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Limit Cycles: Qualitative and Numerical Study

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Dedication

I dedicate this thesis to :

My dear mother for her encouragement me, and support during my whole life.

To the spirit of my dear father who taught me the importance of teaching and learning.

My brothers and their wives and their children.

My sisters and their husbands and their children.

My friends who encourage and support me.

All the people in my life who touch my heart.

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Aziza Berbache, Ahmed Bendjeddou and Sabah Benadouane [19], *Explicit non-algebraic limit cycle of a family of polynomial differential systems of degree even*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.), Tomul LXV, f. 2 (2019).

Sabah Benadouane, Aziza Berbache, Ahmed Bendjeddou [7], *Two non-algebraic limit cycles of a class of polynomial differential systems with non-elementary equilibrium point*, Tatra Mt. Math. Publ. 79, 33–46 (2021).

Aziza Berbache, Ahmed Bendjeddou and Sabah Benadouane [20], *Maximum number of limit cycles for generalized liénard polynomial differential systems*, Mathematica Bohemica, 15 pp (2020).

Sabah Benadouane, Aziza Berbache, Ahmed Bendjeddou [8], *Upper bounds for the number of limit cycles for a class of polynomial differential systems via the averaging method*, Buletinul Academiei De Ştiinţ Ea Republicii Moldova Matematica, Number 3(97), Pages 72–87, 2022.

General Introduction

Dynamical systems model a large family of phenomena which manifest itself in several domains (biology, electronics, mechanics, etc). They have unquestionably become the main mathematical tool for describing a process that evolves over time. The study of these systems is a mathematical domain which historically has been the subject of several works [6],[4],[50]. Many natural phenomena can be modeled by the nonlinear differential systems. Therefore, scientists were very interested in finding explicit solutions to this type of differential systems, as these solutions were often impossible to determine explicitly. Some mathematicians were redirected to a numerical resolution which consists of approaching solutions through different methods. The problem encountered in these methods is that the calculation of the solutions is only done on an interval of finite time. In this case, the global properties of the solutions are not entirely determined. At the end of the 19*th* century, Henri Poincaré [48] in "Mémoire sur les courbes définies par une équation différentielle" introduced the study of qualitative method for differential systems. Using geometric and topological techniques, Poincaré was able to investigate qualitative properties of the solutions of a differential systems without such solutions having to be determined explicitly. Among the contributions of Poincaré, we can mention the concept of phase portrait, the concept such as return map, which are fundamental for classifying orbits with particular behaviors. These results would be the pillars of the "Qualitative Theory of Differential Systems". One of the main important problems in the qualitative theory of differential systems is the study of the integrability and the existence of periodic orbits, more precisely the limit cycles, their number and their stability. A limit cycle is an isolated periodic solution in the set of all periodic solutions. The concept of limit cycles appeared for the first time in the famous articles of H. Poincaré [49]. Many mathematical models in physics, engineering, chemistry, biology, and economics have been displayed as autonomous planar polynomial differential systems with limit cycles. Recently, the notion of limit cycles has become well known, and has attracted the attention of many pure and applied mathematicians, see for instance ([1], [3], [25], [33], [35], [42]) and the book of Ye

Yanqian [6], [50], [54] .

In 1900, David Hilbert [39], in International Congress of Mathematics proposed 23 problems. Among these problems, The 16*th* Hilbert problem, whose second part asking about the maximum number $H(n)$ and the relative position of the limit cycles of a planar polynomial differential systems of degree n of the form

$$\begin{cases} \dot{x} = \dfrac{dx}{dt} = P(x(t), y(t)), \\ \dot{y} = \dfrac{dy}{dt} = Q(x(t), y(t)), \end{cases} \tag{1}$$

where P and Q are real polynomials in the variables x and y. Until nowadays, the 16*th* Hilbert problem remains unsolved, because it is a very difficult problem. Since any linear system in $\mathbb{R}^2$ has no limit cycles, it follows that $H(1) = 0$. But the finiteness of $H(n)$ with $n \geq 2$ is still an open problem. Dulac [28] in 1923 claimed that any polynomial differential systems (1) always has finitely many limit cycles. Ilyashenko [41] in 1985 found an error in Dulac's paper. Later on, two long works have appeared, independently, providing proofs of Dulac's assertion, one due to Écalle [30] in 1992 and the other to Ilyashenko [40] in 1991. As Smale mentioned in [52] these two papers have yet to be thoroughly digested by the mathematical community. We usually only ask for the number of such limit cycles, but their location as orbits of the system is also an interesting problem, and an even more difficult problem is to give an explicit expression of it. In general, it is not easy to distinguish when a limit cycle is algebraic or not. For example, the limit cycle of the van der Pol differential system discovered in 1927 (see [53]) was not proved until 1995 by Odani [45] that it was non-algebraic. The van der Pol system can be written as a polynomial differential systems (1) of degree 3, but its limit cycle is not known explicitly.

In the last years, several papers were published exhibiting polynomial differential systems for which non-algebraic limit cycles are explicitly known. The first explicit non-algebraic limit cycle, due to Gasull, Giacomini and Torregrosa [32], was for a polynomial differential system of degree 5. Afterwards, Al-Dosary [2] gave a family of polynomial differential systems, with an explicit non-algebraic limit cycle, which generalizes the system studied in [32]. In 2006 Giné and Grau [37] shown the simultaneous existence of two limit cycles one algebraic and an explicit non-algebraic in a polynomial differential system of degree 9. In [16], Benterki and Llibre gaves an example of a cubic system with an explicit non-algebraic limit cycle and it was posed an open problem about provide an explicit non-algebraic limit cycle for a polynomial differential system of degree 2, which it remains open until now. Later, Bendjeddou and all published many papers providing an explicit non-algebraic limit cycles for several polynomial differential systems of degree larger than or equal to 3 see [9],[11],[12],[14], [15],[17],[18].

Our contribution in this thesis, is to study the integrability and the existence of limit cycles of some classes of polynomial differential systems of the form (1), more precisely, we present two new results about this topic, firstly, we determine the explicit expression of non algebraic limit cycle of a family of polynomial differential systems of degree even. As a second result, we construct a class of polynomial differential systems of degree $6k+1, (k \in \mathbb{N}^*)$, with explicit two non-algebraic limit cycles surrounding a non-elementary equilibrium point at origin.

This thesis is divided into four chapters:

Chapter 1, we briefly present some basic concepts, definitions and results that are used in the other chapters, for example the definition of autonomous polynomial differential system, periodic solution and limit cycle, integrability of differential system. We introduce also some theorems of existence and non existence of limit cycles.

In **chapter 2,** we study the integrability and the existence of non algebraic limit cycle of a family of polynomial differential systems of degree even, we give the explicit expression of first integral and limit cycle.

Chapter 3 is devoted to provide the integrability and the existence of two non-algebraic limit cycles surrounding a non-elementary equilibrium point at origin for a class of polynomial differential systems of degree $6k + 1, (k \in \mathbb{N}^*)$.

And finally in **chapter 4**, we present a software called P4 (Polynomial Planar Phase Portraits), we use to draw the phase portraits of all polynomial differential systems in this thesis. We finish this our thesis by a conclusion and outlook.

Chapter 1

Preliminary notions

Contents

1.1 Introduction

In this chapter we present some basic notions about qualitative study of differential systems, we start to present the general form of an autonomous polynomial differential systems, next, we give some definitions about: vector field, solution and periodic solution of differential system, invariant curve, integrability, limit cycles. We introduce also some theorems of existence and non existence of limit cycles. Finally, we present the $16th$ Hilbert problem. We use some of these notions in the next chapters.

1.2 Autonomous polynomial differential systems

Definition 1.1. *An autonomous polynomial differential systems is a system of the form*

$$\begin{cases} \dot{x} = \dfrac{dx}{dt} = P\left(x\left(t\right), y\left(t\right)\right), \\ \dot{y} = \dfrac{dy}{dt} = Q\left(x\left(t\right), y\left(t\right)\right), \end{cases} \tag{1.1}$$

where P and Q are real polynomials in the variables x and y. The system (1.1) is of degree n where $n = \max(\deg(P), \deg(Q))$.

Definition 1.2. *In the plane, a homogeneous polynomial differential systems is a system written as follows*

$$\begin{cases} \dot{x} = P\left(x\left(t\right), y\left(t\right)\right) = \sum\limits_{i+j=0}^{i+j=m} \alpha_{ij} x^i y^{m-j}, \\ \dot{y} = Q\left(x\left(t\right), y\left(t\right)\right) = \sum\limits_{i+j=0}^{i+j=m} \beta_{ij} x^i y^{m-j}. \end{cases}$$

1.2.1 Vector field

It is convenient to represent graphically the vector field, before starting the study of a differential system, because it gives us precious information on the different forms of the possible solutions as well as their asymptotic behavior.

Definition 1.3. *A vector field X is a region of the plane in which there exists at every point A of $\Omega \subseteq \mathbb{R}^2$ a vector $\overrightarrow{V}(A,t)$, i.e an application:*

$$X : \Omega \subseteq \mathbb{R}^2 \longmapsto \mathbb{R}^2$$
$$A\left(x, y\right) \longmapsto \overrightarrow{V}\left(A\right) = \begin{pmatrix} P\left(x, y\right) \\ Q\left(x, y\right) \end{pmatrix},$$

where P and Q are of class $\mathbb{C}^1$ on $\Omega \subset \mathbb{R}^2$. The vector field associated to system (1.1) can be represented by the following differential operator

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y},$$

then for the following we consider the vector field X associated with the differential polynomial planar system (1.1)

$$\frac{d\overrightarrow{A}}{dt} = \overrightarrow{V}(A) \Longleftrightarrow \begin{cases} \frac{dx}{dt} = P(x(t), y(t)), \\ \frac{dy}{dt} = Q(x(t), y(t)). \end{cases}$$

Remark 1.1. 1. *In this thesis, we assume that P and Q are $\mathbb{C}^1$ functions, then the Cauchy Lipchitz conditions are satisfied at any point of system* (1.1)*. So in each initial condition (x_0, y_0), the system* (1.1) *has a unique solution.*

2. *The plane of the x and y variables is called the phase plane.*

3. *On the curve $P(x, y) = 0$, called vertical isocline, the vector field is parallel to the $y-$axis, and on the curve $Q(x, y) = 0$, called horizontal isocline, the vector field is parallel to the $x-$axis.*

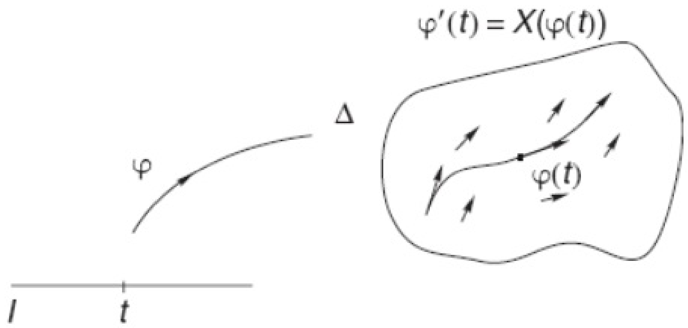


Figure 1.1: Vector field.

1.2.2 Solution and periodic solution

Solution of differential system

Definition 1.4. *A solution to differential system* (1.1) *is an application*

$$\varphi : I \subseteq \mathbb{R} \to \mathbb{R}^2$$
$$t \to \varphi(t) = (x(t), y(t)),$$

where I is a non-empty interval such that, for all $t \in I$, the solution $\varphi(t) = (x(t), y(t))$ satisfies this system i.e: $\dot{\varphi}(t) = X(\varphi(t))$. If $\varphi_1(t) = (x_1(t), y_1(t))$ and $\varphi_2(t) = (x_2(t), y_2(t))$ are two solutions on I_1 and I_2 respectively. We say that $\varphi_2(t)$ is an extension of $\varphi_1(t)$ if $I_1 \subset I_2$ and $\forall t \in I$, $\varphi_1(t) = \varphi_2(t)$. The solution $(x(t), y(t))$ is called maximal solution on I if it does not admit any extension on I.

Periodic solution

Definition 1.5. *We say that the solution* $\varphi(t) = (x(t), y(t))$ *is a periodic solution of the system* (1.1) *if there exists a real number* $T > 0$ *such that* $\forall t \in I, \varphi(t+T) = \varphi(t)$. *The smallest number* T *is called the period of the solution* φ.

Flow

Definition 1.6. *[46] We recall flow associated with the vector field* (P, Q) *the application*

$$\Phi : \mathbb{R} \times \mathbb{R}^2 \mapsto \mathbb{R}^2$$
$$(t, (x, y)) \mapsto \Phi_t(x, y),$$

satisfied the following three properties:

i) $\frac{d}{dt}\Phi_t(x, y) = (P(\Phi_t(x, y)), Q(\Phi_t(x, y)))$,

ii) $\Phi_0(x, y) = (x, y)$,

iii) $\Phi_{t+s}(x, y) = \Phi_t(\Phi_s(x, y))$, *for all* $(x, y) \in \mathbb{R}^2$ *and* $s, t \in \mathbb{R}$.

1.3 Phase portrait

Every solution of (1.1), say $\varphi(x(t), y(t))$, can be represented as a curve in the plane. The solution curves are called trajectories or orbits. The equilibrium points of this system are constants solutions and the complete figure of the orbits of this system as well as its points of equilibrium represented in the plane (x, y) is called phase portrait, and the plane (x, y) is called the phase plane.

Definition 1.7. *The phase portrait is the set of orbits that represent the solutions of system* (1.1) *in phase plan, at each set of initial conditions corresponds to a curve or a point.*

1.4 Equilibrium points

The equilibrium points (or a singular points) play a crucial role in the study of dynamical systems. Henri Poincaré $(1854 - 1912)$ [49] showed that to characterize a dynamical systems with several variables it is not necessary to calculate the detailed solutions, it suffices to know the equilibrium points and their stabilities. This result of great importance considerably simplifies the study of nonlinear systems in the neighborhood of these points. So to know the local behavior of the trajectories of the system (1.1), we have to look for its equilibrium points.

Definition 1.8. *A point (x_0, y_0) is called an equilibrium point (or a singular point) of the system (1.1) if*

$$\begin{cases} P(x_0, y_0) = 0, \\ Q(x_0, y_0) = 0. \end{cases}$$

Remark 1.2. *The equilibrium points lie at the intersection of the horizontal isocline and the vertical isocline.*

Proposition 1.1. *All periodic solution contains at least one equilibrium point.*

1.4.1 Jacobian Matrix and Linearization

In general, for studying the behavior of trajectories near of the equilibrium points, we write the linearized system associated with the system (1.1). Then to make the link between the trajectories of the two systems.

Definition 1.9. *Let $J(x_0, y_0)$ denote the Jacobian matrix associated with the vector field in the neighborhood of an equilibrium point (x_0, y_0) defined by:*

$$J(x_0, y_0) = \begin{pmatrix} \frac{dP}{dx}(x_0, y_0) & \frac{dP}{dy}(x_0, y_0) \\ \frac{dQ}{dx}(x_0, y_0) & \frac{dQ}{dy}(x_0, y_0) \end{pmatrix}$$

The linearized of the system (1.1) near the equilibrium point (x_0, y_0) is given in matrix form by:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \frac{dP}{dx}(x_0, y_0) & \frac{dP}{dy}(x_0, y_0) \\ \frac{dQ}{dx}(x_0, y_0) & \frac{dQ}{dy}(x_0, y_0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.2}$$

Topological equivalence

Firstly, we start by defining what is a homeomorphism

Definition 1.10. *A homeomorphism of* $\mathbb{R}^2$ *is a continuous bijective map* $h : \mathbb{R}^2 \to \mathbb{R}^2$*, whose inverse bijection is continuous.*

Definition 1.11. *Two autonomous polynomial differential systems*

$$(S_1)\begin{cases} \dot{x} = P_1(x(t), y(t)), \\ \dot{y} = Q_1(x(t), y(t)), \end{cases} \qquad (S_2)\begin{cases} \dot{x} = P_2(x(t), y(t)), \\ \dot{y} = Q_2(x(t), y(t)), \end{cases}$$

defined on two open sets U *and* V *of* $\mathbb{R}^2$ *respectively are said to be topologically equivalent if there exists a homeomorphism* $h : U \to V$*, such that* h *transforms the orbits of* (S_1) *into orbits of* (S_2) *and preserves their orientation.*

The Hartman Grobman theorem

This theorem allows us to reduce the study of a dynamical system (1.1) in the neighborhood of a hyperbolic singular point, to the study of a linear system (1.2) topologically equivalent to (1.1), in the neighborhood from the origin.

