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THÈSE

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par
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Analyse de quelques équations differentielles opérationnelles, autonomes, non-autonomes et stochastiques

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Abstact

This thesis focuses basically on the proof of maximal regularity in the non autonomous case, i.e. that we prove the existence and uniqueness of the solution to the problem

$$\begin{cases} u'(t) + A(t)u(t) = f(t) \\ \\ u(0) = 0 \end{cases} \tag{P}$$

We shall allow considerably less restrictive assumptions on f and the initial data u_0. Here, f belongs to the weighted Hilbert space $L^2\left(0, \tau, t^\beta dt; \mathscr{H}\right)$, with $\beta \in [0, 1[$ and the initial data u_0 takes its values in a certain interpolation space $(\mathscr{H}, D(A(0)))_{\frac{1-\beta}{2},2}$ between $\mathscr{H}$ and $D(A(0))$. we establish weighted L^2-maximal regularity for linear autonomous and non autonomous Cauchy problems. The weights we consider are power weights in time $(w(t) = t^\beta, \beta \in (-1, 1))$, and yield optimal regularity for the solutions.

In the non-autonomous case we prove that if $f \in L^2\left(0, \tau, t^\beta dt; \mathscr{H}\right)$ and $u_0 \in (\mathscr{H}, D(A(0)))_{\frac{1-\beta}{2},2}$ for arbitrary $\beta \geqslant 0$ with the assumption that the operator $A(\cdot)$ belongs to the space $W^{1/2,2}\left(0, \tau; \mathcal{L}\left(\mathcal{V}, \mathcal{V}'\right)\right) \cap C^\varepsilon\left([0, \tau], \mathcal{L}\left(\mathcal{V}, \mathcal{V}'\right)\right)$ for some $\varepsilon > 0$, then problem has a unique solution u such that $\dot{u}, A(\cdot)u \in L^2\left(0, \tau, t^\beta dt; \mathscr{H}\right)$. Throughout this thesis we assume that the Kato square root property is satisfied. To prove our results we appeal to classical tools from harmonic analysis such as square function estimate or functional calculus and from functional analysis such as interpolation theory or operator theory. The main Concerns of the second part of this thesis is to study a kind of stochastic evolution equation with drift part α and the diffusion part σ.We proof existence and uniqueness of the mild solution to the relevent integral equation in martingale type 2 Banach space and by extending some result from scalar valued setting to the vector valued setting we are able to show that this solution is Malliavin differentiable. Finally, we prove the existence of the right inverse operator where we suppose such condition to be satisfied .

Résumé

Cette thèse se concentre sur la preuve de la régularité maximale dans le cas non autonome, c'est-à-dire que nous prouvons l'existence et l'unicité de la solution au problème

$$\begin{cases} u'(t) + A(t)u(t) = f(t) \\ \\ u(0) = 0 \end{cases} \tag{P}$$

Nous permettons des hypothèses beaucoup moins restrictives sur f et les données initiales u_0. Ici, f appartient à l'espace de Hilbert pondéré $L^2\left(0, \tau, t^\beta dt; \mathscr{H}\right)$, avec $\beta \in [0,1[$ et les données initiales u_0 prennent ses valeurs dans un certain espace d'interpolation $(\mathscr{H}, D(A(0)))_{\frac{1-\beta}{2},2}$ entre $\mathscr{H}$ et $D(A(0))$. nous établissons L^2 - régularité maximale pondérée pour des problèmes de Cauchy linéaires autonomes et non autonomes. Les poids que nous considérons sont des poids de puissance en temps ($w(t) = t^\beta, \beta \in (-1,1)$), et donnent une régularité optimale pour les solutions.

Dans le cas non autonome on montre que si $f \in L^2\left(0, \tau, t^\beta dt; \mathscr{H}\right)$ et $u_0 \in (\mathscr{H}, D(A(0)))_{\frac{1-\beta}{2},2}$ avec $\beta \geqslant 0$ arbitraire et avec l'hypothèse que l'opérateur $A(\cdot)$ appartient à l'espace $W^{1/2,2}\left(0, \tau; \mathcal{L}\left(\mathcal{V}, \mathcal{V}'\right)\right) \cap C^\varepsilon\left([0,\tau], \mathcal{L}\left(\mathcal{V}, \mathcal{V}'\right)\right)$ pour certains $\varepsilon > 0$, alors le problème a une solution unique u telles que $\dot{u}, A(\cdot)u \in L^2\left(0, \tau, t^\beta dt; \mathscr{H}\right)$. Tout au long de cette thèse, nous supposons que la propriété de racine carrée de Kato est satisfaite. Pour prouver nos résultats, nous faisons appel à des outils classiques d'analyse harmonique tels que l'estimation de la fonction carrée ou le calcul fonctionnel et de l'analyse fonctionnelle telle que la théorie de l'interpolation ou la théorie des opérateurs.

Le souci principal de la deuxième partie de cette thèse est d'étudier une sorte d'équation d'évolution stochastique avec la partie dérive α et la partie diffusion σ .Nous prouvons l'existence et l'unicité de la solution de l'équation intégrale dans l' 'espace martingale type 2 . De plus, nous étendons certains résultats du cas scalaire au cas vectorielle.

Symbols and notations

$MR-$ Maximal Regularity

$\Sigma_\omega, S_\theta-$ open sectors of angle ω respectively θ

$\gamma(\mathscr{H}, E)-$ space of $\gamma-$radonifying operators, where $\mathscr{H}$ is a Hilbert space and E a Banach space

$[E, F]_\theta-$ complex interpolation space between E and F.

$(E, F)_{\theta,p}-$ real interpolation space between E and F.

$\mathcal{L}(E, F)-$ space of bounded linear operators

$W^{k,p}-$ Sobolev space

$\mathbb{E}-$ expectation

$\mathcal{F}-$ $\sigma-$algebra

$\mathbb{E}(\cdot|\cdot)-$ conditional expectation

γ_n- Gaussian variables

$\mathscr{D}(A)-$ domain of A

$a \lesssim b-$ $\exists C > 0$ such that $a \leqslant Cb$

$f * g-$ convolution

$\rho(A)-$ resolvent of the operator A

$L_{1,c}-$ space of locally integrable functions

$C([0;\tau], \mathscr{H})-$ space of continuous functions form $[0,\tau]$ into $\mathscr{H}$

$C_c^\infty([0;\tau], X)-$ space of test functions

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Introduction

Differential equations (DEs) of evolution type are usually viewed as an ordinary differential equations (ODEs) in an finite-dimensional state space. In many examples such as the heat and wave equation, this point of view may lead to existence and uniqueness results and regularity properties. To model the equation in such a way one needs an integration theory for functions with values in an finite or infinite-dimensional space. Since real-valued integration theory extends directly to functions with values in Hilbert spaces, this is the class of spaces in which DEs are usually modelled. This approach has been considered by many authors using semigroup or forms methods . There are situations where it is more natural to model the DE in a function space which is not a Hilbert space but only a Banach space.
During the last 25 years, the theory of maximal regularity turned out to be an important tool in the theory of nonlinear PDEs.
As an application of his operator-valued Fourier multiplier theorem, Weis [36] characterized maximal L^p-regularity for abstract Cauchy problems in UMD Banach spaces in terms of an $\mathcal{R}$-boundedness condition on the operator under consideration. A second approach to the maximal L^p-regularity problem is via the operator sum method, as initiated by Da Prato & Grisvard [16] and extended by Dore & Venni [18] and Kalton & Weis [47]. For more details on these approaches and for more information on (the history of) the maximal L^p-regularity problem in general, we refer to [55, 51].
We consider the non autonomous Cauchy problem written as the following

$$\begin{cases} u'(t) + A(t)u(t) = f(t) \\ \\ u(0) = 0 \end{cases} \tag{P}$$

where $t \in (0; \infty)$.Let $(\mathscr{H}, (\cdot, \cdot), \|\cdot\|)$ be a separable Hilbert space over $\mathbb{R}$ or $\mathbb{C}$. We consider another separable Hilbert space $\mathcal{V}$ which is densely and continuously embedded into $\mathscr{H}$. We denote by $\mathcal{V}'$ the (anti-) dual space of $\mathcal{V}$ so that

$$\mathcal{V} \hookrightarrow_d \mathscr{H} \hookrightarrow_d \mathcal{V}'.$$

to every form $\mathfrak{a}(t)$, we associate two operators $A(t), \mathcal{A}(t)$ on $\mathscr{H}$ and $\mathcal{V}'$ respectively. The operator $A(t)$ is the part of $\mathcal{A}(t)$ in $\mathscr{H}$. J.L.Lions proves that the Cauchy problem (P) has the maximal L^2-regularity

in $\mathcal{V}'$, i.e for all $f \in L^2(0,\tau;\mathcal{V}')$ and $x \in \mathscr{H}$, there exists a unique $u \in H^1(0,\tau;\mathcal{V}') \cap L^2(0,\tau;\mathcal{V})$ satisfies (P).
He ask the question, what would be the right condition on the form $\mathfrak{a}(.)$ that give us the maximal regularity L^2 in $\mathscr{H}$.
For the problem

Problem 0.0.1. Let $f \in L^2(0,\tau;\mathscr{H})$. Under which conditions on the forms $\mathfrak{a}(\cdot)$ the solution $u \in MR(\mathcal{V},\mathcal{V}')$ of (P) satisfies $u \in H^1(0,\tau;\mathscr{H})$.

One has to distinguish the two cases $u_0 = 0$ and $u_0 \neq 0$. For $u_0 = 0$ Problem (P) is explicitly asked by Lions and seems to be open up to today. A positive answer is given by Lions if $\mathfrak{a}$ is symmetric (i.e. $\mathfrak{a}(t,u,v) = \overline{\mathfrak{a}(t,v,u)}$) and $\mathfrak{a}(.,u,v) \in C^1[0,T]$ for all $u,v \in \mathcal{V}$. By a completely different approach a positive answer is also given by E. M. Ouhabaz and C. Spina for general forms such that $\mathfrak{a}(.,u,v) \in C^\alpha[0,T]$ for all $u,v \in V$ and some $\alpha > \frac{1}{2}$. Again, the result in work of E. M. Ouhabaz and C. Spina concerns the case $u_0 = 0$ Concerning $u_0 \neq 0$ it seems natural to assume $u_0 \in \mathcal{V}$ as we did in Problem (P). However, already in the autonomous case, i.e. $A(t) \equiv A$, the solution is in $MR(\mathcal{V},\mathscr{H})$ if and only if $u_0 \in D\left(A^{1/2}\right)$, and it may happen that $\mathcal{V} \not\subset D\left(A^{1/2}\right)$. So one has to impose a stronger condition on the initial value u_0 or the form (e.g. symmetry). Lions gave a positive answer for $u_0 \in D(A(0))$ provided that $\mathfrak{a}(.,u,v) \in C^2[0,T]$ for all $u,v \in \mathcal{V}$ and $f \in H^1(0,T;\mathscr{H})$.
Dier [6] observes that the answer to Lions problem is negative in general. He gave a counterexample using non symmetric forms where the Kato square root condition fails . Dier [7], proves also that the L^2-maximal regularity holds for symmetric forms such that $t \mapsto \mathfrak{a}(.)$ has bounded variation .
Mahdi Achache, El Maati Ouhabaz see [40] gave a positive answer to this problem under minimal regularity assumptions on the forms . In particular they assume that the forms are piecewise $H^{\frac{1}{2}}$ with respect to the variable t. This regularity assumption is optimal and their results are the most general ones on this problem.
Dier et Zacher [8], proves that if $t \mapsto \mathcal{A}(t)$ lies in the fractional Sobolev space $H^{\frac{1}{2}+\epsilon}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}'))$, for certain $\epsilon > 0$ then the L^2-maximal regularity holds.
For the case of Banach spaces for this result we refer to [58] .
This thesis focuses on proving the maximal regularity in the non-autonomous case, i.e. we prove the existence and the uniqueness of solution to Problem (P). We shall allow considerably less restrictive assumptions on f and the initial data u_0. Here, f belongs to the weighted Hilbert space $L^2\left(0,\tau,t^\beta dt;\mathscr{H}\right)$, with $\beta \in [0,1[$ and the initial data u_0 takes its values in a certain interpolation space $(\mathscr{H},D(A(0)))_{\frac{1-\beta}{2},2}$ between $\mathscr{H}$ and $D(A(0))$. we establish weighted L^2-maximal regularity for linear autonomous and non autonomous Cauchy problems. The weights we consider are power weights in time $(w(t) = t^\beta, \beta \in (-1,1))$, and yield optimal regularity for the solutions.
In the non-autonomous case we prove that if $f \in L^2\left(0,\tau,t^\beta dt;\mathscr{H}\right)$ and $u_0 \in (\mathscr{H},D(A(0)))_{\frac{1-\beta}{2},2}$ for

arbitrary $\beta \geqslant 0$ with the assumption that the operator $A(\cdot)$ belongs to the space $W^{1/2,2}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}'))\cap C^{\varepsilon}([0,\tau],\mathcal{L}(\mathcal{V},\mathcal{V}'))$ for some $\varepsilon > 0$, then problem has a unique solution u such that $\dot{u}, A(\cdot)u \in L^2\left(0,\tau,t^{\beta}dt;\mathscr{H}\right)$. Throughout this thesis we assume that the Kato square root property is satisfied. To prove our results we appeal to classical tools from harmonic analysis such as square function estimate or functional calculus and from functional analysis such as interpolation theory or operator theory.
In the following I describe shortly the contents of the individual chapters:

(1) **Chapter 01** We introduce the machinery object of this thesis namely forms in Hilbert space and several result from operator theory and functional analysis we recall also some result on Sobolev weighted spaces .

(2) **Chapter 02** We study the maximal regularity property for autonomous problems, i.e. existence and uniqueness of the solution to the relevent problem P where the operators $A(t)$ are independent of t .Based on some complex interpolation techniques adapted in the theory of operator spaces. Next, we treat the maximal regularity for the non-autonomous problem (which is our main topic), i.e. we prove the existence and the uniqueness of the solution to Problem (P) in the weighted space $W^{1,2}_{\beta}(0,\tau;\mathscr{H})$.

(3) **Chapter 03** We consider in this chapter the problem of maximal regularity for the semilinear non-autonomous evolution equations

$$u'(t) + A(t)u(t) = F(t,u),\ t\text{-a.e},\ u(0) = u_0.$$

where the time dependent operators $A(t)$ are associated with (time dependent) sesquilinear forms on a Hilbert space $\mathscr{H}$. We prove the maximal regularity result in temporally weighted L^2-spaces and other regularity properties for the solution of our problem under minimal regularity assumptions on the forms, the initial value u_0 and the inhomogeneous term F and applied those results to some boundary value problems.

(4) **Chapter 04** The main Concerns of this chapter is to study a kind of stochastic evolution equation with drift part α and the diffusion part σ. We proof existence and uniqueness of the mild solution to the relevent integral equation in martingale type 2 space. Moreover we extend some result from scalar valued setting to the vector valued setting . Finally we prove the existence of the right inverse operator where we suppose an additional condition to be satisfied.

Chapter 1

Preliminaries

We introduce the machinery object of this thesis namely forms in Hilbert spaces. They are so popular in analysis because the Lax-Milgram lemma and Lions representation theorem yields properties of existence and uniqueness which are best adapted for establishing weak solutions of elliptic and parabolic partial differential equations. For more details, see the monograph [13].

1.1 Forms and their operators

Let $(\mathscr{H}, (\cdot,\cdot), \|\cdot\|)$ be a separable Hilbert space over $\mathbb{R}$ or $\mathbb{C}$. We consider another separable Hilbert space $\mathcal{V}$ which is densely and continuously embedded into $\mathscr{H}$. We denote by $\mathcal{V}'$ the (anti-) dual space of $\mathcal{V}$ so that

$$\mathcal{V} \hookrightarrow_d \mathscr{H} \hookrightarrow_d \mathcal{V}'.$$

Hence there exists a constant $C > 0$ such that

$$\|u\| \leqslant C\|u\|_{\mathcal{V}} \quad u \in \mathcal{V},$$

where $\|\cdot\|_{\mathcal{V}}$ denotes the norm of $\mathcal{V}$. Similarly, there exists a constant $C' > 0$ such that

$$\|\psi\|_{\mathcal{V}'} \leqslant C'\|\psi\| \quad \psi \in \mathscr{H}.$$

We denote by $\langle , \rangle$ the duality $\mathcal{V}'$-$\mathcal{V}$ and note that $\langle \psi, v\rangle = (\psi, v)$ if $\psi, v \in \mathscr{H}$. As domain we consider a vector space $\mathcal{V}$ over $\mathbb{K}$. A sesquilinear form on $\mathcal{V}$ is a mapping $a : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{K}$ such that

$$a(u+v, w) = a(u, w) + a(v, w), a(\lambda u, w) = \lambda a(u, w)$$

$$a(u, w+u) = a(u, v) + a(u, w), a(u, \lambda w) = \overline{\lambda} a(u, w)$$

for all $u, v, w \in \mathcal{V}, \lambda \in \mathbb{K}$. If $\mathbb{K} = \mathbb{R}$, then a sesquilinear form is the same as a bilinear form. If $\mathbb{K} = \mathbb{C}$, then a is antilinear in the second variable: it is additive in the second variable but not homogeneous. Thus the form is linear in the first variable, whereas only half of the linearity conditions are fulfilled for the second variable. The form is $1\frac{1}{2}$ -linear; or sesquilinear since the Latin 'sesqui' means 'one and a half'. For simplicity we will mostly use the terminology form instead of sesquilinear form. A form a is called symmetric if

$$a(u, v) = \overline{a(v, u)} \quad (u, v \in \mathcal{V})$$

and a is called accretive if

$$\operatorname{Re} a(u, u) \geqslant 0 \quad (u \in \mathcal{V})$$

A symmetric form is also called positive if it is accretive. In the following we will also use the notation

$$a(u) := a(u, u) \quad (u \in \mathcal{V})$$

for the associated quadratic form.

We consider a form

$$\mathfrak{a} : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$$

be sesquilinear and $\mathcal{V}$-bounded, i.e.

$$|\mathfrak{a}(u, v) \leqslant M\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}, \ (u, v \in \mathcal{V})$$

for some constant $M > 0$. The form $\mathfrak{a}$ is called quasi-coercive if there exist constants $\nu \in \mathbb{R}$ and $\delta > 0$ such that

$$\operatorname{Re}\ \mathfrak{a}(u, u) + \nu\|u\|^2_{\mathscr{H}} \geqslant \delta\|u\|^2_{\mathcal{V}}, \quad (u \in \mathcal{V}).$$

If $\nu = 0$ we say that the form $\mathfrak{a}$ is coercive.

Finally, we introduce the adjoint form. Let $a : \mathcal{V} \times \mathcal{V} \to \mathbb{K}$ be a form. Then

$$a^*(u, v) := \overline{a(v, u)} \quad (u, v \in \mathcal{V})$$

defines a form $a^* : \mathcal{V} \times \mathcal{V} \to \mathbb{K}$. Thus a is symmetric if and only if $a = a^*$. In the case of complex scalars, the forms

$$\operatorname{Re}\ a := \frac{1}{2}\left(a + a^*\right) \text{ and } \operatorname{Im} a := \frac{1}{2\mathrm{i}}\left(a - a^*\right)$$

are symmetric and

$$a = \operatorname{Re} a + \mathrm{i} \operatorname{Im} a$$

We call Re a the real part and Im a the imaginary part of a. Note that $(\operatorname{Re} a)(u) = \operatorname{Re} a(u)$ and $(\operatorname{Im} a)(u) = \operatorname{Im} a(u)$ for all $u \in V$ There is another algebraic notion - only used for the case $\mathbb{K} = \mathbb{C}$ - that will play a role in this thesis. A form $a : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ is sectorial if there exists $\theta \in [0, \pi/2)$ such that $a(u) \in \{z \in \mathbb{C}\backslash\{0\}; |\operatorname{Arg} z| \leqslant \theta\} \cup \{0\}$ for all $u \in \mathcal{V}$. If we want to specify the angle, we say that a is sectorial of angle θ. It is obvious that a form $a : \mathcal{V} \times \mathcal{V} \to \mathrm{C}$ is sectorial if and only if there exists a constant $c \geqslant 0$ such that

$$|\operatorname{Im} a(u)| \leqslant c \operatorname{Re} a(u) \quad (u \in \mathcal{V}) \tag{1.1.1}$$

(The angle θ and the constant c are related by $c = \tan\theta$.)

Representation theorems

Now, we consider the case where the underlying form domain is a Hilbert space $\mathcal{V}$ over $\mathbb{K}$. An important result is the classical representation theorem of Riesz-Fréchet: If η is a continuous linear functional on $\mathcal{V}$, then there exists a unique $u \in \mathcal{V}$ such that

$$\eta(v) = (v \mid u)_{\mathcal{V}} \quad (v \in \mathcal{V})$$

The theorem of Riesz-Fréchet can be reformulated by saying that for each $\eta \in \mathcal{V}^*$ there exists a unique $u \in \mathcal{V}$ such that

$$\eta(v) = (u \mid v)_{\mathcal{V}} \quad (v \in \mathcal{V})$$

We will also need the Riesz isomorphism $\Phi : \mathcal{V} \to \mathcal{V}^*, u \mapsto (u \mid \cdot)$.

Next we derive a slight generalisation of the Riesz-Fréchet theorem, the present Lax-Milgram lemma. A form $a : \mathcal{V} \times \mathcal{V} \to \mathbb{K}$ is called bounded if there exists $M \geqslant 0$ such that

$$|a(u, v)| \leqslant M\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}} \quad (u, v \in \mathcal{V}) \tag{1.1.2}$$

If $a : \mathcal{V} \times \mathcal{V} \to \mathbb{K}$ is a bounded form, then

$$\langle \mathcal{A}u, v\rangle := a(u, v) \quad (u, v \in \mathcal{V})$$

defines a bounded operator $\mathcal{A} : \mathcal{V} \to \mathcal{V}^*$ with $\|\mathcal{A}\|_{\mathcal{L}(\mathcal{V},\mathcal{V}^*)} \leqslant M$, where M is the constant from 1.1.2.

Remark 1.1.1. *Let $\mathcal{A} \in \mathcal{L}(\mathcal{V})$ be coercive, i.e.*

$$\operatorname{Re}(\mathcal{A}u \mid u) \geqslant \alpha\|u\|^2_{\mathcal{V}} \quad (u \in \mathcal{V})$$

with some $\alpha > 0$. Then, obviously, $\mathcal{A} - \alpha I$ is accretive.

In this thesis the form domain is a Hilbert space. Let $\mathcal{V}, \mathscr{H}$ be Hilbert spaces over $\mathbb{K}$ and let $a : \mathcal{V} \times \mathcal{V} \to \mathbb{K}$ be a bounded form. Let $j \in \mathcal{L}(\mathcal{V}, \mathscr{H})$ be an operator with dense range. We consider the condition that

$$u \in \mathcal{V}, j(u) = 0, a(u) = 0 \text{ implies } u = 0 \tag{1.1.3}$$

Let

$$A := \{(x, y) \in \mathscr{H} \times \mathscr{H}; \exists u \in \mathcal{V} : j(u) = x, a(u, v) = (y \mid j(v))(v \in \mathcal{V})\}$$

Proposition 1.1.1. *(a) Assume 1.1.3. Then the relation A defined above is an operator in $\mathscr{H}$. We call A the operator associated with (a, j) and write $A \sim (a, j)$*

(b) If a is accretive, then A is accretive.

(c) If $\mathbb{K} = \mathbb{C}$ and a is sectorial, then A is sectorial of the same angle as a.

Proof. (a) It is easy to see that A is a subspace of $\mathscr{H} \times \mathscr{H}$. Let $(0, y) \in A$. We have to show that $y = 0$. By definition there exists $u \in V$ such that $j(u) = 0$ and $a(u, v) = (y \mid j(v))_H$ for all $v \in \mathcal{V}$. In particular, $a(u) = 0$. Assumption 1.1.3 implies that $u = 0$. Hence $(y \mid j(v))_{\mathscr{H}} = 0$ for all $v \in \mathcal{V}$. since j has dense range, it follows that $y = 0$

(b), (c) If $x \in \mathrm{dom}(A)$, then there exists $u \in \mathcal{V}$ such that $j(u) = x$ and such that $a(u, v) = (Aj(u) \mid j(v))$ for all $v \in \mathcal{V}$, and then $a(u, u) = (Aj(u) \mid j(u)) = (Ax \mid x)$

If $\mathrm{Re}\, a(u, u) \geqslant 0 (u \in \mathcal{V})$, then $\mathrm{Re}(Ax \mid x) \geqslant 0$ for all $x \in \mathrm{dom}(A)$, and this proves (b). Also, in the complex case, $\mathrm{num}(A)$ is contained in $\{a(v); v \in \mathcal{V}\}$, and this proves (c). ☐

Remark 1.1.2. *Let $\mathcal{V}, \mathscr{H}, a, j$ be as above, and let $\omega \in \mathbb{R}$. Then*

$$b(u, v) := a(u, v) + \omega(j(u) \mid j(v)) \quad (u, v \in \mathcal{V})$$

defines a form satisfying 1.1.3 as well (with a replaced by b). Let B be the operator associated with (b, j). Let $x, y \in \mathscr{H}$. Then for all $u, v \in \mathcal{V}$ with $j(u) = x$ we have

$$a(u, v) = (y \mid j(v)) \Longleftrightarrow b(u, v) = (y + \omega x \mid j(v))$$

This shows that

$$(x, y) \in A \Longleftrightarrow (x, y + \omega x) \in B$$

Therefore $B = A + \omega I$. Note that coercivity implies 1.1.3.

We denote by $\langle , \rangle$ the duality $\mathcal{V}' - \mathcal{V}$ and note that $\langle \psi, v \rangle = (\psi, v)$ if $\psi, v \in \mathscr{H}$.
In this thesis we consider a family of sesquilinear forms

$$\mathfrak{a} : [0, \tau] \times \mathcal{V} \times \mathcal{V} \to \mathbb{C}$$

such that

(H1) $D(\mathfrak{a}(t)) = \mathcal{V}$ (constant form domain),

(H2) $|\mathfrak{a}(t, u, v)| \leqslant M\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}$ (uniform boundedness),

(H3) $\operatorname{Re} \mathfrak{a}(t, u, u) + \nu\|u\|^2 \geqslant \delta\|u\|_{\mathcal{V}}^2$ for all $u \in \mathcal{V}$, for some $\delta > 0$ and some $\nu \in \mathbb{R}$ (uniform quasi-coercivity).

We denote by $A(t), \mathcal{A}(t)$ the usual associated operators with $\mathfrak{a}(t)$ (as operators on $\mathscr{H}$ and $\mathcal{V}'$).
.

Proposition 1.1.2. *If $\mathfrak{a}$ is a $\mathcal{V}$-bounded quasi-coercive form then $\mathfrak{a} + \nu I$ is a sectorial form and the numerical range $\mathcal{N}$ of $a + \nu I$ is given by:*

$$\mathcal{N}(\mathfrak{a} + \nu I) = \{z \in \mathbb{C}^*, |\arg z| \leqslant \arctan(\frac{M}{\delta})\}.$$

Proof. Let $u \in \mathcal{V}$, we have that

$$\begin{aligned} |\mathfrak{Im}(\mathfrak{a} + \nu I)(u, u)| &\leqslant |\mathfrak{a}(u, u)| \leqslant M\|u\|_{\mathcal{V}}^2 \\ &\leqslant \frac{M}{\delta}[\operatorname{Re}\, \mathfrak{a}(u, u) + \nu\|u\|_{\mathscr{H}}^2]. \end{aligned}$$

This proves the proposition. □

Let $\mathfrak{a}$ be a sesquilinear form $\mathcal{V}$-bounded and quasi-coercive. The operator $\mathcal{A} \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$ associated with $\mathfrak{a}$ is defined by

$$\langle Au, v \rangle = \mathfrak{a}(u, v), \ (u, v \in \mathcal{V}).$$

Seen as an unbounded operator on $\mathcal{V}'$ with domain $\mathscr{D}(A) = \mathcal{V}$. One can define also an unbounded operator A on $\mathscr{H}$, it is the part of $\mathcal{A}$ on $\mathscr{H}$, i.e.

$$\begin{aligned} \mathscr{D}(A) &:= \{v \in \mathcal{V} : \mathcal{A}v \in \mathscr{H}\} \\ Av &:= \mathcal{A}v. \end{aligned}$$

Observe also that $\mathscr{D}(A)$ is the set of vectors $u \in \mathcal{V}$ for which the map $v \mapsto \mathfrak{a}(u, v)$ is continuous on $\mathcal{V}$ with respect to the norm of $\mathscr{H}$.

Proposition 1.1.3. *Denote by A the operator associated with a sesquilinear $\mathcal{V}$-bounded and quasi-coercive form $\mathfrak{a}$. Then A is densely defined and for every $\lambda > \nu$, the operator $\lambda + A$ is invertible (from $\mathscr{D}(A)$ into $\mathscr{H}$) and its inverse $(\lambda + A)^{-1}$ is a bounded operator on $\mathscr{H}$.*

Definition 1.1.1. *A scalar $\lambda \in \mathbb{C}$ is in the resolvent set of A if $\lambda - A$ is invertible (from $\mathscr{D}(A)$ into $\mathscr{H}$) and its inverse $(\lambda - A)^{-1}$ is a bounded operator on $\mathscr{H}$. For such λ, the operator $(\lambda - A)^{-1}$ is called the resolvent of A at λ. The set*

$$\rho(A) := \{\lambda \in \mathbb{C}, \lambda - A \text{ is invertible and } (\lambda - A)^{-1} \in \mathcal{L}(\mathscr{H})\}.$$

is called the resolvent set of A. The complement of $\rho(A)$ in $\mathbb{C}$ is the spectrum of A.

Definition 1.1.2. *A family $S = \{S(t)\}_{t \geqslant 0}$ of bounded linear operators acting on a Banach space E is called a C_0 -semigroup if the following three properties are satisfied:*

(S_1) $S(0) = I$

(S_2) $S(t)S(s) = S(t+s)$ *for all* $t, s \geqslant 0$

(S_3) $\lim_{t \downarrow 0} \|S(t)x - x\| = 0$ *for all* $x \in E$.

The generator of $(S(t))_{t \geqslant 0}$ is the operator B defined by

$$\mathscr{D}(B) := \{x \in E, \ s.t \lim_{t \downarrow 0} \Big(\frac{S(t)x - x}{t} \Big) \text{ exists}\}.$$
$$Bx := \lim_{t \downarrow 0} \Big(\frac{S(t)x - x}{t} \Big).$$

$(S(t))_{t \geqslant 0}$ *is called a bounded semigroup on the sector Σ_θ.*

Definition 1.1.3. *Let X be a complex Banach space. For $\theta \in (0, \pi]$ we define the (open) sector $\Sigma_\theta := \{z \in \mathbb{C} \backslash \{0\}; |\operatorname{Arg} z| < \theta\} = \{re^{i\alpha}; r > 0, |\alpha| < \theta\}$ We will also use the notation $\Sigma_{\theta,0} := \Sigma_\theta \cup \{0\}$. A holomorphic semigroup (of angle θ if we want to make precise the angle) is a function $S : \Sigma_{\theta,0} \to \mathcal{L}(X)$, holomorphic on Σ_θ, satisfying*

(i) $S(z_1 + z_2) = S(z_1) S(z_2)$ *for all $z_1, z_2 \in \Sigma_{\theta,0}$. If additionally*

(ii) $\lim_{z \to 0, z \in \Sigma_{\theta'}} S(z)x = x$ *for all $x \in X$ and all $\theta' \in (0, \theta)$, then S will be called a holomorphic C_0 -semigroup (of angle θ).*

We turn to the theory of C_0 -semigroups. We review their basic properties (see Appendix 5)and show how semigroups are used to solve the (deterministic) inhomogeneous abstract Cauchy problem

$$u'(t) = Au(t) + f(t)$$

Here A generates a C_0 -semigroup on $\mathscr{H}$ and the term f is a locally integrable $\mathscr{H}$ -valued function.

Lemma 1.1.4. *Let T be a one-parameter semigroup on X, and assume that there exists $\delta > 0$ such that $M := \sup_{0 \leqslant t < \delta} \|T(t)\| < \infty$. Then there exists $\omega \in \mathbb{R}$ such that*

$$\|T(t)\| \leqslant M e^{\omega t} \text{ for all } t \geqslant 0$$

Proof. If $T(0) = 0$, then $M = 0$, and the assertion is trivial. Otherwise $T(0)$ is a non-zero projection, and therefore $M \geqslant \|T(0)\| \geqslant 1$. Let $\omega := \frac{1}{\delta} \ln M$; then $M = e^{\omega\delta}$. For $t \geqslant 0$ there exists $n \in \mathbb{N}_0$ such that $n\delta \leqslant t < (n+1)\delta$. The semigroup property (i) implies $T(t) = T\left(\frac{t}{n+1}\right)^{n+1}$, and therefore

$$\|T(t)\| \leqslant \left\|T\left(\frac{t}{n+1}\right)\right\|^{n+1} \leqslant M^{n+1} = M e^{\omega\delta n} \leqslant M e^{\omega t}$$

□

Lemma 1.1.5. *Let X, Y be Banach spaces, and let (B_n) be a sequence in $\mathcal{L}(X, Y), B \in \mathcal{L}(X, Y)$, and $B = \lim_{n \to \infty} B_n$. Let (x_n) in $X, x_n \to x \in X (n \to \infty)$ Then $B_n x_n \to Bx$ as $n \to \infty$*

Proof. The uniform boundedness theorem implies that $\sup_{n \in \mathbb{N}} \|B_n\| < \infty$. Therefore

$$\begin{aligned} \|Bx - B_n x_n\| &\leqslant \|Bx - B_n x\| + \|B_n (x - x_n)\| \\ &\leqslant \|Bx - B_n x\| + \left(\sup_{j \in \mathbb{N}} \|B_j\|\right) \|x - x_n\| \to 0 \quad (n \to \infty) \end{aligned}$$

□

Proposition 1.1.4. *See [23] Let S be a C_0 -semigroup on E with generator A.*

(1) For all $x \in E$ the orbit $t \mapsto S(t)x$ is continuous for $t \geqslant 0$.

(2) For all $x \in \mathscr{D}(A)$ and $t \geqslant 0$ we have $S(t)x \in \mathscr{D}(A)$ and $AS(t)x = S(t)Ax$

(3) For all $x \in E$ we have $\int_0^t S(s)xds \in \mathscr{D}(A)$ and

$$A\int_0^t S(s)xds = S(t)x - x$$

If $x \in \mathscr{D}(A)$, then both sides are equal to $\int_0^t S(s)Axds$

(4) The generator A is a closed and densely defined operator.

