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Presented by:

Hizia BOUNADJA

Jury

Mr. Boubakeur MEROUANI	President	Prof.	UFA Setif 1
Mr. Belkacem SAID-HOUARI	Supervisor	Prof.	UOS Sharjah
Mr. Salim MESSAOUDI	Examiner	Prof.	UOS Sharjah
Mr. Saïd MAZOUZI	Examiner	Prof.	UBM Annaba
Mr. Amar YOUKANA	Examiner	Prof.	UHL Batna 2
Mr. Abdelfateh FAREH	Examiner	Prof.	UHL El-Oued
Mr. Salah DRABLA	Guest	Prof.	LaMA, UFA Setif 1

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Dedication

Gratefully and thankfully, I dedicate this thesis to:

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INTRODUCTION

This thesis is devoted to the study of the well-posedness and stability of the linearized version of a nonlinear model in acoustics known as the Jordon–Moore–Gibson–Thompson equation (JMGT). We employ various types of dissipation mechanisms and show their effects on the stability of such equation.

Modeling and previous works

Nonlinear acoustics is a branch of physics dealing with sound waves of sufficiently large amplitudes that require using full systems of governing equations of fluid dynamics (for sound waves in liquids and gases) and elasticity (for sound waves in solids). These equations are naturally nonlinear and, therefore, the theory of nonlinear partial differential equations can provide valuable insight into the behavior of their solution for a long time.
Many researchers have investigated nonlinear acoustics due to the increasing number of applications of high-frequency sound waves in medicine and industry such as lithotripsy, thermotherapy, ultrasound cleaning, and welding, see for instance [1, 32, 39, 40, 46, 57, 79, 80, 92, 100].

Modeling

One of the classical model equations in nonlinear acoustics is the Kuznetsov equation [30, 61, 70]:

$$u_{tt} - c^2\Delta u - \delta\Delta u_t = \left(\frac{1}{c^2}\frac{B}{2A}(u_t)^2 + |\nabla u|^2\right)_t, \tag{1}$$

where the constants δ, c being the diffusivity and the speed of sound, respectively, and B/A denotes the parameter of nonlinearity. This equation can be derived from the equation of fluid dynamics by using the Fourier law of heat conduction in the equation of the conservation of energy. However, it is also known that by using Fourier's law we obtain an infinite signal speed paradox of the energy propagation. That is, any thermal disturbance at a single point has an instantaneous effect everywhere in the medium. To overcome this drawback in the Fourier law, one suggestion is to replace the Fourier law with the Maxwell-Cattaneo law. This process then gives rise to an extra term in (1) which contains a third time derivative term, with a small constant coefficient τ, referred to as a relaxation time. As a consequence, the mathematical structure of the underlying model changes drastically from the parabolic character of the Kuznetsov model (1) to the hyperbolic-like character of the Jordan–Moore–Gibson–Thompson model (JMGT) [52, 54, 56, 103, 110]:

$$\tau u_{ttt} + u_{tt} - c^2\Delta u - b\Delta u_t = \left(\frac{1}{c^2}\frac{B}{2A}(u_t)^2 + |\nabla u|^2\right)_t, \tag{2}$$

where $b = \delta + \tau c^2$.
Our goal in this section is to give a brief derivation of equation (2) from the basic equations of fluid dynamics. The conservation of mass (the continuity equation), the conservation of

momentum (Newton's second law), conservation of energy (first law of thermodynamics or entropy balance) and the Maxwell-Cattaneo law are given, respectively, by:

- The conservation of mass

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho v) = 0. \tag{3a}$$

- The conservation of momentum

$$\varrho\Big[v_t + (v \cdot \nabla)v\Big] + \nabla p = \mu_\nu \Delta v + \Big(\frac{\mu_\nu}{3} + \eta_\nu\Big)\nabla(\nabla \cdot v), \tag{3b}$$

  where μ_ν is the shear viscosity (the first coefficient of viscosity) and η_ν is the second coefficient of the viscosity (the bulk viscosity).

- The conservation of energy

$$\varrho\theta\Big[S_t + (\nabla \cdot v)S\Big] = -\nabla \cdot q + 2\mu_\nu \mathbb{D} : \mathbb{D} + \Big(\eta_\nu - \frac{2}{3}\mu_\nu\Big)(\nabla \cdot v)^2 \tag{3c}$$

  where $\mathbb{D}$ is the deformation tensor given by

$$\mathbb{D} = \frac{1}{2}(\nabla v + (\nabla v)^T)$$

  and the components of $\mathbb{D} : \mathbb{D}$ are $D_{ij}D_{ij}$ where D_{ij} denote the components of the matrix $\mathbb{D}$.

- The Maxwell-Cattaneo law

$$q(x,t) + \tau q_t(x,t) = -k\nabla\theta(x,t), \tag{4}$$

  where $\tau > 0$ is relaxation time and $k > 0$ is the heat conductivity.

Let us first start by defining the main physical quantities involved in the description of the Navier-Stokes equations above (3) and the Maxwell-Cattaneo law (4):

- the velocity v, which is assumed to be irrotational, that is $\nabla \times v = 0$, and could be expressed via a scalar acoustic velocity potential i.e., $v = -\nabla u$, the acoustic pressure p, the mass density ϱ, the temperature θ, the specific entropy S and the heat flux q.

All these physical quantities are decomposed into their mean (which is typically constant in space and time) and fluctuating components, such that

$$v = v_0 + \tilde{v}, \tag{5a}$$
$$p = p_0 + \tilde{p}, \qquad \text{with} \quad \nabla p_0 = 0 \tag{5b}$$
$$\varrho = \varrho_0 + \tilde{\varrho} \tag{5c}$$
$$\theta = \theta_0 + \tilde{\theta} \tag{5d}$$
$$S = S_0 + \tilde{S}. \tag{5e}$$

Following [52, 108], we substitute (5c) into (3a) and assume ϱ_0 to be a constant, we find

$$\tilde{\varrho}_t + \varrho_0 \nabla \cdot v = -\tilde{\varrho} \nabla \cdot v - v \cdot \nabla \tilde{\varrho}. \tag{6}$$

Now, using the following vector identities

$$\nabla(\nabla \cdot v) = \Delta v + \nabla \times \nabla \times v \tag{7}$$

$$v \cdot \nabla v = \frac{1}{2} \nabla (v \cdot v) - v \times \nabla \times v, \tag{8}$$

in equation (3b) and exploiting (5b) and (5c) with p_0 taken to be a constant, we obtain

$$\begin{aligned}(\varrho_0 + \tilde{\varrho}) \Big[v_t + \frac{1}{2} \nabla (v \cdot v) - v \times \nabla \times v \Big] + \nabla \tilde{p} \\ = \mu_\nu \Delta v + \Big(\frac{\mu_\nu}{3} + \eta_\nu \Big) (\Delta v + \nabla \times \nabla \times v).\end{aligned} \tag{9}$$

Since the acoustic velocity v is irrotational, that is $\nabla \times v = 0$, as well as, we assume zero time-mean velocity v_0, which implies the identity $v = \tilde{v}$, and omit the third order fluctuating term $\frac{\tilde{\varrho}}{2} \nabla (v \cdot v)$, equation (9) can be rewritten in the following form

$$\varrho_0 \tilde{v}_t + \frac{\varrho_0}{2} \nabla (\tilde{v} \cdot \tilde{v}) + \tilde{\varrho} \tilde{v}_t + \nabla \tilde{p} = \Big(\frac{4\mu_\nu}{3} + \eta_\nu \Big) \Delta \tilde{v}. \tag{10}$$

The equation of state is given by

$$p = p(\varrho, S). \tag{11}$$

Expanding (11) in a Taylor series around the values (ϱ_0, S_0), and neglecting the third-order terms, we arrive at

$$\begin{aligned} p - p_0 = & \varrho_0 \Big(\frac{\partial p}{\partial \varrho} \Big)_{\varrho_0, S_0} \frac{\varrho - \varrho_0}{\varrho_0} + \frac{\varrho_0^2}{2} \Big(\frac{\partial^2 p}{\partial \varrho^2} \Big)_{\varrho_0, S_0} \Big(\frac{\varrho - \varrho_0}{\varrho_0} \Big)^2 \\ & + \Big(\frac{\partial p}{\partial S} \Big)_{\varrho_0, S_0} (S - S_0) + \dots . \end{aligned} \tag{12}$$

By defining

$$A := \varrho_0 \Big(\frac{\partial p}{\partial \varrho} \Big)_{\varrho_0, S_0} = \varrho_0 c^2, \quad B := \varrho_0^2 \Big(\frac{\partial^2 p}{\partial \varrho^2} \Big)_{\varrho_0, S_0}$$

and using (5b), (5c) and (5e), then, equation (12) takes the form

$$\tilde{p} := A \frac{\tilde{\varrho}}{\varrho_0} + \frac{B}{2} \Big(\frac{\tilde{\varrho}}{\varrho_0} \Big)^2 + \Big(\frac{\partial p}{\partial S} \Big)_{\varrho_0, S_0} \tilde{S}. \tag{13}$$

Now, it only remains to combine the three equations (6), (10) and (13):

$$\begin{cases} \tilde{\varrho}_t + \varrho_0 \nabla \cdot \tilde{v} = -\tilde{\varrho} \nabla \cdot \tilde{v} - \tilde{v} \cdot \nabla \tilde{\varrho} & \text{(14a)} \\ \varrho_0 \tilde{v}_t + \frac{\varrho_0}{2} \nabla (\tilde{v} \cdot \tilde{v}) + \tilde{\varrho} \tilde{v}_t + \nabla \tilde{p} = \left(\frac{4\mu_\nu}{3} + \eta_\nu \right) \Delta \tilde{v} & \text{(14b)} \\ \tilde{p} = A \frac{\tilde{\varrho}}{\varrho_0} + \frac{B}{2} \left(\frac{\tilde{\varrho}}{\varrho_0} \right)^2 + \left(\frac{\partial p}{\partial S} \right)_{\varrho_0, S_0} \tilde{S}, & \text{(14c)} \end{cases}$$

respectively, into a single equation. To achieve this combination, we will substitute any physical quantity in a second-order term with its linearized one, since the resulting errors will be of third-order. Presicely we will express the second-order terms in (14) by its linearizations:

$$\nabla \cdot \tilde{v} \approx -\frac{1}{\varrho_0} \tilde{\varrho}_t \qquad \text{linear continuity equation,} \tag{15}$$

$$\tilde{v}_t \approx -\frac{1}{\varrho_0} \nabla \tilde{p} \qquad \text{linear Euler equation,} \tag{16}$$

$$\tilde{\varrho} \approx \frac{\tilde{p}}{c^2} \qquad \text{linear state equation,} \tag{17}$$

respectively. Using now the approximations (15) and (17) to rewrite (14a) as

$$\tilde{\varrho}_t + \varrho_0 \nabla \cdot \tilde{v} = \frac{\tilde{p}}{\varrho_0 c^4} \tilde{p}_t - \frac{1}{c^2} \tilde{v} \cdot \nabla \tilde{p}. \tag{18}$$

Furthermore, equation (14c) can be reformulated, by using (17), as

$$\tilde{\varrho} = \frac{\tilde{p}}{c^2} - \frac{1}{\varrho_0 c^4} \frac{B}{2A} \tilde{p}^2 - \frac{1}{c^2} \left(\frac{\partial p}{\partial S} \right)_{\varrho_0, S_0} \tilde{S}. \tag{19}$$

Next, differentiating (19) with respect to t and using (18), we obtain

$$\frac{\tilde{p}_t}{c^2} - \frac{1}{\varrho_0 c^4} \frac{B}{2A} (\tilde{p}^2)_t + \varrho_0 \nabla \cdot \tilde{v} - \frac{1}{2\varrho_0 c^4} (\tilde{p}^2)_t + \frac{1}{c^2} \tilde{v} \cdot \nabla \tilde{p} = \frac{1}{c^2} \left(\frac{\partial p}{\partial S} \right)_{\varrho_0, S_0} \tilde{S}_t. \tag{20}$$

In what follows, the balance law expressing the rate of entropy production, reduces to (see [31, 46]):

$$\varrho_0 \theta_0 \tilde{S}_t = -\nabla \cdot q, \tag{21}$$

where only the first-order terms of (3c) have been retained. Further, the heat flux equation (4) takes the form

$$(1 + \tau \partial_t) q = -k \nabla \tilde{\theta}. \tag{22}$$

Plugging equation (22) into (21), we arrive to

$$\varrho_0 \theta_0 (1 + \tau \partial_t) \tilde{S}_t = k \Delta \tilde{\theta}. \tag{23}$$

Therefore, from the approximation [105], we have

$$\tilde{\theta} \approx \left(\frac{\partial \theta}{\partial p}\right)_{p_0,S_0} \tilde{p},$$

which gives when using it in (23):

$$(1+\tau\partial_t)\tilde{S}_t = \frac{k}{\varrho_0 c^2 \theta_0}\left(\frac{\partial \theta}{\partial p}\right)_{p_0,S_0} \Delta \tilde{p}. \tag{24}$$

Now, to eliminate $\tilde{S}_t$ in (20), we apply the operator $(1+\tau\partial_t)$ in (20), followed by the use of equation (24), we obtain

$$\begin{aligned}(1+\tau\partial_t)\Big[\frac{\tilde{p}_t}{c^2} - \frac{1}{\varrho_0 c^4}\frac{B}{2A}(\tilde{p}^2)_t + \varrho_0 \nabla\cdot\tilde{v} - \frac{1}{2\varrho_0 c^4}(\tilde{p}^2)_t + \frac{1}{c^2}\tilde{v}\cdot\nabla\tilde{p}\Big] \\ = \frac{k}{\varrho_0 c^2\theta_0}\left(\frac{\partial p}{\partial S}\right)_{\varrho_0,S_0}\left(\frac{\partial \theta}{\partial p}\right)_{p_0,S_0}\Delta\tilde{p}.\end{aligned} \tag{25}$$

Next, to compute the coefficient $\frac{1}{\theta_0}\left(\frac{\partial p}{\partial S}\right)_{\varrho_0,S_0}\left(\frac{\partial \theta}{\partial p}\right)_{S_0,p_0}$ in (25), we use the equation of state for a perfect gas [105], to get,

$$\frac{1}{\theta_0}\left(\frac{\partial p}{\partial S}\right)_{\varrho_0,S_0}\left(\frac{\partial \theta}{\partial p}\right)_{S_0,p_0} = \frac{1}{c_\nu} - \frac{1}{c_p},$$

where c_p and c_v are the specific heats at constant pressure and volume, respectively. Hence, equation (25) recast as

$$\begin{aligned}\tau\frac{\tilde{p}_{tt}}{c^2} + \frac{\tau}{\varrho_0 c^4}\frac{B}{2A}(\tilde{p}^2)_{tt} - \frac{\tau}{2\varrho_0 c^4}(\tilde{p}^2)_{tt} + \frac{\tilde{p}_t}{c^2} - \frac{1}{\varrho_0 c^4}\frac{B}{2A}(\tilde{p}^2)_t \\ - \frac{1}{2\varrho_0 c^4}(\tilde{p}^2)_t + \tau\varrho_0\nabla\cdot\tilde{v}_t + \varrho_0\nabla\cdot\tilde{v} = \frac{k}{\varrho_0 c^2}\left(\frac{1}{c_\nu} - \frac{1}{c_p}\right)\Delta\tilde{p},\end{aligned} \tag{26}$$

where we neglected the second order terms.
On the other hand, by using (15)-(17), one can reformulate equation (14b) as

$$\varrho_0\nabla\cdot\tilde{v}_t + \Delta\tilde{p} = -\frac{\varrho_0}{2}\Delta(\tilde{v}\cdot\tilde{v}) + \frac{1}{2\varrho_0 c^2}\Delta\tilde{p}^2 - \frac{1}{\varrho_0 c^2}\left(\frac{4\mu_v}{3} + \eta_v\right)\Delta\tilde{p}_t, \tag{27}$$

where, we also applied the divergence operator. Differentiating again (26) with respect to time t and then subtracting the resulting equation from (27), we find

$$\begin{aligned}\Delta\tilde{p} - \frac{\tilde{p}_{tt}}{c^2} - \frac{\tau}{c^2}\tilde{p}_{ttt} + \frac{1}{c^2}\Big[\frac{1}{\varrho_0}\left(\frac{4\mu_v}{3} + \eta_v\right) + \frac{k}{\varrho_0}\left(\frac{1}{c_\nu} - \frac{1}{c_p}\right) + \tau c^2\Big]\Delta\tilde{p}_t \\ = -\frac{\varrho_0}{2}\Delta(v\cdot v) - \frac{1}{\varrho_0 c^4}\frac{B}{2A}(\tilde{p}^2)_{tt}.\end{aligned} \tag{28}$$

In addition, we use the relations between time and spatial derivatives according to the linear wave equation for sound pressure and velocity

$$\Delta \tilde{v} = \frac{\tilde{v}_{tt}}{c^2},$$
$$\Delta \tilde{p} = \frac{\tilde{p}_{tt}}{c^2},$$

in the higher order terms, we therefore arrive at

$$\frac{\tau}{c^2}\tilde{p}_{ttt} + \frac{\tilde{p}_{tt}}{c^2} - \Delta \tilde{p} - \frac{b}{c^2}\tilde{p}_{ttt} = \Big(\frac{1}{\varrho_0 c^4}\frac{B}{2A}\tilde{p}^2 + \frac{\varrho_0}{c^2}|v|^2\Big)_{tt}, \tag{29}$$

where $\varrho_0 v_t = -\nabla \tilde{p}$, and $b > 0$ denotes the diffusivity of sound which is given, as presented in [71], by

$$b = \Big[\frac{K}{\varrho_0}\Big(\frac{1}{c_\nu} - \frac{1}{c_p}\Big) + \frac{1}{\varrho_0}\Big(\frac{4\mu_v}{3} + \eta_v\Big) + \tau c^2\Big].$$

Moreover by recalling that the acoustic velocity v is irrotational i.e., $\nabla \times v = 0$, and by introducing the acoustic velocity potential u by $v = -\nabla u$, therefore, the equation (29) reads for the potential u as

$$\tau u_{ttt} + u_{tt} - c^2\Delta u - b\Delta u_t = \Big(\frac{1}{c^2}\frac{B}{2A}(u_t)^2 + |\nabla u|^2\Big)_t, \qquad u_t := \frac{\tilde{p}}{\varrho_0} \tag{30}$$

and referred to as the Jordan–Moore–Gibson–Thompson equation given in (2).
We point out that, the linearized part of (30) can also be derived when modeling with mechanical components, such as springs and dashpots, and known as the standard linear solid model of viscoelasticity, for more detail see [15, 41]. In fact, these springs represent the elastic component of the model response and obey Hooke's law in which the stress σ at any point is simply proportional to the strain e, i.e.,

$$\sigma = Ee,$$

where E is the Young modulus of the elastic structure. The dashpots represent the viscous component of a viscoelastic material. Therefore, the vibrations are governed by the following wave equation

$$u_{tt}(x,t) = a^2\Delta u(x,t),$$

where $a > 0$ is the constant wave velocity. Here, we consider vibrations modeled by the standard linear solid model of viscoelasticity. In this case, a linear spring is connected in series with a combination of another linear spring and a dashpot in parallel. The stress σ and strain e are then related by the differential equation

$$\sigma + \tau\sigma' = E(e + \beta e'),$$

where the constants τ, β are very small satisfying $0 < \tau < \beta$. Consequently, the vibrations of flexible structures are governed by the linear differential equation

$$\tau u_{ttt} + u_{tt} - a^2 \Delta u - a^2 \beta \Delta u_t = 0 \tag{31}$$

which is generally called the standard linear solid model. Equation (31) was first derived by Bose and Gorain [41] to model flexible structural systems possessing internal damping. See also [15] and [43] for further information and a complete description of the model.
It is significant to mention that a coupled standard linear solid model with a heat equation modeling an expected dissipative effect through heat conduction has received a lot of attention, which is reflected in the appearance of many works in the literature treating the well-posedness, the regularity and the stability of the solution [6, 96, 97].

Previous works

We briefly cite previous work on the above-mentioned equation in bounded and unbounded domains.

Moore–Gibson–Thompson equation (MGT)

The linearized version of (30) is known as the Moore–Gibson–Thompson equation (MGT):

$$\tau u_{ttt} + u_{tt} - c^2 \Delta u - b \Delta u_t = 0 \tag{32}$$

and plays a crucial role in the investigation of the nonlinear problem. As a result of this interest, many papers related to the well-posedness and asymptotic behavior of the so-called Moore–Gibson–Thompson equation, considering different types of dissipation, have appeared. In [54], Kaltenbacher *et al.* investigated the following equation:

$$\tau u_{ttt} + \alpha u_{tt} + c^2 \mathcal{A} u + b \mathcal{A} u_t = 0, \tag{33}$$

where $\mathcal{A}$ is a positive self-adjoint operator, and defined a critical parameter

$$\gamma = \alpha - \frac{c^2 \tau}{b}, \tag{34}$$

that controls the behavior of the solution. More precisely, they showed that when $b = 0$, there is a lack of existence of a semigroup associated with the linear dynamics. While, for $b > 0$, they proved that the problem is well-posed and its solution energy

- is exponentially stable if $\gamma > 0$ (the non-critical case), i.e.,

$$\hat{E}(t) \leq C e^{-\omega t} \hat{E}(0), \qquad \omega, t > 0$$

- remains constant if $\gamma = 0$ (the critical case), where $\hat{E}(t) = E(t) + E_0(t)$, such that

$$E(t) = \frac{b}{2}|\mathcal{A}^{1/2}(u_t + c^2b^{-1}u)|^2 + \frac{\tau}{2}|u_{tt} + c^2b^{-1}u_t|^2 + \frac{c^2}{2b}\gamma|u_t|^2$$

and

$$E_0(t) = \frac{\alpha}{2}|u_t|^2 + \frac{c^2}{2}|\mathcal{A}^{1/2}u_t|^2.$$

Spectral analysis for this model was performed in [42] and [81]. In fact, for a clear picture and under a convenient change of variable, Marchand *et al.* [81] showed that the equation (33) has a particular structural decomposition. Accordingly, by using the abstract semigroup approach with a refined spectral analysis, they succeeded in establishing the well-posedness of (33) and identified a point of accumulation of eigenvalues which essentially theoretically connects to the exponential decay of energy. The exponential decay rate obtained in [81] are completed in [98], where the obtention of an explicit scalar product for which the operator is normal allows the authors to obtain the optimal exponential decay rate of the solutions. Conejero *et al.* [28] considered the one-dimensional version of equation (32) and proved a chaotic behavior of the solution when $\gamma < 0$. Their arguments were analytical, not numerical, and offered new insights on dynamical behavior in more general situations.

On the other hand, to our knowledge, no previous research has investigated the equation (32) in $\mathbb{R}^N$ before the work of Pellicer and Said-Houari in [95], where the authors showed the well-posedness and established the decay rate of the solutions of (32) in the whole space $\mathbb{R}^N$ (note that one of the difficulties encountered in $\mathbb{R}^N$ space is the lack of Poincaré inequality). They used the energy method in the Fourier space to show, under the assumption $\beta - \tau > 0$ (same condition given in (34)), that the L^2-norm of the vector $V = (u_t + \tau u_{tt}, \nabla(u + \tau u_t), \nabla u_t)$, and of its higher-order derivatives obey the decay rate:

$$\|\nabla^j V(t)\|_{L^2(\mathbb{R}^N)} \leq C(1+t)^{-N/4-j/2}\|V_0\|_{L^1(\mathbb{R}^N)} + Ce^{-ct}\|\nabla^j V_0\|_{L^2(\mathbb{R}^N)}, \tag{35}$$

with initial data in $L^1(\mathbb{R}^N) \cap H^s(\mathbb{R}^N)$, $s \geq 1$ and j is a positive integer satisfying $0 \leq j \leq s$. Moreover, they established the optimality of the decay rate in (35) by exploiting the eigenvalues expansion method. In recent papers [96, 97], the authors of [95] showed that the condition $\beta - \tau > 0$ is also a necessary condition for stability. The authors of [26] improved slightly the results in [95] and showed that the derived estimates in the low-dimensional cases $N = 1$ and $N = 2$ are a little sharper than those in [95], by preparing representation of solutions in the Fourier space and using asymptotic expansions of eigenvalues together with WKB analysis. They also derived asymptotic profiles for the linear MGT equation in the dissipative case in a framework of L^1 space, by estimating upper bounds and lower bounds of the solution itself with $u_2 \in L^2(\mathbb{R}^N) \cap L^{1,1}(\mathbb{R}^N)$, and showed an approximate relation between the linear MGT equation and the linear viscoelastic damped wave equation (or the strongly damped wave equation).

The memory-type MGT equation has drawn the attention of many researchers lately and many studies have appeared showing various types of asymptotic stability results that have been established depending on the values of parameters in the equation and the decay rate of the relaxation functions. This kind of problems are motivated by a numerous papers on the wave equation with memory (or its abstract version):

$$u_{tt} - \Delta u + \int_0^t g(s)\Delta u(t-s)ds = 0, \tag{36}$$

where the convolution term $\int_0^t g(s)\Delta u(t-s)ds$ represents the memory effects of materials. Physically, the damping mechanism caused by viscoelasticity implies the appearance of the memory term in a system. Indeed, the memory term plays a critical role in the wave equation and is directly related to whether or how the energy decays. The researchers' interest led to understand of what would happen if two different types of damping terms appeared simultaneously in the same system. Because of this interest, they concluded that the two terms help the system to dissipate energy, but they do not necessarily improve the decay rate. As an example, in [25] whereas frictional damping alone yields an exponential decay, only the polynomial decay can be achieved when an additional memory term with a polynomially decaying kernel is present. Let us also cite the work of Conti *et al.* in [29] where they studied (36) on the whole space $\mathbb{R}^N$ and obtained some results concerning the decay of the energy, as time goes to infinity, when g decays either exponentially or polynomially. Some improvements of the decay rates were also given in [102]. Dell'Oro and Pata in [38] discussed the relationship between the MGT equation (32) and the equation of linear viscoelasticity:

$$u_{tt} - \kappa(0)\Delta u + \int_0^\infty \kappa'(s)\Delta u(t-s)ds = 0,$$

for the specific choice of the exponential kernel

$$\kappa(s) = ae^{-bs} + c \qquad a, b, c > 0.$$

For an in-depth study on wave-memory system, we refer the reader to [3, 4, 23, 24, 63, 83, 84], and references therein.

Now let us shade some light on the study of memory damping in the context of a third order in time system. Precisely, we discuss some results obtained for the MGT equation with a memory term of the form:

$$\tau u_{ttt} + \alpha u_{tt} - b\Delta u_t - c^2\Delta u + \int_0^t g(t-s)\Delta u(s)ds = 0. \tag{37}$$

In this regard, the first work studied the equation (37) is established by Lasiecka and Wang in [65], by replacing $-\Delta u$ with $\mathcal{A}$ and the convolution term $\int_0^t g(s)\mathcal{A}u(t-s)ds$ with $\int_0^t g(s)\mathcal{A}z(t-s)ds$, and considered three case: $z = u$ (type I memory), $z = u_t$ (type II memory) or $z = u+u_t$ (type III memory), imposing the assumption on the relaxation function g:

$$g'(t) \leq -c_0 g(t),$$

where c_0 is a positive constant. They studied the effect of the memory on the decay of the energy in the sub-critical case $\gamma > 0$, where γ is given in (34), and showed how the damping mechanism induced by the memory term leads to an exponential decay of the solution. In [5], Alves *et al.* proved the uniform stability of the MGT model encompassing the three different types of memory introduced by Lasiecka and Wang [65] in a history space setting and they showed some stability results, using the linear semigroup theory. Further, Lasiecka and Wang [64] looked into (37) and extended their work [65] by allowing the memory kernel to satisfy a more general decay estimate of the form

$$g'(t) + H(g(t)) \leq 0, \qquad \text{for all} \quad t > 0, \tag{38}$$

with $H(0) = 0$ and H is linear or strictly increasing and strictly convex C^2 function on $(0, r]$, $r < 1$. They proved that the decay rate of the memory kernel is transferred to a decay rate of the solution when $\gamma > 0$. In [75], the authors considered (37) for a class of relaxation functions satisfying

$$g'(t) \leq -\zeta(t) H(g(t)), \qquad \text{for all} \quad t > 0, \tag{39}$$

and established, by using some properties of the convex functions together with the generalized Young inequality, new optimal explicit decay results for the solution in the sub-critical case $\gamma > 0$, depending on g and H, that is

$$E(t) \leq k_2 H_1^{-1}\left(k_1 \int_{g^{-1}(r)}^{t} \zeta(s) ds\right), \tag{40}$$

where $H_1(t) = \int_t^r H'(s)/s\, ds$ and H_1 is strictly decreasing and convex on $(0, r]$, with $\lim\limits_{t\to 0} H_1(t) = +\infty$. One can notice, from (40), that the usual exponential and polynomial decay rates are only special cases. For instance, if we assume $H(s) = s^p$, $1 \leq p < 2$ in (39), then by direct calculations, we see that the decay of the energy is given by

$$E(t) \leq \begin{cases} C_1 e^{-c' \int_0^t \zeta(s) ds}, & if \quad p = 1, \\ C_2 \left(1 + \int_0^t \zeta(s) ds\right)^{-1/(p-1)}, & if \quad 1 < p < 2, \end{cases}$$

for some constants C_1, C_2 and c'.
The critical case $\gamma = 0$ has been investigated by Dell'Oro *et al.* [37] in a bounded domain and proved that system (37), replacing $-\Delta u$ with $\mathcal{A}$ and taking $\tau = 1$, is exponentially stable if and only if $\mathcal{A}$ is a bounded operator. In addition, in the case of an unbounded operator $\mathcal{A}$, they only proved that the corresponding energy decays polynomially, with the rate $1/t$, at least for regular initial data.

The Jordan– Moore–Gibson–Thompson equation (JMGT) with and without memory term, has been studied by many researchers. Indeed, Kaltenbacher *et al.*[55] considered equation (2) without the term of local nonlinear effects $(|\nabla u|^2)_t$, known as the Westervelt equation [108]:

$$\tau u_{ttt} + u_{tt} - c^2 \Delta u - b\Delta u_t = \frac{1}{c^2}\left(1 + \frac{B}{2A}\right)(u_t)^2_t, \tag{41}$$

and established the local and global well-posedness and the exponential decay for the energy

$$\mathcal{E}(t) \equiv |\mathcal{A}u(t)|^2 + |\mathcal{A}^{1/2} u_t(t)|^2 + |u_{tt}(t)|^2,$$

under a certain parameter range $\gamma > 0$. Additionally, Kaltenbacher and Nikolić in [56] established the well-posedness with and without the quadratic velocity term $(|\nabla u|^2)_t$ on the right-hand side of (2) corresponding to the Kuznetsov and the Westervelt nonlinearities, respectively. In addition, they also proved, as $\tau \to 0$, that the limit of (2) leads to the classical Kuznetsov (1) model.
Racke and Said-Houari [103] treated the initial value problem associated with (2) in $\mathbb{R}^3$. First, they proved a local existence result in appropriate functional spaces by employing the contraction mapping theorem as well as a global existence result for small data exploiting the energy method with a bootstrap argument. They also proved a polynomial decay rate of the solution which depends on the space dimension as long as the condition $\gamma > 0$ is satisfied. Quite recently, Said-Houari [106] improved the global existence result in [103] by assuming only the lower-order Sobolev norms of the initial data to be small, whereas the higher-order norms could be arbitrarily large.
Later, the memory-type JMGT equation has also received its share of attention. Lasiecka [62] considered the following JMGT equation with memory and in the absence of quadratic gradient nonlinearity:

$$\tau u_{ttt} + \alpha u_{tt} + c^2 \mathcal{A} u + b\mathcal{A} u_t - \int_0^t g(t-s)\mathcal{A} w(s) ds = \frac{1}{c^2}\left(1 + \frac{B}{2A}\right)(u_t)_t^2 \tag{42}$$

and proved the local and global existence (in time) of smooth solutions. Precisely, her study indicates that with appropriate calibration of the memory kernel, solutions exist globally for sufficiently small and regular initial data. These solutions exhibit exponential decay rates with an exponentially decaying memory kernel. Nikolić and Said-Houari [90] proved, under the assumption that the relaxation kernel decays exponentially, the local well-posedness in unbounded two- and three-dimensional domains. They also showed that the solution of the three-dimensional model exists globally in time and the energy of the system decays polynomially. Further, in $\mathbb{R}^N$ for all $N \geq 3$, the authors in their recent work [91] extended the analysis of [90] by taking into account the local effects in nonlinear sound propagation $|\nabla u_t|^2$, which leads to a quadratic gradient nonlinearity. Global existence is shown by exploiting a sequence of high-order energy uniform in time bounds and is derived under the assumption of an exponentially decaying memory kernel and sufficiently small and regular initial data. Moreover, recently, Nikolić and Said-Houari in [89], studied the asymptotic behavior of the nonlocal Jordan–Moore–Gibson–Thompson equation:

$$\tau u_{ttt} + \alpha u_{tt} - c^2 \Delta u + -b\Delta u_t - \int_0^t g(s)\Delta u(t-s) ds = \left(k u_t^2 + |\nabla u|^2\right)_t, \tag{43}$$

in the critical case $b = \tau c^2$ and where g is decaying exponentially. The authors are the first to have investigated the nonlinear problem (43) in critical case. They demonstrated the global existence and the asymptotic decay of the solutions in $\mathbb{R}^N, N \geq 3$, for smooth and small initial data. Knowing previously that the solutions of the linearized version of (43) for the critical case had the regularity-loss property, which thus presented an obstacle to the proof of nonlinear stability because the classical energy method fails. To deal with this issue, the authors proved the decay estimates for a solution with lower Sobolev regularity, by constructing appropriate time-weighted standards, where weights can have negative exponents.

MGT equation with heat conduction

We end this section by recalling some related work with the so-called standard linear solid model coupled to the different thermal conductions in the bounded or unbounded domain. In this regard, Alves *et al.* in [6] studied the standard linear solid model coupled with the classical Fourier law of heat conduction:

$$\begin{aligned} &\tau u_{ttt} + u_{tt} - a^2\Delta u - a^2\beta\Delta u_t + \eta\Delta\theta = 0, \\ &\theta_t - \Delta\theta - \tau\eta\Delta u_{tt} - \eta\Delta u_t = 0, \end{aligned} \tag{44}$$

where $x \in \Omega$ (a bounded open connected set in $\mathbb{R}^N$). They proved, in the case of $\gamma > 0$ (same γ given in (34)), that the energy decays exponentially with respect to time by using multiplier techniques. Pellicer and Said-Houari in [97] treated (44) in the whole space $\mathbb{R}^N$ and under the assumption $\gamma \geq 0$, (44) indicates the following decay estimates:

- Sub-critical case, $\gamma > 0$:

$$\|\nabla^k V_F(t)\|_{L^2(\mathbb{R}^N)} \leq C(1+t)^{-\frac{N}{4}-\frac{k}{2}}||V_F^0||_{L^1(\mathbb{R}^N)} + Ce^{-ct}\|\nabla^k V_F^0\|_{L^2(\mathbb{R}^N)}, \tag{45}$$

  where, $V_F(x,t) = (\tau u_{tt} + u_t, \nabla(\tau u_t + u), \nabla u_t, \theta)^T(x,t)$ and $V_F^0 = V_F(x,0)$.

- Critical case, $\gamma = 0$:

$$\|\nabla^k W_F(t)\|_{L^2(\mathbb{R}^N)} \leq C(1+t)^{-\frac{N}{8}-\frac{k}{4}}||W_F^0||_{L^1(\mathbb{R}^N)} + Ce^{-ct}\|\nabla^k W_F^0\|_{L^2(\mathbb{R}^N)}, \tag{46}$$

  where, $W_F(x,t) = (\tau u_{tt} + u_t, \nabla(\tau u_t + u), \theta)^T(x,t)$ and $W_F^0 = W_F(x,0)$.

In addition, using the eigenvalues expansion method, they confirmed the optimality of the previous decay rates. Observing that, in the sub-critical case, the decay rate obtained by coupling the standard linear solid model with Fourier law of heat conduction is the same as in the Cauchy problem (MGT) without heat conduction (see [95]). While, for the critical case they showed a slower decay rate, since heat conduction is the only cause of dissipation and the problem will be asymptotically stable. Pellicer and Said-Houari, in their recent work [96], studied the following standard linear model coupled with the Cattaneo law of heat conduction:

$$\begin{aligned} &\tau u_{ttt} + u_{tt} - a^2\Delta u - a^2\beta\Delta u_t + \eta\Delta\theta = 0, \\ &\theta_t + \gamma\nabla.q - \tau\eta\Delta u_{tt} - \eta\Delta u_t = 0, \\ &\tau_0 q_t + q + \kappa\nabla\theta = 0, \end{aligned} \tag{47}$$

where, $x \in \mathbb{R}^N$, and showed that the decay rates obtained are the same as those achieved when the model is coupled by Fourier heat conduction. The only difference lies in the requirement for more regularity of the initial data to obtain these decay rates (see (45) and (46)).

The coupling of the standard linear solid model of viscoelasticity with the Gurtin and Pipkin heat conduction reads as follows:

$$\begin{aligned} &\tau u_{ttt} + u_{tt} - a^2\Delta u - a^2\beta\Delta u_t + \delta\Delta\theta = 0, \\ &\theta_t - \frac{1}{\kappa}\int_0^\infty g(s)\Delta\theta(t-s)\, ds - \tau\delta\Delta u_{tt} - \delta\Delta u_t = 0, \end{aligned} \tag{48}$$

where $a^2 > 0$ is the constant wave velocity, and $\tau, \beta, \delta, \kappa$ are positive constants. The memory kernel $g(s)$ is a convex summable function on $[0, \infty)$ with total mass of

$$\int_0^\infty g(s)\, ds = 1.$$

Very recently, Wang and Liu [111] studied, in the whole space $\mathbb{R}^N$, the system (48) and established the estimate of the decay of the solution in the sub-critical case $0 < \tau < \beta$ (which corresponds to condition $\gamma > 0$), by employing the method of [96] and [97]. They showed the following decay estimate:

$$\|\nabla^k V(t)\|_{L^2(\mathbb{R}^N)} \leq C(1+t)^{-\frac{N}{12}-\frac{k}{6}}\|V_0\|_{L^1(\mathbb{R}^N)} + C(1+t)^{-\frac{\ell}{6}}\|\nabla^{k+\ell}V_0\|_{L^2(\mathbb{R}^N)},$$

for all $0 \leq k+l \leq s$ and where $V_0 \in H^s(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$.

Our contributions

In this thesis, taking inspiration from all the results cited above, we have dealt with the existence and stability of the solutions of the Moore–Gibson–Thompson (MGT) equation with and without a memory term. We summarize our contribution as follows:

▶ **In chapter 1:** We give some preliminaries, then we discuss, in $\mathbb{R}^N$, the well-posedness and the decay rate of the solution of the MGT equation with memory

$$\tau u_{ttt}(t) + \alpha u_{tt}(t) - \beta\Delta u_t(t) - \gamma\Delta u(t) + \int_0^t g(s)\Delta u(t-s)ds = 0, \tag{49}$$

where g satisfies the following differential inequality

$$g'(t) \leq -\delta g^{\frac{p+1}{p}}(t), \quad \text{for all} \quad t \geq 0, \quad p > 1.$$

We prove that the problem (49) is well-posed in some appropriate functional spaces using the semigroup approach. By applying the energy method in Fourier space we build appropriate Lyapunov functionals that are used to prove the asymptotic stability and to give the decay rate of an energy norm for the solution of the problem (49). More precisely, for the asymptotic behavior we prove the following:

- First, for the sub-critical case $\alpha\beta - \tau\gamma > 0$ (which corresponds to (34)) and when g decays either exponentially or polynomially, we give the decay rate of the solution and its higher-order derivatives (see Theorem 1.1 below). For instance if g decays exponentially, we prove that the solution decays exactly as in (35).

- Second, in the critical case $\alpha\beta - \tau\gamma = 0$, we discuss the stability of (49) with a memory term of the form:
$$\int_0^t g(s)\Delta z(t-s)ds$$
with $z = u$ (type I memory) or $z = \alpha u + \tau u_t$ (type III memory). In the type I memory, we show that the decay results obtained are slightly different from the previous ones. In fact, if g decays polynomially, the stability problem for the type I memory is left open. If g decays exponentially, the solution of our system is of regularity-loss type, (see Theorem 1.2 below). A similar dissipative form of our results was obtained in Timoshenko dissipative systems studied by many authors, see for instance, Hosono *et al.* [49], Ide *et al.* [51], Kawashima [58] and references therein.
For the type III memory, we proved the decay rate in both case, when g decays exponentially and polynomially (Theorem 1.3). The main idea that we applied here is that for the critical case $\alpha\beta - \tau\gamma = 0$, we succeeded to rewrite (49) as a second order in "time" wave equation for the unknown $\tau u_t + \alpha u$ with a "nice" memory term that allows us to apply the energy method. Our result in this case agrees with those in [29]. For the type II memory (i.e., $z = u_t$) and if g decays exponentially, then it seems that a decay result under the new assumption
$$\alpha\beta - \tau\gamma - \alpha(\gamma - \ell) > 0$$
is possible. However, the case where g is decaying polynomially seems more challenging.

Note that these results were published in [16].

▶ **In chapter 2:** We investigate the following in a viscoelastic Moore–Gibson–Thompson equation with a type-II memory term in $\mathbb{R}^N$:

$$u_{ttt}(t) + \alpha u_{tt}(t) - \beta\Delta u_t(t) - \gamma\Delta u(t) + \int_0^t g(t-s)\Delta u_t(s)ds = 0, \quad t > 0,$$

together with initial data

$$u(0) = u_0, \qquad u_t(0) = u_1, \qquad u_{tt}(0) = u_2,$$

for relaxation functions g satisfying

$$g'(t) \le -\eta(t)g(t) \qquad \forall t \ge 0$$

and establish a general decay rate result under the condition

$$\beta - \frac{\gamma}{\alpha} - \frac{\varrho}{2} > 0,$$

where $\varrho = \int_0^\infty g(s)ds$. Then, we find the following decay rate of the L^2-norm of the vector $U = (u_{tt} + \alpha u_t, \nabla(u_t + \alpha u), \nabla u_t)$ and those of its higher-order derivatives:

$$\begin{aligned}\|\nabla^j U(t)\|_{L^2(\mathbb{R}^N)} \leq & c_1\Big(1 + \int_0^t \eta(s)ds\Big)^{-\frac{j}{2}-\frac{N}{4}} \|U_0\|_{L^1(\mathbb{R}^N)} \\ & + c_1 e^{-c_2 \int_0^t \eta(s)ds} \|\nabla^j U_0\|_{L^2(\mathbb{R}^N)}.\end{aligned}$$

This result was published in [18].

- **In chapter 3:** Our interest lies in the coupling of the standard linear solid model of viscoelasticity with the Gurtin and Pipkin heat conduction in whole space, namely,

$$\begin{aligned}&\tau u_{ttt} + u_{tt} - a^2\Delta u - a^2\beta\Delta u_t + \delta\Delta\theta = 0, \\ &\theta_t - \frac{1}{\kappa}\int_0^\infty g(s)\Delta\theta(t-s)\ ds - \tau\delta\Delta u_{tt} - \delta\Delta u_t = 0,\end{aligned} \tag{50}$$

where $g'(s) = -\mu(s)$ and $\mu(s)$ satisfies

$$\mu'(s) \leq -\nu\mu(s) \qquad \forall s > 0.$$

We begin by stating the well-posedness of the system and then the asymptotic behavior, where we improve the result in [111] for the sub-critical case and obtain the optimal decay estimate. In addition, we investigate the critical case. In fact, the decay rates are of regularity-loss type and are given as follows, for any $t \geq 0$,

- Sub-critical case: $0 < \tau < \beta$:

$$\begin{aligned}\|\nabla^k V(t)\|_{L^2(\mathbb{R}^N)} \leq & C(1+t)^{-\frac{N}{4}-\frac{k}{2}}\|V_0\|_{L^1(\mathbb{R}^N)} \\ & + C(1+t)^{-\frac{\ell}{2}}\|\nabla^{k+\ell}V_0\|_{L^2(\mathbb{R}^N)},\end{aligned}$$

  where $V(\xi, t) = (u_t + \tau u_{tt}, \nabla(u + \tau u_t), \nabla u_t, \theta)$.

- Critical case: $0 < \tau = \beta$:

$$\begin{aligned}\|\nabla^k V_c(t)\|_{L^2(\mathbb{R}^N)} \leq & C(1+t)^{-\frac{N}{4}-\frac{k}{2}}\|V_c^0\|_{L^1(\mathbb{R}^N)} \\ & + C(1+t)^{-\frac{\ell}{4}}\|\nabla^{k+\ell}V_c^0\|_{L^2(\mathbb{R}^N)},\end{aligned}$$

  where $V_c(\xi, t) = (u_t + \tau u_{tt}, \nabla(u + \tau u_t), \theta)$.

Hence, for the sub-critical case the L^2–norm of the solution decays with rate $(1+t)^{-N/4}$ which is faster than that of $(1+t)^{-N/12}$ obtained in [111]. This result extends those in [95], [96] and [97]. While for the critical case $0 < \tau = \beta$, the decay rates obtained by adding the Fourier or Cattaneo law to the standard linear solid model are slower than that obtained in our case coupled with Gurtin-Pipkin heat conduction.
This work is published in the Journal of Mathematical Analysis and Applications [17].

▶ **In chapter 4:** Under certain setting, we consider the third order in time abstract evolution equation set in the Hilbert space H:

$$\begin{cases} u_{ttt} + Bu_{tt} + \mathcal{A}_0 u + \mathcal{A}_1 u_t = 0, \\ u(0) = u_0, u_t(0) = u_1, u_{tt}(0) = u_2, \end{cases} \tag{51}$$

where u_0, u_1, and u_2 are initial data in appropriated Hilbert spaces. Our first goal is to show that problem (51) is well-posed. We further find sufficient conditions, that guarantee the exponential decay of the energy. Since this sufficient condition is quite strong, we concentrate on the degenerate case

$$\alpha - \beta \geq 0, \text{ a. e. in } \Omega,$$

for which, as we will show, exponential, polynomial or even logarithmic decays are available. This is performed by comparing the resolvent of our operator with the one of the wave equation with frictional interior damping

$$\begin{cases} u_{tt} - \Delta u + (\alpha - \beta) u_t = 0 & \text{in } \Omega \times (0, +\infty), \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(\cdot, 0) = u_0 \text{ and } u_t(\cdot, 0) = u_1 & \text{in } \Omega. \end{cases}$$

Indeed we show that the same behavior of the resolvent of the following standard Moore–Gibson–Thompson system

$$\begin{cases} u_{ttt} + \alpha u_{tt} + \beta \Delta u + \Delta u_t = 0, & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(0, \cdot) = u_0, u_t(0, \cdot) = u_1, u_{tt}(0, \cdot) = u_2, & \text{in } \Omega, \end{cases} \tag{52}$$

is the square of the behavior of the resolvent of this damped wave equation under condition:

$$\alpha - \beta \geq \kappa > 0 \text{ a. e. in } \omega_0, \tag{53}$$

where ω_0 is non empty open subset of Ω. Hence under some geometrical condition on ω_0, system (52) is proved to be exponential, polynomial or even logarithmic decaying. All these results have been published in [87].

▶ **In chapter 5:** We consider the well-posedness and the long time behavior of some evolution equations with memory,

$$\begin{cases} u_{ttt} + \alpha u_{tt} - \Delta u_t - \gamma \Delta u + \int_0^t g(s) \Delta u(t-s) ds = 0, & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(0,.) = u_0, \quad u_t(0,.) = u_1, \quad u_{tt}(0,.) = u_2, & \text{in } \Omega, \end{cases}$$

where g is a memory kernel that is exponentially decaying, namely

$$g'(s) \leq -\delta g(s)$$

and α is a function that is supposed to be in $L^\infty(\Omega)$. The purpose of this work is to combine ideas from [37] and [87] to prove polynomial stability results for $\alpha \in L^\infty(\Omega)$ satisfying (5.2). With no further assumption, we prove a semi-uniform stability result by using the frequency domain approach, the decay reveals to be optimal for kernels for which the absolute value of its derivatives up to order 3 are exponentially decaying at infinity. On the contrary if the wave equation in Ω with a frictional interior damping in a non empty subset ω_0 is exponentially stable or polynomially stable with a decay rate in $t^{-\frac{1}{m}}$ with initial data in the domain of the wave operator with $0 < m < 1$, we prove a better decay rate of the energy if $\alpha - \gamma$ is uniformly bounded from below in ω_0.
The results obtained in this chapter have been published in [88].

Notation and some technical inequality

We indicate in this section the notations and some technical inequality used throughout the thesis and all the important tools and lemmas, which are necessary to obtain various estimates.

- Throughout this thesis, we denote by c a generic positive constant.
- We used the following standard Hilbert space

$$H^1(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^N), \forall i = 1, ..., N \right\},$$

  where the inner product and norm are denoted by

$$\langle u, v \rangle_{H^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N} uv dx + \int_{\mathbb{R}^N} \nabla u \nabla v dx,$$

$$\|u\|^2_{H^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

  We use the notations $\langle u, v \rangle_1 = \langle u, v \rangle_{H^1(\mathbb{R}^N)}$, $\|u\|_1^2 = \|u\|^2_{H^1(\mathbb{R}^N)}$ and $\langle x, \bar{y} \rangle = x\bar{y}$.

- For $1 \leq p < \infty$, we define:

$$u \in L^p(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{C};\ \text{u is measurable and} \int_{\mathbb{R}^N} |u(x)|^p \, dx < \infty \right\}.$$

- For $p=\infty$, we have:

$$u \in L^{\infty}(\mathbb{R}^N) = \Big\{ u : \mathbb{R}^N \to \mathbb{C}; \text{u is measurable and} \, \exists c \geq 0, \text{ such that } |u| \leq c, \text{ for almost } x \in \mathbb{R}^N \Big\}.$$

- Let $T>0$ and let X be a Hilbert space. Denote by $C([0,T];X)$ the space of continuous functions defined on $[0,T]$ with values in X. It is known that $C([0,T];X)$ is a Banach space.

- We are using the standard notation for the partial derivatives, gradient, and Laplace:

$$u_t = \frac{\partial u}{\partial t}, \quad u_{tt} = \frac{\partial^2 u}{\partial t^2}, \quad u_{ttt} = \frac{\partial^3 u}{\partial t^3},$$
$$\nabla u = \left(\frac{\partial u}{\partial x_i} \right)_{1 \leq i \leq n}, \textit{and} \quad \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

Let V and W be two Banach spaces, A linear operator $A : V \to W$ is said:

- Self-adjoint: $A = A^{\star}$, where $A^{\star}$ is called the adjoint of A.

