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par

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Thème

Logarithmic Barrier Interior Point Method for Linearly Constrained Convex Programming

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Dedication

There are a number of people without whom this thesis might not have been written, and to whom I am greatly indebted.
Thanks most of all to my inspiring parents **"Mustapha"** and **"Aljia"**, without whom none of my success would be possible.
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ملخص

في هذه الاطروحة, نهتم بحل مسألة الأمثلية المحدبة بالقيود الخطية. في الجزء الأول قدمنا طريقة اللوغاريتم الحدية مع عامل الثقل متبوعة بدراسة التقارب . الكفاءة العددية لهذه الطريقة المقترحة تظهر من خلال التجارب العددية المنجزة. في الجزء الثاني, نقترح طريقة أولية ثنوية لحل البرمجة التربيعية المحدبة, في هذا الجزء استخدمنا تقنية دوال الحدية العليا لحساب خطوة الانتقال و قد تم اثراء هذا العمل بتجارب عددية قيمة .

الكلمات المفتاحية : برمجة الأمثلية المحدبة بالقيود , البرمجة التربيعية , الدوال الحدية العليا , طريقة النقاط الداخلية .

Resumé :

Dans cette thèse, on s'intéresse a la résolution du problème d'optimisation convexe avec contraintes linéaire. En première partie, on présente une méthode barrière logarithmique avec poids suivie de l'étude de la convergence, l'efficacité de la méthode proposée est confirmée par des tests numériques qui sont encouragée. En deuxième partie on propose une méthode primale duale pour la résolution d'un programme quadratique convexe, la technique des fonctions majorantes est utilisée pour le calcul du pas de déplacement, ce travail est enrichi par des simulations numériques importantes.

Mots clés : Programmation convexe avec contraintes linéaire; Programmation quadratique; Méthodes de points intérieurs; Fonctions majorantes.

Abstract:

In this thesis, we are interested solving the linearly constrained convex optimization problem. In the first part, we present a weighted logarithmic barrier method followed by a study of convergence, the efficiency of the proposed method is shown by presenting numerical experiments. In the second part, we propose a primal-dual method to solving convex quadratic programming, the technique of majorant function is used to compute the step size, this work is enriched by important numerical simulations.

Keywords: Linearly constrained convex programming; Quadratic programming; A primal-dual interior-point method; Logarithmic barrier methods; A Majorant functions.

Title: Logarithmic Barrier Interior-Point Method for Linearly Constrained Convex Programming

Abstract:

This thesis develops two interior-point algorithms, a weighted logarithmic barrier algorithm and a primal-dual logarithmic barrier algorithm, for solving more general groups of convex optimization problems: quadratic and linearly constrained programs.

In chapter 1: we provide an introduction to convexity, necessary notions in convex optimization, duality and we present Newton's method with a variant of the interior-point method for convex programming.

In chapter 2: a weighted logarithmic barrier interior-point method for solving the linearly convex constrained optimization problems is presented. Unlike the classical central path, the barrier parameter associated with the perturbed barrier problem is not a scalar but is a weighted positive vector. This modification gives theoretical flexibility to its convergence and its numerical performance. In addition, this method is of a Newton descent direction and the computation of the step size along this direction is based on a new efficient technique called the tangent method. The practical efficiency of our approach is shown by giving some numerical results.

In chapter 3: we provide a new variant of primal-dual interior-point method for solving a quadratic optimization problem with linear constraint, the method based to solve an equivalent problem of **KKT** condition, this problem is linearly constrained convex problem. Our proposed method uses a classical Newton descent method to compute the descent direction. The difficulty is in the computation of the step size, it induces high computational costs. Here, we use the majorant function which is more efficient than classical line searches. The numerical results are reported to show the efficiency of the method.

Keywords: Linearly constrained convex programming; Quadratic programming; A primal-dual interior-point method; Logarithmic barrier methods; Majorant functions.

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Glossary of Notation

Problem Classes

(CP)	:	Convex Programmation;
(LP)	:	Linear Programmation;
(QP)	:	Quadratic Programmation;
(LCCO)	:	Linearly Constrained Convex Optimization;
(P)	:	Primal of a mathematics problem;
(D)	:	Dual of **(P)**;
IPC	:	Interior-Point Condition;
(K.K.T)	:	Karush-Kuhn-Tucker.

Spaces

$\mathbb{R}$	:	The set of real numbers;
$\mathbb{R}^n$	:	The real n-dimensional space;
$\mathbb{R}^n_+$	:	The nonnegative orthant of $\mathbb{R}^n$;
$\mathbb{R}^{n+m}$	:	we write points in $\mathbb{R}^{n+m}$ in the form (x, y) where $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_m)^T \in \mathbb{R}^m$;
$\mathbb{R}^{n\times n}$	:	The set of all $n \times n$ square matrices;
$\mathbb{R}^{n\times m}$	:	The space of $n \times m$ real matrices.

Vectors

Vectors are denoted by	:	x and y;
$x \in \mathbb{R}^n$	:	$n-$ dimensional real vector (is a column vector of $\mathbb{R}^n$);
$x \geq 0$	:	means $x \in \mathbb{R}^n_+$;
$x > 0$	:	means the components $x_i > 0,$ for $i = 1, \ldots, n$;
e	:	$(1, \ldots, 1)^T$;
x^T	=	$(x_1, \ldots, x_n)$ the transpose of a vector x;
$\langle x, y\rangle$ or $x^T y$	=	$\sum\limits_{i=1}^{n} x_i y_i$ the standard inner product of vectors in $\mathbb{R}^n$;
$\log(x)$	=	$(\log(x_1), \ldots, \log(x_n))^T (x > 0)$;
$\frac{x}{y}$	=	$(\frac{x_1}{y_1}, \frac{x_2}{y_2}, \ldots, \frac{x_n}{y_n})^T (y \neq 0)$;

$\sqrt{x}$	$=$	$(\sqrt{x_1}, \ldots, \sqrt{x_n}),^T \ (x \geq 0);$
x^{-1}	$=$	$(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}),^T \ (x \neq 0);$

Matrices are denoted by capitals;

A^T	:	The transpose of A;
$X = diag(x)$	:	Denote the $n \times n$ diagonal matrix whose (i,i)th entry is x_i for $1 \leq i \leq n$ and is zero on all other entries.

Functions and Norms

$\nabla f(x) = (\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x))^T$	:	The gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$;
$\nabla^2 f(x) = \left(\dfrac{\partial^2 f}{\partial x_i \partial x_j}(x) \right)_{1 \leq i, j \leq n}$	:	The Hessian matrix of $f : \mathbb{R}^n \to \mathbb{R}$;
$\lVert x \rVert$	$=$	$\sqrt{\langle x, x \rangle}$ Euclidean norm;
$\lVert x \rVert_1$	$=$	$\sum_{i=1}^n \lvert x_i \rvert$ Lebesgue norm.

Introduction

An optimization problem consists of maximizing or minimizing a real function by systematically choosing input values from within an allowed set and computing the value of the function. More generally, optimization includes finding "best available" values of some objective function given a defined domain. Optimization or mathematical programming is an important tool in decision science, where optimization problems arise in all quantitative disciplines, from computer science and engineering to operational research and economics, in the words of Leonhard Euler,{ ...Nothing whatsoever takes place in the universe in which some relation of maximum or minimum does not appear}.

Linearly constrained convex optimization problems are classified among the nicest class of optimization problems, where one seeks to minimize a convex function over a linear space. The existence and uniqueness of solutions and on optimality conditions are available as soon as the problem is known to be convex. This problem has many important applications in theory as well as in practice. In particular, it includes linear and quadratic optimization. Feasible logarithmic barrier interior-point methods gained much more attention than others. Their derived algorithms enjoy some interesting results such as polynomial complexity and numerical efficiency. However, these algorithms require that the starting point must be strictly feasible and close to the central-path. This is a hard practical task to release and even impossible. On the other hand, at each iteration, they compute a descent direction and determine a step size in this direction. It is known that computing the latter is very expensive while using classical line search methods. In order to overcome these two difficulties, we suggest for the first, a weighted-path (see [1],[2],[3],[4],[17],[22],[23]) where a relaxation parameter associated with perturbed problems is introduced in order to give more flexibility on the numerical aspects. Besides, we propose a new numerical efficient procedure called the tangent method for the computation of the step size. The numerical tests show the efficiency of our approach compared to the classical logarithmic barrier method.

Quadratic programming is knowledging significant progress, thanks to its various applications, we cite: agriculture, economics, production, operations, finance (problem of portfolio selection). In the last decade, many primal-dual interior-point methods for linear programming (LP) have been extended successfully to quadratic programming and (LCCO) problems. In this thesis, we propose a primal-dual logarithmic barrier interior-point method for quadratic programming (QP) based on solving an equivalent problem of **KKT** optimality conditions. At each iteration, the Newton method and the technique of majorant function are used to compute the descent direction and the step size. The technique of majorant functions solves an equation of a single variable unlike the procedure of classical line-search. The numerical result proved that the method proposed is suitable for (QP).

Structure of thesis

The structure of the thesis is as follows. It consists of three chapters.

Chapter 1. is devoted to convex optimization problems. In the beginning, we introduce the terminology and the notions of convexity for sets and functions and duality. Next, we present a different classes of convex optimization problems. Then we present Newton's method and a variant of interior-point method for linearly constrained convex programming.

Chapter 2. this chapter is organized as follows.

Firstly, perturbed relaxation problems based on the weighted barrier penalization are given where the convergence to the original problem is studied. The computation of the direction and of the step size is stated and a weighted-path interior-point algorithm is presented. Next, some numerical results are given to show the efficiency of our approach. Finally, a conclusion and remarks end the chapter.

Chapter 3. in this chapter, a primal-dual logarithmic barrier method is proposed for solving a convex quadratic optimization problems. The basic idea of this method is to solve an equivalent problem of **KKT** optimality conditions, this problem is a linearly constrained convex problem. The algorithm uses the Newton method to compute the descent direction

and an efficient alternative to determine the step size called the majorant function. Numerical results are reported to show the efficiency of the algorithm.
Finally, we end the thesis with a conclusion and we provide the reader with an important bibliography that should facilitate the reading of the subject.

Chapter 1

Convex Optimization

Contents

In this chapter we consider convex optimization problems. We start with the basic concept of convexity for sets and functions, then, we present different classes of convex optimization problems and we discuss the theoretical aspects of these problems. Finally, we provide Newton's method and we present linearly constrained convex problem with a variant of Interior-Point-Method.

