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## **Dedication**

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to whom I owe a lot

To my inspired parents, without whom my success would not be possible.

To my husband and my children, and my friends who encourage and support me.

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# Abstract

It is well known that a wide class of problems, which arises in pure and applied sciences can be studied in the unified framework of the system of absolute value equations (AVE) of the type:

$$Ax - B|x| = b, \quad A, B \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n,$$

where  $|x|$  is the vector in  $\mathbb{R}^n$  with absolute values of components of  $x$ . In this thesis, we highlight, on one hand the unique solvability of the AVEs where some sufficient weaker conditions are given. These results are also generalized to the unique solvability of horizontal and standard linear complementarity problems. On the other hand, for its numerical solution, Picard's iterative and preconditioned conjugate gradient methods are proposed. Finally, some numerical results are given to confirm the efficiency of our suggested approaches for solving the AVEs.

**Keywords:** Absolute value equations, Linear complementarity problems, Linear system, Singular value, Iterative methods, Unconstrained convex quadratic optimization.

# List of publications

## Paper 1

M. Achache, N. Anane. On unique solvability and Picard's iterative method for absolute value equations. **Bulletin of the Transilvania University of Brasov. Series III: Mathematics and Computer Science**, Vol. 1(63), No. 1 - 2021, 13-26. <https://doi.org/10.31926/but.mif.2021.1.63.1.2>. ( Journal in alphabetical order)

## Paper 2

Anane, N., Achache, M., Preconditioned conjugate gradient methods for absolute value equations. **J. Numer. Anal. Approx. Theory**, vol. 48 (2020) no. 1, 3-14.

# Glossary of Notation

## Problem Classes

- AVE : Absolute value equations .  
SLCP : Standard linear complementarity problem.  
HLCP : Horizontal linear complementarity problem .  
CG : Conjugate gradient.  
PCG : Preconditioned conjugate gradient.  
SPD : symmetric positive definite.  
ICF : Incomplete Cholesky Factorization.

## Spaces

- $\mathbb{R}^n$  : The real  $n$ -dimensional space.  
 $\mathbb{R}_+^n$  : The nonnegative orthant of  $\mathbb{R}^n$ .  
 $\mathbb{R}_{++}^n$  : The positive orthant of  $\mathbb{R}^n$ .  
 $\mathbb{R}^{n \times n}$  : The set of all  $n \times n$  squared matrices.

## Vectors

- $x_i$  : The  $i$ -th component of  $x$ .  
 $x^T$  :  $(x_1, \dots, x_n)$  the transpose of a vector  $x$ , with components  $x_i$   
 $x \geq 0$  =  $x_i \geq 0$  ,  $\forall i$ .  
 $x > 0$  :  $x_i > 0$ ,  $\forall i$ .  
 $e$  =  $(1, \dots, 1)^T \in \mathbb{R}^n$ .  
 $|x|$  = Absolute value of a vector  $x \in \mathbb{R}^n$ .  $|x| = (|x_i|), i = 1, \dots, n$ .  
 $x^T y$  =  $\sum_{i=1}^n x_i y_i$  the standard inner product in  $\mathbb{R}^n$ .  
 $sign(x)$  = denotes a vector with the components equal to -1, 0 or 1.  
 $\|x\|$  : denote the Euclidean norm  $(x^T x)^{1/2}$ .

## Matrices

$I$	:	Identity matrix .
$0_{n \times n}$	:	The null matrix of type $(n, n)$ .
$A^T$	:	The transposed matrix of $A$ .
$A^{-1}$	:	The inverse of a regular matrix $A$ .
$A$ invertible	$\cong$	$A$ non singular.
$ A $	$=$	$ (a_{ij}) $ , the absolute value of $a_{ij}$ .
$\lambda_i(A), i = 1, \dots, n$	:	The eigenvalues of $A \in \mathbb{R}^{n \times n}$ .
$\lambda_{\max}(A), i = 1, \dots, n$	:	The largest eigenvalues of $A \in \mathbb{R}^{n \times n}$ .
$\lambda_{\min}(A), i = 1, \dots, n$	:	The smallest eigenvalues of $A \in \mathbb{R}^{n \times n}$ .
$\sigma_{\max}(A)$	:	The largest singular value of matrix $A$ .
$\sigma_{\min}(A)$	:	The smallest singular value of matrix $A$ .
$tr(A)$	$=$	$\sum_{i=1}^n a_{ii} = \sum_{i=1}^n \lambda_i(A)$ , (the trace of a matrix $A \in \mathbb{R}^{n \times n}$ ).
$\rho(A)$	$=$	$\max  \lambda_i(A) $ , ( the spectral radius of $A$ ).
$\ A\ _2$	$=$	$\sqrt{\rho(A^T A)}$ , (the spectral norm of $A$ ).
$D(x)$	$=$	diagonal matrix whose diagonal elements are $\pm 1$ and $0$ .
$\kappa(M)$	$=$	$\ M\  \ M^{-1}\ $ , condition real number of an invertible matrix $M$ .
$\kappa(M)$	$=$	$\frac{ \lambda_{\max}(M) }{ \lambda_{\min}(M) }$ , if $M$ is symmetric and invertible.

# Introduction

In the last decade the problem of studying the existence and uniqueness and finding a solution of the generalized nonlinear system of absolute value equations (AVE) of the form:

$$Ax - B|x| = b, \quad (0.1)$$

where  $A, B \in \mathbb{R}^{n \times n}$  and  $b \in \mathbb{R}^n$  are given, and  $|x|$  denote the component wise absolute values of the vector  $x \in \mathbb{R}^n$ , has been received a great importance from the searchers in the domain of numerical analysis and optimization. The AVE (0.1) includes the two following important problems. If  $B = I$  the identity matrix, then the AVE reduces to the system of absolute value equations of the type:

$$Ax - |x| = b. \quad (0.2)$$

Furthermore, if  $B = 0$  (null matrix), then the AVE (0.1) is equivalent to the classical system of linear equations:

$$Ax = b.$$

The importance of the AVEs is due to its broad applications in many domains of mathematics and applied sciences. For instance the linear complementarity problem (LCP) including the linear (LO) and convex quadratic (CQO) optimization, bimatrix games, system of linear interval matrix, the boundary value problems, and the hydrodynamic equation can be formulated as AVEs. (For more details the reader can consult the following references [12, 18, 24, 30, 31] and the references therein).

The AVE (0.1) was first introduced by Rohn [33] in 2004. He used the theorem of the alternatives for solving it. There is no direct method for solving system of AVE because these systems are nonlinear. Rohn [33] proved the equivalence between system of AVE and linear complementarity problems. Besides, in 2006 Mangasarian and Meyer [24] studied the AVE (0.2), where several results of existence and nonexistence of solutions are provided. Besides, in 2009, Rhon [31], generalized certain results of existence of AVE (0.2) to AVE (0.1) where a theorem of the alternatives is stated. The problem of checking whether system of AVE has a unique solution is NP-hard [21, 30]. If the system of AVE is solvable, then either it has a unique solution or

multiple solutions (exponentially many). We do not know about the exact number of solutions of the system of AVE. The system of AVE can be solved iteratively. Several iterative methods were proposed for solving system of AVE, for example, generalized Newton method [22], minimization iterative methods [24, 26, 27, 28] and the methods based on linear complementarity problems [24, 30].

Now, we turn to give some sufficient conditions posed by several researchers which play an important role in the unique solvability of system of AVEs. We cite the most frequently used.

- The system of AVE (0.2) has a unique solution when  $\|A^{-1}\| < 1$ , or  $1 < \sigma_{\min}(A)$  ( $\sigma_{\min}(A)$  denotes the smallest singular values of matrix  $A$ ).
- Rohn [31], generalized this result to unique solvability of AVEs (0.1) where he imposed the following sufficient condition

$$\sigma_{\max}(|B|) < \sigma_{\min}(A),$$

where  $\sigma_{\max}(|B|)$  denotes the maximal singular value of matrix  $|B| = (|b_{ij}|)$  and the Furthermore, Lotfi and Veisheh [20], imposed other sufficient condition that if the following matrix

$$A^T A - \|B\|^2 I,$$

is positive definite, then the AVE is uniquely solvable for any  $b \in \mathbb{R}^n$ .

In this thesis, we address two points. The first point is the unique solvability of the AVEs (0.1) and the second point is its numerical solution.

In the **first chapter**, a mathematical background is stated and which will be useful throughout the thesis.

In the **second chapter**, we discuss unique solvability and numerical solution of AVE (0.1) including the AVEs (0.2). So our main principal result is: if matrices  $A$  and  $B$  satisfy either of the following conditions:

- $\sigma_{\min}(A) > \sigma_{\max}(B)$ ,
- $\|A^{-1}B\| < 1$ , provided  $A$  is non singular,
- The matrix  $A^T A - \|B\|^2 I$  is positive definite,

then the AVE is uniquely solvable for any  $b$ . To do so, a reformulation of the AVE as a standard linear system of equations and then we show under our conditions that its matrix of coefficients is non singular. It is worth mentioning that our proof differs for the first condition from Rhon's proof [33], which is based on the alternative theorem and also for the second condition from the proof of Lotfi and Veisheh [20],

which is inspired by the regularity of the interval matrix. Also it differs for the first condition from the proofs of Achache [1], Achache and Hazzam [3] and Wu and Li [36], where they involved the theory of linear complementarity problems with the class of  $P$ -matrix, i.e., a matrix with the determinants of all principal submatrices are positive (Theorem 3.3.7 in [10]). In addition, across an example of AVE, we demonstrate that our obtained results are reliable to detect unique solvability of AVE rather than those stated in [20] and [31]. These results are also extended to standard LCP and its generalization horizontal LCP where through their reformulation as AVE, we deduce their unique solvability. And present, a Picard iterative method is suggested to compute numerically an approximated solution for some uniquely solvable AVE problems. The globally linear convergence of the latter is guaranteed via the sufficient condition  $\|A^{-1}B\| < 1$ .

In the **third chapter**, we suggest an reformulating the AVE into an equivalent unconstrained quadratic optimization problem, we prove first under the condition that the smallest singular value (eigenvalue) of symmetric  $A$  is greater than the largest singular value of  $B$ , the AVE is uniquely solvable for any  $b$ . We show that the unique minimum of the corresponding unconstrained quadratic problem is the unique solution of the AVE. Then, across the latter, we apply the conjugate gradient algorithms to approximate numerically the solution of the AVE. In the presence of the ill-conditioned, preconditioned conjugate gradient methods can be used to ensure and to accelerate the convergence of the basic conjugate gradient algorithms. We show across some examples of the AVE, the efficiency of these approaches. Finally, we end the thesis by a conclusion and future work.

# Chapter 1

## Mathematical background

In this chapter, we present some mathematical background in matrix and convex analysis, differential calculus, unconstrained optimization and the definition of the absolute value equations.

### 1.1 Matrix Analysis

We denote by  $\mathbb{R}^n$ , the finite-dimensional Euclidean vector space  $n \in \mathbb{N} - \{0\}$  and by  $\mathbb{R}^{n \times n}$ , the set of real square matrices of order  $n$ .

- For all  $x$  and  $y \in \mathbb{R}^n$ , we denote by:

$$x^T y = \sum_{i=1}^n x_i y_i,$$

the usual scalar product of  $x$  and  $y$ . The vector  $x^T$  denotes the transpose of the vector column  $x$  of  $\mathbb{R}^n$ .

- Two vectors  $x$  and  $y$  are orthogonal if  $x^T y = 0$ .
- The Euclidean norm associated with the usual scalar product, is given by:

$$\|x\|_2 = \sqrt{x^T x} = \sqrt{\sum_{i=1}^n x_i^2}.$$

**Definition 1.1.** An application  $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be a vectorial norm if it satisfies the following conditions:

- $\|x\| \geq 0, \forall x \in \mathbb{R}^n, \|x\| = 0 \Leftrightarrow x = 0$ .
- $\|\alpha x\| = |\alpha| \|x\|, \forall \alpha \in \mathbb{R}, \forall x \in \mathbb{R}^n$ .
- $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in \mathbb{R}^n$ .

- The Cauchy-Schwarz inequality:

$$|x^T y| \leq \|x\| \|y\| \quad \forall x, y \in \mathbb{R}^n.$$

**Definition 1.2.** Let  $\mathbb{C}$  be the set of complex numbers and let  $A$  be a real square matrix of  $\mathbb{R}^{n \times n}$ . We recall that  $\lambda \in \mathbb{C}$  is an eigenvalue of the matrix  $A$  if there exists a vector  $x \in \mathbb{C}^n$  with  $x \neq 0$  such that:

$$Ax = \lambda x,$$

we call  $x$  the eigenvector of  $A$  associated to the eigenvalue  $\lambda$ .

**Definition 1.3.** We say that the matrix  $A$  is positive semi-definite if:

$$x^T A x \geq 0, \quad \forall x \in \mathbb{R}^n,$$

and we say that  $A$  is positive definite if:

$$x^T A x > 0, \quad \forall x \in \mathbb{R}^n (x \neq 0).$$

**Definition 1.4.** We call the singular value of a matrix  $A \in \mathbb{R}^{m \times n}$  any square root of an eigenvalue of the semi-definite symmetric matrix  $A^T A$ , that is to say:

$$\sigma(A) = \sqrt{\lambda(A^T A)},$$

such that  $\lambda(A^T A) \geq 0$ , is the eigenvalue of  $A^T A$ .