(5) For all $x \in \mathscr{D}(A)$ the orbit $t \mapsto S(t)x$ is continuously differentiable for $t \geqslant 0$ and

$$\frac{d}{dt}S(t)x = AS(t)x = S(t)Ax, \quad t \geqslant 0$$

Proof. 1) The right continuity of $t \mapsto S(t)x$ follows from the right continuity at $t = 0$ (S3) and the semigroup property (S2). For the left continuity, observe that

$$\|S(t)x - S(t-h)x\| \leqslant \|S(t-h)\|\|S(h)x - x\| \leqslant \sup_{s\in[0,t]} \|S(s)\|\|S(h)x - x\|$$

where the supremum is finite by Lemma 1.1.4

2) This follows from the semigroup property:

$$\lim_{h\downarrow 0} \frac{1}{h}(S(t+h)x - S(t)x) = S(t)\lim_{h\downarrow 0} \frac{1}{h}(S(h)x - x) = S(t)Ax$$

(3) The first identity follows from

$$\begin{aligned}\lim_{h\downarrow 0} \frac{1}{h}(S(h) - I)\int_0^t S(s)xds &= \lim_{h\downarrow 0} \frac{1}{h}\left(\int_0^t S(s+h)xds - \int_0^t S(s)xds\right)\\ &= \lim_{h\downarrow 0} \frac{1}{h}\left(\int_t^{t+h} S(s)xds - \int_0^h S(s)xds\right)\\ &= S(t)x - x\end{aligned}$$

where we used the continuity of $t \mapsto S(t)x$. The identity for $x \in \mathscr{D}(A)$ will follow from the second part of the proof of (4).

(4) Denseness of $\mathscr{D}(A)$ follows from the first part of (3), since by (1) we have $\lim_{t\downarrow 0}\frac{1}{t}\int_0^t S(s)xds = x$. To prove that A is closed we must check that the graph $\mathcal{G}(A) = \{(x, Ax) : x \in \mathscr{D}(A)\}$ is closed in $E\times E$. Suppose that $(x_n)_{n=1}^{\infty}$ is a sequence in $\mathscr{D}(A)$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Ax_n = y$ in E. We must show that $x \in \mathscr{D}(A)$ and $Ax = y$. Using that $\lim_{t\downarrow 0}\frac{1}{t}(S(t)-I)S(s)x_n = S(s)Ax_n$ uniformly for $s \in [0, h]$, we obtain

$$\begin{aligned}\frac{1}{h}(S(h)x - x) &= \lim_{n\to\infty}\frac{1}{h}\left(S(h)x_n - x_n\right)\\ &= \lim_{n\to\infty}\frac{1}{h}\left(A\int_0^h S(s)x_n ds\right)\\ &= \lim_{n\to\infty}\frac{1}{h}\lim_{t\downarrow 0}\frac{1}{t}(S(t)-I)\int_0^h S(s)x_n ds\\ &= \lim_{n\to\infty}\frac{1}{h}\lim_{t\downarrow 0}\int_0^h\frac{1}{t}(S(t)-I)S(s)x_n ds\\ &= \lim_{n\to\infty}\frac{1}{h}\int_0^h S(s)Ax_n ds\\ &= \frac{1}{h}\int_0^h S(s)y ds\end{aligned}$$

Passing to the limit for $h\downarrow 0$ this gives $x \in \mathscr{D}(A)$ and $Ax = y$. The above identities also prove the second part of (3) .

(5) For $t \geqslant 0, h > 0$ one has

$$h^{-1}(S(t+h)x - S(t)x) = h^{-1}(S(h)-I)S(t)x = S(t)h^{-1}(S(h)x - x)$$

As $h \to 0$, the third of these expressions converges to $S(t)Ax$. Looking at the second term, one obtains $T(t)x \in \text{dom}(A)$, and looking at the first term one concludes that $t \mapsto S(t)x$ is right-sided differentiable, with right-sided derivative

$$\left(\frac{\mathrm{d}}{\mathrm{dt}}\right)_r S(t)x = AS(t)x = S(t)Ax.$$

So we have shown that the continuous function $t \mapsto AS(t)x = S(t)Ax$ is the derivative of $t \mapsto S(t)x$

□

Theorem 1.1.6. *(Hille)(See [23]). Let $f : A \to E$ be μ-integrable and let T be a closed linear operator with domain $\mathscr{D}(T)$ in E taking values in a Banach space F. Assume that f takes its values in $\mathscr{D}(T)$ μ -almost everywhere and the μ -almost everywhere defined function $Tf : A \to F$ is μ -Bochner integrable. Then $\int_A f d\mu \in \mathscr{D}(T)$ and*

$$T\int_A f d\mu = \int_A T f d\mu$$

Proposition 1.1.5. *Let $\mathfrak{a}$ be sesquilinear form $\mathcal{V}$-bounded and coercive. Denote by A the operator associated with $\mathfrak{a}$. Let $\theta = \arctan \frac{M}{\delta}$. Then $\Sigma_{\pi-\theta} \subset \rho(-A)$ and there exists constants $C_\theta, C'_\theta > 0$ depending on θ, such that*

1- $\|(\lambda + A)^{-1}\|_{\mathcal{L}(\mathscr{H})} \leqslant \frac{C_\theta}{|\lambda|}$.

2- $\|(\lambda + A)^{-1}\|_{\mathcal{L}(\mathscr{H},\mathcal{V})} \leqslant \frac{C'_\theta}{\sqrt{|\lambda|}}$.

Proof. 1- Let $u \in \mathscr{D}(A), \lambda \in \mathbb{C}$. We get

$$\begin{aligned}\|(\lambda - A)u\|\|u\| &\geqslant |((\lambda - A)u, u)| \\ &= |\lambda - \frac{(Au, u)}{\|u\|^2}|\|u\|^2 \\ &= |\lambda - \frac{\mathfrak{a}(u, u)}{\|u\|^2}|\|u\|^2 \\ &= |\lambda - \mathfrak{a}(\frac{u}{\|u\|}, \frac{u}{\|u\|})|\|u\|^2.\end{aligned}$$

Therefore

$$\|(\lambda - A)u\| \geqslant \operatorname{dist}(\lambda, \overline{\Sigma_{\arctan \frac{M}{\delta}}})\|u\|.$$

This implies that $\lambda - A$ is injective and has closed range for $\lambda \notin \overline{\Sigma_{\arctan \frac{M}{\delta}}}$. In order to prove that $\lambda - A$ is invertible it remains to prove that it has dense range. By duality, one has to prove that the adjoint is injective and this true by the same argument as before. Therefore

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(\mathscr{H})} \leqslant \frac{1}{\operatorname{dist}(\lambda, \overline{\Sigma_{\arctan \frac{M}{\delta}}})}$$

for all $\lambda \notin \overline{\Sigma_{\arctan \frac{M}{\delta}}}$. Now we set $\theta = \arctan \frac{M}{\delta}$, then there exists a constant C_θ such that

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(\mathscr{H})} \leqslant \frac{C_\theta}{|\lambda|}.$$

In other words, $\lambda + A$ is invertible for $\lambda \in \Sigma_{\pi-\theta}$ and

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(\mathscr{H})} \leqslant \frac{C_\theta}{|\lambda|}.$$

2- Now let $x \in \mathscr{H}$ and $\lambda \in \Sigma_{\pi-\theta}$. We have

$$\delta\|(\lambda + A)^{-1}x\|_{\mathcal{V}}^2 \leqslant \operatorname{Re}\,(A(\lambda + A)^{-1}x, (\lambda + A)^{-1}x)$$

$$\leqslant \|A(\lambda + A)^{-1}x\| \|(\lambda + A)^{-1}x\|$$
$$\leqslant (1 + C_\theta)\frac{C_\theta}{|\lambda|}\|x\|^2.$$

Therefore

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(\mathscr{H},\mathcal{V})} \leqslant \frac{C'_\theta}{\sqrt{|\lambda|}}.$$

□

Proposition 1.1.6. *Let A as in the previous proposition. Then $-A$ is a generator of a bounded holomorphic contraction semigroup on $\mathscr{H}$ and we have*

1- For all $t \in (0,\infty), n \in \mathbb{N}$, there exists a constant $C > 0$ such that

$$\|A^n e^{-tA}\|_{\mathcal{L}(\mathscr{H})} \leqslant \frac{C}{t^n}.$$

2- For all $t \in (0,\infty)$,

$$\|e^{-tA}\|_{\mathcal{L}(\mathscr{H},\mathcal{V})} \leqslant \frac{C}{\sqrt{t}}.$$

Proof. 1- We use the Cauchy's integral formula .

2- Since $\Sigma_{\frac{\pi}{2}+(\frac{\pi}{2}-\arctan\frac{M}{\delta})} \subset \rho(-A)$ and

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(\mathscr{H})} \leqslant \frac{C_\theta}{|\lambda|},$$

then $-A$ is the generator of a bounded holomorphic semigroup on $\Sigma_{(\frac{\pi}{2}-\arctan\frac{M}{\delta})}$ and for $z \in \Sigma_{(\frac{\pi}{2}-\arctan\frac{M}{\delta})}$ we have $e^{-zA}x \in \mathscr{D}(A)$ where $x \in \mathscr{H}$. So for all $x \in \mathscr{H}$ we obtain

$$\frac{\partial}{\partial z}\|e^{-zA}x\|^2 = -2\text{Re }(Ae^{-zA}x, e^{-zA}x) < 0.$$

Therefore $\|e^{-zA}\|_{\mathcal{L}(\mathscr{H})} \leqslant 1$. For all $x \in \mathscr{H}$ and $t > 0$ we get

$$\begin{aligned}\delta\|e^{-tA}x\|^2_{\mathcal{V}} &\leqslant \text{Re } a(e^{-tA}x, e^{-tA}x)\\ &= \text{Re }(Ae^{-tA}x, e^{-tA}x)\\ &\leqslant \|Ae^{-tA}x\|\|e^{-tA}x\|\\ &\leqslant \frac{C}{t}.\end{aligned}$$

This shows the second assertion.

□

1.2 Fractional Powers

The next definition is motivated by the trivial identity

$$c^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-ct} dt, \quad c > 0 \tag{1.2.1}$$

where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$ is the Euler gamma function (case $c = 1$).

Definition 1.2.1. *For $0 < \alpha < 1$ we define the fractional power $(-A)^{-\alpha}$ of $-A$ by the formula*

$$(-A)^{-\alpha} x := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} S(t) x dt, \quad x \in E$$

Note that $(-A)^{-\alpha}$ is well-defined and bounded on X. Sometimes it is useful to extend the definition to the limiting values $\alpha \in \{0, 1\}$ by putting $(-A)^0 = I$ and $(-A)^{-1} = -A^{-1}$

Lemma 1.2.1. *For all $0 < \alpha, \beta < 1$ satisfying $0 < \alpha + \beta < 1$ we have*

$$(-A)^{-\alpha}(-A)^{-\beta} = (-A)^{-\beta}(-A)^{-\alpha} = (-A)^{-\alpha-\beta}$$

Proof. It suffices to prove that $(-A)^{-\alpha}(-A)^{-\beta} = (-A)^{-\alpha-\beta}$; the other identity follows upon interchanging α and β.

For all $x \in E$ we have

$$\begin{aligned}
(-A)^{-\alpha}(-A)^{-\beta} x &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} S(s+t) x ds dt \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \int_t^\infty t^{\alpha-1} (s-t)^{\beta-1} S(s) x ds dt \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \left(\int_0^s t^{\alpha-1} (s-t)^{\beta-1} dt \right) S(s) x ds \\
&=^* \frac{1}{\Gamma(\alpha+\beta)} \int_0^\infty s^{\alpha+\beta-1} S(s) x ds = (-A)^{-\alpha-\beta} x
\end{aligned}$$

where the identity ($*$) follows from $\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^s t^{\alpha-1}(s-t)^{\beta-1} dt = \frac{s^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \tau^{\alpha-1}(1-\tau)^{\beta-1} d\tau = \frac{s^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}$ Indeed, computing as above,

$$\begin{aligned}
\Gamma(\alpha+\beta) \int_0^1 \tau^{\alpha-1}(1-\tau)^{\beta-1} d\tau &= \int_0^\infty \int_0^1 s^{\alpha+\beta-1} \tau^{\alpha-1} (1-\tau)^{\beta-1} e^{-s} d\tau ds \\
&= \int_0^\infty \int_0^s t^{\alpha-1} (s-t)^{\beta-1} e^{-s} dt ds \\
&= \int_0^\infty \int_t^\infty t^{\alpha-1} (s-t)^{\beta-1} e^{-s} ds dt \\
&= \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-s-t} ds dt \\
&= \Gamma(\alpha)\Gamma(\beta)
\end{aligned}$$

□

In the next lemma we assume that the C_0-semigroup S, in addition to being uniformly exponentially stable, is analytic.

Lemma 1.2.2. *For all* $0 < \alpha < 1$ *and* $t > 0$ *the operator* $(-A)^\alpha S(t)$ *is bounded and we have*

$$\sup_{t>0} t^\alpha \|(-A)^\alpha S(t)\| < \infty$$

Proof. Since S is analytic, $S(t)$ maps E into $\mathscr{D}(A)$ and $\sup_{t>0} t\|AS(t)\| < \infty$ The boundedness of $(-A)^\alpha S(t)$ follows from the boundedness of $AS(t)$ by the identity $(-A)^\alpha S(t) = -(-A)^{\alpha-1}AS(t)$ To prove the estimate, note that for all $x \in E$ we have

$$(-A)^\alpha S(t)x = \frac{-1}{\Gamma(1-\alpha)} \int_0^\infty s^{-\alpha} AS(t+s)xds$$

so, for $t > 0$

$$\begin{aligned} \|(-A)^\alpha S(t)x\| &\leqslant \frac{C}{\Gamma(1-\alpha)} \int_0^\infty s^{-\alpha}(t+s)^{-1}\|x\|ds \\ &= \frac{Ct^{-\alpha}}{\Gamma(1-\alpha)} \int_0^\infty \tau^{-\alpha}(1+\tau)^{-1}\|x\|d\tau \end{aligned}$$

□

The fractional power A^α with $0 < \alpha < 1$, equivalently defined by

$$A^\alpha = -\frac{\sin \pi\alpha}{\pi} \int_0^\infty \mu^\alpha(\mu + A)^{-1} d\mu. \tag{A1}$$

Let A be the operator associated with a $\mathcal{V}$-bounded coercive sesquilinear form $\mathfrak{a}$. We consider $0 < \alpha < 1$ and the complex interpolation space $[\mathscr{H}, \mathcal{V}]_\alpha$.

Proposition 1.2.2. *1-* $\mathcal{V} \hookrightarrow \mathscr{D}(A^{\frac{1}{2}})$ *if and only if* $D(A^{*\frac{1}{2}}) \hookrightarrow \mathcal{V}$.

2- *If* $A = A^*$ *we have* $\mathscr{D}(A^{\frac{1}{2}}) = \mathscr{D}(A^{*\frac{1}{2}}) = \mathcal{V}$ *and*

$$\sqrt{\delta}\|u\|_{\mathcal{V}} \leqslant \|A^{\frac{1}{2}}u\| \leqslant \sqrt{M}\|u\|_{\mathcal{V}}.$$

3- $\mathscr{D}(A^\alpha) = [\mathscr{H}, \mathcal{V}]_{2\alpha}$ *for all* $\alpha < \frac{1}{2}$.

4- $\mathscr{D}(A^{1-\alpha}) \hookrightarrow \mathcal{V}$ *for all* $\alpha < \frac{1}{2}$.

5- *For all* $t \in [0, \tau]$ *we have* $\mathscr{D}(\mathcal{A}(t)^{\frac{1}{2}}) = \mathscr{H}$ *and* $D([A(t)^*]^{\frac{1}{2}}) = \mathcal{V}$.

Proof. Let $u \in \mathscr{D}(A^*)$. If $\mathcal{V} \hookrightarrow \mathscr{D}(A^{\frac{1}{2}})$ we get

$$\begin{aligned}\|u\|_{\mathcal{V}}^2 &\leqslant \frac{1}{\delta}\text{Re }(A^{\frac{1}{2}}u, A^{*\frac{1}{2}}u)\\ &\leqslant \frac{1}{\delta}\|A^{\frac{1}{2}}u\|\|A^{*\frac{1}{2}}u\|\\ &\leqslant C\|u\|_{\mathcal{V}}\|A^{*\frac{1}{2}}u\|.\end{aligned}$$

Then by the density of $\mathscr{D}(A^*)$ on $\mathscr{D}(A^{*\frac{1}{2}})$ we obtain

$$\|u\|_{\mathcal{V}} \leqslant C\|A^{*\frac{1}{2}}u\|$$

for all $u \in \mathscr{D}(A^{*\frac{1}{2}})$. Then $\mathscr{D}(A^{*\frac{1}{2}}) \hookrightarrow \mathcal{V}$.
Now, we assume that $\mathscr{D}(A^{*\frac{1}{2}}) \hookrightarrow \mathcal{V}$. It follows that $A^{*-\frac{1}{2}} \in \mathcal{L}(\mathscr{H}, \mathcal{V})$. Let $x \in \mathscr{H}$ and we write $\mathcal{A}^{*\frac{1}{2}}x = \mathcal{A}^* A^{*-\frac{1}{2}}x$. Then we get

$$\|\mathcal{A}^{*\frac{1}{2}}x\|_{\mathcal{V}'} \leqslant \|\mathcal{A}^*\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}\|A^{*-\frac{1}{2}}x\|_{\mathcal{V}} \leqslant M\|A^{*-\frac{1}{2}}\|_{\mathcal{L}(\mathscr{H},\mathcal{V})}\|x\|.$$

The boundedness implies $\mathcal{A}^{*\frac{1}{2}} \in \mathcal{L}(\mathscr{H}, \mathcal{V}')$ and by duality we have $A^{\frac{1}{2}} \in \mathcal{L}(\mathcal{V}, \mathscr{H})$. Then $\mathcal{V} \subseteq \mathscr{D}(A^{\frac{1}{2}})$ and we get for all $x \in \mathcal{V}$

$$\begin{aligned}\|x\|^2_{\mathscr{D}(A^{\frac{1}{2}})} &= \|x\|^2 + \|A^{\frac{1}{2}}x\|^2_{\mathscr{H}}\\ &\leqslant (C^2_{\mathscr{H}} + \|A^{\frac{1}{2}}\|^2_{\mathcal{L}(\mathcal{V},\mathscr{H})})\|x\|^2_{\mathcal{V}}.\end{aligned}$$

Thus, $\mathcal{V} \hookrightarrow \mathscr{D}(A^{\frac{1}{2}})$. This shows the first assertion.
We assume that $A = A^*$. By the density of $\mathscr{D}(A)$ in $\mathcal{V}$, we get for all $u \in \mathcal{V}$

$$\begin{aligned}\delta\|u\|^2_{\mathcal{V}} &\leqslant \text{Re } \mathfrak{a}(u, u)\\ &= \|A^{\frac{1}{2}}u\|^2\\ &\leqslant M\|u\|^2_{\mathcal{V}}.\end{aligned}$$

This shows second assertion. For the third assertion, we refer to [?] (Theorem 3.1).
Let $\alpha < \frac{1}{2}$ and $u \in \mathscr{D}(A)$. We have

$$\begin{aligned}\|u\|^2_{\mathcal{V}} &\leqslant \frac{1}{\delta}\|A^{1-\alpha}u\|\|A^{*\alpha}u\|\\ &\leqslant \frac{1}{\delta}\|A^{1-\alpha}u\|\|u\|_{[\mathscr{H},\mathcal{V}]_{2\alpha}}\\ &\leqslant \frac{C(\alpha)}{\delta}\|A^{1-\alpha}u\|\|u\|^{2\alpha}_{\mathcal{V}}\|u\|^{1-2\alpha},\end{aligned}$$

where $C(\alpha) > 0$ depending on α. Thus, for all $u \in D(A^{1-\alpha})$ we get

$$\|u\|_{\mathcal{V}} \leqslant \frac{C_{\mathscr{H}}^{1-2\alpha} C(\alpha)}{\delta} \|A^{1-\alpha} u\|.$$

This shows the assertion 4.
We write

$$\mathcal{A}(t)^{\frac{1}{2}} u = \mathcal{A}(t) A(t)^{-\frac{1}{2}} u.$$

Therefore

$$\frac{\alpha}{c^1} \|u\| \leqslant \|\mathcal{A}(t)^{\frac{1}{2}} u\|_{V'} \leqslant \frac{M}{c_1} \|u\|.$$

So that $\mathcal{A}(t)^{\frac{1}{2}} \in \mathcal{L}(\mathscr{H}, \mathcal{V}')$. By duality we have $A(t)^{*\frac{1}{2}} \in \mathcal{L}(\mathcal{V}, \mathscr{H})$. This gives the last assertion. □

For the next result we refer to [2] (Theorem 4.3.5 and Proposition 5.1.1) . In what follows E is a Banach space and B is a generator of holomorphic semigroup.

Proposition 1.2.3. *For $0 < \theta < 1$, $1 \leqslant p < \infty$, we have*

$$(E, \mathscr{D}(B))_{\theta,p} = \{x \in E : \phi(t) = t^{1-\theta} \|B e^{tB} x\|_E \in L^p(0, \infty; \frac{dt}{t})\}$$

with norm

$$\|x\|^p_{(E,\mathscr{D}(B))_{\theta,p}} = \|x\|^p_E + \int_0^\infty \|\phi(t)\|^p_E \frac{dt}{t}.$$

$(E, D(B))_{\theta,p}$ is the real interpolation space between the space E and the domaine of B with parameters θ and p. Holomorphic semigroups and interpolation spaces play an important role in the theory of evolution equations. In particular, if the semigroup generated by B is holomorphic, then the problem

$$\begin{cases} u'(t) = Bu(t) \\ \\ u(0) = x \end{cases} \tag{1.2.2}$$

has a unique solution $u \in W^{1,2}(0, \tau; E) \cap L^2(0, \tau; D(B)))$ for every initial data $x \in (E, \mathscr{D}(B))_{\frac{1}{2},2}$. In fact, it is very known that the solution of the Problem (1.2.2) is giving by $u(t) = e^{tB} x$ and $u'(t) = B e^{tB} x$. Therefore

$$\|u'\|^2_{L^2(0,\tau;E)} = \int_0^\tau \|B e^{tB} x\|^2_E dt \leqslant \|x\|^2_{(E,\mathscr{D}(B))_{\frac{1}{2},2}}.$$

Definition 1.2.4. *We say that $(A(t))$ (or the corresponding forms $\mathfrak{a}(t)$) satisfy the uniform Kato square root property if $\mathscr{D}(A(t)^{\frac{1}{2}}) = \mathcal{V}$ for all $t \in [0, \tau]$ and there exist constants $C_1, C_2 > 0$ such that following condition holds, i.e*

$$C_1 \|u\|_{\mathcal{V}} \leqslant \|A(t)^{\frac{1}{2}} u\| \leqslant C_2 \|u\|_{\mathcal{V}} \text{ for all } u \in \mathcal{V} \text{ and all } t \in [0, \tau]. \tag{1.2.3}$$

1.3 Sobolev spaces

We present the definition and some basic properties of the Sobolev space H^1. This treatment is prepared by several important tools from analysis.

Convolution

We recall the definition of locally integrable functions on an open subset Ω of $\mathbb{R}^n$,
$L_{1,\mathrm{loc}}(\Omega) := \{f : \Omega \to \mathbb{K}$; for all $x \in \Omega$ there exists $r > 0$ such that

$$B(x,r) \subseteq \Omega \text{ and } \left. f\right|_{B(x,r)} \in L_1(B(x,r))\Big\}$$

Moreover, $C_{\mathrm{c}}^{\infty}(\Omega) := C^{\infty}(\Omega) \cap C_{\mathrm{c}}(\Omega)$ is the space of infinitely differentiable functions with compact support.

Lemma 1.3.1. *Let $u \in L_{1,\mathrm{loc}}(\mathbb{R}^n), \rho \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$. We define the convolution of ρ and u*

$$\rho * u(x) := \int_{\mathbb{R}^n} \rho(x-y)u(y)\mathrm{d}y = \int_{\mathbb{R}^n} \rho(y)u(x-y)\mathrm{d}y \quad (x \in \mathbb{R}^n)$$

Then $\rho * u \in C^{\infty}(\mathbb{R}^n)$, and for all $\alpha \in \mathbb{N}_0^n$ one has

$$\partial^{\alpha}(\rho * u) = (\partial^{\alpha}\rho) * u$$

Proof. Note that the integral exists because ρ is bounded and has compact support. (i) Continuity of $\rho * u$: There exists $R > 0$ such that $\operatorname{spt}\rho \subseteq B(0,R)$. Let $R' > 0$ $\delta > 0$. For $x, x' \in B(0,R'), |x-x'| < \delta$, one obtains

$$\begin{aligned} |\rho * u(x) - \rho * u(x')| &= \left| \int_{B(0,R+R')} (\rho(x-y) - \rho(x'-y))\, u(y)\mathrm{d}y \right| \\ &\leqslant \sup\{|\rho(z) - \rho(z')| ; |z-z'| < \delta\} \int_{B(0,R+R')} |u(y)|\mathrm{d}y \end{aligned}$$

The second factor in the last expression is finite because u is locally integrable, and the first factor becomes small for small δ because ρ is uniformly continuous.
(ii) Induction shows the assertion for all $\alpha \in \mathbb{N}_0^{\alpha}$

□

Proposition 1.3.1. *Let (ρ_k) be a δ -sequence in $C_c(\mathbb{R}^n)$*

(a) Let $f \in C(\mathbb{R}^n)$. Then $\rho_k * f \to f$ uniformly on compact subsets of $\mathbb{R}^n$ as $k \to \infty$.

(b) Let $1 \leqslant p \leqslant \infty, f \in L_p(\mathbb{R}^n)$. Then $\rho_k * f \in L_p(\mathbb{R}^n)$

$$\|\rho_k * f\|_p \leqslant \|f\|_p \quad \text{for all } k \in \mathbb{N}$$

If $1 \leqslant p < \infty$, then

$$\|\rho_k * f - f\|_p \to 0 \quad (k \to \infty)$$

Corollary 1.3.2. *Let $\Omega \subseteq \mathbb{R}^n$ be open, $1 \leqslant p < \infty$. Then $C_c^\infty(\Omega)$ is dense in $L_p(\Omega)$*

Proof. Let (ρ_k) be a δ -sequence in $C_c^\infty(\mathbb{R}^n)$. Let $g \in C_c(\Omega)$, and extend g by zero to a function in $C_c(\mathbb{R}^n)$. Then $\rho_k * g \in C^\infty(\mathbb{R}^n)$ for all $k \in \mathbb{N}$, by Lemma 1.3.1 If $\frac{1}{k} < \operatorname{dist}(\operatorname{spt} g, \mathbb{R}^n \backslash \Omega)$, then $\operatorname{spt}(\rho_k * g) \subseteq \operatorname{spt} g + B[0, 1/k] \subseteq \Omega$ and therefore $\rho_k * g \in C_c^\infty(\Omega)$. From proposition 1.3.1 we know that $\rho_k * g \to g(k \to \infty)$ in $L_p(\mathbb{R}^n)$. So, we have shown that $C_c^\infty(\Omega)$ is dense in $C_c(\Omega)$ with respect to the L_p -norm. Now the denseness of $C_c(\Omega)$ in $L_p(\Omega)$ yields the assertion.

□

Lemma 1.3.3. *Let $\Omega \subseteq \mathbb{R}^n$ be open, $f \in L_{1,loc}(\Omega)$*

$$\int f \varphi \mathrm{d}x = 0 \quad (\varphi \in C_c^\infty(\Omega))$$

Then $f = 0$

The statement $'f = 0$ ' means that f is the zero element of $L_{1,\mathrm{loc}}(\Omega)$, i.e., if f is a representative, then $f = 0$ a.e.

Definition 1.3.4. *Let $\Omega \subseteq \mathbb{R}^n$ be open. We define the Sobolev space*

$$H^1(\Omega) := \{f \in L_2(\Omega); \partial_j f \in L_2(\Omega)(j \in \{1, \ldots, n\})\}$$

with scalar product

$$(f \mid g)_1 := (f \mid g) + \sum_{j=1}^n (\partial_j f \mid \partial_j g)$$

(where

$$(f \mid g) := \int_\Omega f(x) \overline{g(x)} \mathrm{d}x$$

denotes the usual scalar product in $L_2(\Omega)$) and associated norm

$$\|f\|_{2,1} := \left(\|f\|_2^2 + \sum_{j=1}^{n} \|\partial_j f\|_2^2\right)^{1/2}$$

Proposition 1.3.2. *The space $C^\infty([0,\tau];\mathcal{V})$ is dense in $MR(\mathcal{V},\mathcal{V}')$.*

Theorem 1.3.5. *(a) One has $MR(\mathcal{V},\mathcal{V}') \hookrightarrow C([0,\tau];\mathscr{H})$.*

(b) If $u \in MR(\mathcal{V},\mathcal{V}')$, then the function $\|u(\cdot)\|_{\mathscr{H}}^2$ is in $W^{1,1}(0,\tau;\mathbb{R})$ and

$$\left(\|u(\cdot)\|_{\mathscr{H}}^2\right)' = 2\operatorname{Re}\langle u'(\cdot), u(\cdot)\rangle$$

Proof. If $u \in C^1([0,\tau];\mathcal{V})$, then it is immediate that $\frac{d}{dt}\|u(t)\|_{\mathscr{H}}^2 = (u'(t) \mid u(t))_{\mathscr{H}} + (u(t) \mid u'(t))_{\mathscr{H}} = 2\operatorname{Re}\langle u'(t), u(t)\rangle$

(a) For $u \in C^1([0,\tau];\mathcal{V})$ we deduce that

$$\begin{aligned}
\|u\|_{C([0,\tau];\mathscr{H})}^2 &\leqslant \inf_{t\in(0,\tau)} \|u(t)\|_{\mathscr{H}}^2 + \int_0^\tau \frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{\mathscr{H}}^2 \,\mathrm{d}t \\
&\leqslant \frac{1}{\tau}\int_0^\tau \|u(t)\|_{\mathscr{H}}^2 dt + 2\int_0^\tau \|u'(t)\|_{\mathcal{V}'} \|u(t)\|_{\mathcal{V}} \,\mathrm{d}t \\
&\leqslant \frac{c}{\tau}\|u\|_{L_2(0,\tau,\mathcal{V})}^2 + 2\|u'\|_{L_2\left(0,\tau;\mathcal{V}'\right)} \|u\|_{L_2(0,\tau;\mathcal{V})}
\end{aligned}$$

with the embedding constant $c \geqslant 0$ of $\mathcal{V} \overset{\mathrm{d}}{\hookrightarrow} \mathscr{H}$. As $C^1([0,\tau];\mathcal{V})$ is dense in $MR(\mathcal{V},\mathcal{V}')$, by Proposition 1.3.2 , this inequality shows that $MR(\mathcal{V},\mathcal{V}') \hookrightarrow C([0,\tau];\mathscr{H})$, and the inequality carries over to all $u \in MR(\mathcal{V},\mathcal{V}')$.

b) Initially we have shown 1.3.2 for $u \in C^1([0,\tau];\mathcal{V})$. Let now $u \in MR(\mathcal{V},\mathcal{V}')$. By Proposition 1.3.2 there exists a sequence (u_n) in $C^1([0,\tau];\mathcal{V})$ converging to u in $MR(\mathcal{V},\mathcal{V}')$. Then

$$\left(\|u_n(\cdot)\|_{\mathscr{H}}^2\right)' = 2\operatorname{Re}\langle u_n'(\cdot), u_n(\cdot)\rangle \to 2\operatorname{Re}\langle u'(\cdot), u(\cdot)\rangle$$

in $L_1(0,\tau)$. Moreover, $u_n \to u$ in $C([0,\tau];\mathscr{H})$ by part (a), and therefore $\|u_n(\cdot)\|_{\mathscr{H}}^2 \to \|u(\cdot)\|_{\mathscr{H}}^2$ in $C[0,\tau]$. This implies that $2\operatorname{Re}\langle u'(\cdot), u(\cdot)\rangle$ is the distributional derivative of $\|u(\cdot)\|_{\mathscr{H}}^2$

□

1.4 Stochastic preliminaries

All vector spaces are assumed to be real. We will always identify Hilbert spaces with their duals by means of the Riesz representation theorem.

1.4.1 γ-Boundedness and γ-Radonifying operators

Let X and Y be Banach spaces and let $\{\gamma_n\}_{n\geqslant 1}$ be Gaussian sequence (i.e., a sequence of independent real-valued standard Gaussian random variables).

Definition 1.4.1. *A family $\mathcal{T}$ of bounded linear operators from X to Y is called* γ-bounded *if there exists a constant $C \geqslant 0$ such that for all finite sequences $\{x_n\}_{n=1}^N$ in X and $\{T_n\}_{n=1}^N$ in $\mathcal{T}$ we have*

$$\mathbb{E}\Big\|\sum_{n=1}^N \gamma_n T_n x_n\Big\|^2 \leqslant C^2 \mathbb{E}\Big\|\sum_{n=1}^N \gamma_n x_n\Big\|^2.$$

Clearly, every γ-bounded family of bounded linear operators from X to Y is uniformly bounded and $\sup_{T\in\mathcal{T}} \|T\|_{\mathscr{L}(X;Y)} \leqslant C$, the constant appearing in the above definition. In the setting of Hilbert spaces both notions are equivalent and the above inequality holds with $C = \sup_{T\in\mathcal{T}} \|T\|_{\mathscr{L}(X;Y)}$. γ-Boundedness is the Gaussian analogue of R-boundedness, obtained by replacing Gaussian variables by Rademacher variables.

1.4.2 γ-Radonifying operators

Let $\mathscr{H}$ be a Hilbert space with inner product $(\cdot|\cdot)$ and X a Banach space. Let $\mathscr{H}\otimes X$ denote the linear space of all finite rank operators from $\mathscr{H}$ to X. Every element in $\mathscr{H}\otimes X$ can be represented in the form $\sum_{n=1}^N h_n \otimes x_n$, where $h_n\otimes x_n$ is the rank one operator mapping the vector $h\in\mathscr{H}$ to $(h|h_n)x_n \in X$. By a Gram-Schmidt orthogonalisation argument we may always assume that the sequence $\{h_n\}_{n=1}^N$ is orthonormal in $\mathscr{H}$.

Definition 1.4.2. *The Banach space $\gamma(\mathscr{H}, X)$ is the completion of $\mathscr{H}\otimes X$ with respect to the norm*

$$\Big\|\sum_{n=1}^N h_n\otimes x_n\Big\|_{\gamma(\mathscr{H},X)} := \Big(\mathbb{E}\Big\|\sum_{n=1}^N \gamma_n x_n\Big\|^2\Big)^{1/2}, \tag{1.4.1}$$

where $\{h_n\}_{n=1}^N$ is orthonormal in $\mathscr{H}$ and $\{\gamma_n\}_{n=1}^N$ is a Gaussian sequence.

Since the distribution of a Gaussian vector in $\mathbb{R}^N$ is invariant under orthogonal transformations, the quantity on the right-hand side of (1.4.1) is independent of the representation of the operator as a finite sum of the form $\sum_{n=1}^N h_n\otimes x_n$ as long as $\{h_n\}_{n=1}^N$ is orthonormal in $\mathscr{H}$. Therefore, the norm $\|\cdot\|_{\gamma(\mathscr{H},X)}$ is well defined.