Theorem 1.2. *Let* λ_1 *and* λ_2 *be the two eigenvalues of Jacobian matrix at the singular point* (x_0, y_0)*, such that* $\operatorname{Re}(\lambda_1) \neq 0$ *and* $\operatorname{Re}(\lambda_2) \neq 0$*, then the solutions of the system* (1.1) *are given approximately by the solutions of the linearized system* (1.2) *in the neighborhood of the singular point. In other words, the phase portrait of the Linearized system constitutes, in the near of this equilibrium point, a good approximation of that of system* (1.1).

Remark 1.3. *In the case where* $\operatorname{Re}(\lambda_{1,2}) = 0$*, the singular point* (x_0, y_0) *is called center for the linearized system. The nature of* (x_0, y_0) *requires further investigation: this is the "problem of the center".*

Remark 1.4. *1. We say that a singular point is non-elementary if both of the eigenvalues of the linear part of the* $J(x_0, y_0)$ *at that point are zero, and elementary otherwise. A non-elementary singular point is called degenerate if its linear part is identically zero, otherwise it is called nilpotent.*

2. The point (x_0, y_0) *is said to be a hyperbolic singular point if the real part of all the eigenvalues of* $J(x_0, y_0)$ *have non-zero. Otherwise, the singular point is said to be non-hyperbolic.*

3. The semi-hyperbolic are the singular points having a unique eigenvalue equal to zero, their phase portraits are well known, see for instance ([29], Theorem 2.19*).*

4. The nilpotent singular points have both eigenvalues zero but their linear part is not identically zero. See for example ([29], Theorem 3.5) for the classification of their local phase portraits.

1.4.2 Classification of equilibrium points

Consider the differential system (1.1) and let $J(x_0, y_0)$ be the Jacobian matrix associated with the system (1.1) at the equilibrium point (x_0, y_0) and let λ_1 and λ_2 be the eigenvalues of this matrix. We distinguish the different cases according to the eigenvalues λ_1 and λ_2 of the matrix $J(x_0, y_0)$.

1. If λ_1 and λ_2 are real nonzero and of different sign, then the equilibrium point (x_0, y_0) is a saddle point, it is always unstable.

2. If λ_1 and λ_2 are real with the same sign, we have three cases:

   a. If $\lambda_1 < \lambda_2 < 0$, the equilibrium point (x_0, y_0) is a stable node.

   b. If $0 < \lambda_1 < \lambda_2$, the equilibrium point (x_0, y_0) is an unstable node.

   c. If $\lambda_1 = \lambda_2 = \lambda \neq 0$, the equilibrium point (x_0, y_0) is a proper node, it is stable if $\lambda < 0$ and unstable if $\lambda > 0$.

3. If λ_1 and λ_2 are conjugate complexes and $\text{Im}(\lambda_{1,2}) \neq 0$, then the equilibrium point (x_0, y_0) is a focus. It is stable if $\text{Re}(\lambda_{1,2}) < 0$ and unstable if $\text{Re}(\lambda_{1,2}) > 0$.

4. If λ_1 and λ_2 are pure imaginary, then the equilibrium point (x_0, y_0) is a center. Figure 1.2 summarizes the types of equilibria.

Stability of equilibrium points

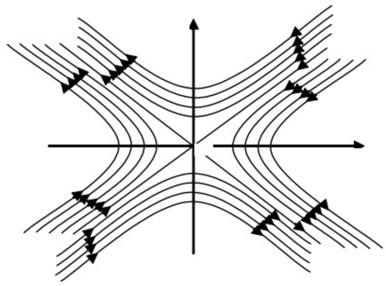
A nonlinear system can have several equilibrium positions which can be stable or unstable. In some situations, equilibrium stability is required, which is defined as follows:

Let (x_0, y_0) be an equilibrium point of the system (1.1). Note by

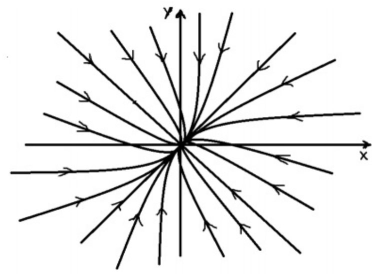
$$X(t) = (P(x(t), y(t)), Q(x(t), y(t))),$$

and

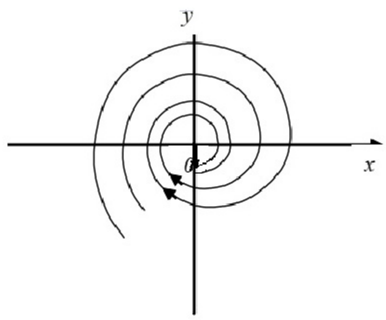
$$X_0 = (P(x_0, y_0), Q(x_0, y_0)).$$



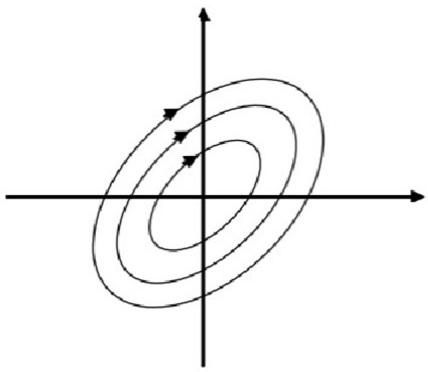
(a) Saddle Point



(b) Node Point



(c) Focus Point



(d) Center Point

Figure 1.2: Phase portraits of equilibrium points.

Definition 1.12. *The equilibrium points* (x_0, y_0) *of system is*

1. *Stable if and only if*

$$\forall \varepsilon > 0, \exists \eta > 0, \|(x, y) - (x_0, y_0)\| < \eta \Longrightarrow (\forall t > 0 : \|X(t) - X_0\| < \varepsilon).$$

2. *The point* (x_0, y_0) *is asymptotically stable if and only if* (x_0, y_0) *is stable and*

$$\lim_{t \longrightarrow \infty} \|X(t) - X_0\| = 0.$$

Remark 1.5. *Asymptotic stability imposes that the limit of the trajectories when* $t \to \infty$ *be the equilibrium point, while the neutral stability (stable but not asymptotically stable) only imposes that the trajectories remain in a neighborhood of the equilibrium point without necessarily tending towards this point.*

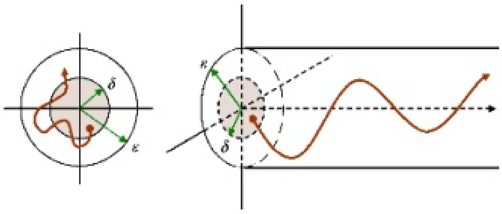


Figure 1.3: Stability of equilibrium points.

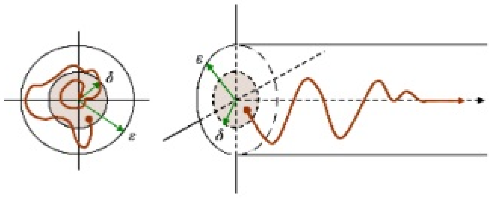


Figure 1.4: Asymptotic stability of equilibrium points.

1.5 Invariant curves

Invariant algebraic curves play an important role in the integrability of differential planar polynomial systems, see for example ([24] [26]), and also are used in the study of the existence and non-existence of limit cycles.

Definition 1.13. *We say that a curve defined by $U(x, y) = 0$ is an invariant curve for (1.1) if there exists a polynomial $K(x, y)$ satisfying*

$$P(x,y)\frac{\partial U(x,y)}{\partial x} + Q(x,y)\frac{\partial U(x,y)}{\partial y} = K(x,y)U(x,y). \tag{1.3}$$

The polynomial K is called a cofactor of the curve U.

Algebraic and non algebraic invariant curve

Definition 1.14. *An invariant curve $U(x, y) = 0$ is said to be algebraic curve of degree n, if $U(x, y)$ is a polynomial of degree n, otherwise we say that the curve is a non algebraic curve.*

Remark 1.6. *In the case where the polynomial differential system (1.1) has an algebraic invariant curve $U(x, y) = 0$ of degree n, then any cofactor is of degree at most $n - 1$.*

Theorem 1.3. *[38] We consider the system (1.1) and $\Gamma(t)$ a periodic orbit of period $T > 0$. We assume that $U : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is an invariant curve, $\Gamma(t) = \{(x, y) \in \Omega :$*

$U(x,y)=0\}$, *and* $K(x,y)\in\mathbb{C}^1$ *is the cofactor given in equation* (1.3)*, of the invariant curve* $U(x,y)=0$*. Suppose that* $p\in\Omega$ *such that* $U(p)=0$ *and* $\nabla U(p)\neq 0$*, then*

$$\int_0^T \operatorname{div}\left(\Gamma\left(t\right)\right)dt=\int_0^T K\left(\Gamma\left(t\right)\right)dt.$$

Remark 1.7. *The assumption* $\nabla U(p)\neq 0$ *means that* U *does not contain singular points.*

1.6 Integrability of polynomial differential systems

In the qualitative study of polynomial differential system, the notion of integrability plays an essential role. We say that a polynomial differential system is integrable if it admits a first integral, see the definition below. We note that the determination of a first integral for a given differential system is not an easy task. The importance of the existence of first integral for a differential system is that it completely determines its phase portrait see [23],[26],[34].

1.6.1 First integral

Definition 1.15. *We say that a non-locally constant* $\mathbb{C}^1$ *function* $H:\Omega\to\mathbb{R}$ *is a first integral of the differential system* (1.1) *in* Ω *if* H *is constant on the trajectories of the system* (1.1) *contained in* Ω *, i.e, if*

$$\frac{dH(x,y)}{dt}=P(x,y)\frac{\partial H(x,y)}{\partial x}+Q(x,y)\frac{\partial H(x,y)}{\partial y}\equiv 0.$$

Moreover, $H=k$ *is the general solution of this equation, where* k *is an arbitrary constant. And a system* (1.1) *is integrable in* Ω *if it has a first integral* H *in* Ω*.*

1.6.2 Darboux integrability

Definition 1.16. *A Darboux function is a function of the form*

$$f_1\left(x\right)^{\lambda_1}...f_p\left(x\right)^{\lambda_p}\exp\left(\frac{g\left(x\right)}{h\left(x\right)}\right),$$

where $f_i\left(x\right)$ *for* $i=1,...,p$*,* $g\left(x\right)$ *and* $h\left(x\right)$ *are polynomials in* $\mathbb{C}[x,y]$ *and the* λ_i *for* $i=1,...,p$ *are complex numbers. For more details see [43][44].*

Definition 1.17. *System* (1.1) *is called Darboux integrable if it has a first integral which is a Darboux function.*

Definition 1.18. *A Liouvillian function is a function which can be expressed by quadratures of elementary functions. For more details see [31].*

The study of the integrability problem consists also in deducing the membership of integrating factor or inverse integrating factor for a given class of functions.

1.6.3 Integrating factors

The notions of integrating factor and inverse integrating factor make it possible to deduce the expression of the first integral.

Definition 1.19. *The function $R(x,y)$ is an integrating factor of differential system (1.1) on the open subset $U \subseteq R^2$ if $R \in \mathbb{C}^1(U), R \neq 0$ on U and*

$$\frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y}, \qquad \operatorname{div}(RP, RQ) = 0 \quad \text{or} \quad P\frac{\partial R}{\partial x} + Q\frac{\partial R}{\partial y} = -R\operatorname{div}(P,Q)$$

As usual the divergence of the vector field X is defined by

$$\operatorname{div}(X) = \operatorname{div}(P,Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

It is clear that the function H satisfying

$$\begin{cases} \dfrac{\partial H}{\partial x} = RQ, \\ \dfrac{\partial H}{\partial y} = -RP, \end{cases}$$

is a first integral, then the first integral H associated to the integrating factor R is given by

$$H(x,y) = -\int R(x,y)\,P(x,y)\,dy + h(x).$$

Or

$$H(x,y) = \int R(x,y)\,Q(x,y)\,dx + h(y).$$

Inverse integrating factor

Definition 1.20. *A non-zero function $V : \Omega \to \mathbb{R}$ is said to be an inverse integrating factor of system (1.1) if of class $\mathbb{C}^1(\Omega)$, not locally null and satisfies the following linear partial differential equation*

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)V.$$

It is easy to verify that the function $R = \dfrac{1}{V}$ defines an integrating factor in $\Omega \backslash \{V = 0\}$ of the system (1.1).

The inverse integrating factor is among the tools that are used in the study of the existence and non-existence of limit cycles. The following theorem, proved in [35], gives an important relation between a limit cycle and an inverse integrating factor.

Theorem 1.4. *[35] Let $V : \Omega \to \mathbb{R}$ be an inverse integrating factor of* (1.1)*. If Γ is a limit cycle of* (1.1)*, then Γ is contained in the set*

$$\Gamma = \{(x, y) \in \Omega : V(x, y) = 0\}$$

Exponential factor

There is another object, which is called exponential factor, which plays the same role as the invariant algebraic curves to obtain a first integral of a polynomial differential system (1.1).

Definition 1.21. *Let $h, g \in \mathbb{R}[x, y]$ either be coprime polynomials in the ring $\mathbb{R}[x, y]$, or $h \equiv 1$. Then, for the polynomial differential system* (1.1)*, the function $\exp(g/h)$ is an exponential factor if there is a polynomial $K \in \mathbb{R}[x, y]$ of degree at most $m-1$, such that:*

$$X\left(\exp\left(\frac{g}{h}\right)\right) = K \exp\left(\frac{g}{h}\right)$$

Then, for the exponential factor $\exp(g/h)$ we stated that K was its cofactor. Since the exponential factor cannot vanish, it does not define invariant curves of the polynomial system (1.1)*.*

1.7 Limit cycles

We have seen that the solutions tend towards a singular point. Another possible behavior for a trajectory is to tend towards a periodic movement, in the case of a planar system, this means that the trajectories tend towards what is called a limit cycle. Limit cycles were introduced for the first time by H. Poincaré in 1881 [47] in his "Mémoire sur les courbes définies par une équation différentielle".

Definition 1.22. *A limit cycle of system (1.1) is an isolated periodic solution in the set of all periodic solutions of system* (1.1)*. Isolated means that neighborhood trajectories are note closed, they will either spiral away or towards the limit cycle.*

Remark 1.8.

1. *If all neighborhood trajectories approach the limit cycle, then the limit cycle is stable, otherwise the limit cycle is unstable.*

2. *If a limit cycle is contained in an algebraic curve of the plane, then we say that it is algebraic, otherwise it is called non-algebraic limit cycle.*

3. *Limit cycles are nonlinear phenomena, they do not occur in linear system. Because in linear system if $x(t)$ is a periodic solution, then $cx(t)$ will be a solution for any constant $c \neq 0$.*

1.7.1 Existence and non existence of limit cycles in the plane

In this subsection, we give some results which allow us it possible to prove the existence or the non existence of limit cycles for a polynomial differential system (1.1).