- The minimal singular value of  $A$  is given through :

$$\sigma_{\min}(A) = \min_{\|x\|_2=1} \sqrt{x^T A^T A x},$$

likewise the maximal singular value of  $A$ , is given by:

$$\sigma_{\max}(A) = \max_{\|x\|_2=1} \sqrt{x^T A^T A x}.$$

We note if the matrix  $A$  is symmetric i.e.,  $A = A^T$ , then the minimal singular value of  $A$  coincides with the minimal eigenvalue of  $A$  and we have:

$$\lambda_{\min}(A) = \min_{\|x\|_2=1} \|Ax\|,$$

likewise the maximal singular value of  $A$  coincides with the maximal eigenvalue of  $A$ , and we have:

$$\lambda_{\max}(A) = \max_{\|x\|_2=1} \|Ax\|.$$

**Lemma 1.5.** Any positive definite matrix  $A$  is invertible i.e. there exists a matrix denoted by  $A^{-1}$  such that  $AA^{-1} = I$  where  $I$  denotes the identity matrix in  $\mathbb{R}^{n \times n}$ .

**Lemma 1.6.** Any real symmetric matrix  $A$ , has real eigenvalues and there exist  $n$  eigenvectors of  $A$  forming a basis of  $\mathbb{R}^n$ .

**Lemma 1.7.** If  $A$  is a real symmetric matrix. Then

$$x^T A x \geq \lambda_{\min}(A) \|x\|_2^2, \forall x \in \mathbb{R}^n,$$

where  $\lambda_{\min}(A)$  denotes the smallest eigenvalue of  $A$ .

**Lemma 1.8.** If the matrix  $A \in \mathbb{R}^{n \times n}$  is symmetric, then  $A$  is positive semi-definite if and only if  $\lambda_{\min}(A) \geq 0$ , and  $A$  is positive definite if and only if  $\lambda_{\min}(A) > 0$ .

**Definition 1.9.** Let  $A, B \in \mathbb{R}^{n \times n}$ , the following application:

$$\|\cdot\| : \mathbb{R}^{n \times n} \rightarrow \mathbb{R},$$

is said to be a matrix norm if it satisfies the following properties:

- $\|A\| \geq 0, \forall A \in \mathbb{R}^{n \times n}, \|A\| = 0 \Leftrightarrow A = 0$ .
- $\|\alpha A\| = |\alpha| \|A\|, \forall \alpha \in \mathbb{R}, \forall A \in \mathbb{R}^{n \times n}$ .
- $\|A + B\| \leq \|A\| + \|B\|, \forall A, B \in \mathbb{R}^{n \times n}$ .
- $\|AB\| \leq \|A\| \|B\|, \forall A, B \in \mathbb{R}^{n \times n}$ .

**Definition 1.10.** - Recall that a subordinate matrix norm is defined as follows:

$$\|A\| = \sup_{x \neq 0, x \in \mathbb{R}^n} \frac{\|Ax\|}{\|x\|}.$$

- If the vector norm, is the Euclidean norm, then:

$$\|A\|_2 = \sqrt{|\lambda_{\max}(A^T A)|},$$

where  $\lambda_{\max}(A^T A)$  denotes the largest eigenvalue of the matrix  $A^T A$ .

- If  $A$  is real symmetric and positive definite, then

$$\|A\|_2 = \lambda_{\max}(A).$$

**Definition 1.11.** The matrix  $A \in \mathbb{R}^{n \times n}$  is said to be  $\mathcal{P}$ -matrix if and only if its principal minors are positive.

**Definition 1.12.** Any positive definite matrix is a  $\mathcal{P}$ -matrix.

**Definition 1.13.** The absolute value of a matrix  $A$  is defined by

$$|A| = |(a_{ij})|,$$

where  $A = (a_{ij})$ .

## 1.2 Differential calculus

In this paragraph, we introduce the notion of differentiability of a function. We start to give some basic topological notions.

- For all  $x \in \mathbb{R}^n$  and  $r > 0$ , the open ball of  $\mathbb{R}^n$ , denoted by  $\mathcal{B}(x, r)$ , is given through:

$$\mathcal{B}(x, r) = \{y \in \mathbb{R}^n : \|x - y\| < r\}.$$

- If  $\{x_k\}$  is a sequence of  $\mathbb{R}^n$  and  $x^*$  is an element of  $\mathbb{R}^n$ . We say that  $\{x_k\}$  converges to  $x^*$  (denoted  $x_k \rightarrow x^*$ ) if  $\|x_k - x^*\| \rightarrow 0$  when  $k \rightarrow \infty$ .

- That is  $\mathcal{D} \subset \mathbb{R}^n$ .

1- We define the interior of  $\mathcal{D}$  as the set of elements  $x \in \mathcal{D}$  for which there exists an  $r > 0$  such that  $\mathcal{B}(x, r) \subset \mathcal{D}$ .

2-  $\mathcal{D}$  is said to be open if  $\forall x \in \mathcal{D}$  exists  $\mathcal{B}(x, r) \subset \mathcal{D}$ .

3-  $\mathcal{D}$  is said to be closed if for any sequence  $\{x_k\}$  from  $\mathcal{D}$  such that  $x_k \rightarrow x^*$  we have:  $x^* \in \mathcal{D}$ .

- Let  $a$  and  $b$ , the set noted  $[a, b]$  of  $\mathbb{R}^n$ , given by:

$$[a, b] = \{a + t(b - a) = (t - 1)a + tb : t \in [0, 1]\}.$$

The set  $[a, b]$  is called the segment connecting  $a$  and  $b$ .

**Definition 1.14.** Let  $\mathcal{D}$  be a subset of  $\mathbb{R}^n$  and  $f : \mathcal{D} \rightarrow \mathbb{R}$ . We say that  $f$  is continuous in  $x \in \mathcal{D}$ , if  $f(x_k) \rightarrow f(x^*)$  for any sequence  $\{x_k\}$  of  $\mathcal{D}$  such that  $x_k \rightarrow x^*$ . So we say that  $f$  is continuous over all  $\mathcal{D} \subset \mathbb{R}^n$ , if it is continuous at any point of  $\mathcal{D}$ .

Suppose now that the set  $\Omega$ , is an open set of  $\mathbb{R}^n$  and  $f$  is a function  $f : \Omega \rightarrow \mathbb{R}$ .

**Definition 1.15.** For all  $x \in \Omega$ , and  $h \in \mathbb{R}^n$ , the directional derivative of  $f$  at  $x$  in the direction  $h$ , is given by

$$\frac{\partial f}{\partial h}(x) = \lim_{t \rightarrow 0} \frac{f(x + th) - f(x)}{t}.$$

The gradient of  $f$  at  $x$ , denoted by  $\nabla f(x)$ , is the column vector of  $\mathbb{R}^n$  given by:

$$\nabla f(x) = \left( \frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x) \right)^T.$$

Recall the formula:

$$\frac{\partial f}{\partial h}(x) = h^T \nabla f(x), \quad \forall x \in \Omega, \quad \forall h \in \mathbb{R}^n.$$

- We say that  $x^*$  is a critical or stationary point of the function  $f$  if

$$\nabla f(x^*) = 0.$$

- We denote for all  $x \in \Omega$ ,  $\nabla^2 f(x)$  the symmetric square matrix of order  $n$ , given by

$$(\nabla^2 f(x))_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}(x), \forall i, j = 1, \dots, n.$$

$\nabla^2 f(x)$  is called the Hessian matrix  $H$  from  $f$  to  $x$ . We also have:

$$H = \nabla^2 f(x)h = \frac{\partial}{\partial x_i} h^T \nabla f(x).$$

- We say that  $f$  is of class  $\mathcal{C}^2$  on  $\Omega$  if the partial derivatives of order 2 of  $f$ , exist and are continuous.

Some important examples:

- Consider the following quadratic function:

$$f(x) = x^T A x.$$

Then

$$\nabla f(x) = (A + A^T)x \quad \text{et} \quad \nabla^2 f(x) = A + A^T.$$

In particular, if  $A$  is real symmetric, then

$$\nabla f(x) = 2Ax \quad \text{et} \quad \nabla^2 f(x) = 2A.$$

- Let  $f(x) = b^T x$ , then:

$$\nabla f(x) = b \quad \text{et} \quad \nabla^2 f(x) = 0.$$

## 1.3 Convex analysis

In this paragraph, we cite some basic notions of convex analysis which will be useful afterwards.

### Convex functions

**Definition 1.16.** A set  $\mathcal{D} \subset \mathbb{R}^n$  is said to be convex if  $\forall x, y \in \mathbb{R}^n$  the segment  $[x, y] \subset \mathcal{D}$ .

**Definition 1.17.** Let  $\mathcal{D} \subset \mathbb{R}^n$  be a convex set and  $f : \mathcal{D} \rightarrow \mathbb{R}$ .

1-  $f$  is said to be convex on  $\mathcal{D}$  if:

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y), \forall x, y \in \mathcal{D}, \forall t \in [0, 1].$$

2-  $f$  is said to be strictly convex if:

$$f((1-t)x + ty) < (1-t)f(x) + tf(y), \forall x, y \in \mathcal{D}, x \neq y, \forall t \in ]0, 1[.$$

**Definition 1.18.** If  $f$  is of class  $\mathcal{C}^2$  on  $\mathcal{D}$ , then  $f$  is convex over  $\mathcal{D}$  if and only if the matrix hessian  $\nabla^2 f(x)$  is semi-definite positive. Likewise,  $f$  is said to be strictly convex on  $\mathcal{D}$  if and only if the hessian matrix is positive definite.

### Coercive functions

**Definition 1.19.** Let  $\mathcal{D}$  be an unbounded set of  $\mathbb{R}^n$  and  $f : \mathcal{D} \rightarrow \mathbb{R}$ .  $f$  is said to be coercive on  $\mathcal{D}$  if:

$$\lim_{\|x\| \rightarrow +\infty} f(x) = +\infty.$$

### Proper function

**Definition 1.20.** We say that  $f$ , is an eigenfunction (proper function) on  $\text{dom} f$  if  $f(x) < +\infty$ . The set  $\text{dom} f \subset \mathbb{R}^n$ , denotes the effective domain of the function  $f$ .

### Generalized Jacobian in the sense of Clarke

In this paragraph, one gives the definition of the generalized Jacobian matrix within the meaning of Clarke [29]. Indeed, we define the generalized Jacobian for non-differentiable functions. For this purpose, we begin to give the definition of a Lipschitzian function.

**Definition 1.21.** Let  $f : \mathcal{D} \subset \mathbb{R}^n \rightarrow \mathbb{R}$  be a function,  $f$  is said to be Lipschitzian on  $\mathcal{D}$  if there exists  $L > 0$ , such as :

$$\|f(x) - f(y)\| \leq L \|x - y\|, \forall x, y \in \mathcal{D}.$$

**Definition 1.22.** Let  $f : \mathcal{D} \rightarrow \mathbb{R}$  be a Lipschitzian function. So the set

$$\partial f_B(x) = \{V \in \mathbb{R}^{m \times n} : \exists \{x_k\} \subset \mathcal{D} \text{ with } x_k \rightarrow x, \nabla f(x_k) \rightarrow V\}.$$

is called the  $B$ -sub-differential of  $f$  at  $x$ .

**Definition 1.23.** The generalized Jacobian matrix  $\partial f$  is defined as follows:

$$\partial f(x) = \text{conv}(\partial f_B(x)),$$

where  $\text{conv}$  denotes the convex hull.

**Definition 1.24.** Let the function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ .

$$f(x) = |x|,$$

then the Jacobian generalized is given by:

$$D(x) = \text{diag}(\text{sign}(x)),$$

where  $D$  is the diagonal matrix whose elements are equal to 1, 0 or  $-1$ , with

$$\text{sign}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases}$$

### Strongly convex function

Let  $\mathcal{D}$  be an unbounded set of  $\mathbb{R}^n$  and  $f : \mathcal{D} \rightarrow \mathbb{R}$ ,  $f$  is said to be strongly convex if and only if there exists  $m > 0$  such that:

$$\langle f(x) - f(y), x - y \rangle \geq m \|x - y\|^2, \forall x, y \in \mathcal{D}.$$

## 1.4 Unconstrained optimization problem

In this section, we only consider unconstrained optimization problem. We give the existence and the uniqueness of a minimum thus the characterization of the latter through its optimality conditions.

A unconstrained optimization problem  $(P)$  is defined by

$$(P) \quad \min_{x \in \mathbb{R}^n} f(x).$$

**Definition 1.25.** If  $\mathcal{D} \subset \mathbb{R}^n$ ,  $x^* \in \mathcal{D}$  and  $f : \mathcal{D} \rightarrow \mathbb{R}$ .

1- We say that  $x^*$  is a global minimum of  $f$  on  $\mathcal{D}$  if:

$$f(x) \geq f(x^*), \forall x \in \mathcal{D}.$$

2- We say that  $x^*$  is a local minimum of  $f$  on  $\mathcal{D}$  if there exists a neighborhood  $\mathcal{V}$  of  $x^*$  such than :

$$f(x) \geq f(x^*), \forall x \in \mathcal{D} \cap \mathcal{V}.$$

**Remark 1.26.** Any global minimum is a local minimum.

### 1.4.1 Results of existence and uniqueness of the solution

In order to be able to easily calculate or approach the solution of an optimization problem, it is interesting to know the hypotheses guaranteeing the existence and uniqueness of this solution.

**Existence.**

**Theorem 1.27.** Let  $\mathcal{D} \subset \mathbb{R}^n$ . If  $f : \mathcal{D} \rightarrow \mathbb{R}$ , is continuous and if  $\mathcal{D}$  is a compact set (closed and bounded), then  $f$  has a minimum on  $\mathcal{D}$ .

**Theorem 1.28.** We suppose that:

1. The set  $\mathcal{D}$  is closed
2.  $f$  is lower semi-continuous on  $\mathcal{D}$ .
3.  $\lim_{\|x\| \rightarrow +\infty} f(x) = +\infty$ , (i.e.  $f$  is coercive).