Remark 1.4.1. *By the Kahane-Khintchine inequalities , for all $0<p<\infty$ there exists a universal constant κ_p, depending only on p, such that for all Banach spaces X and all finite sequences $\{x_n\}_{n=1}^N$ in X we have*

$$\frac{1}{\kappa_p}\Big(\mathbb{E}\Big\|\sum_{n=1}^N \gamma_n x_n\Big\|^p\Big)^{1/p} \leqslant \Big(\mathbb{E}\Big\|\sum_{n=1}^N \gamma_n x_n\Big\|^2\Big)^{1/2} \leqslant \kappa_p\Big(\mathbb{E}\Big\|\sum_{n=1}^N \gamma_n x_n\Big\|^p\Big)^{1/p}.$$

As a consequence, for $1 \leqslant p < \infty$ the norm

$$\Big\| \sum_{n=1}^{N} h_n \otimes x_n \Big\|_{\gamma^p(\mathscr{H},X)} := \Big(\mathbb{E} \Big\| \sum_{n=1}^{N} \gamma_n x_n \Big\|^p \Big)^{1/p},$$

with $\{h_n\}_{n=1}^N$ orthonormal in $\mathscr{H}$, is an equivalent norm on $\gamma(\mathscr{H}, X)$. Endowed with this equivalent norm, the space is denoted by $\gamma^p(\mathscr{H}, X)$.

For any Hilbert space $\mathscr{H}$ we have a natural isometric isomorphism

$$\gamma(\mathscr{H}, X) = HS(\mathscr{H}, X),$$

where $HS(\mathscr{H}, X)$ is the space of all Hilbert-Schmidt operators from $\mathscr{H}$ to X. Furthermore, for $1 \leqslant p < \infty$ and σ-finite measures μ we have an isometric isomorphism of Banach spaces

$$\gamma^p(\mathscr{H}, L^p(\mu; X)) \simeq L^p(\mu; \gamma^p(\mathscr{H}; X)) \tag{1.4.2}$$

which is obtained by associating with $f \in L^p(\mu; \gamma(\mathscr{H}; X))$ the mapping $h' \mapsto f(\cdot)h'$ from $\mathscr{H}$ to $L^p(\mu; X)$ [**?**, Theorem 9.4.8]. In particular, upon identifying $\gamma(\mathscr{H}, \mathbb{R})$ with $\mathscr{H}$, we obtain an isomorphism of Banach spaces

$$\gamma(\mathscr{H}, L^p(\mu)) \simeq L^p(\mu; \mathscr{H}).$$

Proposition 1.4.3 (Ideal property)**.** *Suppose that H_0 and H_1 are Hilbert spaces and E_0 and E_1 are Banach spaces. Let $R \in \gamma(H_0, E_0)$, $T \in \mathscr{L}(H_1, H_0)$ and $U \in \mathscr{L}(E_0, E_1)$, then $URT \in \gamma(H_1, E_1)$ and*

$$\|URT\|_{\gamma(H_1,E_1)} \leqslant \|U\| \|R\|_{\gamma(H_0,E_0)} \|T\|.$$

The following lemma is called the γ-Fubini Lemma, and is taken from [32].

Lemma 1.4.1. *Let (S, Σ, μ) be a σ-finite measure space and let $1 \leqslant p < \infty$. The mapping $U : L^p(S; \gamma(\mathscr{H}, E)) \to \mathscr{L}(H, L^p(S; E))$, given by $((UF)h)s = F(s)h$ for $s \in S$ and $h \in \mathscr{H}$, defines an isomorphism $L^p(S; \gamma(\mathscr{H}, E)) \eqsim \gamma(\mathscr{H}, L^p(S; E))$.*

1.4.3 Malliavin calculus

1.4.4 The Malliavin derivative operator

In this section we recall some of the basic elements of Malliavin calculus. We refer to [9] for details in the scalar situation.

Definition 1.4.2. *An* $\mathscr{H}$-isonormal process *on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a mapping* $W : \mathscr{H} \to L^2(\Omega)$ *with the following properties:*

(i) for all $h \in \mathscr{H}$ *the random variable* $W(h)$ *is Gaussian;*

(ii) for all $h_1, h_2 \in \mathscr{H}$ *we have* $\mathbb{E}W(h_1)W(h_2) = [h_1, h_2]$.

Let $\{W(h), h \in \mathscr{H}\}$ be an *isonormal Gaussian process* associated with $\mathscr{H}$, that is $\{Wh : h \in \mathscr{H}\}$ is a centered family of Gaussian random variable and

$$\mathbb{E}(Wh_1 Wh_2) = \langle h_1, h_2 \rangle, \qquad h_1, h_2 \in \mathscr{H}.$$

We will assume $\mathscr{F}$ is generated by W. Let $1 \leqslant p < \infty$, and let E be a Banach space. Let us define the *Gaussian Sobolev space* $\mathbb{D}^{1,p}(E)$ of E-valued random variables in the following way. Consider the class $\mathscr{S} \otimes E$ of *smooth* E*-valued random variables* $F : \Omega \to E$ of the form

$$F = f(W(h_1), \dots, W(h_n)) \otimes x,$$

where $f \in C_b^\infty(\mathbb{R}^n)$, $h_1, \dots, h_n \in \mathscr{H}$, $x \in E$, and linear combinations thereof. Since $\mathscr{S}$ is dense in $L^p(\Omega)$ and $L^p(\Omega) \otimes E$ is dense in $L^p(\Omega; E)$, it follows that $\mathscr{S} \otimes E$ is dense in $L^p(\Omega; E)$. For $F \in \mathscr{S} \otimes E$, define the *Malliavin derivative* DF as the random variable $DF : \Omega \to \gamma(H, \gamma(H, E))$ given by

$$DF = \sum_{i=1}^{n} \partial_i f(W(h_1), \dots, W(h_n)) \otimes (h_i \otimes x).$$

If $E = \mathbb{R}$, we can identify $\gamma(\mathscr{H}, \mathbb{R})$ with $\mathscr{H}$ and in that case for all $F \in \mathscr{S}$, $DF \in L^p(\Omega; \mathscr{H})$ coincides with the Malliavin derivative in [9]. Recall from [9, Proposition 1.2.1] that D is closable as an operator from $L^p(\Omega)$ into $L^p(\Omega; \mathscr{H})$, and this easily extends to the vector-valued setting (see [25, Proposition 3.3]).

Proposition 1.4.4 (Closability)**.** *For all* $1 \leqslant p < \infty$*, the Malliavin derivative* D *is closable as an operator from* $L^p(\Omega; E)$ *into* $L^p(\Omega; \gamma(\mathscr{H}, E))$.

Proof. Let $(F_n)_{n \geqslant 1}$ in $\mathscr{S} \otimes E$ and $G \in L^p(\Omega; \gamma(\mathscr{H}, E))$ be such that $\lim_{n\to\infty} F_n = 0$ in $L^p(\Omega; X)$ and $\lim_{n\to\infty} DF_n = G$ in $L^p(\Omega; \gamma(\mathscr{H}, E))$. We need to show that $G = 0$. Since G is strongly measurable, it suffices to check that for any $h \in \mathscr{H}$ and $x^* \in E^*$ one has $\langle Gh, x^* \rangle = 0$. By the closability of D in the scalar case one obtains

$$\langle Gh, x^* \rangle = \lim_{n\to\infty} \langle DF_n h, x^* \rangle = \lim_{n\to\infty} D(\langle F_n, x^* \rangle)(h) = 0.$$

□

The closure of the operator D is denoted by D again. The domain of the closure is denoted by $\mathbb{D}^{1,p}(E)$ and endowed with the norm

$$\|F\|_{\mathbb{D}^{1,p}(E)} := (\|F\|^p_{L^p(\Omega;E)} + \|DF\|^p_{L^p(\Omega;\gamma(\mathscr{H},E))})^{1/p}$$

it becomes a Banach space. Similarly, for $k \geqslant 2$ and $p \geqslant 1$ we let $\mathbb{D}^{k,p}(E)$ be the closure of $\mathscr{S} \otimes E$ with respect to the norm

$$\|F\|_{\mathbb{D}^{k,p}(E)} := (\|F\|^p_{L^p(\Omega;E)} + \sum_{i=1}^{k} \|D^i F\|^p_{L^p(\Omega;\gamma(\mathscr{H},E))})^{1/p}.$$

1.5 Maximal Regularity for autonomous problem in Hilbert space

Let $\mathscr{H}$ be an Hilbert space and A be a closed (unbounded) operator with domain $D(A)$ dense in $\mathscr{H}$. Let $f : [0, \infty[\to \mathscr{H}$ be a measurable function and $x \in \mathscr{H}$. We consider the problem of existence and regularity of solution to the following equation

$$\begin{cases} u'(t) + Au(t) = f(t) \\ \\ u(0) = x. \end{cases} \tag{1.5.1}$$

We define the maximal regularity space

$$MR(2, \mathscr{H}) = W^{1,2}(0, \infty; \mathscr{H}) \cap L^2(0, \infty; D(A))$$

endowed with norm

$$\|u\|_{MR(2,\mathscr{H})} = \|u\|_{W^{1,2}(0,\infty;\mathscr{H})} + \|Au\|_{L^2(0,\infty;\mathscr{H})}.$$

We define the associated trace space by

$$TR(2, \mathscr{H}) := \{u(0) : u \in MR(2, \mathscr{H})\},$$

with norm

$$\|x\|_{TR(2,\mathscr{H})} = \inf\{\|u\|_{MR(2,\mathscr{H})} : u \in MR(2, \mathscr{H}), u(0) = x\}.$$

Definition 1.5.1. *Let $p \in (1, \infty)$ We say that A has the (parabolic) L^p-maximal regularity property if there exists a constant $C > 0$ such that for all $f \in L^p(0, \infty; \mathscr{H})$ and $x \in TR(p, \mathscr{H})$, there is a unique $u \in MR(p, \mathscr{H})$ satisfying (1.5.1) for almost every $t \in [0, \infty[$ and*

$$\|u\|_{MR(p,\mathscr{H})} \leqslant C[\|x\|_{TR(p,\mathscr{H})} + \|f\|_{L^p(0,\infty;\mathscr{H})}].$$

Proposition 1.5.1. *See [56] If A has the maximal regularity property, then $-A$ generates a bounded holomorphic semigroup on $\mathscr{H}$.*

1.6 Maximal regularity for non-autonomous problems in $\mathcal{V}'$

We consider a family of sesquilinear forms

$$\mathfrak{a} : [0,\tau] \times \mathcal{V} \times \mathcal{V} \to \mathbb{C}.$$

We recall the following usual assumptions.

- [H1]: $\mathscr{D}(\mathfrak{a}(t)) = \mathcal{V}$ (constant form domain),
- [H2]: $|\mathfrak{a}(t,u,v)| \leqslant M\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}$ (uniform boundedness),
- [H3]: Re $\mathfrak{a}(t,u,u) + \nu\|u\|^2 \geqslant \delta\|u\|_{\mathcal{V}}^2$ $(\forall u \in \mathcal{V})$ for some $\delta > 0$ and some $\nu \in \mathbb{R}$ (uniform quasi-coercivity).

We suppose that $t \mapsto \mathfrak{a}(t,u,u)$ is measurable for all $u \in \mathcal{V}$. We denote by $A(t), \mathcal{A}(t)$ the usual associated operators with $\mathfrak{a}(t)$ as operators on $\mathscr{H}$ and $\mathcal{V}'$, respectively. In particular, $\mathcal{A}(t) : \mathcal{V} \to \mathcal{V}'$ as a bounded operator and

$$\mathfrak{a}(t,u,v) = \langle \mathcal{A}(t)u, v\rangle, \text{ for all } u, v \in \mathcal{V}.$$

The operator $A(t)$ is the part of $\mathcal{A}(t)$ on $\mathscr{H}$.

Theorem 1.6.1 (Lions' theorem)**.** *For every $f \in L^2(0,\tau;\mathcal{V}')$ and $u_0 \in \mathscr{H}$ there exists a unique $u \in MR_2(\mathcal{V},\mathcal{V}') = H^1(0,\tau;\mathcal{V}') \cap L^2(0,\tau;\mathcal{V})$ which solves the equation*

$$\begin{cases} u'(t) + \mathcal{A}(t)\,u(t) &= f(t),\ t \in (0,\tau] \\ u(0) &= u_0. \end{cases} \tag{P'}$$

Lions' proof is based on the following representation result

Proposition 1.6.1 (Lions' representation theorem)**.** *Let $\mathscr{H}$ be a Hilbert space, $\mathcal{V}$ a pre-Hilbert space such that $\mathcal{V} \hookrightarrow \mathscr{H}$. Let $E : \mathscr{H} \times \mathcal{V} \to \mathbb{C}$ be a sesquilinear such that*

- *For all $v \in \mathcal{V}, E(.,v)$ is a continuous linear functional on $\mathscr{H}$.*
- *$|E(v,v)| \geqslant \alpha\|v\|_{\mathcal{V}}^2$ for all $v \in \mathcal{V}$ and some $\alpha > 0$.*

Let $L \in \mathcal{V}'$. Then there exists $u \in \mathscr{H}$ such that

$$Lv = E(u,v)$$

for all $v \in \mathcal{V}$.

The previous proposition was proved in [27] (p. 61).

Lemma 1.6.2. *Let $\tau > 0$ and $u \in MR_2(\mathcal{V},\mathcal{V}') := H^1(0,\tau;\mathcal{V}') \cap L^2(0,\tau;\mathcal{V})$. We have $u \in C([0,\tau];\mathscr{H}) \cap H^{\frac{1}{2}}(0,\tau;\mathscr{H})$ and*

$$2Re \int_0^t \langle u'(s), u(s)\rangle ds = \|u(t)\|^2 - \|u(0)\|^2,$$

with $t \in [0,\tau]$.

Proof. By [54] (Theorem 1, p. 473) we obtain $MR_2(\mathcal{V},\mathcal{V}') \hookrightarrow C([0,\tau];\mathscr{H})$ and for all $u \in MR_2(\mathcal{V},\mathcal{V}')$ we have

$$2\mathrm{Re} \int_0^t \langle u'(s), u(s)\rangle ds = \|u(t)\|^2 - \|u(0)\|^2.$$

By [54] (Lemma 2, p. 473) there exists a continuous extension operator
$P : MR_2(\mathcal{V},\mathcal{V}') \to H^1(\mathbb{R};\mathcal{V}') \cap L^2(\mathbb{R};\mathcal{V})$. Now, let $u \in MR_2(\mathcal{V},\mathcal{V}')$ we get

$$\begin{aligned}\|Pu\|^2_{H^{\frac{1}{2}}(\mathbb{R};\mathscr{H})} &= \|Pu\|^2_{L^2(\mathbb{R};\mathscr{H})} + \int_{\mathbb{R}} \|\sqrt{|\xi|}\mathcal{F}Pu(\xi)\|^2 d\xi \\ &= \|Pu\|^2_{L^2(\mathbb{R};\mathscr{H})} + \int_{\mathbb{R}} \langle |\xi|\mathcal{F}Pu(\xi), \mathcal{F}Pu(\xi)\rangle d\xi \\ &\leqslant \|Pu\|^2_{L^2(\mathbb{R};\mathscr{H})} + \|Pu\|_{H^1(\mathbb{R};\mathcal{V}')}\|Pu\|_{L^2(\mathbb{R};\mathcal{V})} \\ &\leqslant 2\|Pu\|_{H^1(\mathbb{R};\mathcal{V}')\cap L^2(\mathbb{R};\mathcal{V})} \\ &\leqslant 2C\|u\|_{MR_2(\mathcal{V},\mathcal{V}')}.\end{aligned}$$

Since

$$\|u\|^2_{H^{\frac{1}{2}}(0,\tau;\mathscr{H})} \leqslant \|Pu\|^2_{H^{\frac{1}{2}}(\mathbb{R};\mathscr{H})}.$$

Thus, $MR_2(\mathcal{V},\mathcal{V}') \hookrightarrow H^{\frac{1}{2}}(0,\tau;\mathscr{H})$. □

Proof of Theorem 1.6.1. Let $\mathbb{H} = L^2(0,\tau;\mathcal{V})$ endowed with norm $\|g\|^2_{\mathbb{H}} = \int_0^\tau \|g(t)\|^2_{\mathcal{V}} dt$ and

$$\mathbb{V} = \{v \in L^2(0,\tau;\mathcal{V}) \cap H^1(0,\tau;\mathcal{V}') \text{ s.t } v(\tau) = 0\}$$

with norm

$$\|v\|^2_{\mathbb{V}} = \int_0^\tau \|v(t)\|^2_{\mathcal{V}} dt + \|v(0)\|^2.$$

For all $v \in \mathcal{V}$ from the equation $(\dot{u}(t), v(t)) + a(t, u(t), v(t)) = (f(t), v(t))$ which is equivalent to $\int_0^\tau [(\dot{u}(t), v(t)) + a(t, u(t), v(t))]dt = \int_0^\tau (f(t), v(t))dt$ integration by parts gives

$$(u(t), v(t))\Big|_0^\tau - \int_0^\tau (u(t), \dot{v}(t))dt + \int_0^\tau a(t, u(t), v(t))dt = \int_0^\tau (f(t), v(t))dt \tag{1.6.1}$$

hence

$$[(u(\tau), v(\tau)) - (u(0), v(0))] - \int_0^\tau (u(t), \dot{v}(t))dt + \int_0^\tau a(t, u(t), v(t))dt = \int_0^\tau (f(t), v(t))dt$$

Finally

$$-(u(0),v(0))-\int_0^\tau (u(t),\dot v(t))dt+\int_0^\tau a(t,u(t),v(t))dt=\int_0^\tau (f(t),v(t))dt$$

the last equation is equivalent to

$$\underbrace{-\int_0^\tau (u(t),\dot v(t))dt+\int_0^\tau a(t,u(t),v(t))dt}_{E(u,v)}=\underbrace{(u(0),v(0))+\int_0^\tau (f(t),v(t))dt}_{L(v)}$$

because $v(\tau)=0$ according to the definition of the space $\mathbb{V}$.

We set

$$E(u,v)=\int_0^\tau \mathfrak{a}(t,u(t),v(t))-\langle u(t),\dot v(t)\rangle dt$$

and

$$L(v)=\int_0^\tau \langle f(t),v(t)\rangle dt+(u_0,v(0)).$$

For all $v\in\mathcal{V}$, is clear that the form $u\mapsto E(u,v)$ is continuous on $\mathscr{H}$.
For $v\in\mathcal{V}$ we have

$$\text{Re } E(v,v)=\int_0^\tau \mathfrak{a}(t,v(t),v(t))dt-\int_0^\tau \frac{d}{dt}\|v(t)\|^2dt \tag{1.6.2}$$

$$\geqslant \min\{\delta,1\}(\int_0^\tau \|v(t)\|_{\mathcal{V}}^2dt+\|v(0)\|^2). \tag{1.6.3}$$

Finally, it easy to show that $v\mapsto L(v)$ is continuous on $\mathcal{V}$. Therefore by applying Proposition 1.6.1, there exists $u\in\mathscr{H}$ such that $E(u,v)=Lv$ for all $v\in\mathcal{V}$, i.e. a solution of the Problem (P').
By Lemma 1.6.2, we get

$$\begin{aligned}&\|u(t)\|^2-\|u(0)\|^2+2\delta\int_0^\tau \|u(t)\|_{\mathcal{V}}^2dt\\&\leqslant 2\text{Re }\int_0^t\langle \dot u(t),u(t)\rangle dt+2\text{Re }\int_0^t \mathfrak{a}(t,u(t),u(t))dt\\&=2\text{Re }\int_0^\tau \langle f(t),u(t)\rangle dt\\&\leqslant \frac{1}{\delta}\int_0^\tau \|f(t)\|_{\mathcal{V}'}^2dt+\delta\int_0^\tau \|u(t)\|_{\mathcal{V}}^2dt.\end{aligned}$$

Therefore

$$\|u(t)\|^2+\delta\int_0^\tau \|u(t)\|_{\mathcal{V}}^2dt\leqslant \|u(0)\|^2+\frac{1}{\delta}\int_0^\tau \|f(t)\|_{\mathcal{V}'}^2dt.$$

So that, there exists a constant $C>0$, such that

$$\|u\|_{C([0,\tau];\mathscr{H})}+\|u\|_{L^2(0,\tau;\mathcal{V})}\leqslant C(\|u(0)\|^2+\|f\|_{L^2(0,\tau;\mathcal{V}')}).$$

For the uniqueness we suppose there are two solutions u_1,u_2 and we set $w=u_1-u_2$. So w is the solution of the Problem (P') with $f=u_0=0$. Then by the previous estimate we have $w=0$ and $u_1=u_2$. □

1.7 Weighted spaces

Properties of the weighted spaces

In this section we briefly recall the definitions and we give the basic properties of vector-valued function spaces with temporal weights.
Let $(X, \|\cdot\|_X)$ be a Banach space over $\mathbb{R}$ or $\mathbb{C}$. For $-1 < \beta < 1$ we set $L^2_\beta(0,\tau;X) = L^2(0,\tau,t^\beta dt;X)$, endowed with the norm

$$\|u\|^2_{L^2_\beta(0,\tau,X)} = \int_0^\tau \|u(t)\|^2_X t^\beta \, dt.$$

It's very seen that $L^2_\beta(0,\tau;X) \hookrightarrow L^1_{loc}(0,\tau;X)$. Indeed, for all $u \in L^2_\beta(0,\tau;X)$, we have

$$\int_0^\tau \|u(t)\|_X \, dt \leqslant (\int_0^\tau t^{-\beta} \, dt)^{\frac{1}{2}} \|u\|_{L^2_\beta(0,\tau;X)}.$$

It clearly holds $L^2(0,\tau;X) \hookrightarrow L^2_\beta(0,\tau;X)$ for $\beta > 0$ and $L^2_\beta(0,\tau;X) \hookrightarrow L^2(0,\tau;X)$ for $\beta < 0$.
We define the corresponding weighted Sobolev spaces $W^{1,2}_\beta(0,\tau;X) := \{u \in W^{1,1}(0,\tau;X) \text{ s.t } u, \ \dot{u} \in L^2_\beta(0,\tau;X)\}$,

$$W^{1,2}_{\beta,0}(0,\tau;X) := \{u \in W^{1,2}_\beta(0,\tau;X), \ \text{s.t } u(0) = 0\},$$

which are Banach spaces endowed with norms, respectively

$$\|u\|^2_{W^{1,2}_\beta(0,\tau;X)} = \|u\|^2_{L^2_\beta(0,\tau;X)} + \|\dot{u}\|^2_{L^2_\beta(0,\tau;X)},$$

$\|u\|^2_{W^{1,2}_{\beta,0}(0,\tau;X)} = \|\dot{u}\|^2_{L^2_\beta(0,\tau;X)}$. We set also

$$L^\infty_\beta(0,\tau;X) := \{u \in L^1(0,\tau;X) \text{ s.t } s \to s^{\frac{\beta}{2}} u(s) \in L^\infty(0,\tau;X)\},$$

endowed with the norm $\|u\|_{L^\infty_\beta(0,\tau;X)} = \|.^{\frac{\beta}{2}} u(.)\|_{L^\infty(0,\tau;X)}$. We define also the fractional weighted Sobolev space $W^{s,2}_\beta(0,\tau;X)$, where $W^{s,2}_\beta(0,\tau;X) = (L^2_\beta(0,\tau;X); W^{1,2}_\beta(0,\tau;X))_{s,2}$, endowed with norm

$$\|u\|^2_{W^{s,2}_\beta(0,\tau;X)} = \|u\|^2_{L^2_\beta(0,\tau;X)} + \int_0^\tau \int_0^t \frac{\|u(t)-u(s)\|^2_X}{|t-s|^{1+2s}} s^\beta \, ds \, dt,$$

with $s \in (0,1)$. Here $(.;.)_{s,2}$ is the real interpolation space. For more details see [44](2.6).

Lemma 1.7.1 (Weighted Hardy inequality)**.** *For all $f \in L^2_\beta(0,\tau,X)$, we have the following inequality*

$$\int_0^\tau (\frac{1}{t} \int_0^t \|f(s)\|_X \, ds)^2 t^\beta \, dt \lesssim \|f\|^2_{L^2_\beta(0,\tau;X)}.$$

The Lemma 1.7.1 is proved in ([26], Lemma 6).

Proposition 1.7.2. *We have the following properties*

1- (a) *For $p > 2$ and $\beta > \frac{2}{p} - 1$, we have $L^p(0,\tau;X) \hookrightarrow L^2_\beta(0,\tau,X)$,*

(b) *For $p < 2$ and $\beta < \frac{2}{p} - 1$, we obtain $L^2_\beta(0,\tau,X) \hookrightarrow L^p(0,\tau;X)$.*

2- *For all $u \in L^2_\beta(0,\tau,X)$, we have $t \to v(t) = \frac{1}{t}\int_0^t u(s)\,ds \in L^2_\beta(0,\tau,X)$.*

3- *We define the operator $\Phi : L^2_\beta(0,\tau;X) \to L^2(0,\tau;X)$, such that $(\Phi f)(t) = t^{\frac{\beta}{2}} f(t)$ for all $f \in L^2_\beta(0,\tau;X)$ and $t \in [0,\tau]$. Then Φ is an isometric isomorphism.*
Note also that $\Phi \in \mathcal{L}(L^2(0,\tau;X), L^2_{-\beta}(0,\tau;X))$ and $\Phi \in \mathcal{L}(W^{1,2}_{\beta,0}(0,\tau;X), W^{1,2}_0(0,\tau;X))$.

4- *We have $W^{1,2}_{\beta,0}(0,\tau;X) \hookrightarrow L^2_{\beta-2}(0,\tau;X)$.*

5- *$L^2_{-\beta}(0,\tau;\mathcal{V}')$ is the dual space of $L^2_\beta(0,\tau;X)$ by the duality defined in $L^2(0,\tau;\mathscr{H})$.*

6- *If $u \in W^{1,2}_\beta(0,\tau;X)$, we obtain that u has a continuous extension on X and we have*

$$W^{1,2}_\beta(0,\tau;X) \hookrightarrow C([0,\tau];X).$$

7- *$C^\infty_c((0,\tau);X)$ is dense in $L^2_\beta(0,\tau;X)$ and $C^\infty([0,\tau];X)$ is dense in $W^{s,2}_\beta(0,\tau;X)$ for all $s \in [0,1]$.*

Proof. 1- (a) Let $p > 2$ and $\beta > \frac{2}{p} - 1$, we set $p' = \frac{p}{2} > 1$, $\frac{1}{p'} + \frac{1}{q} = 1$ and this implies that $q = \frac{p}{p-2}$. By using the Hölder's inequality and the condition above we get

$$\begin{aligned}
\|u\|^2_{L^2_\beta(0,\tau;X)} &= \int_0^\tau \|u(t)\|^2_X t^\beta\,dt \\
&\leqslant \left(\int_0^\tau \|u(t)\|^p_X\,dt\right)^{\frac{2}{p}} \left(\int_0^\tau t^{\frac{\beta p}{p-2}}\,dt\right)^{\frac{p-2}{p}} \\
&= \left(\frac{1}{\frac{\beta p}{p-2}+1}\tau^{\frac{\beta p}{p-2}+1}\right)^{\frac{p-2}{p}} \|u\|^2_{L^p(0,\tau;X)}.
\end{aligned}$$

(b) Similarly, by the above applied to the case $p' = \frac{2}{p} > 1$, we have

$$\begin{aligned}
\|u\|^p_{L^p(0,\tau;X)} &= \int_0^\tau \|u(t)\|^p_X t^{-\frac{\beta p}{2}} t^{\frac{\beta p}{2}}\,dt \\
&\leqslant \left(\int_0^\tau \|u(t)\|^2_X t^\beta\,dt\right)^{\frac{p}{2}} \left(\int_0^\tau t^{\frac{\beta p}{p-2}}\,dt\right)^{\frac{2-p}{2}} \\
&= C\|u\|^p_{L^2_\beta(0,\tau;X)}.
\end{aligned}$$

2- Using the previous Hardy inequality we have
$\|v\|^2_{L^2_\beta(0,\tau;X)} = \int_0^\tau \|\frac{1}{t}\int_0^t u(t)\,ds\|^2_X t^\beta\,dt \lesssim \|u\|^2_{L^2_\beta(0,\tau;X)}$.
Now, since $u \in L^2_\beta(0,\tau;X)$, we get the result.

3- Note that $\|\Phi f\|_{L^2(0,\tau;X)} = \|f\|_{L^2_\beta(0,\tau;X)}$ and we have
$\Phi^{-1} : L^2(0,\tau;\mathscr{H}) \to L^2_\beta(0,\tau;X)$ such that $(\Phi^{-1}g)(t) = t^{-\frac{\beta}{2}}g(t)$ for all $g \in L^2(0,\tau;X)$.

4- Let $u \in W^{1,2}_\beta(0,\tau;X)$ such that $u(0)=0$. We write $u(t) = \int_0^t \dot{u}(l)\,dl$. Then

$$\|u(t)\|^2_X t^{\beta-2} = \|\int_0^t \dot{u}(l)\,dl\|^2_X t^{\beta-2}.$$

This implies that

$$\begin{aligned}
\|u\|^2_{L^2_{\beta-2}(0,\tau;X)} &= \int_0^\tau \|u(t)\|^2_X t^{\beta-2}\,dt \\
&= \int_0^\tau \frac{1}{t^2}\|\int_0^t \dot{u}(s)\,ds\|^2_X t^\beta\,dt \\
&\leqslant \int_0^\tau \Big(\frac{1}{t}\int_0^t \|\dot{u}(s)\|_X\,ds\Big)^2 t^\beta\,dt \\
&\lesssim \|\dot{u}\|_{L^2_\beta(0,\tau;X)} \leqslant \|u\|_{W^{1,2}_\beta(0,\tau;X)},
\end{aligned}$$

where here we used the Hardy inequality (1.7.1).

5- Use simple functions in $L^2_{-\beta}(0,\tau;X)$ which norm simple functions in $L^2_\beta(0,\tau;X)$ and the Cauchy-Schwartz inequality (the proof is similar to the non weighted case see ([28], p.98)).

6- Let $u \in W^{1,2}_\beta(0,\tau;X)$ and for $(t,s) \in [0,\tau]^2$. We obtain

$$\begin{aligned}
\|u(t)-u(s)\|_X &= \|\int_s^t \dot{u}(l)\,dl\|_X \\
&\leqslant \Big(\int_s^t l^{-\beta}\,dl\Big)^{\frac{1}{2}}\|\dot{u}\|_{L^2_\beta(0,\tau;X)} \\
&= \frac{1}{\sqrt{1-\beta}}\Big(t^{-\beta+1} - s^{-\beta+1}\Big)^{\frac{1}{2}}\|\dot{u}\|_{L^2_\beta(0,\tau;X)}.
\end{aligned}$$

By letting $s \to t$ we get $u(s) \to u(t)$ in X. Therefore u has a continuous extension on X. Thus we can always identify a function in $W^{1,2}_\beta(0,\tau;X)$ by its continuous representative.

7- First we note that $C^\infty_c((0,\tau);X)$ is dense $L^2(0,\tau;X)$. Then for all $f \in L^2_\beta(0,\tau;X)$ and for any given $\varepsilon > 0$ there thus exists a function $\psi \in C^\infty_c((0,\tau);X)$ such that

$$\|(\Phi f) - \psi\|^2_{L^2(0,\tau;X)} \leqslant \varepsilon.$$

It follows that

$$\begin{aligned}&\|f-(\Phi^{-1}\psi)\|^2_{L^2_\beta(0,\tau;X)}\\&\leqslant\|\Phi\|_{\mathcal{L}(L^2_\beta(0,\tau;X);L^2(0,\tau;X))}\|(\Phi f)-\psi\|^2_{L^2(0,\tau;X)}\\&\leqslant\varepsilon.\end{aligned}$$

Therefore $C_c^\infty((0,\tau);X)$ is dense in $L^2_\beta(0,\tau;X)$.
As in ([21], Theorem 2.9.1) for the scalar-valued case, one sees that the space of all function f in $C^\infty([0,\tau];X)$ such that $f(0)=0$, is dense in $W_0^{1,2}(0,\tau;X)$. Then for all $g\in W^{1,2}_{\beta,0}(0,\tau;X)$ and $\epsilon>0$ there exists $\phi\in C^\infty([0,\tau];X)$ with $\phi(0)=0$ such that

$$\|\phi-\Phi g\|^2_{W^{1,2}(0,\tau;X)}\leqslant\varepsilon.$$

Then $\|\Phi^{-1}\phi-g\|^2_{W^{1,2}_\beta(0,\tau;X)}\leqslant\varepsilon$. This shows that the space of all function f in $C^\infty([0,\tau];X)$ such that $f(0)=0$, is dense in $W^{1,2}_{\beta,0}(0,\tau;X)$. Let $f\in W^{1,2}_\beta(0,\tau;X)$ and $\phi\in C^\infty([0,\tau];X)$ such that $\phi(0)=f(0)$. Then $f-\phi\in W^{1,2}_{\beta,0}(0,\tau;X)$ and there is $\xi\in C^\infty([0,\tau];X)$ with $\xi(0)=0$, such that $\|f-\xi-\phi\|^2_{W^{1,2}_\beta(0,\tau;X)}\leqslant\varepsilon$. Since $\xi+\phi\in C^\infty([0,\tau];X)$, then $C^\infty([0,\tau];X)$ is dense in $W^{1,2}_\beta(0,\tau;X)$.
Since $C^\infty([0,\tau];X)$ is dense in $W^{1,2}_\beta(0,\tau;X)$ and

$$W^{s,2}_\beta(0,\tau;X)=(L^2_\beta(0,\tau;X);W^{1,2}_\beta(0,\tau;X))_{s,2},$$

we obtain that $C^\infty([0,\tau];X)$ is also dense in $W^{s,2}_\beta(0,\tau;X)$ by ([21], p.39).

