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Asymptotic analysis for some boundary value problems in thin domains with friction laws

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Contents

Introduction

Contact problems with friction typically appear in everyday life and play a crucial role in a wide range of applications, engineering systems and in the flow of fluids, such as brakes, machine tools, motors, turbines, or wheel-rail systems. As a result of this interest, over the last decades, there has been a lot of work on the mathematical theory of friction contact. The well-known and predominant friction laws used in the mathematical literature are Tresca's and Coulomb's laws.
In 1972, Duvaut and Lions [21] investigated the dynamic and static contact problems with Tresca's friction law involving linearly elastic and fluid rigid bodies.

In the literature, it can be noted that there are a many research done on frictional contact problems involving elastic and piezoelectric materials, see for instance [20, 17, 22, 29] and the references cited therein. Demkowicz et *al.* [17] established the some existence and uniqueness result in contact problem with non linear friction. In addition, El-H-Essofi et *al.* [22, 42] studied static frictional contact models between a piezoelectric body and conductive foundation using Coulomb's model for the contact. For the dynamic evolution with frictional contact of an elastic body, Ionescu et *al.* [29] proved the existence of solution in the two-dimensional case and the uniqueness for one-dimensional shearing problem. Further, in [32, 43] the authors studied a several classes of variational inequalities and showed the existence and uniqueness of the solution. Besides, a frictionless contact between an elastic body and a rigid foundation when the contact is described by the normal compliance with adhesion and that it is bilateral was regarded by Hemici and Matei [28].

In the study of fluid flow, the first mathematical results noticed are with the boundary conditions of the friction type which were proposed by Fujita et *al.* [23, 24] for stationary Stokes flows. A detailed study of the stationary solution for Brigham flow with non-local friction is given in [15]. Recently, Khan et *al.* [30] considered the fractional Brinkman type fluid in the channel under the effect of MHD with Caputo-Fabrizio fractional derivative.

Asymptotic methods have been used to reduce the problems of three-dimensional friction contact to the one-dimensional or two-dimensional models on thin domains. The importance of the latter which are obtained resides in the fact that they can be used instead of full three-dimensional models, the thickness to be small. It is obvious that, the two-dimensional models are simpler than their three-dimensional counterparts and this makes it easier to study. In applied sciences, the problems posed in a thin domain are widely used in several fields such as mechanics, chemistry, aeronautics, and even civil engineering.

In Solid mechanic, the elasticity models for beams, rods and plates have been obtained from a priori hypotheses on the displacement, which upon substitution in the equilibrium and constitutive equations three-dimensional elasticity lead to useful simplifications. Nevertheless, both from a constitutive and geometric point of view, the validity of most of the models thus obtained must be mathematically justified.

In this regard, a considerable effort has been made over the past decades by many authors to derive new models and justify existing ones using the asymptotic expansion method, the foundations of which can be found in [35]. Previous work by Ciarlet and Destuynder [14] showed the equilibrium states of a thin plate $\Omega \times (-\varepsilon, \varepsilon)$ under an external force where Ω is a smooth domain in $\mathbb{R}^2$ and ε is a small parameter, to justify the two-dimensional model of plate bending. A similar method is used recently for problems of general elastic shells in unilateral contract without friction with an obstacle (see for instance [31, 38]). In addition, the piezoelectric beam models were justified in [46] using the asymptotic expansion method followed by rigorous convergence results.

Asymptotic methods represent a powerful tool for studying and modeling thin elastic bodies. In recent decades, many related articles have been written for similar problems with the friction boundary. For instance, the work of Bayada and Lhalouani [4] where they investigated the asymptotic analysis for a unilateral contact problem with Coulomb friction law between an elastic body and a thin elastic soft layer. Likewise, the authors in [5, 33] considered the theoretical analysis of a friction contact between two elastic bodies in a stationary regime in three-dimensional thin domain with Tresca friction law. Additionally, the asymptotic analysis of dynamical problems of isothermal and non-isothermal with nonlinear friction law are proved in [6, 40]. Ultimately, Dilmi and Benseridi [18] showed the asymptotic behavior of an electro-viscoelastic quasi-static problem in a thin domain with friction modeled by Tresca's law.

On the other hand, several works are interested in the study of the asymptotic analysis of incompressible Newtonian and non-Newtonian fluids in stationary and dynamic regime in three-dimensional thin domains. For this, the authors of [4, 9] studied the asymptotic behavior of a Stokes flow with boundary friction conditions and obtained the generalized Reynolds equation for the limit problem. The asymptotic convergence of the Stokes flow with the Fourier and Tresca boundary conditions when one dimension of the fluid domain tends to zero has been studied in [10]. Recently, some authors have proved the asymptotic analysis of the isothermal thin-film Bingham fluid with Fourier and Tresca boundary conditions see [7, 8, 19].

The objective of this thesis is the study of the asymptotic behavior of some boundary value problems for a fluid, piezoelectric and elastic materials in the stationary and dynamic regime occupying a bounded three-dimensional thin domain Ω^ε, when the thickness ε tends to zero. The boundary Γ^ε of this thin domain consists of three parts: the bottom, the later part, and the top surface.

It is then considered that at the bottom, the normal displacement or the normal velocity is equal to zero. However, the displacement or the tangential velocity is unknown and satisfies the boundary conditions of type Tresca or Coulomb. On the later and top parts, we have Dirichlet boundary conditions. The weak form of problems is derived from a variational inequality [21].

The main parts of this thesis can be summarized as follows.

Chapter 1. In the first chapter, we introduce some necessary notations and recall some fundamental definitions and theorems on functional analysis, including, Sobolev's embedding theorem, the basic semi-continuous lower convex definitions, differentiability functions, and Gronwall's lemma, which will be needed to prove our main results. The basic tools presented in this chapter are standard and can be found in many functional analysis books. For further information in the field we suggest to the reader to books [1, 12, 16, 26, 39, 43, 47].

Chapter 2. In this chapter, we are interested in the asymptotic analysis of an incompressible fluid in stationary regime in the thin domain $\Omega^\varepsilon \subset \mathbb{R}^3$ governed by the Brinkman equation ([2, 13]) with the Tresca nonlinear friction condition type.

The complete problem we are studying can be formulated as follows.

$$\begin{cases} -\mu\Delta u^\varepsilon + \nabla p^\varepsilon + \mu(\alpha^\varepsilon)^2 = f^\varepsilon \quad \text{in } \Omega^\varepsilon, \\ div(u^\varepsilon) = 0 \quad \text{in } \Omega^\varepsilon, \\ u^\varepsilon = 0 \quad \text{on } \Gamma_u^\varepsilon, \\ u^\varepsilon = 0 \quad \text{on } \Gamma_l^\varepsilon, \\ u^\varepsilon.\eta = 0 \quad \text{on } \Gamma_b, \\ \left. \begin{array}{l} |\sigma_\tau^\varepsilon| < k^\varepsilon \Longrightarrow u_\tau^\varepsilon = s, \\ |\sigma_\tau^\varepsilon| = k^\varepsilon \Longrightarrow \exists \lambda \geq 0, \ \ u_\tau^\varepsilon = s - \lambda\sigma_\tau^\varepsilon \end{array} \right\} \text{on } \Gamma_b, \end{cases}$$

where f^ε is the external force.

First, we give the related weak formulation of the problem. Then, we discuss the existence and uniqueness theorem of the weak solution. Next, we study the asymptotic analysis according to the change of the variables $x_3 = \frac{z}{\varepsilon}$ to transform the initial problem posed in the domain Ω^ε which depends on a small parameter ε into a new problem posed on a fixed domain Ω which is independent of ε. Then, we find some estimates on the velocity and pressure which are independent of the parameter ε. We obtain further the main results concerning the existence of a weak limit (u^*, p^*) of $(u^\varepsilon, p^\varepsilon)$ such that (u^*, p^*) satisfies the weak form of the Reynolds equation

$$\int_{\Gamma_b} \left[\frac{h^3}{12\mu}\nabla p^* + h\widetilde{u_i}^*(x) - \frac{h}{2}s^*(x) + \widehat{\alpha}\left(\frac{h}{2}U^*(x,h) - \int_0^h U^*(x,t)dt\right) \right.$$
$$\left. + \widetilde{F}(x) \right] \nabla\varphi(x)dx = 0, \ \ \forall\varphi \in H^1(\Omega),$$

where

$$\widetilde{F}(x) = \frac{1}{\mu}\int_0^{h(x)} F(x,t)dt - \frac{h}{2\mu}F(x,h),$$
$$U^*(x,t) = \int_0^t \int_0^\zeta u_i^*(y,\theta)d\theta d\zeta, \quad F(y,t) = \int_0^t \int_0^\zeta \widehat{f_i}(x,\theta)d\theta d\zeta,$$

and, we give precise characterization of the limit form of Tresca boundary conditions

$$\left. \begin{array}{l} \mu\,|\tau^*| < \widehat{k} \Longrightarrow s^* = s \\ \mu\,|\tau^*| = \widehat{k} \Longrightarrow \exists\lambda \geq 0, s^* = s + \lambda\tau^* \end{array} \right\} \text{ a.e. on } \Gamma_b,$$

where

$$s^*(x) = u^*(x,0) \quad \textit{and} \quad \tau^*(x) = \frac{\partial u^*}{\partial z}(x,0).$$

In thin ends, the uniqueness of the limit velocity and pressure distributions are established. This result has been published in the paper [37].

Chapter 3. We consider a piezoelectric body with Coulomb free boundary friction conditions in the stationary regime occupying a bounded domain of $\mathbb{R}^3$. The electro-elastic constitutive law is given by

$$\begin{cases} \sigma^\varepsilon(u^\varepsilon,\varphi^\varepsilon) = 2\mu^\varepsilon d(u^\varepsilon) + \lambda^\varepsilon tr(d(u^\varepsilon))I_3 + (\mathfrak{F}^\varepsilon)^T \nabla\varphi^\varepsilon & \text{in} \quad \Omega^\varepsilon, \\ D^\varepsilon(u^\varepsilon,\varphi^\varepsilon) = -\beta^\varepsilon \nabla\varphi^\varepsilon + \mathfrak{F}^\varepsilon d(u^\varepsilon) & \text{in} \quad \Omega^\varepsilon, \end{cases}$$

where μ^ε, λ^ε are the coefficients of Lamé, I_3 is the identity, $\mathfrak{F}^\varepsilon = (e^\varepsilon_{ijk})$ is the third order piezoelectric tensor $(\mathfrak{F}^\varepsilon)^T$ is the transpose of the tensor $\mathfrak{F}^\varepsilon$, β^ε denotes the electric coefficient of permittivity and $d(u^\varepsilon) = (d_{ij}(u^\varepsilon))$ is the linearized strain tensor, $d_{ij}(u^\varepsilon) = \frac{1}{2}\left(\frac{\partial u_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i}\right)$.
In the first step, we show the variational formulation of the problem and prove its unique weak solution, the theorem of existence was established in [22]. After that, we study the asymptotic behavior of the displacement and the electric potential according the change of variable $x_3 = \frac{z}{\varepsilon}$, we transform the initial problem posed in the domain Ω^ε into a new problem posed on a fixed domain Ω independent of the parameter ε. Using the scaling change we find some estimates and prove that the limit solution satisfies also a variational inequality. Moreover, these estimates will be useful to obtain a weak form of the limit problem and the limit of Coulomb boundary conditions

$$\int_\omega \left[\int_0^h \int_0^z \widehat{\mu}\frac{\partial u^*}{\partial \xi} d\xi dz + \int_0^h \int_0^z \widehat{e}_{3i3}\frac{\partial \varphi^*}{\partial \xi} d\xi dz\right] \nabla \Psi dx$$
$$+ \int_\omega \left[-\frac{h}{2}\left(\int_0^h \widehat{\mu}\frac{\partial u^*}{\partial \xi} d\xi + \int_0^h \widehat{e}_{3i3}\frac{\partial \varphi^*}{\partial \xi} d\xi\right) + \widetilde{F}\right] \nabla \Psi dx = 0 \quad \forall \Psi \in H^1(\omega),$$

$$\begin{cases} |\widehat{\mu}\tau^* + \widehat{e}_{3i3}\pi^*| < \widehat{k}\left|R(\widehat{e}_{333}\pi^*)\right| \Longrightarrow s^* = s, \\ \\ |\widehat{\mu}\tau^* + \widehat{e}_{3i3}\pi^*| = \widehat{k}\left|R(\widehat{e}_{333}\pi^*)\right| \Longrightarrow \exists \lambda \geq 0 \text{ such that } s^* = s + \lambda\left(\widehat{\mu}\tau^* + \widehat{e}_{3i3}\pi^*\right), \end{cases} \quad \text{a.e. on } \omega$$

where

$$\widetilde{F} = \int_0^h F_i(x,z)dz - \frac{h}{2}F_i(x,h), \quad F_i(x,z) = \int_0^z \int_0^\xi \widehat{f}_i(x,\theta)d\theta d\xi, \quad i = 1,2,$$
$$s^* = u^*(x,0), \quad \tau^* = \frac{\partial u^*}{\partial z}(x,0), \quad \rho^* = \varphi^*(x,0), \quad \pi^* = \frac{\partial \varphi^*}{\partial z}(x,0).$$

Finally, we prove the uniqueness of solution of the limit problem.

Chapter 4 This chapter is devoted to the study of the asymptotic behavior for transmission problem in a dynamic regime in a three dimensional thin domain $\Omega^\varepsilon = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon$ with Tresca friction law. In fact, we take materials with Hooke constructive laws with nonlinear dissipative term. We give the related weak formulation of the problem and explore the theorem of existence and uniqueness of the weak solution. By the same change of scale as in the two previous chapters we carry out the asymptotic analysis. Then, we established some estimates independent of the small parameter ε by using the Korn and Poincaré inequality also we prove convergence theorem. Furthermore, we obtain the following limit problem with a specific weak form of the generalized equation

$$\begin{cases} \mu_l|\pi_l^\star| < \hat{\kappa} \Rightarrow \dfrac{\partial s_1^\star}{\partial t} - \dfrac{\partial s_2^\star}{\partial t} = s, \\ \\ \mu_l|\pi_l^\star| = \hat{\kappa} \Rightarrow \exists\beta > 0 \; such \; that \; \dfrac{\partial s_1^\star}{\partial t} - \dfrac{\partial s_2^\star}{\partial t} = s + \beta\mu_l\pi_l^\star, \end{cases} \quad a.e. \text{ on } \omega \times]0, T[$$

$$\int_\omega \left(\tilde{F} + \tilde{G} + \mu_1 \int_0^h \mathbf{u}_1^\star(x', y, t)dy + \mu_2 \int_{-h}^0 \mathbf{u}_2^\star(x', y, t)dy \right) \nabla\psi(x')dx' = 0, \; \forall \psi \in H^1(\omega),$$

where

$$s_l^\star(x,t) = \mathbf{u}_l^\star(x',0,t) \text{ and } \pi_l^\star(x,t) = \frac{\partial \mathbf{u}_l^\star}{\partial z}(x',0,t) \;,\; l = 1,2,$$
$$\tilde{F} = \int_0^h F(x', y, t)dy - hF(x', y, t), \; \tilde{G} = -\int_0^h G(x', y, t)dy + hG(x', y, t).$$

The analysis of this problem has been recently published in [36].

Chapter 1

Mathematical tools

The objective of this chapter is to recall the knowledge of functional analysis which will be used in subsequent chapters in this thesis. The results are stated without proof, since they are standard and can be found in many references. Nevertheless, we pay particular attention to the results which are repeatedly used in the next chapters, including the Hölder inequality, the weak convergence, and among others. Most of them are concerned with different spaces of functions.

For this chapter, we can take designating a bounded Lipschitzian domain of $\mathbb{R}^3$ provided with the Lebesgue measure dx.

1.1 Some functional spaces

Sobolev spaces play an important role in the study of elliptic and hyperbolic partial differential equations. In the theory of hyperbolic evolution equations, it is necessary in general to make the time variable and the distinct roles to the time variable and the space variables.

A Lipschitz domain or domain with Lipschitz boundary, is a domain in Euclidean space whose boundary is "sufficiently regular" in the sense that it can be thought of as locally being the graph of a Lipschitz continuous function. Many of the Sobolev embedding theorems require that the domain of study is a Lipschitz domain. Consequently, many partial differential equations and variational problems are defined on Lipschitz domains.

1.1.1 $L^p(\Omega)$ Spaces

In this subsection, we consider the Lebesgue space $L^p(\Omega)$

Definition 1.1.1 *[12] Let $p \in [1, \infty[$, we call the Lebesgue space $L^p(\Omega)$, the set*

$$L^p(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} \text{ is measurable and } \int_\Omega |u|^p \, dx < \infty \right\}.$$

This is a normed space, with the norm is denoted by $\|.\|_p$ $\left(\text{or } \|.\|_{L^p(\Omega)}\right)$ is defined by

$$\|u\|_p = \left(\int_\Omega |u(x)|^p \, dx \right)^{\frac{1}{p}}.$$

If $p = \infty$ and $u : \Omega \longrightarrow \mathbb{R}$ measurable, then we define the space $L^\infty(\Omega)$ by

$$L^\infty(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} \;\middle|\; \begin{array}{l} u \text{ is measurable and there exist a constant } C \\ \text{such that } |u(x)| \leq C \, a.e. \text{ on } \Omega \end{array} \right\}.$$

This is normed space with norm

$$\|u\|_{L^\infty(\Omega)} = \inf \left\{ C; \;\; |u(x)| \leq C \quad a.e. \text{ on } \Omega \right\}.$$

Notation. Let $1 \leq p \leq \infty$, we denote by q the conjugate exponent of p i.e. $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.1.2 (Fisher-Riesz) *[12] L^p is a Banach space for any p, $1 \leq p \leq \infty$.*

Corollary 1.1.3 *If $p = 2$, $L^2(\Omega)$ equipped with the inner product*

$$\langle u, v \rangle = \int_\Omega uv dx,$$

is a Hilbert space.

Theorem 1.1.4 (Separability) *[12] $L^p(\Omega)$ is separable space for any $p \in [1, \infty[$.*

Theorem 1.1.5 (Reflexivity,) *[12] $L^p(\Omega)$ is reflexive space for any $p \in]1, \infty[$.*

Definition 1.1.6 *[12] We denote by C_c the space of continuous function on Ω with compact support in Ω, i.e.*

$$C_c(\Omega) = \{ f \in C(\Omega); f(x) = 0 \;\; \forall x \in \Omega \setminus K \;\; \text{where } K \subset \Omega \;\; \text{is compact} \}.$$

$C^k(\Omega)$ is the space of function k times continuously differentiable on Ω ($k \geq 1$ is an integer)

$$C^\infty(\Omega) = \cap_k C^k(\Omega),$$

$$C^k_c(\Omega) = C^k(\Omega) \cap C_c(\Omega),$$

$$C^\infty_c(\Omega) = C^\infty(\Omega) \cap C_c(\Omega).$$

In this thesis, we write $D(\Omega)$ instead of $C_c^\infty(\Omega)$.

Theorem 1.1.7 (density) *[12] The space $D(\Omega)$ is dense in $L^p(\Omega)$ for any $1 \leq p < \infty$.*

Theorem 1.1.8 (Dual of $L^p(\Omega)$) *[1] Let p be a real number such that $p \in \,]1, \infty[$. The topological dual of $L^p(\Omega)$ is $(L^p(\Omega))' = L^q(\Omega)$.*

Proposition 1.1.9 *[1] The dual of $L^1(\Omega)$ is $L^\infty(\Omega)$.*

Theorem 1.1.10 (Young's inequality) *[12] Assume that $1 < p, q < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$ we have*

$$ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q \;\; \forall a,\, b \geq 0.$$

Theorem 1.1.11 (Hölder's inequality) *[12] Assume that $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$ $1 \leq p \leq \infty$. Then $uv \in L^1(\Omega)$ and moreover*

$$\|uv\|_{L^1(\Omega)} \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)} \,.$$

In the case $p = 2$ and hence $q = 2$, Hölder inequality is nothing else than Cauchy-Schwarz inequality

$$\|uv\|_{L^1(\Omega)} \leq \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \,.$$

1.1.2 The Sobolev spaces and embedding theorem

Definition 1.1.12 *[12] Let $p \in \mathbb{R}$, with $1 \leq p \leq \infty$, the Sobolev space $W^{1,P}(\Omega)$ is defined to be*

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \;\middle|\; \begin{array}{l} \exists \widehat{g}_1, \widehat{g}_2, \widehat{g}_3 \in L^p(\Omega) \text{ such that} \\ \int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega \widehat{g}_i \varphi \;\; \forall \varphi \in D(\Omega) \;\; i = 1, 2, 3 \end{array} \right\},$$

for $u \in W^{1,p}(\Omega)$ denote

$$\frac{\partial u}{\partial x_i} = \widehat{g}_i \quad \text{and } \nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3} \right).$$

The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{i=1}^{3} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)} \quad \text{If } 1 \leq p \leq \infty,$$

or sometimes, if $1 \leq p < \infty$, *with the equivalent norm*

$$\|u\|_{W^{1,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + \sum_{i=1}^{3} \left\| \frac{\partial u}{\partial x_i} \right\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}.$$

We set

$$H^1(\Omega) = W^{1,2}(\Omega).$$

The space $H^1(\Omega)$ *is equipped with the scalar product*

$$(u, v)_{H^1} = (u, v)_{L^2(\Omega)} + \sum_{i=1}^{3} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)_{L^2(\Omega)},$$

and with the associated norm

$$\|u\|_{H^1(\Omega)} = \left(\|u\|_2^2 + \sum_{i=1}^{3} \left\| \frac{\partial u}{\partial x_i} \right\|_2^2 \right)^{\frac{1}{2}}.$$

The space of Sobolev $H^1(\Omega)$ *is Hilbert space.*

Proposition 1.1.13 *[12] The space* $W^{1,p}(\Omega)$ *is a Banach space for* $1 \leq p \leq \infty$. *It is reflexive for* $1 < p < \infty$ *and separable for* $1 \leq p < \infty$. *The space* $H^1(\Omega)$ *is a separable Hilbert space.*

Definition 1.1.14 *[12] Let* $1 \leq p < \infty$. *The space* $W_0^{1,p}(\Omega)$ *desing the closure of* $C_c^1(\Omega)$ *in* $W^{1,p}(\Omega)$ *(we do not define* $W_0^{1,p}$ *for* $p = \infty$*). Set*

$$H_0^1(\Omega) = W_0^{1,2}(\Omega).$$

The space $W_0^{1,p}(\Omega)$, *is equipped with the norm of* $W^{1,p}(\Omega)$ *is a separable Banach space, it is reflexive for* $(1 \leq p < \infty)$. *The space* $H_0^1(\Omega)$ *is equipped with the scalar product of* $H^1(\Omega)$ *is a separable Hilbert space.*

Definition 1.1.15 *[12] The dual space of* $W_0^{1,p}(\Omega)$ $(1 \leq p < \infty)$ *is denoted by* $W^{-1,q}(\Omega)$, *and the dual space of* $H_0^1(\Omega)$ *is denoted by* $H^{-1}(\Omega)$.

Remark 1.1.16 *The dual of* $L^2(\Omega)$ *is identified with* $L^2(\Omega)$. *We have inclusion*

$$H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega),$$

where these injection are continuous and dense.

Theorem 1.1.17 (Poincaré's inequality) *[12] There exists a constant $C_\Omega > 0$ such that*

$$\|u\|_{L^p(\Omega)} \leq C_\Omega \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W_0^{1,p}(\Omega).$$

where C depending on Ω.

Theorem 1.1.18 (Korn's inequality) *[39] In $L^2(\Omega)$ version, can be stated as follows. There exist $C_K > 0$, such that*

$$\sum_{i,j=1}^{3} \|d_{ij}(v)\|_{0,\Omega}^2 + \|v\|_{0,\Omega}^2 \geq C_K \|v\|_{1,\Omega}^2 \qquad \forall v \in V,$$

where

$$V = \left\{v \in (L^2(\Omega))^3 : d_{ij}(v) \in L^2(\Omega)\right\}, \quad d_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

We now recall some results concerning Sobolev embeddings.

Theorem 1.1.19 (Sobolev embedding Theorem) *[16] Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. We have*

- *If $p \in [1, n[$, then $W^{1,p}(\Omega) \subset L^q(\Omega)$ for every $q \in \left[1, \frac{np}{n-p}\right]$.*
- *If $p = n$ then $W^{1,n}(\Omega) \subset L^q(\Omega)$ for every $q \in [1, \infty)$.*
- *If $p \in]n, \infty[$, then $W^{1,p}(\Omega) \subset L^\infty(\Omega)$.*

all these injection are continuous.

Theorem 1.1.20 (Rellich-Kondrachov Theorem) *[16] Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. We have*

- *If $p \in [1, n[$, then the injection of $W^{1,p}(\Omega)$ in $L^q(\Omega)$ is compact for every $q \in \left[1, \frac{np}{n-p}\right]$.*
- *If $p = n$ then the injection of $W^{1,n}(\Omega)$ in $L^q(\Omega)$ is compact for every $q \in [1, \infty)$.*
- *If $p > n$, then the injection of $W^{1,p}(\Omega)$ in $C(\overline{\Omega})$ is compact.*

In the next paragraph, we will define the Sobolev space consists of the function in $L^p(\Omega)$ whose partial derivation reaches the order of m (with $m \in \mathbb{N}$) in the sense of distribution. For these derivatives, we set $\alpha = (\alpha_1, ..., \alpha_N)$ and $|\alpha| = \sum_{i=1}^{N} \alpha_i$. Moreover, we use the notation

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_i} x_i ... \partial^{\alpha_N} x_N}.$$

Definition 1.1.21 *[43] Let $m \in \mathbb{N}$ and $1 \leq p \leq \infty$ the Sobolev space $W^{m,p}(\Omega)$ defined by*

$$W^{m,p}(\Omega) = \{u \in L^p : \ D^\alpha u \in L^p(\Omega) \ \forall \alpha \ \text{with} \ |\alpha| \leq m\}.$$

The space $W^{m,p}(\Omega)$ endowed with norm

$$\|u\|_{W^{m,p}} = \begin{cases} \left[\sum_{0\leq|\alpha|\leq m} \|D^\alpha u\|^p_{L^p(\Omega)}\right]^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \max_{0\leq|\alpha|\leq m} \|D^\alpha u\|_{L^\infty(\Omega)} & \text{if } \ p = \infty, \end{cases}$$

is a Banach space. For $p \in \,]1, \infty[$ space is reflexive.

Proposition 1.1.22 *[12] The space $H^m(\Omega) = W^{m,2}(\Omega)$ and endowed with the scalar product*

$$(u, v)_{H^m(\Omega)} = \sum_{0\leq|\alpha|\leq m} (D^\alpha u, D^\alpha u)_{L^2(\Omega)}.$$

Supplied with the norm

$$\|u\|_{H^m(\Omega)} = \left(\sum_{0\leq|\alpha|\leq m} \|D^\alpha u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}.$$

Trace operator

Now, we study the trace theory for function belonging to a Sobolev space $H^1(\Omega)$.

Theorem 1.1.23 *[26] Let Ω be a Lipschitz domain of $\mathbb{R}^3$. There exists unique linear continuous operator*

$$\gamma_0 : H^1(\Omega) \longrightarrow L^2(\partial\Omega).$$

The function $\gamma_0 u$, also denoted by $u_{|\partial\Omega}$ defined on a dense subspace $C^1(\Omega)$. $u_{|\partial\Omega}$ is called the trace of u on $\partial\Omega$.
Now, we define

$$H^{\frac{1}{2}}(\partial\Omega) = \{u_{|\partial\Omega} : u \in H^1(\Omega)\},$$

and can be endowed with the norm

$$\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} = \inf\{\|u\|_{H^1(\Omega)} : u \in H^1(\Omega) : u_{|\partial\Omega} = g\}.$$

The most important properties of the trace are the following *(i) If* $u \in H^1(\Omega)$, *the in fact* $u_{|\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega)$ *is surjective from* $H^1(\Omega)$ *and*

$$\left\| u_{|\partial\Omega} \right\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C \left\| u \right\|_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega).$$

(ii) We have Green's formula

$$\int_\Omega \frac{\partial u}{\partial x_i} v dx = -\int_\Omega u \frac{\partial u}{\partial x_i} dx + \int_{\partial\Omega} uv(\vec{\eta}.\vec{e_i}) d\delta, \ \ \forall u, \ \ v \in H^1(\Omega),$$

where $d\delta$ *is the surface measure on* $\partial\Omega$ *and* $\vec{\eta}$ *is the outward unit normal to* $\partial\Omega$. *Note that the surface integral has a meaning since* u, $v \in L^2(\partial\Omega)$.

In particular, if $u \in H^1(\Omega)$, $v \in H^2(\Omega)$, *we have half Green formula*

$$\int_\Omega u \Delta v dx + \int_\Omega \nabla u \nabla v dx = \int_{\partial\Omega} \frac{\partial u}{\partial \eta} v d\delta.$$

1.1.3 Vector valued function spaces

We shall need the spaces of vector-valued functions in studying time-dependent variational problems.
Let X is a Banach space and $[0,T]$ will denote the time interval of interest, for $T > 0$.

Definition 1.1.24 *[43] For* $1 \leq p \leq \infty$. $L^p(0,T;X)$ *is the set of (class of almost every where equal) measurable function* $u :]0,T[\longrightarrow X$ *such that* $\|u\|_X \in L^p(0,T)$, *and*

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{\frac{1}{p}} \quad \text{if } 1 \leq p < \infty,$$

$$\|u\|_{L^\infty(0,T;X)} = ess \sup_{t \in]0,T[} \|u(t)\|_X \quad \text{if } p = \infty.$$

When $(X, (.,.)_X)$ *is a Hilbert space,* $L^2(0,T;X)$ *is also a Hilbert space with the inner product given by*

$$(u,v)_{L^2(0,T;X)} = \int_0^T (u(t), v(t))_X dt.$$

Definition 1.1.25 *[43] The space* $C([0,T];X)$ *comprises all continuous function* $u : [0,T] \longrightarrow X$ *with the norm*

$$\|u\|_{C([0,T];X)} = \max_{0 \leq t \leq T} \|u(t)\|.$$

$C([0,T];X)$ *is Banach space.*

Proposition 1.1.26 *We have*

1. $L^p(0,T;X)$ *is a Banach space for* $1 < p < \infty$.

2. *If* X *is a reflexive space for* $1 \leq p < \infty$, *if* q *satisfy* $\frac{1}{p} + \frac{1}{q} = 1$, *then* $L^p(0,T;X)$ *is a reflexive and its dual is identify algebraically* $L^q(0,T;X')$.

3. *If* X *is separable space for* $1 \leq p < \infty$, *then* $L^p(0,T;X)$ *is separable.*

Theorem 1.1.27 (Aubin) *[41] Let* $X_0,\ X,\ X_1$ *be three Banach spaces and suppose that* X_0 *and* X_1 *are reflexive.* $X_0 \subset X$ *with compact injection and* $X \subset X_1$ *with continuous injection. Let* $p_0, p_1 > 1$. *Consider the space*

$$\mathcal{B} = \left\{ u \in L^{p_0}(0,T;X_0);\ \frac{du}{dt} \in L^{p_1}(0,T;X_1) \right\},$$

endowed with the norm

$$\|u\|_{\mathcal{B}} = \|u\|_{L^{p_0}(0,T;X_0)} + \left\| \frac{du}{dt} \right\|_{L^{p_1}(0,T;X_1)}.$$

Then the injection of $\mathcal{B}$ *into* $L^{p_0}(0,T;X)$ *is compact.*

Lemma 1.1.28 (Simon) *[41] Let* X_0, X_1 X *be three Banach space such that* $X_0 \subset X$ *with compact injection and* $X \subset X_1$ *with continuous injection.*

- *If* u *is bounded in* $L^p(0,T;X_0)$ *and* $\frac{\partial u}{\partial t}$ *is bounded in* $L^p(0,T;X_1)$ *where* $1 < p < \infty$. *Then* u *is relatively compact in* $L^p(0,T;X)$.

- *If* u *is bounded in* $L^\infty(0,T;X_0)$, $\frac{\partial u}{\partial t}$ *is bounded in* $L^r(0,T;X_1)$ *where* $r > 1$. *Then* u *is relatively compact in* $C([0,T];X)$.

Theorem 1.1.29 (Bochner) *[47] A strongly measurable function* $u : [0,T] \longrightarrow X$ *is integrable if and only if* $\|u\|_X$ *is integrable, and*

$$\left\| \int_0^T u(t)dt \right\|_X \leq \int_0^T \|u(t)\|_X\, dt.$$

Theorem 1.1.30 *Let* $(X,(.,.)_X)$ *is a Hilbert space and* $u :]0,T[\longrightarrow X$ *is the function such that* $u \in L^p(0,T;X)$ *and* $\frac{du}{dt} \in L^p(0,T;X)$ *for some* $1 \leq p \leq \infty$. *Then*

1. *The mapping* $t \longrightarrow \|u(t)\|_X$ *is absolutely continuous. With*

$$\frac{d}{dt}\|u(t)\|_X = 2\left(\frac{du}{dt}(t), u(t)\right)_X \quad a.e.\ in\ ]0,T[.$$

2. $\frac{1}{2}\|u(t)\|_X^2 = \frac{1}{2}\|u(0)\|_X^2 + \int_0^t \left(\frac{du}{dt}(s), u(s)\right)_X ds\ \ \forall t \in]0,T[.$

1.2 Functional analysis reminders

In this section, we give some preliminaries on weak convergence and weak star convergence, convex functions, and lower semi-continuous functions. We also present some notions related to the differential and sub-differential.

1.2.1 Weak convergence and Weak star convergence

we now turn our attention to the notion of convergence. Let X is a Banach space and X' the dual space of X and denote $\langle ., . \rangle$ the duality product between X and its topological dual space X'.

Definition 1.2.1 (Strongly convergence) *A sequence u_n is said to strongly converge to u if $u_n, u \in X$ and if*

$$\lim_{n \longrightarrow \infty} \|u_n - u\|_X = 0,$$

we will denoted this convergence by $u_n \longrightarrow u$ in X.

Now, Let's recall some definitions and results on weak topology

Definition 1.2.2 (Weakly convergence) *[47] A sequence u_n is said to weakly converge to u if $u_n, u \in X$ and if*

$$\lim_{n \longrightarrow \infty} \langle u_n, v \rangle_{X \times X'} = \langle u, v \rangle_{X \times X'}, \quad \forall v \in X'.$$

This convergence will be dented by $u_n \rightharpoonup u$ in X.

Proposition 1.2.3 *[12] Let u_n be a sequence in X. Then*

1. *If $u_n \longrightarrow u$ strongly in X, then $u_n \rightharpoonup u$ weakly in X.*
2. *If $u_n \rightharpoonup u$ weakly in X, then $\|u_n\|_X$ is bounded and $\|u\|_X \leq \lim_{n \longrightarrow \infty} \inf \|u_n\|_X$.*
3. *If $u_n \rightharpoonup u$ weakly in X and if $v_n \longrightarrow v$ strongly in X', then $\langle u_n, v_n \rangle_{X \times X'} \longrightarrow \langle u, v \rangle_{X \times X'}$.*

Theorem 1.2.4 (The compactness properties of reflexive space) *[12] Assume that X is a reflexive Banach space and let (u_n) be bounded sequence in X. Then there exists a subsequence (u_{nk}) that converges weakly to u in X.*

Definition 1.2.5 (Weak * Convergence) *[47] A sequence u_n in the dual space X' of a normed linear space X is said to be Weakly* converge to $u \in X'$ if:*

$$\lim_{n \longrightarrow \infty} \langle v, u_n - u \rangle_{X \times X'} = 0, \quad \forall v \in X,$$

will denoted by $u_n \overset{*}{\rightharpoonup} u$ in X'.

Proposition 1.2.6 *[12] Let (u_n) be a sequence in X'. Then*

1. *If $u_n \longrightarrow u$ strongly in X', then $u_n \overset{*}{\rightharpoonup} u$ weakly* in X'.*
2. *If $u_n \overset{*}{\rightharpoonup} u$ weakly* in X', then $\|u_n\|_{X'}$ is bounded and $\|u\|_{X'} \leq \lim_{n \longrightarrow \infty} \inf \|u_n\|_{X'}$.*
3. *If $u_n \overset{*}{\rightharpoonup} u$ weakly * in X' and if $v_n \longrightarrow v$ strongly in X, then $\langle u_n, v_n \rangle_{X' \times X} \longrightarrow \langle u, v \rangle_{X' \times X}$.*

Corollary 1.2.7 *[12] Let X be a separable Banach space and let (u_n) be a bounded sequence in X'. Then, there exists a subsequence (u_{nk}) that converges weakly* to u in X'.*

1.2.2 Lax-Milgram theorem

We introduce in what follows some useful results which are valid in Hilbert spaces. This concerns the Riesz representation theorem and Lax-Milgram Theorem.

Definition 1.2.8 (Bilinear form) *[12] Let X be a vector space. A scalar product (u, v) is a bilinear from $X \times X$ with values in $\mathbb{R}$ (i.e., a map $X \times X$ to $\mathbb{R}$ that is linear in both variables) such that*

$$\begin{aligned} (u,v) &= (v,u) \qquad \forall u, v \in X \qquad \textit{(symetry)}, \\ (u,u) &\geq 0 \qquad \forall u \in X \qquad \textit{(positive)}, \\ (u,u) &\neq 0 \qquad \forall u \neq 0 \qquad \textit{(definite)}. \end{aligned}$$

Theorem 1.2.9 (Riesz representation theorem) *[12] Let X be a Hilbert space and $\varphi \in X'$ there exist unique $f \in X$ such that*

$$\langle \varphi, u \rangle = (f, u) \ \ \forall u \in X.$$

Moreover,

$$\|\varphi\|_{X'} = \|f\|_X .$$

Definition 1.2.10 *[12] Let X be a Hilbert space. A bilinear form $a : X \times X \longrightarrow \mathbb{R}$ is said to be*

1. *Continuous, if there is a constant C such that*

$$|a(u,v)| \leq C\,|u|\,|v| \quad \forall u,v \in X.$$

2. *Coercive, if there exists a constant $\gamma > 0$ such that*

$$a(u,u) \geq \gamma\,|u|^2 \quad \forall u \in X.$$

Theorem 1.2.11 (Lax-Milgram) *[12] Assume that $a(u,v)$ is a continuous coercive bilinear form on X. Then, given any $\varphi \in X'$, there exist a unique element $u \in X$ such that*

$$a(u,v) = \langle \varphi, v \rangle \quad \forall v \in X.$$

1.2.3 Schauder fixed point theorem

In this subsection, we discuss a fundamental fixed theorem on Banach spaces-complete normed spaces, the Schauder fixed point theorem.
The Schauder fixed-point theorem is an extension of the Brouwer fixed-point theorem to infinite-dimensional spaces.

Theorem 1.2.12 *[25] Let C be a compact convex set in a Banach space X and let T be a continuous mapping of C into itself. Then has a fixed point, that is $Tx = x$ for some $x \in C$.*

We note the following extension of Theorem 1.2.12.

Corollary 1.2.13 *[26] Let C be a closed convex set in Banach space X and let T be a continuous mapping of C into C such that the image $T(C)$ is precompact. Then T has a fixed point.*

1.2.4 Convex lower semi-continuous function

The convex lower semi-continuous function represents a crucial ingredient in the study of variational inequalities. Let $j(.)$ defined on a vector space X with values in $]-\infty, +\infty]$. Recall the following definition.

Definition 1.2.14 *[12] A function $j(.)$ is said to be convex if*

$$j((1-t)u+tv) \leq (1-t)j(u)+tj(v), \text{ for all } u,v \in X, \ t \in [0,1] \, .$$

the function $j(.)$ strongly convex if the last inequality is strict for $u \neq v$ and $t \in (0,1)$.
For all function $j : X \longrightarrow \,]-\infty,+\infty]$ so that j take the value $+\infty$ (but $-\infty$ is excluded). We denote by $D(j)$ the domain of j, that is

$$D(j) = \{v \in X; \ \ j(v) < +\infty\}.$$

The epigraph of j is the set

$$epi(j) = \{[v,\lambda] \in X \times \mathbb{R}; \ \ j(v) \leq \lambda\}.$$

It is clear that we can establish the following prosperity

1. *$j(.)$ is proper if and only if $D(j) \neq \emptyset$.*
2. *The domain of $j(.)$ is a convex set of X if $j(.)$ is convex.*
3. *If $j(.)$ is a convex function, then epi(j) is a convex set in $X \times \mathbb{R}$.*

Definition 1.2.15 *[12] Let X is topological space. A function $j : X \longrightarrow \,]-\infty,+\infty]$ is said to be lower semi-continuous (I.s.c) if for every $\lambda \in \mathbb{R}$ the set*

$$\{v \in X; \ \ j(v) \leq \lambda\} \quad \text{is closed.}$$

We shall use some elementary prosperity of I.s.c functions.

Proposition 1.2.16 *[12] Let $j : X \longrightarrow \mathbb{R}$. Then*

(i) If j is I.s.c, the epi(j) is closed in $X \times \mathbb{R}$, and conversely.

(ii) If j is I.s.c, then for every $v \in X$ and for every $\varepsilon > 0$ there is some neighborhood V_v of v such that

$$j(u) \geq j(v) - \varepsilon \ \ \forall u \in V_v.$$

(iii) If j is I.s.c, then for every (u_n) in X such that $u_n \longrightarrow u$, we have

$$\liminf_{n \longrightarrow \infty} j(u_n) \geq j(u).$$

1.2.5 Differentiability

Hereafter X denotes a Banach space and $\langle ., . \rangle$ is the duality product between X and its topological dual space X'. We now recall the definition of Gâteaux differentiable function.

Definition 1.2.17 *[43] A function $j : X \longrightarrow]-\infty, +\infty]$ is Gâteaux differentiable at $u \in X$ if exists an element $\nabla j(u) \in X'$ such that*

$$\lim_{t \longrightarrow 0} \frac{j(u+tv) - j(u)}{t} = \langle \nabla j(u), v \rangle_{X' \times X} \quad \forall v \in X.$$

The element $\nabla j(u)$ is called the differential in the sense of Gâteaux of j at u. The function $j(.)$ is said to be Gâteaux differential if it is Gâteaux differentiable at every point of X. In this case the operator $\nabla j : X \longrightarrow X'$ which maps every element $u \in X$ into the element $\nabla j(u)$ is called the gradient of j.

