\varepsilon, \\ u^\varepsilon = g & \text{on } \Gamma_L^\varepsilon, \\ u^\varepsilon.n = 0 & \text{on } \omega, \\ \left\{\begin{array}{c} |\sigma_\tau^\varepsilon| < \digamma^\varepsilon |\sigma_n^\varepsilon| \Rightarrow u_\tau^\varepsilon = s, \\ |\sigma_\tau^\varepsilon| = \digamma^\varepsilon |\sigma_n^\varepsilon| \Rightarrow \exists \beta \geq 0 \text{ such that } u_\tau^\varepsilon = s - \beta\sigma_\tau^\varepsilon, \end{array}\right\} & \text{on } \omega, \\ T^\varepsilon = 0 & \text{on } \Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon, \\ \dfrac{\partial T^\varepsilon}{\partial n} = 0 & \text{on } \omega \text{ .} \end{cases} \tag{3.1.9}$$

3.2 Variational formulation

In this section we define a functional framework in which we are going to work, and we obtain a weak formulation of the problem $(3.1.9)$.

For the open Ω^ε, define the spaces and sets:

Define spaces and sets of open Ω^ε.

$$H^1(\Omega^\varepsilon)^3 = \left\{ v \in L^2(\Omega^\varepsilon)^3 : \frac{\partial v_i}{\partial x_j} \in L^2(\Omega^\varepsilon)^3, \forall i, j = 1, ..., 3 \right\},$$

Defined the closed convex nonempty of $H^1(\Omega^\varepsilon)^3$:

$$V^\varepsilon = \left\{ \varphi \in \left(H^1(\Omega^\varepsilon)\right)^3; \quad \varphi = g \text{ on } \Gamma_L^\varepsilon \ , \ \varphi = 0 \text{ on } \Gamma_1^\varepsilon \text{ and } \varphi.n = 0 \text{ on } \omega \right\}.$$

Denote by $H^1_{\Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon}$ the vector subspace of $H^1(\Omega^\varepsilon)$:

$$H^1_{\Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon}(\Omega^\varepsilon) = \left\{ \varphi \in H^1(\Omega^\varepsilon) : \varphi = 0 \text{ on } \Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon \ \right\}.$$

The spaces $\Omega^\varepsilon, H^1(\Omega^\varepsilon)^3, V^\varepsilon$ and $H^1_{\Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon}(\Omega^\varepsilon)$ are the domains in which we study the asymptotic behavior of elasticity.

The **Sobolev** spaces, the closed convex, and vector subspaces of $H^1(\Omega^\varepsilon)$ are given with their natural norms and inner products.

In order to simplify drafting, we note:

$$a(T^\varepsilon, u^\varepsilon, v) = \sum_{i,j=1}^{3} \int_{\Omega^\varepsilon} 2\mu^\varepsilon(T^\varepsilon) d_{ij}(u^\varepsilon) d_{ij}(v) d\acute{x} dx_3 + \int_{\Omega^\varepsilon} \lambda^\varepsilon(T^\varepsilon) \ div(u^\varepsilon) \ div(v) d\acute{x} dx_3, \quad (3.2.1)$$

$$(f^\varepsilon, v) = \int_{\Omega^\varepsilon} f^\varepsilon v d\acute{x} dx_3 = \sum_{i=1}^{3} \int_{\Omega^\varepsilon} {f_i}^\varepsilon v_i d\acute{x} dx_3, \quad (3.2.2)$$

$$b(T^\varepsilon, \psi) = \int_{\Omega^\varepsilon} K^\varepsilon \frac{\partial T^\varepsilon}{\partial x_i} \frac{\partial \psi}{\partial x_i} d\acute{x} dx_3, \quad (3.2.3)$$

$$c(u^\varepsilon, T^\varepsilon, \psi) = \sum_{i=1}^{3} \int_{\Omega^\varepsilon} 2\mu^\varepsilon(T^\varepsilon) d_{ij}^2(u^\varepsilon) \psi d\acute{x} dx_3 + \int_{\Omega^\varepsilon} \lambda^\varepsilon(T^\varepsilon) \ div(u^\varepsilon) \ div(u^\varepsilon) \psi d\acute{x} dx_3 + \int_{\Omega^\varepsilon} r^\varepsilon(T^\varepsilon) \psi d\acute{x} dx_3. \quad (3.2.4)$$

$$j^\varepsilon(v) = \int_\omega F^\varepsilon S(\sigma_n^\varepsilon) \left| v_\tau - s \right| d\acute{x}. \quad (3.2.5)$$

Since the integral $j^\varepsilon(v)$ has no meaning for $v \in V^\varepsilon$, we replace σ_n^ε by $S\left(\sigma_n^\varepsilon\right)$, where S is the regularization operator from $H^{-\frac{1}{2}}(\omega)$ into $L^2_+(\omega)$ is defined by

$$\forall\tau \in H^{-\frac{1}{2}}(\omega), S(\tau) \in L^2_+(\omega), S(\tau)(x) = \left| \langle \tau, \varrho(x-\tau)\rangle_{H^{-\frac{1}{2}}(\omega), H^{\frac{1}{2}}_{00}(\omega)} \right|,$$

where ϱ is a given positive function of the class C^∞ with compact support on ω, here $H^{-\frac{1}{2}}(\omega)$ is the dual space to the

$$H^{\frac{1}{2}}_{00}(\omega) = \left\{ \varrho_{|\omega} : \varrho \in H^1(\Omega^\varepsilon);\ \varrho = 0 \quad \text{on} \quad \Gamma^\varepsilon_L \cup \Gamma^\varepsilon_1 \right\}.$$

is the subspace of $L^2(\omega)$ of non-negative functions from $([20],[21])$.

Lemma 3.2.1 *if u^ε and T^ε are solutions of the problem* $(3.1.9)$ *then they verify the following variational problem:*

$$\left\{ \begin{array}{l} \textit{To find } \ u^\varepsilon \in V^\varepsilon(\Omega^\varepsilon) \ \textit{ and find } \ T^\varepsilon \in H^1_{\Gamma^\varepsilon_L \cup \Gamma^\varepsilon_1}(\Omega^\varepsilon) \ \textit{ such that:} \\ a(T^\varepsilon, u^\varepsilon, \varphi - u^\varepsilon) + j^\varepsilon(\varphi) - j^\varepsilon(u^\varepsilon) \geq (f^\varepsilon, \varphi - u^\varepsilon), \quad \forall \varphi \in V^\varepsilon. \\ \qquad b(T^\varepsilon, \psi) = c(u^\varepsilon, T^\varepsilon, \psi), \quad \forall \psi \in H^1_{\Gamma^\varepsilon_L \cup \Gamma^\varepsilon_1}. \end{array} \right. \tag{3.2.6}$$

Proof. By multiplying the equation $(3.1.1)$ by $(\varphi - u^\varepsilon)$ where $\varphi \in V^\varepsilon$ and using the Green formula we get

$$\int_{\Omega^\varepsilon} \sigma^\varepsilon_{ij} \frac{\partial}{\partial x_i}(\varphi_i - u^\varepsilon_i) d\acute{x} dx_3 - \int_{\Gamma^\varepsilon} \sigma^\varepsilon_{ij} n_j (\varphi_i - u^\varepsilon_i) ds = \int_{\Omega^\varepsilon} f_i{}^\varepsilon (\varphi_i - u^\varepsilon_i) d\acute{x} dx_3.$$

According to the boundary conditions $(3.1.4) - (3.1.5)$ we find

$$\int_{\Gamma^\varepsilon} \sigma^\varepsilon_{ij} n_j (\varphi_i - u^\varepsilon_i) ds = \int_{\omega} \sigma^\varepsilon_{ij} n_j (\varphi_i - u^\varepsilon_i) d\acute{x}.$$

We know that: $\sigma^\varepsilon_{ij} n_j = \sigma^\varepsilon_{\tau_i} + \sigma^\varepsilon_n n_i$ then we get

$$\int_{\Gamma^\varepsilon} \sigma^\varepsilon_{ij} n_j (\varphi_i - u^\varepsilon_i) ds = \int_{\omega} \sigma^\varepsilon_{\tau_i} (\varphi_i - u^\varepsilon_i) d\acute{x} + \int_{\omega} \sigma^\varepsilon_n n_i (\varphi_i - u^\varepsilon_i) d\acute{x}$$

as $n_i(\varphi_i - u^\varepsilon_i) = 0,$ we have

$$\int_{\Omega^\varepsilon} \sigma^\varepsilon_{ij} \frac{\partial}{\partial x_i}(\varphi_i - u^\varepsilon_i) d\acute{x} dx_3 - \int_{\omega} \sigma^\varepsilon_{\tau_i} (\varphi_i - u^\varepsilon_i) d\acute{x} = \int_{\Omega^\varepsilon} f_i{}^\varepsilon (\varphi_i - u^\varepsilon_i) d\acute{x} dx_3, \ \varphi \in V^\varepsilon. \tag{3.2.7}$$

In (3.2.7), we add and subtract the term $\int_\omega F^\varepsilon S(\sigma_n^\varepsilon)(|\varphi_\tau - s| - |u_\tau^\varepsilon - s|)d\acute{x}$, we obtain

$$\begin{aligned}
&\int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon \frac{\partial}{\partial x_i}(\varphi_i - u_i^\varepsilon)d\acute{x}dx_3 - \int_\omega \sigma_{\tau_i}^\varepsilon(\varphi_i - u_i^\varepsilon)d\acute{x} + \\
&\int_\omega F^\varepsilon S(\sigma_n^\varepsilon)(|\varphi_\tau - s| - |u_\tau^\varepsilon - s|)d\acute{x} - \int_\omega F^\varepsilon S(\sigma_n^\varepsilon)(|\varphi_\tau - s| - |u_\tau^\varepsilon - s|)d\acute{x} \\
= &\int_{\Omega^\varepsilon} f_i^{\ \varepsilon}(\varphi_i - u_i^\varepsilon)d\acute{x}dx_3.
\end{aligned}$$

Putting

$$B = \int_\omega \sigma_{T_i}^\varepsilon((\varphi_\tau - s) - (u_\tau^\varepsilon - s))d\acute{x} + \int_\omega F^\varepsilon S(\sigma_n^\varepsilon)(|\varphi_\tau - s| - |u_\tau^\varepsilon - s|)d\acute{x},$$

then

$$B = \int_\omega \sigma_{\tau_i}^\varepsilon(\varphi_\tau - s)d\acute{x} + \int_\omega F^\varepsilon S(\sigma_n^\varepsilon)\,|\varphi_\tau - s|\,d\acute{x},$$

thus

$$\sigma_{\tau_i}^\varepsilon(\varphi_\tau - s) \geq -\left|\sigma_{\tau_i}^\varepsilon\right||\varphi_\tau - s| \geq -F^\varepsilon S(\sigma_n^\varepsilon)\,|\varphi_\tau - s| > 0, \text{ on } \omega,$$

and

$$B = \int_\omega \sigma_{\tau_i}^\varepsilon(\varphi_\tau - s)d\acute{x} + \int_\omega F^\varepsilon S(\sigma_n^\varepsilon)\,|\varphi_\tau - s|\,d\acute{x} \geq 0,$$

which gives

$$\int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon \frac{\partial}{\partial x_i}(\varphi_i - u_i^\varepsilon)d\acute{x}dx_3 + \int_\omega F^\varepsilon S(\sigma_n^\varepsilon)\,|\varphi_\tau - s|\,d\acute{x} - \int_\omega F^\varepsilon S(\sigma_n^\varepsilon)\,|u_\tau^\varepsilon - s|\,d\acute{x} \geq \int_{\Omega^\varepsilon} f_i^{\ \varepsilon}(\varphi_i - u_i^\varepsilon)d\acute{x}dx_3. \tag{3.2.8}$$

