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Dedication

This thesis is dedicated first to my idol ***prophet Mohammed*** *peace be upon him.*

To my parents who have helped me keep patient and strong, and without whom I would have never been the person I am today.

Especialy, to my dear husband ***Smail****. He always believes in me and his faith keeps me strong and going.*

To my beloved children; ***Soundouss*** *and* ***Mohammed****, who witnessed so many hours of my absence.*

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Contents

Notations

0.1 For abbreviations and expressions

The abbreviation	The meaning
$s.t.$	such that
iff	if and only if
RHS	right hand side
LHS	left hand side
ONB	orthogonormal basis
”$i.i.d$”	independent identically distributed
UMD	unconditional for martingale difference
$PDEs$	partial differential equations
$SDEs$	stochastic differential equations
$SPDEs$	stochastic partial differential equations
$FSPDEs$	fractional stochastic partial differential equations

The expression	The meaning
$x := y$ or $y := x$	x is equal to y by definition
$a \wedge b$	$\min(a, b)$
$a \vee b$	$\max(a, b)$
a^-, for $a \in \mathbb{R}_+^*$	the positive number $a - \kappa$, for any, $\kappa > 0$
$\arg z$	argument of the complex number z
$Eq.(n.m)$	an equation of number m exists in chapter n
$Prb.(n.m)$	a problem of number m exists in chapter n
$IVP.(n.m)$	an initial value problem of number m exists in chapter n
$Est.(n.m)$	an estimate of number m exists in chapter n
$Cond.(n.m)$	a condition of number m exists in chapter n

0.2 For sets and functions

The symbol	The meaning
$\mathbb{N}_0$	$\mathbb{N} - \{0\}$
$\mathbb{R}_+$	the interval $[0, +\infty)$
$\mathbb{R}_+^*$	the interval $(0, +\infty)$
$\mathbb{R}_+^d$	$\{t = (t_1, ..., t_d) \in \mathbb{R}^d,\ s.t.,\ t_i \geq 0,\ \forall\ i\}$
Domain D	a non empty open set
∂D	the boundary of the domain D
$(a, b),$ For $a < b \in \mathbb{R},$	an open interval
$supp(f)$	support of the function f
1_B	the indecator function of the set B
$f = \mathcal{O}(g),$ as $x \to x_0$	$\exists\ C > 0$ s.t. $\vert f(x)\vert \leq C\vert g(x)\vert,$ for all x sufficiently close to x_0
Γ	gamma function defined by $\Gamma(\alpha) := \int_0^\infty e^{-t} t^{\alpha-1} dt,$ for all $\alpha > 0$
$D(A)$	domain of definition of the operator A
A^*	the adjoint of the operator A
I_E	the identity operator defined on some space E . In chapters 6 and 7 we omit the subscript E for simplify
$\mathcal{F}$ and $\mathcal{F}^{-1}$	Fourier transform and its inverse respectively

0.3 For Stochastic analysis

The symbol	The meaning
$(\Omega, \mathcal{F}, \mathbb{P})$	Probability space
$\mathcal{N}$	The normal law
$\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$	Normal filtration
$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$	Filtered probability space
$(\beta_t)_{t\in[0,T]}$	Brownian motion
$W := (W_t)_{t\in[0,T]}$	Wiener process
$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W)$	Stochastic basis
$\mathbb{E}(X) := \int_\Omega X(w)\ d\,\mathbb{P}(w)$	Expectation of the random variable X
$L^p(\Omega, E),$ for a Banach space E	Space of all $p-$th integrable $E-$valued random variables on Ω
$\mathcal{M}_T^2(E)$	Space of all continuous square integrable $E-$valued martingales
$\mathbb{P}-$a.s.	Property holds exept for $\mathbb{P}-$null sets

0.4 For functional spaces

The symbol	The meaning
$\mathcal{B}(E)$	Borel $\sigma-$ algebra generated by all open sets of the topological space E
$(E, \|.\|_E)$	Banach space with its norm $\|.\|_E$
$E^{'}$	The dual space of E
$(.,.)_{E'\times E}$	the pairing of E and $E^{'}$
$(H, \langle .,. \rangle_H)$	Hilbert space with its inner product $\langle .,. \rangle_H$
L^p	Lebesgue space, for the special case $p = 2$ we will denote it in chapter 6 by H
C and C^m, for $m \in \mathbb{N}_0$	space of continuous functions and space of all functions of class m respectively
C_0^m, for $m \in \mathbb{N}_0$	space of functions of class m with compact support
W_p^m, $m \in \mathbb{N}, 1 \leq p \leq \infty$	Sobolev space
W_p^s, $s \in \mathbb{R}_+^* - \mathbb{N}_0, 1 \leq p < \infty$	fractional Sobolev space
H_2^α and H_0^α, for $\alpha > 0$	fractional Sobolev space for $p = 2$ and the closure of C_0^∞ in H_2^α respectively
B_{pq}^s	Besov space
C^δ, for $\delta \in (0, 1)$	Hölder space
$\mathcal{C}^s$, for $s \in (0, 1)$	Zygmund space
$\mathcal{L}(E_1, E_2)$	Banach space of linear bounded operators from E_1 to E_2 with its norm $\|.\|_{\mathcal{L}(E_1,E_2)}$. For $E_1 = E_2$ we simply write $\mathcal{L}(E_1)$
$E_1 \hookrightarrow E_2$	embedding of E_1 in E_2
$(HS(H_1, H_2), \|.\|_{HS(H_1,H_2)})$	space of Hilbert-Schmidt operators from $H_1$to H_2
HS	space of Hilbert-Schmidt operators from $L^2(0, 1)$ to it self
$\mathcal{L}_\mathcal{N}(H_1, H_2)$	space of nuclear operators from H_1 to H_2
$\mathcal{S}$ and $\mathcal{S}'$	the Schwartz space and its dual respectively

- **Comment on the Constants.** We employ the later C to denote any constant that can be explicitly computed in terms of known quantities. The exact value denoted by C may therfore change from line to line in a given computation. The big advantage is that our calculations will be simpler looking. If we write for example $C = C_{d,p}$ this means that C depends on d and p.

Abstract

This dissertation has provided a rigorous analysis of various numerical schemes for a class of local and global Lipschitz nonlinear fractional stochastic partial differential equations, driven by the fractional Laplacian in the one dimensional space and perturbed by a Gaussian noise. The study contains the elaboration of time, space and space-time schemes and their different convergences. Specially, we express for every scheme the rate of convergence in terms of the fractional power of the Laplacian.

The first contribution (Chapter 5) concerned the study of the fractional stochastic Burgers-type equation in Hölder space $C^{\delta}(0,1)$, with diffusion dissipation index $\alpha \in (\frac{7}{4}, 2]$. We have proved the existence and the uniqueness of a space-time Hölder mild solution. Moreover, we have fulfilled the pathwise convergence of the spacial and the full approximations, where the spectral Galerkin method has been used for the spacial approximation, whereas the exponential Euler scheme has been used for the temporal approximation.

In the second contribution (Chapter 6), we have considered the fractional stochastic nonlinear heat equation in the Hilbert space $L^2(0,1)$, with diffusion dissipation index $\alpha \in (1,2]$. By using the spectral Galerkin method for the spacial approximation and the implicit Euler scheme for the temporal approximation, we have established the strong convergence in the space $L^p(\Omega, L^2(0,1))$ of the temporal, the spacial and the full ap-

proximations of the mild solution.

In the third contribution (Chapter 7), we have improved the diffusion dissipation index obtained in Chapter 5 to $\alpha \in (\frac{3}{2}, 2]$, by proving a weaker convergence (i.e. convergence in probability) of the temporal approximation of the fractional stochastic Burgers equation in the Hilbert space $L^2(0,1)$.

Key words and phraces: Fractional stochastic Burgers-type equation, fractional stochastic nonlinear heat equation, fractional Laplacian, Gaussian noise, multiplicative space-time white noise, additive space-time white noise, Hölder spaces, fractional Sobolev spaces, Hilbert-Schmidt operator, mild solutions, spectral Galerkin approximation, implicit Euler scheme, exponential Euler scheme, full approximation, order of convergence.

Introduction

As early as 1695, when Leibniz and Newton had just established the classical calculus (i.e. calculus of derivatives and integrals of integer order), Leibniz and L'Hôpital had correspondence where they discussed the meaning of the derivative of order one half. Since then, many famous mathematicians like Euler (1738), Laplace (1820), Fourier (1822) and Lagrange (1849) have worked on this and related questions creating the field which is known today as **fractional calculus**. Such calculus is a name for the theory of derivatives and integrals of arbitrary order (fraction, rational, irrational, complex,...). It generalizes the notions of integer-order differentiation and n-fold integration, where the term **fractional** can be misleading and the fact that L'Hôpital specifically asked for the order $n = \frac{1}{2}$ (i.e. a fraction), actually gave rise to the name of this field of mathematics.

As well known, in the classical analysis the integer-order derivative and integral are uniquely determined, which is not the same case for the fractional derivative and fractional integral. Indeed, there are many different definitions that do not coincide in general, it is due to the fact that the different authors try to keep different properties of the classical integer-order derivative and integral. However, the fact that there are obviously more than one way to define such notions in the fractional calculus, is one of the challenges of this mathematical field. Generally, there are two main approaches to the fractional calculus. Namely, the continuous and the discrete. The former is based on the Riemann-

Liouville fractional integral, which is based also on the Cauchy integral formula, and the latter is based on the Grünwald-Letnikov fractional derivative, as a generalization of the fact that ordinary derivatives are limits of difference quotients. The Grünwald-Letnikov derivative is defined as a limit of a fractional-order backward differences. For more details on the fractional calculus we refer to [70, 80, 93].

In recent years, considerable interest in fractional differential equations has been simulated by the applications that they found in different areas of applied sciences. For example, modelling of biological tissues [79] and modelling of neuron systems [81] in medicine, and application of fractional calculus in continuous-time finance [94, 95] in financial markets. Further, fractional Burgers equation has been introduced as a relevant model for anomalous diffusions, like diffusion in complex phenomena, relaxations in viscoelastic mediums, propagation of acoustic waves in gaz-filled tube, see e.g. [75, 97, 98]. The analytic study for such equation has been investigated e.g. in [5, 19, 71].

The numerical study of the fractional differential equations is still a modest field, due to the fact that the fractional operator is nonlocal, see for a short list [21, 22, 42, 100]. For example, in [42] a discretization of the fractional operator, which is represented via an integro-differential operator, is based on the disctretization of the integral, which leads to a slow convergence of the elaborated scheme.
In contrast to the second order differential operators, to apply the finite-difference method we need more than three points, and all the points of the grid have to be used in every step. Further, it is not easy to find a concrete form to the discretized fractional operator, and the discretization has been used for the Liouville and Riemann fractional operators, see e.g [22, 100]. However, Debbi L. in [38] introduced a new idea to discretize the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$, by discretizing first the operator $(-\Delta)$ then taking the fractional power of the discrete operator.

The main task of numerical study for stochastic partial differential equations (briefly SPDEs) is to elaborate schemes, generally based on the deterministic numerical methods,

such as finite difference and Galerkin methods, in order to provide approximations with respect to time, to space or to both simultaneously, and proving convergence of such schemes with or without rates. The classical results state that space or time discretization schemes convergence strongly in the case of coefficients are globaly Lipschitz and/or have linear growth property, see e.g. [38, 73, 87]. Wereas, the weak convergence has been proved when the coefficients are only locally Lipschitz or have nonlinear growth see e.g. [56, 58] for time discretization and see e.g. [54, 55] for space and full discretizations.
One of the first results about the pathwise convergence for the stochastic Burgers equation known to us is [2]. In this work, the authors used the finite difference method and proved that the discretized trajectories converge almost surely to the solution in C_tL^2-topology with rate $\gamma < 1/2$.

Recently, the field of numerical approximations for the stochastic differential equations (briefly SDEs) attracts another attention, due to the strange phenomena particularly emerging in this case. For example in [60], Hairer et al. constructed a SDE for which, nevertheless, the rate 1 is well known for the Euler approximation for the deterministic version, the Euler approximation for the stochastic version converges to the solution, in the strong and in the numerically weak sense without any arbitrarily small polynomial rate. In [62] the authors showed that different finite difference schemes to stochastic Burgers equation driven by space-time white noise converge to different limiting processes. Divergence of schemes can also occur if the stochastic noise is rougher than the space-time white noise. One of the explanations for such strange behaviours is the loss of regularity of the solutions of the stochastic differential equations (briefly SDEs), which arises due to the roughness of the random noise. This loss of regularity even yields for some cases the illposedness of the equations, see [59, 60, 61, 62]. Furthermore, a new tendency for the numerical study of the SDEs has been developed, based on the idea to elaborate numerical approximations for an abstract SDE with coefficients satisfying some conditions, and then to show that this study covers some specific equations. For example, in [12] the authors used Galerkin approximation to prove the wellposedness of an abstract evolution stochastic differential equation, and calculated the rate of convergence to the solution in abstract spaces. The authors applied this abstract theory on the multidimentional heat equation, reaction diffusion equation and on classical Burgers equation deriven by additive space-time white noise. In particular, they proved that the rate of the pathwise convergence of the Galerkin approximation is of order $\gamma < 1/2$ in the C_tC_x-topology.

The numerical study of the nonlinear fractional SPDEs has been affected by many factors. For example, in the case of Burgers equation driven by the fractional Laplacian and perturbed by a space-time white noise, we are going to face extra problems than the classical deterministic case. Namely, on the one hand, the fractional dissipation is not even strong enough to control the steepening of the nonlinear term, and on the other hand, there is a difficulty to approximate the fractional operator, besides the negative effects of the stochastic noise not only on the convergence of the schemes but on the rates of convergences as well.

Our first contribution in this thesis makes part of this direction of study. In fact, we consider a class of the fractional version of the stochastic Burgers-type equation studied in [12, 59, 61, 62], which is given by the following evolution form

$$\begin{cases} du(t) = (-A_\alpha u(t) + F(u(t)))\,dt + dW(t), & t \in (0,T], \\ u(0) = u_0, \end{cases} \tag{1}$$

where $A_\alpha := (-\Delta)^{\frac{\alpha}{2}}$, for $\alpha \in (1,2]$ is the fractional power of the minus Laplacian on $D := (0,1)$ with homogeneous Dirichlet boundary conditions, F is a nonlinear operator given by $F(u(t,x)) := \partial_x f(u(t,x))$, $x \in (0,1)$ with f being locally Lipschitz. Specially, the case when f is a polynomial; $f(x) := a_0 + a_1 x + a_2 x^2 + ...a_n x^n$, with $a_0, a_1, ..., a_n \neq 0 \in \mathbb{R}$, $n \in \mathbb{N}$. It is easily seen that for $n = 2$ (i.e. $a_2 \neq 0$) and $a_1 = 0$, we recuperate the fractional stochastic Burgers equation. This fact justifies the name of Burgers-type, W is an $L^2(0,1)$-valued $I_{L^2(0,1)}$-cylindrical Wiener process, u_0 is an $L^2(0,1)$-valued $\mathcal{F}_0$-measurable random variable and $(u(t))_{t\in[0,T]}$ is an $L^2(0,1)$-valued stochastic process.

By applying the method used in [12] we generalize their results. Precisely, we prove the existence and the uniquness of the mild solution of Prb.(1) in the $C_t^\gamma C_x^\delta$-topology , where $\gamma < \frac{\alpha-1-2\delta}{2\alpha}$ and $\delta \in (1-\frac{\alpha}{2}, \frac{2\alpha-3}{2})$, for $\alpha \in (\frac{7}{4}, 2)$. In addition, we prove the pathwise convergence of the obtained spacial and full approximations with the orders of convergence

$(\frac{\alpha-1-2\delta}{2})-\kappa$ in space and $(\frac{\alpha-1-2\delta}{2\alpha})-\kappa$ in time, for any $\kappa>0$.
To the best of our knowledge, this contribution is the first one proving these results in the Hölder spaces $C^\delta(0,1)$ not only for the fractional stochastic Burgers-type equations, but for the fractional and classical stochastic Burgers equations as well. Moreover, the exponential Euler scheme has been applied here for the first time for the fractional stochastic equations. Recall that this method has been introduced in [69] and used in the approximation of the stochastic heat and reaction diffusion equations, see [68, 69].

In order to improve either the orders of convergence and the constraint on the diffusion dissipation index α, we relax the condition imposed on the nonlinear term and we study such equation in the Hilbert space $L^2(0,1)$ instead of the Hölder space $C^\delta(0,1)$. Precisely, in our second contribution, we deal with the fractional stochastic nonlinear heat equation with multiplicative noise. This means, we consider Eq.(1) with the nonlinear operator F being globally Lipschitz with nonlinear growth condition instead of locally Lipschitz. We prove the strong convergence in the space $L^p(\Omega, L^2(0,1))$ of the temporal, the spacial and the full approximations of the mild solution with orders of convergence $\frac{\alpha}{2}-\kappa$ in space and $\frac{\alpha-1}{2\alpha}-\kappa$ in time, for any $\kappa>0$. In this contribution, we fill the gap $\alpha\in(1,\frac{7}{4}]$, but for the heat equation only. For Burgers equation, we prove the convergence in probability, which is weaker than the L^p and the pathwise convergences, of the temporal approximation with the same order $\frac{\alpha-1}{2\alpha}-\kappa$ provided that $\alpha\in(\frac{3}{2},2]$.

We close this introduction by a summary of the thesis.
In **chapter 1**, we collect some preliminaries related to the functional analysis. First, we gather some notions and results of the linear spaces that are used frequently in this thesis, namely fractional Sobolev spaces and Hölder spaces. Second, we introduce some definitions and basic results of the Linear operators such as the Hilbert-Schmidt operator, besides we give a short review on the semigroup theory. Since the classical calculus can be regarded as a special case of the fractional calculus and some results in it should contain the classical case in a certain way, we present the standard Laplace operator by recalling its definition and some of its properties before introducing the fractional Laplacian.

In **chapter 2**, some well known definitions and results about the stochastic processes and stochastic integrals in Hilbert spaces have been collected, namely the Wiener processes and the stochastic integrals with respect to them, because they are used in the next chapters.

In **chapter 3**, we introduce some results about the classical and the fractional SPDEs. As a starting point, we motivate our study by two important models from physics; heat equation and Burgers equation, we move from the deterministic case to the stochastic one, in order to introduce the stochastic nonlinear heat equation and the stochastic Burgers equation. Next, we deal with an abstract parabolic SPDEs perturbed by a multiplicative noise by giving different concepts of their solutions, after that we introduce the wellposedness results in the two cases; globally Lipschitz and locally Lipschitz nonlinearities. We close this chapter by one of the recent developments; the SPDEs driven by fractional differential operators. Precisely, we deal with the SPDEs driven by the fractional Laplacian, where we introduce the fractional analogs of stochastic nonlinear heat and stochastic Burgers equations and we introduce some results about their wellposedness.

In **chapter 4**, some numerical approximations and results concerning the abstract parabolic SPDEs introduced in Chapter 3 have been collected. In the first part, we deal with the full approximation of the deterministic version of such problem by using the finite difference method for temporal discretization and the spectral Galerkin method for spacial discretization. In the second part, we use the same methods to approximate spacially and temporally the stochastic term of the problem, after that we introduce different notions of convergence in the probability context. We close this chapter by showing an important results concerned the spacial approximation of the fractional sthochastic heat equation.

In **chapter 5** (our first main contribution), we consider the fractional stochastic Burgers-type equation with additive noise in the Hölder space $C^{\delta}(0,1)$. The scheme

obtained by using the spectral Galerkin method, on the one hand, is used to prove the analytic result concerns the wellposedness, the pathwise existence and uniqueness of the mild solution, and on the other hand, to give the space approximation of the mild solution. Moreover, we prove the pathwise convergence of such spacial approximation with a precise order of convergence. The second numerical result concerns the space-time approximation, which converges almost surely with precise orders of convergence. Such approximation is achieved by combining spectral Galerkin and exponential-Euler methods.
The results of this chapter make the subject of the following paper:

- Arab Z. and Debbi L., *Fractional stochastic Burgers-type equation in Hölder space -Wellposedness and Approximations-*. Math Meth Appl Sci. Vol. 44, Issue .1, pp.705-736. (2021).

In **chapter 6** (our second main contribution), we deal with the full approximation for the fractional stochastic nonlinear heat equation in the Hilbert space $L^2(0,1)$, perturbed by a multiplicative noise. Precisely, we establish the temporal, the spacial and the full schemes by using the semi-implicit scheme in time and the spectral Galerkin method in space. After that, we prove the strong convergence (i.e. in the space $L^p(\Omega, L^2(0,1))$) of such schemes with orders of convergence.

In **chapter 7** (our third main contribution), we consider the fractional stochastic Burgers equation in the Hilbert space $L^2(0,1)$. We prove the convergence in probability of the temporal approximation obtained via implicit Euler scheme, with the same order of convergence obtained in Chapter 6.

The results of Chapter 6 and Chapter 7 make the subject of the following manuscript:

- Arab Z. and Debbi L., *Numerical approximations for some fractional stochastic partial differential equations.* To be submitted.

We close this thesis by some appendices. In Appendix A.1, we deal with several versions of Grönwall Lemma. In Appendix A.2, we collect some useful results. Some elementary inequalities have been gathered in Appendix A.3. In Appendix A.4, we recall some basic definitions on functional spaces.

Chapter 1

On some functional aspects

The purpose of this chapter is to collect some concepts and results concerning various aspects of functional analysis. In Section 1.1 we gather some notions and results of linear spaces, especially some of the more important linear spaces such as Banach spaces, Hilbert spaces, and certain functional spaces that are used frequently in this thesis. Namely, fractional Sobolev spaces and Hölder space. In Section 1.2 we introduce some definitions and basic results of Linear operators such as Hilbert-Schmidt operator. Section 1.2.2 focuses on some generalities of the semigroup theory. The standard Laplace operator is postponed to Section 1.2.3 for introducing it specifically in order to present the fractional Laplace operator in Section 1.2.4 which is our case of interest in this thesis. The main references of the presented chapter are [83, 92, 105, 106, 107].

1.1 Functional spaces and some of their properties.

We define functional spaces on $\mathbb{R}$ and on bounded domain $D \subset \mathbb{R}$. Let $0 < p, q \leq \infty,\ \ s \in \mathbb{R}$, for simplicity reasons, we sometimes restrict these parameters for the required cases.

1.1.1 Functional spaces on $\mathbb{R}$ and on bounded domain

For the following definitions see [92, Section 2.1.2, ps. 11-12] and [105, Section 2.2.2, ps. 35-36], we omit $\mathbb{R}$ from the notations for the sack of simplicity.

Definition 1.1 ***(Lebesgue space)***. *The L^p-**Lebesgue space** is defined by:*

$$L^p := \{f \ \ measurable \ \ s.t. \ \ |f|_{L^p}^p := \int_{\mathbb{R}} |f(x)|^p dx < \infty\},$$

for $0 < p < \infty$ and by

$$L^\infty := \{f \ \ measurable \ \ s.t. \ \ |f|_{L^\infty} := ess \ sup_{\mathbb{R}} |f(x)| < \infty\},$$

where " ess sup " means the essential supremum, i.e.

$$ess \ supf := \inf_{c \in \mathbb{R}} \{c, \ \mu(f^{-1}(c, +\infty)) = 0\},$$

with μ the Lebesgue measure.

Definition 1.2 ***(Sobolev spaces)***. *For $1 \leq p \leq \infty$ and $m \in \mathbb{N}$, we define the **Sobolev space**:*

$$W_p^m := \{f \in L^p, \ \ s.t. \ \ |f|_{W_p^m}^p := \sum_{k=0}^{m} |D^k f|_{L^p}^p < \infty\},$$

where $D^k f$ represents the derivative of f of order k in the distributional sense.

Definition 1.3 ***(Fractional Sobolev space)***. *Let $1 \leq p < \infty$ and $0 < s \neq integer$, with $[s], \{s\}$ are respectively the integer and the fractional parts of s. The **fractional Sobolev space** (also called Aronszajn, Gagliardo or Slobodockij space) is given by:*

$$W_p^s := \{f \in W_p^{[s]}, \ \ s.t. \ \ |f|_{W_p^s}^p := |f|_{W_p^{[s]}}^p + \sum_{k=0}^{[s]} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|D^k f(x) - D^k f(y)|^p}{|x-y|^{1+\{s\}p}} dx \, dy < \infty\}.$$

Remark 1.4 *Let us mention here that for $p = 2$, the spaces W_2^m and W_2^s are Hilbert spaces and we will denote them by H_2^m and H_2^s respectively.*

In order to introduce Besov spaces we need first the following definition.

Definition 1.5 ***(Systems of functions)***, *[92, Definition 1, p.7]. Let* $\phi = (\phi_j)_{j=0}^{\infty} \subset \mathcal{S}$ *be a system s.t.*

1. *for every* $x \in \mathbb{R}$, $\sum_{j=0}^{\infty} \phi_j(x) = 1$,

2. *there exist constants* $c_1, c_2, c_3 > 0$ *with*

$$supp\phi_0 \subset \{x \in \mathbb{R} : |x| \leq c_1\},$$

   and

$$supp\phi_j \subset \{x \in \mathbb{R} : c_2\, 2^{j-1} \leq |x| \leq c_3\, 2^{j+1}\},\ for\, j = 1, 2, ...$$

3. *for every nonnegative integer* k *there exists* $c_k > 0$ *s.t.,*

$$\sup_{x\in\mathbb{R}} \sup_{j=0,1,...} 2^{jk}|D^k\phi_j(x)| \leq c_k.$$

Definition 1.6 ***(Besov spaces)***, *[92, Convention 1., p.11]. Let* $s \in \mathbb{R}$ *and* $0 < p, q \leq \infty$, *we define the* ***Besov space***:

$$B^s_{pq} := \{f \in \mathcal{S}',\ \ s.t.\ |f|^q_{B^s_{pq}} := |2^{sj}\mathcal{F}^{-1}[\phi_j \mathcal{F} f](.)|_{l^q(L^p)} < \infty\},$$

where $(\phi_j)_{j=0}^{\infty}$ *is given by Definition 1.5, and* l^q *is the space of* q*-summable real sequences.*

Definition 1.7 ***(Besov spaces on Bounded domain*** D***)***, *[92, Section 4.1.2, Definition 2., p.74]. For* $s \in \mathbb{R}$ *and* $0 < p, q \leq \infty$,

$$B^s_{pq}(D) := \{f \in \mathcal{D}'(D),\quad \exists\, g \in B^s_{pq}\ with\ g|_D = f,\ \ s.t.\ |f|_{B^s_{pq}(D)} := \inf |g|_{B^s_{pq}} < \infty\},$$

where $\mathcal{D}'(D)$ *is the collection of all complex-valued distributions on the domain* D.

Definition 1.8 (Spaces of continuous functions). *The* ***space of continuous function*** *is defined by:*

$$C := \{f\ bounded\ and\ continuous,\ s.t.\ |f|_C := \sup_{x\in\mathbb{R}} |f(x)| < \infty\}.$$

Definition 1.9 (Spaces of differentiable functions of order m). *Let $m \in \mathbb{N}$. The* ***spaces of differentiable functions of order*** *m is defined by:*

$$C^m := \{f \in C \ s.t. \ d^\alpha f \in C, \ \ for \ all \ \ \alpha \leq m\},$$

endowed with the norm

$$|f|_{C^m} := \sum_{\alpha \leq m} |d^\alpha f|_C,$$

where d^α means the classical derivative of order α, with the convention $C^0 = C$. The notation C_0^m is reserved for the space of all functions in C^m with compact support.

Definition 1.10 (Hölder spaces). *For $\delta \in (0,1)$, we define the* ***Hölder space*** *by*

$$C^\delta := \{f \in C, s.t. \ \ |f|_{C^\delta} := |f|_C + \sup_{x,y \in \mathbb{R}, x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\delta} < \infty\}.$$

Definition 1.11 (Zygmund spaces). *Let $s \in (0,1)$, the* ***Zygmund space*** *is defined by:*

$$\mathcal{C}^s := \{f \in C \ s.t. \ |f|_{\mathcal{C}^\delta} := |f|_C + \sup_{h \in \mathbb{R}, h \neq 0} |h|^{-s} |\Delta_h^2 f|_C < \infty\},$$

with $\Delta_h^2 f(x) := \sum_{l=0}^2 (-1)^l \ C_l^2 \ f(x + (2-l)h)$, with $C_l^2 := \frac{2!}{l!(2-l)!}$.

Definition 1.12 ***(Zygmund spaces on Bounded domain*** *D), [106, Section 5.3.2] & [106, Section 1.10.3]. Let D be a bounded C^∞-domain, then for $s > 0$, we define $\mathcal{C}^s(D) := B^s_{\infty\infty}(D)$. In particular, for $s \in (0,1)$*

$$\mathcal{C}^s(D) := \{f \in C(D) \ s.t. \ |f|_{\mathcal{C}^\delta(D)} := |f|_{C(D)} + \sup_{h \in \mathbb{R}, h \neq 0} |h|^{-s} |\Delta_h^2 f(., D)|_{C(D)} < \infty\},$$

with

$$\Delta_h^2 f(x, D) := \begin{cases} \Delta_h^2 f(x), & if \quad x + jh \in D \ for \ j = 0, 1, 2, \\ 0, & otherwise. \end{cases}$$

1.1.2 Some properties of the functional spaces

In this subsection, we present the relationship between the differential spaces and we study under which conditions a space can be a multiplicative algebra, and how to check spaces that are relevant to gather, in order to deal with the factorisation of the pathwise product.

Lemma 1.13 *We have:*

- *The identities, see e.g. [92, p.14] and [105].*
    - *For* $0 < s \neq$ *integer,* $\quad C^s = B^s_{\infty\,\infty} = \mathcal{C}^s$.
    - *For* $0 < s \neq$ *integer and* $1 \leq p < \infty$, $\quad B^s_{pp} = W^s_p$.
    - *For* $1 \leq p \leq \infty$, $W^0_p = L^p$.
- *The continuous embeddings:*

$$B^s_{pq_0} \hookrightarrow B^s_{pq_1}, \tag{1.1}$$

for $s > 0$, $0 < p \leq \infty$ *and* $0 < q_0 < q_1 \leq \infty$, *see e.g. [92, Proposition 2.2.1, p.29].*

$$B^{s_0}_{p_0q_0} \hookrightarrow B^{s_1}_{p_1q_1}, \tag{1.2}$$

provided $s_1 < s_0$, $s_0 - \frac{1}{p_0} > s_1 - \frac{1}{p_1}$ *and* $p_0 \leq p_1$, *see e.g. [92, Remark 2, p.31].*

- *As a consequence of Embedding* (1.2), *we have for* $s_1 < s_0$

$$B^{s_0}_{\infty\infty} \hookrightarrow B^{s_1}_{22}, \tag{1.3}$$

$$\mathcal{C}^{s_0} \hookrightarrow H^{s_1}_2, \tag{1.4}$$

and for $s_0 > \frac{2-\alpha}{2}$ *and* $s_0 \neq$*integer,*

$$\mathcal{C}^{s_0} = C^{s_0} \hookrightarrow H^{1-\frac{\alpha}{2}}_2. \tag{1.5}$$

- *[105, Theorem 2.8.3, ps. 145-146], for* $0 < p, q \leq \infty$, $\ s > \frac{1}{p}$, B^s_{pq} *is a multiplication algebra. In particular, for* $s > 0$, $\mathcal{C}^s$ *is a multiplication algebra.*

Theorem 1.14 . *Let* $s \in \mathbb{R}$, $0 < p, q \leq \infty$ *and* $\delta > max(s, (\frac{1}{min(p,1)} - 1) - s)$. *Then for every* $f \in \mathcal{C}^\delta$ *and every* $g \in B^s_{p,q}$ *there exists* $C > 0$ *s.t.*

$$|f\ g|_{B^s_{p,q}} \leq C\ |f|_{\mathcal{C}^\delta}\ |g|_{B^s_{p,q}}. \tag{1.6}$$

Proof. See [105, Theorem 3.3.2]. ∎

Theorem 1.15 *The results of Theorem 1.14 are still valid for bounded domains, see, e.g. [105, Section 3.3.2] and [106, Section 5.4].*

Moreover, we have

Corollary 1.16 *Let* $0 < s \neq$ *integer and* $\delta > s$ *(in our study* $\delta \in (0,1)$*), then for* $f \in \mathcal{C}^{\delta}(D)$ *and every* $g \in H_2^s(D)$ *there exists a positive constant* C *such that*

$$|f\ g|_{H_2^s} \leq C\ |f|_{\mathcal{C}^{\delta}}\ |g|_{H_2^s}. \tag{1.7}$$

Proof. The result is obtained by application of Theorem 1.14 and [92, Proposition 2.1.2 P.14]. ■

Theorem 1.17 *. Let* $D \subseteq \mathbb{R}$ *be an extension domain for* W_p^r *with no external cups and let* $p \in [1,\infty)$*,* $r \in (0,1)$ *s.t.* $r > \frac{1}{p}$*. Then* $\exists\ C_{D,p,r} > 0$ *s.t.*

$$|f|_{C^{r-\frac{1}{p}}} \leq C_{D,p,r}|f|_{W_p^r},\ \ \textit{for any } f \in L^p(D). \tag{1.8}$$

Proof. See [40, Theorem 8.2]. ■

1.1.3 UMD Banach Space & Banach Space of Type p

In this subsection we recall some notions concerning the UMD Banach Space and Banach Space of type p, because they have properties like Hilbert spaces and Banach spaces.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})$ be a filtered probability space, $(\epsilon_n)_{n\geq 1}$ be a sequence of "i.i.d" symmetric $\{+1,-1\}$-valued random variables and let E be a given Banach space.

Definition 1.18 *(**Martingale difference sequence**, [4, 18, 85]). A sequence* $(d_n)_{n\geq 1}$ *of Bochner-integrable* $E-$*valued random variables is a **martingale difference sequence** relative to the filtration* $(\mathcal{F}_n)_{n\geq 0}$*, if for all* $n \geq 1$*, the function* d_n *is* $\mathcal{F}_n-$*measurable and* $\mathbb{E}(d_n/\mathcal{F}_{n-1}) = 0$.

Definition 1.19 *(**UMD Banach space**, [18, 85, 109, 110]). The Banach space* E *is called **UMD Banach space** if for some (or equivalently for every)* $p \in (1,\infty)$*, there exists* $C_{p,E} \geq 1$*, s.t.* $\forall n \geq 1$*, for all martingale differences* $(d_j)_{j=1}^n$ *and for all* $(\epsilon_j)_{j=1}^n$ *defined above, we have*

$$\Big(\mathbb{E}|\sum_{j=1}^{n}\epsilon_j d_j|_E^p\Big)^{\frac{1}{p}} \leq C_{p,E}\Big(\mathbb{E}|\sum_{j=1}^{n} d_j|_E^p\Big)^{\frac{1}{p}}. \tag{1.9}$$

Definition 1.20 ***(Banach space of type** p , [4, 85, 109, 110]).* *A Banach space E is called of type p, with $1 \leq p \leq 2$ if there exists a constant $C > 0$, s.t. for each finite sequence $(x_i)_{i=1}^n \subset E$ and for all $(\epsilon_j)_{j=1}^n$ defined above,*

$$\Big(\mathbb{E}|\sum_{j=1}^{n} \epsilon_j x_j|_E^2\Big)^{\frac{1}{2}} \leq C\Big(\sum_{j=1}^{n} |x_j|_E^p\Big)^{\frac{1}{p}}. \tag{1.10}$$

Remark 1.21 *Well known examples of UMD Banach spaces are; all Hilbert spaces, The Lebesgue spaces $L^p(S, \mathcal{B}, \mu)$, $1 < p < \infty$, with μ being a $\sigma-$finite measure and $L^p(S; E)$, $1 < p < \infty$, with E being an UMD Banach space. The Hilbert spaces and the L^p-Lebesgue spaces, with $2 \leq p < \infty$ are also Banach spaces of type 2, but L^p-Lebesgue spaces, with $1 \leq p < 2$ are not. More precisely, an UMD Banach space can not be a Banach space of type 2 and vise virsa, see [4, 85, 109, 110].*

1.2 Some useful operators

In this section, we deal with some useful operators like Hilbert-Schmidt operator, which plays an important role in the construction of the stochastic integrals and the fractional Laplacian which is our case of interest in this thesis. For this end, we fix two real-Banach spaces $(X, |.|_X)$, $(Y, |.|_Y)$ and two separable real-Hilbert spaces $(U, \langle ., . \rangle_U, |.|_U)$, $(H, \langle ., . \rangle_H, |.|_H)$.

1.2.1 Basic notions and some useful results

Definition 1.22 ***(Bounded operator).*** *Let $A : D(A) \subseteq X \to Y$ be a linear operator. Then, we say that A is* ***bounded*** *if there exists $C > 0$ s.t.*

$$|A(x)|_Y \leq C|x|_X, \ \forall x \in D(A).$$

Definition 1.23 *We denote by $\mathcal{L}(X, Y)$ for the Banach space of all linear bounded operators defined from X to Y endowed by the norm*

$$\|A\|_{\mathcal{L}(X,Y)} := \sup\{|x|_X^{-1} \ |A(x)|_Y, \ x \in X, \ x \neq 0\}.$$

If $X = Y$, we write $\mathcal{L}(X, X) = \mathcal{L}(X)$. Moreover, if $Y = \mathbb{R}$ we call A a ***linear bounded functional*** *on X. We denote the collection of all such functionals by X', which is the* *dual space of X. The symbols $\|.\|_{X'}$ and $(.,.)_{X',X}$ denote the norm in X' and the duality (pairing) of X' and X respectively.*

Definition 1.24 ***(Compact operator).*** *An operator $A \in \mathcal{L}(X,Y)$ is said to be* ***compact*** *if for all bounded subset $B \subset X$, the closure of $A(B)$ is compact.*

Definition 1.25 ***(Closed operator).*** *We say that, a linear operator $A : D(A) \subseteq X \to Y$ is* ***closed*** *if its graph is a closed subspace of $X \times Y$.*

Theorem 1.26 *Every linear bounded operator A on X satisfies $D(A) = X$. Moreover, A is closed.*

Definition 1.27 ***(Pseudo-inverse).*** *Let $A \in \mathcal{L}(U,H)$. Then, we define the* ***pseudo-inverse*** *of A as*

$$A^{-1} := (A|_{ker(A)^{\perp}})^{-1} : A(ker(A)^{\perp}) = A(U) \to ker(A)^{\perp},$$

where $ker(A)$ denotes the kernel of A and $ker(A)^{\perp}$ its orthogonal complement.

Definition 1.28 ***(Nonnegative operator).*** *We say that, the linear operator $A : D(A) \subseteq U \to U$ is* ***nonnegative*** *if*

$$\langle Au, u\rangle_U \in \ [0, +\infty), \quad \forall\, u \ \in D(A).$$

Definition 1.29 ***(Symmetric operator).*** *A densely defined linear operator $A : D(A) \subseteq U \to U$ is said to be* ***symmetric*** *if*

$$\langle Au, v\rangle_U = \langle u, Av\rangle_U, \quad \forall\, u, v \ \in D(A),$$

where $\langle ., .\rangle_U$ denotes the inner product in U.