Theorem 1.5. *[36] Consider two closed curves C and C', one surrounding the other. If in each point of C, the velocity vector field (P, Q) of the trajectory passing through it is directed towards outside, and if at each point of C' it is directed inwards, then there exists at least a limit cycle between C and C'. See Figure 1.5.*

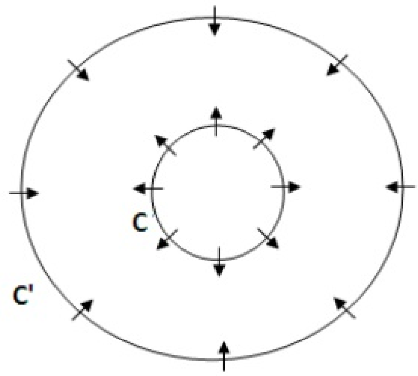


Figure 1.5: Existence of limit cycle located between C and C'

Theorem 1.6. *[35] Let (P, Q) be a vector field of class $\mathbb{C}^1$ defined on a non-empty open set D of $\mathbb{R}^2$, $(x(t), y(t))$ a periodic solution of period T of the system (1.1) and $K : D \to \mathbb{R}$ a $\mathbb{C}^1$ map such that*

$$\int_0^T K(x(t), y(t))\, dt \neq 0,$$

and $U = U(x, y)$ a $\mathbb{C}^1$ solution of the linear partial differential equation

$$P(x, y)\frac{\partial U(x, y)}{\partial x} + Q(x, y)\frac{\partial U(x, y)}{\partial y} = K(x, y)U(x, y),$$

then the closed trajectory

$$\Gamma = \{(x(t), y(t)) \in D : t \in [0, T]\},$$

is contained in

$$\Sigma = \{(x, y) \in D : U(x, y) = 0\},$$

and Γ is not contained in a period annulus of (P, Q). Moreover, if the vector field (P, Q) and the functions K and U are analytic, then Γ is a limit cycle.

Theorem 1.7. ***(Bendixon criterion).*** *Let D be a simply connected domain of $\mathbb{R}^2$. If the quantity $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ is not identically zero and of constant sign on D, then the vector field X does not admit a limit cycle entirely contained in D.*

Theorem 1.8. *[27] If the system (1.1) has no singular point, then it has no limit cycles.*

Example 1.1. *[13] Consider a polynomial differential system*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\left(\left(4x^2 - 1\right) y + x^5 - x^3 + 2x\right). \end{cases} \tag{1.4}$$

It is clear that the curve $F(x,y) = \left(y + x\left(x^2 - 1\right)\right)^2 + 2x^2 - 2 = 0$ *satisfies the partial differential equation*

$$P(x,y)\frac{\partial U}{\partial x} + Q(x,y)\frac{\partial U}{\partial y} = K(x,y)U\left(x,y\right),$$

with cofactor

$$K(x,y) = -2x^2.$$

Moreover, we have $\int_0^T K\left(x\left(t\right), y\left(t\right)\right) dt \neq 0$*, thus the curve* $F(x,y) = 0$ *is hyperbolic limit cycle, see Figure 1.6.*

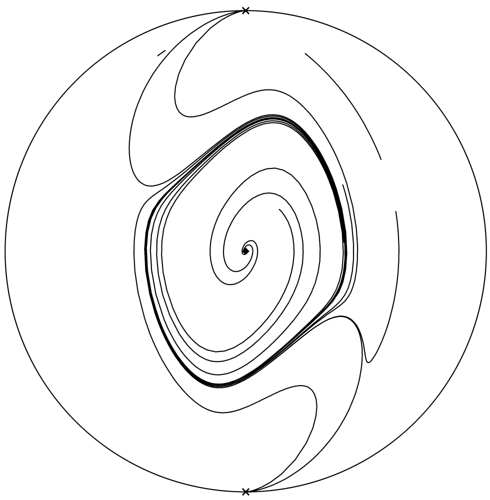


Figure 1.6: Algebraic limit cycle of polynomial differential system (1.4).

1.7.2 Stability of limit cycles

We consider Γ the trajectory corresponding to the limit cycle of system (1.1), the neighboring trajectories are not closed and should be have something like Γ. Neighboring trajectories can either spiral towards Γ or away from Γ, then Γ is a limit cycle: stable, unstable or semi-stable depending on whether the close curves spirals towards Γ, away from Γ, or both respectively.

Theorem 1.9. ***a.*** *The limit cycle* Γ *is stable (or attractive), if the interior and exterior trajectories spirals tend towards the closed orbit when* $t \to +\infty$

b. *The limit cycle* Γ *is unstable (or repulsive), if the interior and exterior trajectories spirals tend towards the closed orbit when* $t \to -\infty$

c. *The limit cycle* Γ *is semi-stable, if the interior spiral trajectories tend towards the closed orbit when* $t \to +\infty$*, the others (external) tend towards when* $t \to -\infty$*, and vice versa.*

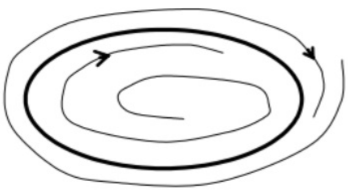


Figure 1.7: Stable limit cycle.

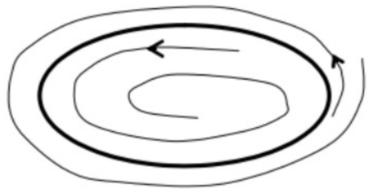


Figure 1.8: Unstable limit cycle.

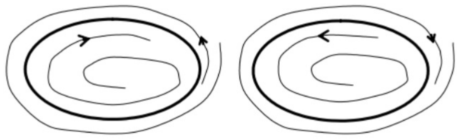


Figure 1.9: Semi-stable limit cycle.

1.8 Return map (Poincaré Map)

One of the most basic tool for studying the stability of periodic orbits is the Poincaré map or first return map, defined by Henri Poincaré in 1881[48]. The idea of the Poincaré map is the following: If Γ is a periodic orbit of the system (1.1) through the point $X_0 = (x_0, y_0)$ and Σ is a hyperplane perpendicular to Γ at X_0, then for any $X = (x, y) \in \Sigma$ sufficiently near X_0, the solution of (1.1) through X at $t = 0$, will cross Σ again at a point $\Pi(X)$ near X_0, (Figure 1.10). The mapping $X \to \Pi(X)$ is called the Poincaré map. The next theorem establishes the existence and continuity of the Poincaré map $\Pi(X)$ and of its first derivative $D\Pi(X)$.

Theorem 1.10. *[46] Let* Ω *be an open subset of* $\mathbb{R}^2$*. Suppose that* $\Phi_t(X_0)$ *is a periodic solution of (1.1) of period* T *and that the cycle*

$$\Gamma = \{X \in \mathbb{R}^2 / X = \Phi_t(X_0),\ 0 \le t \le T\},$$

is contained in Ω*. Let* Σ *be the hyperplane orthogonal to* Γ *at* X_0*; i.e., let*

$$\Sigma = \{X \in \mathbb{R}^n \ / (X - X_0) . (P(X_0), Q(X_0)) = 0\}.$$

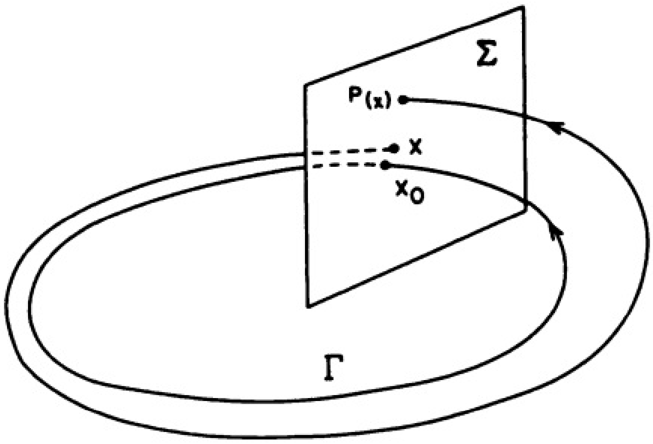


Figure 1.10: The Poincaré map.

Then there is $\delta > 0$ and a unique function $\tau(X)$, defined and continuously differentiable such that $\tau(X_0) = T$ and $\Phi_{\tau(X)}(X) \in \Sigma$ for all $X \in N_\delta(X_0)$.

Definition 1.23. *Let Γ, Σ, δ and $\tau(X)$ be defined as in Theorem 1.10. Then for $X \in N_\delta(X_0) \cap \Sigma$, the function*

$$\Pi(X) = \Phi_{\tau(X)}(X)$$

is called the Poincaré map for Γ at X_0.

Theorem 1.11. *[46] Let $\Gamma(t)$ be a periodic solution of (1.1) of period T. Then the derivative of the Poincaré map $\Pi(s)$ along a straight line Σ normal to $\Gamma = \left\{X \in \mathbb{R}^2 \;/\; X = \Gamma(t) - \Gamma(0) \quad 0 \leq t \leq T\right\}$ at $X = (0,0)$ is given by*

$$\Pi'(0,0) = \exp \int_0^T \operatorname{div}(P(\Gamma(t)), Q(\Gamma(t)))\, dt$$

The following corollary characterizes the stability of a limit cycle

Corollary 1.1. *Under the hypotheses of Theorem 1.11, the periodic solution $\Gamma(t)$ is a stable limit cycle if*

$$\int_0^T \operatorname{div}(P(\Gamma(t)), Q(\Gamma(t)))\, dt < 0,$$

and it is an unstable limit cycle if

$$\int_0^T \operatorname{div}(P(\Gamma(t)), Q(\Gamma(t)))\, dt > 0.$$

It may be a stable, unstable or semi-stable limit cycle or it may belong to a continuous band of cycles if

$$\int_0^T \operatorname{div}(P(\Gamma(t)), Q(\Gamma(t)))\, dt = 0.$$

Remark 1.9. *The limit cycle* $\Gamma(t)$ *is said to be hyperbolic if*

$$\int_0^T \operatorname{div}\left(P\left(\Gamma\left(t\right)\right), Q\left(\Gamma\left(t\right)\right)\right) dt \neq 0.$$

Example 1.2. *Consider the polynomial differential system*

$$\begin{cases} \dot{x} = -y + x\left(1 - x^2 - y^2\right), \\ \dot{y} = x + y\left(1 - x^2 - y^2\right), \end{cases}$$

we remark that this system admit a unique equilibrium point at origin $(0,0)$ *and a limit cycle* Γ *given by* $(x(t), y(t)) = (\cos t,\ \sin t)^T$. *The Poincaré map for* Γ *can be found by solving this system written in polar coordinates*

$$\begin{cases} \dot{r} = r\left(1 - r^2\right), \\ \dot{\theta} = 1, \end{cases}$$

with $r(0) = r_0$ *and* $\theta(0) = \theta_0$. *The first equation can be solved either as a separable differential equation or as a Bernoulli equation. The solution is given by*

$$\begin{aligned} r(t, r_0) &= \left[1 + \left(\frac{1}{r_0^2} - 1\right) e^{-2t}\right]^{\frac{-1}{2}}, \\ \theta(t, \theta_0) &= t + \theta_0. \end{aligned}$$

If Σ *is the ray* $\theta = \theta_0$ *through the origin, then* Σ *is perpendicular to* Γ *and the trajectory through the point* $(r_0, \theta_0) \in \Sigma \cap \Gamma$ *at* $t = 0$ *intersects the ray* $\theta = \theta_0$ *again at* $t = 2\pi$. *It follows that the Poincaré map is given by*

$$\Pi(r_0) = r(2\pi, r_0) = \left[1 + \left(\frac{1}{r_0^2} - 1\right) e^{-4\pi}\right]^{\frac{-1}{2}}.$$

Clearly $\Pi(1) = 1$ *corresponding to the cycle* Γ *and we see that*

$$\Pi'(r_0) = \frac{e^{-4\pi}}{r_0^3}\left[1 + \left(\frac{1}{r_0^2} - 1\right) e^{-4\pi}\right]^{\frac{-3}{2}},$$

and that $\Pi'(1) = e^{-4\pi} < 1$, *then* Γ *is a stable limit cycle.*

1.9 The 16th Hilbert problem

In 1900, David Hilbert [39] presented in International Congress of Mathematicians in Paris, a list of 23 Problems, one of the unsolved problem up to now is the sixteenth problem. The 16*th* Hilbert problem has two parts, the first concerns the number of real branches (ovals) of a plane real algebraic curve and their relative positions. The second part is to decide an upper bound for the number of limit cycles in polynomial vector fields of a given degree and similar to the first part, investigate their relative positions, see [21], [51].

Chapter 2

Explicit non-algebraic limit cycle of a family of polynomial differential systems of degree even

Contents

2.1 Introduction

One of the difficult problems in the qualitative theory is to detect if a planar polynomial differential system (1.1) is integrable or not and also to know if its limit cycles when they exist are algebraic or not, as well as the determination of their explicit expressions. Up to now all the examples of polynomial differential systems of the form (1.1) for which non algebraic limit cycles are known explicitly have degree odd. We can mention for instance in chronological order, the works of J. Giné and M. Grau [37] for $n = 9$, A. Gasull [32] for $n = 5$, Al-Dossary [2] for $n = 5$, Benterki and Llibre [16] for $n = 3$, Bendjeddou and Cheurfa [14] for $n = 5$, Bendjeddou and Berbache [9] for $n = 2k + 1, (k \in \mathbb{N}^*)$.

Our objective in this chapter, is to study a polynomial differential systems of degree even of the form

$$\begin{cases} \dot{x} = n\left(\left(\alpha x - y\right)\left(wx + vy + l\right) + vx^2 - y\left(l + wx\right)\right)\left(\left(a+b\right)x^2 + \left(a-b\right)y^2 + 2cxy\right)^n \\ +x\left(wx + vy + l\right)^{n+1}, \\ \dot{y} = n\left(\left(x + \alpha y\right)\left(wx + vy + l\right) - wy^2 + x\left(l + vy\right)\right)\left(\left(a+b\right)x^2 + \left(a-b\right)y^2 + 2cxy\right)^n \\ +y\left(wx + vy + l\right)^{n+1}, \end{cases} \tag{2.1}$$

where a, b, c, w, v, l and α are real constants and n is strictly positive integer $(n \in \mathbb{N}^*)$. Moreover, under some suitable conditions we prove that this system is integrable and it possess an explicit non-algebraic limit cycle.

Noted that this result was published in the Journal of An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.), Tomul LXV, 2019, f. 2 [19].

In order to present our main result, we use the change of variables in polar coordinates (r, θ) defined by $x = r\cos\theta$ and $y = r\sin\theta$. Then the system (2.1) can be written as follows

$$\begin{cases} \dot{r} = n\left(a + b\cos 2\theta + c\sin 2\theta\right)^n \left(l\alpha + \left(\left(\cos\theta\right)\left(v + w\alpha\right) + \left(\sin\theta\right)\left(v\alpha - w\right)\right)r^{2n+1}\right) \\ \qquad\qquad + \left(l + r\left(w\cos\theta + v\sin\theta\right)\right)^{n+1} r \\ \dot{\theta} = n\left(\left(a + b\cos 2\theta + c\sin 2\theta\right)\right)^n \left(2l + r\left(w\cos\theta + v\sin\theta\right)\right)r^{2n} \end{cases} \tag{2.2}$$

2.2 Statement of the main result

Our main result in this chapter is the following.