□

From now we assume without loss of generality that the forms are coercive, that is $[H3]$ [1.6] holds with $\nu=0$. The reason is that by replacing $A(t)$ by $A(t)+\nu$, the solution v of (P) is $v(t)=e^{\nu t}u(t)$ and it is clear that $u\in W^{1,2}_\beta(0,\tau;\mathscr{H})$ if and only if $v\in W^{1,2}_\beta(0,\tau;\mathscr{H})$.
In this chapter we suppose that $\mathscr{D}(A(t)^{\frac12})=\mathcal{V}$, for all $t\in[0,\tau]$ and there exist $c_1,c^1>0$ such that for all $v\in\mathcal{V}$

$$c_1\|v\|_{\mathcal{V}}\leqslant\|A(t)^{\frac12}v\|\leqslant c^1\|v\|_{\mathcal{V}}.\tag{1.7.1}$$

So that for all $s,t\in[0,\tau]$

$$\frac{c_1}{c^1}\|A(s)^{\frac12}v\|\leqslant\|A(t)^{\frac12}v\|\leqslant\frac{c^1}{c_1}\|A(s)^{\frac12}v\|,$$

this also holds for the adjoint-operator and we have

$$\frac{c_1}{c^1}\|A^*(s)^{\frac12}v\|\leqslant\|A^*(t)^{\frac12}v\|\leqslant\frac{c^1}{c_1}\|A^*(s)^{\frac12}v\|.$$

Proposition 1.7.1. *The solution of problem* (P) *is unique.*

Proof. We suppose that there are two solutions u_1, u_2 to Problem (P). Obviously, $v = u_1 - u_2$ satisfies

$$\begin{aligned} \dot{v}(t) + \mathcal{A}(t)v(t) = 0 \\ v(0) = 0. \end{aligned} \tag{1.7.2}$$

Then for all $t \in [0, \tau]$ we have

$$2 \operatorname{Re} \int_0^t (\dot{v}(s), v(s)) s^\beta \, ds + 2 \operatorname{Re} \int_0^t (\mathcal{A}(s)v(s), v(s)) s^\beta \, ds = 0.$$

Integration by parts gives

$$t^\beta \|v(t)\|^2 - \beta \int_0^t \|v(s)\|^2 s^{\beta-1} \, ds + 2\delta \int_0^t \|v(s)\|_{\mathcal{V}}^2 s^\beta \, ds \leqslant 0.$$

Indeed,

$$2 \operatorname{Re} \int_0^t (\dot{v}(s), v(s)) s^\beta \, ds = \operatorname{Re} \int_0^t \frac{d}{dt} \|v(s)\|^2 s^\beta ds$$

$$= s^\beta \|v(s)\| \Big|_0^t - \beta \int_0^t s^{\beta-1} \|v(s)\|^2 ds = t^\beta \|v(t)\|^2 - \beta \int_0^t s^\beta \|v(s)\|^2 ds \tag{1.7.3}$$

Since $\mathcal{A}$ is coercive we have,

$$2 \operatorname{Re} \int_0^t (\mathcal{A}(s)v(s), v(s)) s^\beta \, ds \geqslant 2\delta \int_0^t \|v(s)\|^2 \, ds \tag{1.7.4}$$

from 1.7.3 and 1.7.4 we get

$$\begin{aligned} 0 &= 2 \operatorname{Re} \int_0^t (\dot{v}(s), v(s)) s^\beta \, ds + 2 \operatorname{Re} \int_0^t (\mathcal{A}(s)v(s), v(s)) s^\beta \, ds \\ &\geqslant 2\delta \int_0^t \|v(s)\|^2 \, ds + t^\beta \|v(t)\|^2 - \beta \int_0^t s^\beta \|v(s)\|^2 ds \end{aligned}$$

this gives the desired result.

It is clear that for the case $\beta \leqslant 0$ we obtain $v(t) = 0$ for all $t \in [0, \tau]$. Therefore $u_1 = u_2$ and then the solution of Problem (P) is unique. For the case $\beta \geqslant 0$ we have

$$t^\beta \|v(t)\|^2 + \int_0^t \|v(s)\|^2 (2\delta C_{\mathscr{H}}^2 s^\beta - \beta s^{\beta-1}) \, ds \leqslant 0.$$

So for the case $t \leqslant \frac{2\delta C_{\mathscr{H}}^2}{\beta}$ we have $v(t) = 0$ for all $t \in [0, \frac{\delta C_{\mathscr{H}}^2}{\beta}]$. Now we proceed inductively to obtain $v = 0$ on $[0, \tau]$. □

We recall by S_θ the open sector $S_\theta = \{z \in \mathbb{C}^* : |arg(z)| < \theta\}$ with vertex 0. It is known that $-A(t)$ is sectorial operator and generates a bounded holomorphic semigroup on $\mathscr{H}$. The same is true for $-\mathcal{A}(t)$ on $\mathcal{V}'$. From [4] (Proposition 2.1), we have the following lemma which point out that the constants involved in the estimates are uniform with respect to t.

Note that this assumption is always true for symmetric forms and we get $c_1 = \sqrt{\delta}$ and $c^1 = \sqrt{M}$. We denote by S_θ the open sector $S_\theta = \{z \in \mathbb{C}^* : |arg(z)| < \theta\}$ with vertex 0.
The following lemma is proved in [4] (Proposition 2.1)

Lemma 1.7.3. *For any $t \in [0, \tau]$, the operators $-A(t)$ and $-\mathcal{A}(t)$ generate strongly continuous analytic semigroups of angle $\gamma = \frac{\pi}{2} - arctan(\frac{M}{\delta})$ on $\mathscr{H}$ and $\mathcal{V}'$.respectively. In addition, there exist constants C and C_θ, independent of t, such that*

1- $\|e^{-zA(t)}\|_{\mathcal{L}(\mathscr{H})} \leqslant 1$ *and* $\|e^{-z\mathcal{A}(t)}\|_{\mathcal{L}(\mathcal{V}')} \leqslant C$ *for all* $z \in S_\gamma$.

2- $\|A(t)e^{-sA(t)}\|_{\mathcal{L}(\mathscr{H})} \leqslant \frac{C}{s}$ *and* $\|\mathcal{A}(t)e^{-s\mathcal{A}(t)}\|_{\mathcal{L}(\mathcal{V}')} \leqslant \frac{C}{s}$ *for all* $s \in (0, \infty)$.

3- $\|e^{-sA(t)}\|_{\mathcal{L}(\mathscr{H},\mathcal{V})} \leqslant \frac{C}{\sqrt{s}}, s \in (0, \infty)$.

4- $\|(z - A(t))^{-1}\|_{\mathcal{L}(\mathscr{H},\mathcal{V})} \leqslant \frac{C_\theta}{\sqrt{|z|}}$ *and* $\|(z - \mathcal{A}(t))^{-1}\|_{\mathcal{L}(\mathcal{V}',\mathscr{H})} \leqslant \frac{C_\theta}{\sqrt{|z|}}$ *for all* $z \notin S_\theta$ *with fixed* $\theta > \frac{\pi}{2} - \gamma$.

The following lemma is proved in [3](Corollary 4.3.12)

Lemma 1.7.4. *Let H_1, H_2 be two Hilbert spaces, with $H_2 \subset H_1$, H_2 dense in H_1. Then for every $\theta \in (0, 1)$,*

$$[H_1, H_2]_\theta = (H_1, H_2)_{\theta,2},$$

with $\|u\|_{[H_1,H_2]_\theta} = C\|u\|_{(H_1,H_2)_{\theta,2}}$, where C is a positive constant independent of H_1 and H_2.

As a consequence from the previous Lemma and ([3], Theorem 4.2.6) we have that for all $\gamma \in (0, 1), t \in [0, \tau]$

$$(\mathscr{H}, \mathscr{D}(A(t)))_{\gamma,2} = [\mathscr{H}, \mathscr{D}(A(t))]_\gamma = \mathscr{D}(A(t)^\gamma).$$

Lemma 1.7.5. *For all $x \in (\mathscr{H}, \mathscr{D}(A(t)))_{\frac{1}{2},2}$, we get*

$$\int_0^\infty \|A(t)e^{-sA(t)}x\|^2 \, ds \leqslant C\|x\|^2_{(\mathscr{H},\mathscr{D}(A(t)))_{\frac{1}{2},2}},$$

where $C > 0$ is independent of t.

Proof. Note that $\|e^{-sA(t)}\|_{\mathcal{L}(\mathscr{H})} \leqslant 1$ and $\|sA(t)e^{-sA(t)}\|_{\mathcal{L}(\mathscr{H})} \leqslant M_1$, where M_1 is independent of t. Let $x \in (H, D(A(t)))_{\frac{1}{2},2}$. We write $x = a + b$, where $a \in \mathscr{H}$ and $b \in \mathscr{D}(A(t))$ to obtain

$$\begin{aligned} s^{\frac{1}{2}}\|A(t)e^{-sA(t)}x\| &\leqslant \inf_{x=a+b;\ a\in\mathscr{H},\ b\in\mathscr{D}(A(t))} M_1 s^{-\frac{1}{2}}\|a\| + s^{\frac{1}{2}}\|b\|_{\mathscr{D}(A(t))} \\ &\leqslant \max\{M_1, 1\}s^{-\frac{1}{2}} \inf_{x=a+b;\ a\in\mathscr{H},\ b\in\mathscr{D}(A(t))} \{\|a\| + s\|b\|_{\mathscr{D}(A(t))}\} \\ &\leqslant \max\{M_1, 1\}s^{-\frac{1}{2}} K(s, x; \mathscr{H}, \mathscr{D}(A(t))). \end{aligned}$$

So $\|A(t)e^{-sA(t)}x\| \leqslant \max\{M_1, 1\}s^{-1}K(s, x; \mathscr{H}, \mathscr{D}(A(t)))$, where

$$K(s, x; \mathscr{H}, \mathscr{D}(A(t))) = \inf_{x=a+b;\ a\in\mathscr{H},\ b\in\mathscr{D}(A(t))} \Big(\|a\| + s\|b\|_{\mathscr{D}(A(t))} \Big).$$

Since $\|x\|^2_{(\mathscr{H},\mathscr{D}(A(t)))_{\frac{1}{2},2}} = \int_0^\infty |K(s, x; \mathscr{H}, \mathscr{D}(A(t)))|^2 \frac{ds}{s^2}$ (See [3], Definition 1.1.1), then

$$\int_0^\infty \|A(t)e^{-sA(t)}x\|^2\, ds \leqslant \max\{M_1, 1\}\|x\|^2_{(\mathscr{H},\mathscr{D}(A(t)))_{\frac{1}{2},2}}.$$

Then we get the desired result. □

In the next lemma we prove the quadratic estimate, this lemma was proved in [40] with the assumption (1.7.1), here we prove this estimate without the assumption.

Lemma 1.7.6. *Let $x \in \mathscr{H}$ and $t \in [0, \tau]$. We have the following estimate*

$$\int_0^\tau \|A(t)^{\frac{1}{2}}e^{-sA(t)}x\|^2\, ds \leqslant c\|x\|^2,$$

where $c > 0$ is independent of t.

Proof. Note that by ([?], (A1) p. 269)

$$A(t)^{-\beta} = \frac{1}{\pi}\int_0^\infty \mu^{-\beta}(\mu + A(t))^{-1}\, d\mu.$$

Then by Lemma 1.7.3 one has $\|A(t)^{-\frac{1}{2}}\|_{\mathcal{L}(\mathscr{H})} \leqslant C'$, with $C' > 0$ independent of t.
Let $x \in \mathscr{H}$ and $t \in [0, \tau]$. We get by the previous lemma

$$\begin{aligned} \int_0^1 \|A(t)^{\frac{1}{2}}e^{-sA(t)}x\|^2 ds &= \int_0^1 \|A(t)e^{-sA(t)}A(t)^{-\frac{1}{2}}x\|^2\, ds \\ &\leqslant \|A(t)^{-\frac{1}{2}}x\|^2_{(\mathscr{H};\mathscr{D}(A(t)))_{\frac{1}{2},2}} \\ &= \|x\|^2_{\mathscr{H}} + \|A(t)^{-\frac{1}{2}}x\|^2 \\ &\leqslant (C'^2 + 1)\|x\|^2. \end{aligned}$$

□

Every $f \in L^2(0,t;\mathscr{H})$, defines an operator by putting

$$(R(t)f) = \int_0^t e^{-(t-s)A(t)} f(s)\, ds.$$

The next lemma shows that $R(t)$ is bounded in $\mathcal{L}(L^2(0,t;\mathscr{H}),\mathcal{V})$.

Lemma 1.7.7. *We have that for all $t \in [0,\tau]$, $R(t) \in \mathcal{L}(L^2(0,t;\mathscr{H}),\mathcal{V})$.*

The Lemma 1.7.7 is proved in ([40], Lemma 4.1).
We define the space

$$L^2_\beta(0,\tau;D(A(.))) = \{u \in L^2_\beta(0,\tau;\mathscr{H}) \text{ s.t } A(.)u \in L^2_\beta(0,\tau;\mathscr{H})\}$$

endowed with the norm

$$\|u\|_{L^2_\beta([0,\tau],D(A(.)))} = \|u\|_{L^2_\beta(0,\tau,\mathscr{H})} + \|A(.)u\|_{L^2_\beta(0,\tau,\mathscr{H})}.$$

Lemma 1.7.8. *We assume that $\mathcal{A}(.) \in C^\varepsilon([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}')), \varepsilon > 0$. Then for all $\lambda \in (0,\infty)$, we have $(\lambda + A(.))^{-1} \in C^\varepsilon([0,\tau];\mathcal{L}(\mathscr{H}))$ and $\|(\lambda + A(.))^{-1}\|_{C^\epsilon([0,\tau];\mathcal{L}(\mathscr{H}))} \leqslant \frac{C}{\lambda}$.*

Proof. Let $\lambda \in (0,\infty), t, s \in [0,\tau]$. We get

$$(\lambda + A(t))^{-1} - (\lambda + A(s))^{-1} = (\lambda + A(t))^{-1}(\mathcal{A}(t) - \mathcal{A}(s))(\lambda + A(s))^{-1}.$$

Therefore by the Lemma (1.7.3) we have

$$\begin{aligned}
&\|(\lambda + A(t))^{-1} - (\lambda + A(s))^{-1}\|_{\mathcal{L}(\mathscr{H})} \\
&\leqslant \|(\lambda + A(t))^{-1}\|_{\mathcal{L}(\mathcal{V}',\mathscr{H})}\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}\|(\lambda + A(s))^{-1}\|_{\mathcal{L}(\mathscr{H},\mathcal{V})} \\
&\leqslant C\frac{|t-s|^\varepsilon}{|\lambda|}.
\end{aligned}$$

□

Lemma 1.7.9. *We suppose that $\mathcal{A}(.) \in C^\epsilon([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))$, then $L^2_\beta(0,\tau;\mathscr{D}(A(.)))$ is dense in $L^2_\beta(0,\tau;\mathscr{H})$.*

Proof. Let $f \in L^2_\beta(0,\tau;\mathscr{H})$, and set $f_n(t) = n(n + A(t))^{-1}f(t)$ for $n \in \mathbb{N}$. Since $t \to (n + A(t))^{-1} \in C^\epsilon([0,\tau];\mathcal{L}(\mathscr{H}))$, then for all $n \in \mathbb{N}$ the function

$f_n : [0, \tau] \to \mathscr{H}$ is measurable and satisfies $f_n(t) \in \mathscr{D}(A(t))$ almost everywhere as well as $\|A(t)f_n(t)\| \leqslant Cn\|f(t)\|$. Moreover

$$\|f_n(t) - f(t)\| = \|\Big(n(n + A(t))^{-1} - I\Big)f(t)\|.$$

Hence, the convergence $f_n \to f$ in $L^2_\beta(0, \tau; \mathscr{H})$ holds by the dominated convergence theorem. □

Proposition 1.7.10. *Assume that $\mathcal{A}(.) \in C^\varepsilon([0, \tau]; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$, for some $\varepsilon > 0$. Then for all $f \in L^2_\beta(0, \tau; \mathscr{H})$, with $\beta < 1$ the operator L defined by*

$$(Lf)(t) = A(t) \int_0^t e^{-(t-s)A(t)} f(s)\, ds$$

is bounded in $L^2_\beta(0, \tau; \mathscr{H})$.

Proof. Let $f \in L^2_\beta(0, \tau; \mathscr{D}(A(.)))$. We split the integral into two parts to get

$$\begin{aligned}(Lf)(t) &= A(t) \int_0^{\frac{t}{2}} e^{-(t-s)A(t)} f(s)\, ds + A(t) \int_{\frac{t}{2}}^t e^{-(t-s)A(t)} f(s)\, ds \\ &:= I_1(t) + I_2(t).\end{aligned}$$

We begin by estimating the first integral

$$\begin{aligned}\|I_1(t)\| = \|A(t) \int_0^{\frac{t}{2}} e^{-(t-s)A(t)} f(s)\, ds\|_{\mathscr{H}} &\lesssim \int_0^{\frac{t}{2}} \frac{1}{t-s} \|f(s)\|\, ds \\ &\lesssim \frac{2}{t} \int_0^{\frac{t}{2}} \|f(s)\|\, ds.\end{aligned}$$

Therefore by Hardy inequality (See Lemma 1.7.1) we have

$$\begin{aligned}&\int_0^\tau \|A(t) \int_0^{\frac{t}{2}} e^{-(t-s)A(t)} f(s)\, ds\|^2_{\mathscr{H}} t^\beta\, dt \\ &\lesssim \int_0^\tau (\frac{2}{t} \int_0^{\frac{t}{2}} \|f(s)\|\, ds)^2 t^\beta\, dt \\ &\lesssim \|f\|^2_{L^2_\beta(0,\tau;\mathscr{H})}.\end{aligned}$$

As before we estimate the second integral, so for all $x \in \mathscr{H}$, we obtain

$$\begin{aligned}|\Big(I_2(t), x\Big)| &= |\int_{\frac{t}{2}}^t \Big(A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)} f(s), A(t)^{\frac{1}{2}*} e^{-\frac{1}{2}(t-s)A(t)^*} x\Big)\, ds| \\ &\leqslant \Big(\int_{\frac{t}{2}}^t \|A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)} f(s)\|^2\, ds\Big)^{\frac{1}{2}} \Big(\int_{\frac{t}{2}}^t \|A(t)^{\frac{1}{2}*} e^{-\frac{1}{2}(t-s)A(t)^*} x\|^2\, ds\Big)^{\frac{1}{2}}\end{aligned}$$

$$\lesssim^{(i)} \left(\int_{\frac{t}{2}}^{t} \|A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)} f(s)\|^2 \, ds\right)^{\frac{1}{2}} \|x\|.$$

In (i) we have used the quadratic estimate (1.7.6).
Taking the supremum over all $x \in \mathscr{H}$, we obtain the following estimate

$$\begin{aligned}\int_0^\tau t^\beta \|I_2(t)\|^2 \, dt &= \int_0^\tau t^\beta \|A(t) \int_{\frac{t}{2}}^{t} e^{-(t-s)A(t)} f(s) ds\|^2 \, dt \\ &\lesssim \int_0^\tau t^\beta \int_{\frac{t}{2}}^{t} \|A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)} f(s)\|^2 \, ds \, dt \\ &\lesssim \int_0^\tau \int_{\frac{t}{2}}^{t} \|A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)} \left(s^{\frac{\beta}{2}} f(s)\right)\|^2 \, ds \, dt.\end{aligned}$$

Let g be the function defined by $g = (\Phi f)$. Using the Fubini theorem and the inequality $(x+y)^2 \leqslant 2x^2 + 2y^2$, we obtain

$$\begin{aligned}&\int_0^\tau \int_{\frac{t}{2}}^{t} \|A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)} [s^{\frac{\beta}{2}} f(s)]\|^2 \, ds \, dt \\ &\leqslant 2 \int_0^\tau \int_{\frac{t}{2}}^{t} \|A(s)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(s)} g(s)\|^2 \, ds \, dt \\ &+ 2 \int_0^\tau \int_{\frac{t}{2}}^{t} \|\left(A(s)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(s)} - A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)}\right) g(s)\|^2 \, ds \, dt \\ &\leqslant 2 \int_0^\tau \int_{s}^{2s} \|A(s)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(s)} g(s)\|^2 \, dt \, ds \\ &+ 2 \int_0^\tau \int_{\frac{t}{2}}^{t} \|\left(A(s)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(s)} - A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)}\right) g(s)\|^2 \, ds \, dt \\ &\lesssim \|g\|^2_{L^2(0,\tau;\mathscr{H})} + \int_0^\tau \int_{\frac{t}{2}}^{t} \|\left(A(s)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(s)} - A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)}\right) g(s)\|^2 \, ds \, dt.\end{aligned}$$

The functional calculus for a sectorial operators gives

$$\begin{aligned}&A(s)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(s)} - A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)} \\ &= \int_\Gamma \lambda^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)\lambda} (\lambda - \mathcal{A}(t))^{-1} (\mathcal{A}(t) - \mathcal{A}(s)) (\lambda - \mathcal{A}(s))^{-1} \, d\lambda.\end{aligned}$$

Hence

$$\begin{aligned}&\|A(s)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(s)} - A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)}\|_{\mathcal{L}(\mathscr{H})} \\ &\leqslant \int_\Gamma |\lambda|^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)\mathrm{Re}\,\lambda} \|(\lambda - \mathcal{A}(t))^{-1}\|_{\mathcal{L}(\mathcal{V}',\mathscr{H})} \|(\mathcal{A}(t) - \mathcal{A}(s))\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} \|(\lambda - \mathcal{A}(s))^{-1}\|_{\mathcal{L}(\mathscr{H},\mathcal{V})} \, |d\lambda|.\end{aligned}$$

Therefore

$$\|A(s)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(s)} - A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)}\|_{\mathcal{L}(\mathscr{H})}$$

$$\leqslant \int_0^\infty |\lambda|^{-\frac{1}{2}} e^{-\frac{1}{2}(t-s)cos(\gamma)\ |\lambda|}\, d|\lambda| \|(\mathcal{A}(t)-\mathcal{A}(s))\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}.$$

Then

$$\|A(s)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(s)} - A(t)^{\frac{1}{2}} e^{-\frac{1}{2}(t-s)A(t)}\|_{\mathcal{L}(\mathscr{H})} \lesssim \frac{\|\mathcal{A}(t)-\mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{(t-s)^{\frac{1}{2}}}.$$

Therefore

$$\begin{aligned}
&\int_0^\tau \int_{\frac{t}{2}}^t \|\Big(A(s)^{\frac{1}{2}} e^{-(t-s)A(s)} - A(t)^{\frac{1}{2}} e^{-(t-s)A(t)}\Big) g(s)\|^2\, ds\, dt \\
&\lesssim \int_0^\tau \int_{\frac{t}{2}}^t \frac{\|\mathcal{A}(t)-\mathcal{A}(s)\|^2_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t-s} \|g(s)\|^2\, ds\, dt \\
&\lesssim \sup_{s\in[0,\tau]} \int_s^\tau \frac{\|\mathcal{A}(t)-\mathcal{A}(s)\|^2_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t-s}\, dt \|g\|^2_{L^2(0,\tau;\mathscr{H})} \\
&\lesssim \tau^{2\epsilon} \|\mathcal{A}\|^2_{C^\varepsilon([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))} \|f\|^2_{L^2_\beta(0,\tau;\mathscr{H})}.
\end{aligned}$$

This completes the proof of the Proposition 1.7.10. □

Proposition 1.7.11. *For $\beta \geqslant 1$ we have that the operator L is not bounded in $L^2_\beta(0,\tau;\mathscr{H})$ in general.*

Proof. Let $u \in \mathcal{H}$ and $g \in L^2_{-\beta}(0,\tau;\mathcal{H})$. Noting that

$$(L^*g)(t) = \int_t^\tau A(s)^* e^{-(s-t)A(s)^*} g(s)\, ds, t \in (0,\tau)$$

and $L \in \mathcal{L}(L^2_\beta(0,\tau;\mathcal{H}))$ if and only if $L^* \in \mathcal{L}(L^2_{-\beta}(0,\tau;\mathcal{H}))$. If $A(s)^* = A(0)^*$ for all $s \in [0,\tau]$, then $(L^*g)(t) = \int_t^\tau A(0)^* e^{-(s-t)A(0)^*} g(s)\, ds$. Assume now that $t < 1 < \tau$ and take $g(s) = \mathbf{1}_{[1,\tau]}(s)u$, so

$$(L^*g)(t) = e^{-(1-t)A(0)^*}u - e^{-(\tau-t)A(0)^*}u,$$

which converges to $e^{-A(0)^*}u - e^{-\tau A(0)^*}u$ as $t \to 0$. We claim that

$$e^{-A(0)^*}u - e^{-\tau A(0)^*}u \neq 0,$$

then

$$\begin{aligned}
\|L^*g\|^2_{L^2_{-\beta}(0,\tau;\mathcal{H})} &\geqslant \|L^*g\|^2_{L^2_{-\beta}(0,1;\mathcal{H})} \\
&= \int_0^1 \|e^{-(1-t)A(0)^*}u - e^{-(\tau-t)A(0)^*}u\|^2 \frac{dt}{t^\beta} = \infty.
\end{aligned}$$

Now, suppose that $e^{-A(0)^*}u - e^{-\tau A(0)^*}u = 0$, thus

$$e^{-A(0)^*}u = e^{-(2\tau-1)A(0)^*}u.$$

Using induction, for all $n \in \mathbb{N}$ we obtain

$$e^{-A(0)^*}u - e^{-(n(\tau-1)+1)A(0)^*}u = 0.$$

Since $\|A(0)^* e^{-(n(\tau-1)+1)A(0)^*} A(0)^{*-1}u\| \lesssim \frac{1}{(n(\tau-1)+1)}\|A(0)^{*-1}u\|$, by letting $n \to \infty$ it follows that $e^{-A(0)^*}u = 0$. Hence $e^{-tA(0)^*}u = 0$ for all $t \geqslant 1$, and we deduce that $u = 0$ by an application of the isolated point theorem and the analyticity of the semigroup. □

Lemma 1.7.12. *For all $f \in L^2_\beta(0,\tau;\mathscr{H})$, $t \in [0,\tau]$ and $\beta < 1$, we have $(L_1 f)(t) \in \mathcal{V}$, where*

$$(L_1 f)(t) = t^{\frac{\beta}{2}} \int_0^t e^{-(t-s)A(t)} f(s)\, ds.$$

Remark 1.7.1. *The operator L is called the maximal regularity operator.*

Proof. We write

$$(L_1 f)(t) = t^{\frac{\beta}{2}} \int_0^{\frac{t}{2}} e^{-(t-s)A(t)} f(s)\, ds + t^{\frac{\beta}{2}} \int_{\frac{t}{2}}^t e^{-(t-s)A(t)} f(s)\, ds.$$

A straightforward computation gives

$$\begin{aligned} \|t^{\frac{\beta}{2}} \int_0^{\frac{t}{2}} e^{-(t-s)A(t)} f(s)\, ds\|_V &\lesssim t^{\frac{\beta}{2}} \int_0^{\frac{t}{2}} \|e^{-(t-s)A(t)}\|_{\mathcal{L}(\mathscr{H},\mathcal{V})} \|f(s)\|\, ds \\ &\lesssim t^{\frac{\beta}{2}} \left(\int_0^{\frac{t}{2}} s^{-\beta-1}\, ds \right)^{\frac{1}{2}} \|f\|_{L^2_\beta(0,\tau;\mathscr{H})} \\ &\lesssim \|f\|_{L^2_\beta(0,\tau;\mathscr{H})}. \end{aligned}$$

Using the Lemma (1.7.6), to deduce

$$\begin{aligned} \|t^{\frac{\beta}{2}} \int_{\frac{t}{2}}^t e^{-(t-s)A(t)} f(s)\, ds\|_V &\lesssim \| \int_{\frac{t}{2}}^t e^{-(t-s)A(t)} \left(s^{\frac{\beta}{2}} f(s) \right) ds\|_V \\ &\lesssim \|f\|_{L^2_\beta(0,\tau;\mathscr{H})}. \end{aligned}$$

Then we get the result. □

Lemma 1.7.13. *For all $u_0 \in (\mathscr{H};\mathscr{D}(A(0)))_{\frac{1-\beta}{2},2}$ and $\beta \in [0,1)$, we have*

$$\int_0^\tau \|t^{\frac{\beta}{2}} A(0) e^{-tA(0)} u_0\|^2_{\mathscr{H}}\, dt \simeq \|u_0\|^2_{(\mathscr{H};\mathscr{D}(A(0)))_{\frac{1-\beta}{2},2}}.$$

Proof. Note that $(\mathscr{H};\mathscr{D}(A(0)))_{\frac{1-\beta}{2},2} = \mathscr{D}(A(0)^{\frac{1-\beta}{2}})$. If $\beta \in [0,1)$, by using the quadratic estimate we obtain

$$\begin{aligned}&\int_0^\tau \|t^{\frac{\beta}{2}} A(0)e^{-tA(0)}u_0\|^2_{\mathscr{H}}\,dt\\ &= \int_0^\tau \|t^{\frac{\beta}{2}} A(0)^{\frac{1+\beta}{2}}e^{-tA(0)}A(0)^{\frac{1-\beta}{2}}u_0\|^2_{\mathscr{H}}\,dt\\ &\lesssim \int_0^\tau \|A(0)^{\frac{1}{2}}e^{-\frac{t}{2}A(0)}A(0)^{\frac{1-\beta}{2}}u_0\|^2_{\mathscr{H}}\,dt\\ &\lesssim \|A(0)^{\frac{1-\beta}{2}}u_0\|^2 = \|u_0\|^2_{[\mathscr{H};\mathscr{D}(A(0))]_{\frac{1-\beta}{2}}}\\ &\lesssim \|u_0\|^2_{(\mathscr{H};\mathscr{D}(A(0)))_{\frac{1-\beta}{2},2}}.\end{aligned}$$

Conversely, we know that (See [3], Definition 1.1.1)

$$\|u_0\|^2_{(\mathscr{H};\mathscr{D}(A(0)))_{\frac{1-\beta}{2},2}} = \int_0^1 t^{\beta-2}\|K(t,u_0)\|^2_{\mathscr{H}}\,dt,$$

where

$$K(t,u_0) = \inf_{u_0=a+b;a\in\mathscr{H},b\in\mathscr{D}(A(0))}\Big(\|a\|_{\mathscr{H}} + t\|b\|_{\mathscr{D}(A(0))}\Big).$$

This allows us to write for $t \in [0,\tau]$

$$\begin{aligned}u_0 &= \Big(u_0 - e^{-tA(0)}u_0\Big) + e^{-tA(0)}u_0\\ &= -\int_0^t A(0)e^{-lA(0)}u_0\,dl + e^{-tA(0)}u_0.\end{aligned}$$

Since $e^{-tA(0)}u_0 \in \mathscr{D}(A(0))$ and $\Big(u_0 - e^{-tA(0)}u_0\Big) \in \mathscr{H}$, it follows moreover that

$$\|K(t,u_0)\|_{\mathscr{H}} \leqslant \int_0^t \|A(0)e^{-lA(0)}u_0\|\,dl + t\|A(0)e^{-tA(0)}u_0\|.$$

Roughly speaking, by Hardy inequality (1.7.1), we have

$$\|u_0\|^2_{(\mathscr{H};\mathscr{D}(A(0)))_{\frac{1-\beta}{2},2}} \lesssim \int_0^\tau \|t^{\frac{\beta}{2}}A(0)e^{-tA(0)}u_0\|^2_{\mathscr{H}}\,dt.$$

This completes the proof of the Lemma. □

Remark 1.7.14. *As consequence of the previous lemma the orbit $t \mapsto e^{-tA(0)}u_0$ belongs to the space $W^{1,2}_\beta(0,\tau;\mathscr{H}) \cap L^2_\beta(0,\tau;\mathscr{D}(A(0)))$ if and only if $u_0 \in (\mathscr{H};\mathscr{D}(A(0)))_{\frac{1-\beta}{2},2}$.*

We define the space

$$W_\beta(\mathscr{D}(A(.)),\mathscr{H}) = \{u \in W^{1,1}(0,\tau;\mathscr{H}),\ \text{s.t } A(.)u \in L^2_\beta(0,\tau;\mathscr{H}),\ \dot{u} \in L^2_\beta(0,\tau;\mathscr{H})\},$$

endowed with the norm

$$\|u\|_{W_\beta(D(A(.),\mathscr{H})} = \|A(.)u\|_{L^2_\beta(0,\tau;\mathscr{H})} + \|\dot{u}\|_{L^2_\beta(0,\tau;\mathscr{H})}.$$

It is easy to see that $W_\beta(D(A(.),\mathscr{H}) \hookrightarrow W^{1,2}_\beta(0,\tau;\mathscr{H})$.

Lemma 1.7.15. *For all $\gamma \leqslant \frac{1}{2}$, we have $(\mathscr{H},\mathscr{D}(A(0)))_{\gamma,2} = [\mathscr{H},\mathcal{V}]_{2\gamma}$ and for $\gamma > \frac{1}{2}$ we get $(\mathscr{H},\mathscr{D}(A(0)))_{\gamma,2} \hookrightarrow \mathcal{V}$.*

Proof. As a consequence of the lation method (See [3], Remark 1.3.6) we have for $\gamma \leqslant \frac{1}{2}$,

$$(\mathscr{H},\mathscr{D}(A(0)))_{\gamma,2} = (\mathscr{H},D(A(0)^{\frac{1}{2}}))_{2\gamma,2} = (\mathscr{H},\mathcal{V})_{2\gamma,2}.$$

Since $\mathscr{H}$ and $\mathcal{V}$ are Hilbert spaces we get by the Lemma 1.7.4

$$(\mathscr{H},\mathscr{D}(A(0)))_{\gamma,2} = (\mathscr{H},\mathcal{V})_{2\gamma,2} = [\mathscr{H},\mathcal{V}]_{2\gamma}.$$

Let $v \in \mathscr{D}(A(0)$ and $\gamma > \frac{1}{2}$. We obtain

$$\begin{aligned}\delta\|v\|^2_{\mathcal{V}} &\leqslant \text{Re } (A(0)v,v)\\ &\lesssim \|A(0)^\gamma v\|_{\mathscr{H}}\|[A(0)^*]^{1-\gamma}v\|_{\mathscr{H}}\\ &\lesssim \|A(0)^\gamma v\|_{\mathscr{H}}\|v\|_{[\mathscr{H},\mathcal{V}]_{2(1-\gamma)}}\\ &\lesssim \|A(0)^\gamma v\|_{\mathscr{H}}\|v\|_{\mathcal{V}}.\end{aligned}$$

Therefore we have that for all $\gamma > \frac{1}{2}$ and $v \in \mathscr{D}(A(0))$

$$\|v\|_V \lesssim \|v\|_{\mathscr{D}(A(0)^\gamma)}.$$

Finally, by the density of $\mathscr{D}(A(0))$ in $\mathscr{D}(A(0)^\gamma)$ we get the desired result. □

Chapter 2

Maximal regularity for autonomous and non-autonomous evolution equations in weighted spaces

In this chapter, we will study the Maximal regularity property for autonomous problems, i.e. existence and uniqueness of the solution to the relevent problem (P) where the operators $A(t)$ are independent of t. Based on some complex interpolation techniques adapted in the theory of operator spaces, we are able to prove the existence and the uniqueness to the problem (P).
We discuss the regularity of the problem

$$\begin{aligned} \dot{u}(t) + \mathcal{A}(0)u(t) &= f(t) \\ u(0) &= u_0. \end{aligned} \tag{2.0.1}$$

2.0.1 Maximal regularity for autonomous problems in weighted spaces

The following is our main result in this section.