Then we get the (3.2.6)

By multiplying the equation (3.1.3) by ψ where, $\psi \in H^1_{\Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon}(\Omega^\varepsilon)$

and using the **Green** formula we get

$$\begin{aligned}
\sum_{i=1}^{3}\int_{\Omega^\varepsilon} K^\varepsilon \frac{\partial T^{\ \varepsilon}}{\partial x_i}\frac{\partial \psi}{\partial x_i}d\acute{x}dx_3 \quad = \quad & \sum_{i,j=1}^{3}\int_{\Omega^\varepsilon} 2\mu^\varepsilon(T^\varepsilon)d_{ij}^2(u^\varepsilon)\psi d\acute{x}dx_3 + \\
&\int_{\Omega^\varepsilon} \lambda^\varepsilon(T^\varepsilon)\ div(u^\varepsilon)\ div(u^\varepsilon)\psi d\acute{x}dx_3 + \\
&\int_{\Gamma^\varepsilon} K^\varepsilon \frac{\partial T^{\ \varepsilon}}{\partial n_i}\psi ds + \int_{\Omega^\varepsilon} r^\varepsilon(T^\varepsilon)\psi d\acute{x}dx_3,
\end{aligned}$$

now the boundary condition (3.1.8), we give

$$\sum_{i=1}^{3}\int_{\Omega^\varepsilon} K^\varepsilon \frac{\partial T^\varepsilon}{\partial x_i}\frac{\partial \psi}{\partial x_i} d\acute{x}dx_3 = \sum_{i,j=1}^{3}\int_{\Omega^\varepsilon} 2\mu^\varepsilon(T^\varepsilon)d_{ij}^2(u^\varepsilon)\psi d\acute{x}dx_3 + \int_{\Omega^\varepsilon}\lambda^\varepsilon(T^\varepsilon)\ div(u^\varepsilon)\ div(u^\varepsilon)\psi d\acute{x}dx_3 + \int_{\Omega^\varepsilon} r^\varepsilon(T^\varepsilon)\psi d\acute{x}dx_3, \tag{3.2.9}$$

that is to say $b(T^\varepsilon,\psi) = c(u^\varepsilon,T^\varepsilon,\psi),\quad \forall\psi\in H^1_{\Gamma_L^\varepsilon\cup\Gamma_1^\varepsilon}$. ∎

Theorem 3.2.1 *If* $f^{\,\varepsilon}\in(L^2(\Omega^\varepsilon))^3$ *and the friction coefficient* F^ε *is a non-negative function in* $L^\infty(\omega)$ *then there exists* $u^\varepsilon\in V^\varepsilon$ *solution to the problem* (3.2.6) *.*

Moreover, for small F^ε *the solution is unique.*

Proof. The proof is similar to that in ([3],) and we shall not reproduce it in full giving only a sketch here, for the existence of a solution u^ε. Firste, we apply **Tichonov's** fixed point theorem ([17]) , then uses the same procedure to prove the uniqueness of u^ε as in ([3], [34]). ∎

3.3 Change of the domain and some estimates

The goal of this section is to make an asymptotic analysis of the previous problem by rescaling the study back to the domain Ω independent of ε, on which we set unknown news.

We obtain a variational formulation in the fixed domain and prove a priori estimates of independent solutions of ε, by various inequalities from (**Korn**, **Poincaré**, **Young**, etc.).

Finally, we derive the convergence results.

3.3.1 Scale change

Asymptotic analysis of the problem (3.1.9), we use an approach that transforms the problem originally posed in the field Ω^ε which depends on a small parameter ε into an equivalent problem in which the fixed domain Ω does not depend on ε.

For this reason, we introduce a small change in variable $z = \dfrac{x_3}{\varepsilon}$, then $(\acute{x}, x_3)$ in Ω^ε become $(\acute{x}, z)$ in Ω, and it's given by

$$\Omega = \{(\acute{x}, z) \in \mathbb{R}^3,\ (\acute{x}, 0) \in \omega,\ 0 < z < h(\acute{x})\}.$$

We denote the fixed domain boundary by $\Gamma = \overline{\omega} \cup \overline{\Gamma_L} \cup \overline{\Gamma_1}$.

Add a new functions in Ω, **independent of** ε :

$$\begin{cases} \hat{u}_i^\varepsilon(\acute{x}, z) = u_i^\varepsilon(\acute{x}, x_3),\ i = 1, 2, \\ \hat{u}_3^\varepsilon(\acute{x}, z) = \varepsilon^{-1} u_3^\varepsilon(\acute{x}, x_3). \end{cases} \tag{3.3.1}$$

$$\begin{cases} \hat{f}^{\,\varepsilon}(\acute{x}, z) = \varepsilon^2\ f^{\,\varepsilon}(\acute{x}, x_3), \\ \hat{F} = \varepsilon^{-1} F^\varepsilon, \\ \hat{T}^\varepsilon(\acute{x}, z) = T^\varepsilon(\acute{x}, x_3), \\ \hat{K} = K^\varepsilon, \hat{r} = \varepsilon^2 r^\varepsilon, \hat{\lambda} = \lambda^\varepsilon, \hat{\mu} = \mu^\varepsilon, \\ \hat{g}(\acute{x}, z) = g^\varepsilon(\acute{x}, x_3). \end{cases} \tag{3.3.2}$$

Let $\hat{G}(\acute{x}, z)$ such as :

$$\hat{G} = \hat{g} \text{ on } \Gamma.$$

The vector $\hat{G}$ introduced earlier is defined as:

$$\begin{aligned} &\hat{G}_i(\acute{x}, z) = G_i^\varepsilon(\acute{x}, x_3),\ i = 1, 2, \\ &\hat{G}_3(\acute{x}, z) = \varepsilon^{-1} G_3^\varepsilon(\acute{x}, x_3). \end{aligned} \tag{3.3.3}$$

3.3.2 Weak formulation in Ω

Here we introduce the functional framework in Ω.

$$\begin{cases} V = \left\{ \varphi \in \left(H^1(\Omega)\right)^3; \quad \varphi = \hat{G} \text{ on } \Gamma_L \text{ , } \varphi = 0 \text{ on } \Gamma_1^\varepsilon \text{ and } \varphi.n = 0 \text{ on } \omega \right\}, \\ \Pi(V) = \left\{ \varphi \in H^1(\Omega)^2 : \varphi = (\varphi_1, \varphi_2) \text{ , } \varphi_i = \hat{G} \text{ on } \Gamma_L \text{ and } \varphi_i = 0 \text{ on } \Gamma_1 \text{ , } i = 1, 2 \right\}, \\ V_z = \left\{ v = (v_1, v_2) \in L^2(\Omega)^2; \quad \dfrac{\partial v_i}{\partial z} \in L^2(\Omega), i = 1, 2; v = 0 \text{ on } \Gamma_1 \right\}, \\ H^1_{\Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon}(\Omega) = \{ \varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma_L \cup \Gamma_1 \}. \end{cases}$$

It is clear that V_z is a **Banach** space with norm:

$$\|v\|_{V_z} = \left(\sum_{i=1}^{2} \|v_i\|^2_{L^2(\Omega)} + \left\| \frac{\partial v_i}{\partial z} \right\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}.$$

The problem (P_v) became with scale change defined in $(3.3.1) - (3.3.3)$.

Find a displacement field $\hat{u}^\varepsilon \in V$ and find a temperature field $\hat{T}^\varepsilon \in H^1_{\Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon}(\Omega)$ such that:

$$a(\hat{T}^\varepsilon, \hat{u}^\varepsilon, \hat{\varphi} - \hat{u}^\varepsilon) + j(\hat{\varphi}) - j(\hat{u}^\varepsilon) \geq (\hat{f}^\varepsilon, \hat{\varphi} - \hat{u}^{\ \varepsilon}), \quad \forall \hat{\varphi} \in V, \tag{3.3.4}$$

$$b(\hat{T}^\varepsilon, \hat{\psi}) = c(\hat{u}^\varepsilon, \hat{T}^\varepsilon, \hat{\psi}), \quad \forall \hat{\psi} \in H^1_{\Gamma_L^\varepsilon \cup \Gamma_1^\varepsilon}(\Omega), \tag{3.3.5}$$

where

$$\begin{aligned} a(\hat{T}^\varepsilon, \hat{u}^\varepsilon, \hat{\varphi} - \hat{u}^\varepsilon) \quad = \quad & \varepsilon^2 \sum_{i,j=1}^{2} \int_\Omega \hat{\mu}(\hat{T}^\varepsilon) \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i} \right) \frac{\partial}{\partial x_j} (\hat{\varphi}_i - \hat{u}_i^\varepsilon) d\acute{x} dz + \\ & \sum_{i=1}^{2} \int_\Omega \hat{\mu}(\hat{T}^\varepsilon) \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \right) \left[\frac{\partial}{\partial z} (\hat{\varphi}_i - \hat{u}_i^\varepsilon) + \varepsilon^2 \frac{\partial}{\partial x_i} (\hat{\varphi}_i - \hat{u}_3^\varepsilon) \right] d\acute{x} dz + \\ & \varepsilon^2 \int_\Omega 2\hat{\mu}(\hat{T}^\varepsilon) \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \frac{\partial}{\partial z} (\hat{\varphi}_3 - \hat{u}_3^\varepsilon) d\acute{x} dz + \varepsilon^2 \int_\Omega \hat{\lambda}(\hat{T}^{\ \varepsilon}) \ div(\hat{u}^\varepsilon) \ div(\hat{\varphi} - \hat{u}^\varepsilon) d\acute{x} dz. \end{aligned}$$

$$(\hat{f}^\varepsilon, \hat{\varphi} - \hat{u}^\varepsilon) = \sum_{i=1}^{2} \int_\Omega \hat{f}_i^\varepsilon \left(\hat{\varphi}_i - \hat{u}_i^\varepsilon\right) d\acute{x} dz + \varepsilon \int_\Omega \hat{f}_3^{\ \varepsilon} \left(\hat{\varphi}_3 - \hat{u}_3^\varepsilon\right) d\acute{x} dz.$$

$$j(\hat{\varphi}) = \int_\omega \hat{F} S(\sigma_n^\varepsilon) \left| \hat{\varphi}_\tau - s \right| d\acute{x}.$$

$$b(\hat{T}^\varepsilon, \hat{\psi}) = \sum_{i=1}^{2} \int_\Omega \hat{K}\varepsilon^2 \frac{\partial \hat{T}^\varepsilon}{\partial x_i} \frac{\partial \hat{\psi}}{\partial x_i} d\acute{x} dz + \int_\Omega \hat{K} \frac{\partial \hat{T}^\varepsilon}{\partial z} \frac{\partial \hat{\psi}}{\partial z} d\acute{x} dz.$$

$$\begin{aligned} c(\hat{u}^\varepsilon, \hat{T}^\varepsilon, \hat{\psi}) \quad = \quad & \sum_{i,j=1}^{2} \frac{1}{2} \int_\Omega \varepsilon^2 \hat{\mu}(\hat{T}^\varepsilon) \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i} \right)^2 \hat{\psi} d\acute{x} dz + \sum_{i=1}^{2} \int_\Omega \hat{\mu}(\hat{T}^\varepsilon) \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \right)^2 \hat{\psi} d\acute{x} dz + \\ & \int_\Omega 2\varepsilon^2 \hat{\mu}(\hat{T}^\varepsilon) \left(\frac{\partial \hat{u}_3^\varepsilon}{\partial z} \right)^2 \hat{\psi} d\acute{x} dz + \int_\Omega \varepsilon^2 \hat{\lambda}(\hat{T}^\varepsilon)\ div(\hat{u}^\varepsilon)\ div(\hat{u}^\varepsilon) \hat{\psi} d\acute{x} dz + \int_\Omega \hat{r}(\hat{T}^\varepsilon) \hat{\psi} d\acute{x} dz. \end{aligned}$$

In the next, we make an estimate of the displacement u^ε and temperature T^ε solution of the variational problem in fixed domain.