The convexity of Gâteaux differentiable function can be characterized as follows.

Proposition 1.2.18 *[43] Let $j : X \longrightarrow \mathbb{R}$ be Gâteaux differentiable function. Then the following statement are equivalent.*

(i) j is a convex function.

(ii) j satisfies the inequality

$$j(v) - j(u) \geq \langle \nabla j(u), v - u \rangle_{X' \times X} \quad \forall u, v \in X.$$

(iii) The gradient of $j(.)$ is a monotone operator, that is

$$\langle \nabla j(u) - \nabla j(v), u - v \rangle_{X' \times X} \geq 0 \quad \forall u, v \in X.$$

Definition 1.2.19 *We say that the function $j : X \longrightarrow]-\infty, +\infty]$ is sub-differential at a point $w \in X'$ such that*

$$j(u) - j(v) \geq \langle w, u - v \rangle_{X' \times X} \quad \forall v \in X.$$

The element w is then called a sub-gradient of j in u and the set of sub-gradients of j in u is called the sub-differential of j in u and is noted $\partial j(u)$, with

$$\partial j(u) = \{ w \in X' ; j(u) - j(v) \geq \langle w, u - v \rangle_{X' \times X}, \ \forall v \in X \}.$$

We note by $D(\partial j)$ the set define by

$$D(\partial j) = \{u \in X;\ \ \partial j(u) \neq \emptyset\}.$$

Using the two last equations and the definition of the domain of a function, we obtain

$$D(\partial j) \subset D(j).$$

The function $j(.)$ is said to be sub-differentiable if it is sub-differentiable at any point of X, i.e. $D(\partial j) = X$.

Corollary 1.2.20 *[43] Let $j : X \longrightarrow]-\infty, +\infty]$ be a convex Gâteaux differentiable function. Then, j is lower semicontinuous.*

Corollary 1.2.21 *Let $j : X \longrightarrow]-\infty, +\infty]$ be a convex Gâteaux differentiable function. Then, $j(.)$ is a sub-differentiable, and we have*

$$\partial j(u) = \{\nabla j(u)\},\ \ \forall u \in X.$$

1.3 Gronwall's Lemma

At the end of this chapter, we review the Gronwall lemmas which are used in many boundary problems, in particular to establish the uniqueness of the solution, as well as to form and to form the a priori estimates.

Lemma 1.3.1 *[27] Let $u(t)$ and $\varphi(t)$ be nonnegative, continuous function on $0 \leq t \leq T$, for which inequality*

$$\varphi(t) \leq a + \int_0^t v(s)\varphi(s)ds,\ \ \forall t \in [0, T].$$

holds, where a is a nonnegative constant. Then

$$\varphi(t) \leq a \exp\left(\int_0^t v(s)ds\right),\ \ \forall t \in [0, T].$$

Lemma 1.3.2 *Let $u(t)$ and $v(t)$ be nonnegative function for all $t \in [0, T]$, such that $u, v \in C(0, T; \mathbb{R})$ and let a nonnegative constant. Consider a function $\varphi \in C(0, T; \mathbb{R})$ such that*

$$\varphi(t) \leq a + \int_0^t u(s)ds + \int_0^t v(s)\varphi(s)ds,\ \ \forall t \in [0, T].$$

Then,

$$\varphi(t) \leq \left(a + \int_0^t u(s)ds\right) \exp\left(\int_0^t v(s)ds\right),\ \ \forall t \in [0, T].$$

Also, if $u \equiv 0$ the Lemma 1.3.2 we get the Lemma 1.3.1

Lemma 1.3.3 *Let $u, v \in C(0,T;\mathbb{R})$ such that $u \geq 0$ and $v \geq 0$ for all $t \in [0,T]$, a is nonnegative constant. Let also $\varphi : [0,T] \longrightarrow \mathbb{R}$ be a function such that*

$$\frac{1}{2}\varphi^2(t) \leq a + \int_0^t u(s)\varphi(s)ds + \int_0^t v(s)\varphi^2(s)ds, \ \ \forall t \in [0,T]\,.$$

Then,

$$|\varphi(t)| \leq \left(a + \int_0^t u(s)ds\right)\exp\left(\int_0^t v(s)ds\right), \ \ \forall t \in [0,T]\,.$$

In particular, If $u \equiv 0$ le Lemma 1.3.3 becomes.

Corollary 1.3.4 *Let $v \in C(0,T;\mathbb{R})$ such that $v \geq 0$ for all $t \in [0,T]$ and $a \geq 0$ constant. Let also $\varphi : [0,T] \longrightarrow \mathbb{R}$ be function such that*

$$\frac{1}{2}\varphi^2(t) \leq a + \int_0^t v(s)\varphi^2(s)ds \ \ \forall t \in [0,T]\,.$$

Then,

$$|\varphi(t)| \leq a\exp\left(\int_0^t v(s)ds\right) \ \ \forall t \in [0,T]\,.$$

Chapter 2

Analysis for flow of an incompressible Brinkman-type fluid in thin domain with friction

In this chapter, we study the convergence asymptotic of an incompressible Brinkman-type fluid in a thin domain with Tresca friction on the bottom surface. In Section 2.1, we discussed the variational formulation of the problem and the results of the existence in addition to the uniqueness of the weak solution. Following [3, 6, 8], we introduce a new scaling taking into account the small parameter ε. In section 2.2, using the scale change and unknown news to conduct the study on a fixed domain Ω. Then we prove after an explicit work different inequalities for the solution $(u^\varepsilon, p^\varepsilon)$ which the thickness becomes infinitely small in the formulation variational. Finally, these estimates allow us to have the limit problem, the Reynolds equation and establish the uniqueness of the solution.

2.1 The problem statement

In this section, we give an overview of the thin domain. Next, we introduce the problem considered in this domain. Finally, we explore the theorem of existence and uniqueness of the weak solution.

2.1.1 The domain

We denote by (x, x_3) the vector of $\mathbb{R}^3$ whose $x = (x_1, x_2)$ is the generic vector of $\mathbb{R}^2$ and $x_3 \in \mathbb{R}$.

Let Γ_b a domain of the x plane and h a bounded continuous function defined on Γ_b, with h of class C^1 such that $0 < h_m \leq h(x) \leq h_M$, $\forall (x,0) \in \Gamma_b$. The fluid is contained between the lower Γ_b and the upper surface $\overline{\Gamma}_u^\varepsilon$ defined by $x_3 = \varepsilon h(x)$. Let

$$\Omega^\varepsilon = \{(x, x_3) \in \mathbb{R}^3 \text{ such that } x \in \Gamma_b, \text{ and } 0 < \frac{x_3}{\varepsilon} < h(x)\},$$

where $\varepsilon \in]0,1[$ is a small parameter that will tend to zero. The boundary of Ω^ε is $\Gamma^\varepsilon = \overline{\Gamma}_b \cup \overline{\Gamma}_u^\varepsilon \cup \overline{\Gamma}_l^\varepsilon$, where Γ_l^ε is the lateral boundary.

Let $\nu = (\nu_1, \nu_2, \nu_3) = (0, 0, -1)$ the unit outward normal to the boundary Γ^ε. The normal and tangential components of u^ε are given by

$$u_\nu^\varepsilon = u^\varepsilon.\nu = u_i^\varepsilon \nu_i, \qquad u_{\tau_i}^\varepsilon = u_i^\varepsilon - u_\nu^\varepsilon \nu_i.$$

Similarly, for a regular tensor field σ^ε, we denote by σ_ν^ε and σ_τ^ε the normal and tangential components of σ^ε given by

$$\sigma_\nu^\varepsilon = (\sigma^\varepsilon.\nu).\nu = \sigma_{ij}^\varepsilon \nu_i \nu_j, \qquad \sigma_{\tau_i}^\varepsilon = \sigma_{ij}^\varepsilon \nu_j - \sigma_\nu^\varepsilon \nu_i.$$

2.1.2 Basic Equations

The boundary-value problem describing the stationary flow for incompressible Brinkman fluid is described by:
The law of conservation of momentum

$$-div(\sigma^\varepsilon) + \mu(\alpha^\varepsilon)^2 u^\varepsilon = f^\varepsilon \text{ in } \Omega^\varepsilon, \tag{2.1}$$

where $f^\varepsilon : \Omega^\varepsilon \to \mathbb{R}^3$ is the external force and $(\alpha^\varepsilon)^{-2}$ is the permeability.
The stress tensor σ^ε is decomposed as follows

$$\sigma_{ij}^\varepsilon = -p^\varepsilon \delta_{ij} + 2\mu d_{ij}(u^\varepsilon), \; d_{ij}(u^\varepsilon) = \frac{1}{2}\left(\frac{\partial u_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i}\right) \; (1 \leq i, j \leq 3), \tag{2.2}$$

where u^ε, p^ε, and μ are the velocity field of the fluid, the pressure, the viscosity, and δ_{ij} is the Krönecker symbol, given by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The incompressibility equation:

$$div(u^\varepsilon) = 0 \text{ in } \Omega^\varepsilon. \tag{2.3}$$

Our boundary conditions is described as.

• On Γ_u^ε, no-slip condition is given. The upper surface is assumed to be fixed as

$$u^\varepsilon = 0. \tag{2.4}$$

• On Γ_l^ε, the velocity is known and chosen parallel to the Γ_b-plane

$$u^\varepsilon = 0. \tag{2.5}$$

• On Γ_b, there is no-flux condition across Γ_b so that

$$u^\varepsilon.\nu = 0. \tag{2.6}$$

The tangential velocity on Γ_b is unknown and satisfies Tresca friction law with k^ε (see [20]) upper limit for stress

$$\left.\begin{aligned} &|\sigma_\tau^\varepsilon| < k^\varepsilon \Longrightarrow u_\tau^\varepsilon = s, \\ &|\sigma_\tau^\varepsilon| = k^\varepsilon \Longrightarrow \exists\lambda \geq 0,\ \ u_\tau^\varepsilon = s - \lambda\sigma_\tau^\varepsilon, \end{aligned}\right\} \text{on } \Gamma_b \tag{2.7}$$

where $|.|$ denotes the $\mathbb{R}^2$ Euclidean norm, s is the velocity of lower surface.

In order to give the variational formulation of the strong problem, we establish the following lemma.

Lemma 2.1.1 *The boundary condition of Tresca* (2.7) *equivalent to the following punctuated relation*

$$|\sigma_\tau^\varepsilon| \leq k^\varepsilon, \quad (u^\varepsilon - s)\sigma_\tau^\varepsilon + k^\varepsilon\,|u^\varepsilon - s| = 0 \quad on\ \Gamma_b. \tag{2.8}$$

The proof of the Lemma 2.1.1 can be found in [21].

2.1.3 Weak formulation

We introduce the following functional framework:

$$E^\varepsilon = \{v \in (H^1(\Omega^\varepsilon))^3 \text{ such that } v = 0 \text{ on } \Gamma_u^\varepsilon \cup \Gamma_l^\varepsilon,\ \ v.\nu = 0 \text{ on } \Gamma_b\},$$

$$E_{div}^\varepsilon = \{v \in E^\varepsilon \text{ such that } \ div(v) = 0 \text{ in } \Omega^\varepsilon\},$$

$$L_0^2(\Omega^\varepsilon) = \{q \in L^2(\Omega^\varepsilon) \text{ such that } \int_{\Omega^\varepsilon} q dx dx_3 = 0\}.$$

To simplify the writing, we note

$$a(u^\varepsilon,\varphi)=\int_{\Omega^\varepsilon}2\mu d_{ij}(u^\varepsilon)d_{ij}(\varphi)dxdx_3+\mu\int_{\Omega^\varepsilon}(\alpha^\varepsilon)^2u_i^\varepsilon\varphi_i dxdx_3, \tag{2.9}$$

$$\Big(p^\varepsilon,div\varphi\Big)=\int_{\Omega^\varepsilon}p^\varepsilon\delta_{ij}\frac{\partial}{\partial x_j}(\varphi_i-u_i^\varepsilon)dxdx_3, \tag{2.10}$$

we define the functional j by

$$j(\varphi)=\int_{\Gamma_b}k^\varepsilon\left|\varphi-s\right|dx, \tag{2.11}$$

j is continuous and convex.

Finally, we note

$$\Big(f^\varepsilon,\varphi\Big)=\int_{\Omega^\varepsilon}f_i^\varepsilon\varphi_i dxdx_3. \tag{2.12}$$

Lemma 2.1.2 *Let u^ε, and p^ε be the solution of* (2.1)-(2.7), *then it checks the following variational formulation problem*

$$\begin{cases}\text{Find the pair } (u^\varepsilon,p^\varepsilon)\in E^\varepsilon_{div}\times L^2_0(\Omega^\varepsilon), \text{ such that}\\ a(u^\varepsilon,\varphi-u^\varepsilon)-\Big(p^\varepsilon,div\varphi\Big)+j(\varphi)-j(u^\varepsilon)\geq\Big(f^\varepsilon,\varphi-u^\varepsilon\Big),\ \forall\varphi\in E^\varepsilon.\end{cases} \tag{2.13}$$

Proof of Lemma 2.1.2. Multiplying equation (2.1) by $(\varphi-u^\varepsilon)$, where $\varphi\in E^\varepsilon$ then integrating over Ω^ε and using the Green's formula, we get

$$\begin{aligned}&\int_{\Omega^\varepsilon}\sigma_{ij}^\varepsilon\frac{\partial}{\partial x_j}\left(\varphi_i-u_i^\varepsilon\right)dxdx_3-\int_{\Gamma^\varepsilon}\sigma_{ij}^\varepsilon\nu_j\left(\varphi_i-u_i^\varepsilon\right)d\delta\\ &+\int_{\Omega^\varepsilon}\mu(\alpha^\varepsilon)^2u_i^\varepsilon\left(\varphi_i-u_i^\varepsilon\right)dxdx_3=\int_{\Omega^\varepsilon}f_i^\varepsilon\left(\varphi_i-u_i^\varepsilon\right)dxdx_3.\end{aligned} \tag{2.14}$$

According the boundary conditions (2.4) and (2.5), yields to

$$\int_{\Gamma^\varepsilon}\sigma_{ij}^\varepsilon\nu_j\left(\varphi_i-u_i^\varepsilon\right)d\delta=\int_{\Gamma_b}\sigma_{ij}^\varepsilon\nu_j(\varphi_i-u_i^\varepsilon)dx.$$

Besides, as $\sigma_{ij}^\varepsilon\nu_j=\sigma_\tau^\varepsilon+\sigma_\nu^\varepsilon\nu_i$ and $\left(\varphi_i-u_i^\varepsilon\right)\nu_i=0$ on Γ_b^ε, we have

$$\int_{\Gamma^\varepsilon}\sigma_{ij}^\varepsilon\nu_j\left(\varphi_i-u_i^\varepsilon\right)d\delta=\int_{\Gamma_b}\sigma_\tau^\varepsilon\left(\varphi-u^\varepsilon\right)dx.$$

The equation (2.14) be written as

$$\begin{aligned}&\int_{\Omega^\varepsilon}\sigma_{ij}^\varepsilon\frac{\partial}{\partial x_j}\left(\varphi_i-u_i^\varepsilon\right)dxdx_3-\int_{\Gamma_b}\sigma_\tau^\varepsilon\left(\varphi-u^\varepsilon\right)dx\\ &+\int_{\Omega^\varepsilon}\mu(\alpha^\varepsilon)^2u_i^\varepsilon\left(\varphi_i-u_i^\varepsilon\right)dxdx_3=\int_{\Omega^\varepsilon}f_i^\varepsilon\left(\varphi_i-u_i^\varepsilon\right)dxdx_3,\end{aligned}$$

by adding and subtracting the term $\int_{\Gamma_b} k^\varepsilon \left(|\varphi - s| - |u^\varepsilon - s|\right) dx$, we find

$$\begin{aligned} &\int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon \frac{\partial}{\partial x_j} \left(\varphi_i - u_i^\varepsilon\right) dx dx_3 + \int_{\Gamma_b} k^\varepsilon \left(|\varphi - s| - |u^\varepsilon - s|\right) dx \\ &- \int_{\Gamma_b} \sigma_\tau^\varepsilon \left(\varphi - u^\varepsilon\right) dx - \int_{\Gamma_b} k^\varepsilon \left(|\varphi - s| - |u^\varepsilon - s|\right) dx \\ &+ \int_{\Omega^\varepsilon} \mu(\alpha^\varepsilon)^2 u_i^\varepsilon \left(\varphi_i - u_i^\varepsilon\right) dx dx_3 = \int_{\Omega^\varepsilon} f_i^\varepsilon \left(\varphi_i - u_i^\varepsilon\right) dx dx_3, \end{aligned} \tag{2.15}$$

we pose

$$\beta = \int_{\Gamma_b} \sigma_\tau^\varepsilon \left(\varphi - u^\varepsilon\right) + \int_{\Gamma_b^\varepsilon} k^\varepsilon \left(|\varphi - s| - |u^\varepsilon - s|\right) dx,$$

we use now Lemma 2.1.1, we show that

$$\beta = \int_{\Gamma_b} \sigma_\tau^\varepsilon \left(\varphi - s\right) + \int_{\Gamma_b} k^\varepsilon |\varphi - s| \, dx \geq 0,$$

$$\int_{\Gamma_b} \sigma_\tau^\varepsilon \left(\varphi - s\right) dx \geq - \int_{\Gamma_b} |\sigma_\tau^\varepsilon| \, |\varphi - s| \, dx \geq - \int_{\Gamma_b} k^\varepsilon |\varphi - s| \, dx.$$

Therefor, the equation (2.15) can be written as follows

$$\begin{aligned} &\int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon \frac{\partial}{\partial x_j} \left(\varphi_i - u_i^\varepsilon\right) dx dx_3 + \int_{\Gamma_b} k^\varepsilon \left(|\varphi - s| - |u^\varepsilon - s|\right) dx \\ &+ \int_{\Omega^\varepsilon} \mu(\alpha^\varepsilon)^2 u_i^\varepsilon \left(\varphi_i - u_i^\varepsilon\right) dx dx_3 \geq \int_{\Omega^\varepsilon} f_i^\varepsilon \left(\varphi_i - u_i^\varepsilon\right) dx dx_3, \end{aligned}$$

We replaced σ^ε by its expression (2.2) and as $div(u^\varepsilon) = 0$ in Ω^ε, then we obtain the Lemma 2.1.2. ■

Theorem 2.1.3 *If $k^\varepsilon \in L_+^\infty(\Gamma_b)$ and $f^\varepsilon \in (L^2(\Omega^\varepsilon))^3$, then there exists a unique $u^\varepsilon \in E_{div}^\varepsilon$ and $p^\varepsilon \in L_0^2(\Omega^\varepsilon)$ (to an additive constant) solution to problem* (2.13).

Proof of Theorem 2.1.3. Since our goal in this work is to prove the asymptotic convergence of the problem posed, we only give the steps followed for the proof of this theorem. Let $\varphi \in E_{div}^\varepsilon$ in (2.13), we have:
Find $u^\varepsilon \in E_{div}^\varepsilon$, such that

$$a\Big(u^\varepsilon, \varphi - u^\varepsilon\Big) + j(\varphi) - j(u^\varepsilon) \geq \Big(f^\varepsilon, \varphi - u^\varepsilon\Big), \forall \varphi \in E_{div}^\varepsilon. \tag{2.16}$$

Using the Cauchy-Schwartz inequality, and $\sum_{i,j=1}^3 |d_{ij}(u^\varepsilon)|^2 \leq |\nabla u^\varepsilon|^2$, we obtain that the bilinear from $a(.,.)$ is continuous

$$\begin{aligned} |a(u^\varepsilon, v)| &= \int_{\Omega^\varepsilon} 2\mu d_{ij}(u^\varepsilon) d_{ij}(v) dx dx_3 + \int_{\Omega^\varepsilon} \mu(\alpha^\varepsilon)^2 u^\varepsilon v dx dx_3 \\ &\leq \left(2\mu + \mu(\alpha^\varepsilon)^2\right) \|u^\varepsilon\|_{H^1(\Omega^\varepsilon)} \|v\|_{H^1(\Omega^\varepsilon)}, \quad \forall (u^\varepsilon, v) \in (E_{div}^\varepsilon)^2. \end{aligned}$$

By Korn's inequality, we obtain

$$
\begin{aligned}
a(u^\varepsilon, u^\varepsilon) &= 2\mu \sum_{i,j=1}^{3} \|d_{ij}(u^\varepsilon)\|^2_{L^2(\Omega^\varepsilon)} + \mu(\alpha^\varepsilon)^2 \|u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} \\
&\geq \min(2\mu C_k, \mu(\alpha^\varepsilon)^2) \|u^\varepsilon\|^2_{H^1(\Omega^\varepsilon)}, \quad \forall u^\varepsilon \in E^\varepsilon_{div},
\end{aligned}
$$

where $C_k > 0$ independent of ε. We deduce that $a(.,.)$ is coercive on $E^\varepsilon_{div} \times E^\varepsilon_{div}$. Moreover j is convex, continuous on E^ε_{div}. This ensures the existence and uniqueness of $u^\varepsilon \in E^\varepsilon_{div}$ satisfying the variational inequality (2.16). In addition, using the techniques of [44], we can prove the existence of $p^\varepsilon \in L^2_0(\Omega^\varepsilon)$ for which $(u^\varepsilon, p^\varepsilon)$ is a solution of (2.13). ■

2.2 Dilatation in the variable x_3

For the asymptotic analysis we will use the dilatation in the variable x_3 given by $x_3 = z\varepsilon$, then our problem is defined on a domain Ω does not depend on ε given by:

$$
\Omega = \{(x, z) \in \mathbb{R}^3 \text{ such that } x \in \Gamma_b \text{ and } 0 < z < h(x)\},
$$

and its boundary $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_L \cup \overline{\Gamma}_b$.
After this change, here are the new functions defined on the fixed domain Ω.

$$
\begin{cases}
\widehat{u}^\varepsilon_i(x, z) = u^\varepsilon_i(x, x_3) \quad (i = 1, 2), \\
\widehat{u}^\varepsilon_3(x, z) = \varepsilon^{-1} u^\varepsilon_3(x, x_3), \\
\widehat{p}^\varepsilon(x, z) = \varepsilon^2 p^\varepsilon(x, x_3).
\end{cases}
\tag{2.17}
$$

Likewise for new data

$$
\begin{cases}
\widehat{f}(x, z) = \varepsilon^2 f^\varepsilon(x, x_3), \\
\widehat{\alpha} = \varepsilon \alpha^\varepsilon, \\
\widehat{k} = \varepsilon k^\varepsilon.
\end{cases}
\tag{2.18}
$$

Following [9], we say that $v = (v_1, v_2) \in (L^2(\Omega))^2$ satisfies condition (D) if

$$
\int_\Omega \left(v_1 \frac{\partial \theta}{\partial x_1} + v_2 \frac{\partial \theta}{\partial x_2} \right) dxdz = 0 \qquad \forall \theta \in C^\infty_0(\Gamma_b). \tag{D}
$$

Now, we define function spaces and sets on Ω we need in our considerations

$$
E = \left\{ \widehat{v} \in (H^1(\Omega))^3 : \widehat{v} = 0 \text{ on } \Gamma_u \cup \Gamma_l, \ \widehat{v}.\nu = 0 \text{ on } \Gamma_b \right\},
$$

$$
E_{div} = \{ \widehat{v} \in E : \ div(\widehat{v}) = 0 \text{ in } \Omega \},
$$

$$
H^1_{\Gamma_u \cup \Gamma_l}(\Omega) = \left\{ \widehat{v} \in H^1(\Omega) : \widehat{v} = 0 \text{ on } \Gamma_u \cup \Gamma_l \right\},
$$

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q dx dz = 0 \right\},$$

and

$$\Pi(E) = \left\{ \overline{\psi} = (\widehat{\psi}_1, \widehat{\psi}_2) \in (H^1(\Omega))^2 : \exists \widehat{\psi}_3 \in H^1(\Omega),\, \widehat{\psi} = (\widehat{\psi}_1, \widehat{\psi}_2, \widehat{\psi}_3) \in E \right\},$$

$$\Sigma(E) = \left\{ \overline{\psi} \in \Pi(E) : \overline{\psi} \text{ satisfies condition (D)} \right\},$$

$$V_z = \left\{ \Phi = (\Phi_1, \Phi_2) \in (L^2(\Omega))^2 : \ \frac{\partial \Phi_i}{\partial z} \in L^2(\Omega),\ \Phi = 0 \text{ on } \Gamma_u \cup \Gamma_l \right\}.$$

V_z be the Banach space with norm

$$\|\Phi\|_{V_z} = \left(\sum_{i=1}^{2} \left(\|\Phi_i\|^2_{L^2(\Omega)} + \left\| \frac{\partial \Phi_i}{\partial z} \right\|^2_{L^2(\Omega)} \right) \right)^{\frac{1}{2}}.$$

We define its linear subspace

$$\widetilde{V}_z = \{ \Phi \in V_z : \ \Phi \ \text{ satisfies condition (D)} \}.$$

According (2.17) and (2.18), then the problem (2.13) leads to the following:

Problem 2.2.1 *Find $(\widehat{u}^\varepsilon, \widehat{p}^\varepsilon) \in E \times L_0^2(\Omega)$, such that*

$$\int_\Omega q \, div(\widehat{u}^\varepsilon) dx dz = 0, \ \ \forall q \in L_0^2(\Omega), \tag{2.19}$$

$$\begin{aligned}
&\sum_{i,j=1}^{2} \int_\Omega \left(\varepsilon^2 \mu \left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \widehat{u}_j^\varepsilon}{\partial x_i} \right) - \widehat{p}^\varepsilon \delta_{ij} \right) \frac{\partial}{\partial x_j} (\widehat{\varphi}_i - \widehat{u}_i^\varepsilon) dx dz \\
&+ \sum_{i=1}^{2} \int_\Omega \mu \left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_i} \right) \left(\frac{\partial}{\partial z} (\widehat{\varphi}_i - \widehat{u}_i^\varepsilon) + \varepsilon^2 \frac{\partial}{\partial x_i} (\widehat{\varphi}_3 - \widehat{u}_3^\varepsilon) \right) dx dz \\
&+ \int_\Omega \left(2\mu\varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial z} - \widehat{p}^\varepsilon \right) \frac{\partial}{\partial z} (\widehat{\varphi}_3 - \widehat{u}_3^\varepsilon) dx dz \\
&+ \sum_{i=1}^{2} \mu \widehat{\alpha}^2 \int_\Omega \widehat{u}_i^\varepsilon (\widehat{\varphi}_i - \widehat{u}_i^\varepsilon) dx dz + \varepsilon^2 \mu \widehat{\alpha}^2 \int_\Omega \widehat{u}_3^\varepsilon (\widehat{\varphi}_3 - \widehat{u}_3^\varepsilon) dx dz + j(\widehat{\varphi}) - j(\widehat{u}^\varepsilon) \\
&\geq \sum_{i=1}^{2} \int_\Omega \widehat{f}_i (\widehat{\varphi}_i - \widehat{u}_i^\varepsilon) dx dz + \varepsilon \int_\Omega \widehat{f}_3 (\widehat{\varphi}_3 - \widehat{u}_3^\varepsilon) dx dz, \ \forall \widehat{\varphi} \in E,
\end{aligned} \tag{2.20}$$

where

$$j(\widehat{\varphi}) = \int_{\Gamma_b} \widehat{k} \, |\widehat{\varphi} - s| \, dx.$$

Next, we present the lemma which is useful in establishing the priori estimates.

Lemma 2.2.2 *We have*

$$\sum_{i,j=1}^{2}\int_{\Omega}\frac{\partial\widehat{v}_j}{\partial x_i}\frac{\partial\widehat{v}_i}{\partial x_j}dxdz+\sum_{i=1}^{2}\int_{\Omega}\frac{\partial\widehat{v}_3}{\partial x_i}\frac{\partial\widehat{v}_i}{\partial z}dxdz+\sum_{j=1}^{2}\int_{\Omega}\frac{\partial\widehat{v}_j}{\partial z}\frac{\partial\widehat{v}_3}{\partial x_j}dxdz+\int_{\Omega}\frac{\partial\widehat{v}_3}{\partial z}\frac{\partial\widehat{v}_3}{\partial z}dxdz=0, \tag{2.21}$$

for any $\widehat{v}\in E_{div}$.

Proof of Lemma 2.2.2. Let us denote for all $\widehat{v}\in\left(C^{\infty}_{\Gamma_u\cup\Gamma_l}(\Omega)\right)^3$,

$$I=\sum_{i,j=1}^{2}\int_{\Omega}\frac{\partial\widehat{v}_j}{\partial x_i}\frac{\partial\widehat{v}_i}{\partial x_j}dxdz+\sum_{i=1}^{2}\int_{\Omega}\frac{\partial\widehat{v}_3}{\partial x_i}\frac{\partial\widehat{v}_i}{\partial z}dxdz+\sum_{j=1}^{2}\int_{\Omega}\frac{\partial\widehat{v}_j}{\partial z}\frac{\partial\widehat{v}_3}{\partial x_j}dxdz+\int_{\Omega}\frac{\partial\widehat{v}_3}{\partial z}\frac{\partial\widehat{v}_3}{\partial z}dxdz.$$

Using the density of $C^{\infty}_{\Gamma_u\cup\Gamma_l}(\Omega)$ in $H^1_{\Gamma_u\cup\Gamma_l}(\Omega)$, and the Green's formula, we get

$$I=I_1-I_2,$$

where

$$I_1=\sum_{i,j=1}^{2}\int_{\Gamma}\frac{\partial\widehat{v}_j}{\partial x_i}\widehat{v}_i\nu_j d\delta+\sum_{i=1}^{2}\int_{\Gamma}\frac{\partial\widehat{v}_3}{\partial x_i}\widehat{v}_i\nu_3 d\delta+\sum_{j=1}^{2}\int_{\Gamma}\frac{\partial\widehat{v}_j}{\partial z}\widehat{v}_3\nu_j d\delta+\int_{\Gamma}\frac{\partial\widehat{v}_3}{\partial z}\widehat{v}_3\nu_3 d\delta,$$

$$I_2=\sum_{i,j=1}^{2}\int_{\Omega}\frac{\partial^2\widehat{v}_j}{\partial x_i\partial x_j}\widehat{v}_i dxdz+\sum_{i=1}^{2}\int_{\Omega}\frac{\partial^2\widehat{v}_3}{\partial x_i\partial z}\widehat{v}_i dxdz+\sum_{j=1}^{2}\int_{\Omega}\frac{\partial^2\widehat{v}_j}{\partial z\partial x_j}\widehat{v}_3 dxdz+\int_{\Omega}\frac{\partial\widehat{v}_3}{\partial z^2}\widehat{v}_3 dxdz.$$

Since $div(\widehat{v})=0$ in Ω, then $I_2=0$ and as $\widehat{v}=0$ on $\Gamma_u\cup\Gamma_l$, we have

$$I_1=\sum_{i,j=1}^{2}\int_{\Gamma_b}\frac{\partial\widehat{v}_j}{\partial x_i}\widehat{v}_i\nu_j d\delta+\sum_{i=1}^{2}\int_{\Gamma_b}\frac{\partial\widehat{v}_3}{\partial x_i}\widehat{v}_i\nu_3 d\delta+\sum_{j=1}^{2}\int_{\Gamma_b}\frac{\partial\widehat{v}_j}{\partial z}\widehat{v}_3\nu_j d\delta+\int_{\Gamma_b}\frac{\partial\widehat{v}_3}{\partial z}\widehat{v}_3\nu_3 d\delta.$$

On Γ_b, we have $\nu=(0,0,-1)$ and from $\widehat{v}.\nu=\widehat{v}_3\nu_3=0$, we have

$$\frac{\partial\widehat{v}_3}{\partial x_i}\nu_3=-\widehat{v}_3\frac{\partial\nu_3}{\partial x_i}=0,\ \ i=1,2,$$
$$\frac{\partial\widehat{v}_3}{\partial z}\nu_3=-\widehat{v}_3\frac{\partial\nu_3}{\partial z}=0,$$

as ν_3 independent of (x_1,x_2,z). Then $I_2=0$, we deduce that $I=0$. ■

Now, we do the estimates of velocity $\widehat{u}^{\varepsilon}$ then on the pressure $\widehat{p}^{\varepsilon}$ solution of our variational problem in fixed domain.

Theorem 2.2.3 *Assuming* (2.17)-(2.18) *the following estimate on* $\widehat{u}^{\varepsilon}$ *is satisfied*

$$\varepsilon^2\sum_{i,j=1}^{2}\left\|\frac{\partial\widehat{u}_i^{\varepsilon}}{\partial x_j}\right\|^2_{L^2(\Omega)}+\frac{1}{2}\sum_{i=1}^{2}\left\|\frac{\partial\widehat{u}_i^{\varepsilon}}{\partial z}\right\|^2_{L^2(\Omega)}+\varepsilon^4\sum_{i=1}^{2}\left\|\frac{\partial\widehat{u}_3^{\varepsilon}}{\partial x_i}\right\|^2_{L^2(\Omega)}+\frac{1}{2}\varepsilon^2\left\|\frac{\partial\widehat{u}_3^{\varepsilon}}{\partial z}\right\|^2_{L^2(\Omega)} \tag{2.22}$$
$$+\widehat{\alpha}^2\left\|\widehat{u}_i^{\varepsilon}\right\|^2_{L^2(\Omega)}+\varepsilon^2\widehat{\alpha}^2\left\|\widehat{u}_3^{\varepsilon}\right\|^2_{L^2(\Omega)}\leq C.$$

Proof of Theorem 2.2.3. Putting $\widehat{\varphi}=0$ in (2.20), leads to

$$\sum_{i,j=1}^{2}\int_{\Omega}\left(\varepsilon^{2}\mu\left(\frac{\partial\widehat{u}_i^{\varepsilon}}{\partial x_j}+\frac{\partial\widehat{u}_j^{\varepsilon}}{\partial x_i}\right)-\widehat{p}^{\varepsilon}\delta_{ij}\right)\frac{\partial\widehat{u}_i^{\varepsilon}}{\partial x_j}dxdz$$
$$+\sum_{i=1}^{2}\int_{\Omega}\mu\left(\frac{\partial\widehat{u}_i^{\varepsilon}}{\partial z}+\varepsilon^{2}\frac{\partial\widehat{u}_3^{\varepsilon}}{\partial x_i}\right)\left(\frac{\partial\widehat{u}_i^{\varepsilon}}{\partial z}+\varepsilon^{2}\frac{\partial\widehat{u}_3^{\varepsilon}}{\partial x_i}\right)dxdz$$
$$+\int_{\Omega}\left(2\mu\varepsilon^{2}\frac{\partial\widehat{u}_3^{\varepsilon}}{\partial z}-\widehat{p}^{\varepsilon}\right)\frac{\partial\widehat{u}_3^{\varepsilon}}{\partial z}dxdz+\sum_{i=1}^{2}\mu\widehat{\alpha}^{2}\int_{\Omega}\widehat{u}_i^{\varepsilon}\widehat{u}_i^{\varepsilon}dxdz$$
$$+\varepsilon^{2}\mu\widehat{\alpha}^{2}\int_{\Omega}\widehat{u}_3^{\varepsilon}\widehat{u}_3^{\varepsilon}dxdz+j(\widehat{u}^{\varepsilon})\leq\sum_{i=1}^{2}\int_{\Omega}\widehat{f}_i\widehat{u}_i^{\varepsilon}dxdz+\varepsilon\int_{\Omega}\widehat{f}_3\widehat{u}_3^{\varepsilon}dxdz.$$

After that, using $\int_{\Omega}\widehat{p}^{\varepsilon}div(\widehat{u}^{\varepsilon})dxdz=0$ and $j(\widehat{u}^{\varepsilon})\geq 0$, we have

$$\sum_{i,j=1}^{2}\int_{\Omega}\mu\varepsilon^{2}\left(\frac{\partial\widehat{u}_i^{\varepsilon}}{\partial x_j}+\frac{\partial\widehat{u}_j^{\varepsilon}}{\partial x_i}\right)\frac{\partial\widehat{u}_i^{\varepsilon}}{\partial x_j}dxdz+\sum_{i=1}^{2}\int_{\Omega}\mu\left(\frac{\partial\widehat{u}_i^{\varepsilon}}{\partial z}+\varepsilon^{2}\frac{\partial\widehat{u}_3^{\varepsilon}}{\partial x_i}\right)\left(\frac{\partial\widehat{u}_i^{\varepsilon}}{\partial z}+\varepsilon^{2}\frac{\partial\widehat{u}_3^{\varepsilon}}{\partial x_i}\right)dxdz$$
$$+\int_{\Omega}2\mu\varepsilon^{2}\frac{\partial\widehat{u}_3^{\varepsilon}}{\partial z}\frac{\partial\widehat{u}_3^{\varepsilon}}{\partial z}dxdz+\sum_{i=1}^{2}\mu\widehat{\alpha}^{2}\int_{\Omega}\widehat{u}_i^{\varepsilon}\widehat{u}_i^{\varepsilon}dxdz$$
$$+\mu\varepsilon^{2}\widehat{\alpha}^{2}\int_{\Omega}\widehat{u}_3^{\varepsilon}\widehat{u}_3^{\varepsilon}dxdz\leq\sum_{i=1}^{2}\int_{\Omega}\widehat{f}_i\widehat{u}_i^{\varepsilon}dxdz+\varepsilon\int_{\Omega}\widehat{f}_3\widehat{u}_3^{\varepsilon}dxdz.$$

According to the Lemma 2.2.2, we get

$$\mu\varepsilon^{2}\sum_{i,j=1}^{2}\left\|\frac{\partial\widehat{u}_i^{\varepsilon}}{\partial x_j}\right\|_{L^2(\Omega)}^{2}+\mu\sum_{i=1}^{2}\left\|\frac{\partial\widehat{u}_i^{\varepsilon}}{\partial z}\right\|_{L^2(\Omega)}^{2}+\mu\varepsilon^{4}\sum_{i=1}^{2}\left\|\frac{\partial\widehat{u}_3^{\varepsilon}}{\partial x_i}\right\|_{L^2(\Omega)}^{2}$$
$$+\mu\varepsilon^{2}\left\|\frac{\partial\widehat{u}_3^{\varepsilon}}{\partial z}\right\|_{L^2(\Omega)}^{2}+\mu\widehat{\alpha}^{2}\left\|\widehat{u}_i^{\varepsilon}\right\|_{L^2(\Omega)}^{2}+\mu\varepsilon^{2}\widehat{\alpha}^{2}\left\|\widehat{u}_3^{\varepsilon}\right\|_{L^2(\Omega)}^{2} \tag{2.23}$$
$$\leq\sum_{i=1}^{2}\int_{\Omega}\widehat{f}_i\widehat{u}_i^{\varepsilon}dxdz+\varepsilon\int_{\Omega}\widehat{f}_3\widehat{u}_3^{\varepsilon}dxdz.$$

We Apply the Cauchy-Schwarz inequality then the Young inequality, we obtain the following

$$\int_{\Omega}\widehat{f}_i\widehat{u}_i^{\varepsilon}dxdz+\varepsilon\int_{\Omega}\widehat{f}_3\widehat{u}_3^{\varepsilon}dxdz\leq\frac{h_M^2}{2\mu}\left\|\widehat{f}_i\right\|_{L^2(\Omega)}^{2}+\frac{\mu}{2h_M^2}\left\|\widehat{u}_i^{\varepsilon}\right\|_{L^2(\Omega)}^{2}+\frac{h_M^2}{2\mu}\left\|\widehat{f}_3\right\|_{L^2(\Omega)}^{2}+\frac{\mu}{2h_M^2}\varepsilon^{2}\left\|\widehat{u}_3^{\varepsilon}\right\|_{L^2(\Omega)}^{2}.$$

The Poincaré inequality, we give

$$\int_{\Omega}\widehat{f}_i\widehat{u}_i^{\varepsilon}dxdz+\varepsilon\int_{\Omega}\widehat{f}_3\widehat{u}_3^{\varepsilon}dxdz\leq\frac{h_M^2}{2\mu}\left\|\widehat{f}\right\|_{L^2(\Omega)}^{2}+\frac{\mu}{2}\left\|\frac{\partial\widehat{u}_i^{\varepsilon}}{\partial z}\right\|_{L^2(\Omega)}^{2}+\frac{\mu}{2}\varepsilon^{2}\left\|\frac{\partial\widehat{u}_3^{\varepsilon}}{\partial z}\right\|_{L^2(\Omega)}^{2}.$$

Also, the equation (2.23) can be written as

$$\mu\varepsilon^2\sum_{i,j=1}^{2}\left\|\frac{\partial\widehat{u}_i^\varepsilon}{\partial x_j}\right\|_{L^2(\Omega)}^2+\frac{\mu}{2}\sum_{i=1}^{2}\left\|\frac{\partial\widehat{u}_i^\varepsilon}{\partial z}\right\|_{L^2(\Omega)}^2+\mu\varepsilon^4\sum_{i=1}^{2}\left\|\frac{\partial\widehat{u}_3^\varepsilon}{\partial x_i}\right\|_{L^2(\Omega)}^2+\frac{\mu}{2}\varepsilon^2\left\|\frac{\partial\widehat{u}_3^\varepsilon}{\partial z}\right\|_{L^2(\Omega)}^2$$
$$+\mu\widehat{\alpha}^2\left\|\widehat{u}_i^\varepsilon\right\|_{L^2(\Omega)}^2+\mu\varepsilon^2\widehat{\alpha}^2\left\|\widehat{u}_3^\varepsilon\right\|_{L^2(\Omega)}^2\leq\frac{h_M^2}{2\mu}\left\|\widehat{f}\right\|_{L^2(\Omega)}^2,$$

thus (2.22) follows. ■

Theorem 2.2.4 *Assuming* (2.17) *the following estimate on* p^ε *are satisfied*

$$\left\|\frac{\partial\widehat{p}^\varepsilon}{\partial z}\right\|_{H^{-1}(\Omega)}\leq\varepsilon C_1, \tag{2.24}$$

$$\left\|\frac{\partial\widehat{p}^\varepsilon}{\partial x_i}\right\|_{H^{-1}(\Omega)}\leq C_2,\ \ (i=1,2), \tag{2.25}$$

where C_1 *and* C_2 *the constants denote independent of* ε.