Then  $f$  has a minimum on  $\mathcal{D}$ .

### Uniqueness.

The following result shows the impact of convexity in optimization problems.

**Proposition 1.29.** *Let  $f$  be a convex function defined on a convex set  $\mathcal{D}$ . Then any local minimum of  $f$  on  $\mathcal{D}$  is a global minimum. If  $f$  is strictly convex, it there is at most a global minimum.*

## 1.4.2 Necessary and sufficient conditions for optimality

### Necessary conditions

**Theorem 1.30.** *(Necessary condition of first order optimality) Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , a differentiable functional. If  $x^*$  achieves a minimum (global or local) of  $f$  on  $\mathbb{R}^n$  then:*

$$\nabla f(x^*) = 0.$$

**Theorem 1.31.** *(Necessary condition of second-order optimality) Let  $x^*$  a minimum of  $f$ . If  $f$  is twice differentiable at the point  $x^*$ , then:*

$$\langle \nabla^2 f(x^*)y, y \rangle \geq 0, \quad \forall y \in \mathbb{R}^n.$$

### Sufficient conditions

**Theorem 1.32.** *(First order sufficient condition ) If  $f$  is convex and if:*

$$\nabla f(x^*) = 0.$$

*then  $x^*$  is a global minimum of  $f$ .*

**Theorem 1.33.** *(Second order sufficient condition) If  $f$  is twice differentiable and if  $\nabla f(x^*) = 0$  and  $\nabla^2 f(x^*)$  positive definite then  $x^*$  is a global minimum of  $f$ .*

### 1.4.3 Convergence of a sequence of points

Let  $x^*$  be the limit of a sequence  $\{x_k\}_{k \geq 0}$  produced by an iterative algorithm. We say that  $x_k \rightarrow x^*$  when  $k \rightarrow +\infty \Leftrightarrow x_k - x^* \rightarrow 0$  when  $k \rightarrow +\infty \Leftrightarrow \|x_k - x^*\| = 0$  when  $k \rightarrow +\infty$ . Now we try to characterize the speed of convergence of the quantity  $x_k - x^* \rightarrow 0$  when  $k \rightarrow +\infty$ . We say that:

- $x_k$  converges linearly to  $x^*$ , if there is a  $c_k (0 < c_k < 1)$  such that:

$$\|x_{k+1} - x^*\| \leq c_k \|x_k - x^*\| \quad \text{from a certain rank } k_0.$$

- $x_k$  converges superlinear to  $x^*$  if  $\lim_{k \rightarrow +\infty} c_k = 0$ .
- $x_k$  converges quadratically to  $x^*$ , if there is a  $c (0 < c < 1)$  such that:

$$\|x_{k+1} - x^*\| \leq c \|x_k - x^*\|^2 \quad \text{from a certain rank } k_0.$$

## 1.5 Equation in absolute values

In this section, we give the definition of an equation in absolute values, and some results of existence and uniqueness of the solution.

An absolute value equations (abbreviated as AVE) is defined as follows:

$$Ax - B|x| = b, \tag{1.1}$$

where  $A, B \in \mathbb{R}^{n \times n}$  are given,  $b \in \mathbb{R}^n$ , and  $|x|$  is a vector whose  $i$ -th entry is the absolute value of the  $i$ -th entry of  $x$ . If  $B = I$ , the identity matrix, then the AVE (1.1) can be reduced to the type:

$$Ax - |x| = b. \tag{1.2}$$

## 1.6 The standard Linear Complementarity Problem

The standard linear complementarity problem (abbreviated by SLCP) consists to find a couple of vectors  $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$  such that

$$x \geq 0, y \geq 0, y = Mx + q, x^T y = 0. \tag{1.3}$$

where  $M \in \mathbb{R}^{n \times n}$  and  $q \in \mathbb{R}^n$  are given. We will denote by

$$\mathcal{F} = \{(x, y) \in \mathbb{R}^{2n} : y = Mx + q, x \geq 0, y \geq 0\}$$

the feasible set of the (SLCP) (1.3). The solution set of the (SLCP) is given by:

$$\text{Sol}(M, q) = \{(x, y) \in \mathcal{F} : x^T y = 0\}.$$

### 1.6.1 Reformulation of SLCP as an AVE

Let us make the following variable change:

$$x = |z| + z, \quad y = |z| - z,$$

it is easy to verify that:

$$x = |z| + z \geq 0, \quad y = |z| - z \geq 0,$$

and

$$x^T y = (|z| + z)^T (|z| - z) = 0.$$

With this change of variable, the (SLCP) turns into an equation in absolute values:

$$(I + M)z - (I - M)|z| = -q.$$

Using the previous results for the AVE, we deduce that:

**Lemma 1.34.** *If the matrix  $(I - M)$  is invertible, then the SLCP becomes:*

$$(I - M)^{-1}(I + M)z - |z| = -(I - M)^{-1}q, \quad (1.4)$$

where

$$A = (I - M)^{-1}(I + M), \quad b = -(I - M)^{-1}q.$$

**Remark 1.35.** The solution of (SLCP) is therefore computed via the formula:

$$x = |z| + z,$$

with  $z$  is the solution of the equation in absolute values (1.4).

## 1.7 The horizontal Linear Complementarity Problem

Given two square matrices  $M, N \in \mathbb{R}^{n \times n}$  and a vector  $q \in \mathbb{R}^n$ , the horizontal linear complementarity problem (HLCP) consists in finding a pair  $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$  such that:

$$Ny - Mx = q, \quad x \geq 0, \quad y \geq 0, \quad x^T y = 0, \quad (1.5)$$

We will denote

$$\mathcal{F}^* = \{(x, y) \in \mathbb{R}^{2n} : Ny = Mx + q, x \geq 0, y \geq 0\}.$$

The feasible set of points of the HLCP as defined in (1.5). The solution set of the HLCP is given by:

$$\text{Sol}(M, N, q) = \{(x, y) \in \mathcal{F}^* : x^T y = 0\}.$$

**Remark 1.36.** If  $N = I$  or  $N^{-1}$  exists, the HLCP reduced to the standard LCP in 1.6.1.

With the same change of variable for SLCP as in (1.6.1), the HLCP is transformed into an equation of absolute values as follows.

$$(N + M)z - (N - M)|z| = -q,$$

with  $A = N + M$ ,  $B = N - M$  and  $b = -q$ .

**Lemma 1.37.** *If the matrix  $(N - M)$  is invertible, then the (HLCP) becomes:*

$$(N - M)^{-1}(N + M)z - |z| = -(N - M)^{-1}q. \quad (1.6)$$

So we set

$$A = (N - M)^{-1}(N + M), b = -(N - M)^{-1}q.$$

and  $z$  is the solution of the equation in absolute values (1.6) such that

$$x = |z| + z.$$

## 1.8 Some results of existence and uniqueness of the solution of the AVE

For unique solvability of AVE, we cite the most well-known established results until today. In [24], Mangasarian and Meyer presented a sufficient condition, namely,  $1 < \sigma_{\min}(A)$  for AVE (1.2). In [31], Rohn generalized this result to unique solvability of AVE (1.1) where he imposed the following sufficient condition

$$\sigma_{\max}(|B|) < \sigma_{\min}(A), \quad (1.7)$$

where  $\sigma_{\max}(|B|)$  denotes the maximal singular value of matrix  $|B| = (|b_{ij}|)$  and the  $\sigma_{\min}(A)$  denotes the smallest singular values of matrix  $A$ .

**Theorem 1.38** (Theorem 2 [31]). *Under Assumption 1.7, the AVE (1.1) is uniquely solvable for every  $b \in \mathbb{R}^n$ .*

*Proof.* The proof proceeds by contradiction. Assume to the contrary that the equation AVE (1.1) does not have a unique solution. Then Theorem 1 [31] with  $D = |B|$  implies existence of a  $\lambda \in [0, 1]$  and of a  $\pm 1$ -vector  $y$  such that the equation

$$Ax - \lambda \text{diag}(y)|B||x| = 0.$$

has a nontrivial solution  $x$  which can be normalized so that  $\|x\|_2 = 1$ . So

$$|Ax| = |\lambda \text{diag}(y)|B||x|| \leq |B||x|,$$

hence

$$x^T A^T A x \leq |(Ax)^T(Ax)| \leq |Ax|^T |Ax| \leq (|B||x|)^T (|B||x|) = |x|^T |B|^T |B| |x|,$$

and

$$\begin{aligned} \sigma_{\min}(A) &= \min_{\|z\|_2=1} z^T A^T A z \leq x^T A^T A x \leq |x|^T |B|^T |B| |x| \\ &\leq \max_{\|z\|_2=1} z^T |B|^T |B| z = \sigma_{\max}(|B|), \end{aligned}$$

which contradicts (1.7). □

In [3], M. Achache and N. Hazzam generalized this result, where assumed that the AVE (1.1) satisfies the following condition.

• **Assumption 1.** The pair of the matrices  $[A, B]$  satisfies

$$\sigma_{\max}(B) < \sigma_{\min}(A),$$

From the theory of the HLCP and Assumption 1, we will show, based on the following theorem, that the AVE (1.1) is uniquely solvable for every  $b \in \mathbb{R}^n$ .

**Theorem 1.39** (Theorem 2 [3]). *Under Assumption 1, the AVE (1.1) is uniquely solvable for every  $b \in \mathbb{R}^n$ .*

*Proof.* We prove under Assumption 1 that HLCP (1.5) is reduced to a standard  $\mathcal{P}$ -LCP with  $M = (A - B)^{-1}(A + B)$  and  $q = (A - B)^{-1}b$ . Then we only prove that  $M$  is positive definite. First, we show that  $(A - B)$  is invertible. If not, for some nonzero  $x \in \mathbb{R}^n$ , we have that  $(A - B)x = 0$ , which derives a contradiction. This implies that  $Ax = Bx$ . Hence,

$$\begin{aligned} \sigma_{\min}(A) &= \min_{\|z\|=1} \langle A^T A z, z \rangle \leq \langle A^T A x, x \rangle = \langle B^T B x, x \rangle \\ &\leq \max_{\|z\|=1} \langle B^T B z, z \rangle = \sigma_{\max}(B), \end{aligned}$$

which contradicts our condition. Hence  $(A - B)^{-1}$  is invertible. Now, we prove that  $M$  is positive definite. To this end, we have, for all nonzero  $x \in \mathbb{R}^n$ .

$$\langle (A - B)^{-1}(A + B)x, x \rangle = \langle (A + B)x, (A^T - B^T)^{-1}x \rangle.$$

Letting  $(A^T - B^T)^{-1}x = z$ , one has

$$\begin{aligned} \langle (A - B)^{-1}(A + B)x, x \rangle &= \langle (A + B)(A^T - B^T)z, z \rangle \\ &= \langle (A^T A - B^T B - AB^T + BA^T)z, z \rangle \\ &= \langle (A^T A - B^T B)z, z \rangle + \langle (BA^T - AB^T)z, z \rangle. \end{aligned}$$

Note that  $\langle (BA^T - AB^T)z, z \rangle = 0$ .  $(A - B)^{-1}(A + B)$  is positive definite if and only if  $(AA^T - BB^T)$  is positive definite. Indeed, by Weyl's inequalities [Matrix Analysis: Theorem 4.3.1 in [13]], we find that

$$\lambda_{\min}(AA^T - BB^T) \geq \lambda_{\min}(AA^T) + \lambda_{\min}(-BB^T) = \lambda_{\min}(AA^T) - \lambda_{\max}(BB^T).$$

From Assumption 1,  $\lambda_{\min}(AA^T) - \lambda_{\max}(BB^T) > 0$ ,  $(AA^T - BB^T)$  is positive definite. Hence  $(A - B)^{-1}(A + B)$  is positive definite. Consequently,  $M$  is a  $P$ -matrix. Thus, from Theorem 3.3.7, [8], the SLCP has a unique solution for any  $q \in \mathbb{R}^n$ , so is the AVE (1.1) for any  $b \in \mathbb{R}^n$ . This completes the proof.  $\square$

Furthermore, Lotfi and Veisheh [20], imposed other sufficient condition that if the following matrix

$$A^T A - \|B\|^2 I, \quad (1.8)$$

is positive definite, then AVE (1.1) is uniquely solvable for any  $b \in \mathbb{R}^n$ . Their proof is based on the regularity of the system of an interval matrix. To simplify this concept, we give a short overview on this topic. We will give first the notion of an interval matrix.

**Definition 1.40.** Given matrices  $A$  and  $B$  in  $\mathbb{R}^{n \times n}$ , the set of matrices

$$\mathbf{I} = [A - |B|, A + |B|] := \{S : |S - A| \leq |B|\} = \{S : A - |B| \leq S \leq A + |B|\}$$

is called an interval matrix (with midpoint  $A$  and radius  $|B|$ ).

Now we have this definition introducing the an important distinction:

**Definition 1.41.** A square interval matrix  $A$  is called regular if each  $S \in \mathbf{I}$  is regular (invertible) and singular otherwise, i.e.  $\mathbf{I}$  contains a singular matrix.

The following result concerning the unique solvability of AVE (1.1).

**Lemma 1.42.** *If the interval matrix  $\mathbf{I}$  is regular, then the AVE (1.1) for any  $b$  has a unique solution.*

For more details we adress to the reference (Lemma 2.3 Lotfi and Veisheh [20]).