Theorem 2.1. *Consider a polynomial differential system (2.1), the following statement hold:*

- *If $\alpha < 0, w \geq 0, l > 0, a > |b| + |c|$ and $b^2 + c^2 \neq 0$, then system (2.1) has an explicit non algebraic limit cycle, given in polar coordinates (r, θ) by*

$$r\left(\theta, r_*\right) = \frac{1}{2}\left(g\left(\theta\right)\rho^{\frac{1}{n}}\left(\theta, r_*\right) + \sqrt{\left(g\left(\theta\right)\rho^{\frac{1}{n}}\left(\theta, r_*\right)\right)^2 + 4l\rho^{\frac{1}{n}}\left(\theta, r_*\right)}\right),$$

where

$$\rho\left(\theta, r_*\right) = e^{n\theta\alpha}\left(\frac{r_*^{2n}}{\left(wr_* + l\right)^n} + f\left(\theta\right)\right),$$

$$f\left(\theta\right) = \int_0^\theta \frac{e^{-n\alpha s}}{\left(a + b\cos 2s + c\sin 2s\right)^n} ds,$$

$$g\left(\theta\right) = w\cos\theta + v\sin\theta, \text{ and}$$

$$r_* = \frac{1}{2}\left(\sqrt{\left(we^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2(\pi n\alpha)}-1}\right)^{\frac{1}{n}}\right)^2 + 4le^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2(\pi n\alpha)}-1}\right)^{\frac{1}{n}}} + we^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}}\right).$$

Moreover, this limit cycle is a stable hyperbolic limit cycle.

The following lemmas also gives some important results. The first lemma is concerned by the equilibrium points of system (2.1).

Lemma 2.1. *If $a > |b| + |c|$, then the origin is an unstable node and if exist another equilibrium point then they are present in the straight line*

$$vy + wx + 2l = 0.$$

The second lemma determine the integrability of differential system (2.1)

Lemma 2.2. *System (2.1) is integrable with the first integral*

$$H(x,y) = \left(\frac{x^2+y^2}{(wx+vy+l)}\right)^n e^{-\alpha n \arctan\frac{y}{x}} - \int_0^{\arctan\frac{y}{x}} \frac{e^{-\alpha ns}}{(a+b\cos 2s + c\sin 2s)^n}\, ds$$

2.3 Proof of the main result

First, we start by the proof of lemmas

Proof of lemma 2.1. Firstly, we have

$$x\dot{y} - y\dot{x} = n\left(x^2+y^2\right)(2l+vy+wx)\left((a+b)x^2+(a-b)y^2+2cxy\right)^n$$

Thus, the equilibrium points of system (2.1) are present in the equation curve's

$$n\left(x^2+y^2\right)(2l+vy+wx)\left((a+b)x^2+(a-b)y^2+2cxy\right)^n = 0.$$

Since $a > |b| + |c|$, then the equilibrium points of system (2.1) are present in the straight line

$$vy + wx + 2l = 0, \tag{2.3}$$

and the origin of coordinates which is an unstable node because its eigenvalues are $l^{n+1} > 0$ with multiplicity two, for more details see for instance [[29], Theorem 2.15]. □

Proof of lemma 2.2. The differential system (2.1) where $\dot{\theta} \neq 0$ can be written as the equivalent differential equation

$$\begin{aligned} \frac{dr}{d\theta} &= \frac{r(l+r(w\cos\theta+v\sin\theta))^{n+1}}{n((a+b\cos 2\theta+c\sin 2\theta))^n(2l+r(w\cos\theta+v\sin\theta))r^{2n}} \\ &+ \frac{nr^{2n+1}(a+b\cos 2\theta+c\sin 2\theta)^n(l\alpha+((\cos\theta)(v+w\alpha)+(\sin\theta)(v\alpha-w))r)}{n((a+b\cos 2\theta+c\sin 2\theta))^n(2l+r(w\cos\theta+v\sin\theta))r^{2n}}. \end{aligned} \tag{2.4}$$

Via the change of variables $\rho = \dfrac{r^{2n}}{(l + r(w\cos\theta + v\sin\theta))^n}$, then

$$\frac{d\rho}{d\theta} = \frac{n((w\cos\theta + v\sin\theta)r + 2l)r^{2n-1}}{((w\cos\theta + v\sin\theta)r + l)^{n+1}}\frac{dr}{d\theta} - \frac{n(v\cos\theta - w\sin\theta)r^{2n+1}}{((w\cos\theta + v\sin\theta)r + l)^{n+1}},$$

and the equation (2.4) is transformed into the linear equation

$$\frac{d\rho}{d\theta} = n\alpha\rho + \frac{1}{(a + b\cos 2\theta + c\sin 2\theta)^n}. \tag{2.5}$$

The general solution of linear equation (2.5) is

$$\rho(\theta) = e^{n\theta\alpha}(k + f(\theta)),$$

where $k \in \mathbb{R}$, $f(\theta)$ is the function defined in the statement of Theorem 2.1. Consequently, the general solution of (2.4) is

$$F(\theta, r) = \frac{r^{2n}}{(l + rg(\theta))^n} - e^{n\theta\alpha}(k + f(\theta)) = 0, \tag{2.6}$$

where $k \in \mathbb{R}, f(\theta), g(\theta)$ are the functions defined in the previous theorem. Going back through the changes of variables $r^2 = x^2 + y^2$ and $\theta = \arctan\dfrac{y}{x}$, we obtain the first integral

$$H(x, y) = \left(\frac{x^2 + y^2}{(wx + vy + l)}\right)^n e^{-\alpha n\arctan\frac{y}{x}} - \int_0^{\arctan\frac{y}{x}} \frac{e^{-\alpha ns}}{(a + b\cos 2s + c\sin 2s)^n}\,ds.$$

□

Proof of Theorem 2.1. From solution (2.6) we can obtain a two different values of r: one of them is equal to

$$r_1(\theta, k) = \frac{1}{2}\left(g(\theta)\rho^{\frac{1}{n}}(\theta, k) - \sqrt{\left(g(\theta)\rho^{\frac{1}{n}}(\theta, k)\right)^2 + 4l\rho^{\frac{1}{n}}(\theta, k)}\right)$$

and we do not consider this case, because since $-(|w| + |v|) < g(\theta) < (|w| + |v|)$, then $r_1(\theta)$ can not be strictly positive.
The second value is

$$r_1(\theta, k) = \frac{1}{2}\left(g(\theta)\rho^{\frac{1}{n}}(\theta, k) + \sqrt{\left(g(\theta)\rho^{\frac{1}{n}}(\theta, k)\right)^2 + 4l\rho^{\frac{1}{n}}(\theta, k)}\right)$$

where

$$\rho(\theta, k) = e^{\alpha n\theta}(k + f(\theta)).$$

Notice that system (2.1) has a periodic orbit if and only if equation (2.4) has a strictly positive $2\pi-$ periodic solution. This, moreover, is equivalent to the existence of a solution of (2.4) that satisfies $r(0,r_*) = r(2\pi, r_*)$ and $r(\theta, r_*) > 0$ for any θ in $[0, 2\pi]$. To go a step further, we remark that the solution such as $r(0,r_0) = r_0 > 0$, corresponds to the value

$$k = \frac{r_0^{2n}}{(l + wr_0)^n}$$

The solution $r(\theta, r_0)$ of the differential equation (2.4) such that $r(0,r_0) = r_0$ is

$$r(\theta, r_0) = \frac{1}{2}\left(g(\theta)\rho^{\frac{1}{n}}(\theta, r_0) + \sqrt{\left(g(\theta)\rho^{\frac{1}{n}}(\theta, r_0)\right)^2 + 4l\rho^{\frac{1}{n}}(\theta, r_0)}\right)$$

where

$$\rho(\theta, r_0) = e^{n\theta\alpha}\left(\frac{r_0^{2n}}{(l + wr_0)^n} + f(\theta)\right).$$

A periodic solution of system (2.1) must satisfy the condition $r(2\pi, r_0) = r(0, r_0)$. Since $g(0) = g(2\pi) = w$, then $r(2\pi, r_0) = r(0, r_0)$ if and only if $\rho(2\pi, r_0) = \rho(0, r_0)$ and there are two different values with the property $r(2\pi, r_0) = r_0$, so one of them is equal to

$$-\frac{1}{2}\left(\sqrt{\left(we^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}}\right)^2 + 4le^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}}} - we^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}}\right)$$

and we do not consider this case because since $a > |b|+|c|$ then $\dfrac{e^{-\alpha ns}}{(a + b\cos 2s + c\sin 2s)^n} >$ 0, for all $\theta \in \mathbb{R}$, thus $\displaystyle\int_0^\theta \frac{e^{-n\alpha s}}{(a + b\cos 2s + c\sin 2s)^n}\,ds > 0$ for all $\theta \in \mathbb{R}$.

Since $\alpha < 0, w \geq 0, l > 0$, it follows that $\left(\dfrac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}} > 0$ and

$$\sqrt{\left(we^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}}\right)^2 + 4le^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}}} > we^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}},$$

thus

$$-\frac{1}{2}\left(\sqrt{\left(we^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}}\right)^2 + 4le^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}}} - we^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}}\right) < 0.$$

We only take into consideration the following value r_* which satisfies $r(2\pi, r_*) = r_* > 0$

$$r_* = \frac{1}{2}\left(\sqrt{\left(we^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}}\right)^2 + 4le^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}}} + we^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}}\right).$$

After the substitution of these value r_* into $r(\theta, r_0)$ we obtain

$$r(\theta, r_*) = \frac{1}{2}\left(g(\theta)\rho^{\frac{1}{n}}(\theta, r_*) + \sqrt{\left(g(\theta)\rho^{\frac{1}{n}}(\theta, r_*)\right)^2 + 4l\rho^{\frac{1}{n}}(\theta, r_*)}\right), \qquad (2.7)$$

where $\rho(\theta, r_*)$ and $g(\theta)$ are the functions defined in the statement of Theorem 2.1. Clearly the curve $(r(\theta)\cos\theta, r(\theta)\sin\theta)$ in the (x, y) plane with

$$\frac{r^{2n}}{(l + rg(\theta))^n} - e^{n\theta\alpha}\left(\frac{r_*^{2n}}{(l + wr_*)^n} + f(\theta)\right) = 0, \tag{2.8}$$

is not algebraic, due to the expression $e^{n\theta\alpha}\frac{r_*^{2n}}{(l + wr_*)^n}$. In what follows we prove that the straight line (2.3) does not intersect the orbit (2.8). This straight line in polar coordinates becomes $rg(\theta) + 2l = 0$ where $g(\theta)$ is the function defined in the statement of Theorem 2.1. To show this, we have to show that the system

$$\begin{cases} \left(\dfrac{r^2}{(l + rg(\theta))}\right)^n - e^{n\theta\alpha}\left(\dfrac{r_*^{2n}}{(l + wr_*)^n} + f(\theta)\right) = 0, \\ rg(\theta) + 2l = 0, \end{cases} \tag{2.9}$$

has no solution.

Indeed the system (2.9) can be written as

$$\left(\frac{r^2}{-l}\right)^n - e^{n\theta\alpha}\left(\frac{r_*^{2n}}{(l + wr_*)^n} + f(\theta)\right) = 0,$$

which leads

$$r^2 = -le^{n\theta\alpha}\left(\frac{r_*^{2n}}{(l + wr_*)^n} + f(\theta)\right)^{\frac{1}{n}}. \tag{2.10}$$

Since $r_* > 0, w \geq 0, l > 0$ and $f(\theta) > 0$ for all $\theta \in \mathbb{R}$, then

$$\rho(\theta, r_*) = e^{n\theta\alpha}\left(\frac{r_*^{2n}}{(l + wr_*)^n} + f(\theta)\right) > 0, \tag{2.11}$$

for all $\theta \in \mathbb{R}$. By (2.11) and taking into $l > 0$ singular points satisfying (2.10) do not exist. Hence, (2.9) has no solution. To show that $r(\theta, r_*)$ is a periodic solution, we have to show that:

i) the function $\theta \to r(\theta, r_*)$ is $2\pi-$periodic.

ii) $r(\theta, r_*) > 0$, for all $\theta \in [0, 2\pi[$. This last condition ensures that $r(\theta, r_*)$ is well defined for all $\theta \in [0, 2\pi[$ and the periodic solution do not pass through the equilibrium point $(0, 0)$ of system (2.1).

Periodicity. Since the function $g(\theta)$ is 2π -periodic, it follows that $r(\theta, r_*)$ is $2\pi-$periodic if and only if $\rho(\theta, r_*)$ is 2π-periodic and we have

$$\begin{aligned} \rho(\theta + 2\pi, r_*) &= e^{\alpha n(\theta+2\pi)}\left(\frac{r_*^{2n}}{(l + wr_*)^n}\right) + e^{\alpha n(\theta+2\pi)} f(\theta + 2\pi) \\ &= e^{\alpha n(\theta+2\pi)}\left(\frac{-e^{2\pi n\alpha}}{e^{2\pi n\alpha} - 1} f(2\pi)\right) + e^{\alpha n(\theta+2\pi)} f(\theta + 2\pi), \end{aligned} \tag{2.12}$$

we have

$$\begin{aligned} f(\theta+2\pi) &= \int_0^{\theta+2\pi} \frac{e^{-\alpha ns}}{(a+b\cos 2s+c\sin 2s)^n}\,ds \\ &= f(2\pi)+\int_{2\pi}^{\theta+2\pi} \frac{e^{-\alpha ns}}{(a+b\cos 2s+c\sin 2s)^n}\,ds, \end{aligned}$$

we make the change of variable $u = s-2\pi$ in the integral $\int_{2\pi}^{\theta+2\pi} \frac{e^{-\alpha ns}}{(a+b\cos 2s+c\sin 2s)^n}\,ds$ we get

$$\begin{aligned} f(\theta+2\pi) &= f(2\pi)+\int_0^{\theta} \frac{e^{-\alpha n(u+2\pi)}}{(a+b\cos 2(u+2\pi)+c\sin 2(u+2\pi))^n}\,du \\ &= f(2\pi)+e^{-2\alpha n\pi} f(\theta). \end{aligned}$$

We replace $f(\theta+2\pi)$ by $f(2\pi)+e^{-2\alpha n\pi} f(\theta)$ in (2.12), and after some calculations we obtain that

$$\rho(\theta+2\pi, r_*) = \rho(\theta, r_*).$$

Since $g(\theta)$ and $\rho(\theta, r_*)$ are a $2\pi-$periodic functions, then $r(\theta, r_*)$ is also $2\pi-$periodic function.

Strict positivity of $r(\theta, r_*)$ for all $\theta \in [0, 2\pi[$. Since $l > 0$, and taking into (2.11), then $4l\rho^{\frac{1}{n}}(\theta, r_*) > 0$, thus

$$r(\theta, r_*) = \frac{1}{2}\left(g(\theta)\rho^{\frac{1}{n}}(\theta, r_*) + \sqrt{\left(g(\theta)\rho^{\frac{1}{n}}(\theta, r_*)\right)^2 + 4l\rho^{\frac{1}{n}}(\theta, r_*)} \right) > 0,$$

for all $\theta \in [0, 2\pi[$. In order to prove that the periodic orbit is hyperbolic limit cycle, we consider (2.7), and introduce the Poincaré return map $\lambda \to \Pi(2\pi, \lambda) = r(2\pi, \lambda)$. Therefore, a limit cycles of system (2.1) is hyperbolic if, and only if $\left.\frac{dr(2\pi,\lambda)}{d\lambda}\right|_{\lambda=r_*} \neq 1$, where

$$r_* = \frac{1}{2}\left(\sqrt{\left(we^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}}\right)^2 + 4le^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}}} + we^{2\pi\alpha}\left(\frac{-f(2\pi)}{e^{2\pi n\alpha}-1}\right)^{\frac{1}{n}} \right),$$

after some calculations we obtain that

$$\left.\frac{dr(2\pi,\lambda)}{d\lambda}\right|_{\lambda=r_*} = e^{2\pi\alpha} < 1.$$

Consequently the limit cycle of the differential equation (2.4) is hyperbolic and stable, for more details see [46]. This completes the proof of Theorem 2.1. □

2.4 Applications

In this section, we present some examples to illustrate the applicability of our main result. In addition, the phase portraits of all systems in the Poincaré disk drawn by P4 software.