Lemma 1.30 *[45, Lemma 13.3, p.488]. Let $A : D(A) \subseteq U \to H$ s.t. $D(A)$ is dense in U. Then, A admits a closed operator A^* called the* ***adjoint****, which is defined on $D(A^*)$ into U where*

$$D(A^*) := \{v \in H,\ s.t.\ D(A) \ni u \mapsto \langle Au, v\rangle_H \ is\ continuous\},$$

such that for all $u \in D(A)$ and all $v \in D(A^*)$ it holds

$$\langle u, A^*v\rangle_U = \langle Au, v\rangle_H.$$

Definition 1.31 ***(Self-adjoint operator).*** *Let $A : D(A) \subseteq U \to U$ be a densely defined linear operator. Then, we say that A is* ***self-adjoint*** *if $D(A) = D(A^*)$ and $A = A^*$.*

Corollary 1.32 *[114, p.199]. Let $(A, D(A))$ be a symmetric operator on the Hilbert space U. If $D(A) = U$, then A is self-adjoint.*

Next, we introduce a useful result concerning a linear, self-adjoint and nonnegative operator and its spectrum. It is necessary to define first such spectrum.

Definition 1.33 ***(Resolvent set of an operator).*** *Let the operator $A \in \mathcal{L}(U)$. The* ***resolvent set*** *of A denoted by $\rho(A)$ is defined by*

$$\rho(A) := \{\lambda \in \mathbb{C} \text{ s.t. } (A - \lambda I_U) \text{ is invertible}\}.$$

Definition 1.34 ***(Spectrum of an operator).*** *Let the operator $A \in \mathcal{L}(U)$. The* ***spectrum*** *of A, denoted by $\sigma(A)$ is the complement of the resolvent set in $\mathbb{C}$.*

The spectrum of A is subdivided as follows

Definition 1.35 *Let the operator $A \in \mathcal{L}(U)$.*

1. *The* ***discrete spectrum*** *of A consists of all $\lambda \in \sigma(A)$ s.t. $(A - \lambda I_U)$ is not one-to-one. In this case λ is called an* ***eigenvalue*** *of A.*

2. *The* ***continuous spectrum*** *of A consists of all $\lambda \in \sigma(A)$ s.t. $(A - \lambda I_U)$ is one-to-one but not onto and range$(A - \lambda I_U)$ is dense in U.*

3. *The* ***residual spectrum*** *of A consists of all $\lambda \in \sigma(A)$ s.t. $(A - \lambda I_U)$ is one-to-one but not onto and range$(A - \lambda I_U)$ is not dense in U.*

Lemma 1.36 *[104, Chapter 7; Est.(7.5) and Est.(7.6), p.112]. Let A be a linear (not necessarily bounded), self-adjoint and nonnegative operator defined on $D(U) \subseteq U$, which has eigenvalues $\{\mu_j\}_{j=1}^N$, for $1 < N \leq \infty$ corresponding to a basis of orthogonormal* *eigenfunctions* $\{\varphi_j\}_{j=1}^N$*. Then, for an arbitrary function* $\mathcal{G}$ *defined on the spectrum* $\sigma(A) = \{\mu_j\}_{j=1}^N$ *of* A*, it holds*

$$\mathcal{G}(A)v = \sum_{j=1}^{N} \mathcal{G}(\mu_j)\langle v, \varphi_j\rangle_U \varphi_j, \quad \forall v \in U. \tag{1.11}$$

and

$$\|\mathcal{G}(A)\|_{\mathcal{L}(U)} = \sup_{1\le j\le N} |\mathcal{G}(\mu_j)|. \tag{1.12}$$

Definition 1.37 ***(Nuclear operator)****. Let* $A \in \mathcal{L}(X, Y)$*, if there exist two sequences* $(x_n)_{n\in\mathbb{N}} \subset X'$ *and* $(y_n)_{n\in\mathbb{N}} \subset Y$ *s.t.*

$$\sum_{n\in\mathbb{N}_0} \|x_n\|_{X'} |y_n|_Y < +\infty.$$

Then, A *defined by*

$$A(x) := \sum_{n\in\mathbb{N}_0} y_n \times x_n(x), \ \forall \ x \in X$$

is called a ***nuclear*** *operator, where "*$\times$*" denotes the scalar multiplication operation.*

We denote by $\mathcal{L}_{\mathcal{N}}(X, Y)$ the space of all nuclear operators from X to Y, which is a Banach space (see [31, Appendix C., p. 436]) endowed with the norm

$$\|A\|_{\mathcal{L}_{\mathcal{N}}(U,H)} := \inf_{(x_n)_{n\in\mathbb{N}}\subset X',\,(y_n)_{n\in\mathbb{N}}\subset Y} \{\sum_{n\in\mathbb{N}} \|x_n\|_{X'} |y_n|_Y, \quad \forall \ x \in X\}.$$

Definition 1.38 ***(Finite trace operator)****. Let* $(e_n)_{n\in\mathbb{N}_0}$ *be an ONB of* U*, and let* $A \in \mathcal{L}(U)$ *if*

$$trA := \sum_{n\in\mathbb{N}_0} \langle Ae_n, e_n\rangle_U < +\infty.$$

Then, we call A *a* ***finite trace*** *operator, where* trA *is called* ***the trace*** *of* A*, which is independent of the choice of the ONB* $(e_n)_{n\in\mathbb{N}_0}$*.*

Proposition 1.39 *[31, Proposition C.1, p.436]. Let* $A \in \mathcal{L}_{\mathcal{N}}(U)$*. Then,* A *is finite trace operator satisfies*

$$|trA| \le \|A\|_{\mathcal{L}_{\mathcal{N}}(U)},$$

Proposition 1.40 *[101, Proposition.6.6, p.497]. Let* $A \in \mathcal{L}(U)$ *be a compact and self-adjoint operator. Then, the spectral theorem for compact and self-adjoint operators ensures the existence of an orthonormal basis* $(e_n)_{n\in\mathbb{N}}$ *of* U *and a sequence of nonnegative real numbers* $(\lambda_n)_{n\in\mathbb{N}}$ *s.t.* $Ae_n = \lambda_n e_n$, $\forall n \in \mathbb{N}$, *with only* 0 *as an accumulation point.*

Definition 1.41 ***(Hilbert-Schmidt operator).*** *Let $(e_n)_{n\in\mathbb{N}_0}$, be an ONB of U. An operator $A \in \mathcal{L}(U,H)$ is said to be* ***Hilbert-Schmidt*** *if*

$$\sum_{n\in\mathbb{N}_0} |Ae_n|_H^2 < +\infty.$$

We denote by $HS(U,H)$ the set of all Hilbert-Schmidt operators from U to H. In the special case $U = H$ we shortly write $HS(U,U) = HS(U)$. The definition of a Hilbert-Schmidt operator and the induced Hilbert-Schmidt norm in $HS(U,H)$

$$\|A\|_{HS(U,H)} := \Big(\sum_{n\in\mathbb{N}_0} |Ae_n|_H^2\Big)^{\frac{1}{2}}, \tag{1.13}$$

are independent of the choice of the basis $(e_n)_{n\in\mathbb{N}_0}$ (see [86, Remark. B.0.6(i)]).

Proposition 1.42 *[86, Prop. B.0.7]. The space $HS(U,H)$ endowed with the inner product*

$$\langle \Psi, \Phi\rangle_{HS(U,H)} := \sum_{n\in\mathbb{N}_0} \langle \Psi e_n, \Phi e_n\rangle_H, \ \forall\, \Psi, \Phi \in HS(U,H)$$

is a separable Hilbert space.

In the next proposition we collect some important properties of such type of operators, for more details see for instance [86, Remark. B.0.6].

Proposition 1.43 *[86]. Let $\Psi \in HS(U,H)$ be a Hilbert-Schmidt operator. Then, the following statements hold*

1. $\|\Psi\|_{\mathcal{L}(U,H)} \le \|\Psi\|_{HS(U,H)}$.
2. *Let V be another separable Hilbert space and $\Psi_1 \in \mathcal{L}(H,V)$, $\Psi_2 \in \mathcal{L}(V,U)$ and $\Psi \in HS(U,H)$. Then, $\Psi_1 \circ \Psi \in HS(U,V)$ and $\Psi \circ \Psi_2 \in HS(V,H)$. Moreover*

$$\|\Psi_1 \circ \Psi\|_{HS(U,V)} \le \|\Psi\|_{HS(U,H)} \|\Psi_1\|_{\mathcal{L}(H,V)},$$

$$\|\Psi \circ \Psi_2\|_{HS(V,H)} \le \|\Psi_2\|_{\mathcal{L}(V,U)} \|\Psi\|_{HS(U,H)}.$$

Regarding the importance of the finite trace operators and their role in the construction of the stochastic integral, we add the following results.

Proposition 1.44 *[86, Proposition. 2.3.4, p.28]. Let $Q \in \mathcal{L}(U)$ be a symmetric and nonnegative operator. Then, there exists a unique symmetric and nonnegative operator $Q^{\frac{1}{2}} \in \mathcal{L}(U)$ satisfies $Q^{\frac{1}{2}} \circ Q^{\frac{1}{2}} = Q$. Moreover, if Q with finite trace, then $Q^{\frac{1}{2}} \in HS(U)$, s.t. $\|Q^{\frac{1}{2}}\|_{HS}^2 = trQ$ and for all $\Psi \in \mathcal{L}(U,H)$ it holds $\Psi \circ Q^{\frac{1}{2}} \in HS(U,H)$.*

Corollary 1.45 *Let $Q \in \mathcal{L}(U)$ be a symmetric and nonnegative operator and $\{e_n,\ n \in \mathbb{N}\}$ be an ONB of U, consisting of eigenvectors of Q with corresponding eigenvalues $\{\lambda_n,\ n \in \mathbb{N}\}$. Then, the operator $Q^{\frac{1}{2}}$ admits the family $\{(e_n, \lambda_n^{\frac{1}{2}}),\ n \in \mathbb{N}\}$ as an eigenpairs.*

Proof. Let $\{e_n,\ n \in \mathbb{N}\}$ be an ONB of U, consisting of eigenvectors of Q with corresponding eigenvalues $\{\lambda_n,\ n \in \mathbb{N}\}$. Let $A \in \mathcal{L}(U)$ defined by $Ae_n := \lambda_n^{\frac{1}{2}} e_n$, for any $n \in \mathbb{N}$. The operator A is symmetric. Indeed, for all $u, v \in U$ we have

$$\begin{aligned} \langle Au, v\rangle_U &= \langle A \sum_{n\in\mathbb{N}} \langle u, e_n\rangle_U e_n, \sum_{m\in\mathbb{N}} \langle v, e_m\rangle_U e_m\rangle_U = \sum_{n,m\in\mathbb{N}} \langle u, e_n\rangle_U \langle v, e_m\rangle_U \langle Ae_n, e_m\rangle_U \\ &= \sum_{n,m\in\mathbb{N}} \langle u, e_n\rangle_U \langle v, e_m\rangle_U \langle \lambda_n^{\frac{1}{2}} e_n, e_m\rangle_U = \sum_{n\in\mathbb{N}} \lambda_n^{\frac{1}{2}} \langle u, e_n\rangle_U \langle v, e_n\rangle_U. \end{aligned}$$

By the same manner it holds that

$$\langle u, Av\rangle_U = \sum_{n\in\mathbb{N}} \lambda_n^{\frac{1}{2}} \langle u, e_n\rangle_U \langle v, e_n\rangle_U,$$

In Addition, A is nonnegative since we can get easily for all $u \in U$,

$$\langle Au, u\rangle_U = \sum_{n\in\mathbb{N}} \lambda_n^{\frac{1}{2}} \langle u, e_n\rangle_U^2 \geq 0.$$

Moreover, for any $\in \mathbb{N}$ we have

$$(A \circ A)e_n = A(\lambda_n^{\frac{1}{2}} e_n) = \lambda_n^{\frac{1}{2}} A(e_n) = \lambda_n e_n = Qe_n.$$

Consequently, for all $u \in U$ it holds $(A \circ A)u = Qu$. Thus, by virtue of Proposition 1.44 we obtian that $A = Q^{\frac{1}{2}}$. ■

The next proposition introduce an important linear subspace of U, namely the image of $Q^{\frac{1}{2}}$, which is also a separable Hilbert space when equipped with an appropriately chosen inner product.

Proposition 1.46 *[31, p.96] and [86, Proposition. C.0.3, p.147]. Let $Q \in \mathcal{L}(U)$ be a symmetric, nonnegative and finite trace operator and let $\{e_n,\ n \in \mathbb{N}\}$ be an ONB of U,* *consisting of eigenvectors of* Q *with corresponding eigenvalues* $\{\lambda_n,\ n \in \mathbb{N}\}$. *We define* $S := \{n \in \mathbb{N},\ \lambda_n > 0\}$ *the index set of non-zero eigenvalues. Then, the space* $U_0 := Q^{\frac{1}{2}}(U)$ *defined by*

$$U_0 = \{u \in ker\left(Q^{\frac{1}{2}}\right)^{\perp},\ \sum_{n \in S} \lambda_n^{-1} \langle u, e_n \rangle_U^2 < +\infty\},$$

is a Hilbert space endowed with the following inner product

$$\langle u, v \rangle_{U_0} := \langle Q^{-\frac{1}{2}} u, Q^{-\frac{1}{2}} v \rangle_U = \sum_{n \in S} \lambda_n^{-1} \langle u, e_n \rangle_U\ \langle v, e_n \rangle_U,\ \forall\ u, v \in U_0,$$

where $Q^{-\frac{1}{2}}$ *is the pseudo-inverse of* $Q^{\frac{1}{2}}$. *In addition, the space* U_0 *admits the family* $\{\lambda_n^{\frac{1}{2}} e_n,\ n \in S\}$ *as an ONB.*

Corollary 1.47 *For any* $A \in HS(U_0, H)$, *it holds*

$$\|A\|_{HS(U_0,H)} = \|A \circ Q^{\frac{1}{2}}\|_{HS(U,H)}.$$

Moreover, let the space $\mathcal{L}_0(U, H) := \left\{A_{|U_0},\ A \in \mathcal{L}(U, H)\right\}$. *Then, we have* $\mathcal{L}_0(U, H) \subset HS(U_0, H)$.

Proof. Let $A \in HS(U_0, H)$. Then,

$$\|A\|_{HS(U_0,H)} := \sum_{n \in S} |A(\lambda_n^{\frac{1}{2}} e_n)|_H^2 = \sum_{n \in S} |A(Q^{\frac{1}{2}} e_n)|_H^2.$$

As $\{Q^{\frac{1}{2}} e_n,\ n \in S\}$ is an ONB of U_0 and $Q^{\frac{1}{2}} e_n = 0_U$ for all n not in S yields,

$$\|A\|_{HS(U_0,H)} = \sum_{n \in S} |A \circ Q^{\frac{1}{2}}(e_n)|_H^2 + \sum_{n \text{ not in } S} |A \circ Q^{\frac{1}{2}}(e_n)|_H^2 = \|A \circ Q^{\frac{1}{2}}\|_{HS(U,H)}.$$

Moreover, from the statement 2. in Proposition 1.43 it holds

$$\|A\|_{HS(U_0,H)} \leq \|A\|_{\mathcal{L}(U,H)} \|Q^{\frac{1}{2}}\|_{HS(U)}.$$

The fact that $\|Q^{\frac{1}{2}}\|_{HS(U)}^2 := \sum_{n \in S} \|Q^{\frac{1}{2}} e_n\|_U^2 = trQ < \infty$ leads to $A \in HS(U_0, H)$. ■

1.2.2 Generalities on the semigroup theory

The semigroup theory became an essential tool in a great number of areas of mathematical analysis. The aim of this section is to present basic notions and useful results on this theory. Throughout this section we fix a Banach space $(X, |.|_X)$ and a linear operator $(A, D(A))$.

Definition 1.48 *(**Semigroup**). A collection $(S(t))_{t\in\mathbb{R}_+}$ in $\mathcal{L}(X)$ is called a **semigroup** if*

1. *$S(0) = I_X$, where I_X is the identity operator on x.*

2. *$S(t+s) = S(t)S(s)$, for all $t, s \in \mathbb{R}_+$.*

*Further, if the family satisfies the above conditions for all $t \in \mathbb{R}$, we call it a **group**.*

Definition 1.49 *(**Uniformly continuous semigroup**). A semigroup $(S(t))_{t\in\mathbb{R}_+}$ on X is said to be **uniformly continuous** if*

$$\lim_{t\to 0^+} S(t) = I_X,$$

where the limit is taken in the topology of $\mathcal{L}(X)$.

Definition 1.50 *(**Strongly continuous or C_0-semigroup**). A semigroup $(S(t))_{t\in\mathbb{R}_+}$ on X is called **strongly continuous** or C_0 for short if for every $x \in X$ it holds*

$$\lim_{t\to 0^+} S(t)x = x.$$

The limit is taken with respect to the norm $|.|_X$.

Theorem 1.51 *Let $(S(t))_{t\in\mathbb{R}_+}$ be a C_0-semigroup. Then, there exist two constants $a \geq 0$ and $C \geq 1$ s.t.*

$$\|S(t)\|_{\mathcal{L}(X)} \leq C\, e^{at}, \text{ for all } t \in \mathbb{R}_+.$$

Proof. See [83, Theorem 2.2, p.4]. ■

Definition 1.52 *(**Semigroup of contraction**). We call $(S(t))_{t\in\mathbb{R}_+}$ a **semigroup of contraction** if*

$$\|S(t)\|_{\mathcal{L}(X)} \leq 1, \text{ for every } t \in \mathbb{R}_+.$$

Definition 1.53 *(**Analytic semigroup**). Let $\theta > 0$. A collection $(S(z))_{z\in\mathcal{Z}} \in \mathcal{L}(X)$, where*

$$\mathcal{Z} := \{z \in \mathbb{C},\ |arg z| < \theta\},$$

*is said to be **analytic semigroup** if*

1. $S(0) = I_X$ *and* $S(z_1 + z_2) = S(z_1)S(z_2)$, *for all* $z_1, z_2 \in \mathcal{Z}$.

2. $\lim_{z\to 0} S(z)x = x$, *for every* $x \in X$.

3. $z \mapsto S(z)$ *is analytic in* $\mathcal{Z}$.

Definition 1.54 ***(Infinitesimal generator of semigroup)****. Let* $(S(t))_{t\in\mathbb{R}_+}$ *be a semigroup. We define an operator* A *as the* ***infinitesimal generator*** *of* $(S(t))_{t\in\mathbb{R}_+}$ *by*

$$Ax := \lim_{t\to 0^+} t^{-1}(S(t)x - x),$$

where, $D(A) = \{x \in X, \text{ s.t. } \lim_{t\to 0^+} t^{-1}(S(t)x - x), \text{ exists in } X\}$.

Remark 1.55 *Due to the fact that* $D(A)$ *contains at least zero, it is evidently non-empty.*

Theorem 1.56 *Let* A *be a linear operator on* X*. Then,* A *is an infinitesimal generator of uniformly continuous semigroup* $(S(t))_{t\in\mathbb{R}_+}$ *iff* A *is bounded, (see [83, Theorem 1.2, p.2]). Moreover, any* $A \in \mathcal{L}(X)$ *is a generator of* ***unique*** *uniformly continuous semigroup, (see [83, Theorem 1.3, p.3]).*

Theorem 1.57 ***(Hille-Yosida)*** *Let* $(A, D(A))$ *be a linear (not necessarily bounded) operator on* X*. Then, the following assertions are equivalent*

1. $(A, D(A))$ *generates a* C_0*-semigroup of contraction* $(S(t))_{t\in\mathbb{R}_+}$.

2. $(A, D(A))$ *is densely defind, closed and for every* $\lambda > 0$ *one has* $\lambda \in \rho(A)$ *and*

$$\|\lambda(\lambda I_X - A)^{-1}\|_{\mathcal{L}(X)} \leq 1. \tag{1.14}$$

Proof. See [83, Theorem 3.1, p.8]. ■

Proposition 1.58 *[101, Proposition 9.4, p. 519]. Let* $(A, D(A))$ *be a nonnegative and self-adjoint operator. Then,* $(-A)$ *is an infinitesimal generator of semigroup of contraction* $(S(t) := e^{-tA})_{t\in\mathbb{R}_+}$.

1.2.3 Laplace operator

In this section, we deal with the **Laplace operator** (or Laplacian). The Laplacian is a differential operator represents the simplest elliptic operators occur in differential equations that describe many physical phenomena, such as the diffusion equation for heat and fluid flow. It is denoted by Δ and is given in the d-dimensional case by

$$\Delta u(x) := \sum_{i=1}^{d} \frac{\partial^2 u(x)}{\partial x_i^2},\ x \in D \subseteq \mathbb{R}^d,$$

where the derivatives are given either in the classical sense for a class of sufficiently smooth function or in the weak sense for a class of less smooth functions.
In our study we will concentrate on the one-dimensional case, for this let $D := (0,1)$, and let the space $H_0^1(D)$, which is the closure of $C_0^1(D)$ in $H_2^1(D)$ (see [14, Section 8.3, p.217]). Now, let us denote the minus Laplacian on D with homogeneous Dirichlet boundary conditions (i.e., the function is nul on the boundary ∂D) by $A := -\Delta$. Then, we can define A as an operator from $D(A) := H_0^1(D) \cap H_2^2(D)$ to $L^2(D)$.

Proposition 1.59 *[14]. The Laplacian $A : D(A) \to L^2(D)$, is unbounded, nonnegative and self adjoint operator.*

Proposition 1.60 *[101, Section 5.1, ps. 303-304]. The Laplacian $A : D(A) \to L^2(D)$, is an isomorphism, its inverse A^{-1} is self-adjoint and compact on $L^2(D)$.*

Proposition 1.61 *[101, Proposition 6.6, p. 497] and [14, Theorem 6.1, p. 167]. There exists an orthonormal basis $(e_j)_{j\in\mathbb{N}_0}$ of $L^2(D)$ consisting of eigenfunctions of A^{-1} and such that the sequence of eigenvalues $(\lambda_j^{-1})_{j\in\mathbb{N}_0}$ with $\lambda_j > 0$, converges to zero.*
Moreover, $(e_j)_{j\in\mathbb{N}_0}$ is also a sequence of eigenfunctions of A corresponding to the eigenvalues $(\lambda_j)_{j\in\mathbb{N}_0}$.

Remark 1.62 *Throughout this thesis, we choose the basis $(e_j(.) := \sqrt{2}\sin(j\pi.))_{j\in\mathbb{N}_0}$. Then the corresponding eigenvalues of the operator A are given by $(\lambda_j = (j\pi)^2)_{j\in\mathbb{N}_0}$.*

1.2.4 Fractional Laplacian

The presence of the long range interactions appear in various applications like nonlocal heat conduction allows to arise the **nonlocal diffusion operators** to replace the standard

Laplace operator. The new operators act by a global integration with respect to a singular kernel instead of acting by pointwise differentiation, in that way the nonlocal character of the process is preserved. The **fractional Laplacian** denoted here by $A_\alpha := (-\Delta)^{\frac{\alpha}{2}}$, for $\alpha > 0$ is one of the famous nonlocal diffusion operators. We can find in the literature many definitions of A_α which reflects its extensive use in applications. Throughout this section we let $\alpha \in (0, 2]$.

Fractional Laplacian on $\mathbb{R}$

The fractional Laplacian can be defined in several equivalent ways in the whole space $\mathbb{R}$, see for example [76]. However, when these definitions are restricted to bounded domains, the associated boundary conditions lead to different operators. Here, we introduce two equivalent definitions of the fractional Laplacian, the first is represented via Fourier transform and its inverse, whereas the second is based on the singular integral representation.

Definition 1.63 ***(Pseudo-differential representation**, [40, 76]). The fractional Laplacian A_α is defined as a pseudo-differential operator,*

$$A_\alpha u(x) := \mathcal{F}^{-1}(|\xi|^\alpha \mathcal{F}(u(x); \xi); x), \tag{1.15}$$

where $u \in L^p(\mathbb{R})$, for $p > 1$.

Definition 1.64 ***(Singular integral representation**, [40, 76]). We define the fractional Laplacian A_α as a singular integral operator*

$$A_\alpha u(x) := C_\alpha \lim_{r \to 0^+} \int_{\mathbb{R} \setminus B(x,r)} k_\alpha(x, y)(u(x) - u(y))dy, \tag{1.16}$$

for any $u \in \mathcal{S}$, where $k_\alpha(x, y) := |x - y|^{-(\alpha+1)}$ for any $x \in \mathbb{R}$ and any $y \in \mathbb{R} \setminus B(x, r)$, with $B(x, r)$ is the open ball of center x and radius r, and $C_\alpha := \frac{\alpha 2^{\alpha-1}\Gamma(\frac{\alpha+1}{2})}{\pi^{\frac{1}{2}}\Gamma(1-\frac{\alpha}{2})}$ is a constant with Γ is the gamma function.

Theorem 1.65 *The two defintions Identity.*(1.15) *and Identity.*(1.16) *of the fractional Laplacian A_α are equivalent.*

Proof. See [40, Proposition 3.3, p.530] and [76, Theorem 1.1, p.9]. ∎

Fractional Laplacian on bounded domains

On bounded domains there are many non-equivalent definitions of the fractional Laplacian, according to the particular boundary conditions used. However, it still of great importance, not only from the mathematical point of view, but from the applied sciences part as well. To shed further light on this operator, we deal in this subsection with the fractional Laplacian represented via the spectral approach. Precisely, we focus on the case of homogeneous Dirichlet boundary conditions, because it is our case of interest in this thesis. The case of inhomogeneous versions is considered recently in the literature, e.g., [1, 7].
Let us mention that, the spectral approach to define the fractional Laplacian on $D := (0,1)$ is based on the spectrum of A and Lemma 1.36 . To this end, first we need some additional functional spaces.

Definition 1.66 *Let $\alpha \in (0,2]$. Then, we have*

$$\hat{H}_2^\alpha(D) := \{u \in H_2^\alpha(\mathbb{R}),\ s.t.\ supp(u) \subset \bar{D}\}, \tag{1.17}$$

and

$$\mathcal{H}_2^\alpha(D) := \{u \in L^2(D),\ s.t.\ |u|^2_{\mathcal{H}^\alpha(D)} := \sum_{j\in\mathbb{N}_0} \lambda_j^\alpha \langle u, e_j\rangle^2_{L^2(D)} < \infty\}, \tag{1.18}$$

where $(\lambda_j, e_j)_{j\in\mathbb{N}_0}$ are the eigenpairs of the operator A.
In Addition, the space denoted by $H_0^\alpha(D)$ is the closure of $C_0^\infty(D)$ in the Hilbert space $H_2^\alpha(D)$.

Theorem 1.67 *It holds that, the two spaces $H_0^\alpha(D)$ and $H_2^\alpha(D)$ are equals iff $\alpha \leq \frac{1}{2}$.*

Proof. See [107, Theorem 1.4.3.2]. ■

As noted in [107, Section 4.4.3], the following relationship exists between the spaces $H_2^\alpha(D)$, $H_0^\alpha(D)$, $H_0^1(D)$, $\hat{H}_2^\alpha(D)$ and $\mathcal{H}_2^\alpha(D)$.

Lemma 1.68 *Let $\alpha \in (0,2]$. Then*

$$\mathcal{H}_2^\alpha(D) = \begin{cases} H_2^\alpha(D), & \text{if } 0 < \alpha < \frac{1}{2}, \\ \hat{H}_2^\alpha(D), & \text{if } \alpha = \frac{1}{2}, \\ H_0^\alpha(D), & \text{if } \frac{1}{2} < \alpha \leq 1, \\ H_2^\alpha(D) \cap H_0^1(D), & \text{if } 1 < \alpha \leq 2. \end{cases} \tag{1.19}$$

Definition 1.69 *(**Spectral fractional Laplacian**, [1, 7]). The spectral fractional Laplacian is an operator $A_\alpha : D(A_\alpha) := \mathcal{H}_2^\alpha(D) \to L^2(D)$ defined in terms of eigen expansion,*

$$A_\alpha u := \sum_{k=1}^{\infty} \lambda_k^{\frac{\alpha}{2}} \langle u, e_k \rangle_{L^2(D)} e_k, \quad \textit{for all}\, u \in D(A_\alpha). \tag{1.20}$$

From the definition (1.20), it can easily see that for all $n \in \mathbb{N}_0$,

$$A_\alpha e_n = \sum_{k=1}^{\infty} \lambda_k^{\frac{\alpha}{2}} \langle e_n, e_k \rangle_{L^2(D)} e_k = \lambda_n^{\frac{\alpha}{2}} e_n,$$

and so, $(e_n, \lambda_n^{\frac{\alpha}{2}})_{n\in\mathbb{N}_0}$ represents the eigenpairs of the fractional Laplacian A_α.

Remark 1.70 *The spectral fractional Laplacian coincides with the standard Laplacian if $\alpha = 2$. Indeed, for all $u \in D(A_2) = D(A)$ we have,*

$$Au = \sum_{k=1}^{\infty} \lambda_k \langle u, e_k \rangle_{L^2(D)} e_k. \tag{1.21}$$

Lemma 1.71 *[47]. The operator A_α, for $\alpha \in (0, 2]$ is the infinitesimal generator of an analytic semigroup $(e^{-tA_\alpha})_{t\geq 0}$ on $L^2(D)$ satisfies for all $v \in L^2(D)$,*

$$(e^{-tA_\alpha} v)(x) = \sum_{k=1}^{\infty} e^{-\lambda_k^{t\frac{\alpha}{2}}} \langle v, e_k \rangle_{L^2(D)} e_k(x). \tag{1.22}$$

Furthermore, such semigroup on the bounded interval $(0, T]$, for $T > 0$ be fixed, has the following no classical results.

Lemma 1.72 *Let $\alpha \in (1, 2]$ and $t \in (0, T]$. Then*

- *For all $\beta \geq 0$, there exists $C_{\alpha,\beta} > 0$, s.t.*

$$\|A^{\frac{\beta}{2}} e^{-tA_\alpha}\|_{\mathcal{L}(L^2(D))} \leq C_{\alpha,\beta} t^{-\frac{\beta}{\alpha}}, \tag{1.23}$$

- *For all $\zeta \in [0, \alpha]$, there exists $C_{\alpha,\zeta} > 0$, s.t.*

$$\|A^{-\frac{\zeta}{2}} (I - e^{-tA_\alpha})\|_{\mathcal{L}(L^2(D))} \leq C_{\alpha,\zeta} t^{\frac{\zeta}{\alpha}}. \tag{1.24}$$

Proof. For the first estimate, let $\beta \geq 0$, the use of Identity.(1.12) in Lemma 1.36, and Lemma A.8 (with $\gamma = \frac{\beta}{\alpha}$), leads to

$$\|A^{\frac{\beta}{2}} e^{-tA_\alpha}\|_{\mathcal{L}(L^2(D))} = \sup_{1\leq j\leq\infty} |\lambda_j^{\frac{\beta}{2}} e^{-t\lambda_j^{\frac{\alpha}{2}}}| = t^{-\frac{\beta}{\alpha}} \sup_{1\leq j\leq\infty} |(t\lambda_j^{\frac{\alpha}{2}})^{\frac{\beta}{\alpha}} e^{-t\lambda_j^{\frac{\alpha}{2}}}| \leq C_{\alpha,\beta} t^{-\frac{\beta}{\alpha}}.$$

About the second estimate, we argue as above. Then, we use Identity.(1.12), and Lemma A.9 (with $\beta = \frac{\zeta}{\alpha}$), to get

$$\|A^{-\frac{\zeta}{2}}(I-e^{-tA_\alpha})\|_{\mathcal{L}(L^2(D))} = \sup_{1\leq j\leq\infty} |\lambda_j^{-\frac{\zeta}{2}}(1-e^{-t\lambda_j^{\frac{\alpha}{2}}})| = t^{\frac{\zeta}{\alpha}} \sup_{1\leq j\leq\infty} |(t\lambda_j^{\frac{\alpha}{2}})^{-\frac{\zeta}{\alpha}}(1-e^{-t\lambda_j^{\frac{\alpha}{2}}})| \leq C_{\alpha,\zeta} t^{\frac{\zeta}{\alpha}}.$$

■

Chapter 2

Introduction to stochastic calculus on functional spaces

We start this chapter by some useful preliminaries. In Section 2.2 we recall some basic notions and facts about the stochastic processes in Hilbert spaces. In Section 2.3 we give definitions of the stochastic Itô integral in Hilbert spaces, which allows us to introduce the concept of stochastic differential equations in the next chapter. The main references for the material presented here are [31, 86].

2.1 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(E, |.|_E)$ be a separable Banach space and $\mathcal{B}(E)$ be the σ-field of its Borel subsets. We fix $T > 0$.

Definition 2.1 ***(Normal filtration)***. *Let $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ be a filtration (i.e., an increasing family of σ-fields defined on $(\Omega, \mathcal{F}, \mathbb{P})$). We say that $(\mathcal{F}_t)_{t\in[0,T]}$ is a* ***normal filtration*** *or say that, it satisfies the usual conditions if*

- *for all $B \in \mathcal{F}$ s.t. $\mathbb{P}(B) = 0$, then $B \in \mathcal{F}_0$.*
- *for all $t \in [0, T]$, $\mathcal{F}_{t^+} := \mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$.*

*Moreover, we call the space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a **filtered probability** space.*

Definition 2.2 ***(E-valued random variable)****. Let the mapping $\mathcal{X} : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$. We say that $\mathcal{X}$ is an **E-valued random variable** if it is measurable, i.e. for any $B \in \mathcal{B}(E)$ it holds $\mathcal{X}^{-1}(B) \in \mathcal{F}$.*

Definition 2.3 ***(Gaussian random variable)****. Let $(U, \langle ., . \rangle_U)$ be a separable Hilbert space. An U-valued random variable $\mathcal{X}$ on Ω is said to be **Gaussian** if the $\mathbb{R}$-valued random variable $\langle \mathcal{X}, u \rangle_U$, for any $u \in U$ is Gaussian.*
*Hence, $\exists\, m \in U$ called the **mean** and a nonnegative and symmetric operator $Q : U \to U$ called the **covariance operator** s.t. the law $\mathbb{P}^{\mathcal{X}} := \mathbb{P} \circ \mathcal{X}^{-1} : \mathcal{B}(U) \to [0, 1]$ of $\mathcal{X}$ is denoted by $\mathcal{N}(m, Q)$.*

Definition 2.4 ***(Stopping time)****. Let Θ be an $\mathbb{R}_+$-valued random variable defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Then, we say that Θ is an **$\mathcal{F}_t$-stopping time** if*

$$\{w \in \Omega,\ \Theta(w) \leq t\} \in \mathcal{F}_t, \ \ \textit{for every}\, t \in [0, T]. \tag{2.1}$$

Definition 2.5 ***(p-th integrability)****. Let $p \in [1, \infty)$ and $\mathcal{X}$ be an E-valued random variable defined on Ω. We say that $\mathcal{X}$ is **p-th integrable** if,*

$$\mathbb{E}\left(|\mathcal{X}|_E^p\right) := \int_\Omega |\mathcal{X}(w)|_E^p \, d\, \mathbb{P}(w) < \infty.$$

The space of all p-th integrable E-valued random variables on Ω is denoted by $L^p(\Omega, E)$ and it is a Banach space equipped with the norm $\|\mathcal{X}\|_{L^p(\Omega,E)} := (\mathbb{E}\left(|\mathcal{X}|_E^p\right))^{\frac{1}{p}}$. For the special cases $p = 1$ and $p = 2$ we say that $\mathcal{X}$ is **integrable** respectively **square integrable**.

Definition 2.6 ***(Stochastic process)****. A family $X := (X_t)_{t\in[0,T]}$ of E-valued random variables X_t, $t \in [0, T]$ defined on Ω is called a **stochastic process**.*

The stochastic process X depends on two variables, the temporal variable $t \in [0, T]$ and the probabilistic variable $w \in \Omega$.

Definition 2.7 ***(Trajectory)****. Let X be an E-valued stochastic process. The sample path $X_.(w) := X(., w)$ depends on $t \in [0, T]$, for fixed $w \in \Omega$ is called the **trajectory** of X.*

Definition 2.8 *Let $X := (X_t)_{t\in[0,T]}$ and $Y := (Y_t)_{t\in[0,T]}$ be two E-valued stochastic processes defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then,*

- *the process Y is called a* ***modification*** *of X if,*

$$\mathbb{P}\{w \in \Omega,\ s.t.\ X_t(w) \neq Y_t(w)\} = 0,\ \forall\, t \in [0,T],$$

- *The processes X and Y are* ***indistinguishable*** *if,*

$$\mathbb{P}\{w \in \Omega,\ s.t.\ X_t(w) = Y_t(w),\ \forall\, t \in [0,T]\} = 1.$$

Of course, the last property implies the first one.

Definition 2.9 *(**Adaptation, continuity** and **measurability of stochastic process**.) Let $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ be a normal filtration and $X = (X_t)_{t\in[0,T]}$ be a stochastic process.*

1. *X is $\mathbb{F}$**-adapted** (or simply adapted) if each X_t is measurable with respect to $\mathcal{F}_t$ for every $t \in [0,T]$,*
2. *X is **continuous** with probability one if its trajectories are continuous almost surely, i.e.*

$$\mathbb{P}\{w \in \Omega,\ \ t \to X_t(w) is\ continuous\ on\ [0,T]\} = 1,$$

3. *X is **measurable** if the mapping $X(.,.) : ([0,T] \times \Omega,\ \mathcal{B}([0,T]) \otimes \mathcal{F}) \to (E, \mathcal{B}(E))$ is measurable with respect to $\mathcal{B}([0,T]) \otimes \mathcal{F}$ (which is σ-algebra generated by the product).*

The regularity of stochastic process has been first studied by Kolmogorov in the 1930's, and extended by Centsov in the 1950's, as the following Kolmogorov-Centsov theorem stated.