Proof of Theorem 2.2.4. Let $\psi\in H_0^1(\Omega)$, putting in (2.20) $\widehat{\varphi}_i=\widehat{u}_i^\varepsilon$ (for $i=1,2$) and $\widehat{\varphi}_3=\widehat{u}_3^\varepsilon\pm\psi$, we deduce

$$\begin{aligned}&\sum_{i=1}^{2}\int_\Omega\varepsilon^2\mu\left(\varepsilon^2\frac{\partial\widehat{u}_3^\varepsilon}{\partial x_j}+\frac{\partial\widehat{u}_i^\varepsilon}{\partial z}\right)\frac{\partial\psi}{\partial x_i}dxdz\\&+\int_\Omega\left(2\varepsilon^2\mu\frac{\partial\widehat{u}_3^\varepsilon}{\partial z}-\widehat{p}^\varepsilon\right)\frac{\partial\psi}{\partial z}dxdz+\varepsilon^2\mu\widehat{\alpha}^2\int_\Omega\widehat{u}_3^\varepsilon\psi dxdz=\varepsilon\int_\Omega\widehat{f}_3\psi dxdz.\end{aligned} \tag{2.26}$$

Therefore,

$$\begin{aligned}\int_\Omega\widehat{p}^\varepsilon\frac{\partial\psi}{\partial z}dxdz=&\sum_{i=1}^{2}\int_\Omega\varepsilon^2\mu\left(\varepsilon^2\frac{\partial\widehat{u}_3^\varepsilon}{\partial x_j}+\frac{\partial\widehat{u}_i^\varepsilon}{\partial z}\right)\frac{\partial\psi}{\partial x_i}dxdz\\&+\int_\Omega2\varepsilon^2\mu\frac{\partial\widehat{u}_3^\varepsilon}{\partial z}\frac{\partial\psi}{\partial z}dxdz+\varepsilon^2\mu\widehat{\alpha}^2\int_\Omega\widehat{u}_3^\varepsilon\psi dxdz-\varepsilon\int_\Omega\widehat{f}_3\psi dxdz.\end{aligned} \tag{2.27}$$

Using the Green's formula and Cauchy-schwarz inequality, we obtain

$$\begin{aligned}\left|\int_\Omega\frac{\partial\widehat{p}^\varepsilon}{\partial z}\psi dxdz\right|\leq&\left[\varepsilon^4\mu\left(\sum_{i=1}^{2}\left\|\frac{\partial\widehat{u}_3^\varepsilon}{\partial x_j}\right\|_{L^2(\Omega)}^2\right)^{\frac12}+\varepsilon^2\mu\left(\sum_{i=1}^{2}\left\|\frac{\partial\widehat{u}_j^\varepsilon}{\partial z}\right\|_{L^2(\Omega)}^2\right)^{\frac12}\right.\\&\left.+2\varepsilon^2\mu\left\|\frac{\partial\widehat{u}_3^\varepsilon}{\partial z}\right\|_{L^2(\Omega)}+\varepsilon^2\mu\widehat{\alpha}^2\left\|\widehat{u}_3^\varepsilon\right\|_{L^2(\Omega)}+\varepsilon\left\|\widehat{f}_3\right\|_{L^2(\Omega)}\right]\|\psi\|_{H_0^1(\Omega)}.\end{aligned} \tag{2.28}$$

From Theorem 2.2.3, we deduce the inequality (2.24).

Taking in (2.20), $\widehat{\varphi}_1^\varepsilon=\widehat{u}_1^\varepsilon\pm\psi$, ψ in $H_0^1(\Omega)$; $\widehat{\varphi}_2=\widehat{u}_2^\varepsilon$, $\widehat{\varphi}_3=\widehat{u}_3^\varepsilon$, we have

$$\begin{aligned}\int_\Omega\widehat{p}^\varepsilon\frac{\partial\psi}{\partial x_1}dxdz=&\int_\Omega\mu\left(\frac{\partial\widehat{u}_1^\varepsilon}{\partial z}+\varepsilon^2\frac{\partial\widehat{u}_3^\varepsilon}{\partial x_1}\right)\frac{\partial\psi}{\partial z}dxdz+\int_\Omega2\varepsilon^2\mu\frac{\partial\widehat{u}_1^\varepsilon}{\partial x_1}\frac{\partial\psi}{\partial x_1}dxdz\\&+\int_\Omega\varepsilon^2\mu\left(\frac{\partial\widehat{u}_1^\varepsilon}{\partial x_2}+\frac{\partial\widehat{u}_2^\varepsilon}{\partial x_1}\right)\frac{\partial\psi}{\partial x_2}dxdz+\mu\widehat{\alpha}^2\int_\Omega\widehat{u}_1^\varepsilon\psi dxdz-\int_\Omega\widehat{f}_1\psi dxdz.\end{aligned} \tag{2.29}$$

In the same way, the chose $\widehat{\varphi}_1 = \widehat{u}_1^\varepsilon$, $\widehat{\varphi}_3 = \widehat{u}_3^\varepsilon$, and $\widehat{\varphi}_2 = \widehat{u}_2^\varepsilon \pm \psi$, ψ in $H_0^1(\Omega)$, we have

$$\begin{aligned} \int_\Omega \widehat{p}^\varepsilon \frac{\partial \psi}{\partial x_2} dxdz &= \int_\Omega \mu \left(\frac{\partial \widehat{u}_2^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_2} \right) \frac{\partial \psi}{\partial z} dxdz + \int_\Omega 2\varepsilon^2 \mu \frac{\partial \widehat{u}_2^\varepsilon}{\partial x_2} \frac{\partial \psi}{\partial x_2} dxdz \\ &+ \int_\Omega \varepsilon^2 \mu \left(\frac{\partial \widehat{u}_1^\varepsilon}{\partial x_2} + \frac{\partial \widehat{u}_2^\varepsilon}{\partial x_1} \right) \frac{\partial \psi}{\partial x_1} dxdz + \mu \widehat{\alpha}^2 \int_\Omega \widehat{u}_2^\varepsilon \psi dxdz - \int_\Omega \widehat{f}_2 \psi dxdz. \end{aligned} \tag{2.30}$$

We use the same technical in (2.29) and (2.30) to obtain inequality (2.25). ■

Thanks to the estimations (2.22), (2.24) and (2.25), we have the following convergence result:

Theorem 2.2.5 *Suppose that the estimations* (2.22), (2.24) *and* (2.25) *hold. There exists* $(u^*, p^*) = ((u_1^*, u_2^*), p^*)$ *in* $\widetilde{V}_z \times L_0^2(\Omega)$ *such that we have the following:*

$$\widehat{u}_i^\varepsilon \rightharpoonup u_i^* \quad \text{weakly in} \quad \widetilde{V}_z, \qquad 1 \le i \le 2. \tag{2.31}$$

$$\varepsilon \frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j} \rightharpoonup 0 \quad \text{weakly in} \quad L^2(\Omega), \qquad 1 \le i, j \le 2. \tag{2.32}$$

$$\varepsilon \frac{\partial \widehat{u}_3^\varepsilon}{\partial z} \rightharpoonup 0 \quad \text{weakly in} \quad L^2(\Omega). \tag{2.33}$$

$$\varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_i} \rightharpoonup 0 \quad \text{weakly in} \quad L^2(\Omega), \qquad 1 \le i \le 2. \tag{2.34}$$

$$\varepsilon \widehat{u}_3^\varepsilon \rightharpoonup 0 \quad \text{weakly in} \quad L^2(\Omega). \tag{2.35}$$

$$\widehat{p}^\varepsilon \rightharpoonup p^* \quad \text{weakly in} \quad L_0^2(\Omega). \tag{2.36}$$

Proof of Theorem 2.3.2. Using the estimate (2.22), there exist a constant C independent of ε such that

$$\left\| \frac{\partial \widehat{u}_i^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 + \|\widehat{u}_i^\varepsilon\|_{L^2(\Omega)}^2 \le C \qquad (i = 1, 2).$$

we deduce that the $(\widehat{u}_1^\varepsilon, \widehat{u}_2^\varepsilon)$ is bounded in V_z. From thin one we get the existence of (u_1^*, u_2^*) in V_z such as

$$(\widehat{u}_1^\varepsilon, \widehat{u}_2^\varepsilon) \rightharpoonup (u_1^*, u_2^*) \text{ in } V_z.$$

To prove that $u_i^* \in \widetilde{V}_z$ we have to check condition (D). As $div(\widehat{u}^\varepsilon) = 0$ in Ω, we have for all $q \in C_0^\infty(\omega)$

$$0 = \int_\Omega q(x) \left(\frac{\partial \widehat{u}_1^\varepsilon}{\partial x_1} + \frac{\partial \widehat{u}_2^\varepsilon}{\partial x_2} + \frac{\partial \widehat{u}_3^\varepsilon}{\partial z} \right) dxdz = -\int_\Omega \left(\widehat{u}_1^\varepsilon \frac{\partial q}{\partial x_1} + \widehat{u}_2^\varepsilon \frac{\partial q}{\partial x_2} \right) dxdz,$$

because $\widehat{u}^\varepsilon.\nu = 0$ on Γ, and $\widehat{u}_i^\varepsilon \rightharpoonup u_i^*$ in V_z, $i = 1, 2$, we obtain the condition (D) for u^*

$$\int_\Omega \left(u_1^* \frac{\partial q}{\partial x_1} + u_2^* \frac{\partial q}{\partial x_2} \right) dxdz = 0, \ \ \forall q \in C_0^\infty(\omega).$$

Now, the convergences (2.32)-(2.35) follows from (2.22).

Finally, from (2.24) and (2.25), there exist a constant $C > 0$ independent of ε such that

$$\|\nabla \widehat{p}^\varepsilon\|_{H^{-1}(\Omega)} \leq C.$$

Then, according [45, Proposition 1.2, p.14-15], there exist a constant $C' > 0$ such that

$$\|\widehat{p}^\varepsilon\|_{L^2(\Omega)} \leq C' \|\nabla \widehat{p}^\varepsilon\|_{H^{-1}(\Omega)} \leq C'C.$$

So, there exists a subsequence $(\widehat{p}^\varepsilon)$ converging weakly in $L^2(\Omega)$ to some p^*, and

$$\|p^*\|_{L^2(\Omega)} \leq \liminf_{\varepsilon \longrightarrow 0} \|\widehat{p}^\varepsilon\|_{L^2(\Omega)} \leq C'C.$$

Moreover, $p^* \in L_0^2(\Omega)$ as $L_0^2(\Omega)$ is weakly closed in $L^2(\Omega)$, also $\widehat{p}^\varepsilon \rightharpoonup p^*$ weakly in $L_0^2(\Omega)$. ■

2.3 Study of the limit problem

To reach the desired goal, we need the results of previous convergences.

We go to the limit in (2.20) and using (2.19) ($\varepsilon \longrightarrow 0$), we find

$$\begin{aligned} &\sum_{i=1}^{2} \int_\Omega \mu \frac{\partial u_i^*}{\partial z} \frac{\partial}{\partial z} (\widehat{\varphi}_i - u_i^*)\, dxdz - \int_\Omega p^* \left(\frac{\partial \widehat{\varphi}_1}{\partial x_1} + \frac{\partial \widehat{\varphi}_2}{\partial x_2} \right) dxdz \\ &+ \sum_{i=1}^{2} \mu \widehat{\alpha} \int_\Omega u_i^* (\widehat{\varphi}_i - u_i^*) dxdz + \int_{\Gamma_b} \widehat{k} \left(|\widehat{\varphi} - s| - |u^* - s| \right) dx \\ &\geq \sum_{i=1}^{2} \int_\Omega \widehat{f}_i (\widehat{\varphi}_i - u_i^*) dxdz, \ \ \forall \widehat{\varphi} \in \Pi(E). \end{aligned} \tag{2.37}$$

If $\widehat{\varphi}$ satisfy condition (D), the inequality (2.37) is reduced as follows

$$\begin{aligned} &\sum_{i=1}^{2} \int_\Omega \mu \frac{\partial u_i^*}{\partial z} \frac{\partial}{\partial z} (\widehat{\varphi}_i - u_i^*) dxdz + \sum_{i=1}^{2} \mu \widehat{\alpha}^2 \int_\Omega u_i^* (\widehat{\varphi}_i - u_i^*) dxdz \\ &+ \widehat{k} \int_{\Gamma_b} (|\widehat{\varphi} - s| - |u^* - s|) dx \geq \sum_{i=1}^{2} \int_\Omega \widehat{f}_i (\widehat{\varphi}_i - u_i^*) dxdz, \ \ \forall \widehat{\varphi} \in \Sigma(E), \end{aligned} \tag{2.38}$$

Theorem 2.3.1 *With the same assumptions as Theorems 2.2.3 and 2.2.4, the pair* (u^*, p^*) *satisfies*

$$p^*(x_1, x_2, z) = p^*(x_1, x_2) \text{ a.e. in } \Omega, \; p^* \in H^1(\Gamma_b), \tag{2.39}$$

$$-\mu \frac{\partial^2 u_i^*}{\partial z^2} + \frac{\partial p^*}{\partial x_i} + \mu \widehat{\alpha}^2 u_i^* = \widehat{f}_i \text{ in } L^2(\Gamma_b). \tag{2.40}$$

Proof of Theorem 2.3.1. We choose in (2.20), $\widehat{\varphi}_3 = \widehat{u}_3^\varepsilon \pm \psi$ with ψ in $H_0^1(\Omega)$ and $\widehat{\varphi}_i = \widehat{u}_i^\varepsilon$, (for $i = 1, 2$) we give

$$\mu \int_\Omega \varepsilon^2 \left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_i} \right) \frac{\partial \psi}{\partial x_i} dxdz + \int_\Omega \left(2\mu\varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial z} - \widehat{p}^\varepsilon \right) \frac{\partial \psi}{\partial z} dxdz$$
$$+\varepsilon \int_\Omega \mu \widehat{\alpha}^2 \widehat{u}_3^\varepsilon \psi dxdz = \varepsilon \int_\Omega \widehat{f}_3 \psi dxdz.$$

Using the convergence limit (2.31), (2.34), (2.33), and the hypothesis of this theorem, we find

$$\int_\Omega p^* \frac{\partial \psi}{\partial z} dxdz = 0 \;\; \forall \psi \in H_0^1(\Omega).$$

The Green's formula, we obtain

$$-\int_\Omega \frac{\partial p^*}{\partial z} \psi dxdz = 0 \;\; \forall \psi \in H_0^1(\Omega),$$

then

$$\frac{\partial p^*}{\partial z} = 0 \text{ in } H^{-1}(\Omega). \tag{2.41}$$

Now, we take $\widehat{\varphi}_i = \widehat{u}_i^\varepsilon \pm \psi_i$ (for $i = 1, 2$) with ψ_i in $H_0^1(\Omega)$ and $\widehat{\varphi}_3 = \widehat{u}_3^\varepsilon$ in (2.20) to get

$$\sum_{i,j=1}^{2} \int_\Omega \left(\varepsilon^2 \mu \left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \widehat{u}_j^\varepsilon}{\partial x_i} \right) - \widehat{p}^\varepsilon \delta_{ij} \right) \frac{\partial \psi}{\partial x_j} dxdz + \sum_{i=1}^{2} \int_\Omega \mu \left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_i} \right) \frac{\partial \psi}{\partial z} dxdz$$
$$+ \sum_{i=1}^{2} \int_\Omega \mu \widehat{\alpha}^2 \widehat{u}_i^\varepsilon \psi_i dxdz = \sum_{i=1}^{2} \int_\Omega \widehat{f}_i \psi dxdz.$$

Using Theorem 2.2.5, we deduce first with $\psi_1 = 0$ and $\psi_2 \in H_0^1(\Omega)$, and then with $\psi_2 = 0$ and ψ_1 in $H_0^1(\Omega)$, the following equality

$$\begin{aligned} &\sum_{i=1}^{2} \int_\Omega -p^* \frac{\partial \psi_i}{\partial x_i} dxdz + \sum_{i=1}^{2} \int_\Omega \mu \frac{\partial u_i^*}{\partial z} \frac{\partial \psi_i}{\partial z} dxdz \\ &+ \sum_{i=1}^{2} \int_\Omega \mu \widehat{\alpha}^2 u_i^* \psi_i dxdz = \sum_{i=1}^{2} \int_\Omega \widehat{f}_i \psi_i dxdz. \end{aligned} \tag{2.42}$$

With Green's formula, we have

$$\begin{aligned} &\int_\Omega -\mu \frac{\partial^2 u_i^*}{\partial z^2}\psi_i dxdz + \int_\Omega \frac{\partial p^*}{\partial x_i}\psi_i dxdz \\ +\mu\widehat{\alpha}^2 &\int_\Omega u_i^*\psi_i dxdz = \int_\Omega \widehat{f}_i\psi_i dxdz \ \ \forall \psi_i \in H_0^1(\Omega). \end{aligned} \tag{2.43}$$

Then,

$$-\mu\frac{\partial^2 u_i^*}{\partial z^2} + \frac{\partial p^*}{\partial x_i} + \mu\widehat{\alpha}^2 u_i^* = \widehat{f}_i, \ \text{ in } H^{-1}(\Omega) \ \ (i=1,2).$$

To prove $p^* \in H^1(\Gamma_b)$, we see from (2.41) that p^* is functions does not depend on z. Then as in [3], we choose $\psi_i(x,z) = z(z-h(x))\gamma(x)$ in (2.43), with $\gamma \in H_0^1(\Gamma_b)$, we obtain

$$\begin{aligned} &\int_{\Gamma_b}\int_0^{h(x)} \frac{\partial p^*}{\partial x_i}\left(z(z-h(x))\gamma(x)\right)dzdx - \int_{\Gamma_b}\int_0^{h(x)} \mu\frac{\partial^2 u_i^*}{\partial z^2}\left(z(z-h(x))\gamma(x)\right)dzdx \\ +&\int_{\Gamma_b}\int_0^{h(x)} \mu\widehat{\alpha}^2 u_i^*(z(z-h(x))\gamma(x))dzdx = \int_{\Gamma_b}\int_0^{h(x)} \widehat{f}_i(x,z)(z(z-h(x))\gamma(x))dzdx. \end{aligned}$$

Using the Green formula, we deduce

$$\begin{aligned} &\frac{1}{6}\int_{\Gamma_b} p^*\frac{\partial(h^3\gamma(x))}{\partial x_i}dx - 2\mu\int_{\Gamma_b}\int_0^{h(x)} u_i^*(x,z)\gamma(x)dzdx \\ +&\int_{\Gamma_b}\int_0^{h(x)} \mu\widehat{\alpha}^2 u_i^*(z(z-h(x))\gamma(x))dzdx = \int_{\Gamma_b}\int_0^{h(x)} \widehat{f}_i(x,z)z(z-h(x))\gamma(x)dzdx. \end{aligned}$$

Thus,

$$\frac{1}{6}\int_{\Gamma_b} p^*\frac{\partial(h^3\gamma)}{\partial x_i}dx - 2\mu\int_{\Gamma_b} h(x)\widetilde{u}_i^*\gamma(x)dx + \mu\widehat{\alpha}^2\int_{\Gamma_b}\widehat{u}_i^*\gamma(x)dx = \int_{\Gamma_b}\widetilde{f}_i\gamma(x)dx,$$

where

$$\widetilde{u}_i^*(x) = \frac{1}{h(x)}\int_0^{h(x)} u_i^*(x,z)dz, \qquad \widehat{u}_i^* = \int_0^{h(x)} u_i^*(x,z)z(z-h(x))dz,$$
$$\widetilde{f}_i = \int_0^{h(x)} \widehat{f}_i(z(z-h(x)))dz.$$

Whence

$$-\frac{h^3}{6}\frac{\partial p^*}{\partial x_i} - 2\mu h(x)\widetilde{u}_i^* + \mu\widehat{\alpha}^2\widehat{u}_i^* = \widetilde{f}_i \quad \text{in } H^{-1}(\Gamma_b). \tag{2.44}$$

As $f_i \in L^2(\Omega)$ and $u_i^* \in V_z$ then in $L^2(\Omega)$, therefore $\widetilde{f}_i$, $\widetilde{u}_i^*$, and $\widehat{u}_i^*$ are in $L^2(\Gamma_b)$. In addition, from (2.44) we get p^* in $H^1(\Gamma_b)$. So, as f_i belong to $L^2(\Omega)$ from (2.39) we have $\frac{\partial^2 u_i^*}{\partial z^2} \in L^2(\Omega)$. Whence (2.40) holds, and we also have $\frac{\partial u_i^*}{\partial z}$ in V_z. ■

Theorem 2.3.2 *Let*

$$s^*(x) = u^*(x,0) \quad and \quad \tau^*(x) = \frac{\partial u^*}{\partial z}(x,0).$$

Under the same assumption of Theorem 2.2.3, the trace (τ^*, s^*) *satisfy the following limit from of Tresca boundary conditions*

$$\int_{\Gamma_b} \widehat{k}\left(|\psi + s^* - s| - |s^* - s|\right) dx - \int_{\Gamma_b} \widehat{k}\tau^*\psi dx \geq 0, \;\; \forall \psi \in (L^2(\Gamma_b))^2, \tag{2.45}$$

and

$$\left.\begin{array}{l} \mu\,|\tau| < \widehat{k} \Longrightarrow s^* = s \\ \mu\,|\tau^*| = \widehat{k} \Longrightarrow \exists\lambda \geq 0, s^* = s + \lambda\tau^* \end{array}\right\} \; a.e. \;\; on \; \Gamma_b. \tag{2.46}$$

Proof of Theorem 2.3.2. In the same spirit in [3, 9], we choose in (2.20), $\widehat{\varphi} = (\widehat{\varphi}_1, \widehat{\varphi}_2, \widehat{u}_3^\varepsilon)$, with $\widehat{\varphi}_i = \widehat{u}_i^\varepsilon + \psi_i$, (for $i = 1, 2$) and $\psi_i \in H^1_{\Gamma_u \cup \Gamma_l}(\Omega)$, we get

$$\sum_{i,j=1}^{2} \int_\Omega \left(\varepsilon^2 \mu \left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \widehat{u}_j^\varepsilon}{\partial x_i}\right) - \widehat{p}^\varepsilon \delta_{ij}\right) \frac{\partial \psi_i}{\partial x_j} dxdz + \sum_{i=1}^{2} \int_\Omega \mu \left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_i}\right) \frac{\partial \psi_i}{\partial z} dxdz$$
$$+ \sum_{i=1}^{2} \mu \widehat{\alpha}^2 \int_\Omega \widehat{u}_i^\varepsilon \psi_i dxdz + \int_{\Gamma_b} \widehat{k}\left(|\widehat{u}^\varepsilon + \psi - s| - |\widehat{u}^\varepsilon - s|\right) dx \geq \sum_{i=1}^{2} \int_\Omega \widehat{f}_i \psi_i dxdz.d$$

Firstly, by the convergence result in the Theorem 2.2.5, we find

$$\sum_{i=1}^{2} \int_\Omega -p^* \frac{\partial \psi_i}{\partial x_i} dxdz + \sum_{i=1}^{2} \mu \int_\Omega \frac{\partial u_i^*}{\partial z} \frac{\partial \psi_i}{\partial z} dxdz + \sum_{i=1}^{2} \int_\Omega \mu \widehat{\alpha}^2 u_i^* \psi_i dxdz$$
$$+ \int_{\Gamma_b} \widehat{k}\left(|s^* + \psi - s| - |s^* - s|\right) dx \geq \sum_{i=1}^{2} \int_\Omega \widehat{f}_i \psi_i dxdz.$$

Now, using the Green formula, we obtain

$$\sum_{i=1}^{2} \int_\Omega \frac{\partial p^*}{\partial x_i} \psi_i dxdz - \sum_{i=1}^{2} \mu \int_\Omega \frac{\partial^2 u_i^*}{\partial z^2} \psi_i dxdz + \sum_{i=1}^{2} \int_\Omega \mu \widehat{\alpha}^2 u_i^* \psi_i dxdz$$
$$+ \int_{\Gamma_b} \mu \frac{\partial u_i^*}{\partial z} \psi_i \nu_i dx + \int_{\Gamma_b} \widehat{k}\left(|s^* + \psi - s| - |s^* - s|\right) dx \geq \sum_{i=1}^{2} \int_\Omega \widehat{f}_i \psi_i dxdz.$$

From equation (2.40) and as $\nu = (0, 0, -1)$, we deduce

$$\int_{\Gamma_b} \widehat{k}\left(|\psi + (s^* - s)| - |s^* - s|\right) dx - \int_{\Gamma_b} \mu \tau_i^* \psi_i dx \geq 0, \;\; \forall \psi \in (H^1_{\Gamma_u \cup \Gamma_l}(\Omega))^2.$$

This inequality remains valid for any $\psi \in (D(\Gamma_b))^2$ and as $D(\Gamma_b)$ dense in $L^2(\Gamma_b)$ for any $\psi \in (L^2(\Gamma_b))^2$. Then (2.45) follows.

Next, to prove (2.46), we take $\psi_i = \pm(s_i^* - s)$ in (2.45), we obtain

$$\int_{\Gamma_b} \widehat{k}\, |s^* - s|\, dx - \int_{\Gamma_b} \mu\tau^*(s^* - s) dx = 0, \tag{2.47}$$

taking $\psi = \phi - (s^* - s)$, with ϕ in $(L^2(\Gamma_b))^2$, we get

$$\int_{\Gamma_b} \left(\widehat{k}\, |\phi| - \mu\tau^*\phi\right) dx \geq \int_{\Gamma_b} \left(\widehat{k}\, |s^* - s| - \mu\tau^*(s^* - s)\right) dx.$$

From (2.47), we deduce

$$\int_{\Gamma_b} \left(\widehat{k}\, |\phi| - \mu\tau^*\phi\right) dx \geq 0, \;\; \forall \phi \in (L^2(\Gamma_b))^2. \tag{2.48}$$

Now, in (2.48), we take $\phi = (\varphi_1, \varphi_2)$ such that $\varphi_i \geq 0$ for $i = 1, 2$ in (2.48), we obtain

$$\int_{\Gamma_b} \left(\widehat{k} - \mu\, |\tau^*| \cos(\tau^*, \phi)\right) |\phi| \geq 0.$$

Thus

$$\widehat{k} \geq \mu\, |\tau^*| \cos(\tau^*, \phi) \;\; \text{a.e. on } \Gamma_b. \tag{2.49}$$

After that, taking $-\phi$, with $\phi = (\varphi_1, \varphi_2)$ such that $\varphi_i \geq 0$ for $i = 1, 2$ in (2.48), we obtain

$$\int_{\Gamma_b} \left(\widehat{k} + \mu\, |\tau^*| \cos(\tau^*, \phi)\right) |\phi| \geq 0,$$

hence

$$-\widehat{k} \leq \mu\, |\tau^*| \cos(\tau^*, \phi) \;\; \text{a.e. on } \Gamma_b. \tag{2.50}$$

Going back to (2.49) and (2.50), we get

$$\mu\, |\tau^*| \leq \widehat{k} \;\; \text{a.e. on } \Gamma_b. \tag{2.51}$$

Then

$$\widehat{k}\, |s^* - s| \geq \mu\, |\tau^*|\, |s^* - s| \geq \mu\tau^*\, (s^* - s) \;\; \text{a.e. on } \Gamma_b,$$

From (2.47), we deduce that

$$\widehat{k}\, |s^* - s| - \mu\tau^*(s^* - s) = 0 \;\; \text{a.e. on } \Gamma_b. \tag{2.52}$$

If $\mu\, |\tau^*| = \widehat{k}$, then from (2.52), we have $\mu\, |\tau^*|\, |s^* - s| = \mu\tau^*(s^* - s)$ a.e. on Γ_b. So, there exist $\lambda \geq 0$ such that $(s^* - s) = \lambda\mu\tau^*$.

If $\mu\, |\tau^*| < \widehat{k}$, then from (2.52) we have

$$0 = \widehat{k}\, |s^* - s| - \mu\tau^*(s^* - s) \geq \left(\widehat{k} - \mu\, |\tau^*|\right) (s^* - s) \;\; \text{a.e. on} \Gamma_b,$$

since $(\widehat{k} - \mu\, |\tau^*|) > 0$, hence $s^* - s = 0$ a.e. on Γ_b. ■

Theorem 2.3.3 *Suppose that the assumptions of the previous Theorem hold, then the solution (u^*, p^*) satisfies the weak generalized equation of Reynolds:*

$$\int_{\Gamma_b} \left[\frac{h^3}{12\mu} \nabla p^* + h\widetilde{u_i}^*(x) - \frac{h}{2} s^*(x) + \widehat{\alpha} \left(\frac{h}{2} U^*(x,h) - \int_0^h U^*(x,t)dt \right) \right. \\ \left. + \widetilde{F}(x) \right] \nabla \varphi(x) dx = 0, \ \ \forall \varphi \in H^1(\Omega), \tag{2.53}$$

where

$$\widetilde{F}(x) = \frac{1}{\mu} \int_0^{h(x)} F(x,t)dt - \frac{h}{2\mu} F(x,h),$$

$$U^*(x,t) = \int_0^t \int_0^\zeta u_i^*(x,\theta) d\theta d\zeta, \quad F(x,t) = \int_0^t \int_0^\zeta \widehat{f}_i(x,\theta) d\theta d\zeta.$$

Proof of Theorem 2.3.3. By integrating twice the expression (2.40) from 0 to z, it comes:

$$-\mu u_i^*(x,z) + \mu u_i^*(x,0) + \mu z \frac{\partial u_i^*}{\partial z}(x,0) + \frac{z^2}{2} \frac{\partial p^*}{\partial x_i} \\ + \mu \widehat{\alpha}^2 \int_0^z \int_0^\zeta u_i^*(x,t) dt d\zeta = \int_0^z \int_0^t \widehat{f}_i dt d\zeta. \tag{2.54}$$

In particular, for $z = h$, we obtain

$$\mu s^*(x) + \mu h \tau^*(x) + \frac{h^2}{2} \frac{\partial p^*}{\partial x_i} + \mu \widehat{\alpha}^2 \int_0^h \int_0^\zeta u_i^*(x,t) dt d\zeta = \int_0^h \int_0^\zeta \widehat{f}_i(x,t) dt d\zeta. \tag{2.55}$$

Integrating (2.54) from 0 to h, we get

$$-\mu \int_0^h u_i^*(x,t)dt + \mu h s^*(x) + \frac{h^2 \mu}{2} \tau^*(x) + \frac{h^3}{6} \frac{\partial p^*}{\partial x_i} \\ + \mu \widehat{\alpha}^2 \int_0^h \int_0^\zeta \int_0^t u_i^*(x,\theta) d\theta dt d\zeta = \int_0^h \int_0^\zeta \int_0^t \widehat{f}_i(x,\theta) d\theta dt d\zeta.$$

As $\widetilde{u}_i^*(x) = \frac{1}{h} \int_0^{h(x)} u_i^*(x,t)dt$, we have

$$-\mu h \widetilde{u}_i^*(x) + \mu h s^*(x) + \mu \frac{h^2}{2} \tau^*(x) + \frac{h^3}{6} \frac{\partial p^*}{\partial x_i} \\ + \mu \widehat{\alpha}^2 \int_0^h \int_0^\zeta \int_0^t u_i^*(x,\theta) d\theta dt d\zeta = \int_0^{h(x)} F(x,t)dt. \tag{2.56}$$

From(2.55)-(2.56), we deduce (2.53). Which ends the proof requested. ■

Theorem 2.3.4 *Suppose that the assumptions of the previous Theorem hold, then the solution (u^*, p^*) of the limit problem (2.37) is unique in $\widetilde{V}_z \times L_0^2(\Omega)$.*

Proof of Theorem 2.3.4. Suppose that the problem (2.38) admits two solutions that we denote by $u^{*,1}$ and $u^{*,2}$. Using classical techniques, we choose in (2.38) $\widehat{\varphi} = u^{*,1}$ then $\widehat{\varphi} = u^{*,2}$ as test functions, then by summing the two inequalities obtained, we find

$$\sum_{i=1}^{2} \mu \int_{\Omega} \left| \frac{\partial}{\partial z}(u_i^{*,1} - u_i^{*,2}) \right|^2 dxdz + \sum_{i=1}^{2} \mu \widehat{\alpha}^2 \int_{\Omega} \left| u_i^{*,1} - u_i^{*,2} \right|^2 dxdz \leq 0. \tag{2.57}$$

Then

$$\left\| \frac{\partial}{\partial z}(u^{*,1} - u^{*,2}) \right\|_{L^2(\Omega)}^2 + \left\| u^{*,1} - u^{*,2} \right\|_{L^2(\Omega)}^2 \leq 0. \tag{2.58}$$

Consequently, this last inequality ensures the uniqueness of the solution u^* in $\widetilde{V}_z$.
Similarly, we take in the Reynolds equation (2.53) the pressure value $p^* = p^{*,1}$ then $p^* = p^{*,2}$ respectively, at the end by subtracting the equations obtained, it becomes:

$$\int_{\Gamma_b} \frac{h^3}{12\mu} \nabla (p^{*,1} - p^{*,2}) \nabla \varphi dx = 0.$$

Finally, by performing the change of variable $\varphi = p^{*,1} - p^{*,2}$ then by Poincaré inequality, we obtain

$$\left\| p^{*,1} - p^{*,2} \right\|_{L^2(\Gamma_b)} = 0,$$

as $p^{*,i} \in L_0^2(\Omega)$. We deduce the desired result. ■

Chapter 3

Asymptotic analysis of a static Electro-Elastic problem with Coulomb's Law

In this chapter, we use asymptotic analysis to derive and justify a two-dimensional limit problem for an electro-elastic problem on a thin domain which is in frictional contact with a foundation. In modeling the contact, the Coulomb free boundary friction condition is used. This chapter is divided into four sections; in section 3.1, the basic equations and some notations are presented. The related weak formulation is given in section 3.2. The results on the existence of the weak solution when the proof is based on arguments of a variational inequality and Schauder's fixed point developed recently in [22]. In section 3.3, we use the change of the domain to find some estimates on displacement and electric potential. Finally, we state the convergence and the limit problem when the small parameter ε close to zero. Moreover, the uniqueness of (u^*, φ^*) are treated.

3.1 Formulation of the problem

Let ω be a fixed bounded domain of $\mathbb{R}^3$ with $x_3 = 0$. The upper surface Γ_1^ε is defined by $x_3 = \varepsilon h(x)$. Now, introducing a small parameter ε which will be close to zero, and the smooth bounded function h such that

$$0 \leq h_m \leq h(x) \leq h_M, \;\; \forall (x, 0) \in \omega.$$

We denote by Ω^ε the following domain

$$\Omega^\varepsilon = \{(x, x_3) \in \mathbb{R}^3, (x, 0) \in \omega, 0 < x_3 < \varepsilon h(x)\},$$

with its boundary is $\Gamma^\varepsilon = \overline{\omega} \cup \overline{\Gamma}_1^\varepsilon \cup \overline{\Gamma}_L^\varepsilon$, where Γ_L^ε is a lateral boundary.
In what follows, $\mathbb{S}^3$ denotes the space of second order symmetric tensors on $\mathbb{R}^3$ and $|.|$ denotes the Euclidean norm on $\mathbb{R}^3$. Indeed, for every $u, v \in \mathbb{R}^3$, $u.v = u_i v_i$, $|v| = (v.v)^{1/2}$, and for every $\sigma, \tau \in \mathbb{S}^3$

$$\sigma.\tau = \sigma_{ij}\tau_{ij}, \qquad |\tau| = (\tau, \tau)^{1/2}.$$

Besides, we denote by $u^\varepsilon : \Omega^\varepsilon \longrightarrow \mathbb{R}^3$ the displacement field, $\sigma^\varepsilon : \Omega^\varepsilon \longrightarrow \mathbb{S}^3$ and $\sigma^\varepsilon = (\sigma_{ij}^\varepsilon)$ the stress tensor, and D^ε the electric displacement field. We also denote $E(\varphi^\varepsilon) = -\nabla\varphi^\varepsilon$ the electric fields, where $\varphi^\varepsilon : \Omega^\varepsilon \longrightarrow \mathbb{R}$ the electric potential. Moreover, let $d(u^\varepsilon) = (d_{ij}(u^\varepsilon))$ is the linearized strain tensors given by

$$d_{ij}(u^\varepsilon) = \frac{1}{2}\left(\frac{\partial u_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i}\right) \quad 1 \leq i, j \leq 3.$$

The electro-elastic constructive law given by

$$\sigma^\varepsilon(u^\varepsilon, \varphi^\varepsilon) = 2\mu^\varepsilon d(u^\varepsilon) + \lambda^\varepsilon tr(d(u^\varepsilon))I_3 - (\mathfrak{F}^\varepsilon)^T E(\varphi^\varepsilon) \quad \text{in} \quad \Omega^\varepsilon, \tag{3.1}$$

$$D^\varepsilon(u^\varepsilon, \varphi^\varepsilon) = \beta^\varepsilon E(\varphi^\varepsilon) + \mathfrak{F}^\varepsilon d(u^\varepsilon) \quad \text{in} \quad \Omega^\varepsilon, \tag{3.2}$$

where μ^ε, λ^ε are the coefficients of Lamé, I_3 is the identity. The operator $\mathfrak{F}^\varepsilon = (e_{ijk}^\varepsilon)$ is the third order piezoelectric tensor and β^ε denotes the electric coefficient of permittivity. We use here $(\mathfrak{F}^\varepsilon)^T$ to denote the transpose of the tensor $\mathfrak{F}^\varepsilon$, given by

$$(\mathfrak{F}^\varepsilon)^T = ({e_{ijk}^\varepsilon}^T) = (e_{kij}^\varepsilon) \text{ for all } i, j, k \in \{1, 2, 3\},$$

and, we have

$$\mathfrak{F}^\varepsilon\sigma.v = \sigma.(\mathfrak{F}^\varepsilon)^T v \quad \forall\sigma \in \mathbb{S}^3, \ \forall v \in \mathbb{R}^3. \tag{3.3}$$

Let $\eta = (\eta_1, \eta_2, \eta_3)$ be the unit outward normal to Γ^ε, we define the normal and the tangential components of displacement and stress tensors on the boundary, respectively, as follow

$$v_\eta^\varepsilon = v^\varepsilon.\eta \quad v_\tau^\varepsilon = v^\varepsilon - v_\eta^\varepsilon.\eta, \quad \sigma_\eta^\varepsilon = (\sigma^\varepsilon.\eta).\eta, \quad \sigma_\tau^\varepsilon = \sigma^\varepsilon\eta - \sigma_\eta^\varepsilon.\eta.$$

Therefore, the classical model for the electro-elastic problem is as follows.
Problem $\mathcal{P}^\varepsilon$. Find a displacement field $u^\varepsilon : \Omega^\varepsilon \to \mathbb{R}^3$, and electric potential $\varphi^\varepsilon : \Omega^\varepsilon \to \mathbb{R}$ such that

$$-div(\sigma^\varepsilon(u^\varepsilon,\varphi^\varepsilon)) = f^\varepsilon \quad \text{in} \quad \Omega^\varepsilon, \tag{3.4}$$

$$div(D^\varepsilon(u^\varepsilon,\varphi^\varepsilon)) = q^\varepsilon \quad \text{in} \quad \Omega^\varepsilon, \tag{3.5}$$

$$u^\varepsilon = 0 \quad \text{on} \quad \Gamma_1^\varepsilon, \tag{3.6}$$

$$u^\varepsilon = g \quad \text{with } g_3 = 0 \quad \text{on} \quad \Gamma_L^\varepsilon, \tag{3.7}$$

$$u_\eta^\varepsilon = 0 \quad \text{on} \quad \omega, \tag{3.8}$$

$$\begin{cases} |\sigma_\tau^\varepsilon| < k^\varepsilon|\sigma_\eta^\varepsilon| \Longrightarrow u_\tau^\varepsilon = s, \\ \\ |\sigma_\tau^\varepsilon| = k^\varepsilon|\sigma_\eta^\varepsilon| \Longrightarrow \exists\alpha > 0 \quad \text{such that} \quad u_\tau^\varepsilon = s - \alpha\sigma_\tau^\varepsilon, \end{cases} \quad \text{on} \quad \omega \tag{3.9}$$

$$\varphi^\varepsilon = 0 \quad \text{on} \quad \Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon, \tag{3.10}$$

$$D^\varepsilon.\eta = 0 \quad \text{on} \quad \omega. \tag{3.11}$$

Expressions (3.4)-(3.5) represent the equilibrium equation and (3.6)-(3.8) represent the boundary condition for the displacement and normal displacement. The Coulomb's friction law with friction coefficient k^ε is given by condition (3.9) (see [21]). The electric boundary conditions are given by (3.10)-(3.11). We note that f^ε and q^ε are the densities of the forces and the densities of electric charge, respectively. Recalling also that $g = (g_i)_{1\le i\le 3}$ is a vector function in $(H^{1/2}(\Gamma^\varepsilon))^3$ such that

$$\int_{\Gamma^\varepsilon} g.\eta d\sigma = 0, \quad g_3 = 0 \text{ on } \Gamma_L^\varepsilon, \quad g = 0 \text{ on } \Gamma_1^\varepsilon, \quad g.\eta = 0 \text{ on } \omega,$$

then, according [26, Lemma 2.2, p.24] there exists a function G^ε such that

$$G^\varepsilon \in (H^1(\Omega^\varepsilon))^3 \quad \text{with} \quad G^\varepsilon = g \text{ on } \Gamma^\varepsilon.$$

For (H_0) we have

$$u_3^\varepsilon = g_3 = 0 \text{ and } s = g \text{ on } \omega.$$

Furthermore, we need the following assumptions.

(H_1) Lamé's coefficient satisfy

$$\mu^\varepsilon, \lambda^\varepsilon \in L^\infty(\Omega^\varepsilon), \ \exists\mu^*, \lambda_* > 0 \text{ such that } \ \mu^\varepsilon(y) \ge \mu^*, \ 0 \le \lambda^\varepsilon(y) \le \lambda_*, \ \forall y \in \Omega^\varepsilon.$$

(H_2) The electric coefficient of permittivity, satisfies

$$\beta^\varepsilon \in L^\infty(\Omega^\varepsilon), \ \exists\beta^* > 0 \text{ such that } \beta^\varepsilon(y) \ge \beta^*, \ \forall y \in \Omega^\varepsilon.$$

(H_3) The piezoelectric tensor $\mathfrak{F}^\varepsilon = (e^\varepsilon_{ijk}) : \Omega^\varepsilon \times \mathbb{S}^3 \longrightarrow \mathbb{R}^3$, satisfies

$e^\varepsilon_{ijk} = e^\varepsilon_{ikj} \in L^\infty(\Omega^\varepsilon),\ \exists e_* > 0$ such that $\ e^\varepsilon_{ijk}(y) \le \sup|e^\varepsilon_{ijk}(y)| = e_*,\ \ i,j,k = 1,2,3,\ \ \forall y \in \Omega^\varepsilon.$

Let us recall an equivalent formulation to (3.9).