3.3.3 A priori estimates of the displacement

Lemma 3.3.1 *Assume that $f \in (L^2(\Omega))^3$, the coefficient of friction $F^\varepsilon > 0$ in $L^\infty(w)$ and there is a strictly positive constants $\mu_*, \mu^*, \lambda_*, \lambda^*$ such that*

$$0 < \mu_* \leq \mu(a) \leq \mu^* \quad \textit{and}\ 0 < \lambda_* \leq \lambda(b) \leq \lambda^* \quad \forall a, b \in \mathbb{R}. \tag{3.3.6}$$

Then there is a strictly positive constant C that does not depend on ε such that

$$\sum_{i,j=1}^{2} \left\| \varepsilon \frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} \right\|_{L^2(\Omega)}^2 + \left\| \varepsilon \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 + \sum_{i=1}^{2} \left(\left\| \frac{\partial \hat{u}_i^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 + \left\| \varepsilon^2 \frac{\partial \hat{u}_i^\varepsilon}{\partial x_i} \right\|_{L^2(\Omega)}^2 \right) \leq C. \tag{3.3.7}$$

Proof. Let be u^ε the solution of the problem (3.2.6) so

$$a(T^\varepsilon, u^\varepsilon, u^\varepsilon - \varphi) + j^\varepsilon(\varphi) - j^\varepsilon(u^\varepsilon) \geq (f^{\ \varepsilon}, u^\varepsilon) - (f^{\ \varepsilon}, \varphi), \forall \varphi \in V^\varepsilon. \tag{3.3.8}$$

Because $j^\varepsilon(u^\varepsilon)$ is positive

$$a(T^\varepsilon, u^\varepsilon, u^\varepsilon) \leq a(T^\varepsilon, u^\varepsilon, \varphi) + j^\varepsilon(\varphi) + (f^{\ \varepsilon}, u^\varepsilon) - (f^{\ \varepsilon}, \varphi), \forall \varphi \in V^\varepsilon.$$

with

$$a(T^\varepsilon, u^\varepsilon, u^\varepsilon) = \int_{\Omega^\varepsilon} 2\mu^\varepsilon(T^\varepsilon) d_{ij}(u^\varepsilon) d_{ij}(u^\varepsilon) d\acute{x} dx_3 + \int_{\Omega^\varepsilon} \lambda^\varepsilon(T^\varepsilon)\ div(u^\varepsilon)\ div(u^\varepsilon) d\acute{x} dx_3,$$

and as
$$\sum_{i,j=1}^{2} |d_{ij}(u^\varepsilon)|^2 \leq |\nabla u^\varepsilon|^2 \text{ and } |\ div(u^\varepsilon)|^2 \leq |\nabla u^\varepsilon|^2 ,$$

so, according to the inequality of **Korn** (from [33]), there exist C_K independent of ε such that:
$$a(T^\varepsilon, u^\varepsilon, u^\varepsilon) \geq 2\mu_* C_K \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} . \tag{3.3.9}$$

Applying the inequalities of **Hölder** and **Young** we find the following inquiries
$$\begin{aligned}
a(T^\varepsilon, u^\varepsilon, \varphi) &\leq \int_{\Omega^\varepsilon} 2\mu^\varepsilon(T^\varepsilon) |d_{ij}(u^\varepsilon)| |d_{ij}(\varphi)| \, d\acute{x}dx_3 + \int_{\Omega^\varepsilon} \lambda^\varepsilon(T^\varepsilon) |\ div(u^\varepsilon)| |\ div(\varphi)| \, d\acute{x}dx_3 \\
&\leq \int_{\Omega^\varepsilon} 2\mu^*(T^\varepsilon) |d_{ij}(u^\varepsilon)| |d_{ij}(\varphi)| \, d\acute{x}dx_3 + \int_{\Omega^\varepsilon} \lambda^*(T^\varepsilon) |\ div(u^\varepsilon)| |\ div(\varphi)| \, d\acute{x}dx_3 \\
&\leq \int_{\Omega^\varepsilon} \left(\sqrt{\frac{\mu_* C_K}{2}} |d_{ij}(u^\varepsilon)| \right) \left(\frac{2\ \mu^* \sqrt{2}}{\sqrt{\mu_* C_K}} |d_{ij}(\varphi)| \right) d\acute{x}dx_3 + \\
&\quad \int_{\Omega^\varepsilon} \left(\frac{\sqrt{\mu_* C_K}}{2} |\ div(u^\varepsilon)| \right) \left(\frac{2\ \lambda^*}{\sqrt{\mu_* C_K}} |\ div(\varphi)| \right) d\acute{x}dx_3 \\
&\leq \frac{\mu_* C_K}{4} \|d_{ij}(u^\varepsilon)\|^2_{L^2(\Omega^\varepsilon)} + \frac{4\ (\mu^*)^2}{\mu_* C_K} \|d_{ij}(\varphi)\|^2_{L^2(\Omega^\varepsilon)} + \\
&\quad \frac{\mu_* C_K}{8} \int_{\Omega^\varepsilon} |\ div(u^\varepsilon)|^2 \, d\acute{x}dx_3 + \frac{2(\lambda^*)^2}{\mu_* C_K} \int_{\Omega^\varepsilon} |\ div(\varphi)|^2 \, d\acute{x}dx_3
\end{aligned}$$

so
$$a(T^\varepsilon, u^\varepsilon, \varphi) \leq \frac{3\ \mu_* C_K}{8} \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + \left(\frac{4\ (\mu^*)^2}{\mu_* C_K} + \frac{2(\lambda^*)^2}{\mu_* C_K} \right) \|\nabla \varphi\|^2_{L^2(\Omega^\varepsilon)} . \tag{3.3.10}$$

Since
$$(f^{\ \varepsilon}, u^\varepsilon) \leq \frac{(\varepsilon h^*)^2}{2\ \mu_* C_K} \|\nabla f^{\ \varepsilon}\|^2_{L^2(\Omega^\varepsilon)} + \frac{\mu_* C_K}{2} \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} . \tag{3.3.11}$$
$$(f^{\ \varepsilon}, \varphi) \leq \frac{(\varepsilon h^*)^2}{2\ \mu_* C_K} \|\nabla f^{\ \varepsilon}\|^2_{L^2(\Omega^\varepsilon)} + \frac{\mu_* C_K}{2} \|\nabla \varphi\|^2_{L^2(\Omega^\varepsilon)} . \tag{3.3.12}$$

We used $(3.3.9) - (3.3.12)$, and we choose $\varphi = G^\varepsilon$ we get the variational equation equivalent to $(2.3.8)$
$$\begin{aligned}
2\mu_* C_K \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} &\leq a(T^\varepsilon, u^\varepsilon, u^\varepsilon) \leq \frac{7}{8}\mu_* C_K \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + \frac{(\varepsilon h^*)^2}{\mu_* C_K} \|\nabla f^{\ \varepsilon}\|^2_{L^2(\Omega^\varepsilon)} + \\
&\quad \left(\frac{4\ (\mu^*)^2}{\mu_* C_K} + \frac{2(\lambda^*)^2}{\mu_* C_K} + \frac{\mu_* C_K}{2} \right) \|\nabla G^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} .
\end{aligned}$$

$$\frac{9}{8}\mu_* C_K \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} \leq \frac{(\varepsilon h^*)^2}{\mu_* C_K} \|\nabla f^{\ \varepsilon}\|^2_{L^2(\Omega^\varepsilon)} + \left(\frac{4\ (\mu^*)^2}{\mu_* C_K} + \frac{2(\lambda^*)^2}{\mu_* C_K} + \frac{\mu_* C_K}{2} \right) \|\nabla G^\varepsilon\|^2_{L^2(\Omega^\varepsilon)}$$

As

$$\varepsilon^2 \|\nabla f^{\ \varepsilon}\|^2_{L^2(\Omega^\varepsilon)} = \varepsilon^{-1} \left\| \nabla \hat{f}^{\ \varepsilon} \right\|^2_{L^2(\Omega^\varepsilon)} \text{ and } \quad \varepsilon \|\nabla G^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} = \left\| \nabla \hat{G} \right\|^2_{L^2(\Omega^\varepsilon)},$$

then

$$\begin{aligned} \frac{9}{8}\mu_* \varepsilon C_K \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} &\leq \frac{(h^*)^2}{\mu_* C_K} \left\| \nabla \hat{f}^{\ \varepsilon} \right\|^2_{L^2(\Omega^\varepsilon)} + \varepsilon \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + \\ &\quad \left(\frac{4\ (\mu^*)^2}{\mu_* C_K} + \frac{2(\lambda^*)^2}{\mu_* C_K} + \frac{\mu_* C_K}{2} \right) \left\| \nabla \hat{G} \right\|^2_{L^2(\Omega^\varepsilon)} \\ &\leq C, \end{aligned}$$

thuse

$$\varepsilon \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} = \varepsilon^2 \sum_{i,j=1}^{2} \left\| \frac{\partial \hat{u}_i^\varepsilon}{\partial x_J} \right\|^2_{L^2(\Omega)} + \varepsilon^2 \left\| \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \right\|^2_{L^2(\Omega)} + \sum_{i=1}^{2} \left(\left\| \frac{\partial \hat{u}_i^\varepsilon}{\partial z} \right\|^2_{L^2(\Omega)} + \varepsilon^4 \left\| \frac{\partial \hat{u}_i^\varepsilon}{\partial x_i} \right\|^2_{L^2(\Omega)} \right) \leq C,$$

with

$$C = \frac{8}{9\mu_* C_K} c_0 \text{ and } c_0 = \frac{(h^*)^2}{\mu_* C_K} \left\| \nabla \hat{f}^{\ \varepsilon} \right\|^2_{L^2(\Omega^\varepsilon)} + \left(\frac{4\ (\mu^*)^2}{\mu_* C_K} + \frac{2(\lambda^*)^2}{\mu_* C_K} + \frac{\mu_* C_K}{2} \right) \left\| \nabla \hat{G} \right\|^2_{L^2(\Omega^\varepsilon)}.$$

■

3.3.4 A priori estimate of the temperature

In this subsection, we look for a priori estimate of the temperature $\hat{T}^\varepsilon$. For that, we need to state the following lemma, which is a direct result of **Poincaré**'s inequality.