Theorem 2.10 *(**Regularity Theorem**). Let $X = (X_t)_{t\in[0,T]}$ be an E-valued stochastic process which satisfies for some $C, \gamma > 0$ and $\beta > 1$,*

$$\mathbb{E}\left(|X(t) - X(s)|_E^{\beta}\right) \leq C|t-s|^{1+\gamma},\ for\ every\, t, s\ \in [0,T].$$

Then, X has a modification $\tilde{X}$ satisfying for every $\delta \in (0, \frac{\gamma}{\beta})$, $\mathbb{P}$-a.s.

$$|\tilde{X}(t)(w) - \tilde{X}(s)(w)|_E \leq C|t-s|^{\delta},\ for\ every\, t, s\ \in [0,T].$$

Proof. See [33, Theorem. 3.3, p.73]. ■

Definition 2.11 ***(Gaussian process)***. *Let* $(U, \langle .,. \rangle_U)$ *be a separable Hilbert space. An* U*-valued stochastic process* X *on* Ω *is called* ***Gaussian*** *if for any* $k \in \mathbb{N}_0$ *and any* $t_1, ..., t_k \in [0,T]$*, the* U^k*-valued random variable* $(X_{t_1}, ..., X_{t_k})$ *is Gaussian.*

Definition 2.12 ***(Predictable σ-field).*** *Let* $\Omega_T := [0,T] \times \Omega$ *endowed with the* σ*-field* $\mathcal{B}([0,T]) \otimes \mathcal{F}$*. The* σ*-field* P_T *generated by the sets of the form*

$$((s,t] \times F_s; 0 \leq s < t \leq T, F_s \in \mathcal{F}_s) \ \textit{and} \ (\{0\} \times F_0), F_0 \in \mathcal{F}_0,$$

is called a ***predictable σ-field*** *and its elements are called* ***predictable sets****.*

Definition 2.13 ***(Predictable process).*** *An arbitrary measurable mapping* $Y : (\Omega_T, P_T) \to (E, \mathcal{B}(E))$ *is called a* ***predictable process****.*

Definition 2.14 ***(p-integrable process).*** *An* E*-valued stochastic process* $X = (X_t)_{t\in[0,T]}$ *is called* p***-integrable****, for* $p \geq 1$ *if the random variable* $X(t)$*, for all* $t \in [0,T]$ *is* p*-th integrable.*

Definition 2.15 ***(Martingale).*** *An* E*-valued stochastic process* $M := (M_t)_{t\in[0,T]}$ *is called an* $\mathbb{F}$***-martingale*** *(or simply martingale), if* M *is* $\mathbb{F}$*-adapted, integrable and satisfies*

$$\mathbb{E}[M(t)|\mathcal{F}_s] = M(s), \ \mathbb{P} - a.s \ \textit{for all} \ 0 \leq s < t \leq T,$$

whith $\mathbb{E}[M(t)|\mathcal{F}_s]$ *is the conditional expectation.*

Definition 2.16 ***(Submartingale/Supermartingale).*** *An* $\mathbb{F}$*-adapted and integrable stochastic process* $M = (M_t)_{t\in[0,T]}$ *is called* ***submartingale*** *(respectively,* ***supermartingale****) if*

$$\mathbb{E}[M(t)|\mathcal{F}_s] \geq M(s), \ \mathbb{P} - a.s \ \textit{for all} \ 0 \leq s < t \leq T,$$

respectively

$$\mathbb{E}[M(t)|\mathcal{F}_s] \leq M(s), \ \mathbb{P} - a.s \ \textit{for all} \ 0 \leq s < t \leq T.$$

Proposition 2.17 *[86, Proposition.2.2.9, p.24]. The space of all continuous square integrable E-valued martingales denoted by $\mathcal{M}_T^2(E)$ is a Banach space endowed with the norm*

$$\|M\|_{\mathcal{M}_T^2(E)} := \sup_{t\in[0,T]} \|M(t)\|_{L^2(\Omega,E)}. \tag{2.2}$$

Definition 2.18 ***(Random field)****. A family $X := (X_\xi)_{\xi\in\mathbb{R}^d}$ of E-valued random variables X_ξ, $\xi \in \mathbb{R}^d$ defined on Ω is called a* ***random field****.*

Definition 2.19 ***(Strongly continuous operator)****. Let $(U, \langle .,.\rangle_U, |.|_U)$, $(H, \langle .,.\rangle_H, |.|_H)$ be two separable Hilbert spaces, and let $G : H \to \mathcal{L}(U,H)$ be an operator. We say that G is* ***strongly continuous****, if for any $u \in U$, the mapping $h \mapsto G(h)(u)$ is continuous from H to H.*

2.2 Wiener processes on Hilbert spaces

There are many types of stochastic processes, among them Wiener process, Markov process and Poisson Process. However, Wiener process without any doubt is one of the most important processes both in the theory and in the applications. Originally it was introduced by the mathematician Norbert Wiener in 1920 as a mathematical model of the **Brownian motion** [1]. The current section gives a short review on such process which is a generalization of Brownian motion taking values in a general functional space. To do so, we need first to fix some tools; let $T > 0$, $(U, \langle .,.\rangle_U, |.|_U)$ be a separable real Hilbert space and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.

2.2.1 Q-Wiener processes

Fix $Q \in \mathcal{L}(U)$ be a symmetric, nonnegative and finite trace operator.

Definition 2.20 ***(Standard Q-Wiener process)****. Let $W := (W_t)_{t\in[0,T]}$ be an U-valued stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$. Then, W is said to be a* ***standard Q-Wiener process*** *if the following holds*

1. $\mathbb{P}(W(0) = 0) = 1$,

[1]The Brownian motion is named after the biologist Robert Brown who observed in 1827 the irregular motion of pollen particles floating in water.

2. *the trajectories of* W *are continuous,*

3. *for any finite increasing sequence* $(t_i)_{i=0}^{k} \subset [0,T]$*, the increments* $(W_{t_{i+1}} - W_{t_i})_{i=0}^{k-1}$ *are independent,*

4. *for any* $0 \leq s \leq t \leq T$*, it holds* $\mathbb{P}^{(W_t - W_s)} = \mathcal{N}(0,(t-s)Q)$*.*

Remark 2.21 *Let us mention that, the* $\mathbb{R}$*-valued random variable* $\langle W(t)-W(s), u\rangle_U$*, for all* $0 \leq s < t \leq T$ *and all* $u \in U$ *is Gaussian, with mean zero and variance* $(t-s)\langle Qu,u\rangle_U$*, and so* $\mathbb{E}\left(|W(t)|_U^2\right) = t\ trQ$*. Hence, this is a reason why the assumption* $trQ < \infty$ *is essential.*

Proposition 2.22 *[31, Proposition 4.3.(i), p.81]. An* U*-valued* Q*-Wiener process,* $W = (W_t)_{t\in[0,T]}$ *is a Gaussian process with mean zero and covariance operator* tQ*,* $\forall\, t \geq 0$*.*

Proposition 2.23 *[31, Proposition 4.4., p.82]. For any symmetric, nonnegative and finite trace operator* $Q \in \mathcal{L}(U)$*, there exists a* Q*-Wiener process on* U*.*

A question may arises here about the construction of such Q-Wiener process. The next proposition answers to this question. Moreover, it gives the presentation of the Q-Wiener process.

Theorem 2.24 *Let* $(e_n)_{n\in\mathbb{N}}$ *be an ONB of* U *consisting of eigenvectors of* Q *corresponding to the nonnegative eigenvalues* $(\lambda_n)_{n\in\mathbb{N}}$*, and let* $S := \{n \in \mathbb{N},\ \ \lambda_n > 0\}$ *be the index set of non-zero eigenvalues. Then, an* U*-valued stochastic process* W*, is a* Q*-Wiener process iff it can be written for any* $t \in [0,T]$ *as*

$$W(t) = \sum_{n\in S} \sqrt{\lambda_n}\beta_n(t)e_n, \tag{2.3}$$

where $(\beta_n)_{n\in S}$ *is a sequence of independent* $\mathbb{R}$*-valued Brownian motions on* $(\Omega,\mathcal{F},\mathbb{P})$ *s.t.*

$$\beta_n(t) := \frac{1}{\sqrt{\lambda_n}}\langle W(t), e_n\rangle_U, \ \ \textit{for any}\ n \in S. \tag{2.4}$$

The series in (2.3) *converges in* $L^2(\Omega, C([0,T],U))$*, where* $C([0,T],U)$ *is equipped with the supremum norm.*

Proof. See [86, Proposition 2.1.10, p.17]. ■

In the case of filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, we need to know how the Q-Wiener process behaves in connection with the filtration $\mathbb{F}$. This is leads to the next definition.

Definition 2.25 *(Q-**Wiener process with respect to a filtration**). Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be a normal filtration on $(\Omega, \mathcal{F}, \mathbb{P})$, and let W be an U-valued Q-Wiener process. We say that W is a Q-Wiener process with respect to $\mathbb{F}$ if the following statements hold*

1. *W is $\mathbb{F}$-adapted,*

2. *for all $0 \leq s \leq t \leq T$ the increment $(W(t) - W(s))$ is independent of $\mathcal{F}_s$.*

Theorem 2.26 *Every U-valued Q-Wiener process with respect to a normal filtration $\mathbb{F}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is an element in the space $\mathcal{M}_T^2(U)$.*

Proof. See [86, Proposition 2.2.10, p.25]. ■

Definition 2.27 ***(Stochastic basis)**. Let $(U, \langle ., . \rangle_U, |.|_U)$ be a separable Hilbert space. We call $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W)$ a **stochastic basis** if, $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered complete probability space with respect to the normal filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$, and $(W_t)_{t\in[0,T]}$ be a U-valued Wiener process on it.*

2.2.2 Cylindrical Wiener processes

Cylindrical Wiener process appears in many models in infinite dimensional spaces as a source of random noise or random perturbation. In this subsection, we introduce by following [86] a result which ensures the existence of such type of processes. To do so, let $(U, \langle ., . \rangle_U)$ be a separable Hilbert space, $Q \in \mathcal{L}(U)$ be a symmetric and nonnegative operator, possibly with $trQ = +\infty$ and let $(e_n)_{n\in\mathbb{N}}$ be an ONB of U that consists of eigenvectors of Q with corresponding eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$. Additionally, let $(U_1, \langle ., . \rangle_{U_1}, |.|_{U_1})$ be an arbitrary separable Hilbert space with $U \subset U_1$ embeded continuously and let $J : (U_0, \langle ., . \rangle_{U_0}) \to (U_1, \langle ., . \rangle_{U_1})$ be a Hilbert-Schmidt embedding, besides we define the operator $Q_1 := JJ^* : U_1 \to U_1$.

Remark 2.28 *It is important to note here that the space U_1 and the operator J as above always exist as explained in [86, Remark 2.5.1].*

Proposition 2.29 *[86, Proposition 2.5.2, p.50]. Let $Q_1 := JJ^*$. Then, Q_1 is a linear, bounded, nonnegative, symmetric and finite trace operator on U_1, and the operator $J : U_0 \to J(U_0) = Q_1^{\frac{1}{2}}(U_1)$ is an isometry, i.e.*

$$\|u_0\|_{U_0} = \|Q_1^{-\frac{1}{2}} J(u_0)\|_{U_1} = \|J(u_0)\|_{Q_1^{\frac{1}{2}}(U_1)}, \text{ for all } u_0 \in U_0.$$

Moreover, let $\tilde{e}_n := Q^{\frac{1}{2}} e_n$, where $(e_n)_{n\in\mathbb{N}}$ be an ONB of U and let $(\beta_n)_{n\in\mathbb{N}}$ be a sequence of independent, real-valued Brownian motions. Then

$$W(t) := \sum_{n=1}^{\infty} B_n(t) J(\tilde{e}_n), \ \ \forall\, t \in [0,T], \tag{2.5}$$

is a Q_1-Wiener process on U_1 with $trQ_1 < +\infty$, where the series in (2.5) is convergent in $\mathcal{M}_T^2(U_1)$.

Remark 2.30 *Let us mention that, in Proposition 2.29 we distinguish two cases, namely if the operator Q is finite trace, and so $Q^{\frac{1}{2}}$ is Hilbert-Schmidt operator, we can choose $U_1 = U$ to arrive at the classical concept of the Q-Wiener process. Whereas, in the case if $trQ = +\infty$ (e.g. $Q = I_U$) we will call the constructed process a **cylindrical Wiener process on** U.*

2.2.3 Some notions in the one dimensional case

In this subsection we recall two usefull notions of real-valued processes.

Definition 2.31 ***(White noise).*** *Let $(\mathcal{E}, \mathcal{B}(\mathcal{E}), \mu)$ be a σ-finite measurable space and let the random set function $\mathbb{W}$ defined on $\{B \in \mathcal{B}(\mathcal{E}),\ s.t.,\ \mu(B) < \infty\}$. Then, we call $\mathbb{W}$ a **white noise** [2] if it satisfies*

1. *for any $B \in \mathcal{B}(\mathcal{E})$, $\mathbb{W}(B)$ is a Gaussian (or normal) random variable with mean 0 and variance $\mu(B)$.*

[2] The term "white noise" arises from the spectral theory of stationary random processes, according to which white noise has a "flat" power spectrum that is uniformly distributed over all frequencies like white light.

2. for any two disjoint sets $B_1, B_2 \in \mathcal{B}(\mathcal{E})$, the random variables $\mathbb{W}(B_1)$ and $\mathbb{W}(B_2)$ are independent and $\mathbb{W}(B_1 \cap B_2) = \mathbb{W}(B_1) + \mathbb{W}(B_2)$.

Definition 2.32 ***(Brownian sheet).*** *Let $d \in \mathbb{N}_0$, $\mathcal{E} = \mathbb{R}_+^d := \{t = (t_1, ..., t_d),\ s.t.,\ t_i \geq 0,\ \forall\ i \in \{1, ..., d\}\}$ and μ be a Lebesgue measure on $\mathbb{R}^d$. The process $(\beta_t)_{t \in \mathbb{R}_+^d}$ is said to be* ***Brownian sheet*** *if it is defined by*

$$\beta_t = \mathbb{W}(\prod_{i=1}^{d}]0, t_i]),$$

where $\mathbb{W}$ is a white noise. This means that, it is a zero-mean Gaussian process with covariance function defined for $t = (t_1, ..., t_d)$ and $s = (s_1, ..., s_d)$ by

$$\mathbb{E}(\beta_t \beta_s) = \prod_{i=1}^{d} t_i \wedge s_i.$$

Remark 2.33 *There is another way to define white noise. In the special case; $\mathcal{E} = \mathbb{R}$ and μ is Lebesgue measure, it is informally described as the weak derivative of Brownian motion, since such motion is nowhere-differentiable in the classical sense. Such description is also possible in higher dimensions, but it involves the Brownian sheet instead of Brownian motion.*

2.3 Stochastic integrals in Hilbert spaces

In this section, we deal with the stochastic integral in Hilbert spaces with respect a Q-Wiener process and a cylindrical Wiener process. We take a close look at these notions by following the approach of Section 4.2 in [31] and Chapter 2 in [86].

2.3.1 Stochastic integral with respect to Q-Wiener process

Throughout this subsection, we fix $T > 0$ and a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Further, we consider two separable Hilbert spaces $(U, \langle ., . \rangle_U, |.|_U)$ and $(H, \langle ., . \rangle_H, |.|_H)$. Let $Q \in \mathcal{L}(U)$ be a symmetric, nonnegative and finite trace operator and let $W = (W(t))_{t \in [0,T]}$ be a U-valued Q-Wiener process with respect to the normal filtration $\mathbb{F}$. In order to shed a light on the notion of **stochastic integral** with respect to W, we give first the meaning of an elementary process.

Definition 2.34 ***(Elementary process)***. *Let* $\varphi = (\varphi(t))_{t\in[0,T]}$ *be a* $\mathcal{L}(U,H)$*-valued process defined on* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$*. We say that* φ *is an* ***elementary process*** *if there exists a partition* $0 = t_0 < t_1 < ... < t_m = T$*,* $m \in \mathbb{N}$ *and a sequence* $(\varphi_k)_{k=0}^{m-1}$ *of* $\mathcal{L}(U,H)$*-valued random variables that are taking only a finite number of values in* $L(U,H)$ *s.t.* φ_k *is* $\mathcal{F}_{t_k}$*-measurable for any* $k \in \{0, ..., m-1\}$ *and* $\varphi(t) = \varphi_k$*, for* $t \in]t_k, t_{k+1}]$ *with the convention* $\varphi(0) = 0$*. Mathematically, we write*

$$\varphi(t) = \sum_{k=0}^{m-1} \varphi_k \; 1_{]t_k,t_{k+1}]}(t), \quad for\, t \in [0,T].$$

We denote the class of such processes by $\mathbf{E}$*.*

Now, we are ready to give the meaning of the stochastic integral on $\mathbf{E}$.

Definition 2.35 *(**stochastic integral on** $\mathbf{E}$). Let* $\varphi \in \mathbf{E}$*. Then, the stochastic integral* $\mathcal{I}(\varphi) := ((\mathcal{I}(\varphi))(t))_{t\in[0,T]}$ *of* φ *with respect to* W *is an* H*-valued stochastic process defined for all* $t \in [0,T]$ *by*

$$(\mathcal{I}(\varphi))(t) := \int_0^t \varphi(s) dW(s) := \sum_{k=0}^{m-1} \varphi_k \left(W(t_{k+1} \wedge t) - W(t_k \wedge t) \right), \tag{2.6}$$

such that φ_k *is acting on* $(W(t_{k+1} \wedge t) - W(t_k \wedge t))$ *as an operator in* $\mathcal{L}(U,H)$*.*

Proposition 2.36 *Let* $\varphi \in \mathbf{E}$*. Then, the stochastic integral* $\mathcal{I}(\varphi)$ *defined by* (2.6) *belongs to* $\mathcal{M}_T^2(H)$ *(see [86, Proposition 2.3.2, p.26] and [31, Proposition 4.20, p.96]), where its norm is given by*

$$\|(\mathcal{I}(\varphi))\|^2_{\mathcal{M}_T^2(H)} = \mathbb{E}\left(\int_0^T \|\varphi(s) \circ Q^{\frac{1}{2}}\|^2_{HS(U,H)} ds \right), \tag{2.7}$$

see [86, Proposition 2.3.5, p.29]. Moreover, if φ *be a* $\mathcal{L}_0(U,H)$*-valued elementary process, then according to Corollary 1.47 it holds that*

$$\|(\mathcal{I}(\varphi))\|^2_{\mathcal{M}_T^2(H)} = \mathbb{E}\left(\int_0^T \|\varphi(s)\|^2_{HS(U_0,H)} ds \right). \tag{2.8}$$

Remark 2.37 *The mapping defined on the class* $\mathbf{E}$ *by*

$$\|\varphi\|^2_{\mathbf{E}} := \mathbb{E}\left(\int_0^T \|\varphi(s) \circ Q^{\frac{1}{2}}\|^2_{HS(U,H)} ds \right), \; for\ any\ T > 0,$$

is just a seminorm.

In order to define a norm on $\mathbf{E}$ we define the equivalence relation $\mathcal{R}$ between two elementary processes φ and ψ as $\varphi \ \mathcal{R} \ \psi$ iff $\|\varphi - \psi\|_{\mathbf{E}} = 0$, and for simplicity sake, we denote the space of equivalence classes on which $\|.\|_{\mathbf{E}}$ is norm by $\mathbf{E}$ as well, and we write φ for its corresponding equivalence class. With this norm on $\mathbf{E}$ the linear mapping $\mathcal{I} : \mathbf{E} \to \mathcal{M}_T^2(H)$ defined for any $\varphi \in \mathbf{E}$ by

$$\mathcal{I}(\varphi) := \int_0^{\cdot} \varphi(s)\, dW(s),$$

is an isometry (see [31, Remark 4.21, p.97]).

It is well known that, since $\mathcal{I}$ is linear and bounded operator, then it admits an unique extension defined on the closure $\bar{\mathbf{E}}$, which is also isometric.

Our object now is to make precise the right representation of $\bar{\mathbf{E}}$. We remark from Identity.(2.8) in Proposition 2.36 that the integrand must be a random variable defined on $\Omega_T := [0,T] \times \Omega$ endowed with the product σ-field $\mathcal{B}([0,T]) \otimes \mathcal{F}$, without forgetting its adaptability. For this, the best guess is to take the integrand to belong in to the predictable σ-field P_T. Hence, the convenient choice of the class of integrands is the space of the $HS(U_0, H)$-valued predictable processes, let us denote it by $\mathcal{P}_W^2$. Then

$$\mathcal{P}_W^2 := \{\varphi : \Omega_T \to HS(U_0, H);\ \text{s.t. } \varphi \text{ is predictable with } \|\varphi\|_{\mathbf{E}} < \infty\}.$$

The fact that $(HS(U_0, H), \langle ., . \rangle_{HS(U_0,H)})$ is a Hilbert space ensures that, the space $\mathcal{P}_W^2$ is also a Hilbert space its inner product defined for all $\varphi, \psi \in \mathcal{P}_W^2$ by

$$\langle \varphi, \psi \rangle_{\mathcal{P}_W^2} := \mathbb{E}\left(\int_0^T \langle \varphi(s), \psi(s) \rangle_{HS(U_0,H)}\right),$$

Moreover, $\mathcal{P}_W^2$ represents the closure of $\mathbf{E}$ (i.e. $\mathcal{P}_W^2 = \bar{\mathbf{E}}$) as it is proved in the following result.

Proposition 2.38 *[31, Proposition 4.22, ps.97-98].*

1. *Let φ ba an elementary process defined from Ω_T into $\mathcal{L}(U, H)$. If φ is predictable then it is $HS(U_0, H)$-valued predictable process as well.*

2. *For any $\varphi \in \mathcal{P}_W^2$ there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of $\mathcal{L}_0(U, H)$-valued elementary processes s.t. $\varphi_n \to \varphi$ as $n \to \infty$ with respect to the norm $\|.\|_{\mathbf{E}}$.*

Definition 2.39 ***(stochastic integral on*** $\mathcal{P}_W^2$*). Let* $\varphi \in \mathcal{P}_W^2$*. Then, the stochastic integral of* φ *with repect to* W *is defined for every* $t \in [0, T]$ *by*

$$(\mathcal{I}(\varphi))(t) := \int_0^t \varphi(s)dW(s) := \lim_{n\to\infty} \int_0^t \varphi_n(s)dW(s), \tag{2.9}$$

where the limit is taken with respect to the norm $\|.\|_{\mathcal{M}_t^2(H)}$*.*

Theorem 2.40 *Let* $\varphi \in \mathcal{P}_W^2$*. Then,* $\mathcal{I}(\varphi) \in \mathcal{M}_T^2(H)$*.*

Proof. See [31, Theorem 4.27, p.103]. ■

Last but not least, by following [31, Section. 4.2, ps.99-100]) we introduce the definition of the stochastic integral for a richer class of $HS(U_0, H)$-valued predictable processes which satisfy the following weak condition

$$\mathbb{P}\left(\int_0^T \|\varphi(s)\|_{HS(U_0,H)}^2 ds < \infty\right) = 1. \tag{2.10}$$

Let us denote such class by $\mathcal{P}_W$.

Lemma 2.41 *[31, Lemma. 4.24, p. 99]. Fix* $T > 0$*. Let* Θ *be an* $\mathcal{F}_t$*-stopping time s.t.* $\mathbb{P}(\Theta \leq T) = 1$ *and let* $\varphi \in \mathcal{P}_W^2$*. Then, the following equality holds* $\mathbb{P}$*-a.s.*

$$\int_0^t (1_{[0,\Theta]}\varphi)(s)dW(s) = \int_0^{\Theta\wedge t} \varphi(s)dW(s), \quad \textit{for all } t \in (0, T].$$

Let $\varphi \in \mathcal{P}_W$ and let for all $n \in \mathbb{N}$ the sequence of stopping time

$$\Theta_n := \inf\{t \in [0, T], \int_0^t \|\varphi(s)\|_{HS(U_0,H)}^2 ds \geq n\},$$

such that

$$\mathbb{E}\left(\int_0^t \|(1_{[0,\Theta_n]}\varphi)(s)\|_{HS(U_0,H)}^2 ds\right) < \infty.$$

Then, for all $n \in \mathbb{N}$ the process $\left((1_{[0,\Theta_n]}\varphi)(t)\right)_{t\in[0,T]}$ is an element in $\mathcal{P}_W^2$, and so for all $t \in (0, T]$ the following stochastic integral

$$\int_0^t (1_{[0,\Theta_n]}\varphi)(s)dW(s),$$

is well defined for all $n \in \mathbb{N}$.

Definition 2.42 ***(Stochastically integrable process).*** *Let $\varphi \in \mathcal{P}_W$. Then, the stochastic integral of φ with respect to W is defined for every $t \in [0,T]$ by*

$$\int_0^t \varphi(s)dW(s) := \int_0^t (1_{[0,\Theta_n]}\varphi)(s)dW(s), \tag{2.11}$$

*on the set $\{\Theta_n \geq t\}$, for every $n \in \mathbb{N}$, and we say that φ is **stochastically integrable**.*

The consistency of definition (2.11) is ensured. Indeed, let $m \in \mathbb{N}$ s.t. $m > n$. If $\Theta_m \geq t$, then

$$\int_0^t (1_{[0,\Theta_n]}\varphi)(s)dW(s) = \int_0^t \left(1_{[0,\Theta_n]}\left(1_{[0,\Theta_m]}\varphi\right)\right)(s)dW(s).$$

According to Lemma 2.41 we have

$$\int_0^t (1_{[0,\Theta_n]}\varphi)(s)dW(s) = \int_0^{\Theta_n \wedge t} (1_{[0,\Theta_m]}\varphi)(s)dW(s),$$

and since $\Theta_n \geq t$ we get

$$\int_0^t (1_{[0,\Theta_n]}\varphi)(s)dW(s) = \int_0^t (1_{[0,\Theta_m]}\varphi)(s)dW(s),$$

for every $n, m \in \mathbb{N}$ s.t. $m > n$.

2.3.2 Stochastic integral with respect to cylindrical Wiener process

Let W be a cylindrical Wiener process given by (2.5), and let $(H, \langle .,. \rangle_H)$ be a separable Hilbert space. A predictable process φ will be integrable with respect to W if it takes values in $HS(Q_1^{\frac{1}{2}}(U_1), H)$ and

$$\mathbb{P}\left(\int_0^T \|\varphi(s)\|^2_{HS(Q_1^{\frac{1}{2}}(U_1),H)} ds\right) = 1.$$

Lemma 2.43 *Let $(\tilde{e}_k)_{k\in S}$ be the ONB of U_0. Then, $(J\tilde{e}_k)_{k\in S}$ is an ONB of $J(U_0) = Q_1^{\frac{1}{2}}(U_1)$. Further, for all $\varphi \in HS(U_0, H)$ it holds*

$$\|\varphi\|_{HS(U_0,H)} = \|\varphi \circ J^{-1}\|_{HS(Q_1^{\frac{1}{2}}(U_1),H)}.$$

Proof. Let $(\tilde{e}_k)_{k\in S}$ be the ONB of U_0 and let $u, v \in U_0$. Then, owing to the polarization identity (A.14) besides the isometry property of J it holds

$$\langle u_0, v_0\rangle_{U_0} = \frac{1}{4}\left(\|J(u_0+v_0)\|_{Q_1^{\frac{1}{2}}(U_1)} - \|J(u_0-v_0)\|_{Q_1^{\frac{1}{2}}(U_1)}\right) = \langle Ju_0, Jv_0\rangle_{Q_1^{\frac{1}{2}}(U_1)}.$$

Consequently, the orthogonormality of the family $(J\tilde{e}_k)_{k\in S}$ is ensured. To prove that such family is ONB of $J(U_0)$ it remains to prove that is generated it. To do so, let $J(u_0) \in J(U_0)$ where $u_0 := \sum_{n\in S}\langle u_0, \tilde{e}_n\rangle_{U_0}\tilde{e}_n \in U_0$. The linearity of J yields

$$J(u_0) = \sum_{n\in S}\langle u_0, \tilde{e}_n\rangle_{U_0} J(\tilde{e}_n).$$

In addition, let $\varphi \in HS(U_0, H)$ then

$$\|\varphi\|^2_{HS(U_0,H)} = \sum_{k\in S}\|\varphi\tilde{e}_k\|^2_H = \sum_{k\in S}\|\varphi \circ J^{-1}(J\tilde{e}_k)\|^2_H = \|\varphi \circ J^{-1}\|^2_{HS(Q_1^{\frac{1}{2}}(U_1),H)}.$$

Thus, it follows that $\varphi \in HS(U_0, H)$ is equivalent to $\varphi \circ J^{-1} \in HS(Q_1^{\frac{1}{2}}(U_1), H)$. ■

By the help of Lemma 2.43, we are able to introduce the notion of stochastic integral with respect to cylindrical Wiener process as follows

Definition 2.44 *Let $\varphi \in \mathcal{P}_W$. Then the H-valued stochastic integral of φ with respect to W the Q-cylindrical Wiener process is defined by*

$$\int_0^t \varphi(s)dW(s) := \int_0^t \varphi \circ J^{-1}(s)dW(s),\ \forall\, t \in [0,T], \tag{2.12}$$

where the RHS of (2.12) *is understood as the stochastic integral of φ with respect to W the Q_1-Wiener process defined by* (2.11).

Remark 2.45 *All the stochastic integrals introduced in this thesis are called **stochastic Itô integrals** since all have a Wiener process as integrator.*

We close this chapter by an important result; the **Burkholder-Davis-Gundy** inequality, which is very useful in the estimation of higher moments of stochastic integrals.

Theorem 2.46 . *Let $p \geq 2$, and let $T > 0$. For any $0 \leq s < t \leq T$, and any $\varphi \in \mathcal{P}^2_W$, it holds*

$$\|\int_s^t \varphi(r)dW(r)\|^p_{L^p(\Omega,H)} \leq C_p\, \mathbb{E}\left(\int_s^t \|\varphi(r)\|^2_{HS(U_0,H)}dr\right)^{\frac{p}{2}}, \tag{2.13}$$

where $C_p := \left(\frac{p}{2}(p-1)\right)^{\frac{p}{2}}\left(\frac{p}{p-1}\right)^{p(\frac{p}{2}-1)}$.

Proof. See [74, Proposition 2.12, p.24]. ■

There is another version of Burkholder-Davis-Gundy inequality, as the following result proves.

Theorem 2.47 *Let $p \geq 2$, and let $t \geq 0$. For any $\varphi \in \mathcal{P}_W^2$, we have*

$$\mathbb{E}\left(\sup_{s\in[0,t]} |\int_0^t \varphi(s)dW(s)|_H^p\right) \leq c_p \left(\int_0^t \left(\mathbb{E}\|\varphi(s)\|_{HS(U_0,H)}^p\right)^{\frac{2}{p}} ds\right)^{\frac{p}{2}}, \tag{2.14}$$

where $c_p := \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}$.

Proof. See [31, Theorem 4.37, p.114]. ■

Chapter 3

Stochastic differential equations in infinite dimensions

In this chapter, we deal with the stochastic differential equations in infinite dimensions. After some motivation, we introduce in Section 3.2, two important models from physics; heat and Burgers equations in the one dimensional case. We deal in Section 3.3 with the stochastic analog of nonlinear heat and Burgers equations. In Section 3.4, we introduce a class of SPDEs perturbed by a cylindrical Wiener process, namely an abstract parabolic stochastic partial differential equations. As a start point, we give different concepts of solutions for such class of equations. After that, we introduce its wellposedness result in Subsection 3.4.1. Regarding the importance of the results in [12], and its relationship with our results, we cite [12, Theorem 3.1.] in Subsection 3.4.2. We close this chapter by one of the recent developments; the FSPDEs. Precisely, we deal with SPDEs driven by the fractional Laplacian. In Subsection 3.5.1 we deal with the wellposedness and regularity of the fractional stochastic nonlinear heat equation in the random-field setting. Whereas, in Subsection 3.5.2, we deal with the wellposedness of the fractional stochastic Burgers equation in the functional-spaces setting.

3.1 Motivation

Partial differential equation (briefly PDE), is a mathematical equation that involves two or more independent variables, an unknown function, and partial derivatives of such function with respect to the independent variables. It is used to model many phenomena, such as the physical ones. In most cases, the coefficients of such PDE and/or initial data are not known with complete certainty, due to the lack of information and/or incertainty in the measurements. Hence, if we allow for some randomness in those coefficients, we often obtain a more realistic mathematical model of the phenomenon, and the resulting equation becomes a stochastic partial differential equation (briefly SPDE).

3.2 Deterministic heat and Burgers equations

The analysis of the physical phenomena has remained up to the present day one of the fundamental concerns of the development of PDEs. In this section, we restrict ourselves with two examples of PDEs that appear in fundamental applications; the heat and Burgers equations in the one-dimensional case. For more examples and their role in applied sciences we refer to [46].

3.2.1 Heat equation

The heat equation was introduced by Fourier J. in his celebrated memoir " Théorie analytique de la chaleur" (1810-1822). It is a linear parabolic PDE which describes the distribution of heat (or variation in temperature) over time in a given region $D \subset \mathbb{R}^d$. For example, in the time interval $[0, T]$, for $T > 0$ be fixed, and in a one-dimensional rod of length L (i.e., $D := [0, L]$) made of single homogeneous conducting material, the diffusion of the heat over the time from regions of higher temperature to regions of lower temperature without external force, is modelled by the following homogeneous heat equation

$$\frac{\partial u(t,x)}{\partial t} = \nu \Delta u(t,x), \ (t,x) \in (0,T] \times [0,L], \tag{3.1}$$

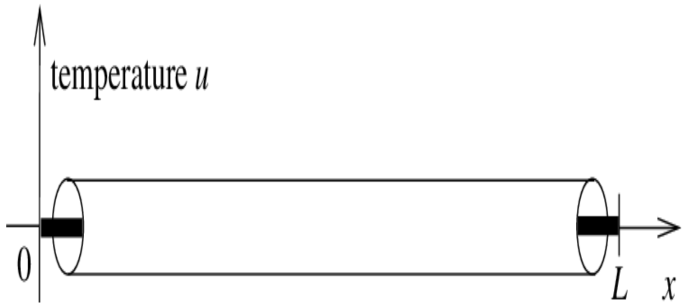
where Δ is the Laplacian and $\nu = \frac{\kappa}{C\rho}$ is called the diffusivity of the rod, with κ is the thermal conductivity, C the thermal capacity and ρ the density of the rod.

To make use of the heat equation, we need more information:

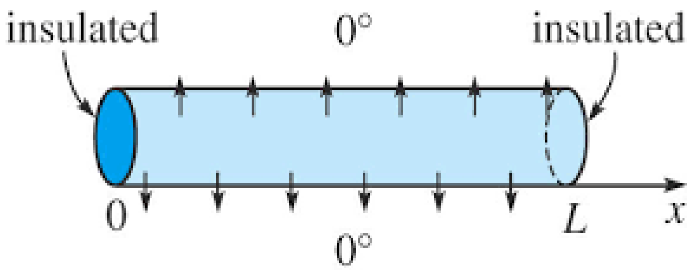
1. The initial temperature distribution in the rod, which described as an **initial condition** $u(0, x) = u_0(x)$.

2. The temperature of the rod is affected by what happens at the ends, $x = 0, L$, which must be specified and described as a **boundary conditions**. Here we consider two simple cases for the boundaries.

   - **Dirichlet type:** represents the temperature prescribed at the boundary. Mathematically, we write $u(t, 0) = u_1(t)$, and $u(t, L) = u_2(t)$ for $t \in (0, T]$ (see the figure taken from the website: math.libretexts.org).



   - **Neumann type:** represents the tempreture which is specified via surrounding medium, for the special case of insulated boundaries; the system is not allowed to have any exchange with the medium, i.e. the heat flow can not pass throughout the boundaries. Mathematically, we write $\frac{\partial u(t,x)}{\partial x}|_{x=0} = \frac{\partial u(t,x)}{\partial x}|_{x=L} = 0$, for $t \in (0, T]$ (see the figure taken from the website: math.upenn.edu).



3.2.2 Forced nonlinear heat equation

In the heat equation Eq.(3.1), the thermel conductivity of the medium is assumed to be constant. In some medium such as gases, this parameter is connected to the temperature of the medium, which allows to arrise a nonlinear heat equation. After adding an external force $B : (0, T] \times [0, L] \to \mathbb{R}$, the homogeneous Dirichlet boundary conditions, and the initial condition, this equation can be written as follows

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} &= \nu \Delta u(t,x) + f(u(t,x)) + B(t,x), \quad \forall\, (t,x) \in (0,T] \times (0,L), \\ u(t,0) &= u(t,L) = 0, \quad \forall\, t \in (0,T], \\ u(0,x) &= u_0(x),\ \forall\, x \in [0,L], \end{cases} \tag{3.2}$$

where $f : \mathbb{R} \to \mathbb{R}$ be a nonlinear globally Lipschitz function.

3.2.3 Forced Burgers equation

If the nonlinear term in Eq.(3.2) is only locally Lipschitz, precisely for $f(u(t,x)) = u(t,x)\dfrac{\partial u(t,x)}{\partial x}$, we obtain the forced Burgers equation, where $u : (0,T] \times [0,L] \to \mathbb{R}$ is the velocity field and $\nu := \frac{\mu}{\rho}$ is the coefficient of kinematic viscosity, with μ and ρ denote respectively the viscosity and the density of the fluid.

Burgers equation was first introduced by Bateman [11] who gave its steady solutions. It was later treated by Burgers [17] as a mathematical model for turbulent fluid motion, later on the equation has been referred to **Burgers equation**. Moreover, it has been found to describe various kinds of phenomena such as the approximation theory of flow through a shock wave traveling in a viscous fluid. It can also be considered as the celebrated Navier-Stokes equation in the one-dimentional case.

3.3 Stochastic nonlinear heat and stochastic Burgers equations

The theory of the SPDEs is very complicated, due to the fact that its study is based on the concept of stochastic integration, which can be treated by different approaches. For example, the martingale approach is based on Walsh's theory [111], where it emphasizes integration with respect to martingale measures, and leads to the concept of random field solutions; which means that, the solution $u = \{u(t,x),\ (t,x) \in (0,T] \times D\}$ is seen as a real-valued function of the two variables (t, x).

To see that, we consider the forcing term as a random perturbation, especially as a space-time white noise which is realized as the weak derivative of the Brownian sheet $\beta(t, x)$ (see Subsection 2.2.3), i.e., $B(t, x) = \dfrac{\partial^2 \beta(t, x)}{\partial t \partial x}$. Then, we can write the stochastic analog of forced nonlinear heat and forced Burgers equations as follows

$$\begin{cases} \dfrac{\partial u(t, x)}{\partial t} & = \nu \Delta u(t, x) + f(u(t, x)) + \dfrac{\partial^2 \beta(t, x)}{\partial t \partial x}, \quad \forall\, (t, x) \in (0, T] \times (0, L), \\ u(t, 0) & = u(t, L) = 0, \quad \forall\, t \in (0, T], \\ u(0, x) & = u_0(x),\ \forall\, x \in [0, L], \end{cases} \tag{3.3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear globally Lipschitz function in the case of heat equation. Whereas, in the case of Burgers equation we have $f(u(t, x)) = u(t, x)\dfrac{\partial u(t, x)}{\partial x}$. The initial condition u_0 is supposed to be $\mathcal{F}_0 \times \mathcal{B}(\mathbb{R})$ measurable.

In otherwise, we can consider the solution as a random function of time, with values in an appropriate (infinite dimensional) functional space. This is called the semigroup approach, which is based on the theory of integration with respect to Hilbert space-valued processes, as expounded in [31]. Here, we consider the Hilbert space $L^2(D)$. Besides, we write $(-\Delta)$ with the homogenuous Dirichlet boundary conditions as an abstract operator A defined from $H_2^2(D) \cap H_0^1(D)$ into $L^2(D)$ (see Subsection 1.2.3). Then, Prb.(3.3) can be rewritten by the following evolution form

$$\begin{cases} \dfrac{du(t)}{dt} & = -\nu A u(t) + F(u(t)) + \dfrac{dW(t)}{dt},\ t \in (0, T], \\ u(0) & = u_0(x),\ \forall\, x \in D, \end{cases} \tag{3.4}$$

where the initial condition u_0 is supposed to be $\mathcal{F}_0$-measurable $L^2(D)$-valued random variable. The term $\frac{dW(t)}{dt}$ is called an **additive noise**, with $(W(t))_{t \in [0,T]}$ be a $L^2(D)$-valued cylindrical Wiener process, and $F : H \to X$ is a nonlinear operator, where $(X, \|.\|_X)$ be a Banach space.

In this case, the solution $u = \{u(t), t \in [0, T]\}$ is an $L^2(D)$-valued stochastic process. Thus, such SPDE can be considered as a SDE in the functional space $L^2(D)$.

3.4 Abstract parabolic stochastic partial differential equations.

In this section, we give by following [90], a short review about the theory of solvability for a class of SPDEs with globally Lipschitz nonlinearities. To do so, let $(H, \langle ., . \rangle_H), |.|_H)$ and

$(U, \langle ., . \rangle_U, |.|_U)$ be two real separable Hilbert spaces, and let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W)$ be a stochastic basis, where $W = (W_t)_{t\in[0,T]}$ be a U-valued cylindrical Wiener process. We consider the following stochastic partial differential equation, perturbed by a multiplicative noise,

$$\begin{cases} \dfrac{du(t)}{dt} & = Au(t) + F(u(t)) + G(u(t))\dfrac{dW(t)}{dt}, \ t \in (0,T], \\ u(0) & = u_0, \end{cases} \tag{3.5}$$

where

- $A : D(A) \subset H \to H$, is in general an unbounded Linear operator (not necessary the Laplacian),
- $F : H \to H$ is $\mathcal{B}(H) \setminus \mathcal{B}(H)$-measurable operator,
- $G : H \to \mathcal{L}(U, H)$ be an operator,
- u_0 be an H-valued, $\mathcal{F}_0$-measurable random variable.