Lemma 3.1.1 *[21] For sufficiently smooth values of σ^ε_η and σ^ε_τ on ω. Coulomb boundary condition* (3.9) *are equivalent to*

$$|\sigma^\varepsilon_\tau| \le k^\varepsilon \left|\sigma^\varepsilon_\eta\right|,\ \ (u^\varepsilon_\tau - \mathbf{s})\sigma^\varepsilon_\tau + k^\varepsilon \left|\sigma^\varepsilon_\eta\right| |u^\varepsilon_\tau - \mathbf{s}| = 0. \tag{3.12}$$

Proof of Lemma 3.1.1. If (3.9) holds. Then, if $|\sigma^\varepsilon_\tau| = k^\varepsilon \left|\sigma^\varepsilon_\eta\right|$ we have $u^\varepsilon_\tau = \mathbf{s} - \alpha\sigma^\varepsilon_\tau$ for somme $\alpha \ge 0$, so

$$\begin{aligned}(u^\varepsilon_\tau - \mathbf{s})\sigma^\varepsilon_\tau + k^\varepsilon \left|\sigma^\varepsilon_\eta\right| |u^\varepsilon_\tau - \mathbf{s}| &= (-\alpha\sigma^\varepsilon_\tau)(\sigma^\varepsilon_\tau) + k^\varepsilon \left|\sigma^\varepsilon_\eta\right| |u_\tau - \mathbf{s}| \\ &= -\alpha\left|\sigma^\varepsilon_\tau\right|^2 + \alpha\left|\sigma^\varepsilon_\tau\right|^2 = 0.\end{aligned}$$

If $|\sigma^\varepsilon_\tau| < k^\varepsilon \left|\sigma^\varepsilon_\eta\right|$ by (3.9) $u^\varepsilon_\tau = \mathbf{s}$, hence (3.12) holds.
Conversely, we assume that (3.12) take place.
if $|\sigma^\varepsilon_\tau| = k^\varepsilon \left|\sigma^\varepsilon_\eta\right|$, then

$$(u^\varepsilon_\tau - \mathbf{s})\sigma^\varepsilon_\tau + k^\varepsilon \left|\sigma^\varepsilon_\eta\right| |u^\varepsilon_\tau - \mathbf{s}| = (u^\varepsilon_\tau - \mathbf{s})\sigma^\varepsilon_\tau + |\sigma^\varepsilon_\tau| \, |u^\varepsilon_\tau - \mathbf{s}| = 0,$$

whence

$$(u^\varepsilon_\tau - \mathbf{s})\sigma^\varepsilon_\tau = -\left|\sigma^\varepsilon_\tau\right| \left|u^\varepsilon_\tau - \mathbf{s}\right|.$$

So, there exist $\alpha \ge 0$ such that $u^\varepsilon_\tau - \mathbf{s} = -\alpha\sigma^\varepsilon_\tau$.
If $|\sigma^\varepsilon_\tau| < k^\varepsilon \left|\sigma^\varepsilon_\eta\right|$, then

$$\begin{aligned}0 = (u^\varepsilon_\tau - \mathbf{s})\sigma^\varepsilon_\tau + k^\varepsilon \left|\sigma^\varepsilon_\eta\right| |u^\varepsilon_\tau - \mathbf{s}| &\ge -\left|u^\varepsilon_\tau - \mathbf{s}\right| |\sigma^\varepsilon_\tau| + k^\varepsilon \left|\sigma^\varepsilon_\eta\right| \left|u^\varepsilon_\tau - \mathbf{s}\right|, \\ &\ge |u^\varepsilon_\tau - \mathbf{s}| \, (k^\varepsilon \left|\sigma^\varepsilon_\eta\right| - |\sigma^\varepsilon_\tau|),\end{aligned}$$

as $(k^\varepsilon \left|\sigma^\varepsilon_\eta\right| - |\sigma^\varepsilon_\tau|) > 0$, so

$$u^\varepsilon_\tau - \mathbf{s} = 0, \Longrightarrow u^\varepsilon_\tau = \mathbf{s}.$$

Our proof is now complete. ∎

3.2 Weak formulation

We turn to formulation of Problem $\mathcal{P}^\varepsilon$, we introduce the necessary functional spaces. Let $L^2(\Omega^\varepsilon)$ be the usual Lebesgue space with the norm denoted by $\|.\|_{L^2(\Omega^\varepsilon)}$ and $\left(H^1(\Omega^\varepsilon)\right)^3$ be the Sobolev space

$$H^1(\Omega^\varepsilon)^3 = \left\{ v \in (L^2(\Omega^\varepsilon))^3 : \frac{\partial v_i}{\partial x_j} \in L^2(\Omega^\varepsilon), \forall i,j = 1,2,3 \right\}.$$

Moreover, we need the following functional spaces

$$K^\varepsilon = \{v \in (H^1(\Omega^\varepsilon))^3 : \quad v = G^\varepsilon \text{ on } \Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon, \;\; v.\eta = 0 \text{ on } \omega\},$$

$$W^\varepsilon = \{\psi \in H^1(\Omega^\varepsilon) : \quad \psi = 0 \quad \text{on } \Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon\}.$$

Multiplying (3.4) by $(v - u^\varepsilon) \in K^\varepsilon$ and (3.5) by $\psi \in W^\varepsilon$ performing an integration by parts on Ω^ε with the use of Green's formula, the conditions (3.6)-(3.11), and Lemma 3.1.1 yield the following variational problem.

Problem$\mathcal{P}_\mathcal{V}^\varepsilon$. Find a displacement field $u^\varepsilon \in K^\varepsilon$, and electric potential $\varphi^\varepsilon \in W^\varepsilon$ such that

$$A(u^\varepsilon, v - u^\varepsilon) + A_{ele}(\varphi^\varepsilon, v - u^\varepsilon) + \widetilde{j}(u^\varepsilon, \varphi^\varepsilon, v) - \widetilde{j}(u^\varepsilon, \varphi^\varepsilon, u^\varepsilon) \geq (f^\varepsilon, v - u^\varepsilon), \quad \forall v \in K^\varepsilon, \tag{3.13}$$

$$A_{coef}(\varphi^\varepsilon, \psi) - A_{\underline{ele}}(u^\varepsilon, \psi) = (q^\varepsilon, \psi), \quad \forall \psi \in W^\varepsilon, \tag{3.14}$$

where

$$A(u^\varepsilon, v) = 2\sum_{i,j=1}^{3} \int_{\Omega^\varepsilon} \mu^\varepsilon d_{ij}(u^\varepsilon) d_{ij}(v) dx dx_3 + \int_{\Omega^\varepsilon} \lambda^\varepsilon div(u^\varepsilon) div(v) dx dx_3,$$

$$A_{ele}(\varphi^\varepsilon, v) = \int_{\Omega^\varepsilon} (\mathfrak{F}^\varepsilon)^T \nabla \varphi^\varepsilon d(v) dx dx_3,$$

$$A_{\underline{ele}}(u^\varepsilon, \psi) = \int_{\Omega^\varepsilon} (\mathfrak{F}^\varepsilon d(u^\varepsilon)) \nabla \psi dx dx_3,$$

$$A_{coef}(\varphi^\varepsilon, \psi) = \int_{\Omega^\varepsilon} \beta^\varepsilon \nabla \varphi^\varepsilon \nabla \psi dx dx_3,$$

$$\widetilde{j}(u^\varepsilon, \varphi^\varepsilon, v) = \int_\omega k^\varepsilon |\sigma_\eta^\varepsilon| |v - s| dx,$$

$$(f^\varepsilon, v) = \sum_{i=1}^{3} \int_{\Omega^\varepsilon} f_i^\varepsilon v_i dx dx_3,$$

$$(q^\varepsilon, \psi) = \int_{\Omega^\varepsilon} q^\varepsilon \psi dx dx_3.$$

The integral $\widetilde{j}(u^\varepsilon, \varphi^\varepsilon, v)$ has no meaning for $u^\varepsilon \in K^\varepsilon, \;\; v \in K^\varepsilon$. Indeed, σ_η^ε is defined by duality as an element of $H^{-\frac{1}{2}}(\omega)$. Therefore, $\left|\sigma_\eta^\varepsilon\right|$ is not well defined on ω. In the same spirit of [15, 20] we replace σ_η^ε by some regularization $R(\sigma_\eta^\varepsilon)$, where R is a regularization operator from $H^{-\frac{1}{2}}(\omega)$ into $L^2(\omega)$ defined by

$$\forall \tau \in H^{-\frac{1}{2}}(\omega),\;\; R(\tau) \in L^2(\omega),\;\; R(\tau)(x) = \langle \tau, \phi_x \rangle_{H^{-\frac{1}{2}}(\omega), H_{00}^{\frac{1}{2}}(\omega)} \quad \forall x \in \omega,$$

with $t \longrightarrow \phi_x(t) = \phi(x-t)$ is a given positive function of class C^∞ with compact support in ω and $H^{-\frac{1}{2}}(\omega)$ is the dual space of

$$H_{00}^{\frac{1}{2}}(\omega) = \{v_{|\omega} : v \in H^1(\Omega), v = 0 \text{ on } \Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon\}.$$

Remark 3.2.1 *From the definition of R, then it is an application that retains positivity and can be obtained by convolution with a positive regular function whose integral is equal to unity.*

After this regularization, we obtain the following new problem:
Problem $\mathcal{P}_\mathcal{V}^{\varepsilon,\phi}$. Find $u^\varepsilon \in K^\varepsilon$, and $\varphi^\varepsilon \in W^\varepsilon$ such that

$$A(u^\varepsilon, v - u^\varepsilon) + A_{elec}(\varphi^\varepsilon, v - u^\varepsilon) + j(u^\varepsilon, \varphi^\varepsilon, v) - j(u^\varepsilon, \varphi^\varepsilon, u^\varepsilon) \geq (f^\varepsilon, v - u^\varepsilon), \quad \forall v \in K^\varepsilon \tag{3.15}$$

$$A_{coef}(\varphi^\varepsilon, \psi) - A_{\underline{ele}}(u^\varepsilon, \psi) = (q^\varepsilon, \psi), \quad \forall \psi \in W^\varepsilon, \tag{3.16}$$

where

$$j(u^\varepsilon, \varphi^\varepsilon, v) = \int_\omega k^\varepsilon |R(\sigma_\eta^\varepsilon(u^\varepsilon, \varphi^\varepsilon))| |v - s| dx.$$

Theorem 3.2.2 *If $f^\varepsilon \in (L^2(\Omega^\varepsilon))^3$, $q^\varepsilon \in L^2(\Omega^\varepsilon)$, the friction coefficient k^ε is non negative function in $L^\infty(\omega)$ and $(H_1) - (H_3)$ hold, then there exists $(u^\varepsilon, \varphi^\varepsilon) \in K^\varepsilon \times W^\varepsilon$, solution of problem $\mathcal{P}_\mathcal{V}^{\varepsilon,\phi}$. Moreover, for small k^ε, the solution is unique.*

Proof. The proof of Theorem 3.2.2 will be carried out in several steps and it is based on arguments of variational inequalities and fixed point techniques. Then, we define the non empty closed convex set of $L^2(\omega)$ by

$$\mathcal{K}^\varepsilon = \{h \in L^2(\omega)\; h \geq 0 \text{ and } \|h\|_{L^2(\omega)} \leq k_1^\varepsilon\},$$

where k_1^ε is positive constant to be specified.
We start with the following equivalence result.

Lemma 3.2.3 *The couple $(u^\varepsilon, \varphi^\varepsilon) \in K^\varepsilon \times W^\varepsilon$ is a solution to $\mathcal{P}_\mathcal{V}^{\varepsilon,\phi}$ if and only if*

$$\begin{aligned} A(u^\varepsilon, v - u^\varepsilon) + A_{ele}(\varphi^\varepsilon, v - u^\varepsilon) + j(u^\varepsilon, \varphi^\varepsilon, v) - j(u^\varepsilon, \varphi^\varepsilon, u^\varepsilon) + A_{coef}(\varphi^\varepsilon, \psi - \varphi^\varepsilon) \\ -A_{\underline{ele}}(u^\varepsilon, \psi - \varphi^\varepsilon) \geq (f^\varepsilon, v - u^\varepsilon) + (q^\varepsilon, \psi - \varphi^\varepsilon), \;\; \forall (v, \psi) \in K^\varepsilon \times W^\varepsilon. \end{aligned} \tag{3.17}$$

Proof of Lemma 3.2.3. Let $(u^\varepsilon, \varphi^\varepsilon) \in K^\varepsilon \times W^\varepsilon$ be a solution to Problem $\mathcal{P}_{\mathcal{V}}^{\varepsilon,\phi}$ and let $(v, \psi) \in K^\varepsilon \times W^\varepsilon$. Using the test function $(\psi - \varphi^\varepsilon)$ in (3.16), add the corresponding inequality to (3.15) to obtain (3.17). Conversely, let $(u^\varepsilon, \varphi^\varepsilon) \in K^\varepsilon \times W^\varepsilon$ satisfies (3.17). For any $v \in K^\varepsilon$, we take $(v, \psi) = (v, \varphi^\varepsilon)$ in (3.17) to obtain (3.15). Then, for any $\psi \in W^\varepsilon$, we take $(v, \psi) = (u^\varepsilon, \varphi^\varepsilon + \psi)$ and $(v, \psi) = (u^\varepsilon, \varphi^\varepsilon - \psi)$, respectively in (3.17) to obtain (3.16).

Now, to prove the existence and uniqueness result of (3.17). The major point of the proof is the existence of a fixed point for the mapping defined from $L^2(\omega)$ into $L^2(\omega)$ by $h \longrightarrow -k^\varepsilon R\left(\sigma_\eta^\varepsilon\right)\left((u^\varepsilon, \varphi^\varepsilon)(h)\right)$, we consider the following intermediate Problem

$$\begin{aligned} &A(u^\varepsilon, v - u^\varepsilon) + A_{ele}(\varphi^\varepsilon, v - u^\varepsilon) + \int_\omega h(|v - s| - |u^\varepsilon - s|)dx + A_{coef}(\varphi^\varepsilon, \psi - \varphi^\varepsilon) \\ &\quad -A_{\underline{ele}}(u^\varepsilon, \psi - \varphi^\varepsilon) \geq (f^\varepsilon, v - u^\varepsilon) + (q^\varepsilon, \psi - \varphi^\varepsilon), \;\; \forall (v, \psi) \in K^\varepsilon \times W^\varepsilon. \end{aligned} \tag{3.18}$$

We put the function $j_h(u^\varepsilon, \varphi^\varepsilon, v) = \int_\omega h\,|v - s|\,dx, \;\; \forall ((u^\varepsilon, \varphi^\varepsilon) \in K^\varepsilon \times W^\varepsilon$. It is easy to see that j_h is a proper, convex and continuous function on $K^\varepsilon \times W^\varepsilon$. The existence and uniqueness result of (3.18) follow from elliptic variational inequalities (see [11]). More precisely, we apply Schauder's fixed point theorem to get the existence of a fixed point for a weakly continuous mapping T from $\mathcal{K}^\varepsilon$ into $\mathcal{K}^\varepsilon$ the proof can be found in [22].

Uniqueness: Let $(u_1^\varepsilon, \varphi_1^\varepsilon), \;\; (u_2^\varepsilon, \varphi_2^\varepsilon) \in K^\varepsilon \times W^\varepsilon$ be two different solution for (3.17). Taking $(v, \psi) = (u_2^\varepsilon, \varphi_2^\varepsilon)$ and $(v, \psi) = (u_1^\varepsilon, \varphi_1^\varepsilon)$ respectively, as test function in (3.17), we obtain

$$\begin{aligned} &A(u_1^\varepsilon, u_2^\varepsilon - u_1^\varepsilon) + A_{ele}(\varphi_1^\varepsilon, u_2^\varepsilon - u_1^\varepsilon) + j(u_1^\varepsilon, \varphi_1^\varepsilon, u_2^\varepsilon) - j(u_1^\varepsilon, \varphi_1^\varepsilon, u_1^\varepsilon) + A_{coef}(\varphi_1^\varepsilon, \varphi_2^\varepsilon - \varphi_1^\varepsilon) \\ &\qquad -A_{\underline{ele}}(u_1^\varepsilon, \varphi_2^\varepsilon - \varphi_1^\varepsilon) \geq (f^\varepsilon, u_2^\varepsilon - u_1^\varepsilon) + (q^\varepsilon, \varphi_2^\varepsilon - \varphi_1^\varepsilon), \\ &A(u_2^\varepsilon, u_1^\varepsilon - u_2^\varepsilon) + A_{ele}(\varphi_2^\varepsilon, u_1^\varepsilon - u_2^\varepsilon) + j(u_2^\varepsilon, \varphi_2^\varepsilon, u_1^\varepsilon) - j(u_2^\varepsilon, \varphi_2^\varepsilon, u_2^\varepsilon) + A_{coef}(\varphi_2^\varepsilon, \varphi_1^\varepsilon - \varphi_2^\varepsilon) \\ &\qquad -A_{\underline{ele}}(u_2^\varepsilon, \varphi_1^\varepsilon - \varphi_2^\varepsilon) \geq (f^\varepsilon, u_1^\varepsilon - u_2^\varepsilon) + (q^\varepsilon, \varphi_1^\varepsilon - \varphi_2^\varepsilon). \end{aligned}$$

Adding both inequality, we get

$$A(u_1^\varepsilon - u_2^\varepsilon, u_1^\varepsilon - u_2^\varepsilon) + A_{coef}(\varphi_1^\varepsilon - \varphi_2^\varepsilon, \varphi_1^\varepsilon - \varphi_2^\varepsilon) \leq \int_\omega k^\varepsilon\,|u_1^\varepsilon - u_2^\varepsilon|\,\left|R(\sigma_\eta^\varepsilon(u_1^\varepsilon, \varphi_1^\varepsilon) - R(\sigma_\eta^\varepsilon(u_2^\varepsilon, \varphi_2^\varepsilon))\right|\,dx. \tag{3.19}$$

Next, using Korn's inequality, there exist a Constant $C_K > 0$ independent of ε, such that

$$A(u_1^\varepsilon - u_2^\varepsilon, u_1^\varepsilon - u_2^\varepsilon) \geq \mu^* C_K \left\|\nabla\left(u_1^\varepsilon - u_2^\varepsilon\right)\right\|_{L^2(\Omega^\varepsilon)}^2,$$

and by (H_2), yields

$$A_{coef}(\varphi_1^\varepsilon - \varphi_2^\varepsilon, \varphi_1^\varepsilon - \varphi_2^\varepsilon) \geq \beta^* \left\|\nabla(\varphi_1^\varepsilon - \varphi_2^\varepsilon)\right\|_{L^2(\Omega^\varepsilon)}^2.$$

Again, using theorem trace, there exist a constant $C_0 > 0$. Then, equation (3.19) can be written as

$$\frac{min(\mu^* C_K, \beta^*)}{2}\left(\|\nabla(u_1^\varepsilon - u_2^\varepsilon)\|^2_{L^2(\Omega^\varepsilon)} + \|\nabla(\varphi_1^\varepsilon - \varphi_2^\varepsilon)\|^2_{L^2(\Omega^\varepsilon)}\right)^{\frac{1}{2}}$$
$$\leq \|k^\varepsilon\|_{L^\infty(\omega)} C_0 \left\|R(\sigma_\eta^\varepsilon(u_1^\varepsilon, \varphi_1^\varepsilon)) - R\left(\sigma_\eta^\varepsilon(u_2^\varepsilon, \varphi_2^\varepsilon)\right)\right\|_{L^2(\omega)}.$$

Denoting by C_2 the norm of the linear and continuous mapping R from $H^{\frac{1}{2}}(\omega)$ into $L^2(\omega)$, and by C_1 the one of σ_η^ε from K^ε into $H^{-\frac{1}{2}}(\omega)$ [4], we obtain

$$\left(\|\nabla(u_1^\varepsilon - u_2^\varepsilon)\|^2_{L^2(\Omega^\varepsilon)} + \|\nabla(\varphi_1^\varepsilon - \varphi_2^\varepsilon)\|^2_{L^2(\Omega^\varepsilon)}\right)^{\frac{1}{2}}$$
$$\leq \frac{2\|k^\varepsilon\|_{L^\infty(\omega)} C_0 C_1 C_2}{min(\mu^* C_K, \beta^*)}\left(\|u_1^\varepsilon - u_2^\varepsilon\|_{L^2(\omega)} + \|\varphi_1^\varepsilon - \varphi_2^\varepsilon\|_{L^2(\omega)}\right).$$

Setting: $k^* = \frac{min(\mu^* C_K, \beta^*)}{2C_0C_1C_2}$. Then if $\|k^\varepsilon\|_{L^\infty(\omega)} < k^*$, therefore $(u_1^\varepsilon, \varphi_1^\varepsilon) = (u_2^\varepsilon, \varphi_2^\varepsilon)$. ■

3.3 Change of the domain and some estimates

In this section, we study the asymptotic analysis of Problem $\mathcal{P}_{\mathcal{V}}^{\varepsilon,\phi}$. For that, we use the approach which consists in transposing the problem initially posed in the domain Ω^ε which depend on a small parameter ε in an equivalent problem posed in the fixed domain Ω which is independent of ε.

First, by the change of variable $z = \frac{x_3}{\varepsilon}$, we define the fixed domain

$$\Omega = \{(x, z) \in \mathbb{R}^3 : (x, 0) \in \omega, \ \ 0 < z < h(x)\},$$

we denote by $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_L \cup \overline{\omega}$ its boundary.

Now, we define the following functions in Ω

$$\begin{cases} \widehat{u}_i^\varepsilon(x, z) = u_i^\varepsilon(x, x_3), & i = 1, 2, \\ \widehat{u}_3^\varepsilon(x, z) = \varepsilon^{-1} u_3^\varepsilon(x, x_3), \\ \widehat{\varphi}^\varepsilon(x, z) = \varphi^\varepsilon(x, x_3). \end{cases}$$

We assume the following dependence of data on ε

$$\begin{cases} \widehat{f}(x, z) = \varepsilon^2 f^\varepsilon(x, x_3), \ \ \widehat{q}(x, z) = \varepsilon^2 q^\varepsilon(x, x_3), \\ \widehat{k} = \varepsilon^{-1} k^\varepsilon, \ \ \widehat{\beta}(x, z) = \beta^\varepsilon(x, x_3), \ \ \widehat{\lambda} = \lambda^\varepsilon, \ \ \widehat{\mu} = \mu^\varepsilon, \\ \widehat{g}(x, z) = g(x, x_3), \\ \widehat{e}_{kij}(x, z) = e_{kij}^\varepsilon(x, x_3), \ \ 1 \leq k, i, j \leq 3, \end{cases}$$

with $\widehat{f} \in (L^2(\Omega))^3$, $\widehat{q} \in L^2(\Omega)$, $\widehat{g} \in (H^{\frac{1}{2}}(\Gamma))^3$ and $\widehat{k}$, $\widehat{\beta}$, $\widehat{\mu}$, $\widehat{\lambda}$, and $\widehat{e}_{ijk}$, $(1 \le i,j,k \le 3)$, independent of ε while the first assumption means that the body forces cannot be too big. Next, we define the stress tensor $\widehat{\sigma}^{\varepsilon}_{ij}$, as follows

$$\begin{cases} \widehat{\sigma}^{\varepsilon}_{ij} = \varepsilon^2 \sigma^{\varepsilon}_{ij} & 1 \le i,j \le 2, \\ \widehat{\sigma}^{\varepsilon}_{i3} = \varepsilon \sigma^{\varepsilon}_{i3} & i = 1,2,3. \end{cases}$$

and for the vector G^{ε} introduced in Section 2 will be defined as

$$\begin{cases} \widehat{G}_i(x,z) = G^{\varepsilon}_i(x,x_3), \;\; i = 1,2, \\ \widehat{G}_3(x,z) = \varepsilon^{-1} G^{\varepsilon}_3(x,x_3). \end{cases}$$

Now, we introduce the functional frameworks on Ω

$$K = \left\{ v \in (H^1(\Omega))^3 : v = \widehat{G} \text{ on } \Gamma_1 \cup \Gamma_L, \;\; v.\eta = 0 \quad on \quad \omega \right\},$$

$$\Pi(K) = \left\{ \overline{\chi} \in (H^1(\Omega))^2 : \overline{\chi} = (\chi_1, \chi_2), \;\; \chi_i = \widehat{G}_i \quad \text{on} \quad \Gamma_1 \cup \Gamma_L \quad \text{for} \quad i = 1,2 \right\},$$

$$W = \left\{ \psi \in H^1(\Omega) : \psi = 0 \text{ on } \Gamma_1 \cup \Gamma_L \right\},$$

$$W_z = \left\{ \widehat{\psi} \in L^2(\Omega) : \frac{\partial \widehat{\psi}}{\partial z} \in L^2(\Omega), \text{ and } \widehat{\psi} = 0 \text{ on } \Gamma_1 \right\},$$

and, let

$$V_z = \left\{ \widehat{v} = (\widehat{v}_1, \widehat{v}_2) \in (L^2(\Omega))^2 : \frac{\partial \widehat{v}_i}{\partial z} \in L^2(\Omega), \text{ and } \widehat{v} = 0 \text{ on } \Gamma_1 \right\},$$

V_z is a Banach space for the norm

$$\|\widehat{v}\|^2_{V_z} = \sum_{i=1}^{2} \left(\|\widehat{v}_i\|^2_{L^2(\Omega)} + \left\| \frac{\partial \widehat{v}_i}{\partial z} \right\|^2_{L^2(\Omega)} \right).$$

Consequently, the variational problem $\mathcal{P}^{\varepsilon,\phi}_{\mathcal{V}}$ is equivalent to the following problem.

Problem $\mathcal{P}_{\mathcal{V}}$. Find $(\widehat{u}^{\varepsilon}, \widehat{\varphi}^{\varepsilon}) \in K \times W$ such that:

$$\begin{aligned} &\widehat{A}(\widehat{u}^{\varepsilon}, \widehat{v} - \widehat{u}^{\varepsilon}) + \widehat{A}_{ele}(\widehat{\varphi}^{\varepsilon}, \widehat{v} - \widehat{u}^{\varepsilon}) + \widehat{j}(\widehat{u}^{\varepsilon}, \widehat{\varphi}^{\varepsilon}, \widehat{v}) \\ &-\widehat{j}(\widehat{u}^{\varepsilon}, \widehat{\varphi}^{\varepsilon}, \widehat{u}^{\varepsilon}) \ge \sum_{i=1}^{2} (\widehat{f}_i, \widehat{v}_i - \widehat{u}^{\varepsilon}_i) + (\widehat{f}_3, \varepsilon \widehat{v}_3 - \varepsilon \widehat{u}^{\varepsilon}_3) \qquad \forall \widehat{v} \in K, \end{aligned} \tag{3.20}$$

$$\widehat{A}_{coef}(\widehat{\varphi}^{\varepsilon}, \widehat{\psi}) - \widehat{A}_{\underline{ele}}(\widehat{u}^{\varepsilon}, \widehat{\psi}) = (\widehat{q}, \widehat{\psi}) \qquad \forall \widehat{\psi} \in W, \tag{3.21}$$

where

$$\widehat{A}(\widehat{u}^\varepsilon, \widehat{v}-\widehat{u}^\varepsilon) = \sum_{i,j=1}^{2} \varepsilon^2 \int_\Omega \widehat{\mu}\left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \widehat{u}_j^\varepsilon}{\partial x_i}\right)\left(\frac{\partial(\widehat{v}_i - \widehat{u}_i^\varepsilon)}{\partial x_j}\right) dxdz$$
$$+\sum_{i=1}^{2} \int_\Omega \widehat{\mu}\left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_i}\right)\left(\frac{\partial(\widehat{v}_i - \widehat{u}_i^\varepsilon)}{\partial z} + \varepsilon^2 \frac{\partial(\widehat{v}_3 - \widehat{u}_3^\varepsilon)}{\partial x_i}\right) dxdz$$
$$+2\varepsilon^2 \int_\Omega \widehat{\mu}\frac{\partial \widehat{u}_3^\varepsilon}{\partial z}\frac{\partial(\widehat{v}_3 - \widehat{u}_3^\varepsilon)}{\partial z} dxdz + \varepsilon^2 \int_\Omega \widehat{\lambda} div(\widehat{u}^\varepsilon) div(\widehat{v}-\widehat{u}^\varepsilon) dxdz,$$

$$\widehat{A}_{ele}(\widehat{\varphi}^\varepsilon, \widehat{v}-\widehat{u}^\varepsilon) = \varepsilon^2 \sum_{i,j=1}^{2} \int_\Omega \left(\sum_{k=1}^{2} \widehat{e}_{kij} \frac{\partial \widehat{\varphi}^\varepsilon}{\partial x_k}\right) \frac{1}{2}\left(\frac{\partial(\widehat{v}_i - \widehat{u}_i^\varepsilon)}{\partial x_j} + \frac{\partial(\widehat{v}_j - \widehat{u}_j^\varepsilon)}{\partial x_i}\right) dxdz$$
$$+\varepsilon \sum_{i,j=1}^{2} \int_\Omega \left(\widehat{e}_{3ij} \frac{\partial \widehat{\varphi}^\varepsilon}{\partial z}\right) \frac{1}{2}\left(\frac{\partial(\widehat{v}_i - \widehat{u}_i^\varepsilon)}{\partial x_j} + \frac{\partial(\widehat{v}_j - \widehat{u}_j^\varepsilon)}{\partial x_i}\right) dxdz$$
$$+\sum_{i=1}^{2} \int_\Omega \left(\widehat{e}_{3i3} \frac{\partial \widehat{\varphi}^\varepsilon}{\partial z} + \varepsilon \sum_{k=1}^{2} \widehat{e}_{ki3} \frac{\partial \widehat{\varphi}^\varepsilon}{\partial x_k}\right)\left(\frac{\partial(\widehat{v}_i - \widehat{u}_i^\varepsilon)}{\partial z} + \varepsilon^2 \frac{\partial(\widehat{v}_3 - \widehat{u}_3^\varepsilon)}{\partial x_i}\right) dxdz$$
$$+\varepsilon \int_\Omega \left(\widehat{e}_{333} \frac{\partial \widehat{\varphi}^\varepsilon}{\partial z} + \varepsilon \sum_{k=1}^{2} \widehat{e}_{k33} \frac{\partial \widehat{\varphi}^\varepsilon}{\partial x_k}\right) \frac{\partial(\widehat{v}_3 - \widehat{u}_3^\varepsilon)}{\partial z} dxdz,$$

and

$$\widehat{A}_{coef}(\widehat{\varphi}^\varepsilon, \widehat{\psi}) = \varepsilon^2 \sum_{i=1}^{2} \int_\Omega \widehat{\beta}\frac{\partial \widehat{\varphi}^\varepsilon}{\partial x_i}\frac{\partial \widehat{\psi}}{\partial x_i} dxdz + \int_\Omega \widehat{\beta}\frac{\partial \widehat{\varphi}^\varepsilon}{\partial z}\frac{\partial \widehat{\psi}}{\partial z} dxdz,$$

$$\widehat{A}_{\underline{ele}}(\widehat{u}^\varepsilon, \widehat{\psi}) = \varepsilon^2 \sum_{k=1}^{2}\sum_{i,j=1}^{2} \int_\Omega \widehat{e}_{kij}\left(\frac{1}{2}\left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \widehat{u}_j^\varepsilon}{\partial x_i}\right)\right)\frac{\partial \widehat{\psi}}{\partial x_k} dxdz$$
$$+\varepsilon \sum_{i,j=1}^{2} \int_\Omega \widehat{e}_{3ij}\left(\frac{1}{2}\left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \widehat{u}_j^\varepsilon}{\partial x_i}\right)\right)\frac{\partial \widehat{\psi}}{\partial z} dxdz$$
$$+\sum_{k=1}^{2}\sum_{i=1}^{2} \int_\Omega \widehat{e}_{ki3}\left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_i}\right)\varepsilon\frac{\partial \widehat{\psi}}{\partial x_k} dxdz + \sum_{i=1}^{2} \int_\Omega \widehat{e}_{3i3}\left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_i}\right)\frac{\partial \widehat{\psi}}{\partial z} dxdz$$
$$+\varepsilon \sum_{k=1}^{2} \int_\Omega \widehat{e}_{k33}\frac{\partial \widehat{u}_3^\varepsilon}{\partial z}\frac{\partial \widehat{\psi}}{\partial x_k} dxdz + \varepsilon \int_\Omega \widehat{e}_{333}\frac{\partial \widehat{u}_3^\varepsilon}{\partial z}\frac{\partial \widehat{\psi}}{\partial z} dxdz,$$

and

$$\widehat{j}(\widehat{u}^\varepsilon, \widehat{\varphi}^\varepsilon, \widehat{v}) = \int_\omega \widehat{k}|R(\widehat{\sigma}_\eta^\varepsilon(\widehat{u}^\varepsilon, \widehat{\varphi}^\varepsilon))||\widehat{v} - s| dx.$$

Now, we establish the a priori estimates for the displacement $\widehat{u}^\varepsilon$ and the electric potential $\widehat{\varphi}^\varepsilon$. For that, we have the following identity.

Lemma 3.3.1 (Poincaré inequality) *[9] Recall that* $0 \le h(x) \le h_M,\ \forall x \in \omega$. *We have* *the following inequality*

$$\|\widehat{u}_i^\varepsilon\|_{L^2(\Omega)} \leq h_M \left\| \frac{\partial \widehat{u}_i^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}.$$

Lemma 3.3.2 (Korn inequality) *[39] For all $v \in K^\varepsilon$. We have the inequality*

$$C_K \|\nabla v\|^2_{L^2(\Omega^\varepsilon)} \leq \|d(v)\|_{L^2(\Omega^\varepsilon)},$$

where C_K is constant independent of ε and v.

Theorem 3.3.3 *Assuming that $(\widehat{f}, \widehat{q}) \in (L^2(\Omega))^3 \times L^2(\Omega)$, then there exists a constant C independent of ε such that*

$$\varepsilon^2 \sum_{i,j=1}^{2} \left\| \frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j} \right\|^2_{L^2(\Omega)} + \sum_{i=1}^{2} \left\| \frac{\partial \widehat{u}_i^\varepsilon}{\partial z} \right\|^2_{L^2(\Omega)} + \sum_{i=1}^{2} \varepsilon^4 \left\| \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_i} \right\|^2_{L^2(\Omega)} + \varepsilon^2 \left\| \frac{\partial \widehat{u}_3^\varepsilon}{\partial z} \right\|^2_{L^2(\Omega)} \leq C, \tag{3.22}$$

$$\sum_{i=1}^{2} \varepsilon^2 \left\| \frac{\partial \widehat{\varphi}^\varepsilon}{\partial x_i} \right\|^2_{L^2(\Omega)} + \left\| \frac{\partial \widehat{\varphi}^\varepsilon}{\partial z} \right\|^2_{L^2(\Omega)} \leq C. \tag{3.23}$$

Proof of Theorem 3.3.3. Choosing $(v, \psi) = (G^\varepsilon, 0)$ in (3.17) and using (3.3), we obtain

$$A(u^\varepsilon, u^\varepsilon - G^\varepsilon) + A_{coef}(\varphi^\varepsilon, \varphi^\varepsilon) - A_{ele}(\varphi^\varepsilon, G^\varepsilon) + j(u^\varepsilon, \varphi^\varepsilon, u^\varepsilon) - j(u^\varepsilon, \varphi^\varepsilon, G^\varepsilon) \leq (f^\varepsilon, u^\varepsilon - G^\varepsilon) + (q^\varepsilon, \varphi^\varepsilon).$$

As $G^\varepsilon = s$ on ω, and $j(u^\varepsilon, \varphi^\varepsilon, u^\varepsilon) \geq 0$, we have

$$A(u^\varepsilon, u^\varepsilon) + A_{coef}(\varphi^\varepsilon, \varphi^\varepsilon) \leq A(u^\varepsilon, G^\varepsilon) + A_{ele}(\varphi^\varepsilon, G^\varepsilon) + (f^\varepsilon, u^\varepsilon - G^\varepsilon) + (q^\varepsilon, \varphi^\varepsilon). \tag{3.24}$$

Now, using Korn's inequality, there exists a constant $C_K > 0$ independent of ε, such that

$$A(u^\varepsilon, u^\varepsilon) \geq \mu^* C_K \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)}. \tag{3.25}$$

By (H_2), we get

$$A_{coef}(\varphi^\varepsilon, \varphi^\varepsilon) \geq \beta^* \|\nabla \varphi^\varepsilon\|^2_{L^2(\Omega^\varepsilon)}. \tag{3.26}$$

Similarly, applying Hölder's and Young's inequalities, we get

$$A(u^\varepsilon, G^\varepsilon) \leq \frac{\mu^* C_K}{4} \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + \left(\frac{8\|\mu^\varepsilon\|^2_{L^\infty(\Omega^\varepsilon)}}{\mu^* C_K} + \frac{6\lambda_*^2}{\mu^* C_K} \right) \|\nabla G^\varepsilon\|^2_{L^2(\Omega^\varepsilon)}, \tag{3.27}$$

$$A_{ele}(\varphi^\varepsilon, G^\varepsilon) \leq \frac{e_*^2}{\beta^*} \|\nabla G^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + \frac{\beta^*}{4} \|\nabla \varphi^\varepsilon\|^2_{L^2(\Omega^\varepsilon)}. \tag{3.28}$$

Using Poincaré's inequality

$$\|u^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq \varepsilon h_M \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}, \quad h_M = \sup_{x\in\omega} h(x).$$

Thus, we can write

$$\left|\int_{\Omega^\varepsilon} f^\varepsilon (u^\varepsilon - G^\varepsilon) dx dx_3\right| \leq 2\frac{(\varepsilon h_M)^2}{\mu^* C_K} \|f^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + \frac{\mu^* C_K}{4} \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + \frac{\mu^* C_K}{4} \|\nabla G^\varepsilon\|^2_{L^2(\Omega^\varepsilon)}, \tag{3.29}$$

$$\left|\int_{\Omega^\varepsilon} q^\varepsilon \varphi^\varepsilon dx dx_3\right| \leq \frac{(\varepsilon h_M)^2}{\beta^*} \|q^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + \frac{\beta^*}{4} \|\nabla \varphi^\varepsilon\|^2_{L^2(\Omega^\varepsilon)}. \tag{3.30}$$

Next, using (3.25)-(3.30) in the variational inequality (3.24), we find

$$\left(\mu^* C_K \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + \beta^* \|\nabla \varphi^\varepsilon\|^2_{L^2(\Omega^\varepsilon)}\right) \leq \left(\frac{16\,\|\mu^\varepsilon\|^2_{L^\infty(\Omega^\varepsilon)}}{\mu^* C_K} + \frac{12\lambda_*^2}{\mu^* C_K} + \frac{\mu^* C_K}{2} + 2\frac{e_*^2}{\beta^*}\right) \|\nabla G^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + 2\frac{(\varepsilon h_M)^2}{\beta^*} \|q^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + 4\frac{(\varepsilon h_M)^2}{\mu^* C_K} \|f^\varepsilon\|^2_{L^2(\Omega^\varepsilon)}. \tag{3.31}$$

Since $\varepsilon^2 \|f^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} = \varepsilon^{-1} \left\|\widehat{f}\right\|^2_{L^2(\Omega)}$ and $\varepsilon^2 \|q^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} = \varepsilon^{-1} \|\widehat{q}\|^2_{L^2(\Omega)}$, multiplying (3.31) by ε, we get

$$\min(\mu^* C_K, \beta^*) \left(\varepsilon \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + \varepsilon \|\nabla \varphi^\varepsilon\|^2_{L^2(\Omega^\varepsilon)}\right) \leq C, \tag{3.32}$$

where C does not depend of ε, with

$$C = \left(\frac{16\,\|\widehat{\mu}\|^2_{L^\infty(\Omega)}}{\mu^* C_K} + \frac{12\lambda_*^2}{\mu^* C_K} + \frac{\mu^* C_K}{2} + 2\frac{e_*^2}{\beta^*}\right) \left\|\nabla \widehat{G}\right\|^2_{L^2(\Omega)} + 2\frac{h_M^2}{\beta^*} \|\widehat{q}\|^2_{L^2(\Omega)} + 4\frac{h_M^2}{\mu^* C_K} \left\|\widehat{f}\right\|^2_{L^2(\Omega)},$$

and,

$$\varepsilon \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} = \varepsilon^2 \sum_{i,j=1}^{2} \left\|\frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j}\right\|^2_{L^2(\Omega)} + \sum_{i=1}^{2} \left\|\frac{\partial \widehat{u}_i^\varepsilon}{\partial z}\right\|^2_{L^2(\Omega)} + \sum_{i=1}^{2} \varepsilon^4 \left\|\frac{\partial \widehat{u}_3^\varepsilon}{\partial x_i}\right\|^2_{L^2(\Omega)} + \varepsilon^2 \left\|\frac{\partial \widehat{u}_3^\varepsilon}{\partial z}\right\|^2_{L^2(\Omega)}$$

$$\varepsilon \|\nabla \varphi^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} = \sum_{i=1}^{2} \varepsilon^2 \left\|\frac{\partial \widehat{\varphi}^\varepsilon}{\partial x_i}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial \widehat{\varphi}^\varepsilon}{\partial z}\right\|^2_{L^2(\Omega)}.$$

■

3.4 A convergence results

Theorem 3.4.1 *Under the assumptions of Theorem 3.3.3, there exists* $u^* = (u_1^*, u_2^*)$ *in* V_z *and* φ^* *in* W_z*, such that*

$$\widehat{u}_i^\varepsilon \rightharpoonup u^* \ (1 \le i \le 2) \quad \textit{weakly in} \quad V_z. \tag{3.33}$$

$$\widehat{\varphi}^\varepsilon \rightharpoonup \varphi^* \quad \textit{weakly in} \quad W_z. \tag{3.34}$$

$$\left.\begin{array}{l} \varepsilon \frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j} \rightharpoonup 0 \ (i,j=1,2) \quad \textit{weakly in } L^2(\Omega), \\ \varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_i} \rightharpoonup 0 \ (1 \le i \le 2) \quad \textit{weakly in } L^2(\Omega), \\ \varepsilon \frac{\partial \widehat{u}_3^\varepsilon}{\partial z} \rightharpoonup 0 \quad \textit{weakly in } L^2(\Omega). \end{array}\right\} \tag{3.35}$$

$$\varepsilon \frac{\partial \widehat{\varphi}^\varepsilon}{\partial x_i} \rightharpoonup 0 \ (1 \le i \le 2) \quad \textit{weakly in } L^2(\Omega). \tag{3.36}$$

Proof of Theorem 3.4.1. According to the Theorem 3.3.3 there exists a constant C independent of ε such that

$$\sum_{i=1}^{2} \left\| \frac{\partial \widehat{u}_i^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 \le C \quad i = 1, 2.$$

Using this estimates and Poincaré's inequality, yields

$$\|\widehat{u}_i^\varepsilon\|_{L^2(\Omega)} \le h_M \left\| \frac{\partial \widehat{u}_i^\varepsilon}{\partial z} \right\|_{L^2(\Omega)} \quad i = 1, 2,$$

we deduce that $\widehat{u}_i^\varepsilon$ is bounded in V_z, and as this space is reflexive we obtain the weak convergence result (3.33) in V_z. Similarly, from (3.24) and Poincaré's inequality in the domain Ω, we deduce that $\widehat{\varphi}^\varepsilon$ is bounded in W_z, this implies the existence of φ^* in W_z such that $\widehat{\varphi}^\varepsilon$ convergences weakly to φ^* in W_z.
Also, the convergences (3.35)-(3.36) result from (3.22)-(3.23). ■

To be able to pass to the limit in the Problem $\mathcal{P}_V$, we must prove the convergence of the integral term defined on ω.
For our case, we adopt the following lemma of ([4]).