Theorem 2.0.2. *Let $f \in L^2_\beta(0, \tau, \mathscr{H})$ and $u_0 \in (\mathscr{H}; \mathscr{D}((A(0)))_{\frac{1-\beta}{2},2}$ for $\beta \geqslant 0$ and $u_0 = 0$ if $\beta < 0$. There exists a unique $u \in W_\beta(\mathscr{D}((A(0)), \mathscr{H}) \cap L^\infty_\beta(0, \tau; \mathcal{V})$ be the solution to Problem (2.0.1). Moreover, we have the following embeddings*

$$W_\beta(\mathscr{D}((A(0)), \mathscr{H}) \hookrightarrow C([0, \tau]; (\mathscr{H}; \mathscr{D}((A(0)))_{\frac{1-\beta}{2},2})$$

$$W_\beta(\mathscr{D}((A(0)), \mathscr{H}) \hookrightarrow W^{\frac{1}{2},2}_\beta(0, \tau; \mathcal{V}), \beta \in [0, 1[.$$

Proof. Since $A(0)$ is a generator of an analytic semigroup in $\mathscr{H}$, it is well known that by the variation of constants formula the solution of Problem (2.0.1) is

$$u(t) = e^{-tA(0)}u_0 + \int_0^t e^{-(t-s)A(0)} f(s)\, ds.$$

Thus,

$$\begin{aligned} A(0)u(t) &= A(0)e^{-tA(0)}u_0 + A(0)\int_0^t e^{-(t-s)A(0)} f(s)\, ds \\ &:= (Fu_0)(t) + (Lf)(t). \end{aligned}$$

Lemmas 1.7.12, 1.7.13 and Proposition 1.7.10 gives

$$\begin{aligned} \|A(0)u\|_{L^2_\beta(0,\tau;\mathscr{H})} &\leqslant \|Fu_0\|_{L^2_\beta(0,\tau;\mathscr{H})} + \|Lf\|_{L^2_\beta(0,\tau;\mathscr{H})} \\ &\leqslant C\Big(\|u_0\|_{(\mathscr{H};\mathscr{D}((A(0)))_{\frac{1-\beta}{2},2}} + \|f\|_{L^2_\beta(0,\tau;\mathscr{H})}\Big). \end{aligned}$$

Since $\dot{u} = f - A(0)u \in L^2_\beta(0,\tau;\mathscr{H})$, we obtain finally

$$\|u\|_{W_\beta(\mathscr{D}((A(0)),\mathscr{H})} \leqslant C'\Big(\|u_0\|_{(\mathscr{H};\mathscr{D}((A(0)))_{\frac{1-\beta}{2},2}} + \|f\|_{L^2_\beta(0,\tau;\mathscr{H})}\Big). \tag{2.0.2}$$

Using Proposition 2.0.4 and (2.0.2), for all $t \in [0,\tau]$ we obtain

$$\begin{aligned} \|u(t)\|_{(\mathscr{H};\mathscr{D}((A(0)))_{\frac{1-\beta}{2},2}} &\lesssim \|u\|_{W_\beta(\mathscr{D}((A(0)),\mathscr{H})\cap L^\infty_\beta(0,\tau;\mathcal{V})} \\ &\lesssim \|u_0\|_{(\mathscr{H};\mathscr{D}((A(0)))_{\frac{1-\beta}{2},2}} + \|f\|_{L^2_\beta(0,\tau;\mathscr{H})}. \end{aligned} \tag{2.0.3}$$

For $0 \leqslant s \leqslant l \leqslant t \leqslant \tau$, we set $v(l) = e^{-(t-l)A(0)}u(l)$. This yields

$$\begin{aligned} u(t) - u(s) &= v(s) - u(s) + \int_s^t \dot{v}(l)\, dl \\ &= (e^{-(t-s)A(0)} - I)u(s) + \int_s^t e^{-(t-l)A(0)} f(l)\, dl. \end{aligned} \tag{2.0.4}$$

Observe that $e^{-(t-s)A(0)}$ is strongly continuous on $(\mathscr{H};\mathscr{D}((A(0)))_{\frac{1-\beta}{2},2}$. In particular, this ensures that

$$\|(e^{-(t-s)A(0)} - I)u(s)\|_{(\mathscr{H};\mathscr{D}((A(0)))_{\frac{1-\beta}{2},2}} \to 0 \quad \text{as } t \to s.$$

The estimate (2.0.3) for the case $u_0 = 0$ gives that

$$\|\int_s^t e^{-(t-l)A(0)} f(l)\, dl\|_{(\mathscr{H};\mathscr{D}((A(0)))_{\frac{1-\beta}{2},2}} \lesssim \|f\|_{L^2_\beta(s,t;\mathscr{H})}.$$

It follows that $u(t)$ is right continuous on $(\mathscr{H};\mathscr{D}((A(0)))_{\frac{1-\beta}{2},2}$. Now, set $v(l) = e^{-(l-s)A(0)}u(l)$, for $0 \leqslant s \leqslant l \leqslant t$. Then

$$u(s) - u(t) = v(t) - u(t) - \int_s^t \dot{v}(l)\, dl$$

$$= (e^{-(t-s)A(0)} - I)u(t) - \int_s^t e^{-(l-s)A(0)}(f(l) - 2A(0)u(l))\,dl.$$

The same argument shows that u is left continuous in $(\mathscr{H}; \mathscr{D}((A(0)))_{\frac{1-\beta}{2},2}$.
Thus, $u \in C([0,\tau]; (\mathscr{H}; \mathscr{D}((A(0)))_{\frac{1-\beta}{2},2})$.

Now, we prove that $W_\beta(\mathscr{D}((A(0)), \mathscr{H}) \hookrightarrow W_\beta^{\frac{1}{2},2}(0,\tau;\mathcal{V})$.
Indeed, let $\beta \in [0,1[$ and $u \in C^\infty([0,\tau]; \mathscr{D}((A(0)))$. We recall that

$$\|u\|^2_{W_\beta^{\frac{1}{2},2}(0,\tau;\mathcal{V})} = \|u\|^2_{L^2_\beta(0,\tau;\mathcal{V})} + \int_0^\tau \int_0^t \frac{\|u(t)-u(s)\|^2_{\mathcal{V}}}{|t-s|^2} s^\beta\,ds\,dt.$$

By (2.0.4) it holds that for all $0 \leqslant s \leqslant t \leqslant \tau$

$$\begin{aligned} u(t) - u(s) &= (e^{-(t-s)A(0)}u(s) - u(s)) + \int_s^t e^{-(t-l)A(0)} f(l)\,dl \\ &:= L_1(t,s) + L_2(t,s), \end{aligned}$$

where $f(l) = A(0)u(l) + \dot{u}(l)$. So

$$\begin{aligned} \|u\|^2_{W_\beta^{\frac{1}{2},2}(0,\tau;\mathcal{V})} &\leqslant \|u\|^2_{L^2_\beta(0,\tau;\mathcal{V})} + 2\int_0^\tau \int_0^t \frac{\|L_1(t,s)\|^2_{\mathcal{V}}}{|t-s|^2} s^\beta\,ds\,dt \\ &\quad + 2\int_0^\tau \int_0^t \frac{\|L_2(t,s)\|^2_{\mathcal{V}}}{|t-s|^2} s^\beta\,ds\,dt. \end{aligned}$$

We write

$$L_1(t,s) = e^{-(t-s)A(0)}u(s) - u(s) = -\int_0^{t-s} e^{-lA(0)}A(0)u(s)\,dl.$$

Lemma 1.7.1 and the quadratic estimate gives

$$\begin{aligned} \int_0^\tau \int_0^t \frac{\|L_1(t,s)\|^2_{\mathcal{V}}}{|t-s|^2} s^\beta\,ds\,dt &\leqslant \int_0^\tau \int_s^\tau \left(\frac{\int_0^{t-s} \|e^{-lA(0)}A(0)u(s)\|_{\mathcal{V}} dl}{|t-s|} \right)^2 dt s^\beta\,ds \\ &\leqslant C \int_0^\tau \int_s^\tau \|e^{-tA(0)}A(0)u(s)\|^2_{\mathcal{V}}\,dt s^\beta\,ds \\ &\leqslant C' \int_0^\tau \|A(0)u(s)\|^2 s^\beta\,ds \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} \int_0^\tau \int_0^t \frac{\|L_2(t,s)\|^2_{\mathcal{V}}}{|t-s|^2} s^\beta\,ds\,dt &\leqslant \int_0^\tau \int_0^t \left(\frac{\int_s^t \|e^{(t-l)A(0)}(\Phi f)(l)\|_{\mathcal{V}}\,dl}{|t-s|} \right)^2 ds\,dt \\ &\leqslant C \int_0^\tau \int_0^t \|e^{(t-s)A(0)}(\Phi f)(s)\|^2_{\mathcal{V}}\,ds\,dt \end{aligned}$$

$$= C\int_0^\tau\int_s^\tau \|e^{(t-s)A(0)}(\Phi f)(s)\|_{\mathcal{V}}^2\,dt\,ds$$
$$\leqslant C\|\Phi f\|^2_{L^2(0,\tau;\mathscr{H})} = C\|f\|^2_{L^2_\beta(0,\tau;\mathscr{H})}.$$

Therefore,

$$\|u\|_{W_\beta^{\frac{1}{2},2}(0,\tau;\mathcal{V})} \lesssim \|A(0)u\|_{L^2_\beta(0,\tau;\mathscr{H})} + \|f\|_{L^2_\beta(0,\tau;\mathscr{H})} \lesssim \|u\|_{W_\beta(\mathscr{D}((A(0)),\mathscr{H})}.$$

We note that $C^\infty([0,\tau];\mathscr{D}((A(0)))$ is dense in $W_\beta(\mathscr{D}((A(0)),\mathscr{H})$. This shows that

$$W_\beta(\mathscr{D}((A(0)),\mathscr{H}) \hookrightarrow W_\beta^{\frac{1}{2},2}(0,\tau;\mathcal{V}).$$

which completes the proof □

Remark 2.0.2. The following embeddings hold

(1) $W_\beta(\mathscr{D}((A(0)),\mathscr{H}) \hookrightarrow C([0,\tau];[\mathscr{H},\mathcal{V}]_{1-\beta})$, for $0 \leqslant \beta < 1$.

(2) $W_\beta(\mathscr{D}((A(0)),\mathscr{H}) \hookrightarrow C([0,\tau];\mathcal{V})$, for $\beta \leqslant 0$.

Theorem 2.0.3. *For all $f \in W^{1,2}_{\beta,0}(0,\tau,\mathscr{H})$, there exists a unique*

$$u \in C^1([0,\tau];(\mathscr{H};\mathscr{D}((A(0)))_{\frac{1-\beta}{2},2}) \cap C([0,\tau];\mathscr{D}((A(0))),$$

which satisfies the equation

$$\begin{aligned}\dot{u}(t) + \mathcal{A}(0)u(t) &= f(t)\\ u(0) &= 0.\end{aligned} \tag{2.0.5}$$

In addition,

$$\|u\|_{C^1([0,\tau];(\mathscr{H};\mathscr{D}((A(0)))_{\frac{1-\beta}{2},2})\cap C([0,\tau];\mathscr{D}((A(0)))} \leqslant C\|f\|_{W_\beta^{1,2}(0,\tau;\mathscr{H})}.$$

Assume now that $\tau = +\infty$ and f is a periodic function with period p. Then u satisfies

$$u(t+p) = e^{-t\mathcal{A}(0)}u(p) + u(t),\ t \in [0,\infty),$$

and it is periodic with the same period p if and only if $u(p) = 0$.

Proof. According to Theorem 2.0.2, there exists a unique solution u to Problem (2.0.5) and for all $f \in L^2_\beta(0,\tau;\mathscr{H})$

$$u(t) = \int_0^t e^{-(t-s)\mathcal{A}(0)} f(s)\,ds, \quad t \in [0,\tau]. \tag{2.0.6}$$

Moreover $u \in W_\beta(\mathscr{D}((A(0)),\mathscr{H})$ and

$$\|u\|_{W_\beta(\mathscr{D}((A(0)),\mathscr{H})} \leqslant C\|f\|_{L^2_\beta(0,\tau;\mathscr{H})}. \tag{2.0.7}$$

Integrating by parts, we obtain for $t \in [0, \tau]$ and $f \in W^{1,2}_{\beta,0}(0, \tau, \mathscr{H})$

$$\begin{aligned} \mathcal{A}(0)u(t) &= \mathcal{A}(0) \int_0^t e^{-(t-s)\mathcal{A}(0)} f(s)\, ds \\ &= f(t) - \int_0^t e^{-(t-s)\mathcal{A}(0)} \dot{f}(s)\, ds \\ &= \dot{u}(t) + \mathcal{A}(0)u(t) - \int_0^t e^{-(t-s)\mathcal{A}(0)} \dot{f}(s)\, ds. \end{aligned}$$

Hence,

$$\dot{u}(t) = \int_0^t e^{-(t-s)\mathcal{A}(0)} \dot{f}(s)\, ds = (L\dot{f})(t).$$

Theorem 2.0.2 shows that $u \in C^1([0, \tau]; (\mathscr{H}; \mathscr{D}((A(0)))_{\frac{1-\beta}{2},2})$. Since $A(0)u = f - \dot{u}$ we deduce that $A(0)u \in C([0, \tau]; \mathscr{H})$. As a consequence, we obtain the final estimate

$$\|u\|_{C^1([0,\tau];(\mathscr{H};\mathscr{D}((A(0)))_{\frac{1-\beta}{2},2})\cap C([0,\tau];\mathscr{D}((A(0)))} \leqslant C\|f\|_{W^{1,2}_{\beta}(0,\tau;\mathscr{H})}.$$

Consider now the case where $\tau = +\infty$ and f is a periodic function with some period $p > 0$, i.e. $f(t + p) = f(t)$ for all $t \in [0, +\infty)$. It is clear that if u is periodic with period p, then $u(p) = u(0) = 0$. Formula (2.0.6) yields

$$u(t + p) = \int_0^{t+p} e^{-(t+p-s)\mathcal{A}(0)} f(s)\, ds.$$

Hence,

$$\begin{aligned} u(t + p) &= \int_0^p e^{-(t+p-s)\mathcal{A}(0)} f(s)\, ds + \int_p^{p+t} e^{-(t+p-s)\mathcal{A}(0)} f(s)\, ds \\ &= e^{-t\mathcal{A}(0)} \int_0^p e^{-(p-s)\mathcal{A}(0)} f(s)\, ds + \int_0^t e^{-(t-l)\mathcal{A}(0)} f(l + p)\, dl \\ &= e^{-t\mathcal{A}(0)} u(p) + u(t). \end{aligned}$$

In the previous equality, we made a change of variables, and in the last equality we used the periodicity of f. Then u is periodic with period p if and only if $e^{-t\mathcal{A}(0)}u(p) = 0$ for all $t \in [0, \infty)$. Therefore, the analyticity of the semigroup shows that $u(p) = 0$ is a necessary condition for u to be periodic. □

2.0.2 Maximal regularity for non-autonomous problems in weighted spaces

In this section we focus on the maximal regularity for the non-autonomous problem (which is our main aim), i.e. we prove the existence and the uniqueness of the solution to Problem P in the weighted space $W^{1,2}_\beta(0,\tau;\mathscr{H})$. We start by stating and proving some estimates which we will need in the proof of the main result.

Proposition 2.0.4. *(1) Assume that*

$$\int_0^\tau \frac{\|\mathcal{A}(t)-\mathcal{A}(0)\|^2_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t}\,dt<\infty.$$

Then for all $s\in[0,\tau]$,

$$\begin{aligned} TR_s : W_\beta(\mathscr{D}(A(\cdot)),\mathscr{H})\cap L^\infty_\beta(0,\tau;\mathcal{V}) &\longrightarrow (\mathscr{H};\mathscr{D}(A(s)))_{\frac{1-\beta}{2},2}\\ u &\longmapsto u(s)\end{aligned}$$

is a bounded operator.
(2) For $u_0\in(\mathscr{H};\mathscr{D}(A(0)))_{\frac{1-\beta}{2},2}$, *we have*

$$t\mapsto (Fu_0)(t)=t^{\beta/2}A(t)e^{-tA(t)}u_0\in L^2(0,\tau;\mathscr{H}).$$

Proof. (1) First we consider the case $s=0$. We have

$$\begin{aligned}
&\|u(0)\|^2_{(\mathscr{H};\mathscr{D}(A(0)))_{\frac{1-\beta}{2},2}}\\
&=\int_0^1\|t^{\beta/2}A(0)e^{-tA(0)}u(0)\|^2\,dt+\|u(0)\|^2\\
&\leqslant 2\int_0^1\|t^{\beta/2}A(0)e^{-tA(0)}(u(0)-u(t))\|^2\,dt+\|u(0)\|^2\\
&\quad+2\int_0^1\|t^{\beta/2}A(0)e^{-tA(0)}u(t)\|^2\,dt\\
&\lesssim\int_0^1 t^\beta\Big(\frac{1}{t}\int_0^t\|\dot u(s)\|\,ds\Big)^2dl+\int_0^\tau t^\beta\|A(t)u(t)\|^2\,dt\\
&\quad+\int_0^\tau\|t^{\beta/2}(A(0)e^{-tA(0)}-A(t)e^{-tA(t)})u(t)\|^2\,dt+\|u(0)\|^2\\
&\lesssim\|\dot u\|^2_{L^2_\beta(0,\tau;\mathscr{H})}+\|A(\cdot)u\|^2_{L^2_\beta(0,\tau;\mathscr{H})}\\
&\quad+\int_0^\tau\frac{\|\mathcal{A}(t)-\mathcal{A}(0)\|^2_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t}\,dt\|u\|_{L^\infty_\beta(0,\tau;\mathcal{V})}+\|u(0)\|^2\\
&\lesssim\|u\|^2_{W_\beta(\mathscr{D}(A(\cdot)),\mathscr{H})}+\|u\|^2_{L^\infty_\beta(0,\tau;\mathcal{V})}+\|u(0)\|^2,
\end{aligned}$$

where we have used the quadratic estimate, Hardy inequality and the estimate

$$\|A(0)e^{-tA(0)} - A(t)e^{-tA(t)}\|_{\mathcal{L}(\mathcal{V},\mathscr{H})} \lesssim \frac{\|\mathcal{A}(t) - \mathcal{A}(0)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t^{1/2}}.$$

Now, we prove the result for all $s \in]0,\tau]$. Indeed, let $t \in]0,\tau[$ and set

$$v(t) := \begin{cases} u(t+s), & t \in [0,\tau - s]. \\ u(\frac{\tau}{s}(\tau - t)), & t \in [\tau - s, \tau]. \end{cases}$$

Similarly,

$$B(t) := \begin{cases} A(t+s), & t \in [0,\tau - s]. \\ A(\frac{\tau}{s}(\tau - t)), & t \in [\tau - s, \tau]. \end{cases}$$

Since $v(t) \in W_\beta(\mathscr{D}(B(\cdot), \mathscr{H})$, therefore

$$v(0) = u(s) \in (\mathscr{H}; \mathscr{D}(B(0)))_{\frac{1-\beta}{2},2} = (\mathscr{H}; \mathscr{D}(A(s)))_{\frac{1-\beta}{2},2}.$$

For the case $s = \tau$, we take $v(t) = u(\tau - t)$ and $B(t) = A(\tau - t)$.

(2) Note that

$$\begin{aligned}(Fu_0)(t) &= t^{\beta/2} A(t) e^{-tA(t)} u_0 \\ &= t^{\beta/2}(A(t)e^{-tA(t)}u_0 - A(0)e^{-tA(0)}u_0) + t^{\beta/2} A(0) e^{-tA(0)} u_0.\end{aligned}$$

For $\beta > 0$ we have by interpolation

$$\|(\lambda - A(0))^{-1}\|_{\mathcal{L}((\mathscr{H};\mathscr{D}(A(0)))_{\frac{1-\beta}{2},2},\mathcal{V})} \lesssim \frac{1}{|\lambda|^{1-\frac{\beta}{2}}}.$$

Therefore

$$\|(Fu_0)(t)\| \lesssim \frac{\|\mathcal{A}(0) - \mathcal{A}(t)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t} \|u_0\|_{(\mathscr{H};\mathscr{D}(A(0)))_{\frac{1-\beta}{2}}} + \|t^{\beta/2} A(0) e^{-tA(0)} u_0\|.$$

Hence,

$$\begin{aligned}\|(Fu_0)\|^2_{L^2(0,\tau;\mathscr{H})} &\lesssim \int_0^\tau \frac{\|\mathcal{A}(0) - \mathcal{A}(t)\|^2_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t}\, dt \|u_0\|_{(\mathscr{H};\mathscr{D}(A(0)))_{\frac{1-\beta}{2},2}} \\ &\quad + \int_0^\tau \|t^{\beta/2} A(0) e^{-tA(0)} u_0\|^2 \, dt \\ &\lesssim \|u_0\|^2_{(\mathscr{H};\mathscr{D}(A(0)))_{\frac{1-\beta}{2},2}}.\end{aligned}$$

This shows the second assertion. □

In the sequel we consider only the case $\beta \in [0,1[$.

Proposition 2.0.5. *Suppose $\mathcal{A} \in C^{\varepsilon}([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))$. Then for each $f \in L^2_\beta(0,\tau;\mathscr{H})$, $u_0 \in (\mathscr{H};\mathscr{D}(A(0)))_{\frac{1-\beta}{2},2}$ and for τ small enough, there exists a unique solution u in $L^\infty_\beta(0,\tau;\mathcal{V})$ for (P).*

Proof. Let $f \in L^2_\beta(0,\tau;\mathscr{H})$. We set $v(s) = e^{-(t-s)A(t)}u(s)$. Since $u(t) = e^{-tA(t)}u_0 + \int_0^t \dot{v}(s)\,ds$, therefore

$$\begin{aligned} u(t) &= e^{-tA(t)}u_0 + \int_0^t e^{-(t-s)A(t)}(\mathcal{A}(t)-\mathcal{A}(s))u(s)\,ds \\ &\quad + \int_0^t e^{-(t-s)A(t)} f(s)\,ds \\ &:= (Mu_0)(t) + (M_1u)(t) + (L_1 f)(t). \end{aligned} \tag{2.0.8}$$

For $\beta > 0$ and $u_0 \in (\mathscr{H},\mathscr{D}(A(0)))_{\frac{1-\beta}{2},2}$ we have by interpolation

$$\|e^{-tA(t)}u_0\|_{\mathcal{V}} \lesssim t^{-\beta/2}\|u_0\|_{(\mathscr{H},\mathscr{D}(A(0)))_{\frac{1-\beta}{2},2}}. \tag{2.0.9}$$

In view of Lemma 1.7.12 and (2.0.9), $t^{\beta/2}(Mu_0)(t)$, $t^{\beta/2}(L_1f)(t)$ are bounded in $\mathcal{V}$ for all $t \in [0,\tau]$. Now, we show that $M_1u \in L^\infty_\beta(0,\tau;\mathcal{V})$ for all $u \in L^\infty_\beta(0,\tau;\mathcal{V})$. We write

$$\begin{aligned} (M_1u)(t) &= \int_0^{t/2} e^{-(t-s)A(t)}(\mathcal{A}(t)-\mathcal{A}(s))u(s)\,ds \\ &\quad + \int_{t/2}^t e^{-(t-s)A(t)}(\mathcal{A}(t)-\mathcal{A}(s))u(s)\,ds \\ &:= (M_{11}u)(t) + (M_{12}u)(t). \end{aligned}$$

By taking $x \in \mathcal{V}'$ we obtain

$$\begin{aligned} &|((M_{12}u)(t),x)_{\mathcal{V}'\times\mathcal{V}}| \\ &= \Big|\int_{t/2}^t (e^{-\frac{(t-s)}{2}A(t)}(\mathcal{A}(t)-\mathcal{A}(s))u(s), A(t)^{*\frac{1}{2}}e^{-\frac{(t-s)}{2}A(t)^*}A(t)^{*-\frac{1}{2}}x)\,ds\Big| \\ &\leqslant \Big(\int_{t/2}^t \|e^{-\frac{(t-s)}{2}A(t)}\|^2_{\mathcal{L}(\mathcal{V}',\mathscr{H})}\|[\mathcal{A}(t)-\mathcal{A}(s)]u(s)\|^2_{\mathcal{V}'}\,ds\Big)^{1/2} \\ &\quad\times \Big(\int_{t/2}^t \|A(t)^{*\frac{1}{2}}e^{-\frac{(t-s)}{2}A(t)^*}A(t)^{*-\frac{1}{2}}x\|^2\,ds\Big)^{1/2}. \end{aligned}$$

Now, we estimate the norm of $(M_{11}u)(t)$ in $\mathcal{V}$ as follows

$$\begin{aligned} &t^{\beta/2}\|(M_{11}u)(t)\|_{\mathcal{V}} \\ &\lesssim t^{\beta/2}\int_0^{t/2}\|e^{-\frac{(t-s)}{2}A(t)}\|_{\mathcal{L}(\mathcal{V}',\mathcal{V})}\|\mathcal{A}(t)-\mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}s^{-\beta/2}\,ds\|s \to s^{\beta/2}u(s)\|_{L^\infty(0,\frac{t}{2};\mathcal{V})} \end{aligned}$$

$$\lesssim t^{\beta/2}\int_0^{t/2}\frac{s^{-\beta/2}}{(t-s)^{1-\varepsilon}}\,ds\sup_{s\in[0,t/2]}\frac{\|\mathcal{A}(t)-\mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{(t-s)^{\varepsilon}}\|s\to s^{\beta/2}u(s)\|_{L^\infty(0,\frac{t}{2};\mathcal{V})}.$$

Note that

$$t^{\beta/2}\int_0^{t/2}\frac{s^{-\beta/2}}{(t-s)^{1-\varepsilon}}\,ds = t^{\varepsilon}\int_0^{1/2}\frac{l^{-\beta/2}}{(1-l)^{1-\varepsilon}}\,dl.$$

Therefore,

$$\begin{aligned}
&t^{\beta/2}\|(M_1u)(t)\|_{\mathcal{V}}\\
&\lesssim t^{\epsilon}\|\mathcal{A}\|_{C^{\epsilon}([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))}\|s\to s^{\beta/2}u(s)\|_{L^\infty(0,\frac{t}{2};\mathcal{V})}+\Big(\int_{t/2}^{t}\frac{\|\mathcal{A}(t)-\mathcal{A}(s)\|^2_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t-s}\,ds\Big)^{1/2}\|u\|_{L^\infty_\beta(\frac{t}{2},t;\mathcal{V})}\\
&\lesssim t^{\varepsilon}\|A(\cdot)\|_{C^{\varepsilon}([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))}\|u\|_{L^\infty_\beta(0,t;\mathcal{V})}.
\end{aligned}$$

Choosing τ small enough, $M_1\in\mathcal{L}(L^\infty_\beta(0,\tau;\mathcal{V}))$, with norm $\|M_1\|_{\mathcal{L}(L^\infty_\beta(0,\tau;\mathcal{V}))}<1$. Therefore $(I-M_1)$ is invertible in $L^\infty_\beta(0,\tau;\mathcal{V})$. Hence,

$$u=(I-M_1)^{-1}(Mu_0+L_1f)\in L^\infty_\beta(0,\tau;\mathcal{V}).$$

This completes the proof. ∎

Our main result reads as follows.

Theorem 2.0.6. *Suppose that $\mathcal{A}\in W^{\frac{1}{2},2}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}'))\cap C^{\varepsilon}([0,\tau],\mathcal{L}(\mathcal{V},\mathcal{V}'))$ with $\varepsilon>0$, then for all $f\in L^2_\beta(0,\tau;\mathscr{H})$ and $u_0\in(\mathscr{H};\mathscr{D}(A(0)))_{\frac{1-\beta}{2}}$, there exists a unique $u\in W_\beta(\mathscr{D}(A(\cdot)),\mathscr{H})$ be the solution of (P).*

Proof. Let τ be small enough and $f\in L^2_\beta(0,\tau;\mathscr{H})$, $u_0\in(\mathscr{H};\mathscr{D}(A(0)))_{\frac{1-\beta}{2},2}$. By Proposition 2.0.5, u belongs to $L^\infty_\beta(0,\tau;\mathcal{V})$, where u is the unique solution to the Cauchy problem (P). Using (2.0.8), for $0\leqslant t\leqslant\tau$, we have

$$\begin{aligned}
A(t)u(t)&=A(t)e^{-tA(t)}u_0+A(t)\int_0^t e^{-(t-s)A(t)}[\mathcal{A}(t)-\mathcal{A}(s)]u(s)\,ds\\
&\quad+A(t)\int_0^t e^{-(t-s)A(t)}f(s)\,ds\\
&:=(Fu_0)(t)+(Su)(t)+(Lf)(t).
\end{aligned}$$

Thanks to Propositions 1.7.10, 2.0.4, Fu_0 and Lf are bounded in $L^2_\beta(0,\tau;\mathscr{H})$. Then to prove that $A(\cdot)u\in L^2_\beta(0,\tau;\mathscr{H})$ it is sufficient to show that Su belongs to $L^2_\beta(0,\tau;\mathscr{H})$.

Taking $g \in L^2(0,\tau;\mathscr{H})$ we find that

$$\begin{aligned}
&|(\cdot^{\beta/2} Su, g)_{L^2(0,\tau;\mathscr{H})}| \\
&= \Big| \int_0^\tau t^{\beta/2} \int_0^t \langle (\mathcal{A}(t)-\mathcal{A}(s))u(s), A(t)^* e^{-(t-s)A(t)^*} g(t)\rangle_{\mathcal{V}'\times\mathcal{V}}\, ds\, dt \Big| \\
&\leqslant \Big| \int_0^\tau t^{\beta/2} \int_0^{t/2} \langle (\mathcal{A}(t)-\mathcal{A}(s))u(s), A(t)^* e^{-(t-s)A(t)^*} g(t)\rangle_{\mathcal{V}'\times\mathcal{V}}\, ds\, dt \Big| \\
&\quad + \Big| \int_0^\tau t^{\beta/2} \int_{t/2}^{t} \langle (\mathcal{A}(t)-\mathcal{A}(s))u(s), A(t)^* e^{-(t-s)A(t)^*} g(t)\rangle_{\mathcal{V}'\times\mathcal{V}}\, ds\, dt \Big| \\
&:= I_1 + I_2.
\end{aligned}$$

For I_2 we find,

$$\begin{aligned}
I_2 &\lesssim \int_0^\tau t^{\beta/2} \int_{t/2}^t \|\mathcal{A}(t)-\mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} \|e^{-\frac{(t-s)}{2}A(t)^*}\|_{\mathcal{L}(\mathscr{H},\mathcal{V})} \\
&\times \|A(t)^{*\frac{1}{2}} e^{-\frac{(t-s)}{4}A(t)^*}\|_{\mathcal{L}(\mathscr{H})} \|A(t)^{*\frac{1}{2}} e^{-\frac{(t-s)}{4}A(t)^*} g(t)\| s^{-\frac{\beta}{2}}\, ds\, dt \|\cdot^{\beta/2} u\|_{L^\infty(0,\tau;\mathcal{V})} \\
&\lesssim \int_0^\tau \int_{t/2}^t \frac{\|\mathcal{A}(t)-\mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t-s} \|A(t)^{*\frac{1}{2}} e^{-\frac{(t-s)}{4}A(t)^*} g(t)\|\, ds\, dt \|\cdot^{\beta/2} u\|_{L^\infty(0,\tau;\mathcal{V})} \\
&\lesssim \|\mathcal{A}\|_{W^{\frac{1}{2},2}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}'))} \Big(\int_0^\tau \int_{t/2}^t \|A(t)^{*\frac{1}{2}} e^{-\frac{(t-s)}{4}A(t)^*} g(t)\|^2\, ds\, dt \Big)^{1/2} \|u\|_{L^\infty_\beta(0,\tau;\mathcal{V})} \\
&\lesssim \|\mathcal{A}\|_{W^{\frac{1}{2},2}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}'))} \|g\|_{L^2(0,\tau,\mathscr{H})} \|u\|_{L^\infty_\beta(0,\tau;\mathcal{V})}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_1 &\lesssim \int_0^\tau t^{\beta/2} \int_0^{t/2} \frac{s^{\frac{-\beta}{2}}}{(t-s)^{\frac{3}{2}-\varepsilon}} \|g(t)\|\, ds\, dt \\
&\quad \times \|\mathcal{A}\|_{C^\varepsilon([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))} \|\cdot^{\beta/2} u\|_{L^\infty(0,\tau;\mathcal{V})} \\
&\lesssim \|\mathcal{A}\|_{C^\varepsilon([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))} \|g\|_{L^2(0,\tau,\mathscr{H})} \|u\|_{L^\infty_\beta(0,\tau;\mathcal{V})}.
\end{aligned}$$

Now, we obtain the final estimate

$$\begin{aligned}
\|A(\cdot)u\|_{L^2_\beta(0,\tau;\mathscr{H})} &\lesssim \|Fu_0\|_{L^2_\beta(0,\tau;\mathscr{H})} + \|Su\|_{L^2_\beta(0,\tau;\mathscr{H})} + \|Lf\|_{L^2_\beta(0,\tau;\mathscr{H})} \\
&\lesssim \|u_0\|_{(\mathscr{H};\mathscr{D}(A(0)))_{\frac{1-\beta}{2},2}} + \|u\|_{L^\infty_\beta(0,\tau;\mathcal{V})} + \|f\|_{L^2_\beta(0,\tau;\mathscr{H})} \\
&\lesssim \|u_0\|_{(\mathscr{H};\mathscr{D}(A(0)))_{\frac{1-\beta}{2},2}} + \|f\|_{L^2_\beta(0,\tau;\mathscr{H})}.
\end{aligned}$$

Therefore $A(\cdot)u \in L^2_\beta(0,\tau;\mathscr{H})$ and since $\dot{u} = f - Au$, one has $\dot{u} \in L^2_\beta(0,\tau;\mathscr{H})$. So u belongs to $W_\beta(\mathscr{D}(A(\cdot)),\mathscr{H})$. Moreover, by Proposition 2.0.4 we have $u(t) \in (\mathscr{H};\mathscr{D}(A(t)))_{\frac{1-\beta}{2},2}$ for all $t \in [0,\tau]$.

For arbitrary τ we split the interval $[0,\tau]$ into union of small intervals and argue exactly as before to each subinterval. Finally we stick the solutions and we obtain the desired result. □

Proposition 2.0.7. *For all $g \in L^2(0,\tau;\mathscr{H})$ and $0 \leqslant \beta < 1$ there exists a unique $v \in W_0(D(A(\cdot),\mathscr{H})$ be the solution of the singular equation*

$$\begin{aligned} \dot{v}(t) + \mathcal{A}(t)v(t) + \frac{\beta}{2}\frac{v(t)}{t} &= g(t) \\ v(0) &= 0. \end{aligned} \tag{2.0.10}$$

Proof. We set $f(t) = (\Phi g)(t) = t^{\beta/2}g(t)$ with $t \in [0,\tau]$, so that $f \in L^2_\beta(0,\tau;\mathscr{H})$. Let $u \in W_\beta(\mathscr{D}(A(\cdot),\mathscr{H})$ be the solution to the problem

$$\begin{aligned} \dot{u}(t) + \mathcal{A}(t)u(t) &= f(t) \\ u(0) &= 0. \end{aligned} \tag{2.0.11}$$

Now, set $v = (\Phi^{-1}u)$. Then $v \in W_0(\mathscr{D}(A(\cdot),\mathscr{H})$ and v is the unique solution to Problem (2.0.10). □

2.1 Applications

This section is devoted to some applications of the results given in the previous sections. We give examples illustrating the theory without seeking for generality.