## 1.9 Fixed point method

### 1.9.1 Fixed point principle

Let  $F$  be a function from to  $\mathbb{R}^n \rightarrow \mathbb{R}^n$ , and consider the equation  $F(x) = 0$ . It is clear that  $x^*$  is a solution of  $F(x) = 0$  if and only if  $x^*$  is a root of the equation

$$F(x^*) = 0.$$

The principal of fixed point method is to convert the equation  $F(x) = 0$  to the equivalent equation:

$$\Phi(x) = x,$$

where  $\Phi(x)$  is a given function. So finding the root of  $F(x) = 0$ , is equivalent to finding the unique fixed point of  $\Phi(x) = x$ . Then, the fixed point method for getting the root  $x^*$  of the equation  $F(x) = 0$  is based on the following recurrent iterative scheme:

$$\begin{cases} x_0 \in \mathbb{R}^n \text{ given} \\ x_{k+1} = \Phi(x_k), k \geq 0, \end{cases}$$

For the analysis of the conditions of its convergence, the reader can consult the Theorem 2.8, Chapter II.

## 1.10 Conjugate gradient methods

In numerical analysis, the conjugate gradient method is an algorithm to solve systems of linear equations  $Ax = b$ , whose matrix  $A$  is positive definite. This method was imagined in 1950 simultaneously by Cornelius Lanczos, Magnus Hestnes 1951 and Edward Steifel, is an iterative method, which converges in a finite number of iterations (equal to the size of the system). The conjugate gradient method is also a method for solving a optimization problem, it is a descent method, it comes to remedy the disadvantage of the orthogonality of the directions of the gradient (it breaks this orthogonality) by taking conjugated directions (not orthogonal) so it can be classified between the gradient method and Newton's method.

### 1.10.1 Conjugate directions methods

**Definition 1.43** (Conjugated vectors). Let  $A \in M_n(\mathbb{R})$  be a symmetric positive definite matrix.

- Two vectors  $d_1$  and  $d_2 \in \mathbb{R}^n \setminus \{0\}$  are said to be  $A$ -conjugate (or conjugate with respect to  $A$ ) if

$$\langle Ad_1, d_2 \rangle = d_2^T Ad_1 = 0.$$

- A set of vectors  $\{d_1, d_2, \dots, d_k\}$  of  $\mathbb{R}^n \setminus \{0\}$  is called  $A$ -conjugate if

$$(d_j)^T Ad_i = \langle Ad_i, d_j \rangle = 0, \forall i \neq j \text{ and } 1 \leq i \leq k, 1 \leq j \leq k.$$

- This means that two  $A$ -conjugate vectors are orthogonal for the product associated to the matrix  $A$  defined by:

$$\langle x, y \rangle_A = y^T Ax, \forall x, y \in \mathbb{R}^n.$$

### Description of the method

Let  $\{d_1, d_2, \dots, d_n\}$  a family of  $A$ -conjugated vectors. We then call the conjugate directions method any iterative method applied to a strictly convex quadratic function of  $n$  variables:

$$f(x) = \frac{1}{2}x^T Ax - x^T b + c,$$

with  $x \in \mathbb{R}^n$  and  $A$  an  $n \times n$ , matrix symmetric and positive definite and  $b \in \mathbb{R}^n$  and  $c \in \mathbb{R}$ , leading to the optimum in  $n$  steps at most. This method is of the form:

$$\begin{cases} x_1 \text{ is a given initial point} \\ x_{k+1} = x_k + \alpha_k d_k, \quad k \geq 1, \end{cases} \quad (1.9)$$

where  $\alpha_k$  is optimal and  $\{d_1, d_2, \dots, d_n\}$  have the property of being mutually conjugated with respect to matrix  $A$ . If we note

$$g_k := \nabla f(x_k) = Ax_k - b,$$

the method is constructed as follows:

#### Calculation of $\alpha_k$

Since  $\alpha_k$  minimizes  $f$  in the direction  $d_k$ , then we have  $\forall k$

$$\begin{aligned} f'(\alpha_k) &= d_k^T \nabla f(x_{k+1}) = 0 \\ d_k^T \nabla f(x_{k+1}) &= d_k^T (Ax_{k+1} - b). \end{aligned}$$

If

$$d_k^T (A(x_k + \alpha_k d_k) - b) = 0,$$

from where

$$\alpha_k = -\frac{g_k^T d_k}{d_k^T A d_k}. \quad (1.10)$$

**Theorem 1.44.** For all  $x_1 \in \mathbb{R}^n$  the sequence  $\{x_k\}$  generated by the algorithm (1.9) – (1.10) converges to the solution  $x^*$  of the linear system  $Ax = b$  in at most  $n$  steps.

### 1.10.2 Conjugate gradient method for quadratic problem

In the conjugate gradient method, we start from a point  $x_1 \in \mathbb{R}^n$ ,  $d_1 = -g_1$ , the directions  $d_k$ ,  $k = 2, \dots, n-1$  are calculated at each iteration. The iterative formula of the method is as follows.

$$\begin{cases} x_1 \in \mathbb{R}^n \text{ (initial point);} \\ x_{k+1} = x_k + \alpha_k d_k \quad k \geq 2, \end{cases}$$

or

$$d_k = \begin{cases} -g_k & \text{if } k = 1 \\ -g_k + \beta_k d_{k-1} & \text{if } k \geq 2 \end{cases} : g_k := \nabla f(x_k).$$

The displacement step  $\alpha_k$  is calculated using an exact linear search and the scalar  $\beta_k \in \mathbb{R}$  is obtained so that  $d_k$  is  $A$  conjugated with the other vectors  $d_i, i = 1, \dots, k-1$ . In other words we must have

$$d_k^T A d_i = 0, \quad i = 1, \dots, k-1.$$

In the appellation conjugate gradient, we find the two words: gradient and conjugated.

- The word gradient is used because  $d_k$  is calculated from the gradient at the point  $x_k$ .
- But the conjugated word it concerns the directions  $\{d_k : k \geq 1\}$  and not the gradients  $\{g_k : k \geq 1\}$ , this is why a better name would be method combined directions.

### 1.10.3 Algorithm of the conjugate gradient for quadratic problem

Suppose that at iteration  $k$  we have a direction  $d_k$  of descent, that is to say that  $d_k$  checks

$$g_k^T d_k = (Ax_k - b)^T d_k < 0.$$

- Let  $\alpha_k > 0$  obtained by an exact linear search, i.e.,  $\alpha_k$  satisfies

$$f(x_k + \alpha_k d_k) = \min_{\alpha > 0} f(x_k + \alpha d_k).$$

Then

$$\alpha_k = -\frac{g_k^T d_k}{d_k^T A d_k}.$$

- $\beta_k$  is chosen so that

$$d_k^T A d_{k-1} = 0,$$

since

$$d_k = -g_k + \beta_k d_{k-1}.$$

Then

$$(-g_k + \beta_k d_{k-1})^T A d_{k-1} = 0,$$

i.e.,

$$\beta_k = \frac{g_k^T A d_{k-1}}{d_{k-1}^T A d_{k-1}}.$$

Therefore a simple description of the conjugate gradient algorithm is stated in the following table.

**Step 1.** An arbitrary initial point  $x_1 \in \mathbb{R}^n$ ,  $\epsilon > 0$  and  $g_1 := Ax_1 - b$ ,  $d_1 = -g_1$ ,  $k = 0$ ;

**Step 2.** Compute  $\alpha_k = -\frac{g_k^T d_k}{d_k^T A d_k}$  and set  $x_{k+1} = x_k + \alpha_k d_k$ ;

**Step 3.** If  $\|g_k\| < \epsilon$  then STOP, otherwise compute  $d_k$  according to

$$d_k = -g_k + \beta_k d_{k-1} \text{ with } \beta_k = \frac{g_{k+1}^T A d_k}{d_k^T A d_k};$$

**Step 4.** Set  $k = k + 1$ , and go to **Step 2**.

## Chapter 2

# On unique solvability and Picard's iterative method for absolute value equations

In this chapter we have introduced some weaker sufficient conditions that guarantee the unique solvability of the AVE (1.1). Across an example of AVE (1.1), we have showed the reliability of our weaker sufficient conditions to detect unique solvability of AVE (1.1). These obtained results are also extended to detecting unique solvability of standard and horizontal LCP. Numerically, the proposed Picard's iterative method is efficient to provide an approximated solution of some uniquely solvable AVE including standard and horizontal LCP problems.

## 2.1 On unique solvability for absolute value equations

In this section, we will give our main results. First, for given matrices  $A, B \in \mathbb{R}^{n \times n}$  and for any diagonal matrix  $D \in \mathbb{R}^{n \times n}$  whose diagonal elements are  $\pm 1$  and 0, we define the matrix  $(A - BD) \in \mathbb{R}^{n \times n}$ . Then to achieve our main results, the following lemma is required.

**Lemma 2.1.** *Each of three conditions below implies the non singularity of  $(A - BD)$ .*

1.  $\sigma_{\min}(A) > \sigma_{\max}(B)$ ,
2.  $\|A^{-1}B\| < 1$ , provided  $A$  is non singular,
3. the matrix  $A^T A - \|B\|^2 I$  is positive definite.

*Proof.* For the first claim, assume that  $(A - BD)$  is singular then,

$$(A - BD)x = 0, \text{ for some } x \neq 0.$$

We then have

$$\begin{aligned} \sigma_{\min}^2(A) &= \min_{\|y\|=1} y^T A^T A y \leq x^T A^T A x = x^T D B^T B D x \\ &\leq \max_{\|z\|=1} z^T D B^T B D z = \|BD\|^2. \\ &\leq \|B\|^2 \|D\|^2 \leq \|B\|^2 = \max_{\|z\|=1} z^T B^T B z = \sigma_{\max}^2(B), \end{aligned}$$

which contradicts the first condition. Hence  $(A - BD)$  is non singular.

Next, by the same argument, assume that  $A$  is singular and let a nonzero vector  $x$  with  $\|x\| = 1$  and such that

$$(A - BD)x = 0.$$

Next, because  $x = A^{-1}BDx$ , we then have,

$$\begin{aligned} 1 &= \|x\| = \|A^{-1}BDx\| \\ &\leq \|A^{-1}B\| \|D\| \|x\| \\ &\leq \|A^{-1}B\|. \end{aligned}$$

This leads to a contradiction and hence  $(A - BD)$  is non singular.

For the last claim, assuming the contrary that  $(A - BD)$  is singular, then for a nonzero vector  $x$  with  $\|x\| = 1$ , we have,

$$(A - BD)x = 0.$$

As  $Ax = BDx$ , we then have,

$$\begin{aligned}
 x^T A^T Ax - \|B\|^2 x^T x &= x^T (BD)^T BDAx - \|B\|^2 x^T x \\
 &= \|BDx\|^2 - \|B\|^2 x^T x \\
 &\leq \|B\|^2 \|D\|^2 \|x\|^2 - \|B\|^2 x^T x \\
 &\leq \|B\|^2 - \|B\|^2 = 0,
 \end{aligned}$$

and consequently

$$x^T A^T Ax - \|B\|^2 x^T x \leq 0.$$

This contradicts the fact that the matrix  $A^T A - \|B\|^2 I$  is positive definite. Hence  $(A - BD)$  is non singular for any diagonal matrix  $D$  whose elements are  $\pm 1$  and 0. This completes the proof.  $\square$

Next, according to  $Dx = |x|$  where  $D := \text{Diag}(\text{sign}(x))$ , the AVE (1.1) can be rewritten [18] as the following standard linear system of equations:

$$(A - BD)x = b. \quad (2.1)$$

Then, it is clear that the AVE (1.1) is uniquely solvable for any  $b$  if the matrix of coefficients  $(A - BD)$  of the linear system (2.1) is non singular for all diagonal matrix  $D$  whose diagonal elements are  $\pm 1$  or 0.

**Theorem 2.2.** *If matrices  $A$  and  $B$  satisfy*

1.  $\sigma_{\min}(A) > \sigma_{\max}(B)$ ,
2.  $\|A^{-1}B\| < 1$ , *provided  $A$  is non singular*,
3. *the matrix  $A^T A - \|B\|^2 I$  is positive definite,*

*then the AVE (1.1) is uniquely solvable for any  $b$ .*

*Proof.* Based on the results in Lemma (2.1), the matrix  $(A - BD)$  of coefficients of the linear system (2.1) is non singular for any diagonal matrix  $D$  whose diagonal elements are  $\pm 1$  and 0. Hence the AVE (1.1) is uniquely solvable for any  $b$ . This completes the proof.  $\square$

For AVE (1.2), the results of unique solvability are summarized in the following theorem.

**Theorem 2.3.** *If a matrix  $A$  satisfies either of the following conditions:*

1.  $\sigma_{\min}(A) > 1$ ,
2.  $\|A^{-1}\| < 1$ , *provided  $A$  is non singular*,
3. *the matrix  $A^T A - I$  is positive definite,*

*then the AVE (1.2) has a unique solution for any  $b$ .*

Proof. The proof is straightforwardly from Theorem (2.2), with  $B = I$ .

The following results concerning the unique solvability of the AVE (1.1) were provided in [20, 31].