Lemma 3.3.2 *The temperature $\hat{T}^\varepsilon$ is increased by*

$$\left\| \hat{T}^\varepsilon \right\|_{L^2(\Omega)} \leq h^* \left\| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right\|_{L^2(\Omega)} \tag{3.3.13}$$

Lemma 3.3.3 *Assuming further verification of the hypothesis of the **Lemma** (3.3.1), suppose we have* *two strictly positive constants* K_* *and* K^* *such that:*

$$0 \leq K_* \leq K(\acute{x}, z) \leq K^*, \forall (\acute{x}, z) \in \Omega \tag{3.3.14}$$

a positive constant $\hat{r}^*$, *such that:*

$$\hat{r}(a) \leq \hat{r}^* \tag{3.3.15}$$

then there is a positive constant C_2 *independent of* ε *such that:*

$$\varepsilon^2 \sum_{i=1}^{2} \left\| \frac{\partial \hat{T}^\varepsilon}{\partial x_i} \right\|_{L^2(\Omega)} + \left\| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right\|_{L^2(\Omega)} \leq C_2 \tag{3.3.16}$$

Proof. *Choosing* $\hat{T}^\varepsilon = \psi$, *in the variational equation* (3.2.9), *gives*

$$\sum_{i=1}^{3} I_i = \sum_{i=1}^{2} \int_\Omega \varepsilon^2 \hat{K} \frac{\partial \hat{T}^\varepsilon}{\partial x_i} \frac{\partial \hat{T}^\varepsilon}{\partial x_i} d\acute{x}dz + \int_\Omega \hat{K} \frac{\partial \hat{T}^\varepsilon}{\partial z} \frac{\partial \hat{T}^\varepsilon}{\partial z} d\acute{x}dz,$$

with

$$\begin{aligned} I_1 &= \sum_{i,j=1}^{2} \frac{1}{2} \int_\Omega \varepsilon^2 \hat{\mu}(\hat{T}^\varepsilon) \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i} \right)^2 \hat{T}^\varepsilon d\acute{x}dz + \\ & \sum_{i=1}^{2} \int_\Omega \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \right)^2 \hat{T}^\varepsilon d\acute{x}dz + \int_\Omega 2\varepsilon^2 \hat{\mu}(\hat{T}^\varepsilon) \left(\frac{\partial \hat{u}_3^\varepsilon}{\partial z} \right)^2 \hat{T}^\varepsilon d\acute{x}dz, \end{aligned}$$

$$I_2 = \int_\Omega \hat{r}(\hat{T}^\varepsilon) \hat{T}^\varepsilon d\acute{x}dz,$$

$$I_3 = \int_\Omega \varepsilon^2 \hat{\lambda}(\hat{T}^\varepsilon) \ div(\hat{u}^\varepsilon) \ div(\hat{u}^\varepsilon) \hat{T}^\varepsilon d\acute{x}dz.$$

By the ***Cauchy-Schwartz*** *inequality, we obtain*

$$\begin{aligned} |I_1| \leq & \frac{\varepsilon^2 \mu^*}{2} \sum_{i,j=1}^{2} \left\| \frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} \right\|_{L^2(\Omega)}^2 \left\| \hat{T}^\varepsilon \right\|_{L^2(\Omega)}^2 + \mu^* \sum_{i=1}^{2} \left\| \frac{\partial \hat{u}_i^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 \left\| \hat{T}^\varepsilon \right\|_{L^2(\Omega)}^2 \\ & + 2\varepsilon^2 \mu^* \left\| \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 \left\| \hat{T}^\varepsilon \right\|_{L^2(\Omega)}^2 + \mu^* \sum_{i=1}^{2} \left\| \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \right\|_{L^2(\Omega)}^2 \left\| \hat{T}^\varepsilon \right\|_{L^2(\Omega)}^2 . \end{aligned}$$

From ***Young****'s inequality*

$$\begin{aligned}|I_1| \;\leq\; & 2\varepsilon^2\mu^* \sum_{i,j=1}^{2} \left\| \frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} \right\|_{L^2(\Omega)}^2 \left\| \hat{T}^\varepsilon \right\|_{L^2(\Omega)}^2 + 2\mu^* \sum_{i=1}^{2} \left\| \frac{\partial \hat{u}_i^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 \left\| \hat{T}^\varepsilon \right\|_{L^2(\Omega)}^2 \\ & + 2\varepsilon^4\mu^* \left\| \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \right\|_{L^2(\Omega)}^2 \left\| \hat{T}^\varepsilon \right\|_{L^2(\Omega)}^2 + 2\varepsilon^2\mu^* \left\| \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 \left\| \hat{T}^\varepsilon \right\|_{L^2(\Omega)}^2 .\end{aligned}$$

By the inequality $(3.3.7)$ *and the* ***Lemma*** $(\mathbf{3.3.2})$, *we find*

$$|I_1| \leq 2\mu^* C \left\| \hat{T}^\varepsilon \right\|_{L^2(\Omega)}^2 \leq 2\mu^* C h^* \left\| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right\|_{L^2(\Omega)} . \tag{3.3.17}$$

The analog of I_1 *gives*

$$|I_2| \leq \hat{r}^* \left\| \hat{T}^\varepsilon \right\|_{L^2(\Omega)}^2 \leq \hat{r}^* h^* \left\| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right\|_{L^2(\Omega)} . \tag{3.3.18}$$

$$|I_3| \leq \lambda \left\| \mathrm{div}(\hat{u}^\varepsilon) \right\|^2 \left\| \hat{T}^\varepsilon \right\|_{L^2(\Omega)}^2 ,$$

since $\varepsilon^2 \left\| \nabla \hat{u}^\varepsilon \right\|_{L^2(\Omega)}^2 \leq C$, *then*

$$|I_3| \leq \lambda^* C \left\| \hat{T}^\varepsilon \right\|_{L^2(\Omega)}^2 . \tag{3.3.19}$$

On the other hand, using $(3.3.14) - (3.3.15)$, *gives*

$$b(\hat{T}^\varepsilon, \hat{T}^\varepsilon) = \sum_{i=1}^{2} \int_\Omega \varepsilon^2 \hat{K} \left| \frac{\partial \hat{T}^\varepsilon}{\partial x_i} \right|^2 d\acute{x} dz + \int_\Omega \hat{K} \left| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right|^2 d\acute{x} dz.$$

Which implies

$$K_* \varepsilon^2 \left\| \frac{\partial \hat{T}^\varepsilon}{\partial x_i} \right\|^2 + K_* \left\| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right\|^2 \leq b(\hat{T}^\varepsilon, \hat{T}^\varepsilon)$$

$$\leq 2\mu^* h^* C \left\| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 + \hat{r}^* h^* \left\| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 + \lambda^* C h^* \left\| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2$$

then

$$K_* \varepsilon^2 \left\| \frac{\partial \hat{T}^\varepsilon}{\partial x_i} \right\|_{L^2(\Omega)}^2 + K_* \left\| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 \leq C_1 \left\| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 , \tag{3.3.20}$$

where C_1 *is a constant that does not depend on* ε *and is given by*

$$C_1 = 2\mu^* h^* C + \hat{r}^* h^* + \hat{\lambda}^* C h^*$$

thus

$$\left\| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right\|^2_{L^2(\Omega)} \le K_*^{-1} C_1. \tag{3.3.21}$$

Using this last estimate of (3.3.20) *we derive*

$$\varepsilon^2 \left\| \frac{\partial \hat{T}^\varepsilon}{\partial x_i} \right\|^2_{L^2(\Omega)} + \left\| \frac{\partial \hat{T}^\varepsilon}{\partial z} \right\|^2_{L^2(\Omega)} \le C_2$$

with

$$C_2 = K_*^{-1} (C)^2$$

■

3.4 Convergence results

In this part, we set up the convergence theorem.

Theorem 3.4.1 *Under the same assumptions as in* ***Lemma*** (**3.3.1**) *and* (3.3.3) *there exists*

$u^* = (u_1^*, u_2^*)$ *in* V_z *and* T^* *in* V_z *such as for subsuites of* $\hat{u}^\varepsilon (resp \hat{T}^\varepsilon)$ *noted again* $\hat{u}^\varepsilon (resp \hat{T}^\varepsilon)$ *we get the following convergence results:*

$$\hat{u}_i^\varepsilon \rightharpoonup u_i^* \quad \textit{weakly in} \quad V_z, 1 \le i \le 2 \tag{3.4.1}$$

$$\varepsilon \frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} \rightharpoonup 0 \quad \textit{weakly in} \quad L^2, 1 \le i, j \le 2. \tag{3.4.2}$$

$$\varepsilon \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \rightharpoonup 0 \quad \textit{weakly in} \quad L^2 \tag{3.4.3}$$

$$\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \rightharpoonup 0 \quad \textit{weakly in} \quad L^2, 1 \le i \le 2 \tag{3.4.4}$$

$$\varepsilon \hat{u}_3^\varepsilon \rightharpoonup 0 \quad \textit{weakly in} \quad L^2 \tag{3.4.5}$$

$$\hat{T}^{\varepsilon} \rightharpoonup T^{*} \quad \textit{weakly in} \quad V_z \tag{3.4.6}$$

$$\varepsilon \frac{\partial \hat{T}^{\varepsilon}}{\partial x_i} \rightharpoonup 0 \quad \textit{weakly in} \quad L^2, 1 \le i \le 2 \tag{3.4.7}$$

Proof. According to **Theorem** $(\mathbf{3.3.1})$, there exists a constant C independent of ε, which

$$\varepsilon \left\| \frac{\partial \hat{u}_i^{\varepsilon}}{\partial z} \right\|_{L^2(\Omega)}^2 \le C, i = 1, 2$$

Using this estimate with the **Poincaré** inequality, we get:

$$\|\hat{u}_i^{\varepsilon}\|_{0,\Omega} \le h^{*} \left\| \frac{\partial \hat{u}_i^{\varepsilon}}{\partial z} \right\|, 1 \le i \le 2,$$

which means $\hat{u}_i^{\varepsilon}$ is bounded in V_z for $i = 1, 2,$ this implies the existence of u_i^{*} in V_Z such as $\hat{u}_i^{\varepsilon}$ converges weakly to u_i^{*} in V_z.
Using the **Theorem** $(\mathbf{3.3.1})$:

$$\varepsilon \left\| \frac{\partial \hat{u}_i^{\varepsilon}}{\partial x_j}, \right\|_{0,\Omega} \le C.$$

So $\varepsilon \dfrac{\partial \hat{u}_i^{\varepsilon}}{\partial x_j}$ converges to $\dfrac{\partial u_i^{*}}{\partial x_j}$ on the other hand, we have:

$$\|\hat{u}_i^{\varepsilon}\|_{0,\Omega} \le C,$$

then $\dfrac{\partial \hat{u}_i^{\varepsilon}}{\partial x_j}$ converges to $\dfrac{\partial u_i^{*}}{\partial x_j}$ which makes the weak convergence of $\dfrac{\partial \hat{u}_i^{\varepsilon}}{\partial x_j}$ to $(0,0)$ in $L^2(\Omega_1) \times L^2(\Omega_2)$.
Ditto for the inequality

$$\varepsilon^2 \left\| \frac{\partial \hat{u}_3^{\varepsilon}}{\partial x_i} \right\|_{0,\Omega} \le C,$$

we have the convergences

$$\varepsilon^2 \frac{\partial \hat{u}_3^{\varepsilon}}{\partial x_i} \rightharpoonup \frac{\partial u_3^{*}}{\partial x_i}, \text{ and } \varepsilon \frac{\partial \hat{u}_3^{\varepsilon}}{\partial x_i} \rightharpoonup \frac{\partial u_3^{*}}{\partial x_i}.$$

Which shows that $\varepsilon^2 \dfrac{\partial \hat{u}_3^{\varepsilon}}{\partial x_i}$ converges weakly to $(0,0)$ in $L^2(\Omega)$.
The proof of the convergence temperature is based on this estimate

$$\varepsilon^2 \sum_{i=1}^{2} \left\| \frac{\partial \hat{T}^{\ \varepsilon}}{\partial x_i} \right\| + \left\| \frac{\partial \hat{T}^{\ \varepsilon}}{\partial z} \right\| \le C_2,$$

where C_2 is a constant independent of ε .