2.4.1 Application 1: Differential system of degree 4

Example 2.1. *If we take* $\alpha = v = b = -1, c = 0, w = l = 1, a = 2, n = 1$, *then system* (2.1) *reads*

$$\begin{cases} \dot{x} = x\left(x - y + 1\right)^2 - \left(2x^2 - y^2 + xy + x + 2y\right)\left(x^2 + 3y^2\right), \\ \dot{y} = y\left(x - y + 1\right)^2 - \left(3xy - x^2 + y - 2x\right)\left(x^2 + 3y^2\right). \end{cases} \tag{2.13}$$

It is easy to verify that all conditions of Theorem 2.1 are satisfied. Then system (2.13) *has a non-algebraic limit cycle whose expression in polar coordinates* (r, θ) *is*

$$r\left(\theta, r_*\right) = \frac{1}{2}\left(g\left(\theta\right)\rho\left(\theta, r_*\right) + \sqrt{\left(g\left(\theta\right)\rho\left(\theta, r_*\right)\right)^2 + 4\rho\left(\theta, r_*\right)}\right)$$

where $\rho\left(\theta, r_*\right) = e^{-\theta}\left(\frac{r_*^2}{\left(r_* + 1\right)} + \int_0^\theta \frac{e^s}{\left(2 - \cos 2s\right)}\right), g\left(\theta\right) = \cos\theta - \sin\theta,\ \theta \in \mathbb{R}$ *and the intersection of the limit cycle with the* OX_+ *axis is the point*

$$r_* = 1.1877.$$

Moreover,

$$\left.\frac{dr(2\pi, \lambda)}{d\lambda}\right|_{\lambda = r_*} = e^{-2\pi} < 1.$$

This limit cycle is a stable hyperbolic limit cycle. See Figure 2.1.

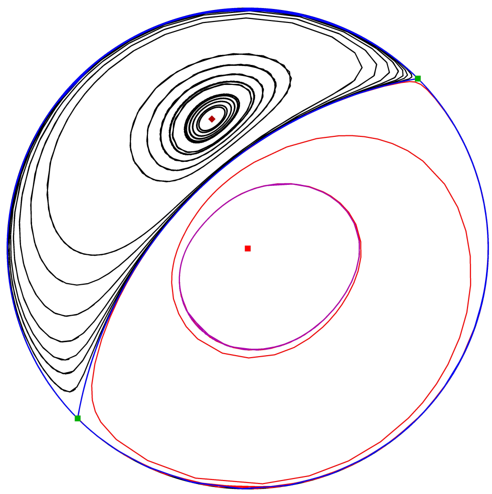


Figure 2.1: Non-algebraic limit cycle of quartic polynomial differential system (2.13).

2.4.2 Application 2: Differential system of degree 6

Example 2.2. *In the system* (2.1)*, we take* $\alpha = -2, w = b = 2, c = l = 1, v = -1, a = 4, n = 2$*, then we obtain*

$$\begin{cases} \dot{x} = 2\left(\left(-2x - y\right)\left(2x - y + 1\right) - x^2 - y\left(1 + 2x\right)\right)\left(6x^2 + 2y^2 + 2xy\right)^2 + x\left(2x - y + 1\right)^3, \\ \dot{y} = 2\left(\left(x - 2y\right)\left(2x - y + 1\right) - 2y^2 + x\left(1 - y\right)\right)\left(6x^2 + 2y^2 + 2xy\right)^2 + y\left(2x - y + 1\right)^3. \end{cases} \tag{2.14}$$

which has a non-algebraic, stable and hyperbolic limit cycle whose expression in polar coordinates (r, θ) *is*

$$r\left(\theta, r_*\right) = \frac{1}{2}\left(g\left(\theta\right)\rho^{\frac{1}{2}}\left(\theta, r_*\right) + \sqrt{\left(g\left(\theta\right)\rho^{\frac{1}{2}}\left(\theta, r_*\right)\right)^2 + 4\rho^{\frac{1}{2}}\left(\theta, r_*\right)}\right)$$

where $\rho\left(\theta, r_*\right) = e^{-4\theta}\left(\frac{r_*^4}{\left(2r_* + 1\right)^2} + \int_0^\theta \frac{e^{4s}}{\left(4 + 2\cos 2s + \sin 2s\right)^2} ds\right)$, $g\left(\theta\right) = 2\cos\theta - \sin\theta$, $\theta \in \mathbb{R}$ *and* $r_* = 0.448\,78$*, see Figure 2.2.*

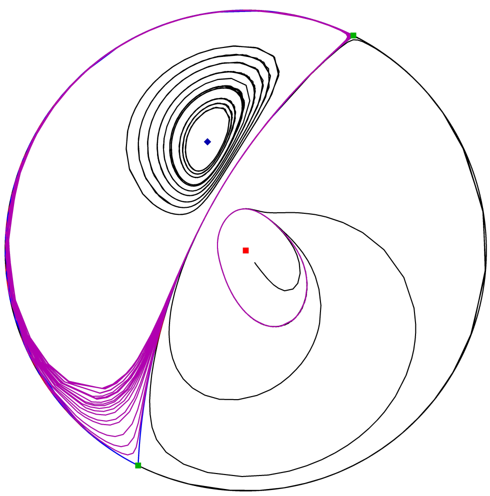


Figure 2.2: Non-algebraic limit cycle of polynomial differential system (2.14).

2.4.3 Application 3: Differential system of degree 8

Example 2.3. *let $\alpha = -1, w = v = 1, b = -2, c = 0, l = 2, a = 3, n = 3$, then the system (2.1) becomes*

$$\begin{cases} \dot{x} = 3\left(\left(-x-y\right)\left(x+y+2\right)+x^2-y\left(2+x\right)\right)\left(x^2+5y^2\right)^3+x\left(x+y+2\right)^4, \\ \dot{y} = 3\left(\left(x-y\right)\left(x+y+2\right)-y^2+x\left(2+y\right)\right)\left(x^2+5y^2\right)^3+\left(x+y+2\right)^4, \end{cases} \tag{2.15}$$

It is easy to verify that all conditions of Theorem 2.1 are satisfied. We conclude that system (2.15) has a non-algebraic limit cycle whose expression in polar coordinates (r,θ) is

$$r\left(\theta,r_*\right) = \frac{1}{2}\left(g\left(\theta\right)\rho^{\frac{1}{3}}\left(\theta,r_*\right)+\sqrt{\left(g\left(\theta\right)\rho^{\frac{1}{3}}\left(\theta,r_*\right)\right)^2+8\rho^{\frac{1}{3}}\left(\theta,r_*\right)}\right)$$

where $\rho\left(\theta,r_*\right) = e^{-\theta}\left(\dfrac{r_*^6}{\left(r_*+2\right)^3}+\displaystyle\int_0^\theta \dfrac{e^{3s}}{\left(3-2\cos 2s\right)^3}\,ds\right)$, $g\left(\theta\right) = \cos\theta+\sin\theta$, $\theta \in \mathbb{R}$ *and the intersection of the limit cycle with the OX_+ axis is the point*

$$r_* = 4.712\,5$$

Moreover,

$$\left.\frac{dr(2\pi;\lambda)}{d\lambda}\right|_{\lambda=r_*} = e^{-4\pi} < 1.$$

This limit cycle is a stable hyperbolic limit cycle, see Figure 2.3.

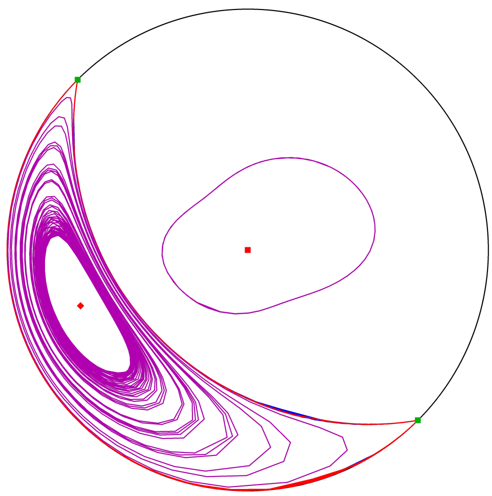


Figure 2.3: Non-algebraic limit cycle of polynomial differential system (2.15).

Chapter 3

Two non-algebraic limit cycles of a class of polynomial differential systems with non-elementary equilibrium point

Contents

3.1 Introduction

One of the important problem in the qualitative theory of differential equations is to solve the second part of the 16*th* problem out of 23 problems that Hilbert presented at the International Congress of Mathematicians in Paris (1900), see [39]. In which Hilbert asked for an upper bound for the maximum number of limit cycles of all polynomial differential systems of degree n of the form (1.1) . We recall that a limit cycle of system (1.1) is an isolated periodic solution in the set of all periodic solutions of system (1.1). If a limit cycle is contained in an algebraic curve of the plane, then we say that it is algebraic, otherwise it is called non-algebraic. The problem of existence of non algebraic limit cycles and their numbers are the most difficult problems in dynamical planar systems. Up to now, all the examples of polynomial differential systems for which non-algebraic limit cycles are known explicitly have one non algebraic limit cycle or two explicit limit cycles one is algebraic and another non-algebraic surronding an elementary equilibrium points, see for instance [2, 9, 10, 11, 12, 15, 14, 22, 32].

In this chapter, we are interested in studying the integrability and the existence of limit cycles for a differential systems with degenerate non-elementary singular point of the form

$$\begin{cases} \dot{x} = 2x\left(x^2+y^2\right)^k + \left(\gamma - \left(x^2+y^2\right)^k\right)\left(2c\left(x^2+y^2\right)^k - bP_{2k}\left(x,y\right)\right) \\ \qquad \times \left(a\gamma x - \left(ax - 4ky\right)\left(x^2+y^2\right)^k\right), \\ \dot{y} = 2y\left(x^2+y^2\right)^k + \left(\gamma - \left(x^2+y^2\right)^k\right)\left(2c\left(x^2+y^2\right)^k - bP_{2k}\left(x,y\right)\right) \\ \qquad \times \left(a\gamma y - \left(ay + 4kx\right)\left(x^2+y^2\right)^k\right), \end{cases} \tag{3.1}$$

where $a, b, c, \gamma \in \mathbb{R}$, and k are positive integer $(k \in \mathbb{N}^*)$, and $P_{2k}(x, y)$ is homogeneous polynomial of degree $2k$ such that

$$P_{2k}(x,y) = \sum_{s=0}^{k-1} (-1)^s \binom{2k}{2s+1} x^{2k-2s-1} y^{2s+1}, \tag{3.2}$$

where $\binom{2k}{2s+1} = \dfrac{2k!}{(2s+1)!\,(2k-2s-1)!}$. Moreover, under some suitable conditions we will show that our systems exhibiting two non algebraic limit cycles surrounding a non elementary equilibrium point at origin or two algebraic limit cycles explicitly given. It should be noted that, to our knowledge, there is no such result in the mathematical literature.

Noted that this result was published in the journal of Tatra Mt. Math. Publ. 79 (2021), 33–46 [7].

Remark 3.1. *We say that a singular point is non-elementary if both of the eigenvalues of the linear part of the vector field at that point are zero, and elementary otherwise. A non elementary singular point is called degenerate if its linear part is identically zero, otherwise it is called nilpotent.*

3.2 Statement of the main result

As a main result, we shall prove the following Theorem

Theorem 3.1. *Consider the polynomial differential system (3.1). Then the following three statements hold*

1. *System (3.1) is Darboux integrable with the Liouvillian first integral*
$$I(x,y) = \left(\left(x^2+y^2\right)^k - \gamma\right)^2 e^{-a\arctan\frac{y}{x}} - \int_0^{\arctan\frac{y}{x}} \frac{e^{-as}}{c-\frac{1}{2}b\sin 2ks}\,ds.$$

2. *If $a<0, b\neq 0, c>\frac{1}{2}|b|, \gamma>0$ and $a\gamma^2\left(c-\frac{1}{2}|b|\right)+1<0$, then the system (3.1) has two explicit non-algebraic limit cycles, given in polar coordinates (r,θ) by*
$$r_1(\theta, r_1^*) = \left(\gamma + \sqrt{\varphi(\theta, r_1^*)}\right)^{\frac{1}{2k}},$$
$$r_2(\theta, r_2^*) = \left(\gamma - \sqrt{\varphi(\theta, r_2^*)}\right)^{\frac{1}{2k}},$$
where
$$\varphi(\theta, r_i^*) = e^{a\theta}\left(\left((r_i^*)^{2k}-\gamma\right)^2 + g(\theta)\right), i=1,2,$$
$$g(\theta) = \int_0^{\theta} \frac{e^{-as}}{c-\frac{1}{2}b\sin 2ks}\,ds,$$
$$r_1^* = \left(\gamma + \sqrt{\frac{e^{2\pi a}}{1-e^{2\pi a}}g(2\pi)}\right)^{\frac{1}{2k}},$$
$$r_2^* = \left(\gamma - \sqrt{\frac{e^{2\pi a}}{1-e^{2\pi a}}g(2\pi)}\right)^{\frac{1}{2k}}.$$

3. *If $b=0, ac<0, \gamma>0$ and $\gamma^2+\frac{1}{ac}>0$, then system (3.1) has two explicit algebraic limit cycles, given in Cartesian coordinates (x,y) by*
$$\left(\left(x^2+y^2\right)^k-\gamma\right)^2+\frac{1}{ac}=0.$$

Next Lemma collects some results which we need to show the statements of Theorem 3.1.

Lemma 3.1. *Let $a<0, c>\frac{1}{2}|b|$, $a\gamma^2\left(c-\frac{1}{2}|b|\right)+1<0$ and $\gamma>0$, then the following statements hold*

1. *The function $\Phi(\theta)=\gamma^2 e^{-a\theta}-g(\theta)$ is strictly increasing, for all $\theta\in[0,2\pi[$, where $g(\theta)$ is a function defined in previous theorem.*

2. $\frac{e^{2\pi a}}{1-e^{2\pi a}}g(2\pi) < \gamma^2 e^{-a\theta}-g(\theta).$

3. $0<\varphi(\theta)<\gamma^2$, *where* $\varphi(\theta)=e^{a\theta}\left(\frac{e^{2\pi a}}{1-e^{2\pi a}}g(2\pi)+g(\theta)\right).$

3.3 Proof of the main result

Proof of statement (a) of Lemma 3.1. We remark that the function Φ is differentiable for all $\theta \in [0, 2\pi[$, then

$$\frac{d\Phi}{d\theta} = -\frac{e^{-a\theta}}{c - \frac{1}{2}b\sin 2k\theta}\left(a\gamma^2\left(c - \frac{1}{2}b\sin 2k\theta\right) + 1\right).$$

Since $a < 0, c > \frac{1}{2}|b|$ and $\gamma > 0$, then

$$0 < c - \frac{1}{2}|b| < c - \frac{1}{2}b\sin 2k\theta < \frac{1}{2}|b| + c$$

and

$$a\gamma^2\left(\frac{1}{2}|b| + c\right) + 1 < a\gamma^2\left(c - \frac{1}{2}b\sin 2k\theta\right) + 1 < a\gamma^2\left(c - \frac{1}{2}|b|\right) + 1.$$

Since $a\gamma^2\left(c - \frac{1}{2}|b|\right) + 1 < 0$, then $a\gamma^2\left(c - \frac{1}{2}b\sin 2k\theta\right) + 1 < 0$, hence $\Phi'(\theta) > 0$. Consequently the function Φ is strictly increasing.