2.1.1 Elliptic operators in the divergence form

Let Ω be a bounded Lipschitz domain of $\mathbb{R}^n$. We set $\mathscr{H} := L^2(\Omega)$ and $\mathcal{V} := H^1(\Omega)$ and we define the sesquilinear forms

$$\mathfrak{a}(t,u,v) := \int_\Omega C(t,x)\nabla u\overline{\nabla v}\,dx$$

where here $u, v \in \mathcal{V}$ and $C : [0,\tau] \times \Omega \to \mathbb{C}^{n\times n}$ is a bounded and measurable function for which there exists $\alpha, M > 0$ such that

$$\alpha|\xi|^2 \leqslant \mathrm{Re}(C(t,x)\xi.\bar{\xi}) \quad \text{and} \quad |C(t,x)\xi.\nu| \leqslant M|\xi||\nu|$$

for all $t \in [0,\tau]$ and a.e $x \in \Omega$, and all $\xi, \nu \in \mathbb{C}^n$. We define the gradient operator $\nabla : \mathcal{V} \to \mathscr{H}$ and $\nabla^* : \mathscr{H} \to \mathcal{V}'$. The non-autonomous form $\mathfrak{a}(t)$ induces the operators

$$\mathcal{A}(t) := -\nabla^* C(t,x)\nabla \in \mathcal{L}(\mathcal{V},\mathcal{V}').$$

The form $\mathfrak{a}(t)$ is $H^1(\Omega)$-bounded and coercive. The part of $\mathcal{A}(t)$ in $\mathscr{H}$ is the operator

$$A(t) := -\mathrm{div}\ C(t,x)\nabla$$

under Neumann boundary conditions.

We note that

$$\|\mathcal{A}(t)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} \simeq \|C(t,\cdot)\|_{L^\infty(\Omega;\mathbb{C}^{n\times n})} = M.$$

Next, we suppose that $C \in W^{\frac{1}{2},2}(0,\tau;L^\infty(\Omega;\mathbb{C}^{n\times n})) \cap C^\varepsilon([0,\tau];L^\infty(\Omega;\mathbb{C}^{n\times n}))$, with $\varepsilon > 0$, which is equivalent to

$$\int_0^\tau \int_0^\tau \sup_{x\in\Omega} \frac{\|C(t,x)-C(s,x)\|^2_{\mathbb{C}^{n\times n}}}{|t-s|^2}\, ds\, dt < \infty,$$
$$\|C(t,x)-C(s,x)\|_{\mathbb{C}^{n\times n}} < C|t-s|^\varepsilon$$

a.e. for $x \in \Omega$ and $t,s \in [0,\tau]$. Note that

$$\|\mathcal{A}(t)-\mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} \lesssim \|C(t,.)-C(s,.)\|_{L^\infty(\Omega;\mathbb{C}^{n\times n})}.$$

Hence $\mathcal{A} \in W^{\frac{1}{2},2}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}')) \cap C^\epsilon([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))$.

Remark 2.1.1. $\mathscr{D}(A(t)^{1/2}) = \mathcal{V} = H^1(\Omega)$ for all $t \in [0,\tau]$ and

$$c_1\|u\|_{H^1(\Omega)} \leqslant \|u\|_{D(A(t)^{1/2})} \leqslant c^1\|u\|_{H^1(\Omega)}$$

where c_1, c^1 are two positive constants independents of t [48, Theorem 1].

In the next proposition we assume that $\beta \in [0,1[$.

Proposition 2.1.1. *For all $f \in L^2_\beta(0,\tau;L^2(\Omega)), u_0 \in H^{1-\beta}(\Omega)$ there is a unique $u \in W_\beta(\mathscr{D}(A(\cdot),L^2(\Omega))$, be the solution of the problem*

$$\begin{aligned} \dot u(t) - \operatorname{div} C(t,x)\nabla u(t) &= f(t)\\ \frac{\partial u(t,\sigma)}{\partial n} &= 0 \ (\sigma \in \partial\Omega)\\ u(0) &= u_0. \end{aligned} \tag{2.1.1}$$

The above proposition follows by Theorem 2.0.6.

2.1.2 Robin boundary conditions

Let Ω be a bounded domain of $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$. We denote by Tr the classical trace operator. Let $\beta : [0,\tau]\times\partial\Omega \to [0,\infty)$ be a bounded function and $\mathscr{H} := L^2(\Omega)$. We define the form

$$\mathfrak{a}(u,v) := \int_\Omega \nabla u.\nabla v\, dx + \int_{\partial\Omega} \beta(\cdot)\mathrm{Tr}(u)\mathrm{Tr}(v)\, d\sigma,$$

for all $u,v \in \mathcal{V} := H^1(\Omega)$.

The form $\mathfrak{a}$ is $H^1(\Omega)$-bounded, symmetric and quasi-coercive. The first statement follows readily from the continuity of the trace operator and the boundedness of β. The second one is a consequence of the inequality

$$\int_{\partial\Omega} |u|^2 d\sigma \leqslant \delta \|u\|^2_{H^1(\Omega)} + C_\delta \|u\|^2_{L^2(\Omega)}$$

which is valid for all $\delta > 0$ (C_δ is a constant depending on δ). Note that this is a consequence of compactness of the trace as an operator from $H^1(\Omega)$ into $L^2(\partial\Omega, d\sigma)$. Formally, the associated operator A is (minus) the Laplacian with the time dependent Robin boundary condition

$$\tfrac{\partial u}{\partial n} + \beta(\cdot)u = 0 \text{ on } \partial\Omega.$$

Here, $\frac{\partial u}{\partial n}$ denotes the normal derivative in the weak sense. For more general boundary conditions with an indefinite weight we refer the reader to the recent paper [43].

Theorems 2.0.2 combined with Theorem 2.0.3 yields the following result.

Proposition 2.1.2. *Let $\beta \in]-1,1[$ and $f \in L^2_\beta(0,\tau; L^2(\Omega))$. There exists a unique $u \in W_\beta(\mathscr{D}(A), L^2(\Omega)) \cap C([0,\tau], (L^2(\Omega); \mathscr{D}(A))_{\frac{1-\beta}{2},2})$ be the solution to the problem*

$$\begin{aligned} \dot{u}(t) - \Delta u(t) &= f(t) \\ \tfrac{\partial u}{\partial n} + \beta(\cdot)u &= 0 \quad \text{on } \partial\Omega \\ u(0) &= 0. \end{aligned} \tag{2.1.2}$$

If we assume moreover that $f \in W^{1,2}_{\beta,0}(0,\tau; L^2(\Omega))$, then the solution u belongs to the space $C^1([0,\tau]; (L^2(\Omega); \mathscr{D}(A))_{\frac{1-\beta}{2},2}) \cap C([0,\tau]; \mathscr{D}(A))$.

Remark 2.1.2. Note that for all $\beta \in [0,1[$ we have

$$(L^2(\Omega); \mathscr{D}(A))_{\frac{1-\beta}{2},2} = [L^2(\Omega); \mathscr{D}(A)]_{\frac{1-\beta}{2}} = [L^2(\Omega); H^1(\Omega)]_{1-\beta} = H^{1-\beta}(\Omega).$$

Chapter 3

Maximal regularity for semilinear non-autonomous evolution equations in weighted Hilbert spaces

3.1 Introduction

The present chapter deals with maximal L^2-regularity for non-autonomous evolution equations in the setting of Hilbert spaces. Before explaining our results we recall some notations and assumptions. Let $(\mathcal{H}, (\cdot,\cdot), \|\cdot\|)$ be a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$. We consider another Hilbert space $\mathcal{V}$ which is densely and continuously embedded into $\mathcal{H}$. We denote by $\mathcal{V}'$ the (anti-) dual space of $\mathcal{V}$ so that

$$\mathcal{V} \hookrightarrow_d \mathcal{H} \hookrightarrow_d \mathcal{V}'.$$

We denote by $\langle , \rangle$ the duality $\mathcal{V}$-$\mathcal{V}'$ and note that $\langle \psi, v\rangle = (\psi, v)$ if $\psi, v \in \mathcal{H}$. Given $\tau \in (0,\infty)$ and consider a family of sesquilinear forms

$$\mathfrak{a} : [0,\tau] \times \mathcal{V} \times \mathcal{V} \to \mathbb{C}$$

such that

- [H1]: $\mathscr{D}(\mathfrak{a}(t)) = \mathcal{V}$ (constant form domain),
- [H2]: $|\mathfrak{a}(t,u,v)| \leqslant M\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}$ (uniform boundedness),
- [H3]: Re $\mathfrak{a}(t,u,u) + \nu\|u\|^2 \geqslant \delta\|u\|_{\mathcal{V}}^2$ $(\forall u \in \mathcal{V})$ for some $\delta > 0$ and some $\nu \in \mathbb{R}$ (uniform quasi-coercivity).

Here and throughout this paper, $\|\cdot\|_{\mathcal{V}}$ denotes the norm of $\mathcal{V}$.

To each form $\mathfrak{a}(t)$ we can associate two operators $A(t)$ and $\mathcal{A}(t)$ on $\mathcal{H}$ and $\mathcal{V}'$, respectively. Recall that $u \in \mathcal{H}$ is in the domain $\mathscr{D}(A(t))$ if there exists $h \in \mathcal{H}$ such that for all $v \in \mathcal{V}$: $\mathfrak{a}(t,u,v) = (h,v)$. We then set $A(t)u := h$. The operator $\mathcal{A}(t)$ is a bounded operator from $\mathcal{V}$ into $\mathcal{V}'$ such that $\mathcal{A}(t)u = \mathfrak{a}(t,u,\cdot)$. The operator $A(t)$ is the part of $\mathcal{A}(t)$ on $\mathcal{H}$. It is a classical fact that $-A(t)$ and $-\mathcal{A}(t)$ are both generators of holomorphic semigroups $(e^{-rA(t)})_{r\geqslant 0}$ and $(e^{-r\mathcal{A}(t)})_{r\geqslant 0}$ on $\mathcal{H}$ and $\mathcal{V}'$, respectively. The semigroup $e^{-rA(t)}$ is the restriction of $e^{-r\mathcal{A}(t)}$ to $\mathcal{H}$. In addition, $e^{-rA(t)}$ induces a holomorphic semigroup on $\mathcal{V}$ (see, e.g., Ouhabaz [13, Chapter 1]). We consider the non-homogeneous Cauchy problem

$$\begin{cases} u'(t) + A(t)\,u(t) &= f(t), \; t \in (0,\tau] \\ u(0) &= u_0, \end{cases} \tag{P}$$

Definition 3.1.1. *The Cauchy problem (P) has maximal L^2-regularity in $\mathcal{H}$ if for every $f \in L^2(0,\tau,\mathcal{H})$, there exists a unique $u \in H^1(0,\tau,\mathcal{H})$ with $u(t) \in \mathscr{D}(A(t))$ for a.e. $t \in [0,\tau]$ and u is a solution of (P) in the L^2-sense.*

By a very well known result of J.L. Lions, maximal L^2-regularity always holds in the space $\mathcal{V}'$. That is for every $f \in L^2(0,\tau;\mathcal{V}')$ and $u_0 \in \mathcal{H}$ there exists a unique $u \in H^1(0,\tau;\mathcal{V}') \cap L^2(0,\tau;\mathcal{V})$ which solves the equation (P).

It has been shown in [57] that the maximal regularity may fail for forms $C^{\frac{1}{2}}$ in time. We note also that for $\mathcal{A}(.) \in W^{s,p}(0,\tau;\mathcal{L}(V,V'))$, with $s < \frac{1}{2}$ the maximal regularity may fail and this follows from the inclusion $C^{\frac{1}{2}}(0,\tau;\mathcal{L}(V,V')) \subset W^{s,p}(0,\tau;\mathcal{L}(V,V'))$.
For $p > 2$ and $\mathcal{A}(.) \in W^{\frac{1}{2},p}(0,\tau;\mathcal{L}(V,V'))$, the maximal regularity may fail also and this follows from the counterexample in [?]. The open problem is does we have the maximal regularity with $\mathcal{A}(.) \in W^{\frac{1}{2},2}(0,\tau;\mathcal{L}(V,V'))$. It is proved in [40] that maximal L^2-regularity holds if $t \mapsto \mathcal{A}(t) \in H^{\frac{1}{2}}([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}'))$ (with some integrability conditions). This regularity assumption is optimal and this results are the most general ones on this problem. For the case of weighted spaces we refer the reader to the recent paper [42]. The choice of the weighted spaces has a big advantages. One of them is to reduce the necessary regularity for initial conditions of evolution equations. Time-weights can be used also to expploit parabolic regularization which is typical for quasilinear parabolic problems. The main focus of this chapter is to consider the semilinear equation

$$u'(t) + \mathcal{A}(t)u(t) = F(t,u),\ t\text{-a.e},\ u(0) = u_0.$$

Here the inhomogeneous term F satisfies some continuity condition. Our main result shows that for forms satisfying the uniform Kato square root property (see the next section) then we have the maximal regularity result in temporally weighted L^2-spaces if $u_0 \in [\mathcal{H};\mathscr{D}(A(0))]_{\frac{1-\beta}{2}}$ and $\mathcal{A} \in W^{\frac{1}{2},2}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}')) \cap C^{\varepsilon}([0,\tau],\mathcal{L}(\mathcal{V},\mathcal{V}'))$. The Kato square root property plays an important role in the questions of (non-autonomous) maximal regularity and optimal control. To prove our results we appeal to classical tools from harmonic analysis such as square function estimate or functional calculus and from functional analysis such as interpolation theory or operator theory.

3.2 Preliminaries

In this section we briefly recall the definitions and we give the basic properties of vector-valued function spaces with temporal weights. For more details we refer to [42].
Let $(X, \|\cdot\|_X)$ be a Banach space over $\mathbb{R}$ or $\mathbb{C}$. For $-1<\beta<1$, we recall that $L^2_\beta(0,\tau;X) = L^2(0,\tau,t^\beta dt;X)$, endowed with the norm

$$\|u\|^2_{L^2_\beta(0,\tau,X)} := \int_0^\tau \|u(t)\|_X^2\, t^\beta\, dt.$$

and that $L^2_\beta(0,\tau;X) \hookrightarrow L^1_{loc}(0,\tau;X)$. We recall also the corresponding weighted Sobolev spaces

$$W^{1,2}_\beta(0,\tau;X) := \{u \in W^{1,1}(0,\tau;X) \text{ s.t } u, u' \in L^2_\beta(0,\tau;X)\},$$

$$W^{1,2}_{\beta,0}(0,\tau;X) := \{u \in W^{1,2}_\beta(0,\tau;X),\ \text{s.t } u(0)=0\},$$

which are Banach spaces for the norms, respectively

$$\|u\|^2_{W^{1,2}_\beta(0,\tau;X)} := \|u\|^2_{L^2_\beta(0,\tau;X)} + \|u'\|^2_{L^2_\beta(0,\tau;X)},$$

$$\|u\|^2_{W^{1,2}_{\beta,0}(0,\tau;X)} := \|u'\|^2_{L^2_\beta(0,\tau;X)}.$$

Remark 3.2.1. *The restriction on β comes from several facts. The first one is the embedding $L^2_\beta(0,\tau;\mathcal{H}) \hookrightarrow L^1(0,\tau;\mathcal{H})$. The second one is due to Hardy' inequality and the third reason comes from the fact that functions in $W^{1,2}_\beta(0,\tau;\mathcal{H})$ have a well-defined trace in case that $-1<\beta<1$.*

Lemma 3.2.2. *Let $u \in W^{1,2}_{\beta,0}(0,\tau;X)$. We have*

$$\|u\|^2_{L^2_\beta(0,\tau;X)} \lesssim \int_0^\tau s \int_0^s \|u'(r)\|_X^2\, r^\beta\, dr\, ds. \tag{3.2.1}$$

Proof. Let $u \in W^{1,2}_{\beta,0}(0,\tau;X), s \in (0,\tau)$. Due to Hölder's inequality we get

$$\begin{aligned}\|u(s)\|_X = \|\int_0^s u'(r)\, dr\|_X &\leqslant \int_0^s \|u'(r)\|_X\, dr\\ &\leqslant (\int_0^s r^{-\beta}\, dr)^{\frac{1}{2}}(\int_0^s \|u'(r)\|_X^2\, r^\beta\, dr)^{\frac{1}{2}}\end{aligned}$$

$$\lesssim s^{\frac{1-\beta}{2}} (\int_0^s \|u'(r)\|_X^2 \, r^\beta \, dr)^{\frac{1}{2}}.$$

Therefore, (3.2.1) follows immediately.

□

Let us define the space

$$W_\beta(\mathscr{D}(A(.)),\mathcal{H}) := \{u \in W^{1,1}(0,\tau;\mathcal{H}), \text{ s.t } A(.)u \in L^2_\beta(0,\tau;\mathcal{H}), \, \dot{u} \in L^2_\beta(0,\tau;\mathcal{H})\},$$

with norm

$$\|u\|_{W_\beta(\mathscr{D}(A(.),\mathcal{H})} = \|A(.)u\|_{L^2_\beta(0,\tau;\mathcal{H})} + \|\dot{u}\|_{L^2_\beta(0,\tau;\mathcal{H})}.$$

It is easy to see that $W_\beta(\mathscr{D}(A(.),\mathcal{H}) \hookrightarrow W^{1,2}_\beta(0,\tau;\mathcal{H})$. For $s \in (0,\tau)$ we define the associated trace space to $W_\beta(\mathscr{D}(A(.)),\mathcal{H})$ by

$$TR(s,\beta) := \{u(s) : u \in W_\beta(\mathscr{D}(A(.)),\mathcal{H})\},$$

endowed with norm

$$\|u(s)\|_{TR(s,\beta)} := \inf\{\|v\|_{W_\beta(\mathscr{D}(A(.)),\mathcal{H})} : v(s) = u(s)\}.$$

Note that $\Big(TR(s,\beta), \|\cdot\|_{TR(s,\beta)}\Big)$ is a Banach space.

From now we assume without loss of generality that the forms are coercive, that is $[H3]$ holds with $\nu = 0$. The reason is that by replacing $A(t)$ by $A(t)+\nu$, the solution v of (P) is $v(t) = e^{-\nu t}u(t)$ and it is clear that $u \in W^{1,2}_\beta(0,\tau;\mathcal{H}) \cap L^2_\beta(0,\tau;\mathcal{V})$ if and only if $v \in W^{1,2}_\beta(0,\tau;\mathcal{H}) \cap L^2_\beta(0,\tau;\mathcal{V})$.

In the statements below we shall need the following square root property (called Kato's square root property)

$$\mathscr{D}(A(t)^{1/2}) = \mathcal{V} \text{ and } c_1\|A(t)^{1/2}v\| \leqslant \|v\|_{\mathcal{V}} \leqslant c_2\|A(t)^{1/2}v\| \tag{3.2.2}$$

for all $v \in \mathcal{V}$ and $t \in [0,\tau]$, where the positive constants c_1 and c_2 are independent of t. Note that this assumption is always true for symmetric forms when $\nu = 0$ in [H3]. It is also valid for uniformly elliptic operator on $\mathbb{R}^n$, see [48].

3.3 Main results

In this section we state explicitly our main results. Assume in addition that $t \mapsto F(t,x), x \in \mathcal{H}$ satisfies $F_0(.) = F(.,0) \in L^2_\beta(0,\tau;\mathcal{H})$ and the following continuity property: for any $\varepsilon > 0$ there exists a constant $N_\varepsilon > 0$ such that

$$\|F(.,u) - F(.,v)\|^2_{L^2_\beta(0,\tau;\mathcal{H})} \leqslant \varepsilon\|u-v\|^2_{W_\beta(\mathscr{D}(A(.),\mathcal{H})} + N_\varepsilon\|u-v\|^2_{L^2_\beta(0,\tau;\mathcal{H})}, \tag{3.3.1}$$

for any $u, v \in W_\beta(\mathscr{D}(A(.),\mathcal{H})$.

Example 3.3.1.

1- If we assume that $\|F(t,x)-F(t,y)\| \leqslant K\|x-y\|_{\mathcal{V}}, K>0, x,y\in\mathcal{V}, t\in(0,\tau)$ *then the conditions* (3.3.1) *is satisfied. Indeed, let* $u,v\in W_\beta(D(A(.),\mathcal{H})$ *one has*

$$\begin{aligned}\|F(.,u)-F(.,v)\|^2_{L^2_\beta(0,\tau;\mathcal{H})} &\leqslant K^2\|u-v\|^2_{L^2_\beta(0,\tau;\mathcal{V})}\\ &= \frac{K^2}{\delta}\int_0^\tau \Big(\delta\|u(t)-v(t)\|^2_{\mathcal{V}}\Big)t^\beta\,dt\\ &\leqslant \frac{K^2}{\delta}\int_0^\tau \Big(Re\,\Big(A(t)(u(t)-v(t)),u(t)-v(t)\Big)\Big)t^\beta\,dt\\ &\leqslant \frac{K^2}{\delta}\int_0^\tau \|A(t)(u(t)-v(t))\|\|u(t)-v(t)\|t^\beta\,dt\\ &\leqslant \epsilon\|A(.)(u-v)\|^2_{L^2_\beta(0,\tau;\mathcal{H})}+N_\epsilon\|u-v\|^2_{L^2_\beta(0,\tau;\mathcal{H})},\end{aligned}$$

where $N_\epsilon=\frac{K^4}{\delta^2\epsilon}$.

The following theorem is proved in the previous chapter see Theorem 2.0.6

Theorem 3.3.2. *Suppose that* $\mathcal{A}\in W^{\frac{1}{2},2}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}'))\cap C^\varepsilon([0,\tau],\mathcal{L}(\mathcal{V},\mathcal{V}'))$, *with* $\varepsilon>0$, *then for all* $f\in L^2_\beta(0,\tau;\mathcal{H})$ *and* $u_0\in[\mathcal{H};\mathscr{D}(A(0))]_{\frac{1-\beta}{2}}$, *there exists a unique* $u\in W_\beta(\mathscr{D}(A(.),\mathcal{H})$ *be the solution to* (P). *Moreover, there exists a positive constant* $c>0$ *such that*

$$\|u\|_{W_\beta(\mathscr{D}(A(.),\mathcal{H})}\leqslant c\Big[\|u_0\|_{[\mathcal{H};\mathscr{D}(A(0))]_{\frac{1-\beta}{2}}}+\|f\|_{L^2_\beta(0,\tau;\mathcal{H})}\Big].$$

The following proposition gives a characterization of the trace space $TR(s,\beta)$.

Proposition 3.3.1. *For all* $\beta\in(0,1), s\in(0,\tau)$ *we have*

$$TR(s,\beta)=[\mathcal{H};\mathscr{D}(A(s))]_{\frac{1-\beta}{2}}\quad \textit{with equivalent norms.}$$

Proof. The first injection $TR(s,\beta)\hookrightarrow[\mathcal{H};\mathscr{D}(A(s))]_{\frac{1-\beta}{2}}$ is obtained by Proposition 2.0.4. The second injection "$\hookleftarrow$" follows by Theorem 3.3.2. □

The following is our main result

Theorem 3.3.3. *Suppose that $\mathcal{A} \in W^{\frac{1}{2},2}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}')) \cap C^{\varepsilon}([0,\tau],\mathcal{L}(\mathcal{V},\mathcal{V}'))$, with $\varepsilon > 0$, then for all $u_0 \in [\mathcal{H};\mathscr{D}(A(0))]_{\frac{1-\beta}{2}}$, there exists a unique $u \in W_\beta(\mathscr{D}(A(.),\mathcal{H})$ be the solution to*

$$u'(t) + \mathcal{A}(t)u(t) = F(t,u),\ t\text{-}a.e,\ u(0) = u_0. \tag{3.3.2}$$

Moreover there exists a positive constants $c > 0$ such that

$$\|u\|_{W_\beta(\mathscr{D}(A(.),\mathcal{H})} \leqslant c\Big[\|u_0\|_{[\mathcal{H};\mathscr{D}(A(0))]_{\frac{1-\beta}{2}}} + \|F_0\|_{L^2_\beta(0,\tau;\mathcal{H})}\Big].$$

Proof. First, let us define the space $W_{\beta,0}(D(A(.),\mathcal{H}) := W_\beta(D(A(.),\mathcal{H}) \cap W^{1,2}_{\beta,0}(0,\tau;X)$. For $v \in W_\beta(D(A(.),\mathcal{H})$ consider the linear equation

$$w' + A(.)w = F(.,v), w(0) = 0. \tag{3.3.3}$$

Thanks to Theorem 3.3.2, (3.3.3) has a unique solution $w \in W_{\beta,0}(D(A(.),\mathcal{H})$.

We define

$$\begin{array}{rcl} S : W_{\beta,0}(\mathscr{D}(A(.),\mathcal{H}) & \to & W_{\beta,0}(\mathscr{D}(A(.),\mathcal{H}) \\ v & \mapsto & w \end{array}$$

Let $v_1, v_2 \in W_{\beta,0}(\mathscr{D}(A(.),\mathcal{H})$. Obviously, $x = Sv_1 - Sv_2$ satisfies $x' + A(.)x = F(.,v_1) - F(.,v_2), x(0) = 0$ and we have by Theorem 3.3.2 and Lemma 3.2.2

$$\begin{aligned} \|Sv_1 - Sv_2\|^2_{W_\beta(\mathscr{D}(A(.),\mathcal{H})} &\leqslant N\|F(.,v_1) - F(.,v_2)\|^2_{L^2_\beta(0,\tau;\mathcal{H})} \\ &\leqslant N\epsilon\|v_1 - v_2\|^2_{W_\beta(\mathscr{D}(A(.),\mathcal{H})} + NN_\epsilon\|v_1 - v_2\|^2_{L^2_\beta(0,\tau;\mathcal{H})} \\ &\leqslant N\epsilon\|v_1 - v_2\|^2_{W_\beta(\mathscr{D}(A(.),\mathcal{H})} + C'NN_\epsilon\int_0^\tau s\int_0^s \|(v_1 - v_2)'(r)\|^2 r^\beta\, dr\, ds \\ &\leqslant N\epsilon\|v_1 - v_2\|^2_{W_\beta(\mathscr{D}(A(.),\mathcal{H})} + C'NN_\epsilon\int_0^\tau s\|v_1' - v_2'\|^2_{L^2_\beta(0,s;\mathcal{H})}\, ds. \end{aligned}$$

Set $K_0 := N\epsilon$ and $K_1 := C'NN_\epsilon$. Then repeating the above inequality and using the identity

$$\int_0^t s_1\int_0^{s_1} s_2 ... \int_0^{s_{n-1}} s_n\, ds_n...ds_1 = \frac{1}{\Gamma(2n+1)}t^{2n},$$

we obtain

$$\|S^n v_1 - S^n v_2\|^2_{W_\beta(\mathscr{D}(A(.),\mathcal{H})} \leqslant \sum_0^n \binom{n}{k} K_0^{n-k}(K_1\tau^2)^k\frac{1}{\Gamma(2k+1)}\|v_1 - v_2\|^2_{W_\beta(\mathscr{D}(A(.),\mathcal{H})}$$

$$\leqslant (2K_0)^n \left[\max_{k=0,..,n} \left(\frac{\left(K_0^{-1}\tau^2 K_1\right)^k}{\Gamma(2k+1)} \right) \right] \|v_1 - v_2\|^2_{W_\beta(\mathscr{D}(A(.),\mathcal{H})}.$$

For the second inequality we use $\sum_0^n \begin{pmatrix} n \\ k \end{pmatrix} = 2^n$. Note that $\max_{k=0,..,n}\left(\frac{\left(K_0^{-1}\tau^2 K_1\right)^k}{\Gamma(2k+1)} \right)$ is bounded for all $n \in \mathbb{N}^*$.

Now, we take $\epsilon < \frac{1}{4N}$, which gives $K_0 < \frac{1}{4}$ and n sufficiently large to get

$$\begin{aligned} \|S^n v_1 - S^n v_2\|^2_{W_\beta(\mathscr{D}(A(.),\mathcal{H})} &< \frac{1}{2^n} \left[\max_{k=0,..,n} \left(\frac{\left(K_0^{-1}\tau^2 K_1\right)^k}{\Gamma(2k+1)} \right) \right] \|v_1 - v_2\|^2_{W_\beta(\mathscr{D}(A(.),\mathcal{H})} \\ &< \|v_1 - v_2\|^2_{W_\beta(\mathscr{D}(A(.),\mathcal{H})}. \end{aligned}$$

Then S^n is a contraction on $W_\beta(\mathscr{D}(A(.),\mathcal{H})$ and this yields the existence and uniqueness of a solution $w \in W_\beta(\mathscr{D}(A(.),\mathcal{H})$ to the equation (3.3.3). Therefore it only remains to prove the a priori estimate. From the linear equation and (3.3.1) we have for all $\epsilon > 0$

$$\begin{aligned} \|w\|^2_{W_\beta(\mathscr{D}(A(.),\mathcal{H})} &\leqslant N\|F(.,w)\|^2_{L^2_\beta(0,\tau;\mathcal{H})} \\ &\leqslant N\|F(.,w) - F_0(.)\|^2_{L^2_\beta(0,\tau;\mathcal{H})} + 2N\|F_0(.)\|^2_{L^2_\beta(0,\tau;\mathcal{H})} \\ &\leqslant 2N\epsilon\|w\|^2_{W_\beta(\mathscr{D}(A(.),\mathcal{H})} + 2NN_\epsilon\|w\|^2_{L^2_\beta(0,s;\mathcal{H})} + 2N\|F_0(.)\|^2_{L^2_\beta(0,s;\mathcal{H})} \\ &\leqslant 2N\epsilon\|w\|^2_{W_\beta(\mathscr{D}(A(.),\mathcal{H})} + 2C'NN_\epsilon \int_0^\tau s\|w'\|^p_{L^2_\beta(0,s;\mathcal{H})}\, ds + 2N\|F_0(.)\|^2_{L^2_\beta(0,s;\mathcal{H})}. \end{aligned}$$

Now, taking $\epsilon = \frac{1}{8N}$ and applying Gronwall's lemma gives that there exists $C > 0$ such that

$$\|w\|_{W_\beta(\mathscr{D}(A(.),\mathcal{H})} \leqslant C\|F_0\|_{L^2_\beta(0,s;\mathcal{H})}.$$

Now, we consider the non homogeneous equation . Let $u_0 \in [\mathcal{H}; \mathscr{D}(A(0))]_{\frac{1-\beta}{2}}$. Since by Theorem 2.0.6, $Tr(0,\beta) = [\mathcal{H}; \mathscr{D}(A(0))]_{\frac{1-\beta}{2}}$, then there exists $v \in W_\beta(\mathscr{D}(A(.),\mathcal{H})$ (with minimal norm) such that $v(0) = u_0$ and

$$\|v\|_{W_\beta(\mathscr{D}(A(.),\mathcal{H})} = \|u_0\|_{[\mathcal{H};\mathscr{D}(A(0))]_{\frac{1-\beta}{2}}}. \tag{3.3.4}$$

For $w \in W_\beta(\mathscr{D}(A(.),\mathcal{H})$ we define the function

$$G(t,w,w') = F(t,w+v,w'+v') - \Big(v'(t) + \mathcal{A}(t)v(t)\Big),\ t \in (0,\tau).$$

It easy to see that G satisfies the condition (3.3.1), $t \mapsto G(t,w,w') \in L^2_\beta(0,\tau;\mathcal{H})$, $G(t,0,0) = F(t,v,v') - \Big(v'(t) + \mathcal{A}(t)v(t)\Big)$. Moreover,

$$\|G(.,0,0)\|_{L^2_\beta(0,\tau;\mathcal{H})} \leqslant \|F(.,v,v') - F(.,0,0)\|_{L^2_\beta(0,\tau;\mathcal{H})} + \|F(.,0,0)\|_{L^2_\beta(0,\tau;\mathcal{H})}$$

$$
\begin{aligned}
&+ \|v' + \mathcal{A}(.)v\|_{L^2_\beta(0,\tau;\mathcal{H})} \\
&\leqslant C_1 \|v\|_{W_\beta(D(A(.),\mathcal{H})} + \|F_0\|_{L^2_\beta(0,\tau;\mathcal{H})} \\
&\leqslant C\Big[\|F_0\|_{L^2_\beta(0,\tau;\mathcal{H})} + \|u_0\|_{(\mathcal{H};D(A(0)))_{\frac{1-\beta}{2}}} \Big].
\end{aligned} \tag{3.3.5}
$$

Now, we follow the same procedure as before we get the existence and the uniqueness of the solution for the equation

$$w' + A(.)w = G(.,w), w(0) = 0.$$

Set $u = v + w$. Hence, u is the unique solution to (3.3.2). □

3.4 Applications

This section is devoted to application of our results on existence and maximal regularity to concrete evolution equations. We show how they can be applied to both linear and semilinear evolution equations.

3.4.1 Elliptic operators.

Define on $\mathcal{H} = L^2(\mathbb{R}^d)$ the sesquilinear forms

$$\mathfrak{a}(t,u,v) = \sum_{k,j=1}^{d} \int_{\mathbb{R}^d} a_{kj}(t,x)\partial_k u \overline{\partial_j v}\, dx + \sum_{j=1}^{d} \int_{\mathbb{R}^d} b_j(t,x)\partial_j u \bar{v}\, dx + \int_{\mathbb{R}^d} c(t,x) u\bar{v}\, dx, u,v \in H^1(\mathbb{R}^d).$$

We assume that $a_{kj}, b_j, c : [0,\tau] \times \mathbb{R}^d \to \mathbb{C}$ such that:

$$a_{kj}, b_j, c \in L^\infty([0,\tau] \times \mathbb{R}^d) \text{ for } 1 \leqslant k,j \leqslant d,$$

and

$$\operatorname{Re} \sum_{k,j=1}^{d} a_{kj}(t,x)\xi_k \bar{\xi_j} \geqslant \delta |\xi|^2 \text{ for all } \xi \in \mathbb{C}^d \text{ and a.e. } (t,x) \in [0,\tau] \times \mathbb{R}^d.$$

Here $\delta > 0$ is a constant independent of t.
It easy to check that $\mathfrak{a}(t,.,.)$ is $H^1(\mathbb{R}^d)$-bounded and quasi-coercive. The associated operator with $\mathfrak{a}(t,.,.)$ is elliptic operator given by the formal expression

$$A(t)u = -\sum_{k,j=1}^{d} \partial_j(a_{kj}(t,.)\partial_k u) + \sum_{j=1}^{d} b_j(t,.)\partial_j u + c(t,.)u.$$

In addition to the above assumptions we assume that $C = (a_{kj})_{k,j} \in W^{\frac{1}{2},2}(0,\tau; L^\infty(\Omega;\mathbb{C}^{n\times n})) \cap C^\varepsilon([0,\tau]; L^\infty(\Omega;\mathbb{C}^{n\times n}))$, with $\varepsilon > 0$. which is equivalent to

$$\int_0^\tau \int_0^\tau \sup_{x\in\Omega} \frac{\|C(t,x) - C(s,x)\|^2_{\mathbb{C}^{n\times n}}}{|t-s|^2}\, ds\, dt < \infty,$$

$$\|C(t,x)-C(s,x)\|_{\mathbb{C}^{n\times n}} < C|t-s|^{\varepsilon}$$

a.e. for $x\in\Omega$ and $t,s\in[0,\tau]$.
Note that

$$\|\mathcal{A}(t)-\mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} \lesssim \|C(t,.)-C(s,.)\|_{L^\infty(\Omega;\mathbb{C}^{n\times n})}.$$

Hence

$$\mathcal{A}\in W^{\frac{1}{2},2}(0,\tau;\mathcal{L}(\mathcal{V},\mathcal{V}'))\cap C^{\epsilon}([0,\tau];\mathcal{L}(\mathcal{V},\mathcal{V}')).$$

Let $F(t,x):(0,\tau)\times\mathcal{H}\to\mathcal{H}$ and $F_0(t)=F(t,0)$. Assume that $F_0\in L^2_\beta(0,\tau;\mathcal{H})$ and F satisfies the following continuity property:

$$\|F(t,x)-F(t,y)\|_{L^2(\mathbb{R}^d)} \leqslant K\|x-y\|_{H^1(\mathbb{R}^d)}, K>0, x,y\in H^1(\mathbb{R}^d), t\in(0,\tau). \tag{3.4.1}$$

Therefore, applying Theorem 3.3.2 we conclude that for every $u_0\in[\mathcal{H};D(A(0))]_{\frac{1-\beta}{2}}$ the problem

$$u'(t)-\sum_{k,j=1}^{d}\partial_j(a_{kj}(t,.)\partial_k u(t))+\sum_{j=1}^{d}b_j(t,.)\partial_j u(t)+c(t,.)u(t)=F(t,u(t)), t-\text{a.e.},\ u(0)=u_0$$

has a unique solution $u\in L^2_\beta(0,\tau;H^1(\mathbb{R}^d))$ such that $A(.)u\in L^2_\beta(0,\tau;\mathcal{H}), u'\in L^2_\beta(0,\tau;\mathcal{H})$.