Lemma 3.4.2 *There exists a subsequence of* $R(\widehat{\sigma}_\eta(\widehat{u}^\varepsilon, \widehat{\varphi}^\varepsilon))$ *converging strongly towards* $R\left(\widehat{e}_{333} \frac{\partial \varphi^*}{\partial z}\right)$ *in* $L^2(\omega)$.

Proof of Lemma 3.4.2. From the equilibrium equation (3.4), we have

$$-div(\sigma^\varepsilon(u^\varepsilon,\varphi^\varepsilon)) = f^\varepsilon \text{ in } \Omega^\varepsilon,$$

with $f^\varepsilon \in (L^2(\Omega^\varepsilon))^3$. According to the bounds of $(\widehat{u}^\varepsilon,\widehat{\varphi}^\varepsilon)$ in $V_z \times W$ given in the Proof of Theorems 3.4.1, we deduce that $\widehat{\sigma}^\varepsilon$ is bounded in

$$H_{div}(\Omega) = \{\tau = (\tau_{ij}) \in (L^2(\Omega))^9, \text{ and } \ div(\tau) \in (L^2(\Omega))^3\},$$

hence, there exists a subsequence converging weakly towards σ^*. Using the continuity of the trace operator from $H_{div}(\Omega)$ into $H^{-\frac{1}{2}}(\omega)$ (see [26]) $\widehat{\sigma}_\eta(\widehat{u}^\varepsilon,\widehat{\varphi}^\varepsilon)$ weakly converges towards $(\widehat{e}_{333})\frac{\partial\varphi^*}{\partial z}$ in $H^{-\frac{1}{2}}(\omega)$ for almost all x in ω, and

$$\langle\widehat{\sigma}_\eta(\widehat{u}^\varepsilon,\widehat{\varphi}^\varepsilon),\phi(x-t)\rangle_{H^{-\frac{1}{2}}(\omega),H^{\frac{1}{2}}_{00}(\omega)} \text{converges towards } \left\langle \widehat{e}_{333}\frac{\partial\varphi^*}{\partial z},\phi(x-t)\right\rangle_{H^{-\frac{1}{2}}(\omega),H^{\frac{1}{2}}_{00}(\omega)}.$$

Which implies by definition of R that $R(\widehat{\sigma}_\eta(\widehat{u}^\varepsilon,\widehat{\varphi}^\varepsilon))$ converges almost everywhere towards $R\Big(\widehat{e}_{333}\frac{\partial\varphi^*}{\partial z}\Big)$ on ω. On the other hand, using the Cauchy−Schwartz inequality and the fact that the support of ϕ is compact in ω, we obtain

$$\begin{aligned}|R(\widehat{\sigma}_\eta(\widehat{u}^\varepsilon,\widehat{\varphi}^\varepsilon)(x))| &= \left|\langle\widehat{\sigma}_\eta(\widehat{u}^\varepsilon,\widehat{\varphi}^\varepsilon),\phi(x-t)\rangle_{H^{-\frac{1}{2}}(\omega),H^{\frac{1}{2}}_{00}(\omega)}\right| \\ &\leq \|\widehat{\sigma}_\eta(\widehat{u}^\varepsilon,\widehat{\varphi}^\varepsilon)\|_{H^{-\frac{1}{2}}(\omega)}\|\phi\|_{H^{\frac{1}{2}}_{00}(\omega)} \quad \forall x\in\omega.\end{aligned}$$

As $\widehat{\sigma}^\varepsilon_\eta$ is bounded in $H^{-\frac{1}{2}}(\omega)$, we deduce the existence of a constant $C>0$ independent ε and x such that:

$$\big|R(\widehat{\sigma}^\varepsilon_\eta)(x)\big| \leq C\|\phi\|_{H^{\frac{1}{2}}_{00}(\omega)} \quad \forall x\in\omega. \tag{3.37}$$

Applying Lebesgue's convergence dominated theorem, we get that $R(\widehat{\sigma}_\eta(\widehat{u}^\varepsilon,\widehat{\varphi}^\varepsilon))$ strongly converges towards $R\big(\widehat{e}_{333}\frac{\partial\varphi^*}{\partial z}\big)$ in $L^1(\omega)$. Now, we shall prove the convergence with the L^2-norm. From (3.37), and the almost everywhere convergence of $\big|R\big(\widehat{\sigma}_\eta(\widehat{u}^\varepsilon,\widehat{\varphi}^\varepsilon)\big)\big|$ towards $\big|R\left(\widehat{e}_{333}\frac{\partial\varphi^*}{\partial z}\right)\big|$ on ω, there exists a constant $C>0$ such that

$$\left|R\Big(\widehat{e}_{333}\frac{\partial\varphi^*}{\partial z}\Big)\right| \leq C, \quad \forall x\in\omega.$$

So, from (3.37), we get the existence of a constant $C>0$ depending only of ϕ, such that

$$\left\|R(\widehat{\sigma}_\eta(\widehat{u}^\varepsilon,\widehat{\varphi}^\varepsilon)) - R\Big(\widehat{e}_{333}\frac{\partial\varphi^*}{\partial z}\Big)\right\|^2_{L^2(\omega)} \leq C\left\|R(\widehat{\sigma}_\eta(\widehat{u}^\varepsilon,\widehat{\varphi}^\varepsilon)) - R\Big(\widehat{e}_{333}\frac{\partial\varphi^*}{\partial z}\Big)\right\|_{L^1(\omega)}.$$

Due to the L^1 strong convergence, we get the desired result. ■

Lemma 3.4.3 *Let $\mathcal{H} = \{u^* = (u_1^*, u_2^*) : u^*$ is a weak limit of $\widehat{u}^\varepsilon$ in the topology of $V_z\}$. So $\mathcal{H}$ is contained in the closure of $\Pi(K)$ in the norm of V_z.*

The proof of this Lemma can be found in [10].

Theorem 3.4.4 *$\widehat{u}_i^\varepsilon \longrightarrow u_i^*$ strongly in V_z for $i = 1, 2$. Moreover with the same assumptions of Theorem 3.4.1, (u^*, φ^*) satisfies*

$$\begin{aligned} &\sum_{i=1}^{2} \int_\Omega \widehat{\mu} \frac{\partial u_i{}^*}{\partial z} \frac{\partial}{\partial z} (\widehat{v}_i - u_i{}^*)\, dxdz + \sum_{i=1}^{2} \int_\Omega \widehat{e}_{3i3} \frac{\partial \varphi^*}{\partial z} \frac{\partial}{\partial z} (\widehat{v}_i - u_i{}^*)\, dxdz \\ &+ \int_\omega \widehat{k} |R\big(\widehat{e}_{333} \frac{\partial \varphi^*}{\partial z}\big)| \left(|\widehat{v} - s| - |u^* - s|\right) dx \geq \sum_{i=1}^{2} \int_\Omega \widehat{f}_i (\widehat{v}_i - u_i^*) dxdz \ \ \forall \widehat{v} \in \Pi(K), \end{aligned} \tag{3.38}$$

$$\int_\Omega \widehat{\beta} \frac{\partial \varphi^*}{\partial z} \frac{\partial \widehat{\psi}}{\partial z} dxdz - \int_\Omega \left(\widehat{e}_{3i3} \frac{\partial u^*}{\partial z} \right) \left(\frac{\partial \widehat{\psi}}{\partial z} \right) dxdz = \int_\Omega \widehat{q} \widehat{\psi} dxdz \ \ \forall \widehat{\psi} \in W, \tag{3.39}$$

and, the limit problem

$$-\sum_{i=1}^{2} \frac{\partial}{\partial z} \left(\widehat{\mu} \frac{\partial u_i^*}{\partial z} \right) - \sum_{i=1}^{2} \frac{\partial}{\partial z} \left(\widehat{e}_{3i3} \left(\frac{\partial \varphi^*}{\partial z} \right) \right) = \widehat{f}_i \ \ (i = 1, 2) \quad in\ L^2(\Omega), \tag{3.40}$$

$$-\frac{\partial}{\partial z} \left(\widehat{\beta} \frac{\partial \varphi^*}{\partial z} \right) + \sum_{i=1}^{2} \frac{\partial}{\partial z} \left(\widehat{e}_{3i3} \frac{\partial u_i^*}{\partial z} \right) = \widehat{q} \ \ (i = 1, 2) \quad in\ L^2(\Omega). \tag{3.41}$$

Proof of Theorem 3.4.4. We use the same idea in [9, Proof of Theorem 4.2], for the proof of strong convergence of $(\widehat{u}_1^\varepsilon, \widehat{u}_2^\varepsilon)$ to (u_1^*, u_2^*). Indeed, for $(u^\varepsilon, \phi) \in K^\varepsilon \times K^\varepsilon$, and $(\varphi^\varepsilon, \psi) \in W^\varepsilon \times W^\varepsilon$, from inequality (3.17), we obtain

$$\begin{aligned} A(u^\varepsilon - \phi, u^\varepsilon - \phi) + A_{coef}(\varphi^\varepsilon - \psi, \varphi^\varepsilon - \psi) &\leq A_{ele}(\varphi^\varepsilon, \phi - u^\varepsilon) \\ -A_{\underline{ele}}(u^\varepsilon, \psi - \varphi^\varepsilon) &+ \int_{\Omega^\varepsilon} f^\varepsilon (u^\varepsilon - \phi) dx' dx_3 + (q^\varepsilon, \varphi^\varepsilon - \psi) \\ +A(\phi, \phi - u^\varepsilon) + A_{coef}(\psi, \psi - \varphi^\varepsilon) &+ j(u^\varepsilon, \varphi^\varepsilon, \phi) - j(u^\varepsilon, \varphi^\varepsilon, u^\varepsilon). \end{aligned}$$

Using (3.3), we get

$$\begin{aligned} A(u^\varepsilon - \phi, u^\varepsilon - \phi) + A_{coef}(\varphi^\varepsilon - \psi, \varphi^\varepsilon - \psi) \leq A_{ele}(\varphi^\varepsilon, \phi) - A_{\underline{ele}}(u^\varepsilon, \psi) + \int_{\Omega^\varepsilon} f^\varepsilon (u^\varepsilon - \phi) dx dx_3 \\ + \int_{\Omega^\varepsilon} q^\varepsilon (\varphi^\varepsilon - \psi) dx dx_3 + A(\phi, \phi - u^\varepsilon) + A_{coef}(\psi, \psi - \varphi^\varepsilon) + j(u^\varepsilon, \varphi^\varepsilon, \phi) - j(u^\varepsilon, \varphi^\varepsilon, u^\varepsilon). \end{aligned}$$

The use of Korn's inequality and (H_2), yields

$$\begin{aligned}\mu^* C_K \left\|\nabla(u^\varepsilon - \phi)\right\|^2_{L^2(\Omega^\varepsilon)} + \beta^* \left\|\nabla(\varphi^\varepsilon - \psi)\right\|^2_{L^2(\Omega^\varepsilon)} &\leq A_{ele}(\varphi^\varepsilon, \phi) - A_{\underline{ele}}(u^\varepsilon, \psi) \\ &+ \int_{\Omega^\varepsilon} f^\varepsilon (u^\varepsilon - \phi) dx dx_3 + \int_{\Omega^\varepsilon} q^\varepsilon (\varphi^\varepsilon - \psi) dx dx_3 + A(\phi, \phi - u^\varepsilon) \\ &+ A_{coef}(\psi, \psi - \varphi^\varepsilon) + j(u^\varepsilon, \varphi^\varepsilon, \phi) - j(u^\varepsilon, \varphi^\varepsilon, u^\varepsilon).\end{aligned} \tag{3.42}$$

Now, writing (3.42) in new variables, we get

$$\begin{aligned}2\mu^* C_K &\left(\left\| \frac{\partial}{\partial z}(\widehat{u}_1^\varepsilon - \widehat{\phi}_1) \right\|^2_{L^2(\Omega)} + \left\| \frac{\partial}{\partial z}(\widehat{u}_2^\varepsilon - \widehat{\phi}_2) \right\|^2_{L^2(\Omega)} \right) + \beta^* \left\| \frac{\partial}{\partial z}(\widehat{\varphi}^\varepsilon - \widehat{\psi}) \right\|^2_{L^2(\Omega)} \leq \widehat{A}_{ele}(\widehat{\varphi}^\varepsilon, \widehat{\phi}) \\ &- \widehat{A}_{\underline{ele}}(\widehat{u}^\varepsilon, \widehat{\psi}) + \sum_{i=1}^{2} \int_\Omega \widehat{f}_i (\widehat{u}_i^\varepsilon - \widehat{\phi}_i) dx dz + (\widehat{f}_3, \varepsilon(\widehat{u}_3^\varepsilon - \widehat{\phi}_3)) + \int_\Omega \widehat{q}^\varepsilon (\widehat{\varphi}^\varepsilon - \widehat{\psi}) dx dz \\ &+ \widehat{A}(\widehat{\phi}, \widehat{\phi} - \widehat{u}^\varepsilon) + \widehat{A}_{coef}(\widehat{\psi}, \widehat{\psi} - \widehat{\varphi}^\varepsilon) + \widehat{j}(\widehat{u}^\varepsilon, \widehat{\varphi}^\varepsilon, \widehat{\phi}) - \widehat{j}(\widehat{u}^\varepsilon, \widehat{\varphi}^\varepsilon, \widehat{u}^\varepsilon).\end{aligned}$$

Let $\overline{u}^\varepsilon = (\widehat{u}_1^\varepsilon, \widehat{u}_2^\varepsilon)$, $u^* = (u_1^*, u_2^*)$ and φ^* as in Theorem 3.4.1, let $\overline{\phi} = (\widehat{\phi}_1, \widehat{\phi}_2)$, $\overline{\phi} \in \Pi(K)$ and $\widehat{\psi} \in W$, we have

$$\begin{aligned}\lim_{\varepsilon \longrightarrow 0} \sup &\left\{ 2\mu^* C_K \left\|\overline{u}^\varepsilon - \overline{\phi}\right\|^2_{V_z} + \beta^* \left\|\widehat{\varphi}^\varepsilon - \widehat{\psi}\right\|^2_{W_z} + \widehat{j}(\overline{u}^\varepsilon, \widehat{\varphi}^\varepsilon, \overline{u}^\varepsilon) - \widehat{j}(\overline{u}^\varepsilon, \widehat{\varphi}^\varepsilon, \overline{\phi}) \right\} \\ &\leq \sum_{i=1}^{2} (\widehat{f}_i, u_i^* - \widehat{\phi}_i) + (\widehat{q}, \varphi^* - \widehat{\psi}) + \sum_{i=1}^{2} \int_\Omega \widehat{\mu} \frac{\partial \widehat{\phi}_i}{\partial z} \frac{\partial}{\partial z}(\widehat{\phi}_i - u_i^*) dx dz \\ &+ \int_\Omega \widehat{\beta} \frac{\partial \widehat{\psi}}{\partial z} \frac{\partial}{\partial z}(\widehat{\psi} - \varphi^*) dx dz + \sum_{i=1}^{2} \int_\Omega \widehat{e}_{3i3} \frac{\partial \varphi^*}{\partial z} \frac{\partial \widehat{\phi}_i}{\partial z} dx dz + \sum_{i=1}^{2} \int_\Omega \widehat{e}_{3i3} \frac{\partial u_i^*}{\partial z} \frac{\partial \widehat{\psi}}{\partial z} dx dz.\end{aligned}$$

Then

$$\begin{aligned}2\mu^* C_K &\left\|\overline{u}^\varepsilon - \overline{\phi}\right\|^2_{V_z} + \beta^* \left\|\widehat{\varphi}^\varepsilon - \widehat{\psi}\right\|^2_{W_z} + \widehat{j}(\overline{u}^\varepsilon, \widehat{\varphi}^\varepsilon, \overline{u}^\varepsilon) - \widehat{j}(\overline{u}^\varepsilon, \widehat{\varphi}^\varepsilon, \overline{\phi}) \\ &\leq \sum_{i=1}^{2} (\widehat{f}_i, u_i^* - \widehat{\phi}_i) + (\widehat{q}, \varphi^* - \widehat{\psi}) + \int_\Omega \widehat{\mu} \frac{\partial \overline{\phi}}{\partial z} \frac{\partial}{\partial z}(\overline{\phi} - u^*) dx dz + \int_\Omega \widehat{\beta} \frac{\partial \widehat{\psi}}{\partial z} \frac{\partial}{\partial z}(\widehat{\psi} - \varphi^*) dx dz \\ &+ \sum_{i=1}^{2} \int_\Omega \widehat{e}_{3i3} \frac{\partial \varphi^*}{\partial z} \frac{\partial \widehat{\phi}_i}{\partial z} dx dz - \sum_{i=1}^{2} \int_\Omega \widehat{e}_{3i3} \frac{\partial u_i^*}{\partial z} \frac{\partial \widehat{\psi}}{\partial z} dx dz + \delta, \ \forall \varepsilon < \varepsilon(\delta),\end{aligned}$$

where $\delta > 0$ is arbitrary.

Thanks to Lemma 3.4.3, there exists a sequence of function $\overline{\phi} \in \Pi(K)$ which has u^* as a limit in V_z, hence

$$\begin{aligned}2\mu^* C_K &\left\|\overline{u}^\varepsilon - u^*\right\|^2_{V_z} + \beta^* \left\|\widehat{\varphi}^\varepsilon - \widehat{\psi}\right\|^2_{W_z} + \widehat{j}(\overline{u}^\varepsilon, \widehat{\varphi}^\varepsilon, \overline{u}^\varepsilon) - \widehat{j}(u^*, \varphi^*, u^*) \leq (\widehat{q}, \varphi^* - \widehat{\psi}) \\ &+ \int_\Omega \widehat{\beta} \frac{\partial \widehat{\psi}}{\partial z} \frac{\partial}{\partial z}(\widehat{\psi} - \varphi^*) dx dz + \sum_{i=1}^{2} \int_\Omega \widehat{e}_{3i3} \frac{\partial \varphi^*}{\partial z} \frac{\partial u_i^*}{\partial z} dx dz - \sum_{i=1}^{2} \int_\Omega \widehat{e}_{3i3} \frac{\partial u_i^*}{\partial z} \frac{\partial \widehat{\psi}}{\partial z} dx dz + \delta, \ \forall \varepsilon < \varepsilon(\delta).\end{aligned}$$

Next, by density, there exists a sequence of function $\widehat{\psi} \in W$ which has φ^* as a limit in W_z, whence

$$2\mu^* C_K \left\| \overline{u}^\varepsilon - u^* \right\|_{V_z}^2 + \beta^* \left\| \widehat{\varphi}^\varepsilon - \varphi^* \right\|_{W_z}^2 + \widehat{j}(\overline{u}^\varepsilon, \widehat{\varphi}^\varepsilon, \overline{u}^\varepsilon) - \widehat{j}(u^*, \varphi^*, u^*) \leq \delta, \ \ \forall \varepsilon < \varepsilon(\delta).$$

Thus

$$2\mu^* C_K \left\| \overline{u}^\varepsilon - u^* \right\|_{V_z}^2 + \widehat{j}(\overline{u}^\varepsilon, \widehat{\varphi}^\varepsilon, \overline{u}^\varepsilon) - \widehat{j}(u^*, \varphi^*, u^*) \leq \delta, \ \ \forall \varepsilon < \varepsilon(\delta).$$

Now, by weak lower semi-continuity of the functional $\widehat{j}$, as well as $\widehat{j}(\overline{u}^\varepsilon, \widehat{\varphi}^\varepsilon, \overline{u}^\varepsilon) \longrightarrow \widehat{j}(u^*, \varphi^*, u^*)$ $(\varepsilon \longrightarrow 0)$, we obtain the strong convergence of $\overline{u}^\varepsilon$ to u^* in V_z.

Using the convergence results of Theorem 3.4.1 and the lower semi-continuity property in $L^2(\Omega)$, we obtain as $\varepsilon \longrightarrow 0$,

$$\begin{gathered}
\sum_{i=1}^{2} \int_\Omega \widehat{\mu} \frac{\partial u_i^*}{\partial z} \frac{\partial}{\partial z} \left(\widehat{v}_i - u_i^* \right) dxdz + \sum_{i=1}^{2} \int_\Omega \widehat{e}_{3i3} \frac{\partial \varphi^*}{\partial z} \frac{\partial}{\partial z} \left(\widehat{v}_i - u_i^* \right) dxdz \\
+ \int_\omega \widehat{k} |R \left(\widehat{e}_{333} \frac{\partial \varphi^*}{\partial z} \right) | \left(|\widehat{v} - s| - |u^* - s| \right) dx \geq \sum_{i=1}^{2} \int_\Omega \widehat{f}_i (\widehat{v}_i - u_i^*) dxdz, \\
\int_\Omega \widehat{\beta} \frac{\partial \varphi^*}{\partial z} \frac{\partial \widehat{\psi}}{\partial z} dxdz - \sum_{i=1}^{2} \int_\Omega \left(\widehat{e}_{3i3} \frac{\partial u_i^*}{\partial z} \right) \left(\frac{\partial \widehat{\psi}}{\partial z} \right) dxdz = \int_\Omega \widehat{q} \widehat{\psi} dxdz.
\end{gathered}$$

Now, we choose $\widehat{v}_i = u_i^* \pm \chi_i, \ \ i = 1, 2$, with $\chi_i \in H_0^1(\Omega)$, we find

$$\sum_{i=1}^{2} \int_\Omega \widehat{\mu} \frac{\partial u_i^*}{\partial z} \frac{\partial \chi_i}{\partial z} dxdz + \sum_{i=1}^{2} \int_\Omega \widehat{e}_{3i3} \frac{\partial \varphi^*}{\partial z} \frac{\partial \chi_i}{\partial z} dxdz = \sum_{i=1}^{2} \int_\Omega \widehat{f}_i \chi_i dxdz.$$

Using Green's formula, we deduce first with $\chi_1 = 0$ and $\chi_2 \in H_0^1(\Omega)$, then with $\chi_2 = 0$ and $\chi_1 \in H_0^1(\Omega)$, the following equalities

$$-\sum_{i=1}^{2} \int_\Omega \frac{\partial}{\partial z} \left(\widehat{\mu} \frac{\partial u_i^*}{\partial z} + \widehat{e}_{3i3} \frac{\partial \varphi^*}{\partial z} \right) \chi_i dxdz = \sum_{i=1}^{2} (\widehat{f}_i, \chi_i), \quad \forall \chi_i \in H_0^1(\Omega).$$

And

$$\begin{gathered}
-\int_\Omega \frac{\partial}{\partial z} \left(\widehat{\beta} \frac{\partial \varphi^*}{\partial z} \right) \widehat{\psi} dxdz + \sum_{i=1}^{2} \int_\Omega \frac{\partial}{\partial z} \left(\widehat{e}_{3i3} \frac{\partial u_i^*}{\partial z} \right) \widehat{\psi} dxdz \\
+ \int_\omega \left(\widehat{\beta} \frac{\partial \varphi^*}{\partial z} - \sum_{i=1}^{2} \widehat{e}_{3i3} \frac{\partial u_i^*}{\partial z} \right) \eta_3 \widehat{\psi} dx = \int_\Omega \widehat{q} \widehat{\psi} dxdz, \ \ \forall \widehat{\psi} \in W.
\end{gathered}$$

From (3.11), we have $\left(\widehat{\beta}\frac{\partial \varphi^*}{\partial z} - \sum_{i=1}^{2} \widehat{e}_{3i3}\frac{\partial u_i^*}{\partial z}\right)\eta_3 = 0$, which gives

$$-\sum_{i=1}^{2}\int_\Omega \frac{\partial}{\partial z}\left(\widehat{\mu}\frac{\partial u_i^*}{\partial z} + \widehat{e}_{3i3}\frac{\partial \varphi^*}{\partial z}\right)\chi_i dxdz = \sum_{i=1}^{2}(\widehat{f}_i, \chi_i) \quad \forall \chi_i \in H_0^1(\Omega),$$

$$-\int_\Omega \frac{\partial}{\partial z}\left(\widehat{\beta}\frac{\partial \varphi^*}{\partial z}\right)\widehat{\psi} dxdz + \sum_{i=1}^{2}\int_\Omega \frac{\partial}{\partial z}\left(\widehat{e}_{3i3}\frac{\partial u_i^*}{\partial z}\right)\widehat{\psi} dxdz = \int_\Omega \widehat{q}\widehat{\psi} dxdz, \ \forall \widehat{\psi} \in W.$$

Therefore

$$\begin{cases} -\sum_{i=1}^{2} \frac{\partial}{\partial z}\left(\widehat{\mu}\frac{\partial u_i^*}{\partial z} + \widehat{e}_{3i3}\frac{\partial \varphi^*}{\partial z}\right) = \widehat{f}_i \quad i = 1, 2 \text{ in } H^{-1}(\Omega), \\ -\frac{\partial}{\partial z}\left(\widehat{\beta}\frac{\partial \varphi^*}{\partial z}\right) + \sum_{i=1}^{2}\frac{\partial}{\partial z}\left(\widehat{e}_{3i3}\frac{\partial u_i^*}{\partial z}\right) = \widehat{q} \quad \text{in } H^{-1}(\Omega), \end{cases} \tag{3.43}$$

As $\widehat{f}_i$, and $\widehat{q}$ belongs to $L^2(\Omega)$. From (3.43), we have $\frac{\partial^2 u_i^*}{\partial z^2}$ in $L^2(\Omega)$ and $\frac{\partial^2 \varphi^*}{\partial z^2}$ in $L^2(\Omega)$. Whence (3.40)-(3.41) holds, and we also have $\frac{\partial u_i^*}{\partial z}$ in V_z, $\frac{\partial \varphi_i^*}{\partial z}$ in W_z. ■

For convenience, we will denote by $s^*(x) = u^*(x, 0)$, $\tau^*(x) = \frac{\partial u^*}{\partial z}(x, 0)$, $\rho^*(x) = \varphi^*(x, 0)$ and $\pi^*(x) = \frac{\partial \varphi^*}{\partial z}(x, 0)$.Moreover, as $\frac{\partial u_i^*}{\partial z}$ in V_z and $\frac{\partial \varphi^*}{\partial z}$ in W_z, then τ^*, π^* belong to $L^2(\omega)$, and we have

Theorem 3.4.5 *Under the assumptions of preceding theorems, the traces of displacement (s^*, τ^*) and the electric potential (ρ^*, π^*) satisfy the inequality*

$$\int_\omega \widehat{k}\left|R\left(\widehat{e}_{333}\pi^*\right)\right|\left(|\phi + s^* - s| - |s^* - s|\right)dx - \int_\omega \left(\widehat{\mu}\tau^* + \widehat{e}_{3i3}\pi^*\right)\phi dx \geq 0, \ \forall \phi \in (L^2(\omega))^2, \tag{3.44}$$

and the following limit form of the Coulomb boundary conditions

$$\begin{cases} |\widehat{\mu}\tau^* + \widehat{e}_{3i3}\pi^*| < \widehat{k}\left|R\left(\widehat{e}_{333}\pi^*\right)\right| \Longrightarrow s^* = s, \\ \\ |\widehat{\mu}\tau^* + \widehat{e}_{3i3}\pi^*| = \widehat{k}\left|R\left(\widehat{e}_{333}\pi^*\right)\right| \Longrightarrow \exists \lambda \geq 0 \text{ such that } s^* = s + \lambda\left(\widehat{\mu}\tau^* + \widehat{e}_{3i3}\pi^*\right). \end{cases} \quad a.e. \text{ on } \omega \tag{3.45}$$

Moreover, the pair (u^*, φ^*) satisfies the following weak form

$$\begin{aligned} &\int_\omega \left[\int_0^h \int_0^z \widehat{\mu}\frac{\partial u^*}{\partial \xi} d\xi dz + \int_0^h \int_0^z \widehat{e}_{3i3}\frac{\partial \varphi^*}{\partial \xi} d\xi dz\right]\nabla \Psi dx \\ &+ \int_\omega \left[-\frac{h}{2}\left(\int_0^h \widehat{\mu}\frac{\partial u^*}{\partial \xi} d\xi + \int_0^h \widehat{e}_{3i3}\frac{\partial \varphi^*}{\partial \xi} d\xi\right) + \widetilde{F}\right]\nabla \Psi dx = 0 \quad \forall \Psi \in H^1(\omega), \end{aligned} \tag{3.46}$$

where

$$\widetilde{F}_i = \int_0^h F_i(x, z)dz - \frac{h}{2}F_i(x, h), \ \ F_i(x, z) = \int_0^z \int_0^\xi \widehat{f}_i(x, \theta)d\theta d\xi, \ \ i = 1, 2,$$
$$\widetilde{F} = \left(\widetilde{F}_1, \widetilde{F}_2\right).$$

Proof of Theorem 3.4.5. Using the Green's formula in (3.38), and taking $\widehat{v}_i = \phi_i + u_i^*$ (for $i = 1, 2$) where $\phi_i \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega)$ and

$$H^1_{\Gamma_1 \cup \Gamma_L}(\Omega) = \{v \in H^1(\Omega) : v_i = 0 \text{ on } \Gamma_1 \cup \Gamma_L\},$$

then

$$\begin{aligned} -\sum_{i=1}^{2} \left[\int_\Omega \frac{\partial}{\partial z}\left(\widehat{\mu}\frac{\partial u_i{}^*}{\partial z}\right)\phi_i dx dz + \int_\Omega \frac{\partial}{\partial z}\left(\widehat{e}_{3i3}\frac{\partial \varphi^*}{\partial z}\right)\phi_i dx dz \right] \\ + \int_\omega \widehat{k}\left|R\big(\widehat{e}_{333}\pi^*\big)\right| \left(|\phi + s^* - s| - |s^* - s|\right) dx \\ - \int_\omega \left(\widehat{\mu}\tau^* + \widehat{e}_{3i3}\pi^*\right)\phi dx \geq \sum_{i=1}^{2} \int_\Omega \widehat{f}_i \phi_i dx dz. \end{aligned}$$

From (3.40) and for $\phi \in (H^1_{\Gamma_1 \cup \Gamma_L}(\Omega))^2$, we have

$$\int_\omega \widehat{k}\left|R(\widehat{e}_{333}\pi^*)\right| \left(|\phi + s^* - s| - |s^* - s|\right) dx - \int_\omega \left(\widehat{\mu}\tau^* + \widehat{e}_{3i3}\pi^*\right)\phi dx \geq 0.$$

This inequality remains valid for any $\psi \in (D(\omega))^2$, and by the density of $D(\omega)$ in $L^2(\omega)$ we deduce (3.44). For the proof (3.45), refer to Chapter 2 (Proof of Theorem 3.7) .

Now, to prove (3.46) we integrate twice the equation (3.40) between 0 and z, we obtain

$$\begin{aligned} z\widehat{\mu}(x,0)\tau^* + z\widehat{e}_{3i3}\pi^* - \int_0^z (\widehat{\mu}\frac{\partial u_i^*}{\partial \xi})(x,\xi)d\xi \\ - \int_0^z \widehat{e}_{3i3}\frac{\partial \varphi^*}{\partial \xi}(x,\xi)d\xi = \int_0^z \int_0^\xi \widehat{f}_i(x,\theta)d\theta d\xi. \end{aligned} \tag{3.47}$$

For $z = h(x)$, then the equation (3.47) can be rewritten as follows

$$\begin{aligned} h(x)\widehat{\mu}(x,0)\tau^* + h(x)\widehat{e}_{3i3}(x,0)\pi^* = \int_0^h (\widehat{\mu}\frac{\partial u_i^*}{\partial \xi})(x,\xi)d\xi \\ + \int_0^h \widehat{e}_{3i3}\frac{\partial \varphi^*}{\partial \xi}(x,\xi)d\xi + \int_0^h \int_0^\xi \widehat{f}_i(x,\theta)d\theta d\xi. \end{aligned} \tag{3.48}$$

Integrating the equation (3.48) from 0 to h, we get

$$\begin{aligned} \frac{h^2(x)}{2}\widehat{\mu}(x,0)\tau^* + \frac{h^2(x)}{2}\widehat{e}_{3i3}(x,0)\pi^* - \int_0^h \int_0^z (\widehat{\mu}\frac{\partial u_i^*}{\partial \xi})(x,\xi)d\xi dz \\ - \int_0^h \int_0^z \widehat{e}_{3i3}\frac{\partial \varphi^*}{\partial \xi}(x,\xi)d\xi dz = \int_0^h F_i(x,z)dz, \end{aligned} \tag{3.49}$$

where

$$F_i(x,z) = \int_0^z \int_0^\xi \widehat{f}_i(x,\theta)d\theta d\xi.$$

From (3.48) and (3.49), we deduce (3.46). ■

Theorem 3.4.6 *There exists a positive constant sufficiently small k^* such that for $\|\widehat{k}\|_{L^\infty(\omega)} \leq k^*$ the solution (u^*, φ^*) of limit problem* (3.38)-(3.39) *is unique.*

Proof of Theorem 3.4.6. We replace $\widehat{\psi}$ by $(\widehat{\psi} - \varphi^*)$ in (3.39), and the sum with equation (3.38), we give the following variational equation

$$\begin{aligned}
&\sum_{i=1}^{2} \int_\Omega \widehat{\mu} \frac{\partial u_i{}^*}{\partial z} \frac{\partial}{\partial z} \left(\widehat{v}_i - u_i{}^*\right) dxdz + \sum_{i=1}^{2} \int_\Omega \widehat{e}_{3i3} \frac{\partial \varphi^*}{\partial z} \frac{\partial}{\partial z} \left(\widehat{v}_i - u_i{}^*\right) dxdz \\
&+ \int_\omega \widehat{k} |R\big((\widehat{e}_{333}) \frac{\partial \varphi^*}{\partial z}\big)| \left(|\widehat{v} - s| - |u^* - s|\right) dx + \int_\Omega \widehat{\beta} \frac{\partial \varphi^*}{\partial z} \frac{\partial(\widehat{\psi} - \varphi^*)}{\partial z} dxdz \\
&- \sum_{i=1}^{2} \int_\Omega \widehat{e}_{3i3} \frac{\partial u_i{}^*}{\partial z} \frac{\partial}{\partial z} (\widehat{\psi} - \varphi^*) dxdz \geq \sum_{i=1}^{2} \int_\Omega \widehat{f}_i (\widehat{v}_i - u_i^*) dxdz \\
&+ \int_\Omega \widehat{q} (\widehat{\psi} - \varphi^*) dxdz, \quad \forall (\widehat{v}, \widehat{\psi}) \in \Pi(K) \times W.
\end{aligned} \tag{3.50}$$

Let (u^{*1}, φ^{*1}), (u^{*2}, φ^{*2}) be two solution of (3.50). Taking $(\widehat{v}, \widehat{\psi}) = (u^{*2}, \varphi^{*2})$ and $(\widehat{v}, \widehat{\psi}) = (u^{*1}, \varphi^{*1})$ respectively, as test functions in (3.50), we obtain

$$\begin{aligned}
&\sum_{i=1}^{2} \int_\Omega \widehat{\mu} \frac{\partial u_i^{*1}}{\partial z} \frac{\partial}{\partial z} \left(u_i^{*2} - u_i^{*1}\right) dxdz + \sum_{i=1}^{2} \int_\Omega \widehat{e}_{3i3} \frac{\partial \varphi^{*1}}{\partial z} \frac{\partial}{\partial z} \left(u_i^{*2} - u_i^{*1}\right) dxdz \\
&+ \int_\omega \widehat{k} \left| R\left((\widehat{e}_{333}) \frac{\partial \varphi^{*1}}{\partial z}\right) \right| \left(|u_i^{*2} - s| - |u_i^{*1} - s|\right) dx + \int_\Omega \widehat{\beta} \frac{\partial \varphi^{*1}}{\partial z} \frac{\partial(\varphi^{*2} - \varphi^{*1})}{\partial z} dxdz \\
&- \sum_{i=1}^{2} \int_\Omega \widehat{e}_{3i3} \frac{\partial u_i^{*1}}{\partial z} \frac{\partial}{\partial z} (\varphi^{*2} - \varphi^{*1}) dxdz \geq \sum_{i=1}^{2} \int_\Omega \widehat{f}_i (u_i^{*2} - u_i^{*1}) dxdz \\
&+ \int_\Omega \widehat{q} (\varphi^{*2} - \varphi^{*1}) dxdz,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=1}^{2} \int_\Omega \widehat{\mu} \frac{\partial u_i^{*2}}{\partial z} \frac{\partial}{\partial z} \left(u_i^{*1} - u_i^{*2}\right) dxdz + \sum_{i=1}^{2} \int_\Omega \widehat{e}_{3i3} \frac{\partial \varphi^{*2}}{\partial z} \frac{\partial}{\partial z} \left(u_i^{*1} - u_i^{*2}\right) dxdz \\
&+ \int_\omega \widehat{k} \left| R\left((\widehat{e}_{333}) \frac{\partial \varphi^{*2}}{\partial z}\right) \right| \left(|u_i^{*1} - s| - |u_i^{*2} - s|\right) dx + \int_\Omega \widehat{\beta} \frac{\partial \varphi^{*2}}{\partial z} \frac{\partial(\varphi^{*1} - \varphi^{*2})}{\partial z} dxdz \\
&- \sum_{i=1}^{2} \int_\Omega \widehat{e}_{3i3} \frac{\partial u_i^{*2}}{\partial z} \frac{\partial}{\partial z} (\varphi^{*1} - \varphi^{*2}) dxdz \geq \sum_{i=1}^{2} \int_\Omega \widehat{f}_i (u_i^{*1} - u_i^{*2}) dxdz \\
&+ \int_\Omega \widehat{q} (\varphi^{*1} - \varphi^{*2}) dxdz.
\end{aligned}$$

Summing up the two inequalities, we get

$$-\sum_{i=1}^{2}\int_{\Omega}\widehat{\mu}\left|\frac{\partial}{\partial z}(u_i^{*1}-u_i^{*2})\right|^2 dxdz-\int_{\Omega}\widehat{\beta}\left|\frac{\partial}{\partial z}(\varphi^{*1}-\varphi^{*2})\right|^2 dxdz$$
$$+\int_{\omega}\widehat{k}\left(\left|R\Big(\widehat{e}_{333}\frac{\partial\varphi^{*2}}{\partial z}\Big)\right|-\left|R\Big(\widehat{e}_{333}\frac{\partial\varphi^{*1}}{\partial z}\Big)\right|\right)\left(|u^{*1}-s|-|u^{*2}-s|\right)dx\geq 0,$$

so that

$$\mu^*\left\|\frac{\partial}{\partial z}(u_i^{*1}-u_i^{*2})\right\|^2_{L^2(\Omega)}+\beta^*\left\|\frac{\partial}{\partial z}(\varphi^{*1}-\varphi^{*2})\right\|^2_{L^2(\Omega)}$$
$$\leq\int_{\omega}\widehat{k}\left(\left|R\left(\widehat{e}_{333}\frac{\partial\varphi^{*2}}{\partial z}\right)\right|-\left|R\left(\widehat{e}_{333}\frac{\partial\varphi^{*1}}{\partial z}\right)\right|\right)|u^{*1}-u^{*2}|dx.$$

Using the fact that $(a+b)^2\leq 2(a^2+b^2)$, then the last term can be rewritten as:

$$\frac{\min(\mu^*,\beta^*)}{2}\left(\left\|\frac{\partial}{\partial z}(u_i^{*1}-u_i^{*2})\right\|_{L^2(\Omega)}+\left\|\frac{\partial}{\partial z}(\varphi^{*1}-\varphi^{*2})\right\|_{L^2(\Omega)}\right)^2$$
$$\leq\|\widehat{k}\|_{L^\infty(\omega)}\left(\int_{\omega}\left|\left(R\Big(\widehat{e}_{333}\frac{\partial\varphi^{*2}}{\partial z}\Big)-R\Big(\widehat{e}_{333}\frac{\partial\varphi^{*1}}{\partial z}\Big)\right)\right|^2dx\right)^{\frac{1}{2}}\|u_i^{*1}-u_i^{*2}\|_{L^2(\omega)}. \tag{3.51}$$

By Sobolev's trace theorem, there exists a constant $C_0>0$ such that

$$\frac{\min(\mu^*,\beta^*)}{2}\left(\left\|\frac{\partial}{\partial z}(u_i^{*1}-u_i^{*2})\right\|_{L^2(\Omega)}+\left\|\frac{\partial}{\partial z}(\varphi^{*1}-\varphi^{*2})\right\|_{L^2(\Omega)}\right)^2$$
$$\leq C_0\|\widehat{k}\|_{L^\infty(\omega)}\left(\int_{\omega}\left|R\Big(\widehat{e}_{333}\frac{\partial\varphi^{*2}}{\partial z}\Big)-R\Big(\widehat{e}_{333}\frac{\partial\varphi^{*1}}{\partial z}\Big)\right|^2dx\right)^{\frac{1}{2}}\left\|\frac{\partial}{\partial z}\left(u_i^{*1}-u_i^{*2}\right)\right\|_{L^2(\Omega)}.$$

Thus, we obtain

$$\frac{\min(\mu^*,\beta^*)}{2}\left(\left\|\frac{\partial}{\partial z}(u_i^{*1}-u_i^{*2})\right\|_{L^2(\Omega)}+\left\|\frac{\partial}{\partial z}(\varphi^{*1}-\varphi^{*2})\right\|_{L^2(\Omega)}\right)^2$$
$$\leq C_0\|\widehat{k}\|_{L^\infty(\omega)}\left\|R\Big(\widehat{e}_{333}\frac{\partial\varphi^{*2}}{\partial z}\Big)-R\Big(\widehat{e}_{333}\frac{\partial\varphi^{*1}}{\partial z}\Big)\right\|_{L^2(\omega)}$$
$$\times\left(\left\|\frac{\partial}{\partial z}\left(u_i^{*1}-u_i^{*2}\right)\right\|_{L^2(\Omega)}+\left\|\frac{\partial}{\partial z}(\varphi^{*1}-\varphi^{*2})\right\|_{L^2(\Omega)}\right).$$

Consequently, the last equation leads to

$$\left\|\frac{\partial}{\partial z}(u_i^{*1}-u_i^{*2})\right\|_{L^2(\Omega)}+\left\|\frac{\partial}{\partial z}(\varphi^{*1}-\varphi^{*2})\right\|_{L^2(\Omega)}$$
$$\leq\frac{2h_M C_0\|\widehat{k}\|_{L^\infty(\omega)}}{\min(\mu^*,\beta^*)}\left\|R\Big(\widehat{e}_{333}\frac{\partial\varphi^{*2}}{\partial z}\Big)-R\Big(\widehat{e}_{333}\frac{\partial\varphi^{*1}}{\partial z}\Big)\right\|_{L^2(\omega)}. \tag{3.52}$$

Now, as R is a linear continuous operator from $H^{-\frac{1}{2}}(\omega)$ to $L^2(\omega)$, there exists a constant $C_1 \geq 0$, such that

$$\left\| R\left(\widehat{e}_{333}\frac{\partial \varphi^{*2}}{\partial z}\right) - R\left(\widehat{e}_{333}\frac{\partial \varphi^{*1}}{\partial z}\right)\right\|_{L^2(\omega)} \leq C_1 \left\|\widehat{e}_{333}\right\|_{L^\infty(\omega)} \left\| \frac{\partial}{\partial z}(\varphi^{*1} - \varphi^{*2})\right\|_{L^2(\omega)}. \qquad (3.53)$$

From (3.52)-(3.53), we deduce that if $\|\widehat{k}\|_{L^\infty(\omega)} \leq k^*$ for sufficiently small k^*, then we have

$$\|(u_i^{*1} - u_i^{*2})\|_{V_z} + \|(\varphi^{*1} - \varphi^{*2})\|_{W_z} = 0.$$

Hence $\|(u_i^{*1} - u_i^{*2})\|_{V_z} = 0$ and $\|(\varphi^{*1} - \varphi^{*2})\|_{W_z} = 0$.
This shows the uniqueness of solution (u^*, φ^*) in $V_z \times W_z$. ■

Chapter 4

3D-2D asymptotic analysis of an interface problem with a dissipative term in a dynamic regime

In the last chapter, we present the asymptotic analysis of frictional contact between two elastic bodies in a 3D thin domain with a dissipative term $\alpha g\left(\frac{\partial u}{\partial t}\right)$. In section 4.1, we derive the model and its variational formulation of the problem. For this problem, the theorem of existence and uniqueness of the weak solution has been proved recently in [20, 34]. In Section 4.2, we study the asymptotic analysis, in which the small parameter of the domain tends to zero. The next step will be to use Gronwall's lemma and Korn's inequality to establish some estimates independently of the parameter ε. These estimates will be useful to prove the convergence of the solution toward the expected function. Finally, section 4.3 is devoted to obtaining the main results pertaining to the convergence, the limit problem and establish the uniqueness of its solution.