Proof of statement (b) of Lemma 3.1. Because $\Phi(\theta)$ is strictly increasing then we have $\Phi(0) < \Phi(\theta) < \Phi(2\pi)$ equivalent to

$$\gamma^2 < \gamma^2 e^{-a\theta} - g(\theta) < \gamma^2 e^{-2\pi a} - g(2\pi). \tag{3.3}$$

We remark that $\gamma^2 < \gamma^2 e^{-2\pi a} - g(2\pi)$ which implies that

$$g(2\pi) < \gamma^2\left(e^{-2\pi a} - 1\right) = \gamma^2\frac{(1 - e^{2\pi a})}{e^{2\pi a}},$$

since $a < 0$, then

$$\frac{e^{2\pi a}}{1 - e^{2\pi a}}g(2\pi) < \gamma^2, \tag{3.4}$$

Taking into account (3.3) we obtain

$$\frac{e^{2\pi a}}{1 - e^{2\pi a}}g(2\pi) < \gamma^2 e^{-a\theta} - g(\theta).$$

Proof of statement (c) of Lemma 3.1. First, we prove that $\varphi(\theta) < \gamma^2$, from the statement 2 of Lemma **3.1**, we have

$$\varphi(\theta) \quad = \quad e^{a\theta}\left(\frac{e^{2\pi a}}{1 - e^{2\pi a}}g(2\pi) + g(\theta)\right),$$

then

$$\varphi(\theta) \quad < \quad e^{a\theta}\left(\gamma^2 e^{-a\theta} - g(\theta) + g(\theta)\right) = \gamma^2.$$

Consequently

$$\varphi(\theta) < \gamma^2.$$

Since $c > \frac{1}{2}|b|$ we get that $g(\theta) > 0$ and we have $a < 0$, it follows that $\frac{e^{2\pi a}}{1 - e^{2\pi a}} g(2\pi) > 0$, hence $\varphi(\theta) > 0$. Consequently $0 < \varphi(\theta) < \gamma^2$, for all $\theta \in [0, 2\pi[$. This complete proof of Lemma **3.1**. □

Proof of Theorem 3.1. Firstly, we have

$$x\dot{y} - y\dot{x} = -4k\left(\gamma - \left(x^2 + y^2\right)^k\right)\left(2c\left(x^2 + y^2\right)^k - bP_{2k}(x, y)\right)\left(x^2 + y^2\right)^{k+1}.$$

Thus, the equilibrium points of system (3.1) are present in the equation curve's

$$4k\left(\gamma - \left(x^2 + y^2\right)^k\right)\left(2c\left(x^2 + y^2\right)^k - bP_{2k}(x, y)\right)\left(x^2 + y^2\right)^{k+1} = 0. \tag{3.5}$$

In polar coordinates (r, θ) where $x = r\cos\theta$ and $y = r\sin\theta$ the function $P_{2k}(x, y)$ reads as

$$\begin{aligned} P_{2k}(r\cos\theta, r\sin\theta) &= \sum_{s=0}^{k-1} C_{2k}^{2s+1} (-1)^s (r\cos\theta)^{2k-2s-1} (r\sin\theta)^{2s+1} \\ &= r^{2k}\sin(2k\theta). \end{aligned}$$

Then, the curve's (3.5) can be written as

$$4kr^{2k}\left(\gamma - r^{2k}\right)(2c - b\sin(2k\theta))\, r^{2k+2} = 0.$$

From the condition $c > \frac{1}{2}|b|$, we have $2c - b\sin(2k\theta) > 0$, then the equilibrium points of system (3.1) are present on the curve

$$r^{2k}\left(\gamma - r^{2k}\right) = 0,$$

we deduce that the origin is an equilibrium point which is a degenerate non–elementary singular point of system (3.1), because the linear part of this system is identically zero, and any other equilibrium point, if exists must lies on the curve

$$\left(x^2 + y^2\right)^k - \gamma = 0. \tag{3.6}$$

To prove our results we write the polynomial differential system (3.1) in polar coordinates (r, θ) defined by $x = r\cos\theta$ and $y = r\sin\theta$. Then the system become

$$\begin{cases} \dot{r} = r^{2k+1}\left(2 + a(2c - b\sin 2k\theta)\left(\gamma - r^{2k}\right)^2\right), \\ \dot{\theta} = -4k(2c - b\sin 2k\theta)\left(\gamma - r^{2k}\right) r^{4k}. \end{cases} \tag{3.7}$$

The differential system (3.7) where $4k\,(2c - b\sin 2k\theta)\left(\gamma - r^{2k}\right) r^{4k} \neq 0$ can be written as the equivalent differential equation

$$\frac{dr}{d\theta} = \frac{r\left(2 + a\,(2c - b\sin 2k\theta)\left(\gamma - r^{2k}\right)^2\right)}{-4k\,(2c - b\sin 2k\theta)\,(\gamma - r^{2k})\,r^{2k}}. \tag{3.8}$$

Via the change of variables $\varphi = \left(r^{2k} - \gamma\right)^2$, the equation (3.8) is transformed into the linear equation

$$\frac{d\varphi}{d\theta} = a\varphi + \frac{1}{c - \frac{1}{2}b\sin 2k\theta}. \tag{3.9}$$

The general solution of this equation is

$$\varphi\,(\theta, h) = e^{a\theta}\,(h + g\,(\theta))\,, \tag{3.10}$$

where $h \in \mathbb{R}$ and $g\,(\theta) = \int_0^\theta \frac{e^{-as}}{c - \frac{1}{2}b\sin 2ks}\,ds$. Consequently, the implicit solution of equation (3.8) is given by

$$\left(r^{2k} - \gamma\right)^2 = e^{a\theta}\,(h + g\,(\theta))\,.$$

By passing to cartesian coordinates, we deduce the first integral is

$$I\,(x, y) = \left(\left(x^2 + y^2\right)^k - \gamma\right)^2 e^{-a\arctan\frac{y}{x}} - \int_0^{\arctan\frac{y}{x}} \frac{e^{-as}}{c - \frac{1}{2}b\sin 2ks}\,ds.$$

Since this first integral is a function that can be expressed by quadratures of elementary functions, it is a Liouvillian function, and consequently system (3.1) is Darboux integrable. Notice that system (3.1) has a periodic orbit if and only if the equation (3.8) has a strictly positive 2π-periodic solution. The solution satisfying the initial condition $r(0, r_0) = r_0 > 0$ is given by $h = \left(r_0^{2k} - \gamma\right)^2$. Then, the implicit solution of equation (3.8) with this initial condition is

$$\left(r^{2k} - \gamma\right)^2 = e^{a\theta}\left(\left(r_0^{2k} - \gamma\right)^2 + g\,(\theta)\right). \tag{3.11}$$

Since $\varphi = \left(r^{2k} - \gamma\right)^2$, it is clear that $r\,(2\pi, r_0) = r\,(0, r_0)$ if and only if $\varphi\,(2\pi, r_0) = \varphi\,(0, r_0)\,$, we have

$$\varphi_0 = \varphi\,(0, r_0) = \left(r_0^{2k} - \gamma\right)^2 \quad \text{and} \quad \varphi\,(2\pi, r_0) = e^{2\pi a}\left(\left(r_0^{2k} - \gamma\right)^2 + g\,(2\pi)\right).$$

Then, the condition $\varphi\,(2\pi, r_0) = \varphi\,(0, r_0)$ implies that

$$\left(r_0^{2k} - \gamma\right)^2 = \frac{e^{2\pi a}}{1 - e^{2\pi a}}\,g\,(2\pi)\,. \tag{3.12}$$

Thus

$$\varphi\,(\theta) = e^{a\theta}\left(\frac{e^{2\pi a}}{1 - e^{2\pi a}}\,g\,(2\pi) + g\,(\theta)\right).$$

The implicit form of the solution of (3.8) can be written as

$$\left(r^{2k}-\gamma\right)^2 = e^{a\theta}\left(\frac{e^{2\pi a}}{1-e^{2\pi a}}g\left(2\pi\right)+g\left(\theta\right)\right), \tag{3.13}$$

From the expression of the change of variables $\varphi = \left(r^{2k}-\gamma\right)^2$ that transform (3.8) into (3.9), we get

$$\begin{aligned} r_1\left(\theta\right) &= \left(\gamma+\sqrt{\varphi\left(\theta\right)}\right)^{\frac{1}{2k}}, \\ r_2\left(\theta\right) &= \left(\gamma-\sqrt{\varphi\left(\theta\right)}\right)^{\frac{1}{2k}}. \end{aligned}$$

These two solutions are strictly positive because we have $\gamma > 0$ and from the statement (3) of Lemma **3.1** we have $0 < \varphi\left(\theta\right) < \gamma^2$.
From (3.12), there are two different values of r_0 with the property

$$r(2\pi, r_0) = r_0 > 0,$$

given by

$$\begin{aligned} r_1^* &= \left(\gamma+\sqrt{\frac{e^{2\pi a}}{1-e^{2\pi a}}g\left(2\pi\right)}\right)^{\frac{1}{2k}}, \\ r_2^* &= \left(\gamma-\sqrt{\frac{e^{2\pi a}}{1-e^{2\pi a}}g\left(2\pi\right)}\right)^{\frac{1}{2k}}. \end{aligned}$$

Since $a<0, \gamma>0$ and by (3.4) we have $\frac{e^{2\pi a}}{1-e^{2\pi a}}g\left(2\pi\right) < \gamma^2$, then $r_1^* > 0$ and $r_2^* > 0$.
After the substitution of the values $r_i^*, i = 1, 2$ into $r_i\left(\theta\right), i = 1, 2$ we obtain

$$\begin{aligned} r_1\left(\theta, r_1^*\right) &= \left(\gamma+\sqrt{\varphi\left(\theta, r_1^*\right)}\right)^{\frac{1}{2k}}, \\ r_2\left(\theta, r_2^*\right) &= \left(\gamma-\sqrt{\varphi\left(\theta, r_2^*\right)}\right)^{\frac{1}{2k}}. \end{aligned} \tag{3.14}$$

where

$$\varphi\left(\theta, r_i^*\right) = e^{a\theta}\left(\left(\left(r_i^*\right)^{2k}-\gamma\right)^2+g\left(\theta\right)\right), i = 1, 2.$$

To show that $r_i\left(\theta\right), i = 1, 2$ are periodic solutions, we have to show that

a) There does not exist any singular point of (3.13).

b) The functions $\theta \longmapsto r_i\left(\theta\right), i = 1, 2$ are 2π -periodic.

c) $r_i\left(\theta\right) > 0, i = 1, 2$ for all $\theta \in [0, 2\pi[$.

a) We first prove that there is no singular point of (3.13). In particular, we shall prove that the curve (3.6) does not intersect the orbit (3.13). This curve in polar coordinates becomes $r^{2k} - \gamma = 0$. To show this, we have to show that the system

$$\begin{cases} \left(r^{2k} - \gamma\right)^2) - e^{a\theta} \left(\frac{e^{2\pi a}}{1 - e^{2\pi a}} g\left(2\pi\right) + g\left(\theta\right) \right) = 0, \\ r^{2k} - \gamma = 0. \end{cases} \tag{3.15}$$

has no solutions.

From the second equation of system (3.15) we get that $r^{2k} = \gamma$, we replace this value in the first equation we obtain $e^{a\theta} \left(\frac{e^{2\pi a}}{1 - e^{2\pi a}} g\left(2\pi\right) + g\left(\theta\right) \right) = 0$, which is a contradiction because $e^{a\theta} \left(\frac{e^{2\pi a}}{1 - e^{2\pi a}} g\left(2\pi\right) + g\left(\theta\right) \right) = \varphi\left(\theta\right)$ and from Lemma **3.1** we have $0 < \varphi\left(\theta\right) < \gamma^2$. So (3.15) has no solutions.

b) **Periodicity**. From (3.14) we say that $r_i\left(\theta, r_i^*\right), i = 1, 2$ are 2π-periodic if and only if $\varphi\left(\theta\right)$ is 2π-periodic, then we have

$$\varphi\left(\theta + 2\pi\right) = e^{a(\theta+2\pi)} \left(\frac{e^{2\pi a}}{1 - e^{2\pi a}} g\left(2\pi\right) + g\left(\theta + 2\pi\right) \right). \tag{3.16}$$

However,

$$\begin{aligned} g\left(\theta + 2\pi\right) &= \int_0^{\theta+2\pi} \frac{e^{-as}}{c - \frac{1}{2}b\sin 2ks} ds \\ &= g\left(2\pi\right) + \int_{2\pi}^{\theta+2\pi} \frac{e^{-as}}{c - \frac{1}{2}b\sin 2ks} ds. \end{aligned}$$

In the integral $\int_{2\pi}^{\theta+2\pi} \frac{e^{-as}}{c - \frac{1}{2}b\sin 2ks} ds$, we use the change of variable $w = s - 2\pi$, we obtain

$$\begin{aligned} g\left(\theta + 2\pi\right) &= g\left(2\pi\right) + \int_0^{\theta} \frac{e^{-a(w+2\pi)}}{c - \frac{1}{2}b\sin 2k\left(w + 2\pi\right)} dw \\ &= g\left(2\pi\right) + e^{-2\pi a} g\left(\theta\right). \end{aligned}$$

We replace $g\left(\theta + 2\pi\right)$ by $g\left(2\pi\right) + e^{-2\pi a} g\left(\theta\right)$ in (3.16), and after some calculations we obtain $\varphi\left(\theta + 2\pi\right) = \varphi\left(\theta\right)$, hence $\varphi\left(\theta\right)$ is 2π-periodic. Consequently the functions $r_i\left(\theta, r_i^*\right), i = 1, 2$ are also 2π-periodic.

c) **Strict positivity of** $r_i\left(\theta, r_i^*\right), i = 1, 2$, for all $\theta \in [0; 2\pi[$. Since $0 < \varphi\left(\theta\right) < \gamma^2$ and $\gamma > 0$, we have $\gamma^2 > \varphi\left(\theta\right)$, which implies $\gamma > \sqrt{\varphi\left(\theta\right)}$, hence

$$\gamma - \sqrt{\varphi\left(\theta\right)} > 0 \text{ and } \gamma + \sqrt{\varphi\left(\theta\right)} > 0.$$

Therefore $r_i\left(\theta, r_i^*\right), i = 1, 2$ are strictly positive.

In order to prove that the periodic orbit is hyperbolic limit cycles, we introduce the Poincaré return map

$$\lambda \to \Pi(2\pi, \lambda) = r(2\pi, \lambda).$$

Therefore, the periodic orbits of system (3.1) are hyperbolic limit cycles if and only if $\left.\dfrac{dr_i(2\pi,\lambda)}{d\lambda}\right|_{\lambda=r_i^*} \neq 1, i = 1,2$ where

$$r_1^* = \left(\gamma + \sqrt{\frac{e^{2\pi a}}{1-e^{2\pi a}} g\left(2\pi\right)}\right)^{\frac{1}{2k}}, \quad r_2^* = \left(\gamma - \sqrt{\frac{e^{2\pi a}}{1-e^{2\pi a}} g\left(2\pi\right)}\right)^{\frac{1}{2k}},$$

We have

$$\begin{aligned} r_1\left(2\pi,\lambda\right) &= \left(\gamma + \sqrt{e^{2a\pi}\left(\left(\lambda^{2k}-\gamma\right)^2 + g\left(2\pi\right)\right)}\right)^{\frac{1}{2k}}, \\ r_2\left(2\pi,\lambda\right) &= \left(\gamma - \sqrt{e^{2a\pi}\left(\left(\lambda^{2k}-\gamma\right)^2 + g\left(2\pi\right)\right)}\right)^{\frac{1}{2k}}. \end{aligned}$$

After some calculations we obtain

$$\left.\frac{dr_1(2\pi,\lambda)}{d\lambda}\right|_{\lambda=r_1^*} = \left.\frac{dr_2(2\pi,\lambda)}{d\lambda}\right|_{\lambda=r_2^*} = e^{2\pi a} \neq 1.$$

Consequently the limit cycles of the differential equation (3.8) are hyperbolic (see for instance [46].