1.1 Convex Analysis

1.1.1 Convex sets

Definition 1.1.1. *A set C in $\mathbb{R}^n$ is called convex if*

$$\forall x, y \in C \quad \text{and} \quad \lambda \in [0,1] : \lambda x + (1-\lambda) y \in C.$$

Definition 1.1.2. *A set C in $\mathbb{R}^n$ is called affine if*

$$\forall x, y \in C \quad \text{and} \quad \lambda \in \mathbb{R} : \lambda x + (1-\lambda) y \in C.$$

Definition 1.1.3. *The set K in $\mathbb{R}^n$ is a convex cone if K is convex set and $\forall x \in K$, $\forall \lambda \geq 0, \quad \lambda x \in K$.*

Definition 1.1.4. *Let C is a nonempty convex set, we say that d is a direction of recession of C if $x + \alpha d \in C$ for all $x \in C$ and $\alpha \geq 0$.*

Thus, d is a direction of recession of C if starting at any $x \in C$ and going indefinitely along d, we never cross the relative boundary of C to points outside C.
The set of all directions of recession is a cone containing the origin. It is called the recession cone of C and it is denoted by C_∞. An important property of a closed convex set is that test whether $d \in C_\infty$ it is enough to verify the property $x + \alpha d \in C$ for a single $x \in C$.

Proposition 1.1.5. ***Recession cone*** *Let C be a nonempty closed convex set.*

1. *The recession cone C_∞ is closed and convex.*
2. *A vector d belongs to C_∞ if and only if there exist a vector $x \in C$ such that $x + \alpha d \in C$ for all $\alpha \geq 0$.*

Proposition 1.1.6. *Let C is a closed and nonempty convex set, so C is bounded if and only if $C_\infty = \{0\}$.*

1.1.2 Convex functions

Definition 1.1.7. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is called convex if its domain is a convex set and for all x, y in its domain, and $\forall \lambda \in [0, 1]$, we have*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

Definition 1.1.8. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is*

- *Strictly convex if $\forall x, y \in \mathbb{R}^n$, $x \neq y$ and $\lambda \in]0, 1[$*

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y).$$

- *Strongly convex if $\forall x, y \in \mathbb{R}^n$ and $\lambda \in]0, 1[$, $\exists \tau > 0$ such that*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) - \frac{\tau}{2} \parallel x - y \parallel^2 .$$

Remark 1.1.9. *f is strongly convex $\Rightarrow$ f is strictly convex $\Rightarrow$ f is convex.*

Theorem 1.1.10. *[26] Suppose $f : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}$ is twice differentiable over an open domain $\mathcal{D}$. Then, the following are equivalent:*

1. *f is convex,*
2. *$f(y) \geq f(x) + \nabla f(x)^T (y - x)$, for all $x, y \in \mathcal{D}$,*
3. *$\nabla^2 f(x) \succeq 0$, for all $x, y \in \mathcal{D}$.*

Definition 1.1.11. *A function $f : \mathbb{R}^n \to]-\infty, +\infty[$ is said to be lower semi-continuous at $\bar{x}$ if for any sequence $\{x^k\}$ in $\mathbb{R}^n$ with $x^k \to \bar{x}$*

$$f(\bar{x}) \leq \lim_{k \to +\infty} f(x^k).$$

Definition 1.1.12. *A convex function f is proper if its effective domain is nonempty and it never attains $-\infty$.*

Definition 1.1.13. *A full rank matrix is one which has linearly independent rows.*

Description of a system of equations. Suppose that we have p equations depending on $p+1$ variables (parameters) written in implicit form as follows

$$\begin{aligned}
\psi_1(x_1, x_2, \ldots, x_p, y) &= 0, \\
\psi_2(x_1, x_2, \ldots, x_p, y) &= 0, \\
&\vdots \\
\psi_p(x_1, x_2, \ldots, x_p, y) &= 0.
\end{aligned} \tag{1.1}$$

The Jacobian matrix of the system in (1.1) is defined as $p \times p$ matrix of first partials as follows

$$\begin{bmatrix}
\frac{\partial \psi_1}{\partial x_1} & \frac{\partial \psi_1}{\partial x_2} & \cdots & \frac{\partial \psi_1}{\partial x_p} \\
\frac{\partial \psi_2}{\partial x_1} & \frac{\partial \psi_2}{\partial x_2} & \cdots & \frac{\partial \psi_2}{\partial x_p} \\
\vdots & \vdots & \vdots & \vdots \\
\frac{\partial \psi_p}{\partial x_1} & \frac{\partial \psi_p}{\partial x_2} & \cdots & \frac{\partial \psi_p}{\partial x_p}
\end{bmatrix}$$

This matrix will be of rank p if its determinant is not zero.

Theorem 1.1.14. ***Implicit Function Theorem****[34] Suppose that ψ_i are real-valued functions defined on a domain D and continuously differentiable on an open set $D^1 \subset D \subset \mathbb{R}^{p+1}$, where $p > 0$, where*

$$\begin{aligned}
\psi_1(x_1^0, x_2^0, \ldots, x_p^0, y) &= 0, \\
\psi_2(x_1^0, x_2^0, \ldots, x_p^0, y) &= 0, \\
&\vdots \\
\psi_m(x_1^0, x_2^0, \ldots, x_p^0, y) &= 0,
\end{aligned} \tag{1.2}$$

where $(x^0, y^0) \in D^1$. We often write equation (1.2) as follows

$$\psi_i(x^0, y^0) = 0, \ \ i = 1, 2, \ldots, m, \text{ and } (x^0, y^0) \in D^1.$$

Assume the Jacobian matrix $\left[\frac{\partial \psi_i(x^0, y^0)}{\partial x_j}\right]$ has rank p. Then there exists a neighborhood $N_\delta(x^0, y^0) \subset D^1$, an open set $D^2 \subset \mathbb{R}$ containing y^0 and real valued functions ζ_k, $k = 1, 2, \ldots, p$ continu-

ously differentiable on D^2, such that the following conditions are satisfied:

$$\begin{aligned} x_1^0 &= \zeta_1(y^0), \\ x_2^0 &= \zeta_2(y^0), \\ &\vdots \\ x_p^0 &= \zeta_p(y^0) \end{aligned} \tag{1.3}$$

For every $y \in D^2$, we have

$$\psi_i(\zeta(y), y) = 0, \ i = 1, 2, \ldots, p.$$

We also have that for all $(x, y) \in N_\delta(x^0, y^0)$, the Jacobian matrix $\left[\frac{\partial \psi_i(x^0,y^0)}{\partial x_j}\right]$ has rank p. Furthermore for $y \in D^2$, the partial derivatives of $\zeta(y)$ are the solutions of the set of linear equations

$$\begin{aligned} \textstyle\sum_{k=1}^{p} \frac{\partial \psi_1(\zeta(y),y)}{\partial x_k} \frac{\partial \zeta_k(y)}{\partial y} &= \frac{-\partial \psi_1(\zeta(y),y)}{\partial y}, \\ \textstyle\sum_{k=1}^{p} \frac{\partial \psi_2(\zeta(y),y)}{\partial x_k} \frac{\partial \zeta_k(y)}{\partial y} &= \frac{-\partial \psi_2(\zeta(y),y)}{\partial y}, \\ &\vdots \\ \textstyle\sum_{k=1}^{p} \frac{\partial \psi_p(\zeta(y),y)}{\partial x_k} \frac{\partial \zeta_k(y)}{\partial y} &= \frac{-\partial \psi_p(\zeta(y),y)}{\partial y}, \end{aligned}$$

or perhaps most usefully as the following matrix equation

$$\begin{pmatrix} \frac{\partial \psi_1}{\partial x_1} & \frac{\partial \psi_1}{\partial x_2} & \cdots & \frac{\partial \psi_1}{\partial x_p} \\ \frac{\partial \psi_2}{\partial x_1} & \frac{\partial \psi_2}{\partial x_2} & \cdots & \frac{\partial \psi_2}{\partial x_p} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \psi_p}{\partial x_1} & \frac{\partial \psi_p}{\partial x_2} & \cdots & \frac{\partial \psi_m}{\partial x_p} \end{pmatrix} \begin{pmatrix} \frac{\partial \zeta_1(y)}{\partial y} \\ \frac{\partial \zeta_2(y)}{\partial y} \\ \vdots \\ \frac{\partial \zeta_p(y)}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{-\partial \psi_1(\zeta(y),y)}{\partial y} \\ \frac{-\partial \psi_2(\zeta(y),y)}{\partial y} \\ \vdots \\ \frac{-\partial \psi_p(\zeta(y),y)}{\partial y} \end{pmatrix}.$$

1.2 Convex Optimization Problems

1.2.1 The statement of the problem

A convex programming problem is often written as

$$\min_x f(x) \text{ subject to } x \in \mathcal{C}, \tag{CP}$$

where $\mathcal{C} = \{x \in \mathbb{R}^n : h_j(x) \leq 0, j = 1, \ldots, k, \ \ g_i(x) = 0, i = 1, \ldots, m\}$ is a convex set and f is a convex function on $\mathbb{R}^n$ to $\mathbb{R}$. The function f is known as the objective function and the

vector x is called the decision variable, while $\mathcal{C}$ is often termed the set of feasible solutions. In an optimization problem, the types of mathematical relationships between the objective function and constraints and the decision variables determine how hard it is to solve and the solution methods or algorithms that can be used for optimization. The classification of (CP) is based on the fundamental properties of function f and the set of feasible solutions $\mathcal{C}$, namely convexity and differentiability.
Among the most studied special cases we note:

1. Unconstrained convex optimization:

$$f \text{ convex}, \ \mathcal{C} = \mathbb{R}^n \ \text{(noconstraints)}$$

2. Linear programming:

$$(\ f \text{ linear}, \ \mathcal{C} \text{ affine set}).$$

3. Quadratic programming:

$$(f \text{ quadratic}, \ \mathcal{C} \text{ affine set }).$$

4. Linearly Constrained Convex Problems

$$(f \text{ convex}, \ \mathcal{C} \text{ affine set }).$$

5. Convex programming

$$(f \text{ convex}, \ \mathcal{C} \text{ convex set}).$$

Note that cases 2, 3, and 4 are successive generalizations. In fact linear programming is a special case of every other problem type except for case 1.

1.2.2 Existence and uniqueness of optimal solutions

We are interested in identifying points in $\mathcal{C}$ at which the function f attains a (local or global) minimum.

- **Local minimum:** Let $f : \mathcal{C} \to \mathbb{R}$, a point $x^* \in \mathcal{C}$ is local minimum of f if $f(x^*) \leq f(x)$ for all x in the neighborhood of x^* i.e. exists an open ball around x^*, $B_\varepsilon(x^*)$ such that $f(x^*) \leq f(x) \quad \forall x \in B_\varepsilon(x^*)$.

- **Global minimum:** A point $x^* \in \mathcal{C}$ is a global minimum to (CP) if $f(x^*) \leq f(x)$ for all $x \in \mathcal{C}$.

Local and global maximum can be defined similarly by just reverting the inequalities.