4.1 Description of the problem

In this section, we first define the thin domain and some necessary sets to study the asymptotic behavior of the solutions. Next, we introduce the problem considered in the thin domain. We finish this section by giving the weak variational formulations of our problem.

The points $x \in \mathbb{R}^3$ will be decomposed as $x = (x', x_3)$ with $x' \in \mathbb{R}^2$, $x_3 \in \mathbb{R}$. We also use the notation x' to denote a generic vector of $\mathbb{R}^2$. Thus, we define the thin domain $\Omega^\varepsilon = (\Omega_1^\varepsilon \cup \Omega_2^\varepsilon) \in \mathbb{R}^3$, where

$$\Omega_1^\varepsilon = \{(x', x_3) \in \mathbb{R}^3, (x', 0) \in \omega, 0 < x_3 < \varepsilon h(x')\},$$
$$\Omega_2^\varepsilon = \{(x', x_3) \in \mathbb{R}^3, (x', 0) \in \omega, -\varepsilon h(x') < x_3 < 0\}.$$

We use a superscript l to indicate that a quantity is related to the domain $\Omega_l^\varepsilon, l = 1, 2$, where ε is a small parameter that tend to zero. For each domain Ω_l^ε, we assume that its boundary $\partial\Omega_l^\varepsilon$ is the class C^1 and is partitioned into three disjoint measurable parts: $\partial\Omega_1^\varepsilon = \bar{\omega} \cup \bar{\Gamma}_1^\varepsilon \cup \bar{\Gamma}_{L_1}^\varepsilon$, and $\partial\Omega_2^\varepsilon = \bar{\omega} \cup \bar{\Gamma}_2^\varepsilon \cup \bar{\Gamma}_{L_2}^\varepsilon$, where

- ω is a fixed region in the plane $x' = (x_1, x_2) \in \mathbb{R}^2$.
- The upper surface Γ_1^ε is defined by $x_3 = \varepsilon h(x')$, and the upper surface Γ_2^ε is defined by $x_3 = -\varepsilon h(x')$ where h is a positive smooth and bounded function such that

$$0 \leq h_{min} = \underline{h} \leq h(x') \leq h_{Max} = \overline{h}, \ \ \forall (x', 0) \in \omega.$$

- $\Gamma_{L_l}^\varepsilon, l = 1, 2$ is a lateral boundary of the domain Ω_l^ε (see FIG 4.1).

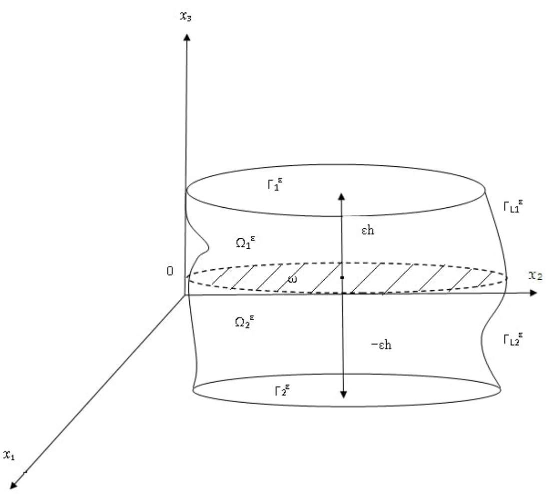

Figure 4.1: *The domain* Ω^ε

In what follows, $\mathbb{S}^3$ denotes the space of second order symmetric tensors on $\mathbb{R}^3$, while "." represent the inner product and $|.|$ the Euclidean norm on $\mathbb{R}^3$ and $\mathbb{S}^3$,

$$u.v = u_i.v_i, \quad |v| = (v.v)^{1/2} \qquad \forall u, v \in \mathbb{R}^3,$$
$$\sigma.\tau = \sigma_{ij}.\tau_{ij}, \quad |\tau| = (\tau.\tau)^{\frac{1}{2}} \qquad \forall \sigma, \tau \in \mathbb{S}^3.$$

The indexes i and j take their values between 1 and 3, and the summation convention over repeated index is adopted.

Let η_l be the unit outward normal to $\partial\Omega_l^\varepsilon$. Now, we define the normal and the tangential components of displacement and stress tensors on the boundary, respectively, as follows

$$\mathbf{v}_{l\eta}^\varepsilon = \mathbf{v}_l^\varepsilon.\eta_l, \quad \mathbf{v}_{l\tau}^\varepsilon = \mathbf{v}_l^\varepsilon - \mathbf{v}_{l\eta}^\varepsilon.\eta_l, \quad \text{with} \quad \eta = \eta_1 = -\eta_2,$$

$$\sigma_{l\eta}^\varepsilon = (\sigma_l^\varepsilon \eta_l).\eta_l, \quad \sigma_{l\tau}^\varepsilon = \sigma_l^\varepsilon \eta_l - (\sigma_{l\eta}^\varepsilon).\eta_l.$$

4.1.1 The model problem

We consider two elastic bodies that occupy the domains Ω_1^ε and Ω_2^ε. The two bodies are in bilateral, frictional, contact along the common part ω.

We denote the displacements field by $\mathbf{u}_l^\varepsilon = (u_{li}^\varepsilon)_{1\le i\le 3}$, $l = 1,2$, and the stress tensor by $\sigma_l^\varepsilon = (\sigma_{lij}^\varepsilon)_{1\le i,j\le 3}$, $l = 1,2$. $d_{ij}(\mathbf{u}_l^\varepsilon)$ designates the components of the linearized strain tensor. We model the materials according to elastic constructive law

$$\sigma_{lij}^\varepsilon(\mathbf{u}_l^\varepsilon) = 2\mu_l d_{ij}(\mathbf{u}_l^\varepsilon) + \lambda_l d_{kk}(\mathbf{u}_l^\varepsilon)\delta_{ij}, \;\; 1 \le i,j,k \le 3, \;\; l = 1,2,$$

where μ_l, λ_l are the Lamé coefficients and (δ_{ij}) is the Krönecker symbol.

• On $\Gamma_l^\varepsilon \times]0,T[\,, l = 1,2$, no slip condition is given. The upper surface is assumed to be fixed. Therefore:

$$\mathbf{u}_l^\varepsilon = 0, \quad l = 1,2.$$

• On $\Gamma_{L_l}^\varepsilon \times]0,T[\,, l = 1,2$ the displacement is known and parallel to the w-plane:

$$\mathbf{u}_l^\varepsilon = 0, \quad l = 1,2.$$

• We describe the conditions on the common surface $\omega \times]0,T[$. We suppose that the normal velocity is bilateral, that is:

$$\frac{\partial \mathbf{u}_1^\varepsilon}{\partial t}.\eta_1 + \frac{\partial \mathbf{u}_2^\varepsilon}{\partial t}.\eta_2 = 0 \quad \text{on} \quad \omega \times]0,T[\,.$$

Therefore,

$$\eta = \eta_1 = -\eta_2 \quad \text{and} \quad \sigma_1^\varepsilon.\eta_1 = -\sigma_2^\varepsilon.\eta_2 \;\; \text{on} \;\; \omega \times]0,T[\,.$$

Consequently,

$$\sigma_\eta^\varepsilon = \sigma_{1\eta}^\varepsilon = \sigma_{2\eta}^\varepsilon \quad \text{and} \quad \sigma_\tau^\varepsilon = \sigma_{1\tau}^\varepsilon = -\sigma_{2\tau}^\varepsilon \;\; \text{on } \omega \times]0,T[\,.$$

However, the tangential velocity on $\omega \times]0,T[$ is unknown and satisfies the Tresca boundary conditions, with friction coefficient κ^ε :

$$\begin{cases} |\sigma_\tau^\varepsilon| < \kappa^\varepsilon \Rightarrow \left(\frac{\partial \mathbf{u}_1^\varepsilon}{\partial t}\right)_\tau - \left(\frac{\partial \mathbf{u}_2^\varepsilon}{\partial t}\right)_\tau = s, \\ |\sigma_\tau^\varepsilon| = \kappa^\varepsilon \Rightarrow \exists \beta \geq 0, \left(\frac{\partial \mathbf{u}_1^\varepsilon}{\partial t}\right)_\tau - \left(\frac{\partial \mathbf{u}_2^\varepsilon}{\partial t}\right)_\tau = s - \beta \sigma_\tau^\varepsilon, \end{cases} \quad \text{on } \omega \times]0,T[\,.$$

The problem consists in finding $\mathbf{u}_l^\varepsilon, l = 1,2$ satisfying limit conditions, using the following initial conditions:

$$\mathbf{u}_l^\varepsilon(x,0) = \mathbf{u}_l^0(x), \quad \frac{\partial \mathbf{u}_l^\varepsilon}{\partial t}(x,0) = \mathbf{u}_l^1(x) \quad \forall x \in \Omega_l^\varepsilon, \ \ l = 1,2.$$

Next, we suppose that the dissipative terms $g_l, \ \ l = 1,2$ are continuous vector functions

$$g_l : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$v \longrightarrow g_l(v),$$

with $g_1 \neq g_2$, we make the following assumptions:

(H_1) Functions $g_l, \ \ l = 1,2$ is monotonic

$$(g_l(u) - g_l(v), u - v) \geq 0, \ \ \forall u, v \in \mathbb{R}^3.$$

(H_2) Functions $g_l, \ \ l = 1,2$ in the origin satisfy,

$$g_l(0,0,0) = (0,0,0) \quad l = 1,2.$$

(H_3) For all $v \in \mathbb{R}, \ \ g_l$ is increasing function

$$g'_{li}(v) \geq 0 \quad i = 1,2,3, \ \ l = 1,2.$$

We consider a transmission problem associated to the non-homogeneous elastic body Ω^ε in a dynamic regime with a nonlinear dissipative term.

Problem $\mathcal{P}^\varepsilon$. *Find a displacement field* $\mathbf{u}_l^\varepsilon = (u_{lij}^\varepsilon)_{1 \leq i,j \leq 3} : \Omega_l^\varepsilon \times]0,T[\to \mathbb{R}^3, \ \ l = 1,2,$ *such that*

$$\frac{\partial^2 \mathbf{u}_1^\varepsilon}{\partial t^2} - \text{div}\sigma_1^\varepsilon + \alpha_1^\varepsilon g_1\left(\frac{\partial \ \mathbf{u}_1^\varepsilon}{\partial t}\right) = \mathbf{f}_1^\varepsilon \ \ \text{in} \ \ \Omega_1^\varepsilon \times]0,T[\,, \tag{4.1}$$

$$\frac{\partial^2 \mathbf{u}_2^\varepsilon}{\partial t^2} - \text{div}\sigma_2^\varepsilon + \alpha_2^\varepsilon g_2\left(\frac{\partial \ \mathbf{u}_2^\varepsilon}{\partial t}\right) = \mathbf{f}_2^\varepsilon \ \ \text{in} \ \ \Omega_2^\varepsilon \times]0,T[\,, \tag{4.2}$$

$$\sigma_1^\varepsilon(\mathbf{u}_1^\varepsilon) = 2\mu_1 d(\mathbf{u}_1^\varepsilon) + \lambda_1 d_{kk}(\mathbf{u}_1^\varepsilon)\delta \quad \text{in} \quad \Omega_1^\varepsilon \times]0,T[\,, \tag{4.3}$$

$$\sigma_2^\varepsilon(\mathbf{u}_2^\varepsilon) = 2\mu_2 d(\mathbf{u}_2^\varepsilon) + \lambda_2 d_{kk}(\mathbf{u}_2^\varepsilon)\delta \quad \text{in} \quad \Omega_2^\varepsilon \times]0,T[\,, \tag{4.4}$$

$$\mathbf{u}_1^\varepsilon = 0 \quad \text{on} \quad \Gamma_1^\varepsilon \times]0,T[\,, \tag{4.5}$$

$$\mathbf{u}_2^\varepsilon = 0 \quad \text{on} \quad \Gamma_2^\varepsilon \times]0,T[\,, \tag{4.6}$$

$$\mathbf{u}_1^\varepsilon = 0 \quad \text{on} \quad \Gamma_{L_1}^\varepsilon \times]0,T[\,, \tag{4.7}$$

$$\mathbf{u}_2^\varepsilon = 0 \quad \text{on} \quad \Gamma_{L_2}^\varepsilon \times]0,T[\,, \tag{4.8}$$

$$\tfrac{\partial \mathbf{u}_1^\varepsilon}{\partial t}.\eta - \tfrac{\partial \mathbf{u}_2^\varepsilon}{\partial t}.\eta = 0 \quad \text{on} \quad \omega \times]0,T[\,, \tag{4.9}$$

$$\sigma_1^\varepsilon.\eta - \sigma_2^\varepsilon.\eta = 0 \quad \text{on} \quad \omega \times]0,T[\,, \tag{4.10}$$

$$\begin{cases} |\sigma_\tau^\varepsilon| < \kappa^\varepsilon \Rightarrow \left(\frac{\partial \mathbf{u}_1^\varepsilon}{\partial t}\right)_\tau - \left(\frac{\partial \mathbf{u}_2^\varepsilon}{\partial t}\right)_\tau = s, \\ |\sigma_\tau^\varepsilon| = \kappa^\varepsilon \Rightarrow \exists \beta \geq 0, \left(\frac{\partial \mathbf{u}_1^\varepsilon}{\partial t}\right)_\tau - \left(\frac{\partial \mathbf{u}_2^\varepsilon}{\partial t}\right)_\tau = s - \beta\sigma_\tau^\varepsilon, \end{cases} \quad \text{on } \omega \times]0,T[\,, \tag{4.11}$$

$$\mathbf{u}_l^\varepsilon(x,0) = \mathbf{u}_l^0(x), \quad \frac{\partial \mathbf{u}_l^\varepsilon}{\partial t}(x,0) = \mathbf{u}_l^1(x) \quad \forall x \in \Omega_l^\varepsilon, \;\; l = 1,2, \tag{4.12}$$

where $f_l^\varepsilon = (f_{1l}^\varepsilon, f_{2l}^\varepsilon, f_{3l}^\varepsilon)$, $l = 1,2$, is the density of the forces. Expressions (4.1)-(4.2) are representing the dynamic elasticity systems with nonlinear dissipative terms in Ω_1^ε and Ω_2^ε. Relations (4.3)-(4.4) are the Hooke constructive laws. Furthermore, equations (4.5)-(4.8) are the displacements on $\Gamma_l^\varepsilon \times]0,T[$ and $\Gamma_{L_l}^\varepsilon \times]0,T[\,$, $l = 1,2$ respectively. Conditions (4.9)-(4.10) are representing the normal velocity. The Tresca friction law is given by (4.11) with friction coefficient κ^ε and s is the velocity of the lower surface ω. Finally, the initial conditions of the displacement and the velocity field are given by (4.12).

Lemma 4.1.1 *[20] The condition of Tresca* (4.11) *is equivalent to*

$$\left(\frac{\partial u_1^\varepsilon}{\partial t} - \frac{\partial u_2^\varepsilon}{\partial t} - s\right)\sigma_\tau^\varepsilon + \kappa^\varepsilon \left|\left(\frac{\partial u_1^\varepsilon}{\partial t} - \frac{\partial u_2^\varepsilon}{\partial t} - s\right)\right| = 0 \;\; on \; \omega.$$

4.1.2 Variational formulation

We end this section by giving the equivalent weak variational formulation of **Problem** $\mathcal{P}^\varepsilon$, which will be useful in the next sections.

First, we use the following notation

$$V(\Omega_l^\varepsilon) = \left\{\mathbf{v} \in (H^1(\Omega_l^\varepsilon))^3 : \mathbf{v} = 0 \quad \text{on} \quad \Gamma_l^\varepsilon \cup \Gamma_{L_l}^\varepsilon\right\}\,, \; l = 1,2,$$

the space $V(\Omega_l^\varepsilon)$ is real Hilbert space endowed with their natural norms $\|.\|_{1,\Omega_l^\varepsilon}$ and scalar product $\langle .,.\rangle_{1,\Omega_l^\varepsilon}$.

Moreover, we need the following functional spaces

$$K^\varepsilon = \{(\mathbf{v_1},\mathbf{v_2}) \in V(\Omega_1^\varepsilon) \times V(\Omega_2^\varepsilon) : \mathbf{v_1}.\eta_1 + \mathbf{v_2}.\eta_2 = 0 \quad on \quad \omega\}.$$

The norm of a real Hilbert space K^ε is

$$\|(\mathbf{v_1},\mathbf{v_2})\|_{K^\varepsilon} = \left(\|\mathbf{v_1}\|^2_{V(\Omega_1^\varepsilon)} + \|\mathbf{v_2}\|^2_{V(\Omega_2^\varepsilon)}\right)^{1/2}.$$

We still denote the norm of space $(H^1(\Omega_1^\varepsilon))^3 \times (H^1(\Omega_2^\varepsilon))^3$ by $\|(.,.)\|_{1,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}$.

Lemma 4.1.2 *Let $(\mathbf{u}_1^\varepsilon,\mathbf{u}_2^\varepsilon)$ be a solution of **Problem** $\mathcal{P}^\varepsilon$, with a sufficient regularity. Then, it checks following variational problem*

$$\left\{\begin{array}{l} Find\ (\mathbf{u}_1^\varepsilon,\mathbf{u}_2^\varepsilon)\ \ where\ \left(\frac{\partial \mathbf{u}_1^\varepsilon}{\partial t}(t), \frac{\partial \mathbf{u}_2^\varepsilon}{\partial t}(t)\right) \in K^\varepsilon,\ \ \forall t\in[0,T], \\ \displaystyle\sum_{l=1}^2 \int_{\Omega_l^\varepsilon} \frac{\partial^2 \mathbf{u}_l^\varepsilon}{\partial t^2}\left(\varphi_l - \frac{\partial \mathbf{u}_l^\varepsilon}{\partial t}\right)dx + \mathcal{A}\left((\mathbf{u}_1^\varepsilon,\mathbf{u}_2^\varepsilon), \left(\varphi_1 - \frac{\partial \mathbf{u}_1^\varepsilon}{\partial t}, \varphi_2 - \frac{\partial \mathbf{u}_2^\varepsilon}{\partial t}\right)\right) \\ \displaystyle +\sum_{l=1}^2 \int_{\Omega_l^\varepsilon} \alpha_l^\varepsilon g_l\left(\frac{\partial \mathbf{u}_l^\varepsilon}{\partial t}\right)\left(\varphi_l - \frac{\partial \mathbf{u}_l^\varepsilon}{\partial t}\right)dx + J^\varepsilon(\varphi_1,\varphi_2) \\ \displaystyle -J^\varepsilon\left(\frac{\partial \mathbf{u}_1^\varepsilon}{\partial t}, \frac{\partial \mathbf{u}_2^\varepsilon}{\partial t}\right) \geq \sum_{l=1}^2 \int_{\Omega_l^\varepsilon} \mathbf{f}_l^\varepsilon.\left(\varphi_l - \frac{\partial \mathbf{u}_l^\varepsilon}{\partial t}\right)dx \qquad \forall(\varphi_1,\varphi_2)\in K^\varepsilon, \end{array}\right. \tag{4.13}$$

$$with\ \ (\mathbf{u}_1^\varepsilon(0),\mathbf{u}_2^\varepsilon(0)) = (\mathbf{u}_1^0,\mathbf{u}_2^0), \quad \left(\frac{\partial \mathbf{u}_1^\varepsilon}{\partial t}(0), \frac{\partial \mathbf{u}_2^\varepsilon}{\partial t}(0)\right) = (\mathbf{u}_1^1,\mathbf{u}_2^1),$$

where

$$\mathcal{A}\Big((\mathbf{u}^\varepsilon{}_1,\mathbf{u}^\varepsilon{}_2),(\varphi_1,\varphi_2)\Big) = \sum_{l=1}^2 \left\{2\mu_l \int_{\Omega_l^\varepsilon} d_{ij}(\mathbf{u}^\varepsilon{}_l)d_{ij}(\varphi_l)dx + \lambda_l \int_{\Omega_l^\varepsilon} div(\mathbf{u}^\varepsilon{}_l)div(\varphi_l)dx\right\},$$

$$J^\varepsilon(\varphi_1,\varphi_2) = \int_\omega \kappa^\varepsilon|(\varphi_{1\tau} - \varphi_{2\tau} - s)|dx'.$$

Proof of Lemma 4.1.2. We multiply (4.1) by $(\varphi_1 - \frac{\partial \mathbf{u}_1^\varepsilon}{\partial t})$ and (4.2) by $(\varphi_2 - \frac{\partial \mathbf{u}_2^\varepsilon}{\partial t})$ where $(\varphi_1,\varphi_2)\in K^\varepsilon$. We perform an integration by parts on Ω_1^ε and Ω_2^ε, using Green's formula leads to .

$$\begin{aligned} \sum_{l=1}^2 \int_{\Omega_l^\varepsilon} \frac{\partial^2 u_{li}^\varepsilon}{\partial t^2}\left(\varphi_{li} - \frac{\partial u_{li}^\varepsilon}{\partial t}\right)dx'dz + \sum_{l=1}^2 \int_{\Omega_l^\varepsilon} \sigma_{lij}^\varepsilon \frac{\partial}{\partial x_j}\left(\varphi_{li} - \frac{\partial u_{li}^\varepsilon}{\partial t}\right)dx'dz \\ -\sum_{l=1}^2 \int_{\partial\Omega_l^\varepsilon} \sigma_{lij}^\varepsilon \eta_j \left(\varphi_{li} - \frac{\partial u_{li}^\varepsilon}{\partial t}\right)ds + \sum_{l=1}^2 \int_{\Omega_l^\varepsilon} \alpha_l^\varepsilon g_l\left(\frac{\partial u_{li}^\varepsilon}{\partial t}\right)dx'dx_3 \\ = \sum_{l=1}^2 \int_{\Omega_l^\varepsilon} f_{li}^\varepsilon\left(\varphi_{li} - \frac{\partial u_{li}^\varepsilon}{\partial t}\right)dx'dz. \end{aligned} \tag{4.14}$$

According to the boundary conditions(4.5)-(4.8), we find

$$\int_{\partial\Omega_1^\varepsilon} \sigma_{1ij}^\varepsilon \eta_{1j}\left(\varphi_{1i}-\frac{\partial u_{1i}^\varepsilon}{\partial t}\right) d\delta+\int_{\partial\Omega_2^\varepsilon} \sigma_{2ij}^\varepsilon \eta_{2j}\left(\varphi_{2i}-\frac{\partial u_{2i}^\varepsilon}{\partial t}\right) d\delta = \int_\omega \sigma_1^\varepsilon \eta_1\left(\varphi_1-\frac{\partial u_1^\varepsilon}{\partial t}\right) dx'+\int_\omega \sigma_2^\varepsilon \eta_2\left(\varphi_2-\frac{\partial u_2^\varepsilon}{\partial t}\right) dx'.$$

On the other hand, we have $\sigma_l^\varepsilon \eta_l=\sigma_{l\tau}^\varepsilon+(\sigma_{l\eta}^\varepsilon).\eta_l,\ \ (l=1,2),\ \ \sigma_\tau^\varepsilon=\sigma_{1\tau}^\varepsilon=-\sigma_{2\tau}^\varepsilon$ and by the condition (4.9)-(4.10), we get

$$\int_{\partial\Omega_1^\varepsilon} \sigma_{1ij}^\varepsilon \eta_{1j}\left(\varphi_{1i}-\frac{\partial u_{1i}^\varepsilon}{\partial t}\right) d\delta+\int_{\partial\Omega_2^\varepsilon} \sigma_{2ij}^\varepsilon \eta_{2j}\left(\varphi_{2i}-\frac{\partial u_{2i}^\varepsilon}{\partial t}\right) d\delta = \int_\omega \sigma_\tau^\varepsilon\left[(\varphi_{1\tau}-\varphi_{2\tau})-\left(\frac{\partial u_{1\tau}^\varepsilon}{\partial t}-\frac{\partial u_{2\tau}^\varepsilon}{\partial t}\right)\right] dx'.$$

Thus, equation (4.14) becomes

$$\begin{aligned}&\sum_{l=1}^2\int_{\Omega_l^\varepsilon}\frac{\partial^2 u_{li}^\varepsilon}{\partial t^2}\left(\varphi_{li}-\frac{\partial u_{li}^\varepsilon}{\partial t}\right)dx'dz+\sum_{l=1}^2\int_{\Omega_l^\varepsilon}\sigma_{lij}^\varepsilon\frac{\partial}{\partial x_j}\left(\varphi_{li}-\frac{\partial u_{li}^\varepsilon}{\partial t}\right)dx'dz\\
&-\sum_{l=1}^2\int_{\partial\Omega_l^\varepsilon}\sigma_{lij}^\varepsilon\eta_j\left(\varphi_{li}-\frac{\partial u_{li}^\varepsilon}{\partial t}\right)ds+\sum_{l=1}^2\int_{\Omega_l^\varepsilon}\alpha_l^\varepsilon g_l\left(\frac{\partial u_{li}^\varepsilon}{\partial t}\right)dx'dx_3\\
&+J^\varepsilon(\varphi_1,\varphi_2)-J^\varepsilon\left(\frac{\partial u_1^\varepsilon}{\partial t},\frac{\partial u_2^\varepsilon}{\partial t}\right)-\sum_{l=1}^2\int_{\Omega_l^\varepsilon}f_{li}^\varepsilon\left(\varphi_{li}-\frac{\partial u_{li}^\varepsilon}{\partial t}\right)dx'dz=\\
&\int_\omega\sigma_\tau^\varepsilon\left[(\varphi_{1\tau}-\varphi_{2\tau})-\left(\frac{\partial u_{1\tau}^\varepsilon}{\partial t}-\frac{\partial u_{2\tau}^\varepsilon}{\partial t}\right)\right]dx'+\int_\omega k^\varepsilon\left[|\varphi_{1\tau}-\varphi_{2\tau}-s|-\left|\frac{\partial u_{1\tau}^\varepsilon}{\partial t}-\frac{\partial u_{2\tau}^\varepsilon}{\partial t}-s\right|\right]dx'.\end{aligned}$$

Using Lemma 4.1.1, we deduce directly the variational inequality (4.13) ■

Theorem 4.1.3 *Suppose that*

$$\begin{cases}(f_1^\varepsilon,f_2^\varepsilon),\ \left(\frac{\partial f_1^\varepsilon}{\partial t},\frac{\partial f_2^\varepsilon}{\partial t}\right),\ \left(\frac{\partial^2 f_1^\varepsilon}{\partial t^2},\frac{\partial^2 f_2^\varepsilon}{\partial t^2}\right)\in L^2\big(0,T;(L^2(\Omega_1^\varepsilon))^3\times(L^2(\Omega_2^\varepsilon))^3\big),\\ \kappa^\varepsilon\in C_0^\infty(\omega),\quad \kappa^\varepsilon>0\quad \textit{is independent of } t,\\ (\mathbf{u}_1^0,\mathbf{u}_2^0)\in(H^2(\Omega_1^\varepsilon))^3\times(H^2(\Omega_2^\varepsilon))^3,\quad (\mathbf{u}_1^1,\mathbf{u}_2^1)\in(H^1(\Omega_1^\varepsilon))^3\times(H^1(\Omega_2^\varepsilon))^3,\\ g_l(\mathbf{u}_l^1)\in(L^2(\Omega_l^\varepsilon))^3,\ \ (\mathbf{u}_l^1)_\tau=0,\quad l=1,2.\end{cases}\tag{4.15}$$

Then, there exists one and only one solution $(\mathbf{u}_1^\varepsilon,\mathbf{u}_2^\varepsilon)$ *satisfying the problem* (4.13) *with*

$$(\mathbf{u}_1^\varepsilon,\mathbf{u}_2^\varepsilon),\ \left(\frac{\partial\mathbf{u}_1^\varepsilon}{\partial t},\frac{\partial\mathbf{u}_2^\varepsilon}{\partial t}\right)\in L^\infty\big(0,T;(H^1(\Omega_1^\varepsilon))^3\times(H^1(\Omega_2^\varepsilon))^3\big),$$
$$\left(\frac{\partial^2\mathbf{u}_1^\varepsilon}{\partial t^2},\frac{\partial^2\mathbf{u}_2^\varepsilon}{\partial t^2}\right)\in L^\infty\big(0,T;(L^2(\Omega_1^\varepsilon))^3\times(L^2(\Omega_2^\varepsilon))^3\big).$$

Proof of Theorem 4.1.3. The proof consists of three steps (see the works of Duvaut and Lions [20, 34]):
First, we regularize the function J^ε by J^ε_ζ, where

$$J^\varepsilon_\zeta(v_1, v_2) = \int_\omega \kappa^\varepsilon(x)\phi_\zeta(|v_{1\tau} - v_{2\tau} - s|^2)dx' \quad \text{with} \quad \phi_\zeta(\lambda) = \frac{1}{1+\zeta}|\lambda|^{(1+\zeta)}, \ \zeta > 0.$$

Next, we formulate the associated approximate problem. Then, the proof is based on the nonlinear operators theory. For that, using Galerkin's method, we show that there exists a unique solution $\mathbf{u}^\varepsilon_\zeta = (\mathbf{u}^\varepsilon_{1\zeta}, \mathbf{u}^\varepsilon_{2\zeta})$ of this approximate problem. Finally, we prove that the limit of $\mathbf{u}^\varepsilon_\zeta$ to $\mathbf{u}^\varepsilon$ when ζ tends to zero is a solution of (4.13). ■

4.2 Scale change in the variable x_3

Our aim is to study the asymptotic behavior of $\mathbf{u}^\varepsilon = (\mathbf{u}^\varepsilon_1, \mathbf{u}^\varepsilon_2)$ when ε tend to zero. For this purpose, as usual when we deal with thin domains, we use the dilatation in the variable x_3 given by $z = \dfrac{x_3}{\varepsilon}$. Thus, for (x', x_3) in Ω^ε_l (for $l = 1, 2$), we have (x', z) in

$$\Omega_1 = \{(x', z) \in \mathbb{R}^3, \ (x', 0) \in \omega, \quad 0 < z < h(x')\},$$
$$\Omega_2 = \{(x', z) \in \mathbb{R}^3, \ (x', 0) \in \omega, \quad -h(x') < z < 0\}.$$

To have the functions defined in Ω_l with boundary $\partial\Omega_l = \overline{\omega} \cup \overline{\Gamma}_l \cup \overline{\Gamma}_{L_l}, l = 1, 2$, we define $(\hat{\mathbf{u}}^\varepsilon_1, \hat{\mathbf{u}}^\varepsilon_2) \in L^\infty\Big(0, T; (H^1(\Omega_1))^3 \times (H^1(\Omega_2))^3\Big)$ by

$$\begin{cases} \hat{u}^\varepsilon_{li}(x', z, t) = u^\varepsilon_{li}(x', x_3, t), & l, i = 1, 2, \\ \hat{u}^\varepsilon_{l3}(x', z, t) = \varepsilon^{-1} u^\varepsilon_{l3}(x', x_3, t), & l = 1, 2. \end{cases} \tag{4.16}$$

For the data of problem (4.1)-(4.12), we suppose that they depend of ε in the following manner

$$\begin{cases} \hat{\mathbf{f}}_l(x', z, t) = \varepsilon^2 \mathbf{f}^\varepsilon_l(x', x_3, t), & l = 1, 2, \\ \hat{\kappa} = \varepsilon \kappa^\varepsilon, & \\ \hat{\alpha}_l = \varepsilon^2 \alpha^\varepsilon_l, & l = 1, 2, \end{cases} \tag{4.17}$$

with $\hat{\mathbf{f}}_l, \hat{\kappa}$ and $\hat{\alpha}_l$ independent of ε.

Now, we introduce the functional framework on $\Omega_1 \cup \Omega_2$. We note

$$V(\Omega_l) = \left\{ \hat{\mathbf{v}} \in (H^1(\Omega_l))^3 : \hat{\mathbf{v}} = 0 \quad \text{on} \quad \Gamma_l \cup \Gamma_{L_l} \right\},$$

$$K = \{(\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2) \in V(\Omega_1) \times V(\Omega_2) : \hat{\mathbf{v}}_1.\eta - \hat{\mathbf{v}}_2.\eta = 0 \quad \text{on} \quad \omega\},$$

$$H_z(\Omega_l) = \left\{ \hat{\mathbf{v}}_l = (\hat{v}_{l1}, \hat{v}_{l2}) \in (L^2(\Omega_l))^2 : \frac{\partial \hat{v}_{li}}{\partial z} \in L^2(\Omega_l), \text{ and } \hat{\mathbf{v}}_l = 0 \text{ on } \Gamma_l, \ i, l = 1, 2 \right\},$$

$$H_z = H_z(\Omega_1) \times H_z(\Omega_2),$$

$$\overline{V(\Omega_l)} = \left\{ \overline{\varphi}_l \in (H^1(\Omega_l))^2 : \overline{\varphi_l} = (\varphi_{l1}, \varphi_{l2}), \ \overline{\varphi}_{li} = 0 \quad \text{on} \quad \Gamma_l \cup \Gamma_{L_l} \quad \text{for} \quad i = 1, 2 \right\}.$$

$H_z(\Omega_l), \ l = 1, 2,$ and H_z be two Banach spaces, with norms

$$\|\hat{\mathbf{v}}_l\|_{H_z(\Omega_l)} = \left\{ \sum_{i=1}^{2} \left(\|\hat{\mathbf{v}}_{li}\|^2_{L^2(\Omega_l)} + \|\frac{\partial \hat{\mathbf{v}}_{li}}{\partial z}\|^2_{L^2(\Omega_l)} \right) \right\}^{\frac{1}{2}},$$

$$\|(\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2)\|_{H_z} = \left(\|\hat{\mathbf{v}}_1\|^2_{H_z(\Omega_1)} + \|\hat{\mathbf{v}}_2\|^2_{H_z(\Omega_2)} \right)^{\frac{1}{2}}.$$

Using the transformation $z = \dfrac{x_3}{\varepsilon}$, the variational problem (4.13) is equivalent to the following problem.

Problem 4.2.1 *Find* $(\hat{\mathbf{u}}_1^\varepsilon, \hat{\mathbf{u}}_2^\varepsilon)$ *where* $\left(\frac{\partial \hat{\mathbf{u}}_1^\varepsilon}{\partial t}(t), \frac{\partial \hat{\mathbf{u}}_2^\varepsilon}{\partial t}(t)\right) \in K$, $\forall t \in [0, T]$, *such that*

$$\begin{cases} \varepsilon^2 \displaystyle\sum_{1 \le i,l \le 2} \int_{\Omega_l} \frac{\partial^2 \hat{u}_{li}^\varepsilon}{\partial t^2} \left(\hat{\varphi}_{li} - \frac{\partial \hat{u}_{li}^\varepsilon}{\partial t} \right) dx' dz + \varepsilon^4 \sum_{l=1}^{2} \int_{\Omega_l} \frac{\partial^2 \hat{u}_{l3}^\varepsilon}{\partial t^2} \left(\hat{\varphi}_{l3} - \frac{\partial \hat{u}_{l3}^\varepsilon}{\partial t} \right) dx' dz \\ + \displaystyle\sum_{1 \le i,l \le 2} \hat{\alpha}_l \int_{\Omega_l} g_{li} \left(\frac{\partial \hat{u}_{li}^\varepsilon}{\partial t} \right) \left(\hat{\varphi}_{li} - \frac{\partial \hat{u}_{li}^\varepsilon}{\partial t} \right) dx' dz + \sum_{l=1}^{2} \hat{\alpha}_l \int_{\Omega_l} \varepsilon g_{l3} \left(\varepsilon \frac{\partial \hat{u}_{l3}^\varepsilon}{\partial t} \right) \left(\hat{\varphi}_{l3} - \frac{\partial \hat{u}_{l3}^\varepsilon}{\partial t} \right) dx' dz \\ + \hat{J}(\hat{\varphi}_1, \hat{\varphi}_2) - \hat{J}\left(\frac{\partial \hat{\mathbf{u}}_1^\varepsilon}{\partial t}, \frac{\partial \hat{\mathbf{u}}_2^\varepsilon}{\partial t} \right) + \mathcal{A}\left((\hat{\mathbf{u}}_1^\varepsilon, \hat{\mathbf{u}}_2^\varepsilon), (\hat{\varphi}_1 - \frac{\partial \hat{\mathbf{u}}_1^\varepsilon}{\partial t}, \hat{\varphi}_2 - \frac{\partial \hat{\mathbf{u}}_2^\varepsilon}{\partial t}) \right) \ge \\ \displaystyle\sum_{1 \le i,l \le 2} \int_{\Omega_l} \hat{\mathbf{f}}_{li} \left(\hat{\varphi}_{li} - \frac{\partial \hat{\mathbf{u}}_{li}^\varepsilon}{\partial t} \right) dx' dz + \sum_{l=1}^{2} \varepsilon \int_{\Omega_l} \hat{\mathbf{f}}_{l3} \left(\hat{\varphi}_{l3} - \frac{\partial \hat{\mathbf{u}}_{l3}^\varepsilon}{\partial t} \right) dx' dz, \ \forall (\varphi_1, \varphi_2) \in K, \\ \hat{\mathbf{u}}_l^\varepsilon(0) = \hat{\mathbf{u}}_l^0, \quad \frac{\partial \hat{\mathbf{u}}^\varepsilon}{\partial t}(0) = \hat{\mathbf{u}}_l^1, \quad l = 1, 2, \end{cases} \tag{4.18}$$

where

$$\hat{J}(\hat{\varphi}_1, \hat{\varphi}_2) = \int_\omega \hat{\kappa} |\hat{\varphi}_{1\tau} - \hat{\varphi}_{2\tau} - s| dx',$$

and

$$\mathcal{A}\left((\hat{u}_1^\varepsilon,\hat{u}_2^\varepsilon),(\varphi_1,\varphi_2)\right)=\varepsilon^2\sum_{1\le i,j,l\le 2}\mu_l\int_{\Omega_l}\left(\frac{\partial \hat{u}_{li}^\varepsilon}{\partial x_j}+\frac{\partial \hat{u}_{lj}^\varepsilon}{\partial x_i}\right)\frac{\partial \hat{\varphi}_{li}}{\partial x_j}dx'dz$$
$$+\sum_{1\le j,l\le 2}\mu_l\int_{\Omega_l}\left(\frac{\partial \hat{u}_{lj}^\varepsilon}{\partial z}+\varepsilon^2\frac{\partial \hat{u}_{l3}^\varepsilon}{\partial x_j}\right)\left(\varepsilon^2\frac{\partial \hat{\varphi}_{l3}}{\partial x_j}+\frac{\partial \hat{\varphi}_{lj}}{\partial z}\right)dx'dz$$
$$+\sum_{1\le l\le 2}\left[2\varepsilon^2\mu_l\int_{\Omega_l}\frac{\partial \hat{u}_{l3}^\varepsilon}{\partial z}\frac{\partial \hat{\varphi}_{l3}}{\partial z}dx'dz+\varepsilon^2\lambda_l\int_{\Omega_l}div(\hat{u}_l^\varepsilon)div\left(\hat{\varphi}_l\right)dx'dz\right].$$

Our goal then is to describe the asymptotic behavior of these new sequences when ε tends to zero. To do this, we need to obtain appropriate estimates on $\hat{\mathbf{u}}^\varepsilon$, $\dfrac{\partial \hat{\mathbf{u}}^\varepsilon}{\partial t}$ and $\dfrac{\partial^2 \hat{\mathbf{u}}^\varepsilon}{\partial t^2}$. These estimates will be useful to prove the convergence of $\hat{\mathbf{u}}^\varepsilon$ toward the expected function. For this, we introduce some results which will be used next. The detailed description can be found in [10].

$$C_K\left\|\nabla v_l\right\|_{0,\Omega}\le\left\|d(v_l)\right\|_{0,\Omega}\quad l=1,2,\quad\text{for }(v_1,v_2)\in K\quad\text{(Korn inequality)},$$

$$\|(\widehat{u}_{1i}^\varepsilon,\widehat{u}_{2i}^\varepsilon)\|_{0,\Omega_1\times\Omega_2}\le\bar{h}\left\|\left(\frac{\partial\widehat{u}_{1i}^\varepsilon}{\partial z},\frac{\partial\widehat{u}_{2i}^\varepsilon}{\partial z}\right)\right\|_{0,\Omega_1\times\Omega_2}\quad\text{(Poincaré inequality)}.$$

4.2.1 A priori estimates

We start this subsection by obtaining some priori estimates given in the following Theorems.