There are different concepts of the solutions, namely strong, weak and mild.

Definition 3.1 ***(Strong solution)***, *[90, Definition G.0.1, p.167]. Let* $u := (u(t))_{t\in[0,T]}$ *be an* $D(A)$*-valued predictable process. We say that* u *is a* ***strong solution*** *of Prb.*(3.5) *if*

- *for all* $t \in [0,T]$, $(0,t) \ni s \mapsto (Au(s) + F(u(s))) \in H$ *is* $\mathbb{P}$*-a.s. Böchner integrable,*
- $G(u) : \Omega_T \to HS(U,H)$ *is continuous predictable s.t.*

$$\mathbb{P}\{\int_0^T \|G(u(s))\|^2_{HS(U,H)} ds < \infty\} = 1, \tag{3.6}$$

- *the following equality holds* $\mathbb{P}$*-a.s. in* H*,*

$$u(t) = u_0 + \int_0^t (Au(s) + F(u(s)))\, ds + \int_0^t G(u(s))\, dW(s), \tag{3.7}$$

  for every $t \in [0,T]$.

Definition 3.2 ***(Weak solution)***, *[90, Definition G.0.5, p.168]. Let* $u := (u(t))_{t\in[0,T]}$ *be an* H*-valued predictable process. We say that* u *is a* ***weak solution*** *of Prb.*(3.5) *if*

- *for all* $t \in [0,T]$, *and all* $\xi \in D(A^*)$, $(0,t) \ni s \mapsto (\langle u(s), A^*\xi\rangle_H + \langle F(u(s)), \xi\rangle_H) \in \mathbb{R}$ *is* $\mathbb{P}$*-a.s. Lebesgue integrable,*

- *the following equality holds* $\mathbb{P}$*-a.s. in* $\mathbb{R}$*,*

$$\begin{aligned}\langle u(t), \xi\rangle_H = \langle u_0, \xi\rangle_H &+ \int_0^t (\langle u(s), A^*\xi\rangle_H + \langle F(u(s)), \xi\rangle_H)\, ds \\ &+ \int_0^t \langle G(u(s))dW(s), \xi\rangle_H,\end{aligned} \tag{3.8}$$

  for every $t \in [0,T]$, *and every* $\xi \in D(A^*)$.

Definition 3.3 ***(Mild solution)***, *[90, Definition 5.1.1, p.115]. Let* $u := (u(t))_{t\in[0,T]}$ *be an* H*-valued predictable process. We say that* u *is a* ***mild solution*** *of Prb.*(3.5) *if*

- *for all* $t \in [0,T]$, $(0,t) \ni s \mapsto S(t-s)F(u(s)) \in H$ *is* $\mathbb{P}$*-a.s. Böchner integrable,*

- *for all* $t \in [0,T]$, $1_{[0,t[}(.)S(t-.)G(u(.)) : \Omega_T \to HS(U,H)$ *is continuous predictable s.t.*

$$\mathbb{P}\{\int_0^T \|1_{[0,t[}(s)S(t-s)G(u(s))\|^2_{HS(U,H)} ds < \infty\} = 1, \tag{3.9}$$

- *the following equality holds in* H, $\mathbb{P}$*-a.s.,*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))\, ds + \int_0^t S(t-s)G(u(s))\, dW(s), \tag{3.10}$$

  for every $t \in [0,T]$, *where* $(S(t))_{t\in[0,T]}$ *is the semigroup generated by the operator* A.

Remark 3.4 *For more results on the relationship between the above concepts of solutions we refer to [90, Appendix G., p.167].*

It is worth noticing here to introduce the notion of the mild solution in a general framework as in [109].

Definition 3.5 *Let* X *be an UMD-Banach space of type 2 and* H *be a Hilbert space. Assume that* $u_0 : \Omega \to X$ *is strongly* $\mathcal{F}_0-$*measurable. A strongly measurable* $\mathcal{F}_t-$*adapted* X*-valued stochastic process,* $(u(t), t \in [0,T])$*, is called a mild solution of Prb.*(3.5) *if*

- *(i) for all* $t \in [0,T]$, $s \mapsto S(t-s)F(u(s))$ *is in* $L^0(\Omega, L^1(0,t;X))$,

- *(ii) for all $t \in [0,T]$, $s \mapsto S(t-s)G(u(s))$ is $H-$strongly measurable $\mathcal{F}_t-$adapted and a.s. in the $\gamma-$Radonifying space; $R(H,X)$,*

- *(iii) $\forall t \in [0,T]$, the equality (3.10) holds in X, $\mathbb{P}-a.s.$*

The following defintion, gives us the meaning of the unicity of the solution.

Definition 3.6 *We say that a pathwise uniqueness holds for Prb.(3.5) if for any two solutions $(u^1(t))_{t\in[0,T]}$ and $(u^2(t))_{t\in[0,T]}$ starting from the same initial data u_0, we have*

$$\mathbb{P}\left(u^1(t) = u^2(t), \ \ \forall\, t \in [0,T]\right) = 1. \tag{3.11}$$

Next, we give an important result in order to understand the stochastic term in the RHS of Eq.(3.10), but first we have to introduce the so-called stochastic convolution.

Definition 3.7 ***(Stochastic convolution).*** *Let $\varphi := (\varphi(t))_{t\in[0,T]}$ be an $\mathcal{L}(U,H)$-valued predictable process and let $(S(t))_{t\in[0,T]}$ be the strongly continuous semigroup generated by the operator A s.t. the following integral*

$$W_S^\varphi(t) := \int_0^t S(t-s)\varphi(s)dW(s), \quad \textit{for every}\, t \in [0,T],$$

*is well defined. Then, the process $W_S^\varphi := (W_S^\varphi(t))_{t\in[0,T]}$ is called **stochastic convolution**.*

Theorem 3.8 ***(factorization-method)****. Let φ be an $\mathcal{L}(U,H)$-valued predictable process and let $(S(t))_{t\in[0,T]}$ be the strongly continuous semigroup generated by the operator A. If for some $\nu \in (0,1)$,*

$$S(t-s)\varphi(s) \textit{ is } HS(U,H) - \textit{valued for all}\, t \in [0,T],$$

and

$$\int_0^t (t-s)^{\nu-1}\left(\int_0^s (s-r)^{-2\nu}\mathbb{E}\left(\|S(t-r)\varphi(r)\|^2_{HS(U,H)}\right)dr\right)^{\frac{1}{2}} ds \ < +\infty.$$

Then,

$$W_S^\varphi(t) = \frac{\sin \nu\pi}{\pi}\int_0^t (t-s)^{\nu-1}S(t-s)Y_\nu(s)ds, \ t \in [0,T],$$

where,

$$Y_\nu(t) = \int_0^t (t-s)^{-\nu}S(t-s)\varphi(s)dW(s), \ t \in [0,T].$$

Proof. See [31, Theorem 5.10, p.130] and [16]. ■

3.4.1 Wellposedness of the problem

In this subsection, we introduce the result concerning the existence and the uniqueness of the mild solution of Prb.(3.5), for more details see [90, Section. 5.1]. First, we need to give the precise formulations of the assumptions which are sufficient for the wellposedness of the problem.

Assumption $\mathcal{H}_A$. (Linear operator A). $A : D(A) \subset H \to H$ is an infinitesimal generator of a C_0-semigroup $(S(t))_{t\in[0,T]}$ on H.

Assumption $\mathcal{H}_F$. (Drift term F). The nonlinear operator $F : H \to H$ satisfies, there exists a constants $C_F > 0$ s.t.

$$|F(x) - F(y)|_H \leq C_F|x - y|_H, \ \forall \, x, y \in H.$$

Assumption $\mathcal{H}_G$. (Diffusion term G). $G : H \to \mathcal{L}(U, H)$ is strongly measurable. Moreover, for any $t \in (0, T]$ and $x \in H$, it holds

$$S(t)G(x) \in HS(U, H)$$

and there exists a square integrable mapping $K : [0, T] \to [0, \infty)$ s.t.

$$\|S(t)G(x)\|_{HS(U,H)} \leq K(t)(1 + |x|_H) \ \text{ and } \ \|S(t)(G(x) - G(y))\|_{HS(U,H)} \leq K(t)|x - y|_H,$$

for all $x, y \in H$ and all $t \in (0, T]$.

Assumption $\mathcal{H}_{u_0}$. (Initial condition u_0). Let $p \geq 2$. The initial condition u_0 is an element in the space $L^p(\Omega, \mathcal{F}_0, \mathbb{P}; H)$.

We need also to introduce the space where the mild solution is living.

Definition 3.9 *Let $T > 0$ and $p \geq 2$. We denote by $\mathcal{H}_T^p(H)$ the space of all H-valued processes v which have a predictable version, equiped with the norm*

$$\|v\|_{\mathcal{H}_T^p(H)} := \sup_{t\in[0,T]} \|v(t)\|_{L^p(\Omega,H)} < \infty.$$

Theorem 3.10 *Assume that the Assumptions $\mathcal{H}_A$, $\mathcal{H}_F$, $\mathcal{H}_G$ and $\mathcal{H}_{u_0}$ are fulfilled. Then, there exists a unique mild solution $(u(t))_{t\in[0,T]}$ of Prb.(3.5). Moreover, there exists a constant $C_{T,p} > 0$ independent of u_0 s.t.*

$$\|u\|_{\mathcal{H}_T^p(H)} \leq C_{T,p}(1 + \|u_0\|_{L^p(\Omega,\mathcal{F}_0,\mathbb{P};H)}).$$

Proof. See [90, Theorem 5.1.3, p.118]. ■

3.4.2 Blömker et al.'s Theorem

Theorem 3.10 ensures the wellposedness of the stochastic nonlinear heat equation introduced in Section 3.3. However, it does not cover the stochastic Burgers equation, for this reason, and regarding the importance of the results in [12], we cite [12, Theorem 3.1.] in this subsection.

Theorem 3.11 . *Let T be fixed, V, U be two $\mathbb{R}-$Banach spaces and let $P_N : V \to V$ be a sequence of linear bounded operators. Assume that the following assumptions are fulfilled:*

- ***Assumption 1.*** *Let $S : (0, T] \to \mathcal{L}(U, V)$ be a continuous map satisfying*

$$\sup_{t\in(0,T]} \Big(t^{\alpha} |S(t)|_{\mathcal{L}(U,V)} \Big) < \infty, \tag{3.12}$$

$$\sup_{N\in\mathbb{N}} \sup_{t\in(0,T]} \Big(N^{\gamma} t^{\alpha} |S(t) - P_N S(t)|_{\mathcal{L}(U,V)} \Big) < \infty, \tag{3.13}$$

where $\alpha \in [0, 1)$ and $\gamma \in (0, \infty)$ are given constants.

- ***Assumption 2.*** *Let $F : V \to U$ be a mapping which satisfies*

$$\sup_{|u|_V,\, |v|_V \leq r,\ u\neq v} \frac{|F(u) - F(v)|_U}{|u - v|_V} < \infty. \tag{3.14}$$

- ***Assumption 3.*** *Let $O : [0, T] \times \Omega \to V$ be a stochastic process with continuous sample paths and*

$$\sup_{N\in\mathbb{N}} \sup_{t\in(0,T]} \Big(N^{\gamma} |(I - P_N) O_t(\omega)|_V \Big) < \infty, \ \ \textit{for every}\, \omega, \tag{3.15}$$

where $\gamma \in (0, \infty)$ is given in Assumption 1.

- ***Assumption 4.*** *Let $X^N : [0, T] \times \Omega \to V, N \in \mathbb{N}$ be a sequence of stochastic processes with continuous sample paths and with*

$$\sup_{N\in\mathbb{N}} \sup_{t\in(0,T]} \Big(|X_t^N(\omega)|_V \Big) < \infty, \tag{3.16}$$

$$X_t^N(\omega) = \int_0^t P_N S(t - s) F(X_s^N(\omega)) ds + P_N(O_t(\omega)), \tag{3.17}$$

for every $\omega \in \Omega$, $t \in [0, T]$ and every $N \in \mathbb{N}$.

Then there exists a unique stochastic process $X : [0,T] \times \Omega \to V$*, with continuous sample paths s.t.*

$$X_t(\omega) = \int_0^t S(t-s)F(X_s(\omega))ds + O_t(\omega), \tag{3.18}$$

for every $\omega \in \Omega, \quad t \in [0,T]$*. Moreover, there exists a* $\mathcal{F}/\mathcal{B}(0,\infty)-$*measurable mapping* $C : \Omega \to [0,\infty)$*, s.t.*

$$\sup_{t\in(0,T]} |X_t(\omega) - X_t^N(\omega)|_V \leq C(\omega).N^{-\gamma}, \tag{3.19}$$

holds for every $N \in \mathbb{N}$ *and every* $\omega \in \Omega$*, where* γ *is given in Assumption 1.*

Proof. See [12, Theorem 3.1]. ■

Remark 3.12 *The random variable* $C(\omega)$ *in Est.*(3.19) *is given by, see [12, Identity (5.7)]:*

$$C(\omega) := 2(R(\omega))^4 E_{(1-\alpha)}\Big(TR(\omega)Z(\omega)\Gamma(1-\alpha))^{\frac{1}{(1-\alpha)}}\Big), \tag{3.20}$$

where for $r \in (0,\infty)$*, the function* $E_r : [0,\infty) \to [0,\infty)$*, is defined by* $E_r(x) := \sum_{n=0}^{\infty} \frac{x^{rn}}{\Gamma(nr+1)}$*,*

$$\begin{aligned} R(\omega) \;&:=\; \sup_{N\in\mathbb{N}_0}\sup_{0\leq t\leq T} |F(X_t^N(\omega))|_U + T + \sup_{N\in\mathbb{N}_0}\sup_{0\leq t\leq T}\Big(N^\gamma |(I-P_N)O_t(\omega)|_V\Big) \\ &+\; \frac{1}{1-\alpha}\sup_{N\in\mathbb{N}_0}\sup_{0\leq t\leq T}\Big(t^\alpha |S(t)|_{\mathcal{L}(U,V)}\Big) \\ &+\; \sup_{N\in\mathbb{N}_0}\sup_{0\leq t\leq T}\Big(N^\gamma t^\alpha |S(t) - P_N S(t)|_{\mathcal{L}(U,V)}\Big), \end{aligned}$$

$$Z(\omega) := L(\sup_{N\in\mathbb{N}_0}\sup_{0\leq t\leq T} |X_t^N(\omega)|_V),$$

$$L(r) := \sup\{\frac{|F(u)-F(v)|_U}{|u-v|_V} : |u|_V,\ |v|_V \leq\ r, \quad u \neq v\}.$$

3.5 Fractional stochastic nonlinear heat and fractional stochastic Burgers equations

In the recent years, the fractional calculus and the stochastic analysis have been used simultaneously in order to describe complex processes, such as anomalous diffusions, e.g. diffusion in disordred or fractal medium. By this, it was created a new powerful topic in the applied mathematics which is represented by the so-called **fractional stochastic partial differential equations**. There is a large volume of literature available about this topic, see for a short list [15, 16, 26, 28, 34, 35, 36, 37, 38, 39] and the references therein. In this section, we deal with the fractional stochastic nonlinear heat equation and the fractional stochastic Burgers equation. To do this, we fix $T > 0$, $\alpha \in (1, 2]$, $p \geq 2$. Let $H := L^2(0, 1)$ and let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered complete probability space.

3.5.1 Wellposedness of the fractional stochastic nonlinear heat equation

As is well known, the nature of the anomalous diffusions needs nonlocal differentional operators instead of the classical ones, like the Laplacian. In this subsection, we go one level up in difficulty by replacing $(-\Delta)$ in the stochastic nonlinear heat equation (3.3) by its fractional counterpart $(-\Delta)^{\frac{\alpha}{2}}$ defined on the whole space $\mathbb{R}$ (see Definition 1.15). Moreover, we suppose that the equation is perturbed by a multiplicative noise instead of the additive noise. Then, we write the fractional stochastic nonlinear heat equation as follows

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} & = -(-\Delta)^{\frac{\alpha}{2}} u(t,x) + f(u(t,x)) + g(u(t,x)) \dfrac{\partial^2 W(t,x)}{\partial t \, \partial x}, \ t \in (0,T], \ x \in \mathbb{R}, \\ u(0,x) & = u_0(x), \ \forall \ x \in \mathbb{R}, \end{cases} \tag{3.21}$$

where $g : \mathbb{R} \to \mathbb{R}$, and $W = \{W(t,x), (t,x) \in [0,T] \times \mathbb{R}\}$ be a space-time white noise.
To recall the results of the wellposedness of Prb.(3.21); in the random field setting we follow [39], and in the L^2-setting we follow [37], but first we need the following assumptions:
Assumption $\mathcal{B}$. For all $T > 0$, $\exists\, C_T^f > 0$ s.t. for all $t \in [0,T]$ and all $x, y \in \mathbb{R}$

$$|f(x) - f(y)|_{\mathbb{R}} \leq C_T^f |x - y|_{\mathbb{R}}, \tag{3.22}$$

and

$$|f(x)|_{\mathbb{R}} \leq C_T^f \left(1 + |x|_{\mathbb{R}}\right). \tag{3.23}$$

Assumption $\mathcal{C}$. For all $T > 0$, $\exists\, C_T^g > 0$ s.t. for all $t \in [0,T]$ and all $x, y \in \mathbb{R}$

$$|g(x) - g(y)|_{\mathbb{R}} \leq C_T^g |x - y|_{\mathbb{R}}, \tag{3.24}$$

and

$$|g(x)|_{\mathbb{R}} \leq C_T^g \left(1 + |x|_{\mathbb{R}}\right). \tag{3.25}$$

Wellposedness and regularity of Prb.(3.21) **on the random-field setting**

Definition 3.13 *Fix $T > 0$, and let $u := \{u(t,x), t \in [0,T], x \in \mathbb{R}\}$ be a measurable stochastic field. We say that, u is a* ***mild solution*** *of Prb.*(3.21) *on $[0,T]$, if for all $x \in \mathbb{R}$, the process $(u(t,x))_{t\in[0,T]}$ is $\mathcal{F}_t$-adapted and if u satisfies the following integral equation*

$$\begin{aligned} u(t,x) &= \int_{-\infty}^{+\infty} G_\alpha(t,x-y)u_0(y)dy + \int_0^t \int_{-\infty}^{+\infty} f(u(s,y))G_\alpha(t-s,x-y)dyds \\ &+ \int_0^t \int_{-\infty}^{+\infty} g(u(s,y))G_\alpha(t-s,x-y)W(dyds), \end{aligned} \tag{3.26}$$

for all $t \in [0,T]$ and all $x \in \mathbb{R}$, where $G_\alpha(t,x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\xi x - t|\xi|^\alpha} d\xi$ is the Green function associated to Prb.(3.21).

The field $\{u(t,x), t \in [0,T], x \in \mathbb{R}\}$ is called a **global mild solution** of Prb.(3.21) if, for all $T > 0$, it is a mild solution of Prb.(3.21) on the interval $[0,T]$.

Definition 3.14 *We say that, the Prb.*(3.21) *has a* ***unique*** *mild solution on $[0,T]$, if for any two solutions u^1 and u^2 starting from the same initial data u_0, it holds*

$$\mathbb{P}\Big(u^1(t,x) = u^2(t,x),\ \ \forall\, t \in [0,T],\ \ \forall\ x \in \mathbb{R}\Big) = 1. \tag{3.27}$$

Theorem 3.15 *Fix $T > 0$, $\alpha \in (1,2]$ and $p \geq 2$. Under Assumptions $\mathcal{B}$, $\mathcal{C}$ and the assumption that the initial condition u_0 satisfies $\sup_{x\in\mathbb{R}} \|u_0(x)\|_{L^p(\Omega,\mathbb{R})} < \infty$, Prb.*(3.21) *has an unique global solution $u = \{u(t,x), t \in [0,T], x \in \mathbb{R}\}$, satisfies*

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} \|u(t,x)\|_{L^p(\Omega,\mathbb{R})} < \infty. \tag{3.28}$$

Proof. See [39, Theorem 1.]. ■

The following result not only ensures the temporal regularity of the mild solution, but the spacial regularity as well.

Theorem 3.16 . *Under the assumptions of Theorem 3.15, it holds*

- *For fixed $x \in \mathbb{R}$, the process $\{u(t,x), t \in [0,T]\}$ has Hölder continuous trajectories with exponent $\frac{\alpha-1}{2\alpha} - \kappa$, for any $\kappa > 0$, $\mathbb{P}$-a.s.*

- *For fixed $t \in [0,T]$, the process $\{u(t,x), x \in \mathbb{R}\}$ has Hölder continuous trajectories with exponent $\frac{\alpha-1}{2} - \kappa$, for any $\kappa > 0$, $\mathbb{P}$-a.s.*

Proof. See [39, Theorem 2.]. ■

Wellposedness of Prb.(3.21) **on the L^2-setting**

It is well known that, the random valued solution is not equivalent to the function-valued solution. However, Debbi L. in [37], has proved under the same conditions (i.e. Assumptions $\mathcal{B}$ and $\mathcal{C}$) that, the mild solution given in Definition 3.13 coincides with an L^2-solution has the following definition.

Definition 3.17 *Fix $T > 0$, and let $u := \{u(t,.), t \in [0,T]\}$ be a L^2-valued $\mathcal{F}_t$-adapted stochastic process. We say that, u is a* ***mild solution*** *of Prb.*(3.21) *on* $[0,T]$, *if it satisfies the following integral equation*

$$\begin{aligned} u(t,.) &= \int_{-\infty}^{+\infty} G_\alpha(t,.-y)u_0(y)dy + \int_0^t \int_{-\infty}^{+\infty} f(u(s,y))G_\alpha(t-s,.-y)dyds \\ &+ \int_0^t \int_{-\infty}^{+\infty} g(u(s,y))G_\alpha(t-s,.-y)W(dyds), \end{aligned} \tag{3.29}$$

for all $t \in [0,T]$, where the equality (3.29) *is taken in the* $L^2(\mathbb{R})$.

Theorem 3.18 *Fix $T > 0$, $\alpha \in (1,2]$ and $p \geq 1$. Under Assumptions $\mathcal{B}$, $\mathcal{C}$ and the assumption that $u_0 \in L^p(\Omega, \mathcal{F}_0, L^2)$, Prb.*(3.21) *has an unique mild solution $u = \{u(t,.), t \in [0,T]\}$ satisfies*

$$\sup_{t\in[0,T]} \|u(t)\|^p_{L^p(\Omega,L^2)} < \infty, \tag{3.30}$$

where the uniquness is taken with respect to the norm in the LHS of (3.30).

Proof. See [37, Theorem 1.]. ■

Besides the L^2-solution given by Definition 3.17, there are also another kind of solutions of variational type of Prb.(3.21); weak solution of the first kind and weak solution of the second kind, which are equivalent as the next theorem proves.

Definition 3.19 *Let $u := \{u(t,.), t \in [0,T]\}$ be a L^2-valued $\mathcal{F}_t$-adapted stochastic process. We say that, u is a* ***weak solution of the first kind*** *of Prb.*(3.21) *on* $[0,T]$, *if it satisfies the condition* (3.30), *and the following integral equation*

$$\begin{aligned} \int_{-\infty}^{+\infty} u(t,x)\varphi(x)dx &= \int_{-\infty}^{+\infty} u_0(x)\varphi(x)dx + \int_0^t \int_{-\infty}^{+\infty} u(s,x)A_\alpha\varphi(x)dxds \\ &+ \int_0^t \int_{-\infty}^{+\infty} f(u(s,x))\varphi(x)dxds \\ &+ \int_0^t \int_{-\infty}^{+\infty} g(u(s,x))\varphi(x)W(dxds), \quad a.s. \end{aligned} \tag{3.31}$$

for all $t \in [0,T]$, and all $\varphi \in C_0^\infty$.

Definition 3.20 *Let $u := \{u(t,.), t \in [0,T]\}$ be a L^2-valued $\mathcal{F}_t$-adapted stochastic process. We say that, u is a* ***weak solution of the second kind*** *of Prb.*(3.21) *on* $[0,T]$ *if it satisfies the condition* (3.30)*, and the following integral equation*

$$\begin{aligned}\int_{-\infty}^{+\infty} u(t,x)\Phi(t,x)dx &= \int_{-\infty}^{+\infty} u_0(x)\Phi(0,x)dx + \int_0^t \int_{-\infty}^{+\infty} u(s,x)\partial_s\Phi(s,x)dxds \\ &+ \int_0^t \int_{-\infty}^{+\infty} u(s,x)A_\alpha\Phi(s,x)dxds \\ &+ \int_0^t \int_{-\infty}^{+\infty} f(u(s,x))\Phi(s,x)dxds \\ &+ \int_0^t \int_{-\infty}^{+\infty} g(u(s,x))\Phi(s,x)W(dxds), \quad a.s. \end{aligned} \tag{3.32}$$

for all $t \in [0,T]$, and all $\Phi \in C^{1,\infty}((0,t)\times\mathbb{R})$ s.t. $\Phi(s,.) \in D(A_\alpha)$, for all $s < t$.

Theorem 3.21 *For $p \geq 2$, the L^2-solution, the weak solution of the first kind and the weak solution of the second kind are equivalent.*

Proof. See [37, Theorem 2.]. ■

3.5.2 Wellposedness of the fractional stochastic Burgers equation

The stochastic Burgers equation (3.5), for the special case $A = -\Delta$ on $D := (0,1)$ with homogeneous Dirichlet boundary conditions, and $F(v) = v\frac{dv}{dx}$, for any $v \in L^2(D)$, is used to model many phenomena in different fields, such as fluid dynamics, hydrodynamics, cosmology and astrophysics (see for a short list [3, 30, 82] and the references therein). Moreover, the fractional Burgers equation can model some anomalous diffusions, like the long time behavior of the acoustic waves propagating in a gas-filled tube and the wave propagation in viscoelastic medium. Then, if we replace $(-\Delta)$ by its fractional counterpart $A_\alpha := (-\Delta)^{\frac{\alpha}{2}}$ (see Definition 1.69), we obtain the following fractional stochastic Burgers equation,

$$\begin{cases} \dfrac{du(t)}{dt} &= -A_\alpha u(t) + F(u(t)) + G(u(t))\,\dfrac{dW(t)}{dt},\ t \in (0,T], \\ u(0) &= u_0, \end{cases} \tag{3.33}$$

where G is a nonlinear operator from $L^2(D)$ to $\mathcal{L}(L^2(D))$ defined by $(G(u)(v))(x) = G(u(x))v(x)$, for all $u, v \in L^2(D)$ and all $x \in D$, which is a Nemytski map associated with the bounded Lipschitz continuous function $G : \mathbb{R} \to \mathbb{R}$, and $W := (W(t))_{t\in[0,T]}$ be an $L^2(D)$-valued cylindrical Wiener process on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

We are motivated here, by the tri-interaction between the wave steepening given by the nonlinearity, and by the small dissipation given by fractional power of the Laplacian $\frac{\alpha}{2}$ and by the irregular random perturbation given by the stochastic noise.

The following theorem ensures the wellposedness of Prb.(3.33).

Theorem 3.22 *Fix $T > 0$, $\alpha \in (\frac{3}{2}, 2)$ and $p > \frac{2\alpha}{\alpha-1}$. Let u_0 be an $\mathcal{F}_0$-measurable random variable satisfying $u_0 \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(D))$. Then, Prb.*(3.33) *admits an unique mild solution $(u(t))_{t\in[0,T]}$ in $L^2(D)$ s.t.*

$$\mathbb{E}\left(\sup_{t\in[0,T]} |u(t)|^p_{L^2}\right) < \infty, \tag{3.34}$$

where the unicity here is understood in the sense of Definition 3.6.

Proof. See [16, Theorem 1.3]. ■

Let us mention that, the proof of Theorem 3.22 is based on different useful results, among them Lemma 2.11 in the same paper [16], and regarding its importance in this thesis, we recall it here.

Lemma 3.23 *[16, Lemma 2.11]. For each $\alpha > \frac{3}{2}$, there exists a constant $C_\alpha > 0$ s.t. for all $t > 0$ and for any bounded and strongly-measurable function $v : (0,t) \to L^1(0,1)$ the following inequality holds*

$$\int_0^t |e^{-A^{\frac{\alpha}{2}}(t-s)} \frac{\partial v(s)}{\partial x}|_{L^2} ds \le C_\alpha t^{1-\frac{3}{2}\alpha} \sup_{s\le t} |v(s)|_{L^1}. \tag{3.35}$$

Chapter 4

Introduction to deterministic and stochastic numerical approximations

4.1 Introduction

The real-world is defined by structure in space and time, it forever changes in complex ways and the most of its problems which can be modelled as SPDEs are complicated. Generally, there is no hope to find their analytical (or exact) solutions, and so to understand their behavior we need the approximations of such solutions instead of the exact ones, which can be implemented via computer simulations. To obtain such approximations we use numerical techniques, like the finite difference method and Galerkin method.

The finite difference method has been used centuries ago, long before computers were available. It is the one of the oldest methods to solve the differential equations, it is used in the work of Euler L., which has been published in the book *Institutionum calculi integralis* in 1768. The analysis of the finite difference method for PDEs has arisen by two works; the early work of Richardson L. F. in 1910 [89], and the work of Courant, Friedrichs and Lewy in 1928 [24]. Such method is based on the taylor development, it has as advantages a low cost of calculations and a great simplicity of writing. But to apply it, we need a high regularity of the solution, which is in general not realistic. Moreover, such method is simple to use for problems defined on regular geometries. Then, for more complicated geometry, it is preferable to use more adapted techniques, like Galerkin method, which is one of the most popular numerical techniques for solving PDEs whith complex geometries. The first mathematical formulation of Galerkin method appeared at the end of 1915 by Galerkin B. in [49]. Modern numerical methods are based on the development of principle ideas around Galerkin method, such as spectral Galerkin method that has been widely used because of its features.

In this chapter, we deal with the full approximation of the parabolic SPDE (3.5) introduced in Section 3.4, for A be the Laplacian endowed with the homogeneous Dirichlet boundary conditions on the spacial domain $D = (0,1)$. In Section 4.2, we treat the full approximation of its deterministic version (i.e. $G = 0$), by using finite difference method for the temporal approximation and spectral Galerkin method for the spacial approximation. In Section 4.3, we use the same methods to approximate spacially and temporally the stochastic term of Eq.(3.5), after that we introduce a different notions of convergence in the probability context. We close this section by showing important results concerning the spacial approximation of the fractional stochastic heat equation.

4.2 Full approximation of the deterministic problem

In this section we deal with the full aproximation of the deterministic version of Prb.(3.5) (i.e., $G = 0$), let us write it here

$$\begin{cases} \dfrac{du(t)}{dt} = \Delta u(t) + F(u(t)), \ \forall\, t \in \ (0,T], \\ u(0) = u_0. \end{cases} \tag{4.1}$$

4.2.1 Spacial approximation via spectral Galerkin method

Due to the fact that, Galerkin method allows greater flexibility in spacial discretization than finite difference method, we use its approach spectral Galerkin method to discretize spacially Prb.(4.1). Such problem represents the so-called **strong form**, which requires that the equation be satisfied point-by-point in the computational domain D. Starting from the strong form, we define the **weak form** as follows.

Definition 4.1 ***(Weak form).*** *Let $V := H_0^1(D) = \{v \in H^1(D) \mid v_{|\partial D} = 0\}$. The* ***weak form*** *of Prb.(4.1) is defined by; for a given $u_0 \in L^2(D)$, find $u \in L^2(0,T;V) \cap C([0,T];L^2(D))$ s.t.*

$$\begin{cases} \frac{d}{dt}\langle u(t), v\rangle = \mathcal{A}(u(t), v) + \langle F(u), v\rangle; \forall\, v \in V, \\ u(0) = u_0, \end{cases} \tag{4.2}$$

where $\langle .,. \rangle$ denotes the scalar product in $L^2(D)$ and $\mathcal{A}(.,.)$ be a bilinear form definded on V by

$$\mathcal{A}(v,w) := \langle \nabla\, v, \nabla\, w\rangle. \tag{4.3}$$

Remark 4.2 *The weak form is also called* ***variational formulation*** *because the function v varies in the space V, where v is called* ***test function****, since it tests the equation that is satisfied by u.*

The weak formulation represents a powerful techniqhe in the process of approximating spacially the PDEs, because we directly can approximate the space V by a finite dimensional space. Indeed, we establish the spectral Galerkin method by taking the finite dimensional subspace $V_N := span\{e_1, ..., e_N\} \subset V$ for some integer $N \geq 1$, where $(e_n)_{n=1}^N$ are the N first eigenvectors of the Laplacian corresponding the eigenvalues $-\lambda_n = -(n\pi)^2$, $n = 1, ..., N$. Then, we seek $u_N = \sum_{n=1}^N \hat{u}_n e_n \in V_N$, where $\hat{u}_n := \langle u_N, e_n\rangle$, satisfying

$$\begin{cases} \frac{d}{dt}\langle u_N, v_N\rangle = \mathcal{A}(u_N, v_N) + \langle F(u_N), v_N\rangle; \quad \forall\, v_N \in V_N, \\ u_N(0) = u_{0,N}. \end{cases} \tag{4.4}$$

Equivalently, we can write Prb.(4.4) as follows

$$\begin{cases} \frac{d}{dt}\langle u_N, e_n\rangle = \mathcal{A}(u_N, e_n) + \langle F(u_N), e_n\rangle, \quad \forall\, n \in \{1, ..., N\}, \\ u_N(0) = u_{0,N}. \end{cases} \tag{4.5}$$

By using the Galerkin projection $P_N : L^2(D) \to V_N$ definded by $P_N v := \sum_{n=1}^N \langle v, e_n\rangle e_n$, the spacial discrete version of Prb.(4.1) is given by the following evolution form

$$\begin{cases} \frac{d}{dt}u_N(t) = \Delta_N u_N(t) + P_N F(u_N(t)), \;\; \forall\, t \in (0,T), \\ u_N(0) = P_N u_0, \end{cases} \tag{4.6}$$

where $\Delta_N := P_N \Delta$.

Since we have, $\mathcal{A}(u_N, e_n) = \langle \nabla u_N, \nabla e_n \rangle = \langle u_N, \Delta e_n \rangle = -\lambda_n \langle u_N, e_n \rangle = -\lambda_n \hat{u}_n$, and by denoting $\langle F(u_N), e_n \rangle =: \hat{F}_n(u_N)$, we can also rewrite Prb.(4.5) as follows

$$\frac{d}{dt}\hat{u}_n = -\lambda_n \hat{u}_n + \hat{F}_n(u_N), \quad \text{with} \quad \hat{u}_n(0) = \langle P_N u_0, e_n \rangle, \tag{4.7}$$

for any $n \in \{1, ..., N\}$. If we denote $\hat{U}_N := (\hat{u}_n)_{n=1}^N$, Prb.(4.7) gives the following system of ODEs

$$\begin{cases} \frac{d}{dt}\hat{U}_N = \mathcal{M}\, \hat{U}_N + \hat{F}_N(\hat{U}_N), \\ \hat{U}_N(0) = (\hat{u}_n(0))_{n=1}^N, \end{cases} \tag{4.8}$$

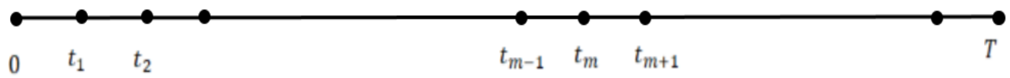
where $\hat{F}_N(\hat{U}_N) := (\hat{F}_n(u_N))_{n=1}^N$ and $\mathcal{M} := (m_{nn})_{n=1}^N$ is a diagonal matrix such that $m_{nn} = -\lambda_n$.

4.2.2 Temporal approximation via Finite difference method

To obtain the full approximation of Prb.(4.1), we discretize the Prb.(4.6) for $u_N(t)$ in time by using the finite difference method. Such method is based on the Taylor expansions of the function u_N around a given time t. To do this, we first fix $M \in \mathbb{N}_0$, and let $\Delta t = \frac{T}{M}$ be the time **step size** (or **step mesh**). Moreover, let the set $\mathbb{S}$ of all **grid points** (or **nodes**), i.e.,

$$\mathbb{S} := \{t_m, \text{ s.t. } t_m = m\Delta t, \ m \in \{0, 1, ..., M\}\},$$

which called the **grid**.



Remark 4.3 *The grid points defined above are equidistant, but we can define points which are unequal spacings, where different values of Δt between each successive pairs of nodes are used. Often problems are solved on a grid which involves uniform spacing, because this simplifies the programming.*

Now, we assume that u_N is of class C^2 with respect to t. Then, for $x \in D$ be fixed, we can express the first derivative by different ways as follows

$$
\begin{aligned}
\frac{du_N(t,x)}{dt} &= \frac{u_N(t+\Delta t,x)-u_N(t,x)}{\Delta t}+\mathcal{O}(\Delta t),\\
&= \frac{u_N(t,x)-u_N(t-\Delta t,x)}{\Delta t}+\mathcal{O}(\Delta t),
\end{aligned} \tag{4.9}
$$

Let us denote the approximate value of u_N at the grid point t_m by $u_{N,M}^m$.
Now, by neglecting the remains in (4.9) we obtain the following commonly used finite differences.

Definition 4.4 ***(Finite differences).*** *The following finite quotients:*

$$
\frac{u_{N,M}^{m+1}-u_{N,M}^{m}}{\Delta t}, \tag{4.10}
$$

and

$$
\frac{u_{N,M}^{m}-u_{N,M}^{m-1}}{\Delta t}, \tag{4.11}
$$

*are called **forward-time** difference and **backward-time** difference respectively.*

Now, to achieve the full approximation of Prb.(4.1), we use the finite differences defined above to formulate some popular difference schemes.

Explicit scheme

In order to obtain the **explicit scheme**, we replace the first time derivative in Eq.(4.6) by the forward-time difference (4.10). Then, we have

$$
\begin{cases}
\dfrac{u_{N,M}^{m+1}-u_{N,M}^{m}}{\Delta t} = \Delta_N u_{N,M}^m + P_N F(u_{N,M}^m), \quad m=0,...,M-1,\\
u_{N,M}^0 = P_N u_0.
\end{cases} \tag{4.12}
$$

This difference equation can be written as

$$
\begin{cases}
u_{N,M}^{m+1} = (I+\Delta t\Delta_N)u_{N,M}^m + \Delta t P_N F(u_{N,M}^m), \ m=0,...,M-1,\\
u_{N,M}^0 = P_N u_0.
\end{cases} \tag{4.13}
$$

Thus, the unknown $u_{N,M}^{m+1}$ at the time level t_{m+1} is obtained directly from the known value $u_{N,M}^m$ at the previous time level t_m, for this reason it is called explicit.
Such scheme is easy and simple in computation, besides it has low cost of calculations. However, it is conditionally stable, for this reason it is not preferable in the temporal approximations.

Semi-implicit scheme

To get the **semi-implicit scheme**, we replace the first time derivative in Eq.(4.6) by the backward-time difference (4.11). Hence, we obtain

$$\begin{cases} \dfrac{u^m_{N,M} - u^{m-1}_{N,M}}{\Delta t} = \Delta_N u^m_{N,M} + P_N F(u^{m-1}_{N,M}), \quad m = 1, ..., M, \\ u^0_{N,M} = P_N u_0. \end{cases} \tag{4.14}$$

We can write such difference equation in the following form

$$\begin{cases} (I - \Delta t \Delta_N) u^m_{N,M} = u^{m-1}_{N,M} + \Delta t P_N F(u^{m-1}_{N,M}), \; m = 1, ..., M, \\ u^0_{N,M} = P_N u_0. \end{cases} \tag{4.15}$$

This equation is useful for proving convergence and examining theoretical properties of the numerical method. However, we can not use it for the implementation, for this it is convenient to write its matrix form. To this end, starting from the Prb.(4.8), and by following the same steps above, we get

$$\begin{cases} (I - \Delta t \mathcal{M}) \hat{U}^m_{N,M} = \hat{U}^{m-1}_{N,M} + \Delta t \hat{F}_N(\hat{U}^{m-1}_{N,M}), \; m = 1, ..., M, \\ \hat{U}^0_{N,M} = (\langle P_N u_0, e_n \rangle)^N_{n=1}, \end{cases} \tag{4.16}$$

where $\hat{U}^m_{N,M}$ is the approximate value of $\hat{U}_N$ at the time level t_m. To calculate it, we have to solve the system of algebraic equations (4.16), for this reason this scheme is called implicit. Moreover, it requires more calculations per time step compared to the explicit scheme, but it is unconditionally stable.