Proposition 1.2.1. *Suppose that $\mathcal{C}$ is a convex set, $f : \mathcal{C} \to \mathbb{R}$ is a convex function, and x^* is a local minimum of (CP). Then x^* is a global minimum of f over $\mathcal{C}$.*

Existence of a solution:

Theorem 1.2.2. *Weierstrass Theorem[19]*
Let Ω be a compact subset of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ be continuous on Ω. Then there exists at least $x^* \in \Omega$ such that $f(x^*) \leq f(x)$ for all $x \in \Omega$.

Uniqueness of a solution:

Theorem 1.2.3. *[19] Suppose $\mathcal{C}$ is a nonempty and convex subset of $\mathbb{R}^n$, suppose f est strictly convex function on $\mathcal{C}$. Then (CP) admits an optimum solution at most.*

1.2.3 Optimality conditions

The constraint qualification

Definition 1.2.4. *Let $\bar{x}$ be feasible for (CP) and put $I(\bar{x}) := \{i \mid h_i(\bar{x}) = 0\}$. We say that the linear independence constraint qualification holds at $\bar{x}$ if the gradients*

$$\nabla g_i(\bar{x}) \ (i = 1, \ldots, m), \quad \nabla h_j(\bar{x}) \ (j \in I(\bar{x})),$$

are linearly independent.

Theorem 1.2.5. ***(Karush-Kuhn-Tucker (K.K.T))****[?] If x^* is a solution of (CP), if the gradients of f, g and h are finite at x^*, and if a constraint qualification is satisfied, then there exists an $y^* \in \mathbb{R}^m$ and $z^* \in \mathbb{R}^k$ such that:*

$\nabla f(x^*) + \sum_{i=1}^m y_i^* \nabla g_i(x^*) + \sum_{j=1}^k z_j^* \nabla h_j(x^*) = 0,$

$z_j^* h_j(x^*) = 0 \quad j = 1, \ldots, k,$

$z_j^* \geq 0 \quad j = 1 \in I(\bar{x}).$

1.2.4 Convergence

Rate of convergence:
If a sequence $\{x_1, x_2, ..., x_n\}$ converges to a value $\tilde{x}$ and if there exist real numbers $\nu > 0$ and $p \geq 1$ such that

$$\lim_{n\to\infty} \frac{|x^{n+1} - \tilde{x}|}{|x^n - \tilde{x}|^p} = \nu,$$

then we say that p is the rate of convergence of the sequence.
When p = 1 we say the sequence converges linearly and when p = 2 we say the sequence converges quadratically. If $1 < p < 2$ then the sequence exhibits superlinear convergence.

1.3 Newton's Method

Our problem consists of finding a vector x^* that will minimize a function f:

$$\begin{cases} f : \mathbb{R}^n \to \mathbb{R}, \\ x^* = argmin_x f(x). \end{cases}$$

Newton's method is a root-finding algorithm used to generate a sequence $\{x_k\}$ that will converge towards x^* $(argminf(x))$. The sequence is built in the following way:

$$x_{k+1} \quad = \quad x_k + d,$$

Where n is the iteration, and d is a vector, same size as x, defined as follow:

$$d = -(\nabla^2 f(x_k))^{-1} \nabla f(x_k),$$

where $\nabla f(x) = (\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x))^T$ is the gradient of the function f and $\nabla^2 f(x)$ is the Hessian matrix of the function f.
In order to use Newton's method, you need to guess a first approximation to the zero of the function and then use the above procedure.

1.4 Convex Optimization with Linear Constraints (LCCO)

1.4.1 The Primal problem

Convex nonlinear optimization problem with linear equality constraints is a problem consisting of minimizing a nonlinear function over a convex set. More explicitly, a convex nonlinear problem is of the form

$$\bar{p} = \min_x f(x) \text{ subject to } x \in \mathcal{F}, \tag{P}$$

where the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable and convex over the feasible set $\mathcal{F} = \{x \in \mathbb{R}^n : x \geq 0,\ Ax = b\}$, A is a given $(m \times n)$ matrix with full rank row m and $b \in \mathbb{R}^m$. This problem has many important applications in theory as well as in practice. In particular, it includes linear and quadratic optimization. The method that works on the original problem directly by searching through the feasible region for the optimal solution called the primal method.

1.4.2 The Dual problem

The dual of the problem (P) in the Wolfe sense is:

$$\bar{d} = \{\max_y b^T y + f(x) - x^T \nabla f(x) | A^T y + z = \nabla f(x), \quad z \geq 0\}. \tag{D}$$

We define the following feasibility sets:

$$\mathcal{F}_p = \{x \in \mathbb{R}^n | Ax = b, \quad x \geq 0\} \ ; \ \mathcal{F}_d = \{y \in \mathbb{R}^m | A^T y - z = \nabla f(x), \quad z \geq 0\},$$

$$\mathcal{F}_p^o = \{x \in \mathbb{R}^n | Ax = b, \quad x > 0\} \ ; \ \mathcal{F}_d^o = \{y \in \mathbb{R}^m | A^T y - z = \nabla f(x), \quad z > 0\}.$$

The method that works simultaneously with primal and dual variables called the primal-dual method.

Throughout the thesis, we focus to convex programming with linear constraints problems when the function f is convex and $\mathcal{F}$ is an affine set.

1.4.3 Weak duality

Theorem 1.4.1. ***(Weak duality theorem)****[19] If x and (y, z) are respectively feasible solutions for (P) and (D) then,*

$$\bar{p} \geq \bar{d}.$$

1.4.4 Strong duality via slater's condition

We have seen how weak duality allows to form a convex optimization problem that provides a lower bound on the original (primal) problem, even when the latter is non-convex. The duality gap is the non-negative number $\bar{p} - \bar{d}$.
Strong duality holds for the above problem if the duality gap is zero: $\bar{p} = \bar{d}$.
Slater's condition
We say that the problem satisfies Slater's condition if it is strictly feasible. We can replace the above by a weak form of Slater's condition, where strict feasibility is not required whenever the set $\mathcal{F}$ is affine.

Theorem 1.4.2. *[19] If the primal problem is convex, and satisfies the weak Slater's condition, then strong duality holds, that is, $\bar{p} = \bar{d}$.*

Remark 1.4.3. *If f is quadratic convex and the set $\mathcal{F}$ is affine, then the duality gap is always zero, provided one of the primal or dual problems is feasible. In particular, strong duality holds for any feasible linear optimization problem.*

1.5 Solving (LCCO) Problems

There is in general no analytical formula for the solution of (LCCO) problems, but there are very effective methods for solving them. Interior-point methods work very well in practice, and in some cases can be proved to solve the problem to a specified accuracy with a number of operations that does not exceed a polynomial of the problem dimensions like methods for solving linear programs.

1.5.1 Interior-point methods

In this section we discuss interior-point methods for solving convex optimization problems that include linear equality constraints,

$$\min_{x\in\mathbb{R}^n} f(x) \text{ subject to } \{Ax = b, x \geq 0\}. \tag{P}$$

Interior-point methods are a certain class of algorithms that solve linear and nonlinear convex optimization problems. Arguably, interior-point methods were known as early as the 1960s in the form of the barrier function methods, but the modern era of interior-point methods dates to 1984 when Karmarkar proposed his algorithm for linear programming. In 1994 algorithms and software for linear programming have become quite sophisticated (more than 1300 published papers). The interior-point methods can be generalized to more general classes of problems, such as convex quadratic programming, semi-definite programming, and nonlinear convex problems.
Interior-point methods solve the problem (P) by applying Newton's method to a sequence of equality constrained problems. We will concentrate on a particular interior-point method, the logarithmic barrier method for which we give proof of convergence.

1.5.2 Logarithmic barrier interior-point method for (LCCO) problems

Idea:
The logarithmic barrier interior-point method is a procedure for approximating constrained optimization problems by unconstrained problems. The approximation is accomplished in the case of logarithmic barrier method by adding to the objective function a logarithmic function that prescribes a high cost for violation of the constraints.
We consider the following (LCCO) problem:

$$\min_{x} f(x) \text{ subject to } x \in \mathcal{F}, \tag{P}$$

with a feasible set $\mathcal{F}$ given by

$$\mathcal{F} = \{Ax = b,\ x \geq 0\},$$

We assume that the optimal set of P is nonempty and compact.
We shall study a family of unconstrained minimization problems of the form

$$\min \varphi_\mu(x), \quad x \in \mathbb{R}^n, \qquad (P_\mu)$$

with

$$\varphi_\mu(x) = \begin{cases} f(x) + \sum_{i=1}^n \theta_\mu(x_i) & \text{if } x \in \mathcal{F}, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\mu > 0$ is a barrier parameter which will ultimately go to zero and the function θ_μ defined by

$$\theta_\mu(x) = \begin{cases} \mu \log x - \mu \log \mu & \text{if} \quad \mu > 0,\ x_i > 0, \\ 0 & \text{if} \quad \mu = 0,\ x_i \geq 0, \\ +\infty & \text{otherwise.} \end{cases}$$

The function θ_μ is convex, proper and lower semi-continuous defined on $\mathbb{R} \times \mathbb{R}^n$.
Newton method is used to find an approximate solution of P_μ for a certain value of μ, and then μ is decreased to zero.
The prototype primal logarithmic barrier algorithm can be specified as follows:

Framework (Log-Barrier)

Given $\mu_0 > 0$, a starting point x_0;
Set $k \leftarrow 0$
Repeat
Starting at x_k , and fixing μ_k;
Obtain x_{k+1} by performing one or more Newton steps;
Choose new barrier parameter $\mu_{k+1} \in (0, \mu_k)$;
$k \leftarrow k + 1$;
Until some termination test is satisfied.

.

Figure 1.1: **Algorithm 1.1**

Chapter 2

A Weighted Logarithmic Barrier Interior-Point Method for Linearly Constrained Optimization

Contents

In this chapter, we consider the linearly convex constrained optimization (LCCO) problem:

$$\bar{p} = \min f(x) \text{ subject to } x \in \mathcal{F}, \tag{P}$$

where the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable and convex over the feasible set $\mathcal{F} = \{x \in \mathbb{R}^n : x \geq 0,\ Ax = b\}$, A is a given $(m \times n)$ matrix with full rank row m and $b \in \mathbb{R}^m$.

2.1 The Weighted Barrier Penalization

Throughout the chapter, we assume that the following assumptions hold.