Using the **lemma** $(\mathbf{2.3.1})$ we deduce

$$\left\|\hat{T}^{\varepsilon}\right\| \leq h^* \left\|\frac{\partial \hat{T}^{\varepsilon}}{\partial z}\right\| \leq h^* C_2.$$

So, $\hat{T}^{\varepsilon}$ is bound in $V_z(\Omega)$ which shows the existence of T^* in $V_z(\Omega)$, such that $\hat{T}^{\varepsilon}$ converges weakly to T^* in $V_z(\Omega)$,

again, using the same estimate to show $\varepsilon \left\|\frac{\partial \hat{T}^{\varepsilon}}{\partial x_i}\right\| \leq C_2$,

so, $\left(\varepsilon \dfrac{\partial \hat{T}^{\varepsilon}}{\partial x_i}\right)$ converge to $\dfrac{\partial T^*}{\partial x_i}$ and like $\hat{T}^{\varepsilon}$ converge to T^* in $V_z(\Omega)$ then $\varepsilon \dfrac{\partial \hat{T}^{\varepsilon}}{\partial x_i}$ weakly converges to 0 in $V_z(\Omega)$. ■

3.5 Study of the limit problem

The rest of this section requires previous convergence results to achieve the desired goal.

Lemma 3.5.1 *There is a subsequence of $S(\sigma_n^{\varepsilon}(u^{\varepsilon}))$ that strongly converges towards $S(\sigma_n^*(u^*))$ in $L^2(\omega)$.*

Proof. To prove this lemma, the use of identical techniques in $([3];\mathbf{Lemma}5.1)$ and in $([10];\mathbf{Lemma}5.2)$. ■

Theorem 3.5.1 *$\hat{u}_i^{\varepsilon} \to u_i^*$ strongly in $V_z, i = 1, 2$ and with the same assumptions of the **Theorem** (3.3.1), the solutions u^* and T^* satisfy*

$$\sum_{i=1}^{2} \int_{\Omega} \hat{\mu}(T^*) \frac{\partial u_i^*}{\partial z} \frac{\partial}{\partial z}(\hat{\varphi}_i - u_i^*) d\acute{x} dz + j(\hat{\varphi}) - j(u^*) \geq \sum_{i=1}^{2} \left(\hat{f}_i^{\,\varepsilon}, \hat{\varphi}_i - u_i^*\right), \quad \forall \hat{\varphi} \in \Pi(V) \quad (3.5.1)$$

$$-\frac{\partial}{\partial z}\left(\hat{\mu}(T^*) \frac{\partial u_i^*}{\partial z}\right) = \hat{f}_i^{\,\varepsilon}, \quad \textit{for } i = 1, 2 \textit{ in } L^2(\Omega) \quad (3.5.2)$$

$$-\frac{\partial}{\partial z}\left(\hat{K} \frac{\partial T^*}{\partial z}\right) = \sum_{i=1}^{2} \hat{\mu}(T^*) \left(\frac{\partial u_i^*}{\partial z}\right)^2 + \hat{r}(T^*) \textit{ in } L^2(\Omega) \quad (3.5.3)$$

Proof. For $u_i^\varepsilon \to u_i^*$ strongly V_z, we use the same methods in ([10]; proof of **Theorem** 4.2). The variational equality (3.3.4) can be written as

$$\sum_{i=1}^{4} I_i(\varepsilon) + \varepsilon^2 \int_\Omega \hat{\lambda}(\hat{T}^\varepsilon)\ div(\hat{u}^\varepsilon)\ div(\hat{\varphi} - \hat{u}^\varepsilon) d\acute{x} dz + \int_\omega \hat{F} S(\sigma_n^\varepsilon) \left|\hat{\varphi} - s\right| d\acute{x} - \int_\omega \hat{F} S(\sigma_n^\varepsilon) \left|\hat{u}^\varepsilon - s\right| d\acute{x}$$

$$\geq \sum_{i=1}^{2} \left(\hat{f}_i^{\ \varepsilon}, \hat{\varphi}_i - \hat{u}_i^\varepsilon\right) + \varepsilon \left(\hat{f}_3^{\ \varepsilon}, \hat{\varphi}_3 - \hat{u}_3^\varepsilon\right),$$

where

$$I_1 = \sum_{i,j=1}^{2} \int_\Omega \varepsilon^2 \hat{\mu}(\hat{T}^\varepsilon) \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i}\right) \frac{\partial}{\partial x_j} (\hat{\varphi}_i - \hat{u}_i^\varepsilon) d\acute{x} dz,$$

$$I_2 = \sum_{i=1}^{2} \int_\Omega \hat{\mu}(\hat{T}^\varepsilon) \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i}\right) \frac{\partial}{\partial z} (\hat{\varphi}_i - \hat{u}_i^\varepsilon) d\acute{x} dz,$$

$$I_3 = \sum_{i=1}^{2} \int_\Omega \varepsilon^2 \hat{\mu}(\hat{T}^\varepsilon) \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i}\right) \frac{\partial}{\partial x_i} (\hat{\varphi}_3 - \hat{u}_3^\varepsilon) d\acute{x} dz,$$

$$I_4 = \int_\Omega 2\varepsilon^2 \hat{\mu}(\hat{T}^\varepsilon) \left(\frac{\partial \hat{u}_3^\varepsilon}{\partial z}\right) \frac{\partial}{\partial z} \left(\hat{\varphi}_3 - \hat{u}_3^\varepsilon\right) d\acute{x} dz.$$

By the convergence **Theorem** (**3.3.1**), we find

$$\lim_{\varepsilon \to 0} \sum_{i=1}^{4} I_i(\varepsilon) = \sum_{i=1}^{2} \int_\Omega \hat{\mu}(T^*) \left(\frac{\partial u_i^*}{\partial z}\right) \frac{\partial}{\partial z} (\hat{\varphi}_i - u_i^*) d\acute{x} dz,$$

$$\lim_{\varepsilon \to 0} \varepsilon^2 \int_\Omega \hat{\lambda}(\hat{T}^\varepsilon)\ div(\hat{u}^\varepsilon)\ div(\hat{\varphi} - \hat{u}^\varepsilon) d\acute{x} dz = 0,$$

$$\lim_{\varepsilon \to 0} \varepsilon \int_\Omega \hat{f}_3^{\ \varepsilon} (\hat{\varphi}_3 - \hat{u}_3^\varepsilon) d\acute{x} dz = 0,$$

and as j is convex and lower semi-continuous, we obtain

$$\lim_{\varepsilon \to 0} \left(\inf \int_\omega \hat{F} S(\sigma_n^\varepsilon) \left|\hat{u}^\varepsilon - s\right| d\acute{x}\right) \geq \int_\omega \hat{F} S(\sigma_n^\varepsilon) \left|\hat{u}^\varepsilon - s\right| d\acute{x}$$

so we have

$$\sum_{i=1}^{2} \int_\Omega \hat{\mu}(T^*) \frac{\partial u_i^*}{\partial z} \frac{\partial}{\partial z} (\hat{\varphi}_i - u_i^*) d\acute{x} dz + j(\hat{\varphi}) - j(u^*) \geq \sum_{i=1}^{2} \left(\hat{f}_i^{\ \varepsilon}, \hat{\varphi}_i - u_i^*\right). \tag{3.5.4}$$

From ([11]; **Lemma**5.3) ,we choose in (3.5.4)

$$\hat{\varphi}_i = u_i^* \pm \psi_i, \psi_i \in H_0^1(\Omega) \text{ for } i = 1, 2 \text{ and } \hat{\varphi}_3 = u_3^*$$

then we get

$$\sum_{i=1}^{2}\int_{\Omega}\hat{\mu}(T^*)\frac{\partial u_i^*}{\partial z}\frac{\partial \psi_i}{\partial z}d\acute{x}dz=\sum_{i=1}^{2}\left(\hat{f}_i^{\,\varepsilon},\psi_i\right).$$

Using **Green**'s formula and choosing $\psi_1=0$ and $\psi_2\in H_0^1(\Omega)$ then $\psi_2=0$ and $\psi_1\in H_0^1(\Omega)$ we obtain:

$$-\int_{\Omega}\hat{\mu}(T^*)\frac{\partial}{\partial z}\left(\frac{\partial u_i^*}{\partial z}\right)\psi_i d\acute{x}dz=\int_{\Omega}\hat{f}_i^{\,\varepsilon}\psi_i d\acute{x}dz,$$

thus:

$$-\hat{\mu}(T^*)\frac{\partial}{\partial z}\left(\frac{\partial u_i^*}{\partial z}\right)=\hat{f}_i^{\,\varepsilon},\text{for } i=1,2 \text{ in } H^{-1}(\Omega), \tag{3.5.5}$$

and as $\hat{f}_i^{\,\varepsilon}\in L^2(\Omega)$, then (3.5.5) is true in $L^2(\Omega)$.