4.2.3 Basic notions about the convergence

The **convergence** represents the most important criterium for the numerical approximation, due to the fact that it demands that, the approximate solution gets closer to the exact one as the discretization is made finer. Then a difference scheme is useful only if it is convergent. Such notion is very connected with the discretization error, which has the following meaning.

Definition 4.5 ***(Discretization error).*** *Let u be the exact solution of Prb.(4.1) and u^m_n its approximation obtained by a difference scheme at the time level t_m. The* ***discretization error*** *denoted by e^m_n, is defined by*

$$e^m_n := |u(t_m) - u^m_n|_{L^2(D)}. \tag{4.17}$$

Definition 4.6 ***(Convergence).*** *Let e_n^m be the discretization error of a difference scheme at the time level t_m. Then, such scheme is said to be convergent with respect to the norm $|.|_{L^2(D)}$ if*

$$\lim_{N,M\to+\infty}\left(\max_{1\le m\le M}|e_n^m|_{L^2(D)}\right)=0. \tag{4.18}$$

The speed of convergence is one of the factors of the efficiency of a numerical method. Such speed is expressed by the so-called order of convergence.

Definition 4.7 ***(order of convergence).*** *Let p, q be two positive integers and e_n^m be the discretization error of a difference scheme at the time level t_m. If p and q are the largest integers for which*

$$\max_{1\le m\le M}|e_n^m|_{L^2(D)}\le C(N^{-p}+M^{-q}), \tag{4.19}$$

for some positive constant C independent of N and M, the scheme is said to have ***order of convergence*** *p in space and q in time.*

From this definition we can conclude that, a higher order of convergence leads to a faster convergence of the scheme.
There is a huge number of works about the numerical approximations of the PDEs, see for example the books [6, 10, 63, 88]. In this section, we recall the results concerned the spacial and the full approximations of Prb.(3.5), for the special case $F(u(t))=f(t)$, $\forall\, t\in(0,T]$.

Theorem 4.8 *Fix $N\ge 1$. Let u be the solution of the weak problem (4.2), s.t. $\frac{\partial u}{\partial t}\in L^1(0,T;H_2^s(D))$ and $u_0\in H_2^s(D)$, for some $s\ge 1$, and let u_N be its approximate solution obtained via spectral Galerkin method. For all $t\in[0,T]$, it holds*

$$|u(t)-u_N(t)|_{L^2(D)}\le C_1N^{-s}, \tag{4.20}$$

for some positive constant C_1.

Proof. See [88, Section 11.2.2] ■

Theorem 4.9 *Fix $N, M \geq 1$. Let u_N be the solution of the spacial approximate problem (4.6), s.t. $\frac{\partial u_N(0)}{\partial t} \in L^2(D)$, and let $(u_{N,M}^m)_{m=1}^M$ be its temporal approximation obtained via the semi-implicit scheme. If $f, \frac{\partial f}{\partial t} \in L^2((0,T] \times D)$, then for any $m \in \{0, ..., M\}$ it holds*

$$|u_{N,M}^m - u_N(t_m)|_{L^2(D)} \leq C_2 M^{-1}, \tag{4.21}$$

for some positive constant C_1.

Proof. See [88, Section 11.3.2] ■

4.3 Full approximation of the stochastic problem

In order to discretize Prb.(3.5), for the special case A be the Laplacian endowed with the homogeneous Dirichlet boundary conditions on the spacial domain $D = (0, 1)$, we just discretize its stochastic term $G(u)\frac{dW(t)}{dt}$, due to the fact that the deterministic term has been discretized already in Section 4.2. Then, by using the Galerkin projection P_N, the stochastic term is discretized by using spectral Galerkin method as follows

$$P_N G(u_N(t))\frac{dW(t)}{dt}, \quad \forall\, t \in (0, T). \tag{4.22}$$

As a result, from (4.6) and (4.22), the spacial discrete version of Prb.(3.5) is given by

$$\begin{cases} \frac{d}{dt}u_N(t) = \Delta_N u_N(t) + P_N F(u_N(t)) + P_N G(u_N(t))\frac{dW(t)}{dt}, \quad \forall\, t \in (0, T), \\ u_N(0) = P_N u_0, \end{cases} \tag{4.23}$$

where $\Delta_N := P_N \Delta$.

Now, we use the semi-implicit scheme to achieve the full discretization of Prb.(3.5) as follows

$$\begin{cases} u_{N,M}^m = (I - \Delta t \Delta_N)^{-1}\left(u_{N,M}^{m-1} + \Delta t P_N F(u_{N,M}^{m-1}) + P_N G(u_{N,M}^{m-1})\Delta W_m\right), \; m = 1, ..., M, \\ u_{N,M}^0 = P_N u_0, \end{cases} \tag{4.24}$$

where $\Delta W_m := W(t_m) - W(t_{m-1})$.

4.3.1 Notions of convergence in the probability context

In the probability context, we can consider different notions of boundedness for random variables, like the almost surely and boundedness in probability. And so, different notions

of order of convergence naturally arise. Regarding the importance of these notions, and our need to use them in the approximation results, we recall them in this section.
Fix $N \geq 1$, and let u be the analytic solution of Prb.(3.5), and u_N be its spacial approximation obtained by some numerical scheme Σ.

Definition 4.10 ***(Convergence in*** $L^p(\Omega, H)$***).*** *Let* $p \geq 1$*. We say that* u_N ***converges strongly*** *(or in* $L^p(\Omega, H)$*) to* u*, if*

$$\lim_{N\to+\infty} \left(\mathbb{E}\left(\sup_{t\in[0,T]} |u_N(t) - u(t)|_H^p \right) \right)^{\frac{1}{p}} = 0. \tag{4.25}$$

Definition 4.11 ***(almost sure convergence).*** *We say that* u_N ***converges almost surely*** *in the space* H *to* u*, if*

$$\lim_{N\to+\infty} \left(\sup_{t\in[0,T]} |u_N(t) - u(t)|_H \right) = 0, \ a.s.\ w \in \Omega. \tag{4.26}$$

The next notion is weaker than the previous two, but it is still connected with pathwise approximation and corresponds to the convergence in probability.

Definition 4.12 ***(Convergence in probability).*** *We say that* u_N ***converges in probability*** *to* u*, if for all* $\epsilon > 0$*, it holds*

$$\lim_{\epsilon\to 0} \mathbb{P} \left\{ \sup_{t\in[0,T]} |u_N(t) - u(t)|_H \geq \epsilon \right\} = 0. \tag{4.27}$$

Their corresponding orders of convergence

Definition 4.13 ***(***L^p ***order).*** *Let* $p \geq 1$*. We say that, the scheme* Σ *is of* L^p ***order*** $\xi > 0$ *in* H*, if there exists a constant* $c_p > 0$ *s.t.*

$$\left(\mathbb{E}\left(\sup_{t\in[0,T]} |u_N(t) - u(t)|_H^p \right) \right)^{\frac{1}{p}} \leq c_p\, \tau^{\xi}. \tag{4.28}$$

Definition 4.14 ***(almost surely (a.s.) order).*** *We say that, the scheme* Σ *is of* ***a.s. order*** $\xi > 0$ *in* H*, if for a.s.* $w \in \Omega$*, there exists a constant* $C(w) > 0$ *s.t.*

$$\sup_{t\in[0,T]} |u_N(t) - u(t)|_H \leq C(w)\tau^{\xi}. \tag{4.29}$$

Definition 4.15 ***(order in probability)***. *We say that, the scheme* Σ *is of order in probability* $\xi > 0$ *in* H*, if*

$$\lim_{C\to\infty} \limsup_{\tau\to 0} \mathbb{P}\left\{ \sup_{t\in[0,T]} |u_N(t) - u(t)|_H \geq C\tau^{\xi} \right\} = 0. \tag{4.30}$$

Remark 4.16 *If we have a temporal approximation* $(u^m)_{m=0}^M$*, for some integer* $M \geq 1$*, which is obtained by some numerical scheme* $\tilde{\Sigma}$*, instead of the spacial approximation* u_N*, the above notions of convergence and order of convergence are still valid, we just replace* u_N *by* u^m*,* $\sup_{t\in[0,T]}$ *by* $\max_{0\leq m\leq M}$ *and* Σ *by* $\tilde{\Sigma}$*.*

Lemma 4.17 *. Let* $(u^m)_{m=0}^M$ *be the temporal approximation of* u *defined by some numerical scheme* $\tilde{\Sigma}$*. If such scheme is of* L^p*-order* β *in* H *with* $p > \frac{1}{\beta}$*. Then, it is of the same order in probability. Moreover, it is of a.s. order* $\bar{\beta} < \beta - \frac{1}{p}$*.*

Proof. See [87, Lemma 2.8]. ∎

In the literature, we can find many results concerned the numerical approximations of the SPDEs, see for a short list [54, 55, 56, 58, 73, 87]. The classical results state that space or time discretization schemes convergence strongly in the case of coefficients are globally Lipschitz and/or have linear growth property, and weakly when the coefficients are only locally Lipschitz or have nonlinear growth. For example, in [87] the temporal approximation achived via the semi-implicit scheme of Prb.(3.5) has been proved. Moreover, the spacial approximation is achieved by using; the finite difference method in [2], the Galerkin finite element method in [73] and the spectral Galerkin method in [12] (see Theorem 3.11 in Subsection 3.4.2).

We close this chapter by recalling the results concerning the spacial approximation of the fractional stochastic heat equation, i.e. Eq.(3.21) on the spacial domain $D := (0, 1)$, where $f = 0$ and the operator $g : L^2(D) \to \mathcal{L}(L^2(D))$ is the Nemytski map associated with the Lipschitz function $g : \mathbb{R} \to \mathbb{R}$.

First, we need to fix some parameters; for $\alpha > 1$, let $\eta \in (\frac{1}{4} + \frac{\alpha}{4}, \frac{1}{4} + \frac{3\alpha}{4})$, $\delta \in (\max\{\frac{1}{2}, \frac{1}{4} + \frac{\alpha}{8}\}, \frac{1}{4} + \frac{3\alpha}{4})$ and $p > \max\{\frac{2\alpha}{\alpha-2}, \frac{\alpha}{2\delta-1}, \frac{2\alpha}{8\delta-\alpha-2}\}$.

Theorem 4.18 *Let* u *be the mild solution of Prb.*(3.21)*, and let* u_N *be its spacial approximation obtained by the spectral Galerkin method. Under the assumptions that;* $g \in H_2^{\delta}$ *s.t.*

$b_\delta := \sup_{x\in\mathbb{R}} |(-\Delta)^\delta g(x)| < \infty$, *and the initial condition* u_0 *is an* H_2^η*-valued* L^p *random variable, i.e.* $u_0 \in L^p(\Omega, H_2^\eta))$ *it holds*

$$\mathbb{E} \sup_{t\in[0,T]} |u_N(t) - u(t)|^p_{L^2(D)} < C_{(T,|u_0|_{H_2^\eta},b_\delta)}\ N^{-\min\{\frac{\alpha}{2},2\delta\}}. \tag{4.31}$$

For $\alpha \in (1, 2]$*, the order of convergence is* $\frac{\alpha}{2}$*.*

Proof. See [38, Theorem 5.1]. ■

Let us mention that, this result also ensures the spacial convergence of the stochastic heat equation (i.e. Eq.(3.21) on the spacial domain $D := (0, 1)$, where $\alpha = 2$ and $f = 0$, equivalently Eq.(3.3) where $f = 0$, $\nu = 1$ and $L = 1$), where the order of convergence is 1. Moreover, for $\alpha > 1$ the regularity of the diffusion term has been required to ensure the convergence. However, for $\alpha > 2$ the spacial convergence has been obtained without regularity of the diffusion term as the following result proves.

Theorem 4.19 *Let* $\alpha > 2$ *and* $p > \frac{2\alpha}{\alpha-2}$*. Under the assumptions of Theorem 4.18, with* $\delta = 0$*. We have*

$$\mathbb{E} \sup_{t\in[0,T]} |u_N(t) - u(t)|^p_{L^2(D)} < C_{(T,|u_0|_{H_2^\eta},b_0)}\ N^{-\left(\frac{\alpha}{4}-\frac{1}{2}-\frac{\alpha}{2p}\right)}. \tag{4.32}$$

Proof. See [38, Theorem 5.2]. ■

Chapter 5

Fractional stochastic Burgers-type Equation in Hölder space -Wellposedness and Approximations-

This chapter is our first scientific contribution in this thesis. We study a class of one dimensional fractional stochastic Burgers-type equation given by the following evolution form

$$\begin{cases} du(t) = \left(-A^{\alpha/2}u(t) + F(u(t))\right) dt + dW(t), \quad t \in (0, T], \\ u(0) = u_0, \end{cases} \tag{5.1}$$

where $A^{\alpha/2}$ is the fractional power of the Laplacian $A = -\Delta$ endowed with Dirichlet boundary conditions defined in Subsection 1.2.4, F is a nonlinear operator given by $F(u(t,x)) := \partial_x f(u(t,x))$, $x \in (0,1)$ with f being locally Lipschitz, W is a Wiener process and u_0 is a L^2-valued random variable. These equations represent typical examples for locally Lipschitz nonlinear growth equations. The fractional stochastic Burgers equation is recuperated by taking $f(u) := u^2$ and if in addition $\alpha = 2$, we obtain the classical one. To the contrary to the fractional and Burgers-type equations, the classical stochastic Burgers equation has been extensively studied, see for a short list, [2, 12, 30, 32, 41, 66, 84, 87] and the references therein. Fractional Burgers equation has been introduced as a relevant model for anomalous diffusions such as diffusion in complex phenomena, relaxations in viscoelastic mediums, propagation of acoustic waves in gaz-filled tube, see e.g. [75, 97, 98]. Analytical studies for the deterministic fractional Burgers equation have been carried out e.g. in [5, 19, 71]. The wellposedness of the L^2-solution and the ergodic properties of the fractional stochastic Burgers equation have been obtained in [15, 16], see also [108, 112]. The numerical study of the fractional equations is still at the earliest stage due to the difficulties to approximate the fractional operator, to the complexity of the schemes and to the control of the nonlinearity. In [99], the authors partially circumvent the first difficulty, by using the Monte Carlo method. In [38], the authors constructed a concrete form to the discretized fractional operator and used it in the approximation of the fractional stochastic heat equation. In [52], the authors considered one dimensional stochastic hyperdissipation Burgers equation, with $-A^{\alpha/2}$ being replaced by $(-1)^{p+1}\nu\frac{\partial^{2p}}{\partial x^{2p}}, p \geq 2, \nu > 0$ and W being a colored noise. Based on the discretization of the random noise, the authors elaborated an approximation for the solution of the equation and proved that it converges to the solution in $L^m(\Omega; L^\infty(0,T) \times L^2(0,1)),\ m \in \mathbb{N}$ and in $L^1(\Omega; L^2(0,T) \times L^\infty(0,1))$.

To the best of our knowledge, Göyngy was the first to introduce stochastic Burgers-type equation [53], then several works followed, see e.g. [20, 57, 113]. Other kind of generalization has been investigated by L. Debbi in [34]. In this work, the author introduced a new class of multi-dimensional stochastic active scalar equations covering among others the multidimensional quasi-geostrophic, $2D$-Navier-Stokes and the fractional stochastic Burgers equation on the torus with $\alpha \in (1, 2]$. The author established thresholds guaranteeing the wellposedness of these equations, according to the kind of solutions (strong, weak, martingale) and according to the Sobolev and the integrability regularities required. A generalization of this class of equations on $D \subseteq \mathbb{R}^d$ and their wellposedness have also been studied in [35].

In general, the goal of the numerical study of stochastic partial differential equations is to elaborate schemes providing approximations with respect to time, to space or to both simultaneously, to prove convergence of these schemes and to establish the rate of convergence. The classical results state that, in the case the coefficients are globally Lipschitz and/or have linear growth property, space or time discretization schemes convergence in expectation. In the cases, the coefficients are locally Lipschitz or have nonlinear growth, only weak convergence has been proven, see e.g. [58, 87, 54, 55, 56, 73] for time discretization and see e.g. [54, 55, 56, 58, 73] for space and full discretizations. One of the first results about the pathwise convergence for the stochastic Burgers equation known to us is the work [2]. In this work, the authors used the finite difference method and proved that the discretized trajectories converge almost surely to the solution in C_tL^2-topology with rate $\gamma < 1/2$.

Our aim in this chapter is to prove the wellposedness of a space-time Hölder solution for the fractional stochastic Burgers-type Eq.(5.1) and to establish the rate of convergence of both the Galerkin approximation and the full discretization schemes. To the best of our knowledge, the current contribution is the first one proving these results not only for the fractional stochastic Burgers-type equations but for the fractional and classical stochastic Burgers equations as well. The exponential Euler scheme has been applied here for the first time for the fractional stochastic equations. Recall that this method has been introduced in [69] and used to approximate the solution of the stochastic heat and reaction diffusion equations, see [68, 69]. In [12], the wellposedness and the Galerkin approximation have been obtained for the stochastic classical Burgers-type equations. To elaborate the full discretization scheme, we combined the spectral Galerkin method and a version of the exponential Euler scheme. Furthermore, we have established estimates for the product of functions in specific Sobolev and Hölder spaces and also some sufficient conditions to prove the existence of the Galerkin approximation.

This chapter is presented in the following plan. In Section 5.1, we describe our problem and we introduce the resolution ingredients. Our main results are given in Section 5.2. Section 5.3 is devoted to the study of the Ornstein-Uhlenbeck stochastic process defined via the fractional semigroup. We study the wellposedenss and the properties of the solutions of pathwise fractional equations of Burgers-type in Section 5.4. The proofs of our results are given in Section 5.5.

5.1 Formulation of the problem

5.1.1 Properties of the linear drift term.

The fractional Laplacian denoted by A_α is defined in Subsection 1.2.4. In this subsection we add to the list of the properties of its semigroup $(e^{-A^{\alpha/2}t})_{t\in[0,T]}$ stated in Lemma 1.71 the following non classical results.

Lemma 5.1 *Let $0 < T < \infty$ and $1 < \alpha \leq 2$ be fixed and let $\delta \in [0,1)$ and $\beta \in \mathbb{R}$, such that $\delta - \beta < \alpha - \frac{1}{2}$. Then for all $\eta \in (\frac{1+2\delta-2\beta}{2\alpha}, 1)$, there exists a positive constant $C_{\alpha,\delta,\eta,\beta} > 0$ s.t. for all $t \in (0,T]$,*

$$\|e^{-A^{\alpha/2}t}\|_{\mathcal{L}(H_2^\beta, C^\delta)} \leq C_{\delta,\eta,\alpha,\beta} t^{-\eta}. \tag{5.2}$$

In particular, for $\beta > \frac{1}{2}$ and $\delta < \beta - \frac{1}{2}$, there exists a positive constant $C_{\delta,\beta} > 0$ s.t.

$$\|e^{-A^{\alpha/2}t}\|_{\mathcal{L}(H_2^\beta, C^\delta)} \leq C_{\delta,\alpha,\beta}. \tag{5.3}$$

Proof.

Let $\delta \in [0,1)$ and $\beta \in \mathbb{R}$, satisfying $\delta - \beta < \alpha - \frac{1}{2}$ and let $v \in H_2^\beta$. Then it is easy to see that

$$|e^{-A^{\alpha/2}t}v|_{C^\delta} = |\sum_{k=1}^{\infty} e^{-\lambda_k^{\frac{\alpha}{2}} t}\langle v, e_k\rangle e_k|_{C^\delta} \leq \sum_{k=1}^{\infty} e^{-\lambda_k^{\frac{\alpha}{2}} t}|\langle v, e_k\rangle||e_k|_{C^\delta}. \tag{5.4}$$

Thanks to Lemma A.7, we infer the existence of a constant $c_\delta > 0$, *s.t.* $|e_k|_{C^\delta} \leq c_\delta k^\delta$ and thanks to Lemma A.8, there exists a constant $c_\eta > 0$, *s.t.* $e^{-\lambda_k^{\frac{\alpha}{2}} t} \leq c_\eta(\lambda_k^{\frac{\alpha}{2}} t)^{-\eta}$. From Remark 1.62, we have $\lambda_k = (k\pi)^2$, hence $e^{-\lambda_k^{\frac{\alpha}{2}} t} \leq c_\eta (k\pi)^{-\alpha\eta} t^{-\eta}$. Using these estimates for (5.4), we obtain

$$|e^{-A^{\alpha/2}t}v|_{C^\delta} \leq C_{\delta,\eta}\sum_{k=1}^{\infty} t^{-\eta}(k\pi)^{-\eta\alpha}|\langle v, e_k\rangle|k^\delta \leq C_{\delta,\eta,\alpha} t^{-\eta}\sum_{k=1}^{\infty} k^{-\eta\alpha+\delta-\beta}|\langle v, e_k\rangle|(k\pi)^\beta.$$

Now, using Hölder inequality, we get

$$\begin{aligned} |e^{-A^{\alpha/2}t}v|_{C^\delta} &\leq C_{\delta,\eta,\alpha,\beta} t^{-\eta}\Big(\sum_{k=1}^{\infty} k^{2(-\eta\alpha+\delta-\beta)}\Big)^{\frac{1}{2}}\Big(\sum_{k=1}^{\infty}(\langle v, e_k\rangle)^2(k\pi)^{2\beta}\Big)^{\frac{1}{2}} \\ &\leq C_{\delta,\eta,\alpha,\beta} t^{-\eta}|v|_{H_2^\beta}, \end{aligned} \tag{5.5}$$

provided $2(-\eta\alpha + \delta - \beta) < -1$. This last is equivalent to $\eta > \frac{1+2\delta-2\beta}{2\alpha}$. A sufficient condition for η to be in $[0,1)$ is that $\delta - \beta < \alpha - \frac{1}{2}$. The proof of the Est.(5.2) is then completed.

Now if $\beta > \frac{1}{2}$, we use the estimate; $e^{-\lambda_k^{\frac{\alpha}{2}} t} \leq 1$ in Est.(5.4) and than we follow the same steps as to get (5.5). The condition $\delta < \beta - \frac{1}{2}$ emerges as a consequence for the above calculus. ■

Corollary 5.2 *Let $0 < T < \infty$ and $1 < \alpha \leq 2$ be fixed and let $\delta \in [0, \frac{\alpha-1}{2})$. Then for all $\eta \in (\frac{1+2\delta+\alpha}{2\alpha}, 1)$, there exists a positive constant $C_{\alpha,\delta,\eta} > 0$ s.t. for all $t \in (0,T]$,*

$$\|e^{-A^{\alpha/2}t}\|_{\mathcal{L}(H_2^{-\frac{\alpha}{2}}, C^\delta)} \leq C_{\alpha,\delta,\eta} t^{-\eta}. \tag{5.6}$$

Proof. Applying Lemma 5.1 for $1 < \alpha \leq 2$, $\beta = -\frac{\alpha}{2}$ and $\delta < \frac{\alpha-1}{2}$, we conclude that Est.(5.6) is fulfilled

■

Lemma 5.3 *Let $0 < T < \infty$ and $1 < \alpha \leq 2$ be fixed and let $\beta, \gamma \in \mathbb{R}$ s.t. $0 < \gamma - \beta < \alpha$. Then for all $\eta \in (\frac{\gamma-\beta}{\alpha}, 1)$, there exists a positive constant $C_{\alpha,\gamma,\eta,\beta} > 0$ s.t. for all $t \in (0,T]$,*

$$\|e^{-A^{\alpha/2}t}\|_{\mathcal{L}(H_2^\beta, H_2^\gamma)} \leq C_{\alpha,\gamma,\eta,\beta} t^{-\eta}. \tag{5.7}$$

In particular, for $\beta > \gamma$, we have

$$\|e^{-A^{\alpha/2}t}\|_{\mathcal{L}(H_2^\beta, H_2^\gamma)} \leq 1. \tag{5.8}$$

Proof. Let $\beta, \gamma \in \mathbb{R}$ s.t. $0 < \gamma - \beta < \alpha$ and let $v \in H_2^\beta$. A simple calculus yields to

$$|e^{-A^{\alpha/2}t}v|^2_{H_2^\gamma} = |A^{\frac{\gamma}{2}} e^{-A^{\alpha/2}t} v|^2_{L^2} \leq C_{\alpha,\gamma} \sum_{k=1}^\infty k^{2\gamma} e^{-2\lambda_k^{\frac{\alpha}{2}} t} \langle v, e_k\rangle^2. \tag{5.9}$$

Using Lemma A.8, we get for a given $\eta > 0$,

$$\begin{aligned} |e^{-A^{\alpha/2}t}v|^2_{H_2^\gamma} &\leq C_{\alpha,\gamma,\eta} t^{-2\eta} \sum_{k=1}^\infty k^{2(\gamma-\alpha\eta)} \langle v, e_k\rangle^2 \leq C_{\alpha,\gamma,\eta,\beta} t^{-2\eta} \sum_{k=1}^\infty k^{2(\gamma-\alpha\eta-\beta)} \langle A^{\frac{\beta}{2}} v, e_k\rangle^2 \\ &\leq C_{\alpha,\gamma,\eta,\beta} t^{-2\eta} |v|^2_{H_2^\beta}, \end{aligned} \tag{5.10}$$

provided $2(\gamma - \alpha\eta - \beta) < 0$ which is equivalent to $\eta > \frac{\gamma-\beta}{\alpha}$.
If $\beta > \gamma$, we get Est.(5.40) by following the same steps of the proof above and using of the estimate $e^{-2\lambda_k^{\frac{\alpha}{2}} t} < 1$ in stead of Lemma A.8. ■

Lemma 5.4 *Let $0 < T < \infty$ and $1 < \alpha \leq 2$ be fixed. Then, for any $\kappa > 0$ there exists a positive constant $C_\alpha > 0$ s.t. for all $t \in (0, T]$,*

$$\|e^{-A^{\alpha/2}t}\|_{\mathcal{L}(L^2,L^4)} \leq C_\alpha t^{-\frac{1}{4\alpha}-\kappa}. \tag{5.11}$$

Proof. Let $v \in L^2$. Using the boundedness of the semigroup $e^{-A^{\alpha/2}t}$ on $L^2(0,1)$ and Lemma 5.1 (with $\beta = \delta = 0$), we can easily deduce that

$$\begin{aligned} |e^{-A^{\alpha/2}t}v|_{L^4}^4 &\leq \int_0^1 |(e^{-A^{\alpha/2}t}v)(x)|^4 dx \leq \sup_{x\in[0,1]} |(e^{-A^{\alpha/2}t}v)(x)|^2 \int_0^1 |(e^{-A^{\alpha/2}t}v)(x)|^2 dx \\ &\leq |e^{-A^{\alpha/2}t}|^2_{\mathcal{L}(L^2,C^0)}|v|^2_{L^2}|e^{-A^{\alpha/2}t}v|^2_{L^2} \leq |e^{-A^{\alpha/2}t}|^2_{\mathcal{L}(L^2,C^0)}|v|^4_{L^2} \leq C_\alpha t^{-\frac{1}{\alpha}-\kappa}|v|^4_{L^2}. \end{aligned} \tag{5.12}$$

■

Lemma 5.5 *Let $0 < T < \infty$ and $1 < \alpha \leq 2$ be fixed and let $\delta \in [0, \frac{\alpha-1}{2})$. Then for all $\gamma \in (\frac{1+\alpha+2\delta}{2\alpha}, 1)$ and all $\eta \in (0, \gamma - (\frac{1+\alpha+2\delta}{2\alpha}))$, there exists a positive constant $C_{\alpha,\delta,\gamma,\eta} > 0$ s.t. for all $t, s \in (0, T]$,*

$$\|e^{-A^{\alpha/2}t} - e^{-A^{\alpha/2}s}\|_{\mathcal{L}(H_2^{-\frac{\alpha}{2}},C^\delta)} \leq C_{\alpha,\delta,\gamma,\eta} s^{-\gamma}|t-s|^\eta. \tag{5.13}$$

Proof. Let $v \in H_2^{-\frac{\alpha}{2}}(0,1)$. Using the semigroup property, Lemmas A.7, A.8 and A.9, we infer, for $\gamma, \eta \in (0,1)$, that

$$\begin{aligned} |(e^{-A^{\alpha/2}t} - e^{-A^{\alpha/2}s})v|_{C^\delta} &= |e^{-A^{\alpha/2}s}(e^{-A^{\alpha/2}(t-s)} - I)v|_{C^\delta} \\ &\leq \sum_{k=1}^\infty e^{-\lambda_k^{\alpha/2}s}(1 - e^{-\lambda_k^{\alpha/2}(t-s)})|\langle v, e_k\rangle||e_k|_{C^\delta} \\ &\leq C_{\alpha,\delta,\gamma} \sum_{k=1}^\infty k^{-\alpha\gamma}s^{-\gamma}(1 - e^{-\lambda_k^{\alpha/2}(t-s)})|\langle v, e_k\rangle|k^\delta \\ &\leq C_{\alpha,\delta,\gamma,\eta} \sum_{k=1}^\infty k^{-\alpha\gamma}s^{-\gamma}k^{\alpha\eta}(t-s)^\eta|\langle v, e_k\rangle|k^\delta \\ &\leq C_{\alpha,\delta,\gamma,\eta}s^{-\gamma}(t-s)^\eta \sum_{k=1}^\infty k^{-\alpha\gamma}k^{\alpha\eta}|\langle v, e_k\rangle|k^{\delta+\alpha/2}k^{-\alpha/2}. \end{aligned} \tag{5.14}$$

Now, we apply Hölder inequality, we get

$$\begin{aligned} |(e^{-A^{\alpha/2}t} - e^{-A^{\alpha/2}s})v|_{C^\delta} &\leq C_{\alpha,\delta,\gamma,\eta}s^{-\gamma}(t-s)^\eta(\sum_{k=1}^\infty k^{2(\alpha(1/2-\gamma+\eta)+\delta)})^{1/2}(\sum_{k=1}^\infty k^{-\alpha}\langle v, e_k\rangle^2)^{1/2} \\ &\leq C_{\alpha,\delta,\gamma,\eta}s^{-\gamma}(t-s)^\eta|v|_{H_2^{-\frac{\alpha}{2}}}, \end{aligned} \tag{5.15}$$

provided that $\delta \in [0, \frac{\alpha-1}{2})$, $\gamma \in (\frac{1+\alpha+2\delta}{2\alpha}, 1)$ and $\eta \in (0, \gamma - (\frac{1+\alpha+2\delta}{2\alpha}))$. ■

Corollary 5.6 *Assume that the conditions of Lemma 5.5 are satisfied. Then, for any* $\kappa > 0$ *there exists a positive function* $C_{\alpha,\delta}(\cdot)$ *on* $(0,T]$*, s.t. for all* $t \in (0,T]$*,*

$$\|e^{-A^{\alpha/2}t} - e^{-A^{\alpha/2}t_0}\|_{\mathcal{L}(H_2^{-\frac{\alpha}{2}}, C^\delta)} \leq C_{\alpha,\delta}(t_0)|t - t_0|^{(\frac{\alpha-1-2\delta}{2\alpha})-\kappa}. \tag{5.16}$$

Proof. Est.(5.16) can be easily obtained from Est.(5.13) with $\gamma = 1 - \kappa$ and $\eta = \frac{\alpha-1-2\delta}{2\alpha} - \kappa$ for any $\kappa > 0$. ■

5.1.2 Definition and properties of the nonlinear drift term

The general form of the nonlinear part of the drift term is given by a function F satisfying the condition (*Assumption 2.* in Blömker et al.' Theorem above; Subsection 3.4.2):

$$\sup_{|u|_V,\ |v|_V \leq r,\ u \neq v} \frac{|F(u) - F(v)|_U}{|u - v|_V} < \infty,$$

with $V := C^\delta[0,1]$, for a relevant $\delta \in (0,1)$ and $U := H_2^{-\alpha/2}(0,1)$. Our typical example is,

$$F(u)(x) = \frac{df(u(x))}{dx}, \quad \text{with } f : H_2^{1-\frac{\alpha}{2}}(0,1) \to C^\delta[0,1] \quad \text{being locally Lipschitz.} \tag{5.17}$$

Specially, the case when f is a polynomial; $f(x) := a_0 + a_1x + a_2x^2 + ...a_nx^n$, with $a_0, a_1, ..., a_n \neq 0 \in \mathbb{R},\ n \in \mathbb{N}$. It is easily seen that for $n = 2$ (i.e. $a_2 \neq 0$) and $a_1 = 0$, we recuperate the fractional stochastic Burgers equation. This fact justifies the name of Burgers-type. For $n = 1$, the drift is then linear and the equation is nothing than the fractional stochastic heat equation with globally Lipschitz coefficients. This study covers, with slight modifications, the case when the coefficients $(a_j(\cdot))_{j=0}^n$ are differentiable real functions with bounded derivatives. Moreover, the proofs can be easily extended to the case of non autonomous function $f(t,x,u)$, under suitable conditions.

Lemma 5.7 *Let* $1 < \alpha \leq 2$*,* $\delta \in (1 - \frac{\alpha}{2}, 1)$ *and let* F *be given by* (5.17)*, with* f *being a polynomial of order* n*, then the following mapping*

$$\begin{aligned} F \ &: \ C^\delta[0,1] \to H_2^{-\frac{\alpha}{2}}(0,1) \\ & u \mapsto F(u) = \frac{df(u(\cdot))}{dx}, \end{aligned} \tag{5.18}$$

is well defined. Moreover, for all $R > 0$*, there exists a positive constant* C_R *such that for every* $u,\ v \in C^\delta[0,1]$ *with* $|u|_{C^\delta}$ *,* $|v|_{C^\delta} \leq R$*, the following inequality holds*

$$|F(u) - F(v)|_{H_2^{-\frac{\alpha}{2}}} \leq C_R|u - v|_{C^\delta}. \tag{5.19}$$

Proof. Let $u,\ v \in C^\delta[0,1]$ such that $|u|_{C^\delta}$, $|v|_{C^\delta} \leq R$ for a given $R > 0$, thanks to the Imbedding (1.5) in Lemma 1.13, we get

$$|F(u) - F(v)|_{H_2^{-\frac{\alpha}{2}}} \leq C|f(u) - f(v))|_{H_2^{1-\frac{\alpha}{2}}} \leq C|f(u) - f(v)|_{C^\delta}.$$

First, we consider the case $n = 1$. Then

$$|F(u) - F(v)|_{H_2^{-\frac{\alpha}{2}}} \leq C|a_1||u - v|_{C^\delta}. \tag{5.20}$$

In this case F is globally Lipschitz. Now For $n \geq 2$, we use the definition of f and the fact that C^δ is a multiplication algebra, see Lemma 1.13, we get

$$|F(u) - F(v)|_{H_2^{-\frac{\alpha}{2}}} \leq C\sum_{k=1}^{n}\sum_{j=0}^{k-1}|a_k||u^{k-1-j}v^j|_{C^\delta}|u - v|_{C^\delta} \leq n^2CR^n|u - v|_{C^\delta}. \tag{5.21}$$

■

Corollary 5.8 *For* $1 < \alpha \leq 2$*,* $\delta \in (1 - \frac{\alpha}{2}, 1)$*,* F *given by* (5.17)*, with* f *being a polynomial,* $U = H_2^{-\frac{\alpha}{2}}(0,1)$ *and* $V = C^\delta[0,1]$*, then thanks to Lemma 5.7, Assumption 2. is fulfilled.*

Corollary 5.9 *For* $\alpha \in (\frac{3}{2}, 2]$*,* $\delta \in (1 - \frac{\alpha}{2}, \frac{\alpha-1}{2})$*,* F *given by* (5.17)*, with* f *being a polynomial,* $U = H_2^{-\frac{\alpha}{2}}(0,1)$ *and* $V = C^\delta[0,1]$ *then thanks to Lemmas 5.1 and 5.7, the first part of Assumption 1. and Assumption 2. are simultaneously fulfilled.*

Corollary 5.10 *For* $R > 0$*, there exists a positive constant* C_R *such that for every* $u \in C^\delta[0,1]$ *with* $|u|_{C^\delta} \leq R$*,*

$$|F(u)|_{H_2^{-\frac{\alpha}{2}}} \leq C_R(1 + |u|_{C^\delta}). \tag{5.22}$$

Proof. It is sufficient to take $v = 0$, in the Est.(5.19).

■

In [12], the authors assumed the existence of a family $(X_N)_N$ satisfying suitable conditions, see *Assumption 4.* in Blömker et al.' Theorem above; Subsection 3.4.2; Cond.(3.16) & Cond.(3.17). In our work, we give sufficient conditions for the existence of such family. We assume that:

- There exists a function $g : \mathbb{R}^2 \to \mathbb{R}$, such that

$$f(x)-f(y) = (x-y)g(x,y) \ \ \& \ \ \forall R > 0, \ \exists\, C_R, \ s.t. \ \forall x, y, |x|, |y| \leq R, \ \ |g(x,y)| \leq C_R. \tag{5.23}$$

- There exist $m \in \mathbb{N}_0$, $(c_j)_{j=1}^m, c_j > 0$ and $(\mu_j)_{j=1}^m$, with $0 < \mu_j < 2$, such that for all $v \in H_2^{\frac{\alpha}{2}}(0,1)$, $\xi \in C^\delta[0,1]$,

$$|\langle F(v+\xi), v\rangle| \leq \sum_{j=1}^m c_j |v|_{H_2^{\frac{\alpha}{2}}}^{\mu_j} |\xi|_{C^\delta}^j + |v|_{L^2} |v|_{H_2^{\frac{\alpha}{2}}} (\sum_{j=1}^m c_j |\xi|_{C^\delta}^j). \tag{5.24}$$

5.1.3 Definition of the stochastic term.

We fix a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, W)$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a normal filtration. $W := (W(t), t \in [0,T])$ is a cylindrical Wiener process defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$. W can be formally written as

$$W(t) = \sum_{k=1}^\infty \beta_k(t) e_k, \quad \mathbb{P} - a.s, \tag{5.25}$$

where $(\beta_k)_{k\geq 1}$ is a family of independent standard Brownian motions and $(e_k(.) = \sqrt{2}\sin(k\pi .))_{k\in\mathbb{N}_0}$ is an orthonormal basis in the space $L^2(0,1)$. We introduce the following Ornstein-Uhlenbeck stochastic process (OU)

$$\mathcal{W}(t) := \int_0^t e^{-A^{\alpha/2}(t-s)} W(ds). \tag{5.26}$$

It is easy to see that $\mathcal{W}$ is well defined for all $\alpha > 1$, see e.g. [39].