- Bounded: There exists some $M>0$ such that for all v in V, $\|Av\|_W \leq M\|v\|_V$.

- Compact: If A maps the unit ball of V into a relatively compact subset of W (that is, a subset of W with compact closure).

- Range of A : Ran $A = \{Ax : x \in D(A)\}$, kernel of $A : \ker A = \{x \in D(A) : Ax = 0\}$, the resolvent set of $A : \varrho(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is bijective}\}$, spectrum of $A : \sigma(A) = \mathbb{C}/\varrho(A)$.

- Injective: A is said to be injective, if and only if $\ker A = \{0\}$.

- Isomorphism: A bijective bounded linear operator A is an isomorphism if and only if A^{-1} is continuous.

- Dissipative: if for every $x \in D(A)$ there is a $x^{\star} \in F(x)$(duality set), such that $\text{Re}\,\langle Ax, x^{\star}\rangle \leq 0$.

- Fredholm operator: One says that $A \in \mathcal{L}(V,W)$ is a Fredholm operator if it satisfies:

  i) $\ker A$ is finite-dimensional.

  ii) Ran A is closed and has finite co-dimension.

  The index of A is defined by

$$\text{ind}A = \dim\ker A - \text{codim Ran}A.$$

Either in $\mathbb{R}^N$, or assuming that Ω is an open and bounded subset of $\mathbb{R}^N$, and $\partial\Omega$ is C^1, we have the following identities and inequalities:

1. **Young's Inequality**: Let $1 < p, q < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then
$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \qquad (a, b > 0).$$

2. **Cauchy-Schwarz's inequality**: For all vectors u and v in Hilbert space H, then
$$|\langle u, v\rangle|^2 \leq \langle u, u\rangle \langle v, v\rangle .$$

3. **Hölder's Inequality**: Assume $1 \leq p, q \leq \infty$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then if $u \in L^p(\Omega)$, $v \in L^q(\Omega)$, we have
$$\int_\Omega |uv| \; dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.$$

The following technical lemmas will be needed in the proof of the our main results. No proofs are included but references are provided.

Lemma 0.1. *[29, Lemma 1.1]. Let $k \geq 1, t \geq 0$. Then the following estimate holds:*
$$\int_0^1 r^{k-1} e^{-r^2 t} dr \leq c(k)(1+t)^{-k/2}. \tag{54}$$

Lemma 0.2. *[29, Lemma 1.2]. Let $k \geq 1, m > 0$ and $t \geq 0$. Then*
$$\int_0^1 \frac{r^{k-1}}{(1+rt)^m} dr \leq \begin{cases} c(m,k)(1+t)^{-\min(m,k)}, & \text{if} \quad m \neq k, \\ c(k)(1+t)^{-k} \log(2+t), & \text{if} \quad m = k. \end{cases} \tag{55}$$

Lemma 0.3. *[112, Lemma 4.24]. Assume that $\eta(t) > 0$, for all $t \geq 0$. Then we have:*
$$\| |\xi|^\ell e^{-c|\xi|^2 \int_0^t \eta(s) ds} \|_{L^p} \leq c\Big(1 + \int_0^t \eta(s) ds\Big)^{-\frac{\ell}{2} - \frac{n}{2p}}, \qquad \forall t \geq 0. \tag{56}$$

We close the part of the lemmas with the following lemma which is an important tool to obtain various estimates below.

Lemma 0.4. *[8, page.17].(classical differential version of Gronwall lemma) We assume that $u \in C([0,T); \mathbb{R})$, $T \in (0, \infty)$ satisfies the differential inequality:*
$$u' \leq a(t) u + b(t), \qquad a.e. \ in \quad (0,T),$$
for some $a, b \in L^1(0,T)$. Then, u satisfies the pointwise estimate:
$$u(t) \leq e^{A(t)} u(0) + \int_0^t b(s) e^{A(t) - A(s)} ds, \qquad \forall t \in [0,T),$$

where, $A(t) = \int_0^t a(s)ds.$

In the sequel, we state the main definitions, corollaries and theorems useful to obtain our results thereafter.

Definition 0.5. *A family of bounded linear operators* $(S(t))_{t\geq 0}$ *on a Banach space* X *is a strongly continuous semigroup (or* C_0-*semigroup on* X*) if it satisfies the following properties:*

(i) $S(0) = I$,

(ii) $S(t+s) = S(t)S(s), \qquad \forall t, s \geq 0,$

(iii) $\lim_{t\to 0} \|S(t)x - x\|_X = 0, \qquad \forall x \in X.$

Definition 0.6. *A strongly continuous semigroup* $(S(t))_{t\geq 0}$ *on* X *is a semigroup of contractions if:*

$$\|S(t)\| \leq 1, \qquad \text{for all} \quad t \geq 0.$$

Corollary 0.7. *[34] A necessary and sufficient condition that a strongly continuous semigroup* $T(t)$ *of class* C_0 *defined on a complex Hilbert space satisfies the condition* $\|T(t)\| \leq M_0 e^{-\alpha t}$, *where* $\alpha > 0$, *is that for each* x *in* X *the integral* $\int_0^\infty \|T(t)x\|^2 dt$ *be convergent.*

Theorem 0.8. *[19, Corollary 5.8](Lax-Milgram) Let* $(H, \|.\|_H, (.,.)_H)$ *be a Hilbert space and* $a(x, y)$ *is a bilinear, continuous and coercive functional, i.e. there exist* $C, \alpha > 0$, *such that:* $\forall x, y \in H$,

$$|a(x, y)| \leq C\|x\|_H \|y\|_H \qquad \text{and} \quad a(x, x) \geq \alpha \|x\|_H^2,$$

then for any continuous linear form L *of* H *there exists a unique* $u \in H$, *such that:*

$$a(u, x) = L(x), \qquad \forall x \in H.$$

Theorem 0.9. *[94, Theorem 4.3](Lumer-Phillips) Let* A *be a linear operator with dense domain* $D(A)$ *in* X

(a) If A *is dissipative and there is a* $\lambda_0 > 0$ *such that the range,* $R(\lambda_0 I - A)$, *of* $\lambda_0 I - A$ *is* X, *then* A *is the infinitesimal generator of a* C_0-*semigroup of contractions on* X.

(b) If A *is the infinitesimal generator of a* C_0-*semigroup of contractions on* X *then* $R(\lambda_0 I - A) = X$ *for all* $\lambda > 0$ *and* A *is dissipative. Moreover, for every* $x \in D(A)$ *and every* $x^\star \in F(x)$, $\mathrm{Re}\,\langle Ax, x^\star\rangle \leq 0$.

Theorem 0.10. *[14, Theorem 2.4](Borichev-Tomilov theorem) Let* $(T(t))_{t\leq 0}$ *be a bounded* C_0-*semigroup on a Hilbert space* H *with generator* A, *such that* $i\mathbb{R} \subset \varrho(A)$. *Then for a fixed* $\alpha > 0$, *the following conditions are equivalent:*

i: $\|R(is, A)\| = O(|s|^\alpha), \qquad s \to \infty.$

ii: $\|T(t)(-A)^{-\alpha}\| = O(t^{-1}), \qquad t \to \infty.$

iii: $\|T(t)(-A)^{-\alpha}x\| = O(t^{-1}), \qquad t \to \infty,\ x \in H.$

iv: $\|T(t)A^{-1}\| = O(t^{-1/\alpha}), \qquad t \to \infty.$

v: $\|T(t)A^{-1}x\| = O(t^{-1/\alpha}), \qquad t \to \infty,\ x \in H.$

Theorem 0.11. *[101](Huang and Prüss theorem) Let $S(t) = e^{At}$ be a C_0-semigroup of contractions on Hilbert space. Then $S(t)$ is exponentially stable iff:*

$$\varrho(A) \supseteq \{i\omega : \omega \in \mathbb{R}\} \equiv i\mathbb{R}, \qquad \text{and} \qquad \overline{\lim_{|\omega|\to\infty}} \|i\omega I - A)^{-1}\| < \infty$$

hold, where $\varrho(A)$ is the resolvent set of A.

Theorem 0.12. *[66, Theorem 4.2](Calderón theorem) Let g be such that $|g(y)| \leq C|y|$. Let Ω be a connected open set in $\mathbb{R}^N$ and let $\omega \Subset \Omega$, with $\omega \neq \emptyset$. If $u \in H^2(\Omega)$ satisfies $Pu = g(u)$ in Ω and $u(x) = 0$ in ω, then u vanishes in Ω.*

Theorem 0.13. *[48](Plancherel theorem) If $f \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, then $\hat{f}$ is in $L^2(\mathbb{R}^N)$ and the following formula of Plancherel holds:*

$$\|\hat{f}\|_2 = \|f\|_2, \tag{57}$$

where $\hat{f}(\xi) = \int_{\mathbb{R}^N} f(x)e^{-2i\pi x.\xi}\,dx$, with $x.\xi$ is the scalar product of x by ξ in $\mathbb{R}^N$.
The map $f \to \hat{f}$ has a unique extension to a continuous, linear map from $L^2(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ which is an isometry, i.e., Plancheral's formula (57) holds for this extension. We continue to denote this map by $f \to \hat{f}$ (even if $f \notin L^1(\mathbb{R}^N)$).

CHAPTER 1
DECAY RATES FOR THE MOORE–GIBSON–THOMPSON EQUATION WITH MEMORY

1.1 Introduction

The mathematical community has got interested in the third-order PDE model for acoustic wave propagation as a consequence of their physical justification. The interest was certainly motivated at least initially by the difficulties and challenges encountered in the study and analysis of the MGT equation, differently from the linearization of the Kuznetsov and Westervelt equations, which is the well-known, parabolic-like, strongly damped wave equation. As a result of this interest, a large number of works have studied the asymptotic behavior of the Moore–Gibson–Thompson equation (MGT) equation, using different types of dissipation. See, for instance [5, 6, 26, 35, 36, 54, 55, 64, 65, 95] and references therein for more information. In this regard, the main goal of this chapter is to investigate the existence and stability of the solutions for the MGT equation with a memory term in the whole space $\mathbb{R}^N$, namely,

$$\tau u_{ttt}(t) + \alpha u_{tt}(t) - \beta \Delta u_t(t) - \gamma \Delta u(t) + \int_0^t g(s)\Delta z(t-s)ds = 0, \tag{1.1}$$

together with the following initial data

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad u_{tt}(x,0) = u_2(x), \tag{1.2}$$

where $z = u$ (type I memory) and $z = \alpha u + \tau u_t$ (type III memory). First, following [37] and [95], we show that the problem is well-posed under an appropriate assumption on the coefficients of the system. Then, we built some Lyapunov functionals by using the energy method in Fourier space. These functionals allows us to get control estimates on the Fourier image of the solution. These estimates of the Fourier image together with some integral inequalities lead to the decay rate of the L^2-norm of the solution. We use two types of memory term here: type I memory term and type III memory term. Decay rates are obtained in both types. More precisely, decay rates of the solution are obtained depending on the exponential or polynomial decay of the memory kernel. More importantly, we show stability of the solution in both cases: a sub-critical range of the parameters and a critical range. However for the type I memory we show in the critical case that the solution has the regularity-loss property.
We briefly state the main results of this chapter. First, we state the decay results for the sub-critical case $\alpha\beta > \tau\gamma$.

Theorem 1.1. *Let u be the solution of (1.1), (1.2). Assume that $\alpha\beta > \tau\gamma$. Let $U = (\alpha u_t + \tau u_{tt}, \nabla(\alpha u + \tau u_t), \nabla u_t)$ and assume in addition that $U_0 \in L^1(\mathbb{R}^N) \cap H^s(\mathbb{R}^N)$. Then, for all $0 \le j \le s$, we have:*

(i) *For $p = \infty$:*

$$\|\nabla^j U(t)\|_{L^2(\mathbb{R}^N)} \le C(1+t)^{-N/4-j/2}\|U_0\|_{L^1(\mathbb{R}^N)} + Ce^{-ct}\|\nabla^j U_0\|_{L^2(\mathbb{R}^N)}. \tag{1.3}$$

(ii) *For $p^\star < p < \infty$:*

$$\|\nabla^j U(t)\|_{L^2(\mathbb{R}^N)} \le C I_0^j \begin{cases} (1+t)^{-\min(\frac{1}{2p}, \frac{N}{4}+\frac{j}{2})}, & \text{if} \quad 2j+N \neq \frac{2}{p}, \\ (1+t)^{-\frac{1}{2p}} \log(2+t), & \text{if} \quad 2j+N = \frac{2}{p}, \end{cases} \tag{1.4}$$

where C, c are two positive constants independent of t and U_0 and $I_0^j = \|\nabla^j U_0\|_{L^2(\mathbb{R}^N)} + \|U_0\|_{L^1(\mathbb{R}^N)}$.

The proof of the above theorem will be given in Section 1.5 and it is based on Proposition 1.9 below. As we have said in the introduction, in the absence of the memory term, that is for $g = 0$ the assumption $\alpha\beta > \tau\gamma$ has been proved in [95] to be a necessary condition for the stability result. Here, we show that in the presence of the memory damping term, then we can push the stability result even to the critical case $\alpha\beta = \tau\gamma$. Hence, we have the following theorem. Our second main result reads as follows.

Theorem 1.2. *Let u be the solution of (1.1), (1.2). Assume that $\alpha\beta = \tau\gamma$. Let $U = (\alpha u_t + \tau u_{tt}, \nabla(\alpha u + \tau u_t), \nabla u_t)$ and assume in addition that $U_0 \in L^1(\mathbb{R}^N) \cap H^s(\mathbb{R}^N)$. Then for all $k + j \le s$ and for $p = \infty$, we have*

$$\|\nabla^j U(t)\|_{L^2(\mathbb{R}^N)} \le C(1+t)^{-N/4-j/4}\|U_0\|_{L^1(\mathbb{R}^N)} + C(1+t)^{-k/2}\|\nabla^{j+k} U_0\|_{L^2(\mathbb{R}^N)}, \tag{1.5}$$

where C, c are two positive constants independent of U_0 and t.

The proof of Theorem 1.2 is based on Proposition 1.14. We have $\frac{|\xi|^2}{(1+|\xi|^2)^2} \sim c|\xi|^{-2}$ for $|\xi| \to \infty$, this behavior in high frequency region causes the worse decay (1.5) of the regularity-loss type. Indeed, the second term on the right-hand side of (1.5), which comes from the high frequency part, shows that we can get the decay rate $t^{-k/2}$ of the solution only under k*th* order of regularity on the initial data.

Now, we present the main result of the energy decay in the critical case for the memory type III (the sub-critical case is easier). We can easily see that for $\alpha\beta = \tau\gamma$, we can rewrite the problem (1.1) with the type III memory as a second order in "time" wave equation with a memory term of the form

$$\tau z_{tt} - \beta\Delta z + \tau \int_0^t g(s)\Delta z(t-s)ds = 0, \tag{1.6}$$

where $z = \alpha u + \tau u_t$. Now, we state the main result of the decay rate of the L^2-norm of the energy

$$\mathcal{E}(t, \nabla z, z_t) = \frac{1}{2}\Big[\tilde{\ell}\|\nabla z\|_{L^2}^2 + \tau\|z_t\|_{L^2}^2 + \tau\int_0^\infty g(s)\|\nabla\mu^t(s)\|_{L^2}^2 ds\Big].$$

where $z(x,t)$ is the solution of (1.6) and $\mu_t^t(s) = -\mu_s^t(s) + z_t(t)$.

Theorem 1.3. *Let z be the solution of (1.6). Assume that $Z(0) = (z_t(0), \nabla z(0)) \in L^1(\mathbb{R}^N) \cap H^s(\mathbb{R}^N)$ for all $0 \le j \le s$. Then, we have for $Z(t) = (z(t), \nabla z(t))$*

(i) *for* $p = \infty$:

$$\|\nabla^j Z(t)\|_{L^2} \leq C(1+t)^{-N/4-j/2}\|Z_0\|_{L^1(\mathbb{R}^N)} + Ce^{-ct}\|\nabla^j Z_0\|_{L^2(\mathbb{R}^N)}, \tag{1.7}$$

(ii) *for* $p^\star < p < \infty$:

$$\|\nabla^j Z(t)\|_{L^2} \leq C J_0^j \begin{cases} (1+t)^{-\min(1/2p, N/4+j/2)}, & \text{if} \quad 2j+N \neq \frac{2}{p}, \\ (1+t)^{-1/2p}\log(2+t), & \text{if} \quad 2j+N = \frac{2}{p}, \end{cases} \tag{1.8}$$

where

$$J_0^j = \|\nabla^j Z_0\|_{L^2(\mathbb{R}^N)} + \|Z_0\|_{L^1(\mathbb{R}^N)},$$

C *and* c *are two positive constants independent of* t *and the initial data.*

The proof of Theorem 1.3 is a result of the pointwise estimates in Proposition 1.18.

1.2 Preliminaries and well-posedness of the problem

In this section, we state some preliminaries and assumptions, then we prove the existence and uniqueness of the solutions of the MGT equation (1.1). We impose now the basic assumptions on g as follows:

(G1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing twice differentiable function, such that

$$g(0) > 0, \qquad \gamma - \int_0^\infty g(s)ds = \ell > 0$$

and $g'(s) \leq 0$ for every $s > 0$.

(G2) For some $\delta > 0$ the function g satisfies the following differential inequality:

$$g'(t) \leq -\delta g^{\frac{p+1}{p}}(t), \quad \text{for all} \quad t \in (0,\infty) \quad \text{and} \quad p^\star < p \leq \infty,$$

where $p^\star = \frac{1+\sqrt{5}}{2}$.

(G3) $g'' \geq 0$ almost everywhere.

Remark 1.4. *There are many functions satisfying (G1) and (G2). Examples of such functions are, for* $b > 0, q > 0, \nu > 1,$ *and* $a > 0$ *small enough,*

$$g_1(t) = \frac{a}{(1+t)^\nu}, \qquad g_2(t) = ae^{-b(t+1)^q}.$$

We immediately adopt the following lemma from [78] without proof which will be used in the sequel.

Lemma 1.5. *Assume that the assumption (G2) is satisfied. Then for any $\nu \in (1/p, 1)$, there exists a positive constant $c_0 = c_0(p, \nu, \gamma, \ell)$, such that*

$$\int_0^\infty [g(s)]^\nu \, ds \le c_0 < \infty.$$

Now, we use a change of variables from Dafermos [33] and extend the solution of (1.1) for all times, by setting $u(x,t) = 0$ when $t < 0$ and considering for $t \ge 0$ the auxiliary past history variable $\eta(t,s) = \eta^t(s)$, defined as:

$$\eta^t(s) = u(t) - u(t-s), \qquad t \ge 0, \quad s \in \mathbb{R}^+. \tag{1.9}$$

Consequently, the problem (1.1) together with (1.2) read as:

$$\begin{cases} \tau u_{ttt}(t) + \alpha u_{tt}(t) - \beta\Delta u_t(t) - \ell\Delta u(t) - \int_0^\infty g(s)\Delta\eta^t(s)ds = 0, \\ \eta^t_t(x,s) + \eta^t_s(x,s) = u_t(x,t), \end{cases} \tag{1.10}$$

with the initial data

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad u_{tt}(x,0) = u_2(x), \quad \eta^0(x,s) = \eta^0(s). \tag{1.11}$$

Introducing now the $(-g')$-weighted L^2-space

$$\mathcal{M} = L^2_{(-g')}(\mathbb{R}^+, H^1(\mathbb{R}^N)),$$

endowed with the inner product

$$\langle \eta^t, \tilde{\eta}^t \rangle_{\mathcal{M}} = \int_0^\infty -g'(s)\langle \nabla\eta^t(s), \nabla\tilde{\eta}^t(s)\rangle_{L^2(\mathbb{R}^N)} ds$$

and with the following associated norm

$$\|\eta^t\|^2_{\mathcal{M}} = \int_0^\infty -g'(s)\|\nabla\eta^t(s)\|^2_{L^2(\mathbb{R}^N)} ds,$$

for all $\eta^t, \tilde{\eta}^t \in \mathcal{M}$. In addition, we consider the infinitesimal generator of the right-translation C_0-semigroup on $\mathcal{M}$, i.e., the linear operator T given by

$$T\eta^t = -(\eta^t)' \quad \text{with} \quad D(T) = \{\eta^t \in \mathcal{M} : (\eta^t)' \in \mathcal{M}, \eta^t(0) = 0\},$$

where the prime stands for the distributional derivative with respect to the variable $s > 0$. In order to write the problem (1.10) as a first-order "in time" evolution equation, we introduce the change of variables:

$$v = u_t, \quad \text{and} \quad w = u_{tt},$$

and hence, we rewrite (1.10) as a first order system of the form:

$$\begin{cases} u_t = v, \\ v_t = w, \\ \tau w_t = -\alpha w + \beta \Delta v + \ell \Delta u + \int_0^\infty g(s) \Delta \eta^t(s) ds, \\ \eta_t^t = v - \eta_s^t. \end{cases} \tag{1.12}$$

Now, the problem (1.12) with initial data (1.11) can be reduced to

$$\begin{cases} \frac{d}{dt} U(t) = AU(t), \qquad t \in (0, +\infty), \\ U(x, 0) = U_0. \end{cases} \tag{1.13}$$

where $U(t) = (u(t), v(t), w(t), \eta^t), U_0 = (u_0, v_0, w_0, \eta^0)$, and A is the linear operator given by

$$AU = A \begin{pmatrix} u \\ v \\ w \\ \eta^t \end{pmatrix} = \begin{pmatrix} v \\ w \\ -\frac{\alpha}{\tau} w + \frac{\beta}{\tau} \Delta v + \frac{\ell}{\tau} \Delta u + \frac{1}{\tau} \int_0^\infty g(s) \Delta \eta^t(s) ds \\ v + T\eta^t \end{pmatrix}. \tag{1.14}$$

Inspired by [37], we define the Hilbert space

$$\mathcal{H} = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \times \mathcal{M},$$

with the following inner product

$$\begin{aligned} (U, \tilde{U})_{\mathcal{H}} =& \frac{\ell}{\alpha} \left\langle (\alpha u + \tau v), (\alpha \tilde{u} + \tau \tilde{v}) \right\rangle_1 + \frac{\tau}{\alpha} (\alpha\beta - \tau\ell) \left\langle v, \tilde{v} \right\rangle_1 \\ &+ \langle (\alpha v + \tau w), (\alpha \tilde{v} + \tau \tilde{w}) \rangle_{L^2(\mathbb{R}^N)} + \tau \langle \eta^t(s), \tilde{\eta}^t(s) \rangle_{\mathcal{M}} \\ &+ \alpha \int_0^\infty g(s) \langle \nabla \eta^t(s), \nabla \tilde{\eta}^t(s) \rangle_{L^2(\mathbb{R}^N)} ds \\ &+ \tau \int_0^\infty g(s) \left[\langle \nabla \eta^t(s), \nabla \tilde{v} \rangle_{L^2(\mathbb{R}^N)} + \langle \nabla v, \nabla \tilde{\eta}^t(s) \rangle_{L^2(\mathbb{R}^N)} \right] ds \end{aligned} \tag{1.15}$$

and the corresponding norm

$$\begin{aligned} \|U\|_{\mathcal{H}}^2 =& \frac{\ell}{\alpha} \|(\alpha u + \tau v)\|_1^2 + \frac{\tau}{\alpha} (\alpha\beta - \tau\ell) \|v\|_1^2 + \|\alpha v + \tau w\|_{L^2(\mathbb{R}^N)}^2 + \tau \|\eta^t(s)\|_{\mathcal{M}}^2 \\ &+ \alpha \int_0^\infty g(s) \|\nabla \eta^t(s)\|_{L^2(\mathbb{R}^N)}^2 ds + 2\tau \int_0^\infty g(s) \langle \nabla \eta^t(s), \nabla v \rangle_{L^2(\mathbb{R}^N)} ds. \end{aligned} \tag{1.16}$$

For all vectors $U = (u, v, w, \eta^t)$ and $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{\eta}^t)$ in $\mathcal{H}$. We consider (1.13) in the Hilbert space $\mathcal{H}$, with the following domain of the operator A:

$$D(A) = \left\{ (u, v, w, \eta) \in \mathcal{H} \left| \begin{array}{c} w \in H^1(\mathbb{R}^N) \\ \frac{\beta}{\tau}v + \frac{\ell}{\tau}u + \frac{1}{\tau}\int_0^\infty g(s)\eta^t(s)ds \in H^2(\mathbb{R}^N) \\ \eta^t \in D(T) \end{array} \right. \right\}.$$

Theorem 1.6. *Assume that (G1)-(G3) are satisfied. Then under the assumption* $\alpha\beta - \tau\gamma \geq 0$, *the linear operator* A *is the infinitesimal generator of a* C_0*-semigroup* $S(t) = e^{tA}$ *of contractions on* $\mathcal{H}$.

Proof. Instead of considering our problem (1.13), we follow [95] and consider the perturbed problem

$$\begin{cases} \frac{d}{dt}U(t) = A_B U(t), & t \in (0, +\infty), \\ U(x, 0) = U_0, \end{cases} \tag{1.17}$$

where A_B is given by

$$A_B \begin{pmatrix} u \\ v \\ w \\ \eta \end{pmatrix} = (A + B) \begin{pmatrix} u \\ v \\ w \\ \eta \end{pmatrix}$$
$$= \begin{pmatrix} v \\ w \\ -\frac{\alpha}{\tau}w + \frac{\beta}{\tau}\Delta v + \frac{\ell}{\tau}\Delta u + \frac{1}{\tau}\int_0^\infty g(s)\Delta\eta^t(s)ds - \ell u - \frac{\tau\ell}{\alpha}v - (\beta - \frac{\tau\ell}{\alpha})v \\ v + T\eta^t \end{pmatrix}.$$

Remark 1.7. *Here, due to the standard semigroup theory, when we prove that* A_B *generates a* C_0-*semigroup of contractions on* $\mathcal{H}$, *we can say that* A *generates a* C_0-*semigroup on* $\mathcal{H}$, *because we have* A_B *is a bounded perturbation of* A, *(see [94], Theorem 1.1 in Chap. 3).*

The first thing to notice is that, as we are in a Hilbert space, the operator is densely defined. Hence, we only need to prove that the operator A_B is dissipative and 0 belongs to the resolvent set of A_B, denoted by $\varrho(A_B)$. Then our conclusion will follow using the well known Lumer–Phillips theorem [94]. Following the same steps as in [6], Proof of Proposition

2.1) and using our new inner product, we can see that we have by direct calculations

$$
\begin{aligned}
\langle A_B U, U\rangle_{\mathcal{H}} =& \frac{\tau}{\alpha}(\alpha\beta-\tau\ell)\int_{\mathbb{R}^N}\nabla w.\nabla v dx + \langle v+T\eta^t,\eta^t\rangle_{\mathcal{M}} \\
&+\frac{\tau}{\alpha}(\alpha\beta-\tau\ell)\int_{\mathbb{R}^N} w.v dx + \frac{\ell}{\alpha}\int_{\mathbb{R}^N}(\alpha v+\tau w).(\alpha u+\tau v)dx \\
&-\int_{\mathbb{R}^N}\left(\beta\nabla v+\ell\nabla u+\int_0^\infty g\nabla\eta^t(s)ds\right).(\alpha\nabla v+\tau\nabla w)\,dx \\
&+\frac{\ell}{\alpha}\int_{\mathbb{R}^N}(\alpha\nabla v+\tau\nabla w).(\alpha\nabla u+\tau\nabla v)dx \\
=& -(\alpha\beta-\tau\gamma)\|\nabla v\|^2_{L^2(\mathbb{R}^N)} - \frac{\alpha}{2}\|\eta^t\|^2_{\mathcal{M}} \\
&-\frac{\tau}{2}\int_0^\infty g''(s)\|\nabla\eta^t(s)\|^2_{L^2(\mathbb{R}^N)}ds \\
\leq& 0.
\end{aligned}
$$

Therefore, according to the assumption (G3) and since $\alpha\beta-\tau\gamma\geq 0$, then the operator A is dissipative.
Next, we show that for all $\mathcal{F}=(f,g,h,p)\in\mathcal{H}$, there exists a unique solution $U=(u,v,w,\eta^t)\in D(A)$ to the equation

$$
U - A_B U = \mathcal{F}, \tag{1.18}
$$

such that in terms of its components, we have

$$
u - v = f, \tag{1.19a}
$$

$$
v - w = g, \tag{1.19b}
$$

$$
(\tau+\alpha)w - \beta\Delta v - \ell\Delta u - \int_0^\infty g(s)\Delta\eta^t(s)ds + \ell u + \frac{\tau\ell}{\alpha}v + (\beta-\frac{\tau\ell}{\alpha})v = \tau h, \tag{1.19c}
$$

$$
\eta^t - v - T\eta^t = p. \tag{1.19d}
$$

Integrating (1.19d) with the condition $\eta^t(0)=0$, we obtain

$$
\eta^t(s) = (1-e^{-s})v + \int_0^s e^{-(s-y)}p(y)dy. \tag{1.20}
$$

Plugging (1.19a), (1.19b) and (1.20) into (1.19c), we arrive at an elliptic problem of the form

$$
-\kappa\Delta v + \sigma v = q, \tag{1.21}
$$

such that,

$$
\kappa = \beta + \ell + \int_0^\infty g(s)(1-e^{-s})ds,
$$

$$
\sigma = \tau + \alpha + \beta + \ell
$$

and

$$
q = \ell(\Delta f + f) + (\tau+\alpha)g + \tau h + \int_0^\infty g(s)\left[\int_0^s e^{-(s-y)}\Delta p(y)dy\right]ds.
$$

The constant $\kappa > 0$ and the functional q belongs to $H^{-1}(\mathbb{R}^N)$. Thus, in view of the Lax-Milgram Theorem, problem (1.21) has a unique solution $v \in H^1(\mathbb{R}^N)$. From (1.19a) and (1.19b) we see that $u, w \in H^1(\mathbb{R}^N)$. Furthermore, from (1.20), one can also show that $\eta^t \in \mathcal{M}$, and checking that $\eta(0) = 0$ and consequently, $T\eta^t = \eta^t - p - v \in \mathcal{M}$ as well. Finally, going back to (1.19c), we conclude that

$$\beta v + \ell u + \int_0^\infty g(s)\eta^t(s)ds = (-\Delta)^{-1}\left(\tau h - (\tau+\alpha)w - \ell u - \beta v\right) \in H^2(\mathbb{R}^N),$$

which implies that $U = (u, v, w, \eta^t) \in D(A)$ is the unique solution of (1.18). As we said, as A is a bounded perturbation of A_B, we therefore obtain that A is the generator of a C_0-semigroup on $\mathcal{H}$. This finishes the proof. □

Corollary 1.8. *Under the dissipativity condition $\alpha\beta \geq \tau\gamma$ with the results obtained above and for any $U_0 \in D(A)$, the problem (1.10),(1.11) has a unique classical solution such that*

$$(u, u_t, u_{tt}, \eta^t) \in C^1\left([0,\infty);\mathcal{H}\right) \cap C\left([0,\infty);D(A)\right).$$

1.3 Decay estimates–the sub-critical case

In this section, we apply the energy method in the Fourier space to get some pointwise estimate of $\hat{U}(\xi, t)$ with $U = (\alpha u_t + \tau u_{tt}, \nabla(\alpha u + \tau u_t), \nabla u_t)$, where $u(x,t)$ is the solution of (1.1), (1.2). First, taking the Fourier transform of (1.12), we obtain

$$\begin{cases} \hat{u}_t = \hat{v}, \\ \hat{v}_t = \hat{w}, \\ \tau\hat{w}_t = -\alpha\hat{w} - \beta|\xi|^2\hat{v} - \ell|\xi|^2\hat{u} - |\xi|^2\int_0^\infty g(s)\hat{\eta}^t(s)ds, \\ \hat{\eta}^t_t = \hat{v} - \hat{\eta}^t_s. \end{cases} \tag{1.22}$$

with

$$\hat{U}(\xi, 0) = \left(\hat{u}_0, \hat{v}_0, \hat{w}_0, \hat{\eta}^0\right)(\xi). \tag{1.23}$$

Our main goal is to build an appropriate Lyapunov functional $\hat{\mathcal{L}}(\xi, t)$ which is equivalent to the energy functional associated with (1.22):

$$\begin{aligned} \hat{E}(\xi, t) =& \frac{1}{2}\left[\frac{\ell}{\alpha}|\xi|^2|\alpha\hat{u} + \tau\hat{v}|^2 + \frac{\tau}{\alpha}(\alpha\beta - \tau\ell)|\xi|^2|\hat{v}|^2 + |\alpha\hat{v} + \tau\hat{w}|^2 + \tau\|\hat{\eta}^t\|^2_{\mathcal{M}}\right. \\ &\left. + \alpha|\xi|^2\int_0^\infty g(s)|\hat{\eta}^t(s)|^2ds + 2\tau|\xi|^2\,\mathrm{Re}\left(\int_0^\infty g(s)\left\langle\hat{\eta}^t(s), \bar{\hat{v}}\right\rangle ds\right)\right] \end{aligned} \tag{1.24}$$

in the sense that there exist two positive constants d_1 and d_2 such that

$$d_1\hat{E}(\xi, t) \leq \hat{\mathcal{L}}(\xi, t) \leq d_2\hat{E}(\xi, t),$$

for all $\xi \in \mathbb{R}^N$ and for all $t \geq 0$ and also satisfies (1.51). Hence, we have the following result.

Proposition 1.9. *Let $\hat{U}(\xi,t)=(\hat{u},\hat{v},\hat{w},\hat{\eta}^t)(\xi,t)$ be the solution of (1.22)-(1.23). Assume that (G1)-(G3) hold and $\alpha\beta>\tau\gamma$. Then, $\hat{U}(\xi,t)$ satisfies the following estimates:*

$$|\hat{U}(\xi,t)|^2 \le C|\hat{U}(\xi,0)|^2 e^{-c\rho(\xi)t}, \qquad \textit{for} \quad p=\infty, \tag{1.25}$$

$$|\hat{U}(\xi,t)|^2 \le C|\hat{U}(\xi,0)|^2\Big(1+\rho(\xi)t\Big)^{-\frac{1}{p}}, \qquad \textit{for} \quad p^\star<p<\infty, \tag{1.26}$$

for all $t>0$, where

$$\rho(\xi)=\frac{|\xi|^2}{1+|\xi|^2}. \tag{1.27}$$

Proposition 1.9 is main ingredient in proving the decay estimates in Theorem 1.1. Its proof will be given through several lemmas. First, we have the following lemma.

Lemma 1.10. *Under the condition $\alpha\beta\ge\tau\gamma$, the energy functional defined in (1.24) satisfies for all $t\ge 0$ and for all $\xi\in\mathbb{R}^N$ the inequality:*

$$\frac{d}{dt}\hat{E}(\xi,t)\le -(\alpha\beta-\tau\gamma)|\xi|^2|\hat{v}|^2-\frac{\alpha}{2}\|\hat{\eta}^t\|^2_{\mathcal{M}}\le 0. \tag{1.28}$$

Proof. In the same spirit of [95], multiplying the first equation in (1.22) by α and the second by τ, adding the results to get:

$$(\alpha\hat{u}+\tau\hat{v})_t=(\alpha\hat{v}+\tau\hat{w}). \tag{1.29}$$

Multiplying (1.29) by $\ell\left(\alpha\bar{\hat{u}}+\tau\bar{\hat{v}}\right)$ and taking the real part, we get

$$\frac{1}{2}\frac{d}{dt}\ell|\alpha\hat{u}+\tau\hat{v}|^2=\alpha\tau\ell|\hat{v}|^2+\ell\alpha^2\operatorname{Re}(\hat{v}\bar{\hat{u}})+\tau^2\ell\operatorname{Re}(\hat{v}\bar{\hat{w}})+\alpha\tau\ell\operatorname{Re}(\hat{w}\bar{\hat{u}}). \tag{1.30}$$

Next, we multiply the second equation in (1.22) by $\tau(\alpha\beta-\tau\ell)\bar{\hat{v}}$ and taking the real part, we get

$$\frac{1}{2}\tau(\alpha\beta-\tau\ell)\frac{d}{dt}|\hat{v}|^2=\tau(\alpha\beta-\tau\ell)\operatorname{Re}(\hat{w}\bar{\hat{v}}). \tag{1.31}$$

Furthermore, multiplying the second equation in (1.22) by α and adding the result to the third equation we obtain

$$(\alpha\hat{v}+\tau\hat{w})_t=-|\xi|^2\beta\hat{v}-\ell|\xi|^2\hat{u}-|\xi|^2\int_0^\infty g(s)\eta^t(s)ds. \tag{1.32}$$

Multiplying (1.32) by $\alpha(\alpha\bar{\hat{v}}+\tau\bar{\hat{w}})$ and taking the real part, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\alpha|\alpha\hat{v}+\tau\hat{w}|^2=&-\alpha^2\beta|\xi|^2|\hat{v}|^2-\tau\alpha\beta|\xi|^2\operatorname{Re}(\hat{v}\bar{\hat{w}})-\ell\alpha^2|\xi|^2\operatorname{Re}(\hat{u}\bar{\hat{v}})\\&-\ell\alpha\tau|\xi|^2\operatorname{Re}(\hat{u}\bar{\hat{w}})-\alpha^2|\xi|^2\operatorname{Re}\left(\int_0^\infty g(s)\left\langle\hat{\eta}^t(s),\bar{\hat{v}}\right\rangle ds\right)\\&-\tau\alpha|\xi|^2\operatorname{Re}\left(\int_0^\infty g(s)\left\langle\hat{\eta}^t(s),\bar{\hat{w}}\right\rangle ds\right).\end{aligned} \tag{1.33}$$

Using the fact that $\hat{\eta}_t^t + \hat{\eta}_s^t = \hat{v}$, then the term

$$-\alpha^2|\xi|^2 \operatorname{Re}\left(\int_0^\infty g(s)\left\langle \hat{\eta}^t(s), \bar{\hat{v}}\right\rangle ds\right)$$

can be rewritten as:

$$\begin{aligned}-\alpha^2|\xi|^2 \operatorname{Re}\left(\int_0^\infty g(s)\left\langle \hat{\eta}^t(s), \bar{\hat{v}}\right\rangle ds\right) =& -\alpha^2|\xi|^2 \operatorname{Re}\left(\int_0^\infty g(s)\left\langle \hat{\eta}^t(s), \bar{\hat{\eta}}_t^t(s)\right\rangle ds\right)\\ &-\alpha^2|\xi|^2 \operatorname{Re}\left(\int_0^\infty g(s)\left\langle \hat{\eta}^t(s), \bar{\hat{\eta}}_s^t(s)\right\rangle ds\right).\end{aligned}$$

Integrating by parts with respect to s (the boundary terms vanish see [93]), then we obtain

$$\begin{aligned}-\alpha^2|\xi|^2 \operatorname{Re}\left(\int_0^\infty g(s)\left\langle \hat{\eta}^t(s), \bar{\hat{v}}\right\rangle ds\right) =& -\frac{1}{2}\frac{d}{dt}\alpha^2|\xi|^2\int_0^\infty g(s)|\hat{\eta}^t(s)|^2 ds\\ &+\frac{\alpha^2}{2}|\xi|^2\int_0^\infty g'(s)|\hat{\eta}^t(s)|^2 ds.\end{aligned}$$

Hence, we get

$$\begin{aligned}-\alpha^2|\xi|^2 \operatorname{Re}\left(\int_0^\infty g(s)\left\langle \hat{\eta}^t(s), \bar{\hat{v}}\right\rangle ds\right) =& -\frac{1}{2}\frac{d}{dt}\alpha^2|\xi|^2\int_0^\infty g(s)|\hat{\eta}^t(s)|^2 ds\\ &-\frac{\alpha^2}{2}\|\hat{\eta}^t\|^2_{\mathcal{M}}.\end{aligned}$$

Applying the same computation to the term

$$-\tau\alpha|\xi|^2 \operatorname{Re}\left(\int_0^\infty g(s)\left\langle \hat{\eta}^t(s), \bar{\hat{w}}\right\rangle ds\right)$$

and using the equation $\hat{\eta}_{tt}^t + \hat{\eta}_{ts}^t = \hat{w}$, to obtain:

$$\begin{aligned}-\tau\alpha|\xi|^2 \operatorname{Re}\left(\int_0^\infty g(s)\left\langle \hat{\eta}^t(s), \bar{\hat{w}}\right\rangle ds\right) =& -\tau\alpha|\xi|^2 \operatorname{Re}\left(\int_0^\infty g(s)\left\langle \hat{\eta}^t(s), \bar{\hat{\eta}}_{tt}^t(s)\right\rangle ds\right)\\ &-\tau\alpha|\xi|^2 \operatorname{Re}\left(\int_0^\infty g(s)\left\langle \hat{\eta}^t(s), \bar{\hat{\eta}}_{ts}^t(s)\right\rangle ds\right).\end{aligned}$$

Then, we have

$$\begin{aligned}-\tau\alpha|\xi|^2 \operatorname{Re}\left(\int_0^\infty g(s)\left\langle \hat{\eta}^t(s), \bar{\hat{w}}\right\rangle ds\right) =& -\tau\alpha|\xi|^2\frac{d}{dt} \operatorname{Re}\left(\int_0^\infty g(s)\left\langle \hat{\eta}^t(s), \bar{\hat{\eta}}_t^t(s)\right\rangle ds\right)\\ &+\tau\alpha|\xi|^2 \operatorname{Re}\left(\int_0^\infty g(s)\left\langle \hat{\eta}_t^t(s), \bar{\hat{\eta}}_t^t(s)\right\rangle ds\right)\\ &-\tau\alpha|\xi|^2\frac{d}{dt} \operatorname{Re}\left(\int_0^\infty g(s)\left\langle \hat{\eta}^t(s), \bar{\hat{\eta}}_s^t(s)\right\rangle ds\right)\\ &+\tau\alpha|\xi|^2 \operatorname{Re}\left(\int_0^\infty g(s)\left\langle \hat{\eta}_t^t(s), \bar{\hat{\eta}}_s^t(s)\right\rangle ds\right).\end{aligned}$$

Moreover, we obtain

$$-\tau\alpha|\xi|^2 \operatorname{Re}\left(\int_0^\infty g(s)\left\langle\hat{\eta}^t(s),\bar{\hat{w}}\right\rangle ds\right) = -\tau\alpha|\xi|^2\frac{d}{dt}\operatorname{Re}\left(\int_0^\infty g(s)\left\langle\hat{\eta}^t(s),\bar{\hat{v}}\right\rangle ds\right) + \tau\alpha|\xi|^2 \operatorname{Re}\left(\int_0^\infty g(s)\left\langle\hat{\eta}_t^t(s),\bar{\hat{v}}\right\rangle ds\right).$$

Inserting the estimates above into (1.33), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}I_1 =& -\alpha^2\beta|\xi|^2|\hat{v}|^2 - \tau\alpha\beta|\xi|^2\operatorname{Re}(\hat{v}\bar{\hat{w}}) - \ell\tau\alpha|\xi|^2\operatorname{Re}(\hat{u}\bar{\hat{w}}) - \ell\alpha^2|\xi|^2\operatorname{Re}(\hat{u}\bar{\hat{v}})\\ &- \frac{\alpha^2}{2}\|\hat{\eta}^t\|^2_{\mathcal{M}} + \tau\alpha|\xi|^2\operatorname{Re}\left(\int_0^\infty g(s)\left\langle\hat{\eta}_t^t(s),\bar{\hat{v}}\right\rangle ds\right), \end{aligned}\tag{1.34}$$

where

$$\begin{aligned}I_1 =& \alpha|\alpha\hat{v}+\tau\hat{w}|^2 + \alpha^2|\xi|^2\int_0^\infty g(s)|\hat{\eta}^t(s)|^2 ds\\ &+ 2\tau\alpha|\xi|^2\operatorname{Re}\left(\int_0^\infty g(s)\left\langle\hat{\eta}^t(s),\bar{\hat{v}}\right\rangle ds\right).\end{aligned}$$

Now, computing $|\xi|^2$(1.30)+$|\xi|^2$(1.31)+(1.34) and dividing the result by α, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}I_2 =& -(\alpha\beta-\tau\ell)|\xi|^2|\hat{v}|^2 - \frac{\alpha}{2}\|\hat{\eta}^t\|^2_{\mathcal{M}}\\ &+\tau|\xi|^2\operatorname{Re}\left(\int_0^\infty g(s)\left\langle\hat{\eta}_t^t(s),\bar{\hat{v}}\right\rangle ds\right),\end{aligned}\tag{1.35}$$

with

$$I_2 = \frac{I_1}{\alpha} + \frac{\tau}{\alpha}(\alpha\beta-\tau\ell)|\xi|^2|\hat{v}|^2 + \frac{\ell}{\alpha}|\xi|^2|\alpha\hat{u}+\tau\hat{v}|^2.$$

Here, substituting $\hat{\eta}_t^t$ by $\hat{v}-\hat{\eta}_s^t$, we get

$$\frac{1}{2}\frac{d}{dt}I_2 = -(\alpha\beta-\tau\ell)|\xi|^2|\hat{v}|^2 - \frac{\alpha}{2}\|\hat{\eta}^t\|^2_{\mathcal{M}} + \tau|\xi|^2\operatorname{Re}\left(\int_0^\infty g(s)\left\langle\hat{v}-\hat{\eta}_s^t(s),\bar{\hat{v}}\right\rangle ds\right).$$

Hence, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}I_2 =& -(\alpha\beta-\tau\ell)|\xi|^2|\hat{v}|^2 - \frac{\alpha}{2}|\|\hat{\eta}^t\|^2_{\mathcal{M}} + \tau(\gamma-\ell)|\xi|^2|\hat{v}|^2\\ &+\tau|\xi|^2\operatorname{Re}\left(\int_0^\infty g'(s)\left\langle\hat{\eta}^t(s),\bar{\hat{v}}\right\rangle ds\right).\end{aligned}$$

Again substituting $\bar{\hat{v}}$ by $\bar{\hat{\eta}}_t^t + \bar{\hat{\eta}}_s^t$, then we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}I_2 =& -(\alpha\beta - \tau\gamma)|\xi|^2|\hat{v}|^2 - \frac{\alpha}{2}\|\hat{\eta}^t\|_{\mathcal{M}}^2 + \tau|\xi|^2 \operatorname{Re}\left(\int_0^\infty g'(s)\left\langle \hat{\eta}^t(s), \bar{\hat{\eta}}_t^t(s)\right\rangle ds\right) \\ &+ \tau|\xi|^2 \operatorname{Re}\left(\int_0^\infty g'(s)\left\langle \hat{\eta}^t(s), \bar{\hat{\eta}}_s^t(s)\right\rangle ds\right).\end{aligned}$$

Integrating by parts once again with respect to s, then we obtain

$$\frac{1}{2}\frac{d}{dt}\hat{E}(\xi,t) = -(\alpha\beta - \tau\gamma)|\xi|^2|\hat{v}|^2 - \frac{\alpha}{2}\|\hat{\eta}^t\|_{\mathcal{M}}^2 - \frac{\tau}{2}|\xi|^2\int_0^\infty g''(s)|\hat{\eta}^t(s)|^2 ds,$$

with

$$\hat{E}(\xi,t) = I_2 + \tau\|\hat{\eta}^t\|_{\mathcal{M}}^2.$$

Consequently, we obtain the energy (1.24) and the use of assumptions (G2) with the condition $\alpha\beta - \tau\gamma \geq 0$, we obtain the desired result (1.28). □

Now, we define for all $t \geq 0$,

$$\begin{aligned}|\hat{V}(\xi,t)|^2 =& |\xi|^2|\alpha\hat{u} + \tau\hat{v}|^2 + |\xi|^2|\hat{v}|^2 + |\alpha\hat{v} + \tau\hat{w}|^2 \\ &+ \|\hat{\eta}^t\|_{\mathcal{M}}^2 + |\xi|^2\int_0^\infty g(s)|\hat{\eta}^t(s)|^2 ds.\end{aligned} \tag{1.36}$$

Hence, we have the following result.

Lemma 1.11. *Assume that $\alpha\beta > \tau\gamma$, then there exist two positive constants C_1 and C_2 such that*

$$C_1|\hat{V}(\xi,t)|^2 \leq \hat{E}(\xi,t) \leq C_2|\hat{V}(\xi,t)|^2, \tag{1.37}$$

for all $\xi \in \mathbb{R}^N$ and for all $t \geq 0$.

Proof. To show (1.37), we infer from Young's inequality that for every $\epsilon > 0$,

$$\left|2\tau|\xi|^2 \operatorname{Re}\left(\int_0^\infty g(s)\left\langle \hat{\eta}^t(s), \bar{\hat{v}}\right\rangle ds\right)\right| \leq \frac{2\tau^2(\gamma - \ell)}{\alpha\epsilon}|\xi|^2|\hat{v}|^2 + \frac{\alpha\epsilon}{2}|\xi|^2\int_0^\infty g(s)|\hat{\eta}^t(s)|^2 ds.$$

Hence, we obtain from (1.24)

$$\begin{aligned}\hat{E}(\xi,t) \geq& \frac{1}{2}\left[\frac{\ell}{\alpha}|\xi|^2|\alpha\hat{u} + \tau\hat{v}|^2 + |\alpha\hat{v} + \tau\hat{w}|^2 + \tau\|\hat{\eta}^t\|_{\mathcal{M}}^2\right. \\ &+ \left(\frac{\tau(\alpha\beta - \tau\ell)}{\alpha} - \frac{2\tau^2(\gamma - \ell)}{\alpha\epsilon}\right)|\xi|^2|\hat{v}|^2 \\ &\left.+\left(\alpha - \frac{\alpha\epsilon}{2}\right)|\xi|^2\int_0^\infty g(s)|\hat{\eta}^t(s)|^2 dsds\right].\end{aligned} \tag{1.38}$$

Recall that the assumption $\alpha\beta > \tau\gamma$ implies that $\alpha\beta > \tau\ell$. This yields

$$\frac{\tau(\gamma - \ell)}{\alpha\beta - \tau\ell} < 1.$$

Thus, we fix $\epsilon > 0$, such that

$$\frac{\tau(\gamma-\ell)}{\alpha\beta-\tau\ell} < \frac{\epsilon}{2} < 1.$$

Consequently, the left-hand side inequality in (1.37) holds. For the other side, we have

$$\begin{aligned}\hat{E}(\xi,t) \leq & \frac{1}{2}\left[\frac{\ell}{\alpha}|\xi|^2|\alpha\hat{u}+\tau\hat{v}|^2 + |\alpha\hat{v}+\tau\hat{w}|^2 + \tau\|\hat{\eta}^t\|^2_{\mathcal{M}}\right.\\ & + \left(\frac{\tau(\alpha\beta-\tau\ell)}{\alpha} + \frac{2\tau^2(\gamma-\ell)}{\alpha\epsilon}\right)|\xi|^2|\hat{v}|^2 \\ & \left. + \left(\alpha + \frac{\alpha\epsilon}{2}\right)|\xi|^2\int_0^\infty g(s)|\hat{\eta}^t(s)|^2 ds\right]. \end{aligned} \tag{1.39}$$

This yields the right-hand side inequality in (1.37). □

Now, following [95], we define the functional $\hat{F}_1(\xi,t)$ as:

$$\hat{F}_1(\xi,t) = \operatorname{Re}\left\{\alpha(\alpha\bar{\hat{u}}+\tau\bar{\hat{v}})(\alpha\hat{v}+\tau\hat{w})\right\}. \tag{1.40}$$

Then, we have the following lemma.