1. There exist a strictly feasible point $x_0 > 0$ such that $Ax_0 = b$.

2. The set of optimal solutions of P is non empty bounded set.

It follows from the second hypothesis that the recession cone

$$\mathcal{K}_\infty = \{d \in \mathbb{R}^n :\ f_\infty(d) \leq 0,\ d \geq 0,\ Ad = 0\} = \{0\},$$

where f_∞ denote the recession function of f. We deduce from the optimality conditions that x^* is a solution of P if and only if there exists an $y^* \in \mathbb{R}^m$ and $z^* \in \mathbb{R}^n$ such that

$$\nabla f(x^*) + A^T y^* = z^* \geq 0, \quad Ax^* = b, \quad \langle z^*, x^* \rangle = 0, x^* \geq 0. \tag{2.1}$$

2.1.1 The weighted perturbed problems

Let us define the function $\theta : \mathbb{R} \times \mathbb{R} \to (-\infty, +\infty]$ by

$$\theta(t, w) = \begin{cases} t(\log t - \log w) & \text{if} \quad t > 0,\ w > 0, \\ 0 & \text{if} \quad t = 0,\ w \geq 0, \\ +\infty & \text{otherwise.} \end{cases}$$

The function θ is convex, lower semi-continuous and proper. We consider now the following function defined on $\mathbb{R}^n_+ \times \mathbb{R}^n_+$ by

$$\varphi(\mu r, x) = \begin{cases} f(x) + \sum_{i=1}^n \theta(\mu r_i, x_i) & \text{if } x \in \mathcal{F}, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\mu > 0$ is the barrier parameter and $r = (r_1, r_2, \ldots, r_n)^T \in \mathbb{R}^n_+$, the vector of the weight associated with the barrier function.
Finally, we introduce the function p^r defined by

$$p^r(\mu) = \inf_x \left[\varphi^r_\mu(x) = \varphi(\mu r, x) \, : \, x \in \mathbb{R}^n \right]. \tag{P^r_μ}$$

The function p^r is convex since φ^r_μ is convex. By construction, P^r_0 is only the problem P with $\bar{p} = p^r(0)$. The function φ^r_μ is convex, lower semi-continuous and proper, its recession function is given by

$$(\varphi^r_\mu)_\infty(d) = \lim_{\alpha \to +\infty} \frac{\varphi^r_\mu(x_0 + \alpha d) - \varphi^r_\mu(x_0)}{\alpha}.$$

We obtain

$$(\varphi^r_\mu)_\infty(d) = \begin{cases} f_\infty(d) & \text{if} \qquad d \geq 0, \; Ad = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Then

$$\{d \in \mathbb{R}^n \, : \, (\varphi^r_\mu)_\infty(d) \leq 0\} = \{d \in \mathbb{R}^n \, : \, f_\infty(d) \leq 0, \, d \geq 0, \, Ad = 0\},$$

where d is a recession direction and $\alpha \geq 0$.
Since this set is reduced to $\{0\}$ then the problem P^r_μ admits an optimal solution for each $\mu > 0$. The function φ^r_μ is strictly convex for all $\mu > 0$ and $r \geq 0$, then P^r_μ has an unique optimal solution denoted by x^r_μ.

2.1.2 Convergence of the weighted perturbed solutions to the optimal solution of P

The necessary and sufficient optimality conditions of (P^r_μ) imply that there exits $y^r_\mu = y(\mu, r) \in \mathbb{R}^m$, such that

$$\nabla f(x^r_\mu) - \mu X^{-1} r + A^T y^r_\mu = 0, \tag{2.2}$$

$$Ax^r_\mu = b, \tag{2.3}$$

where $X = Diag(x^r_\mu)$.
Note that y^r_μ is uniquely defined since A is of full rank row. In fact, the couple (x^r_μ, y^r_μ) is the solution of the system $H(x,y) = 0$ where

$$H(x,y) = \begin{pmatrix} \nabla f(x) - \mu X^{-1} r + A^T y \\ Ax - b \end{pmatrix}.$$

By the implicit function theorem, the functions $\mu \mapsto x(\mu, r) = x^r_\mu$ and $\mu \mapsto y(\mu,\, r) = y^r_\mu$ are differentiable on $(0, \infty)$ and we have,

$$\begin{pmatrix} \nabla^2 f(x^r_\mu) + \mu R X^{-2} & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x'(\mu, r) \\ y'(\mu, r) \end{pmatrix} = \begin{pmatrix} X^{-1} r \\ 0 \end{pmatrix}, \tag{2.4}$$

where R=Diag(r), it follows that the function p^r is differentiable on $(0, \infty)$. Recall that

$$p^r(\mu) = f(x^r_\mu) + \mu \sum_{i=1}^n r_i (\ln \mu r_i - \ln (x_i)^r_\mu),$$

and then

$$(p^r(\mu))' = \sum_{i=1}^n r_i (1 + \ln \mu r_i - \ln (x_i)^r_\mu) + \langle \nabla f(x^r_\mu) - \mu X^{-1} r, x'(\mu, r) \rangle.$$

In view of (2.2) and (2.4)

$$\begin{aligned} (p^r(\mu))' &= \sum_{i=1}^n r_i (1 + \ln \mu r_i - \ln (x_i)^r_\mu) - \langle A^T y^r_\mu, x'(\mu, r) \rangle, \\ &= \sum_{i=1}^n r_i (1 + \ln \mu r_i - \ln (x_i)^r_\mu) - \langle y^r_\mu, A x'(\mu, r) \rangle, \\ &= \sum_{i=1}^n r_i (1 + \ln \mu r_i - \ln (x_i)^r_\mu). \end{aligned}$$

Since $x^r_\mu \in \mathcal{F}$ and p^r is convex we obtain:

$$f(x^r_\mu) \geq \bar{p} = p^r(0) \geq p^r(\mu) + (0 - \mu)(p^r(\mu))' = f(x^r_\mu) - \mu \|r\|_1.$$

Consequently, we have

$$\bar{p} \leq f(x_\mu^r) \leq \bar{p} + \mu\|r\|_1.$$

Then if

$$\mu \mapsto 0, f(x_\mu^r) = \bar{p}.$$

Now, we interested to the weighted-path of $\{x_\mu^r\}$ when $\mu \mapsto 0$.
i) Case where f is strongly convex with coefficient $\tau > 0$. Hence P has a unique optimal solution x^*, and we have

$$\mu\|r\|_1 \geq f(x_\mu^r) - f(x^*) \geq \langle \nabla f(x^*), x_\mu^r - x^* \rangle + \frac{\tau}{2}\|x_\mu^r - x^*\|^2.$$

In view of (2.1), we deduce

$$\mu\|r\|_1 \geq \langle z^*, x_\mu^r \rangle + \frac{\tau}{2}\|x_\mu^r - x^*\|^2 \geq \frac{\tau}{2}\|x_\mu^r - x^*\|^2,$$

$$\|x_\mu^r - x^*\| \leq \sqrt{\frac{2\mu\|r\|_1}{\tau}}.$$

ii) For the case where f is only convex is more complicated case. Note first that for $\mu \leq 1$,

$$x_\mu^r \in \{x \,:\, x \geq 0,\ Ax = b,\ f(x) \leq \|r\|_1 + \bar{p}\}.$$

This set is closed convex and non empty. Its recession cone is

$$\{d \in \mathbb{R}^n :\ f_\infty(d) \leq 0,\ d \geq 0,\ Ad = 0\} = \{0\}.$$

By the second assumption the set of optimal solutions of P is bounded which implies that each adherence value of $\{x_\mu^r\}$ when $\mu \mapsto 0$ is an optimal solution of P.

Remark 2.1.1. *If $r = e$, where e is the vector of ones, then the weighted-path coincides with the classical central path.*

2.2 The Description of The Method

Letting $\mathcal{F}^o = \{x \in \mathbb{R}^n \,:\, x > 0,\ Ax = b\}$ the set of strictly feasible points. The principle of the method is as follows: Let $(\mu_k r, x_k) \in \mathbb{R}_+^n \times \mathcal{F}^o$, the current iterate.

1. We make an approximated minimization of the weighted perturbed $P^r_{\mu_k}$ which gives a new point x_{k+1} such that $\varphi(\mu_{k+1}r, x_{k+1}) < \varphi(\mu_k r, x_k)$.

2. We take $\mu_{k+1} < \mu_k$.

We iterated until we obtained an approximated optimal solution of the original problem. The weighted perturbed problem is defined by

$$\min_x \varphi^r_\mu(x) = \min_x [\, f(x) + \sum_{i=1}^{n} \theta(\mu r_i, x_i) \ : x \in \mathcal{F}]. \qquad (P^r_\mu)$$

2.3 The Newton Descent Direction

At $x \in \mathcal{F}^\circ$, the Newton descent direction d is given by solving the following quadratic convex program:

$$\min_d [\, \langle \nabla \varphi^r_\mu(x), d \rangle + \frac{1}{2} \langle \nabla^2 \varphi^r_\mu(x) d, d \rangle : Ad = 0\,].$$

It suffices to solve the linear system with $n + m$ equations

$$\begin{pmatrix} \nabla^2 f(x) + \mu R X^{-2} & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix} = \begin{pmatrix} \mu X^{-1} r - \nabla f(x) \\ 0 \end{pmatrix}, \qquad (2.5)$$

where $s \in \mathbb{R}^m$.

It easy to prove that the linear system (2.5) has a unique solution. The descent direction being thus obtained, it is now question of minimizing a function of one real variable to obtain the step size α.

$$\gamma^r(\alpha) = \varphi^r_\mu(x + \alpha d) - \varphi^r_\mu(x) = f(x + \alpha d) - f(x) \ - \mu \sum_{i=1}^{n} r_i \ln(1 + \alpha t_i),$$

where $t = X^{-1}d$, the function γ^r is convex.

Next task, we propose a new method to determine the step size .

2.4 A Tangent Method for Determining The Step Size

Our approach is try out a sequence of candidate values for α, when the condition $(\gamma^r(\alpha))' \leq \epsilon$ is satisfied, stopping and accept this value. We can say that this technique is done in two phases.

1. The first phase finds an interval containing the required step size, the choice of the bounds of the interval is similar to the bisection method, when we restrict the value of α until we find the required value.

2. The second phase computes the optimal step size within this interval, in this phase we determine the tangents T_1 and T_2 in the bounds of the interval and we select the value corresponding to the intersection of the tangents T_1 and T_2.

The upper bound on the step size α is given by

$$\alpha_{max} = \min \left\{ -\frac{x_i}{d_i}; \quad i \in \hat{I} \right\},$$

where

$$\hat{I} = \{ i \ : \quad d_i < 0 \} .$$

Because the convexity of the function $\gamma^r(\alpha)$, this technique will be more efficiency in practice, the next figures shows clearly this idea:

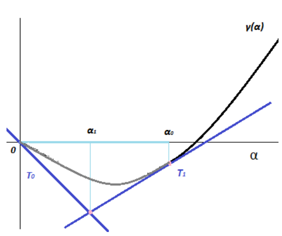

Figure 2.1: Step 1.

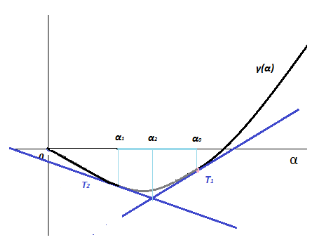

Figure 2.2: Step 2.

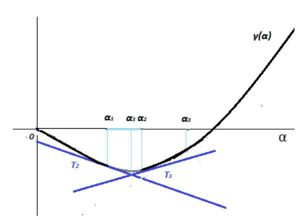

Figure 2.3: Step 3.