5.1.4 Definition of the Galerkin approximation.

We fix $N \geq 1$. We denote by P_N, the Galerkin projection on the finite dimensional space H_N generated by the N first eigenvectors $(e_k)_{k=1}^N$, i.e. for $\delta > 0$, $v \in C^\delta[0,1] \subset L^2(0,1)$ and for all $x \in [0,1]$,

$$P_N v(x) = \sum_{k=1}^N \langle v, e_k\rangle e_k(x). \tag{5.27}$$

Lemma 5.11
- *P_N and $e^{-A^{\alpha/2}t}$ commute.*
- *Let $\delta \in [0,1)$ and $\eta > \delta + \frac{1}{2}$, then there exists $C_{\delta,\eta} > 0$, s.t.*

$$\|P_N\|_{\mathcal{L}(H_2^\eta, C^\delta)} \leq C_{\delta,\eta,}. \tag{5.28}$$

- *Let $\beta \leq \gamma \in \mathbb{R}$, then there exists $C_{\gamma,\beta} > 0$, s.t.*

$$\|I - P_N\|_{\mathcal{L}(H_2^\gamma, H_2^\beta)} \leq C_{\gamma,\beta} N^{-(\gamma-\beta)}. \tag{5.29}$$

- *Let $\alpha \in (1,2]$ and $\delta \in [0, \frac{\alpha-1}{2})$. Then, for any $\kappa > 0$ there exists $C_{\gamma,\beta} > 0$, s.t.*

$$N^{(\frac{\alpha-1}{2}-\delta)-\kappa} \|(I - P_N)e^{-A^{\alpha/2}t}\|_{\mathcal{L}(H_2^{\alpha/2}, C^\delta)} \leq C_{\alpha,\delta}. \tag{5.30}$$

- *Let $\beta \in \mathbb{R}$, then there exists $C_\beta > 0$, s.t.*

$$\|P_N\|_{\mathcal{L}(H_2^\beta)} \leq C_\beta. \tag{5.31}$$

Proof. We omit the proofs of the first and last statements as they are easy. Now, we prove the second one. Let $v \in C^\delta[0,1]$, thanks to Identity (5.27) and Lemma A.7, it is easy to see that

$$\begin{aligned} |P_N v|_{C^\delta} &\leq \sum_{k=1}^N |\langle v, e_k\rangle| |e_k|_{C^\delta} \leq C_\delta \sum_{k=1}^N |\langle v, e_k\rangle| k^\delta \\ &\leq C_\delta \sum_{k=1}^N |\langle v, k^\eta e_k\rangle| k^{\delta-\eta} \leq C^\delta \sum_{k=1}^N |\langle A^{\eta/2} v, e_k\rangle| k^{\delta-\eta}. \end{aligned} \tag{5.32}$$

Using Hölder inequality and the condition $\eta > \delta + \frac{1}{2}$, we deduce that

$$|P_N v|_{C^\delta} \leq C_\delta (\sum_{k=1}^\infty \langle A^{\eta/2} v, e_k\rangle^2)^{\frac{1}{2}} (\sum_{k=1}^\infty k^{2(\delta-\eta)})^{\frac{1}{2}} \leq C_{\delta,\eta} |v|_{H_2^\eta}. \tag{5.33}$$

For the third estimate, we consider $v \in H_2^\gamma$, then

$$\begin{aligned} |(I - P_N)v|^2_{H_2^\beta} &= |A^{\beta/2}(I - P_N)v|^2_{L^2} = \sum_{k=1}^\infty \langle A^{\beta/2}(I - P_N)v, e_k\rangle^2 \leq \sum_{k=N+1}^\infty \lambda_k^\beta \langle v, e_k\rangle^2 \\ &\leq \sum_{k=N+1}^\infty \lambda_k^{\beta-\gamma} \langle A^{\gamma/2} v, e_k\rangle^2 \leq \pi^{2(\beta-\gamma)} \sum_{k=N+1}^\infty k^{2(\beta-\gamma)} \langle A^{\gamma/2} v, e_k\rangle^2 \\ &\leq \pi^{2(\beta-\gamma)} N^{2(\beta-\gamma)} \sum_{k=N+1}^\infty \langle A^{\gamma/2} v, e_k\rangle^2 \leq \pi^{2(\beta-\gamma)} N^{2(\beta-\gamma)} |v|^2_{H_2^\gamma}. \end{aligned} \tag{5.34}$$

For the fourth estimate, we assume that $v \in H_2^{\alpha/2}$ and we prove that for any $\kappa > 0$ there exists $C_{\alpha,\delta} > 0$, such that

$$\sup_{N\in\mathbb{N}} \sup_{t\in(0,T]} \Big(N^{(\frac{\alpha-1}{2}-\delta)-\kappa} |(I - P_N)e^{-tA^{\alpha/2}} v|_{C^\delta}\Big) < \infty. \tag{5.35}$$

In fact, by application of Lemma 5.11, Est.(5.3) and Est.(5.29), we infer the existence of $C_{\alpha,\delta} > 0$, such that

$$\begin{aligned} |(I-P_N)e^{-A^{\alpha/2}t}v|_{C^\delta} &= |e^{-A^{\alpha/2}t}(I-P_N)v|_{C^\delta} \\ &\le |e^{-A^{\alpha/2}t}|_{\mathcal{L}(H_2^{(\delta+\frac{1}{2})+\kappa},C^\delta)}|I-P_N|_{\mathcal{L}(H_2^{\alpha/2},H_2^{(\delta+\frac{1}{2})+\kappa})}|v|_{H_2^{\alpha/2}} \\ &\le C_{\alpha,\delta}N^{-(\frac{\alpha-1}{2}-\delta)+\kappa}|v|_{H_2^{\alpha/2}}. \end{aligned} \tag{5.36}$$

■

Corollary 5.12 *Let $0 < T < \infty$, $1 < \alpha \le 2$, $\delta \in [0,1)$.*

- *Let $\beta \in \mathbb{R}$, such that $\delta - \beta < \alpha - \frac{1}{2}$. Then, for all $\eta \in (\frac{1+2\delta-2\beta}{2\alpha}, 1)$, there exists a positive constant $C_{\alpha,\delta,\beta,\eta} > 0$ s.t. for all $t \in (0,T]$,*

$$\|e^{-A^{\alpha/2}t}P_N\|_{\mathcal{L}(H^\beta,C^\delta)} \le C_{\alpha,\delta,\beta,\eta}t^{-\eta}, \tag{5.37}$$

and for $\beta > \frac{1}{2}$ and $\delta < \beta - \frac{1}{2}$, there exists a positive constant $C_{\delta,\beta} > 0$ s.t.

$$\|e^{-A^{\alpha/2}t}P_N\|_{\mathcal{L}(H^\beta,C^\delta)} \le C_{\delta,\beta}. \tag{5.38}$$

- *Let $\beta \le \gamma \in \mathbb{R}$. Then for all $\eta' \in (\frac{\gamma-\beta}{\alpha}, 1)$, there exists a positive constant $C_{\alpha,\beta,\gamma} > 0$ s.t. for all $t \in (0,T]$,*

$$\|e^{-A^{\alpha/2}t}P_N\|_{\mathcal{L}(H_2^\beta,H_2^\gamma)} \le C_{\alpha,\beta,\gamma}t^{-\eta'}. \tag{5.39}$$

In particular, for $\beta > \gamma$, we have

$$\|e^{-A^{\alpha/2}t}P_N\|_{\mathcal{L}(H_2^\beta,H_2^\gamma)} \le 1. \tag{5.40}$$

Proof. Combining Lemma 5.1 and Lemma 5.11, we conclude the first statement and similarly, combining Lemma 5.3 and Lemma 5.11 we get the second one. ■

We introduce the following discretized version of Eq.(5.1), using the Galerkin spectral method:

$$\begin{cases} du_N(t) = [-A^{\alpha/2}u_N(t) + P_NF(u_N(t))]dt + dW_N(t), \quad t \in (0,T], \\ u_N(0) = P_Nu_0, \end{cases} \tag{5.41}$$

where

$$W_N(t) := P_NW(t) = \sum_{k=1}^{N}\beta_k(t)e_k. \tag{5.42}$$

5.1.5 Full Discretization

Let us fix $M \geq 1$ and consider the uniform step subdivision of the time interval $[0,T]$, with time step $\Delta t = \frac{T}{M}$. We define $t_m = m\Delta t,\ for\ m = 1,...,M$. We construct the sequence of random variables $(u^m_{N,M})^M_{m=0}$ as:

$$\begin{cases} u^0_{N,M} & := P_N u_0, \\ u^{m+1}_{N,M} & := e^{-A^{\alpha/2}T/M}\Big(u^m_{N,M} + \frac{T}{M}(P_N F)(u^m_{N,M})\Big) + P_N\Big(\mathcal{W}((m+1)\frac{T}{M}) - e^{-A^{\alpha/2}T/M}\mathcal{W}((m)\frac{T}{M})\Big), \end{cases} \tag{5.43}$$

where $\mathcal{W}$ is the Ornstein-Uhlenbeck stochastic process given by Eq.(5.26). Let us mention here that we can also rewrite $u^m_{N,M}$ as

$$\begin{cases} u^0_{N,M} & := P_N u_0, \\ u^m_{N,M} & := e^{-A^{\alpha/2}t_m}u^0_{N,M} + \Delta t\sum_{k=0}^{m-1} e^{-A^{\alpha/2}(t_m-t_k)}P_N F(u^k_{N,M}) + \mathcal{W}_N(t_m), \end{cases} \tag{5.44}$$

where $\mathcal{W}_N := P_N\mathcal{W}$.

The idea of the construction of (5.43) is based on the representation of the solution of the Initial Problem (5.41) via Duhamel's formula and the application of the semigroup property. More precisely, the solution of IPblm (5.41) at time t_{m+1} is given by:

$$u_N(t_{m+1}) = e^{-A^{\alpha/2}t_{m+1}}P_N u_0 + \int_0^{t_{m+1}} e^{-A^{\alpha/2}(t_{m+1}-s)}P_N F(u_N(s))ds + \int_0^{t_{m+1}} e^{-A^{\alpha/2}(t_{m+1}-s)}dW_N(s).$$

Using the semigroup property, we can rewrite

$$e^{-A^{\alpha/2}t_{m+1}} = e^{-A^{\alpha/2}T/M}e^{-A^{\alpha/2}t_m} \quad \text{and} \quad e^{-A^{\alpha/2}(t_{m+1}-s)} = e^{-A^{\alpha/2}T/M}e^{-A^{\alpha/2}(t_m-s)}.$$

Therefore,

$$\begin{aligned} u_N(t_{m+1}) &= e^{-A^{\alpha/2}T/M}[e^{-A^{\alpha/2}t_m}P_N u_0 + \int_0^{t_m} e^{-A^{\alpha/2}(t_m-s)}P_N F(u_N(s))ds + \int_0^{t_m} e^{-A^{\alpha/2}(t_m-s)}dW_N(s)] \\ &+ \int_{t_m}^{t_{m+1}} e^{-A^{\alpha/2}(t_{m+1}-s)}P_N F(u_N(s))ds + \int_{t_m}^{t_{m+1}} e^{-A^{\alpha/2}(t_{m+1}-s)}dW_N(s) \\ &= e^{-A^{\alpha/2}T/M}u_N(t_m) + e^{-A^{\alpha/2}T/M}\int_{t_m}^{t_{m+1}} e^{-A^{\alpha/2}(t_m-s)}P_N F(u_N(s))ds \\ &+ \int_0^{t_{m+1}} e^{-A^{\alpha/2}(t_{m+1}-s)}dW_N(s) - e^{-A^{\alpha/2}T/M}\int_0^{t_m} e^{-A^{\alpha/2}(t_m-s)}dW_N(s). \end{aligned}$$

Thanks to Identities (5.42) and (5.26), we write

$$\begin{aligned} \int_0^{t_{m+1}} e^{-A^{\alpha/2}(t_{m+1}-s)}dW_N(s) &- e^{-A^{\alpha/2}T/M}\int_0^{t_m} e^{-A^{\alpha/2}(t_m-s)}dW_N(s) \\ &= P_N\Big(\mathcal{W}((m+1)\frac{T}{M}) - e^{-A^{\alpha/2}T/M}\mathcal{W}((m)\frac{T}{M})\Big). \end{aligned}$$

Now, it is quiet standard to consider the approximation:

$$\int_{t_m}^{t_{m+1}} e^{-A^{\alpha/2}(t_m-s)} P_N F(u_N(s))ds \simeq \frac{T}{M} P_N F(u_N(t_m))$$

and hence the approximation in (5.43).

The sequence $(u^m_{N,M})_m$ has the following property (see the proof in Subsection 5.5.3).

Lemma 5.13 *Let $\frac{7}{4} < \alpha < 2$, $\delta \in (1-\frac{\alpha}{2}, \frac{2\alpha-3}{2})$, F defined as in Subsection 5.1.2 and u_0 satiesfies Assumption $\mathcal{A}$ below with $s > \delta + \frac{1}{2}$. Then there exists a finite $\mathcal{F}_0$-random variable $C_{\alpha,\delta}$, s.t. for all $\omega \in \Omega$,*

$$\sup_{m,N,M} |u^m_{N,M}(\omega)|_{C^\delta} < C_{\alpha,\delta}(\omega). \tag{5.45}$$

Assumption $\mathcal{A}$. For $s > \frac{1}{2}$, we have $u_0 : \Omega \to H^s_2(0,1)$ is a $\mathcal{F}_0$ random variable.

5.2 The main results

In this section we present the main results. First of all, we give the following auxiliary one,

Theorem 5.14 *Let $T > 0$, $\alpha \in (\frac{3}{2}, 2)$, $\delta \in (1-\frac{\alpha}{2}, \frac{\alpha-1}{2})$ and let u_0 satisfies Assumption $\mathcal{A}$ with $s > \delta + \frac{1}{2}$. Then Eq.(5.41) admits a unique L^2-mild solution $u_N := (u_N(t), t \in [0,T])$ satisfying $\sup_N \sup_{t\in[0,T]} |u_N(t)|_{L^2} < \infty$. Moreover, for $\frac{7}{4} < \alpha < 2$ and $\delta \in (1-\frac{\alpha}{2}, \frac{2\alpha-3}{2})$, for almost all ω, the map $u_N : [0,T] \to C^\delta[0,1]$ is Hölder continuous of index $(\frac{\alpha-1-2\delta}{2\alpha}) - \kappa$, for any $\kappa > 0$ and satisfies Est.(3.16), with $V := C^\delta[0,1]$, i.e. for all most all $\omega \in \Omega$,*

$$\sup_N \sup_{t\in[0,T]} |u_N(t,\omega)|_{C^\delta} < \infty. \tag{5.46}$$

Our main results are obtained under the conditions $0 < T < \infty$, $\alpha \in (\frac{7}{4}, 2)$, $\delta \in (1-\frac{\alpha}{2}, \frac{2\alpha-3}{2})$ and that u_0 satisfies Assumption $\mathcal{A}$ with $s > \delta + \frac{1}{2}$. We have

Theorem 5.15 *The fractional stochastic Burgers-type equation* (5.1) *with initial condition u_0, admits a unique mild solution $u : [0,T] \times \Omega \to C^\delta[0,1]$. Moreover, almost surely, the paths of u are Hölder continuous of order $(\frac{\alpha-1-2\delta}{2\alpha}) - \kappa$, for any $\kappa > 0$.*

Theorem 5.16 *$\forall\ \kappa > 0$, there exists a $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable mapping $C : \Omega \to \mathbb{R}_+^*$, such that for almost surely,*

$$\sup_{t\in[0,T]} |u(t) - u_N(t)|_{C^\delta} \leq C(\omega)\ N^{-(\frac{\alpha-1}{2}-\delta)+\kappa}, \tag{5.47}$$

where u is the unique solution of Eq.(5.1) *with initial condition u_0 and u_N is the solution of the Galerkin approximation Eq.*(5.41)*.*

Theorem 5.17 *$\forall\ \kappa > 0$, there exists a $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable mapping $C : \Omega \to \mathbb{R}_+^*$, such that for all $N, M \geq 1$ we have almost surely,*

$$\sup_{t_m\in[0,T]} |u(t_m) - u^m_{N,M}|_{C^\delta} \leq C(w)\ \left((\Delta t)^{(\frac{\alpha-1-2\delta}{2\alpha})-\kappa} + N^{-(\frac{\alpha-1}{2}-\delta)+\kappa}\right), \tag{5.48}$$

where $u^m_{N,M}$ is the solution of the problem (5.43)*.*

Remark 5.18 *The $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable mappings $C(\omega)$ in Theorems 5.16 & 5.17 depend on α, T& δ. In fact, the mapping in Theorem 5.16 emerges from the application of Theorem [12, Theorem 3.1.], see Theorem 3.11, Remark 3.12 and the proof of Theorems 5.15 & 5.16 in Subsection 5.5.2. As a consequence of the dependence of $C(\omega)$ in Theorems 5.16 on α, T& δ, the Estimates* (5.118) *and* (5.135)*, the mapping $C(\omega)$ in Theorem 5.17 depends also on these parameters.*

5.3 Some estimates for the stochastic terms

This section is mainly devoted to study the rate, the different kinds of convergence, in particular the pathwise convergence, and the regularities of the Galerkin approximation of the stochastic terms; W_N given by (5.42) and

$$\mathcal{W}_N(t) := P_N\mathcal{W}(t) = \int_0^t e^{-A^{\alpha/2}(t-s)} W_N(ds) = \sum_{k=1}^N \int_0^t e^{-\lambda_k^{\alpha/2}(t-s)} d\beta_k(t) e_k, \tag{5.49}$$

where $\mathcal{W}$ is the Ornstein-Uhlenbeck stochastic process given by (5.26). The process $\mathcal{W}$ has been studied, for example, in [16, 37, 39], where the authors proved that for $\alpha \in (1,2]$, the Ornstein-Uhlenbeck process (5.26) is well defined as an L^2-valued stochastic process with $C_t^{\frac{\alpha-1}{2\alpha}} C_x^{\frac{\alpha-1}{2}}-$Hölder continuous trajectories. In [34], the author proved that for $\alpha \in (1,2]$, $\mathcal{W} \in L^p(\Omega; C_t \times H_q^\beta)(c(0,1))$, $2 \leq q < \infty$, $\beta \geq 0$, $p \geq 2$ and $c(0,1)$ is the unit circle. The following Lemma is a generalization of [12, Proposition 4.2.],

Lemma 5.19 *Let $\alpha \in (1,2]$, $0<\beta<\frac{\alpha-1}{2}$, $q\geq 2$ and let $p_0 > \frac{2\alpha}{\alpha-1-2\beta}$ be fixed. Then for $p\geq 1$ and for any $\kappa>0$, there exists $C_{\alpha,\beta,q,p}>0$, s.t.*

$$\sup_{N\in\mathbb{N}} \mathbb{E}\Big[|\mathcal{W}_N|^p_{C_tH^\beta_q} + N^{p(\frac{\alpha-1}{2}-(\beta+\frac{\alpha}{p_0})-\kappa)}|(I-P_N)\mathcal{W}|^p_{C_tH^\beta_q} + |\mathcal{W}|^p_{C_tH^\beta_q}\Big] < C_{\alpha,\beta,q,p}. \quad (5.50)$$

Proof. To prove Lemma 5.19, it is sufficient to prove the result for the second term in the LHS of (5.50). The remaining estimates can be easily obtained by following a similar steps, but simple calculus without considering any power of N.

The proof is given in two steps. In the first one, we prove that for $\alpha\in(1,2]$, $0<\beta<\frac{\alpha-1}{2}$, $q\geq 2$ and $p>\frac{2\alpha}{\alpha-1-2\beta}$, there exist $C_{\alpha,\beta,q,p}>0$ and $\xi_p\in(0,\frac{\alpha-1}{2}-(\beta+\frac{\alpha}{p}))$, such that the following estimate holds:

$$\sup_{N\in\mathbb{N}} \mathbb{E}\Big[N^{\xi_p p}|(I-P_N)\mathcal{W}|^p_{C_tH^\beta_q}\Big] < C_{\alpha,\beta,q,p}. \quad (5.51)$$

In the second step, we show that Est.(5.51) is true for all $p\geq 1$ and for a universal ξ.

Step1. Recall that

$$(I-P_N)\mathcal{W}(t) = \int_0^t e^{-A^{\alpha/2}(t-s)}(I-P_N)W(ds). \quad (5.52)$$

Using the factorization method (see Theorem 3.8), we represent $(I-P_N)\mathcal{W}$ as,

$$(I-P_N)\mathcal{W}(t) = \int_0^t (t-s)^{\nu-1}e^{-A^{\alpha/2}(t-s)}Y^N(s)ds, \quad (5.53)$$

$$Y^N(t) = \int_0^t (t-s)^{-\nu}e^{-A^{\alpha/2}(t-s)}(I-P_N)W(ds), \quad (5.54)$$

with $\nu\in(0,1)$. Thanks to Est.(1.23) and by application of Hölder inequality and the fact that we can choose $\frac{1}{p}+\frac{\beta}{\alpha}<\nu<1$, we get

$$\begin{aligned}
\mathbb{E}|(I-P_N)\mathcal{W}|^p_{C_tH^\beta_q} &= \mathbb{E}\Big(\sup_{t\in[0,T]}|A^{\frac{\beta}{2}}\int_0^t (t-s)^{\nu-1}e^{-A^{\alpha/2}(t-s)}Y^N(s)ds|^p_{L^q}\Big)\\
&\leq \mathbb{E}\sup_{t\in[0,T]}\Big(\int_0^t (t-s)^{\nu-1}|A^{\frac{\beta}{2}}e^{-A^{\alpha/2}(t-s)}Y^N(s)|_{L^q}ds\Big)^p\\
&\leq C\mathbb{E}\sup_{t\in[0,T]}\Big(\int_0^t (t-s)^{\nu-1-\frac{\beta}{\alpha}}|Y^N(s)|_{L^q}ds\Big)^p\\
&\leq C\sup_{t\in[0,T]}\Big(\int_0^t (t-s)^{(\nu-1-\frac{\beta}{\alpha})\frac{p}{p-1}}ds\Big)^{(p-1)}\mathbb{E}\int_0^T |Y^N(s)|^p_{L^q}ds\\
&\leq CT^{(1+(\nu-1-\frac{\beta}{\alpha})\frac{p}{p-1})(p-1)}\mathbb{E}\int_0^T |Y^N(s)|^p_{L^q}ds. \quad (5.55)
\end{aligned}$$

Moreover, using the stochastic isometry, the estimate $|e_k|_{L^q} \leq 1$ and Lemma A.8, with $\frac{1}{\alpha} < \gamma < 1 - 2\nu$ and $\nu < \frac{1}{2} - \frac{1}{2\alpha}$, we obtain

$$\begin{aligned}
\mathbb{E}\int_0^T |Y^N(s)|^p_{L^q} ds &\leq \mathbb{E}\int_0^T |\int_0^t (t-s)^{-\nu} e^{-A^{\alpha/2}(t-s)}(I-P_N)W(ds)|^p_{L^q} ds \\
&\leq C\int_0^T (\int_0^t (t-s)^{-2\nu} \sum_{k=N+1}^{\infty} e^{-2\lambda_k^{\frac{\alpha}{2}}(t-s)} |e_k|^2_{L^q} ds)^{\frac{p}{2}} dt \\
&\leq C\int_0^T (\int_0^t (t-s)^{-2\nu} \sum_{k=N+1}^{\infty} e^{-2\lambda_k^{\frac{\alpha}{2}}(t-s)} ds)^{\frac{p}{2}} dt \\
&\leq C\int_0^T (\int_0^t (t-s)^{-2\nu} \sum_{k=N+1}^{\infty} (2\lambda_k)^{\frac{-\gamma\alpha}{2}} (t-s)^{-\gamma} ds)^{\frac{p}{2}} dt \\
&\leq C_{\alpha,\gamma}\int_0^T (\int_0^t (t-s)^{-2\nu-\gamma} ds \sum_{k=N+1}^{\infty} k^{-\gamma\alpha})^{\frac{p}{2}} dt. \qquad (5.56)
\end{aligned}$$

Remark that γ exists thanks to the condition $\nu < \frac{1}{2} - \frac{1}{2\alpha}$ and $\alpha > 1$. It is also easy to see that, thanks to the choice of ν and γ, the integral in the RHS of the last inequality of Est.(5.56) converges. Now, let $\xi \in (0, \alpha\gamma - 1)$, then $\sum_{k=N+1}^{\infty} k^{-\gamma\alpha} \leq N^{-\xi} \sum_{k=N+1}^{\infty} k^{-\gamma\alpha+\xi} \leq C_{\alpha,\gamma,\xi} N^{-\xi}$. Hence

$$\begin{aligned}
\mathbb{E}\int_0^T |Y^N(s)|^p_{L^q} ds &\leq C_{\alpha,\gamma,\beta} N^{-\xi\frac{p}{2}} (\int_0^T t^{(1-2\nu-\gamma)\frac{p}{2}} dt)(\sum_{k=N+1}^{\infty} k^{-\gamma\alpha+\xi})^{\frac{p}{2}} \\
&\leq C_{\alpha,\gamma,\beta} N^{-\xi\frac{p}{2}} T^{(1-2\nu-\gamma)\frac{p}{2}+1} (\sum_{k=1}^{\infty} k^{-\gamma\alpha+\xi})^{\frac{p}{2}} \\
&\leq C_{\alpha,\gamma,\beta,p,\nu,\xi,T} N^{-\xi\frac{p}{2}}. \qquad (5.57)
\end{aligned}$$

Now to combine (5.55) and (5.57), we have to assume that $\frac{1}{p} + \frac{\beta}{\alpha} < \nu < \frac{1}{2} - \frac{1}{2\alpha}$. The existence of ν is guaranted thanks to the conditions $0 < \beta < \frac{\alpha-1}{2}$ and $p > \frac{2\alpha}{\alpha-1-2\beta}$. Therefore,

$$\mathbb{E}|(I-P_N)\mathcal{W}|^p_{C_t H_q^\beta} \leq C_{\alpha,\gamma,\beta,p,\nu,\xi,T} N^{-p\frac{\xi}{2}}. \qquad (5.58)$$

According to the values of the parameters ξ, γ and ν, we deduce that $\frac{\xi}{2} \in (0, \frac{\alpha-1}{2} - (\beta + \frac{\alpha}{p}))$, hence ξ depends, in particular, on p, so let us denote $\frac{\xi}{2}$ by ξ_p.

Step2. Let us fix $p_0 > \frac{2\alpha}{\alpha-1-2\beta}$, using Hölder inequality, we deduce Est.(5.51), for all $p \leq p_0$, with ξ_p being replaced by ξ_{p_0} and the constant depends also on p_0 and p. Now, for $p \geq p_0$, then Est.(5.51) holds and thanks to the inclusion $(0, \frac{\alpha-1}{2} - (\beta + \frac{\alpha}{p_0})) \subset (0, \frac{\alpha-1}{2} - (\beta + \frac{\alpha}{p}))$, it is sufficient to take ξ_p equal to ξ_{p_0}. ∎

Corollary 5.20 *Let $\alpha \in (1,2]$, $0 < \delta < \frac{\alpha-1}{2}$ and $p_0 > \frac{2(\alpha+1)}{\alpha-1-2\delta}$. Then for $p \geq 1$ and for any $\kappa > 0$, there exists $C_{\alpha,\delta,p} > 0$, s.t.*

$$\sup_{N\in\mathbb{N}} \mathbb{E}\Big[|\mathcal{W}_N|^p_{C_tC^\delta} + N^{p(\frac{\alpha-1}{2}-(\delta+\frac{\alpha+1}{p_0})-\kappa)}|(I-P_N)\mathcal{W}|^p_{C_tC^\delta} + |\mathcal{W}|^p_{C_tC^\delta}\Big] < C_{\alpha,\delta,p}. \tag{5.59}$$

Proof. It is easy to see that Est.(5.50) is valid for the special case $\beta = \delta + \frac{1}{p_0}$, with $\delta < \frac{\alpha-1}{2}$, $p_0 > \frac{2(\alpha+1)}{\alpha-1-2\delta}$ and $q = p_0$. Hence, we get

$$\sup_{N\in\mathbb{N}} \mathbb{E}\Big[|\mathcal{W}_N|^p_{C_tH_{p_0}^{\delta+\frac{1}{p_0}}} + N^{p(\frac{\alpha-1}{2}-(\delta+\frac{\alpha+1}{p_0})-\kappa)}|(I-P_N)\mathcal{W}|^p_{C_tH_{p_0}^{\delta+\frac{1}{p_0}}} + |\mathcal{W}|^p_{C_tH_{p_0}^{\delta+\frac{1}{p_0}}}\Big] < C_{\alpha,\delta,p}. \tag{5.60}$$

Thanks to the embedding $H_{p_0}^{\delta+\frac{1}{p_0}} \hookrightarrow C^\delta$, see Theorem 1.17, we get Est.(5.59). ■

Corollary 5.21 *Let $\alpha \in (1,2]$ and $\delta \in (0, \frac{\alpha-1}{2})$. Then, for any $\kappa > 0$ there exists a finite positive random variable $C_{\alpha,\delta}$, such that for almost surely,*

$$\sup_{N\in\mathbb{N}} \Big[N^{(\frac{\alpha-1}{2}-\delta)-\kappa}|(I-P_N)\mathcal{W}(\omega)|_{C_tC^\delta}\Big] \leq C_{\alpha,\delta}(\omega). \tag{5.61}$$

Proof. Using Corollary 5.20 and Lemma A.3, we deduce for a given $p_0 > \frac{2(\alpha+1)}{\alpha-1-2\delta}$, that almost surely,

$$\sup_{N\in\mathbb{N}} \sup_{t\in(0,T]} \Big(N^{(\frac{\alpha-1}{2}-(\delta+\frac{\alpha+1}{p_0})-\kappa)-\epsilon}|(I-P_N)\mathcal{W}(\omega,t)|_{C^\delta}\Big) < \infty. \tag{5.62}$$

For p_0 large ($\frac{\alpha+1}{p_0} = \epsilon$) then there exists a random variable $C_{\alpha,\delta,\epsilon}$, such that (5.61) is fulfilled.

■

Lemma 5.22 *Let $\alpha \in (1,2]$ and $0 < \delta < \frac{\alpha-1}{2}$. Then there exists a finite positive random variable $C_{\alpha,\delta}$, such that for almost surely,*

$$\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} |\mathcal{W}_N(t,\omega)|_{C^\delta} < C_{\alpha,\delta}(\omega). \tag{5.63}$$

Proof. Thanks to Lemma A.7, we have

$$\begin{aligned} |\mathcal{W}_N(t,\omega)|_{C^\delta} &= |\sum_{k=1}^N (\int_0^t e^{-(t-s)\lambda_k^{\alpha/2}} d\beta_k(s))(\omega) e_k|_{C^\delta} \leq \sum_{k=1}^N |(\int_0^t e^{-(t-s)\lambda_k^{\alpha/2}} d\beta_k(s))(\omega)||e_k|_{C^\delta} \\ &\leq C_\delta \sum_{k=1}^\infty |(\int_0^t (k\pi)^\delta e^{-(t-s)(k\pi)^\alpha} d\beta_k(s))(\omega)|. \end{aligned} \tag{5.64}$$

We define

$$C(t,\omega) := C_\delta \sum_{k=1}^{\infty} |(\int_0^t (k\pi)^\delta e^{-(t-s)(k\pi)^\alpha} d\beta_k(s))(\omega)|. \tag{5.65}$$

It is well known that the process $C(t,\omega)$ is well defined provided that

$$\sum_{k=1}^{\infty} \int_0^t (k\pi)^{2\delta} e^{-2(t-s)(k\pi)^\alpha} ds < \infty.$$

This last condition is satisfied by using Lemma A.8, with $\frac{2\delta+1}{\alpha} < \gamma < 1$. Moreover, $C(\cdot,\omega)$ has continuous trajectories on $[0,T]$. Therefore, the random variable:

$$C_{\alpha,\delta}(\omega) := \sup_{t\in[0,T]} C(t,\omega) \tag{5.66}$$

exists, is positive and finite and we have,

$$\sup_{t\in[0,T]} |\mathcal{W}_N(t,\omega)|_{C^\delta} \quad \leq \quad C_{\alpha,\delta}(\omega). \tag{5.67}$$

■

Lemma 5.23 *Let $\alpha \in (1,2]$, $0 \leq \delta < \frac{\alpha-1}{2}$ and fix $N \in \mathbb{N}$. The stochastic process: $\mathcal{W}_N : [0,T] \times \Omega \to C^\delta[0,1]$ has Hölder continuous sample paths of degree $\frac{1}{2} - \kappa$, for any $\kappa > 0$.*

Proof. Our main tool here is Kolmogorov-Centsov Theorem 2.10. First, we prove that for large $p \in [1,\infty)$ and for $t_1, t_2 \in (0,T]$, there exist positive constants $C_{N,p}$ and $\tau' \in (0,1)$ s.t.

$$\left(\mathbb{E}|\mathcal{W}_N(t_2) - \mathcal{W}_N(t_1)|^p_{C^\delta}\right)^{\frac{1}{p}} \leq C_{N,p}|t_1 - t_2|^{\frac{\tau'}{2}}, \tag{5.68}$$

Let $x, y \in [0,1]$ and $t_1, t_2 \in (0,T]$. Then

$$\begin{aligned} ((\mathcal{W}_N(t_2)(x) \quad &- \quad \mathcal{W}_N(t_1)(x)) - (\mathcal{W}_N(t_2)(y) - \mathcal{W}_N(t_1)(y))) \\ &= \sum_{k=1}^{N} \left(\int_0^{t_2} e^{-\lambda_k^{\frac{\alpha}{2}}(t_2-s)} d\beta_k(s) - \int_0^{t_1} e^{-\lambda_k^{\frac{\alpha}{2}}(t_1-s)} d\beta_k(s)\right)(e_k(x) - e_k(y)). \end{aligned} \tag{5.69}$$

Thanks to the fact that the elements of the sequence $(\beta_k)_k$ are independent and to Lemma A.5, we get for every $\tau' \in (0,1)$,

$$\begin{aligned} \mathbb{E}((\mathcal{W}_N(t_2)(x) \quad &- \quad \mathcal{W}_N(t_1)(x)) - (\mathcal{W}_N(t_2)(y) - \mathcal{W}_N(t_1)(y)))^2 \\ &= \sum_{k=1}^{N} |e_k(x) - e_k(y)|^2 \mathbb{E}\left(\int_0^{t_2} e^{-\lambda_k^{\frac{\alpha}{2}}(t_2-s)} dB_k(s) - \int_0^{t_1} e^{-\lambda_k^{\frac{\alpha}{2}}(t_1-s)} dB_k(s)\right)^2 \\ &\leq \sum_{k=1}^{N} |e_k(x) - e_k(y)|^2 \lambda_k^{-\frac{\alpha}{2}(1-\tau')} |t_2 - t_1|^{\tau'}. \end{aligned} \tag{5.70}$$

Let $0 < \epsilon < \frac{1}{p}$. Using the properties of the trigonometric function $sin(k\pi x)$, a simple calculus yields to

$$\begin{aligned}
\mathbb{E}((\mathcal{W}_N(t_2)(x) &- \mathcal{W}_N(t_1)(x)) - (\mathcal{W}_N(t_2)(y) - \mathcal{W}_N(t_1)(y)))^2 \\
&\leq \sum_{k=1}^{N} |e_k(x) - e_k(y)|^{2(\delta+\frac{1}{p}+\epsilon)}(|e_k(x)| + |e_k(y))|^{2-2(\delta+\frac{1}{p}+\epsilon)}\lambda_k^{-\frac{\alpha}{2}(1-\tau')}|t_2 - t_1|^{\tau'} \\
&\leq 4\sum_{k=1}^{N} k^{2(\delta+\frac{1}{p}+\epsilon)}|x - y|^{2(\delta+\frac{1}{p}+\epsilon)}\lambda_k^{-\frac{\alpha}{2}(1-\tau')}|t_2 - t_1|^{\tau'} \\
&\leq C\Big(\sum_{k=1}^{N} k^{-\alpha(1-\tau'-\frac{2}{\alpha}(\delta+\frac{1}{p}+\epsilon))}\Big)|t_2 - t_1|^{\tau'}|x - y|^{2(\delta+\frac{1}{p}+\epsilon)} \\
&\leq C\Big(\sum_{k=1}^{N} k^{-\alpha(1-\tau'-\frac{2}{\alpha}(\delta+\frac{1}{p}+\epsilon))}\Big)|t_2 - t_1|^{\tau'}|x - y|^{2(\delta+\frac{1}{p}+\epsilon)} \\
&\leq C_N|t_2 - t_1|^{\tau'}|x - y|^{2(\delta+\frac{1}{p}+\epsilon)}. \qquad (5.71)
\end{aligned}$$

Furthermore, using Lemma A.5 and the properties of the trigonometric functions, we get

$$\begin{aligned}
\mathbb{E}(\mathcal{W}_N(t_2)(x) &- \mathcal{W}_N(t_1)(x))^2 \\
&= \sum_{k=1}^{N} |e_k(x)|^2\mathbb{E}\left(\int_0^{t_2} e^{-\lambda_k^{\frac{\alpha}{2}}(t_2-s)}dB_k(s) - \int_0^{t_1} e^{-\lambda_k^{\frac{\alpha}{2}}(t_1-s)}dB_k(s)\right)^2 \\
&\leq C\sum_{k=1}^{N} k^{-\alpha(1-\tau')}|t_2 - t_1|^{\tau'} \leq C_N|t_2 - t_1|^{\tau'}. \qquad (5.72)
\end{aligned}$$

Thus, using Lemma A.4, we infer that

$$\begin{aligned}
\mathbb{E}((\mathcal{W}_N(t_2)(x) &- \mathcal{W}_N(t_1)(x)) - (\mathcal{W}_N(t_2)(y) - \mathcal{W}_N(t_1)(y)))^p \\
&\leq p!\Big(\mathbb{E}((\mathcal{W}_N(t_2)(x) - \mathcal{W}_N(t_1)(x)) - (\mathcal{W}_N(t_2)(y) - \mathcal{W}_N(t_1)(y)))^2\Big)^{\frac{p}{2}} \\
&\leq C_{N,p}|t_2 - t_1|^{\tau'\frac{p}{2}}|x - y|^{p(\delta+\frac{1}{p}+\epsilon)}. \qquad (5.73)
\end{aligned}$$

And

$$\mathbb{E}(\mathcal{W}_N(t_2)(x) - \mathcal{W}_N(t_1)(x))^p \leq p!\Big(\mathbb{E}(\mathcal{W}_N(t_2)(x) - \mathcal{W}_N(t_1)(x))^2\Big)^{\frac{p}{2}} \leq C_{N,p}|t_2 - t_1|^{\tau'\frac{p}{2}}. \qquad (5.74)$$

Thanks to the Sobolev embedding $H_p^{\delta+\frac{1}{p}} \hookrightarrow C^{\delta}$, see e.g. Theorem 1.17, Est.(5.73), Est.(5.74) and Lemma A.6, we obtain

$$\begin{aligned}
\mathbb{E}|\mathcal{W}_N(t_2) &- \mathcal{W}_N(t_1)|^p_{C^\delta} \leq C_N\Big(\int_0^1 \mathbb{E}(\mathcal{W}_N(t_2)(x) - \mathcal{W}_N(t_1)(x))^p dx \\
&+ \int_0^1\int_0^1 \frac{\mathbb{E}((\mathcal{W}_N(t_2)(x) - \mathcal{W}_N(t_1)(x)) - (\mathcal{W}_N(t_2)(y) - \mathcal{W}_N(t_1)(y)))^p}{|x-y|^{2+\delta p}} dx\, dy\Big) \\
&\leq C_{N,p}|t_2 - t_1|^{\frac{\tau' p}{2}}\left(1 + \int_0^1\int_0^1 |x-y|^{-(1-p\epsilon)} dx\, dy\right) \\
&\leq C_{N,p}(t_2 - t_1)^{\frac{\tau' p}{2}}. \qquad (5.75)
\end{aligned}$$