On the other hand, going to the limit in (3.3.5) and using the convergence results of the theorem, we find

$$-\int_{\Omega}\hat{K}\frac{\partial T^*}{\partial z}\frac{\partial \psi}{\partial z}d\acute{x}dz=\sum_{i=1}^{2}\int_{\Omega}\hat{\mu}(T^*)\left(\frac{\partial u_i^*}{\partial z}\right)^2\psi d\acute{x}dz+\int_{\Omega}\hat{r}(T^*)\psi d\acute{x}dz,\forall\psi\in H^1_{\Gamma_L\cup\Gamma_1}(\Omega).$$

Now the formula of **Green**, we give

$$-\int_{\Omega}\frac{\partial}{\partial z}\left(\hat{K}\frac{\partial T^*}{\partial z}\right)\psi d\acute{x}dz=\sum_{i=1}^{2}\int_{\Omega}\hat{\mu}(T^*)\left(\frac{\partial u_i^*}{\partial z}\right)^2\psi d\acute{x}dz+\int_{\Omega}\hat{r}(T^*)\psi d\acute{x}dz,\forall\psi\in H^1_{\Gamma_L\cup\Gamma_1}(\Omega).$$

Consequently

$$-\frac{\partial}{\partial z}\left(\hat{K}\frac{\partial T^*}{\partial z}\right)=\sum_{i=1}^{2}\hat{\mu}(T^*)\left(\frac{\partial u_i^*}{\partial z}\right)^2+\hat{r}(T^*),\text{in}\in H^1_{\Gamma_L\cup\Gamma_1}(\Omega). \tag{3.5.6}$$

The formula (3.5.6) is valid in $L^2(\Omega)$, since $\hat{\mu}$ and $\hat{r}$ are two bounded functions in $\mathbb{R}$ and $\left(\frac{\partial u_i^*}{\partial z}\right)^2$ is an element of $L^2(\Omega)$. ■

Theorem 3.5.2 *Under the same assumptions as in the previous* ***Theorem****, we have*

$$\int_{\omega}\hat{F}\left|S(\sigma_n^*(s^*))\right|\left(|\psi+s^*-s|-|s^*-s|\right)d\acute{x}-\int_{\omega}\hat{\mu}(\varsigma^*)\hat{\xi}^*\psi d\acute{x}\geq 0,\forall\psi\in L^2(\omega) \tag{3.5.7}$$

and

$$\begin{cases}\hat{\mu}(\varsigma^*)\hat{\xi}^*<\hat{F}(S(\sigma_n^*(s^*))\Rightarrow s^*=s\\ \hat{\mu}(\varsigma^*)\hat{\xi}^*=\hat{F}(S(\sigma_n^*(s^*))\Rightarrow\exists\beta\geq 0 \text{ such that } s^*=s-\beta\hat{\xi}^*\end{cases} \tag{3.5.8}$$

Another u^* and T^* satisfies the following weak form of the Reynolds equation:

$$\int_\omega \left(\tilde{F} - \int_0^h \int_0^y \hat{\mu}(T^*(\acute{x},\xi)\frac{\partial u^*(\acute{x},\xi)}{\partial \xi}\partial\xi dy\right)\nabla\psi d\acute{x} = \int_\omega \frac{h}{2}\int_0^h \hat{\mu}(T^*(\acute{x},\xi)\frac{\partial u^*(\acute{x},\xi)}{\partial \xi}\partial\xi\nabla\psi d\acute{x}, \forall\psi \in H^1(\omega) \tag{3.5.9}$$

where

$$\tilde{F}(\acute{x},z) = \int_0^h F(\acute{x},y)dy - \frac{h}{2}F(\acute{x},h) \text{ and } F(\acute{x},z) = \int_0^h \int_0^\xi \hat{f}_i^{\ \varepsilon}(\acute{x},\alpha)d\alpha d\xi$$

Proof. The variational inequality (3.3.4) becomes

$$\begin{aligned}
&\varepsilon^2 \sum_{i,j=1}^{2} \int_\Omega \hat{\mu}(\hat{T}^\varepsilon)\left(\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i}\right)\frac{\partial}{\partial x_j}(\hat{\varphi}_i - \hat{u}_i^\varepsilon)d\acute{x}dz + \\
&\sum_{i=1}^{2} \int_\Omega \hat{\mu}(\hat{T}^\varepsilon)\left(\frac{\partial \hat{u}_i^\varepsilon}{\partial z} + \varepsilon^2\frac{\partial \hat{u}_3^\varepsilon}{\partial x_i}\right)\left[\frac{\partial}{\partial z}(\hat{\varphi}_i - \hat{u}_i^\varepsilon) + \varepsilon^2\frac{\partial}{\partial x_i}(\hat{\varphi}_i - \hat{u}_3^\varepsilon)\right]d\acute{x}dz + \\
&\varepsilon^2 \int_\Omega 2\hat{\mu}(\hat{T}^\varepsilon)\frac{\partial \hat{u}_3^\varepsilon}{\partial z}\frac{\partial}{\partial z}(\hat{\varphi}_3 - \hat{u}_3^\varepsilon)d\acute{x}dz + \varepsilon^2\int_\Omega \hat{\lambda}(\hat{T}^{\ \varepsilon})\ div(\hat{u}^\varepsilon)\ div(\hat{\varphi} - \hat{u}^\varepsilon)d\acute{x}dz + \\
&\int_\omega \hat{F}S(\sigma_n^\varepsilon)\left|\hat{\varphi}_T - s\right| d\acute{x} - \int_\omega \hat{F}S(\sigma_n^\varepsilon)\left|\hat{u}_T^\varepsilon - s\right| d\acute{x} \\
\geq\ &\sum_{i=1}^{2}\int_\Omega \hat{f}_i^\varepsilon(\hat{\varphi}_i - \hat{u}_i^\varepsilon)\,d\acute{x}dz + \varepsilon\int_\Omega \hat{f}_3^{\ \varepsilon}(\hat{\varphi}_3 - \hat{u}_3^\varepsilon)\,d\acute{x}dz,
\end{aligned}$$

choosing the next change

$$\hat{\varphi}_i = u_i^* + \psi_i,\ i = 1,2,\ \text{with } \psi_i \in H^1_{\Gamma_1\cup\Gamma_L}(\Omega),$$

then

$$\mu \sum_{1\le i\le 2}\int_\Omega \frac{\partial u_i^*}{\partial z}\frac{\partial \psi_i}{\partial z}\ d\acute{x}dz + +j(\psi_1 + s^*) - j(s^*) \geq \sum_{1\le i\le 2}\int_\Omega \hat{f}_i\psi_i\ d\acute{x}dz$$

Using **Green** 's formula on the domain Ω we find

$$\begin{aligned}
&\sum_{1\le i\le 2}\int_\Omega\left(-\mu_1\frac{\partial^2 u_i^*}{\partial z^2}\right)\psi_i\ d\acute{x}dz + \int_\Gamma \mu_1\frac{\partial u_{1i}^*}{\partial z}n\psi_i ds \\
&+\int_\omega \hat{F}S(\sigma_n^\varepsilon)\left(|\psi_1 + s^* - s| - |s^* - s|\right)d\acute{x} \\
\geq\ &\sum_{1\le i\le 2}\int_\Omega \hat{f}_i\psi_i\ d\acute{x}dz.
\end{aligned}$$

We know that
$$\int_\Gamma \mu \frac{\partial u^*}{\partial z} n\psi ds = -\int_\omega \mu \frac{\partial u^*}{\partial z}\psi d\acute{x}.$$
So
$$\begin{aligned} &\sum_{1\le i\le 2}\int_\Omega \left(-\mu\frac{\partial^2 u_i^*}{\partial z^2}\right)\psi_i \; d\acute{x}dz - \int_\omega \mu\xi^*\psi_i d\acute{x} \\ &+\int_\omega \hat{F}\, S(\sigma_n^\varepsilon)\left(|\psi_1+s^*-s| - |s^*-s|\right)d\acute{x} \\ \geq \; & \sum_{1\le i\le 2}\int_\Omega \hat{f}_i\psi_i \; d\acute{x}dz. \end{aligned}$$
On the other hand
$$\frac{\partial}{\partial z}\left(\mu\frac{\partial u_i^*}{\partial z}\right) = \hat{f}_i,\, i=1,2.$$
According to (3.4.4) and for $\psi \in H^1_{\Gamma_1\cup\Gamma_L}(\Omega)^2$, getting
$$\int_\omega \hat{F}\, S(\sigma_n^\varepsilon)\left(|\psi_1+s^*-s| - |s^*-s|\right)d\acute{x} - \int_\omega \mu\xi^*\psi d\acute{x} \geq 0. \tag{3.4.10}$$
Inequality (3.5.10) is also valid for all $\psi \in D(\omega)$, and by the density of $D(\omega)$ in $L^2(\omega)$, we deduce
$$\int_\omega \hat{F}\, S(\sigma_n^\varepsilon)\left(|\psi_1+s^*-s| - |s^*-s|\right)d\acute{x} - \int_\omega \mu\xi^*\psi d\acute{x} \geq 0,\ \forall\psi\in L^2(\omega).$$
Which implies (3.2.7).

To demonstrate (3.5.9) by integrating twice the equation (3.5.2) from 0 to z ,we find
$$-\int_0^z \hat{\mu}(T^*(\acute{x},\xi)\frac{\partial u^*(\acute{x},\xi)}{\partial\xi}\partial\xi + \hat{\mu}(\varsigma^*)\tau^* = \int_0^z \hat{f_i}^\varepsilon(\acute{x},\alpha)d\alpha, \tag{3.5.10}$$
with
$$\varsigma^* = T^*(\acute{x},0) \text{ and } \tau^* = \frac{\partial u^*}{\partial z}(\acute{x},0).$$
By integrating (3.5.10) from 0 to h, we obtain
$$-\int_0^h \hat{\mu}(T^*(\acute{x},\xi)\frac{\partial u^*(\acute{x},\xi)}{\partial\xi}\partial\xi + \hat{\mu}(\varsigma^*)\tau^* h = \int_0^h\int_0^\xi \hat{f_i}^\varepsilon(\acute{x},\alpha)d\alpha d\xi, \tag{3.5.11}$$

integrating $(3.5.11)$ between 0 and h, we have

$$-\int_0^h \int_0^y \hat{\mu}(T^*(\acute{x},\xi) \frac{\partial u^*(\acute{x},\xi)}{\partial \xi} \partial\xi \partial y + \hat{\mu}(\varsigma^*)\tau^* \frac{h^2}{2} = \int_0^h \int_0^y \int_0^\xi \hat{f}_i^{\,\varepsilon}(\acute{x},\alpha) d\alpha d\xi dy. \quad (3.5.12)$$

From $(3.5.11)$ we derive

$$\hat{\mu}(\varsigma^*)\tau^* h = \int_0^h \hat{\mu}(T^*(\acute{x},\xi) \frac{\partial u^*(\acute{x},\xi)}{\partial \xi} \partial\xi + \int_0^h \int_0^\xi \hat{f}_i^{\,\varepsilon}(\acute{x},y) dy d\xi. \quad (3.5.13)$$

By $(3.5.12)$ and $(3.5.13)$, we obtain

$$\tilde{F} - \int_0^h \int_0^y \hat{\mu}(T^*(\acute{x},\xi) \frac{\partial u^*(\acute{x},\xi)}{\partial \xi} \partial\xi dy - \frac{h}{2} \int_0^h \hat{\mu}(T^*(\acute{x},\xi) \frac{\partial u^*(\acute{x},\xi)}{\partial \xi} \partial\xi = 0.$$

■

Before giving the existence and uniqueness of the solution, we need the following sets

$$\begin{aligned} W_z &= \left\{ v \in V_z; \frac{\partial^2 v}{\partial^2 z} \in L^2(\Omega) \right\}, \\ B_c &= \left\{ v \in W_z \times W_z; \left\| \frac{\partial v}{\partial z} \right\|_{V_z} \leq c \right\}. \end{aligned}$$

Theorem 3.5.3 *Under the assumptions of* ***Theorem*** $(3.3.1)$ *and if there is a small enough positive constant* F^* *such that* $\|\hat{F}\|_{L^\infty(\omega)} < F^*$, *then the solution* (u^*, T^*) *of the problem limit* $(3.5.1) - (3.5.3)$ *is unique in* $B_c \times W_z$.