The tangent algorithm for determining the step size is follows.

Algorithm

Input

An accuracy parameter $\epsilon > 0$;

A threshold parameter $0 < \beta < 1$;

$a = 0, \quad b = \beta\alpha_{max}$, such that $(\gamma^r(b))' > 0$;

$\alpha = \frac{b}{2}$;

While$|(\gamma^r(\alpha))'| > \epsilon$ **do**

if $(\gamma^r(\alpha))' > 0$ then

$b = \alpha$;

if not

$a = \alpha$;

end if

$\alpha = \frac{-(\gamma^r(b))'b+\gamma^r(b)+(\gamma^r(a))'a-\gamma^r(a)}{(\gamma^r(a))'-(\gamma^r(b))'}$;

End While

.

Figure 2.4: **Algorithm 2.1**

We are now ready to state the generic algorithm for solving LCCO.

The Generic interior-point algorithm for LCCO:

Threshold parameters $\epsilon > 0, \bar{\mu} > 0$ and $\lambda \in [0, 1]$, are given;

Start with $x_0 \in \mathcal{F}^o$, $\mu > \bar{\mu}$ and a weight vector $r > 0$;

1) Solve the linear system (2.5) to obtain d;

2) Take $t = X^{-1}d$;

If $\|t\| \geq \epsilon$

- Determinate $\tilde{\alpha}$ with tangent method ;

- Update $x_{k+1} = x_k + \tilde{\alpha}d_k$, $\mu_{k+1} = \lambda\mu_k$ and return to 1;

If $\|t\| \leq \epsilon$

Case 1. $\mu_k \leq \bar{\mu}$

STOP we have obtained a good approximation of the optimal solution of P;

Case 2. $\mu_k > \bar{\mu}$

We have obtained a good approximation of $p^r(\mu)$, do $\mu_{k+1} = \lambda\mu_k$ and go to 1;

.

Figure 2.5: **Algorithm 2.2**

2.5 Numerical Results

In the following section, we apply our algorithm on some different examples of LCCO. A comparative numerical tests with a classical line search are presented. Our implementation is done by the Scilab 5.4.1. We use in the sequel the following notation.
Method 1: the first alternative uses the tangent technique.
Method 2: the second alternative uses the Wolfe method.
Outer: the number of outer iterations.
Inner: the number of inner iterations.
Objective: the optimal value of the objective function $\bar{p}$.
Time: the time measured in seconds. Our tolerance is $\epsilon = 10^{-6}$ in all our testing examples.
We consider three examples that are written in the following form:

$$(QP)\begin{cases} \bar{p} = \min_{x\in\mathbb{R}^n} \frac{1}{2}\langle x, Qx\rangle + \langle c, x\rangle \\ \text{s.t } Ax = b, \\ \ x \geq 0. \end{cases}$$

Example 1.

$$\begin{aligned} Q &= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}, \\ c &= (0,0,2)^T,\ b = (6,\ 4)^T. \end{aligned}$$

The initial point is

$$x^0 = \begin{pmatrix} 1, & 1.5, & 3.5 \end{pmatrix},^T$$

and the optimal solution is:

$$x^* = \begin{pmatrix} 1, & 1.00028, & 3.99971 \end{pmatrix}.^T$$

with $\bar{p} = 10$, such as $r_1 = (0.9,\ 1,\ 0.03)^T$.

Example 2.

$$Q = \begin{pmatrix} 20 & 1.2 & 0.5 & 0.5 & -1 \\ 1.2 & 32 & 1 & 1 & 1 \\ 0.5 & 1 & 14 & 1 & 1 \\ 0.5 & 1 & 1 & 15 & 1 \\ -1 & 1 & 1 & 1 & 16 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1.2 & 1 & 1.8 & 0 \\ 3 & -1 & 1.5 & -2 & 1 \\ -1 & 2 & -3 & 4 & 2 \end{pmatrix},$$

$$c = (1, -1.5, 2, 1.5, 3)^T, \ b = (9.31,\ 5.45,\ \ 7.06)^T.$$

The initial point is

$$x^0 = \begin{pmatrix} 2.42, & 1, & 1.55, & 2.3, & 1.465 \end{pmatrix}^T,$$

and an optimal solution is:

$$x^* = \begin{pmatrix} 2.6603177, & 0.7036150, & 1.3245096, & 2.4893527, & 1.1646029 \end{pmatrix}^T,$$

with $\bar{p} = 175.24586$, and $r_2 = (2,\ 1,\ 3,\ 1,\ 4)^T$.

Example 3.

$$Q = \begin{pmatrix} 30 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 21 & 0 & 1 & -1 & 1 & 0 & 1 & 0.5 & 1 \\ 1 & 0 & 15 & -0.5 & -2 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & -0.5 & 30 & 3 & -1 & 1 & -1 & 0.5 & 1 \\ 1 & -1 & -2 & 3 & 27 & 1 & 0.5 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 16 & -0.5 & 0.5 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0.5 & -0.5 & 8 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 0.5 & 1 & 24 & 1 & 1 \\ 1 & 0.5 & 1 & 0.5 & 1 & 0 & 1 & 1 & 39 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 11 \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & -1 & 1.9 & 1.25 & 1.2 & 0.4 & -0.7 & 1.06 & 1.5 & 1.05 \\ 1.3 & 1.2 & 0.15 & 2.15 & 1.25 & 1.5 & 0.4 & 1.52 & 1.3 & 1 \\ 1.5 & -1.1 & 3.5 & 1.25 & 1.8 & 2 & 1.95 & 1.2 & 1 & -1 \end{pmatrix},$$

$$b = \begin{pmatrix} 11.64771 \\ 16.66587 \\ 21.28925 \end{pmatrix}, c = \begin{pmatrix} -0.5, & -1, & 0, & 0, & -0.5, & 0, & 0, & -1, & -0.5, & -1 \end{pmatrix}.^T$$

The strictly feasible starting point is taken as:

$$x^0 = \begin{pmatrix} 1.475, & 0.868, & 1.752, & 1.637, & 1.560, & 1.926, & 1.538, & 1.416, & 1.223, & 0.728 \end{pmatrix}^T.$$

An optimal solution is:

$$x^* = \begin{pmatrix} 0.9635696, & 0.5093399, & 1.7397881, & 1.9042982, & 1.2431635, \\ 2.6257839, & 1.3225584, & 1.6164774, & 0.8237509, & 0.8972367 \end{pmatrix}^T,$$

with the optimal value is $\bar{p} = 263.96783$. The value of the weight vector is taken as: $r_3 = (1,\ 1,\ 0.4,\ 1,\ 0.4,\ 1,\ 0.4,\ 0.4,\ 1,\ 0.96)^T$.

		Method 1		**Method 2**	
Example	Size (m, n)	Inner	Outer	Inner	Outer
1	(2,3)	7	7	162	7
2	(3,5)	24	6	46	8
3	(3,10)	20	7	40	8

Table 2.1: Comparative numerical tests

In the following, we compare our approach with the classical path method (non weighted case).

Problem with variable size. We consider the following LCCO problem:

$$\bar{p} = \min_{x \in \mathbb{R}^n} [f(x) : x \geq 0,\ Ax = b],$$

where $f(x) = \sum_{i=1}^{n} x_i \ln x_i$, $b_i = 1$ and

$$A[i,j] = \begin{cases} 1 & \text{if } i = j \text{ or } \ j = i + m), \\ 0 & \text{else,} \end{cases} \quad \text{with}\, n = 2m.$$

The strictly feasible starting point is:

$$x^0 = \begin{pmatrix} 0.7, & \dots, & 0.7, & 0.3, & \dots, & 0.3 \end{pmatrix}.^T$$

An optimal solution is:

$$x^* = \begin{pmatrix} 0.5, & 0.5, & \dots & ,0.5 \end{pmatrix}.^T$$

The optimal values with different size of n are:

n	20	400	900
Objective	-6.9314718	-138.62944	-311.911623

The obtained numerical results with different size of n and barrier parameter μ are stated in tables 2 and 3.

Weighted case:

The weighted vector is $r = (0.011, \dots, 0.011, 0.022 \dots, 0.022).^T$

	$n = 20$		$n = 400$		$n = 900$	
μ	Outer	Time	Outer	Time	Outer	Time
0.01	2	0.296×10^{-3}	2	0.17160	2	23.476212
0.25	4	0.316×10^{-3}	4	0.28762	4	32.40515
1	5	0.499×10^{-3}	5	0.355702	5	41.84014
5	6	0.665×10^{-3}	6	0.400961	6	52.96016

Table 2.2: Numerical results (Weighted case)

Non weighted case

	$n = 20$		$n = 400$		$n = 900$	
μ	Outer	Time	Outer	Time	Outer	Time
0.01	4	0.325×10^{-3}	4	0. 2915518	4	33.147112
0.25	6	0.482×10^{-3}	6	0.3832547	6	40.40114
1	7	0.591×10^{-3}	7	0.4512415	7	49.88745
5	8	0.835×10^{-3}	8	0.6001489	8	58.91456

Table 2.3: Numerical results (Non weighted case)

2.6 Conclusion and Remarks

In this chapter we have introduced a relaxation of the classical path of the perturbed LCCO problem and we have presented a new technique for determining the step size. These have a great influence on the acceleration of the convergence of the algorithm i.e., the number of iterations and the time produced are reduced significantly.

Chapter 3

A Primal-Dual Logarithmic Barrier Interior-Point Method for Linearly Constrained Quadratic Programming

Contents

The purpose of this chapter is to solve the following optimization problem

$$\bar{p} = \min_{x \in \mathbb{R}^n} \{c^T x + \frac{1}{2} x^T Q x \text{ subject to } Ax = b,\ \ x \geq 0\}, \tag{QP}$$

and its dual

$$\bar{d} = \max_{y \in \mathbb{R}^m} \{b^T y - \frac{1}{2} x^T Q x \text{ subject to } A^T y + z = Qx + c,\ z \in \mathbb{R}^n_+\}, \tag{QD}$$

with $Q \in \mathbb{R}^{n \times n}$ is symmetric positive semi-definite matrix, c is an n real vector, b is an m real vector.
The feasible sets of (QP) and (QD) problems given by:

$$\mathcal{F}_p = \{x \in \mathbb{R}^n, Ax = b, x \geq 0\},\ \mathcal{F}_d = \{(y,z) \in \mathbb{R}^{m+n}, A^T y + z = Qx + c, z \geq 0\},$$

$$\mathcal{F}_p^o = \{x \in \mathbb{R}^n, Ax = b, x > 0\},\ \mathcal{F}_d^o = \{(y,z) \in \mathbb{R}^{m+n}, A^T y + z = Qx + c, z > 0\}.$$

Without loss of generality, we assume that (QP) and (QD) satisfy the interior-point condition (IPC), i.e., there exist x_0, y_0 and z_0 such that $x_0 \in \mathcal{F}_p^o$ and $(y_0, z_0) \in \mathcal{F}_d^o$. It is well known that searching a pair of optimal solution of (QP) and (QD) is equivalent to solving the following system, which represents the (K.K.T) optimality conditions

$$(KKT) \begin{cases} Ax = b, \quad x \geq 0, & (\text{ Primal feasibility}) \\ A^T y + z - Qx = c, \quad z \geq 0,\ y \in \mathbb{R}^m, & (\text{ Dual feasibility}) \\ x^T z = 0. & (\text{ Complementarity condition}) \end{cases}$$

Generally the idea of the classical path-following method is to replace the complementarity equation $x^T z = 0$ by $xz = \mu$. Thus we consider the nonlinear system parameterized by μ.

$$\begin{cases} Ax = b, \quad x \geq 0, \\ A^T y + z - Qx = c, \quad z \geq 0,\ y \in \mathbb{R}^m, \\ x_i z_i = \mu, \quad i = 1, \ldots, n, \end{cases}$$

when $\mu > 0$.
In the next section, we shall present an other strategy for solving the primal-dual problems by logarithmic penalty methods.