Remark 3.4.1. *Note that for all* $\beta\in[0,1[$ *we have*

$$[L^2(\mathbb{R}^d);\mathscr{D}(A(t))]_{\frac{1-\beta}{2}}=[L^2(\mathbb{R}^d);H^1(\mathbb{R}^d)]_{1-\beta}=H^{1-\beta}(\mathbb{R}^d).$$

The maximal regularity we proved here holds also in the case of elliptic operators on Lipschitz domains with Dirichlet or Neumann boundary conditions. The arguments are the same. One define the previous forms $\mathfrak{a}(t)$ with domain $\mathcal{V}=H^1_0(\Omega)$ (for Dirichlet boundary conditions) or $\mathcal{V}=H^1(\Omega)$ (for Neumann boundary conditions).
Assume now that

$$F(t,y)=f(t,x)+g(t,x)|y(x)|^{\alpha}+h(t,x)\sum_{i=1}^{i=n}|\frac{\partial y(x)}{\partial x_i}|^{\gamma},$$

such that $\alpha,\gamma\in[0,1]$ and $f\in L^2_\beta(0,\tau;\mathcal{H}), h\in L^\infty(0,\tau;L^{\frac{2}{1-\gamma}}(\mathbb{R}^d)), y\in H^1(\mathbb{R}^d)$ with

- $g\in L^\infty(0,\tau;L^2(\mathbb{R}^d))$ for $d=1$.
- $g\in L^\infty(0,\tau;L^{\frac{2q}{q-1}}(\mathbb{R}^d))$ for $q\geqslant\frac{1}{\alpha}$ and $d=2$.
- $g\in L^\infty(0,\tau;L^{\frac{2q}{q-1}}(\mathbb{R}^d))$ for $\frac{1}{\alpha}\leqslant q\leqslant\frac{d}{\alpha(d-2)}$ and $d>2$.

Then the finction F satisfies the condition (3.3.1).

Chapter 4

Stochastic evolution equation

Our plan for this chapter is as follow:
First, we discuss existence of the mild solution in the setting of martingale type 2 in the space $S_2(E)$ under Lipschitz and growth linear conditions using fixed point argument. Moreover, (see section 4.2) we extend some lemma to the vector valued setting which allow us to ensure the existence of the malliavin derivative and the adaptness of such a process .Under some additional condition we proof the uniqueness of the solution to an SDE 4.3.1.
Finally, we proof by using assumption 4.4 the exitence of the right inverse oprator, which is essentially important to prove the existence of the malliavin matrix.

4.1 Existence of mild solution

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, W_t be an $\mathscr{H}$-cylindrical Brownian motion, and $\mathcal{F}_t$ be the filtration generated by W_t. We consider stochastic evolution equation 4.1.2 in a $M-$type 2 Banach space E ,and denote by $L_2([0,T], \mathscr{H})$ the space where $\mathscr{H} \subset E$ is a Hilbert space dense in E and the canonical embedding $\mathscr{H} \hookrightarrow E$ is continuous . We recall the definition of $M-$ type p property for a Banach space X.

Definition 4.1.1. *Let $p \in [1,2]$. A Banach space X has* martingale type p *if there exists a constant $\mu \geqslant 0$ such that for all finite X-valued martingale difference sequences $(d_n)_{n=1}^N$ we have*

$$\mathbb{E}\Big\| \sum_{n=1}^{N} d_n \Big\|^p \leqslant \mu^p \sum_{n=1}^{N} \mathbb{E}\|d_n\|^p. \tag{4.1.1}$$

The least admissible constant in this definition is denoted by $\mu_{p,X}$. We consider the stochastic evolution equation in E:

$$\begin{cases} dX_t = (AX_t + \alpha(X_t))dt + \sigma(X_t)dW_t, \\ \\ X_0 = x, \end{cases} \tag{4.1.2}$$

where A is the generator of C_0-semigroup on E , $\alpha : E \longrightarrow E$ and $\sigma : E \longrightarrow \gamma(\mathscr{H}, E)$, further let $\{e_i\}_{i=1}^{\infty}$ denote an orthonormal basis in $\mathscr{H}$.

We prove the existence of mild solution to 4.1.2 on the interval $[0,T]$, $T > 0$ i.e. an $\mathcal{F}-$adapted stochastic process X_t satisfying

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A}\alpha(X_s)ds + \int_0^t e^{(t-s)A}\sigma(X_s)dW_s. \tag{4.1.3}$$

where $x \in E$, $(e^{tA})_{t\geqslant 0}$ is the semigroup generated by A. α and σ satisfy the following Lipschitz and linear growth conditions:

(H1) $\|\alpha(x) - \alpha(y)\|_E + \|\sigma(x) - \sigma(y)\|_{\gamma(\mathscr{H},E)} \leqslant c_1\|x - y\|_E$, for all $x, y \in E$

(H2) $\|\alpha(x)\|_E + \|\sigma(x)\|_{\gamma(\mathscr{H},E)} \leqslant c_2(1 + \|x\|_E)$, for all $x \in E$.

(H3) The functions $\alpha : E \to E$, $\sigma_i : E \to E$, $i = 1, .., n$ have a bounded Fréchet derivatives.

On the Banach space E of $\mathcal{F}$- adapted stochastic process we define the norm:

$$\|\alpha\|^2_{S_2(E)} = \sup_{t\in[0,T]} \mathbb{E}\|\alpha(t)\|^2_E = \|\alpha\|^2_{L^\infty(0,T);L^2(\Omega,E)}. \tag{4.1.4}$$

and let $\mathbb{F}$ be defined by $\mathbb{F} = \{\mathcal{F}_t : t \in [0,T]\}$.

Theorem 4.1.1. *Let E be a martingale type 2 Banach space. Assume that (H_1) and (H_2) hold. Then equation (4.1.2) has a unique solution in the space $S_2(E)$. Furthermore, this solution has a continuous path modification i.e. $C([0,T]; L^2(\Omega, E))$.*

Proof. We consider the map $\Gamma : C([0,T]; L^2(\Omega, E)) \to C([0,T]; L^2(\Omega, E))$,

$$\Gamma(X_t) = e^{tA}x + \int_0^t e^{(t-s)A}\alpha(X_s)\, ds + \int_0^t e^{(t-s)A}\sigma(X_s)\, dW_s. \tag{4.1.5}$$

By the results of [61] , $\int_0^t e^{(t-s)A}\sigma(X_s)\,dW_s$ is E-valued, and

$$\mathbb{E}\Big\|\int_0^t e^{(t-s)A}\sigma(X_s)\,dW_s\Big\|_E^2 \leqslant C_2\mathbb{E}\int_0^t \|e^{(t-s)A}\sigma(X_s)\|^2_{\gamma(H,E)}ds. \tag{4.1.6}$$

Thanks to (H_2) and (4.1.6), the map Γ is well-defined. Then, (H_1) and usual stochastic integral estimates imply that there exists a constant $K > 0$ so that for each pair X and X' from $S_2(E)$

$$\sup_{t\in[0,T]} \mathbb{E}\|\Gamma^n(X_t) - \Gamma^n(X'_t)\|_E^2 \leqslant \frac{K^n(T+1)^nT^n}{n!} \sup_{t\in[0,T]} \mathbb{E}\|X_t - X'_t\|_E^2. \tag{4.1.7}$$

Indeed, for $t \in [0,T]$ one obtains

$$\begin{aligned}
&\mathbb{E}\|\Gamma(X_t) - \Gamma(X'_t)\|_E^2 \\
&\leqslant \mathbb{E}\|\int_0^t e^{(t-s)A}(\alpha(X_s)-\alpha(X'_s))\,ds\|_E^2 + \mathbb{E}\|\int_0^t e^{(t-s)A}(\sigma(X_s)-\sigma(X'_s))\,dW_s\|_E^2 \\
&\leqslant C(\int_0^t ds\int_0^t \mathbb{E}\|\alpha(X_s)-\alpha(X'_s)\|_E^2\,ds + \mathbb{E}(\int_0^t \|\sigma(X_s)-\sigma(X'_s)\|^2_{\gamma(H,E)}\,ds)) \\
&\leqslant K(t\int_0^t \mathbb{E}\|X_s - X'_s\|^2\,ds + \int_0^t \mathbb{E}\|X_s - X'_s\|_E^2\,ds) \\
&\leqslant K(T+1)\int_0^t \mathbb{E}\|X_s - X'_s\|^2\,ds.
\end{aligned}$$

Here, $K = c_1^2(C_2+1)\sup_{t\in[0,T]}\|e^{tA}\|^2_{\mathcal{L}(E)}$. Iterating $(n-1)$ times and using the identity:

$$\int_0^t\int_0^{s_1}...\int_0^{s_{n-1}} ds_n...ds_1 = \frac{1}{n!}t^n,$$

we obtain (4.1.7). Choose the integer n so that $\frac{K^n(T+1)^nT^n}{n!} < 1$. Then $\Gamma^n : C([0,T];L^2(\Omega,E)) \to C([0,T];L^2(\Omega,E))$ is a contraction map. Using Banach fixed point theorem, the map Γ has a unique fixed point in the space $C([0,T];L^2(\Omega,E))$. This fixed point is a unique solution to (4.1.2). □

4.2 The malliavin derivative of the solution

Definition 4.2.1. *Let G a Borel set in $[0,T]$. We define $\mathcal{F}_G$ to be the completed $\sigma-$algebra generated by all random variables of the form*

$$F = \int_0^T \chi_A(t)dW_t$$

for all Borel sets $A \subseteq G$.

Definition 4.2.2. *A real function $f : [0,T] \to \mathbb{R}$ is called symmetric if*

$$f(t_{\sigma_1},\cdots,t_{\sigma_n}) = f(t_1,\cdots,t_n)$$

for all permutation $(\sigma_1,\cdots,\sigma_n)$ of $(1,2,3,...,n)$.

Let $L_2([0,T]^n)$ to be the standard space of square integrable real functions on $[0,T]^n$ such that

$$\|f\|^2_{L_2([0,T]^n)} = \int_{[0,T]^n} f^2(t_1,\cdots,t_n)\, dt_1\cdots dt_n < \infty$$

Let $\tilde{L}_2([0,T]^n)$ be the subspace of $L_2([0,T]^n)$ of symmetric square integrable Borel real functions on $[0,T]^n$. If f is a real function on $[0,T]^n$ then its symmetrization $\tilde{f}$ is defined by

$$\tilde{f} = \frac{1}{n!}\sum_{\sigma} f(t_{\sigma_1},\cdots,t_{\sigma_n})$$

where the sum is taken over all permutations σ of $(1,2,3,...,n)$. Note that $f=\tilde{f}$ iff f is symmetric.

Definition 4.2.3.

(1) If $f\in\tilde{L}_2([0,T]^n)$ we define

$$I_n(f) = \int_{[0,T]^n} f(t_1,\cdots,t_n)dW(t_1)\cdots dW(t_n) = n!J_n(f)$$

where $J_n(f) = \int_0^T\int_0^{t_n}\cdots\int_0^{t_3}\int_0^{t_2} f(t_1,\cdots,t_n)dW_{t_1}\cdots dW_{t_n}$.

(2) Let F be $\mathcal{F}_T-$measurable with chaos expansion

$$F=\sum_{n=0}^{\infty} I_n(f_n)$$

where $f\in\tilde{L}_2([0,T]^n)$, then $F\in\mathbb{D}^{1,2}$ if

$$\|F\|_{\mathbb{D}^{1,2}} = \sum_{n=1}^{\infty} nn!\|f_n\|^2_{L_2([0,T]^n)} < +\infty \tag{4.2.1}$$

(3) If $F\in\mathbb{D}^{1,2}$ we define the Malliavin derivative D_tF of F at time t as the expansion

$$D_tF = \sum_{n=0}^{\infty} nI_{n-1}(f_n(\cdot,t)), \qquad t\in[0,T]. \tag{4.2.2}$$

Lemma 4.2.1. *let E be vector valued space,let $G\subseteq[0,T]$ be a Borel set and $v:[0,T]\to E$, be a stochastic process such that*

(1) for all t , $v(t)$ is measurable with respect to $\mathcal{F}_t\cap\mathcal{F}_G=\mathcal{F}_{[0,t]\cap G}$.

(2) $\mathbb{E}\Big[\int_0^T v^2(t)dt\Big]<\infty$.

Then

$$\int_G v(t)dW(t)$$

is $\mathcal{F}_G-$ measurable.

Proof. By a standard approximation procedure it is sufficient to consider v to be an elementary process of the form

$$v(t) = \sum_{i=1}^{\infty} v_i \chi_{(t_i,t_{i+1}]}(t)$$

where $0 = t_0 < t_1 < \cdots < t_n = T$ and v_i are $\mathcal{F}_{t_i} \cap \mathcal{F}_G-$ measurable random variables such that (2) is satisfied .For such v we have , for all $x^* \in E^*$,

$$\begin{aligned}
\mathbb{E}[\langle \int_0^T \chi_G(t)v(t)dW(t), x^*\rangle|\mathcal{F}_G] &= \mathbb{E}[\int_0^T \chi_G(t)\langle v(t), x^*\rangle dW(t)|\mathcal{F}_G] \\
&= \mathbb{E}[\int_0^T \chi_G(t)\langle \sum_{i=1}^{\infty} v_i \chi_{[t_i,t_{i+1}]}(t), x^*\rangle dW(t)|\mathcal{F}_G] \\
&= \sum_{i=1}^{\infty} \mathbb{E}[\int_{t_i}^{t_{i+1}} \chi_G(t)\langle v_i, x^*\rangle dW(t)|\mathcal{F}_G] \\
&= \sum_{i=1}^{\infty} \int_{t_i}^{t_{i+1}} \chi_G(t)\langle v_i, x^*\rangle dW(t) \\
&= \int_0^T \chi_G(t)\langle v(t), x^*\rangle dW(t) \\
&= \int_G \langle v(t), x^*\rangle dW(t)
\end{aligned}$$

which imply that $\int_G v(t)dW(t)$ is weak F_G-measurable functions and hence F_G-measurable by Pettis measurability theorem. □

Remark 4.2.1. *Note that if A and B are Borel sets in $[0, T]$ then $\mathcal{F}_A \cap \mathcal{F}_B = \mathcal{F}_{A\cap B}$.*

Lemma 4.2.2. *Let $u : [0, T] \to E$ be an $\mathbb{F}-$adapted E-valued stochastic process in $L_2(P \times \lambda)$. Then*

$$\mathbb{E}\Big[\int_0^T u(t)dW(t)|\mathcal{F}_G\Big] = \int_G \mathbb{E}\Big[u(t)|\mathcal{F}_G\Big]dW(t)$$

Proof. Lemma (4.2.1) guarantees that $\int_G \mathbb{E}\Big[u(t)|\mathcal{F}_G\Big]dW(t)$ is $\mathcal{F}_G-$ measurable .Then it suffices to verify that

$$\mathbb{E}\Big[F\int_0^T u(t)dW(t)\Big] = \mathbb{E}\Big[F\int_G \mathbb{E}\Big[u(t)|\mathcal{F}_G\Big]dW(t)\Big] \tag{4.2.3}$$

for all F of the form $F = \int_A dW(t)$, where $A \subseteq G$ is a Borel set.

Then by Itô isometry

$$\mathbb{E}[F\int_0^T g(t)dW(t)] = \mathbb{E}[\int_0^T \chi_A(t)g(t)dt]$$

hence

$$\langle \mathbb{E}\Big[F\int_0^T u(t)dW(t)\Big], x^*\rangle = \mathbb{E}\Big[\int_0^T \chi_A(t)\langle u(t), x^*\rangle dt\Big] = \int_A \mathbb{E}\Big[\langle u(t), x^*\rangle\Big]dt \tag{4.2.4}$$

for all $x^* \in E^*$.
and

$$\langle \mathbb{E}\Big[F\int_G \mathbb{E}\big[u(t)|\mathcal{F}_G\big]dW(t)\Big], x^*\rangle = \mathbb{E}\Big[\int_0^T \chi_A(t)\chi_G(t)\langle\mathbb{E}\big[u(t)|\mathcal{F}_G\big], x^*\rangle dt \qquad (4.2.5)$$

$$= \int_0^T \chi_A(t)\mathbb{E}\Big[\mathbb{E}\big[\langle u(t), x^*\rangle|\mathcal{F}_G\big]\Big]dt$$

$$= \int_A \mathbb{E}\big[\langle u(t), x^*\rangle\big]dt.$$

for all $x^* \in E^*$
from 4.2.4 and 4.2.5 we get 4.2.3 .To conclude we prove that 4.2.3 is equivalent to

$$\mathbb{E}\Big[\chi_A\int_0^T u(t)dW(t)\Big] = \mathbb{E}\Big[\chi_A\int_G \mathbb{E}\big[u(t)|\mathcal{F}_G\big]dW(t)\Big] \qquad (4.2.6)$$

hence by Itô isometry applied to both sides of 4.2.3 we get

$$\mathbb{E}\Big[\chi_A\int_0^T \langle u(t), x^*\rangle dt\Big] = \mathbb{E}\Big[\chi_A\int_0^T \chi_G\mathbb{E}(\langle u(t), x^*\rangle|\mathcal{F}_G)dt\Big] \qquad (4.2.7)$$

$$= \mathbb{E}\Big[\chi_A\int_G \mathbb{E}(\langle u(t), x^*\rangle|\mathcal{F}_G)dt$$

for all $x^* \in E^*$. By definition of the conditional expectation we get

$$\mathbb{E}\Big[\int_0^T \langle u(t), x^*\rangle dt|\mathcal{F}_G\Big] = \int_G \mathbb{E}(\langle u(t), x^*\rangle|\mathcal{F}_G)dt$$

$x^* \in E^*$ density argument completes the proof . □

Proposition 4.2.4. *[15] Let $f_n \in \tilde{L}_2([0,T]^n)$, $n = 1, 2, ...,$ Then*

$$\mathbb{E}\Big[I_n(f_n)|\mathcal{F}_G\Big] = I_n\Big[f_n\chi_G^{\otimes n}\Big] \qquad (4.2.8)$$

where $(f_n\chi_G^{\otimes n})(t_1, ..., t_n) = f_n(t_1, ..., t_n)\chi_G(t_1)\cdots\chi_G(t_n)$.

Proof. We proceed by induction on n .For $n = 1$ we have

$$\mathbb{E}\Big[I_1(f_1)|\mathcal{F}_G\Big] = \mathbb{E}\Big[\int_0^T f_1(t_1)dW(t_1)|\mathcal{F}_G\Big]$$

$$= \int_0^T f_1(t_1)\chi_G(t_1)dW_{t_1} = I_1\Big[f_1\chi_G^{\otimes 1}\Big]$$

By lemma (4.2.2) assume that (4.2.8) holds for $n = k$ then again by lemma 4.2.2 we have

$$\mathbb{E}\Big[I_{k+1}(f_{k+1})|\mathcal{F}_G\Big] =$$

$$(k+1)!\mathbb{E}\Big[\int_0^T\int_0^{t_{k+1}}\cdots\int_0^{t_2} f_{k+1}(t_1,.....,t_{k+1})dW(t_1)\cdots dW(t_k)dW_{t_{k+1}}|\mathcal{F}_G\Big]$$
$$=(k+1)!\int_0^T\mathbb{E}\Big[\int_0^{t_{k+1}}\cdots\int_0^{t_2} f_{k+1}(t_1,.....,t_{k+1})dW(t_1)\cdots dW(t_k)|\mathcal{F}_G\Big]\cdot\chi_G(t_{k+1})dW(t_{k+1})$$
$$=...=(k+1)!\int_0^T\int_0^{t_{k+1}}\cdots\int_0^{t_2} f_{k+1}(t_1,...,t_{k+1})\chi_G(t_1)\cdots\chi_G(t_{k+1})dW(t_1)\cdots dW(t_{k+1}).$$
$$=I_{k+1}\Big[f_{k+1}\chi_G^{\otimes(k+1)}\Big].$$

and the proof is complete. □

Proposition 4.2.5. *Let E be separable Banach space ,and let $F\in\mathbb{D}^{1,2}(E)$, then $\mathbb{E}\Big[F|\mathcal{F}_G\Big]\in\mathbb{D}^{1,2}(E)$ and*

$$D_t\mathbb{E}\Big[F|\mathcal{F}_G\Big]=\mathbb{E}\Big[D_tF|\mathcal{F}_G\Big]\chi_G(t)$$

Proof. First assume that $F=I_n(f_n)$ for some $f_n\in\tilde{L_2}([0,T]^n)$. We have , for all $x^*\in E^*$

$$D_t\mathbb{E}\Big[\langle F,x^*\rangle|\mathcal{F}_G\Big]=D_t\mathbb{E}\Big[\langle I_n(f_n),x^*\rangle|\mathcal{F}_G\Big]$$
$$\overset{(4.2.8)}{=}D_tI_n(\langle f_n,x^*\rangle\chi_G^{\otimes n})$$
$$=nI_{n-1}\Big[\langle f_n,x^*\rangle(\cdot,t)\chi_G^{\otimes(n-1)}(\cdot)\chi_G(t)\Big]$$
$$=nI_{n-1}\Big[\langle f_n(\cdot,t),x^*\rangle\chi_G^{\otimes(n-1)}(\cdot)\Big]\chi_G(t)$$
$$=\mathbb{E}\Big[\langle D_tF,x^*\rangle|\mathcal{F}_G\Big]\chi_G(t) \tag{4.2.9}$$

Next, let $F=\sum_{n=0}^{\infty}I_n(f_n)$ belong to $\mathbb{D}^{1,2}(E)$.Let $F_k=\sum_{n=0}^{k}I_n(f_n)$.Then

$$\langle F_k,x^*\rangle\to\langle F,x^*\rangle \quad \text{in} \quad L_2(\Omega) \quad \text{and } \langle D_tF_k,x^*\rangle\to\langle D_tF,x^*\rangle \quad \text{in} \quad L_2(P\times\lambda)$$

as $k\to\infty$.By (4.2.9) we have

$$D_t\mathbb{E}\Big[\langle F_k,x^*\rangle|\mathcal{F}_G\Big]=\mathbb{E}\Big[\langle D_tF_k,x^*\rangle)|\mathcal{F}_G\Big]\chi_G(t)$$

for all k , and taking the limit with convergence in $L_2(P\times\lambda)$ of this , as $k\to\infty$ we obtain the result. □

Corollary 4.2.2. *Let E be a Banach space and $u(s)$, $s\in[0,T]$ be an $\mathbb{F}$-adapted E-valued stochastic process , assume that $u(s)\in\mathbb{D}^{1,2}(E)$ for all s.Then*

(1) $D_tu(s)$, $s\in[0,T]$, *is $\mathbb{F}$-adapted for all t;*

(2) $D_tu(s)=0$, *for $t>s$.*

Proof. By proposition (4.2.5) we have that

$$\langle D_t u(s), x^* \rangle = D_t \mathbb{E}\Big[\langle u(s), x^* \rangle | \mathcal{F}_s\Big] \overset{4.2.5}{=} \mathbb{E}\Big[D_t \langle u(s), x^* \rangle | \mathcal{F}_s\Big] \chi_{[0,s]}(t)$$

$$= \mathbb{E}\Big[\langle D_t u(s), x^* \rangle | \mathcal{F}_s\Big] \chi_{[t,T]}(s)$$

from which (1) and (2) follow immediately. □

Proposition 4.2.6. *If $u \in L_2(T \times \Omega)$ is an $E-$valued adapted process then $u \in D(\delta)$ .Moreover $\delta(u)$ coincides with the Itô integral with respect to the Brownian motion , that is*

$$\delta(u) = \int_0^T u(s) dW_s$$

here δ denote the Skorohod integral. By $\mathcal{S}$ we denote the class of random variables X such that there exists an $n \in \mathbb{N}$, vectors $h_1, \cdots, h_n$ and a function $f \in C_p^\infty(\mathbb{R}^n)$ such that

$$X = f(W(h_1), \cdots, X(h_n))$$

where $f \in C_p^\infty(\mathbb{R}^n)$ is the space of infinitely differentiable functions on $\mathbb{R}^d$ which, together with all their partial derivatives, have polynomial growth. The elements of $\mathcal{S}$ are called smooth random variables. We consider the space $S(V)$, consisting of V-valued random vectors X of the form

$$X = \sum_{j=1}^m Y_j v_j$$

where $Y_j \in S$ and

$$Y_j = f_j(W(h_1), \cdots, X(h_{n_j}))$$

for certain

$$n_j \in \mathbb{N}, f_j \in C_p^\infty(\mathbb{R}^{n_j})$$

and
$h_1, \cdots, h_j \in H$ and $v_j \in V$.

Theorem 4.2.3. *Let $u \in \mathcal{S}$ and $h \in \mathscr{H}$. Then*

$$\langle D\delta(u), h \rangle - \delta(Dh \cdot u) = \langle u, h \rangle \tag{4.2.10}$$

Lemma 4.2.3. *[45] Let E be a reflexive Banach space and let $p \in (1, \infty)$. Let $(F_n)_{n \geqslant 1}$ be a sequence in $\mathbb{D}^{1,p}(E)$ and $F \in L_p(\Omega; E)$. Assume $F_n \rightharpoonup F$ in $L_p(\Omega; E)$ and that there is a constant C such that for all $n \geqslant 1$,*

$$\|DF_n\|_{L_p(\Omega;\gamma(\mathscr{H},E))} \leqslant C. \tag{4.2.11}$$

Then $F \in \mathbb{D}^{1,p}(E)$ and $\|DF\|_{L_p(\Omega;\gamma(\mathscr{H},E))} \leqslant C$. Moreover, there exists a subsequence $(n_k)_{k \geqslant 1}$ such that $DF_{n_k} \rightharpoonup DF$.

Since martingale type 2 Banach space are reflexive we have the following lemma based on the lemma 4.2.3

Lemma 4.2.4. *[10] Let E be Banach space , $\xi \in \mathbb{D}^{1,2}(E)$ and let $F : E \to E$ have a bounded continuous Fréchet derivative. Then, $F(\xi) \in \mathbb{D}^{1,2}(E)$, and*

$$DF(\xi) = F'(\xi)D\xi. \tag{4.2.12}$$

Theorem 4.2.4. *[29] Suppose that the Banach space X has martingale type 2 and let $\phi : \mathbb{R}_+ \times \Omega \to \mathscr{H} \otimes X$ be an adapted elementary process. Then*

$$\mathbb{E}\| \int_0^\infty \phi \, dW\|^2 \leqslant \mu_{2,X}^2 \mathbb{E} \int_0^\infty \|\phi_t\|^2_{\gamma(\mathscr{H},X)} \, dt.$$

where $\mu_{2,X}$ is the least constant in 4.1.1 in case p=2.

Lemma 4.2.5. *We have $D_r(W_t e_i) = e_i \chi_{[0,t]}(r)$,*

Proof. Indeed , Let F be a smooth random variable ,ie $F : \Omega \to \mathbb{R}$ of the form $F = f(W(h_1), \cdots, W(h_n))$ with $f \in C_b^\infty(\mathbb{R})$(vector space of real valued $C^\infty-$functions with bounded derivatives of all orders), and $h_i \in H, i = 1, .., n$, by definition of the Malliavin derivative

$$DF(W(h_1), \cdots, W(h_n)) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1), \cdots, W(h_n))h_i$$

it follows that if we take $h_1 \in L_2(\mathbb{R}_+, \mathscr{H})$, with $e_i \in \mathscr{H}$ and $n = 1$, $h_1 = \chi_{[0,t]} \otimes e_i$ then $D_r F(W(\chi_{[0,t]} \otimes e_i)) = F'(W(\chi_{[0,t]} \otimes e_i))(\chi_{[0,t]} \otimes e_i)(r)$, we get the result by taking and $F(x) = x$. □

Theorem 4.2.5. *Let E be martingale type 2 space and suppose (H_3) 4.1 is fulfilled. and X_t as defined in 4.1.3 , then $X_t \in \mathbb{D}^{1,2}(E)$ for all $t \in [0,T]$. Moreover, $DX_t \in L_2(\Omega \times [0,T], \gamma(\mathscr{H}, E))$, and for $r \leqslant t$, $D_r X_t$ satisfies the following equation in $\gamma(\mathscr{H}, E)$*

$$D_r X_t = e^{(t-r)A}\sigma(X_r) + \int_r^t e^{(t-s)A}\alpha'(X_s)\, D_r X_s\, ds$$

$$+ \int_r^t e^{(t-s)A}\sigma'(X_s)\, D_r X_s\, dW_s, \text{For } r > t, D_r X_t = 0. \tag{4.2.13}$$

Proof. First we note that $\gamma(\mathscr{H}, E)$ is a M-type 2 Banach space, since

$$\gamma(\mathscr{H}, E) \subseteq L_2(\Omega, E)$$

therefore by 4.2.3 and 4.2.2 , 4.2.5 is well-defined. We construct iterations by setting $X_t^{(0)} = e^{tA}x$, and $X_t^{(n+1)} = \Gamma(X_t^{(n)})$, where Γ is defined by 4.1.5. Notice that each successive iteration $X_t^{(n)}$ has a continuous version, since by the results of [22], the stochastic convolution process has a continuous version. We are going to prove by induction on n that all successive iterations $X_t^{(n)}$ are in the domain $\mathbb{D}^{1,2}(E)$. Clearly, $X_t^{(0)} \in \mathbb{D}^{1,2}(E)$, and $DX_t^{(0)} = 0$. As the induction hypothesis, we assume the following
1) $X_t^{(n)} \in \mathbb{D}^{1,2}(E)$,
2)$DX_t^{(n)} \in L_2(\Omega \times [0,T], \gamma(\mathscr{H}, E))$,
3)for each fixed $r > 0$ the path of $D_r X_t^{(n)}$ is uniformly continuous on $[r, T]$ in the mean-square sense, ie $\mathbb{E} \sup_{r \leqslant t \leqslant T} \|D_r X_t^n\|^2_{\gamma(\mathscr{H},E)} < \infty$.
4) $D_r X_t^{(n)} = 0$ for $r > t$,
5)$\mathbb{E}\|D_r X_t^{(n)}\|^4_{\gamma(\mathscr{H},E)}$ is bounded, ie $\sup_{t\in[0,T]} \mathbb{E}\|D_r X_t^{(n)}\|^4_{\gamma(\mathscr{H},E)} < \infty$, for fixed $r > 0$.