Lemma 1.12. *For any $\epsilon_0, \epsilon_1 > 0$, we have*

$$\begin{aligned}&\frac{d}{dt}\hat{F}_1(\xi,t) + (\ell - \epsilon_0 - (\gamma-\ell)\epsilon_1)|\xi|^2|\alpha\hat{u}+\tau\hat{v}|^2 \\ &\qquad \leq \alpha|\alpha\hat{v}+\tau\hat{w}|^2 + C(\epsilon_0)|\xi|^2|\hat{v}|^2 + C(\epsilon_1)|\xi|^2\int_0^\infty g(s)|\hat{\eta}^t(s)|^2 ds. \end{aligned} \tag{1.41}$$

Proof. Multiplying (1.32) by $\alpha(\alpha\bar{\hat{u}}+\tau\bar{\hat{v}})$ and (1.29) by $\alpha(\alpha\bar{\hat{v}}+\tau\bar{\hat{w}})$ we obtain, respectively,

$$\begin{aligned}\alpha(\alpha\hat{v}+\tau\hat{w})_t(\alpha\bar{\hat{u}}+\tau\bar{\hat{v}}) &= \left(-|\xi|^2\alpha\beta\hat{v} - \alpha\ell|\xi|^2\hat{u} - \alpha|\xi|^2\int_0^\infty g(s)\hat{\eta}^t(s)ds\right)\left(\alpha\bar{\hat{u}}+\tau\bar{\hat{v}}\right)\\ &= \left(-|\xi|^2\alpha\beta\hat{v} - \alpha\ell|\xi|^2\hat{u} + \tau\ell|\xi|^2\hat{v} - \tau\ell|\xi|^2\hat{v}\right.\\ &\qquad \left. -\alpha|\xi|^2\int_0^\infty g(s)\hat{\eta}^t(s)ds\right)\left(\alpha\bar{\hat{u}}+\tau\bar{\hat{v}}\right)\end{aligned}$$

and

$$\alpha(\alpha\hat{u}+\tau\hat{v})_t(\alpha\bar{\hat{v}}+\tau\bar{\hat{w}}) = \alpha(\alpha\hat{v}+\tau\hat{w})(\alpha\bar{\hat{v}}+\tau\bar{\hat{w}}).$$

Summing up the above equations and taking the real part, we obtain

$$\begin{aligned}&\frac{d}{dt}\hat{F}_1(\xi,t) + \ell|\xi|^2|\alpha\hat{u}+\tau\hat{v}|^2 - \alpha|\alpha\hat{v}+\tau\hat{w}|^2\\ &\quad = |\xi|^2(\tau\ell-\alpha\beta)\operatorname{Re}(\hat{v}(\alpha\bar{\hat{u}}+\tau\bar{\hat{v}}) - \alpha|\xi|^2\operatorname{Re}\left(\int_0^\infty g(s)\left\langle\hat{\eta}^t(s), \alpha\bar{\hat{u}}+\tau\bar{\hat{v}}\right\rangle ds\right).\end{aligned}$$

Applying Young's inequality, we obtain the estimate (1.41) for any $\epsilon_0, \epsilon_1 > 0$. □

Next, we set the functional $\hat{F}_2(\xi,t)$ as

$$F_2(\xi,t) = -\tau\alpha\,\mathrm{Re}(\bar{\hat{v}}(\alpha\hat{v}+\tau\hat{w})). \tag{1.42}$$

Lemma 1.13. *For any $\epsilon_2, \epsilon_3 > 0$, we have*

$$\begin{aligned}\frac{d}{dt}\hat{F}_2(\xi,t) + (\alpha-\epsilon_3)|\alpha\hat{v}+\tau\hat{w}|^2 \le\ & \epsilon_2|\xi|^2|\alpha\hat{u}+\tau\hat{v}|^2 + C(\epsilon_3,\epsilon_2)(1+|\xi|^2)|\hat{v}|^2 \\ & + \frac{1}{2}|\xi|^2\int_0^\infty g(s)|\hat{\eta}^t(s)|^2 ds.\end{aligned} \tag{1.43}$$

Proof. Multiplying the second equation in (1.22) by $-\tau\alpha(\alpha\bar{\hat{v}}+\tau\bar{\hat{w}})$ and the equation (1.32) by $-\tau\alpha\bar{\hat{v}}$, then we have, respectively,

$$-\alpha\tau(\alpha\bar{\hat{v}}+\tau\bar{\hat{w}})\hat{v}_t = -\tau\alpha\hat{w}(\alpha\bar{\hat{v}}+\tau\bar{\hat{w}})$$

and

$$\begin{aligned}-\tau\alpha\bar{\hat{v}}(\alpha\hat{v}+\tau\hat{w})_t &= \left(|\xi|^2\alpha\beta\tau\hat{v} + \alpha\ell\tau|\xi|^2\hat{u} + \tau\alpha|\xi|^2\int_0^\infty g(s)\hat{\eta}^t(s)ds\right)\bar{\hat{v}} \\ &= \Big(|\xi|^2\alpha\beta\tau\hat{v} + \alpha\ell\tau|\xi|^2\hat{u} + \tau^2\ell|\xi|^2\hat{v} - \tau^2\ell|\xi|^2\hat{v} \\ &\quad + \alpha^2(\alpha\hat{v}+\tau\hat{w}) - \alpha^2(\alpha\hat{v}+\tau\hat{w}) + \tau\alpha|\xi|^2\int_0^\infty g(s)\hat{\eta}^t(s)ds\Big)\bar{\hat{v}}.\end{aligned}$$

Summing up the above two equations and taking the real parts, we obtain

$$\begin{aligned}&\frac{d}{dt}\hat{F}_2(\xi,t) + \alpha|\alpha\hat{v}+\tau\hat{w}|^2 - \tau(\alpha\beta-\tau\ell)|\xi|^2|\hat{v}|^2 \\ &\qquad = \tau\ell|\xi|^2\,\mathrm{Re}(\bar{\hat{v}}(\alpha\hat{u}+\tau\hat{v})) + \alpha^2\,\mathrm{Re}(\bar{\hat{v}}(\alpha\hat{v}+\tau\hat{w})) \\ &\qquad\quad + \alpha\tau|\xi|^2\,\mathrm{Re}\left(\int_0^\infty g(s)\left\langle\hat{\eta}^t(s),\bar{\hat{v}}\right\rangle ds\right).\end{aligned}$$

Applying Young's inequality, we obtain the estimate (1.43) for any $\epsilon_2, \epsilon_3 > 0$. □

Proof of Proposition 1.9

We define the Lyapunov functional $\hat{\mathcal{L}}_1(\xi,t)$ as:

$$\hat{\mathcal{L}}_1(\xi,t) = N_0\hat{E}(\xi,t) + \rho(\xi)\hat{F}_1(\xi,t) + N_1\rho(\xi)\hat{F}_2(\xi,t), \tag{1.44}$$

where N_0 and N_1 are positive constants that will be fixed later on. Taking the derivative of (1.44) with respect to t and making use of (1.28), (1.41) and (1.43), we obtain

$$\begin{aligned}
&\frac{d}{dt}\hat{\mathcal{L}}_1(\xi,t) + \Big[N_0(\alpha\beta - \tau\gamma) - C(\epsilon_0) - N_1 C(\epsilon_2,\epsilon_3)\Big]|\xi|^2|\hat{v}|^2 \\
&\quad + \frac{\alpha N_0}{2}|\xi|^2\int_0^\infty -g'(s)|\hat{\eta}^t(s)|^2 ds \\
&\quad + \Big[(\ell - \epsilon_0 - (\gamma-\ell)\epsilon_1) - \epsilon_2 N_1\Big]\rho(\xi)|\xi|^2|\alpha\hat{u} + \tau\hat{v}|^2 \\
&\quad + \Big[N_1(\alpha - \epsilon_3) - \alpha\Big]\rho(\xi)|\alpha\hat{v} + \tau\hat{w}|^2 \\
&\quad - \Big[C(\epsilon_1) + \frac{N_1}{2}\Big]\rho(\xi)|\xi|^2\int_0^\infty g(s)|\hat{\eta}^t(s)|^2 ds \le 0,
\end{aligned} \tag{1.45}$$

where we used the fact that $\rho(\xi) \le 1$. Now, we discuss the case where g is decaying exponentially (i.e., $p = \infty$) and then the case where g is decaying polynomially (i.e., $p^\star < p < \infty$) since the proof is slightly different.

1. **Exponentially decaying kernel:** Taking $p = \infty$, then the assumption (G2) can be rewritten as follows:

$$g'(t) \le -\delta g(t). \tag{1.46}$$

Therefore, making use of (1.46), we can now estimate the last term in (1.45) as:

$$\Big[C(\epsilon_1) + \frac{N_1}{2}\Big]\int_0^\infty g(s)|\hat{\eta}^t(s)|^2 ds \le \Big[\frac{C(\epsilon_1)}{\delta} + \frac{N_1}{2\delta}\Big]\int_0^\infty -g'(s)|\hat{\eta}^t(s)|^2 ds. \tag{1.47}$$

Plugging (1.47) into (1.45), we get

$$\begin{aligned}
&\frac{d}{dt}\hat{\mathcal{L}}_1(\xi,t) + \Big[N_0(\alpha\beta - \tau\ell) - C(\epsilon_0) - N_1 C(\epsilon_2,\epsilon_3)\Big]\rho(\xi)|\xi|^2|\hat{v}|^2 \\
&\quad + \Big[\frac{\alpha N_0}{2} - \frac{C(\epsilon_1)}{\delta} - \frac{N_1}{2\delta}\Big]\rho(\xi)|\xi|^2\int_0^\infty (-g')(s)|\hat{\eta}^t(s)|^2 ds \\
&\quad + \Big[(\ell - \epsilon_0 - (\gamma-\ell)\epsilon_1) - \epsilon_2 N_1\Big]\rho(\xi)|\xi|^2|\alpha\hat{u} + \tau\hat{v}|^2 \\
&\quad + \Big[N_1(\alpha - \epsilon_3) - \alpha\Big]\rho(\xi)|\alpha\hat{v} + \tau\hat{w}|^2 \le 0, \qquad \forall t \ge 0.
\end{aligned} \tag{1.48}$$

In the above estimate, we can fix our constants in such a way that the coefficients in (1.48) are positive. This can be achieved as follows: we pick ϵ_3 small enough such that $\epsilon_3 < \alpha$ and $\epsilon_0 = \epsilon_1$, then we can select ϵ_0 small enough such that

$$\epsilon_0 < \frac{\ell}{1 + (\gamma - \ell)}.$$

After that, we take N_1 large enough such that

$$N_1 > \frac{\alpha}{\alpha - \epsilon_3}.$$

Once N_1 and $\epsilon_0 = \epsilon_1$ are fixed, we select ϵ_2 small enough such that

$$\epsilon_2 < \frac{\ell - \epsilon_0(1 + (\gamma - \ell))}{N_1}.$$

Finally, keeping in mind the assumption $\tau\gamma < \alpha\beta$, we take N_0 large enough such that

$$N_0 = \max\left\{\frac{C(\epsilon_0) + N_1 C(\epsilon_2, \epsilon_3)}{\alpha\beta - \tau\gamma}, \frac{2C(\epsilon_1) + N_1}{\alpha\delta}\right\}.$$

Consequently, from above and (1.37), we deduce that there exists a positive constant Λ_0 such that for all $t \geq 0$,

$$\frac{d}{dt}\hat{\mathcal{L}}_1(\xi, t) + \Lambda_0 \rho(\xi)\hat{E}(\xi, t) \leq 0. \tag{1.49}$$

It is not difficult to see that from (1.37), (1.40), (1.42), (1.44) and for N_0 large enough, there exist two positive constants d_1 and d_2, such that for all $\xi \in \mathbb{R}^N$ and for all $t \geq 0$, it holds that

$$d_1 \hat{E}(\xi, t) \leq \hat{\mathcal{L}}_1(\xi, t) \leq d_2 \hat{E}(\xi, t). \tag{1.50}$$

Consequently, the last estimate together with (1.49), leads to

$$\frac{d}{dt}\mathcal{L}_1(\xi, t) + \Lambda_1 \rho(\xi)\hat{\mathcal{L}}_1(\xi, t) \leq 0, \qquad \text{for all} \quad t \geq 0, \tag{1.51}$$

and for some $\Lambda_1 > 0$. A simple application of Gronwall's lemma to the estimate (1.51) yields

$$\hat{\mathcal{L}}_1(\xi, t) \leq \hat{\mathcal{L}}_1(\xi, 0)e^{-c\rho(\xi)t}, \qquad \text{for all} \quad t \geq 0. \tag{1.52}$$

Once again, (1.50) and (1.52) yield for some $\Lambda_2 > 0$,

$$\hat{E}(\xi, t) \leq \Lambda_2 \hat{E}(\xi, 0)e^{-c\rho(\xi)t}, \qquad \text{for all} \quad t \geq 0. \tag{1.53}$$

On the other hand, from (1.37), we have

$$\hat{E}(\xi, t) \geq C_3 |\hat{U}(\xi, t)|^2 \tag{1.54}$$

and the use of $\hat{\eta}^0(s) = \hat{u}_0$, yields

$$\hat{E}(\xi, 0) \leq C_4 |\hat{U}(\xi, 0)|^2. \tag{1.55}$$

This leads to the estimate (1.25).

2. **Polynomially decaying kernel:** Now, we assume that g is decaying polynomially and we try to absorb the integral term $\left(\int_0^\infty g(s)|\hat{\eta}^t(s)|^2 ds\right)$ in (1.45) by the term $\left(\int_0^\infty -g'(s)|\hat{\eta}^t(s)|^2 ds\right)$ for $p^\star < p < \infty$.
As we did above in the first case, we take the same coefficients of (1.45) which are already fixed in such a way they are positive except the coefficient of $\left(\int_0^\infty g(s)|\hat{\eta}^t(s)|^2 ds\right)$.

We therefore deduce that for all $t \geq 0$, (1.45) can be rewritten as:

$$\frac{d}{dt}\hat{\mathcal{L}}_1(\xi,t) \leq -\alpha_1\rho(\xi)\hat{E}(\xi,t) + \alpha_2\rho(\xi)|\xi|^2 \int_0^\infty g(s)|\hat{\eta}^t(s)|^2 ds, \tag{1.56}$$

where

$$\alpha_2 = C(\epsilon_1) + \frac{N_1}{2} + \alpha_1\alpha$$

and for some $\alpha_1 > 0$. Our main goal now is to estimate the last term in (1.56). Indeed, we have by using Hölder's inquality,

$$\begin{aligned}\left(|\xi|^2 \int_0^\infty g(s)|\hat{\eta}^t(s)|^2 ds\right) &= |\xi|^2 \int_0^\infty [g(s)]^{\frac{1}{p}} |\hat{\eta}^t(s)|^{\frac{2}{p+1}} g(s)^{1-\frac{1}{p}} |\hat{\eta}^t(s)|^{2(1-\frac{1}{p+1})} ds \\ &\leq \left(|\xi|^2 \int_0^\infty [g(s)]^{\frac{p+1}{p}} |\hat{\eta}^t(s)|^2 ds\right)^{\frac{1}{p+1}} \left(|\xi|^2 \int_0^\infty [g(s)]^{\frac{(p+1)(p-1)}{p^2}} |\hat{\eta}^t(s)|^2 ds\right)^{\frac{p}{p+1}}.\end{aligned}$$

The use of (1.9), (1.24) and (1.28) lead to

$$\begin{aligned}|\xi|^2 \int_0^\infty [g(s)]^{\frac{(p+1)(p-1)}{p^2}} |\hat{\eta}^t(s)|^2 ds \leq& \int_0^\infty [g(s)]^{\frac{(p+1)(p-1)}{p^2}} |\xi|^2 |\hat{u}(t) - \hat{u}(t-s)|^2 ds \\ \leq& 2\int_0^\infty [g(s)]^{\frac{(p+1)(p-1)}{p^2}} |\xi|^2 |\hat{u}(t)|^2 ds \\ &+ 2\int_0^\infty [g(s)]^{\frac{(p+1)(p-1)}{p^2}} |\xi|^2 |\hat{u}(t-s)|^2 ds.\end{aligned}$$

Now, using the fact that (see (1.38)),

$$\hat{E}(\xi,t) \geq \frac{1}{2}\frac{\ell}{\beta}(\alpha\beta - \ell\tau)|\hat{u}|^2$$

then we obtain from above

$$\begin{aligned}|\xi|^2 \int_0^\infty [g(s)]^{\frac{(p+1)(p-1)}{p^2}} |\hat{\eta}^t(s)|^2 ds \leq& \frac{4\beta}{\ell(\alpha\beta - \tau\ell)} \int_0^\infty [g(s)]^{\frac{(p+1)(p-1)}{p^2}} \hat{E}(\xi,t) ds \\ &+ \frac{4\beta}{\ell(\alpha\beta - \tau\ell)} \int_0^t [g(s)]^{\frac{(p+1)(p-1)}{p^2}} \hat{E}(\xi,t-s) ds \\ \leq& \frac{8\beta}{\ell(\alpha\beta - \tau\ell)} \hat{E}(\xi,0) \int_0^\infty [g(s)]^{\frac{(p+1)(p-1)}{p^2}} ds.\end{aligned}$$

Consequently, Lemma 1.5 leads to

$$\left(|\xi|^2 \int_0^\infty [g(s)]^{\frac{(p+1)(p-1)}{p^2}} |\hat{\eta}^t(s)|^2 ds\right)^{\frac{p}{p+1}} \leq \left(\frac{8\beta c_0}{\ell(\alpha\beta - \tau\ell)} \hat{E}(\xi,0)\right)^{\frac{p}{p+1}}.$$

Let $\nu = \frac{(p+1)(p-1)}{p^2}$, it is clear that the assumption $p > p^\star$ implies that $\nu \in (\frac{1}{p}, 1)$. Hence, due to (1.28) and $(G2)$, we have

$$\left(|\xi|^2 \int_0^\infty g(s)|\hat{\eta}^t(s)|^2 ds\right) \leq d_3 \left(\hat{E}(\xi,0)\right)^{\frac{p}{p+1}} \left(-\frac{d}{dt}\hat{E}(\xi,t)\right)^{\frac{1}{p+1}}, \tag{1.57}$$

where

$$d_3 = 2c_1 \left(\frac{1}{\delta\alpha}\right)^{\frac{1}{p+1}}.$$

Combining now (1.57) and (1.56), we get

$$\frac{d}{dt}\hat{\mathcal{L}}_1(\xi,t) \leq -\alpha_1\rho(\xi)\hat{E}(\xi,t) + d_4\rho(\xi)\left(\hat{E}(\xi,0)\right)^{\frac{p}{p+1}} \left(-\frac{d}{dt}\hat{E}(\xi,t)\right)^{\frac{1}{p+1}},$$

with $d_4 = \alpha_2 d_3$. In the same spirit of [84], multiplying the last inequality by $\hat{E}^p(\xi,t)$, gives

$$\begin{aligned}\hat{E}^p(\xi,t)\left(\frac{d}{dt}\hat{\mathcal{L}}_1(\xi,t)\right) \leq &-\alpha_1\rho(\xi)\hat{E}^{p+1}(\xi,t)\\ &+ d_4\rho(\xi)\hat{E}^p(\xi,t)\left(\hat{E}(\xi,0)\right)^{\frac{p}{p+1}}\left(-\frac{d}{dt}\hat{E}(\xi,t)\right)^{\frac{1}{p+1}}.\end{aligned}$$

The use of Young's inequality, yields

$$\begin{aligned}\hat{E}^p(\xi,t)\left(\frac{d}{dt}\hat{\mathcal{L}}_1(\xi,t)\right) \leq &-\alpha_1\rho(\xi)\hat{E}^{p+1}(\xi,t)\\ &+ d_4\rho(\xi)\left[\epsilon\hat{E}^{p+1}(\xi,t) - C(\epsilon)\hat{E}^p(\xi,0)\frac{d}{dt}\hat{E}(\xi,t)\right]\\ = &-(\alpha_1 - \epsilon d_4)\,\rho(\xi)\hat{E}^{p+1}(\xi,t) - d_5\rho(\xi)\hat{E}^p(\xi,0)\frac{d}{dt}\hat{E}(\xi,t).\end{aligned}$$

Now, choosing $\epsilon < \frac{\alpha_1}{d_4}$ and recalling that $\frac{d}{dt}\hat{E}(\xi,t) \leq 0$ and using the fact that $\rho(\xi) \leq 1$, to get

$$\begin{aligned}\frac{d}{dt}\left[\hat{E}^p(\xi,t)\hat{\mathcal{L}}_1(\xi,t)\right] &\leq \hat{E}^p(\xi,t)\frac{d}{dt}\hat{\mathcal{L}}_1(\xi,t)\\ &\leq -d_6\rho(\xi)\hat{E}^{p+1}(\xi,t) - \frac{d}{dt}\left[d_5\hat{E}^p(\xi,0)\hat{E}(\xi,t)\right],\end{aligned}$$

for some $d_6 > 0$. This implies

$$\frac{d}{dt}\left[\hat{E}^p(\xi,t)\hat{\mathcal{L}}_1(\xi,t) + d_5\hat{E}^p(\xi,0)\hat{E}(\xi,t)\right] \leq -d_6\rho(\xi)\hat{E}^{p+1}(\xi,t). \tag{1.58}$$

Now, we define

$$\hat{\mathcal{E}}(\xi,t) = \hat{E}^p(\xi,t)\hat{\mathcal{L}}_1(\xi,t) + d_5\hat{E}^p(\xi,0)\hat{E}(\xi,t),$$

which satisfies, by using (1.50)

$$d_5\hat{E}^p(\xi,0)\hat{E}(\xi,t) \leq \hat{\mathcal{E}}(\xi,t) \leq d_7\hat{E}^p(\xi,0)\hat{E}(\xi,t), \tag{1.59}$$

for some $d_7 > 0$. Therefore, we obtain from (1.58) the estimate

$$\frac{d}{dt}\hat{\mathcal{E}}(\xi,t) \leq -d_8\rho(\xi)\left(\frac{\hat{\mathcal{E}}(\xi,t)}{\hat{E}^p(\xi,0)}\right)^{p+1},$$

for some $d_8 > 0$. A simple application of Gronwall's lemma leads to the estimate:

$$\hat{\mathcal{E}}(\xi,t) \leq \Lambda_3\hat{E}^{p+1}(\xi,0)\Big(1+\rho(\xi)t\Big)^{-\frac{1}{p}}, \tag{1.60}$$

for $\Lambda_3 > 0$. Using (1.59) we obtain

$$\hat{E}(\xi,t) \leq \Lambda_4\hat{E}(\xi,0)\Big(1+\rho(\xi)t\Big)^{-\frac{1}{p}}, \tag{1.61}$$

for $\Lambda_4 > 0$. This leads to the desired estimate (1.26) using (1.54) and (1.55). Thus the proof of Proposition 1.9 is finished.

1.4 Decay estimates–the critical case

In this section, we assume that $\alpha\beta = \tau\gamma$ and derive the pointwise estimate (1.62) for the Fourier transform of the solution. This estimate is the key ingredient in proving the decay rate stated in Theorem 1.2.

Proposition 1.14. *Let $\hat{U}(\xi,t)$ be the solution of (1.22)-(1.23). Assume that (G1)-(G3) and $\alpha\beta = \tau\gamma$ hold. Then, $\hat{U}(\xi,t)$ satisfies the following estimate*

$$|\hat{U}(\xi,t)|^2 \leq C|\hat{U}(\xi,0)|^2 e^{-c\frac{|\xi|^2}{(1+|\xi|^2)^2}t}, \qquad \text{for} \qquad p=\infty, \tag{1.62}$$

for all $t > 0$.

Remark 1.15. *It is clear from the form of $\frac{|\xi|^2}{(1+|\xi|^2)^2}$ obtained in the estimate (1.62) that $\frac{|\xi|^2}{(1+|\xi|^2)^2} \sim |\xi|^{-2}$ for $|\xi| \to \infty$. This leads to the decay rate of regularity-loss type shown in Theorem 1.2. On the other hand, for $|\xi| \to 0$, we have $\frac{|\xi|^2}{(1+|\xi|^2)^2} \sim |\xi|^2$, this means that the dissipation is still effective for low frequencies and prevents any loss in the regularity.*

To prove Proposition 1.14, we need to define another functional $\hat{F}_3(\xi,t)$ (which is unnecessary in the first case) in such a way to recover the dissipation of $|\hat{v}|^2$, since for $\alpha\beta = \tau\gamma$, we lose the dissipation of $|\hat{v}|^2$ in (1.28). In this case (1.28) becomes:

$$\frac{d}{dt}\hat{E}(\xi,t) \leq -\frac{\alpha}{2}\|\hat{\eta}(s)\|^2_{\mathcal{M}}, \qquad \forall t \geq 0. \tag{1.63}$$

Now, we show that if g decays exponentially, that is

$$g'(t) \leq -\delta g(t), \tag{1.64}$$

which holds for $p = \infty$ in the assumption (G2), then the vector $|\hat{V}(\xi,t)|^2$ defined in (1.36) is equivalent to $\hat{E}(\xi,t)$, as we did in Lemma 1.11. Thus we have the following lemma.

Lemma 1.16. *Let $\alpha\beta = \tau\gamma$ and assume that (1.64) holds. Then there exist two positive constants C_5 and C_6, such that*

$$C_5|\hat{V}(\xi,t)|^2 \leq \hat{E}(\xi,t) \leq C_6|\hat{V}(\xi,t)|^2, \tag{1.65}$$

for all $\xi \in \mathbb{R}^N$ and for all $t \geq 0$.

Proof. The right-hand side in (1.65) is estimated just in the same way as in (1.39). For the left-hand side, we follow [37, Lemma 3.1] to have from Young's inequality the estimate:

$$\left|2\tau|\xi|^2 \operatorname{Re}\left(\int_0^\infty g(s)\left\langle \hat{\eta}^t(s), \bar{\hat{v}}\right\rangle ds\right)\right| \leq \frac{\tau^2}{2\epsilon_0}|\xi|^2|\hat{v}|^2 + 2\epsilon_0(\gamma-\ell)|\xi|^2\int_0^\infty g(s)|\hat{\eta}^t(s)|^2 ds.$$

Taking $2\epsilon_0 = \frac{\alpha(\epsilon_1+1)}{\gamma-\ell}$, yields

$$\left|2\tau|\xi|^2 \operatorname{Re}\left(\int_0^\infty g(s)\left\langle \hat{\eta}^t(s), \bar{\hat{v}}\right\rangle ds\right)\right| \leq \frac{\tau^2(\gamma-\ell)}{\alpha(\epsilon_1+1)}|\xi|^2|\hat{v}|^2 + \alpha(\epsilon_1+1)|\xi|^2\int_0^\infty g(s)|\hat{\eta}^t(s)|^2 ds.$$

Hence, we have, as in [37]

$$\begin{aligned}
\alpha|\xi|^2\int_0^\infty g(s)|\hat{\eta}^t(s)|^2 ds &+ 2\tau|\xi|^2 \operatorname{Re}\left(\int_0^\infty g(s)\left\langle \hat{\eta}^t(s), \bar{\hat{v}}\right\rangle ds\right)\\
&\geq -\frac{\tau^2(\gamma-\ell)}{\alpha(\epsilon_1+1)}|\xi|^2|\hat{v}|^2 - \alpha\epsilon_1|\xi|^2\int_0^\infty g(s)|\hat{\eta}^t(s)|^2 ds\\
&\geq -\frac{\tau^2(\gamma-\ell)}{\alpha(\epsilon_1+1)}|\xi|^2|\hat{v}|^2 - \frac{\alpha\epsilon_1}{\delta}|\xi|^2\int_0^\infty -g'(s)|\hat{\eta}^t(s)|^2 ds\\
&= -\frac{\tau^2(\gamma-\ell)}{\alpha(\epsilon_1+1)}|\xi|^2|\hat{v}|^2 - \frac{\alpha\epsilon_1}{\delta}\|\hat{\eta}^t\|^2_{\mathcal{M}}.
\end{aligned}$$

Consequently, from (1.24) and the above estimate, we obtain

$$\hat{E}(\xi,t) \geq \frac{1}{2}\left[\frac{\ell}{\alpha}|\xi|^2|\alpha\hat{u}+\tau\hat{v}|^2 + |\alpha\hat{v}+\tau\hat{w}|^2 + \frac{\epsilon_1\tau^2(\gamma-\ell)}{\alpha(\epsilon_1+1)}|\xi|^2|\hat{v}|^2 + (\tau - \frac{\alpha\epsilon_1}{\delta})\|\hat{\eta}^t\|^2_{\mathcal{M}}\right].$$

It is clear that for ϵ_1 small enough, that is for $\epsilon_1 < \frac{\delta\tau}{\alpha}$, we get the left-hand side of (1.65). This finishes the proof of Lemma 1.16. □

Now, we define the functional $\hat{F}_3(\xi, t)$ as

$$\hat{F}_3(\xi,,t) = -\tau|\xi|^2 \operatorname{Re}\left\{\int_0^\infty g(s)\left\langle \hat{\eta}^t(s), \bar{\hat{v}}\right\rangle ds\right\}. \tag{1.66}$$

Then, we have the following result.

Lemma 1.17. *For any $\epsilon_4, \epsilon_5, \epsilon_6 > 0$, it holds that*

$$\begin{aligned}
&\frac{d}{dt}\hat{F}_3(\xi,t) + (\tau(\gamma-\ell) - \epsilon_4 g_0 - \epsilon_6(\gamma-\ell))|\xi|^2|\hat{v}|^2 \\
&\qquad \le \epsilon_5 \frac{|\xi|^2}{1+|\xi|^2}(\gamma-\ell)|\alpha\hat{v}+\tau\hat{w}|^2 + C(\epsilon_4)|\xi|^2\int_0^\infty -g'(s)|\hat{\eta}^t(s)|^2 ds \\
&\qquad\quad + C(\epsilon_5,\epsilon_6)|\xi|^2(1+|\xi|^2)\int_0^\infty g(s)|\hat{\eta}^t(s)|^2 ds.
\end{aligned} \tag{1.67}$$

Proof. Multiplying the second equation in (1.22) by $\tau|\xi|^2\int_0^\infty g(s)\bar{\hat{\eta}}^t(s)ds$, we get

$$|\xi|^2\int_0^\infty g(s)\left\langle \tau\hat{v}_t, \bar{\hat{\eta}}^t(s)\right\rangle ds = |\xi|^2\int_0^\infty g(s)\left\langle \tau\hat{w}, \bar{\hat{\eta}}^t(s)\right\rangle ds.$$

Consequently,

$$\frac{d}{dt}|\xi|^2\int_0^\infty g(s)\left\langle \tau\hat{v}, \bar{\hat{\eta}}^t(s)\right\rangle ds - |\xi|^2\int_0^\infty g(s)\left\langle \tau\hat{v}, \bar{\hat{\eta}}^t_t(s)\right\rangle ds = |\xi|^2\int_0^\infty g(s)\left\langle \tau\hat{w}, \bar{\hat{\eta}}^t(s)\right\rangle ds.$$

Using the fact that $\hat{\eta}^t_t + \hat{\eta}^t_s = \hat{v}$ and integrating by parts with respect to s and taking the real part we obtain

$$\begin{aligned}
\frac{d}{dt}\hat{F}_3(\xi,t) + \tau(\gamma-\ell)|\xi|^2|\hat{v}|^2 =& \tau|\xi|^2 \operatorname{Re}\left\{\int_0^\infty -g'(s)\left\langle \hat{v}, \bar{\hat{\eta}}^t(s)\right\rangle ds\right\} \\
&- |\xi|^2 \operatorname{Re}\left\{\int_0^\infty g(s)\left\langle \tau\hat{w}+\alpha\hat{v}, \bar{\hat{\eta}}^t(s)\right\rangle ds\right\} \\
&+ |\xi|^2 \operatorname{Re}\left\{\int_0^\infty g(s)\left\langle \alpha\hat{v}, \bar{\hat{\eta}}^t(s)\right\rangle ds\right\}.
\end{aligned}$$

Applying Young's inequality (we write the coefficient of the second term on the right as $-|\xi|^2 = \dfrac{-|\xi|^2}{\sqrt{1+|\xi|^2}}\sqrt{1+|\xi|^2}$ and then we applied Young's inequality), we obtain the estimate (1.67) for any $\epsilon_4, \epsilon_5, \epsilon_6 > 0$. □

Proof of Proposition 1.14

Now, we define the new Lyapunov functional $\hat{\mathcal{L}}_2(\xi, t)$ associated with the critical case as follows:

$$\hat{\mathcal{L}}_2(\xi,t) = N\hat{E}(\xi,t) + \frac{|\xi|^2}{(1+|\xi|^2)^2}\hat{F}_1(\xi,t) + 2\frac{|\xi|^2}{(1+|\xi|^2)^2}\hat{F}_2(\xi,t) + M\frac{1}{1+|\xi|^2}\hat{F}_3(\xi,t), \tag{1.68}$$

for some positive constants N and M that have to be chosen later. Taking the derivative of (1.68) with respect to t, and using (1.41), (1.43), (1.63) and (1.67), since g decays exponentially (i.e., satisfies (1.64)), we get

$$\begin{aligned}
&\frac{d}{dt}\hat{\mathcal{L}}_2(\xi,t)+\frac{|\xi|^2}{(1+|\xi|^2)^2}\Big[(\ell-\epsilon_0-(\gamma-\ell)\epsilon_1)-2\epsilon_2\Big]|\xi|^2|\alpha\hat{u}+\tau\hat{v}|^2\\
&\quad+\frac{1}{1+|\xi|^2}\Big[M(\tau(\gamma-\ell)-\epsilon_4 g_0-\epsilon_6(\gamma-\ell))-2C(\epsilon_0)-C(\epsilon_2,\epsilon_3)\Big]|\xi|^2|\hat{v}|^2\\
&\quad+\frac{|\xi|^2}{(1+|\xi|^2)^2}\Big[2(\alpha-\epsilon_3)-\alpha-M\epsilon_5(\gamma-\ell)\Big]|\alpha\hat{v}+\tau\hat{w}|^2\\
&\quad+(N-\Lambda)|\xi|^2\int_0^\infty -g'(s)|\hat{\eta}^t(s)|^2ds\le 0, \qquad (1.69)
\end{aligned}$$

where for the coefficient of $|\hat{v}|^2$ in F_1, we used the fact that $\frac{|\xi|^4}{(1+|\xi|^2)^2}\le\frac{|\xi|^2}{1+|\xi|^2}$. Here Λ is a constant that depends on all the other constants. In the above estimate, we can fix our constants in such a way that the previous coefficients are positive. We take $\epsilon_0=\epsilon_1$, then we pick ϵ_0 small enough such that

$$\epsilon_0<\frac{\ell}{1+(\gamma-\ell)}.$$

Once ϵ_0,ϵ_1 are fixed, we select ϵ_2 small enough such that

$$\epsilon_2<\frac{\ell-\epsilon_0(1+(\gamma-\ell))}{2}.$$

Now, we pick $\epsilon_4=\epsilon_6$, that gives to us

$$\epsilon_4<\frac{\tau(\gamma-\ell)}{g_0+(\gamma-\ell)}.$$

Next, we pick $\epsilon_3<\dfrac{\alpha}{4}$ and we take M large enough such that

$$M>\frac{C(\epsilon_0)+2C(\epsilon_2,\epsilon_3)}{\tau(\gamma-\ell)-\epsilon_4(g_0+(\gamma-\ell))}.$$

Then, we can select ϵ_5 small enough such that

$$\epsilon_5<\frac{2(\alpha-\epsilon_3)-\alpha}{M(\gamma-\ell)}.$$

Finally, we take

$$N>\Lambda.$$

Consequently, we deduce that there exists a constant $R_1>0$ such that for all $t>0$

$$\frac{d}{dt}\hat{\mathcal{L}}_2(\xi,t)+R_1\frac{|\xi|^2}{(1+|\xi|^2)^2}|\hat{V}(\xi,t)|^2\le 0. \qquad (1.70)$$

Thanks to (1.65), there exists a constant $R_2 > 0$ such that for all $t > 0$, we deduce

$$\frac{d}{dt}\hat{\mathcal{L}}_2(\xi,t) + R_2\frac{|\xi|^2}{(1+|\xi|^2)^2}\hat{E}(\xi,t) \leq 0. \tag{1.71}$$

On the other hand, for N large enough, there exist two positive constants R_3, R_4 such that

$$R_3\hat{E}(\xi,t) \leq \hat{\mathcal{L}}_2(\xi,t) \leq R_4\hat{E}(\xi,t). \tag{1.72}$$

Consequently, (1.71) with (1.72) lead to

$$\frac{d}{dt}\hat{\mathcal{L}}_2(\xi,t) + R_5\frac{|\xi|^2}{(1+|\xi|^2)^2}\hat{\mathcal{L}}_2(\xi,t) \leq 0. \tag{1.73}$$

For some $R_5 > 0$. Integrating (1.73) with respect to t yields

$$\hat{\mathcal{L}}_2(\xi,t) \leq \hat{\mathcal{L}}_2(\xi,0)e^{-c\frac{|\xi|^2}{(1+|\xi|^2)^2}t}, \tag{1.74}$$

This last estimate leads, for $R_6 > 0$, to

$$\hat{E}(\xi,t) \leq R_6\hat{E}(\xi,0)e^{-c\frac{|\xi|^2}{(1+|\xi|^2)^2}t}, \tag{1.75}$$

Therefore, by (1.65), (1.54) and (1.55) we get the desired result (1.62).

1.5 Proof of the decay estimates of Theorems 1.1 and 1.2

In this section, we prove the decay rate both in the sub-critical case (Theorem 1.1) and in the critical case (Theorem 1.2), for $p = \infty$ and $p^\star < p < \infty$.

1.5.1 Proof of Theorem 1.1.

We proceed with the proof of Theorem 1.1, starting with the first case when $p = \infty$. To show (1.3) we have by Plancherel's theorem and the estimate (1.25) that (the constant C here is a generic positive constant that may take different values in difference places)

$$\|\nabla^j U(t)\|^2_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N}|\xi|^{2j}|\hat{U}(\xi,t)|^2 d\xi \leq C\int_{\mathbb{R}^N}|\xi|^{2j}e^{-c\rho(\xi)t}|\hat{U}(\xi,0)|^2 d\xi. \tag{1.76}$$

It is obvious that the term on the right-hand side of (1.76) depends on the behavior of the function (1.27). Since

$$\rho(\xi) \geq \begin{cases} \frac{1}{2}|\xi|^2, & \text{if} \quad |\xi| \leq 1, \\ \frac{1}{2}, & \text{if} \quad |\xi| \geq 1. \end{cases} \tag{1.77}$$

Then we write the integral on the right-hand side of (1.76) as

$$\begin{aligned}\int_{\mathbb{R}^N} |\xi|^{2j} e^{-c\rho(\xi)t} |\hat{U}(\xi,0)|^2 d\xi =& \int_{|\xi|\leq 1} |\xi|^{2j} e^{-c\rho(\xi)t} |\hat{U}(\xi,0)|^2 d\xi \\ &+ \int_{|\xi|\geq 1} |\xi|^{2j} e^{-c\rho(\xi)t} |\hat{U}(\xi,0)|^2 d\xi \\ =& I_1 + I_2. \end{aligned} \tag{1.78}$$

Concerning the integral I_1, we have by exploiting (54) that

$$I_1 \leq \|\hat{U}_0\|^2_{L^\infty(\mathbb{R}^N)} \int_{|\xi|\leq 1} |\xi|^{2j} e^{-\frac{c}{2}|\xi|^2 t} d\xi \leq C(1+t)^{-N/2-j} \|U_0\|^2_{L^1(\mathbb{R}^N)}. \tag{1.79}$$

On the other hand, in the high frequency region ($|\xi| \geq 1$), we have

$$I_2 \leq e^{-\frac{c}{2}t} \int_{|\xi|\geq 1} |\xi|^{2j} |\hat{U}(\xi,0)|^2 d\xi \leq e^{-\frac{c}{2}t} \|\nabla^j U_0\|^2_{L^2(\mathbb{R}^N)}. \tag{1.80}$$

Collecting the above two estimates give the desired decay estimate (1.3). Now, we show the decay rate of the vector U in the case of polynomially decaying kernel as follows: to establish (1.4), we have by Plancherel's theorem and the estimate (1.26) that

$$\begin{aligned}\|\nabla^j U(t)\|^2_{L^2(\mathbb{R}^N)} =& \int_{\mathbb{R}^N} |\xi|^{2j} |\hat{U}(\xi,t)|^2 d\xi \\ \leq& C \int_{\mathbb{R}^N} |\xi|^{2j} |\hat{U}(\xi,0)|^2 \Big(1+\rho(\xi)t\Big)^{-\frac{1}{p}} d\xi \\ \leq& C(1+t)^{-\frac{1}{p}} \int_{|\xi|\geq 1} |\xi|^{2j} |\hat{U}(\xi,0)|^2 d\xi \\ &+ C\|\hat{U}_0\|^2_{L^\infty(\mathbb{R}^N)} \int_{|\xi|\leq 1} |\xi|^{2j} \Big(1+|\xi|^2 t\Big)^{-\frac{1}{p}} d\xi. \end{aligned} \tag{1.81}$$

According to (1.77), the integral was split into two parts. For the last term above, the use of polar coordinates leads to

$$\int_{|\xi|\leq 1} |\xi|^{2j} \Big(1+|\xi|^2 t\Big)^{-\frac{1}{p}} d\xi \leq \int_0^1 \frac{r^{j+\frac{N}{2}-1}}{(1+rt)^{1/p}} dr. \tag{1.82}$$

Plugging (1.82) into (1.81), then exploiting Lemma 0.2, we deduce the desired result (1.4). This completes the proof of Theorem 1.1.

1.5.2 Proof of Theorem 1.2.

To show (1.5), we have:

$$\frac{|\xi|^2}{(1+|\xi|^2)^2} \geq \begin{cases} \frac{1}{4}|\xi|^2, & \text{if} \quad |\xi| \leq 1, \\ \frac{1}{4}|\xi|^{-2}, & \text{if} \quad |\xi| \geq 1. \end{cases} \tag{1.83}$$

By Plancherel's theorem, we obtain

$$\begin{aligned}\|\nabla^j U(t)\|_{L^2(\mathbb{R}^N)}^2 =& \int_{\mathbb{R}^N} |\xi|^{2j} |\hat{U}(\xi,t)|^2 d\xi \\ \leq& C\int_{\mathbb{R}^N} |\xi|^{2j} e^{-c\frac{|\xi|^2}{(1+|\xi|^2)^2}t} |\hat{U}(\xi,0)|^2 d\xi \\ =& C\int_{|\xi|\leq 1} |\xi|^{2j} e^{-c\frac{|\xi|^2}{(1+|\xi|^2)^2}t} |\hat{U}(\xi,0)|^2 d\xi \\ &+ C\int_{|\xi|\geq 1} |\xi|^{2j} e^{-c\frac{|\xi|^2}{(1+|\xi|^2)^2}t} |\hat{U}(\xi,0)|^2 d\xi \\ =& J_1 + J_2. \end{aligned} \tag{1.84}$$

Concerning the integral J_1, we have by using (1.83)

$$J_1 \leq C\|\hat{U}_0\|_{L^\infty(\mathbb{R}^N)}^2 \int_{|\xi|\leq 1} |\xi|^{2j} e^{-\frac{c}{4}|\xi|^2 t} d\xi \leq C(1+t)^{-N/2-j/2} \|U_0\|_{L^1(\mathbb{R}^N)}^2. \tag{1.85}$$

Using (1.83) and the estimate

$$\sup_{|\xi|\geq 1} \left\{ |\xi|^{-2k} e^{-c|\xi|^{-2}t} \right\} \leq C(1+t)^{-k}.$$

Then, the high frequency part J_2 is estimated as follow

$$J_2 \leq \sup_{|\xi|\geq 1} \left\{ |\xi|^{-2k} e^{-c|\xi|^{-2}t} \right\} \int_{|\xi|\geq 1} |\xi|^{2(j+k)} |\hat{U}(\xi,0)|^2 d\xi \leq C(1+t)^{-k} \|\nabla^{j+k} U_0\|_{L^2(\mathbb{R}^N)}^2. \tag{1.86}$$

Collecting the above two estimates yields (1.5).

1.6 Decay rates for the type III memory–the critical case

As in the previous sections, we first, need to find the decay rate of the Fourier image of the solution. Taking Fourier's transform of (1.6) for all $t \geq 0$ and for all $\xi \in \mathbb{R}^N$, we get

$$\tau \hat{z}_{tt} + \beta|\xi|^2 \hat{z} - \tau|\xi|^2 \int_0^t g(s)\hat{z}(t-s)ds = 0. \tag{1.87}$$

In this case the second assumption in (G1), is replaced by

(G4) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing twice differentiable function, such that

$$g(0) > 0, \qquad \beta - \tau\int_0^\infty g(s)ds = \tilde{\ell} > 0$$

and $g'(s) \leq 0$ for every $s > 0$.

As we have seen before in Section 1.2, we extend the solution of (1.87) for all times, by setting $z(x,t) = 0$ when $t < 0$ and considering for $t \geq 0$ the auxiliary past history variable

$\mu^t(s)$, defined as:

$$\mu^t(s) = z(t) - z(t-s), \qquad t \geq 0, \quad s \in \mathbb{R}^+. \tag{1.88}$$

Consequently, problem (1.87) can be recast as follows:

$$\begin{cases} \tau \hat{z}_{tt} + \tilde{\ell}|\xi|^2 \hat{z} + \tau|\xi|^2 \int_0^\infty g(s)\hat{\mu}^t(s)ds = 0, \\ \hat{\mu}_t^t(s) + \hat{\mu}_s^t(s) = \hat{z}_t. \end{cases} \tag{1.89}$$

Hence, we have the following result.

Proposition 1.18. *Let $\hat{z}(\xi,t)$ be the solution of (1.89). Assume that (G2) and (G4) hold. Then, $\hat{\mathcal{E}}(\xi,t)$ satisfies the following estimates:*

$$\hat{\mathcal{E}}(\xi,t) \leq C|\hat{Z}(\xi,0)|^2 e^{-c\rho(\xi)t}, \qquad \text{for} \quad p = \infty, \tag{1.90}$$

$$\hat{\mathcal{E}}(\xi,t) \leq C|\hat{Z}(\xi,0)|^2 \Big(1+\rho(\xi)t\Big)^{-\frac{1}{p}}, \qquad \text{for} \quad p^\star < p < \infty, \tag{1.91}$$

for all $t > 0$, where

$$\hat{\mathcal{E}}(\xi,t) = \frac{1}{2}\Big[\tilde{\ell}|\xi|^2|\hat{z}|^2 + \tau|\hat{z}_t|^2 + \tau|\xi|^2 \int_0^\infty g(s)|\hat{\mu}^t(s)|^2 ds\Big]. \tag{1.92}$$

The proof of Proposition 1.18 will be given through several lemmas. It is clear that $\hat{\mathcal{E}}(\xi,t)$ satisfies for all $t \geq 0$ and for all $\xi \in \mathbb{R}^N$,

$$\frac{d}{dt}\hat{\mathcal{E}}(\xi,t) = \frac{\tau}{2}|\xi|^2 \int_0^\infty g'(s)|\hat{\mu}^t(s)|^2 ds. \tag{1.93}$$

Since $g(s)$ is nonincreasing, then $\hat{\mathcal{E}}(\xi,t)$ is nonincreasing and dissipative.