Proof. For solution uniqueness, follow the sam procedure, in $([3]$ and $[12])$.

Assuming that for each $\psi \in H^1_{\Gamma_L \cup \Gamma_1}(\Omega)$ there are (u^1, T^1) and (u^2, T^2) solutions to the problem limit $(3.5.1)$ and $(3.5.3)$ we have

$$\int_\Omega -\hat{K} \frac{\partial T^1}{\partial z} \frac{\partial \psi}{\partial z} d\acute{x} dz = \sum_{i=1}^2 \int_\Omega \hat{\mu}(T^1) \left(\frac{\partial u_i^1}{\partial z} \right)^2 \psi d\acute{x} dz + \int_\Omega \hat{r}(T^1) \psi d\acute{x} dz. \quad (3.5.14)$$

$$\int_\Omega -\hat{K} \frac{\partial T^2}{\partial z} \frac{\partial \psi}{\partial z} d\acute{x} dz = \sum_{i=1}^2 \int_\Omega \hat{\mu}(T^2) \left(\frac{\partial u_i^2}{\partial z} \right)^2 \psi d\acute{x} dz + \int_\Omega \hat{r}(T^2) \psi d\acute{x} dz. \quad (3.5.15)$$

By subtracting of $(3.5.14) - (3.5.15)$ we get

$$\int_\Omega -\hat{K} \frac{\partial}{\partial z} \left(T^1 - T^2 \right) \frac{\partial \psi}{\partial z} d\acute{x} dz \sum_{i=1}^2 \int_\Omega \left[\hat{\mu}(T^1) \left(\frac{\partial u_i^1}{\partial z} \right)^2 - \hat{\mu}(T^2) \left(\frac{\partial u_i^2}{\partial z} \right)^2 \right] \psi d\acute{x} dz$$
$$+ \int_\Omega \left[\hat{r}(T^1) - \hat{r}(T^2) \right] \psi d\acute{x} dz. \quad (3.5.16)$$

In (3.5.16), we add and subtract the term $\hat{\mu}(T^1)\left(\frac{\partial u_i^2}{\partial z}\right)^2$, we find

$$\begin{aligned}\int_\Omega \hat{K}\frac{\partial}{\partial z}\left(T^1-T^2\right)\frac{\partial\psi}{\partial z}d\acute{x}dz &= \sum_{i=1}^{2}\int_\Omega\left[\hat{\mu}(T^1)\frac{\partial}{\partial z}\left(u_i^1+u_i^2\right)\frac{\partial}{\partial z}\left(u_i^1-u_i^2\right)\right]\psi d\acute{x}dz+\\ &\sum_{i=1}^{2}\int_\Omega\left[\hat{\mu}(T^1)-\hat{\mu}(T^2)\right]\left(\frac{\partial u_i^2}{\partial z}\right)^2\psi d\acute{x}dz+\int_\Omega\left[\hat{r}(T^1)-\hat{r}(T^2)\right]\psi d\acute{x}dz.\end{aligned}$$

By choosing $\psi = T^1 - T^2 \in H^1_{\Gamma_L\cup\Gamma_1}(\Omega)$ we get

$$\int_\Omega \hat{K}\frac{\partial}{\partial z}\left|T^1-T^2\right|^2 d\acute{x}dz = \sum_{i=1}^{3} R_k, \tag{3.5.17}$$

where

$$\begin{aligned}R_1 = \sum_{i=1}^{2} R_1^i &= \sum_{i=1}^{2}\int_\Omega\left[\hat{\mu}(T^1)\frac{\partial}{\partial z}\left(u_i^1+u_i^2\right)\frac{\partial}{\partial z}\left(u_i^1-u_i^2\right)\right]\left(T^1-T^2\right)d\acute{x}dz,\\ R_2 = \sum_{i=1}^{2} R_2^i &= \sum_{i=1}^{2}\int_\Omega\left[\hat{\mu}(T^1)-\hat{\mu}(T^2)\right]\left(\frac{\partial u_i^2}{\partial z}\right)^2\left(T^1-T^2\right)d\acute{x}dz,\\ R_3 &= \int_\Omega\left[\hat{r}(T^1)-\hat{r}(T^2)\right]\left(T^1-T^2\right)d\acute{x}dz,\end{aligned}$$

as

$$\int_\Omega \hat{K}\left|\frac{\partial}{\partial z}\left(T^1-T^2\right)\right|^2 d\acute{x}dz \geq K_*\left[1+(h^*)^2\right]^{-1}\left\|T^1-T^2\right\|_{V_z}. \tag{3.5.18}$$

On the other hand

$$\begin{aligned}|R_1| &\leq \mu^*\left(\int_\Omega\left|\frac{\partial}{\partial z}\left(u_i^1+u_i^2\right)\right|^4 d\acute{x}dz\right)^{\frac14}\left(\int_\Omega\left|\frac{\partial}{\partial z}\left(u_i^1-u_i^2\right)\right|^2 d\acute{x}dz\right)^{\frac12}\left(\int_\Omega\left|T^1-T^2\right|^4 d\acute{x}dz\right)^{\frac14}\\ &\leq \mu^*\left\|\frac{\partial}{\partial z}\left(u_i^1+u_i^2\right)\right\|_{L^4(\Omega)}\left\|\frac{\partial}{\partial z}\left(u_i^1-u_i^2\right)\right\|_{L^2(\Omega)}\left\|T^1-T^2\right\|_{L^4(\Omega)},\end{aligned}$$

using the **Hölder** inequality, and as the compact injection of $V_z(\Omega)$ in $L^4(\Omega)$ is continuous, then there is a constant $\alpha > 0$, such that

$$\begin{aligned}|R_1| &\leq \mu^*\alpha^2\left\|\frac{\partial}{\partial z}\left(u_i^1+u_i^2\right)\right\|_{V_z}\left\|\frac{\partial}{\partial z}\left(u_i^1-u_i^2\right)\right\|_{L^2(\Omega)}\left\|T^1-T^2\right\|_{V_z},\\ &\leq \mu^*\alpha^2\left\|\frac{\partial}{\partial z}\left(u_i^1+u_i^2\right)\right\|_{V_z}\left\|\left(u_i^1-u_i^2\right)\right\|_{V_z}\left\|T^1-T^2\right\|_{V_z}.\end{aligned}$$

And since u_i^1 and u_i^2 are two elements of B_c then

$$|R_1| \leq 2\mu^* \alpha^2 c \left\| \left(u_i^1 - u_i^2\right) \right\|_{V_z} \left\| T^1 - T^2 \right\|_{V_z},$$

using inequality $\alpha_1 + \alpha_2 \leq \sqrt{2} \left(\alpha_1 + \alpha_2\right)^{\frac{1}{2}}$ for $\alpha_1, \alpha_2 \geq 0$,
we have

$$\begin{aligned} |R_1| &\leq 2\mu^* \alpha^2 c \left\| T^1 - T^2 \right\|_{V_z} \sum_{i=1}^{2} \left\| \left(u_i^1 - u_i^2\right) \right\|_{V_z}, \\ &\leq 2\sqrt{2}\mu^* \alpha^2 c \left\| T^1 - T^2 \right\|_{V_z} \left(\sum_{i=1}^{2} \left\| \left(u_i^1 - u_i^2\right) \right\|_{V_z}^2 \right)^{\frac{1}{2}}, \\ &\leq 2\sqrt{2}\mu^* \alpha^2 c \left\| T^1 - T^2 \right\|_{V_z} \left\| \left(u_i^1 - u_i^2\right) \right\|_{V_z \times V_z}, \end{aligned}$$

then

$$|R_1| \leq 2\sqrt{2}\mu^* \alpha^2 c \left\| T^1 - T^2 \right\|_{V_z} \left\| \left(u^1 - u^2\right) \right\|_{V_z \times V_z}. \tag{3.5.19}$$

And

$$\begin{aligned} |R_2| &\leq C_{\hat{\mu}} \int_{\Omega} \left| T^1 - T^2 \right|^2 \left| \frac{\partial u_i^2}{\partial z} \right| d\acute{x}dz \\ &\leq C_{\hat{\mu}} \left(\int_{\Omega} \left| T^1 - T^2 \right|^4 \right)^{\frac{1}{2}} \left(\int_{\Omega} \left| \frac{\partial u_i^2}{\partial z} \right|^4 \right)^{\frac{1}{2}} d\acute{x}dz \\ &\leq C_{\hat{\mu}} \left\| T^1 - T^2 \right\|_{L^4(\Omega)}^2 \left\| \frac{\partial u_i^2}{\partial z} \right\|_{L^4(\Omega)}^2 \\ &\leq C_{\hat{\mu}} \alpha^4 \left\| T^1 - T^2 \right\|_{V_z}^2 \left\| \frac{\partial u_i^2}{\partial z} \right\|_{V_z}^2 \\ &\leq C_{\hat{\mu}} \alpha^4 \left\| T^1 - T^2 \right\|_{V_z}^2 \left\| u_i^2 \right\|_{W_z}^2 \\ &\leq C_{\hat{\mu}} \alpha^4 c^2 \left\| T^1 - T^2 \right\|_{V_z}^2, \end{aligned}$$

then

$$|R_2| \leq 2C_{\hat{\mu}} \alpha^4 c^2 \left\| T^1 - T^2 \right\|_{V_z}^2, \tag{3.5.20}$$

as the function $\hat{r}$ is Lipschitzian on $\mathbb{R}$ report $C_{\hat{r}}$ then

$$|R_3| \leq C_{\hat{r}} \left\| T^1 - T^2 \right\|_{L^2(\Omega)}^2.$$

$$|R_3| \leq C_{\hat{r}} \left\|T^1 - T^2\right\|_{V_z}^2 . \tag{3.5.21}$$

By injecting (3.5.16) – (3.5.21) in (3.5.15) we have

$$\begin{aligned} K_* \left[1 + (h^*)^2\right]^{-1} \left\|T^1 - T^2\right\|_{V_z}^2 \quad &\leq \quad \left(2C_{\hat{\mu}}\alpha^4 c^2 + C_{\hat{r}}\right) \left\|T^1 - T^2\right\|_{V_z}^2 \\ &\quad +2\sqrt{2}\mu^* \alpha^2 c \left\|T^1 - T^2\right\|_{V_z} \left\|\left(u^1 - u^2\right)\right\|_{V_z \times V_z} , \end{aligned}$$

then

$$\left(K_* \left[1 + (h^*)^2\right]^{-1} - \left(2C_{\hat{\mu}}\alpha^4 c^2 + C_{\hat{r}}\right)\right) \left\|T^1 - T^2\right\|_{V_z}^2 \leq 2\sqrt{2}\mu^* \alpha^2 c \left\|T^1 - T^2\right\|_{V_z} \left\|\left(u^1 - u^2\right)\right\|_{V_z \times V_z} ,$$

we suppose that

$$c < c_0 = \left[2C_{\hat{\mu}}\alpha^4\right]^{\frac{-1}{2}} \left(K_* \left[1 + (h^*)^2\right]^{-1} - C_{\hat{r}}\right)^{\frac{1}{2}} ,$$

provided that

$$K_* > \left[1 + (h^*)^2\right] C_{\hat{r}},$$

then

$$\left\|T^1 - T^2\right\|_{V_z}^2 \leq \sqrt{2}\mu^* \alpha^{-2} C_{\hat{\mu}}^{-1} c(c_0^2 - c^2)^{-1} \left\|\left(u^1 - u^2\right)\right\|_{V_z \times V_z} . \tag{3.5.22}$$