Theorem 4.2.2 *Under the assumptions of Theorem 4.1.3, there exists a constant C independent of ε, such that the following estimates hold*

$$\sum_{i=1}^{2}\left[\left\|\varepsilon\left(\frac{\partial \hat{u}_{1i}^\varepsilon}{\partial t},\frac{\partial \hat{u}_{2i}^\varepsilon}{\partial t}\right)\right\|_{0,\Omega_1\times\Omega_2}^2+\left\|\varepsilon^2\left(\frac{\partial \hat{u}_{13}^\varepsilon}{\partial x_i},\frac{\partial \hat{u}_{23}^\varepsilon}{\partial x_i}\right)\right\|_{0,\Omega_1\times\Omega_2}^2+\left\|\left(\frac{\partial \hat{u}_{1i}^\varepsilon}{\partial z},\frac{\partial \hat{u}_{2i}^\varepsilon}{\partial z}\right)\right\|_{0,\Omega_1\times\Omega_2}^2\right]$$
$$+\sum_{1\le i,j\le 2}\left\|\varepsilon\left(\frac{\partial \hat{u}_{1i}^\varepsilon}{\partial x_j},\frac{\partial \hat{u}_{2i}^\varepsilon}{\partial x_j}\right)\right\|_{0,\Omega_1\times\Omega_2}^2+\left\|\varepsilon^2\left(\frac{\partial \hat{u}_{13}^\varepsilon}{\partial t},\frac{\partial \hat{u}_{23}^\varepsilon}{\partial t}\right)\right\|_{0,\Omega_1\times\Omega_2}^2+\left\|\varepsilon\left(\frac{\partial \hat{u}_{13}^\varepsilon}{\partial z},\frac{\partial \hat{u}_{23}^\varepsilon}{\partial z}\right)\right\|_{0,\Omega_1\times\Omega_2}^2\le C. \tag{4.19}$$

$$\sum_{1\le l,i\le 2}\int_0^t\int_{\Omega_l}g_l\left(\frac{\partial \hat{u}_{li}^\varepsilon}{\partial t}\right)\frac{\partial \hat{u}_{li}^\varepsilon}{\partial t}dx'dz+\sum_{l=1}^{2}\varepsilon\int_0^t\int_{\Omega_l}g_l\left(\varepsilon\frac{\partial \hat{u}_{l3}^\varepsilon}{\partial t}\right)\frac{\partial \hat{u}_{l3}^\varepsilon}{\partial t}dx'dz\le C. \tag{4.20}$$

Theorem 4.2.3 *Let the assumptions of Theorem 4.1.3 hold, there exists a constant C independent of ε, such that*

$$\sum_{i=1}^{2}\left[\left\|\varepsilon^2\frac{\partial}{\partial x_i}\left(\frac{\partial \hat{u}_{13}^\varepsilon}{\partial t},\frac{\partial \hat{u}_{23}^\varepsilon}{\partial t}\right)\right\|_{0,\Omega_1\times\Omega_2}^2+\left\|\varepsilon\left(\frac{\partial^2 \hat{u}_{1i}^\varepsilon}{\partial t^2},\frac{\partial^2 \hat{u}_{2i}^\varepsilon}{\partial t^2}\right)\right\|_{0,\Omega_1\times\Omega_2}^2\right]$$
$$+\sum_{i=1}^{2}\left\|\frac{\partial}{\partial z}\left(\frac{\partial \hat{u}_{1i}^\varepsilon}{\partial t},\frac{\partial \hat{u}_{2i}^\varepsilon}{\partial t}\right)\right\|_{0,\Omega_1\times\Omega_2}^2+\sum_{1\leq i,j\leq 2}\left\|\varepsilon\frac{\partial}{\partial x_j}\left(\frac{\partial \hat{u}_{1i}^\varepsilon}{\partial t},\frac{\partial \hat{u}_{2i}^\varepsilon}{\partial t}\right)\right\|_{0,\Omega_1\times\Omega_2}^2 \tag{4.21}$$
$$+\left\|\varepsilon^2\left(\frac{\partial^2 \hat{u}_{13}^\varepsilon}{\partial t^2},\frac{\partial^2 \hat{u}_{23}^\varepsilon}{\partial t^2}\right)\right\|_{0,\Omega_1\times\Omega_2}^2+\left\|\varepsilon\frac{\partial}{\partial z}\left(\frac{\partial \hat{u}_{13}^\varepsilon}{\partial t},\frac{\partial \hat{u}_{23}^\varepsilon}{\partial t}\right)\right\|_{0,\Omega_1\times\Omega_2}^2\leq C.$$

Proof of Theorem 4.2.2. Let $(\mathbf{u}_1^\varepsilon,\mathbf{u}_2^\varepsilon)$ be a solution to the problem (4.13) , we choose $(\varphi_1,\varphi_2)=(0,0)$. Then, we have

$$\int_{\Omega_1^\varepsilon}\frac{\partial^2\mathbf{u}_1^\varepsilon}{\partial t^2}\frac{\partial\mathbf{u}_1^\varepsilon}{\partial t}dx+\int_{\Omega_2^\varepsilon}\frac{\partial^2\mathbf{u}_2^\varepsilon}{\partial t^2}\frac{\partial\mathbf{u}_2^\varepsilon}{\partial t}dx+\mathcal{A}\left((\mathbf{u}_1^\varepsilon(t),\mathbf{u}_2^\varepsilon(t)),\left(\frac{\partial\mathbf{u}_1^\varepsilon}{\partial t},\frac{\partial\mathbf{u}_2^\varepsilon}{\partial t}\right)\right)$$
$$+J^\varepsilon\left(\frac{\partial u_1^\varepsilon}{\partial t},\frac{\partial u_2^\varepsilon}{\partial t}\right)+\sum_{l=1}^{2}\alpha_l^\varepsilon\int_{\Omega_l^\varepsilon}g_l\left(\frac{\partial\mathbf{u}_l^\varepsilon}{\partial t}\right)\frac{\partial\mathbf{u}_l^\varepsilon}{\partial t}dx'dx_3\leq\sum_{l=1}^{2}\int_{\Omega_l^\varepsilon}\mathbf{f}_l^\varepsilon\frac{\partial\mathbf{u}_l^\varepsilon}{\partial t}dx'dx_3.$$

As $J^\varepsilon\left(\frac{\partial u_1^\varepsilon}{\partial t},\frac{\partial u_2^\varepsilon}{\partial t}\right)\geq 0$, hence

$$\frac{1}{2}\frac{d}{dt}\left[\left\|\frac{\partial\mathbf{u}_1^\varepsilon}{\partial t}\right\|_{L^2(\Omega_1^\varepsilon)}^2+\left\|\frac{\partial\mathbf{u}_2^\varepsilon}{\partial t}\right\|_{L^2(\Omega_2^\varepsilon)}^2+\mathcal{A}\left((\mathbf{u}_1^\varepsilon(t),\mathbf{u}_2^\varepsilon(t)),(\mathbf{u}_1^\varepsilon(t),\mathbf{u}_2^\varepsilon(t))\right)\right]$$
$$+\sum_{l=1}^{2}\alpha_l^\varepsilon\int_{\Omega_l^\varepsilon}g_l\left(\frac{\partial\mathbf{u}_l^\varepsilon}{\partial t}\right)\frac{\partial\mathbf{u}_l^\varepsilon}{\partial t}dx'dx_3\leq\sum_{l=1}^{2}\int_{\Omega_l^\varepsilon}\mathbf{f}_l^\varepsilon\frac{\partial\mathbf{u}_l^\varepsilon}{\partial t}dx'dx_3.$$

By integration in times on $[0,t]$, and as $\sum_{1\leq i,j\leq 2}|d_{ij}(\mathbf{v})|^2\leq|\nabla\mathbf{v}|^2$, $|div(\mathbf{v})|^2\leq 3|\nabla\mathbf{v}|^2$, we find

$$\left\|\left(\frac{\partial\mathbf{u}_1^\varepsilon}{\partial t},\frac{\partial\mathbf{u}_2^\varepsilon}{\partial t}\right)\right\|_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}^2+\mathcal{A}\left((\mathbf{u}_1^\varepsilon,\mathbf{u}_2^\varepsilon),(\mathbf{u}_1^\varepsilon,\mathbf{u}_2^\varepsilon)\right)+2\alpha_1^\varepsilon\int_0^t\int_{\Omega_1^\varepsilon}g_1\left(\frac{\partial\mathbf{u}_1^\varepsilon}{\partial t}\right)\frac{\partial\mathbf{u}_1^\varepsilon}{\partial t}dxds$$
$$+2\alpha_2^\varepsilon\int_0^t\int_{\Omega_2^\varepsilon}g_2\left(\frac{\partial\mathbf{u}_2^\varepsilon}{\partial t}\right)\frac{\partial\mathbf{u}_2^\varepsilon}{\partial t}dxds\leq\left\|(\mathbf{u}_1^1,\mathbf{u}_2^1)\right\|_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}^2+\sum_{l=1}^{2}(2\mu_l+3\lambda_l)\left\|(\nabla\mathbf{u}_1^0,\nabla\mathbf{u}_2^0)\right\|_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}^2$$
$$+2\int_0^t\int_{\Omega_1^\varepsilon}\mathbf{f}_1^\varepsilon\frac{\partial\mathbf{u}_1^\varepsilon}{\partial t}dxds+2\int_0^t\int_{\Omega_2^\varepsilon}\mathbf{f}_2^\varepsilon\frac{\partial\mathbf{u}_2^\varepsilon}{\partial t}dxds. \tag{4.22}$$

By Korn's inequality, there exists a constant $C_K>0$ independent of ε, such that

$$\mathcal{A}\Big((\mathbf{u}_1^\varepsilon(t),\mathbf{u}_2^\varepsilon(t)),(\mathbf{u}_1^\varepsilon(t),\mathbf{u}_2^\varepsilon(t))\Big)\geq C_K\left(2\mu_1\|\nabla\mathbf{u}_1^\varepsilon(t)\|_{0,\Omega_1^\varepsilon}^2+2\mu_2\|\nabla\mathbf{u}_2^\varepsilon(t)\|_{0,\Omega_2^\varepsilon}^2\right). \tag{4.23}$$

By applying the integration by parts, we obtain

$$2\left[\int_0^t\int_{\Omega_1^\varepsilon}\mathbf{f}_1^\varepsilon(s)\frac{\partial\mathbf{u}_1^\varepsilon}{\partial t}dxds+\int_0^t\int_{\Omega_2^\varepsilon}\mathbf{f}_2^\varepsilon(s)\frac{\partial\mathbf{u}_2^\varepsilon}{\partial t}dxds\right]=$$
$$2\sum_{l=1}^{2}\left[\int_{\Omega_l^\varepsilon}\mathbf{f}_l^\varepsilon(t)\mathbf{u}_l^\varepsilon(t)dx-\int_{\Omega_l^\varepsilon}\mathbf{f}_l^\varepsilon(0)\mathbf{u}_l^\varepsilon(0)dx-\int_0^t\int_{\Omega_l^\varepsilon}\frac{\partial\mathbf{f}_l^\varepsilon}{\partial t}(s)\mathbf{u}_l^\varepsilon(s)dxds\right].$$

Using Poincaré's inequality, we obtain

$$\|\mathbf{u}_l^\varepsilon\|_{L^2(\Omega_l^\varepsilon)}\le\varepsilon\overline{h}\,\|\nabla\mathbf{u}_l^\varepsilon\|_{L^2(\Omega_l^\varepsilon)},\quad l=1,2,\qquad \overline{h}=\sup_{x\in\omega}h(x).$$

Applying Cauchy-Schwartz and Young's inequalities, we obtain

$$2\left|\int_0^t\int_{\Omega_1^\varepsilon}\mathbf{f}_1^\varepsilon(s)\frac{\partial\mathbf{u}_1^\varepsilon}{\partial t}dxds+\int_0^t\int_{\Omega_2^\varepsilon}\mathbf{f}_2^\varepsilon(s)\frac{\partial\mathbf{u}_2^\varepsilon}{\partial t}dxds\right|\le\mu_1C_K\|\nabla\mathbf{u}_1^\varepsilon\|_{0,\Omega_1^\varepsilon}^2+\frac{\varepsilon^2\overline{h}^2}{\mu_1C_K}\|\mathbf{f}_1^\varepsilon(t)\|_{0,\Omega_1^\varepsilon}^2$$
$$+\mu_2C_K\|\nabla\mathbf{u}_2^\varepsilon\|_{0,\Omega_2^\varepsilon}^2+\frac{\varepsilon^2\overline{h}^2}{\mu_2C_K}\|\mathbf{f}_2^\varepsilon(t)\|_{0,\Omega_2^\varepsilon}^2+\|\nabla\mathbf{u}_1^0,\nabla\mathbf{u}_2^0\|_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}^2+\varepsilon^2\overline{h}^2\|\mathbf{f}_1^\varepsilon(0),\mathbf{f}_2^\varepsilon(0)\|_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}^2$$
$$+C_K\left(\mu_1\int_0^t\|\nabla\mathbf{u}_1^\varepsilon(s)\|_{0,\Omega_1^\varepsilon}^2ds+\mu_2\int_0^t\|\nabla\mathbf{u}_2^\varepsilon(s)\|_{0,\Omega_2^\varepsilon}^2ds\right)+\sum_{l=1}^{2}\frac{(\varepsilon\overline{h})^2}{\mu_lC_K}\int_0^t\left\|\frac{\partial f_l^\varepsilon}{\partial t}(s)\right\|_{0,\Omega_l^\varepsilon}^2ds. \tag{4.24}$$

From (4.22)-(4.24), we deduce

$$\left\|\left(\frac{\partial\mathbf{u}_1^\varepsilon}{\partial t},\frac{\partial\mathbf{u}_2^\varepsilon}{\partial t}\right)\right\|_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}^2+C_K\left(\mu_1\|\nabla\mathbf{u}_1^\varepsilon(t)\|_{0,\Omega_1^\varepsilon}^2+\mu_2\|\nabla\mathbf{u}_2^\varepsilon(t)\|_{0,\Omega_2^\varepsilon}^2\right)$$
$$+2\alpha_1^\varepsilon\int_0^t\int_{\Omega_1^\varepsilon}g_1\left(\frac{\partial\mathbf{u}_1^\varepsilon}{\partial t}\right)\frac{\partial\mathbf{u}_1^\varepsilon}{\partial t}dxds+2\alpha_2^\varepsilon\int_0^t\int_{\Omega_2^\varepsilon}g_2\left(\frac{\partial\mathbf{u}_2^\varepsilon}{\partial t}\right)\frac{\partial\mathbf{u}_2^\varepsilon}{\partial t}dxds\le\left\|(\mathbf{u}_1^1,\mathbf{u}_2^1)\right\|_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}^2$$
$$+(2\mu_1+3\lambda_1)\left\|\nabla\mathbf{u}_1^0\right\|_{0,\Omega_1^\varepsilon}^2+(2\mu_2+3\lambda_2)\left\|\nabla\mathbf{u}_2^0\right\|_{0,\Omega_2^\varepsilon}^2+\|(\nabla\mathbf{u}_1^0,\nabla\mathbf{u}_2^0)\|_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}^2$$
$$+\varepsilon^2\overline{h}^2\|(\mathbf{f}_1^\varepsilon(0),\mathbf{f}_2^\varepsilon(0))\|_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}^2+C_K\int_0^t\left(\mu_1\|\nabla\mathbf{u}_1^\varepsilon(s)\|_{0,\Omega_1^\varepsilon}^2+\mu_2\|\nabla\mathbf{u}_2^\varepsilon(s)\|_{0,\Omega_2^\varepsilon}^2\right)ds$$
$$+\sum_{l=1}^{2}\left[\frac{(\varepsilon\overline{h})^2}{\mu_lC_K}\int_0^t\left\|\frac{\partial f_l^\varepsilon}{\partial t}(s)\right\|_{0,\Omega_l^\varepsilon}^2ds+\frac{\varepsilon^2\overline{h}^2}{\mu_lC_K}\|\mathbf{f}_l^\varepsilon(t)\|_{0,\Omega_l^\varepsilon}^2\right]. \tag{4.25}$$

Since $\varepsilon^2\|\mathbf{f}_l^\varepsilon\|_{L^2(\Omega_l^\varepsilon)}^2=\varepsilon^{-1}\|\hat{\mathbf{f}}_l\|_{L^2(\Omega_l)}^2$, $l=1,2$, multiplying (4.25) by ε and for $0<\varepsilon<1$, we see that

$$\varepsilon\left[\left\|\left(\frac{\partial \mathbf{u}_1^\varepsilon}{\partial t},\frac{\partial \mathbf{u}_2^\varepsilon}{\partial t}\right)\right\|^2_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}+C_K\left(\mu_1\|\nabla\mathbf{u}_1^\varepsilon(t)\|^2_{0,\Omega_1^\varepsilon}+\mu_2\|\nabla\mathbf{u}_2^\varepsilon(t)\|^2_{0,\Omega_2^\varepsilon}\right)\right]$$
$$+2\varepsilon\sum_{l=1}^{2}\alpha_l^\varepsilon\int_0^t\int_{\Omega_l^\varepsilon}g_l\left(\frac{\partial \mathbf{u}_l^\varepsilon}{\partial t}\right)\frac{\partial \mathbf{u}_l^\varepsilon}{\partial t}dxds\le$$
$$\int_0^t\varepsilon\left[\left\|\left(\frac{\partial \mathbf{u}_1^\varepsilon}{\partial t},\frac{\partial \mathbf{u}_2^\varepsilon}{\partial t}\right)\right\|^2_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}+C_K\left(\mu_1\|\nabla\mathbf{u}_1^\varepsilon(t)\|^2_{0,\Omega_1^\varepsilon}+\mu_2\|\nabla\mathbf{u}_2^\varepsilon(t)\|^2_{0,\Omega_2^\varepsilon}\right)\right]ds$$
$$+\sum_{l=1}^{2}\varepsilon\left[\frac{(\varepsilon\overline{h})^2}{\mu_l C_K}\int_0^t\left\|\frac{\partial f_l^\varepsilon}{\partial t}(s)\right\|^2_{0,\Omega_l^\varepsilon}ds+\frac{\varepsilon^2\overline{h}^2}{\mu_l C_K}\|\mathbf{f}_l^\varepsilon(t)\|^2_{0,\Omega_l^\varepsilon}+(2\mu_l+3\lambda_l)\|\nabla\mathbf{u}_{l0}^\varepsilon\|^2_{0,\Omega_l^\varepsilon}\right]$$
$$+\varepsilon\left\|(\mathbf{u}_1^1,\mathbf{u}_2^1)\right\|^2_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}+\varepsilon\|(\nabla\mathbf{u}_1^0,\nabla\mathbf{u}_2^0)\|^2_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}+\varepsilon^3\overline{h}^2\|(\mathbf{f}_1^\varepsilon(0),\mathbf{f}_2^\varepsilon(0))\|^2_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon},$$

with $\mu_-=\min(\mu_1,\mu_2)$, $\mu_+=\max(\mu_1,\mu_2)$ and $\lambda_+=\max(\lambda_1,\lambda_2)$. Therefore

$$\varepsilon\left(\left\|\left(\frac{\partial \mathbf{u}_1^\varepsilon}{\partial t},\frac{\partial \mathbf{u}_2^\varepsilon}{\partial t}\right)\right\|^2_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}+C_K\left(\mu_1\|\nabla\mathbf{u}_1^\varepsilon(t)\|^2_{0,\Omega_1^\varepsilon}+\mu_2\|\nabla\mathbf{u}_2^\varepsilon(t)\|^2_{0,\Omega_2^\varepsilon}\right)\right)$$
$$+2\varepsilon\sum_{l=1}^{2}\alpha_l^\varepsilon\int_0^t\int_{\Omega_l^\varepsilon}g_l\left(\frac{\partial \mathbf{u}_l^\varepsilon}{\partial t}\right)\frac{\partial \mathbf{u}_l^\varepsilon}{\partial t}dxds\le\mathbf{A}\tag{4.26}$$
$$+\int_0^t\varepsilon\left(\left\|\left(\frac{\partial \mathbf{u}_1^\varepsilon}{\partial t},\frac{\partial \mathbf{u}_2^\varepsilon}{\partial t}\right)\right\|^2_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}+C_K\left(\mu_1\|\nabla\mathbf{u}_1^\varepsilon(t)\|^2_{0,\Omega_1^\varepsilon}+\mu_2\|\nabla\mathbf{u}_2^\varepsilon(t)\|^2_{0,\Omega_2^\varepsilon}\right)\right)ds,$$

where $\mathbf{A}$ does not depend on ε

$$\mathbf{A}=\left\|(\hat{\mathbf{u}}_1^1,\hat{\mathbf{u}}_2^1)\right\|^2_{0,\Omega_1\times\Omega_2}+(1+2\mu_++3\lambda_+)\|(\nabla\hat{\mathbf{u}}_1^0,\nabla\hat{\mathbf{u}}_2^0)\|^2_{0,\Omega_1\times\Omega_2}+\overline{h}^2\|(\hat{\mathbf{f}}_1(0),\hat{\mathbf{f}}_2(0)\|^2_{0,\Omega_1\times\Omega_2}$$
$$+\frac{\overline{h}^2}{\mu_-C_K}\left(\|(\hat{\mathbf{f}}_1,\hat{\mathbf{f}}_2)\|^2_{L^\infty(0,T;(L^2(\Omega_1))^3\times(L^2(\Omega_2))^3)}+\left\|\left(\frac{\partial\hat{\mathbf{f}}_1}{\partial t},\frac{\partial\hat{\mathbf{f}}_2}{\partial t}\right)\right\|^2_{L^2(0,T;(L^2(\Omega_1))^3\times(L^2(\Omega_2))^3)}\right).$$

Using now Gronwall's Lemma, we have

$$\varepsilon\left(\left\|\left(\frac{\partial \mathbf{u}_1^\varepsilon}{\partial t},\frac{\partial \mathbf{u}_2^\varepsilon}{\partial t}\right)\right\|^2_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}+\|(\nabla\mathbf{u}_1^\varepsilon(t),\nabla\mathbf{u}_2^\varepsilon(t))\|^2_{0,\Omega_1^\varepsilon\times\Omega_2^\varepsilon}\right)\le C,$$

from which (4.19)-(4.20) follows. ■

Proof of Theorem 4.2.3. The functional J^ε is convex but non differentiable. To overcome this difficulty, we shall use the following approach. We regularize the function J^ε by J^ε_ζ, where

$$J^\varepsilon_\zeta(v_1,v_2)=\int_\omega\kappa^\varepsilon(x)\phi_\zeta(|v_{1\tau}-v_{2\tau}-s|^2)dx'\quad\text{with}\quad\phi_\zeta(\lambda)=\frac{1}{1+\zeta}|\lambda|^{(1+\zeta)},\ \ \zeta>0.$$

Then, we build the approximate problem

$$\left\{\begin{array}{l} \displaystyle\sum_{l=1}^{2}\left[\int_{\Omega_l^\varepsilon}\frac{\partial^2\mathbf{u}_{l\zeta}^\varepsilon}{\partial t^2}\varphi_l dx+\alpha_l^\varepsilon\int_{\Omega_l^\varepsilon}g\left(\frac{\partial\mathbf{u}_{l\zeta}^\varepsilon}{\partial t}\right)\varphi_l dx\right]+\mathcal{A}\left((\mathbf{u}_{1\zeta}^\varepsilon,\mathbf{u}_{2\zeta}^\varepsilon),(\varphi_1,\varphi_2)\right)\\ \qquad +\left((J_\zeta^\varepsilon)'\left(\frac{\partial\mathbf{u}_{1\zeta}^\varepsilon}{\partial t},\frac{\partial\mathbf{u}_{2\zeta}^\varepsilon}{\partial t}\right),(\varphi_1,\varphi_2)\right)=\displaystyle\sum_{l=1}^{2}\int_{\Omega_l^\varepsilon}\mathbf{f}_l^\varepsilon\varphi_l dx,\\ \qquad \text{with}\quad \mathbf{u}_{l\zeta}^\varepsilon(0)=\mathbf{u}_l^0,\ \ \frac{\partial\mathbf{u}_{l\zeta}^\varepsilon}{\partial t}(0)=\mathbf{u}_l^1,\ \ l=1,2. \end{array}\right. \tag{4.27}$$

We derive (4.27) in t and take $\varphi_1=\dfrac{\partial^2\mathbf{u}_{1\zeta}^\varepsilon}{\partial t^2}$, $\varphi_2=\dfrac{\partial^2\mathbf{u}_{2\zeta}^\varepsilon}{\partial t^2}$, then

$$\begin{aligned} &\sum_{l=1}^{2}\int_{\Omega_l^\varepsilon}\frac{\partial^3\mathbf{u}_{l\zeta}^\varepsilon}{\partial t^3}\frac{\partial^2\mathbf{u}_{l\zeta}^\varepsilon}{\partial t^2}dx+\mathcal{A}\left(\left(\frac{\partial\mathbf{u}_{1\zeta}^\varepsilon}{\partial t},\frac{\partial\mathbf{u}_{2\zeta}^\varepsilon}{\partial t}\right),\left(\frac{\partial^2\mathbf{u}_{1\zeta}^\varepsilon}{\partial t^2},\frac{\partial^2\mathbf{u}_{2\zeta}^\varepsilon}{\partial t^2}\right)\right)\\ &\qquad+\sum_{l=1}^{2}\alpha_l^\varepsilon\int_{\Omega_l^\varepsilon}g_l'\left(\frac{\partial\mathbf{u}_{l\zeta}^\varepsilon}{\partial t}\right)\frac{\partial^2\mathbf{u}_{l\zeta}^\varepsilon}{\partial t^2}.\frac{\partial^2\mathbf{u}_{l\zeta}^\varepsilon}{\partial t^2}dx\\ &+\left(\frac{d}{dt}(J_\zeta^\varepsilon)'\left(\frac{\partial\mathbf{u}_{1\zeta}^\varepsilon}{\partial t},\frac{\partial\mathbf{u}_{2\zeta}^\varepsilon}{\partial t}\right),\left(\frac{\partial^2\mathbf{u}_{1\zeta}^\varepsilon}{\partial t^2},\frac{\partial^2\mathbf{u}_{2\zeta}^\varepsilon}{\partial t^2}\right)\right)=\sum_{l=1}^{2}\int_{\Omega_l^\varepsilon}\frac{\partial\mathbf{f}_l^\varepsilon}{\partial t}\frac{\partial^2\mathbf{u}_{l\zeta}^\varepsilon}{\partial t^2}dx, \end{aligned}$$

where

$$\left(\frac{d}{dt}(J_\zeta^\varepsilon)'\left(\frac{\partial\mathbf{u}_{1\zeta}^\varepsilon}{\partial t},\frac{\partial\mathbf{u}_{2\zeta}^\varepsilon}{\partial t}\right),\left(\frac{\partial^2\mathbf{u}_{1\zeta}^\varepsilon}{\partial t^2},\frac{\partial^2\mathbf{u}_{2\zeta}^\varepsilon}{\partial t^2}\right)\right)\geq 0.$$

Due to (H_3), we have

$$\alpha_1^\varepsilon\int_{\Omega_1^\varepsilon}g_1'\left(\frac{\partial\mathbf{u}_{1\zeta}^\varepsilon}{\partial t}\right)\frac{\partial^2\mathbf{u}_{1\zeta}^\varepsilon}{\partial t^2}.\frac{\partial^2\mathbf{u}_{1\zeta}^\varepsilon}{\partial t^2}dx+\alpha_2^\varepsilon\int_{\Omega_2^\varepsilon}g_2'\left(\frac{\partial\mathbf{u}_{2\zeta}^\varepsilon}{\partial t}\right)\frac{\partial^2\mathbf{u}_{2\zeta}^\varepsilon}{\partial t^2}.\frac{\partial^2\mathbf{u}_{2\zeta}^\varepsilon}{\partial t^2}dx\geq 0,$$

then, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\left\|\frac{\partial^2\mathbf{u}_{1\zeta}^\varepsilon}{\partial t^2}\right\|_{L^2(\Omega_1^\varepsilon)}^2+\left\|\frac{\partial^2\mathbf{u}_{2\zeta}^\varepsilon}{\partial t^2}\right\|_{L^2(\Omega_2^\varepsilon)}^2+\mathcal{A}\left(\left(\frac{\partial\mathbf{u}_{1\zeta}^\varepsilon}{\partial t},\frac{\partial\mathbf{u}_{2\zeta}^\varepsilon}{\partial t}\right),\left(\frac{\partial\mathbf{u}_{1\zeta}^\varepsilon}{\partial t},\frac{\partial\mathbf{u}_{2\zeta}^\varepsilon}{\partial t}\right)\right)\right]\\ &\qquad\qquad\leq\int_{\Omega_1^\varepsilon}\frac{\partial\mathbf{f}_1^\varepsilon}{\partial t}\frac{\partial^2\mathbf{u}_{1\zeta}^\varepsilon}{\partial t^2}dx+\int_{\Omega_2^\varepsilon}\frac{\partial\mathbf{f}_2^\varepsilon}{\partial t}\frac{\partial^2\mathbf{u}_{2\zeta}^\varepsilon}{\partial t^2}dx. \end{aligned}$$

Using Korn's inequality, we get

$$\left\| \left(\frac{\partial^2 \mathbf{u}_{1\zeta}^\varepsilon}{\partial t^2}, \frac{\partial^2 \mathbf{u}_{2\zeta}^\varepsilon}{\partial t^2} \right) \right\|_{0,\Omega_1^\varepsilon \times \Omega_2^\varepsilon}^2 + 2C_K \sum_{l=1}^{2} \mu_l \left\| \nabla \left(\frac{\partial \mathbf{u}_{l\zeta}^\varepsilon}{\partial t} \right) \right\|_{L^2(\Omega_l^\varepsilon)}^2 \leq \left\| \left(\frac{\partial^2 \mathbf{u}_{1\zeta}^\varepsilon}{\partial t^2}(0), \frac{\partial^2 \mathbf{u}_{2\zeta}^\varepsilon}{\partial t^2}(0) \right) \right\|_{0,\Omega_1^\varepsilon \times \Omega_2^\varepsilon}^2$$
$$+ \sum_{l=1}^{2} \left[(2\mu_l + 3\lambda_l) \left\| \nabla \left(\frac{\partial \mathbf{u}_{l\zeta}^\varepsilon}{\partial t} \right)(0) \right\|_{L^2(\Omega_l^\varepsilon)}^2 + \frac{\varepsilon^2 \overline{h}^2}{\mu_l C_K} \left\| \frac{\partial \mathbf{f}_l^\varepsilon}{\partial t}(t) \right\|_{L^2(\Omega_l^\varepsilon)}^2 \right]$$
$$+ \sum_{l=1}^{2} \left[C_K \mu_l \left\| \nabla \left(\frac{\partial \mathbf{u}_{l\zeta}^\varepsilon}{\partial t} \right)(t) \right\|_{L^2(\Omega_l^\varepsilon)}^2 + \frac{\varepsilon^2 \overline{h}^2}{\mu_l C_K} \left\| \frac{\partial \mathbf{f}_l^\varepsilon}{\partial t}(0) \right\|_{L^2(\Omega_l^\varepsilon)}^2 + \mu_l C_K \left\| \nabla \left(\frac{\partial \mathbf{u}_{l\zeta}^\varepsilon}{\partial t} \right)(0) \right\|_{L^2(\Omega_l^\varepsilon)}^2 \right]$$
$$+ \sum_{l=1}^{2} \left[\frac{\varepsilon^2 \overline{h}^2}{\mu_l C_K} \int_0^t \left\| \frac{\partial^2 \mathbf{f}_l^\varepsilon}{\partial t^2}(s) \right\|_{L^2(\Omega_l^\varepsilon)}^2 ds + \int_0^t \mu_l C_K \left\| \nabla \frac{\partial \mathbf{u}_{l\zeta}^\varepsilon}{\partial t}(s) \right\|_{L^2(\Omega_l^\varepsilon)}^2 ds \right]. \tag{4.28}$$

Now, let us estimate $\left(\frac{\partial^2 \mathbf{u}_{1\zeta}^\varepsilon}{\partial t^2}(0), \frac{\partial^2 \mathbf{u}_{2\zeta}^\varepsilon}{\partial t^2}(0) \right)$. From (3.15) and (4.27), we deduce

$$\int_{\Omega_1^\varepsilon} \frac{\partial^2 \mathbf{u}_{1\zeta}^\varepsilon}{\partial t^2}(0) \varphi_1 dx + \int_{\Omega_2^\varepsilon} \frac{\partial^2 \mathbf{u}_{2\zeta}^\varepsilon}{\partial t^2}(0) \varphi_2 + A\left((u_{1\zeta}(0), u_{2\zeta}(0)), (\varphi_1, \varphi_2) \right)$$
$$+ \sum_{l=1}^{2} \int_{\Omega_l^\varepsilon} g\left(\frac{\partial u_{l\zeta}^\varepsilon}{\partial t}(0) \right) \varphi_l dx = \sum_{l=1}^{2} f_l^\varepsilon(0) \varphi_l dx.$$

So,

$$\left| \int_{\Omega_1^\varepsilon} \frac{\partial^2 \mathbf{u}_{1\zeta}^\varepsilon}{\partial t^2}(0) \varphi_1 dx + \int_{\Omega_2^\varepsilon} \frac{\partial^2 \mathbf{u}_{2\zeta}^\varepsilon}{\partial t^2}(0) \varphi_2 dx \right| \leq \varepsilon \overline{h} \sum_{l=1}^{2} \| \mathbf{f}_l^\varepsilon(0) \|_{L^2(\Omega_l^\varepsilon)} \| \nabla \varphi_l \|_{L^2(\Omega_l^\varepsilon)}$$
$$+ \sum_{l=1}^{2} \left[(2\mu_l + 3\lambda_l) \| \nabla \mathbf{u}_l^0 \|_{L^2(\Omega_l^\varepsilon)} \| \nabla \varphi_l \|_{L^2(\Omega_1^\varepsilon)} + \alpha_l^\varepsilon \| g_l(\mathbf{u}_l^1) \|_{L^2(\Omega_l^\varepsilon)} \| \varphi_l \|_{L^2(\Omega_l^\varepsilon)} \right] \leq$$
$$\left(\sum_{l=1}^{2} \left[\varepsilon \overline{h} \| \mathbf{f}_l^\varepsilon(0) \|_{L^2(\Omega_l^\varepsilon)} + (2\mu_l + 3\lambda_l) \| \mathbf{u}_l^0 \|_{H^1(\Omega_l^\varepsilon)} + \frac{\hat{\alpha}_l}{\varepsilon} \overline{h} \| g_l(\mathbf{u}_l^1) \|_{L^2(\Omega_l^\varepsilon)} \right] \right) \| (\varphi_1, \varphi_2) \|_{1,\Omega_1^\varepsilon \times \Omega_2^\varepsilon}.$$

Multiplying this inequality by $\sqrt{\varepsilon}$, we get

$$\sqrt{\varepsilon} \left\| \frac{\partial^2 \mathbf{u}_{1\zeta}^\varepsilon}{\partial t^2}(0), \frac{\partial^2 \mathbf{u}_{2\zeta}^\varepsilon}{\partial t^2}(0) \right\|_{0,\Omega_1^\varepsilon \times \Omega_2^\varepsilon} \leq c, \tag{4.29}$$

where $c = \sum_{l=1}^{2} \left[\overline{h} \| \hat{\mathbf{f}}_l^\varepsilon(0) \|_{L^2(\Omega_l)} + (2\mu_l + 3\lambda_l) \| \hat{\mathbf{u}}_l^0 \|_{H^1(\Omega_l)} + \hat{\alpha}_l \overline{h} c^\star \| g_l(\mathbf{u}_l^1) \|_{L^2(\Omega_l)} \right]$ is independent of ε.

If we pass to the limit in (4.28) as ζ tends to zero, then

$$\left\| \left(\frac{\partial^2 \mathbf{u}_1^\varepsilon}{\partial t^2}, \frac{\partial^2 \mathbf{u}_2^\varepsilon}{\partial t^2} \right) \right\|^2_{0,\Omega_1^\varepsilon \times \Omega_2^\varepsilon} + C_K \left(\mu_1 \left\| \nabla \left(\frac{\partial \mathbf{u}_1^\varepsilon}{\partial t} \right) \right\|^2_{L^2(\Omega_1^\varepsilon)} + \mu_2 \left\| \nabla \left(\frac{\partial \mathbf{u}_2^\varepsilon}{\partial t} \right) \right\|^2_{L^2(\Omega_2^\varepsilon)} \right)$$
$$\leq \left\| \left(\frac{\partial^2 \mathbf{u}_1^\varepsilon}{\partial t^2}(0), \frac{\partial^2 \mathbf{u}_2^\varepsilon}{\partial t^2}(0) \right) \right\|^2_{0,\Omega_1^\varepsilon \times \Omega_2^\varepsilon} + \sum_{l=1}^{2} (2\mu_l + 3\lambda_l + \mu_l C_K) \left\| \nabla \mathbf{u}_l^1 \right\|^2_{L^2(\Omega_l^\varepsilon)}$$
$$+ \sum_{l=1}^{2} \left[\frac{\varepsilon^2 \overline{h}^2}{\mu_l C_K} \left\| \frac{\partial \mathbf{f}_l^\varepsilon}{\partial t}(t) \right\|^2_{L^2(\Omega_l^\varepsilon)} + \frac{\varepsilon^2 \overline{h}^2}{\mu_l C_K} \left\| \frac{\partial \mathbf{f}_l^\varepsilon}{\partial t}(0) \right\|^2_{L^2(\Omega_l^\varepsilon)} + \frac{\varepsilon^2 \overline{h}^2}{\mu_l C_K} \int_0^t \left\| \frac{\partial^2 \mathbf{f}_l^\varepsilon}{\partial t^2}(s) \right\|^2_{L^2(\Omega_l^\varepsilon)} ds \right]$$
$$+ \int_0^t \left[\left\| \left(\frac{\partial^2 \mathbf{u}_1^\varepsilon}{\partial t^2}, \frac{\partial^2 \mathbf{u}_2^\varepsilon}{\partial t^2} \right) \right\|^2_{0,\Omega_1^\varepsilon \times \Omega_2^\varepsilon} + C_K \left(\mu_1 \left\| \nabla \left(\frac{\partial \mathbf{u}_1^\varepsilon}{\partial t} \right) \right\|^2_{L^2(\Omega_1^\varepsilon)} + \mu_2 \left\| \nabla \left(\frac{\partial \mathbf{u}_2^\varepsilon}{\partial t} \right) \right\|^2_{L^2(\Omega_2^\varepsilon)} \right) \right] ds. \tag{4.30}$$

Multiplying now (4.30) by ε, we have

$$\varepsilon \left[\left\| \left(\frac{\partial^2 \mathbf{u}_1^\varepsilon}{\partial t^2}, \frac{\partial^2 \mathbf{u}_2^\varepsilon}{\partial t^2} \right) \right\|^2_{0,\Omega_1^\varepsilon \times \Omega_2^\varepsilon} + C_K \sum_{l=1}^{2} \left(\mu_l \left\| \nabla \left(\frac{\partial \mathbf{u}_l^\varepsilon}{\partial t} \right) \right\|^2_{L^2(\Omega_l^\varepsilon)} \right) \right]$$
$$\leq B + \int_0^t \varepsilon \left[\left\| \left(\frac{\partial^2 \mathbf{u}_1^\varepsilon}{\partial t^2}, \frac{\partial^2 \mathbf{u}_2^\varepsilon}{\partial t^2} \right) \right\|^2_{0,\Omega_1^\varepsilon \times \Omega_2^\varepsilon} + C_K \sum_{l=1}^{2} \left(\mu_l \left\| \nabla \left(\frac{\partial \mathbf{u}_l^\varepsilon}{\partial t} \right) \right\|^2_{L^2(\Omega_l^\varepsilon)} \right) \right] ds,$$

where

$$B = c^2 + (2\mu_+ + 3\lambda_+ + \mu_+ C_K) \left\| (\nabla \hat{\mathbf{u}}_1^1, \nabla \hat{\mathbf{u}}_2^1) \right\|^2_{0,\Omega_1 \times \Omega_2} + \frac{\overline{h}^2}{\mu_- C_K} \left\| \left(\frac{\partial \hat{\mathbf{f}}_1}{\partial t}(t), \frac{\partial \hat{\mathbf{f}}_2}{\partial t}(t) \right) \right\|^2_{0,\Omega_1 \times \Omega_2}$$
$$+ \frac{\overline{h}^2}{\mu_- C_K} \left\| \left(\frac{\partial \hat{\mathbf{f}}_1}{\partial t}(0), \frac{\partial \hat{\mathbf{f}}_2}{\partial t}(0) \right) \right\|^2_{0,\Omega_1 \times \Omega_2} + \frac{\overline{h}^2}{\mu_- C_K} \int_0^t \left\| \left(\frac{\partial \hat{\mathbf{f}}_1}{\partial t}(s), \frac{\partial \hat{\mathbf{f}}_2}{\partial t}(s) \right) \right\|^2_{0,\Omega_1 \times \Omega_2} ds.$$

By Gronwall's inequality there exists a constant C independent of ε, such that

$$\varepsilon \left[\left\| \left(\frac{\partial^2 \mathbf{u}_1^\varepsilon}{\partial t^2}, \frac{\partial^2 \mathbf{u}_2^\varepsilon}{\partial t^2} \right) \right\|^2_{0,\Omega_1^\varepsilon \times \Omega_2^\varepsilon} + \left\| \nabla \left(\frac{\partial \mathbf{u}_1^\varepsilon}{\partial t}, \frac{\partial \mathbf{u}_2^\varepsilon}{\partial t} \right) \right\|^2_{0,\Omega_1^\varepsilon \times \Omega_2^\varepsilon} \right] \leq C.$$

Thus we obtain (4.21). ■

4.3 Convergence results and limit problem

In this section, we prove the convergence results and give the limit problem in the fixed domain Ω.