Lemma 1.19. *For all $t \geq 0$ and for all $\xi \in \mathbb{R}^N$, the functional*

$$\hat{\Phi}_1(\xi,t) = \tau \operatorname{Re}(\bar{\hat{z}}\hat{z})$$

satisfies, for any $\lambda_1 > 0$,

$$\frac{d}{dt}\hat{\Phi}_1(\xi,t) + \Big[\tilde{\ell} - \lambda_1(\beta - \tilde{\ell})\Big]|\xi|^2|\hat{z}|^2 \leq \tau|\hat{z}_t|^2 + C(\lambda_1)|\xi|^2 \int_0^\infty g(s)|\hat{\mu}^t(s)|^2 ds. \tag{1.94}$$

Proof. Multiplying (1.89) by $\bar{\hat{z}}$ and taking the real part, we obtain

$$\tau\bar{\hat{z}}\hat{z}_{tt} + \tilde{\ell}|\xi|^2|\hat{z}|^2 + \tau|\xi|^2 \operatorname{Re}\left(\int_0^\infty g(s)\left\langle \hat{\mu}(s), \bar{\hat{z}}\right\rangle ds\right) = 0.$$

Hence,

$$\tau\frac{d}{dt}\hat{\Phi}_1(\xi,t) - \tau|\hat{z}_t|^2 + \tilde{\ell}|\xi|^2|\hat{z}|^2 + \tau|\xi|^2 \operatorname{Re}\left(\int_0^\infty g(s)\left\langle \hat{\mu}(s), \bar{\hat{z}}\right\rangle ds\right) = 0.$$

Then by applying Young's inequality, we get the estimate (1.94) for any $\lambda_1 > 0$. □

Lemma 1.20. *For all $t \geq 0$ and for all $\xi \in \mathbb{R}^N$, the functional*

$$\hat{\Phi}_2(\xi,t) = -\tau \operatorname{Re}\left(\int_0^\infty g(s) \left\langle \hat{z}_t, \bar{\hat{\mu}}^t(s) \right\rangle ds \right)$$

satisfies for any $\lambda_2, \lambda_3 > 0$, the following estimate

$$\begin{aligned} \frac{d}{dt}\hat{\Phi}_2(\xi,t) + &\Big((\beta - \tilde{\ell}) - \lambda_3 g(0) \Big) |\hat{z}_t|^2 \\ &\leq \lambda_2 \frac{(\beta - \tilde{\ell})}{\tau} |\xi|^2 |\hat{z}|^2 + C(\lambda_3)|\xi|^2 \int_0^\infty -g'(s)|\hat{\mu}^t(s)|^2 ds \\ &\quad + \Big(C(\lambda_2) + (\beta - \tilde{\ell}) \Big) |\xi|^2 \int_0^\infty g(s)|\hat{\mu}^t(s)|^2 ds. \end{aligned} \tag{1.95}$$

Proof. Multiplying (1.89) by $\int_0^\infty g(s)\bar{\hat{\mu}}^t(s)ds$ and taking the real part, we obtain

$$\begin{aligned} \tau \operatorname{Re}\Big(\int_0^\infty g(s) \left\langle \hat{z}_{tt}, \bar{\hat{\mu}}^t(s) \right\rangle ds \Big) &+ \tilde{\ell}|\xi|^2 \operatorname{Re}\left(\int_0^\infty g(s) \left\langle \hat{z}, \bar{\hat{\mu}}^t(s) \right\rangle ds \right) \\ &+ \tau|\xi|^2 \Big(\int_0^\infty g(s)\hat{\mu}(s) ds \Big)^2 = 0. \end{aligned}$$

Hence, we have

$$\begin{aligned} \tau \frac{d}{dt} \operatorname{Re}\left(\int_0^\infty g(s) \left\langle \hat{z}_t, \bar{\hat{\mu}}^t(s) \right\rangle ds \right) &- \tau \operatorname{Re}\left(\int_0^\infty g(s) \left\langle \hat{z}_t, \bar{\hat{\mu}}^t_t(s) \right\rangle ds \right) \\ &+ \tilde{\ell}|\xi|^2 \operatorname{Re}\left(\int_0^\infty g(s) \left\langle \hat{z}, \bar{\hat{\mu}}^t(s) \right\rangle ds \right) + \tau|\xi|^2 \Big(\int_0^\infty g(s)\hat{\mu}(s) ds \Big)^2 = 0. \end{aligned}$$

Using the fact that $\hat{\mu}^t_t + \hat{\mu}^t_s = \hat{z}_t$ and integrating by parts with respect to s, we obtain

$$\begin{aligned} \frac{d}{dt}\hat{\Phi}_2(\xi,t) = &-(\beta - \tilde{\ell})|\hat{z}_t|^2 + \tau \operatorname{Re}\left(\int_0^\infty -g'(s) \left\langle \hat{z}_t, \bar{\hat{\mu}}^t(s) \right\rangle ds \right) \\ &+ \tilde{\ell}|\xi|^2 \operatorname{Re}\left(\int_0^\infty g(s) \left\langle \hat{z}, \bar{\hat{\mu}}^t(s) \right\rangle ds \right) + \tau|\xi|^2 \Big(\int_0^\infty g(s)\hat{\mu}(s) ds \Big)^2 = 0. \end{aligned}$$

Then, to estimate the last three terms we apply Young's inequality and Hölder's inquality, we get the estimate (1.95) for any $\lambda_2, \lambda_3 > 0$. □

Now, we define the Lyapunov functional $\hat{\mathcal{L}}_3(\xi,t)$ as:

$$\hat{\mathcal{L}}_3(\xi,t) = N_2 \hat{\mathcal{E}}(\xi,t) + \rho(\xi)\hat{\Phi}_1(\xi,t) + N_3 \rho(\xi)\hat{\Phi}_2(\xi,t), \tag{1.96}$$

where N_2 and N_3 are positive constants that will be fixed later on. Taking the derivative of (1.96) with respect to t and making use of (1.93), (1.94) and (1.95), we obtain

$$\begin{aligned}
\frac{d}{dt}\hat{\mathcal{L}}_3(\xi,t) &+ \left[\left(\tilde{\ell} - \lambda_1(\beta - \tilde{\ell})\right) - N_3\lambda_2 \frac{(\beta - \tilde{\ell})}{\tau}\right]|\xi|^2\rho(\xi)|\hat{z}|^2 \\
&+ \left[N_3\left((\beta - \tilde{\ell}) - \lambda_3 g(0)\right) - \tau\right]|\xi|^2\rho(\xi)|\hat{z}_t|^2 \\
&- \left[C(\lambda_1) + N_3\left(C(\lambda_2) + (\beta - \tilde{\ell})\right)\right]|\xi|^2\rho(\xi)\int_0^\infty g(s)|\hat{\mu}^t(s)|^2 ds \\
&+ \left[N_2\frac{\tau}{2} - N_3 C(\lambda_3)\right]|\xi|^2 \int_0^\infty -g'(s)|\hat{\mu}^t(s)|^2 ds \leq 0. \qquad (1.97)
\end{aligned}$$

Now, we discuss the case where g is decaying exponentially (i.e., $p = \infty$) and then the case where g is decaying polynomially (i.e., $p^\star < p < \infty$).

1.6.1 Exponentially decaying kernel

Taking $p = \infty$, according to the assumption (G2), (1.97) can be written as

$$\begin{aligned}
\frac{d}{dt}\hat{\mathcal{L}}_3(\xi,t) &+ \left[\left(\tilde{\ell} - \lambda_1(\beta - \tilde{\ell})\right) - N_3\lambda_2 \frac{(\beta - \tilde{\ell})}{\tau}\right]|\xi|^2\rho(\xi)|\hat{z}|^2 \\
&+ \left[N_3\left((\beta - \tilde{\ell}) - \lambda_3 g(0)\right) - \tau\right]|\xi|^2\rho(\xi)|\hat{z}_t|^2 \\
&+ \left[N_2\frac{\tau}{2} - N_3 C(\lambda_3) - \frac{1}{\delta}\left(C(\lambda_1) + N_3\left(C(\lambda_2) + (\beta - \tilde{\ell})\right)\right)\right] \\
&\quad |\xi|^2\rho(\xi)\int_0^\infty -g'(s)|\hat{\mu}^t(s)|^2 ds \leq 0, \qquad t \geq 0. \qquad (1.98)
\end{aligned}$$

In the above estimate, we can fix our constants in such a way that the coefficients in (1.98) are positive. This can be achieved as follows:

$$\lambda_1 < \frac{\tilde{\ell}}{\beta - \tilde{\ell}}$$

and we fix λ_3 small enough such that

$$\lambda_3 < \frac{\beta - \tilde{\ell}}{g(0)}.$$

Then, we can select N_3 large enough

$$N_3 > \frac{\tau}{(\beta - \tilde{\ell}) - \lambda_3 g(0)}.$$

Once N_3 is fixed, we select λ_2 small enough such that

$$\lambda_2 < \frac{\tau\left(\tilde{\ell} - \lambda_1(\beta - \tilde{\ell})\right)}{N_3(\beta - \tilde{\ell})}.$$

Finally, we take N_2 large enough such that

$$N_2 > \frac{2}{\tau}\Big(N_3 C(\lambda_3) + \frac{1}{\delta}\Big(C(\lambda_1) + N_3\Big(C(\lambda_2) + (\beta - \tilde{\ell})\Big)\Big)\Big).$$

Consequently, from above we deduce that there exists a positive constant Λ_5 such that for all $t \geq 0$ and for all $\xi \in \mathbb{R}^N$,

$$\frac{d}{dt}\hat{\mathcal{L}}_3(\xi,t) + \Lambda_5 \rho(\xi)\hat{\mathcal{E}}(\xi,t) \leq 0. \tag{1.99}$$

It is clear that for N_2 large enough, there exist two positives constants d_9 and d_{10}, for all $t \geq 0$ and for all $\xi \in \mathbb{R}^N$, such that

$$d_9\hat{\mathcal{E}}(\xi,t) \leq \hat{\mathcal{L}}_3(\xi,t) \leq d_{10}\hat{\mathcal{E}}(\xi,t). \tag{1.100}$$

Consequently, the last estimate lead to

$$\frac{d}{dt}\hat{\mathcal{L}}_3(\xi,t) + \Lambda_6 \rho(\xi)\hat{\mathcal{L}}_3(\xi,t) \leq 0, \qquad \text{for all} \quad t \geq 0, \tag{1.101}$$

and for some $\Lambda_6 > 0$. A simple application of Gronwall's lemma to the estimate (1.101) yields

$$\hat{\mathcal{L}}_3(\xi,t) \leq \hat{\mathcal{L}}_3(\xi,0)e^{-c\rho(\xi)t}, \qquad \text{for all} \quad t \geq 0. \tag{1.102}$$

Once again, (1.100) and (1.102) yield for some $\Lambda_7 > 0$,

$$\hat{\mathcal{E}}(\xi,t) \leq \Lambda_7\hat{\mathcal{E}}(\xi,0)e^{-c\rho(\xi)t}, \qquad \text{for all} \quad t \geq 0. \tag{1.103}$$

The use of $\hat{\mu}^0(s) = \hat{z}_0$, yields the desired estimate (1.90).

1.6.2 Polynomially decaying kernel

In this section, we assume that g is decaying polynomially for $p^\star < p < \infty$. As we did above, taking the same coefficients of (1.97) which are already fixed in such a way they are positive except the coefficient of $\left(\int_0^\infty g(s)|\hat{\mu}^t(s)|^2 ds\right)$. Then for all $t \geq 0$, and $\xi \in \mathbb{R}^N$, the inequality (1.97) can be rewritten as:

$$\frac{d}{dt}\hat{\mathcal{L}}_3(\xi,t) \leq -m_1\rho(\xi)\hat{\mathcal{E}}(\xi,t) + m_2\rho(\xi)|\xi|^2 \int_0^\infty g(s)|\hat{\mu}^t(s)|^2 ds, \tag{1.104}$$

with

$$m_2 = \Big[C(\epsilon_1) + N_1 C(\epsilon_2)\Big] + \tau m_1$$

and for some $m_1 > 0$. To establish the result (1.91), we apply the same techniques used in Section 1.3. We estimate the integral term of (1.104) by using Hölder inequality, Lemma 1.5, (1.92), (1.93) and applying the method used in Section 1.3, we obtain the desired result. We omit the details. This finishes the proof of Proposition 1.18.

CHAPTER 2
A GENERAL STABILITY RESULT FOR VISCOELASTIC MOORE–GIBSON–THOMPSON EQUATION IN THE WHOLE SPACE

2.1 Introduction

In this chapter, we are interested in a viscoelastic Moore–Gibson–Thompson equation with a type II memory term, in the whole space $\mathbb{R}^N$:

$$u_{ttt}(t) + \alpha u_{tt}(t) - \beta \Delta u_t(t) - \gamma \Delta u(t) + \int_0^t g(t-s)\Delta u_t(s)ds = 0, \quad t > 0, \tag{2.1}$$

together with initial data

$$u(0) = u_0, \qquad u_t(0) = u_1, \qquad u_{tt}(0) = u_2, \tag{2.2}$$

where α, β, γ are positive constants. It is worth mentioning that the MGT model (2.1) with convolution memory term $\int_0^t g(t-s)\mathcal{A}z(s)ds$ was first provided by Lasiecka and Wang [65], in a bounded domain Ω, by considering z in the three classes $z = u$, $z = u_t$ and $z = \zeta u + u_t$ (with $\zeta > 0$), and memory kernel g with an exponential behavior. Thus, working on the non-critical regime, the authors proved that the corresponding energy functional is exponentially stable, see [65, Subsect. 1.2].
Our aim in this work is to investigate (2.1) and (2.2) for relaxation functions g satisfying

$$g' \leq -\eta(t)g(t)$$

and establish a general decay rate result under the condition

$$\beta - \frac{\gamma}{\alpha} - \frac{\varrho}{2} > 0, \tag{2.3}$$

where $\varrho = \int_0^\infty g(s)ds$. Then, we find the decay rate of the L^2-norm of the vector $U = (u_{tt}+\alpha u_t, \nabla(u_t+\alpha u), \nabla u_t)$ and those of its higher-order derivatives by applying Plancherel's theorem. To the best of our knowledge, the MGT equation with a type II memory satisfying the general condition (2.5) has never been discussed in the whole space $\mathbb{R}^N$. Our main result, which establishes the decay estimates of the L^2-norm of U, as well as the L^2-norm of its derivatives, is given in the following theorem.

Theorem 2.1. *Assume that (2.3) holds. We assume further that $U_0 \in L^1(\mathbb{R}^N) \cap H^s(\mathbb{R}^N)$. Then, there exist positive constants c_1, c_2, such that, for all $t \geq 0$ and all $j \leq s$, we obtain*

$$\begin{aligned}\|\nabla^j U(t)\|_{L^2(\mathbb{R}^N)} \leq & c_1\Big(1 + \int_0^t \eta(s)ds\Big)^{-\frac{j}{2}-\frac{N}{4}}\|U_0\|_{L^1(\mathbb{R}^N)} \\ & + c_1 e^{-c_2\int_0^t \eta(s)ds}\|\nabla^j U_0\|_{L^2(\mathbb{R}^N)}.\end{aligned} \tag{2.4}$$

To prove our main result, we first establish estimates on the Fourier image of U.

The essential of our work is organized as follows. In section 2, we give the hypotheses, then we state the well-posedness. The proofs of the main results and some illustrative examples are given in Section 4.

2.2 Preliminaries and well-posedness of the problem

In this section, we present some preliminaries and hypotheses necessary to prove our main results. Then we state the well-posedness of the problem (2.1). We first state the assumptions on the relaxation function.

(H1) $g : \mathbb{R}_+ \to (0, \infty)$ is a non-increasing differentiable function, such that

$$g(0) > 0, \qquad \varrho = \int_0^\infty g(s)ds < \beta.$$

(H2) There exists a non-increasing differentiable function $\eta : \mathbb{R}_+ \to (0, \infty)$, such that

$$g'(t) \leq -\eta(t)g(t) \qquad \forall t \geq 0. \tag{2.5}$$

We adopt the following lemma without proof which will be used in our work.

Lemma 2.2. *[86, Lemma 3.2]. Assume that **(H1)** holds. Then, there exists a constant* $c > 0$, *such that*

$$\left| \int_0^t g(t-s)(u(t)-u(s))ds \right|^2 \leq c(g \circ u)(t), \qquad \forall t \geq 0, \quad \forall u \in L^2(\mathbb{R}^N), \tag{2.6}$$

where,

$$(g \circ v)(t) = \int_0^t g(t-s)|v(t)-v(s)|^2 ds.$$

Now, by taking $u_t(x, \tau) = 0$, for all $\tau < 0$ and using the fact that

$$\int_0^\infty g(s)\Delta u_t(t-s)ds = \int_0^t g(s)\Delta u_t(t-s)ds + \int_t^\infty g(s)\Delta u_t(t-s)ds,$$

we deduce that our problem (2.1) together with (2.2) can be read as:

$$\begin{cases} u_{ttt}(t) + \alpha u_{tt}(t) - \beta\Delta u_t(t) - \gamma\Delta u(t) + \int_0^\infty g(s)\Delta u_t(t-s)ds = 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad u_{tt}(x,0) = u_2(x), \end{cases} \tag{2.7}$$

for all $x \in \mathbb{R}^N$ and $t \geq 0$. Therefore, thanks to [5, Theorem 3.2], we have the following lemma.

Lemma 2.3. *Assume that **(H1)** and **(H2)** are satisfied. Given* $U_0 = (u_0, u_1, u_2) \in \mathcal{H} = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, *then problem (2.7) has a unique solution such that*

$$(u, u_t, u_{tt}) \in C([0, +\infty); \mathcal{H}).$$

Proof. For more details on the proof, we refer the reader to [5, Theorem 3.2]. □

Now, by taking the Fourier transform of the problem (2.1)-(2.2), we obtain

$$\begin{cases} \hat{u}_{ttt}(t) + \alpha\hat{u}_{tt}(t) + \beta|\xi|^2\hat{u}_t(t) + \gamma|\xi|^2\hat{u} - |\xi|^2\int_0^t g(t-s)\hat{u}_t(s)ds = 0, \\ \hat{u}(0) = \hat{u}_0, \qquad \hat{u}_t(0) = \hat{u}_1, \qquad \hat{u}_{tt}(0) = \hat{u}_2, \end{cases} \tag{2.8}$$

for all $\xi \in \mathbb{R}^N$ and for all $t \geq 0$. Now, we define the vector U as

$$U = (u_{tt} + \alpha u_t, \nabla(u_t + \alpha u), \nabla u_t).$$

In addition, the energy functional associated with problem (2.8) is given by

$$\hat{E}(\xi,t) = \frac{1}{2}\left[|\hat{u}_{tt} + \alpha\hat{u}_t|^2 + \frac{\gamma}{\alpha}|\xi|^2|\hat{u}_t + \alpha\hat{u}|^2 + \left(\beta - \frac{\gamma}{\alpha} - \varrho\right)|\xi|^2|\hat{u}_t|^2 + |\xi|^2(g \circ \hat{u}_t)(t)\right]. \tag{2.9}$$

Thus, one can easily verify that, for some two positive constants λ_1, λ_2, we have

$$\hat{E}(\xi,t) \geq \lambda_1|\hat{U}(\xi,t)|^2 \tag{2.10}$$

and

$$\hat{E}(\xi,0) \leq \lambda_2|\hat{U}(\xi,0)|^2, \tag{2.11}$$

for all $\xi \in \mathbb{R}^N$ and for all $t \geq 0$. Hence, we have the following results.

Lemma 2.4. *Under the condition (2.3), the energy functional, defined in (2.9), satisfies, for all $t \geq 0$ and all $\xi \in \mathbb{R}^N$,*

$$\frac{d}{dt}\hat{E}(\xi,t) \leq \frac{1}{2}|\xi|^2(g' \circ \hat{u}_t)(t) - \alpha(\beta - \frac{\gamma}{\alpha} - \frac{\varrho}{2})|\xi|^2|\hat{u}_t|^2 \leq 0. \tag{2.12}$$

Proof. Multiplying the equation in (2.8) by $(\bar{\hat{u}}_{tt} + \alpha\bar{\hat{u}}_t)$ and taking the real part, we get

$$\begin{aligned} &\mathrm{Re}\left((\hat{u}_{tt} + \alpha\hat{u}_t)_t(\bar{\hat{u}}_{tt} + \alpha\bar{\hat{u}}_t)\right) + |\xi|^2\,\mathrm{Re}\left((\beta\hat{u}_t + \gamma\hat{u})(\bar{\hat{u}}_{tt} + \alpha\bar{\hat{u}}_t)\right) \\ &\quad - |\xi|^2\,\mathrm{Re}\left(\int_0^t g(t-s)\hat{u}_t(s)(\bar{\hat{u}}_{tt} + \alpha\bar{\hat{u}}_t)ds\right) = 0. \end{aligned} \tag{2.13}$$

Adding and subtracting $\frac{\gamma}{\alpha}|\xi|^2\hat{u}_t$ to the second summand, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[|\hat{u}_{tt} + \alpha\hat{u}_t|^2 + \frac{\gamma}{\alpha}|\xi|^2|\hat{u}_t + \alpha\hat{u}|^2 + \left(\beta - \frac{\gamma}{\alpha}\right)|\xi|^2|\hat{u}_t|^2\right] + (\alpha\beta - \gamma)|\xi|^2|\hat{u}_t|^2 \\ &\quad - |\xi|^2\,\mathrm{Re}\left(\int_0^t g(t-s)\hat{u}_t(s)(\bar{\hat{u}}_{tt} + \alpha\bar{\hat{u}}_t)ds\right) = 0. \end{aligned} \tag{2.14}$$

Now, we estimate the last term in (2.14) as follows

$$
\begin{aligned}
-|\xi|^2 \operatorname{Re}\left(\int_0^t g(t-s)\hat{u}_t(s)(\bar{\hat{u}}_{tt}+\alpha\bar{\hat{u}}_t)ds\right)
&\le |\xi|^2 \operatorname{Re}\left(\int_0^t g(t-s)(\hat{u}_t(t)-\hat{u}_t(s))\bar{\hat{u}}_{tt}ds\right)\\
&\quad+\alpha|\xi|^2 \operatorname{Re}\left(\int_0^t g(t-s)(\hat{u}_t(t)-\hat{u}_t(s))\bar{\hat{u}}_{t}ds\right)\\
&\quad-\varrho|\xi|^2 \operatorname{Re}\left(\hat{u}_t(t)(\bar{\hat{u}}_{tt}+\alpha\bar{\hat{u}}_t)\right).
\end{aligned}
\tag{2.15}
$$

For the first term of the right-hand side of (2.15), we have

$$
\begin{aligned}
|\xi|^2 \operatorname{Re}\left(\int_0^t g(t-s)(\hat{u}_t(t)-\hat{u}_t(s))\bar{\hat{u}}_{tt}ds\right)
=&\frac{1}{2}\frac{d}{dt}|\xi|^2\int_0^t g(t-s)|\hat{u}_t(t)-\hat{u}_t(s)|^2ds\\
&-\frac{1}{2}|\xi|^2\int_0^t g'(t-s)|\hat{u}_t(t)-\hat{u}_t(s)|^2ds.
\end{aligned}
$$

For the second term of the right-hand side in (2.15), we use Hölder's and Young's inequalities to get

$$
\begin{aligned}
\alpha|\xi|^2 \operatorname{Re}\left(\int_0^t g(t-s)(\hat{u}_t(t)-\hat{u}_t(s))\bar{\hat{u}}_{t}ds\right)
\le&\frac{\alpha}{2}|\xi|^2\int_0^t g(t-s)|\hat{u}_t(t)-\hat{u}_t(s)|^2ds\\
&+\frac{\alpha\varrho}{2}|\xi|^2|\hat{u}_t|^2.
\end{aligned}
$$

Finally, for the last term in (2.15), we have

$$
-\varrho|\xi|^2 \operatorname{Re}\left(\hat{u}_t(t)(\bar{\hat{u}}_{tt}+\alpha\bar{\hat{u}}_t)\right)\le -\frac{1}{2}\frac{d}{dt}|\xi|^2|\hat{u}_t|^2-\alpha\varrho|\xi|^2|\hat{u}_t|^2.
$$

Inserting the last three estimates above into (2.15), we arrive at

$$
\begin{aligned}
-|\xi|^2 \operatorname{Re}\left(\int_0^t g(t-s)\hat{u}_t(s)(\bar{\hat{u}}_{tt}+\alpha\bar{\hat{u}}_t)ds\right)
\le&\frac{1}{2}\frac{d}{dt}|\xi|^2\left[(g\circ\hat{u}_t)(t)-\varrho|\hat{u}_t|^2\right]\\
&-\frac{1}{2}|\xi|^2(g'\circ\hat{u}_t)(t)+\frac{\alpha}{2}|\xi|^2(g\circ\hat{u}_t)(t)\\
&-\frac{\alpha\varrho}{2}|\xi|^2|\hat{u}_t|^2.
\end{aligned}
\tag{2.16}
$$

Combining (2.16) and (2.14), with the use of **(H1)**, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[|\hat{u}_{tt}+\alpha\hat{u}_t|^2+\frac{\gamma}{\alpha}|\xi|^2|\hat{u}_t+\alpha\hat{u}|^2+\left(\beta-\frac{\gamma}{\alpha}-\varrho\right)|\xi|^2|\hat{u}_t|^2+|\xi|^2(g\circ\hat{u}_t)(t)\right]\\
&\quad\le-\alpha(\beta-\frac{\gamma}{\alpha}-\frac{\varrho}{2})|\xi|^2|\hat{u}_t|^2+\frac{1}{2}|\xi|^2(g'\circ\hat{u}_t)(t).
\end{aligned}
$$

Consequently, the desired result is established. □

2.3 Decay estimate

In this section, we apply the energy method in the Fourier space to get some pointwise estimate of $\hat{U}(\xi,t)$, with $U = (\alpha u_t + u_{tt}, \nabla(\alpha u + u_t), \nabla u_t)$, and present two illustrative examples. We first state the following result.

Proposition 2.5. *Let $\hat{U}(\xi,t)$ be the solution of (2.8). Assume that (2.3) and **(H1),(H2)** hold. Then, $\hat{U}(\xi,t)$ satisfies the following estimates, for positive constants c_0, c_0',*

$$|\hat{U}(\xi,t)|^2 \leq c_0|\hat{U}(\xi,0)|^2 e^{-c_0'\rho(\xi)\int_0^t \eta(s)ds}, \qquad \forall \xi \in \mathbb{R}^N, \quad \forall t \geq 0, \tag{2.17}$$

where $\rho(\xi) = \frac{|\xi|^2}{1+|\xi|^2}$.

Proposition 2.5 is the main ingredient in proving the decay estimates in Theorem 2.1. Before going on, we first establish some technical lemmas which help to construct the appropriate Lyapunov functionals in the Fourier space and, ultimately, lead to the proof of our main results.

Lemma 2.6. *Under the assumption **(H1)**, the functional $\hat{\Psi}_1$ defined by*

$$\hat{\Psi}_1(\xi,t) = \mathrm{Re}\left((\hat{u}_{tt} + \alpha\hat{u}_t)(\bar{\hat{u}}_t + \alpha\bar{\hat{u}})\right)$$

satisfies, along the solution of (2.8) and for any $\epsilon_0, \epsilon_1 > 0$, the estimate

$$\begin{aligned}\frac{d}{dt}\hat{\Psi}_1(\xi,t) \leq &-\left(\frac{\gamma}{\alpha} - \epsilon_0 - \epsilon_1\right)|\xi|^2|\hat{u}_t + \alpha\hat{u}|^2 + |\hat{u}_{tt} + \alpha\hat{u}_t|^2 \\ &+ C(\epsilon_0)|\xi|^2|\hat{u}_t|^2 + C(\epsilon_1)|\xi|^2(g\circ\hat{u}_t)(t).\end{aligned} \tag{2.18}$$

Proof. Differentiating $\hat{\Psi}_1$ with respect to t, we get

$$\frac{d}{dt}\hat{\Psi}_1(\xi,t) = \mathrm{Re}\left((\hat{u}_{ttt} + \alpha\hat{u}_{tt})(\hat{u}_t + \alpha\hat{u})\right) + |\hat{u}_{tt} + \alpha\hat{u}_t|^2.$$

Using (2.8), we obtain

$$\begin{aligned}\frac{d}{dt}\hat{\Psi}_1(\xi,t) =& \mathrm{Re}\left((-\beta|\xi|^2\hat{u}_t - \gamma|\xi|^2\hat{u} + |\xi|^2\int_0^t g(t-s)\hat{u}_t(s)ds)(\hat{u}_t + \alpha\hat{u})\right) + |\hat{u}_{tt} + \alpha\hat{u}_t|^2 \\ =& -\beta|\xi|^2\,\mathrm{Re}\left(\hat{u}_t(\hat{u}_t + \alpha\hat{u})\right) - \frac{\gamma}{\alpha}|\xi|^2|\hat{u}_t + \alpha\hat{u}|^2 + \frac{\gamma}{\alpha}|\xi|^2\,\mathrm{Re}\left(\hat{u}_t(\hat{u}_t + \alpha\hat{u})\right) \\ &+ |\hat{u}_{tt} + \alpha\hat{u}_t|^2 + |\xi|^2\,\mathrm{Re}\left((\bar{\hat{u}}_t + \alpha\bar{\hat{u}})\int_0^t g(t-s)(\hat{u}_t(s) - \hat{u}_t(t))ds\right) \\ &+ \varrho|\xi|^2\,\mathrm{Re}(\hat{u}_t(\bar{\hat{u}}_t + \alpha\bar{\hat{u}})) \\ \leq& |\hat{u}_{tt} + \alpha\hat{u}_t|^2 - \left(\beta - \frac{\gamma}{\alpha} - \varrho\right)|\xi|^2\,\mathrm{Re}\left(\hat{u}_t(\hat{u}_t + \alpha\hat{u})\right) - \frac{\gamma}{\alpha}|\xi|^2|\hat{u}_t + \alpha\hat{u}|^2 \\ &+ |\xi|^2\,\mathrm{Re}\left((\bar{\hat{u}}_t + \alpha\bar{\hat{u}})\int_0^t g(t-s)(\hat{u}_t(s) - \hat{u}_t(s))ds\right).\end{aligned}$$

Applying Young's inequality and using (2.6), we arrive at the estimate (2.18). □

Lemma 2.7. *Assume that **(H1)** holds. Then the functional $\hat{\Psi}_2$ defined by*

$$\hat{\Psi}_2(\xi,t) = -\operatorname{Re}(\bar{\hat{u}}_t(\hat{u}_{tt}+\alpha\hat{u}_t))$$

satisfies, along the solution of (2.8) *and for any $\epsilon_2 > 0$, the estimate*

$$\begin{aligned}\frac{d}{dt}\hat{\Psi}_2(\xi,t) \leq & -\frac{1}{2}|\hat{u}_{tt}+\alpha\hat{u}_t|^2 + \Big(1+\beta-\frac{\gamma}{\alpha}-\varrho+C(\epsilon_2)\Big)(1+|\xi|^2)|\hat{u}_t|^2 \\ & + \epsilon_2|\xi|^2|\hat{u}_t+\alpha\hat{u}|^2 + \frac{1}{2}|\xi|^2(g\circ\hat{u}_t)(t). \end{aligned} \tag{2.19}$$

Proof. Taking the derivative of $\hat{\Psi}_2$ with respect to t and exploiting (2.8), we get

$$\frac{d}{dt}\hat{\Psi}_2(\xi,t) = -\operatorname{Re}\Big(\bar{\hat{u}}_{tt}(\hat{u}_{tt}+\alpha\hat{u}_t)\Big) - \operatorname{Re}\Big(\bar{\hat{u}}_t(\hat{u}_{ttt}+\alpha\hat{u}_{tt})\Big).$$

Adding and subtracting the term $\alpha\bar{\hat{u}}_t$, we obtain

$$\begin{aligned}\frac{d}{dt}\hat{\Psi}_2(\xi,t) =& -\operatorname{Re}\Big((\bar{\hat{u}}_{tt}+\alpha\bar{\hat{u}}_t-\alpha\bar{\hat{u}}_t)(\hat{u}_{tt}+\alpha\hat{u}_t)\Big) \\ & -\operatorname{Re}\Big(\bar{\hat{u}}_t(-\beta|\xi|^2\hat{u}_t-\gamma|\xi|^2\hat{u}+|\xi|^2\int_0^t g(t-s)\hat{u}_t(s)ds)\Big) \\ =& -|\hat{u}_{tt}+\alpha\hat{u}_t|^2+\alpha\operatorname{Re}(\bar{\hat{u}}_t(\hat{u}_{tt}+\alpha\hat{u}_t)) \\ & -\operatorname{Re}\Big(\bar{\hat{u}}_t(-\beta|\xi|^2\hat{u}_t-\gamma|\xi|^2\hat{u}+|\xi|^2\int_0^t g(t-s)\hat{u}_t(s)ds)\Big).\end{aligned}$$

Again, adding and subtracting the term $\frac{\gamma}{\alpha}|\xi|^2\hat{u}_t$ to the last term above, we get

$$\begin{aligned}\frac{d}{dt}\hat{\Psi}_2(\xi,t) =& -|\hat{u}_{tt}+\alpha\hat{u}_t|^2+\alpha\operatorname{Re}(\bar{\hat{u}}_t(\hat{u}_{tt}+\alpha\hat{u}_t)) + \Big(\beta-\frac{\gamma}{\alpha}-\varrho\Big)|\xi|^2|\hat{u}_t|^2 \\ & +|\xi|^2\frac{\gamma}{\alpha}\operatorname{Re}(\bar{\hat{u}}_t(\hat{u}_t+\alpha\hat{u})) - |\xi|^2\operatorname{Re}\Big(\bar{\hat{u}}_t\int_0^t g(t-s)(\hat{u}_t(s)-\hat{u}_t(t))ds\Big).\end{aligned}$$

Applying Young's inequality and using (2.6), we obtain the estimate (2.19). □

Now, we define the Lyapunov functional $\hat{L}$ as

$$\hat{L}(\xi,t) = N_0\hat{E}(\xi,t)+\rho(\xi)\hat{\Psi}_1(\xi,t)+N_1\rho(\xi)\hat{\Psi}_2(\xi,t), \tag{2.20}$$

where N_0 and N_1 are positive constants to be specified later. Taking the derivative of (2.20) with respect to t and making use of (2.12), (2.18) and (2.19), we arrive at

$$\begin{aligned}\frac{d}{dt}\hat{L}(\xi,t) &+ \frac{N_0}{2}|\xi|^2(-g'\circ\hat{u}_t)(t) + \Big[\Big(\frac{\gamma}{\alpha}-\epsilon_0-\epsilon_1\Big)-N_1\epsilon_2\Big]\rho(\xi)|\xi|^2|\hat{u}_t+\alpha\hat{u}|^2 \\ &+\Big[N_0\alpha(\beta-\frac{\gamma}{\gamma}-\frac{\varrho}{2})-C(\epsilon_0)-N_1\Big(1+\beta-\frac{\gamma}{\alpha}-\varrho+C(\epsilon_2)\Big)\Big]\rho(\xi)|\xi|^2|\hat{u}_t|^2 \\ &+\Big[\frac{N_1}{2}-1\Big]\rho(\xi)|\hat{u}_{tt}+\alpha\hat{u}_t|^2 - \Big[C(\epsilon_1)+\frac{N_1}{2}\Big]\rho(\xi)|\xi|^2(g\circ\hat{u}_t)(t) \leq 0, \end{aligned} \tag{2.21}$$

for all $t \geq 0$ where we used the fact that $\rho(\xi) \leq 1$. Now, we choose our constants in such a way so that the coefficients in (2.21) are positive. First, we take $\epsilon_0 = \epsilon_1$ then select ϵ_0 small enough such that

$$\epsilon_0 < \frac{\gamma}{2\alpha}.$$

After that, we set $N_1 > 2$, then pick ϵ_2 small enough such that

$$\epsilon_2 < \frac{1}{N_1}\Big(\frac{\gamma}{\alpha} - 2\epsilon_0\Big).$$

Finally, as $\beta - \frac{\gamma}{\gamma} - \frac{\varrho}{2} > 0$, we take N_0 large enough, precisely such that

$$N_0 > \frac{N_1(1 + \beta - \gamma/\alpha - \varrho + C(\epsilon_2)) + C(\epsilon_0)}{\alpha(\beta - \frac{\gamma}{\gamma} - \frac{\varrho}{2})}.$$

We therefore deduce that for all $\xi \in \mathbb{R}^N$ and for all $t \geq 0$, (2.21) reads as:

$$\frac{d}{dt}\hat{L}(\xi, t) \leq -m_1\rho(\xi)\hat{E}(\xi, t) + m_2\rho(\xi)|\xi|^2(g \circ \hat{u}_t)(t), \tag{2.22}$$

where $m_1 > 0$ is a constant and $m_2 = m_1 + C(\epsilon_1) + \dfrac{N_1}{2}$. Notice that, for N_0 large enough we have $\hat{L} \sim \hat{E}$, which means that there exist two positives constant λ_3, λ_4, such that

$$\lambda_3\hat{E}(\xi, t) \leq \hat{L}(\xi, t) \leq \lambda_4\hat{E}(\xi, t) \tag{2.23}$$

2.3.1 Proof of Proposition 2.5

Proof. First, multiplying (2.22) by $\eta(t)$, we get

$$\eta(t)\frac{d}{dt}\hat{L}(\xi, t) \leq -m_1\rho(\xi)\eta(t)\hat{E}(\xi, t) + m_2\rho(\xi)|\xi|^2\eta(t)(g \circ \hat{u}_t)(t). \tag{2.24}$$

Hence, for the last term in (2.24), using the assumption **(H2)** and (2.12) with the fact that η is non-increasing, we easily conclude that, for all $\xi \in \mathbb{R}^N$ and for any $t \geq 0$,

$$\begin{aligned} |\xi|^2\eta(t)\int_0^t g(t-s)|\hat{u}_t(t) - \hat{u}_t(s)|^2 ds |\xi|^2 \int_0^t \eta(t-s)g(t-s)|\hat{u}_t(t) - \hat{u}_t(s)|^2 ds \\ \leq |\xi|^2 \int_0^t -g'(t-s)|\hat{u}_t(t) - \hat{u}_t(s)|^2 ds \\ \leq -2\frac{d}{dt}\hat{E}(\xi, t). \end{aligned}$$

Plugging the estimate above in (2.24), we get for some constant $m_3 > 0$, for all $\xi \in \mathbb{R}^N$ and all $t \geq 0$,

$$\eta(t)\frac{d}{dt}\hat{L}(\xi, t) \leq -m_1\rho(\xi)\eta(t)\hat{E}(\xi, t) - m_3\rho(\xi)\frac{d}{dt}\hat{E}(\xi, t).$$

Again, using the monotonicity of η and the fact that $\rho(\xi) \leq 1$, we get

$$\frac{d}{dt}\Big[\eta(t)\hat{L}(\xi,t) + m_3\hat{E}(\xi,t)\Big] \leq -m_1\rho(\xi)\eta(t)\hat{E}(\xi,t).$$

Let

$$\hat{\mathcal{L}}(\xi,t) = \eta(t)\hat{L}(\xi,t) + m_3\hat{E}(\xi,t), \qquad \forall \xi \in \mathbb{R}^N \quad \text{and} \quad \forall t \geq 0,$$

which is clearly equivalent to $\hat{E}(\xi,t)$. In fact, as $\eta'(t) < 0$ and recalling (2.23), we infer

$$m_3\hat{E}(\xi,t) \leq \hat{\mathcal{L}}(\xi,t) \leq (\eta(0)\lambda_4 + m_3)\hat{E}(\xi,t). \tag{2.25}$$

A simple integration with respect to t yields, for a constant $c_0' > 0$,

$$\hat{\mathcal{L}}(\xi,t) \leq \hat{\mathcal{L}}(\xi,0)e^{-c_0'\rho(\xi)\int_0^t \eta(s)ds}, \qquad \forall \xi \in \mathbb{R}^N \quad \text{and} \quad \forall t \geq 0. \tag{2.26}$$

Thus, thanks to (2.25), we have

$$\hat{E}(\xi,t) \leq \hat{E}(\xi,0)e^{-c_0'\rho(\xi)\int_0^t \eta(s)ds}, \qquad \forall \xi \in \mathbb{R}^N \quad \text{and} \quad \forall t \geq 0.$$

Owing to (2.10) and (2.11), the estimate (2.17) is obtained. □

2.3.2 Proof of Theorem 2.1

Proof. Now, we proceed with the proof of Theorem 2.1. To show (2.4), we have by Plancherel's theorem and the estimate (2.17)

$$\begin{aligned}\|\nabla^j U(t)\|^2_{L^2(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} |\xi|^{2j}|\hat{U}(\xi,t)|^2 d\xi \\ &\leq c_0 \int_{\mathbb{R}^N} |\xi|^{2j}|\hat{U}(\xi,0)|^2 e^{-c_0'\rho(\xi)\int_0^t \eta(s)ds} d\xi.\end{aligned} \tag{2.27}$$

Since

$$\rho(\xi) \geq \begin{cases} \frac{1}{2}|\xi|^2 & \text{if} \quad |\xi| \leq 1, \\ \frac{1}{2} & \text{if} \quad |\xi| \geq 1, \end{cases} \tag{2.28}$$

then we rewrite the right-hand side of (2.27) as

$$\begin{aligned} & c_0 \int_{\mathbb{R}^N} |\xi|^{2j}|\hat{U}(\xi,t)|^2 e^{-c_0'\rho(\xi)\int_0^t \eta(s)ds} d\xi \\ \leq & c_0 \int_{|\xi|\leq 1} |\xi|^{2j}|\hat{U}(\xi,0)|^2 e^{-\frac{c_0'}{2}|\xi|^2\int_0^t \eta(s)ds} d\xi \\ & + c_0 \int_{|\xi|\geq 1} |\xi|^{2j}|\hat{U}(\xi,0)|^2 e^{-\frac{c_0'}{2}\int_0^t \eta(s)ds} d\xi \\ = & I_1 + I_2. \end{aligned}$$

For the integral I_1, we have, by (56),

$$I_1 \leq c_0 \|\hat{U}(\xi,0)\|^2_{L^\infty(\mathbb{R}^N)} \int_{|\xi|\leq 1} |\xi|^{2j} e^{-\frac{c_0'}{2}|\xi|^2 \int_0^t \eta(s)ds} d\xi$$
$$\leq c_1 \|U_0\|^2_{L^1(\mathbb{R}^N)} \left(1 + \int_0^t \eta(s)ds\right)^{-j-\frac{N}{2}}. \tag{2.29}$$

On the other hand, for the high frequency region, we have

$$I_2 \leq c_0 e^{-\frac{c_0'}{2}\int_0^t \eta(s)ds} \int_{|\xi|\geq 1} |\xi|^{2j} |\hat{U}(\xi,0)|^2 d\xi$$
$$\leq c_1 e^{-c_2 \int_0^t \eta(s)ds} \|\nabla^j U_0\|^2_{L^2(\mathbb{R}^N)}. \tag{2.30}$$

Combining (2.29) with (2.30), we obtain (2.4). □

Remark 2.8. *Condition (2.3) appears to be purely technical and dictated by the method used in the proofs. It would be more natural if it can be replaced by $\alpha\beta - \gamma > 0$.*

Remark 2.9. *In our work, we established a decay estimate result for (2.1),(2.2), where the kernel satisfies*

$$g'(t) \leq -\zeta(t)g(t).$$

A question remains open, whether we can obtain a similar result for kernel satisfying (39) with more general convex functions H as in the case of bounded domains, see [75].

We end our chapter by presenting two illustrative examples.

Example 1: Let $\eta(s) = 1$. In this case, g decays exponentially and (2.4) gives a decay estimate similar to the one we obtained in the first chapter with $p = \infty$, i.e.

$$\|\nabla^j U(t)\| \leq c_1(1+t)^{-\frac{j}{2}-\frac{N}{4}} \|U_0\|_{L^1(\mathbb{R}^N)} + c_1 e^{-c_2 t} \|\nabla^j U_0\|_{L^2(\mathbb{R}^N)},$$

for $c_1, c_2 > 0$.

Example 2: Let $\eta(s) = \dfrac{1}{1+s}$. In this case, g decays polynomially and (2.4) gives the following decay estimate, $\forall t > 0$,

$$\|\nabla^j U(t)\| \leq c_1(1+\ln t)^{-\frac{j}{2}-\frac{N}{4}} \|U_0\|_{L^1(\mathbb{R}^N)} + \frac{c_1}{t^{c_2}} \|\nabla^j U_0\|_{L^2(\mathbb{R}^N)},$$

where $c_1, c_2 > 0$.

CHAPTER 3
OPTIMAL DECAY RATE FOR THE CAUCHY PROBLEM OF THE STANDARD LINEAR SOLID MODEL WITH GURTIN–PIPKIN THERMAL LAW

3.1 Introduction

In this chapter, we are concerned with the following Cauchy problem of the standard linear solid model with Gurtin–Pipkin Thermal Law in $\mathbb{R}^N$, that is,

$$\begin{aligned} &\tau u_{ttt} + u_{tt} - a^2\Delta u - a^2\beta\Delta u_t + \delta\Delta\theta = 0, \\ &\theta_t - \frac{1}{k}\int_0^\infty g(s)\Delta\theta(t-s)\, ds - \tau\delta\Delta u_{tt} - \delta\Delta u_t = 0, \end{aligned} \tag{3.1}$$

with the following initial conditions:

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad u_{tt}(x,0) = u_2(x), \quad \theta(x,t)\mid_{t\le 0} = \theta_0(x,t), \tag{3.2}$$

where $\theta_0(x,t)$ is a prescribed past history of $\theta(x,t)$ for $t \le 0$.
In this context, we recall different thermal conduction models coupled with the standard solid model (3.1) which appeared in the literature. The Fourier law is a frequently used model for the heat conduction, that is for the temperature difference $\theta(x,t)$ and the heat flux $q(x,t)$, this law reads as:

$$q(x,t) = -\kappa\nabla\theta(x,t). \tag{3.3}$$

Equation (3.3) together with the energy equation:

$$\theta_t(x,t) + \gamma\nabla\cdot q(x,t) = 0, \tag{3.4}$$

where, $\gamma > 0$ and the thermal conductivity $\kappa > 0$, leads to the parabolic heat equation:

$$\theta_t(x,t) - \kappa\gamma\Delta\theta(x,t) = 0, \tag{3.5}$$

also known as the diffusion equation. In the diffusion equation, if a sudden change of temperature is inflicted at some point on the body, it will be felt instantly everywhere. Hence, we can say that the diffusion allows an infinite speed of propagation. To reduce this difficulty in the Fourier law, we may consider Cattaneo law of heat conduction given by:

$$\tau_0 q_t(x,t) + q(x,t) + \kappa\nabla\theta(x,t) = 0, \tag{3.6}$$

where $\tau_0 > 0$ is the relaxation time of the heat flux. Equation (3.6) is different from Fourier's by the existence of relaxation term $\tau_0 q_t(x,t)$ and model the heat propagation equation as a damped wave equation. The Cattaneo equation (3.6) can be expressed as an integral over the history of the temperature gradient, i.e.,

$$q(x,t) = -\frac{\kappa}{\tau_0}\int_{-\infty}^t e^{(s-t)/\tau_0}\nabla\theta(x,s)\, ds. \tag{3.7}$$

If we consider the relaxation function $g(s)$ instead of the exponential function in (3.7), we have

$$q(x,t) = -\kappa \int_{-\infty}^{t} g(t-s)\nabla\theta(x,s)\ ds. \tag{3.8}$$

Equation (3.8) is known as the Gurtin–Pipkin heat conduction law [45]. We may construct different models by different choices of the heat flux kernel $g(s)$, for example equation (3.7) can be restored from equation (3.8) by considering

$$g(s) = \frac{1}{\tau_0} e^{-s/\tau_0}.$$

Summarizing the main part of the work as: The state of the problem is presented in section 3.2. Section 3.3 is devoted to the energy method in the Fourier space and the construction of the Lyapunov functionals. We ultimately, in Section 3.4, prove the main estimates of the solution in the energy space.

3.2 Preliminaries and Main Results

In this section, we are going to state some preliminaries and assumptions, then state the main results associated with our problem (3.1). Following the approach of Dafermos [33], we introduce the new variable $\eta = \eta(x,t,s)$:

$$\eta(x,t,s) = \int_0^s \theta(x,t-\sigma)d\sigma = \int_{t-s}^{t} \theta(x,\sigma)d\sigma \qquad s \geq 0,\ t \geq 0. \tag{3.9}$$

Differentiating (3.9) with respect to t yields that η satisfies the supplementary equation

$$\eta_t(s) = -\eta_s(s) + \theta(t), \qquad \lim_{s\to 0} \eta(x,s) = 0, \qquad x \in \mathbb{R}^N, \quad \forall\, t \geq 0, \tag{3.10}$$

which has to be added to system (3.1). Then, we define the operator $T\eta = -\eta'$. From (3.10), we get the following equation:

$$\eta_t = T\eta + \theta. \tag{3.11}$$

Also, we define $\mu(s) = -g'(s)$ and assume that μ satisfies the following two assumptions:

(M1) μ is a nonnegative, nonincreasing and absolutely continuous function on $\mathbb{R}^+$ such that:

$$\mu(0) = \lim_{s\to 0} \mu(s) \in (0,\infty).$$

(M2) There exists $\nu > 0$, such that the differential inequality

$$\mu'(s) + \nu\mu(s) \leq 0$$

holds for almost every $s > 0$.

Remark 3.1. *In particular, μ is summable on $\mathbb{R}^+$ with*

$$\int_0^\infty \mu(s)ds = g(0) > 0. \tag{3.12}$$

With all these new variables and without loss of generality, we take $a = 1$, hence, we rewrite system (3.1) as:

$$\begin{aligned} &\tau u_{ttt} + u_{tt} - \Delta u - \beta\Delta u_t + \delta\Delta\theta = 0, \\ &\theta_t - \frac{1}{k}\int_0^\infty \mu(s)\Delta\eta(s)\, ds - \tau\delta\Delta u_{tt} - \delta\Delta u_t = 0, \\ &\eta_t = T\eta + \theta, \end{aligned} \tag{3.13}$$

with the conditions:

$$\begin{cases} (u, u_t, u_{tt}, \theta)(x, 0) = (u_0, u_1, u_2, \theta_0)(x), \\ \lim_{s\to 0} \eta(x, t, s) = 0, \\ \eta(x, 0, s) = \eta_0(x, s) = \int_0^s \theta_0(x, \sigma)\, d\sigma. \end{cases} \tag{3.14}$$

Applying the change of variables $v = u_t$ and $w = u_{tt}$, system (3.13) can be written as

$$\begin{aligned} &u_t - v = 0, \\ &v_t - w = 0, \\ &\tau w_t + w - \Delta u - \beta\Delta v + \delta\Delta\theta = 0, \\ &\theta_t - \frac{1}{k}\int_0^\infty \mu(s)\Delta\eta(s, t)ds - \tau\delta\Delta w - \delta\Delta v = 0, \\ &\eta_t + \eta_s = \theta. \end{aligned} \tag{3.15}$$

Consequently, problem (3.15) with initial data (3.14) is of the form:

$$\begin{cases} \dfrac{d}{dt}U(t) = \mathcal{A}U(t), \qquad t \in (0, +\infty), \\ U(x, 0) = U_0(x) = (u_0, u_1, u_2, \theta_0, \eta_0)^T, \end{cases} \tag{3.16}$$

where $U = (u, v, w, \theta, \eta)^T$ and $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ being a linear operator defined by

$$\mathcal{A}U = \mathcal{A}\begin{pmatrix} u \\ v \\ w \\ \theta \\ \eta \end{pmatrix} = \begin{pmatrix} v \\ w \\ -\dfrac{1}{\tau}\Big(w + \Delta(\beta v + u + \delta\theta)\Big) \\ \dfrac{1}{k}\displaystyle\int_0^\infty \mu(s)\Delta\eta(s)ds + \delta\Delta(v + \tau w) \\ \theta + T\eta \end{pmatrix}. \tag{3.17}$$

Now, we introduce the space $\mathcal{M}$ as:

$$\mathcal{M} = L^2_\mu(\mathbb{R}^+, H^1(\mathbb{R}^N)),$$

endowed with the following norm:

$$\|\bar{\eta}\|_{\mathcal{M}}^2 = \int_0^\infty \mu(s)\|\nabla\bar{\eta}(s)\|_{L^2(\mathbb{R}^N)}^2 ds,$$

for all $\bar{\eta} \in \mathcal{M}$. Then, following [111], we define the Hilbert space as:

$$\mathcal{H} = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \times \mathcal{M},$$

with the following inner product:

$$\begin{aligned}(U,\tilde{U})_{\mathcal{H}} =& \langle (u+\tau v),(\bar{u}+\tau\bar{v})\rangle_1 + \tau(\beta-\tau)\langle v,\bar{v}\rangle_1 + \langle (v+\tau w),(\bar{v}+\tau\bar{w})\rangle_{L^2(\mathbb{R}^N)} \\ &+ \left\langle \theta,\bar{\theta}\right\rangle_{L^2(\mathbb{R}^N)} + \frac{1}{k}\int_0^\infty \mu(s)\langle \nabla\eta(s),\nabla\bar{\eta}(s)\rangle_{L^2(\mathbb{R}^N)} ds,\end{aligned}$$

and the corresponding norm:

$$\|U\|_{\mathcal{H}}^2 = \|u+\tau v\|_1^2 + \tau(\beta-\tau)\|v\|_1^2 + \|v+\tau w\|_{L^2(\mathbb{R}^N)}^2 + \|\theta\|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{k}\|\eta\|_{\mathcal{M}}^2,$$

for all vectors $U = (u,v,w,\theta,\eta)^T$ and $\tilde{U} = (\bar{u},\bar{v},\bar{w},\bar{\theta},\bar{\eta})^T$ in $\mathcal{H}$.
After that, we will give the domain of operator $\mathcal{A}$ as follows:

$$D(\mathcal{A}) = \left\{ U = (u,v,w,\theta,\eta) \in \mathcal{H} \;\middle|\; \begin{array}{c} w,\theta \in H^1(\mathbb{R}^N) \\ \beta v + u - \delta\theta \in H^2(\mathbb{R}^N) \\ \frac{1}{k}\int_0^\infty \mu(s)\eta(s)ds + \tau\delta w + \delta v \in H^2(\mathbb{R}^N) \\ \eta \in D(T) \end{array} \right\}.$$

Define

$$V = (u_t + \tau u_{tt}, \nabla(u+\tau u_t), \theta, \nabla\eta)^T,$$

where, $(u(x,t),\theta(x,t))$ is the solution of (3.13)-(3.14). Now, we state the well-posedness result of system (3.16).