■

Corollary 5.24 *Let $\alpha \in (1,2]$ and $0 \leq \delta < \frac{\alpha-1}{2}$. The Ornstein-Uhlenbeck stochastic process $\mathcal{W}$ has a continuous version, we still denote by $\mathcal{W} : [0,T] \times \Omega \to C^\delta[0,1]$ with Hölder continuous sample paths of degree $\left(\frac{\alpha-1-2\delta}{2\alpha}\right) - \kappa$, for any $\kappa > 0$.*

Proof. First, we find τ' such that Est.(5.68) holds with a constant in the RHS which is independent of N, i.e. we prove that for large $p \in [1,\infty)$ and for $t_1, t_2 \in (0,T]$, there exist positive constants $C_{\alpha,\delta,p}$ and $\tau' \in (0,1)$ s.t.

$$\left(\mathbb{E}|\mathcal{W}_N(t_2) - \mathcal{W}_N(t_1)|^p_{C^\delta}\right)^{\frac{1}{p}} \leq C_{\alpha,\delta,p}|t_1 - t_2|^{\frac{\tau'}{2}}, \qquad (5.76)$$

To this aim, it is sufficient to follow the same calculus as in the proof of Lemma 5.23 and to choose τ' such that $\sum_{k=1}^\infty k^{-\alpha(1-\tau'-\frac{2}{\alpha}(\delta+\frac{1}{p}+\epsilon))} < \infty$. We consider p large and ϵ small such that $0 < \frac{1}{\alpha}(\frac{2}{p} + 2\epsilon) < \epsilon'$ for a given ϵ'. We have

$$\sum_{k=1}^\infty k^{-\alpha(1-\tau'-\frac{2}{\alpha}(\delta+\frac{1}{p}+\epsilon))} \leq \sum_{k=1}^\infty k^{-\alpha(1-\tau'-\frac{2\delta}{\alpha}+\epsilon'))} < \infty, \qquad (5.77)$$

provided $\alpha(1-\tau'-\frac{2\delta}{\alpha}-\epsilon') > 1$. So it is sufficient to take $0 < \tau' < \frac{\alpha-1-2\delta}{\alpha}$.
Now, we take $p_0 > \frac{2(\alpha+1)}{\alpha-1-2\delta}$ and we use Corollary 5.20, and Est.(5.76), then for any $\kappa > 0$ there exists $C_{\delta,p} > 0$ s.t for all $N \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{E}|\mathcal{W}(t_2) - \mathcal{W}(t_1)|^p_{C^\delta} &\leq \mathbb{E}|(I-P_N)(\mathcal{W}(t_2) - \mathcal{W}(t_1))|^p_{C^\delta} + \mathbb{E}|\mathcal{W}_N(t_2) - \mathcal{W}_N(t_1)|^p_{C^\delta} \\
&\leq C_{\alpha,\delta,p}N^{-p(\frac{\alpha-1}{2}-(\delta+\frac{\alpha+1}{p_0})-\kappa)} + C_{\delta,p}|t_1 - t_2|^{\frac{\tau' p}{2}}. \qquad (5.78)
\end{aligned}$$

The result is easily deduced making $N \to \infty$ and applying Kolomogorov-Centsov Theorem 2.10. ■

5.4 Some auxiliary results

In this section we provide non classical results to estimate the nonlinear term. We mainly focus on nonlinear term of Burgers equation, i.e. for F given by $f(x) = x^2$.
Let $v^N : (0,T) \to L^2(0,1)$ be a sequence of continuous functions. We define

$$y^N(t) := \int_0^t e^{-A^{\alpha/2}(t-s)} P_N F(v^N(s))ds. \tag{5.79}$$

Lemma 5.25 *Assume $\alpha \in (\frac{7}{4}, 2)$ and that the sequence $(v^N)_N$ satisfies*

$$\sup_N \sup_{t\in[0,T]} |v^N(t)|_{L^2} < \infty. \tag{5.80}$$

Then

$$\sup_N \sup_{t\in[0,T]} |y^N(t)|_{L^4} < \infty. \tag{5.81}$$

Proof. Using Lemma 5.4, Lemma 5.11 and [16, Lemma 2.11](see e.g. Lemma 3.23), we get

$$\begin{aligned} |y^N(t)|_{L^4} &\leq \int_0^t |e^{-A^{\alpha/2}(t-s)} P_N F(v^N(s))|_{L^4} ds \\ &\leq \int_0^t \|e^{-A^{\alpha/2}\frac{(t-s)}{2}}\|_{\mathcal{L}(L^2,L^4)} \|P_N\|_{\mathcal{L}(L^2)} |e^{-A^{\alpha/2}\frac{(t-s)}{2}} F(v^N(s))|_{L^2} ds \\ &\leq C\int_0^t (t-s)^{-\frac{1}{4\alpha}-\kappa_1}(t-s)^{-\frac{3}{2\alpha}-\kappa_2}|v^N(s)^2|_{L^1} ds \leq C\int_0^t (t-s)^{-\frac{7}{4\alpha}-\kappa}|v^N(s)|^2_{L^2} ds, \end{aligned} \tag{5.82}$$

for any $\kappa_1, \kappa_2 > 0$ and $\kappa := \kappa_1 + \kappa_2$.
Finally, using Assumption (5.80), we end up with

$$|y^N(t)|_{L^4} \leq C \sup_N \sup_{t\in[0,T]} |v^N(s)|^2_{L^2} T^{1-\frac{7}{4\alpha}-\kappa} < \infty, \text{ for any } \kappa > 0. \tag{5.83}$$

■

Lemma 5.26 *Assume $\alpha \in (\frac{3}{2}, 2)$, $\delta \in [0, \frac{2\alpha-3}{2})$. Let $v^N : (0,T) \to L^4(0,1)$ satisfying*

$$\sup_N \sup_{t\in[0,T]} |v^N(t)|_{L^4} < \infty. \tag{5.84}$$

Then

$$\sup_N \sup_{t\in[0,T]} |y^N(t)|_{C^\delta} < \infty. \tag{5.85}$$

Proof. Using Lemma 5.1, in particular Est.(5.2) and Lemma 5.11, we infer that

$$\begin{aligned}
|y^N(t)|_{C^\delta} &\leq \int_0^t |e^{-A^{\alpha/2}(t-s)} P_N F(v^N(s))|_{C^\delta} ds \\
&\leq \int_0^t \|e^{-A^{\alpha/2}(t-s)}\|_{\mathcal{L}(H_2^{-1},C^\delta)} \|P_N\|_{\mathcal{L}(H_2^{-1})} |F(v^N(s))|_{H_2^{-1}} ds \\
&\leq C_{\alpha,\beta,\delta} \int_0^t (t-s)^{-\frac{3+2\delta}{2\alpha}-\kappa} |v^N(s)|^2_{L^4} ds \leq C_{\alpha,\beta,\delta} T^{1-\frac{3+2\delta}{2\alpha}-\kappa} \sup_N \sup_{t\in[0,T]} |v^N(s)|^2_{L^4} < \infty,
\end{aligned} \tag{5.86}$$

for any $\kappa > 0$.

■

Corollary 5.27 *Assume $\alpha \in (\frac{7}{4}, 2)$, $\delta \in (0, \frac{2\alpha-3}{2})$ and that $(v^N)_N$ satisfies Cond.(5.80). Then*

$$\sup_N \sup_{t\in[0,T]} |y^N(t)|_{C^\delta} < \infty. \tag{5.87}$$

Proof. An application of Lemmas 5.25 and 5.26 leads to the result. ■

Lemma 5.28 *Let $\alpha \in (\frac{3}{2}, 2)$ and $\delta \in (1-\frac{\alpha}{2}, \frac{\alpha-1}{2})$. We introduce the following initial value problems*

$$\begin{cases} \frac{\partial}{\partial t} v^N(t) &= -A^{\alpha/2} v^N(t) + P_N F(v^N(t) + \xi_N(t)), \\ v^N(0) &= P_N v_0, \end{cases} \tag{5.88}$$

where $v_0 \in L^2(0,1)$ and $\xi_N : (0,T) \to C^\delta[0,1]$ is continuous with

$$\sup_N \sup_{[0,T]} |\xi_N(t)|_{C^\delta} < \infty. \tag{5.89}$$

Assume that F is given by f satisfying Cond.(5.23). Then for all $N \in \mathbb{N}_0$, IVP.(5.88) admits a local solution. Moreover, if F satisfies Cond.(5.24), then the local solution v^N becomes global, unique and it satisfies

$$v^N \in C(0,T; L^2(0,1)) \cap L^2(0,T; H_2^{\frac{\alpha}{2}}(0,1)), \tag{5.90}$$

and

$$\sup_N \sup_{t\in[0,T]} |v^N(t)|_{L^2} < \infty. \tag{5.91}$$

In particular, this result is true for Burgers equation.

Proof. To prove the existence of the local solution it is sufficient to prove that there exists $T_0 \le T$, such that the application $\varphi^N : C(0,T_0;C^\delta[0,1]) \to C(0,T_0;C^\delta[0,1])$ is welldefined and it is a contraction, where φ^N is given by

$$(\varphi^N v)(t) = e^{-A^{\alpha/2}t} P_N v_0 + \int_0^t e^{-A^{\alpha/2}(t-s)} P_N F(v(s)+\xi_N(s))ds. \tag{5.92}$$

In fact, let $u, v \in C(0,T_0;C^\delta[0,1])$ such that $|u|_{C^\delta}, |v|_{C^\delta} < R$. Using Corollary 5.2, the embedding $C^\delta[0,1] \hookrightarrow H_2^{1-\frac{\alpha}{2}}(0,1)$, see Lemma 1.13, Cond.(5.23) and the fact that $C^\delta[0,1]$ is a multiplication algebra, for any $\kappa > 0$ we obtain

$$\begin{aligned}
|(\varphi^N v - \varphi^N u)(t)|_{C^\delta} &\le \int_0^t |e^{-A^{\alpha/2}(t-s)} P_N(F(v(s)+\xi_N(s)) - F(u(s)+\xi_N(s)))|_{C^\delta} ds \\
&\le \int_0^t \|e^{-A^{\alpha/2}(t-s)}\|_{\mathcal{L}(H_2^{-\frac{\alpha}{2}},C^\delta)} \|P_N\|_{\mathcal{L}(H_2^{-\frac{\alpha}{2}})} |f(v(s)+\xi_N(s)) - f(u(s)+\xi_N(s))|_{H_2^{1-\frac{\alpha}{2}}} ds \\
&\le C_{\alpha,\delta} \int_0^t (t-s)^{-\frac{1+2\delta+\alpha}{2\alpha}-\kappa} |(v(s)-u(s))\,(g(v(s)+\xi_N(s), u(s)+\xi_N(s)))\,|_{C^\delta} ds \\
&\le C_{\alpha,\delta} \int_0^t (t-s)^{-\frac{1+2\delta+\alpha}{2\alpha}-\kappa} |(v(s)-u(s))|_{C^\delta} |\,(g(v(s)+\xi_N(s), u(s)+\xi_N(s)))\,|_{C^\delta} ds \\
&\le C_{\alpha,\delta,R} \sup_{s\in[0,T_0]} |v(s)-u(s)|_{C^\delta} \int_0^t (t-s)^{-\frac{1+2\delta+\alpha}{2\alpha}-\kappa} ds \\
&\le C_{\alpha,\delta,R} T_0^{1-\frac{1+2\delta+\alpha}{2\alpha}-\kappa} \sup_{s\in[0,T_0]} |v(s)-u(s)|_{C^\delta}.
\end{aligned} \tag{5.93}$$

Then, we choose T_0, such that $C_{\alpha,\delta,R} T_0^{1-\frac{1+2\delta+\alpha}{2\alpha}-\kappa} < 1$.
Now, we prove that for any solution of IVP(5.88) on $[0,T_0]$, we have

$$\sup_N \sup_{t\in[0,T_0]} |v^N(t)|_{L^2} < \infty. \tag{5.94}$$

This last condition is sufficient to guarantee the global existence of the solution. In fact, we multiply the two sides of the first equation in IVP(5.88) by v^N and we use [103] and we integrate, we get

$$|v^N(t)|^2_{L^2} \;+\; 2\int_0^t |v^N(s)|^2_{H_2^{\frac{\alpha}{2}}} ds = |v^N(0)|^2_{L^2} + 2\int_0^t \langle F(v^N(s)+\xi_N(s)), P_N v^N(s)\rangle ds. \tag{5.95}$$

We use Cond.(5.24), Young inequality with $\epsilon_1, \epsilon_2 > 0$ and Lemma 5.11, we obtain

$$
\begin{aligned}
|v^N(t)|^2_{L^2} &+ 2\int_0^t |v^N(s)|^2_{H_2^{\frac{\alpha}{2}}} ds \leq |P_N v_0|^2_{L^2} + \sum_{j=1}^m c_j \int_0^t |v^N(s)|^{\mu_j}_{H_2^{\frac{\alpha}{2}}} |\xi_N(s)|^j_{C^\delta} ds \\
&+ \int_0^t |v^N(s)|_{L^2} |v^N(s)|_{H_2^{\frac{\alpha}{2}}} (\sum_{j=1}^m c_j |\xi_N(s)|^j_{C^\delta}) ds \\
&\leq |v_0|^2_{L^2} + m\epsilon_1 \int_0^t |v^N(s)|^2_{H_2^{\frac{\alpha}{2}}} ds + \frac{1}{\epsilon_1} (\sum_{j=1}^m c_j \int_0^t |\xi_N(s)|^{\frac{2}{2-\mu_j}}_{C^\delta} ds) \\
&+ \epsilon_2 \int_0^t |v^N(s)|^2_{H_2^{\frac{\alpha}{2}}} ds + \frac{1}{\epsilon_2} \int_0^t |v^N(s)|^2_{L^2} (\sum_{j=1}^m c_j |\xi_N(s)|^j_{C^\delta})^2 ds. \qquad (5.96)
\end{aligned}
$$

We choose $\epsilon_1, \epsilon_2 > 0$ such that $m\epsilon_1 + \epsilon_2 < 2$ and we use Cond.(5.89) we end up with

$$
\begin{aligned}
|v^N(t)|^2_{L^2} + (2 - m\epsilon_1 - \epsilon_2) \int_0^t |v^N(s)|^2_{H_2^{\frac{\alpha}{2}}} ds &\leq |v_0|^2_{L^2} + \frac{1}{\epsilon_1} T (\sum_{j=1}^m c_j \sup_N \sup_s |\xi_N(s)|^{\frac{2}{2-\mu_j}}_{C^\delta}) \\
&+ \frac{1}{\epsilon_2} sup_N \sup_s (\sum_{j=1}^m c_j |\xi_N(s)|^j_{C^\delta})^2) \int_0^t |v^N(s)|^2_{L^2} ds \\
&\leq C_1 + C_2 \int_0^t |v^N(s)|^2_{L^2} ds. \qquad (5.97)
\end{aligned}
$$

In particular, as the first term in the LHS of Est.(5.97) is bounded by the RHS of Est.(5.97) and by application of Gronwall lemma, we deduce that $|v^N(t)|^2_{L^2} \leq C_1 e^{C_2 T}$ and consequently that $\sup_N \int_0^t |v^N(s)|^2_{H_2^{\frac{\alpha}{2}}} ds < \infty$. Thus, conditions (5.90) & (5.91) are fulfilled.

To clarify further that our study covers the fractional stochastic Burgers equation, we independently develop below this later case. In fact, using the fact that for all y, we have $\langle \partial_x y^2, y \rangle = 0$, [16, Lemma 11], the embedding $C^\delta[0,1] \hookrightarrow H_2^{1-\frac{\alpha}{2}}(0,1)$, see Lemma 1.13, and the fact that $C^\delta[0,1]$ is a multiplication algebra, we get

$$
\begin{aligned}
|v^N(t)|^2_{L^2} &+ 2\int_0^t |v^N(s)|^2_{H_2^{\frac{\alpha}{2}}} ds = |v^N(0)|^2_{L^2} + 2\int_0^t \langle F(v^N(s) + \xi_N(s)), P_N v^N(s) \rangle ds \\
&\leq |v^N(0)|^2_{L^2} + 2\int_0^t \left(|\langle \partial_x (v^N(s))^2, v^N(s) \rangle| + |\langle \partial_x (\xi_N(s))^2, v^N(s) \rangle| \right) ds \\
&+ 4\int_0^t |\langle \partial_x (v^N(s)\xi_N(s)), v^N(s) \rangle| ds \\
&\leq |v^N(0)|^2_{L^2} + 2\int_0^t |\xi_N(s))^2|_{H_2^{1-\frac{\alpha}{2}}} |v^N(s)|_{H_2^{\frac{\alpha}{2}}} ds + 4\int_0^t |v^N(s)|_{H_2^{\frac{\alpha}{2}}} |v^N(s)\xi_N(s))|_{H_2^{1-\frac{\alpha}{2}}} ds \\
&\leq |P_N v(0)|^2_{L^2} + 2\int_0^t |\xi_N(s))|^2_{C^\delta} |v^N(s)|_{H_2^{\frac{\alpha}{2}}} ds + 4\int_0^t |v^N(s)|_{H_2^{\frac{\alpha}{2}}} |v^N(s)|_{H_2^{1-\frac{\alpha}{2}}} |\xi_N(s))|_{C^\delta} ds.
\end{aligned}
$$
(5.98)

Using the following interpolation $|v^N(s)|_{H_2^{1-\frac{\alpha}{2}}} \leq c |v^N(s)|^{2\frac{\alpha-1}{\alpha}}_{L^2} |v^N(s)|^{\frac{2-\alpha}{\alpha}}_{H_2^{\frac{\alpha}{2}}}$, Young inequality, Lemma 5.11 and Cond.(5.89), we deduce that

$$\begin{aligned}
|v^N(t)|^2_{L^2} &+ 2\int_0^t |v^N(s)|^2_{H_2^{\frac{\alpha}{2}}} ds \le |v_0|^2_{L^2} + \int_0^t (\frac{1}{\epsilon_1}|\xi_N(s))|^4_{C^\delta} + \epsilon_1^2 |v^N(s)|^2_{H_2^{\frac{\alpha}{2}}})ds \\
&+ 4c\int_0^t |v^N(s)|^{\frac{2}{\alpha}}_{H_2^{\frac{\alpha}{2}}} |v^N(s)|^{2\frac{\alpha-1}{\alpha}}_{L^2} |\xi_N(s))|_{C^\delta} ds \\
&\le |v_0|^2_{L^2} + \int_0^t (\frac{1}{\epsilon_1}|\xi_N(s))|^4_{C^\delta} + \epsilon_1^2 |v^N(s)|^2_{H_2^{\frac{\alpha}{2}}})ds \\
&+ 4c\int_0^t (\epsilon_2 |v^N(s)|^2_{H_2^{\frac{\alpha}{2}}} + \frac{1}{\epsilon_2}(|v^N(s)|^2_{L^2} |\xi_N(s))|^2_{C^\delta} ds \\
&\le |v_0|^2_{L^2} + \frac{Tc}{\epsilon_1} + (\epsilon_1^2 + 4c\epsilon_2)\int_0^t |v^N(s)^2_{H_2^{\frac{\alpha}{2}}})ds + \frac{4c}{\epsilon_2}\int_0^t |v^N(s)|^2_{L^2} ds. \quad (5.99)
\end{aligned}$$

the choice of ϵ_1 and ϵ_2 such that $\epsilon_1^2 + 4c\epsilon_2 \le 2$ gives us

$$|v^N(t)|^2_{L^2} + (2 - \epsilon_1^2 - 4c\epsilon_2)\int_0^t |v^N(s)|^2_{H_2^{\frac{\alpha}{2}}} ds \le (|v_0|^2_{L^2} + \frac{Tc}{\epsilon_1}) + \frac{4c}{\epsilon_2}\int_0^t |v^N(s)|^2_{L^2} ds. \quad (5.100)$$

Now, arguing as above and we apply Gronwall lemma A.1 , we infer that

$$|v^N(t)|^2_{L^2} \le (|v_0|^2_{L^2} + \frac{Tc}{\epsilon_1})e^{\frac{4c}{\epsilon_2}T}. \quad (5.101)$$

Thus, Cond.(5.91) is fulfilled and consequently, $v^N \in L^2(0,T; H_2^{\frac{\alpha}{2}}(0,1))$. Thus the proof is achieved. ■

5.5 Proof of Theorems

5.5.1 Proof of Theorem 5.14

Existence. We understand equation (5.41) in the integral form as

$$u_N(t) = e^{-A^{\alpha/2}t} P_N u_0 + \int_0^t e^{-A^{\alpha/2}(t-s)} P_N F(u_N(s))ds + \mathcal{W}_N(t), \quad t \in [0,T], \quad (5.102)$$

where $\mathcal{W}_N(t)$ is given by (5.49). Remark that if u_N is solution of Eq.(5.41), then $v_N := u_N - \mathcal{W}_N$ is a mild solution of the pathwise IVP.(5.88), with ξ_N by $\mathcal{W}_N$ and vice versa. To prove the existence of the solution u_N satisfying Eq.(5.102) and Cond.(5.91), we apply Lemma 5.22 and Lemma 5.28.

Uniform boundedness of $(u_N)_N$. Now, we assume that $\frac{7}{4} < \alpha < 2$, $\delta \in (1-\frac{\alpha}{2}, \frac{2\alpha-3}{2})$ and we prove that the solutions $(u_N)_N$ satisfy Est.(5.46). In fact, using Identity (5.102), it is obvious that

$$|u_N(t)|_{C^\delta} \leq |e^{-A^{\alpha/2}t}P_N u_0|_{C^\delta} + |\int_0^t e^{-A^{\alpha/2}(t-s)}P_N F(u_N(s))ds|_{C^\delta} + |\mathcal{W}_N(t)|_{C^\delta}. \quad (5.103)$$

Let us remark that the second term respectively the third one in Est.(5.103), are bounded thanks to Corollary 5.27 and Lemma 5.28 respectively to Lemma 5.22. To estimate the first term in Est.(5.103), we use Lemma 5.1, Lemma 5.11 and Assumption $\mathcal{A}$ with $s \geq \beta > \delta + \frac{1}{2}$, we get

$$|e^{-A^{\alpha/2}t}P_N u_0|_{C^\delta} \leq \|e^{-A^{\alpha/2}t}\|_{\mathcal{L}(H_2^\beta, C^\delta)}\|P_N\|_{\mathcal{L}(H_2^\beta)}|u_0|_{H_2^\beta} \leq C_{\alpha,\delta,\beta}|u_0|_{H_2^\beta} < \infty. \quad (5.104)$$

The proof is then achieved.

Hölder Regularity of u_N. We prove that each term of Identity (5.102) is Hölder continuous of index $\frac{\alpha-1-2\delta}{2\alpha} - \kappa$, for any $\kappa > 0$. In fact, the regularity of $\mathcal{W}_N$ follows from Lemma 5.23. To get the regularity of the first term, we use Corollary 5.6, Lemma 5.11, the embedding $H_2^\beta(0,1) \hookrightarrow H_2^{-\frac{\alpha}{2}}(0,1)$ and the Assumption $\mathcal{A}$ with $s \geq \beta > \delta + \frac{1}{2}$. Then, for $\tau < t \in (0,T)$, we have

$$\begin{aligned} |(e^{-A^{\alpha/2}t} - e^{-A^{\alpha/2}\tau})P_N u_0|_{C^\delta} &\leq C\|e^{-A^{\alpha/2}t} - e^{-A^{\alpha/2}\tau}\|_{\mathcal{L}(H_2^{-\frac{\alpha}{2}}, C^\delta)}\|P_N\|_{\mathcal{L}(H_2^{-\frac{\alpha}{2}})}|u_0|_{H_2^{-\frac{\alpha}{2}}} \\ &\leq C_{\alpha,\delta,\tau}|t-\tau|^{(\frac{\alpha-1-2\delta}{2\alpha})-\kappa}|u_0|_{H_2^\beta}, \end{aligned} \quad (5.105)$$

for any $\kappa > 0$.

For the second term, we have,

$$\begin{aligned} &|\int_0^t e^{-A^{\alpha/2}(t-s)}P_N F(u_N(s))ds - \int_0^\tau e^{-A^{\alpha/2}(\tau-s)}P_N F(u_N(s))ds|_{C^\delta} \\ &\leq |\int_0^\tau [e^{-A^{\alpha/2}(t-s)} - e^{-A^{\alpha/2}(\tau-s)}]P_N F(u_N(s))ds|_{C^\delta} + |\int_\tau^t e^{-A^{\alpha/2}(t-s)}P_N F(u_N(s))ds|_{C^\delta} \\ &\leq \int_0^\tau \|e^{-A^{\alpha/2}(t-s)} - e^{-A^{\alpha/2}(\tau-s)}\|_{\mathcal{L}(H_2^{-\frac{\alpha}{2}}, C^\delta)}\|P_N\|_{\mathcal{L}(H_2^{-\frac{\alpha}{2}})}|F(u_N(s))|_{H_2^{-\frac{\alpha}{2}}}ds \\ &+ \int_\tau^t \|e^{-A^{\alpha/2}(t-s)}\|_{\mathcal{L}(H_2^{-\frac{\alpha}{2}}, C^\delta)}\|P_N\|_{\mathcal{L}(H_2^{-\frac{\alpha}{2}})}|F(u_N(s))|_{H_2^{-\frac{\alpha}{2}}}ds. \end{aligned} \quad (5.106)$$

Using Lemma 5.5 and Lemma 5.11, we infer that, for $\epsilon_1, \epsilon_2 \in (0, \frac{\alpha-1-2\delta}{2\alpha} - \kappa)$, for any $\kappa > 0$,

$$\begin{aligned} &|\int_0^t e^{-A^{\alpha/2}(t-s)}P_N F(u_N(s))ds - \int_0^\tau e^{-A^{\alpha/2}(\tau-s)}P_N F(u_N(s))ds|_{C^\delta} \\ &\leq C_\alpha \int_0^\tau (t-\tau)^{(\frac{\alpha-1-2\delta}{2\alpha}-\kappa)-\epsilon_1}(\tau-s)^{-1+\epsilon_2}|F(u_N(s))|_{H_2^{-\frac{\alpha}{2}}}ds \\ &+ C_\alpha \int_\tau^t (t-s)^{-1+\epsilon_2}|F(u_N(s))|_{H_2^{-\frac{\alpha}{2}}}ds. \end{aligned} \quad (5.107)$$

Let us now, remark that thanks to the uniform boundedness of $(u_N)_N$ with respect to t and N, we can choose $R(\omega) = \sup_N \sup_{t\in[0,T]} |u_N(t,\omega)|_{C^\delta}$, for $a.s.\omega \in \Omega$ and by application of Corollary 5.10, we infer the existence of a random variable $C_{F,\alpha,\delta}(\omega)$, such that

$$|F(u_N(s,\omega))|_{H_2^{-\frac{\alpha}{2}}} \leq C_{F,\alpha,\delta}(\omega)(1 + |u_N(s,\omega)|_{C^\delta}) \leq C_{F,\alpha,\delta}(\omega). \tag{5.108}$$

Thus

$$\begin{aligned} &|\int_0^t e^{-A^{\alpha/2}(t-s)} P_N F(u_N(s))ds - \int_0^\tau e^{-A^{\alpha/2}(\tau-s)} P_N F(u_N(s))ds|_{C^\delta} \\ &\leq \; C_{F,\alpha,\delta}(\cdot)\Big[\int_0^\tau (t-\tau)^{(\frac{\alpha-1-2\delta}{2\alpha}-\kappa)-\epsilon_1}(\tau-s)^{-1+\epsilon_2}ds + \int_\tau^t (t-s)^{-1+\epsilon_2}ds\Big] \\ &\leq \; C_{F,\alpha,\delta,T}(\cdot)\Big[(t-\tau)^{(\frac{\alpha-1-2\delta}{2\alpha}-\kappa)-\epsilon_1} + (t-\tau)^{\epsilon_2}\Big]. \end{aligned} \tag{5.109}$$

Now, it is easy to get the Hölder index, by taking $\epsilon_1 \to 0$ and $\epsilon_2 \to \frac{\alpha-1-2\delta}{2\alpha} - \kappa$.

Uniqueness. Assume that there exist two solutions u_N^1 and u_N^2 of Eq.(5.102) starting from the same initial condition u_0 and satisfying the boundedness, the regularity properties above, then using Corollary 5.2, the boundedness property of u_N^1 and u_N^2 and Lemma 5.7, we obtain $\mathbb{P}-a.s.$, for all $t \in (0,T]$

$$\begin{aligned} |u_N^1(t,\omega) \;&-\; u_N^2(t,\omega)|_{C^\delta} \leq \int_0^t |e^{-A^{\alpha/2}(t-s)} P_N(F(u_N^1(s,\omega)) - F(u_N^2(s,\omega)))|_{C^\delta} ds \\ &\leq \; \int_0^t \|e^{-A^{\alpha/2}(t-s)}\|_{\mathcal{L}(H_2^{-\frac{\alpha}{2}}, C^\delta)} \|P_N\|_{\mathcal{L}(H_2^{-\frac{\alpha}{2}})} |F(u_N^1(s,\omega)) - F(u_N^2(s,\omega))|_{H_2^{-\frac{\alpha}{2}}} ds \\ &\leq \; C_{F,\alpha,\delta}(\omega) \int_0^t (t-s)^{-(\frac{1+2\delta+\alpha}{2\alpha})-\kappa} |u_N^1(s,\omega) - u_N^2(s,\omega)|_{C^\delta} ds, \end{aligned} \tag{5.110}$$

for any $\kappa > 0$. By application of Gronwall lemma A.1 we get, $\mathbb{P}-a.s.$,

$$|u_N^1(t,\omega) - u_N^2(t,\omega)|_{C^\delta} = 0, \; \forall t \in (0,T].$$

Thus the uniqueness is proved.

5.5.2 Proof of Theorems 5.15 & 5.16

To prove theorems 5.15 & 5.16, we will mainly check that Assumptions 1-4 of Theorem 3.11 hold, with $V := C^\delta[0,1]$, $U := H_2^{-\alpha/2}(0,1)$ and $S(t) = e^{-A^{\alpha/2}t}$.

Assumption 1. Lemma 5.5 states that for $\alpha \in (1,2]$ and $\delta \in [0,1)$ the semigroup $e^{-A^{\alpha/2}\cdot} : [0,T] \to \mathcal{L}(H_2^{-\frac{\alpha}{2}}(0,1), C^\delta[0,1])$ is Hölderian so it is continuous. Moreover, for

$\delta < \frac{\alpha-1}{2}$ and thanks to Corollary 5.2, we have for all $\eta' \in (\frac{1+2\delta+\alpha}{2\alpha}, 1)$,

$$\sup_{t\in(0,T]} \left(t^{\eta'} \|e^{-A^{\alpha/2}t}\|_{\mathcal{L}(H_2^{-\alpha/2}, C^\delta)} \right) < \infty. \tag{5.111}$$

Now, we introduce the auxiliary parameter $\beta \in (\frac{\alpha}{2}, \alpha - \delta - \frac{1}{2})$. Thanks to Lemma 5.1 and Lemma 5.11, we infer that for all $\eta'' \in (\frac{1+2\delta+2\beta}{2\alpha}, 1)$, there exists $C_{\delta,\beta,\eta''} > 0$, s.t.

$$\begin{aligned} \|(I-P_N)e^{-A^{\alpha/2}t}\|_{\mathcal{L}(H_2^{-\frac{\alpha}{2}}, C^\delta)} &= \|e^{-A^{\alpha/2}t}(I-P_N)\|_{\mathcal{L}(H_2^{-\frac{\alpha}{2}}, C^\delta)} \\ &\leq \|e^{-A^{\alpha/2}t}\|_{\mathcal{L}(H_2^{-\beta}, C^\delta)} \|I-P_N\|_{\mathcal{L}(H_2^{-\frac{\alpha}{2}}, H_2^{-\beta})} \\ &\leq C_{\delta,\beta,\eta''} t^{-\eta''} N^{-(\beta-\frac{\alpha}{2})}. \end{aligned} \tag{5.112}$$

We consider $\beta = (\alpha - \delta - \frac{1}{2}) - \kappa$, for any $\kappa > 0$ and we take $\eta' = \eta'' := 1 - \epsilon$, with $\epsilon \in (0, \frac{\alpha-1-2\delta}{2\alpha})$, then we get the estimate

$$\|(I-P_N)e^{-A^{\alpha/2}t}\|_{\mathcal{L}(H_2^{-\frac{\alpha}{2}}, C^\delta)} \leq C_{\delta,\beta,\eta''} t^{-1+\epsilon} N^{-(\frac{\alpha-1}{2}-\delta)+\kappa}. \tag{5.113}$$

Consequently, Assumption 1 is satisfied.

Assumption 2. This assumption is satisfied, for the fractional stochastic Burgers type equations, thanks to Corollary 5.8, provided that $1 < \alpha \leq 2$ and $\delta > 1 - \frac{\alpha}{2}$.

As a result, Assumptions 1 & 2 are simultaneously satisfied for $3/2 < \alpha \leq 2$ and $\delta \in (1 - \frac{\alpha}{2}, \frac{\alpha-1}{2})$, see also Corollary 5.9.

Assumption 3. Corollary 5.24 shows the continuity of the process $\mathcal{W}$. Moreover, it is easy to see that Est.(5.105) is still valid without P_N. Thus, the process $e^{-A^{\alpha/2}t}u_0 : \Omega \to C^\delta[0,1]$ and consequently the process $O(t) := e^{-tA^{\alpha/2}}u_0 + \mathcal{W}(t) : \Omega \to C^\delta[0,1]$ are continuous. In addition, thanks to Lemma 5.11, the fact that $\alpha < 2 + \delta$, Assumption $\mathcal{A}$ with $s \geq \beta > \delta + \frac{1}{2}$ and the embeddings $H_2^\beta(0,1) \hookrightarrow H_2^{\delta+\frac{1}{2}}(0,1) \hookrightarrow H_2^{\alpha/2}(0,1)$, we have $\mathbb{P} - a.s.$

$$\begin{aligned} |(I-P_N)e^{-A^{\alpha/2}t}u_0(\omega)|_{C^\delta} &\leq \|(I-P_N)e^{-A^{\alpha/2}t}\|_{\mathcal{L}(H_2^{\alpha/2}, C^\delta)} |u_0(\omega)|_{H_2^{\alpha/2}} \\ &\leq C_{\alpha,\delta}(\omega) N^{-(\frac{\alpha-1}{2}-\delta)+\kappa} |u_0(\omega)|_{H_2^\beta}. \end{aligned} \tag{5.114}$$

Now, Corollary 5.21 and Est.(5.114) together show that $O(t)$ satisfies $\mathbb{P} - a.s.$

$$\sup_{N\in\mathbb{N}} \sup_{t\in(0,T]} \left(N^{(\frac{\alpha-1}{2}-\delta)-\kappa} |(I-P_N)O(t,\omega)|_{C^\delta} \right) < \infty, \tag{5.115}$$

For any $\kappa > 0$. Thus Assumption 3 is fulfilled.

Assumption 4. This assumption is satisfied thanks to Theorem 5.14.

5.5.3 Proof of Lemma 5.13

Using Eq.(5.44), we rewrite $u^m_{N,M}$ as

$$\begin{cases} u^0_{N,M} & := P_N u_0, \\ u^m_{N,M} & = e^{-A^{\alpha/2}t_m} u^0_{N,M} + \int_0^{t_m} e^{-A^{\alpha/2}(t_m-s)} \Delta t \sum_{k=0}^{m-1} \delta_{t_k}(s) P_N F(u^k_{N,M}) ds + \mathcal{W}_N(t_m). \end{cases} \tag{5.116}$$

We introduce the following problem:

$$\begin{cases} Z_{N,M}(0) & := P_N u_0, \\ \quad for & t \in (t_{m-1}, t_m] : \\ Z_{N,M}(t) & = e^{-A^{\alpha/2}t} P_N u_0 + \int_0^t e^{-A^{\alpha/2}(t-s)} \Delta t \sum_{k=0}^{m-1} \delta_{t_k}(s) P_N F(Z_{N,M}(s) + \mathcal{W}_N(t_m)) ds. \end{cases} \tag{5.117}$$

We argue as in the proof of Lemma 5.28 using Lemma 5.22, we prove the existence of a stochastic process $Z_{N,M}$ solution of Eq.(5.117) and satisfying

$$\sup_{N,M} \sup_{t\in[0,T]} |y_{N,M}(t,\omega)|_{C^\delta} \leq C_{\alpha,\delta}(\omega).$$

It is easy to see that $Z_{N,M}(t_m) = u^m_{N,M} - \mathcal{W}_N(t_m)$, where $u^m_{N,M}$ is solution of Eq.(5.116). Now, argue as in the proof of Theorem 5.14, we infer that $u^m_{N,M}$ exists and fulfill Est.(5.45).