3.1 The Primal-Dual Logarithmic Barrier Method

The basic idea of our approach is to solve an equivalent system of (K.K.T) optimality conditions defined as

$$(S)\begin{cases}\min x^T z\\ \text{s.t } Ax = b,\\ A^T y + z - Qx = c,\\ x \geq 0,\ z \geq 0,\ y \in \mathbb{R}^m,\end{cases}$$

we denote by $\mathcal{F}_s$ the set of optimal solutions of this problem. Now, we make assumptions about the problem (S):

Assumption 1. The m rows of the matrix A are linearly independent.

Assumption 2. The set of optimal solution $\mathcal{F}_s$ is nonempty and bounded set.

Throughout all the following, we assume that the two assumptions hold and we set $\chi = (x, z, y)$, $\mathcal{D} = \mathcal{F}_p \times \mathcal{F}_d$, $\mathcal{D}_{int} = \mathcal{F}_p^o \times \mathcal{F}_d^o$.

Firstly let us define the function

$$g(\chi) = \begin{cases} x^T z & \text{if} \quad \chi \in \mathcal{D},\\ +\infty & \text{otherwise.}\end{cases}$$

Then the function g is convex, lower semi-continuous and proper on $\mathcal{D}$.

Proof. To prove that g is a convex function on affine space

$$\mathcal{D} = \{(x, z, y) \in \mathbb{R}^{2n+m} : Ax = b,\ A^T y + z - Qx = c,\ x \geq 0,\ z \geq 0\},$$

we shall prove that the restriction of the function g to any straight-line of $\mathcal{D}$ is convex.

Let

$$(x, z, y) \in \mathbb{R}^{2n+m} \text{such that } Ax = b,\ A^T y + z - Qx = c,\ x \geq 0,\ z \geq 0,$$

and

$$(dx, dz, dy) \in \mathbb{R}^{2n+m} \text{ such that } \ Adx = 0, A^T dy + dz = Qdx.$$

Define the function

$$\zeta(t) = g(x + tdx, z + tdz, y + tdy),$$

then

$$\begin{aligned} \zeta(t) &= (x+tdx)^T(z+tdz), \\ &= t^2 dx^T dz + t(dx^T z + x^T dz) + x^T z, \end{aligned}$$

then if $dx^T dz \geq 0$ the function g is convex, we have

$$Adx = 0, \quad A^T dy + dz = Qdx,$$

since Q is a symmetric positive semi-definite matrix, we have

$$dx^T dz = dx^T(Qdx - A^T dy) = dx^T Q dx \geq 0.$$

□

3.2 Penalization

Now let us introduce the function $\vartheta : \mathbb{R}^2 \to (-\infty, +\infty)$

$$\vartheta(t,w) = \begin{cases} t(\frac{1}{2}\log t - \log w) & \text{if} \quad t > 0,\ w > 0, \\ 0 & \text{if} \quad t = 0,\ w \geq 0, \\ +\infty & \text{otherwise}, \end{cases}$$

given $\mu \geq 0$, we set

$$\phi(\chi,\mu) = \begin{cases} g(\chi) + \sum_{i=1}^{2n} \vartheta(\mu, \chi_i), & \text{if } \chi \in \mathcal{D}_{int}, \\ +\infty & \text{if not.} \end{cases}$$

We consider the perturbed problem

$$l(\mu) = \inf_{\chi}[\phi(\chi,\mu) = \phi_\mu(\chi), \quad \chi \in \mathbb{R}^{2n+m}], \tag{S_μ}$$

ϕ_μ is convex lower semi-continuous and proper on $\mathbb{R}^{2n+m}$. In addition if $\mu > 0$ then ϕ_μ is strictly convex function and (S_μ) has at most one optimal solution. Our next result concerns the existence of such solution.

Recall that the recession function of ϕ_μ is defined by

$$[\phi_\mu]_\infty(d) = \lim_{\alpha \to \infty} \frac{\phi_\mu(\chi_0 + \alpha d) - \phi_\mu(\chi_0)}{\alpha},$$

where $\chi_0 = (x_0, z_0, y_0)$ and $d = (dx, dz, dy)$.
We set

$$\mathcal{A} = \begin{pmatrix} A & 0 & 0 \\ -Q & I & A^T \end{pmatrix},$$

the recession functions of ϕ and g are related by

$$[\phi_\mu]_\infty(d) = \begin{cases} g_\infty(d) & \text{if} \quad d \geq 0, \mathcal{A}d = 0, \\ +\infty & \text{if not.} \end{cases}$$

Then

$$\{d \in \mathbb{R}^{2n+m} : [\phi_\mu]_\infty(d) \leq 0\} = \{d \in \mathbb{R}^{2n+m} : [g]_\infty(d) \leq 0,\ d \geq 0, \mathcal{A}d = 0\}.$$

On the other hand, assumption 2 is equivalent to

$$\{d \in \mathbb{R}^{2n+m} : [g]_\infty(d) \leq 0,\ d \geq 0, \mathcal{A}d = 0\} = \{0\},$$

hence,

$$\{d \in \mathbb{R}^{2n+m} : [\phi_\mu]_\infty(d) \leq 0\} = \{0\},$$

this condition implies that the set of optimal solutions of problem (S_μ) is a nonempty closed convex and bounded set.
Now, we show that the problem (S_μ) converges to the problem (S) as $\mu \to 0$.

3.3 Convergence of S_μ to S

We have the following lemma

Lemma 3.3.1. *For $\mu > 0$, let the problem (S_μ) have $\chi_\mu = (x_\mu, z_\mu, y_\mu)$ as an optimal solution, then the problem (S) has $\chi^* = \lim_{\mu \to 0} \chi_\mu$ as an optimal solution.*

Proof. Let μ a positive parameter be given and $\chi \in \mathcal{D}_{int}$ be arbitrary. We set $h(\chi) = x^T z$

and recall that

$$\phi(\chi,\mu)=\begin{cases} h(\chi)-\mu\sum_{i=1}^{2n}\ln(\chi_i)+n\mu\ln\mu & \text{if} \quad \chi\in\mathcal{D}_{int},\\ +\infty & \text{if not.}\end{cases}$$

Since the function $\phi(.,.)$ is differentiable at the point (χ,μ), we have

$$\begin{aligned} h(\chi) &= \phi(\chi,0)\\ &\geq \phi(\chi_\mu,\mu)+(\chi-\chi_\mu)^T\nabla_\chi\phi(\chi_\mu,\mu)+(0-\mu)\frac{\partial}{\partial\mu}\phi(\chi_\mu,\mu),\\ &= \phi(\chi_\mu,\mu)-\mu\frac{\partial}{\partial\mu}\varphi(\chi_\mu,\mu),\\ &= h(\chi_\mu)-\mu\sum_{i=1}^{2n}\ln(\chi_i)+n\mu\ln\mu-\mu(-\sum_{i=1}^{2n}\ln(\chi_i)+n+n\ln\mu),\\ &= h(\chi_\mu)-n\mu, \end{aligned}$$

where the second equality follows from the fact that

$$\nabla_\chi\phi(\chi_\mu,\mu)=0,$$

because the point (χ_μ,μ) is optimal. Since $\chi\in\mathcal{D}_{int}$ was arbitrary, we then have

$$h(\chi_\mu)-n\mu\leq\min_{\chi\in\mathcal{F}_s^o}h(\chi)\leq h(\chi_\mu).$$

Letting μ approach zero and $\chi^*=\lim_{\mu\to 0}\chi_\mu$, we get

$$h(\chi^*)=h(\lim_{\mu\to 0}\chi_\mu)=\lim_{\mu\to 0}h(\chi_\mu)=\min_{\chi\in\mathcal{D}_{int}}h(\chi).$$

The result is established. $\square$

3.4 Solving The Perturbed Problems (S_μ)

The focus of this section is on the numerical solution of perturbed problem (S_μ). Recall that the perturbed problem is

$$l(\mu)=\min_\chi\phi_\mu(\chi)=\min_\chi\left[g(\chi)+\sum_{i=1}^{2n}\vartheta(\mu,\chi_i),\ \chi\in\mathcal{D}_{int}\right], \qquad (S_\mu)$$

3.4.1 The descent direction

At $\chi \in \mathcal{D}$, the Newton descent direction d is given by solving the following quadratic convex optimization problem:

$$\min_{d \in \mathbb{R}^{2n+m}} \left[\langle \nabla \phi_\mu(\chi), d \rangle + \frac{1}{2} \langle \nabla^2 \phi_\mu(\chi) d, d \rangle, \quad \mathcal{A} d = 0 \right].$$

It suffices to solve the linear system

$$\begin{pmatrix} \nabla^2_\chi \phi_\mu & \mathcal{A}^T \\ \mathcal{A} & 0 \end{pmatrix} \begin{pmatrix} d \\ \lambda \end{pmatrix} = \begin{pmatrix} -\nabla_\chi \phi_\mu \\ 0 \end{pmatrix}, \tag{L}$$

with

$$\nabla^2_\chi \phi_\mu = \begin{pmatrix} \mu X^{-2} & I & 0 \\ I & \mu Z^{-2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \nabla_\chi \phi_\mu = \begin{pmatrix} z - \mu X^{-1} e \\ x - \mu Z^{-1} e \\ 0 \end{pmatrix},$$

where $X = Diag(x)$, $Z = Diag(z)$.
It easy to show that the system (L) is non singular, the most desirable methods for solving (L) are the Cholesky methods. The descent direction being obtained, the next task consists in a line search.