Now we prove that these two limit cycles are not algebraic for $b \neq 0$. The curve defined by these two limit cycles is

$$\left(r^{2k}-\gamma\right)^2 - e^{a\theta}\left(\frac{e^{2\pi a}}{1-e^{2\pi a}} g\left(2\pi\right) + g\left(\theta\right)\right) = 0. \tag{3.17}$$

More precisely, in Cartesian coordinates $r^2 = x^2 + y^2, \theta = \arctan\dfrac{y}{x}$, this curve can be written as

$$G\left(x,y\right) = \left(\left(x^2+y^2\right)^k - \gamma\right)^2 - e^{a\arctan\frac{y}{x}}\left(\frac{e^{2\pi a} g\left(2\pi\right)}{1-e^{2\pi a}} + \int_0^{\arctan\frac{y}{x}} \frac{e^{-as}}{c-\frac{1}{2}b\sin 2ks}\, ds\right) = 0. \tag{3.18}$$

We remark that $G\left(x,y\right) = 0$ does not a polynomial because there is no integer n for which both $\dfrac{\partial^n G}{\partial x^n}$ and $\dfrac{\partial^n G}{\partial y^n}$ vanish identically, for example when calculating $\dfrac{\partial G}{\partial x}$ note that the expression $e^{a\arctan\frac{y}{x}}\left(\dfrac{e^{2\pi a}}{1-e^{2\pi a}} g\left(2\pi\right) + \displaystyle\int_0^{\arctan\frac{y}{x}} \frac{e^{-as}}{c-\frac{1}{2}b\sin 2ks}\, ds\right)$ appear again, so for any order of derivation this expression will appear. Therefore the curve $G\left(x,y\right) = 0$ is non-algebraic and the limit cycles of the system (3.1) will also be non-algebraic.

Proof of statement (3) of Theorem 3.1. If we take $b = 0$, we get $g\left(\theta\right) = \dfrac{1}{ac}\left(1-e^{-a\theta}\right)$ and $g\left(2\pi\right) = \dfrac{1}{ac}\left(1-e^{-2\pi a}\right)$. Then

$$\varphi\left(\theta\right) = \frac{-1}{ac},$$

Going back through the changes of variables $\left(r^{2k}-\gamma\right)^2=\varphi$ and by passing to Cartesian coordinates (x,y), we obtain

$$\left(\left(x^2+y^2\right)^k-\gamma\right)^2+\frac{1}{ac}=0.$$

The system (3.1) has two algebraic limit cycles if and only if $ac<0$, $\gamma>\sqrt{-\frac{1}{ac}}$. This complete the proof of Theorem **3.1**. □

3.4 Applications

The following examples illustrate our result

For $k=1$, we have $P_2(x,y)=2xy$.

For $k=2$, we have

$$\begin{aligned} P_4(x,y) &= \binom{4}{1}x^3y+(-1)\binom{4}{3}xy^3 \\ &= 4x^3y-4xy^3. \end{aligned}$$

3.4.1 Differential systems with two non-algebraic limit cycles

Example 3.1. *If we take $\gamma=c=2, b=1, a=-1$, then system (3.1) reads*

$$\begin{cases} \dot{x}=2x\left(x^2+y^2\right)^k+\left(2-\left(x^2+y^2\right)^k\right)\left(4\left(x^2+y^2\right)^k-P_{2k}(x,y)\right) \\ \qquad\times\left(-2x+(x+4ky)\left(x^2+y^2\right)^k\right), \\ \dot{y}=2y\left(x^2+y^2\right)^k+\left(2-\left(x^2+y^2\right)^k\right)\left(4\left(x^2+y^2\right)^k-P_{2k}(x,y)\right) \\ \qquad\times\left(-2y-(4kx-y)\left(x^2+y^2\right)^k\right). \end{cases} \tag{3.19}$$

This system has two non algebraic limit cycles whose expressions in polar coordinates (r,θ) are

$$\begin{aligned} r_1(\theta,r_1^*) &= 2+\sqrt{e^{-\theta}\left(\left(\left(r_1^*\right)^2-2\right)^{2k}+g(\theta)\right)}, \\ r_2(\theta,r_2^*) &= 2-\sqrt{e^{-\theta}\left(\left(\left(r_2^*\right)^{2k}-2\right)^2+g(\theta)\right)} \end{aligned}$$

where

$$\begin{aligned} g(\theta) &= \int_0^\theta \frac{e^s}{2-\frac{1}{2}\sin 2ks}\,ds \\ r_1^* &= \left(2+\sqrt{\frac{e^{-2\pi}}{1-e^{-2\pi}}g(2\pi)}\right)^{\frac{1}{2k}}, \\ r_2^* &= \left(2-\sqrt{\frac{e^{-2\pi}}{1-e^{-2\pi}}g(2\pi)}\right)^{\frac{1}{2k}}. \end{aligned}$$

Case k=1 Case k=2

Figure 3.1: Two non-algebraic limit cycles for system (3.19) in Poincaré disc.

3.4.2 Differential systems with two algebraic limit cycles

Example 3.2. *Let $\gamma = c = 2, b = 0$ and $a = -1$, then the system (3.1) becomes*

$$\begin{cases} \dot{x} = 2x\left(x^2+y^2\right)^k - 4\left(x^2+y^2\right)^k\left(2-\left(x^2+y^2\right)^k\right)\left(2x-(x+4ky)\left(x^2+y^2\right)^k\right), \\ \dot{y} = 2y\left(x^2+y^2\right)^k - 4\left(x^2+y^2\right)^k\left(2-\left(x^2+y^2\right)^k\right)\left(2y+(-y+4kx)\left(x^2+y^2\right)^k\right). \end{cases} \tag{3.20}$$

*We remark that this system satisfy the conditions of statement (3) of Theorem **3.1**, hence system (3.20) possess two algebraic limit cycles, these two limit cycles given in Cartesian coordinates by the expression*

$$\left(\left(x^2+y^2\right)^k - 2\right)^2 - \frac{1}{2} = 0.$$

Case k=1 Case k=2

Figure 3.2: Two algebraic limit cycles for system (3.20) in Poincaré disc.

Chapter 4

Limit cycles: A numerical approach using the P4 software

Contents

4.1 Introduction

The existence of limit cycle is one of the important and difficult studies of planar polynomial differential systems. In the last years several research papers have appeared on this topic. But there are a few works which study the explicit expression of these limit cycles. This thesis is part of this perspective. Although theorems can be applied to prove the existence of a limit cycle, giving the explicit expression of these limit cycles is a difficult task. In this situation, requiring to use the numerical method is useful, to have an idea on the configuration of the planar polynomial differential systems which admits this limit cycle. Recently a software has been designed to plot the phase portrait of any polynomial differential systems. Moreover, and if we do not know a theorem of existence of a limit cycle, for a given differential systems, this software can determine if it exists or not.

In this chapter, we will present a software based on the theory of differential polynomial planar systems. This software is used to draw the phase portrait of any polynomial differential system on the plane. This software is called **P4** it is abbreviation of **Polynomial Planar Phase Portraits.**

Our main reference in this part is based on chapter 9 of the book "Qualitative Theory of Planar Differential Systems" by Dumortier, Llibre and Artés [29].

4.2 History of P4

P4 [5] is a software designed by A.C Artés, C. Herssens, P. De Maesschalck, F. Dumortier and J. Llibre. The first version of P4, It only works on UNIX or LINUX, It was written partly in C and partly in REDUCE. It ran only under a UNIX or LINUX system and its developer was mainly C. Herssens. The new version of P4 has changed the symbolic language from REDUCE to MAPLE, and can now be implemented more easily in any system, either WINDOWS, UNIX or MACINTOSH OS-X, as long as MAPLE is available. The new version has been developed by De Maesschalck. It is possible to work in numerical mode or in mixed mode, i.e, if possible, the calculations are done in algebraic mode.

4.3 Functionality of P4

In this section, we present the essential structure and possibilities of P4 for study a polynomial differential systems.

1. P4 draws the phase portraits on either the Poincaré disc, or on a Poincaré-Lyapunov disc for all polynomial differential system of any degree.

2. P4 determines all finite and infinite singularieties of the vector field. It distinguishes between the different type of singularities (saddle, node, focus, center, semi hyperbolic saddle-nodes, degenerate). To distinguish between focus behavior and weak focus behavior in the singularities of purely imaginary eigenvalues, P4 uses the Lyapunov coefficients method.

3. P4 has the possibility to find the limit cycles (i.e. isolated periodic orbits for all periodic solution) of any degree of polynomial differential system, and it able to draw the separatrix of vector field.

4.4 P4 Algorithm

We consider a polynomial differential system of the form

$$\begin{cases} \dot{x} = \dfrac{dx}{dt} = P(x,y), \\ \dot{y} = \dfrac{dy}{dt} = Q(x,y), \end{cases} \tag{4.1}$$

where P and Q are real polynomials in the variables x and y of degree $n = \max(\deg(P), \deg(Q))$. The vector field of this system is $P(x,y)\dfrac{\partial}{\partial x} + Q(x,y)\dfrac{\partial}{\partial y}$. For studying the phase portrait of a polynomial differential system (4.1), P4 uses the following steps :

Step1: Eliminated the Greatest Common Factor (GCF) of a differential system, using Maple. In the following steps we will assume that the GCF has been eliminated.

Step 2: Studying the system at the finite region, i.e. determining the real finite singularities. This is done evaluate algebraically or numerically. Suppose (x_0, y_0) is a singular of system (4.1) then $P(x_0, y_0) = Q(x_0, y_0) = 0$. Define the Jacobian matrix

$$J(x_0,y_0) = \begin{pmatrix} \dfrac{\partial P}{\partial x}(x_0,y_0) & \dfrac{\partial P}{\partial y}(x_0,y_0) \\ \dfrac{\partial Q}{\partial x}(x_0,y_0) & \dfrac{\partial Q}{\partial y}(x_0,y_0) \end{pmatrix}$$

The linearization of the system (4.1) is given in matrix form by:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial P}{\partial x}(x_0,y_0) & \dfrac{\partial P}{\partial y}(x_0,y_0) \\ \dfrac{\partial Q}{\partial x}(x_0,y_0) & \dfrac{\partial Q}{\partial y}(x_0,y_0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

We distinguish the different cases according to the eigenvalues λ_1 and λ_2 of the matrix $J(x_0, y_0)$:

a . The eigenvalues λ_1, λ_2 are the opposite sign, then (x_0, y_0) is a saddle.

b . The eigenvalues $\lambda\ _1$, λ_2 are the same sign and nonzero, then (x_0, y_0) is a node.

c . The eigenvalues are complex conjugates with $\lambda_j = \alpha \pm i\beta$ and $\alpha \neq 0$, $\beta \neq 0$, then (x_0, y_0) is a focus.

d . The eigenvalues are pure imaginary i.e. $\mathrm{Im}(\lambda_j) \neq 0$ and $\mathrm{Re}(\lambda_j) = 0, \forall j = 1, 2$, then (x_0, y_0) is a center.

e . The eigenvalues λ_1, 0 with $\lambda_1 \neq 0$, $\lambda_2 = 0$, then (x_0, y_0) is a semi-elementary.

f . The eigenvalues $\lambda_1 = \lambda_2 = 0$ and $J(x_0, y_0) \neq 0$, then (x_0, y_0) is a nilpotent or degenerate.

Step 3: Calculating the invariant separatrices.

Step 4: Studying the system at infinity, we make use of the Poincaré disc. Let $\chi(x,y) = (P(x,y), Q(x,y))$ be a polynomial vector field of degree n. Consider the Poincaré sphère $S^2 = \left\{(y_1; y_2; y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\right\}$. The Poincaré compactification of χ denoted by $p(\chi)$. The singular points of $p(\chi)$ are called the infinite singular points of χ or $p(\chi)$. To study the vector field $p(\chi)$ we use six local charts on S^2 given by

$$U_k = \left\{y \in S^2 : y_k > 0\right\}; V_k = \left\{y \in S^2 : y_k < 0\right\} \text{ for } k = 1,2,3.$$

The corresponding local maps $\Phi_k : U_k \to \mathbb{R}^2$ and $\Psi_k : V_k \to \mathbb{R}^2$. The expression for $p(\chi)$ in local chart (U_1, Φ_1) is given by

$$\begin{cases} \dot{u} = v^n \left[-uP\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right)\right], \\ \dot{v} = -v^{n+1} P\left(\frac{1}{v}, \frac{u}{v}\right). \end{cases}$$

The expression for $p(\chi)$ in local chart (U_2, Φ_2) is given by

$$\begin{cases} \dot{u} = v^n \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right)\right], \\ \dot{v} = -v^{n+1} Q\left(\frac{u}{v}, \frac{1}{v}\right). \end{cases}$$

The expression for (U_3, Φ_3) is

$$\begin{cases} \dot{u} = P(u,v), \\ \dot{v} = Q(u,v). \end{cases} \tag{4.2}$$

Remark 4.1. *The expression for $p(\chi)$ in the charts (V_k, Ψ_k) is the same as for (U_k, Φ_k) multiplied by $(-1)^{n-1}$ for $k = 1,2,3$. Geometrically the coordinates (u,v) can be expressed as in Figure 4.1. For more details on the Poincaré compactification see Chapter 5 of [29].*

4.5 Examples for using the P4

In this section we will present some examples about the use of P4, see [5]. For these examples we will use the program running on a WINDOWS system, and using MAPLE as symbolic language.

Example 4.1. *Consider a quadratic polynomial differential system*

$$\begin{cases} \dot{x} = y(x + 2y + 3) + ax + x^2, \\ \dot{y} = -x(x + 2y + 3) + ay + y^2. \end{cases} \tag{4.3}$$

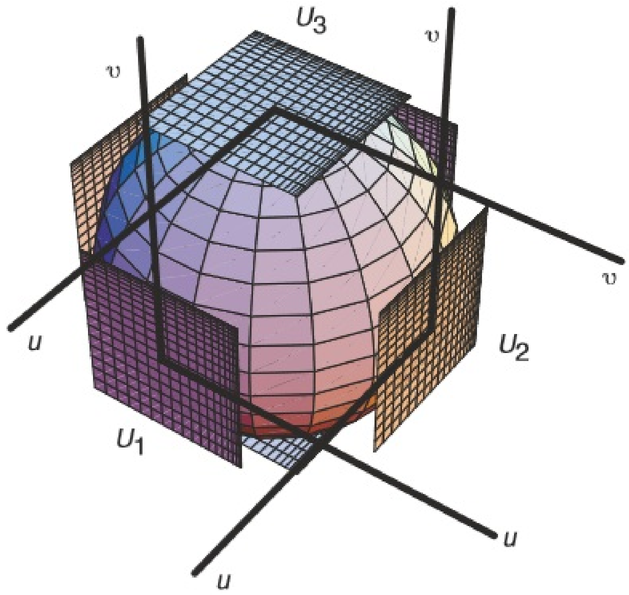


Figure 4.1: The local charts (U_k, Φ_k) for k=1,2,3 of the Poincaré sphere.