Theorem 3.2. *Suppose that $0 < \tau < \beta$. Let $U_0 \in \mathcal{H}$, then system (3.16) has a unique weak solution $U \in C([0,\infty),\mathcal{H})$. Moreover, if $U_0 \in D(\mathcal{A})$, then $U \in C\big([0,\infty); D(\mathcal{A})\big) \cap C^1\big([0,\infty);\mathcal{H}\big)$.*

For a complete proof, see [111, Theorem 2.1].
Now, we are going to state lemma which we will adapt later in proving the main results.

Lemma 3.3. *Under the assumption $(M1)$, the following inequality holds*

$$\left|\int_0^\infty \mu(s)\hat{\eta}(s,t)ds\right|^2 \le g(0)\int_0^\infty \mu(s)|\hat{\eta}(s,t)|^2 ds. \tag{3.18}$$

Proof. We have by using Hölder's inequality

$$\begin{aligned}\left|\int_0^\infty \mu(s)\hat{\eta}(s,t)ds\right|^2 &= \left|\int_0^\infty (\mu(s))^{\frac{1}{2}}(\mu(s))^{\frac{1}{2}}\hat{\eta}(s,t)\ ds\right|^2 \\ &\le \left|\left(\int_0^\infty \mu(s)ds\right)^{\frac{1}{2}}\left(\int_0^\infty \mu(s)(\hat{\eta}(s,t))^2 ds\right)^{\frac{1}{2}}\right|^2 \\ &= \left(\int_0^\infty \mu(s)ds\right)\int_0^\infty \mu(s)|\hat{\eta}(s,t)|^2 ds \\ &= g(0)\int_0^\infty \mu(s)|\hat{\eta}(s,t)|^2 ds.\end{aligned}$$

This completes the proof of Lemma 3.3. □

We close this section with our main results and we will give their proof in Section 4.

Theorem 3.4 (Sub-critical case). *Suppose that* $0 < \tau < \beta$. *Let* $V(\xi,t) = (u_t + \tau u_{tt}, \nabla(u + \tau u_t), \nabla u_t, \theta)$ *and* $V_0 = V(0,x) \in H^s(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, *where s is a non–negative integer, then for all* $0 \le k + \ell \le s$, *V satisfies the following decay estimate:*

$$||\nabla^k V(t)||_{L^2(\mathbb{R}^N)} \le C(1+t)^{-\frac{N}{4}-\frac{k}{2}}||V_0||_{L^1(\mathbb{R}^N)} + C(1+t)^{-\frac{\ell}{2}}||\nabla^{k+\ell}V_0||_{L^2(\mathbb{R}^N)}, \quad (3.19)$$

where, C is a positive constant independent of V_0 *and t.*

Theorem 3.5 (Critical case). *Suppose that* $\beta = \tau$. *Let* $V_c(\xi,t) = (u_t + \tau u_{tt}, \nabla(u + \tau u_t), \theta)$ *and assume in addition that* $V_c^0 \in H^s(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. *Then, for all* $k + \ell \le s$, *we have:*

$$\|\nabla^k V_c(t)\|_{L^2(\mathbb{R}^N)} \le C(1+t)^{-\frac{N}{4}-\frac{k}{2}}\|V_c^0\|_{L^1(\mathbb{R}^N)} + C(1+t)^{-\frac{\ell}{4}}\|\nabla^{k+\ell}V_c^0\|_{L^2(\mathbb{R}^N)}, \quad (3.20)$$

where, C is a positive constant independent of V_0 *and t.*

Remark 3.6. *As a consequence of the results obtained in this work, we notice that the exponent of the decay rate when the thermal law of Gurtin–Pipkin heat conduction is added to the standard linear solid model is similar to the one obtained when the model is coupled by Fourier or Cattaneo heat conduction. We further notice that it is the same as that obtained in the Cauchy problem without heat conduction, which is of the form* $(1+t)^{-N/4}$, *[98, Theorem 3.6]. The difference lies only in the requirement of a higher regularity of the initial data for the decay to be achieved (see Remark 4.4 in [98])). It should also be noted that the coupling of heat conduction under the Gurtin-Pipkin law to the standard linear solid model leads to the regularity-loss phenomenon in both cases;* $(0 < \tau < \beta)$ *and* $(0 < \tau = \beta)$.

Remark 3.7. *One can observe that, we did not improve the decay rate of the Cauchy problem without thermal conduction even if we added the Gurtin–pipkin law, because, following the mathematical literature, it is obvious that for many mathematical models, the dissipation caused by thermal conduction (Fourier's, Cattaneo's and Pipkin–Gurtin's models) is weaker than the viscoelastic or frictional dissipation, for a clear picture, see for instance [59] and [85], where the authors found similar decay rates obtained for the Timoshenko system coupled to thermal conduction.*

Remark 3.8. *With reference to [95], [96] and [97], it has been proven by using the eigenvalues expansion and energy method in the Fourier space, that the decay rate* $(1+t)^{-N/4}$ *is optimal when* $(0<\tau<\beta)$ *and* $(0<\tau=\beta)$. *Since the Fourier and Cattaneo models are only particular cases of the result of this chapter. Thus, we are able to deduce from the aforementioned works the optimality of our decay rate.*

3.3 The energy method in the Fourier space

Our intention in this section is to achieve some decay estimates of the Fourier image of the energy for system (3.15) in both cases sub-critical, where $0<\tau<\beta$ and critical, where $0<\tau=\beta$. Before going on, we state the pointwise estimates.

First, for the sub-critical case, where $0<\tau<\beta$, we have

Proposition 3.9. *(Pointwise estimate). Let* $(\hat{u},\hat{v},\hat{w},\hat{\theta},\hat{\eta})(\xi,t)$ *be the solution of* (3.25a)-(3.25e). *Assume that (M1) and (M2) hold and* $0<\tau<\beta$. *Then,* $\hat{V}(\xi,t)$ *satisfies the following estimate:*

$$|\hat{V}(\xi,t)|^2 \leq Ce^{-c\rho_f(\xi)t}|\hat{V}_0(\xi)|^2, \tag{3.21}$$

for any $t\geq 0$, *where*

$$\rho_f(\xi)=\frac{|\xi|^2}{1+|\xi|^2+|\xi|^4}. \tag{3.22}$$

Proposition 3.9 is the main key to prove the decay estimates in theorem 3.4. Its proof will be given through several lemmas.
For the critical case $0<\tau=\beta$, we have the following result.

Proposition 3.10. *Let* $\hat{V}_c(\xi,t)$ *be the solution of* (3.58a)-(3.58e). *Assume that (M1), (M2) and* $\beta=\tau$ *hold. Then,* $\hat{V}_c(\xi,t)$ *satisfies the following estimate:*

$$|\hat{V}_c(\xi,t)|^2 \leq C|\hat{V}_c(\xi,0)|^2e^{-c\tilde{\rho}_c(\xi)t}, \tag{3.23}$$

for all $t>0$, *where,*

$$\tilde{\rho}_c(\xi)=\frac{|\xi|^2}{(1+|\xi|^2)(1+|\xi|^2+|\xi|^4)}. \tag{3.24}$$

This estimate is the main ingredient in proving the decay rate stated in Theorem 3.5.

3.3.1 Proof of Proposition 3.9

To attain the decay estimate, we apply the energy method in the Fourier space and then form an appropriate Lyapunov functional, which will lead to our desired estimates.

Applying the Fourier transform to (3.15), we have:

$$\hat{u}_t - \hat{v} = 0, \tag{3.25a}$$
$$\hat{v}_t - \hat{w} = 0, \tag{3.25b}$$
$$\tau\hat{w}_t + \hat{w} + |\xi|^2\hat{u} + \beta|\xi|^2\hat{v} - \delta|\xi|^2\hat{\theta} = 0, \tag{3.25c}$$
$$\hat{\theta}_t + \frac{|\xi|^2}{k}\int_0^\infty \mu(s)\hat{\eta}(s,t)ds + \tau\delta|\xi|^2\hat{w} + \delta|\xi|^2\hat{v} = 0, \tag{3.25d}$$
$$\hat{\eta}_t + \hat{\eta}_s = \hat{\theta}, \tag{3.25e}$$

where the initial data are written in terms of the solution vector $\hat{U}(\xi,t) = (\hat{u},\hat{v},\hat{w},\hat{\theta},\hat{\eta})^T$, as:

$$\begin{cases} \hat{U}(\xi,0) = \left(\hat{u}_0,\hat{v}_0,\hat{w}_0,\hat{\theta}_0\right)(\xi), \\ \hat{\eta}_0(\xi,s) = \int_0^s \hat{\theta}_0(\xi,\sigma)d\sigma. \end{cases} \tag{3.26}$$

The energy functional $\hat{E}(\xi,t)$ associated with the system (3.25) is defined as follows:

$$\hat{E}(\xi,t) = |\hat{v}+\tau\hat{w}|^2 + \tau(\beta-\tau)|\hat{v}|^2 + |\xi|^2|\hat{u}+\tau\hat{v}|^2 + |\hat{\theta}|^2 + \frac{|\xi|^2}{k}\int_0^\infty \mu(s)|\hat{\eta}(\xi,t,s)|^2ds. \tag{3.27}$$

Lemma 3.11. *Assume that $0 < \tau \leq \beta$. Let $(\hat{u},\hat{v},\hat{w},\hat{\theta},\hat{\eta})$ be the solution of (3.25), then the energy $\hat{E}(\xi,t)$ given by (3.27) is a nonincreasing function and satisfies, for all $t \geq 0$,*

$$\frac{d}{dt}\hat{E}(\xi,t) = -(\beta-\tau)|\xi|^2|\hat{v}|^2 + \frac{|\xi|^2}{2k}\int_0^\infty \mu'(s)|\hat{\eta}(\xi,t,s)|^2ds. \tag{3.28}$$

Proof. The proof of this Lemma is given in [111] . □

Define the functional $\mathcal{B}_1(\xi,t)$ as in [95]

$$\mathcal{B}_1(\xi,t) = \operatorname{Re}\left((\hat{v}+\tau\hat{w})(\bar{\hat{u}}+\tau\bar{\hat{v}})\right). \tag{3.29}$$

Lemma 3.12. *The functional $\mathcal{B}_1(\xi,t)$ satisfies*

$$\begin{aligned}\frac{d}{dt}\mathcal{B}_1(\xi,t) + |\xi|^2|\hat{u}+\tau\hat{v}|^2 - |\hat{v}+\tau\hat{w}|^2 =& |\xi|^2(\tau-\beta)\operatorname{Re}(\hat{v}(\bar{\hat{u}}+\tau\bar{\hat{v}})) \\ &+ \delta|\xi|^2\operatorname{Re}\left(\hat{\theta}(\bar{\hat{u}}+\tau\bar{\hat{v}})\right).\end{aligned} \tag{3.30}$$

Proof. Adding equations (3.25b) and (3.25c), we have:

$$\frac{d}{dt}(\hat{v}+\tau\hat{w}) = -|\xi|^2\hat{u} - \beta|\xi|^2\hat{v} + \delta|\xi|^2\hat{\theta}. \tag{3.31}$$

Multiplying the above equation (3.31) by $\bar{\hat{u}} + \tau\bar{\hat{v}}$, we have:

$$(\bar{\hat{u}} + \tau\bar{\hat{v}})\frac{d}{dt}(\hat{v} + \tau\hat{w}) = (-|\xi|^2\hat{u} - \beta|\xi|^2\hat{v} - \tau|\xi|^2\hat{v} + \tau|\xi|^2\hat{v} + \delta|\xi|^2\hat{\theta})(\bar{\hat{u}} + \tau\bar{\hat{v}}). \tag{3.32}$$

Adding equation (3.25a) to τ(3.25b), we have:

$$\frac{d}{dt}(\hat{u} + \tau\hat{v}) = \hat{v} + \tau\hat{w}. \tag{3.33}$$

Multiplying the above equation (3.33) by $\bar{\hat{v}} + \tau\bar{\hat{w}}$, we have

$$(\bar{\hat{v}} + \tau\bar{\hat{w}})\frac{d}{dt}(\hat{u} + \tau\hat{v}) = (\bar{\hat{v}} + \tau\bar{\hat{w}})(\hat{v} + \tau\hat{w}). \tag{3.34}$$

Adding equations (3.32) and (3.34) and taking the real part, we obtain (3.30). This completes the proof of Lemma 3.12. □

Define the functional $\mathcal{B}_2(\xi, t)$ as in [95]

$$\mathcal{B}_2(\xi, t) = \mathrm{Re}(-\tau\bar{\hat{v}}(\hat{v} + \tau\hat{w})).$$

Lemma 3.13. *The functional $\mathcal{B}_2(\xi, t)$ satisfies*

$$\frac{d}{dt}\mathcal{B}_2(\xi, t) + (1-\epsilon_2'')|\hat{v}+\tau\hat{w}|^2 \leq C(\epsilon_2, \epsilon_2', \epsilon_2'')(1+|\xi|^2+|\xi|^4)|\hat{v}|^2 + \epsilon_2|\xi|^2|\hat{u}+\tau\hat{v}|^2 + \epsilon_2'|\hat{\theta}|^2, \tag{3.35}$$

where, ϵ_2, ϵ_2' and ϵ_2'' are arbitrary positive constants.

Proof. Multiplying equation (3.25b) by $-\tau(\bar{\hat{v}} + \tau\bar{\hat{w}})$, we have:

$$-\tau(\bar{\hat{v}} + \tau\bar{\hat{w}})\frac{d}{dt}\hat{v} = -\tau\hat{w}(\bar{\hat{v}} + \tau\bar{\hat{w}}). \tag{3.36}$$

Next, multiplying equation (3.31) by $-\tau\bar{\hat{v}}$, we get:

$$-\tau\bar{\hat{v}}\frac{d}{dt}(\hat{v}+\tau\hat{w}) = \bar{\hat{v}}\Big(\tau|\xi|^2\hat{u} + \beta\tau|\xi|^2\hat{v} + \tau^2|\xi|^2\hat{v} - \tau^2|\xi|^2\hat{v} - \delta\tau|\xi|^2\hat{\theta} + (\hat{v}+\tau\hat{w}) - (\hat{v}+\tau\hat{w})\Big). \tag{3.37}$$

Adding the above two equations then taking the real part, we get:

$$\begin{aligned}\frac{d}{dt}\mathcal{B}_2(\xi, t) + |\hat{v} + \tau\hat{w}|^2 - \tau(\beta - \tau)|\xi|^2|\hat{v}|^2 =& \tau|\xi|^2\,\mathrm{Re}\left(\bar{\hat{v}}(\hat{u} + \tau\hat{v})\right) + \mathrm{Re}\left(\bar{\hat{v}}(\hat{v} + \tau\hat{w})\right) \\ &- \delta\tau|\xi|^2\,\mathrm{Re}(\bar{\hat{v}}\theta).\end{aligned} \tag{3.38}$$

Applying Young's inequality, we have, for any $\epsilon_2, \epsilon_2', \epsilon_2'' > 0$,

$$\begin{aligned}|\tau|\xi|^2\,\mathrm{Re}(\bar{\hat{v}}(\hat{u} + \tau\hat{v}))| &\leq \epsilon_2|\xi|^2|\hat{u} + \tau\hat{v}|^2 + C(\epsilon_2)|\xi|^2|\hat{v}|^2, \\ |\,\mathrm{Re}(\bar{\hat{v}}(\hat{v} + \tau\hat{w}))| &\leq \epsilon_2''|\hat{v} + \tau\hat{w}|^2 + C(\epsilon_2'')|\hat{v}|^2, \\ |\delta\tau|\xi|^2\,\mathrm{Re}(\hat{\theta}\bar{\hat{v}})| &\leq \epsilon_2'|\hat{\theta}|^2 + C(\epsilon_2')|\xi|^4|\hat{v}|^2.\end{aligned}$$

Plugging the above estimates into (3.38), then (3.35) is fulfilled. Thus, the proof of Lemma 3.13 is finished. □

Define the functional $\mathcal{B}_3(\xi,t)$.

$$\mathcal{B}_3(\xi,t) = \operatorname{Re}\left(-\delta|\xi|^4(\bar{\hat{u}}+\tau\bar{\hat{v}})\int_0^\infty \mu(s)\hat{\eta}(s,t)ds - |\xi|^2\hat{\theta}\int_0^\infty \mu(s)\bar{\hat{\eta}}(s,t)\ ds \right). \tag{3.39}$$

Lemma 3.14. *The functional $\mathcal{B}_3(\xi,t)$ satisfies*

$$\begin{aligned}\frac{d}{dt}\mathcal{B}_3(\xi,t) + g(0)|\xi|^2|\hat{\theta}|^2 = & \frac{|\xi|^4}{k}\operatorname{Re}\left(\int_0^\infty \mu(s)\hat{\eta}(s,t)\ ds\right)^2 \\ & - \operatorname{Re}\left(|\xi|^2\bar{\hat{\theta}}\int_0^\infty \mu'(s)\hat{\eta}(s,t)\ ds\right) \\ & - \operatorname{Re}\left(\delta|\xi|^4(\bar{\hat{u}}+\tau\bar{\hat{v}})\int_0^\infty \mu'(s)\hat{\eta}(s,t)ds\right) \\ & - g(0)\delta|\xi|^4\operatorname{Re}\left(\hat{\theta}(\bar{\hat{u}}+\tau\bar{\hat{v}})\right).\end{aligned} \tag{3.40}$$

Proof. Multiplying equation (3.25d) by $-|\xi|^2\mu(s)\bar{\hat{\eta}}(s,t)$ and (3.25e) by $-|\xi|^2\mu(s)\bar{\hat{\theta}}$, adding the results and making the integration with respect to s and then taking the real part, we have:

$$\begin{aligned}&-\frac{d}{dt}\operatorname{Re}\left(|\xi|^2\hat{\theta}\int_0^\infty \mu(s)\bar{\hat{\eta}}(s,t)\,ds\right) + \int_0^\infty |\xi|^2\mu(s)|\hat{\theta}|^2\ ds \\ &= \frac{|\xi|^4}{k}\operatorname{Re}\left(\int_0^\infty \mu(s)\hat{\eta}(s,t)\ ds\right)^2 + \operatorname{Re}\left(|\xi|^2\bar{\hat{\theta}}\int_0^\infty \mu(s)\hat{\eta}_s(s,t)\ ds\right) \\ &\quad + \delta|\xi|^4\operatorname{Re}\left((\hat{v}+\tau\hat{w})\int_0^\infty \mu(s)\bar{\hat{\eta}}(s,t)\ ds\right).\end{aligned} \tag{3.41}$$

While, an integration by parts leads to:

$$\operatorname{Re}\left(|\xi|^2\bar{\hat{\theta}}\int_0^\infty \mu(s)\hat{\eta}_s(s,t)ds\right) = -\operatorname{Re}\left(|\xi|^2\bar{\hat{\theta}}\int_0^\infty \mu'(s)\hat{\eta}(s,t)ds\right).$$

Hence, (3.41) can be written as:

$$\begin{aligned}&-\frac{d}{dt}\operatorname{Re}\left(|\xi|^2\hat{\theta}\int_0^\infty \mu(s)\bar{\hat{\eta}}(s,t)\ ds\right) + g(0)|\xi|^2|\hat{\theta}|^2 \\ &= \frac{|\xi|^4}{k}\operatorname{Re}\left(\int_0^\infty \mu(s)\hat{\eta}(s,t)\ ds\right)^2 - \operatorname{Re}\left(|\xi|^2\bar{\hat{\theta}}\int_0^\infty \mu'(s)\hat{\eta}(s,t)\ ds\right) \\ &\quad + \delta|\xi|^4\operatorname{Re}\left((\hat{v}+\tau\hat{w})\int_0^\infty \mu(s)\bar{\hat{\eta}}(s,t)\ ds\right).\end{aligned} \tag{3.42}$$

Next, multiplying equation (3.33) by $-|\xi|^2\mu(s)\bar{\hat{\eta}}(s,t)$ and equation (3.25e) by $-|\xi|^2\mu(s)(\bar{\hat{u}}+$

$\tau\bar{\hat{v}}$), adding the results and making the integration with respect to s, then taking the real parts, we obtain:

$$
\begin{aligned}
-\frac{d}{dt}\operatorname{Re}&\left(|\xi|^2(\bar{\hat{u}}+\tau\bar{\hat{v}})\int_0^\infty \mu(s)\hat{\eta}(s,t)ds\right)\\
&= -\operatorname{Re}\left(|\xi|^2(\hat{v}+\tau\hat{w})\int_0^\infty \mu(s)\bar{\hat{\eta}}(s,t)ds\right)\\
&\quad+\operatorname{Re}\left(|\xi|^2(\bar{\hat{u}}+\tau\bar{\hat{v}})\int_0^\infty \mu(s)\hat{\eta}_s(s,t)ds\right)\\
&\quad-\operatorname{Re}\left(|\xi|^2\hat{\theta}\int_0^\infty \mu(s)(\bar{\hat{u}}+\tau\bar{\hat{v}})ds\right).
\end{aligned}
\tag{3.43}
$$

The last term of the above identity (3.43) can be written as

$$
\operatorname{Re}\left(|\xi|^2\hat{\theta}\int_0^\infty \mu(s)(\bar{\hat{u}}+\tau\bar{\hat{v}})ds\right) = g(0)|\xi|^2\operatorname{Re}\left(\hat{\theta}(\bar{\hat{u}}+\tau\bar{\hat{v}})\right).
$$

Similarly, as above, an integration by parts leads to:

$$
\operatorname{Re}\left(|\xi|^2(\bar{\hat{u}}+\tau\bar{\hat{v}})\int_0^\infty \mu(s)\hat{\eta}_s(s,t)ds\right) = -\operatorname{Re}\left(|\xi|^2(\bar{\hat{u}}+\tau\bar{\hat{v}})\int_0^\infty \mu'(s)\hat{\eta}(s,t)ds\right).
$$

Hence, (3.43) can be written

$$
\begin{aligned}
-\frac{d}{dt}\operatorname{Re}&\left(|\xi|^2(\bar{\hat{u}}+\tau\bar{\hat{v}})\int_0^\infty \mu(s)\hat{\eta}(s,t)ds\right)\\
&= -\operatorname{Re}\left(|\xi|^2(\hat{v}+\tau\hat{w})\int_0^\infty \mu(s)\bar{\hat{\eta}}(s,t)ds\right)\\
&\quad-\operatorname{Re}\left(|\xi|^2(\bar{\hat{u}}+\tau\bar{\hat{v}})\int_0^\infty \mu'(s)\hat{\eta}(s,t)ds\right)-g(0)|\xi|^2\operatorname{Re}\left(\hat{\theta}(\bar{\hat{u}}+\tau\bar{\hat{v}})\right).
\end{aligned}
\tag{3.44}
$$

Now, computing (3.42)+$\delta|\xi|^2$(3.44), we have (3.40). This completes the proof of Lemma 3.14.

Next, define the functional:

$$
\mathcal{B}_4(\xi,t) = g(0)|\xi|^2\mathcal{B}_1(\xi,t)+\mathcal{B}_3(\xi,t).
$$

Then, we have from (3.30) and (3.40):

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{B}_4(\xi,t)+g(0)|\xi|^4|\hat{u}+\tau\hat{v}|^2-g(0)|\xi|^2|\hat{v}+\tau\hat{w}|^2+g(0)|\xi|^2|\hat{\theta}|^2\\
=&g(0)|\xi|^4(\tau-\beta)\operatorname{Re}(\hat{v}(\bar{\hat{u}}+\tau\bar{\hat{v}}))+\frac{|\xi|^4}{k}\operatorname{Re}\left(\int_0^\infty \mu(s)\hat{\eta}(s,t)\;ds\right)^2\\
&-|\xi|^2\operatorname{Re}\left(\bar{\hat{\theta}}\int_0^\infty \mu'(s)\hat{\eta}(s,t)\;ds\right)\\
&-\delta|\xi|^4\operatorname{Re}\left((\bar{\hat{u}}+\tau\bar{\hat{v}})\int_0^\infty \mu'(s)\hat{\eta}(s,t)ds\right).
\end{aligned}
\tag{3.45}
$$

Applying Young's inequality for $\epsilon_1,\epsilon_1'>0$, we have:

$$
\left|g(0)|\xi|^4(\tau-\beta)\operatorname{Re}(\hat{v}(\bar{\hat{u}}+\tau\bar{\hat{v}}))\right|\le\frac{\epsilon_1}{2}|\xi|^4|\hat{u}+\tau\hat{v}|^2+C(\epsilon_1)|\xi|^4|\hat{v}|^2
$$

$$
\left|\operatorname{Re}\left(|\xi|^2\bar{\hat{\theta}}\int_0^\infty \mu'(s)\bar{\hat{\eta}}(s,t)ds\right)\right|\le\epsilon_1'|\xi|^2|\hat{\theta}|^2+C(\epsilon_1')g'(0)\int_0^\infty|\xi|^2\mu'(s)|\hat{\eta}(s,t)|^2ds,
$$

$$
\begin{aligned}
\left|\delta|\xi|^4\operatorname{Re}\left((\bar{\hat{u}}+\tau\bar{\hat{v}})\int_0^\infty \mu'(s)\hat{\eta}(s,t)ds\right)\right|\le&\frac{\epsilon_1}{2}|\xi|^4|\hat{u}+\tau\hat{v}|^2\\
&+C(\epsilon_1)g'(0)|\xi|^2\int_0^\infty|\xi|^2\mu'(s)|\hat{\eta}(s,t)|^2ds,
\end{aligned}
$$

$$
\left|-\frac{|\xi|^4}{k}\operatorname{Re}\left(\int_0^\infty \mu(s)\hat{\eta}(s,t)ds\right)^2\right|\le\frac{1}{k}\int_0^\infty|\xi|^4\mu(s)|\hat{\eta}(s,t)|^2ds.
$$

Substituting the above estimates into (3.45), we get:

$$
\begin{aligned}
\frac{d}{dt}\mathcal{B}_4(\xi,t)&+(g(0)-\epsilon_1)|\xi|^4|\hat{u}+\tau\hat{v}|^2+(g(0)-\epsilon_1')|\xi|^2|\hat{\theta}|^2\\
\le&|\xi|^2g(0)|\hat{v}+\tau\hat{w}|^2+C(\epsilon_1)|\xi|^4|\hat{v}|^2\\
&+C(\epsilon_1,\epsilon_1')g'(0)(1+|\xi|^2)\int_0^\infty|\xi|^2\mu'(s)|\hat{\eta}(s,t)|^2ds\\
&+\frac{|\xi|^2}{k}\int_0^\infty|\xi|^2\mu(s)|\hat{\eta}(s,t)|^2ds.
\end{aligned}
\tag{3.46}
$$

□

Proof of Proposition 3.9. Using the above Lemmas, we are ready to prove Proposition 3.9. We, first, define the Lyapunov functional

$$
L_f(\xi,t)=|\xi|^2\mathcal{B}_2(\xi,t)+\lambda\mathcal{B}_4(\xi,t),
\tag{3.47}
$$

where, λ is a positive constant to be fixed later. Taking the derivative of $L_f(\xi, t)$ with respect to t and using inequalities (3.35) and (3.46), we have:

$$\begin{aligned}
&\frac{d}{dt}L_f(\xi,t) + |\xi|^2\Big[\lambda(g(0)-\epsilon_1)-\epsilon_2\Big]|\xi|^2|\hat{u}+\tau\hat{v}|^2 \\
&+|\xi|^2\Big[(1-\epsilon_2'')-\lambda g(0)\Big]|\hat{v}+\tau\hat{w}|^2 + |\xi|^2\Big[\lambda(g(0)-\epsilon_1')-\epsilon_2'\Big]|\hat{\theta}|^2 \\
&+(1+|\xi|^2+|\xi|^4)\Big[-C(\epsilon_2,\epsilon_2',\epsilon_2'')-\lambda C(\epsilon_1)\Big]|\xi|^2|\hat{v}|^2 \\
&-\lambda\frac{|\xi|^2}{k}\int_0^\infty |\xi|^2\mu(s)|\hat{\eta}(s,t)|^2\,ds \\
&+\lambda C(\epsilon_1,\epsilon_1')g'(0)(1+|\xi|^2)\int_0^\infty |\xi|^2(-\mu'(s))|\hat{\eta}(s,t)|^2\,ds \le 0.
\end{aligned} \tag{3.48}$$

Now, using the assumption $(M2)$, we may write:

$$\int_0^\infty |\xi|^2\mu(s)|\hat{\eta}(s,t)|^2\,ds \le \frac{1}{\nu}\int_0^\infty |\xi|^2(-\mu'(s))|\hat{\eta}(s,t)|^2\,ds. \tag{3.49}$$

Therefore, we get:

$$\begin{aligned}
&\frac{d}{dt}L_f(\xi,t) + |\xi|^2\Big[\lambda\big(g(0)-\epsilon_1\big)-\epsilon_2\Big]|\xi|^2|\hat{u}+\tau\hat{v}|^2 \\
&+|\xi|^2\Big[(1-\epsilon_2'')-\lambda g(0)\Big]|\hat{v}+\tau\hat{w}|^2 \\
&+|\xi|^2\Big[\lambda(g(0)-\epsilon_1')-\epsilon_2'\Big]|\hat{\theta}|^2 \\
&-(1+|\xi|^2+|\xi|^4)\Big[C(\epsilon_2,\epsilon_2',\epsilon_2'')+\lambda C(\epsilon_1)\Big]|\xi|^2|\hat{v}|^2 \\
&-C(\lambda,\epsilon_1,\epsilon_1',\nu)(1+|\xi|^2)\int_0^\infty |\xi|^2(-\mu'(s))|\hat{\eta}(s,t)|^2\,ds \le 0.
\end{aligned} \tag{3.50}$$

Now, we choose the constants in (3.50) very carefully in order to make all coefficients (except the last one) in (3.50) positive. Let us fix ϵ_1, ϵ_1' and ϵ_2'' small enough, such that:

$$\epsilon_1 < g(0), \quad \epsilon_1' < g(0), \quad \epsilon_2'' < 1.$$

After that, owing to (3.12), we choose λ small enough, such that:

$$\lambda < \frac{1-\epsilon_2''}{g(0)}.$$

Next, choose ϵ_2 and ϵ_2' small, such that:

$$\epsilon_2 < \lambda(g(0)-\epsilon_1), \qquad \epsilon_2' < \lambda(g(0)-\epsilon_1').$$

Consequently, we deduce that there exists a positive constant $\eta_1 > 0$, such that:

$$
\begin{aligned}
&\frac{d}{dt} L_f(\xi,t) + \eta_1 Q(\xi,t) \\
\leq &(1+|\xi|^2+|\xi|^4)\big[C(\lambda,\epsilon_1,\epsilon_2,\epsilon_2',\epsilon_2'')\big]|\xi|^2|\hat{v}|^2 \\
&+ C(\lambda,\epsilon_1,\epsilon_1',\nu)(1+|\xi|^2)\int_0^\infty |\xi|^2(-\mu'(s))|\hat{\eta}(s,t)|^2\, ds, \\
\leq &(1+|\xi|^2+|\xi|^4)\big[C(\lambda,\epsilon_1,\epsilon_2,\epsilon_2',\epsilon_2'')\big]|\xi|^2|\hat{v}|^2 \\
&+ C(\lambda,\epsilon_1,\epsilon_1',\nu)(1+|\xi|^2+|\xi|^4)\int_0^\infty |\xi|^2(-\mu'(s))|\hat{\eta}(s,t)|^2\, ds,
\end{aligned}
\tag{3.51}
$$

where,

$$
Q(\xi,t) = |\xi|^2\Bigg(|\xi|^2|\hat{u}+\tau\hat{v}|^2 + |\hat{v}+\tau\hat{w}|^2 + |\hat{\theta}|^2\Bigg).
$$

Now, we define the functional

$$
\mathcal{L}_f(\xi,t) = M(1+|\xi|^2+|\xi|^4)\hat{E}(\xi,t) + L(\xi,t), \tag{3.52}
$$

where, M is a positive constant that will be chosen later. Using (3.28) and (3.51), the functional $\mathcal{L}(\xi,t)$ satisfies the estimate:

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{L}_f(\xi,t) + \eta_1 Q(\xi,t) \\
&+ (1+|\xi|^2+|\xi|^4)\Big[M(\beta-\tau) - C(\lambda,\epsilon_1,\epsilon_2,\epsilon_2',\epsilon_2'')\Big]|\xi|^2|\hat{v}|^2 \\
&+ (1+|\xi|^2+|\xi|^4)\left[\frac{M}{2k} - C(\lambda,\epsilon_1,\epsilon_1',\nu)\right]\int_0^\infty |\xi|^2(-\mu'(s))|\hat{\eta}(s,t)|^2\, ds \leq 0.
\end{aligned}
\tag{3.53}
$$

Now, choose M large enough, such that:

$$
M > \max\left\{\frac{C(\lambda,\epsilon_1,\epsilon_2,\epsilon_2',\epsilon_2'')}{\beta-\tau}, 2kC(\lambda,\epsilon_1,\epsilon_1',\nu)\right\}.
$$

Finally, (3.53) can be written as:

$$
\frac{d}{dt}\mathcal{L}_f(\xi,t) + \eta_2|\xi|^2\hat{E}(\xi,t) \leq 0, \quad \forall t \geq 0. \tag{3.54}
$$

Now, using (3.47), (3.52) and (3.54), together with the definitions of all functionals involved in (3.47) for all $\xi \in \mathbb{R}^N$, we have:

$$
|L_f(\xi,t)| = |\mathcal{L}_f(\xi,t) - M(1+|\xi|^2+|\xi|^4)\hat{E}(\xi,t)| \leq \eta_3\hat{E}(\xi,t).
$$

Hence, we conclude that:

$$(M-\eta_3)(1+|\xi|^2+|\xi|^4)\hat{E}(\xi,t) \le \mathcal{L}_f(\xi,t) \le (M+\eta_3)(1+|\xi|^2+|\xi|^2)\hat{E}(\xi,t).$$

For M large enough, we deduce that there exist two positive constants C_1 and C_2, such that:

$$C_1(1+|\xi|^2+|\xi|^4)\hat{E}(\xi,t) \le \mathcal{L}_f(\xi,t) \le C_2(1+|\xi|^2+|\xi|^4)\hat{E}(\xi,t), \qquad \forall t \ge 0. \tag{3.55}$$

Combining (3.54) and (3.55), we have:

$$\frac{d}{dt}\mathcal{L}_f(\xi,t) \le -\frac{\eta_2}{C_2}\frac{|\xi|^2}{1+|\xi|^2+|\xi|^4}\mathcal{L}(\xi,t), \qquad \forall t \ge 0. \tag{3.56}$$

Applying Gronwall's Lemma, using (3.54) and making use of the equivalence of $\hat{E}(\xi,t)$ and $|\hat{V}(\xi,t)|^2$ then (3.21) holds. This completes the proof of Proposition 3.9. □

3.3.2 Proof of Proposition 3.10

Substituting $\beta=\tau$ in (3.13), we have:

$$\begin{cases} \tau u_{ttt}+u_{tt}-\Delta u-\tau\Delta u_t+\delta\Delta\theta=0, \\ \theta_t-\frac{1}{k}\int_0^\infty \mu(s)\Delta\eta(s)ds-\tau\delta\Delta u_{tt}-\delta\Delta u_t=0, \\ \eta_t=T\eta+\theta, \end{cases} \tag{3.57}$$

Now, equations (3.25) are given as follows:

$$\hat{u}_t-\hat{v}=0, \tag{3.58a}$$

$$\hat{v}_t-\hat{w}=0, \tag{3.58b}$$

$$\tau\hat{w}_t+\hat{w}+|\xi|^2\hat{u}+\tau|\xi|^2\hat{v}-\delta|\xi|^2\hat{\theta}=0, \tag{3.58c}$$

$$\hat{\theta}_t+\frac{|\xi|^2}{k}\int_0^\infty \mu(s)\hat{\eta}(s,t)ds+\tau\delta|\xi|^2\hat{w}+\delta|\xi|^2\hat{v}=0, \tag{3.58d}$$

$$\hat{\eta}_t+\hat{\eta}_s=\hat{\theta}. \tag{3.58e}$$

We redefine the energy functional (3.27) as follows:

$$\hat{E}_c(\xi,t)=\frac{1}{2}\left[|\xi|^2|\hat{u}+\tau\hat{v}|^2+|\hat{v}+\tau\hat{w}|^2+|\hat{\theta}|^2+\frac{\xi^2}{k}\int_0^\infty \mu(s)|\hat{\eta}(\xi,t,s)|^2ds\right], \tag{3.59}$$

for all $\xi\in\mathbb{R}^N$ and all $t\ge 0$ and it is equivalent to the vector $|\hat{V}_c(\xi,t)|^2$, so that, there exist two positive constants γ_2,γ_3, such that, for all $\xi\in\mathbb{R}^N$ and all $t\ge 0$,

$$\gamma_2|\hat{V}_c(\xi,t)|^2 \le \hat{E}_c(\xi,t) \le \gamma_3|\hat{V}_c(\xi,t)|^2, \tag{3.60}$$

where,

$$|\hat{V}_c(\xi,t)|^2=|\xi|^2|\hat{u}+\tau\hat{v}|^2+|\hat{v}+\tau\hat{w}|^2+|\hat{\theta}|^2+\xi^2\int_0^\infty \mu(s)|\hat{\eta}(s)|^2ds.$$

Lemma 3.15. *Assume that $0 < \tau = \beta$. Let $(\hat{u}, \hat{v}, \hat{w}, \hat{\theta}, \hat{\eta})$ be the solution of (3.58), then the energy $\hat{E}_c(\xi, t)$ given by (3.59) is a nonincreasing function and satisfies, for all $t \geq 0$,*

$$\frac{d}{dt}\hat{E}_c(\xi,t) = \frac{|\xi|^2}{2k}\int_0^\infty \mu'(s)|\hat{\eta}(\xi,t,s)|^2 ds. \tag{3.61}$$

As we did before, we are going to prove Proposition 3.10 by modifying the Lemmas above as needed. The functional $\mathcal{B}_1(\xi, t)$ satisfies

$$\frac{d}{dt}\mathcal{B}_1(\xi,t) + |\xi|^2|\hat{u}+\tau\hat{v}|^2 - |\hat{v}+\tau\hat{w}|^2 = \delta|\xi|^2 \operatorname{Re}\left(\hat{\theta}(\bar{\hat{u}}+\tau\bar{\hat{v}})\right). \tag{3.62}$$

Therefore, (3.62) takes the form:

$$\frac{d}{dt}\mathcal{B}_1(\xi,t) + (1-\epsilon_0)|\xi|^2(1+\xi^2)|\hat{u}+\tau\hat{v}|^2 \leq |\hat{v}+\tau\hat{w}|^2 + C(\epsilon_0)|\hat{\theta}|^2, \tag{3.63}$$

for any $\epsilon_0 > 0$. Define the functional $\mathcal{B}_5(\xi, t)$.

$$\mathcal{B}_5(\xi,t) = \operatorname{Re}\left(\bar{\hat{\theta}}(\hat{v}+\tau\hat{w})\right).$$

Lemma 3.16. *The functional $\mathcal{B}_5(\xi, t)$ satisfies*

$$\begin{aligned}
&\frac{d}{dt}\mathcal{B}_5(\xi,t) + \delta|\xi|^2|\hat{v}+\tau\hat{w}|^2 \\
&= -|\xi|^2 \operatorname{Re}\left((\hat{u}+\tau\hat{v})\bar{\hat{\theta}}\right) - |\xi|^2(\beta-\tau)\operatorname{Re}(\hat{v}\bar{\hat{\theta}}) + \delta|\xi|^2|\hat{\theta}|^2 \\
&\quad - \frac{|\xi|^2}{k}\operatorname{Re}\left((\bar{\hat{v}}+\tau\bar{\hat{w}})\int_0^\infty \mu(s)\hat{\eta}(s,t)\, ds\right).
\end{aligned} \tag{3.64}$$

Proof. Write equation (3.31) as follows:

$$\frac{d}{dt}(\hat{v}+\tau\hat{w}) + |\xi|^2\hat{u} + \tau|\xi|^2\hat{v} - \tau|\xi|^2\hat{v} + \beta|\xi|^2\hat{v} - \delta|\xi|^2\hat{\theta} = 0. \tag{3.65}$$

Multiplying equation (3.65) by $\bar{\hat{\theta}}$ and equation (3.58d) by $\bar{\hat{v}} + \tau\bar{\hat{w}}$, adding the results and taking the real parts, we obtain (3.64), which finishes the proof of Lemma 3.16. ☐

Substituting $\beta = \tau$ in (3.64) then applying Young's inequality for any $\tilde{\epsilon}_0, \tilde{\epsilon}_1 > 0$, we obtain

$$\begin{aligned}
&\left|\frac{|\xi|^2}{k}\operatorname{Re}\left((\bar{\hat{v}}+\tau\bar{\hat{w}})\int_0^\infty \mu(s)\hat{\eta}(s,t)\, ds\right)\right| \\
&\qquad \leq \tilde{\epsilon}_0|\hat{v}+\tau\hat{w}|^2 + C(\tilde{\epsilon}_0)|\xi|^4\int_0^\infty \mu(s)|\hat{\eta}(s,t)|^2\, ds
\end{aligned}$$

and

$$\left||\xi|^2 \operatorname{Re}\left((\hat{u}+\tau\hat{v})\bar{\hat{\theta}}\right)\right| \leq \tilde{\epsilon}_1|\xi|^4|\hat{u}+\tau\hat{v}|^2 + C(\tilde{\epsilon}_1)|\hat{\theta}|^2.$$

Now, collecting the similar terms, we have:

$$\begin{aligned}\frac{d}{dt}\mathcal{B}_5(\xi,t)+(\delta-\tilde{\epsilon}_0)(1+|\xi|^2)|\hat{v}+\tau\hat{w}|^2 \leq & C(\tilde{\epsilon}_0)|\xi|^4\int_0^\infty \mu(s)|\hat{\eta}(s,t)|^2\,ds \\ & +\tilde{\epsilon}_1|\xi|^4|\hat{u}+\tau\hat{v}|^2 \\ & +\Big(C(\tilde{\epsilon}_1)+\delta\Big)(1+|\xi|^2)|\hat{\theta}|^2,\end{aligned} \tag{3.66}$$

where, $\tilde{\epsilon}_0$ and $\tilde{\epsilon}_1$ are arbitrary positive constants. Recall equation (3.42) and define the functional

$$\mathcal{B}_6(\xi,t)=-\operatorname{Re}\left(\hat{\theta}\int_0^\infty \mu(s)\bar{\hat{\eta}}(s,t)\,ds\right).$$

Accordingly,

$$\begin{aligned}\frac{d}{dt}\mathcal{B}_6(\xi,t)+g(0)|\hat{\theta}|^2 = & \frac{|\xi|^2}{k}\operatorname{Re}\left(\int_0^\infty \mu(s)\hat{\eta}(s,t)\,ds\right)^2 \\ & -\operatorname{Re}\left(\bar{\hat{\theta}}\int_0^\infty \mu'(s)\hat{\eta}(s,t)\,ds\right) \\ & +\delta|\xi|^2\operatorname{Re}\left((\hat{v}+\tau\hat{w})\int_0^\infty \mu(s)\bar{\hat{\eta}}(s,t)\,ds\right).\end{aligned} \tag{3.67}$$

Applying Young's inequality for $\epsilon_4,\epsilon_5>0$, together with the assumption (M2), we get

$$\begin{aligned}&\frac{d}{dt}\mathcal{B}_6(\xi,t)+(g(0)-\epsilon_5)|\hat{\theta}|^2-\epsilon_4|\hat{v}+\tau\hat{w}|^2 \\ &\quad\leq\left(C(\epsilon_5)+\frac{C(\epsilon_4)}{\nu}+\frac{g(0)}{\nu k}\right)(1+|\xi|^2+|\xi|^4)\int_0^\infty -\mu'(s)|\hat{\eta}(s)|^2 ds.\end{aligned} \tag{3.68}$$

Next, define the functional

$$\mathcal{B}_7(\xi,t)=\frac{\delta}{2}\mathcal{B}_1(\xi,t)+\mathcal{B}_5(\xi,t).$$

Then, by using (3.63) and (3.66), we obtain:

$$\begin{aligned}\frac{d}{dt}\mathcal{B}_7(\xi,t)+&\left(\frac{\delta}{2}(1-\epsilon_0)-\tilde{\epsilon}_1\right)\xi^2(1+\xi^2)|\hat{u}+\tau\hat{v}|^2+\left(\frac{\delta}{2}-\tilde{\epsilon}_0\right)(1+\xi^2)|\hat{v}+\tau\hat{w}|^2 \\ &\leq(\delta+C(\epsilon_0,\tilde{\epsilon}_1))(1+\xi^2)|\hat{\theta}|^2+C(\tilde{\epsilon}_0)\xi^4\int_0^\infty \mu(s)|\hat{\eta}^t(s)|^2.\end{aligned} \tag{3.69}$$

We are now ready to show the demonstration of Proposition 3.10. We define the new Lyapunov functional $\hat{L}_c(\xi,t)$ associated with the critical case, $\beta=\tau$, as follows:

$$\hat{L}_c(\xi,t)=N(1+\xi^2+\xi^4)\hat{E}_c(\xi,t)+M\frac{\xi^2}{1+\xi^2}\mathcal{B}_6(\xi,t)+\frac{\xi^2}{(1+\xi^2)^2}\mathcal{B}_7(\xi,t), \tag{3.70}$$

for some positive constants N and M that have to be chosen later. Taking the derivative of

(3.70) with respect to t and using (3.61), (3.68) and (3.69), we get:

$$\begin{aligned}
&\frac{d}{dt}\hat{L}_c(\xi,t)+\frac{\xi^2}{1+\xi^2}\left[\frac{\delta}{2}\Big(1-\epsilon_0\Big)-\tilde{\epsilon_1}\right]\xi^2|\hat{u}+\tau\hat{v}|^2\\
&\quad+\frac{\xi^2}{1+\xi^2}\left[\left(\frac{\delta}{2}-\tilde{\epsilon}_0\right)-M\epsilon_4\right]|\hat{v}+\tau\hat{w}|^2\\
&\quad+\frac{\xi^2}{1+\xi^2}\Big[M(g(0)-\epsilon_5)-\Big(\delta+C(\epsilon_0,\tilde{\epsilon}_1)\Big)\Big]|\hat{\theta}|^2\\
&\quad+\left[\frac{N}{2k}-M\Lambda\right]|\xi|^2(1+|\xi|^2+|\xi|^4)\int_0^\infty -\mu'(s)|\hat{\eta}(s)|^2ds\leq 0, \qquad (3.71)
\end{aligned}$$

for all $t\geq 0$ and where, Λ is a constant that depends on all the other constants.
Regarding (3.71), we fix our constants so that the above coefficients are positive. Therefore, we pick $\epsilon_0,\tilde{\epsilon}_0$ and ϵ_5 small enough, such that:

$$\epsilon_0<1,\qquad \tilde{\epsilon}_0\leq\delta/2\qquad\text{and}\quad \epsilon_5\leq g(0).$$

Once ϵ_0 is fixed, we select $\tilde{\epsilon}_1$ small enough, such that:

$$\tilde{\epsilon}_1<\frac{\delta(1-\epsilon_0)}{2}.$$

Now, as $\tilde{\epsilon}_1$ and ϵ_5 are fixed, then we take M large enough, such that:

$$M>\frac{\delta+C(\epsilon_0,\tilde{\epsilon}_1)}{g(0)-\epsilon_5}.$$

Finally, we can select ϵ_4 small enough, such that:

$$\epsilon_4<\frac{\delta/2-\tilde{\epsilon}_0}{M}$$

and take N large enough, such that:

$$N>2kM\Lambda.$$

Consequently, we conclude that there exists a constant $\Lambda_0>0$, such that for all $\xi\in\mathbb{R}^N$ and all $t\geq 0$,

$$\frac{d}{dt}\hat{L}_c(\xi,t)+\Lambda_0\frac{|\xi|^2}{1+|\xi|^2}\hat{E}_c(\xi,t)\leq 0. \qquad (3.72)$$

At this point, observe that from the identity (3.70), and for N large enough, there are two positive constants γ_4,γ_5, such that:

$$\gamma_4(1+\xi^2+\xi^4)\hat{L}_c(\xi,t)\leq\hat{E}_c(\xi,t)\leq\gamma_5(1+\xi^2+\xi^4)\hat{L}_c(\xi,t) \qquad (3.73)$$

Therefore, owing to (3.73), the inequality (3.72) reads as:

$$\frac{d}{dt}\hat{L}_c(\xi,t) \leq -\Lambda_1 \tilde{\rho}_c(\xi)\hat{L}_c(\xi,t), \tag{3.74}$$

for some $\Lambda_1 > 0$ and for all $t \geq 0$, with $\tilde{\rho}_c(\xi)$ satisfing (3.24). A simple integration with respect to t leads to:

$$\hat{L}_c(\xi,t) \leq \hat{L}_c(\xi,0)e^{-c\tilde{\rho}_c(\xi)t}. \tag{3.75}$$

Furthermore, for some $\Lambda_2 > 0$, we have:

$$\hat{E}_c(\xi,t) \leq \Lambda_2 \hat{E}_c(\xi,0)e^{-c\tilde{\rho}_c(\xi)t}, \qquad \forall t \geq 0. \tag{3.76}$$

The desired result (3.23) is claimed due to (3.60).

3.4 The Decay Estimate

In this section, we give the proof of main Theorems 3.4 and 3.5 and derive the decay rate of the solution for system (3.15) in both cases.