3.4.2 A majorant function

The function of one real variable to be minimized is

$$\psi(\alpha) = \frac{1}{\mu}[\phi_\mu(\chi + \alpha d) - \phi_\mu(\chi)] = \frac{1}{\mu}[h(\chi + \alpha d) - h(\chi) - \mu \sum_{i=1}^{2n} \ln(1 + \alpha \breve{y}_i)],$$

where

$$\breve{y} = \begin{pmatrix} X^{-1} d_x \\ Z^{-1} dz \end{pmatrix}.$$

This function is convex, we have

$$\psi'(\alpha) = \frac{1}{\mu} \langle d, \nabla h(\chi + \alpha d) \rangle - \sum_{i=1}^{2n} \frac{\breve{y}_i}{1 + \alpha \breve{y}_i},$$

$$\psi''(\alpha) = \frac{1}{\mu}\langle d, \nabla^2 h(\chi + \alpha d)d\rangle + \sum_{i=1}^{2n} \frac{\breve{y}_i^2}{(1+\alpha\breve{y}_i)^2} \geq 0,$$

Our problem consists to find some α giving a significant decrease of the convex function ψ. Crouzeix and Merikhi [16] propose a new technique called **a majorant function** this technique is much less expensive than line search methods. The idea is approach the function $\psi(\alpha)$ by a simple majorant function producing the optimal step size α_k at each iteration k. It is based on the following result.

Theorem 3.4.1 ([16])**.** *Assume that $v_i > 0$ for $i = 1, \ldots, N,$ Then*

$$\sum_{i=1}^{N} \ln v_i \geq A,$$

with

$$A = (N-1)\ln(\bar{v} + \frac{\sigma_v}{\sqrt{N-1}}) + \ln(\bar{v} - \sigma_v\sqrt{N-1}).$$

where

$$\bar{v} = \frac{1}{N}\sum_{i=1}^{N} v_i \quad \text{and} \quad \sigma_v = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(v_i - \bar{v})^2}.$$

Set

$$\bar{y} = \frac{1}{2n}\sum_{i=1}^{2n} \breve{y}_i, \quad \sigma_y = \sqrt{\frac{1}{2n}\sum_{i=1}^{2n}(\breve{y}_i - \bar{y})^2}, v_i = 1 + \alpha\breve{y}_i$$

and

$$\beta_1 = \bar{y} + \frac{\sigma_y}{\sqrt{2n-1}}, \quad \beta_2 = \bar{y} - \sigma_y\sqrt{2n-1}.$$

Then, $\bar{v} = 1 + \alpha\bar{y}$ and $\sigma_v = \alpha\sigma_y$.

For all $\alpha > 0$ we may define a majorant function ψ_1 by

$$\psi(\alpha) \leq \psi_1(\alpha) = \frac{h(\chi + \alpha d) - h(\chi)}{\mu} - (2n-1)\ln(1+\alpha\beta_1) - \ln(1+\alpha\beta_2),$$

$$\psi_1'(\alpha) = \frac{\langle d, \nabla h(\chi + \alpha d)\rangle}{\mu} - (2n-1)\frac{\beta_1}{1+\alpha\beta_1} - \frac{\beta_2}{1+\alpha\beta_2},$$

$$\psi_1''(\alpha) = \frac{\langle d, \nabla^2 h(\chi + \alpha d)d\rangle}{\mu} + (2n-1)\frac{\beta_1^2}{(1+\alpha\beta_1)^2} + \frac{\beta_2^2}{(1+\alpha\beta_2)^2}.$$

The function ψ_1 is strictly convex function and therefore reaches its minimum at one unique point $\bar{\alpha}$. This point $\bar{\alpha}$ is the unique root of the equation $\psi_1'(\alpha) = 0$ which belongs to the domain of $\psi_1(\alpha)$. Since ψ is bounded from above by ψ_1 one has

$$\psi(\bar{\alpha}) \leq \psi_1(\bar{\alpha}) < 0.$$

Thus, with $\bar{\alpha}$, we obtain a significant decrease of the function ψ. Now we are interested to solving the equation $\psi_1'(\alpha) = 0$, we can write ψ_1' as

$$\psi_1'(\alpha) = \frac{\dot{a}\alpha^3 + \dot{b}\alpha^2 + \dot{c}\alpha + \dot{d}}{\mu(1+\alpha\beta_1)(1+\alpha\beta_2)} = 0.$$

This implies that

$$\dot{a}\alpha^3 + \dot{b}\alpha^2 + \dot{c}\alpha + \dot{d} = 0, \tag{3.1}$$

where

$$\begin{cases} \dot{a} = 2\beta_1\beta_2 d_x^T d_z, \\ \dot{b} = 2(\beta_1+\beta_2)d_x^T d_z + \beta_1\beta_2(d_x^T z + x^T d_z), \\ \dot{c} = 2d_x^T d_z + (\beta_1+\beta_2)(d_x^T z + x^T d_z) - 2n\mu\beta_1\beta_2, \\ \dot{d} = -\mu[(2n-1)\beta_1 + \beta_2)] + d_x^T d_z. \end{cases}$$

We may solve the equation (3.1) by Cardon method and we get .

$$\begin{cases} \alpha_1 = -\frac{w+uv^{-\frac{1}{3}}}{1+v^{-\frac{1}{3}}}, \\ \alpha_2 = \frac{2\dot{a}^3 - 9\dot{a}\dot{b} + 27\dot{c}}{27}, \\ \alpha_3 = \frac{3q}{2p} + \sqrt{(\frac{3q}{2p})^2 + \frac{p}{3}}, \end{cases}$$

where

$$\begin{cases} p = \dot{b} - \frac{\dot{a}^3}{3}, \\ q = \frac{2\dot{a}^3 - 9\dot{a}\dot{b} + 27\dot{c}}{27}, \\ u = \frac{3q}{2p} + \sqrt{(\frac{3q}{2p})^2 + \frac{p}{3}}, \\ w = -\frac{p}{3u}, \\ v = -\frac{p}{w}. \end{cases}$$

We choose the root which belongs to the domain of function ψ_1.

Algorithm

Initialization
$(x^{(0)}, z^{(0)}, y^{(0)}) \in \mathcal{D}_{int}$, $\varepsilon > 0$, $\mu^{(0)} > 0$, $\bar{\mu} > 0$ and $\kappa \in [0, 1]$.
Set $x = x^{(0)}$, $z = z^{(0)}$, $\mu = \mu^{(0)}$.
1. Solve the Linear System (L) to obtain (dx,dz,dy).
2. Compute the step size α.
3. Update $x = x + \alpha d_x$, $z = z + \alpha d_z$, $y = y + \alpha d_y$.
If $x^T z > \varepsilon$
Set $\mu = \kappa\mu$, and go to 1.
If not
a.**If** $\mu > \bar{\mu}$ (we have a good approximation of $h(\mu_k)$.)
Set $\mu = \kappa\mu$, and go to 1.
b. **If** $\mu < \bar{\mu}$ Stop we have obtained a good approximation of optimal solution.

Figure 3.1: **Algorithm 3.1**

3.5 Numerical Experiments

In this section, we test the Algorithm (3.1) on some quadratic problems of different size, we compare the performance of our approach with a Primal-dual interior-point approach proposed by Achache and Goutali [6] , the implementation is done by Scilab 5.4.1.
We use the following notation in the following.
Method 2.1: the approach proposed in this chapter,
Method 2.2: the approach proposed in [6],
N.I: the number of iterations,
Time: the time measured in seconds,
$\bar{p}$: the approximation of the optimal value of primal problem obtained by The Algorithm (3.1),
$\bar{d}$: the approximation of the optimal value of the dual problem obtained by The Algorithm (3.1).
The following numerical results obtained with the barrier parameter $\mu_0 = \frac{x_0^T z_0}{n}$, $\kappa = 0.5$ and $\varepsilon = 10^{-6}$.

Problem 1.

We consider the following convex quadratic problem:

$$\bar{p} = \min_x f(x) = \frac{1}{2}\langle x, Qx\rangle + \langle c, x\rangle, \; Ax = b, \; x \geq 0, \; .$$

$$Q = \begin{pmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 5 & 1 & 1 \end{pmatrix},$$

$$c = (1, 2, 0, 0)^T, \; b = (2, \;\; 5).^T$$

The initial point is

$$x^0 = \begin{pmatrix} 0.6, & 0.3, & 1.1, & 1.8 \end{pmatrix},^T$$

$$z^0 = \begin{pmatrix} 3.8, & 5, & 1, & 0.5 \end{pmatrix},^T$$

$$y^0 = \begin{pmatrix} -0.5, & -0.5 \end{pmatrix}.^T$$

The numerical results of this example are configured in the following table:

$x^* = (3.5 \times 10^{-6}, 1.7 \times 10^{-6}, 2, 3)^T$
$z^* = (1, \; 2, \; 1.7 \times 10^{-6}, 1.2 \times 10^{-6})^T$
$y^* = (-5 \times 10^{-7}, -1 \times 10^{-6})^T$
$\bar{p} = 7 \times 10^{-6}$
$\bar{d} = -7 \times 10^{-6}$
$x^T z = 1 \times 10^{-7}$
$N.I = 17$
$Time = 0.00234$

Table 3.1: Problem 1.

Problem 2.

$$Q = \begin{pmatrix} 20 & 1.2 & 0.5 & 0.5 & -1 \\ 1.2 & 32 & 1 & 1 & 1 \\ 0.5 & 1 & 14 & 1 & 1 \\ 0.5 & 1 & 1 & 15 & 1 \\ -1 & 1 & 1 & 1 & 16 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 1.2 & 1 & 1.8 & 0 \\ 3 & -1 & 1.5 & -2 & 1 \\ -1 & 2 & -3 & 4 & 2 \end{pmatrix},$$

$$c = (1, -1.5, 2, 1.5, 3)^T, \; b = (9.31, \; 5.45, \; 7.06) .^T$$

The initial point is

$$x^0 = \begin{pmatrix} 2.420, & 1.000, & 1.550, & 2.300, & 1.465 \end{pmatrix},^T$$

$$z^0 = \begin{pmatrix} 3.060, & 15.719, & 8.175, & 7.225, & 7.870 \end{pmatrix},^T$$

$$y^0 = \begin{pmatrix} 20.000, & 11.000, & 5.000 \end{pmatrix}.^T$$

The numerical results of this example are configured in the following table:

$x^* = (2.6604159 \;\; 0.7034860 \;\; 1.3244184 \;\; 2.4894347 \;\; 1.1644801)^T$
$z^* = (1 \times 10^{-6} \;\; 3.7 \times 10^{-6} \;\; 2 \times 10^{-6} \;\; 1 \times 10^{-6} \;\; 2.2 \times 10^{-6})^T$
$y^* = (25.001084 \;\; 12.153761 \;\; 5.6674206)^T$
$\bar{p} = 175.24586,$
$\bar{d} = 175.245847$
$x^T z = 0.000013$
$N.I = 24$
$Time = 0.004368$

Table 3.2: Problem 2.