We also have the following two inequalities

$$\begin{aligned} &\textstyle\sum_{i=1}^2 \int_\Omega \hat{\mu}(T^1) \dfrac{\partial u_i^1}{\partial z} \dfrac{\partial}{\partial z} \left(\hat{\varphi}_i^1 - u_i^1\right) d\acute{x}dz + \int_\omega \hat{F} S(\sigma_n^*(u_i^1)) \left|\hat{\varphi}_i^1 - s\right| d\acute{x} - \\ &\qquad \textstyle\int_\omega \hat{F} S(\sigma_n^*(u_i^1)) \left|u_i^1 - s\right| d\acute{x} \geq \sum_{i=1}^2 \left(\hat{f}_i^{\,\varepsilon}, \hat{\varphi}_i^1 - u_i^1\right) . \end{aligned} \tag{3.5.23}$$

$$\begin{aligned} &\textstyle\sum_{i=1}^2 \int_\Omega \hat{\mu}(T^2) \dfrac{\partial u_i^2}{\partial z} \dfrac{\partial}{\partial z} \left(\hat{\varphi}_i^2 - u_i^2\right) d\acute{x}dz + \int_\omega \hat{F} S(\sigma_n^*(u_i^2)) \left|\hat{\varphi}_i^2 - s\right| d\acute{x} - \\ &\qquad \textstyle\int_\omega \hat{F} S(\sigma_n^*(u_i^2)) \left|u_i^2 - s\right| d\acute{x} \geq \sum_{i=1}^2 \left(\hat{f}_i^{\,\varepsilon}, \hat{\varphi}_i^2 - u_i^2\right) . \end{aligned} \tag{3.5.24}$$

We choose that $\hat{\varphi}_i^1 = u_i^2$ in (3.5.23) and $\hat{\varphi}_i^2 = u_i^1$ in (3.5.24) and summing up the two inequalities, it comes to $W = u_i^2 - u_i^1$

$$\sum_{i=1}^2 \int_\Omega \left[\hat{\mu}(T^1) \frac{\partial u_i^1}{\partial z} \frac{\partial W}{\partial z} - \hat{\mu}(T^2) \frac{\partial u_i^2}{\partial z} \frac{\partial W}{\partial z}\right] d\acute{x}dz + \int_\omega \hat{F} S(\sigma_n^*(u_i^1) \left(\left|u_i^2 - s\right| - \left|u_i^1 - s\right|\right) d\acute{x} -$$

$$\int_\omega \hat{F} S(\sigma_n^*(u_i^2) \left(\left|u_i^2 - s\right| - \left|u_i^1 - s\right|\right) d\acute{x} \geq 0,$$

so for the next term we have

$$\int_\omega \hat{F}\, S(\sigma_n^*(u_i^1)\left(\left|u_i^2-s\right|-\left|u_i^1-s\right|\right)d\acute{x}-\int_\omega \hat{F}\, S(\sigma_n^*(u_i^2)\left(\left|u_i^2-s\right|-\left|u_i^1-s\right|\right)d\acute{x}$$
$$\leq \int_\omega \left|\hat{F}\left(S(\sigma_n^*(u_i^1)-S(\sigma_n^*(u_i^2))\right|\left|u_i^2-u_i^1\right|d\acute{x}.$$

Using the **Cauchy-Schwarz** inequality, we have

$$\int_\omega \left|\hat{F}\left(S(\sigma_n^*(u_i^1)-S(\sigma_n^*(u_i^2))\right|\left|u_i^2-u_i^1\right|d\acute{x}\leq \left\|\hat{F}\right\|_{L^\infty(\omega)} C\left\|u_i^2-u_i^1\right\|_{V_z}^2\leq F^*C\left\|u_i^2-u_i^1\right\|_{V_z}^2.$$

By the previous **Theorem**, this term is $F^*C\left\|u_i^2-u_i^1\right\|_{V_z}^2$ tends to 0 then we have

$$\sum_{i=1}^{2}\int_\Omega \left[\hat{\mu}(T^1)\frac{\partial u_i^1}{\partial z}\frac{\partial W}{\partial z}-\hat{\mu}(T^2)\frac{\partial u_i^2}{\partial z}\frac{\partial W}{\partial z}\right]d\acute{x}dz, \tag{3.5.25}$$

add and subtract term $\hat{\mu}(T^1)\frac{\partial u_i^2}{\partial z}\frac{\partial W}{\partial z}$ from the equation $(2.5.25)$, we get

$$\sum_{i=1}^{2}\int_\Omega \left[\hat{\mu}(T^1)\frac{\partial u_i^1}{\partial z}\frac{\partial W}{\partial z}-\hat{\mu}(T^2)\frac{\partial u_i^2}{\partial z}\frac{\partial W}{\partial z}\right]d\acute{x}dz+\sum_{i=1}^{2}\int_\Omega \hat{\mu}(T^1)\frac{\partial u_i^2}{\partial z}\frac{\partial W}{\partial z}d\acute{x}dz-$$
$$\sum_{i=1}^{2}\int_\Omega \hat{\mu}(T^1)\frac{\partial u_i^2}{\partial z}\frac{\partial W}{\partial z}d\acute{x}dz\geq 0.$$

So

$$\sum_{i=1}^{2}\int_\Omega -\hat{\mu}(T^1)\frac{\partial W}{\partial z}\frac{\partial W}{\partial z}d\acute{x}dz+\sum_{i=1}^{2}\int_\Omega (\hat{\mu}(T^1)-\hat{\mu}(T^2))\frac{\partial u_i^2}{\partial z}\frac{\partial W}{\partial z}d\acute{x}dz\geq 0, \tag{3.5.26}$$
$$\sum_{i=1}^{2}\int_\Omega \hat{\mu}(T^1)\frac{\partial W}{\partial z}\frac{\partial W}{\partial z}d\acute{x}dz\leq \sum_{i=1}^{2}\int_\Omega (\hat{\mu}(T^1)-\hat{\mu}(T^2))\frac{\partial u_i^2}{\partial z}\frac{\partial W}{\partial z}d\acute{x}dz.$$

As

$$\sum_{i=1}^{2}\int_\Omega \hat{\mu}(T^1)\frac{\partial W}{\partial z}\frac{\partial W}{\partial z}d\acute{x}dz\geq \frac{\mu_*}{2}\left\|W\right\|_{V_z}^2. \tag{3.5.27}$$

Due to **Hölder**'s inequality, and the result of $([10])$ we fined

$$\begin{aligned}
\left|\sum_{i=1}^{2}\int_\Omega (\hat{\mu}(T^1)-\hat{\mu}(T^2))\frac{\partial u_i^2}{\partial z}\frac{\partial W}{\partial z}d\acute{x}dz\right| &\leq C_{\hat{\mu}}\int_\Omega \left|T^1-T^2\right|\left|\frac{\partial u_i^2}{\partial z}\right|\left|\frac{\partial W}{\partial z}\right|d\acute{x}dz\\
&\leq C_{\hat{\mu}}\left\|T^1-T^2\right\|_{L^4(\Omega)}\left\|\frac{\partial u_i^2}{\partial z}\right\|_{L^4(\Omega)}\left\|\frac{\partial W}{\partial z}\right\|_{L^2(\Omega)}\\
&\leq \alpha^2 C_{\hat{\mu}}\left\|T^1-T^2\right\|_{L^4(\Omega)}\left\|\frac{\partial u_i^2}{\partial z}\right\|_{V_z}\left\|\frac{\partial W}{\partial z}\right\|_{L^2(\Omega)}\\
&\leq c\alpha^2 C_{\hat{\mu}}\left\|T^1-T^2\right\|_{V_z}\left\|\frac{\partial W}{\partial z}\right\|_{V_z},
\end{aligned}$$

From **Young**'s inequality

$$\left|\sum_{i=1}^{2}\int_{\Omega}(\hat{\mu}(T^1)-\hat{\mu}(T^2))\frac{\partial u_i^2}{\partial z}\frac{\partial W}{\partial z}d\acute{x}dz\right| \leq \sqrt{2}c\alpha^2 C_{\hat{\mu}}\left\|T^1-T^2\right\|_{V_z}\|W\|_{V_z}. \tag{3.5.28}$$

Inserting (3.5.27) into (3.5.26) gives

$$\frac{\mu_*}{2}\|W\|_{V_z}^2 \leq \sqrt{2}c\alpha^2 C_{\hat{\mu}}\left\|T^1-T^2\right\|_{V_z}\|W\|_{V_z}$$

$$\frac{\mu_*}{2}\|W\|_{V_z} \leq \sqrt{2}c\alpha^2 C_{\hat{\mu}}\left\|T^1-T^2\right\|_{V_z}. \tag{3.5.29}$$

Returning to (3.5.22) we fined

$$\begin{aligned}\left\|T^1-T^2\right\|_{V_z} &\leq 2\sqrt{2}\mu^* c\alpha^{-2}(c_0^2-c^2)^{-1}\left\|u^2-u^1\right\|_{V_z\times V_z}\\ &\leq 8\mu_*^{-1}\mu^* C_{\hat{\mu}}^{-1}c^2(c_0^2-c^2)^{-1}\left\|T^1-T^2\right\|_{V_z},\end{aligned}$$

then

$$\left(8\mu_*^{-1}\mu^* C_{\hat{\mu}}^{-1}c^2(c_0^2-c^2)^{-1}\right)\left\|T^1-T^2\right\|_{V_z} \leq 0,$$

on condition that

$$0 < c < c_1 = (1+8\mu_*^{-1}\mu^*)^{-\frac{1}{2}}c_0,$$

Consequently

$$\|T^1-T^2\|_{V_z} = 0,$$

then we obtain $T^1 = T^2$ almost everywhere in $V_z(\omega)$.

From (3.5.29), we conclude $u^1 = u^2$ almost everywhere in $V_z(\omega)$.

■

Conclusion

In the firste work, we are examined the strong convergence of the velocity of a non-Newtonian incompressible áuid whose viscosity follows the Power law with Coulomb friction.We are assumed that the áuid coefficients of the thin layer vary with respect to the thin layer parameter ε . We are interested for the main convergence results.Finally we give the detail of the proofs of this results.

In the second work, we are focused on the study of the asymptotic behavior of a coupled problem that consists of an elastic body and the change of the heat.We proved some estimates then we gave convergence results. At the end the uniqueness of the weak solution is given and proved.

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