**Theorem 2.4** (Theorem 1.38). *Let  $A$  and  $B$  satisfy*

$$\sigma_{\min}(A) > \sigma_{\max}(|B|),$$

*then the AVE (1.1) has a unique solution for any  $b$ .*

**Theorem 2.5** ([20]). *Let  $A, B \in \mathbb{R}^{n \times n}$  and the matrix*

$$A^T A - \|B\|^2 I,$$

*is positive definite, then the AVE (1.1) has a unique solution for any  $b$ .*

Next example shows that Theorem (2.2) is reliable to detect unique solvability of AVE (1.1). Consider the following AVE (1.1) problem, where  $A, B \in \mathbb{R}^{3 \times 3}$  and  $b \in \mathbb{R}^3$  are given by

$$A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}, B = \begin{bmatrix} 4 & -2 & -2 \\ -2 & -5 & -2 \\ -2 & -2 & 2 \end{bmatrix}, b = \begin{bmatrix} 7 \\ 2 \\ 9 \end{bmatrix}.$$

By simple calculations,  $\sigma_{\min}(A) = 7, \sigma_{\max}(B) = 6.1355, \sigma_{\max}(|B|) = 7.9018$ . Theorem 1.38, is not capable to detect the unique solvability of the AVE since  $\sigma_{\min}(A) = 6 < \sigma_{\max}(|B|) = 7.9018$ . However, the application of Theorem 2.2 shows that  $\sigma_{\min}(A) > \sigma_{\max}(B)$  which implies that the AVE has a unique solution for any  $b \in \mathbb{R}^3$ . Next, checking Theorem (2.5) [20], we have

$$A^T A - \|B\|^2 I = \begin{bmatrix} -13.438 & 0 & 0 \\ 0 & -13.438 & 0 \\ 0 & 0 & -13.438 \end{bmatrix}.$$

So it is clear that  $A^T A - \|B\|^2 I$  is not positive definite. But Theorem (2.2), shows that

$$A^T A - \|B\|^2 I = \begin{bmatrix} 11.356 & 0 & 0 \\ 0 & 11.356 & 0 \\ 0 & 0 & 11.356 \end{bmatrix}$$

is positive definite and the AVE has a unique solution for any  $b$ . The unique solution of this example is  $x^* = [1, -1, 1]^T$ .

## 2.2 Unique solvability of standard and horizontal LCP

### 2.2.1 The standard LCP

In this section, our interest is to deduce sufficient conditions for the existence and uniqueness of the solution of standard LCP via those established in Theorem (2.2) for the AVE.

The linear complementarity problem [8], consists in finding a vector  $x \in \mathbb{R}^n$  such that

$$y = Mx + q \geq 0, x \geq 0, x^T y = 0 \quad (2.2)$$

where  $M \in \mathbb{R}^{n \times n}$  is a given matrix and  $q \in \mathbb{R}^n$ . Letting

$$y = |z| - z, \text{ and } x = |z| + z,$$

then the LCP (2.2) can be reformulated as the following AVE (1.1) as follows:

$$(I + M)z - (I - M)|z| = -q. \quad (2.3)$$

Then the LCP (2.2) has a unique solution for all  $q$  if and only if the AVE in (2.3) has a unique solution for any  $b = -q$ . Using Theorem (2.2) the following results of the unique solvability of the LCP are deduced.

**Theorem 2.6.** *Let  $I + M, I - M \in \mathbb{R}^{n \times n}$  satisfy either of the following conditions:*

- 1-  $\sigma_{\min}(I + M) > \sigma_{\max}(I - M)$ ,
- 2-  $\|((I + M)^{-1}(I - M))\| < 1$ , *provided that the matrix  $I + M$  is non singular,*
- 3-  $(I + M)^T(I + M) - \|I - M\|^2 I$  *is positive definite, then the LCP has a unique solution for any  $q$ .*

### 2.2.2 The horizontal LCP

The generalization of the standard LCP is called the horizontal linear complementarity problem (HLCP) [3, 8], consists also in finding a vector  $x \in \mathbb{R}^n$  such that

$$Ny = Mx + q \geq 0, x \geq 0, x^T y = 0, \quad (2.4)$$

where  $N, M \in \mathbb{R}^{n \times n}$  are given matrices and  $q \in \mathbb{R}^n$ . It is worth noting that the HLCP becomes a standard LCP if  $N = I$ , then the HLCP reduces to an LCP. By using the same change of variables,  $y = |z| - z$  and  $x = |z| + z$ , the HLCP reduced to the following AVE (1.1)

$$(N + M)z - (N - M)|z| = -q. \quad (2.5)$$

According to this relation, then it is clear that the HLCP (2.4) has a unique solution for any  $q$  if and only if the AVE (2.5) is uniquely solvable for any  $b = -q$ . Again by Theorem (2.2), we provide the following results for the unique solvability of HLCP.

**Theorem 2.7.** *Let  $N + M, N - M \in \mathbb{R}^{n \times n}$  satisfy either of the following conditions:*

- 1-  $\sigma_{\min}(N + M) > \sigma_{\max}(N - M)$
  - 2-  $\|(N + M)^{-1}(N - M)\| < 1$ , *provided that the matrix  $(N + M)$  is non singular,*
  - 3-  $(N + M)^T(N + M) - \|N - M\|^2 I$  *is positive definite,*
- then the HLCP has a unique solution for any  $q$ .*

## 2.3 Picard's iterative method for AVE

In this section, in order to provide an approximated solution of some uniquely solvable AVE problems, a simple Picard's iterative method is proposed. First, we state the Banach fixed point theorem which will be used for proving the convergence of the proposed method, one can see [17] and [6] for its details proof.

**Theorem 2.8** (Banach's fixed point theorem). *Let  $(X, d)$  be a non-empty complete metric space,  $0 \leq \alpha < 1$  and  $T : X \rightarrow X$  a mapping satisfying*

$$d(T(x), T(y)) \leq \alpha d(x, y), \text{ for all } x, y \in X.$$

*Then there exists a unique  $x \in X$  such that  $T(x) = x$ . Furthermore,  $x$  can be found as follows: start with an arbitrary element  $x_0 \in X$  and define a sequence  $\{x_k\}$  by*

$$x_{k+1} = T(x_k),$$

*then*

$$\lim_{k \rightarrow \infty} x_k = x,$$

*and the following inequalities hold:*

$$d(x, x_{k+1}) \leq \frac{\alpha}{1 - \alpha} d(x_{k+1}, x_k), \quad d(x, x_{k+1}) \leq \alpha d(x, x_k).$$

Next, based on the fixed point principle, the sequence of iterations for solving the AVE (1.1) is given by

$$x_{k+1} = A^{-1}B|x_k| + A^{-1}b, \quad k = 0, 1, 2, \dots \quad (2.6)$$

Next under the condition 2 (Theorem (2.2)), we provide a sufficient condition for the globally linear convergence of the fixed point iterations (2.6).

**Theorem 2.9.** *Let  $A$  be a non singular matrix and if*

$$\|A^{-1}B\| < 1,$$

*then the sequence  $\{x_k\}$  converges to the unique solution  $x^*$  of the AVE (1.1) for any arbitrary  $x_0 \in \mathbb{R}^n$ . In this case the error bound is given by*

$$\|x_{k+1} - x^*\| \leq \frac{\|A^{-1}B\|}{1 - \|A^{-1}B\|} \|x_k - x^*\|, \quad k = 0, 1, 2, \dots \quad (2.7)$$

Moreover, the sequence  $\{x_k\}$  converges linearly to  $x^*$  as follows

$$\|x_{k+1} - x^*\| \leq \|A^{-1}B\| \|x_k - x^*\|, \quad k = 0, 1, 2, \dots \quad (2.8)$$

*Proof.* First, if the condition  $\|A^{-1}B\| < 1$ , holds then Theorem 2.(2.2), implies that the AVE (1.1) is uniquely solvable for any  $b$ . Next, to prove the convergence of the sequence  $\{x_k\}$  to  $x^*$ , we define the function  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by

$$\varphi(x) = A^{-1}B|x| + A^{-1}b.$$

Then it is easy to see with the help of the following inequality

$$\||x| - |y|\| \leq \|x - y\|, \quad \text{for all } x, y \in \mathbb{R}^n,$$

that

$$\|\varphi(x) - \varphi(y)\| \leq \|A^{-1}B\| \|x - y\|, \quad \text{for all } x, y \in \mathbb{R}^n.$$

Using Theorem (2.8) with  $X = \mathbb{R}^n, T = \varphi, d(x, y) = \|x - y\|$  for all  $x, y \in \mathbb{R}^n$  and  $\alpha = \|A^{-1}B\| < 1$ , we deduce the convergence of the sequence  $\{x_k\}$  given by

$$x_{k+1} = \varphi(x_k), \quad k = 0, 1, 2, \dots$$

to the unique fixed point  $x^*$  to  $\varphi(x)$  which is in turn the unique solution of the AVE (1.1). Moreover, the (2.7) and (2.8) hold which lead to the globally linear convergence of the method.  $\square$

## 2.4 Checking unique solvability of AVE and numerical results

In this section, we present some examples of AVE problems including some examples of LCP where their unique solvability is checked. Also by applying Picard's iterative method, we compute an approximated solution of these examples. Our implementation is done by using the software Matlab. The starting point and the unique solution of the AVE are denoted, respectively, by  $x_0$  and  $x^*$ . In the tables of numerical results we display the following notations: "Iter" and "CPU" state for the number of iterations and the elapsed times. The termination of the algorithm is as the relative residue:

$$\text{RSD} := \frac{\|Ax - B|x| - b\|}{\|b\|}$$

is less than the tolerance  $\varepsilon = 10^{-6}$ .

**Example 1.** Consider the problem of AVE where  $A, B \in \mathbb{R}^{10 \times 10}$  are given by:

$$A = \begin{bmatrix} 101 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 102 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 103 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 104 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 105 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 106 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 107 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 108 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 109 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 110 \end{bmatrix}, B = I.$$

By the help of Matlab software, we get  $\sigma_{\min}(A) = 101.04 > 1$  and  $\|A^{-1}B\| = 0.09897 < 1$ . Hence Theorem (2.3) implies that this problem is uniquely solvable for any  $b \in \mathbb{R}^{10}$ . For  $b = (A - I)e$  and with the starting point

$$x_0 = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]^T,$$

the obtained numerical results by Picard's method are stated in Table 1.

Iter	CPU (time)	RSD
5	0.008021	$3.9734e - 008$

Table 1.

The unique solution of this example is  $x^* = e$ .

**Example 2.** Consider the AVE in (1.1) where  $A, B \in \mathbb{R}^{7 \times 7}$  are given by:

$$A = \begin{bmatrix} 1 & 10 & 1 & 1 & 2 & 0 & 0 \\ 2 & 1 & 6 & 6 & 1 & 1 & 2 \\ 1 & 3 & 5 & 9 & 100 & 1500 & -5 \\ 5 & 1 & 3 & 1 & 0 & 3 & 40 \\ 3 & 3 & 8 & 2 & 2 & 0 & 2 \\ 1 & 5 & 5 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1000 \end{bmatrix},$$

and

$$B = \begin{bmatrix} 0.5 & 0.5 & 0.05 & 0.05 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.25 & 0.5 & 0.25 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0 & 0 & 0.05 & 0 \\ 0.5 & 0.05 & 0 & 0.05 & 0 & 0 & 0 \end{bmatrix}.$$

Applying Theorem (2.2), we have,  $\sigma_{\min}(A) = 1.5029 > \sigma_{\max}(B) = 1.4653$  and  $\|A^{-1}B\| = 0.373 < 1$ , then this problem is uniquely solvable for any  $b$ .

For

$$b = [-16.2, 23, 3206, 79, 13, -1.1, 2004.8]^T.$$

The starting point is taken as:

$$x_0 = [1, 2, 3, 4, 5, 6, 7]^T.$$

Then the obtained numerical results are summarized in Table 2.

Iter	CPU (time)	RSD
6	0.023814	$7.1883e - 007$ .

Table 2.

The exact unique solution of this problem is given by

$$x^* = [-2, -2, 2, 2, 2, 2, 2]^T.$$

**Example 3.** Consider the standard LCP where  $M \in \mathbb{R}^{4 \times 4}$  is given as

$$M = \begin{bmatrix} 0.4974 & -0.0105 & -0.0630 & -0.001 \\ -0.0839 & 0.6642 & -0.0147 & -0.00336 \\ -0.0105 & -0.042 & 0.7482 & -0.0042 \\ -0.001 & -0.0042 & -0.0252 & 0.7996 \end{bmatrix}.$$

By a simple calculation using Theorem (2.6), we get  $\sigma_{\min}(I + M) = 1.4786 > \sigma_{\max}(I - M) = 0.5239$  and  $\|(I + M)^{-1}(I - M)\| = 0.35413 < 1$  then the associated AVE has a unique solution for any  $b = -q$  and consequently the LCP problem has a unique solution for every  $q$ . For example if  $b = -(I + M)^{-1}q$ , where  $q = [-1.5, -2, -3.5, -4.5]^T$  and with the starting point

$$z_0 = [1, 2, 3, 4]^T,$$

the obtained numerical results are stated in Table 3.

Iter	CPU (time)	RSD
13	0.005759	$3.9225e - 007$ .

Table 3.

The exact unique solution for the AVE (2.3) is given by

$$z^* = [1.8664, 1.8110, 2.4831, 2.9040]^T.$$

Hence the unique solution of the LCP is  $x^* = z^* + |z^*|$ . But since  $z^*$  is positive vector i.e.,  $z_i^* > 0$  for all  $i$ , then  $|z^*| = z^*$  and so the unique solution of the LCP is  $x^* = 2z^*$ .

**Example 4.** Consider the standard LCP where  $M \in \mathbb{R}^{n \times n}$  is given by:

$$M = \begin{bmatrix} 0.6 & -0.01 & 0 & \cdots & 0 & 0 \\ -0.01 & 0.6 & -0.01 & \cdots & 0 & 0 \\ 0 & -0.01 & 0.6 & \cdots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -0.01 & 0 \\ 0 & 0 & 0 & \cdots & 0.6 & -0.01 \\ 0 & 0 & \cdots & 0 & -0.01 & 0.6 \end{bmatrix}.$$

By using Matlab software, we get for any size of  $n$  of the matrix  $M$ ,  $\sigma_{\min}(I + M) = 1.5808 > \sigma_{\max}(I - M) = 0.4192$ , and  $\|(I + M)^{-1}(I - M)\| = 0.2652 < 1$ , then the LCP has a unique solution for any  $q \in \mathbb{R}^n$ . For  $b = -(I + M)^{-1}q$  where  $q = -e$ , and the starting point  $z_0 = [1, 2, 3, \dots, n]^T$ , the obtained numerical results with different size of  $n$ , are summarized in Table 4.

size $n$	Iter	CPU(s)	RSD
100	15	0.039046	$5.8061e - 007$
1000	17	4.912430	$4.1563e - 007$
2000	17	35.421034	$8.3189e - 007$
3000	18	119.553462	$3.3178e - 007$

Table 4.