4.3.1 Convergence results

Theorem 4.3.1 *Assume that the assumptions of Theorems 4.2.2 and 4.2.3 hold, then there exists* $(\mathbf{u}_1^\star, \mathbf{u}_2^\star) = (u_{1i}^\star, u_{2i}^\star)$ *in* $L^2(0,T;H_z) \cap L^\infty(0,T;H_z)$, $i = 1, 2$, *such that*

$$\left.\begin{array}{l} (\hat{u}_{1i}^\varepsilon, \hat{u}_{2i}^\varepsilon) \rightharpoonup (u_{1i}^\star,\ u_{2i}^\star) \\ \\ \left(\dfrac{\partial \hat{u}_{1i}^\varepsilon}{\partial t}, \dfrac{\partial \hat{u}_{2i}^\varepsilon}{\partial t}\right) \rightharpoonup \left(\dfrac{\partial u_{1i}^\star}{\partial t}, \dfrac{\partial\ u_{2i}^\star}{\partial t}\right) \end{array} \quad (1 \le i \le 2) \right\} \begin{array}{l} \text{weakly in } L^2(0,T;H_z), \\ \text{weakly} \star \text{ in } L^\infty(0,T;H_z). \end{array} \tag{4.31}$$

$$\left.\begin{array}{l} \left(\varepsilon\dfrac{\partial \hat{u}_{1i}^\varepsilon}{\partial x_j}, \varepsilon\dfrac{\partial \hat{u}_{2i}^\varepsilon}{\partial x_j}\right) \rightharpoonup (0,0) \\ \\ \left(\varepsilon\dfrac{\partial^2 \hat{u}_{1i}^\varepsilon}{\partial x_j \partial t}, \varepsilon\dfrac{\partial^2 \hat{u}_{2i}^\varepsilon}{\partial x_j \partial t}\right) \rightharpoonup (0,0) \end{array} \quad i,j = 1,2 \right\} \begin{array}{l} \text{weakly in } L^2(0,T;L^2(\Omega_1) \times L^2(\Omega_2)), \\ \text{weakly} \star \text{ in } L^\infty(0,T;L^2(\Omega_1) \times L^2(\Omega_2)). \end{array} \tag{4.32}$$

$$\left.\begin{array}{l} \left(\varepsilon\dfrac{\partial \hat{u}_{1i}^\varepsilon}{\partial t}, \varepsilon\dfrac{\partial \hat{u}_{2i}^\varepsilon}{\partial t}\right) \rightharpoonup (0,0) \\ \\ \left(\varepsilon\dfrac{\partial^2 \hat{u}_{1i}^\varepsilon}{\partial t^2}, \varepsilon\dfrac{\partial^2 \hat{u}_{2i}^\varepsilon}{\partial t^2}\right) \rightharpoonup (0,0) \end{array} \quad (1 \le i \le 2) \right\} \begin{array}{l} \text{weakly in } L^2(0,T;L^2(\Omega_1) \times L^2(\Omega_2)), \\ \text{weakly} \star \text{ in } L^\infty(0,T;L^2(\Omega_1) \times L^2(\Omega_2)). \end{array} \tag{4.33}$$

$$\left.\begin{array}{l} \left(\varepsilon^2\dfrac{\partial \hat{u}_{13}^\varepsilon}{\partial x_i}, \varepsilon^2\dfrac{\partial \hat{u}_{23}^\varepsilon}{\partial x_i}\right) \rightharpoonup (0,0) \\ \\ \left(\varepsilon^2\dfrac{\partial^2 \hat{u}_{1i}^\varepsilon}{\partial x_i \partial t}, \varepsilon^2\dfrac{\partial^2 \hat{u}_{2i}^\varepsilon}{\partial x_i \partial t}\right) \rightharpoonup (0,0) \end{array} \quad (1 \le i \le 2) \right\} \begin{array}{l} \text{weakly in } L^2(0,T;L^2(\Omega_1) \times L^2(\Omega_2)), \\ \text{weakly} \star \text{ in } L^\infty(0,T;L^2(\Omega_1) \times L^2(\Omega_2)). \end{array} \tag{4.34}$$

$$\left.\begin{array}{l} \left(\varepsilon^2\dfrac{\partial \hat{u}_{13}^\varepsilon}{\partial t}, \varepsilon^2\dfrac{\partial \hat{u}_{23}^\varepsilon}{\partial t}\right) \rightharpoonup (0,0) \\ \left(\varepsilon\dfrac{\partial^2 \hat{u}_{13}^\varepsilon}{\partial z \partial t}, \varepsilon\dfrac{\partial^2 \hat{u}_{23}^\varepsilon}{\partial z \partial t}\right) \rightharpoonup (0,0) \quad (1 \le i \le 2)) \\ \left(\varepsilon^2\dfrac{\partial^2 \hat{u}_{13}^\varepsilon}{\partial t^2}, \varepsilon^2\dfrac{\partial^2 \hat{u}_{23}^\varepsilon}{\partial t^2}\right) \rightharpoonup (0,0) \end{array} \right\} \begin{array}{l} \text{weakly in} \\ L^2(0,T;L^2(\Omega_1) \times L^2(\Omega_2)), \\ \text{weakly} \star \text{ in} \\ L^\infty(0,T;L^2(\Omega_1) \times L^2(\Omega_2)). \end{array} \tag{4.35}$$

$$\left.\begin{array}{l} \left(g_{1i}\left(\dfrac{\partial \hat{u}_{1i}^\varepsilon}{\partial t}\right), g_{2i}\left(\dfrac{\partial \hat{u}_{2i}^\varepsilon}{\partial t}\right)\right) \rightharpoonup \left(g_{1i}\left(\dfrac{\partial u_{1i}^\star}{\partial t}\right), g_{2i}\left(\dfrac{\partial\ u_{2i}^\star}{\partial t}\right)\right) \\ \left(\varepsilon g_{13}\left(\varepsilon\dfrac{\partial \hat{u}_{13}^\varepsilon}{\partial t}\right), \varepsilon g_{23}\left(\varepsilon\dfrac{\partial \hat{u}_{23}^\varepsilon}{\partial t}\right)\right) \rightharpoonup (0,0) \end{array} \right\} \begin{array}{l} \text{weakly in} \\ L^2(0,T;L^2(\Omega_1) \times L^2(\Omega_2)), \\ \text{weakly} \star \text{ in} \\ L^\infty(0,T;L^2(\Omega_1) \times L^2(\Omega_2)). \end{array} \tag{4.36}$$

Proof of Theorem 4.3.1. According to estimates (4.19) and (4.21), there exists a fixed constant C which does not depend on ε such that

$$\left\|\left(\frac{\partial \hat{u}_{1i}^{\varepsilon}}{\partial z}, \frac{\partial \hat{u}_{2i}^{\varepsilon}}{\partial z}\right)\right\|_{0,\Omega_1\times\Omega_2}^2 \leq C, \quad \left\|\frac{\partial}{\partial z}\left(\frac{\partial \widehat{u}_{1i}^{\varepsilon}}{\partial t}, \frac{\partial \widehat{u}_{2i}^{\varepsilon}}{\partial t}\right)\right\|_{0,\Omega_1\times\Omega_2}^2 \leq C \quad (1 \leq i \leq 2).$$

Using the above estimate and Poincaré's inequality in the domain $\Omega = \Omega_1 \cup \Omega_2$, we obtain

$$\|(\widehat{u}_{1i}^{\varepsilon}, \widehat{u}_{2i})\| \leq \bar{h}\left\|\left(\frac{\partial \hat{u}_{1i}^{\varepsilon}}{\partial z}, \frac{\partial \hat{u}_{2i}^{\varepsilon}}{\partial z}\right)\right\|_{0,\Omega_1\times\Omega_2},$$

$$\left\|\left(\frac{\partial \widehat{u}_{1i}^{\varepsilon}}{\partial t}, \frac{\partial \widehat{u}_{2i}^{\varepsilon}}{\partial t}\right)\right\|_{0,\Omega_1\times\Omega_2} \leq \bar{h}\left\|\frac{\partial}{\partial z}\left(\frac{\partial \widehat{u}_{1i}^{\varepsilon}}{\partial t}, \frac{\partial \widehat{u}_{2i}^{\varepsilon}}{\partial t}\right)\right\|_{0,\Omega_1\times\Omega_2} \qquad (1 \leq i \leq 2).$$

We deduce that the sequences $(\widehat{u}_{1i}^{\varepsilon}, \widehat{u}_{2i}^{\varepsilon})$ and $\left(\frac{\partial \widehat{u}_{1i}^{\varepsilon}}{\partial t}, \frac{\partial \widehat{u}_{2i}^{\varepsilon}}{\partial t}\right)$ are bounded in $L^2(0,T;L^2(\Omega_1)\times L^2(\Omega_2))\cap L^\infty(0,T;L^2(\Omega_1)\times L^2(\Omega_2))$ this implies the existence of(u_{1i}^*, u_{2i}^*), and $\left(\frac{\partial u_{1i}^*}{\partial t}, \frac{\partial u_{2i}^*}{\partial t}\right)$ in $L^2(0,T;L^2(\Omega_1)\times L^2(\Omega_2))\cap L^\infty(0,T;L^2(\Omega_1)\times L^2(\Omega_2))$ such that $(\widehat{u}_{1i}^{\varepsilon}, \widehat{u}_{2i}^{\varepsilon})$, and $\left(\frac{\partial \widehat{u}_{1i}^{\varepsilon}}{\partial t}, \frac{\partial \widehat{u}_{2i}^{\varepsilon}}{\partial t}\right)$ converges weakly to (u_{1i}^*, u_{2i}^*) $\left(\left(\frac{\partial u_{1i}^*}{\partial t}, \frac{\partial u_{2i}^*}{\partial t}\right), \text{ respectively}\right)$ in $L^2(0,T;L^2(\Omega_1)\times L^2(\Omega_2))\cap L^\infty(0,T;L^2(\Omega_1)\times L^2(\Omega_2))$. Also, the convergence (4.32)-(4.36) are deduced from (4.19)-(4.21) and (4.31). ■

In the following subsection, we give the convergence of our problem towards the weak generalized equation.

4.3.2 The limit problem

Theorem 4.3.2 *With the same assumptions of Theorem 4.2.2, the solution* $(\mathbf{u}_1^\star, \mathbf{u}_2^\star)$ *satisfies*

$$\begin{aligned}
&\sum_{1\leq l,i\leq 2} \mu_l \int_{\Omega_l} \frac{\partial u_{li}^\star}{\partial z}\frac{\partial}{\partial z}\left(\hat{\varphi}_{li} - \frac{\partial u_{li}^\star}{\partial t}\right) dx'dz + \hat{J}(\hat{\varphi}_1, \hat{\varphi}_2) \\
&+ \sum_{1\leq l,i\leq 2} \hat{\alpha}_l \int_{\Omega_l} g_{li}\left(\frac{\partial u_{li}^\star}{\partial t}\right)\left(\hat{\varphi}_{li} - \frac{\partial u_{li}^\star}{\partial t}\right) dx'dz - \hat{J}\left(\left(\frac{\partial u_1^\star}{\partial t}\right), \left(\frac{\partial u_2^\star}{\partial t}\right)\right) \\
&\geq \sum_{1\leq l,i\leq 2} \int_{\Omega_l} \hat{f}_{li}\left(\hat{\varphi}_{li} - \frac{\partial u_{li}^\star}{\partial t}\right) dx'dz, \qquad \forall(\hat{\varphi}_1, \hat{\varphi}_2) \in \overline{V(\Omega_1)} \times \overline{V(\Omega_2)}.
\end{aligned} \tag{4.37}$$

$$(u_{1i}^\star(x', z, 0), u_{2i}^\star(x', z, 0)) = (\hat{u}_{1i}^0, \hat{u}_{2i}^0) \qquad \forall i = 1, 2. \tag{4.38}$$

$$\left(-\mu_1\frac{\partial^2 \mathbf{u}_1^\star}{\partial z^2} + \hat{\alpha}_1 g_1\left(\frac{\partial \mathbf{u}_1^\star}{\partial t}\right), -\mu_2\frac{\partial^2 \mathbf{u}_2^\star}{\partial z^2} + \hat{\alpha}_2\, g_2\left(\frac{\partial \mathbf{u}_2^\star}{\partial t}\right)\right) = (\hat{\mathbf{f}}_1, \hat{\mathbf{f}}_2) \quad in\ (L^2(\Omega_1))^2 \times (L^2(\Omega_2))^2. \tag{4.39}$$

Proof of Theorem 4.3.2. Using the convergence result of Theorem 4.3.1 in the variational inequality (4.18), with the fact that $\hat{J}$ is convex and lower semi-continuous, we obtain

$$\sum_{1\leq i,l\leq 2}\mu_l\int_{\Omega_l}\frac{\partial u^*_{li}}{\partial z}\frac{\partial}{\partial z}\left(\widehat{\varphi}_{li}-\frac{\partial u^*_{li}}{\partial t}dx'dz\right)+\sum_{1\leq i,l\leq 2}\widehat{\alpha}_l\int_{\Omega_l}g_{li}\left(\frac{\partial u^*_{li}}{\partial t}\right)+\widehat{J}(\widehat{\varphi}_1,\widehat{\varphi}_2)$$
$$-\widehat{J}\left(\frac{\partial u^*_1}{\partial t},\frac{\partial u^*_2}{\partial t}\right)\geq\sum_{1\leq l,i\leq 2}\int_{\Omega_l}\widehat{f}_{li}\left(\widehat{\varphi}_{li}-\frac{\partial u^*_{li}}{\partial t}\right)dx'dz.$$

By choosing (see [9])

$$\hat{\varphi}_{1i}=\frac{\partial u^{\star}_{1i}}{\partial t}\pm\psi_{1i},\ \ \hat{\varphi}_{2i}=\frac{\partial u^{\star}_{2i}}{\partial t}\pm\psi_{2i},\ \ i=1,2,$$

with $(\psi_{1i},\psi_{2i})_{(1\leq i\leq 2)}\in H^1_0(\Omega_1)\times H^1_0(\Omega_2)$, we find

$$\mu_1\sum_{i=1}^{2}\int_{\Omega_1}\frac{\partial u^{\star}_{1i}}{\partial z}(t)\frac{\partial\psi_{1i}}{\partial z}dx'dz+\mu_2\sum_{i=1}^{2}\int_{\Omega_2}\frac{\partial u^{\star}_{2i}}{\partial z}(t)\frac{\partial\psi_{2i}}{\partial z}dx'dz+\hat{\alpha}_1\sum_{i=1}^{2}\int_{\Omega_1}g_{1i}\left(\frac{\partial u^{\star}_{1i}}{\partial t}\right)\psi_{1i}dx'dz$$
$$+\hat{\alpha}_2\sum_{i=1}^{2}\int_{\Omega_2}g_{2i}\left(\frac{\partial u^{\star}_{2i}}{\partial t}\right)\psi_{2i}dx'dz=\sum_{i=1}^{2}\int_{\Omega_1}\hat{f}_{1i}\psi_{1i}dx'dz+\sum_{i=1}^{2}\int_{\Omega_2}\hat{f}_{2i}\psi_{2i}dx'dz.$$

By employing Green's formula, we deduce

$$-\mu_1\sum_{i=1}^{2}\int_{\Omega_1}\frac{\partial}{\partial z}\left(\frac{\partial u^{\star}_{1i}}{\partial z}\right)(t)\psi_{1i}dx'dz-\mu_2\sum_{i=1}^{2}\int_{\Omega_2}\frac{\partial}{\partial z}\left(\frac{\partial u^{\star}_{2i}}{\partial z}(t)\right)\psi_{2i}dx'dz$$
$$+\hat{\alpha}_1\sum_{i=1}^{2}\int_{\Omega_1}g_{1i}\left(\frac{\partial u^{\star}_{1i}}{\partial t}\right)\psi_{1i}dx'dz+\hat{\alpha}_2\sum_{i=1}^{2}\int_{\Omega_2}g_{2i}\left(\frac{\partial u^{\star}_{2i}}{\partial t}\right)\psi_{2i}dx'dz$$
$$=\sum_{i=1}^{2}\int_{\Omega_1}\hat{f}_{1i}\psi_{1i}dx'dz+\sum_{i=1}^{2}\int_{\Omega_2}\hat{f}_{2i}\psi_{2i}dx'dz.$$

So,

$$\left\langle\left(-\mu_1\left(\frac{\partial^2 u^{\star}_{1i}}{\partial z^2}(t)\right)-\hat{f}_{1i}+\hat{\alpha}_1g_{1i}\left(\frac{\partial u^{\star}_{1i}}{\partial t}\right),-\mu_2\left(\frac{\partial u^{\star}_{2i}}{\partial z}(t)\right)-\hat{f}_{2i}+\hat{\alpha}_2g_{2i}\left(\frac{\partial u^{\star}_{2i}}{\partial t}\right)\right),(\psi_{1i},\psi_{2i})\right\rangle=0,$$
$$i=1,2,\quad\forall(\psi_{1i},\psi_{2i})\in H^1_0(\Omega_1)\times H^1_0(\Omega_2).$$

Thus

$$\left(-\mu_1\left(\frac{\partial^2 u^{\star}_{1i}}{\partial z^2}\right)+\hat{\alpha}_1g_{1i}\left(\frac{\partial u^{\star}_{1i}}{\partial t}\right),-\mu_2\left(\frac{\partial u^{\star}_{2i}}{\partial z}\right)+\hat{\alpha}_2g_{2i}\left(\frac{\partial u^{\star}_{2i}}{\partial t}\right)\right)=\left(\hat{f}_{1i},\hat{f}_{2i}\right),$$
$$i=1,2\quad\text{in}\quad H^{-1}(\Omega_1)\times H^{-1}(\Omega_2),\tag{4.40}$$

and as $\left(\hat{f}_{1i},\hat{f}_{2i}\right)\in L^2(\Omega_1)\times L^2(\Omega_2)$, then (4.40) is valid in $L^2(\Omega_1)\times L^2(\Omega_2)$. ■

Theorem 4.3.3 *Under the assumptions of previous theorems, we have the following equality*

$$\mu_1\pi_1^\star = \mu_2\pi_2^\star \quad in \ (L^2(\omega))^2, \tag{4.41}$$

$$\int_\omega \hat{\kappa}\left(\left|\psi + \frac{\partial s_1^\star}{\partial t} - \frac{\partial s_2^\star}{\partial t} - s\right| - \left|\frac{\partial s_1^\star}{\partial t} - \frac{\partial s_2^\star}{\partial t} - s\right|\right)dx' - \int_\omega \mu_l\pi_l^\star\psi dx' \geq 0, \ \forall\psi \in (L^2(\omega))^2, \tag{4.42}$$

$$\begin{cases} \mu_l|\pi_l^\star| < \hat{\kappa} \Rightarrow \dfrac{\partial s_1^\star}{\partial t} - \dfrac{\partial s_2^\star}{\partial t} = s, \\ \\ \mu_l|\pi_l^\star| = \hat{\kappa} \Rightarrow \exists\beta > 0 \ such \ that \ \dfrac{\partial s_1^\star}{\partial t} - \dfrac{\partial s_2^\star}{\partial t} = s + \beta\mu_l\pi_l^\star, \end{cases} \quad a.e. \ on \ \omega\times]0,T[\tag{4.43}$$

where

$$s_l^\star(x,t) = \mathbf{u}_l^\star(x',0,t) \ \ and \ \ \pi_l^\star(x,t) = \frac{\partial \mathbf{u}_l^\star}{\partial z}(x',0,t) \ , \ l = 1,2.$$

Also $\mathbf{u}_l^\star$ *satisfy the weak generalized equation:*

$$\int_\omega\left(\tilde{F} + \tilde{G} + \mu_1\int_0^h \mathbf{u}_1^\star(x',y,t)dy + \mu_2\int_{-h}^0 \mathbf{u}_2^\star(x',y,t)dy\right)\nabla\psi(x')dx' = 0, \ \forall\psi\in H^1(\omega), \tag{4.44}$$

where

$$\tilde{F} = \int_0^h F(x',y,t)dy - hF(x',y,t), \ \tilde{G} = -\int_0^h G(x',y,t)dy + hG(x',y,t).$$

Proof of Theorem 4.3.3. From [4], we can choose $\widehat{\varphi}_l$ in (4.37) such that $\hat{\varphi}_{1i} = \dfrac{\partial u_{1i}^\star}{\partial t} + \psi_{1i}$, $\hat{\varphi}_{2i} = \dfrac{\partial u_{2i}^\star}{\partial t} + \psi_{2i}$, $i = 1,2$, with $(\psi_{1i},\psi_{2i})_{(1\leq i\leq 2)} \in H^1_{\Gamma_1\cup\Gamma_{L_1}}(\Omega_1)\times H^1_{\Gamma_2\cup\Gamma_{L_2}}(\Omega_2)$, where

$$H^1_{\Gamma_l\cup\Gamma_{L_l}}(\Omega_l) = \{\varphi_l\in H^1(\Omega_l) : \varphi_l = 0 \ \text{on} \ \Gamma_l\cup\Gamma_{L_l}\} \ \ (\text{for } l = 1,2),$$

then, we find

$$\begin{aligned} &\sum_{i=1}^2 \mu_1\int_{\Omega_1}\frac{\partial u_{1i}^\star}{\partial z}\frac{\partial\psi_{1i}}{\partial z}dx'dz + \sum_{i=1}^2\mu_2\int_{\Omega_2}\frac{\partial u_{2i}^\star}{\partial z}\frac{\partial\psi_{2i}}{\partial z}dx'dz \\ &+ \sum_{1\leq l,i\leq 2}\hat{\alpha}_l\int_{\Omega_l} g_{li}\psi_{li}dx'dz + \hat{J}\left(\frac{\partial u_{1i}^\star}{\partial t} + \psi_{1i}, \frac{\partial u_{2i}^\star}{\partial t} + \psi_{2i}\right) - \hat{J}\left(\frac{\partial u_1^\star}{\partial t}, \frac{\partial u_2^\star}{\partial t}\right) \\ &\geq \sum_{1\leq l,i\leq 2}\int_{\Omega_l}\hat{f}_{li}\psi_{li}dx'dz, \qquad \forall(\psi_{1i},\psi_{2i})_{(1\leq i\leq 2)}\in H^1_{\Gamma_1\cup\Gamma_{L_1}}(\Omega_1)\times H^1_{\Gamma_2\cup\Gamma_{L_2}}(\Omega_2). \end{aligned}$$

Using Green's formula on each sub-domain $\Omega_l, l = 1, 2$, and $\eta = (0, 0, -1)$, we get

$$-\sum_{1\le l,i\le 2} \mu_l \int_{\Omega_l} \frac{\partial^2 u^\star_{li}}{\partial z^2}(t)\psi_{li} dx' dz + \int_\omega \mu_1 \pi^\star_1.\psi_1 dx' - \int_\omega \mu_2 \pi^\star_2.\psi_2 dx'$$
$$+ \int_\omega \hat{\kappa}\left(\left|\psi_1 - \psi_2 + \frac{\partial s^\star_1}{\partial t} - \frac{\partial s^\star_2}{\partial t} - s\right| - \left|\frac{\partial s^\star_1}{\partial t} - \frac{\partial s^\star_2}{\partial t} - s\right|\right) dx'$$
$$+ \sum_{1\le l,i\le 2} \hat{\alpha}_l \int_{\Omega_l} g_{li}\left(\frac{\partial u^\star_{li}}{\partial t}\right)\psi_{li} dx' dz \ge \sum_{1\le l,i\le 2} \int_{\Omega_l} \hat{f}_{li}\psi_{li} dx' dz,$$

by (4.39) and for $\psi_l \in (H^1_{\Gamma_l\cup\Gamma_{L_l}}(\Omega_l))^2, \; l = 1, 2,$

$$\int_\omega \hat{\kappa}\left(\left|\psi_1 - \psi_2 + \frac{\partial s^\star_1}{\partial t} - \frac{\partial s^\star_2}{\partial t} - s\right| - \left|\frac{\partial s^\star_1}{\partial t} - \frac{\partial s^\star_2}{\partial t} - s\right|\right) dx' - \int_\omega (\mu_1\pi^\star_1.\psi_1 - \mu_2\pi^\star_2.\psi_2)\, dx' \ge 0.$$

This inequality remains valid for any $\psi_l \in (D(\omega))^2, \; l = 1, 2$, and by the density of $D(\omega)$ in $L^2(\omega)$, we deduce

$$\begin{aligned}&\int_\omega \hat{\kappa}\left(\left|\psi_1 - \psi_2 + \frac{\partial s^\star_1}{\partial t} - \frac{\partial s^\star_2}{\partial t} - s\right| - \left|\frac{\partial s^\star_1}{\partial t} - \frac{\partial s^\star_2}{\partial t} - s\right|\right) dx' \\ &- \int_\omega (\mu_1\pi^\star_1.\psi_1 - \mu_2\pi^\star_2.\psi_2)\, dx' \ge 0, \qquad \forall \psi_1, \psi_2 \in (L^2(\omega))^2.\end{aligned} \tag{4.45}$$

In the particular case for $\psi_1 = \psi_2 = \pm\psi$, we obtain

$$\int_\omega (\mu_1\pi^\star_1 - \mu_2\pi^\star_2) = 0, \qquad \forall \psi \in (L^2(\omega))^2,$$

which implies (4.41).

From (4.41) and (4.45), we deduce the inequality (4.42). The proofs of (4.43) are those given in case of the problem of fluid (see [3]).

To prove (4.44), we integrate twice the first equation of (4.39) between 0 and z, and the second between z and 0, then by setting $z = h$ in the first equation and $z = -h$ in the second equation, we find

$$\begin{cases} \mu_1 s^\star_{1i} + \mu_1 h \pi^\star_{1i} = -\int_0^h \int_0^\xi \hat{\alpha}_1 g_{1i}\left(\frac{\partial u^\star_{1i}}{\partial t}\right)(x, \theta, t) d\theta d\xi + \int_0^h \int_0^\xi \hat{f}_{1i}(x, \theta, t) d\theta d\xi, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2 \\ \mu_2 s^\star_{2i} - \mu_2 h \pi^\star_{2i} = -\int_{-h}^0 \int_\xi^0 \hat{\alpha}_2 g_{2i}\left(\frac{\partial u^\star_{2i}}{\partial t}\right)(x, \theta, t) d\theta d\xi + \int_{-h}^0 \int_\xi^0 \hat{f}_{2i}(x, \theta, t) d\theta d\xi. \end{cases} \tag{4.46}$$

Consequently, (4.46) with (4.41) leads to

$$\begin{aligned}\mu_1 s^\star_{1i} + \mu_2 s^\star_{2i} = &-\int_0^h \int_0^\xi \hat{\alpha}_1 g_{1i}\left(\frac{\partial u^\star_{1i}}{\partial t}\right)(x', \theta, t) d\theta d\xi + \int_0^h \int_0^\xi \hat{f}_{1i}(x', \theta, t) d\theta d\xi \\ &- \int_{-h}^0 \int_\xi^0 \hat{\alpha}_2 g_{2i}\left(\frac{\partial u^\star_{2i}}{\partial t}\right)(x', \theta, t) d\theta d\xi + \int_{-h}^0 \int_\xi^0 \hat{f}_{2i}(x', \theta, t) d\theta \xi.\end{aligned} \tag{4.47}$$

Now, we integrate the first equation of (4.46) between 0 and h, and the second between $-h$ and 0, we obtain

$$\begin{cases} \mu_1 \int_0^h \mathbf{u}_1^\star(x',y,t)dy = \mu_1 h \mathbf{s}_1^\star + \frac{1}{2}\mu_1 h^2 \pi_1^\star + \hat{\alpha}_1 \int_0^h G_1(x',y,t)dy - \int_0^h F_1(x',y,t)dy, \\ \\ \mu_2 \int_{-h}^0 \mathbf{u}_2^\star(x',y,t)dy = \mu_1 h \mathbf{s}_1^\star - \frac{1}{2}\mu_2 h^2 \pi_2^\star + \hat{\alpha}_2 \int_{-h}^0 G_2(x',y,t)dy - \int_{-h}^0 F_2(x',y,t)dy, \end{cases}$$

with

$$F_1(x,y,t) = \int_0^y \int_0^\xi \hat{\mathbf{f}}_1(x',\theta,t)d\theta d\xi \ and \ F_2(x',y,t) = \int_y^0 \int_\xi^0 \hat{\mathbf{f}}_2(x',\theta,t)d\theta d\xi,$$

$$G_1(x',y,t) = \int_0^y \int_0^\xi g_1\left(\frac{\partial \mathbf{u}_1^\star}{\partial t}\right)(x',\theta,t)d\theta d\xi, \ \ G_2(x',y,t) = \int_y^0 \int_\xi^0 g_2\left(\frac{\partial \mathbf{u}_2^\star}{\partial t}\right)(x',\theta,t)d\theta d\xi.$$

Therefore,

$$\begin{aligned} h(\mu_1 s_{1i}^\star + \mu_2 s_{2i}^\star) - \mu_1 \int_0^h \mathbf{u}_1^\star(x',y,t)dy - \mu_2 \int_{-h}^0 \mathbf{u}_2^\star(x',y,t)dy = \\ - \int_0^h G(x',y,t)dy + \int_0^h F_(x',y,t)dy, \end{aligned} \tag{4.48}$$

with

$$F(x',y,t) = F_1(x',y,t) + F_2(x',-y,t) \ and \ G(x',y,t) = \hat{\alpha}_1 G_1(x',y,t) + \hat{\alpha}_2 G_2(x',-y,t).$$

which gives (4.44). ■

Theorem 4.3.4 *The solution* $(\mathbf{u}_1^\star, \mathbf{u}_2^\star)$ *of the limit problem* (4.37)-(4.39) *is unique in* $L^2(0,T;H_z) \cap L^\infty(0,T;H_z)$.

Proof of Theorem 4.3.4. Let $(\mathbf{u}_1^\star, \mathbf{u}_2^\star)$ and $(\mathbf{v}_1^\star, \mathbf{v}_2^\star)$ be two solution of (4.37), then

$$\begin{aligned} &\sum_{1\le l,i\le 2} \mu_l \int_{\Omega_l} \frac{\partial u_{li}^\star}{\partial z} \frac{\partial}{\partial z}\left(\hat{\varphi}_{li} - \frac{\partial u_{li}^\star}{\partial t}\right) dx'dz + \hat{J}(\hat{\varphi}_1, \hat{\varphi}_2) \\ &+ \sum_{1\le l,i\le 2} \hat{\alpha}_l \int_{\Omega_l} g_{li}\left(\frac{\partial u_{li}^\star}{\partial t}\right)\left(\hat{\varphi}_{li} - \frac{\partial u_{li}^\star}{\partial t}\right) dx'dz - \hat{J}\left(\left(\frac{\partial \mathbf{u}_1^\star}{\partial t}\right), \left(\frac{\partial \mathbf{u}_2^\star}{\partial t}\right)\right) \\ &\ge \sum_{1\le l,i\le 2} \int_{\Omega_l} \hat{f}_{li}\left(\hat{\varphi}_{li} - \frac{\partial u_{li}^\star}{\partial t}\right) dx'dz, \qquad \forall(\hat{\varphi}_1, \hat{\varphi}_2) \in \overline{V(\Omega_1)} \times \overline{V(\Omega_2)}. \end{aligned} \tag{4.49}$$

$$\begin{aligned} &\sum_{1\le l,i\le 2} \mu_l \int_{\Omega_l} \frac{\partial v_{li}^\star}{\partial z} \frac{\partial}{\partial z}\left(\hat{\varphi}_{li} - \frac{\partial v_{li}^\star}{\partial t}\right) dx'dz + \hat{J}(\hat{\varphi}_1, \hat{\varphi}_2) \\ &+ \sum_{1\le l,i\le 2} \hat{\alpha}_l \int_{\Omega_l} g_{li}\left(\frac{\partial v_{li}^\star}{\partial t}\right)\left(\hat{\varphi}_{li} - \frac{\partial v_{li}^\star}{\partial t}\right) dx'dz - \hat{J}\left(\left(\frac{\partial \mathbf{v}_1^\star}{\partial t}\right), \left(\frac{\partial \mathbf{v}_2^\star}{\partial t}\right)\right) \\ &\ge \sum_{1\le l,i\le 2} \int_{\Omega_1} \hat{f}_{li}\left(\varphi_{li} - \frac{\partial v_{li}^\star}{\partial t}\right) dx'dz, \qquad \forall(\hat{\varphi}_1, \hat{\varphi}_2) \in \overline{V(\Omega_1)} \times \overline{V(\Omega_2)}. \end{aligned} \tag{4.50}$$

Taking $(\hat{\varphi}_1, \hat{\varphi}_2) = \left(\frac{\partial \mathbf{v}_1^\star}{\partial t}, \frac{\partial \mathbf{v}_2^\star}{\partial t} \right)$ in (4.49), then $(\hat{\varphi}_1, \hat{\varphi}_2) = \left(\frac{\partial \mathbf{u}_1^\star}{\partial t}, \frac{\partial \mathbf{u}_2^\star}{\partial t} \right)$ in (4.50) and summing the two inequalities, we find for $\overline{W}_1 = \mathbf{u}_1^\star - \mathbf{v}_1^\star$ and $\overline{W}_2 = \mathbf{u}_2^\star - \mathbf{v}_2^\star$,

$$\begin{aligned} &\mu_1 \sum_{i=1}^{2} \int_{\Omega_1} \frac{\partial}{\partial z} \left(\frac{\partial}{\partial t} \overline{W}_{1i} \right) \frac{\partial}{\partial z} \overline{W}_{1i} dx' dz + \mu_2 \sum_{i=1}^{2} \int_{\Omega_2} \frac{\partial}{\partial z} \left(\frac{\partial}{\partial t} \overline{W}_{2i} \right) \frac{\partial}{\partial z} \overline{W}_{2i} dx' dz \\ &\quad + \sum_{i,l=1}^{2} \hat{\alpha}_l \int_{\Omega_l} \left(g_{li} \left(\frac{\partial u_{li}^\star}{\partial t} \right) - g_{li} \left(\frac{\partial v_{li}^\star}{\partial t} \right) \right) \left(\frac{\partial u_{li}^\star}{\partial t} - \frac{\partial v_{li}^\star}{\partial t} \right) dx' dz \leq 0. \end{aligned} \tag{4.51}$$

Now, using the assumption (H_2), the equation (4.51) can be rewritten as:

$$\mu_1 \sum_{i=1}^{2} \int_{\Omega_1} \frac{\partial}{\partial z} \left(\frac{\partial}{\partial t} \overline{W}_{1i} \right) \frac{\partial}{\partial z} \overline{W}_{1i} dx' dz + \mu_2 \sum_{i=1}^{2} \int_{\Omega_2} \frac{\partial}{\partial z} \left(\frac{\partial}{\partial t} \overline{W}_{2i} \right) \frac{\partial}{\partial z} \overline{W}_{2i} dx' dz \leq 0,$$

which implies

$$\mu_1 \frac{1}{2} \frac{d}{dt} \left\| \frac{\partial}{\partial z} \overline{W}_1 \right\|_{0,\Omega_1}^2 + \mu_2 \frac{1}{2} \frac{d}{dt} \left\| \frac{\partial}{\partial z} \overline{W}_2 \right\|_{0,\Omega_2}^2 \leq 0.$$

Since $\overline{W}_l(0) = 0, \;\; l = 1, 2$, we have

$$\left\| \frac{\partial}{\partial z} \overline{W}_1 \right\|_{0,\Omega_1}^2 = 0 \quad and \quad \left\| \frac{\partial}{\partial z} \overline{W}_2 \right\|_{0,\Omega_2}^2 = 0.$$

Using Poincaré 's inequality, we find

$$\left\| (\overline{W}_1, \overline{W}_2) \right\|_{L^2(0,T;H_z)}^2 = \left\| (\overline{W}_1, \overline{W}_2) \right\|_{L^\infty(0,T;H_z)}^2 = 0.$$

Finally, we deduce that $(\mathbf{u}_1^\star, \mathbf{u}_2^\star) = (\mathbf{v}_1^\star, \mathbf{v}_2^\star)$ in $L^2(0,T;H_z) \cap L^\infty(0,T;H_z)$. ■

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الملخص: تركز هذه الأطروحة على دراسة التحليل التقاربي لبعض المسائل الحدودية في مجال رقيق ثلاثي الأبعاد Ω^{ε} مع شروط حدية غير خطية لنوع الاحتكاك على جزء من الحافة. الفكرة الرئيسية لهذه الدراسة هي إظهار كيفية استخلاص مسائل حد ثنائية الأبعاد عندما تميل السماكة إلى الصفر لثلاثة أنواع من مسائل الاتصال الثنائية التي تتضمن قانون الاحتكاك تريسكا أو كولومب. نبدأ أولاً بسائل غير قابل للضغط تحكمه معادلة برينكمان. تتعلق المسألة الثانية بنموذج رياضي يصف العملية الثابتة للاتصال بين الجسم الكهروضغطي والأساس. وتم تخصيص العمل الثالث الذي تم تنفيذه لمسألة النقل لمعادلة المرونة الخطية بمصطلح تبديد غير خطي. على وجه التحديد، قمنا بتحويل المسائل الأصلية المطروحة في المجال Ω^{ε} إلى مسائل مكافئة جديدة على مجال ثابت Ω مستقل عن الوسيط الصغيرε باستعمال مقياس جديد والعديد من المتراجحات، نثبت بعض التقديرات ونظريات التقارب ونحصل في النهاية على المسائل الحدية مع المعادلات الضعيفة المعممة ووحدانيتها.

الكلمات المفتاحية: سائل برينكمان؛ قانون كولومب؛ الأجسام المرنة؛الأجسام الكهرومرنة؛ معادلة رينولدز؛ قانون تريسكا.

Abstract: This thesis focuses on the study of the asymptotic analysis of some boundary value problems in a three-dimensional thin domain Ω^{ε} with nonlinear boundary conditions of friction type on a part of the boundary. The main idea of this study is to show how to derive two-dimensional limit problems when the thickness tends to zero for three types of bilateral contacts problems involving Tresca's or Coulomb's friction law. We start first with an incompressible fluid governed by the Brinkman equation. Then the second problem concerns a mathematical model describing the static process of contact between a piezoelectric body and a foundation. Finally, the third work carried out is devoted to the transmission problem for the linear elasticity equation with a nonlinear dissipative term. Precisely, we have transformed the original problems posed in the domain Ω^{ε} into new equivalent problems on a fixed domain Ω independent of a small parameter ε, and by using a new scale and several inequalities we prove some estimates and convergence theorems. Then, we obtain the limit problems with the weak generalized equation and its uniqueness.

Keywords: Brinkman fluid; Coulomb law; Elastic bodies; Electro-Elastic bodies; Reynolds equation; Tresca law.

Résumé : Cette thèse porte sur l'étude de l'analyse asymptotique de quelques problèmes aux limites dans un domaine mince tridimensionnel Ω^{ε} avec des conditions aux limites non linéaires de type frottement sur une partie de la frontière. L'idée principale de cette étude est de montrer comment dériver des problèmes limites bidimensionnels lorsque l'épaisseur tend vers zéro pour trois types des problèmes de contacts bilatéraux mettant en jeu la loi de frottement de Tresca ou de Coulomb. Nous commençons d'abord avec un fluide incompressible régi par l'équation de Brinkman. Ensuite le deuxième problème concerne un modèle mathématique décrivant le processus statique de contact entre un corps piézoélectrique et une fondation. Enfin, le troisième travail réalisé est consacré au problème de transmission de l'équation d'élasticité linéaire à terme dissipatif non linéaire. Précisément, nous transformons les problèmes originaux posés dans le domaine Ω^{ε} en de nouveaux problèmes équivalents sur un domaine fixe Ω indépendant d'un petit paramètre ε. En utilisant un changement d'échelle et plusieurs inégalités nous prouvons quelques estimations et théorèmes de convergence. Grâce à ces estimates, nous obtenons finalement les problèmes limites associes avec les équations généralisées faibles et leurs unicités.

Mots-clés: Corps élastiques; Corps Electro-élastique; Equations de Reynolds; Fluide de Brinkman; Loi de Coulomb; Loi de Tresca.