Proof Theorem 3.4. Using (3.22) we have:

$$\rho_f(\xi) \geq \begin{cases} c|\xi|^2 & \text{for} \quad |\xi| \leq 1, \\ c|\xi|^{-2} & \text{for} \quad |\xi| \geq 1. \end{cases} \tag{3.77}$$

Applying the Plancherel theorem together with inequality (3.21), we have:

$$\begin{aligned} ||\nabla^k V(t)||^2_{L^2(\mathbb{R}^N)} =& \int_{\mathbb{R}^N} |\xi|^{2k}|\hat{V}(\xi,t)|^2 \, d\xi \\ \leq& C\int_{\mathbb{R}^N} |\xi|^{2k}e^{-c\rho_f(\xi)t}|\hat{V}(\xi,0)|^2 \, d\xi \\ =& C\int_{|\xi|\leq 1} |\xi|^{2k}e^{-c\rho_f(\xi)t}|\hat{V}(\xi,0)|^2 \, d\xi \\ &+ C\int_{|\xi|\geq 1} |\xi|^{2k}e^{-c\rho_f(\xi)t}|\hat{V}(\xi,0)|^2 \, d\xi \\ =& I_1(t) + I_2(t). \end{aligned}$$

Here, we split the integral into two parts, so that $I_1(t)$ is the low–frequency part where $|\xi| \leq 1$ and $I_2(t)$ is the high–frequency part where $|\xi| \geq 1$. Using the first inequality in (3.77), we can estimate $I_1(t)$ as:

$$\begin{aligned} I_1(t) &= C\int_{|\xi|\leq 1} |\xi|^{2k}e^{-c\rho_f(t)t}|\hat{V}(\xi,0)|^2 \, d\xi \\ &\leq C\int_{|\xi|\leq 1} |\xi|^{2k}e^{-c|\xi|^2 t}|\hat{V}(\xi,0)|^2 \, d\xi \\ &\leq C||\hat{V}(\xi,0)||^2_{L^\infty(\mathbb{R}^N)} \int_{|\xi|\leq 1} |\xi|^{2k}e^{-c|\xi|^2 t} \, d\xi. \end{aligned} \tag{3.78}$$

Finally, using Lemma 0.1, we obtain

$$\int_{|\xi|\leq 1} |\xi|^{2k} e^{-c|\xi|^2 t}\, d\xi \leq C(1+t)^{-\frac{N}{2}-k}. \tag{3.79}$$

Hence, we have:

$$I_1(t) \leq C\|\hat{V}(\xi,0)\|^2_{L^\infty(\mathbb{R}^N)}(1+t)^{-\frac{N}{2}-k}. \tag{3.80}$$

Using the second inequality of (3.77), we can find the estimate for $I_2(t)$ as follows:

$$\begin{aligned} I_2(t) &= C\int_{|\xi|\geq 1} |\xi|^{2k} e^{-c\rho_f(\xi)t} |\hat{V}(\xi,0)|^2\, d\xi \\ &\leq C \sup_{|\xi|\geq 1} \{|\xi|^{-2\ell} e^{-c|\xi|^{-2}t}\} \int_{|\xi|\geq 1} |\xi|^{2(k+\ell)} |\hat{V}(\xi,0)|^2\, d\xi \\ &\leq C(1+t)^{-\ell} \|\partial_x^{k+\ell}\hat{V}(\xi,0)\|^2_{L^2(\mathbb{R}^N)}. \end{aligned} \tag{3.81}$$

Now, adding estimates (3.80) and (3.81) shows that estimate (3.19) holds. □

Proof Theorem 3.5. As we did before we have from (3.24)

$$\tilde{\rho}_c(\xi) \geq \begin{cases} c|\xi|^2, & \text{if} \quad |\xi|\leq 1, \\ c|\xi|^{-4}, & \text{if} \quad |\xi|\geq 1. \end{cases} \tag{3.82}$$

By exploiting Plancherel's theorem, we get

$$\begin{aligned} \|\nabla^k V_c(t)\|^2_{L^2(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} |\xi|^{2k} |\hat{V}_c(\xi,t)|^2 d\xi \\ &\leq C\int_{\mathbb{R}^N} |\xi|^{2k} e^{-c\tilde{\rho}_c(\xi)t} |\hat{V}_c(\xi,0)|^2 d\xi \\ &= C\int_{|\xi|\leq 1} |\xi|^{2k} e^{-c\tilde{\rho}_c(\xi)t} |\hat{V}_c(\xi,0)|^2 d\xi \\ &\quad + C\int_{|\xi|\geq 1} |\xi|^{2k} e^{-c\tilde{\rho}_c(\xi)t} |\hat{V}_c(\xi,0)|^2 d\xi \\ &= J_1 + J_2. \end{aligned} \tag{3.83}$$

For the law frequency part $|\xi|\leq 1$, we have by using (3.82)

$$\begin{aligned} J_1 &\leq C\|\hat{V}_c(\xi,0)\|^2_{L^\infty(\mathbb{R}^N)} \int_{|\xi|\leq 1} |\xi|^{2k} e^{-c|\xi|^2 t} d\xi \\ &\leq C(1+t)^{-\frac{N}{2}-k} \|V_c(\xi,0)\|^2_{L^1(\mathbb{R}^N)}. \end{aligned} \tag{3.84}$$

Again the use of (3.82) together with the estimate

$$\sup_{|\xi|\geq 1} \left\{|\xi|^{-2\ell} e^{-c|\xi|^{-4}t}\right\} \leq C(1+t)^{-\frac{\ell}{2}},$$

for the high frequency part J_2 yields

$$J_2 \le \sup_{|\xi|\ge 1}\left\{|\xi|^{-2\ell}e^{-c|\xi|^{-4}t}\right\}\int_{|\xi|\ge 1}|\xi|^{2(k+\ell)}|\hat{V}_c(\xi,0)|^2 d\xi$$
$$\le C(1+t)^{-\frac{\ell}{2}}\|\nabla^{k+\ell}V_c^0\|^2_{L^2(\mathbb{R}^N)}. \tag{3.85}$$

The estimate (3.20) is fulfilled by collecting the above two estimates. □

CHAPTER 4
WELL-POSEDNESS AND LONG TIME BEHAVIOR FOR A GENERAL CLASS OF MOORE–GIBSON–THOMPSON EQUATIONS

4.1 Introduction

In this chapter, in Ω, we consider the well-posedness and the long time behavior of third order in time linear (abstract) evolution equations,

$$\begin{cases} u_{ttt} + Bu_{tt} + \mathcal{A}_0 u + \mathcal{A}_1 u_t = 0, \\ u(0) = u_0, u_t(0) = u_1, u_{tt}(0) = u_2. \end{cases} \tag{4.1}$$

This allows to treat concrete examples where the operators $B, \mathcal{A}_0$, and $\mathcal{A}_1$ have space variable coefficients, in particular, we can consider the standard Moore–Gibson–Thompson system

$$\begin{cases} u_{ttt} + \alpha u_{tt} + \beta \Delta u + \Delta u_t = 0, & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(0, \cdot) = u_0, u_t(0, \cdot) = u_1, u_{tt}(0, \cdot) = u_2, & \text{in } \Omega, \end{cases} \tag{4.2}$$

where Ω is a bounded domain of $\mathbb{R}^n$, β is a positive constant and $\alpha \in L^\infty(\Omega)$, a case mentioned in [74, p. 306], for which an existence result is proved in [53]. Our approach does not allow to treat the non-autonomous situation when α may depend on the time variable, for such a situation we refer to [53, 55, 56] for existence and exponential decay.

Before going on, let us formulate the Hilbert setting and the basic assumptions. Let H and V be two Hilbert spaces such that V is continuously and densely embedded into H and let V' be the dual of l (with H as pivot space). We suppose given two sesquilinear and continuous forms a_0 and a_1 on V and such that a_1 is symmetric and coercive, namely

$$a_1(u, v) = \overline{a_1(v, u)}, \forall u, v \in V,$$

and

$$a_1(u, u) \gtrsim \|u\|_V^2, \forall u \in V.$$

Then we introduce the associated (bounded) operators $\mathcal{A}_0$ and $\mathcal{A}_1$ from V into V' defined by

$$\langle \mathcal{A}_i u, u' \rangle_{V-V'} = a_i(u, u'), \forall u, u' \in V, i = 0, 1,$$

where here and below $\langle \cdot, \cdot \rangle$ means the duality pairing between V and V'. Note that $\mathcal{A}_1$ is self-adjoint due to the assumptions on a_1. Note that $\mathcal{A}_1^{-1}\mathcal{A}_0$ is bounded from V into itself, but we suppose that it can be extended into a bounded operator from H into itself. Finally we suppose also given a bounded operator B from H into itself. The notation $a \lesssim b$ means that there exists a constant $C > 0$ independent of a, b, such that $a \leq Cb$, while $a \sim b$ means that both $a \lesssim b$ and $b \lesssim a$ hold. The well-posedness of our problem is proved in section 4.2 by using semigroup theory and an appropriate change of unknowns. An illustrative example is also presented. In section 4.3, we find sufficient conditions that guarantee the exponential decay of the energy of our abstract system and again illlustrate such a result. The link between system (4.2) with the wave equation with a frictional interior damping is extricated in section 4.4, where we show that the decay rate of the wave equation lead to a similar decay for our system (4.2).

4.2 An existence result

In this section we first prove the well-posedness of system (4.1), then we give an illustrative example.

4.2.1 General setting

In order to show that system (4.1) is well-posed, we introduce the following operator $\mathcal{A}$ on the Hilbert space $\mathcal{H} = V \times V \times H$, endowed with the inner product

$$((u,v,w)^\top, (u',v',w'))_{\mathcal{H}} = a_1(u,u') + a_1(v,v') + (w,w'), \forall (u,v,w)^\top, (u',v',w')^\top \in \mathcal{H}.$$

On this space we define the unbounded operator $\mathcal{A}$ by

$$D(\mathcal{A}) = \{(u,v,w)^\top \in V^3 \,|\, \mathcal{A}_0 u + \mathcal{A}_1 v \in H\}, \tag{4.3}$$

and

$$\mathcal{A}(u,v,w)^\top = (v, w, -(\mathcal{A}_0 u + \mathcal{A}_1 v + Bw)), \quad \forall (u,v,w)^\top \in D(\mathcal{A}). \tag{4.4}$$

With that definition we see that formally u is solution of (4.1) if and only if $U = (u, u_t, u_{tt})$ is solution of the first order evolution equation

$$\begin{cases} U_t = \mathcal{A}U, \\ U(0) = U_0, \end{cases} \tag{4.5}$$

where $U_0 = (u_0, u_1, u_2)$. This formal equivalence is correct as soon as strong solutions are concerned. Namely a strong solution of (4.5) yields a solution to (4.1), more precisely we have the next equivalence. Since its proof is immediate we let it to the reader.

Lemma 4.1. *$U = (u,v,w)^\top \in C^1([0,\infty), \mathcal{H}) \cap C^0([0,\infty), D(\mathcal{A}))$ is a solution of (4.5) if and only if $u \in C^2([0,\infty), V) \cap C^3([0,\infty), H)$ is solution of (4.1), with $v(t) = u_t(t)$ and $w(t) = u_{tt}(t)$ $\mathcal{A}_0 u(t) + \mathcal{A}_1 u_t(t) \in H$, for all $t \in [0,\infty)$.*

Now we are left to the existence of a solution to (4.5), which is obtained using semigroup theory after a change of unknowns. For that purpose, according to the standard decomposition of the solution into $z = u_t + \mathcal{A}_1^{-1}\mathcal{A}_0 u$ and u, see [55, §3.2], [54, §2] and [55, §2.1], where z satisfies a wave equation (see Remark 4.3 below) and u an abstract ODE with exponential decay, we introduce the bounded operator $\mathcal{M}$ from $\mathcal{H}$ into itself defined by

$$\mathcal{M}(u,v,w)^\top = \begin{pmatrix} I & 0 & 0 \\ \mathcal{A}_1^{-1}\mathcal{A}_0 & I & 0 \\ 0 & \mathcal{A}_1^{-1}\mathcal{A}_0 & I \end{pmatrix}, \forall (u,v,w)^\top, \forall (u,v,w)^\top \in \mathcal{H}.$$

This operator is even an isomorphism since its inverse is the bounded operator given by

$$\mathcal{M}^{-1}(u,v,w)^\top = \begin{pmatrix} I & 0 & 0 \\ -\mathcal{A}_1^{-1}\mathcal{A}_0 & I & 0 \\ (\mathcal{A}_1^{-1}\mathcal{A}_0)^2 & -\mathcal{A}_1^{-1}\mathcal{A}_0 & I \end{pmatrix} (u,v,w)^\top, \forall (u,v,w)^\top \in \mathcal{H}.$$

Now we prove the following Lemma (compare with section 3 from [54]).

Lemma 4.2. *$U = (u,v,w)^\top \in C^1([0,\infty),\mathcal{H}) \cap C^0([0,\infty), D(\mathcal{A}))$ is a strong solution of (4.5) if and only if $\tilde{U} = (u,z,y)^T = \mathcal{M}U \in C^1([0,\infty),\mathcal{H}) \cap C^0([0,\infty), D(\tilde{\mathcal{A}}))$ is a strong solution of*

$$\begin{cases} \tilde{U}_t = \tilde{\mathcal{A}}\tilde{U}, \\ \tilde{U}(0) = \tilde{U}_0, \end{cases} \tag{4.6}$$

where $\tilde{U}_0 = \mathcal{M}U_0 = (u_0, u_1 + \mathcal{A}_1^{-1}\mathcal{A}_0 u_0, u_2 + \mathcal{A}_1^{-1}\mathcal{A}_0 u_1)^\top$,

$$D(\tilde{\mathcal{A}}) = V \times D(\mathcal{A}_1) \times V, \tag{4.7}$$

and

$$\tilde{\mathcal{A}}(u,z,y)^\top = (z - \mathcal{A}_1^{-1}\mathcal{A}_0 u, y, -\mathcal{A}_1 z - R(u,z,y)^\top), \quad \forall (u,z,y)^\top \in D(\tilde{\mathcal{A}}), \tag{4.8}$$

when

$$R(u,z,y)^\top = (B - \mathcal{A}_1^{-1}\mathcal{A}_0)\left(y - \mathcal{A}_1^{-1}\mathcal{A}_0 z + (\mathcal{A}_1^{-1}\mathcal{A}_0)^2 u\right).$$

Proof. Let us first show that if $U = (u,v,w)^\top$ is solution of (4.5), then $\tilde{U} = (u,z,y)^\top$ is solution of (4.6). Indeed (4.5) directly implies that $v = u_t, w = v_t = u_{tt}$ and u satisfies (4.1). Hence by their definition, $z = u_t + \mathcal{A}_1^{-1}\mathcal{A}_0 u$,

$$z_t = u_{tt} + \mathcal{A}_1^{-1}\mathcal{A}_0 u_t = y,$$

and

$$\begin{aligned} y_t &= w_t + \mathcal{A}_1^{-1}\mathcal{A}_0 v_t \\ &= u_{ttt} + \mathcal{A}_1^{-1}\mathcal{A}_0 w \\ &= -Bu_{tt} - \mathcal{A}_0 u - \mathcal{A}_1 u_t + \mathcal{A}_1^{-1}\mathcal{A}_0 w \\ &= (-Bu_{tt} + \mathcal{A}_1^{-1}\mathcal{A}_0) w - \mathcal{A}_0 u - \mathcal{A}_1 v \\ &= -\mathcal{A}_1 z - R(u,z,y)^\top. \end{aligned}$$

This directly yield (4.6).
Let us notice that the regularity $\tilde{U} = (u,z,y)^\top \in C^1([0,\infty),\mathcal{H}) \cap C^0([0,\infty), D(\tilde{\mathcal{A}}))$ follows from the fact that the operator $\mathcal{A}_1^{-1}\mathcal{A}_0$ is bounded from H into itself as well as from V into itself. The converse implication is proved in a fully similar manner. □

Remark 4.3. *From* (4.6), *we see that* z *satisfies*

$$z_{tt} + \mathcal{A}_1 z - (B - \mathcal{A}_1^{-1}\mathcal{A}_0)z_t + (B - \mathcal{A}_1^{-1}\mathcal{A}_0)(\mathcal{A}_1^{-1}\mathcal{A}_0 z - (\mathcal{A}_1^{-1}\mathcal{A}_0)^2 u) = 0, \tag{4.9}$$

which under the assumption that $B - \mathcal{A}_1^{-1}\mathcal{A}_0$ *is a nonnegative operator from* H *into itself, is an weakly damped wave type equation with lower order term* $(B - \mathcal{A}_1^{-1}\mathcal{A}_0)(\mathcal{A}_1^{-1}\mathcal{A}_0 z - (\mathcal{A}_1^{-1}\mathcal{A}_0)^2 u)$. *Similarly* u *is solution of*

$$u_t + \mathcal{A}_1^{-1}\mathcal{A}_0 u = z.$$

which is a sort of ODE since $\mathcal{A}_1^{-1}\mathcal{A}_0$ *is bounded from* H *into itself.*

Theorem 4.4. *Under the above assumptions, the operator* $\mathcal{A}$ *generates a* C_0*-semigroup on* $\mathcal{H}$.

Proof. We first prove that $\tilde{\mathcal{A}}$ generates a C_0-semigroup on $\mathcal{H}$. For that purpose, we notice that $\tilde{\mathcal{A}}$ can be split up into

$$\tilde{\mathcal{A}} = \tilde{\mathcal{A}}_d + \mathcal{B},$$

where the operator $\mathcal{B}$ defined by

$$\mathcal{B}(u,z,y)^\top = (z - \mathcal{A}_1^{-1}\mathcal{A}_0 u, 0, -R(u,z,y)^\top), \quad \forall (u,z,y)^\top \in \mathcal{H},$$

is a bounded operator (in $\mathcal{H}$) and the unbounded operator $\tilde{\mathcal{A}}_d$ is defined by

$$\tilde{\mathcal{A}}_d(u,z,y)^\top = (0, y, -\mathcal{A}_1 z), \quad \forall (u,z,y)^\top \in D(\tilde{\mathcal{A}}_d) = D(\tilde{\mathcal{A}}). \tag{4.10}$$

Therefore by a standard bounded perturbation Theorem (see for instance [94, Theorem 3.1.1]), we are reduced to proving that $\tilde{\mathcal{A}}_d$ generates a C_0-semigroup on $\mathcal{H}$. This last property holds since $\tilde{\mathcal{A}}_d$ is a maximal dissipative operator, hence, by Lumer-Phillips' theorem it generates a C_0-semigroup of contractions on $\mathcal{H}$ (it even generates a group). The dissipativity is mainly direct because for $U = (u,z,y)^\top \in D(\tilde{\mathcal{A}})$, we have

$$\mathrm{Re}(\tilde{\mathcal{A}}_d U, U)_{\mathcal{H}} = \mathrm{Re}(a_1(y,z) - (\mathcal{A}_1 z, y)) = 0.$$

The maximality is also quite direct. Indeed for $\lambda > 0$ and $F = (f,g,h) \in \mathcal{H}$ fixed, we look for $U = (u,z,y)^\top \in D(\tilde{\mathcal{A}})$ solution of $(\lambda I - \tilde{\mathcal{A}}_d)U = F$, or equivalently

$$\begin{cases} \lambda u = f \text{ in } V, \\ \lambda z - y = g \text{ in } V, \\ \lambda y + \mathcal{A}_1 z = h \text{ in } H. \end{cases} \tag{4.11}$$

This means that $u = f/\lambda \in V$, $y = \lambda z - g$ and

$$\lambda^2 z + \mathcal{A}_1 z = h + \lambda g \text{ in } H.$$

Since $\lambda^2 I + \mathcal{A}_1$ is an isomorphism from $D(\mathcal{A}_1)$ into H, we find a unique solution $z \in D(\mathcal{A}_1)$ of this problem and hence $y = \lambda z - g$ indeed belongs to V. Denote by $(\tilde{T}(t))_{t\geq 0}$ the C_0-

semigroup generated by $\tilde{\mathcal{A}}$, then we define

$$T(t) = \mathcal{M}^{-1}\tilde{T}(t)\mathcal{M}, \forall t \geq 0,$$

and as $\mathcal{M}$ is an isomorphism from $\mathcal{H}$ into itself, we directly deduce that $(T(t))_{t\geq 0}$ is a C_0-semigroup on $\mathcal{H}$. According to Lemma 4.2, its generator is nothing else than $\mathcal{A}$, the proof is then complete. □

Corollary 4.5. *(Existence and uniqueness of the solution) If $U_0 \in \mathcal{H}$, then problem (4.5) admits a unique weak solution $U = (u, v, w)^\top \in C^0([0, \infty), \mathcal{H})$. On the contrary if $U_0 \in D(\mathcal{A})$, then problem (4.5) admits a unique strong solution $U = (u, v, w)^\top \in C^1([0, \infty), \mathcal{H}) \cap C^0([0, \infty), D(\mathcal{A}))$.*

4.2.2 An illustrative example

Let $\Omega \subset \mathbb{R}^d, d \geq 1$ be a bounded open set with a Lipschitz boundary Γ. We take $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$. Now we define the operators $\mathcal{A}_i$, $i = 0$ and 1 and B as follows. For $i = 0, 1$, we suppose given scalar functions $b_i \in L^\infty(\Omega)$, and matrix valued functions $M_i \in L^\infty(\Omega; \mathbb{R}^{d\times d})$. Suppose also given a scalar function $\alpha \in L^\infty(\Omega)$, and a vector field function $\mathbf{c} \in L^\infty(\Omega; \mathbb{R}^d)$. Then we define

$$\begin{aligned} a_1(u, v) &= \int_\Omega (M_1 \nabla u \cdot \nabla \bar{v} + b_1 u \bar{v})\, dx, \\ a_0(u, v) &= \int_\Omega (M_0 \nabla u \cdot \nabla \bar{v} + (\mathbf{c} \cdot \nabla u)\bar{v} + b_0 u \bar{v})\, dx, \end{aligned}$$

for all $u, v \in H_0^1(\Omega)$ and

$$Bu = \alpha u, \forall u \in L^2(\Omega). \tag{4.12}$$

This yields two sesquilinear and continuous forms on $H_0^1(\Omega)$ and a bounded and self-adjoint operator B from $L^2(\Omega)$ into itself. We further assume that a_1 is symmetric and coercive on $H_0^1(\Omega)$. The symmetry of a_1 is clearly guaranteed if and only if M_1 is symmetric. The coerciveness of a_1 holds if we further assume that M_1 is uniformly positive definite, namely for almost all $x \in \Omega$,

$$M_1(x)\xi \cdot \bar{\xi} \geq m\|\xi\|_2^2, \qquad \forall \xi \in \mathbb{C}^d,$$

for some $m > 0$ (independent of x) and if the negative part $b_1^- = \max\{-b_1, 0\}$ of b_1 is small enough (see below). First define

$$B_1 = \sup_{x\in\Omega} b_1^-(x),$$

and let $c_0 > 0$ be the Poincaré constant

$$c_0\|u\|_{L^2(\Omega)}^2 \leq \|\nabla u\|_{L^2(\Omega)^d}^2, \forall u \in H_0^1(\Omega).$$

Since by the above assumption and definition, we have

$$a_1(u,u) \geq (mc_0 - B_1)\|u\|^2_{L^2(\Omega)}, \forall u \in H^1_0(\Omega), \tag{4.13}$$

then, if we assume that

$$B_1 < mc_0, \tag{4.14}$$

then a_1 will be coercive on $H^1_0(\Omega)$. The assumption (4.14) means that the negative part b_1^- of b_1 is small enough with respect to M_1 and is easily checked in practice since c_0 is explicitly known for some domains Ω or different upper bounds are available in the literature, see [60] and the references cited there.
It remains to check the assumption that $\mathcal{A}_1^{-1}\mathcal{A}_0$ can be extended into a bounded operator from $L^2(\Omega)$ into itself. The trivial case is to take $a_0 = a_1$, here is a non trivial one.

Lemma 4.6. *Assume that the boundary Γ is of class $C^{1,1}$ and that $M_0 \in W^{1,\infty}(\Omega;\mathbb{R}^{d\times d})$, as well as $\mathbf{c} \in W^{1,\infty}(\Omega;\mathbb{R}^d)$ then $\mathcal{A}_1^{-1}\mathcal{A}_0$ can be extended into a bounded operator from $L^2(\Omega)$ into itself.*

Proof. Define the unbounded operator A_1 that is the extension of $\mathcal{A}_1$ from $L^2(\Omega)$ into itself defined by

$$D(A_1) := \{u \in H^1_0(\Omega) : \exists g_u \in L^2(\Omega) \text{ such that } a_1(u,v) = \int_\Omega g_u \bar{v}\, dx, \forall v \in H^1_0(\Omega)\},$$

and

$$A_1 u = g_u, \forall u \in D(A_1).$$

From our assumptions, it is well known that this operator is positive and self-adjoint. It is then an isomorphism from $D(A_1^s)$ to $D(A_1^{s-1})$, for all real number s.
Now the assumption on the boundary guarantees that

$$D(A_1) = H^2(\Omega) \cap H^1_0(\Omega),$$

see for instance [44, Theorem 2.2.2.3].
We now show that the mapping $\mathcal{A}_0$ can be extended into a continuous mapping from $L^2(\Omega)$ into $D(A_1^{-1}) = (H^2(\Omega) \cap H^1_0(\Omega))'$, the dual of $H^2(\Omega) \cap H^1_0(\Omega)$. This holds if we can show that

$$a_0(u,v) \lesssim \|u\|_{L^2(\Omega)}\|v\|_{H^2(\Omega)}, \tag{4.15}$$

for any $u \in H^1_0(\Omega)$ and $v \in H^2(\Omega) \cap H^1_0(\Omega)$. Indeed, let $u \in H^1_0(\Omega)$ and $v \in H^2(\Omega) \cap H^1_0(\Omega)$ then by Green's formula (allowed by our assumptions on M_0 and $\mathbf{c}$), we get

$$a_0(u,v) = \int_\Omega \left(-u\, div(M_0^\top \nabla \bar{v}) - u\, div(\mathbf{c}\bar{v}) + b_0 u \bar{v}\right) dx.$$

By Cauchy-Schwarz's inequality we obtain (4.15). In conclusion as the restriction of A_1 to $D(A_1^{\frac{1}{2}}) = H^1_0(\Omega)$ coincides with $\mathcal{A}_1$, the operator $\mathcal{A}_1^{-1}\mathcal{A}_0$ can then be extended from $L^2(\Omega)$

into itself. □

Altogether, this means that the system

$$\begin{cases} u_{ttt} + \alpha u_{tt} + div(M_0 \nabla u) + (\mathbf{c} \cdot \nabla u) + b_0 u + div(M_1 \nabla u_t) + b_1 u_t = 0, & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(0, \cdot) = u_0, u_t(0, \cdot) = u_1, u_{tt}(0, \cdot) = u_2, & \text{in } \Omega \end{cases} \tag{4.16}$$

is well-posed in $H_0^1(\Omega)^2 \times L^2(\Omega)$.

Remark 4.7. *Note that other choices for B are possible for instance an integral operator is possible, namely if a scalar kernel $k \in L^\infty(\Omega \times \Omega)$ is given we may choose*

$$Bu(x) = \int_\Omega k(x, y)u(y)\, dy, \forall u \in L^2(\Omega),$$

that is a bounded operator B from $L^2(\Omega)$ into itself.

4.3 Uniform stability results

In this section, inspired by [53, §4], [55, §4], [54, §3], and [81, §4], we prove that the semigroup $(T(t))_{t\geq 0}$ generated by $\mathcal{A}$ decays exponentially under some additional assumptions. In the whole section we assume that $\mathcal{A}_0$ and B are self-adjoint, that

$$\mathcal{A}_1^{-1}\mathcal{A}_0 = \mathcal{A}_0\mathcal{A}_1^{-1}, \tag{4.17a}$$

$$\mathcal{A}_1^{-1}\mathcal{A}_0 B = B\mathcal{A}_1^{-1}\mathcal{A}_0, \tag{4.17b}$$

and that

$$(Bv, v) \geq 0, \forall v \in H, \tag{4.17c}$$

$$(\mathcal{A}_1^{-1}\mathcal{A}_0(B - \mathcal{A}_1^{-1}\mathcal{A}_0)v, v) \geq 2\delta\|v\|^2, \forall v \in H, \tag{4.17d}$$

$$((B - \mathcal{A}_1^{-1}\mathcal{A}_0)v, v) \geq 2\delta\|v\|^2, \forall v \in H, \tag{4.17e}$$

$$a_0(u, u) \geq \alpha_0\|u\|_V^2, \forall u \in V, \tag{4.17f}$$

for some positive constants δ and α_0. According to Remark 4.3, the assumption (4.17e) is certainly needed to obtain exponential decay of $T(t)$. Let us first define the following energies

$$E(t) = \frac{1}{2}\Big[a_1(u_t + \mathcal{A}_1^{-1}\mathcal{A}_0 u, u_t + \mathcal{A}_1^{-1}\mathcal{A}_0 u) + \|u_{tt} + \mathcal{A}_1^{-1}\mathcal{A}_0 u_t\|^2 \\ + (\mathcal{A}_1^{-1}\mathcal{A}_0(B - \mathcal{A}_1^{-1}\mathcal{A}_0)u_t, u_t)\Big],$$

$$E_0(t) = \frac{1}{2}((Bu_t, u_t) + a_0(u, u)),$$

$$E_{tot}(t) = E(t) + \delta E_0(t), \forall t \geq 0.$$

Note that our assumptions guarantee that $\mathcal{A}_1^{-1}\mathcal{A}_0$ is self-adjoint as well as $\mathcal{A}_1^{-1}\mathcal{A}_0 B$. Notice further that $E(t)$ and $E_0(t)$ are nonnegative, both are not equivalent to $\|(u, u_t, u_{tt})\|^2_{\mathcal{H}}$ in general, but under the previous assumptions, the sum is, as shown in the next Lemma (compare with [53, Remark 4.2],[54, Remark 3.2] and [55, Remark 3.2])

Lemma 4.8. *Let $U = (u, u_t, u_{tt})$ be a strong solution of (4.5) with an initial datum $U_0 = (u_0, u_1, u_2) \in D(\mathcal{A})$. Then it holds*

$$E_{\text{tot}}(t) \sim \|(u, u_t, u_{tt})\|^2_{\mathcal{H}}, \forall t \geq 0. \tag{4.18}$$

Proof. Since the estimate

$$E_{\text{tot}}(t) \lesssim \|(u, u_t, u_{tt})\|^2_{\mathcal{H}},$$

is immediate, let us concentrate on the converse estimation. As

$$\|(u, u_t, u_{tt})\|^2_{\mathcal{H}} = a_1(u, u) + a_1(u_t, u_t) + \|u_{tt}\|^2,$$

by the continuity of a_1 and (4.17f), we get

$$\|(u, u_t, u_{tt})\|^2_{\mathcal{H}} \lesssim E_0(t) + a_1(u_t, u_t) + \|u_{tt}\|^2.$$

For the two last terms of this right-hand side, we insert some zero term to get

$$\begin{aligned}\|(u, u_t, u_{tt})\|^2_{\mathcal{H}} \lesssim & E_0(t) + 2a_1(u_t + \mathcal{A}_1^{-1}\mathcal{A}_0 u, u_t + \mathcal{A}_1^{-1}\mathcal{A}_0 u) + 2\|u_{tt} + \mathcal{A}_1^{-1}\mathcal{A}_0 u_t\|^2 \\ & + 2a_1(\mathcal{A}_1^{-1}\mathcal{A}_0 u, \mathcal{A}_1^{-1}\mathcal{A}_0 u) + 2\|\mathcal{A}_1^{-1}\mathcal{A}_0 u_t\|^2.\end{aligned}$$

By the boundedness properties of $\mathcal{A}_1^{-1}\mathcal{A}_0$ mentioned before, we obtain

$$\|(u, u_t, u_{tt})\|^2_{\mathcal{H}} \lesssim E_0(t) + a_1(u_t + \mathcal{A}_1^{-1}\mathcal{A}_0 u, u_t + \mathcal{A}_1^{-1}\mathcal{A}_0 u) + 2\|u_{tt} + \mathcal{A}_1^{-1}\mathcal{A}_0 u_t\|^2 + \|u_t\|^2.$$

Using the assumption (4.17d), we arrive at

$$\|(u, u_t, u_{tt})\|^2_{\mathcal{H}} \lesssim E(t) + E_0(t),$$

as requested. □

In a first step, we give an explicit expression of the derivative of the energy E, (compare with [53, Lemma 4.3],[54, Lemma 4.1] and [55, Lemma 3.1]).

Lemma 4.9. *Let $U = (u, u_t, u_{tt})$ be a strong solution of (4.5) with an initial datum $U_0 = (u_0, u_1, u_2) \in D(\mathcal{A})$. Then*

$$E'(t) = -((B - \mathcal{A}_1^{-1}\mathcal{A}_0)u_{tt}, u_{tt}). \tag{4.19}$$

In particular the energy E is nonincreasing.

Proof. Introduce the continuous form

$$\begin{aligned}((u,v,w)^\top,(u',v',w'))_{\mathcal{H}_0} =& a_1(v+\mathcal{A}_1^{-1}\mathcal{A}_0u, v'+\mathcal{A}_1^{-1}\mathcal{A}_0u') \\ &+(w+\mathcal{A}_1^{-1}\mathcal{A}_0v, w'+\mathcal{A}_1^{-1}\mathcal{A}_0v') \\ &+(\mathcal{A}_1^{-1}\mathcal{A}_0(B-\mathcal{A}_1^{-1}\mathcal{A}_0)v,v'). \end{aligned} \tag{4.20}$$

for all $(u,v,w)^\top,(u',v',w')^\top \in \mathcal{H}$. As underlined before as $\mathcal{A}_1^{-1}\mathcal{A}_0$ and $\mathcal{A}_1^{-1}\mathcal{A}_0(B-\mathcal{A}_1^{-1}\mathcal{A}_0)$ are self-adjoint, the above form is symmetric. Now we notice that

$$2E(t) = (U(t),U(t))_{\mathcal{H}_0},$$

hence

$$E'(t) = \mathrm{Re}(U'(t),U(t))_{\mathcal{H}_0} = \mathrm{Re}(\mathcal{A}U(t),U(t))_{\mathcal{H}_0}.$$

To get the conclusion it then remains to show that

$$\mathrm{Re}(\mathcal{A}U,U)_{\mathcal{H}_0} = -((B-\mathcal{A}_1^{-1}\mathcal{A}_0)w,w), \forall U=(u,v,w)\in D(\mathcal{A}). \tag{4.21}$$

But in view of the definition of $\mathcal{A}$, for $U=(u,v,w)\in D(\mathcal{A})$, we have

$$\begin{aligned}(\mathcal{A}U,U)_{\mathcal{H}_0} =& a_1(w+\mathcal{A}_1^{-1}\mathcal{A}_0v, v+\mathcal{A}_1^{-1}\mathcal{A}_0u) \\ &+(-\mathcal{A}_0u-\mathcal{A}_1v-Bw+\mathcal{A}_1^{-1}\mathcal{A}_0w, w+\mathcal{A}_1^{-1}\mathcal{A}_0v) \\ &+(\mathcal{A}_1^{-1}\mathcal{A}_0(B-\mathcal{A}_1^{-1}\mathcal{A}_0)w,v).\end{aligned}$$

Using the definition of $\mathcal{A}_1$, we get

$$\begin{aligned}\mathrm{Re}(\mathcal{A}U,U)_{\mathcal{H}_0} =& \mathrm{Re}\Big\{\langle\mathcal{A}_1w+\mathcal{A}_0v, v+\mathcal{A}_1^{-1}\mathcal{A}_0u\rangle - \langle\mathcal{A}_0u, w+\mathcal{A}_1^{-1}\mathcal{A}_0v\rangle \\ &-\langle\mathcal{A}_1v, w+\mathcal{A}_1^{-1}\mathcal{A}_0v\rangle + (-Bw+\mathcal{A}_1^{-1}\mathcal{A}_0w, w+\mathcal{A}_1^{-1}\mathcal{A}_0v) \\ &+(\mathcal{A}_1^{-1}\mathcal{A}_0(B-\mathcal{A}_1^{-1}\mathcal{A}_0)w,v)\Big\}.\end{aligned}$$

Using our assumptions, some terms of this right-hand side cancelled out to reduce to the right-hand side of (4.21). □

In a second step, we need the following identity (compare with [53, Lemma 4.4],[54, (30)] and [55, Lemma 3.2]).

Lemma 4.10. *Let $U=(u,u_t,u_{tt})$ be a strong solution of (4.5) with an initial datum $U_0 = (u_0,u_1,u_2)\in D(\mathcal{A})$, then it holds*

$$a_1(u_t,u_t) = \|u_{tt}\|^2 - \frac{d}{dt}E_0(t) - \frac{d}{dt}\left(\mathrm{Re}(u_{tt},u_t)\right). \tag{4.22}$$

Proof. Taking the inner product of the first identity of (4.1) with u_t we directly get

$$(u_{ttt},u_t) + (Bu_{tt},u_t) + a_0(u,u_t) + a_1(u_t,u_t) = 0.$$

Taking the real part of this identity, we obtain

$$a_1(u_t, u_t) = -\operatorname{Re}(u_{ttt}, u_t) - \frac{d}{dt}E_0(t).$$

As

$$\frac{d}{dt}(u_{tt}, u_t) = (u_{ttt}, u_t) + (u_{tt}, u_{tt}),$$

the two previous identities directly yield (4.22). □

We are ready to state the exponential decay result (compare with [53, Lemma 4.5 and §4.3], [54, steps 3 and 4 pp. 982-983], and [55, steps 3 and 4 pp. 20-21]).

Theorem 4.11. *Under the additional assumptions of this section, the semigroup generated by $\mathcal{A}$ is exponentially stable in $\mathcal{H}$, namely there exist two positive constants M and ω such that*

$$\|e^{t\mathcal{A}}U_0\| \leq Me^{-\omega t}\|U_0\|, \forall U_0 \in \mathcal{H}.$$

Proof. Let us first fix $U_0 \in D(\mathcal{A})$ and let $U(t) = (u(t), v(t), w(t)) = e^{t\mathcal{A}}U_0$ be the strong solution of (4.5) (that satisfies $v = u_t$ and $w = u_{tt}$). For such a solution using the identities (4.19) and (4.22), we have

$$\frac{d}{dt}E_{\text{tot}}(t) = -((B - \mathcal{A}_1^{-1}\mathcal{A}_0)u_{tt}, u_{tt}) + \delta\|u_{tt}\|^2 - \delta a_1(u_t, u_t) - \delta\frac{d}{dt}\left(\operatorname{Re}(u_{tt}, u_t)\right).$$

Hence by our assumption (4.17e), we get

$$\frac{d}{dt}E_{\text{tot}}(t) \leq -\delta\|u_{tt}\|^2 - \delta a_1(u_t, u_t) - \delta\frac{d}{dt}\left(\operatorname{Re}(u_{tt}, u_t)\right).$$

Integrating this estimate in $t \in (0, T)$ for an arbitray $T > 0$, one gets

$$\begin{aligned} E_{\text{tot}}(T) - E_{\text{tot}}(0) + \delta\int_0^T \left(\|u_{tt}\|^2 + a_1(u_t, u_t)\right)\, dt \\ \leq -\delta\operatorname{Re}(u_{tt}(T), u_t(T)) + \delta\operatorname{Re}(u_{tt}(0), u_t(0)). \end{aligned} \tag{4.23}$$

For the second term of this right hand side using Cauchy-Schwarz's inequality and Lemma 4.8, we get

$$\operatorname{Re}(u_{tt}(0), u_t(0)) \lesssim E_{\text{tot}}(0). \tag{4.24}$$

On the contrary for the first term, we write

$$(u_{tt}(T), u_t(T)) = (u_{tt}(T) + \mathcal{A}_1^{-1}\mathcal{A}_0 u_t(T), u_t(T)) - (\mathcal{A}_1^{-1}\mathcal{A}_0 u_t(T), u_t(T)).$$

Using Cauchy-Schwarz's inequality and Young's inequality and the boundedness of $\mathcal{A}_1^{-1}\mathcal{A}_0$ from H into itself, we find

$$(u_{tt}(T), u_t(T)) \lesssim \|u_{tt}(T) + \mathcal{A}_1^{-1}\mathcal{A}_0 u_t(T)\|^2 + \|u_t(T)\|^2,$$

note that here and below the constant involved in $\lesssim$ is independent of T. By the assumption

(4.17d) and the definition of $E(t)$, we get

$$(u_{tt}(T), u_t(T)) \lesssim E(T),$$

and since E is non increasing, we arrive at

$$(u_{tt}(T), u_t(T)) \lesssim E(0).$$

This estimate and (4.24) in (4.23) directly yields

$$E_{\text{tot}}(T) + \delta \int_0^T \left(\|u_{tt}\|^2 + a_1(u_t, u_t)\right) dt \lesssim E_{\text{tot}}(0).$$

Using Lemma 4.8, we arrive at

$$\begin{aligned}\|(u(T), u_t(T), u_{tt}(T))\|_{\mathcal{H}}^2 + \int_0^T \left(\|u_{tt}\|^2 + a_1(u_t, u_t)\right) dt \\ \lesssim \|(u(0), u_t(0), u_{tt}(0))\|_{\mathcal{H}}^2.\end{aligned} \tag{4.25}$$

We are now reduced to estimate $\int_0^T \|u\|_V^2 \, dt$. For that purpose, we take the inner product in H of the first identity of (4.1) with u to get

$$(u_{ttt} + Bu_{tt} + \mathcal{A}_0 u + \mathcal{A}_1 u_t, u) = 0.$$

Taking the real part of this identity, we find

$$a_0(u, u) + \frac{1}{2}\frac{d}{dt} a_1(u, u) = -\operatorname{Re}(u_{ttt} + Bu_{tt}, u).$$

As

$$\operatorname{Re}(u_{ttt}, u) = -\frac{1}{2}\frac{d}{dt}\|u_t\|^2 + \operatorname{Re}\frac{d}{dt}(u_{tt}, u),$$

and

$$(Bu_{tt}, u) = \frac{d}{dt}(Bu_t, u) - (Bu_t, u_t),$$

we find

$$a_0(u, u) + \frac{1}{2}\frac{d}{dt} a_1(u, u) = (Bu_t, u_t) + \frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2 - \operatorname{Re}(u_{tt}, u) - \operatorname{Re}(Bu_t, u)\right).$$

Integrating this estimate between 0 and $T > 0$, we find

$$\begin{aligned}&\int_0^T a_0(u, u)\, dt + \frac{1}{2} a_1(u(T), u(T)) - \frac{1}{2} a_1(u(0), u(0)) \\ &\quad \le \int_0^T (Bu_t, u_t)\, dt + \frac{1}{2}\|u_t(T)\|^2 + |(u_{tt}(T), u(T))| + |(Bu_t(T), u(T))| \\ &\quad + \frac{1}{2}\|u_t(0)\|^2 + |(u_{tt}(0), u(0))| + |(Bu_t(0), u(0))|.\end{aligned}$$

Using Cauchy-Schwarz's inequality, the boundedness of B, the continuous embedding of V into H and the coerciveness of a_1, we obtain

$$\int_0^T a_0(u,u)\,dt \lesssim \int_0^T a_1(u_t,u_t)\,dt \\ + \|(u(T),u_t(T),u_{tt}(T))\|^2_{\mathcal{H}} + \|(u(0),u_t(0),u_{tt}(0))\|^2_{\mathcal{H}}. \tag{4.26}$$

With the help of (4.25), we get

$$\int_0^T a_0(u,u)\,dt \lesssim \|(u(0),u_t(0),u_{tt}(0))\|^2_{\mathcal{H}}.$$

This estimate and again (4.25) leads to

$$\int_0^T \left(\|u_{tt}\|^2 + a_1(u_t,u_t) + a_0(u,u)\right)\,dt \lesssim \|(u(0),u_t(0),u_{tt}(0))\|^2_{\mathcal{H}},$$

and by Lemma 4.8, we finally obtain

$$\int_0^T \|(u(t),u_t(t),u_{tt}(t))\|^2_{\mathcal{H}}\,dt \lesssim \|(u(0),u_t(0),u_{tt}(0))\|^2_{\mathcal{H}}.$$

Since this estimate is valid for all $T>0$ and recalling that $(u(t),u_t(t),u_{tt}(t)) = e^{tA}U_0$, we get

$$\int_0^\infty \|e^{tA}U_0\|^2_{\mathcal{H}}\,dt \lesssim \|U_0\|^2_{\mathcal{H}}.$$

As $D(\mathcal{A})$ is dense in $\mathcal{H}$, this estimate remains valid for all $U_0 \in \mathcal{H}$ and, by [34] (see also [94, Theorem 4.1.4]), we conclude that e^{tA} is exponentially stable. □

Let us end up with some examples.

Example 4.12. *In the setting of subsection 4.2.2, assuming that $M_0 = \beta M_1$, $b_0 = \beta b_1 + r$, $\mathbf{c} = \mathbf{0}$ for some real number r and a positive real number β, the operator $\mathcal{A}_0$ is selfadjoint. Then due to (4.13), (4.17f) will be valid if*

$$r > \beta(B_1 - mc_0).$$

Now denoting by $\mathbb{I}$ the identity operator, as

$$\mathcal{A}_0 = \beta\mathcal{A}_1 + r\mathbb{I},$$

we deduce that

$$\mathcal{A}_1^{-1}\mathcal{A}_0 = \mathcal{A}_0\mathcal{A}_1^{-1} = \beta\mathbb{I} + r\mathcal{A}_1^{-1},$$

and hence (4.17a) holds.

If $r \neq 0$, we take $B = \alpha\mathbb{I}$ with a constant α which guarantees that (4.17b) holds. On the contrary if $r = 0$, we can take $B = \alpha\mathbb{I}$ with $\alpha \in L^\infty(\Omega)$ and (4.17b) remains valid.

In both cases, (4.17c) holds if $\alpha \geq 0$. Hence if remains to examine (4.17d) and (4.17e).

If $r = 0$, as $\mathcal{A}_1^{-1}\mathcal{A}_0 = \beta\mathbb{I}$ with $\beta > 0$, (4.17d) and (4.17e) are equivalent and as

$$B - \mathcal{A}_1^{-1}\mathcal{A}_0 = (\alpha - \beta)\mathbb{I},$$

they holds if and only if

$$\alpha - \beta \geq 2\delta, \text{ a. e. in } \Omega. \tag{4.27}$$

On the contrary if $r \neq 0$, then

$$B - \mathcal{A}_1^{-1}\mathcal{A}_0 = (\alpha - \beta)\mathbb{I} - r\mathcal{A}_1^{-1},$$

while

$$\mathcal{A}_1^{-1}\mathcal{A}_0(B - \mathcal{A}_1^{-1}\mathcal{A}_0) = \beta(\alpha - \beta)\mathbb{I} + (\alpha - 2\beta)r\mathcal{A}_1^{-1} - r^2\mathcal{A}_1^{-2}.$$

As there exists a positive constant C_1 such that

$$\|\mathcal{A}_1^{-1}v\|_{L^2(\Omega)} \leq C_1\|v\|_{L^2(\Omega)}, \forall v \in L^2(\Omega),$$

by Cauchy-Schwarz's inequality we deduce that

$$((B - \mathcal{A}_1^{-1}\mathcal{A}_0)v, v)_{L^2(\Omega)} = (\alpha - \beta)\|v\|^2_{L^2(\Omega)} - r(\mathcal{A}_1^{-1}v, v)_{L^2(\Omega)} \tag{4.28}$$

$$\geq (\alpha - \beta - |r|C_1)\|v\|^2_{L^2(\Omega)}. \tag{4.29}$$

This means that (4.17e) holds if

$$\alpha - \beta - |r|C_1 > 0.$$

Similarly, we have

$$(\mathcal{A}_1^{-1}\mathcal{A}_0(B - \mathcal{A}_1^{-1}\mathcal{A}_0)v, v)_{L^2(\Omega)} \geq (\beta(\alpha - \beta) - |\alpha - 2\beta||r|C_1 - r^2C_1^2)\|v\|^2_{L^2(\Omega)}.$$

Consequently (4.17d) holds if

$$\beta(\alpha - \beta) - |\alpha - 2\beta||r|C_1 - r^2C_1^2 > 0.$$

4.4 A degenerate case

In this section, we present examples where the assumptions (4.17d) and (4.17e) fail and for which exponential, polynomial or logarithmic rate is reached. In the setting of subsection 4.2.2, we assume that Ω is connected and we choose M_1 equal to the identity matrix $= \mathbb{I}_{d\times d}$, $M_0 = \beta\mathbb{I}_{d\times d}$, where β is a positive constant and B in the form (4.12) with $\alpha \in L^\infty(\Omega)$ such that

$$\alpha \geq \beta \text{ a. e. in } \Omega. \tag{4.30}$$

In other words, we study Moore–Gibson–Thompson system (4.2). We further suppose that there exist an non empty open subset ω_0 of Ω and a positive constant κ such that

$$\alpha - \beta \geq \kappa \text{ a. e. in } \omega_0. \tag{4.31}$$

In this case all assumptions of section 4.3 hold except (4.17d) and (4.17e) since

$$\mathcal{A}_1^{-1}\mathcal{A}_0(B - \mathcal{A}_1^{-1}\mathcal{A}_0) = \beta(B - \mathcal{A}_1^{-1}\mathcal{A}_0) = \beta(\alpha - \beta)\mathbb{I},$$

that could be zero on $\Omega \setminus \omega_0$. Hence the sole case of interest here is the case when ω_0 is different from Ω and $\alpha = \beta$ on a non empty open set of Ω.
According to Remark 4.3 and commonly found works about weaker (polynomial or logarithmic) decay rate of the wave equation (see below), we can expect weaker decay rate for system (4.2). In this case our stability result is based on a spectral analysis and a resolvent estimate obtained by a comparison with the resolvent of the wave equation with an interior damping in ω_0.
We then first analyze the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$. Note that the domain of $\mathcal{A}$ is not compactly embedded into $\mathcal{H}$, hence, if it exists, the resolvent of $\mathcal{A}$ is not compact. This renders the analysis more complex and forces us to use a compact perturbation argument (described below).