Remark 4.2.6. *Note that, by the induction hypothesis, we can evaluate $DX_t^{(n)}$ at any point $r \in [0,T]$, and write $D_r X_t^{(n)}$ for this evaluation.*

Let us prove these statements for $n+1$. Assuming these hold for n. We start by showing that $X_t^{n+1} \in \mathcal{D}(D)$, where $\mathcal{D}$ is the domain of D. By lemma 4.2.4 and (H_3) we have

$$D_r(\sigma(X_s^{(n)})) = D_r X_s^{(n)} \sigma'(X_s^{(n)}) \chi_{\{r \leqslant s\}} \text{ and } D_r(\alpha(X_s^{(n)})) = D_r X_s^{(n)} \alpha'(X_s^{(n)}) \chi_{\{r \leqslant s\}}$$

Thus the processes $\{D_r(\sigma(X_s^{(n)})), s \geqslant r\}$ and $\{D_r(\alpha(X_s^{(n)})), s \geqslant r\}$ are adapted and there exists $c_1, c_2 > 0$ such that

$$\|D_r(\sigma(X_s^{(n)}))\| \leqslant c_1 \|D_r X_s^n\| \qquad , \quad \|D_r(\alpha(X_s^{(n)}))\| \leqslant c_2 \|D_r X_s^n\|$$

and Using theorem 4.2.3, we deduce that the Itô integral $\int_0^t e^{(t-s)A}\sigma(X_s^n)dW_s \in \mathbb{D}^{1,2}(E)$, the hypothesis $D_t(e^{(t-s)A}\sigma(X_s^n)) \in D(\delta)$ follow from Itô isometry and the induction hypothesis 3) . Moreover, $\int_0^t e^{(t-s)A}\alpha(X_s^{(n)})\, ds \in \mathbb{D}^{1,2}(E)$ by (H_3) and the closability of D . We have

$$D_r X_t^{(n+1)} = e^{(t-r)A}\sigma(X_r^{(n)}) + \int_r^t e^{(t-s)A}\alpha'(X_s^{(n)})D_r X_s^{(n)}\,ds \qquad (4.2.14)$$
$$+\int_r^t e^{(t-s)A}\sigma'(X_s^{(n)})D_r X_s^{(n)}\,dW_s$$

Indeed, we need to prove the following

$$D_r\int_0^t e^{(t-s)A}\sigma(X_s^{(n)})\,dW_s = e^{(t-r)A}\sigma(X_r^{(n)}) + \int_r^t e^{(t-s)A}\sigma'(X_s^{(n)})D_r X_s^{(n)}\,dW_s \qquad (4.2.15)$$

and

$$D_r\int_0^t e^{(t-s)A}\alpha(X_s^{(n)})\,ds = \int_r^t e^{(t-s)A}\alpha'(X_s^{(n)})\,D_r X_s^{(n)}\,ds. \qquad (4.2.16)$$

By 4.2.2, note that the stochastic integral on the right-hand side of 4.2.15 is well-defined and adapted. Indeed, since $D_r X_s^{(n)}$ takes values in $\gamma(\mathscr{H}, E)$, then, by (H_3) 4.1, the integrand of the stochastic integral takes values in $\gamma(\mathscr{H}, \gamma(\mathscr{H}, E))$. This implies ([31],[61]) that the stochastic integral in 4.2.15 is in $L_2(\Omega, \gamma(\mathscr{H}, E))$, and, moreover, that there exists a constant $C > 0$ so that 4.2.4 implies

$$\mathbb{E}\left\|\int_r^t e^{(t-s)A}\sigma'(X_s^{(n)})D_r X_s^{(n)}\,dW_s\right\|^2_{\gamma(\mathscr{H},E)} \leqslant C\int_r^t \mathbb{E}\|e^{(t-s)A}\sigma'(X_s^{(n)})D_r X_s^{(n)}\|^2_{\gamma(H,\gamma(\mathscr{H},E))}\,ds.$$

To prove 4.2.15 and 4.2.16, suppose first that $r > t$. Fix a partition $\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_N = t\}$ and consider a simple integrand of the form

$$\sigma(X^{(n)}, s) = \sum_{i=1}^N e^{(t-t_{i-1})A}\sigma(X^{(n)}_{t_{i-1}})\chi_{(t_{i-1},t_i]}(s). \qquad (4.2.17)$$

Note that $\sigma(X^{(n)}, s)$ converges to $\sigma(X_s^{(n)})$ in the mean-square sense which is implied by the uniform continuity of paths of $X_s^{(n)}$ in the $L_2(\Omega, E)$-norm.
The latter uniform continuity is implied by the relation $X^{(n)} = \Gamma(X^{(n-1)})$, and by the fact that $\mathbb{E}\|X_t^{(n)}\|_E^2$ is bounded uniformly in n and $t \in [0, T]$ which follows from the same relation and the usual stochastic integral estimates.
Then, from Lemma 4.2.4 and the identity in 4.2.5 ,by taking the limit as the mesh of $\mathcal{P}$ goes to 0, we obtain that $D_r\int_0^t e^{(t-s)A}\sigma(X_s^{(n)})dW_s = 0$.
Also, $D_r\int_0^t e^{(t-s)A}\alpha(X_s^{(n)})ds = 0$ if $D_r X_t^{(n)} = 0$. This proves that for $r > t$, $D_r X_t^{(n+1)} = 0$.
Now take an $r \leqslant t$ and fix a partition $\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_N = t\}$ containing r. We have

$$D_r\int_0^t \sigma(X^{(n)}, s)\,dW_s = e^{(t-r)A}\sigma(X_r^{(n)}) + \int_r^t D_r\sigma(X^{(n)}, s)\,dW_s, \qquad (4.2.18)$$

where $D_r\sigma(X^{(n)}, s)$ is computed using 4.2.17.
We consider the approximations of 4.2.15 by simple integrands of form 4.2.17. The right-hand side of

the above relation, converges to the right-hand side of 4.2.15 in $L_2(\Omega, \gamma(\mathscr{H}, E))$ pointwise in $r \in [0,t]$. Indeed, there exists a constant $C > 0$ so that

$$\mathbb{E}\Big\| \int_r^t e^{(t-s)A}\sigma'(X_s^{(n)})D_rX_s^{(n)}\,dW_s - \int_r^t D_r\sigma(X^{(n)}, s)\,dW_s\Big\|^2_{\gamma(\mathscr{H},E)} \tag{4.2.19}$$

$$= \mathbb{E}\Big\| \int_r^t e^{(t-s)A}\sigma'(X_s^{(n)})D_rX_s^{(n)}\,dW_s - \int_r^t D_r\sum_{i=1}^n e^{t-t_{i-1}}\sigma(X_{t_{i-1}}^{(n)})\chi_{(t_{i-1},t_i]}(s)\,dW_s\Big\|^2_{\gamma(\mathscr{H},E)}$$

$$= \mathbb{E}\Big\| \int_r^t e^{(t-s)A}\sigma'(X_s^{(n)})D_rX_s^{(n)}\,dW_s - \int_r^t D_r\sum_{i=1}^n e^{t-t_{i-1}}\sigma(X_{t_{i-1}}^{(n)})\chi_{(t_{i-1},t_i]}(s)\,dW_s +$$

$$\sum_{i=1}^n\int_{t_{i-1}}^{t_i} e^{t-t_{i-1}}D_rX_s^{(n)}\sigma'(X_{t_{i-1}}^{(n)})dW_s - \sum_{i=1}^n\int_{t_{i-1}}^{t_i} e^{t-t_{i-1}}D_rX_s^{(n)}\sigma'(X_{t_{i-1}}^{(n)})dW_s\Big\|^2_{\gamma(\mathscr{H},E)}$$

$$\leqslant^{(i)} C\Big[\Big(\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\mathbb{E}\|e^{(t-s)A}\sigma'(X_s^{(n)}) - e^{(t-t_{i-1})A}\sigma'(X_{t_{i-1}}^{(n)})\|^4_{\gamma(\mathscr{H},E)}ds\Big)^{\frac{1}{2}}$$

$$\times\Big(\int_r^t\mathbb{E}\|D_rX_s^{(n)}\|^4_{\gamma(\mathscr{H},E)}ds\Big)^{\frac{1}{2}} + \sum_{i=1}^n\int_{t_{i-1}}^{t_i}\mathbb{E}\|D_rX_s^{(n)} - D_rX_{t_{i-1}}^{(n)}\|^2_{\gamma(\mathscr{H},E)}\,ds\Big].$$

where the triangle inequality and the Hölder inequality are used In (i).

The right-hand side of the above inequality converges to zero by the uniform continuity of paths of $X_s^{(n)}$, Lebesgue's theorem, and the induction hypothesis 5) 4.2.
The equality 4.2.15 holds by Itô's isometry, by the continuity of paths, and by the closedness of the Malliavin derivative operator. Equality 4.2.16 follows from Hille theorem. Therefore, $X_t^{(n+1)} \in \mathbb{D}^{1,2}(E)$, $DX_t^{(n+1)} \in L_2(\Omega \times [0,T], \gamma(\mathscr{H}, E))$, and relation 4.2.14 holds.
This relation implies that the paths of $D_rX_t^{(n+1)}$ are continuous in the mean-square sense on $[r,T]$. The same relation and the maximal inequality for stochastic convolutions, proved in [22], imply that $\mathbb{E}\|D_rX_t^{(n+1)}\|^4$ is bounded in time. This completes the induction argument.
We note that, by the ideal property of $\gamma(\mathscr{H}, E)$, (see [31]), any bounded operator $E \to E$ induce a bounded operator $\gamma(\mathscr{H}, E) \to \gamma(\mathscr{H}, E)$. Therefore, the stochastic integral on the right-hand side of 4.2.14 is $\gamma(\mathscr{H}, E)$-valued.
The same applies to the Lebesgue integral on the right-hand side of 4.2.16 (since by the induction hypothesis, $D_rX_s^{(n)}$ is $\gamma(\mathscr{H}, E)$-valued).
also we note that $D_rX_t^{(1)} = e^{(t-r)A}\sigma(x)$ takes values in $\gamma(\mathscr{H}, E)$, we obtain that for each fixed $r > 0$, $D_rX_s^{(n)}$ is a $\gamma(\mathscr{H}, E)$-valued process.

Now, prove 4.2.11 for $\xi_n = X_t^{(n)}$. Relation 4.2.14 implies

$$\mathbb{E}\|D_rX_t^{(n+1)}\|^2_{\gamma(\mathscr{H},E)} = \mathbb{E}\|e^{(t-r)A}\sigma(X_r^{(n)}) + \int_r^t e^{(t-s)A}\alpha'(X_s^{(n)})D_rX_s^{(n)}\,ds$$

$$+\int_r^t e^{(t-s)A}\sigma'(X_s^{(n)})D_rX_s^{(n)}\,dW_s\|^2_{\gamma(\mathscr{H},E)}$$

$$\leqslant \mathbb{E}\|e^{(t-r)A}\sigma(X_r^{(n)})\|^2_{\gamma(\mathscr{H},E)} + \mathbb{E}\|\int_r^t e^{(t-s)A}\alpha'(X_s^{(n)})D_rX_s^{(n)}\,ds\|^2_{\gamma(\mathscr{H},E)}$$

$$+\mathbb{E}\|\int_r^t e^{(t-s)A}\sigma'(X_s^{(n)})D_rX_s^{(n)}\,dW_s\|^2_{\gamma(\mathscr{H},E)}$$

$$\leqslant c_1 + c_2\int_r^t \mathbb{E}\|D_rX_s^n\|^2_{\gamma(\mathscr{H},E)}ds + c_3\int_r^t \mathbb{E}\|D_rX_s^n\|^2_{\gamma(\mathscr{H},E)}ds$$

$$\leqslant C(1+\int_r^t \mathbb{E}\|D_rX_s^n\|^2_{\gamma(\mathscr{H},E)}ds)$$

we get

$$\mathbb{E}\|D_rX_t^{(n+1)}\|^2_{\gamma(\mathscr{H},E)} \leqslant C\Big(1+\int_r^t \mathbb{E}\|D_rX_s^{(n)}\|^2_{\gamma(\mathscr{H},E)}\Big), \tag{4.2.20}$$

where $C>0$ is a constant which does not depend on r. By Gronwall's lemma for all n

$$\mathbb{E}\|D_rX_t^{(n)}\|^2_{\gamma(\mathscr{H},E)} \leqslant Ce^{CT}. \tag{4.2.21}$$

Integrating 4.2.21 from 0 to T and using the fact of the canonical embedding of $L_2([0,T],\gamma(\mathscr{H},E))$ into $\gamma(L_2([0,T],\mathscr{H}),E)$ for type 2 Banach spaces (see [31]), we obtain that $D_rX_t^n$ takes values in $\gamma(L_2([0,T],\mathscr{H}),E)$, and

$$\mathbb{E}\|DX_t^{(n)}\|^2_{\gamma(L_2([0,T],\mathscr{H}),E)} \leqslant J\,C\int_0^T \mathbb{E}\|D_rX_t^{(n)}\|^2_{\gamma(\mathscr{H},E)}\,dt \leqslant J\,C\,T\,e^{CT} \tag{4.2.22}$$

where $J>0$ is the embedding constant. By the results of theorem 4.1.1 , $X_t^{(n)} \to X_t$ in $L_2(\Omega,E)$. Hence, by Lemma 4.2.3 , $X_t \in \mathbb{D}^{1,2}(E)$, and, moreover, there is a weakly convergent subsequence $DX_t^{(n_k)} \to DX_t$. By 4.2.22 , this subsequence contains a further subsequence which converges in $L_2(\Omega\times[0,T],\gamma(\mathscr{H},E)))$, again by the canonical embedding of $L_2([0,T],\gamma(\mathscr{H},E))$ into $\gamma(L_2([0,T],\mathscr{H}),E)$. The latter implies that we can evaluate DX_t at $r\in[0,T]$, and, moreover, D_rX_t takes values in $\gamma(\mathscr{H},E)$.

□

4.3 Differentiability with respect to the initial data

Let $X_t(x) = X(x,t,\omega)$ denote the solution to

$$\begin{cases} dX_t = (AX_t + \alpha(X_t))dt + \sigma(X_t)dW_t \\ X_0 = x. \end{cases}$$

If σ and α are Lipschitz, then $X(x,t)$ will be continuous in x. We have X_t is differentiable with respect to x and its derivative is given by

$$Y_t = e^{tA} + \int_0^t e^{(t-s)A}\alpha'(X_s))Y_s\,ds + \int_0^t e^{(t-s)A}\sigma'(X_s)\,Y_s\,dW_s. \tag{4.3.1}$$

To prove the existence of a solution to 4.3.1 in the space of bounded operators we assume the following:

(H4) $\alpha'(x)$ is bounded in $\mathcal{L}(E)$ and $\gamma(\mathscr{H},E)$, and $\sigma'(x)$ are bounded in $\gamma(\mathscr{H},\gamma(\mathscr{H},E))$ and $\gamma(\mathscr{H},\mathcal{L}(E))$.

(H5) The restriction of the semigroup e^{tA} to $\mathscr{H}$ is a semigroup on $\mathscr{H}$.

For simplicity, we will use the same notations, i.e. $\alpha'(x)$, $\sigma'(x)$, e^{tA}, for the restrictions to $\mathscr{H}$.

Theorem 4.3.1. *Suppose (H3), (H4), and (H5) are fulfilled. Then the solution $X(x,t)$ to 4.1.3, is Fréchet differentiable along $\mathscr{H}$ with respect to the initial data x The derivative operator Y_t takes the form $Y_t = e^{tA} + V_t$, so V_t is given by $V_t = Y_t - e^{tA}$ and takes values in $\gamma(\mathscr{H},E)$. Moreover, Y_t is the unique solution to 4.3.1, and possesses a continuous path modification.*

Proof. for the proof of Fréchet differentiability see[17] First we prove uniqueness for 4.3.1 ,let $t_0 > 0$ and we consider $t \leqslant t_0$, we then write

$$\mathbb{E}\sup_{s\leqslant t}\|Y_t - Y_t'\|^2 = \mathbb{E}\sup_{s\leqslant t}\|\int_0^t e^{(t-s)A}\alpha'(X_s)Y_s\,ds - \int_0^t e^{(t-s)A}\alpha'(X_s)Y_s'\,ds +$$

$$\int_0^t e^{(t-s)A}\sigma'(X_s)Y_s\,dW_s - \int_0^t e^{(t-s)A}\sigma'(X_s)Y_s'\,dW_s\|^2$$

$$= \mathbb{E}\sup_{s\leqslant t}\|\int_0^t e^{(t-s)A}\alpha'(X_s))[Y_s - Y_s']\,ds + \int_0^t e^{(t-s)A}\sigma'(X_s))[Y_s - Y_s']\,dW_s\|^2$$

$$\leqslant 2\mathbb{E}\sup_{s\leqslant t}\|\int_0^t e^{(t-s)A}\alpha'(X_s))[Y_s - Y_s']\,ds\|^2 + 2\mathbb{E}\sup_{s\leqslant t}\|\int_0^t e^{(t-s)A}\sigma'(X_s))[Y_s - Y_s']\,dW_s\|^2$$

$$\leqslant^{(i)} 2C\mathbb{E}\,t\int_0^t \|e^{(t-s)A}\alpha'(X_s))[Y_s - Y_s']\|^2\,ds + 2C\mathbb{E}\int_0^T \|\sigma'(X_s))[Y_s - Y_s']\|^2_{\gamma(\mathscr{H},E)}\,ds$$

$$\leqslant 2C(1+t)\mathbb{E}\int_0^t \|Y_s - Y_s'\|^2_{\gamma(\mathscr{H},E)}\,ds$$

where in (i) we use the maximal inequality see[[22]],using Gronwall's lemma we have uniqueness since t_0 is arbitrary. □

4.4 The right inverse operator

In this section, under some additional assumptions, we prove the existence of the right inverse operator to Y_t. Consider the equation

$$Z_t e^{tA} = I + \int_0^t Z_s\Big(\Sigma(X_s) - \alpha'(X_s)\Big)e^{sA}\,ds - \int_0^t Z_s\sigma'(X_s)e^{sA}\,dW_s \tag{4.4.1}$$

with $\Sigma(x) = \sum_{i=1}^{\infty}\sigma'(x)e_i\sigma'(x)e_i$, which is obtained by a formal derivation of an SDE for Y_t^{-1} and multiplying the both parts by e^{tA} from the right. Introducing the operator $R_t = Z_t e^{tA}$, we obtain the SDE for R_t:

$$R_t = I + \int_0^t R_s e^{-sA}\Big(\Sigma(X_s) - \alpha'(X_s)\Big)e^{sA}\,ds - \int_0^t R_s e^{-sA}\sigma'(X_s)e^{sA}\,dW_s. \tag{4.4.2}$$

we will assume the following:

(H6) $[P_tR_t, e^{-TA}(\Sigma(x) - \alpha'(x))e^{TA}] = 0$

where $[\cdot,\cdot]$ denote the commutator.

Theorem 4.4.1. *Let Assumptions H5–H6 be fulfilled. Then, equation 4.4.2 has a unique solution of the form $R_t = I + U_t$ where U_t is $\mathcal{L}(E,\mathscr{H})$-valued. Moreover, the operator $Z_t = R_te^{-tA}$, defined on $e^{tA}E$, is the right inverse to Y_t.*

Proof. We consider the equation:

$$P_t = I + \int_0^t e^{-sA}\alpha'(X_s)e^{sA}P_s\,ds + \int_0^t e^{-sA}\sigma'(X_s)e^{sA}\,P_s\,dW_s. \tag{4.4.3}$$

where $P_t = e^{-tA}Y_t$. Let us show that $P_tR_t = I$ on $\mathscr{H}$. To compute P_tR_t, we apply Itô's formula for the product to $\langle R_ty, P_t^*y^*\rangle$, where $y \in H$, $y^* \in E^*$ after we take the derivative with respect to t:

$$\begin{aligned}\langle P_tR_ty, y^*\rangle &= \langle y, y^*\rangle + \int_0^t \langle e^{-sA}\alpha'(X_s)e^{sA}P_sR_sy, y^*\rangle ds\\ &+ \int_0^t \langle e^{-sA}\sigma'(X_s)e^{sA}\,P_sR_sy, y^*\rangle dW_s - \int_0^t \langle P_sR_se^{-sA}\sigma'(X_s)e^{sA}y, y^*\rangle dW_s\\ &+ \int_0^t \langle P_sR_se^{-sA}\Big(\Sigma(X_s) - \alpha'(X_s)\Big)e^{sA}y, y^*\rangle ds\\ &- \int_0^t \sum_{k=1}^{\infty}\langle e^{-sA}\sigma_k'(X_s)e^{sA}P_sR_se^{-sA}\sigma_k'(X_s)e^{sA}y, y^*\rangle ds.\end{aligned}$$

we take the derivative we get :

$$
\begin{aligned}
\langle dP_tR_ty, y^*\rangle = \langle e^{-tA}\alpha'(X_t)e^{tA}P_tR_ty, y^*\rangle dt + \langle e^{-tA}\sigma'(X_t)e^{tA}\, P_tR_ty, y^*\rangle dW_t \\
-\langle P_tR_te^{-tA}\sigma'(X_t)e^{tA}y, y^*\rangle dW_t \\
+\langle P_tR_te^{-tA}\Big(\Sigma(X_t) - \alpha'(X_t)\Big)e^{tA}y, y^*\rangle dt - \sum_{k=1}^{\infty}\langle e^{-tA}\sigma_k'(X_t)e^{tA}P_tR_te^{-tA}\sigma_k'(X_t)e^{tA}y, y^*\rangle dt.
\end{aligned}
$$

under the hypothesis (H6)

$$
\langle dP_tR_ty, y^*\rangle = 0 \tag{4.4.4}
$$

for all $y^* \in E^*$ hence $dP_tR_ty = 0$ this imply that $P_tR_t = P_0R_0 = I$,so $P_tR_t = I$. □

Chapter 5

Appendix

5.1 C_0- Semigroups

C_0 -semigroups serve to describe the time evolution of autonomous linear systems. The objective of the present section is to introduce the notion of C_0 -semigroups and their generators, and to derive some basic properties.

5.2 Definition and some basic properties

Let X be a (real or complex) Banach space. A one-parameter semigroup on X is a function $T : [0, \infty) \to \mathcal{L}(X)$ (where $\mathcal{L}(X)$ denotes the space of bounded linear operators in X, with domain all of X), satisfying
(i) $T(t+s) = T(t)T(s)$, for all $t, s \geqslant 0$. If additionally
(ii) $\lim_{t \to 0+} T(t)x = x$ for all $x \in X$ then T is called a C_0 -semigroup (on X) (also a strongly continuous semigroup). If T is defined on $\mathbb{R}$ instead of $[0, \infty)$, and (i) holds for all $t, s \in \mathbb{R}$, then T is called a one-parameter group, and if additionally (ii) holds, then T is called a C_0 -group.

Remark 5.2.1. *(a) Property (i) implies that for $t, s \geqslant 0$ the operators $T(t), T(s)$ commute; also, if $t_1, t_2, \ldots, t_n \geqslant 0$, then $T\left(\sum_{j=1}^{n} t_j\right) = \prod_{j=1}^{n} T(t_j)$*
(b) Property (i) implies that $T(0) = T(0)^2$ is a projection.
(c) If T is a C_0 -semigroup, then $T(0)x = \lim_{t \to 0_+} T(t)T(0)x = \lim_{t \to 0_+} T(t)x = x$ for all $x \in X$, i.e., $T(0) = I$.

Proposition 5.2.1. *Let T be a C_0 -semigroup on X.*
(a) Then there exist $M \geqslant 0$ and $\omega \in \mathbb{R}$ such that

$$\|T(t)\| \leqslant M \mathrm{e}^{\omega t} \text{ for all } t \geqslant 0$$

(b) For all $x \in X$ the function $[0, \infty) \ni t \mapsto T(t) x \in X$ is continuous. In other words, the function T is strongly continuous.
(c) If T is a C_0 -group on X, then there exist $M \geqslant 0$ and $\omega \in \mathbb{R}$ such that

$$\|T(t)\| \leqslant M \mathrm{e}^{\omega|t|} \text{ for all } t \in \mathbb{R}$$

For all $x \in X$ the function $\mathbb{R} \ni t \mapsto T(t) x \in X$ is continuous.

Proof. (a) In view of Lemma 1.1.4 it is sufficient to show that there exists $\delta>0$ such that $\sup_{0 \leqslant t<\delta}\|T(t)\|<\infty$. Assuming that this is not the case we can find a null sequence (t_n) in $(0, \delta)$ such that $\|T(t_n)\| \rightarrow \infty$ as $n \rightarrow \infty$. However, for all $x \in X$ the sequence $(T(t_n) x)$ is convergent (to x), by property (ii) of C_0 -semigroups. Therefore the uniform boundedness theorem (for which we refer to [[33]; II, 1 Corollary 1] or [[20]; Théorème II. 1]) implies that $\sup_{n \in \mathbb{N}}\|T(t_n)\|<\infty$; a contradiction.
(b) Let $x \in X, t>0$. Then $T(t+h) x-T(t) x=T(t)(T(h) x-x) \rightarrow 0$ as $h \rightarrow 0+$, which proves the right-sided continuity of $T(\cdot) x$. In order to prove the left-sided continuity we let $-t \leqslant h<0$ and write $T(t+h) x-T(t) x=T(t+h)(x-T(-h) x)$. Then we obtain

$$\|T(t+h) x-T(t) x\| \leqslant\left(\sup _{0 \leqslant s \in t}\|T(s)\|\right)\|x-T(-h) x\| \rightarrow 0$$

as $h \rightarrow 0-$
(c) First we show that, given $x \in X$, the orbit $T(\cdot) x$ is continuous. As the restriction of T to $[0, \infty)$ is a C_0 -semigroup it follows from (b) that $T(\cdot) x$ is continuous on $[0, \infty)$. Let $t \leqslant 0$. Then $T(t+h) x-T(t) x= T(t-1)(T(1+h) x-T(1) x) \rightarrow 0(h \rightarrow 0)$, and this implies that $T(\cdot) x$ is continuous on $\mathbb{R}$. As a consequence, the function $[0, \infty) \ni t \mapsto T(-t) \in \mathcal{L}(X)$ is a C_0 -semigroup, and therefore satisfies an estimate as in (a). Putting the estimates for the C_0 -semigroups $t \mapsto T(t)$ and $t \mapsto T(-t)$ together one obtains the asserted estimate.

□

5.3 Operators

Let X, Y be two vector spaces over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For a linear relation in $X \times Y$, i.e., a subspace $A \subseteq X \times Y$, we define the domain of A

$$\mathscr{D}(A) = \{x \in X;\ \text{there exists } y \in Y \text{ such that } (x, y) \in A\}$$

the range of A $\operatorname{ran}(A) := \{y \in Y;$ there exists $x \in X$ such that $(x, y) \in A\}$ and the kernel (or null space) of A,

$$\ker(A) := \{x \in X; (x, 0) \in A\}$$

The linear relation

$$A^{-1} := \{(y, x); (x, y) \in A\}$$

in $Y \times X$ is the inverse relation of A. If B is another linear relation in $X \times Y$, satisfying $A \subseteq B$, then B is called an extension of A, and A a restriction of B
In this setting, a linear operator from X to Y is a linear relation in $X \times Y$ satisfying additionally

$$A \cap (\{0\} \times Y) = \{(0, 0)\}$$

Then, for all $x \in \operatorname{dom}(A)$, there exists a unique $y \in Y$, such that $(x, y) \in A$, and we will write $Ax = y$. In this sense, A is also a mapping $A : \mathscr{D}(A) \to Y$. As we will consider only linear operators we will mostly drop 'linear' and simply speak of 'operators'. If the spaces X and Y coincide, then we call A an operator in X. Next, let X and Y be Banach spaces. We define a norm on $X \times Y$ by

$$\|(x, y)\|_{X \times Y} := \|x\|_X + \|y\|_Y \quad ((x, y) \in X \times Y)$$

which makes $X \times Y$ a Banach space. In this context an operator A from X to Y is called closed if A is a closed subset of $X \times Y$, and A is closable if the closure $\bar{A}$ of A in $X \times Y$ is an operator.
For a subspace $D \subseteq \mathscr{D}(A)$, the restriction of A to D is the operator $A|_D := A \cap (D \times \underline{Y})$. The set D is called a core for A if A is a restriction of the closure of $A|_D$ i.e., $A \subseteq \overline{A|_D}$ Finally, if A and B are operators from X to Y, then the sum of A and B is the is the operator defined by

$$\mathscr{D}(A + B) := \mathscr{D}(A) \cap \mathscr{D}(B), \quad (A + B)x := Ax + Bx \quad (x \in \mathscr{D}(A) + \mathscr{D}(B))$$

or, expressed differently,

$$A + B = \{(x, Ax + Bx); x \in \mathscr{D}(A) \cap \mathscr{D}(B)\} \subseteq X \times Y$$

We mention that in most cases of the use of a sum of two operators the domain of one of the operators is a subset of the domain of the other, or more specially, one of the operators is defined everywhere.

5.4 Resolvent set, spectrum and resolvent

Let X be a Banach space over $\mathbb{K}$, and let A be an operator in X. We define the resolvent set of A,

$$\rho(A) := \left\{\lambda \in \mathbb{K}; \lambda I - A : \mathscr{D}(A) \to X \text{ bijective, } (\lambda I - A)^{-1} \in \mathcal{L}(X)\right\}$$

The operator $R(\lambda, A) := (\lambda I - A)^{-1}$ is called the resolvent of A at λ, and the mapping

$$R(\cdot, A) : \rho(A) \to \mathcal{L}(X)$$

is called the resolvent of A. The set

$$\sigma(A) := \mathbb{K} \backslash \rho(A)$$

is called the spectrum of A. The following theorem contains the basic results concerning the resolvent.

Theorem 5.4.1. *Let A be a closed operator in X. (a) If $\lambda \in \rho(A), x \in \mathscr{D}(A)$, then $AR(\lambda, A)x = R(\lambda, A)Ax$ (b) For all $\lambda, \mu \in \rho(A)$ one has the resolvent equation*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\mu, A)R(\lambda, A)$$

Proof. (a) $AR(\lambda, A)x - \lambda R(\lambda, A)x = -x = R(\lambda, A)Ax - R(\lambda, A)\lambda x$
(b) Multiplying the equation

$$(\mu I - A) - (\lambda I - A) = (\mu - \lambda)I|_{\text{dom}(A)}$$

from the right by $R(\lambda, A)$ and from the left by $R(\mu, A)$, one obtains the resolvent equation. □

Remark 5.4.2. *The analyticity of $R(\cdot, A)$ implies that $R(\cdot, A)$ is infinitely differentiable, and from the Neumann power series one can read off the derivatives*

$$\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^n R(\lambda, A) = (-1)^n n! R(\lambda, A)^{n+1} \quad (\lambda \in \rho(A), n \in \mathbb{N}_0)$$

Proposition 5.4.1. *Let X, Y be Banach spaces, $a, b \in \mathbb{R}, a < b$. Let $F : [a, b] \to \mathcal{L}(X, Y)$ be strongly continuous, and assume that $h : [a, b] \to [0, \infty)$ is an integrable function such that $\|F(t)\| \leqslant h(t)(a \leqslant t \leqslant b)$. Then the mapping*

$$X \ni x \mapsto \int_a^b F(t)x \, \mathrm{d}t \in Y$$

belongs to $\mathcal{L}(X, Y)$ and has norm less or equal $\int_a^b h(t)\mathrm{d}t$.

5.5 Characterisation of generators of C_0 -semigroups

In this section let X be a Banach space.

Theorem 5.5.1. *Let T be a C_0 -semigroup on X, and let A be its generator. Let $M \geqslant 1$, $\omega \in \mathbb{R}$ be such that*

$$\|T(t)\| \leqslant M\mathrm{e}^{\omega t} \quad (t \geqslant 0)$$

Then $\{\lambda \in \mathbb{K}; \operatorname{Re}\lambda > \omega\} \subseteq \rho(A)$, and for all $\lambda \in \mathbb{K}$ with $\operatorname{Re}\lambda > \omega$ one has

$$R(\lambda, A) = \textstyle\int_0^\infty \mathrm{e}^{-\lambda t} T(t)\mathrm{d}t \quad (\textit{strong imp}$$
$$\|R(\lambda, A)^n\| \leqslant \tfrac{M}{(\operatorname{Re}\lambda - \omega)^n} \quad (n \in \mathbb{N})$$

In the proof we will use the concept of rescaling. If T is a C_0 -semigroup on X with generator A, and $\lambda \in \mathbb{K}$, then it is easy to see that T_λ, defined by

$$T_\lambda(t) := \mathrm{e}^{-\lambda t} T(t) \quad (t \geqslant 0)$$

is also a C_0 -semigroup, called a rescaled semigroup, and that the generator of T_λ is given by $A - \lambda I$; .

Proof. Let $\lambda \in \mathbb{K}, \operatorname{Re}\lambda > \omega$. Observe that the rescaled semigroup T_λ obeys the estimate

$$\|T_\lambda(t)\| \leqslant M\mathrm{e}^{(\omega - \operatorname{Re}\lambda)t} \quad (t \geqslant 0)$$

and that the resolvent of A at λ corresponds to the resolvent of $A - \lambda I$ at 0. This means that it is sufficient to prove the existence and the formula of the resolvent for the case $\lambda = 0$ and $\omega < 0$ The estimate $\|T(t)\| \leqslant M\mathrm{e}^{-\infty t}(t \geqslant 0)$ implies that the strong improper integral

$$R := \int_0^\infty T(t)\mathrm{d}t$$

defines an operator $R \in \mathcal{L}(X)$. Let $x \in \mathscr{D}(A)$. Then

$$RAx = \int_0^\infty T(t)Ax \; \mathrm{d}t = \lim_{c\to\infty} \int_0^c \frac{\mathrm{d}}{\mathrm{d}t} T(t)x \; \mathrm{d}t = \lim_{c\to\infty} (T(c)x - x) = -x$$

Further, $\|T(t)x\| \leqslant M\mathrm{e}^{\omega t}\|x\|$ and $\|AT(t)x\| \leqslant M\mathrm{e}^{\omega t}\|Ax\|(t \geqslant 0)$, and therefore The(Hille's theorem), which also holds in the present context, implies that $Rx \in \mathscr{D}(A)$ and

$$ARx = \int_0^\infty AT(t)x \; \mathrm{d}t = \int_0^\infty T(t)Ax \; \mathrm{d}t = RAx = -x$$

If $x \in X$, and (x_n) is a sequence in $\mathscr{D}(A)$ with $x = \lim_{n\to\infty} x_n$, then $Rx_n \to Rx$ and $ARx_n = -x_n \to -x(n \to \infty)$, and because A is closed we conclude that $Rx \in \mathscr{D}(A)$ and $ARx = -x$. The two equations

$RA = -I|_{\mathrm{dom}\,(A)}$, $AR = -I$ imply that $0 \in \rho(A)$ and $R = (-A)^{-1}$ For the powers of $R(\lambda, A)$ we now obtain (recall Remark 5.4.2)

$$\begin{aligned} R(\lambda, A)^n &= (-1)^{n-1}\frac{1}{(n-1)!}\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^{n-1}\int_0^\infty \mathrm{e}^{-\lambda t}T(t)\mathrm{d}t \\ &= \frac{1}{(n-1)!}\int_0^\infty t^{n-1}\mathrm{e}^{-\lambda t}T(t)\mathrm{d}t \end{aligned}$$

(The last equality is obtained by differentiation under the integral By Proposition 5.4.1 this yields the estimate

$$\begin{aligned} \|R(\lambda, A)^n\| &\leqslant \frac{1}{(n-1)!}M\int_0^\infty t^{n-1}\mathrm{e}^{(\omega-\mathrm{Re}\,\lambda)t}\,\mathrm{d}t \\ &= \frac{1}{(n-1)!}M\left(\frac{\mathrm{d}}{\mathrm{d}\omega}\right)^{n-1}\int_0^\infty \mathrm{e}^{(\omega-\mathrm{Re}\lambda)t}\,\mathrm{d}t \\ &= \frac{1}{(n-1)!}M\left(\frac{\mathrm{d}}{\mathrm{d}\omega}\right)^{n-1}\frac{1}{\mathrm{Re}\,\lambda-\omega} = \frac{M}{(\mathrm{Re}\,\lambda-\omega)^n} \end{aligned}$$

□

Lemma 5.5.1. *(**Grönwall's Lemma**) If $\psi \geqslant 0$ and ϕ two continuous functions satisfying the following condition :*

$$\forall t \geqslant t_0 \quad \phi(t) \leqslant K + \int_{t_0}^t \psi(s)\phi(s)\,\mathrm{d}s$$

where K is a constant, then we have :

$$\forall t \geqslant t_0 \quad \phi(t) \leqslant K \exp\left(\int_{t_0}^t \psi(s)\,\mathrm{d}s\right)$$

Lemma 5.5.2. *Let $\mathcal{V}, \mathscr{H}$ be Hillert spaces with $\mathcal{V} \hookrightarrow \mathscr{H}$. Let $a : \mathcal{V} \times \mathcal{V} \to \mathbb{K}$ be an coercive continuous form. Let $b : \mathcal{V} \times \mathcal{V} \to \mathbb{K}$ be a continuous form such that*

$$|b(u)| \leqslant M\|u\|_{\mathcal{V}}\|u\|_{\mathscr{H}} \quad (u \in \mathcal{V}).$$

Then $a + b : \mathcal{V} \times \mathcal{V} \to \mathbb{K}$ is coercive

Proof. By the coercivity of a there exist $\omega \in \mathbb{R}$ and $\alpha > 0$ such that

$$\mathrm{Re}\,a(u) + \omega\|u\|_{\mathscr{H}}^2 \geqslant \alpha\|u\|_{\mathcal{V}}^2$$

for all $u \in \mathcal{V}$. By the "Peter-Paul inequality" (i.e., Young's inequality, $ab \leqslant \frac{1}{2}\left(\gamma a^2 + \frac{1}{\gamma} b^2\right)$ for all $a, b \geqslant 0, \gamma > 0$) one has

$$\begin{aligned}\operatorname{Re} a(u) + \operatorname{Re} b(u) + \omega \|u\|_{\mathscr{H}}^2 &\geqslant \alpha \|u\|_{\mathcal{V}}^2 - M \|u\|_{\mathcal{V}} \|u\|_{\mathscr{H}} \\ &\geqslant \alpha \|u\|_{\mathcal{V}}^2 - \frac{1}{2}\left(\alpha \|u\|_{\mathcal{V}}^2 + \frac{1}{\alpha} M^2 \|u\|_{\mathscr{H}}^2\right).\end{aligned}$$

This implies

$$\operatorname{Re}(a(u) + b(u)) + \left(\omega + \frac{M^2}{2\alpha}\right) \|u\|_{\mathscr{H}}^2 \geqslant \frac{\alpha}{2} \|u\|_{\mathcal{V}}^2 \quad (u \in \mathcal{V}).$$

This gives the coercivity of $a + b$. □

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