Therefore the unique solution of the LCP for any size of  $n$ , is deduced from the formula  $x^* = |z^*| + z^*$ . Also since  $z^* > 0$ , then  $x^* = 2z^*$  where

$$z^* = [0.8477, 0.8618, 0.8621, \dots, 0.8621, 0.8618, 0.8477]^T.$$

**Example 5.** Consider the following horizontal LCP where  $M, N \in \mathbb{R}^{n \times n}$  are given by

$$M = \begin{bmatrix} 4 & 2 & 2 & \cdots & 2 & 2 \\ 2 & 4 & 2 & \cdots & 2 & 2 \\ 2 & 2 & 4 & \cdots & 2 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 2 & 2 \\ 2 & 2 & 2 & \cdots & 4 & 2 \\ 2 & 2 & \cdots & 2 & 2 & 4 \end{bmatrix},$$

and

$$N = \begin{bmatrix} 5 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 5 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 5 & \cdots & 1 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 5 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 5 \end{bmatrix}.$$

By simple calculation using Theorem (2.7), and for different size of  $n$ , we have

$$\|(N + M)^{-1}(N - M)\| = 0.3333 < 1,$$

then this problem has a unique solution for any  $b$ . For  $b = -(I + M)^{-1}q$  with  $q = -Me$ , and with the starting point

$$z_0 = [1, 2, 3, \dots, n]^T,$$

the obtained numerical results for different size of  $n$ , are summarized in Table 5.

Size ( $n$ )	Iter	CPU(s)	RSD
100	17	0.042596	$5.9651e - 007$
1000	19	5.722172	$8.7750e - 007$
2000	20	40.834046	$3.7460e - 007$
3000	20	135.925389	$8.9758e - 007$

Table 5.

Also since  $z^*$  is positive, the unique solution of the HLCP is  $x^* = 2z^* = e$  where

$$z^* = [0.5, 0.5, \dots, 0.5]^T.$$

## Chapter 3

# Preconditioned conjugate gradient methods for absolute value equations

In this chapter we have introduced preconditioned conjugate gradient methods for solving the NP-hard absolute value equations. These methods determine new descent search directions and offer solutions for some class of AVE problems, including some interesting instances such as hydrodynamic equations and standard linear complementarity problems.

### 3.1 Basic conjugate gradient methods

Before describing the conjugate gradient algorithm, the following results are useful. For given symmetric matrices  $A$  and  $B$ , we define, for any diagonal matrix  $D$  whose elements are equal to 1, 0 or - 1, the matrix  $Q = A - BD$ . To prove the unique solvability of the AVE (1.1), the following result is required.

**Lemma 3.1.** *If symmetric matrices  $A$  and  $B$  satisfy:*

$$\sigma_{\min}(A) > \sigma_{\max}(B),$$

*then the matrix  $A - BD$  is non-singular for any diagonal matrix  $D$  whose elements are equal to +1, 0 or - 1.*

*Proof.* Assume a contrary that  $A - BD$  is singular, then for some nonzero vector  $x$  with  $\|x\| = 1$ , we then have that  $(A - BD)x = 0$ , which derives a contradiction. This implies that  $Ax = BDx$ . Hence

$$\begin{aligned} \sigma_{\min}(A) &= \min_{\|y\|=1} \|Ay\| \leq \|Ax\| = \|BDx\| \\ &\leq \|B\| \|D\| \|x\| \leq \|B\| = \sigma_{\max}(B). \end{aligned}$$

This contradicts our condition. Hence  $A - BD$  is non-singular.  $\square$

Now according to the equality  $D(x)x = |x|$ , with  $D(x) = \text{diag}(\text{sign}(x))$  the AVE (1.1) can be transformed into the following linear system of equations:

$$Qx = b, \tag{3.1}$$

where  $Q = A - BD$ .

**Lemma 3.2.** *If symmetric matrices  $A$  and  $B$  satisfy*

$$\sigma_{\min}(A) > \sigma_{\max}(B),$$

*then the AVE (1.1) is uniquely solvable for any  $b \in \mathbb{R}^n$ .*

*Proof.* Based on the result of Lemma (3.1), the matrix  $Q$  is non-singular for any arbitrary diagonal matrix  $D$  whose elements are equal to 1, 0 or - 1 and therefore the AVE (1.1) has a unique solution for any  $b$ .  $\square$

One of the important numerical tools to solve the system (3.1) is to transform it into an equivalent convex quadratic optimization problem:

$$\min_{x \in \mathbb{R}^n} f(x) = \frac{1}{2}(Qx - b)^T(Qx - b). \tag{3.2}$$

The gradient and the Hessian matrix of  $f(x)$  are given by:

$$g(x) := \partial f(x) = Q^T(Qx - b),$$

and

$$H(x) := \partial^2 f(x) = Q^T Q.$$

Since  $H(x)$  is positive definite for any diagonal matrix  $D$  whose elements are equal to  $+1$ ,  $0$  or  $-1$ , the problem (3.2) has a unique minimum that satisfies

$$g(x) = 0,$$

or

$$Q^T Q x = Q^T b. \quad (3.3)$$

Since  $Q$  is non-singular therefore (3.3) is equivalent to (3.1) and so is equivalent to AVE (11). Hence solving the AVE (1.1) is equivalent to find the unique minimum of (3.2). The conjugate gradient methods are known to be effective in solving quadratic problems in finite termination [10], [11], [35], [37], [38]. These methods start with an initial point  $x_0$  and generate a sequence  $\{x_k\}$  according to the following recurrence formula:

$$x_{k+1} = x_k + \alpha_k d_k, \quad k = 0, 1, 2, \dots \quad (3.4)$$

where  $\alpha_k > 0$  is the step-size obtained by a line search and the directions  $d_k$  are computed by the rule

$$d_k = -g_k + \beta_k d_{k-1}, \quad k \geq 1, \quad d_0 = -g_0, \quad (3.5)$$

where  $\beta_k$  is a suitable positive scalar known as the conjugate updating parameter and  $g_k$  refers to  $g(x_k)$ .

### 3.1.1 Exact line search

Determining the step-size  $\alpha_k$  in (3.4) along the direction  $d_k$ , for an objective function  $f(x)$ , which is to be minimized, can be simplified to finding the value of  $\alpha_k = \alpha$  which consequently minimizes the function:

$$f(x_{k+1}) = f(x_k + \alpha d_k) = m(\alpha).$$

The function  $m(\alpha)$  is of a single variable, that is  $\alpha$ . Therefore, the  $\alpha_k$  is calculated by an exact line search as follows. Using Taylor's expansion, we have,

$$m(\alpha) = f(x_k + \alpha d_k) = f(x_k) + \alpha g_k^T d_k + \frac{\alpha^2}{2} d_k^T H_k d_k,$$

so

$$\frac{\partial m}{\partial \alpha} = 0 \Leftrightarrow \alpha = -\frac{g_k^T d_k}{d_k^T H_k d_k}.$$

Therefore the exact line search is taken as:

$$\alpha_k = -\frac{g_k^T d_k}{d_k^T H_k d_k}, \quad (3.6)$$

where  $H_k$  refers to  $H(x_k)$ .

### 3.1.2 Computation of $\beta_k$

The coefficients  $\beta_k$  being chosen in such a way that  $d_k$  is conjugated with all the preceding directions, in other word

$$d_k^T Q d_{k-1} = 0,$$

then it implies that:

$$d_k^T Q d_{k-1} = -g_k^T Q d_{k-1} + (\beta_k d_{k-1})^T Q d_{k-1} = 0,$$

and so:

$$\beta_k = \frac{g_k^T Q d_{k-1}}{d_{k-1}^T Q d_{k-1}}. \quad (3.7)$$

### 3.1.3 Basic conjugate gradient algorithms.

We are now ready to state the basic CG algorithms for solving the AVE (1.1).

- Step 1.** Choose an arbitrary initial point  $x_0 \in \mathbb{R}^n$ ,  $\epsilon > 0$  and  $d_0 = -g_0$ ,  $k = 0$ ;  
**Step 2.** Compute  $\alpha_k$  from (3.6) and set  $x_{k+1} = x_k + \alpha_k d_k$ ;  
**Step 3.** If  $\|Ax_k - B|x_k| - b\| < \epsilon$  then STOP, otherwise compute  $d_k$  according to  
 $d_k = -g_k + \beta_k d_{k-1}$  with  $\beta_k$  is computed from (3.7);  
**Step 4.** Set  $k = k + 1$ , and go to **Step 2**.

In [10], and [37], it is shown that the convergence of the *CG* methods is linearly global to the unique minimum  $x^*$ . It is known that the convergence of *CG* methods depends heavily on the condition number  $\kappa(Q)$ . If  $\kappa(Q)$  is close to 1, i.e. if the matrix  $Q$  is well-conditioned then *CG* methods converge fast to the solution. Otherwise, in the presence of ill-conditioned of the matrix  $Q$ , these methods have a very slow convergence.

## 3.2 Preconditioned conjugate gradient algorithms

Preconditioning is mainly used in *CG* methods in order to accelerate their convergence when  $\kappa(Q)$  is very far from 1, i.e. when  $Q$  is ill-conditioned. Based on this fact, we can consider the preconditioned AVE (1.1):

$$PAx - PB|x| = Pb, \quad (3.8)$$

where  $P$  is a non-singular matrix, called the preconditioner. Obviously, the form (3.8) is a general form of the AVE (1.1). For  $P = I$ , the form (3.8) reduced to the AVE

(1.1). Again using  $D(x)x = |x|$ , then (3.8) becomes the following preconditioned linear system:

$$PQx = Pb. \quad (3.9)$$

Hence the system (3.9) has a unique solution if the matrix  $PQ$  is invertible. Since  $P$  is assumed to be non-singular, then we only prove that  $Q$  is non-singular. By Lemma 3.1, the matrix  $Q$  is non-singular for any diagonal matrix  $D$  whose elements are 1, 0, or -1, and consequently, the system (3.9) has a unique solution and so the preconditioned AVE in (3.8) is uniquely solvable for each  $b$ . Based on this observation, therefore, the equivalent preconditioned quadratic optimization problem is:

$$\min_{x \in \mathbb{R}^n} f_P(x) = \frac{1}{2}(PQx - Pb)^T(PQx - Pb). \quad (3.10)$$

The gradient and the Hessian matrix of  $f$  are:

$$g^P(x) := \partial f_P(x) = (PQ)^T(PQx - Pb)$$

and

$$H^P(x) := \partial^2 f_P(x) = (PQ)^T(PQ).$$

It is clear that if  $P = I$ , the problem (3.10) reduces to the original problem (3.2). Also since  $(PQ)^T(PQ)$  is positive definite matrix, the problem (3.10) has a unique minimum that satisfies:

$$g^P(x) = 0,$$

or

$$(PQ)^T(PQx - Pb) = 0,$$

which means that the unique minimum is the unique solution of the preconditioned system and which is in turn the unique solution of the AVE (1.1). For the preconditioned problem (3.9), with same manner as the basic CG algorithms, we compute the exact line search  $\alpha_k$  and the conjugate parameter  $\beta_k$  along the new preconditioned modified search direction by the formulas:

$$\alpha_k = -\frac{(g_k^P)^T d_k}{d_k^T H_k^P d_k}, \quad (3.11)$$

and

$$\beta_k = \frac{(g_k^P)^T Q d_{k-1}}{d_{k-1}^T Q d_{k-1}}. \quad (3.12)$$

Now the preconditioned conjugate gradient (PCG) algorithm for solving the AVE (1.1) is described as follows.

### 3.2.1 Preconditioned conjugate gradient algorithm

**Step 1.** Choose an arbitrary  $x_0 \in \mathbb{R}^n$ , a preconditioner matrix  $P$ ,  $\epsilon > 0$  and  $d_0 = -g_0^P$ ,  $k = 0$ ;

**Step 2.** Compute  $\alpha_k$  from (3.11) and set  $x_{k+1} = x_k + \alpha_k d_k$ ;

**Step 3.** If  $\|PAx_k - PB|x_k| - Pb\| < \epsilon$  then STOP, otherwise compute  $d_k$  according to  $d_k = -g_k^P + \beta_k d_{k-1}$  with  $\beta_k$  is computed from (3.12);

**Step 4.** Set  $k = k + 1$ , and go to **Step 2**.

Note that there is no unique strategy for choosing the preconditioning matrix  $P$  for the conjugate CG methods. In fact, the strategy of choosing  $P$  is based on a such way that the  $\kappa(PQ) \ll \kappa(Q)$ . For more details see [10].