Lemma 4.13. *Under the previous assumptions,*

$$\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subset \rho(\mathcal{A}). \tag{4.32}$$

Proof. Let $\lambda \in \mathbb{C}_+$ and $F = (f, g, h)^\top \in \mathcal{H}$. We look for $U = (u, v, w)^\top \in D(\mathcal{A})$ such that

$$\lambda U - \mathcal{A}U = F, \tag{4.33}$$

or equivalently

$$\lambda u - v = f, \tag{4.34a}$$
$$\lambda v - w = g, \tag{4.34b}$$
$$(\lambda + \alpha)w - \Delta(\beta u + v) = h. \tag{4.34c}$$

Assuming that a solution U exists. Then the two first identities yield

$$v = \lambda u - f, \tag{4.35}$$
$$w = \lambda v - g = \lambda^2 u - \lambda f - g, \tag{4.36}$$

and plugging (4.35) and (4.36) into (4.34c), we find

$$(\lambda + \alpha)\lambda^2 u - (\beta + \lambda)\Delta u = h + (\lambda + \alpha)(\lambda f + g) - \Delta f \text{ in } \mathcal{D}'(\Omega), \tag{4.37}$$

where $\mathcal{D}'(\Omega)$ is the space of Schwartz distributions, the dual of the space $\mathcal{D}(\Omega)$ made of smooth and compactly supported functions in Ω, see [2, p. 19] or [107]. This equivalently means that

$$a_\lambda(u,v) = F_\lambda(v), \forall v \in \mathcal{D}(\Omega), \tag{4.38}$$

where for all $u, v \in H_0^1(\Omega)$

$$\begin{aligned} a_\lambda(u,v) &= \int_\Omega \Big((\lambda+\alpha)\lambda^2 u\bar{v} + (\beta+\lambda)\nabla u \cdot \nabla \bar{v}\Big)\, dx, \\ F_\lambda(v) &= \int_\Omega \left((h + (\lambda+\alpha)(\lambda f + g))\,\bar{v} + \nabla f \cdot \nabla \bar{v}\right) dx. \end{aligned}$$

Since a_λ is continuous on $H_0^1(\Omega) \times H_0^1(\Omega)$ and F_λ is continuous on $H_0^1(\Omega)$ and $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$, the identity (4.38) remains valid for test-functions in $H_0^1(\Omega)$, namely

$$a_\lambda(u,v) = F_\lambda(v), \forall v \in H_0^1(\Omega). \tag{4.39}$$

Let us now show that this problem has a unique solution $u \in H_0^1(\Omega)$. For that purpose, we distinguish two cases:

- If $\lambda = 0$, we see that

$$\begin{aligned} a_0(u,v) &= \beta \int_\Omega \nabla u \cdot \nabla \bar{v}\, dx, \\ F_0(v) &= \int_\Omega \left((h + \alpha g)\,\bar{v} + \nabla f \cdot \nabla \bar{v}\right) dx. \end{aligned}$$

  Since a_0 is a continuous sesquilinear and coercive form on $H_0^1(\Omega)$, problem (4.39) (with $\lambda = 0$) has a unique solution $u \in H_0^1(\Omega)$. Defining $v = -f$ and $w = -g$ (see (4.35) and (4.36)), we easily see that $U = (u, v, w)^\top$ belongs to $D(\mathcal{A})$ and is solution of (4.33) (with $\lambda = 0$). Hence 0 belongs to $\rho(\mathcal{A})$.

- If $\lambda \neq 0$, our argument is more complex and is based on a compact perturbation argument. Namely introduce the sesquilinear and continuous form b_λ on $H_0^1(\Omega)$ defined by

$$b_\lambda(u,v) = (\beta+\lambda) \int_\Omega \nabla u \cdot \nabla \bar{v}\, dx, \forall u, v \in H_0^1(\Omega).$$

  Introduce further the operators A_λ and B_λ by

$$\begin{aligned} A_\lambda : H_0^1(\Omega) \to H^{-1}(\Omega) : u \to A_\lambda u, \\ B_\lambda : H_0^1(\Omega) \to H^{-1}(\Omega) : u \to B_\lambda u, \end{aligned}$$

  with

$$\langle A_\lambda u, v\rangle = a_\lambda(u,v), \quad \langle B_\lambda u, v\rangle = b_\lambda(u,v), \forall u, v \in H_0^1(\Omega).$$

  Since $\operatorname{Re}(\beta+\lambda) \geq \beta$, the form b_λ is coercive, in the sense that

$$\operatorname{Re} b_\lambda(u,u) \geq \beta \int_\Omega |\nabla u|^2\, dx \gtrsim \|u\|_{1,\Omega}^2, \forall u \in H_0^1(\Omega),$$

due to Poincaré inequality. Hence by Lax-Milgram lemma, the operator B_λ is an isomorphism from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$. Since $A_\lambda - B_\lambda = (\lambda+\alpha)\lambda^2\mathbb{I}$ is a compact operator from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$, A_λ is then a Fredholm operator of index zero from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$. Hence it is an isomorphism if and only if it is injective. So let $u \in \ker A_\lambda$, then it is a solution of (4.39) with $F_\lambda = 0$, namely

$$a_\lambda(u,v) = 0, \forall v \in H_0^1(\Omega). \tag{4.40}$$

But then defining (compare with (4.35) and (4.36))

$$v = \lambda u, w = \lambda v,$$

we easily see that the triple $U = (u,v,w)^\top$ belongs to $D(\mathcal{A})$ and is solution of

$$\lambda U - \mathcal{A}U = 0.$$

Now we take advantage of the identity (4.21) that here implies

$$\operatorname{Re}\lambda(U,U)_{\mathcal{H}_0} = -\int_\Omega (\alpha-\beta)|w|^2\,dx.$$

But according to its definition (4.20) and the assumption (4.30), we have

$$(U,U)_{\mathcal{H}_0} \geq 0, \text{ and } \int_\Omega (\alpha-\beta)|w|^2\,dx \geq 0,$$

and therefore the previous identity implies that

$$\int_\Omega (\alpha-\beta)|w|^2\,dx = 0.$$

By (4.30) and (4.31), we conclude that

$$w = 0 \quad \text{on} \quad \omega_0.$$

As $\lambda \neq 0$, we deduce that

$$u = 0 \quad \text{on} \quad \omega_0. \tag{4.41}$$

But $u \in H_0^1(\Omega)$ being solution of (4.40) it satisfies

$$(\lambda+\alpha)\lambda^2 u - (\beta+\lambda)\Delta u = 0 \text{ in } \mathcal{D}'(\Omega).$$

Since the operator Δ is elliptic and u is zero on ω_0 by Calderon uniqueness theorem (see for instance [66, Theorem 4.2]), $u = 0$ on the whole Ω. This obviously implies that $v = w = 0$ and hence $U = (0,0,0)^\top$ and the injectivity of A_λ is proved.

In conclusion, A_λ is an isomorphism, which guarantees that problem (4.39) has a unique solution $u \in H_0^1(\Omega)$. As before defining v by (4.35) and w by (4.36), we easily see that the triple $U = (u,v,w)^\top$ belongs to $D(\mathcal{A})$ and is solution of (4.33). The proof is then complete. $\square$

The main ingredient to obtain the resolvent estimate is to use the decay rate (exponential, polynomial or less) of the semigroup generated by wave equation in Ω with Dirichlet boundary condition and with a frictional interior damping in ω_0:

$$\begin{cases} u_{tt} - \Delta u + (\alpha - \beta) u_t = 0 & \text{in } \Omega \times (0, +\infty), \\ u = 0 & \text{on } \Gamma \times (0, +\infty), \\ u(\cdot, 0) = u_0 \quad \text{and} \quad u_t(\cdot, 0) = u_1 & \text{in } \Omega. \end{cases} \tag{4.42}$$

where α and β are the functions introduced before. More precisely, introduce the Hilbert space $\mathcal{H}_w = H_0^1(\Omega) \times L^2(\Omega)$ with norm

$$\|(u, v)^\top\|_{\mathcal{H}_w}^2 = |u|_{1,\Omega}^2 + \|v\|_\Omega^2, \forall (u, v)^\top \in \mathcal{H}_w,$$

and the operator $\mathcal{A}_w$ defined by

$$D(\mathcal{A}_w) = \{(u, v)^\top \in H_0^1(\Omega) \times H_0^1(\Omega) \,|\, \Delta u \in L^2(\Omega)\}, \tag{4.43}$$

and

$$\mathcal{A}_w (u, v)^\top = (v, \Delta u - (\alpha - \beta) v)), \quad \forall (u, v)^\top \in D(\mathcal{A}_w). \tag{4.44}$$

It is well-known that $\mathcal{A}_w$ generates a C_0-semigroup of contractions $(T_w(t))_{t \geq 0}$, see [9, p. 232]. Hence it is well-known that its resolvent set $\rho(\mathcal{A}_w)$ contains the open right half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\}$ and that

$$\|(\lambda \mathbb{I} - \mathcal{A}_w)^{-1}\|_{\mathcal{L}(\mathcal{H}_w)} \leq \frac{1}{\operatorname{Re} \lambda}, \forall \lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0. \tag{4.45}$$

Since ω_0 is open and non empty, by Holmgren's uniqueness theorem (see above), one can show that the imaginary axis is included into $\rho(\mathcal{A}_w)$ and therefore $\mathbb{C}_+ \subset \rho(\mathcal{A}_w)$. The decay rate of the solution to system (4.1) is based on the following bound on the resolvent of $\mathcal{A}_\omega$ on the imaginary axis

$$\|(i\xi \mathbb{I} - \mathcal{A}_w)^{-1}\|_{\mathcal{L}(\mathcal{H}_w)} \lesssim M(|\xi|), \forall \xi \in \mathbb{R}, \tag{4.46}$$

where M is a continuous, positive, and non decreasing function from $[0, \infty)$ into itself. Before going on, recall that for any $\lambda \in \mathbb{C}_+$ and an arbitrary $F_1 = (f_1, g_1) \in \mathcal{H}_w$, $(u_1, v_1)^\top = (\lambda \mathbb{I} - \mathcal{A}_w)^{-1} F_1 \in D(\mathcal{A}_w)$ satisfies

$$v_1 = \lambda u_1 - f_1, \tag{4.47}$$

$$(\lambda + \alpha - \beta) \lambda u_1 - \Delta u_1 = g_1 + (\lambda + \alpha - \beta) f_1, \tag{4.48}$$

and the estimate

$$|u_1|_{1,\Omega} + \|v_1\|_\Omega \leq \|(\lambda \mathbb{I} - \mathcal{A}_w)^{-1}\|_{\mathcal{L}(\mathcal{H}_w)} (|f_1|_{1,\Omega} + \|g_1\|_\Omega). \tag{4.49}$$

Due to (4.47), this implies

$$|u_1|_{1,\Omega} + |\lambda|\|u_1\|_\Omega \leq \max\{1, \|(\lambda\mathbb{I} - \mathcal{A}_w)^{-1}\|_{\mathcal{L}(\mathcal{H}_w)}\}(|f_1|_{1,\Omega} + \|g_1\|_\Omega). \tag{4.50}$$

Now we are ready to prove the following result.

Theorem 4.14. *There exists a positive constant C such that*

$$\begin{aligned}&\|(\lambda\mathbb{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})}\\ &\quad\leq C\max\left\{1, \|(\lambda\mathbb{I} - \mathcal{A}_w)^{-1}\|_{\mathcal{L}(\mathcal{H}_w)}, \frac{\|(\lambda\mathbb{I} - \mathcal{A}_w)^{-1}\|^2_{\mathcal{L}(\mathcal{H}_w)}}{1+|\lambda|^2}\right\}, \forall\lambda \in \mathbb{C}_+. \end{aligned}\tag{4.51}$$

Proof. 1. For $\lambda \in \mathcal{K} = \{\mu \in \mathbb{C}_+ \,|\, |\mu| \leq 1\}$, the estimate (4.51) is direct using Lemma 4.13 and since the resolvent operator

$$\lambda \to (\lambda\mathbb{I} - \mathcal{A})^{-1}$$

is holomorphic on $\rho(\mathcal{A})$ (see [9, CorollaryB.3]), hence continuous on $\mathcal{K}$. Therefore

$$\|(\lambda\mathbb{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \lesssim 1, \forall\lambda \in \mathcal{K},$$

where from now on the positive constant hidden in $\lesssim$ is independent of λ.

2. Let $\lambda \in \mathbb{C}_+$ satisfy $|\lambda| \geq 1$ and $F = (f, g, h)^\top \in \mathcal{H}$ and let $U = (u, v, w)^\top \in D(\mathcal{A})$ be the unique solution of (4.33). We have seen before that v is given by (4.35), hence the condition from $D(\mathcal{A})$, $\mathcal{A}_0 u + \mathcal{A}_1 v \in L^2(\Omega)$ takes the form

$$\Delta\left((\beta + \lambda)u - f\right) \in L^2(\Omega).$$

This suggests to introduce the new unknow

$$u_1 = (\lambda + \beta)u - f \tag{4.52}$$

that belongs to $H^1_0(\Omega)$ with $\Delta u_1 \in L^2(\Omega)$. Recalling that u satisfies (4.4), we see that u_1 satisfies

$$(\lambda + \alpha)\lambda^2 u - \Delta u_1 = h + (\lambda + \alpha)(\lambda f + g) \text{ in } L^2(\Omega).$$

As $\lambda + \beta \neq 0$, replacing u by $\frac{1}{\lambda+\beta}(u_1 + f)$, we find

$$\frac{\lambda + \alpha}{\lambda + \beta}\lambda^2 u_1 - \Delta u_1 = h + (\lambda + \alpha)g + \frac{\lambda\beta(\lambda + \alpha)}{\lambda + \beta}f \text{ in } L^2(\Omega). \tag{4.53}$$

By setting

$$\begin{aligned} g_1 &= h + \beta g + \frac{\lambda\beta^2}{\lambda+\beta} f + \frac{\lambda\beta(\alpha-\beta)}{\lambda+\beta} u_1, \\ f_1 &= g + \frac{\lambda\beta}{\lambda+\beta} f, \end{aligned}$$

we see that u_1 is solution of (4.48). Hence setting $v_1 = \lambda u_1 - f_1$, we find a pair $(u_1, v_1)^\top \in D(\mathcal{A}_w)$ satisfying (4.47)-(4.48). Consequently, the estimate (4.50) holds for u_1, namely

$$|u_1|_{1,\Omega} + |\lambda| \|u_1\|_\Omega \leq C_\lambda(|f_1|_{1,\Omega} + \|g_1\|_\Omega),$$

where for shortness we have set

$$C_\lambda = \max\{1, \|(\lambda\mathbb{I} - \mathcal{A}_w)^{-1}\|_{\mathcal{L}(\mathcal{H}_w)}\}.$$

Using (4.35) and (4.52), we see that

$$\frac{\lambda}{\lambda+\beta} u_1 = v + \frac{\beta}{\lambda+\beta} f,$$

hence

$$g_1 = \beta(\alpha-\beta)v + h + \beta g + \frac{\beta^2(\lambda+\alpha-\beta)}{\lambda+\beta} f.$$

Using this expression of g_1 and the definition f_1, we get

$$\begin{aligned} |u_1|_{1,\Omega} + |\lambda| \|u_1\|_\Omega \leq C_\lambda \Big(& |g|_{1,\Omega} + \frac{|\lambda|\beta}{|\lambda+\beta|} |f|_{1,\Omega} + \|h\|_\Omega + \beta\|g\|_\Omega \\ & + \frac{\beta^2}{|\lambda+\beta|} \|(\lambda+\alpha-\beta)f\|_\Omega + \beta\|(\beta-\alpha)v\|_\Omega \Big). \end{aligned}$$

As

$$\frac{|\lambda|}{|\lambda+\beta|} \leq 1, \forall \lambda \in \mathbb{C}_+, \tag{4.54}$$

we find that

$$\begin{aligned} |u_1|_{1,\Omega} + |\lambda| \|u_1\|_\Omega \leq & C_\lambda(|g|_{1,\Omega} + \beta|f|_{1,\Omega} + \|h\|_\Omega + \beta\|g\|_\Omega + \beta(\beta+K)\|f\|_\Omega \\ & + \beta\|(\beta-\alpha)v\|_\Omega), \end{aligned} \tag{4.55}$$

where $K = \max_\Omega(\alpha - \beta)$. Now we exploit the dissipativeness relation (4.21) that here implies

$$-\operatorname{Re}((\lambda U - \mathcal{A}U, U)_{\mathcal{H}_0} = \int_\Omega (\alpha-\beta)|w|^2 \, dx.$$

Using Cauchy-Schwarz's inequality and the fact that $(U,U)_{\mathcal{H}_0} \lesssim (U,U)_{\mathcal{H}}$, we find

$$\int_\Omega (\alpha-\beta)|w|^2\,dx \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

By Young's inequality this estimate implies that

$$\|(\alpha-\beta)w\|_\Omega \lesssim \|\sqrt{\alpha-\beta}w\|_\Omega \lesssim \varepsilon^{-1}\|F\|_{\mathcal{H}} + \varepsilon\|U\|_{\mathcal{H}}, \tag{4.56}$$

for all $\varepsilon > 0$. As (4.36) yields $\lambda v = w + g$, we find

$$|\lambda|\|(\alpha-\beta)v\|_\Omega \le \|(\alpha-\beta)w\|_\Omega + \|(\alpha-\beta)g\|_\Omega,$$

and therefore by (4.56)

$$|\lambda|\|(\alpha-\beta)v\|_\Omega \lesssim (1+\varepsilon^{-1})\|F\|_{\mathcal{H}} + \varepsilon\|U\|_{\mathcal{H}},$$

for all $\varepsilon > 0$. As we here assumed that $|\lambda| \ge 1$, this estimate in (4.55) leads to

$$|u_1|_{1,\Omega} + |\lambda|\|u_1\|_\Omega \lesssim C_\lambda\left((1+\varepsilon^{-1}|\lambda|^{-1})\|F\|_{\mathcal{H}} + \varepsilon|\lambda|^{-1}\|U\|_{\mathcal{H}}\right), \tag{4.57}$$

for all $\varepsilon > 0$. Now we come back to u, v and w. First using (4.52), we have (recalling that β is constant)

$$|\lambda+\beta||u|_{1,\Omega} \le |u_1|_{1,\Omega} + |f|_{1,\Omega},$$

which, by (4.57), yields

$$|\lambda+\beta||u|_{1,\Omega} \lesssim C_\lambda\left((1+\varepsilon^{-1}|\lambda|^{-1})\|F\|_{\mathcal{H}} + \varepsilon|\lambda|^{-1}\|U\|_{\mathcal{H}}\right),$$

for all $\varepsilon > 0$. Recalling (4.54), we deduce that

$$|\lambda||u|_{1,\Omega} \lesssim C_\lambda\left((1+\varepsilon^{-1}|\lambda|^{-1})\|F\|_{\mathcal{H}} + \varepsilon|\lambda|^{-1}\|U\|_{\mathcal{H}}\right), \tag{4.58}$$

for all $\varepsilon > 0$. By (4.35), we directly obtain

$$|v|_{1,\Omega} \lesssim C_\lambda\left((1+\varepsilon^{-1}|\lambda|^{-1})\|F\|_{\mathcal{H}} + \varepsilon|\lambda|^{-1}\|U\|_{\mathcal{H}}\right), \tag{4.59}$$

for all $\varepsilon > 0$. It remains to estimate the L^2-norm of w. For that purpose, we notice that $u_1 = \beta u + v$, hence

$$\|v\|_\Omega \le \beta\|u\|_\Omega + \|u_1\|_\Omega.$$

Therefore using (4.57) and (4.58) (with Poincaré inequality), we get

$$\begin{aligned} |\lambda|\|v\|_\Omega &\le \beta|\lambda|\|u\|_\Omega + |\lambda|\|u_1\|_\Omega \\ &\lesssim C_\lambda\left((1+\varepsilon^{-1}|\lambda|^{-1})\|F\|_{\mathcal{H}} + \varepsilon|\lambda|^{-1}\|U\|_{\mathcal{H}}\right), \end{aligned}$$

for all $\varepsilon > 0$. By (4.36), we get that

$$\|w\|_\Omega \lesssim C_\lambda \left((1+\varepsilon^{-1}|\lambda|^{-1})\|F\|_{\mathcal{H}} + \varepsilon|\lambda|^{-1}\|U\|_{\mathcal{H}}\right), \tag{4.60}$$

for all $\varepsilon > 0$. In conclusion using this estimate and (4.58)-(4.60), there exists a positive constant C (independent of λ and ε) such that

$$\|U\|_{\mathcal{H}} = \|(u,v,w)\|_{\mathcal{H}} \leq CC_\lambda \left((1+\varepsilon^{-1}|\lambda|^{-1})\|F\|_{\mathcal{H}} + \varepsilon|\lambda|^{-1}\|U\|_{\mathcal{H}}\right),$$

for all $\varepsilon > 0$. Hence chosing $\varepsilon = \frac{|\lambda|}{2CC_\lambda}$, we conclude that

$$\frac{1}{2}\|U\|_{\mathcal{H}} \leq CC_\lambda(1+2CC_\lambda|\lambda|^2)\|F\|_{\mathcal{H}}, \tag{4.61}$$

which proves (4.51) for $|\lambda| \geq 1$.

□

Corollary 4.15. *Under the previous setting, if we suppose additionaly that (4.46) holds for a continuous, positive, and non decreasing function M from $[0,\infty)$ into itself. Then the semigroup $T(t) = e^{t\mathcal{A}}$ generated by $\mathcal{A}$ is bounded and the bound of the resolvent of $\mathcal{A}$*

$$\|(i\xi\mathbb{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \lesssim \max\{M(|\xi|), \frac{M(|\xi|)^2}{1+|\xi|^2}\},\ \forall \xi \in \mathbb{R} \tag{4.62}$$

holds.

Proof. To prove the first statement, recall (see for instance [9]) that the spectral bound of the operator $\mathcal{A}$ is defined by

$$s(\mathcal{A}) = \sup\ \{\operatorname{Re}\lambda\ :\ \lambda \in Sp(\mathcal{A})\},$$

while

$$s_0(\mathcal{A}) := \inf\ \{x > s(\mathcal{A})\ :\ \exists C_x > 0\ :\ ||(\lambda\mathbb{I} - \mathcal{A})^{-1}|| \leq C_x \text{ whenever } \operatorname{Re}\lambda > x\}.$$

By Theorem 5.2.1 in [9], we know that

$$\begin{aligned} \omega(T) &= \inf \left\{\omega \in \mathbb{R}\ :\ \exists M_\omega > 0 \text{ such that } ||T(t)||_{\mathcal{L}(\mathcal{H})} \leq M_\omega e^{\omega t},\ \forall t \geq 0\right\} \\ &= s_0(\mathcal{A}). \end{aligned} \tag{4.63}$$

First owing to Lemma 4.13, $s(\mathcal{A}) \leq 0$. Secondly combining the estimates (4.45) and (4.51), we directly get

$$\|(\lambda\mathbb{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \lesssim \max\{1, \frac{1}{(\operatorname{Re}\lambda)^2}\}, \forall \lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0.$$

This proves that

$$s_0(\mathcal{A}) \leq 0,$$

and by (4.63), we deduce that $T(t)$ is bounded. Finally the bound (4.62) is a direct consequence of (4.51) and (4.46), since $M(x) \geq M(0) > 0$. □

This result combined with the frequency domain approach yields the following decay rates of the semigroup generated by $\mathcal{A}$. We start with the exponential decay.

Corollary 4.16. *Assume that $\mathcal{A}_w$ generates a C_0-semigroup of contractions $(T_w(t))_{t\geq 0}$ that is exponentially stable, namely*

$$\|T_w(t)U\|_{\mathcal{H}_w} \leq Me^{-\omega t}\|U\|_{\mathcal{H}_w}, \forall U \in \mathcal{H}_w,$$

for some positive constants M and ω. Then the semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ is exponentially stable in $\mathcal{H}$.

Proof. By a well-known result due to Huang and Prüss [50, 101], $(e^{t\mathcal{A}_w})_{t\geq 0}$ is exponentially stable in $\mathcal{H}_w$ if and only if $i\mathbb{R} \cap \sigma(\mathcal{A}_w) = \emptyset$ and (4.46) with $M(x) = 1$ holds. Hence by Corollary 4.15, (4.62) holds with $M(x) = 1$, and again applying Huang/Prüss theorem to $\mathcal{A}$, we conclude. □

For polynomial decays, we replace Huang/Prüss theorem by Borichev-Tomilov theorem [14, Theorem 2.4] to obtain the next result.

Corollary 4.17. *Assume that the semigroup $(e^{t\mathcal{A}_w})_{t\geq 0}$ is polynomially stable in $\mathcal{H}_w$, namely there exists a positive real number ℓ such that*

$$||e^{t\mathcal{A}_w}U_0||_{\mathcal{H}_w} \lesssim t^{-\frac{1}{\ell}}||U_0||_{\mathcal{D}(\mathcal{A}_w)}, \quad \forall U_0 \in \mathcal{D}(\mathcal{A}_w), \quad \forall t > 1. \tag{4.64}$$

Then the semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ is polynomially stable in $\mathcal{H}$, i.e.,

$$||e^{t\mathcal{A}}U||_{\mathcal{H}} \lesssim t^{-\frac{1}{2\ell-2}}||U||_{\mathcal{D}(\mathcal{A})}, \quad \forall U \in \mathcal{D}(\mathcal{A}), \quad \forall t > 1,$$

if $\ell > 2$, while if $\ell \leq 2$, one has

$$||e^{t\mathcal{A}}U||_{\mathcal{H}} \lesssim t^{-\frac{1}{\ell}}||U||_{\mathcal{D}(\mathcal{A})}, \quad \forall U \in \mathcal{D}(\mathcal{A}), \quad \forall t > 1,$$

Proof. As the semigroup generated by $\mathcal{A}_w$ is bounded and $i\mathbb{R}\cap\sigma(\mathcal{A}_w) = \emptyset$, by [14, Theorem 2.4], (4.64) holds if and only if (4.46) holds with $M(x) = 1 + x^\ell$ holds. As before the conclusion follows with the help of Corollary 4.15, and again applying [14, Theorem 2.4] to $\mathcal{A}$ by noticing that

$$\max\{1+|\xi|^\ell, 1+|\xi|^{2\ell-2}\} = \begin{cases} 1+|\xi|^{2\ell-2}, \text{ if } \ell > 2, \\ 1+|\xi|^\ell, \ \ell \leq 2. \end{cases}$$

□

For lower decay, the equivalence between the semigroup decay rate and the asymptotic behavior of the resolvent on the imaginary axis is not guaranteed, but by taking advantage of a result due to Batty and Duyckaerts [13, Theorem 1.5] and our Corollary 4.15, we get as before the next corollary.

Corollary 4.18. *Assume that (4.46) holds with a continuous, positive, and non decreasing function M from $[0,\infty)$ into itself. Then the semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ has the following asymptotic decay in $\mathcal{H}$:*

$$||e^{t\mathcal{A}}U||_{\mathcal{H}} \lesssim \frac{1}{\tilde{M}_{\log}^{-1}\left(\frac{t}{C}\right)}||U||_{\mathcal{D}(\mathcal{A})}, \quad \forall U \in \mathcal{D}(\mathcal{A}), \quad \forall t > 1,$$

for some positive constant C, and $\tilde{M}_{\log}$ is defined by

$$\tilde{M}_{log}(x) = \tilde{M}(x)\left(\log(1+\tilde{M}(x)) + \log(1+x)\right), \forall x \geq 0,$$

with $\tilde{M}(x) = \max\{M(x), \frac{M(x)^2}{1+x^2}\}$.

Let us finish this section with some illustrative examples.

Example 4.19. Let us now mention some examples for which the frictional damping in ω_0 guarantees the polynomial stability of the wave equation (5.55).
1) If Ω is the unit square and ω_0 contains a vertical strip of Ω, then it is proved in [76, 109], a polynomial decay rate. Namely in [76], assuming that

$$(a,b) \times (0,1) \subset \omega_0,$$

for some $0 \leq a < b < 1$, it is shown that (4.64) holds with $\ell = 2$. On the contrary if

$$(0,c) \times (0,1) \subset \omega_0,$$

for $0 < c < 1$, it is swhown in [109] that (4.64) holds with $\ell = 3/2$.
2) If Ω is a partially rectangular domain and ω_0 contains the non-rectangular part of Ω, then it is proved in [20] (combined with [14, Theorem 2.4]) that (4.64) holds with $\ell = 2$.
3) Examples of domains Ω and ω_0 leading to (4.64) holds for $\ell > 0$ can be found in [99].

Example 4.20. It was shown in [68] that if $\alpha - \beta$ is smooth and not identically equal to zero and if the boundary of Ω is smooth or convex, then (4.46) holds with $M(x) = e^{Cx}$, for some positive constant C.This yields

$$||e^{t\mathcal{A}}U||_{\mathcal{H}} \lesssim \frac{1}{\log t}||U||_{\mathcal{D}(\mathcal{A})}, \quad \forall U \in \mathcal{D}(\mathcal{A}), \quad \forall t > 1,$$

since $\tilde{M}_{\log}^{-1}(t) \sim \log t$, for t large (see [13, Example 1.6]).

CHAPTER 5
WELL-POSEDNESS AND LONG TIME BEHAVIOR FOR A GENERAL CLASS OF MOORE–GIBSON–THOMPSON EQUATIONS WITH A MEMORY

5.1 Introduction

In this chapter, we consider the well-posedness and the long time behavior of some evolution equations with memory (see (5.4) below), the following Moore–Gibson–Thompson equation with memory

$$\begin{cases} u_{ttt} + \alpha u_{tt} - \Delta u_t - \gamma \Delta u + \int_0^t g(s)\Delta u(t-s)ds = 0, & \text{in} \quad \Omega \times (0,\infty), \\ u = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(0,.) = u_0, \qquad u_t(0,.) = u_1, \qquad u_{tt}(0,.) = u_2, & \text{in} \quad \Omega, \end{cases} \tag{5.1}$$

being a particular instance. Here and below, Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, γ is a positive constant, g is a memory kernel that is exponentially decaying (more precisely it satisfies the assumptions (A1) to (A3) below) and α is here a function that is supposed to be in $L^\infty(\Omega)$.
The chapter is organized as follows: The well-posedness of our problem is proved in section 5.2 by using semigroup theory and a bounded perturbation argument; further under the assumption

$$\alpha \geq \gamma, \tag{5.2}$$

we show the boundedness of the generated semigroup using a Lyapunov technique. In section 5.3, we prove that the semigroup is semi-uniformly stable using Batty criterion. The lack of exponential decay is shown in section 5.4 by building a sequence of eigenvalues (of the operator associated with our system) that approach the imaginary axis, this construction requires that the absolute value of the derivatives up to order 3 of the memory kernel are exponentially decaying at infinity. In section 5.5, we prove a polynomial decay of the solution in $t^{-\frac{1}{2}}$ for initial data in the domain of the associated operator, and the optimality is derived. Finally a stronger polynomial decay using a resolvent estimate of the wave equation with a frictional interior damping is obtained in section 5.6 and some illlustrative examples are given.
Let us finish this introduction with some notation used in the chapter. The usual norm and semi-norm of $H^s(\Omega)$ ($s \geq 0$) are denoted by $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$, respectively. In the same spirit, we denote by $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_{1,\Omega}$ the inner product in $L^2(\Omega)$ and in $H_0^1(\Omega)$ (the subspace of functions in $H^1(\Omega)$ with a zero trace on the boundary $\partial\Omega$ of Ω), namely

$$\begin{aligned} (u,u')_\Omega &= \int_\Omega u\bar{u}'dx, \forall u, u' \in L^2(\Omega), \\ (u,u')_{1,\Omega} &= \int_\Omega \nabla u \cdot \nabla \bar{u}'dx, \forall u, u' \in H_0^1(\Omega). \end{aligned}$$

For $s = 0$ we drop the index s. By $a \lesssim b$, we mean that there exists a constant $C > 0$ independent of a, b, such that $a \leq Cb$, while $a \sim b$ means that both $a \lesssim b$ and $b \lesssim a$ hold.

5.2 Well-posedness of the problem

In this section, we first state some assumptions, then we prove the well-posedness of system (5.1). The nonzero function g, called the memory kernel, is supposed to be in $W^{1,1}(\mathbb{R}^+)$ with g' absolutely continuous on $\mathbb{R}^+ = (0, \infty)$ and to satisfy

(A1) $g : \mathbb{R}_+ \to (0, \infty)$ is a nonincreasing function such that

$$g(0) > 0, \qquad \gamma - \int_0^\infty g(s)ds = \ell > 0.$$

(A2) For some $\delta > 0$, the functional g satisfies

$$g'(t) \leq -\delta g(t), \qquad \text{for all} \quad t \in (0, \infty).$$

(A3) $g'' \geq 0$ almost everywhere.

Further, note that g' being an absolutely continuous function, it has a continuous extension at 0, and its derivative g'' is integrable over interval $(0, R)$ with $R > 0$, see for instance [69, Corollary 3.9 and Theorem 3.30]. As a consequence of (A1) and (A3), we deduce that g' is nonpositive and nondecreasing and therefore

$$\lim_{R\to\infty} g'(R) < \infty.$$

This implies that

$$\lim_{R\to\infty} \int_0^R g''(t)\,dt = \lim_{R\to\infty} g'(R) - g'0) < \infty,$$

and by Fatou's lemma, g'' is integrable on $\mathbb{R}_+$. In other words, under the previous assumptions, $g \in W^{2,1}(\mathbb{R}^+)$. To prove the well-posedness of the equation (5.1), we adopt the same procedure used in chapter 1 and instead of taking the whole space we redefine the framework of the spaces and their norms in Ω. Indeed, we first set the problem in the so-called history space setting to get

$$\eta(x, t, s) = \eta^t(x, s) = u(x, t) - u(x, t - s). \tag{5.3}$$

Therefore, the problem (5.1) is written as follows

$$\begin{cases} u_{ttt} + \alpha u_{tt} - \Delta u_t - \ell \Delta u + \displaystyle\int_0^\infty g(s)\Delta\eta^t(s)ds = 0, \\ \eta_t^t(x, s) + \eta_s^t(x, s) = u_t(x, t) \\ u = 0 \\ u(0, .) = u_0, \quad u_t(0, .) = u_1, \quad u_{tt}(0, .) = u_2, \quad \eta^0(., s) = \eta^0(s), \end{cases} \tag{5.4}$$

For more details, see Chapter 1 (Section 1.2. The Hilbert space associated with (5.4) is given as

$$\mathcal{H} = H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times \mathcal{M}.$$

Similar inner product as well as the norm corresponding to $\mathcal{H}$ to those defined in Section 1.2, ((1.15) and (1.16), respectively, with $\Omega \in \mathbb{R}^N$), are given in this chapter. By the change of variables $u_t = v$ and $u_{tt} = w$, we rewrite (5.4) as

$$\begin{cases} u_t = v, \\ v_t = w, \\ w_t = -\alpha w + \Delta v + \ell \Delta u + \int_0^\infty g(s)\Delta \eta^t(s) ds, \\ \eta_t^t = v - \eta_s^t. \end{cases} \tag{5.5}$$

Consequently, the system (5.5) is an evolution equation in $\mathcal{H}$, that is,

$$\begin{cases} \dfrac{d}{dt} U(t) = AU(t), \qquad t \in (0, +\infty), \\ U(x,0) = U_0, \end{cases} \tag{5.6}$$

where $U(t) = (u(t), v(t), w(t), \eta^t)$, $U_0 = (u_0, v_0, w_0, \eta^0)$ is arbitrary in $\mathcal{H}$, and A is the linear operator given by

$$AU = A \begin{pmatrix} u \\ v \\ w \\ \eta^t \end{pmatrix} = \begin{pmatrix} v \\ w \\ -\alpha w + \Delta\left(v + \ell u + \int_0^\infty g(s)\eta^t(s) ds\right) \\ v + T\eta^t \end{pmatrix}, \tag{5.7}$$

for all $U = (u, v, w, \eta)^\top \in D(A)$, where its domain $D(A)$ is given by

$$D(A) = \left\{ (u, v, w, \eta)^\top \in \mathcal{H} \;\middle|\; \begin{matrix} w \in H_0^1(\Omega) \\ v + \ell u + \int_0^\infty g(s)\eta^t(s) ds \in H^2(\Omega) \\ \eta^t \in D(T) \end{matrix} \right\}. \tag{5.8}$$

Theorem 5.1. *Assume that (A1)-(A3) are satisfied. Then under the assumption $\alpha \in L^\infty(\Omega)$ the linear operator A is the infinitesimal generator of a C_0-semigroup $(S(t) = e^{tA})_{t \geq 0}$ in $\mathcal{H}$.*

Proof. Denote by A_0 the operator A corresponding to the case $\alpha = \gamma$, namely

$$A_0 \begin{pmatrix} u \\ v \\ w \\ \eta^t \end{pmatrix} = \begin{pmatrix} v \\ w \\ -\gamma w + \Delta\left(v + \ell u + \int_0^\infty g(s)\eta^t(s) ds\right) \\ v + T\eta^t \end{pmatrix},$$

for all $U = (u, v, w, \eta)^\top \in D(A_0) = D(A)$. According to Theorem 3.2 of [37], A_0 generates a contractions semigroup in $\mathcal{H}$. As $A - A_0$ is given by

$$(A - A_0)\begin{pmatrix} u \\ v \\ w \\ \eta^t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ (\gamma - \alpha)w \\ 0 \end{pmatrix}. \tag{5.9}$$

We directly see that $A - A_0$ is a bounded operator from $\mathcal{H}$ into itself and by Theorem 3.1.1 of [94], we conclude that A generates a C_0-semigroup in $\mathcal{H}$. □

Corollary 5.2. *Under the assumptions of Theorem 5.1, for any $U_0 \in D(A)$, the problem (5.4) has a unique classical solution in the class*

$$(u, u_t, u_{tt}, \eta^t) \in C^1([0, \infty); \mathcal{H}) \cap C([0, \infty); D(A)).$$

Corollary 5.3. *In addition to the assumptions of Theorem 5.1, we suppose that:*

(A4) *$\alpha \in L^\infty(\Omega)$ is real valued and satisfies $\alpha - \gamma \geq 0$, a. e. in Ω.*

Then the semigroup $(S(t))_{t\geq 0}$ is bounded in $\mathcal{H}$, namely there exists a positive real number $M \geq 1$ such that

$$\|S(t)\| \leq M.$$

Proof. For $U_0 \in D(A)$, denote by $U(t) = S(t)U_0$, and introduce the modified energy

$$\tilde{E}(t) = \frac{1}{2}\left(\|U(t)\|^2_{\mathcal{H}} + \gamma\|\sqrt{\alpha - \gamma}u_t\|^2_{\Omega}\right).$$

Then by the previous corollary, U is a strong solution of (5.6), and $\tilde{E}$ is differentiable with

$$\begin{aligned} \tilde{E}'(t) &= \mathrm{Re}(U'(t), U(t)_{\mathcal{H}} + \gamma\,\mathrm{Re}((\alpha - \gamma)u_t, u_{tt})_{\Omega} \\ &= \mathrm{Re}(AU(t), U(t))_{\mathcal{H}} + \gamma\,\mathrm{Re}((\alpha - \gamma)u_t, u_{tt})_{\Omega}, \quad \forall t > 0. \end{aligned}$$

But recalling (5.9), we have

$$\begin{aligned} \tilde{E}'(t) &= \mathrm{Re}(A_0U(t), U(t))_{\mathcal{H}} + \mathrm{Re}((\gamma - \alpha)(u_{tt} + \gamma u_t), u_{tt})_{\Omega} + \gamma\,\mathrm{Re}((\alpha - \gamma)u_t, u_{tt})_{\Omega} \\ &= \mathrm{Re}(A_0U(t), U(t))_{\mathcal{H}} + \mathrm{Re}((\gamma - \alpha)u_{tt}, u_{tt})_{\Omega}, \quad \forall t > 0. \end{aligned}$$

Hence using the dissipativity of A_0 and assumption (A4), we deduce that

$$\tilde{E}'(t) \leq 0, \forall t > 0.$$

This means that, this energy is nonincreasing, namely

$$\tilde{E}(t) \leq \tilde{E}(0), \forall t \geq 0.$$

This, in particular, implies that

$$\|U(t)\|^2_{\mathcal{H}} \leq \|U_0\|^2_{\mathcal{H}} + \gamma\|\sqrt{\alpha - \gamma}u_1\|^2_{\Omega},$$

if $U_0 = (u_0, u_1, u_2, \eta^t(t=0))^\top$. But, we clearly have

$$\gamma \|\sqrt{\alpha - \gamma} u_1\|_\Omega \leq C|u_1|_{1,\Omega},$$

for some positive constant C independent of u_1. According to the definition of the norm $\|\cdot\|_{\mathcal{H}}$, we deduce that

$$\|U(t)\|^2_{\mathcal{H}} \leq M^2 \|U_0\|^2_{\mathcal{H}},$$

where $M^2 = 1 + \frac{C\gamma^2}{\gamma - \ell}$. Since $D(A)$ is dense in $\mathcal{H}$, this property remains valid for all $U_0 \in \mathcal{H}$, hence the result is fulfilled. □

5.3 Strong stability

One simple way to prove the strong stability of (5.6) is to use a theorem due to Arendt & Batty and Lyubich & Vũ (see [10, 77]). But here we prove a stronger result based on the following abstract result of Batty [12] (see also [13, Theorem 1.1] or [27, Theorem 3.4]).

Theorem 5.4. *Let X be a Banach space and $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup generated by A on X. Then $\sigma(A) \cap i\mathbb{R} = \emptyset$, if and only if*

$$\lim_{t\to\infty} \|T(t)A^{-1}\| = 0.$$

A bounded semigroup $(T(t))_{t\geq 0}$ satisfying one of these two conditions of this Theorem is called semi-uniformly stable (see [13, Definition 1.2]) and in that case the decay rate may be characterized via the quantity

$$M(s) = \sup\{\|(i\xi\mathbb{I} - A)^{-1}\| : |\xi| \leq s\}.$$

In our case, this decay rate is precisely quantified in Sections 5.5 and 5.6. Note further that a semi-uniformly stable semigroup is strongly stable due to its boundedness and the density of the domain of its generator.
We now want to take advantage of this Theorem. As Corollary 5.3 guarantees the boundedness of the semigroup generated by A, but as the resolvent of our operator is not compact, we need to analyze the full spectrum of A on the imaginary axis. This is the goal of the next Lemma.

Lemma 5.5. *Assume that (A1)-(A4) hold, then we have the following*

$$\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subset \rho(A), \tag{5.10}$$

where $\rho(A)$ is the resolvent set of the operator A.

Proof. Let $\lambda \in \mathbb{C}_+$ and $F = (f, g, h, p) \in \mathcal{H}$, we look for a solution $U = (u, v, w, \eta^t) \in D(A)$ to the equation

$$\lambda U - AU = F, \tag{5.11}$$

which component-wise becomes

$$\lambda u - v = f, \tag{5.12a}$$

$$\lambda v - w = g, \tag{5.12b}$$

$$(\lambda + \alpha)w - \Delta v - \ell \Delta u - \int_0^\infty g(s)\Delta \eta^t(s)ds = h, \tag{5.12c}$$

$$\lambda \eta^t - v - T\eta^t = p. \tag{5.12d}$$

From (5.12a), (5.12b) and (5.12d) we obtain

$$v = \lambda u - f, \tag{5.13a}$$

$$w = \lambda v - g = \lambda^2 u - \lambda f - g, \tag{5.13b}$$

$$\eta^t = (1 - e^{-\lambda s})u - \left(\tfrac{1-e^{-\lambda s}}{\lambda}\right) f + \int_0^s e^{-\lambda(s-\tau)} p(\tau)d\tau, \tag{5.13c}$$

with the convention that $\frac{1-e^{-\lambda s}}{\lambda} = s$ if $\lambda = 0$.

Assuming for the moment that such a U exists, then plugging the three identities above (5.13a), (5.13b) and (5.13c) into (5.12c), we get

$$\begin{aligned} &-\left(\lambda + \ell + \int_0^\infty g(s)(1 - e^{-\lambda s})ds\right)\Delta u + (\alpha + \lambda)\lambda^2 u \\ &= h + (\alpha + \lambda)(\lambda f + g) - \left(1 + \int_0^\infty g(s)\left(\frac{1 - e^{-\lambda s}}{\lambda}\right)ds\right)\Delta f \\ &+ \int_0^\infty g(s)\left[\int_0^s e^{-\lambda(s-\tau)}\Delta p(\tau)d\tau\right] ds, \qquad \text{in} \quad \mathcal{D}'(\Omega). \end{aligned} \tag{5.14}$$

Note that this identity only hold in the distributional sense, since we have sent the contribution of Δf and Δp in the right-hand side. Accordingly (see [2] or [107]), it is equivalent to

$$a_\lambda(u, v) = F_\lambda(v), \qquad \forall v \in \mathcal{D}(\Omega), \tag{5.15}$$

where

$$a_\lambda(u, v) = \int_\Omega \left[(\alpha + \lambda)\lambda^2 u\bar{v} + \left(\lambda + \ell + \int_0^\infty g(s)(1 - e^{-\lambda s})ds\right)\nabla u \nabla \bar{v}\right] dx$$

and

$$\begin{aligned} F_\lambda(v) = \int_\Omega \Bigg[&\left(1 + \int_0^\infty g(s)\left(\frac{1 - e^{-\lambda s}}{\lambda}\right)ds\right)\nabla f \nabla \bar{v} + (h + (\alpha + \lambda)(\lambda f + g))\bar{v} \\ &- \int_0^\infty g(s)\left(\int_0^s e^{-\lambda(s-\tau)} \nabla p \nabla \bar{v}\, d\tau\right)ds\Bigg] dx, \end{aligned}$$

for all $u, v \in H_0^1(\Omega)$. Since $\mathcal{D}(\Omega)$ is dense into $H_0^1(\Omega)$, the identity (5.15) remains valid in the whole $H_0^1(\Omega)$, namely

$$a_\lambda(u,v) = F_\lambda(v), \qquad \forall v \in H_0^1(\Omega). \tag{5.16}$$

Now, we prove that problem (5.16) admits a unique solution $u \in H_0^1(\Omega)$. For that purpose, we deal with the following two cases.

- For $\lambda = 0$. we have

$$a_0(u,v) = \ell \int_\Omega \nabla u \nabla \bar{v} dx,$$

  and

$$F_0(v) = \int_\Omega \Big[\Big(1 + \int_0^\infty sg(s)ds\Big) \nabla f \nabla \bar{v} + (h + \alpha g)\bar{v} \\ - \int_0^\infty g(s) \Big(\int_0^s \nabla p \nabla \bar{v} \, d\tau \Big) ds \Big] dx.$$

  We first observe that a_0 is a sesquilinear, continuous and thanks to Poincaré inequality it satisfies

$$a_0(u,u) = \ell \int_\Omega |\nabla u|^2 \geq \|u\|_\Omega^2, \qquad \forall u \in H_0^1(\Omega),$$

  hence, the coerciveness of a_0. Since F_0 is clearly an antilinear and continuous form on $H_0^1(\Omega)$, by Lax-Milgram's lemma, we deduce that for $\lambda = 0$, (5.16) has a unique solution $u \in H_0^1(\Omega)$. Moreover, by setting $v = -f$, $w = -g$ and

$$\eta^t(s) = sf + \int_0^s p(\tau) d\tau,$$

  we find that $U = (u, v, w, \eta^t)$ belongs to $D(A)$ and is the unique solution of (5.11) for $\lambda = 0$. Consequently $0 \in \rho(A)$.

- For $\lambda \neq 0$. The proof is a slightly different from the previous one. First, we introduce the sequilinear form b_λ by

$$b_\lambda(u,v) = \int_\Omega \Big(\lambda + \ell + \int_0^\infty g(s)(1 - e^{-\lambda s}) ds \Big) \nabla u \nabla \bar{v} dx \qquad \forall u, v \in H_0^1(\Omega).$$

  Moreover, we define the operators A_λ and B_λ as follow

$$A_\lambda : H_0^1(\Omega) \to H^{-1}(\Omega)) = (H_0^1(\Omega))' : u \to A_\lambda u, \\ B_\lambda : H_0^1(\Omega) \to H^{-1}(\Omega)) : u \to B_\lambda u,$$

  where, for all $u, v \in H_0^1(\Omega)$, we set

$$\langle A_\lambda u, v \rangle = a_\lambda(u,v), \\ \langle B_\lambda u, v \rangle = b_\lambda(u,v).$$

  It is easy to check that b_λ is a sesquilinear and continuous form on $H_0^1(\Omega)$ and since

$\operatorname{Re}\left(\lambda+\ell+\int_0^\infty g(s)(1-e^{-\lambda s})ds\right) \geq \ell$, it is coercive. So, according to Lax-Milgram lemma, the operator B_λ is an isomorphism.
Next, as $A_\lambda - B_\lambda = \lambda^2(\alpha+\lambda)\mathbb{I}$, it is a compact operator from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$, we then deduce A_λ is a Fredholm operator of index zero. It is therefore sufficient to show that the operator is injective to deduce that it is an isomorphism.
For this reason, let $u \in \ker A_\lambda$, then, it becomes a solution of (5.16) with $F_\lambda = 0$, which means that

$$a_\lambda(u,v) = 0 \qquad \forall v \in H_0^1(\Omega). \tag{5.17}$$

This implies that

$$\begin{cases} v = \lambda u, \\ w = \lambda^2 u, \\ \eta^t(s) = (1-e^{-\lambda s})u, \end{cases} \tag{5.18}$$

and that $U = (u, v, w, \eta^t)^\top$ belongs to $D(A)$ and satisfies $\lambda U - AU = 0$. In addition, owing to Lemma 3.4 of [37], we have

$$\operatorname{Re}(A_0U,U)_{\mathcal{H}} = -\gamma\|\eta^t\|_{\mathcal{M}}^2 - \int_0^\infty g''(s)|\eta^t(s)|_{1,\Omega}^2\,ds, \tag{5.19}$$

and by (5.9) and (5.18), we get

$$\begin{aligned}\operatorname{Re}(AU,U)_{\mathcal{H}} = -\gamma\|\eta^t\|_{\mathcal{M}}^2 - \int_0^\infty g''(s)|\eta^t(s)|_{1,\Omega}^2\,ds \\ - \operatorname{Re}\left(\lambda^2(\bar\lambda^2+\gamma\bar\lambda)\right)((\alpha-\gamma)u,u)_\Omega.\end{aligned}$$

Using the identity $AU = \lambda U$, we find that

$$\begin{aligned}(\operatorname{Re}\lambda)\|U\|_{\mathcal{H}}^2 = -\gamma\|\eta^t\|_{\mathcal{M}}^2 - \int_0^\infty g''(s)|\eta^t(s)|_{1,\Omega}^2\,ds \\ - \operatorname{Re}\left(\lambda^2(\bar\lambda^2+\gamma\bar\lambda)\right)((\alpha-\gamma)u,u)_\Omega.\end{aligned} \tag{5.20}$$

As

$$\lambda^2(\bar\lambda^2+\gamma\bar\lambda) = |\lambda|^4 + \gamma|\lambda|^2\lambda,$$

we directly see that

$$\operatorname{Re}\left(\lambda^2(\bar\lambda^2+\gamma\bar\lambda)\right) = |\lambda|^4 + \gamma|\lambda|^2\operatorname{Re}\lambda,$$

and is therefore positive (recalling that $\lambda \neq 0$ and $\operatorname{Re}\lambda \geq 0$). As the left-hand side of (5.20) is non-negative and its right-hand side is non-positive, we deduce that both terms are zero, which in particular implies that

$$\eta^t = 0.$$

Finally as $(1 - e^{-\lambda s}) \neq 0$, for all $s > 0$ and $\lambda \neq 0$, we find

$$u = 0, \qquad \text{in} \quad \Omega.$$

By (5.18) this yields $v = w = 0$, thus, the injectivity of A_λ is proved.
In summary, for $\lambda \neq 0$, A_λ is an isomorphism which guarantees that problem (5.16) admits a unique solution $u \in H_0^1(\Omega)$. Defining v, w and η^t by (5.13a), (5.13b) and (5.13c), respectively, we again find $U = (u, v, w, \eta^t)^T \in D(A)$ that is the unique solution of (5.11).