Problem 3.

$$Q = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} \frac{5}{12} & -1 & 1 & 0 & 0 & 0 \\ \frac{5}{2} & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$c = (-30, -30, 0, 0, 0, 0)\,,^T \;\; b = \left(\frac{35}{12},\; \frac{35}{2},\; 5,\; 5\right).^T$$

The initial point is

$$x^0 = \left(4.7399, \;\; 4.2328, \;\; 5.1745, \;\; 1.4173, \;\; 9.3799, \;\; 0.7672 \right),^T$$

$$z^0 = \left(2.2411, \;\; 2.5161, \;\; 2.0281, \;\; 7.4499, \;\; 0.9415, \;\; 13.8886 \right),^T$$

$$y^0 = \left(-2.02809, \;\; -7.4499, \;\; -0.9415, \;\; -13.8886 \right).^T$$

The numerical results of this example are configured in the following table:

$x^* = (4.99994,\;\; 5,\;\; 5.83335,\;\; 9 \times 10^{-7},\;\; 9.63994,\;\; 6 \times 10^{-7})^T$
$z^* = (1.1 \times 10^{-6},\;\; 1.1 \times 10^{-6},\;\; 9 \times 10^{-6},\;\; 5.9999647,\;\; 6 \times 10^{-7}\;\; 8.999897)^T$
$y^* = (9 \times 10^{-6},\;\; -5.999965,\;\; -6 \times 10^{-6}\;\; -8.999897)^T$
$\bar{p} = -224.99909$
$\bar{d} = -224.99796$
$x^T z = 0.0000402$
$N.I = 24$
$Time = 0.005668$

Table 3.3: Problem 3.

Problem 4.

$$Q=\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ A=\begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$c=(-1,-3,1,-1,0,0,0,0,0,0,0)\,,^T\ \ b=(5,\ 4,\ 5,\ 1.5,\ 0,\ 0,\ 0,\ 0\ )\,.^T$$

The initial point is

$$x^0=\begin{pmatrix} 0.130437 & 1.421849 & 0.709742 & 0.00001 & 1.316122 & 0.055277 \\ 5.760817 & 0.130437 & 1.421849 & 0.709742 & 0.00001 & \end{pmatrix},^T$$

$$z^0=\begin{pmatrix} 0.113781 & 0.080468 & 3.031929 & 0.733798 & 0.222491 & 0.716914 \\ 0.016924 & 0.098479 & 0.072834 & 0.394635 & 0.084649 & \end{pmatrix},^T$$

$$y^0=\begin{pmatrix} -0.222487 & -0.71691 & -0.016919 & -0.098474 \\ -0.07283 & -0.394631 & -0.084645 & \end{pmatrix}.^T$$

The numerical results of this example are configured in the following table: **Problem 5.**

x^*	$\begin{pmatrix} 0.2069005 & 1.6896584 & 0.0000002 & 0.0000191 & 1.4137724 & 0.0000003 \\ 3.1896593 & 0.2069005 & 1.6896584 & 0.0000002 & 0.0000009 \end{pmatrix}^T$
z^*	$\begin{pmatrix} 0.0000010 & 0.0000001 & 0.9516398 & 0.0110610 & 0.0000002 & 0.7586191 \\ 6.795 \times 10^{-8} & 0.0000010 & 0.0000001 & 0.9423597 & 0.2524304 \end{pmatrix}^T$
y^*	$\begin{pmatrix} 0.0000038 & -0.7586151 & 0.0000049 & 0.0000040 & 0.0000039 \\ -0.9423557 & -0.2524264 \end{pmatrix}^T$
$\bar{p}$	-4.1552052
$\bar{d}$	-4.155123
$x^T z$	0.0000024
$N.I$	21
$Time$	0.00858

Table 3.4: Problem 4.

$$Q = \begin{pmatrix}
2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4
\end{pmatrix},$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0. & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$c = (-1,-1,-1,-1,-1,-1,-1,-1,0,0,0,0,0,0,0,0)^T, \; b = (2,2,2,2,2,2,2,2)\,.^T$$

The initial point

$$x^0 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},^T$$

$$z^0 = \begin{pmatrix} 5 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 10 & 10 & 10 & 10 & 10 & 8 & 10 & 8 \end{pmatrix},^T$$

$$y^0 = \begin{pmatrix} -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 \end{pmatrix},^T$$

The numerical results of this example are configured in the following table:

$x^* = \begin{pmatrix} 2 & 0.1250718 & 1.9998565 & 0.1250718 & 1.9999999 & 1.848\times10^{-8} \\ 1.75 & 0.7500000 & 1.884\times10^{-8} & 1.8749282 & 0.0001435 & 1.8749282 \\ 7.403\times10^{-8} & 2 & 0.2500000 & 1.25 & & \end{pmatrix}^T$
$z^* = \begin{pmatrix} 1.859\times10^{-8} & 3\times10^{-7} & 1.865\times10^{-8} & 3\times10^{-7} & 1.863\times10^{-8} \\ 1.9999997 & 2.114\times10^{-8} & 4.951\times10^{-8} & 1.999713 & 1.971\times10^{-8} \\ 0.0005735 & 3.665\times10^{-8} & 0.4997133 & 1.852\times10^{-8} & 1\times10^{-7} \\ 2.974\times10^{-8} & & & & \end{pmatrix}^T$
$y^* = \begin{pmatrix} 3.2505845 & 7.5000002 & 7.4988305 & 7.5000002 & 7.2505848 & 4.4999998 \\ 7.5000001 & 5.5 & & & & \end{pmatrix}^T$
$\bar{p} = 46.125$
$\bar{d} = 46.124999$
$x^T z = 0.000001$
$N.I = 28$
$Time = 0.008736$

Table 3.5: Problem 5.

Example with variable size. We consider the following quadratic problem with $n = 2m$:

$$\bar{p} = \min[f(x) : x \geq 0,\ Ax = b],$$

where $f(x) = x^T Qx + c^T x, b_i = 2,\ c(i) = 1$, for $i = 1, \ldots, m$ and $c[m+1 : n] = 0$, with

$$Q[i,j] = \begin{cases} 2 & \text{if } i = j = 1 \text{ or } \ i = j = n-2, \\ 4 & \text{if } i = j \text{ and } \ i \neq \{1, n-2\}, \\ 2 & \text{if } i = j-1 \text{ or } \ i = j+1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$A[i,j] = \begin{cases} 1 & \text{if } i = j \text{ or } \ j = i+m), \\ 0 & \ \text{if not,} \end{cases}$$

The starting point $\chi^0 = (x^0, y^0, z^0)$ given by

$$\begin{aligned} & x^0(i) = 1 \quad \text{for } i = 1, \ldots, n, \\ & y^0(i) = -2 \quad \text{for } i = 1, \ldots, m, \\ & z^0(1) = 5,\ z^0[2 : m] = 9,\ z^0[m+1 : n-3] = 10, \text{ and } z^0[n-2 : n] = (8, 10, 8). \end{aligned}$$

In the following tables we compare the primal-dual logarithmic barrier algorithm with the primal-dual interior-point algorithm proposed by Achache and Goutalli in [6].

$\boxed{\text{n=50}}$

	Method 2.1	Method 2.2
$N.I$	29	67
$\bar{p}$	164.92857	169.475
$\bar{d}$	164.92857	169.475
Time	0.079	0.3806424

Table 3.6: Numerical results with $n = 50$

n=100

	Method 2.1	Method 2.2
$N.I$	30	103
$\bar{p}$	338.8125	342.3533
$\bar{d}$	338.8125	342.3533
Time	0.255204	0. 6162039

Table 3.7: Numerical results with $n = 100$

n=150

	Method 2.1	Method 2.2
$N.I$	31	132
$\bar{p}$	513.13028	515.51894
$\bar{d}$	513.13028	515.51894
Time	1.0728088	1.428969

Table 3.8: Numerical results with $n = 150$

n=200

	Method 2.1	Method 2.2
$N.I$	31	157
$\bar{p}$	687.25	690.65836
$\bar{d}$	687.25	690.65836
Time	2.80489	3.371181

Table 3.9: Numerical results with $n = 200$

n=250

	Method 2.1	Method 2.2
$N.I$	32	180
$\bar{p}$	861.52686	862.66061
$\bar{d}$	861.52686	862.66061
Time	5.113712	7.313326

Table 3.10: Numerical results with $n = 250$

n=300

	Method 2.1	Method 2.2
$N.I$	32	201
$\bar{p}$	1035.6875	1037.5621
$\bar{d}$	1035.6875	1037.5621
Time	8.332013	14.1102

Table 3.11: Numerical results with $n = 300$

Comment

The tables (3.6, 3.7, 3.8, 3.9, 3.10) and 3.11 are showed the performance of our algorithm 3.1, the tables show that the primal-dual logarithmic barrier approach with the majorant function produced the lowest iterations, time and the optimal value compared to the proposed algorithm in [6].

Conclusion

This thesis is concerned with the analysis, implementation of logarithmic barrier interior-point method. In particular, we focus on two type of problems: the linearly constrained convex problems (LCCO) and the quadratic primal-dual problem.
In the first part of study, we have introduced a relaxation of the classical path of the perturbed (LCCO) problem and we have presented a new technique for determining the step size called the tangent technique. The numerical efficiency of this algorithm is confirmed by numerical tests.
In the second part, we presented a theoretical and numerical study of a logarithmic barrier interior-point method of the primal-dual type, to solve a quadratic program with linear constraints and we have used a majorant function to compute the step size. The efficiency of the proposed approach in terms of the quality of the obtained solutions and the speed of convergence.

Annex

The Algorithm proposed by Achache and Goutalli to solve the LCCO problem in the following.

Algorithm

Input

An accuracy parameter $\epsilon > 0$;

a threshold parameter $0 < \beta < 1$;

a fixed barrier update parameter $0 < \theta < 1$;

a feasible point (x^0, y^0, z^0) and μ^0 such that $\delta(x^0 z^0; \mu^0 \leq \beta)$;

begin

$x = x^0$; $y = y^0$; $z = z^0$; $\mu = \mu^0$;

While $n\mu \geq 0$ **do**

begin

- Solve the system (3.2) to obtain: $(\Delta x, \Delta y, \Delta z)$;
- Update $x = x + \Delta x$, $y = y + \Delta y$, $z = z + \Delta z$;
- $\mu = (1 - \theta)\mu$;

end

end

.

Figure 3.2: **Algorithm 3.2**

The system (3.2) given by

$$\begin{pmatrix} A & 0 & 0 \\ -\nabla^2 f(x) & A^T & I \\ Z & 0 & X \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \mu e - Xz \end{pmatrix}, \tag{3.2}$$

we give also a norm-based proximity measure $\delta(xz; \mu)$ to the central-path as follows

$$\delta(xz; \mu) = \frac{1}{2} \left\| \sqrt{\left(\frac{xz}{\mu}\right)^{-1}} - \sqrt{\frac{xz}{\mu}} \right\|.$$

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