### 3.2.2 Another preconditioned conjugate gradient algorithm

In this subsection, we restrict our selves to the AVE (1.2) i.e.,  $B = I$ . Here, we give another choice to the preconditioning  $P$  followed by the obtained numerical results. We have seen that if the matrix  $Q$  is ill-conditioned then CG algorithm converges slowly or even diverges. For this reason, we look to transform the system  $Qx = b$  into an equivalent well-conditioned system. We remind that  $Q = A - D$ . Then it is worth to write  $A$  in the form of a splitting

$$A = M - N$$

where  $M$  is SPD. Since  $M$  is SPD then it has a Cholesky decomposition ,

$$M = L^T L,$$

where  $L$  is a lower triangular matrix. If we let

$$\tilde{A} = L^{-T} A L^{-1}, \tilde{b} = L^{-T} b$$

then solving (1.2) is equivalent to

$$\tilde{Q} \tilde{y} = \tilde{b}$$

where  $\tilde{Q} = \tilde{A} - \tilde{D}$  with  $\tilde{y} = Lx$ . We hope that  $\kappa(\tilde{Q}) \ll \kappa(Q)$ , where  $k(Q)$  denotes the condition number of the corresponding matrix. For  $Q$  its spectral condition number is defined as:

$$k(Q) = \frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)}.$$

Therefore solving (1.2) by performing the conjugate gradient to the SPD linear system  $\tilde{Q} \tilde{y} = \tilde{b}$ . With the new formulation, we have

$$\tilde{y}_k = Lx_k.$$

and

$$\tilde{r}_k = \tilde{b} - \tilde{Q}\tilde{y}_k = L^{-T}(b - Qx_k) = L^{-T}r_k.$$

A conjugate gradient step would be

$$\begin{aligned}\tilde{d}_{k+1} &= \tilde{r}_k + \beta_k \tilde{d}_k \\ y_k &= y_{k-1} + \alpha_k \tilde{d}_k\end{aligned}$$

We prefer to update  $x_k = L^{-1}y_k$ . We have that

$$\tilde{d}_{k+1} = L^{-T}r_k + \beta_k \tilde{d}_k,$$

so

$$x_k = L^{-1}y_k = L^{-1}y_{k-1} + \alpha_k L^{-1}\tilde{d}_k.$$

Thus

$$x_k = x_{k-1} + \alpha_k d_k$$

where

$$d_{k+1} = L^{-1}L^{-T}r_k + \beta_k d_k.$$

Since

$$M^{-1} = L^{-1}L^{-T}$$

if we let

$$z_k = M^{-1}r_k$$

then

$$d_{k+1} = z_k + \beta_k d_k$$

is the update to the descent direction. The formulas for  $\beta_k$  and  $\alpha_k$  update to

$$\begin{aligned}\beta_k &= \frac{r_k^T z_k}{r_{k-1}^T z_{k-1}} \\ \alpha_k &= \frac{r_{k-1}^T z_{k-1}}{d_k^T Q d_k}.\end{aligned}$$

These modifications lead to the another preconditioned conjugate gradient method (PCG).

### 3.2.3 PCG algorithms

The basic PCG algorithms for solving the AVE (1.2) is described as follows.

**Step 1.** Choose an arbitrary initial point  $x_0 \in \mathbb{R}^n$ ,  
 $\epsilon > 0$  and  $d_0 = r_0$ ,  $k = 0$ ,  $Mz_0 = r_0$ ;

**Step 2.** Compute  $\alpha_k$  from  $\alpha_k = \frac{r_{k-1}^T z_{k-1}}{d_k^T Q d_k}$ , and set  $x_k = x_{k-1} + \alpha_k d_k$ ;

**Step 3.** If  $\|Qx_k - b\| < \epsilon$  then STOP,  
otherwise compute  $d_k$  according to  $d_{k+1} = z_k + \beta_k d_k$   
with  $\beta_k = \frac{r_k^T z_k}{r_{k-1}^T z_{k-1}}$ ;

**Step 4.** Set  $k = k + 1$ , and go to **Step 2**.

In this work the choice of the matrix  $M$  is very important, because it has an effect on numerical results. So we applied the Incomplete Cholesky Factorization program to choose  $M$ .

### 3.2.4 Incomplete Cholesky Factorization

We get

$$M = \tilde{L}^T \tilde{L}.$$

where  $L_M = (\tilde{l}_{kj})$  from the following algorithm. Here  $l_{kj} = 0$  if  $a_{kj} = 0$ .

#### Algorithm (Incomplete Cholesky Factorization (ICF))

```

for  $k = 1, \dots, n - 1$ 
     $\tilde{l}_{kk} = \sqrt{a_{kk}}$ ;
    for  $j = k + 1 : n$ 
         $\tilde{l}_{kj} = \frac{a_{kj}}{\tilde{l}_{kk}}$ ;
    end;
    for  $i = k + 1 : n$ 
        for  $j = i : n$  % Upper triangle only
            if  $a_{ij} \neq 0$ 
                 $a_{ij} = a_{ij} - \tilde{l}_{ki} \tilde{l}_{kj}$ ;
            end;
        end;
    end;
end;
 $l_{nn} = \sqrt{a_{nn}}$ ;

```

### 3.3 Numerical experiments

In this section, we present some numerical experiments on some examples of solvable AVE (1.1) to confirm the viability of the *PCG* algorithms. The experiments are performed with **MATLAB 7.9** and carried out on a PC where our tolerance is set to  $\epsilon = 10^{-6}$ . The initial point and the true solution of AVE (1.1) are denoted by  $x_0$  and  $x^*$ , respectively. Meanwhile, the number of iterations, the elapsed times and the residue are denoted by **Iter**, **CPU** and **RSD** =  $\|Ax_k - B|x_k| - b\|$  respectively. In our numerical implementation, the appropriate choice of the preconditioners are  $P = \frac{1}{n}I, n \geq 1$ , and  $P = A^{-1}$ .

**Example 1.** Let the symmetric matrices  $A$ ,  $B$  and the vector  $b$  be given as

$$A = (a_{ij}) = \begin{bmatrix} 4n & n & 0.5 & \cdots & 0.5 & 0.5 \\ n & 4n & n & \cdots & 0.5 & 0.5 \\ 0.5 & n & 4n & \cdots & 0.5 & \vdots \\ \vdots & \vdots & \ddots & \ddots & n & 0.5 \\ 0.5 & 0.5 & 0.5 & \cdots & 4n & n \\ 0.5 & 0.5 & \cdots & 0.5 & n & 4n \end{bmatrix},$$

and

$$B = (b_{ij}) = \begin{bmatrix} n & \frac{1}{n} & 0.125 & \cdots & 0.125 & 0.125 \\ \frac{1}{n} & n & \frac{1}{n} & \cdots & 0.125 & 0.125 \\ 0.125 & \frac{1}{n} & n & \cdots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{1}{n} & 0 \\ 0.125 & 0.125 & 0.125 & \cdots & n & \frac{1}{n} \\ 0.125 & 0.125 & \cdots & 0.125 & \frac{1}{n} & n \end{bmatrix},$$

$$b = [548, 647.5, \dots, 647.5, 548]^T.$$

The two initial points are taken as  $x_1^0 = [0.001, \dots, 0.001]^T$  and  $x_2^0 = [0.9, \dots, 0.9]^T$ , The computational results with different size of  $n$ , are summarized in Table 1.

Size $n$	$x_0$		$P = I$ (basic CGA)	$P = \frac{1}{n}I, n > 1$	$P = A^{-1}$
100	$x_1^0$	Iter	29	21	2
		CPU(s)	0.0186	0.0161	0.0057
		RSD	$6.0870e - 006$	$8.4089e - 006$	$4.5068e - 007$
	$x_2^0$	Iter	27	19	3
		CPU(s)	0.0178	0.0158	0.0074
		RSD	$6.7790e - 006$	$9.1783e - 006$	$6.9333e - 006$
1000	$x_1^0$	Iter	33	21	2
		CPU(s)	3.5787	2.4979	0.2273
		RSD	$5.4336e - 006$	$8.5594e - 006$	$4.5352e - 008$
	$x_2^0$	Iter	31	19	3
		CPU(s)	3.3429	2.0959	0.3293
		RSD	$6.0443e - 006$	$9.3010e - 006$	$8.8665e - 015$
2000	$x_1^0$	Iter	34	21	2
		CPU(s)	26.6083	16.2365	1.5644
		RSD	$5.7908e - 006$	$8.4035e - 006$	$2.2684e - 008$
	$x_2^0$	Iter	32	19	3
		CPU(s)	24.4110	14.6529	2.2716
		RSD	$6.4720e - 006$	$9.2160e - 006$	$1.5029e - 014$
3000	$x_1^0$	Iter	34	21	2
		CPU(s)	90.5026	53.2645	4.9661
		RSD	$8.4709e - 006$	$8.3032e - 006$	$1.5124e - 008$
	$x_2^0$	Iter	32	19	3
		CPU(s)	83.9471	48.7091	5.7692
		RSD	$9.5473e - 006$	$9.1581e - 006$	$2.7556e - 014$

**Table 1.**

The true solution is  $x^* = \left[ \frac{4}{3}, \frac{4}{3}, \dots, \frac{4}{3}, \frac{4}{3} \right]^T$ .

**Example 2.** The hydrodynamic equations (equilibrium problem [18] ), is modeled as the following non-differentiable algebraic equations:

$$Bx + \max(0, x) = c,$$

where  $B \in \mathbb{R}^{n \times n}$ ,  $c \in \mathbb{R}^n$  are given. Using the identity

$$\max(a, b) = \frac{1}{2}(a + b + |a - b|),$$

equality, the hydrodynamic equation can be reformulated as an AVE (1.1). We have,  $Bx + \frac{1}{2}(x + |x|) = c \Leftrightarrow Ax - |x| - b = 0$  where  $A = -(2B + I)$  and  $b = -2c$ .

Consider now, a randomly hydrodynamic equation where  $B \in \mathbb{R}^{n \times n}$  and  $c$  are given by:

$$B = (b_{ij}) = \begin{bmatrix} -25.5 & -2.5 & 0 & \cdots & 0 & 0 \\ -2.5 & -25.5 & -2.5 & \cdots & 0 & 0 \\ 0 & -2.5 & -25.5 & \cdots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -2.5 & 0 \\ 0 & 0 & 0 & \cdots & -25.5 & -2.5 \\ 0 & 0 & \cdots & 0 & -2.5 & -25.5 \end{bmatrix},$$

and

$$c = [-27, -29.5, \dots, -29.5, -27]^T.$$

The initial points are  $x_1^0 = [0.5, \dots, 0.5]^T$  and  $x_2^0 = [0.9, \dots, 0.9]^T$ . The computational results with different size of  $n$ , are summarized in Table 2.

Size $n$	$x_0$		$P = I$ (basic CGA)	$P = \frac{1}{n}I, n > 1$	$P = A^{-1}$
100	$x_1^0$	Iter	11	8	4
		CPU(s)	0.0143	0.0133	0.0081
		RSD	$6.1357e - 006$	$5.7939e - 006$	$4.8013e - 008$
	$x_2^0$	Iter	10	7	3
		CPU(s)	0.0170	0.0113	0.0068
		RSD	$5.6321e - 006$	$5.1941e - 006$	$2.0096e - 006$
1000	$x_1^0$	Iter	11	7	4
		CPU(s)	1.1760	0.7838	1.5308
		RSD	$5.8906e - 006$	$2.5537e - 006$	$4.8112e - 008$
	$x_2^0$	Iter	10	6	3
		CPU(s)	1.1115	0.6666	1.1335
		RSD	$5.4239e - 006$	$2.2815e - 006$	$2.0219e - 006$
2000	$x_1^0$	Iter	11	6	4
		CPU(s)	8.9451	4.7227	5.5383
		RSD	$5.8609e - 006$	$5.7002e - 006$	$4.8116e - 008$
	$x_2^0$	Iter	10	5	3
		CPU(s)	8.1684	4.0932	5.3227
		RSD	$5.4023e - 006$	$5.0782e - 006$	$2.0226e - 006$
3000	$x_1^0$	Iter	11	6	4
		CPU(s)	28.4459	15.5639	13.9749
		RSD	$5.8503e - 006$	$3.7993e - 006$	$4.8117e - 008$
	$x_2^0$	Iter	10	5	3
		CPU(s)	25.7548	12.8931	10.7083
		RSD	$5.3948e - 006$	$3.3853e - 006$	$2.0228e - 006$

**Table 2.**

The true solution of this example is  $x^* = e$ .

**Example 3.** We consider a LCP ( see subsection 2.2.1). The  $M \in \mathbb{R}^{n \times n}$  and  $q$  are given by:

$$M = (m_{ij}) = \begin{bmatrix} 0.6 & -0.01 & 0 & \cdots & 0 & 0 \\ -0.01 & 0.6 & -0.01 & \cdots & 0 & 0 \\ 0 & -0.01 & 0.6 & \cdots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -0.01 & 0 \\ 0 & 0 & 0 & \cdots & 0.6 & -0.01 \\ 0 & 0 & \cdots & 0 & -0.01 & 0.6 \end{bmatrix}, \quad q = -e.$$

The initial points are  $x_1^0 = [0.001, \dots, 0.001]^T$  and  $x_2^0 = [0.9, \dots, 0.9]^T$  and the obtained computational results with different size of  $n$ , are stated in Table 3.

Size $n$	$x_0$		$P = I$ (basic CGA)	$P = \frac{1}{n}I, n > 1$	$P = A^{-1}$
100	$x_1^0$	Iter	7	5	4
		CPU(s)	0.0152	0.0090	0.0083
		RSD	$2.0378e - 006$	$2.3141e - 006$	$9.3375e - 006$
	$x_2^0$	Iter	6	4	4
		CPU(s)	0.0106	0.0081	0.0104
		RSD	$3.3940e - 006$	$3.8226e - 006$	$7.3068e - 006$
1000	$x_1^0$	Iter	7	4	4
		CPU(s)	1.6359	1.0504	0.9420
		RSD	$1.9182e - 006$	$2.4378e - 006$	$9.34412e - 006$
	$x_2^0$	Iter	6	3	4
		CPU(s)	1.4547	0.6995	0.9617
		RSD	$3.2870e - 006$	$4.1163e - 006$	$7.3117e - 006$
2000	$x_1^0$	Iter	7	4	4
		CPU(s)	7.2802	4.1854	4.1506
		RSD	$1.9091e - 006$	$1.2187e - 006$	$9.3440e - 006$
	$x_2^0$	Iter	6	3	4
		CPU(s)	6.1432	3.0415	4.8948
		RSD	$3.2795e - 006$	$2.0590e - 006$	$7.3116e - 006$
3000	$x_1^0$	Iter	7	3	4
		CPU(s)	20.3593	8.6417	11.5123
		RSD	$1.9059e - 006$	$6.7655e - 006$	$9.3440e - 006$
	$x_2^0$	Iter	6	3	4
		CPU(s)	18.1800	8.6375	11.6423
		RSD	$3.2770e - 006$	$1.3729e - 006$	$7.3116e - 006$

**Table 3.**

The true solution is

$$x^* = [0.8477, 0.8618, 0.8621, \dots, 0.8621, 0.8618, 0.8477]^T,$$

and then

$$z^* = [1.6954, 1.7237, 1.7241, \dots, 1.7241, 1.7237, 1.6954]^T$$

is the solution of LCP.