We therefore conclude that (5.10) holds. □

Remark 5.6. *In the framework of the previous Lemma, we may note that by Corollary 5.3 and a well-known criterion on the generators of bounded C_0-semigroup in Hilbert spaces (see for instance [14, Lemma 2.1]), the set $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$ is automatically included into $\rho(A)$. Therefore one could reduce the previous proof to the case $\lambda \in i\mathbb{R}$. We have done it for all $\lambda \in \mathbb{C}_+$ for the sake of completeness.*

As a direct consequence of Corollary 5.3 and Lemma 5.5, Lemma 5.4 yields

Corollary 5.7. *Under the assumptions of Lemma 5.5, $(S(t))_{t\geq 0}$ is semi-uniformly stable, in particular it is stable, i.e.,*

$$S(t)U_0 \to 0 \text{ in } \mathcal{H}, \text{ as } t \to \infty, \ \forall U_0 \in \mathcal{H}.$$

5.4 Lack of exponential decay

It was proved in [37, §4], in the case $\alpha = \gamma$, and $g(s) = \rho e^{-\delta s}$ with $\rho, \delta > 0$ that the energy of the solution of (5.1) does not decay exponentially, which implies that the semigroup $(S(t))_{t\geq 0}$ is not exponentially stable. We here prove a similar property of this semigroup in the case $\alpha = \gamma$, but for any memory kernel satisfying (A1) to (A3) and the additional assumptions that $g \in W^{3,1}(\mathbb{R}_+)$ with

$$|g^{(k)}(s)| \leq K e^{-\delta_0 s}, \text{ for almost all } s \in \mathbb{R}_+, \tag{5.21}$$

for some positive constants K, δ_0 with $\delta_0 \leq \delta$ and for $k = 0, 1, 2, 3$. This condition allows to give a meaning to the Laplace transform $\mathcal{L}g(\lambda)$ of g (as well as of its derivative up to the order 3) for all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > -\delta_0$, where we recall that

$$\mathcal{L}g(\lambda) = \int_0^\infty e^{-\lambda s} g(s)\, ds.$$

It also allows to prove the next technical result.

Lemma 5.8. *If the memory kernel $g \in W^{3,1}(\mathbb{R}_+)$ satisfy assumption (A1) to (A3) and (5.21), then*

$$\mathcal{L}g(\lambda) = \frac{g(0)}{\lambda} + \frac{g'(0)}{\lambda^2} + o\left(\frac{1}{|\lambda|^2}\right), \tag{5.22}$$

for all $\lambda \in \mathbb{C}$ *such that* $\operatorname{Re}\lambda > -\frac{\delta_0}{2}$.

Proof. Owing to the property (5.21) for $k = 0$ and 1, the following identity

$$\lambda \mathcal{L}g(\lambda) = g(0) + \mathcal{L}g'(\lambda), \tag{5.23}$$

holds for all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > -\delta_0$. Using this property to g' and g'', we also have

$$\lambda \mathcal{L}g'(\lambda) = g'(0) + \mathcal{L}g''(\lambda), \tag{5.24}$$

$$\lambda \mathcal{L}g''(\lambda) = g''(0) + \mathcal{L}g'''(\lambda), \tag{5.25}$$

for all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > -\delta_0$, and consequently

$$\mathcal{L}g(\lambda) = \frac{g(0)}{\lambda} + \frac{g'(0)}{\lambda^2} + \frac{\mathcal{L}g''(\lambda)}{\lambda^2}, \tag{5.26}$$

for all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > -\delta_0$. Since (5.21) implies that

$$|\mathcal{L}g'''(\lambda)| \leq \frac{K}{\operatorname{Re}\lambda + \delta_0},$$

for all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > -\delta_0$, we will get

$$|\mathcal{L}g'''(\lambda)| \leq \frac{2K}{\delta_0},$$

for all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > -\frac{\delta_0}{2}$. This estimate and the identity (5.25) allows to deduce that

$$|\lambda \mathcal{L}g''(\lambda)| \leq |g''(0)| + \frac{2K}{\delta_0},$$

for all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > -\frac{\delta_0}{2}$. Using this property in (5.26) leads to (5.22). □

Example 5.9. *Examples of memory kernels* $g \in W^{3,1}(\mathbb{R}_+)$ *that satisfy assumption (A1) to (A3) and* (5.21) *are numerous. The simplest example is* $g(s) = \rho e^{-\delta s}$ *with* $\rho, \delta > 0$. *The second example is kernel of the form*

$$g(t) = \sum_{j=1}^{J} P_j(t) e^{-x_j t},$$

where x_j *are positive real number,* P_j *are a real-valued polynomials of degree* d_j *with coefficients chosen so that* g *is positive,* $P_j \geq 0$ $P_j' + \delta P_j \leq 0$ *for some* $\delta < x_j$ *and* $P_j'' \geq 0$. *As third example, we can mention*

$$g(t) = (a + bt)^{-1} e^{-\delta t},$$

with δ, b *positive real numbers and* a *a non negative real number.*

Finally we denote by $(\lambda_k^2)_{k=1}^{\infty}$ the set of eigenvalues (repeated according to its multiplicity) of the positive Laplace operator with Dirichlet boundary conditions in Ω and let φ_k, the associated orthonomalized eigenvector, namely satisfying

$$-\Delta\varphi_k = \lambda_k^2\varphi_k.$$

Without loss of generality we may assume that λ_k is positive.

Lemma 5.10. *Assume that the memory kernel $g \in W^{3,1}(\mathbb{R}_+)$ satisfies assumptions (A1) to (A3) and (5.21) and that $\alpha = \gamma$, then there exists k_0 large enough such that the operator A has the eigenvalues $\lambda_{k,\pm}$, for all $k \geq k_0$ that satisfy*

$$\lambda_{k,\pm} = \pm i\lambda_k \pm i\frac{g(0)}{2\lambda_k} + \frac{g'(0) - \gamma g(0)}{2\lambda_k^2} + o\left(\frac{1}{\lambda_k^2}\right), \forall k \geq k_0. \tag{5.27}$$

Its associated eigenvector U_k is in the form

$$U_{k,\pm} = c_{k,\pm}\begin{pmatrix} \varphi_k \\ \lambda_{k,\pm}\varphi_k \\ \lambda_{k,\pm}^2\varphi_k \\ (1 - e^{-\lambda_{k,\pm}s})\varphi_k \end{pmatrix}, \tag{5.28}$$

where $c_{k,\pm} \neq 0$ is a normalization factor chosen such that

$$\|U_k^{\pm}\|_{\mathcal{H}} = 1.$$

Proof. Let $U = (u, v, w, \eta^t)^\top \in D(A)$ be an eigenvector of the operator A of eigenvalue $\lambda \in \mathbb{C}$ such that $-\frac{\delta_0}{2} < \operatorname{Re}\lambda < 0$. Then it satisfies $AU = \lambda U$, i.e. it satisfies (5.12a)-(5.12d) with $f = g = h = p = 0$. Arguing as in the proof of Lemma 5.5, we obtain (see (5.13a)-(5.13c) and (5.14))

$$v = \lambda u, \tag{5.29}$$
$$w = \lambda^2 u, \tag{5.30}$$
$$\eta^t(s) = (1 - e^{-\lambda s})u, \tag{5.31}$$

and

$$-\left(\lambda + \ell + \int_0^\infty g(s)(1 - e^{-\lambda s})ds\right)\Delta u + (\alpha + \lambda)\lambda^2 u = 0.$$

By the definition of ℓ, and since $\alpha = \gamma$, this identity is equivalent to

$$-(\lambda + \gamma - \mathcal{L}g(\lambda))\Delta u + (\lambda + \gamma)\lambda^2 u = 0. \tag{5.32}$$

Due to Lemma 5.8, $\lambda + \gamma - \mathcal{L}g(\lambda)$ is invertible for $|\lambda|$ large enough, and for such a λ, (5.32) is equivalent to

$$\Delta u = \frac{(\lambda + \gamma)\lambda^2}{\lambda + \gamma - \mathcal{L}g(\lambda)}u = \lambda^2(1 + r(\lambda))u, \tag{5.33}$$

where

$$r(\lambda) = \frac{\mathcal{L}g(\lambda)}{\lambda + \gamma - \mathcal{L}g(\lambda)}. \tag{5.34}$$

Hence a non trivial solution u of (5.33) exists if and only if

$$\lambda^2(1 + r(\lambda)) = -\lambda_k^2,$$

and in this case u is a multiple of φ_k. Setting $F_k(\lambda) = \lambda^2(1 + r(\lambda)) + \lambda_k^2$, the previous identity is equivalent to

$$F_k(\lambda) = 0. \tag{5.35}$$

Now we set $F_{0,k}(\lambda) = \lambda^2 + \lambda_k^2$, whose roots are

$$\lambda = \pm i\lambda_k.$$

Now we take advantage of Rouché's theorem to find an asymptotic behavior of the roots of F_k. More precisely we want to apply Rouché's theorem in the ball $B(\pm i\lambda_k, \rho_k)$ with $0 < \rho_k < \frac{\delta_0}{2}$ appropriately chosen so that

$$|F_k(z) - F_{0,k}(z)| < |F_0(z)|, \forall z = \pm i\lambda_k + \rho_k e^{i\theta}, \theta \in [0, 2\pi].$$

First due to Lemma 5.8 one has

$$|r(\lambda)| \lesssim |\lambda|^{-2}.$$

Consequently as $F_k(z) - F_{0,k}(z) = z^2 r(z)$, we deduce that

$$|F_k(z) - F_{0,k}(z)| \leq C, \forall z = \pm i\lambda_k + \rho_k e^{i\theta}, \theta \in [0, 2\pi],$$

for k large enough and a positive constant C (independent of k). Now for $z = \pm i\lambda_k + \rho_k e^{i\theta}, \theta \in [0, 2\pi]$, we have

$$\begin{aligned} F_{0,k}(z) &= \pm 2i\lambda_k\rho_k e^{i\theta} + \rho_k^2 e^{2i\theta} \\ &= 2\lambda_k\rho_k e^{i\theta}(\pm i + \frac{\rho_k e^{i\theta}}{2\lambda_k}), \end{aligned}$$

and consequently

$$|F_{0,k}(z)| = 2\lambda_k\rho_k| \pm i + \frac{\rho_k e^{i\theta}}{2\lambda_k}|.$$

As for k large enough, $| \pm i + \frac{\rho_k e^{i\theta}}{2\lambda_k}| \geq \frac{1}{2}$, we deduce for such a k that

$$|F_{0,k}(z)| \geq \lambda_k\rho_k,$$

hence the choice $\rho_k = \frac{C}{2\lambda_k}$, with k large enough, allows to apply Rouché's theorem and conclude that $F_{0,k}$ has a root $\lambda_{k,\pm}$ in $B(\pm i\lambda_k, \frac{C}{2\lambda_k})$. In a second step, we write

$$\lambda_{k,\pm} = \pm i\lambda_k + r_{k,\pm}, \tag{5.36}$$

where the remainder $r_{k,\pm}$ satisfies

$$|r_{k,\pm}| = O(\frac{1}{\lambda_k}), \tag{5.37}$$

which implies that

$$\frac{1}{\lambda_{k,\pm}} = \mp i\frac{1}{\lambda_k} + O(\frac{1}{\lambda_k^3}). \tag{5.38}$$

Plugging this splitting (5.36) in (5.35), we find

$$\pm 2i\lambda_k r_{k,\pm} + r_{k,\pm}^2 + \lambda_{k,\pm}^2 r(\lambda_{k,\pm}) = 0.$$

But using (5.34), (5.22), and (5.38), one direclty sees that

$$\lambda_{k,\pm}^2 r(\lambda_{k,\pm}) = g(0) + O(\frac{1}{\lambda_k}).$$

Inserting this property in the previous identity we get

$$\pm 2i\lambda_k r_{k,\pm} + r_{k,\pm}^2 + g(0) + O(\frac{1}{\lambda_k}) = 0,$$

and using (5.37), we obtain

$$\pm 2i\lambda_k r_{k,\pm} + g(0) + O(\frac{1}{\lambda_k}) = 0.$$

As a consequence, we find

$$r_{k,\pm} = \pm\frac{ig(0)}{2\lambda_k} + O(\frac{1}{\lambda_k^2}),$$

that, owing to (5.36), leads to

$$\lambda_{k,\pm} = \pm i\lambda_k + \pm\frac{ig(0)}{2\lambda_k} + s_k, \tag{5.39}$$

with a remainder s_k satisfying $s_k = O(\frac{1}{\lambda_k^2})$. Finally plugging this second splitting (5.39) in (5.35), the same argument than before leads to (5.27). □

The lack of exponential stability directly follows from this Lemma.

Corollary 5.11. *Under the assumptions of Lemma 5.10, system (5.6) (or equivalently (5.1)) is not exponentially stable.*

Proof. Assume that system (5.6) is exponentially stable, in other words, assume that there exist two positive constants M and ω such that

$$\|S(t)U_0\|_{\mathcal{H}} \leq Me^{-\omega t}\|U_0\|_{\mathcal{H}}, \forall t \geq 0, \tag{5.40}$$

for all $U_0 \in \mathcal{H}$. Now for $k \geq k_0$, we take

$$U_k(t) = S(t)U_{k,+} = e^{\lambda_{k,+}t}U_{k,+}.$$

By our assumption (5.40), we would deduce that

$$|e^{\lambda_{k,+}t}| \leq Me^{-\omega t}, \forall t \geq 0,$$

which leads to a contradiction because (5.27) yields

$$|e^{\lambda_{k,+}t}| = e^{\left(\frac{g'(0)-\gamma g(0)}{2\lambda_k^2}+o(\frac{1}{\lambda_k^2})\right)t}.$$

□

5.5 A first polynomial stability result

Our polynomial stability results are based on a frequency domain approach. Recall that the polynomial decay of the energy can be obtained by using the next result stated in Theorem 2.4 of [14] (see also [13, 76] for weaker variants and [50, 101] for exponential decay):

Lemma 5.12. *Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a Hilbert space H (with norm $\|\cdot\|_H$) with generator $\mathcal{L}$ such that*

$$\rho(\mathcal{L}) \supset i\mathbb{R}. \tag{5.41}$$

Then for a fixed positive real number l, the following conditions are equivalent:
(i) There exists $C > 0$ such that

$$||T(t)U_0||_H \leq C\, t^{-\frac{1}{l}}||U_0||_{\mathcal{D}(\mathcal{L})}, \quad \forall U_0 \in \mathcal{D}(\mathcal{L}), \quad \forall t > 1.$$

(ii) There exists $C_1 > 0$ such that

$$||T(t)U_0||_H \leq C_1 t^{-1}||U_0||_{\mathcal{D}(\mathcal{L}^l)}, \quad \forall U_0 \in \mathcal{D}(\mathcal{L}^l), \quad \forall t > 1.$$

(iii) It holds

$$\limsup_{|\xi|\to\infty} \frac{1}{\xi^l}\|(i\xi - \mathcal{L})^{-1}\| < \infty. \tag{5.42}$$

In view of this result and Corollary 5.3, and Lemma 5.5 it remains to check (5.42) for some $l > 0$, in other words, we look for resolvent estimates on the imaginary axis $\mathrm{Re}\,\lambda = 0$ of the Moore–Gibson–Thompson equation (5.4). Since in the case $\alpha = \gamma$, a polynomial decay rate is proved in [37, Corollary 7.3], we first prove a similar decay with the sole condition (A4).

Lemma 5.13. *Assume that (A1) to (A4) hold, then*

$$||(\lambda \mathbb{I} - A)^{-1}|| \lesssim 1 + |\lambda|^2, \forall \lambda \in i\mathbb{R}. \tag{5.43}$$

Proof. In order to show our result, we distinguish two cases.

- For $\lambda \in \mathcal{K} = \{\mu \in i\mathbb{R} : |\mu| \leq 1\}$. Using Lemma 5.5 and the fact that $(\lambda \mathbb{I} - A)^{-1}$ is holomorphic on the resolvent set $\rho(A)$ ([9]), it is continuous on any compact set of $\mathbb{C}_+$, in particular on $\mathcal{K}$. Hence (5.43) is valid for all $\lambda \in \mathcal{K}$.
- For $\lambda \in i\mathbb{R}$ with $|\lambda| \geq 1$. Let $F = (f, g, h, p)^\top \in \mathcal{H}$, and $U = (u, v, w, \eta^t)^\top \in D(A)$ be the unique solution of (5.11) (or equivalently to (5.12a)-(5.12d)). Then as $\operatorname{Re} \lambda = 0$, we have
$$-\operatorname{Re}(AU, U)_{\mathcal{H}} = \operatorname{Re}(F, U)_{\mathcal{H}}.$$
Using the splitting (5.9), we find
$$-\operatorname{Re}(A_0 U, U)_{\mathcal{H}} - \operatorname{Re}((\gamma - \alpha)w, w + \gamma v) = \operatorname{Re}(F, U)_{\mathcal{H}}.$$
Now using (5.19), (5.12a) and (5.12b), we find
$$\begin{aligned}\gamma \|\eta^t\|^2_{\mathcal{M}} &+ \int_0^\infty g''(s) |\eta^t(s)|^2_{1,\Omega}\, ds + |\lambda|^4((\alpha - \gamma)u, u) \\ &+ \operatorname{Re}((\lambda - \gamma)f + g, (\alpha - \gamma)(\lambda f + g) \\ &- \operatorname{Re}((\lambda - \gamma)\lambda u, (\alpha - \gamma)(\lambda f + g) - \operatorname{Re}((\lambda - \gamma)f + g, (\alpha - \gamma)\lambda^2 u) \\ &= \operatorname{Re}(F, U)_{\mathcal{H}}.\end{aligned}$$
Using the assumption (H3) and Cauchy-Schwarz's inequality we find
$$\begin{aligned}\gamma \|\eta^t\|^2_{\mathcal{M}} + |\lambda|^4 \|\sqrt{\alpha - \gamma} u\|^2_\Omega \lesssim & \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + |\lambda|^2 \|f\|^2_\Omega + \|g\|^2_\Omega \\ &+ |\lambda|^2 \|\sqrt{\alpha - \gamma} u\|_\Omega (|\lambda| \|f\|_\Omega + \|g\|_\Omega).\end{aligned}$$
Young's inequality yields
$$\gamma \|\eta^t\|^2_{\mathcal{M}} + |\lambda|^4 \|\sqrt{\alpha - \gamma} u\|^2_\Omega \lesssim \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + |\lambda|^2 \|f\|^2_\Omega + \|g\|^2_\Omega.$$
In particular, it yields
$$\|\eta^t\|^2_{\mathcal{M}} \lesssim \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + |\lambda|^2 \|f\|^2_\Omega + \|g\|^2_\Omega. \tag{5.44}$$
We now need to estimate the norm of u, v and w. First using (5.13c), we may write
$$(1 - e^{-\lambda s})u = \eta^t + \Big(\frac{1 - e^{-\lambda s}}{\lambda}\Big) f - \int_0^s e^{-\lambda(s - \tau)} p(\tau) d\tau.$$
Hence recalling that $\lambda \in i\mathbb{R}$ and $|\lambda| \geq 1$, we find
$$\|(1 - e^{-\lambda s})u\|_{\mathcal{M}} \leq \|\eta^t\|_{\mathcal{M}} + 2\|f\|_{\mathcal{M}} + \|\int_0^{\cdot} e^{-\lambda(\cdot - \tau)} p(\tau) d\tau\|_{\mathcal{M}}. \tag{5.45}$$

Since we have

$$\begin{aligned}\|(1-e^{-\lambda s})u\|_{\mathcal{M}} =& \left(\int_0^\infty (-g'(s))|1-e^{-\lambda s}|^2\,ds\right)^{\frac{1}{2}} |u|_{1,\Omega},\\ \|f\|_{\mathcal{M}} =& g(0)|f|_{1,\Omega},\end{aligned}$$

and since we easily check that

$$\|\int_0^{\cdot} e^{-\lambda(.-\tau)}p(\tau)d\tau\|_{\mathcal{M}} \lesssim \|p\|_{\mathcal{M}},$$

the previous estimate (5.45) becomes

$$\left(\int_0^\infty (-g'(s))|1-e^{-\lambda s}|^2\,ds\right)^{\frac{1}{2}} |u|_{1,\Omega} \lesssim \|\eta^t\|_{\mathcal{M}} + |f|_{1,\Omega} + \|p\|_{\mathcal{M}}.$$

As the next Lemma 5.15 shows that the factor in front of $|u|_{1,\Omega}$ is uniformly bounded from below, we then find

$$|u|_{1,\Omega} \lesssim \|\eta^t\|_{\mathcal{M}} + |f|_{1,\Omega} + \|p\|_{\mathcal{M}}.$$

By the estimate (5.44), we conclude that

$$|u|^2_{1,\Omega} \lesssim \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + |\lambda|^2\|F\|^2_{\mathcal{H}}. \tag{5.46}$$

At this stage we use the identity (5.12a) to find

$$|v|^2_{1,\Omega} \lesssim |\lambda|^2\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + |\lambda|^4\|F\|^2_{\mathcal{H}}. \tag{5.47}$$

It now remains to estimate the L^2 norm of w. Since the use of (5.12b) gives rise to a sub-optimal estimate, we take advantage of (5.14). First we notice that using the definition of $D(A), \Delta\left(\ell u + v + \int_0^\infty g(s)\eta^t(s)ds\right) \in L^2(\Omega)$, and taking into account (5.12b), we find

$$\Delta\left((\lambda+\ell)u - f + \int_0^\infty g(s)\eta^t(s)ds\right) \in L^2(\Omega).$$

This suggests to define u_w as

$$u_w = (\lambda+\ell)u - f + \int_0^\infty g(s)\eta^t(s)ds, \tag{5.48}$$

that satisfies $u_w \in H^1_0(\Omega)$ and $\Delta u_w \in L^2(\Omega)$. Plugging this expression in (5.14), we obtain

$$(\lambda+\alpha)\lambda^2 u - \Delta u_w = h + (\alpha+\lambda)(\lambda f + g). \tag{5.49}$$

Taking the inner L^2 inner product with u and using Green's formula we find

$$\lambda^2((\lambda+\alpha)u, u))_{\Omega}. + (u_w, u)_{1,\Omega} = (h + (\alpha+\lambda)(\lambda f + g), u)_{\Omega}.$$

Setting $\lambda = i\xi$ with $\xi \in \mathbb{R}$ and taking the imaginary part of this identity, we find

$$-\xi^3\|u\|_\Omega^2 = -\Im(u_w, u)_{1,\Omega} + \Im(h + (\alpha+\lambda)(\lambda f + g), u)_\Omega.$$

Taking the absolute value of this left-hand side, we get

$$|\lambda|^3\|u\|_\Omega^2 \leq |(u_w, u)_{1,\Omega}| + |(h + (\alpha+\lambda)(\lambda f + g), u)_\Omega|.$$

and using Cauchy-Schwarz's inequaltiy, we obtain

$$|\lambda|^3\|u\|_\Omega^2 \lesssim |u_w|_{1,\Omega}|u|_{1,\Omega} + (|\lambda|^2\|f\|_\Omega + |\lambda|\|g\|_\Omega + \|h\|_\Omega)\|u\|_\Omega.$$

Using Young's inequaltiy, we get

$$|\lambda|^3\|u\|_\Omega^2 \lesssim |u_w|_{1,\Omega}|u|_{1,\Omega} + |\lambda|\|f\|_\Omega^2 + \|g\|_\Omega^2 + \|h\|_\Omega^2. \tag{5.50}$$

For the first term of this right hand side, we first use (5.48) to obtain

$$|u_w|_{1,\Omega} \lesssim |\lambda||u|_{1,\Omega} + |f|_{1,\Omega} + \|\eta^t\|_{\mathcal{M}}.$$

Inserting this estimate in (5.50), we find

$$|\lambda|^3\|u\|_\Omega^2 \lesssim |\lambda||u|_{1,\Omega}^2 + (|f|_{1,\Omega} + \|\eta^t\|_{\mathcal{M}})|u|_{1,\Omega} + |\lambda|\|f\|_\Omega^2 + \|g\|_\Omega^2 + \|h\|_\Omega^2.$$

By Young's inequality and the definition of the norm in $\mathcal{H}$, this leads to

$$|\lambda|^3\|u\|_\Omega^2 \lesssim |\lambda||u|_{1,\Omega}^2 + \|\eta^t\|_{\mathcal{M}}^2 + |\lambda|\|F\|_{\mathcal{H}}^2.$$

At this stage the estimates (5.44) and (5.46) allow to obtain

$$|\lambda|^3\|u\|_\Omega^2 \lesssim |\lambda|\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + |\lambda|^3\|F\|_{\mathcal{H}}^2.$$

and then

$$|\lambda|^4\|u\|_\Omega^2 \lesssim |\lambda|^2\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + |\lambda|^4\|F\|_{\mathcal{H}}^2.$$

As $w = \lambda^2 u - \lambda f - g$ (see (5.13b)), we conclude that

$$\|w\|_\Omega^2 \lesssim |\lambda|^2\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + |\lambda|^4\|F\|_{\mathcal{H}}^2. \tag{5.51}$$

Using the estimates (5.44), (5.46), (5.47), and (5.51), we find

$$\|U\|_{\mathcal{H}}^2 \lesssim |\lambda|^2\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + |\lambda|^4\|F\|_{\mathcal{H}}^2,$$

and by Young's inequality we arrive at

$$\|U\|_{\mathcal{H}}^2 \lesssim |\lambda|^4\|F\|_{\mathcal{H}}^2,$$

which proves (5.43) for all $\lambda \in i\mathbb{R}$ large. The proof is complete.

□

Combining this Lemma and Corollary 5.3, Lemma 5.12 allows to obtain the next polynomial decay rate.

Corollary 5.14. *Assume that (A1)-(A4) hold, then system (5.1) is polynomially stable with a decay rate in $t^{-\frac{1}{4}}$ for initial data in $D(A)$, namely we have*

$$\|S(t)U_0\|_{\mathcal{H}} \lesssim t^{-\frac{1}{2}}||U_0||_{D(A)}, \quad \forall U_0 \in D(A), \quad \forall t > 1. \tag{5.52}$$

Lemma 5.15. *If (A1)-(A3) are valid, then*

$$\int_0^{\infty}(-g'(s))|1-e^{-\lambda s}|^2\,ds \gtrsim 1, \forall \lambda \in i\mathbb{R} : |\lambda| \geq 1. \tag{5.53}$$

Proof. We consider the case $\lambda = i\xi$, with $\xi \in \mathbb{R}$, and $\xi \geq 1$. The case $\xi \leq -1$ is treated in the same manner. Direct calculations yield

$$|1-e^{-\lambda s}|^2 = 2(1-\cos(\xi s)).$$

As $g'(s)$ is non positive, then we may write

$$\begin{aligned}\int_0^{\infty}(-g'(s))|1-e^{-\lambda s}|^2\,ds &\geq 2\sum_{k=0}^{\infty}\int_{\xi s\in(\frac{\pi}{4}+2k\pi,\frac{\pi}{3}+2k\pi)}(-g'(s))(1-\cos(\xi s))\,ds \\ &\geq (2-\sqrt{2})\sum_{k=0}^{\infty}\int_{\xi s\in(\frac{\pi}{4}+2k\pi,\frac{\pi}{3}+2k\pi)}(-g'(s))\,ds.\end{aligned}$$

Let us now introduce the unique integer k_ξ such that $2k_\xi\pi \leq \xi < 2(k_\xi+1)\pi$, then we trivially have

$$\int_0^{\infty}(-g'(s))|1-e^{-\lambda s}|^2\,ds \geq (2-\sqrt{2})\sum_{k=0}^{k_\xi}\int_{\xi s\in(\frac{\pi}{4}+2k\pi,\frac{\pi}{3}+2k\pi)}(-g'(s))\,ds.$$

Since $-g'$ is non increasing, we then find

$$\int_0^{\infty}(-g'(s))|1-e^{-\lambda s}|^2\,ds \geq (2-\sqrt{2})(k_\xi+1)\xi^{-1}\frac{\pi}{12}(-g'(\frac{\pi}{3\xi}+\frac{2(k_\xi+1)\pi}{\xi})).$$

As $\frac{2(k_\xi+1)\pi}{\xi} \geq 1$, we obtain

$$\int_0^{\infty}(-g'(s))|1-e^{-\lambda s}|^2\,ds \geq \frac{2-\sqrt{2}}{24}(-g'(1)),$$

which proves (5.53) as $-g'(1) > 0$ due to (A1) and (A2). □

Let us finish this section by proving that the polynomial decay rate from Corollary 5.14 is optimal if $\alpha = \gamma$ and if the memory kernel satisfies the additional assumptions from section 5.4.

Lemma 5.16. *Assume that the memory kernel $g \in W^{3,1}(\mathbb{R}_+)$ satisfies assumptions (A1) to (H3) and (5.21) and that $\alpha = \gamma$. Then the decay rate (5.52) is optimal in the sense that for any $l < 2$, we can not expect the decay rate $t^{-\frac{1}{l}}$ for all initial data $U_0 \in D(A)$.*

Proof. We take advantage of Lemmas 5.12 and 5.10 (see [82, Theorem 2.10] for a similar argument). In the setting of Lemma 5.10, for $k \geq k_0$, we take

$$\xi_k = \Im \lambda_{k,+},$$

that owing to (5.36) and (5.37) satisfies

$$\xi_k \sim \lambda_k. \tag{5.54}$$

Now we consider $(i\xi_k - A)U_{k,+}$, that owing to (5.27), is given by

$$(i\xi_k - A)U_{k,+} = \left(\frac{g'(0) - \gamma g(0)}{2\lambda_k^2} + o(\frac{1}{\lambda_k^2}) \right) U_{k,+}.$$

This means that

$$\|(i\xi_k - A)U_{k,+}\|_{\mathcal{H}} = \left| \frac{g'(0) - \gamma g(0)}{2\lambda_k^2} + o(\frac{1}{\lambda_k^2}) \right| \sim \lambda_k^{-2}.$$

This property combined with (5.54) shows that

$$\xi_k^l \|(i\xi_k - A)U_{k,+}\|_{\mathcal{H}} \sim \lambda_k^{l-2},$$

and implies that

$$\lim_{k\to\infty} \frac{\xi_k^l \|(i\xi_k - A)U_{k,+}\|_{\mathcal{H}}}{\|U_{k,+}\|_{\mathcal{H}}} = \lim_{k\to\infty} \xi^l \|(i\xi_k - A)U_{k,+}\|_{\mathcal{H}} = 0,$$

for any $l < 2$. In other words, property (5.42) cannot holds for $l < 2$ and we conclude by Lemma 5.12. □

5.6 A stronger polynomial decay using a resolvent estimate of the wave equation with a frictional interior damping

In this subsection, we want to take advantage of the decay rate of the semigroup generated by wave equation in Ω with Dirichlet boundary condition and a frictional interior damping, namely

$$\begin{cases} u_{tt} - \Delta u + (\alpha - \gamma)u_t = 0 & \text{in} \quad \Omega \times (0, \infty), \\ u = 0 & \text{on} \quad \Gamma \times (0, \infty), \\ u(.,0) = u_0, \quad u_t(.,0) = u_1 & \text{in} \quad \Omega, \end{cases} \tag{5.55}$$

where α and γ are given before. It is well known that the Hilbert setting of this kind of equation consists in taking

$$\mathcal{H}_w = H_0^1(\Omega) \times L^2(\Omega)$$

with the standard norm

$$\|(u,v)^T\|_{\mathcal{H}_w}^2 = |u|_{1,\Omega}^2 + \|v\|_\Omega^2, \qquad \forall u, v \in \mathcal{H}_w.$$

Moreover, the associated operator is given as

$$A_w(u,v)^T = \begin{pmatrix} v \\ \Delta u - (\alpha - \ell)v \end{pmatrix}, \qquad \forall (u,v)^T \in D(A_w),$$

with domain

$$D(A_w) = \left\{(u,v)^T \in \mathcal{H}_\omega / v \in H_0^1(\Omega), \Delta u \in L^2(\Omega)\right\}.$$

It is well-known that A_w is the generator of a C_0-semigroup of contractions on $\mathcal{H}_w$ (even if $\alpha = \gamma$). Consequently, we have

$$\{\lambda \in \mathbb{C}, \operatorname{Re}\lambda > 0\} \subset \rho(A_w)$$

and

$$\|(\lambda\mathbb{I} - A_w)^{-1}\|_{\mathcal{L}(A_w)} \leq \frac{1}{\operatorname{Re}\lambda} \qquad \forall \lambda \in \mathbb{C}, \quad \operatorname{Re}\lambda > 0. \tag{5.56}$$

Recalling that, for any $\lambda \in \rho(A_w)$ and for $F_w = (f_w, g_w) \in \mathcal{H}_w$, the identity

$$(\lambda\mathbb{I} - A_w)U_w = F_w$$

is equivalent to

$$\begin{cases} v_w = \lambda u_\omega - f_\omega \\ (\lambda + \alpha - \gamma)\lambda u_\omega - \Delta u_\omega = g_w + (\lambda + \alpha - \gamma) f_\omega. \end{cases} \tag{5.57}$$

Therefore, by the estimate

$$|u_w|_{1,\Omega} + \|v_w\|_\Omega \leq \|(\lambda\mathbb{I} - A_w)^{-1}\|_{\mathcal{L}(\mathcal{H}_w)}\Big(|f_w|_{1,\Omega} + \|g_w\|_\Omega\Big), \tag{5.58}$$

and the first equation in (5.57), we get

$$|u_w|_{1,\Omega} + |\lambda|\|u_w\|_\Omega \leq C_\lambda\Big(|f_w|_{1,\Omega} + \|g_w\|_\Omega\Big), \tag{5.59}$$

where $C_\lambda = \max\left\{1, \|(\lambda\mathbb{I} - A_w)^{-1}\|_{\mathcal{L}(\mathcal{H}_w)}\right\}$.
If, in addition to (A4), we suppose that the function α satisfies

(A5) there exist an non empty open subset ω_0 of Ω and a positive constant κ such that $\alpha - \gamma \geq \kappa$ a. e. in ω_0,

then, by Calderon uniqueness Theorem, we deduce that the imaginary axis is in the resolvent set $\rho(A_w)$, hence, $\mathbb{C}_+ \subset \rho(A_w)$.

The idea is now to derive a better decay rate of the solution of system (5.4) from the following bound on the resolvent of A_w on the imaginary axis

$$\|(i\xi\mathbb{I} - A_w)^{-1}\|_{\mathcal{L}(\mathcal{H}_w)} \lesssim 1 + |\xi|^m, \quad \forall \xi \in \mathbb{R}, \tag{5.60}$$

where m is a positive constant such that $0 \leq m < 1$. As before, the remaining task is to estimate the resolvent of A on the imaginary axis.

Lemma 5.17. *Assume that (A1) to (A5) hold with ω_0 so that (5.60) holds with $0 \leq m < 1$, then*

$$||(\lambda\mathbb{I} - A)^{-1}|| \lesssim 1 + |\lambda|^{m+1}, \forall \lambda \in i\mathbb{R}. \tag{5.61}$$

Proof. As in Lemma 5.13, the estimate (5.61) is valid for all $\lambda \in \mathcal{K}_R = \{\mu \in i\mathbb{R} : |\mu| \leq R\}$, with $R \geq 1$ fixed below. Hence we are reduced to the case $\lambda \in i\mathbb{R}$ with $|\lambda| \geq R$. For all $F = (f, g, h, p) \in \mathcal{H}$, let $U = (u, v, w, \eta^t) \in D(A)$ be the unique solution of (5.11) (or equivalently to (5.12a)-(5.12d)). As before, we shall use the new variable u_w defined in (5.48). Note that, by using (5.13c), we see that

$$u_w = (\lambda + \gamma - \mathcal{L}g(\lambda))u - (1 + \lambda^{-1}(\gamma - \ell - \mathcal{L}g(\lambda))f + P, \tag{5.62}$$

where

$$P = \int_0^\infty g(s) \left(\int_0^s e^{-\lambda(s-\tau)} p(\tau)\, d\tau \right) ds,$$

and we recall that $\mathcal{L}g$ is the Laplace transform of g (that has a meaning because g is integrable and $\operatorname{Re}\lambda = 0$). Further direct calculations show that

$$|P|_{1,\Omega} \lesssim \|p\|_{\mathcal{M}}. \tag{5.63}$$

As g and g' are integrable, the following identity

$$\lambda\mathcal{L}g(\lambda) = g(0) + \mathcal{L}g'(\lambda),$$

holds. This implies that

$$|\lambda\mathcal{L}g(\lambda)| \leq |g(0)| - \int_0^\infty g'(s)\, ds =: C. \tag{5.64}$$

For further uses we need that the factor in front of u in the right-hand side of (5.62) is different from zero, hence we need to fix $|\lambda|$ large enough. More precisely if we require

$$\gamma + C < |\lambda|,$$

then, we directly obtain that

$$|\lambda|^{-1}|\gamma - \mathcal{L}g(\lambda)| < 1,$$

which guarantee the invertibility of $\lambda+\gamma-\mathcal{L}g(\lambda)$. This means that we fix R as follows

$$R \geq \max\{1, 2(\gamma+C)\}.$$

From (5.48), we can write u as

$$u=\frac{1}{\lambda+\gamma-\mathcal{L}g(\lambda)}\Big[u_w+(1+\lambda^{-1}(\gamma-\ell-\mathcal{L}g(\lambda))f-P\Big]. \tag{5.65}$$

Inserting this expression in (5.49), we get

$$\begin{aligned}&\frac{(\lambda+\alpha)\lambda^2}{\lambda+\gamma-\mathcal{L}g(\lambda)}u_w-\Delta u_w\\&=h+(\alpha+\lambda)g-\frac{\lambda^2(\lambda+\alpha)}{\lambda+\gamma-\mathcal{L}g(\lambda)}f+\frac{(\lambda+\alpha)\lambda^2}{\lambda+\gamma-\mathcal{L}g(\lambda)}P.\end{aligned} \tag{5.66}$$

Now, we notice that

$$\frac{(\lambda+\alpha)\lambda^2}{\lambda+\gamma-\mathcal{L}g(\lambda)}=\lambda(\lambda+\alpha(\lambda)),$$

where

$$|\alpha(\lambda)-(\alpha-\gamma)|\lesssim|\lambda|^{-1}, \tag{5.67}$$

due to (5.64). This allows to rewrite (5.66) as

$$\lambda(\lambda+\alpha(\lambda))u_w-\Delta u_w=h+(\alpha+\lambda)g-\frac{\lambda^2(\lambda+\alpha)}{\lambda+\gamma-\mathcal{L}g(\lambda)}f+\frac{(\lambda+\alpha)\lambda^2}{\lambda+\gamma-\mathcal{L}g(\lambda)}P,$$

or equivalently, by setting $r(\lambda)=\alpha(\lambda)-(\alpha-\gamma)$, and

$$g_w=h+(\alpha+\lambda)g-\frac{\lambda^2(\lambda+\alpha)}{\lambda+\gamma-\mathcal{L}g(\lambda)}f+\frac{(\lambda+\alpha)\lambda^2}{\lambda+\gamma-\mathcal{L}g(\lambda)}P+\lambda r(\lambda)u_\omega,$$

$$\lambda(\lambda+\alpha-\gamma)u_w-\Delta u_w=g_w. \tag{5.68}$$

By taking $f_w=0$ and setting

$$v_w=\lambda u_w,$$

we see that the pair (u_w,v_w) belongs to $D(A_w)$ and is solution of (5.57). Consequently, by (5.59) and (5.60), we find

$$|u_w|_{1,\Omega}+|\lambda|\|u_w\|_\Omega\lesssim|\lambda|^m\|g_w\|_\Omega.$$

By the definition of g_w, we find

$$|u_w|_{1,\Omega}+|\lambda|\|u_w\|_\Omega\lesssim|\lambda|^m\Big(|g|_{1,\Omega}+|\lambda||f|_{1,\Omega}+|\lambda||P|_{1,\Omega}+\|h\|_\Omega+|\lambda r(\lambda)|\|u_w\|_\Omega\Big).$$

As the estimate (5.67) implies that $|\lambda r(\lambda)| \lesssim 1$, we find that

$$|u_w|_{1,\Omega} + |\lambda| \|u_w\|_\Omega \lesssim |\lambda|^m \Big(|g|_{1,\Omega} + |\lambda||f|_{1,\Omega} + |\lambda||P|_{1,\Omega} + \|h\|_\Omega \Big),$$

if $|\lambda| \geq R_1$ for a positive real number R_1 large enough independent of $|\lambda|$ but depending on m and on the constants from (5.59) and from (5.67) (hence we fix R larger than this R_1 as well). By the definition of the norm in $\mathcal{H}$ and the estimate (5.63), we have found

$$|u_w|_{1,\Omega} + |\lambda| \|u_w\|_\Omega \lesssim |\lambda|^{m+1} \|F\|_{\mathcal{H}}. \tag{5.69}$$

We now come back to the original variables u, v and w. Indeed from (5.65), (5.64) and (5.69), we find

$$|u|_{1,\Omega} + |\lambda| \|u\|_\Omega \lesssim |\lambda|^m \|F\|_{\mathcal{H}}. \tag{5.70}$$

Now the identity (5.13a) and the previous estimate directly yield

$$|v|_{1,\Omega} + |\lambda| \|v\|_\Omega \lesssim |\lambda|^{m+1} \|F\|_{\mathcal{H}}. \tag{5.71}$$

Similarly using (5.13b) and this estimate, we find

$$\|w\|_\Omega \lesssim |\lambda|^{m+1} \|F\|_{\mathcal{H}}. \tag{5.72}$$

Finally using (5.13c) and (5.70), we obtain

$$\|\eta^t\|_{\mathcal{M}} \lesssim |\lambda|^m \|F\|_{\mathcal{H}}. \tag{5.73}$$

The estimates (5.70) to (5.73) show that (5.61) is valid for $\lambda \in i\mathbb{R}$ such that $|\lambda| \geq R$, which completes the proof. □

Combining this Lemma and Corollary 5.3, Lemma 5.12 allows to obtain the next polynomial decay rate.

Corollary 5.18. *Under the assumptions of Lemma 5.17, system (5.1) is polynomially stable with a decay rate in $t^{-\frac{1}{m+1}}$ for initial data in $D(A)$, namely we have*

$$\|S(t)U_0\|_{\mathcal{H}} \lesssim t^{-\frac{1}{m+1}} ||U_0||_{D(A)}, \quad \forall U_0 \in D(A), \quad \forall t > 1. \tag{5.74}$$

Note that the decay rate (5.74) is always better than (5.52) since $m \in [0, 1)$.

Let us finish this section with some examples for which the decay rate (5.74) is available.

Example 5.19. There are many cases for which the frictional damping in ω_0 is sufficient to guarantee the exponential stability of the wave equation (5.55) (hence by Huang-Prüss theorem [50, 101], (5.60) holds with $m = 0$), hence by Corollary 5.18, system (5.1) is polynomially stable with a decay rate in t^{-1} for initial data in $D(A)$. Let us mention a few of them.

1) By [104, Theorem 2] (see also [11, 68]), (5.55) is exponentially stable if the boundary

of Ω is of class C^∞ and ω_0 satisfies the Geometric Control Condition (GCC). Recall that the GCC can be formulated as follows: For a subset ω of Ω, we shall say that ω satisfies the Geometric Control Condition if there exists $T > 0$ such that every geodesic traveling at speed one issued from Ω at time $t = 0$ intersects ω before time T.
2) From [72, Lemma VII.2.4] (see also [113, Theorem 1.1 and Remark 1.2] or [47, Example 3]) (5.55) is exponentially stable if the boundary of Ω is of class C^2 and ω_0 is a neighborhood of $\bar{\Gamma}(x^0)$, for some $x^0 \in \mathbb{R}^d$, where

$$\Gamma(x^0) = \{x \in \partial\Omega \mid (x - x^0) \cdot \nu(x) > 0\},$$

$\nu(x)$ being the unit outward normal vector at $x \in \partial\Omega$.
3) From [47, Example 1], (5.55) is exponentially stable if $d = 1$, $\Omega = (0, \ell)$ for some positive real number ℓ, and ω_0 is a non empty open subset of Ω.
4) In [73, Remark 4.3], further examples of pairs (Ω, ω_0) such that (5.55) is exponentially stable are given.

Example 5.20. To the best of our knowledge, only three papers are considering the estimate (5.60) in the case $0 < m < 1$, namely [21, 22, 67] on a smooth connected compact Riemannian manifold, but the arguments from [7, Remark 2.8] allow to deduce that the same result holds for the damped wave equation in a hyper-cube $(-L, L)^n$ of $\mathbb{R}^n$, with $n \geq 2$ and Dirichlet boundary conditions. More precisely, for $\delta > 0$ and $\Omega = (-L, L)^n$ with $L > 0$, if we take $(\alpha - \gamma)(x_1, \cdots, x_n) = x_1^{2\delta}$ near $x_1 = 0$, positive elsewhere and depending only on x_1, then using [67, Theorem 1.8] or [21, Theorem 1.2] (and [7, Remark 2.8]), the estimate (5.60) holds with $m = \frac{\delta}{\delta+1}$, that indeed belongs to $(0, 1)$. Consequently by Corollary 5.18, system (5.1) is polynomially stable with a decay rate in $t^{-\frac{\delta+1}{2\delta+1}}$ for initial data in $D(A)$.

CONCLUSION AND PERSPECTIVE

CONCLUSION

In this thesis, we studied a linear Cauchy problem for the Moore–Gibson–Thompson (MGT) equation with type I and III memory terms (in whole space $\mathbb{R}^N$), in the presence of viscoelastic dissipation, where the relaxation function decays exponentially and polynomially. We have established existence and stability results which depend on the coefficients of the equation and show that the rate of decay is of the "Loss of regularity" type. We also studied the case of the type II memory term with a large class of relaxation functions and showed a general stability result.

Then, we are interested in a linear Cauchy problem with MGT equation coupled to a Gurtin and Pipkin type heat equation, where the relaxation function decays exponentially. We have shown polynomial stability result with an optimal rate of type "Regularity loss".

We were also interested in third order in time linear evolution equations, general and abstract version of the Moore–Gibson–Thompson system with and without memory term, where existence and uniqueness of a solution has been shown using semigroup theory and several stability rates have been obtained.

Our results generalize and improve many previous results in the literature.

PERSPECTIVE

A research perspective focused on the study of the existence and stability of the solution of the nonlinear JMGT equation in bounded or unbounded domains, with the use of different types of dissipation.

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Abstract: The main goal of this dissertation is to discuss the asymptotic behavior of some hyperbolic PDE systems, precisely, we handled the well-posedness and stability of the solutions for the Moore-Gibson-Thompson equation (MGT) employing various types of dissipation. In fact, under an appropriate assumption on the coefficients of the systems together with the energy method in Fourier space we have proved the well-posedness of the systems and built some Lyapunov functionals which allowed us to get control estimates on the Fourier image of the solution and led to the decay rate of the L^2-norm of the solution.

On the other hand, by comparing the behavior of the resolvent of the Moore-Gibson-Thompson system with the one of the resolvent of the wave equation with a frictional interior damping, we furnish weaker conditions that guarantee exponential, polynomial or even logarithmic decay of the solution of the Moore-Gibson-Thompson system in a bounded domain.

Keywords: Moore-Gibson-Thompson equation, memory kernel, energy method, wave equation, relaxation function, general decay, Fourier space, Gurtin-Pipkin thermal law, stabilization, exponential decay, polynomial decay, regularity loss.

الملخص: الهدف الرئيسي من هذه الأطروحة هو مناقشة السلوك المقارب لبعض أنظمة المعادلات التفاضلية الجزئية الزائدية، على وجه التحديد معادلة مور جيبسون طومسون ، تعاملنا مع الوضع الجيد واستقرار الحلول لمعادلة باستخدام أنواع مختلفة من التبديد. في الواقع ، كلا المشكلتين و في ظل افتراض مناسب لمعاملات الأنظمة مع طريقة الطاقة في فضاء فورييه، قمنا بإثبات الوضع الجيد للأنظمة وقمنا ببناء بعض وظائف ليابونوف التي سمحت لنا بالحصول على تقديرات التحكم على صورة فورييه للحل كما تؤدي إلى استنتاج معدل اضمحلال للحل.

من ناحية أخرى، من خلال مقارنة سلوك مذيب نظام مور جيبسون طومسون بأحد مذيب معادلة الموجة مع التخميد الداخلي الاحتكاكي، وفرنا ظروفا أضعف تضمن تحللا أسيًا أو متعدد الحدود أو حتى لوغاريتميًا للحل في مجال محدد.

الكلمات المفتاحية: معادلة مور جيبسون طومسون، نواة الذاكرة ، طريقة الطاقة ، معادلة الموجة ، وظيفة الاسترخاء ، الاضمحلال العام ، فضاء فورييه ، قانون جورتين-بيبكين الحراري ، استقرار، الاضمحلال الأسي، الاضمحلال متعدد الحدود، فقدان الانتظام.

Résumé: L'objectif principal de cette thèse est de discuter le comportement asymptotique de certains systèmes d'EDP hyperboliques, précisément, nous avons traité la bonne position et la stabilité des solutions de l'équation de Moore-Gibson-Thompson (MGT) employant divers types de dissipation. En fait, sous une hypothèse appropriée sur les coefficients des systèmes avec la méthode de l'énergie dans l'espace de Fourier nous avons prouvé la bonne position des systèmes et construit des fonctionnelles de Lyapunov qui nous ont permis d'obtenir des estimations de contrôle sur l'image de Fourier de la solution et conduisent au taux de décroissance de la norme L^2 de la solution.

D'autre part, en exploitant la résolvante de l'équation des ondes avec amortissement par frottement interne, on fournit des conditions plus faibles qui garantissent une décroissance exponentielle, polynomiale ou même logarithmique de la solution du système de Moore-Gibson-Thompson dans un domaine borné.

Mots-clés: Equation de Moore-Gibson-Thompson, noyau mémoire, méthode énergétique, équation des ondes, fonction de relaxation, décroissance générale, espace de Fourier, loi thermique de Gurtin-Pipkin, stabilisation, perte de régularité.