Using the integrability method, we are not able to find the explicit expression of limit cycle, but we can determine the position of limit cycle and all phase portrait of this system using P4. For the study of this system in P4 we use the following steps.

Firstly, you open the main window of P4. At the button of the window you see two fields where you can introduce the equation of the vector field. In the x' *you write* $y*(x+2*y+3)+a*x+x\hat{}2$ *and in the* y' *field* $-x*(x+2*y+3)+a*y+y\hat{}2$. *Since there is no line of singularities, you can leave the Gcf field equal to* 1. *Next you must click on the Number of parameters option and set how many parameters you are using. Fix it to just one parameter. In the new line opened below write the name of parameter in the left column and fixed number in the second column. In this case you put* $a = 0.25$. *To save this system, press the Save button, then enter a name for it, for example, Example 1. Clicking Evaluate button, P4 using Maple for studying numerically this system, you see in the Output window a message as in Figure 4.2. With a View menu button, you see that the system has two finite singular points one of them is the origin* $(0,0)$ *which is a strong unstable focus, and the second point is* $(1.2961974577354, -1.4632709341136)$ *which is a saddle point. and there also a node in infinite. By clicking the Plot button in the main window you see a green box and a red box which represent a finite saddle and unstable weak focus respectively. On the circle you see one blue box which represents the stable node at infinity, and a red box, which represents the unstable node at infinity. For more information about the symbols that you have in the drawing space you press the Legend*

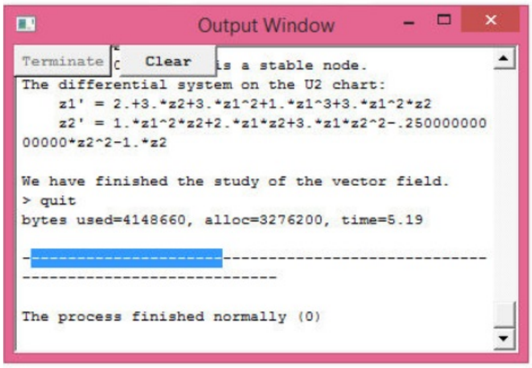


Figure 4.2: The end of the calculations.

button will open the Legend window, see Figure 4.3. Click to the Plot All Separatrices

Figure 4.3: The legend P4 window

.

button there will appear some lines in blue and red. These lines are the stable and unstable separatrices. To obtain a global vision of the phase portrait more orbits have to be drawn, see Figure 4.4. This is conclusive proof of the existence of at least one limit cycle. You may easily remember that if a red separatrix spirals around a red point (or blue around blue) you have a limit cycle.

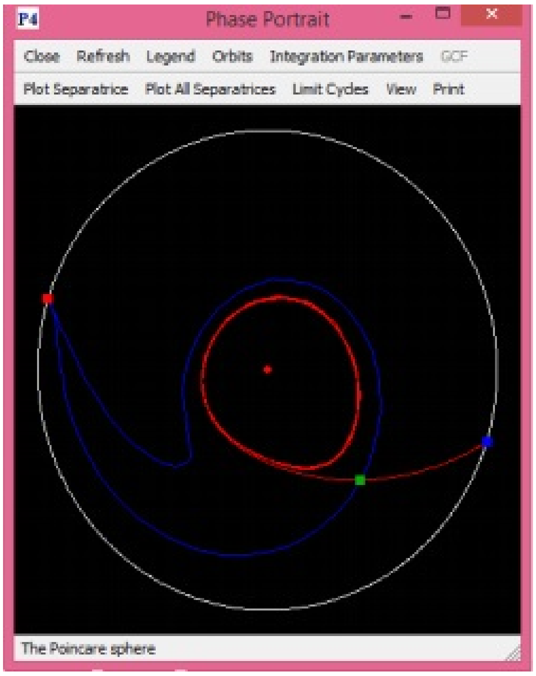


Figure 4.4: Stable and unstable separatrices of system (4.3)

.

You can confirm this existence of limit cycle by another way. Click the Limit Cycles button in the Phase Portrait window. In this window the user has to give two points forming a segment which he suspects is cut by at least one limit cycle. Fill this window as follows

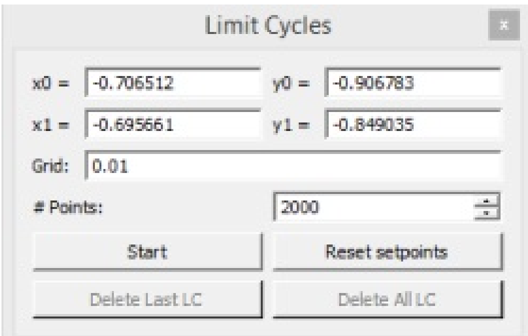


Figure 4.5: Limit Cycle window.

where ***x0, y0****: Defines the first point of the line segment.*
x1, y1: *Defines the last point of the line segment. The user can select these two points by clicking the left button of the mouse on the first point, and while holding the button down, moving the mouse to the second point and releasing the button.*
Grid: *Determines the precision up to which the limit cycles will be determined.*
Points: *This parameter equals the number of steps the Runge–Kutta 7/8 method has to*

do each time we want to integrate an orbit with initial condition a point of the segment. Press starting button in the previous window and after a time you will see the limit cycle portrayed in purple, see Figure 4.6.

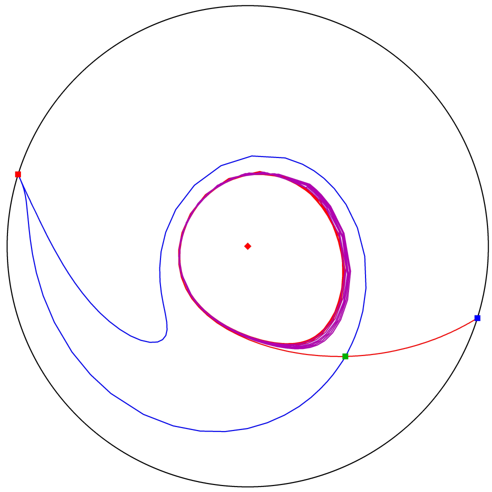


Figure 4.6: The limit cycle of system (4.3) with a=0.25.

Example 4.2. *The van der Pol oscillator, which was discovered by Van der Pol (1926). The Van der Pol equation has a long history of being used in both the physical and biological sciences. The second-order differential equation that governs the Van Der Pol oscillator is:*

$$\ddot{x} + \left(x^2 - 1\right)\dot{x} + x = 0,$$

which is written in phase plan as:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\left(x^2 - 1\right)y - x. \end{cases}$$

By the same steps of study in Example 4.1. Firstly, enter this system into P4, and evaluate it in Numeric mode, click View button you see that the system has one finite singular points which is a strong unstable focus at origin $(0,0)$*. In infinity there are two pairs of singularities, one a semi-hyperbolic saddle at origin in local chart* U_1 *and the other a non-elementary point in local chat* U_2*. By pressing the Plot button of the main window you see one red box which represents the unstable strong focus. On the circle you see two green triangles and two more crosses, which represent a pair of semi-hyperbolic saddles and the non-elementary singular points at infinity, respectively. Now if you press the Plot All Separatrices button, there will appear some lines in red. Two more clicks on the same button give you a better idea of what those separatrices do. All this proof the existence of at least one limit cycle see Figure 4.7.*

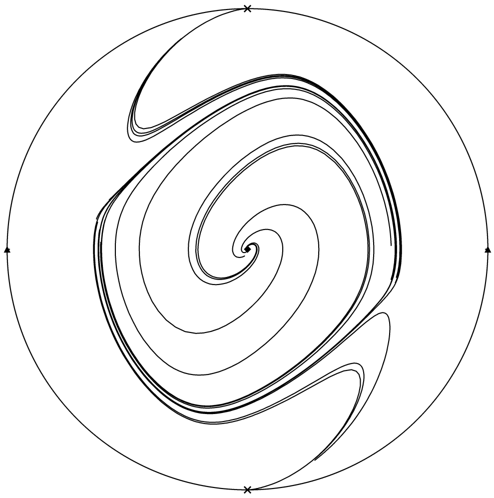


Figure 4.7: Limit cycle of Van Der Pol equation in Poincaré disc.

Example 4.3. *Consider two populations whose sizes at a reference time are denoted by $x(t), y(t)$, respectively. The functions x and y might denote population numbers or concentrations (number per area) or some other scaled measure of the populations sizes, but are taken to be continuous functions. Changes in population size with time are described by the time derivatives $\dot{x} \equiv \frac{dx}{dt}$ and $\dot{y} \equiv \frac{dy}{dt}$, respectively, and a general model of interacting populations is written in terms of two autonomous differential equations*

$$\begin{cases} \dot{x} = xf(x,y), \\ \dot{y} = yg(x,y). \end{cases}$$

This general model is often called Kolmogorov's predator-prey model. We consider cubic Kolmogorov vector field

$$\begin{cases} \dot{x} = (x+p)\left(y(x+2y+3)+ax+x^2\right), \\ \dot{y} = (y+q)\left(-x(x+2y+3)+ay+y^2\right). \end{cases} \tag{4.4}$$

Introduce this system in P4. Firstly using the following values of parameters $a = 0.25$, $p = q = 4$ and evaluate it in Numeric mode. By pressing the View button you see that the system has five finite singular points which are two saddles at the points $(-4., -4)$, $(1.2961974577354, -1.4632709341136)$ and two unstable nodes at the points $(-4., -8.7092802052231), (-4, 0.45928020522306)$ and one an unstable strong focus at origin. At infinity, you see that the system in local chart U_1 has two stable nodes at the points $(0,0), (-2,0)$ and one saddle at the point $(-1,0)$. In local chart U_2 the origin is a saddle. Clicking the Plot button of the main window you see one red box which represents a finite strong unstable focus and two green boxes for the saddles and two blue squares represents a stables nodes. On the circle you see green and blue boxes for the saddles and nodes *respectively. Now if you press the Plot All Separatrices button, there will appear some lines in red and blue. These lines are the unstable and stable separatrices of the finite saddles. you see that the red line of separatrice around the red box which represents the unstable strong focus all this confirm the existence of at least limit cycle surrounding the origin, see Figure 4.8.*

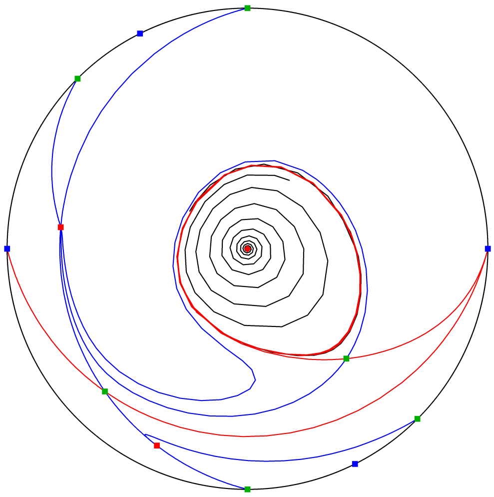


Figure 4.8: The phase portrait in the Poincaré disc of cubic Kolmogorov differential system (4.4), with limit cycle included.

Conclusion and Outlook

In the qualitative theory of differential systems in the plane, a limit cycle is an isolated periodic solution in the set of all periodic solutions of the system. This concept was defined by Poincaré in several papers. Limit cycles have been shown to model many real-world phenomena. The nonexistence, the existence, the maximum number and other properties of limit cycles have been widely studied by mathematicians and physicists and more recently by biologists, economists and engineers. It is interesting to know the number of these limit cycles of a given differential system, but their location in the orbits of the system is also an interesting problem. On the other hand, determining the explicit expression for the limit cycle is a more difficult task. In this thesis, we have studied the integrability and the existence of limit cycles of some classes of planar polynomial differential systems. Our contribution has been materialized by two new results. Firstly, we have determined a sufficient conditions for which a class of planar polynomial differential systems of degree even, have a explicit non-algebraic limit cycle . On the other hand, we have constructed a class of differential systems of degree $6k + 1$,$(k \in \mathbb{N}^*)$, we have studied their integrability, then we have shown that it has a two non-algebraic limit cycles surrounding a non-elementary equilibrium point at origin, these limit cycles are explicitly constructed. To our knowledge, there are no such type of examples in the mathematical literature.

In the outlook:

- It is convenient to construct a class of differential systems of degree $n = 5$ or $n = 3$, with two non algebraic limit cycles surrounding a non elementary equilibriuim point.
- On the other hand, we have considered applying our results to models from other disciplines.
- We search a numerical method for determine a limit cycles of non integrable differential systems.

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ملخص

الهدف من هذه الأطروحة هو الدراسة النوعية لبعض أصناف النظم التفاضلية كثيرة الحدود في المستوي غير الخطية. اولا قمنا بدراسة قابلية التكامل لفئة من النظم التفاضلية من الدرجة الزوجية ، ثم أثبتنا تحت شروط معينة وجود دورة حد غير جبرية لهذه الفئة. ثانيا قدمنا فئة اخرى من النظم التفاضلية من الدرجة $6k+1, (k \in \mathbb{N}^*)$، درسنا قابلية التكامل، ثم اثبتنا انها تقبل دورتين حديتين غير جبريتين، بالإضافة إلى ذلك تمكنا من تحديد المعادلات الصريحة لكل التكاملات الأولية ودورات الحد غير الجبرية التي تم العثور عليها لجميع الأصناف المدروسة. كما قدمنا بعض الأمثلة لتوضيح النتائج التي تم الحصول عليها لكل صنف. وفي الأخير قدمنا لمحة عن برمجية $P4$ التي تساعد على رسم تصرفات الحلول للنظم التفاضلية كثيرة الحدود في المستوي.

الكلمات المفتاحية : دورات حد نهائية ، دورات حد نهائية غير جبرية، حلول دورية، النظم التفاضلية كثيرة الحدود في المستوي، التكامل الأولي.

Résumé

L'objectif de cette thèse est l'étude qualitative de quelques classes de systèmes différentiels planaires polynomiaux non linéaire. Dans un premier temps, nous avons étudié l'intégrabilité et l'existence d'un cycle limite non algébrique d'une classe de systèmes différentiels de degré pair, de plus, on a déterminé leurs expressions explicites. D'autre part, nous avons introduit une autre classe de systèmes différentiels de degré $6k+1, (k \in \mathbb{N}^*)$, nous avons étudié leur intégrabilité, puis nous avons montré qu'elle possède deux cycles limites non algébriques explicitement données. On a donné quelques exemples pour illustrer nos résultats obtenus. Finalement, on a donné une description du fonctionnement du logiciel P4, qui sert à tracer le portrait de phases des solutions des systèmes différentiels planaires polynomiaux.

Mots clés : Cycle limite, cycle limite non algébrique, solution périodique, systèmes différentiels planaires, intégrale première.

Abstract

The objective of this thesis is the qualitative study of some classes of nonlinear planar polynomial differential systems. Firstly, we have studied the integrability and the existence of a non-algebraic limit cycle of a class of differential systems of degree even, moreover, we have determined their explicit expression. On the other hand, we have introduced another class of differential systems of degree $6k+1, (k \in \mathbb{N}^*)$, we have studied their integrability, then we have shown that it has a two non-algebraic limit cycles explicitly given. We have given some examples to illustrate the results obtained for each class. Finally, we gave a description of the operation of the P4 software, which is used to draw the phase portrait of the solutions of planar polynomial differential systems.

Keywords : Limit cycle, non algebraic limit cycle, periodic solution, planar polynomial differential systems, integrability, first integral.