**Example 4.** The matrices  $A$  and  $B$  are given by

$$A = (a_{ij}) = \begin{bmatrix} 10001 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n-1} & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} + 1 & \frac{1}{4} & \cdots & \frac{1}{n} & \frac{1}{n+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} + 1 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 1 \\ \frac{1}{n-1} & \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{2n-4}{1} + 1 & \frac{2n-3}{1} \\ \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-3} & \frac{1}{2n-2} & \frac{1}{2n-1} + 1 \end{bmatrix}, B = I.$$

For example, if  $n = 4$ , then

$$A = \begin{bmatrix} 10001 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{4}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{5}{4} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{7}{6} \end{bmatrix}.$$

The spectrum of  $A$ , is given by  $\{10001, 1.657, 1.0189, 1.0002\}$ . Since  $\lambda_{\min}(A) = 1.0002 > \lambda_{\max}(I) = 1$ , then the AVE is uniquely solvable for any  $b$  and also the matrix  $Q = A - D$  is symmetric positive definite for all diagonal matrix  $D$  whose elements are 1,0, or -1. The matrix  $A$  is ill-conditioned since  $\kappa(A) = \frac{10001}{1.0002} = 9999 \gg 1$ , and consequently,  $Q$  is ill-conditioned for any matrix  $D$  whose diagonal elements are 1,0, or -1.

Next, for  $b = (A - I)e \in \mathbb{R}^n$ , and with the initial point  $x_0 = [0, 0, \dots, 0]^T$ , the computational results for this example with different size of  $n$ , are illustrated in Table 4.

Size $n$		$P = I$ (basic CGA)	$P = \frac{1}{n}I, n > 1$	$P = A^{-1}$
4	Iter	3530	2701	2
	CPU(s)	0.1300	0.0983	0.0050
	RSD	$9.9959e - 006$	$9.9952e - 006$	$4.1168e - 016$
10	Iter	16483	12079	2
	CPU(s)	0.6712	0.4594	0.0051
	RSD	$9.9991e - 006$	$9.9983e - 006$	$6.753e - 016$
1000	Iter	*	*	2
	CPU(s)	*	*	0.2494
	RSD	*	*	$3.9550e - 014$
2000	Iter	*	*	2
	CPU(s)	*	*	1.5669
	RSD	*	*	$7.2000e - 014$

Table 4.

The "\*" means that the basic CG and the preconditioned CG with  $P = \frac{1}{n}I, n > 1$  algorithms failed.

The true solution of this example is  $x^* = e$ .

- For numerical experiments the another preconditioned conjugate gradient algorithms (ICFPCG). Where we set  $\epsilon = 10^{-6}$ . The residue is given by

$$\mathbf{RSD} = \frac{\|Qx - b\|}{\|b\|}, \text{ or } \frac{\|\tilde{Q}x - \tilde{b}\|}{\|\tilde{b}\|} \text{ (for Alg (CG) and Alg (ICFPCG), respectively) .}$$

**Example 5.** Let the symmetric matrices  $A$ ,  $B$  and the vector  $b$  be given by:

$$A = (a_{ij}) = \begin{bmatrix} 4n & n & 0.5 & \cdots & 0.5 & 0.5 \\ n & 4n & n & \cdots & 0.5 & 0.5 \\ 0.5 & n & 4n & \cdots & 0.5 & \vdots \\ \vdots & \vdots & \ddots & \ddots & n & 0.5 \\ 0.5 & 0.5 & 0.5 & \cdots & 4n & n \\ 0.5 & 0.5 & \cdots & 0.5 & n & 4n \end{bmatrix},$$

$$B = I, \quad b = (A - I)e.$$

With the initial points  $x_1^0 = [0.01, \dots, 0.01]^T$  and  $x_2^0 = [0.9, \dots, 0.9]^T$ , the computational results with different size of  $n$ , are summarized in Table 5.

size ( $n$ )	$x_0$		Alg (CG)	Alg (ICFPCG)
100	$x_1^0$	Iter	15	4
		CPU(s)	0.0109	0.0052
		RSD	$6.1097e - 006$	$7.9089e - 006$
	$x_2^0$	Iter	10	4
		CPU(s)	0.0106	0.0052
		RSD	$6.9542e - 006$	$5.7335e - 006$
1000	$x_1^0$	Iter	12	4
		CPU(s)	0.0848	0.0411
		RSD	$7.6005e - 006$	$3.2862e - 006$
	$x_2^0$	Iter	7	4
		CPU(s)	0.0529	0.0310
		RSD	$7.4305e - 007$	$2.8331e - 006$
2000	$x_1^0$	Iter	11	4
		CPU(s)	0.2570	0.0908
		RSD	$7.6835e - 006$	$2.5811e - 006$
	$x_2^0$	Iter	6	4
		CPU(s)	0.1438	0.0954
		RSD	$8.9312e - 006$	$2.4027e - 006$
3000	$x_1^0$	Iter	11	4
		CPU(s)	0.5743	0.2075
		RSD	$6.2286e - 006$	$2.1659e - 006$
	$x_2^0$	Iter	6	4
		CPU(s)	0.3069	0.2025
		RSD	$7.2842e - 006$	$2.0873e - 006$

**Table 5.**

The true solution is  $x^* = e$ .

• Same as the second example. We give the numerical results for the initial points are  $x_1^0 = [0.1, \dots, 0.1]^T$  and  $x_2^0 = [0.9, \dots, 0.9]^T$ . The computational results with different size of  $n$ , are summarized in Table 6.

size ( $n$ )	$x_0$		Alg (CG)	Alg (ICFPCG)
100	$x_1^0$	Iter	7	3
		CPU(s)	0.0088	0.0060
		RSD	$4.6532e - 006$	$2.5617e - 006$
	$x_2^0$	Iter	5	3
		CPU(s)	0.0082	0.0066
		RSD	$6.0301e - 006$	$5.8363e - 006$
1000	$x_1^0$	Iter	3	3
		CPU(s)	0.0470	0.0499
		RSD	$1.4699e - 007$	$7.5929e - 007$
	$x_2^0$	Iter	3	3
		CPU(s)	0.1309	0.0475
		RSD	$1.0384e - 007$	$1.8679e - 006$
2000	$x_1^0$	Iter	6	3
		CPU(s)	0.1515	0.1383
		RSD	$3.2255e - 006$	$5.3482e - 007$
	$x_2^0$	Iter	4	3
		CPU(s)	0.1027	0.1251
		RSD	$3.2648e - 006$	$1.3216e - 006$
3000	$x_1^0$	Iter	5	3
		CPU(s)	0.2477	0.2332
		RSD	$9.4332e - 006$	$4.3611e - 007$
	$x_2^0$	Iter	4	3
		CPU(s)	0.2512	0.2444
		RSD	$3.9532e - 006$	$1.0794e - 006$

**Table 6.**

The true solution is  $x^* = e$ .

• Same as the third example. We give the numerical results of the initial points are  $x_1^0 = [0.1, \dots, 0.1]^T$  and  $x_2^0 = [0.9, \dots, 0.9]^T$  and the obtained computational results with different size of  $n$ , are stated in Table 7.

size ( $n$ )	$x_0$		Alg (CG)	Alg (ICFPCG)
100	$x_1^0$	Iter	5	4
		CPU(s)	0.0074	0.0065
		RSD	$2.8462e - 006$	$4.0077e - 007$
	$x_2^0$	Iter	4	4
		CPU(s)	0.0067	0.0069
		RSD	$9.6956e - 006$	$3.8310e - 007$
1000	$x_1^0$	Iter	4	3
		CPU(s)	0.0531	0.0506
		RSD	$7.3072e - 006$	$5.1550e - 006$
	$x_2^0$	Iter	4	3
		CPU(s)	0.0543	0.0540
		RSD	$2.9346e - 006$	$5.0868e - 006$
2000	$x_1^0$	Iter	4	3
		CPU(s)	0.1870	0.1284
		RSD	$5.1523e - 006$	$3.6387e - 006$
	$x_2^0$	Iter	4	3
		CPU(s)	0.1417	0.1317
		RSD	$2.0693e - 006$	$3.5944e - 006$
3000	$x_1^0$	Iter	4	3
		CPU(s)	0.2730	0.2368
		RSD	$4.2028e - 006$	$2.9693e - 006$
	$x_2^0$	Iter	4	3
		CPU(s)	0.2872	0.2363
		RSD	$1.6879e - 006$	$2.9342e - 006$

Table 7.

The true solution is  $x^* = [0.8477, 0.8618, 0.8621, \dots, 0.8621, 0.8618, 0.8477]^T$ , and then the true solution for LCP is  $z^* = [1.6954, 1.7237, 1.7241, \dots, 1.7241, 1.7237, 1.6954]^T$ .

# General conclusion and perspectives

In this dissertation, , we have presented some weaker sufficient conditions that guarantee the unique solvability of the AVE (1.1). Our proofs are simple and elegant with the advantage that we do not use the theory of linear complementarity to prove unique solvability of the AVE (1.1). Across an example of AVE, we have showed the reliability of our weaker sufficient conditions to detect unique solvability of AVE (1.1). These obtained results are also extended to detecting unique solvability of standard and horizontal LCP. Numerically, the proposed Picard's iterative method is efficient to provide an approximated solution of some uniquely solvable AVE including standard and horizontal LCP problems.

We have also presented preconditioned conjugate gradient methods for solving the absolute value equations. The obtained numerical results with the preconditioned matrix  $P = A^{-1}$  are the best since the number of iterations and the elapsed times are minimum compared with those obtained by the basic conjugate gradient algorithms ( $P = I$ ). We have also used a new choice of preconditioning matrix  $P$  based on the Incomplete Cholesky decomposition. The obtained results are also efficient.

We hope that the preconditioned absolute value equations serves as a basis for future research on other more choice for the preconditioned matrix  $P$  to intend an efficient study of the absolute value equations.

Finally some perspectives are given.

1- A good topic of research is the study of the AVE where there is no conditions are stated on the existence and uniqueness of solutions. For this purpose, we have planned in the futur a DC programming reformulation ( difference of convex functions) for solving the AVE (1.1). Theoretical and numerical study.

2- An important new axis is to consider the matrix absolute value equation of the form (Sylvester):

$$AX - B|X| = C,$$

where  $A, B$  and  $C$  are given matrices and  $X$  is the matrix solution. Here  $|X|$  denotes the absolute value of the matrix  $X$ . This new type of the AVE, generalizes the classical absolute values equation  $Ax - B|x| = b$ . This topic remains a good subject in the future where a theoretical and numerical study is needed.

**3-** An interesting question is the reformulation of the nonlinear LCP as absolute value equations and deducing new result concerning its solvability. In addition its numerical solution through the AVE remains a good subject in the future.

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## ملخص

في هذه الأطروحة، نسلط الضوء، من ناحية على قابلية وحدانية الحل للمعادلات ذات القيمة المطلقة حيث يكون بعضها كافياً، يتم إعطاء شروط أضعف. ويتم تعميم هذه النتائج أيضاً على قابلية وحدانية الحل لمسائل التتام الخطي العادي و الأفقي . من ناحية أخرى ، لحلها العددي يتم اقتراح طريقة النقطة الثابتة التكرارية وطرق التدرج المترافق العادي والمشروط. . أخيراً، يتم إعطاء بعض النتائج العددية لتأكيد فعالية الطرق المقترحة لحل المعادلات ذات القيمة المطلقة.

**الكلمات المفتاح :** معادلات القيمة المطلقة ، مسائل التكامل الخطي ، النظام الخطي ، القيم الذاتية الشاذة ، الطرق التكرارية ، مسائل الأمثلية التربيعية المحدبة غير المقيدة

## Résumé

Dans cette thèse, nous soulignons, d'une part, la solvabilité unique des équations aux valeurs absolues (EVA) lorsque des conditions suffisamment faibles sont données. Ces résultats sont également généralisés à la solvabilité unique des problèmes de complémentarité linéaire horizontale et standard. D'autre part, pour la solution numérique, la méthode itérative point fixe et des méthodes de gradient conjugué classique et pré-conditionné sont proposées. Enfin, quelques résultats numériques sont donnés pour confirmer l'efficacité de nos approches proposées pour résoudre l'EVA.

**Mots clés :** Equations aux valeurs absolues, Problèmes de complémentarité linéaire, Système linéaire, Valeur singulière, Méthodes itératives, Optimisation quadratique convexe sans contrainte

## Abstract

In this thesis, we highlight, on one hand the unique solvability of the absolute value equations where some sufficient weaker conditions are given. These results are also generalized to the unique solvability of horizontal and standard linear complementarity problems. On the other hand, for its numerical solution, Picard's iterative and preconditioned conjugate gradient methods are proposed. Finally, some numerical results are given to confirm the efficiency of our proposed approaches for solving the AVEs.

**Keywords:** Absolute value equations, Linear complementarity problems, Linear system, Singular value, Iterative methods, Unconstrained convex quadratic optimization.