5.5.4 Proof of Theorem 5.17

Using the triangular inequality and Theorem 5.16, in particular Est.(5.47), we get

$$\begin{aligned} |u(t_m) - u^m_{N,M}|_{C^\delta} &\leq |u(t_m) - u_N(t_m)|_{C^\delta} + |u_N(t_m) - u^m_{N,M}|_{C^\delta} \\ &\leq CN^{-(\frac{\alpha-1}{2}-\delta)+\kappa} + |u_N(t_m) - u^m_{N,M}|_{C^\delta}, \end{aligned} \tag{5.118}$$

for any $\kappa > 0$. In the aim to estimate the term $|u_N(t_m) - u^m_{N,M}|_{C^\delta}$, we rewrite $u^m_{N,M}$ as:

$$u^m_{N,M} := e^{-A^{\alpha/2}t_m} u^0_N + \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} e^{-A^{\alpha/2}(t_m-t_k)} P_N F(u^k_{N,M}) ds + \mathcal{W}_N(t_m). \tag{5.119}$$

Then,

$$\begin{aligned} |u_N(t_m) - u^m_{N,M}|_{C^\delta} &\leq \sum_{k=0}^{m-1} |\int_{t_k}^{t_{k+1}} [e^{-A^{\alpha/2}(t_m-s)} P_N F(u_N(s)) - e^{-A^{\alpha/2}(t_m-t_k)} P_N F(u^k_{N,M})] ds|_{C^\delta} \\ &\leq J_1 + J_2 + J_3, \end{aligned} \tag{5.120}$$

where

$$J_1 := \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} |e^{-A^{\alpha/2}(t_m-s)} P_N [F(u_N(s)) - F(u_N(t_k))]|_{C^\delta} ds, \tag{5.121}$$

$$J_2 \;:=\; \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} |[e^{-A^{\alpha/2}(t_m-s)} - e^{-A^{\alpha/2}(t_m-t_k)}]P_N F(u_N(t_k)|_{C^\delta} ds \qquad (5.122)$$

and

$$J_3 \;:=\; \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} |e^{-A^{\alpha/2}(t_m-t_k)}P_N[F(u_N(t_k) - F(u^k_{N,M})]|_{C^\delta} ds. \qquad (5.123)$$

Now, we estimate the terms J_1, J_2, J_3. But, first of all, let us remark that thanks to the uniform boundedness of $(u_N)_N$ with respect to t and N and of $u^m_{N,M}$ with respect to m, N, M, we can choose $R(\omega) = \max\{\sup_N \sup_{t\in[0,T]} |u_N(t,\omega)|_{C^\delta}, \sup_{m,N,M} |u^m_{N,M}(\omega)|_{C^\delta}\}$, for $a.s.\omega \in \Omega$ and by application of Lemma 5.7, we infer the existence of a random variable $C_{F,\alpha,\delta}(\omega)$, such that

$$|F(u_N(s,\omega)) - F(u^m_{N,M}(\omega))|_{H_2^{-\frac{\alpha}{2}}} \leq C_{F,\alpha,\delta}(\omega)(|u_N(s,\omega) - u^m_{N,M}(\omega)|_{C^\delta}) \qquad (5.124)$$

and

$$|F(u_N(s,\omega)) - F(u_N(t,\omega))|_{H_2^{-\frac{\alpha}{2}}} \leq C_{F,\alpha,\delta}(\omega)(|u_N(s,\omega) - u_N(t,\omega)|_{C^\delta}). \qquad (5.125)$$

To estimate J_1, we use Corollary 5.2, Lemma 5.11, in particular, Est.(5.31) and Est.(5.125), we get

$$\begin{aligned} J_1 &\leq \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \|e^{-A^{\alpha/2}(t_m-s)}\|_{\mathcal{L}(H_2^{-\alpha/2},C^\delta)}\|P_N\|_{\mathcal{L}(H_2^{-\frac{\alpha}{2}})}|F(u_N(s)) - F(u_N(t_k)|_{H_2^{-\frac{\alpha}{2}}} ds \\ &\leq C_{F,\alpha,\delta}\Big(\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} (t_m-s)^{-(\frac{1}{2\alpha}(\alpha+2\delta+1)+\kappa')}|u_N(s) - u_N(t_k)|_{C^\delta} ds\Big), \qquad (5.126) \end{aligned}$$

for any $\kappa' > 0$. Thanks to the regularity of the Galerkin solution, see Theorem 5.14, we infer that

$$J_1 \;\leq\; C_{F,\alpha,\delta}\Big(\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} (t_m-s)^{-(\frac{1}{2\alpha}(\alpha+2\delta+1)+\kappa')}(s-t_k)^{(\frac{\alpha-1-2\delta}{2\alpha})-\kappa} ds\Big), \qquad (5.127)$$

for any $\kappa > 0$. As $(s - t_k) \leq \Delta t$ for all $s \in [t_k, t_{k+1}]$ and $\frac{1}{2\alpha}(\alpha + 2\delta + 1) + \kappa' < 1$, we get

$$\begin{aligned} J_1 &\leq C(\Delta t)^{(\frac{\alpha-1-2\delta}{2\alpha})-\kappa}\Big(\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} (t_m-s)^{-(\frac{1}{2\alpha}(\alpha+2\delta+1)+\kappa')} ds\Big) \\ &\leq C(\Delta t)^{(\frac{\alpha-1-2\delta}{2\alpha})-\kappa}\int_0^{t_m} (t_m-s)^{-(\frac{1}{2\alpha}(\alpha+2\delta+1)+\kappa')} ds \\ &\leq C_{\alpha,\delta}T^{1-(\frac{1}{2\alpha}(\alpha+2\delta+1))+\kappa')}(\Delta t)^{(\frac{\alpha-1-2\delta}{2\alpha})-\kappa}. \qquad (5.128) \end{aligned}$$

To estimate J_2, we use Lemma 5.5, Lemma 5.11, in particular, Est.(5.31) and Est.(5.108), then we end up with the following estimate

$$\begin{aligned} J_2 &\leq \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \|e^{-A^{\alpha/2}(t_m-s)} - e^{-A^{\alpha/2}(t_m-t_k)}\|_{\mathcal{L}(H_2^{-\alpha/2},C^\delta)}\|P_N\|_{\mathcal{L}(H_2^{-\frac{\alpha}{2}})}|F(u_N(t_k)|_{H_2^{-\alpha/2}}ds \\ &\leq C_{F,\alpha,\delta}\Big(\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}(s-t_k)^\eta(t_m-s)^{-\gamma}(1+|u_N(t_k)|_{C^\delta})ds\Big), \end{aligned} \tag{5.129}$$

by taking $\gamma = 1-\kappa$ and $\eta = (\frac{\alpha-1-2\delta}{2\alpha})-\kappa$, for any $\kappa>0$ we deduce

$$J_2 \;\leq\; C_{F,\alpha,\delta}T^\kappa(\Delta t)^{(\frac{\alpha-1-2\delta}{2\alpha})-\kappa}. \tag{5.130}$$

Now, arguing as for the estimation of J_1, using Corollary 5.2, Lemma 5.11, in particular, Est.(5.31) and Est.(5.124), we get

$$\begin{aligned} J_3 &\leq \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \|e^{-A^{\alpha/2}(t_m-t_k)}\|_{\mathcal{L}(H_2^{-\alpha/2},C^\delta)}\|P_N\|_{\mathcal{L}H_2^{-\frac{\alpha}{2}}}|F(u_N(t_k)-F(u^k_{N,M})]|_{H_2^{-\alpha/2}}ds \\ &\leq C_{\alpha,\delta}\Big(\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}(t_m-t_k)^{-(\frac{\alpha+1+2\delta}{2\alpha})-\kappa'}|u_N(t_k)-u^k_{N,M}|_{C^\delta}ds\Big), \end{aligned} \tag{5.131}$$

for any $\kappa'>0$. Thus thanks to the estimates (5.120), (5.128), (5.130) and (5.131), we get

$$\begin{aligned} |u_N(t_m) \;-\; & u^m_{N,M}|_{C^\delta} \\ &\leq C_{\alpha,\delta,T}\Big(\sum_{k=0}^{m-1}(\int_{t_k}^{t_{k+1}}(t_m-t_k)^{-(\frac{\alpha+1+2\delta}{2\alpha})-\kappa'}ds)|u_N(t_k)-u^k_{N,M}|_{C^\delta}+(\Delta t)^{(\frac{\alpha-1-2\delta}{2\alpha})-\kappa}\Big), \end{aligned} \tag{5.132}$$

as $t_m - s\leq t_m-t_k$, for all $s\in[t_k,t_{k+1}]$ we get

$$\begin{aligned} |u_N(t_m) \;-\; & u^m_{N,M}|_{C^\delta} \\ &\leq C_{\alpha,\delta,T}\Big(\sum_{k=0}^{m-1}(\int_{t_k}^{t_{k+1}}(t_m-s)^{-(\frac{\alpha+1+2\delta}{2\alpha})-\kappa'}ds)|u_N(t_k)-u^k_{N,M}|_{C^\delta}+(\Delta t)^{(\frac{\alpha-1-2\delta}{2\alpha})-\kappa}\Big) \end{aligned} \tag{5.133}$$

The application of the discretized version of Gronwall lemma A.2 yields

$$\begin{aligned} |u_N(t_m)-u^m_{N,M}|_{C^\delta} &\leq C_{\alpha,\delta,T}(\Delta t)^{(\frac{\alpha-1-2\delta}{2\alpha})-\kappa}\, exp\Big(\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}(t_m-s)^{-(\frac{\alpha+1+2\delta}{2\alpha})-\kappa'}ds\Big) \\ &\leq C_{\alpha,\delta,T}(\Delta t)^{(\frac{\alpha-1-2\delta}{2\alpha})-\kappa}\, exp\Big(\int_0^{t_m}(t_m-s)^{-(\frac{\alpha+1+2\delta}{2\alpha})-\kappa'}ds\Big). \end{aligned} \tag{5.134}$$

Thanks to the fact that $\frac{\alpha+1+2\delta}{2\alpha}+\kappa'<1$, we obtain

$$|u_N(t_m)-u^m_{N,M}|_{C^\delta}\leq C_{\alpha,\delta,T}(\Delta t)^{(\frac{\alpha-1-2\delta}{2\alpha})-\kappa}. \tag{5.135}$$

Chapter 6

Numerical approximations for fractional stochastic nonlinear heat equation in Hilbert space- Temporal, spacial and full approximations-

6.1 Introduction

In this second contribution, we deal with the fractional stochastic nonlinear heat equation (FSNHE) with multiplicative noise. This means, we consider Eq.(5.1), with the nonlinear operator F being globally Lipschitz with nonlinear growth condition instead of locally Lipschitz. Moreover, we study this equation in the Hilbert space $L^2(0,1)$ instead of the Hölder space $C^\delta(0,1)$. We establish temporal, spacial and full schemes for this equation and we prove their convergence. Regarding the relaxed condition imposed on the nonlinear term, we improve either the rate of convergence and the constraint on the diffusion dissipation index α. Recall that, in our first contribution Chapter 5, we studied the additive fractional stochastic Burgers-type equation in the Hölder space $C^\delta(0,1)$. The full approximation has been proved with the rate of convergence $(N^{-\zeta} + (\Delta t)^\xi)$, with $\zeta < \frac{\alpha-1}{2} - \delta$ and $\xi < \frac{\alpha-1}{2\alpha} - \frac{\delta}{\alpha}$, for $\alpha \in (\frac{7}{4}, 2]$ and $\delta \in (1 - \frac{\alpha}{2}, \frac{2\alpha-3}{2})$. In this chapter, we fill the gap $\alpha \in (1, \frac{7}{4}]$ for the heat equation. Specially, we prove the strong convergence (i.e. in the space $L^p(\Omega, L^2(0,1))$) of the temporal implicit scheme via Euler method, the strong convergence of the spacial approximation via spectral Galerkin method and the strong convergence of the space-time approximation obtained as a combination of the temporal and the spacial schemes.

The structure of this chapter is as follows: in Section 6.2, we give the main assumptions and definitions. Sections 6.4, 6.5 and 6.6 are devoted to the temporal, the spacial and the full approximations, respectively.

6.2 Preliminaries and Assumptions

Let us first, recall the formal fractional stochastic nonlinear heat equation on the bounded domain $[0,1]$ with Dirichlet boundary conditions:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} & = -(-\frac{\partial^2}{\partial^2 x})^{\frac{\alpha}{2}} u(t,x) + F(u(t))(x) + G(u(t))(x)\, \dfrac{\partial^2 W(t,x)}{\partial t\, \partial x},\ (t,x) \in (0,1)^2, \\ u(t,0) & = u(t,1) = 0,\ \forall\, t \in (0,1] \quad \text{(Dirichlet boundary conditions)}, \\ u(0,x) & = u_0(x),\ \forall\, x \in [0,1] \quad \text{(initial condition)}. \end{cases} \tag{6.1}$$

The evolution form of Eq.(6.1) is again given by Eq.(5.1), with $A_\alpha := (-\frac{\partial^2}{\partial^2 x})^{\frac{\alpha}{2}} = A^{\frac{\alpha}{2}}$, with $A = -\Delta$ endowed with the Dirichlet boundary conditions, see Section. 1.2.4. Now, let us take $\alpha \in (1,2]$, $p \geq 2$, H denotes the Hilbert space $L^2(0,1)$, $\langle .,. \rangle$ its inner product and we assume that:

Assumption $\mathcal{A}_1$. The initial condition u_0 is an $\mathcal{F}_0$-masurable random variable, satisfies $u_0 \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; H_2^\eta)$, with $\eta > 0$.

Assumption $\mathcal{B}_1$. The nonlinear operator $F : H \to D(A^{-\rho})$ with $0 \leq \rho < \frac{\alpha}{2}$ satisfies the global Lipschitz and the nonlinear growth conditions, i.e.

$$\exists C_{F1} > 0, s.t.\ |A^{-\rho}(F(x) - F(y))|_H \leq C_{F1}|x-y|_H \quad \text{for all } x, y \in H, \tag{6.2}$$

and

$$\exists C_F > 0, s.t.\ |A^{-\rho}F(x)|_H \leq C_F\,(1 + |x|_H) \quad \text{for all } x \in H. \tag{6.3}$$

Assumption $\mathcal{C}_1$. The operator $G : H \to \mathcal{L}(H)$, where $\mathcal{L}(H)$ is the set of linear bounded operators from H to H, (see the Definition 1.23) satisfies the global Lipschitz and the nonlinear growth conditions, i.e. $\exists C_G > 0$, s.t.

$$\|G(x) - G(y)\|_{\mathcal{L}(H)} \leq C_G |x - y|_H \quad \text{for all } x, y \in H, \tag{6.4}$$

and

$$\|G(x)\|_{\mathcal{L}(H)} \leq C_G \left(1 + |x|_H\right) \quad \text{for all } x \in H. \tag{6.5}$$

The random version of Assumptions $\mathcal{A}_1$ and $\mathcal{B}_1$.

In order to make the proofs in this chapter more easier when we use the Assumptions $\mathcal{A}_1$ and $\mathcal{B}_1$, we reformulate them here in the random context as follows; let x, y be two H-valued random variable. Then, we have

Assumption $\mathcal{A}_1^r$. The initial condition u_0 is an $\mathcal{F}_0$-masurable random variable, satisfying

$$\left(\mathbb{E}|A^{\frac{\eta}{2}} u_0|^p\right)^{\frac{1}{p}} < \infty,$$

with $\eta > 0$.

Assumption $\mathcal{B}_1^r$. The nonlinear operator $F : H \to D(A^{-\rho})$ with $0 \leq \rho < \frac{\alpha}{2}$ satisfies the global Lipschitz and the nonlinear growth conditions, i.e.

$$\|A^{-\rho}(F(x) - F(y))\|^p_{L^p(\Omega,H)} = \mathbb{E}|A^{-\rho}(F(x) - F(y))|^p_H \leq C^p_{F1} \mathbb{E}|(x - y)|^p_H \leq C^p_{F1} \|x - y\|^p_{L^p(\Omega,H)}, \tag{6.6}$$

for some positive constant C_{F1}, and

$$\|A^{-\rho} F(x)\|^p_{L^p(\Omega,H)} = \mathbb{E}|A^{-\rho} F(x)|^p_H \leq C^p_{F1} \mathbb{E}\left(1 + |x|_H\right)^p.$$

The fact that, for any $a, b \geq 0$, $(a + b)^p \leq C_p(a^p + b^p)$, for all $p \geq 1$ leads to,

$$C^p_{F,p} \mathbb{E}\left(1 + |x|_H\right)^p \leq C^p_{F1,p}\left(1 + \mathbb{E}|x|^p_H\right) = C^p_{F1,p}\left(1 + \|x\|^p_{L^p(\Omega,H)}\right). \tag{6.7}$$

Moreover, the use of the basic inequality $(a + b)^q \leq C_q(a^q + b^q)$, for any $a, b \geq 0$, and any $0 < q < 1$, yields

$$\|A^{-\rho} F(x)\|_{L^p(\Omega,H)} \leq C_{F1,p}\left(1 + \|x\|^p_{L^p(\Omega,H)}\right)^{\frac{1}{p}} \leq C_{F1,p}\left(1 + \|x\|_{L^p(\Omega,H)}\right), \tag{6.8}$$

for some positive constant $C_{F1,p}$.

Remark 6.1 *To avoid the repetitions, in the rest of this chapter, when we need to use estimations in the random context as it has been proved above for Assumptions $\mathcal{A}_1$ and $\mathcal{B}_1$, we will do it without proof.*

Let us metion here that, the wellposedness of Prb.(6.1) in the whole space $\mathbb{R}$ has been obtained in the random field-setting (see Theorem 3.15). However, to approximate the problem temporally we need its wellposedness in the L^2-setting. For this, in the rest of this chapter we assume that Prb.(6.1) admits an unique mild solution u given $\mathbb{P}$-a.s. by:

$$u(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha(t-s)F(u(s))\,ds + \int_0^t S_\alpha(t-s)G(u(s))\,dW(s), \quad \forall\, t \in [0,1], \tag{6.9}$$

where $S_\alpha(t) := e^{-A_\alpha t}$, for any $t \in [0,1]$.

6.3 Temporal regularity of the mild solution

Due to the fact that, the temporal regulariy of the mild solution of Prb.(6.1) has been proved in the random field setting as mentioned in Theorem 3.16, and regarding the importance of such peoperty in the proof of the temporal approximation, we prove it in this section in the L^2-setting.

Proposition 6.2 *According to Assumptions $\mathcal{A}_1^r$ with $\eta > \frac{\alpha-1}{2}$, $\mathcal{B}_1^r$ with $\rho \in [0,\frac{1}{2}]$, and $\mathcal{C}_1$, the mild solution u of Prb.(5.1) equivalently Prb.(6.1) is time Hölder continuous with exponent $\frac{\alpha-1}{2\alpha} - \kappa$, for any $\kappa > 0$. i.e.*

$$\|u(t) - u(s)\|_{L^p(\Omega,H)} \leq C_{(\|A^{\frac{\eta}{2}}u_0\|_{L^p(\Omega,H)},p,\alpha)}(t-s)^{\frac{\alpha-1}{2\alpha}-\kappa}, \tag{6.10}$$

for all $0 \leq s < t \leq 1$.

In order to prove this proposition, we need the following useful result.

Lemma 6.3 *For all $\xi > \frac{1}{4}$, there exists $C_\xi > 0$, s.t.*

$$\|A^{-\xi}\|_{HS} \leq C_\xi. \tag{6.11}$$

Let $\alpha \in (1,2]$. For any $0 \leq s < t \leq 1$, we have

- *For all $\zeta > 0$ and all $0 < \gamma < \frac{\zeta}{\alpha}$, there exists $C_\gamma > 0$, s.t.*

$$\|A^{-\frac{\zeta}{2}}(S_\alpha(t) - S_\alpha(s))\|_{\mathcal{L}(H)} \leq C_\gamma (t-s)^\gamma. \tag{6.12}$$

- *For all $\xi \geq 0$ and all $\gamma \in (\frac{2}{\alpha}\xi, 1 + \frac{2}{\alpha}\xi)$,there exists $C_{\gamma,\alpha,\xi} > 0$, s.t.*

$$\|A^{\xi}(S_\alpha(t) - S_\alpha(s))\|_{\mathcal{L}(H)} \leq C_{\gamma,\alpha,\xi} s^{-\gamma} (t-s)^{\gamma - \frac{2}{\alpha}\xi}. \tag{6.13}$$

Proof. For the first estimate, let $\xi > \frac{1}{4}$, the use of the definition of the Hilbert-Schmidt norm (see Identity.(1.13)), with the facts that $|e_j|_H = 1$ and $\lambda_j = (j\pi)^2$, leads to

$$\|A^{-\xi}\|_{HS}^2 = \sum_{j\in\mathbb{N}_0} |A^{-\xi} e_j|_H^2 = \sum_{j\in\mathbb{N}_0} |\lambda_j^{-\xi} e_j|_H^2 = \sum_{j\in\mathbb{N}_0} \lambda_j^{-2\xi} |e_j|_H^2 = \sum_{j\in\mathbb{N}_0} \lambda_j^{-2\xi} = \pi^{-4\xi} \sum_{j\in\mathbb{N}_0} j^{-4\xi}.$$

Riemman series $\sum_{j\in\mathbb{N}_0} j^{-4\xi}$ converges, thanks to $\xi > \frac{1}{4}$, and so

$$\|A^{-\xi}\|_{HS} \leq C_\xi.$$

For the second estimate, we use Identity.(1.12) in Lemma 1.36 to get

$$\|A^{-\frac{\zeta}{2}}(S_\alpha(t) - S_\alpha(s))\|_{\mathcal{L}(H)} = \sup_{1\leq j\leq\infty} |\lambda_j^{-\frac{\zeta}{2}}(e^{-t\lambda_j^{\frac{\alpha}{2}}} - e^{-s\lambda_j^{\frac{\alpha}{2}}})|.$$

Since the function $e^{-x} \in C^1$, then it is Lipschitz and γ-Hölder continuous for any $\gamma \in (0,1)$. Therefore, there exists $C_\gamma > 0$ s.t.

$$|\lambda_j^{-\frac{\zeta}{2}}(e^{-t\lambda_j^{\frac{\alpha}{2}}} - e^{-s\lambda_j^{\frac{\alpha}{2}}})| \leq C_\gamma (t-s)^\gamma (\lambda_j^{-\frac{\zeta}{2}} \lambda_j^{\frac{\alpha\gamma}{2}}) = C_\gamma (t-s)^\gamma (\lambda_j^{-(\frac{\zeta}{2} - \frac{\alpha\gamma}{2})}).$$

Let $0 < \gamma < \frac{\zeta}{\alpha}$, then $\zeta - \alpha\gamma > 0$. We have $\lambda_j = (j\pi)^2 > 1$, and so $\lambda_j^{\frac{1}{2}(\zeta - \alpha\gamma)} > 1$. Hence

$$\|A^{-\frac{\zeta}{2}}(S_\alpha(t) - S_\alpha(s))\|_{\mathcal{L}(H)} \leq C_\gamma (t-s)^\gamma \sup_{1\leq j\leq\infty} (\lambda_j^{-\frac{1}{2}(\zeta - \alpha\gamma)}) \leq C_\gamma (t-s)^\gamma.$$

For the last estimate, let $\xi \in [0, \frac{\alpha}{2})$, by using Identity.(1.12) in Lemma 1.36, we have

$$\|A^{\xi}(S_\alpha(t) - S_\alpha(s))\|_{\mathcal{L}(H)} = \sup_{1\leq j\leq\infty} |\lambda_j^{\xi}(e^{-t\lambda_j^{\frac{\alpha}{2}}} - e^{-s\lambda_j^{\frac{\alpha}{2}}})| = \sup_{1\leq j\leq\infty} |\lambda_j^{\xi} e^{-s\lambda_j^{\frac{\alpha}{2}}}(e^{-(t-s)\lambda_j^{\frac{\alpha}{2}}} - 1)|. \tag{6.14}$$

We use Lemma A.8 (with $\gamma \in (\frac{2}{\alpha}\xi, 1)$) and Lemma A.9 (with $\beta = \gamma - \frac{2}{\alpha}\xi$), to get

$$\begin{aligned}
\|A^{\xi}(S_\alpha(t) - S_\alpha(s))\|_{\mathcal{L}(H)} &= \sup_{1\leq j\leq\infty} |\lambda_j^{\xi} e^{-s\lambda_j^{\frac{\alpha}{2}}}(e^{-(t-s)\lambda_j^{\frac{\alpha}{2}}} - 1)| \\
&\leq C_\gamma s^{-\gamma} \sup_{1\leq j\leq\infty} |\lambda_j^{\xi} \lambda_j^{-\frac{\alpha}{2}\gamma}(e^{-(t-s)\lambda_j^{\frac{\alpha}{2}}} - 1)| = C_\gamma s^{-\gamma} \sup_{1\leq j\leq\infty} |\lambda_j^{-(\frac{\alpha}{2}\gamma - \xi)}(e^{-(t-s)\lambda_j^{\frac{\alpha}{2}}} - 1)| \\
&\leq C_{\gamma,\alpha,\xi} s^{-\gamma}(t-s)^{\gamma - \frac{2}{\alpha}\xi} \sup_{1\leq j\leq\infty} |\lambda_j^{-(\frac{\alpha\gamma}{2} - \xi)} \lambda_j^{(\frac{\alpha\gamma}{2} - \xi)}| \leq C_{\gamma,\alpha,\xi} s^{-\gamma}(t-s)^{\gamma - \frac{2}{\alpha}\xi}.
\end{aligned} \tag{6.15}$$

■

Proof of Proposition 6.2

From Eq.(6.9) we have

$$\begin{aligned}\|u(t)-u(s)\|_{L^p(\Omega,H)} &\leq \|(S_\alpha(t)-S_\alpha(s))u_0\|_{L^p(\Omega,H)}\\ &+ \|\int_0^t S_\alpha(t-r)F(u(r))dr - \int_0^s S_\alpha(s-r)F(u(r))dr\|_{L^p(\Omega,H)}\\ &+ \|\int_0^t S_\alpha(t-r)G(u(r))dW(r) - \int_0^s S_\alpha(s-r)G(u(r))dW(r)\|_{L^p(\Omega,H)}.\end{aligned} \tag{6.16}$$

To estimate the first term in the RHS of Est.(6.16), we use the fact that $A^{\frac{\eta}{2}}$ commutes with the semigroup $S_\alpha(t)$ and Assumption $\mathcal{A}_1^r$ with $\eta > 0$, to obtain

$$\begin{aligned}\|(S_\alpha(t)-S_\alpha(s))u_0\|_{L^p(\Omega,H)} &= \|A^{-\frac{\eta}{2}}(S_\alpha(t)-S_\alpha(s))A^{\frac{\eta}{2}}u_0\|_{L^p(\Omega,H)}\\ &\leq \|A^{-\frac{\eta}{2}}(S_\alpha(t)-S_\alpha(s))\|_{\mathcal{L}(H)}\|A^{\frac{\eta}{2}}u_0\|_{L^p(\Omega,H)}.\end{aligned} \tag{6.17}$$

Thus, by replacing Est.(6.12) (with $\zeta = \eta$ and $\gamma = \beta$) in Est.(6.17) it holds

$$\|(S_\alpha(t)-S_\alpha(s))u_0\|_{L^p(\Omega,H)} \leq C_{(\beta,\|A^{\frac{\eta}{2}}u_0\|_{L^p(\Omega,H)})}(t-s)^\beta, \tag{6.18}$$

where $\eta > 0$ and $\beta \in (0, \frac{\eta}{\alpha})$.

To estimate the second term in the RHS of Est.(6.16), first we have

$$\begin{aligned}\|\int_0^t S_\alpha(t-r)F(u(r))dr &- \int_0^s S_\alpha(s-r)F(u(r))dr\|_{L^p(\Omega,H)}\\ &\leq \|\int_0^s (S_\alpha(t-r)-S_\alpha(s-r))F(u(r))dr\|_{L^p(\Omega,H)}\\ &+ \|\int_s^t S_\alpha(t-r)F(u(r))dr\|_{L^p(\Omega,H)} := S1 + S2.\end{aligned} \tag{6.19}$$

Second, we estimate $S1$ by using the commutative property of A^ρ and $S_\alpha(t)$ and Assumption $\mathcal{B}_1^r$ with $0 \leq \rho < \frac{\alpha}{2}$, as follows

$$\begin{aligned}S1 &= \|\int_0^s (S_\alpha(t-r)-S_\alpha(s-r))F(u(r))dr\|_{L^p(\Omega,H)}\\ &= \|\int_0^s A^\rho(S_\alpha(t-r)-S_\alpha(s-r))A^{-\rho}F(u(r))dr\|_{L^p(\Omega,H)}\\ &\leq \int_0^s \|A^\rho(S_\alpha(t-r)-S_\alpha(s-r))\|_{\mathcal{L}(H)}\|A^{-\rho}F(u(r))\|_{L^p(\Omega,H)}dr\\ &\leq C_{F1,p}\int_0^s \|A^\rho(S_\alpha(t-r)-S_\alpha(s-r))\|_{\mathcal{L}(H)}\left(1+\|u(r)\|_{L^p(\Omega,H)}\right)dr\\ &\leq C_{F1,p}\left(1+\sup_{r\in[0,1]}\|u(r)\|_{L^p(\Omega,H)}\right)\int_0^s \|A^\rho(S_\alpha(t-r)-S_\alpha(s-r))\|_{\mathcal{L}(H)}dr.\end{aligned} \tag{6.20}$$

By replacing Est.(6.13) (with $\xi = \rho \in [0, \frac{\alpha}{2})$ and $\gamma \in (\frac{2}{\alpha}\rho, 1)$) in Est.(6.20), we obtain

$$S1 \leq C_{(F1,p,\sup_{r\in[0,1]}\|u(r)\|_{L^p(\Omega,H)},\gamma,\alpha,\rho)}(t-s)^{\gamma-\frac{2}{\alpha}\rho}\int_0^s (s-r)^{-\gamma}dr.$$

The integral $\int_0^s (s-r)^{-\gamma}dr$ is finite, thanks to the condition $\gamma < 1$. And so,

$$S1 \leq C_{(F1,p,\sup_{r\in[0,1]}\|u(r)\|_{L^p(\Omega,H)},\gamma,\alpha,\rho)}(t-s)^{\gamma-\frac{2}{\alpha}\rho}. \tag{6.21}$$

Third, to estimate $S2$ we argue as above, to arrive at

$$\begin{aligned} S2 &= \|\int_s^t S_\alpha(t-r)F(u(r))dr\|_{L^p(\Omega,H)} \leq \int_s^t \|A^\rho S_\alpha(t-r)\|_{\mathcal{L}(H)}\|A^{-\rho}F(u(r))\|_{L^p(\Omega,H)}dr \\ &\leq C_{F1,p}\left(1+\sup_{r\in[0,1]}\|u(r)\|_{L^p(\Omega,H)}\right)\int_s^t \|A^\rho S_\alpha(t-r)\|_{\mathcal{L}(H)}dr. \end{aligned} \tag{6.22}$$

We replace Est.(1.23) (with $\beta = 2\rho$) in Est.(6.22), yields

$$S2 \leq C_{(F1,p,\alpha,\rho,\sup_{r\in[0,1]}\|u(r)\|_{L^p(\Omega,H)})}\int_s^t (t-r)^{-\frac{2}{\alpha}\rho}dr \leq C_{(F1,p,\alpha,\rho,\sup_{r\in[0,1]}\|u(r)\|_{L^p(\Omega,H)})}(t-s)^{1-\frac{2}{\alpha}\rho}. \tag{6.23}$$

Hence, from Est.(6.19), Est.(6.21) and Est.(6.23) we get

$$\begin{aligned} \|\int_0^t S_\alpha(t-r)F(u(r))dr &- \int_0^s S_\alpha(s-r)F(u(r))dr\|_{L^p(\Omega,H)} \leq C_{(p,\gamma,\alpha,\rho)}(t-s)^{\gamma-\frac{2}{\alpha}\rho} \\ &+ C_{(p,\alpha,\rho)}(t-s)^{1-\frac{2}{\alpha}\rho}. \end{aligned} \tag{6.24}$$

We have $\min\{\gamma - \frac{2}{\alpha}\rho, 1-\frac{2}{\alpha}\rho\} = \gamma - \frac{2}{\alpha}\rho$ because $\gamma < 1$, and so

$$\|\int_0^t S_\alpha(t-r)F(u(r))dr - \int_0^s S_\alpha(s-r)F(u(r))dr\|_{L^p(\Omega,H)} \leq C_{(p,\alpha,\rho,\gamma)}(t-s)^{\gamma-\frac{2}{\alpha}\rho}, \tag{6.25}$$

where $\rho \in [0, \frac{\alpha}{2})$ and $\gamma \in (\frac{2}{\alpha}\rho, 1)$.

About the stochastic term in the RHS of Est.(6.16), we have

$$\begin{aligned} \|\int_0^t S_\alpha(t-r)G(u(r))dW(r) &- \int_0^s S_\alpha(s-r)G(u(r))dW(r)\|_{L^p(\Omega,H)} \\ &\leq \|\int_0^s (S_\alpha(t-r)-S_\alpha(s-r))G(u(r))dW(r)\|_{L^p(\Omega,H)} \\ &+ \|\int_s^t S_\alpha(t-r)G(u(r))dW(r)\|_{L^p(\Omega,H)} := S3+S4. \end{aligned} \tag{6.26}$$

In order to estimate $S3$, we use Burkholder-Davis-Gundy inequality with $p \geq 2$, and the definition of the norm in $L^p(\Omega, \mathbb{R})$ to get

$$\begin{aligned} S3 &= \| \int_0^s (S_\alpha(t-r) - S_\alpha(s-r))G(u(r))dW(r)\|_{L^p(\Omega,H)} \\ &\leq C_p \left(\mathbb{E} \left(\int_0^s \|(S_\alpha(t-r) - S_\alpha(s-r))G(u(r))\|_{HS}^2 dr \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \\ &= C_p \| \left(\int_0^s \|(S_\alpha(t-r) - S_\alpha(s-r))G(u(r))\|_{HS}^2 dr \right) \|_{L^{\frac{p}{2}}(\Omega,\mathbb{R})}^{\frac{1}{2}}. \qquad (6.27) \end{aligned}$$

First, we estimate $\|(S_\alpha(t-r) - S_\alpha(s-r))G(u(r))\|_{HS}$ by using the fact that $\|AB\|_{HS} \leq \|A\|_{HS}\|B\|_{\mathcal{L}(H)}$, for every $A \in HS$ and every $B \in \mathcal{L}(H)$, and Assumption $\mathcal{C}_1$ as follows

$$\begin{aligned} \|(S_\alpha(t-r) - S_\alpha(s-r))G(u(r))\|_{HS} &\leq \|(S_\alpha(t-r) - S_\alpha(s-r))\|_{HS}\|G(u(r))\|_{\mathcal{L}(H)} \\ &\leq C_G(1 + |u(r)|_H)\|(S_\alpha(t-r) - S_\alpha(s-r))\|_{HS}. \end{aligned}$$

Now, let $\grave{\xi} \in (\frac{1}{4}, \frac{\alpha}{4})$, we use Est.(6.11) (with $\xi = \grave{\xi}$) and Est.(6.13) (with $\xi = \grave{\xi}$ and $\gamma = \grave{\gamma}$ s.t. $\grave{\gamma} \in (\frac{2}{\alpha}\grave{\xi}, \frac{1}{2})$), to obtain

$$\begin{aligned} \|(S_\alpha(t-r) &- S_\alpha(s-r))G(u(r))\|_{HS} \leq C_G(1 + |u(r)|_H)\|A^{-\grave{\xi}}(A^{\grave{\xi}}(S_\alpha(t-r) - S_\alpha(s-r)))\|_{HS} \\ &\leq C_G(1 + |u(r)|_H)\|A^{-\grave{\xi}}\|_{HS}\|A^{\grave{\xi}}(S_\alpha(t-r) - S_\alpha(s-r))\|_{\mathcal{L}(H)} \\ &\leq C_{G,\grave{\xi}}(1 + |u(r)|_H)\|A^{\grave{\xi}}(S_\alpha(t-r) - S_\alpha(s-r))\|_{\mathcal{L}(H)} \\ &\leq C_{G,\grave{\xi},\grave{\gamma},\alpha}(1 + |u(r)|_H)(s-r)^{-\grave{\gamma}}(t-s)^{\grave{\gamma}-\frac{2}{\alpha}\grave{\xi}}. \qquad (6.28) \end{aligned}$$

We replace Est.(6.28) in Est.(6.27), and use the fact that, for all $a, b > 0$, $(a+b)^2 \leq C(a^2 + a^2)$, for some $C > 0$, to get

$$\begin{aligned} S3 &= \| \int_0^s (S_\alpha(t-r) - S_\alpha(s-r))G(u(r))dW(r)\|_{L^p(\Omega,H)} \\ &\leq C_{p,G,\grave{\xi},\grave{\gamma},\alpha}(t-s)^{\grave{\gamma}-\frac{2}{\alpha}\grave{\xi}} \| \left(\int_0^s (1 + |u(r)|_H^2)(s-r)^{-2\grave{\gamma}} dr \right) \|_{L^{\frac{p}{2}}(\Omega,\mathbb{R})}^{\frac{1}{2}} \\ &\leq C_{p,G,\grave{\xi},\grave{\gamma},\alpha}(t-s)^{\grave{\gamma}-\frac{2}{\alpha}\grave{\xi}} \left(\int_0^s \|1 + |u(r)|_H^2\|_{L^{\frac{p}{2}}(\Omega,\mathbb{R})}(s-r)^{-2\grave{\gamma}} dr \right)^{\frac{1}{2}} \\ &\leq C_{p,G,\grave{\xi},\grave{\gamma},\alpha}(t-s)^{\grave{\gamma}-\frac{2}{\alpha}\grave{\xi}} \left(\int_0^s (1 + \||u(r)|_H^2\|_{L^{\frac{p}{2}}(\Omega,\mathbb{R})})(s-r)^{-2\grave{\gamma}} dr \right)^{\frac{1}{2}} \\ &= C_{p,G,\grave{\xi},\grave{\gamma},\alpha}(t-s)^{\grave{\gamma}-\frac{2}{\alpha}\grave{\xi}} \left(\int_0^s (1 + \|u(r)\|_{L^p(\Omega,H)}^2)(s-r)^{-2\grave{\gamma}} dr \right)^{\frac{1}{2}} \\ &\leq C_{p,G,\grave{\xi},\grave{\gamma},\alpha}(t-s)^{\grave{\gamma}-\frac{2}{\alpha}\grave{\xi}}(1 + \sup_{r\in[0,1]} \|u(r)\|_{L^p(\Omega,H)}^2)^{\frac{1}{2}} \left(\int_0^s (s-r)^{-2\grave{\gamma}} dr \right)^{\frac{1}{2}} \end{aligned}$$

The integral $\int_0^s (s-r)^{-2\grave{\gamma}} dr$ is finite, thanks to the condition $\grave{\gamma} < \frac{1}{2}$, and so

$$S3 \leq C_{(p,G,\grave{\xi},\grave{\gamma},\alpha,\sup_{r\in[0,1]}\|u(r)\|^2_{L^p(\Omega,\mathbb{H})})}(t-s)^{\grave{\gamma}-\frac{2}{\alpha}\grave{\xi}}. \qquad (6.29)$$

In order to estimate $S4$, we argue as above. Then, by using Burkholder-Davis-Gundy inequality with $p \geq 2$, and the definition of the norm in $L^p(\Omega,\mathbb{R})$ we obtain

$$\begin{aligned} S4 &= \| \int_s^t S_\alpha(t-r)G(u(r))dW(r)\|_{L^p(\Omega,H)} \\ &\leq C_p \| \left(\int_s^t \|S_\alpha(t-r)G(u(r))\|^2_{HS} dr \right) \|^{\frac{1}{2}}_{L^{\frac{p}{2}}(\Omega,\mathbb{R})}. \end{aligned} \qquad (6.30)$$

The use of the fact that $\|AB\|_{HS} \leq \|A\|_{HS}\|B\|_{\mathcal{L}(H)}$, for every $A \in HS$ and every $B \in \mathcal{L}(H)$, and Assumption $\mathcal{C}_1$, yields

$$\begin{aligned} \|S_\alpha(t-r)G(u(r))\|_{HS} &\leq \|S_\alpha(t-r)\|_{HS}\|G(u(r))\|_{\mathcal{L}(H)} \\ &\leq C_G(1+|u(r)|_H)\|S_\alpha(t-r)\|_{HS}. \end{aligned}$$

Let $\grave{\xi} \in (\frac{1}{4}, \frac{\alpha}{4})$. We use Est.(6.11) (with $\xi = \grave{\xi}$) and Est.(1.23) (with $\beta = 2\grave{\xi}$), to obtain

$$\begin{aligned} \|S_\alpha(t-r)G(u(r))\|_{HS} &\leq C_G(1+|u(r)|_H)\|A^{-\grave{\xi}}(A^{\grave{\xi}}S_\alpha(t-r))\|_{HS} \\ &\leq C_G(1+|u(r)|_H)\|A^{-\grave{\xi}}\|_{HS}\|A^{\grave{\xi}}S_\alpha(t-r)\|_{\mathcal{L}(H)} \\ &\leq C_{G,\grave{\xi}}(1+|u(r)|_H)\|A^{\grave{\xi}}S_\alpha(t-r))\|_{\mathcal{L}(H)} \\ &\leq C_{G,\grave{\xi},\alpha}(1+|u(r)|_H)(t-r)^{-\frac{2}{\alpha}\grave{\xi}}. \end{aligned} \qquad (6.31)$$

By replacing Est.(6.31) in Est.(6.30), and using the fact that, for all $a, b > 0$, $(a+b)^2 \leq C(a^2+a^2)$, for some $C > 0$, we get

$$\begin{aligned} S4 &= \| \int_s^t S_\alpha(t-r)G(u(r))dW(r)\|_{L^p(\Omega,H)} \\ &\leq C_{p,G,\grave{\xi},\alpha} \| \left(\int_s^t (1+|u(r)|^2_H)(t-r)^{-\frac{4}{\alpha}\grave{\xi}} dr \right) \|^{\frac{1}{2}}_{L^{\frac{p}{2}}(\Omega,\mathbb{R})} \\ &\leq C_{p,G,\grave{\xi},\alpha} \left(\int_s^t \|1+|u(r)|^2_H\|_{L^{\frac{p}{2}}(\Omega,\mathbb{R})}(t-r)^{-\frac{4}{\alpha}\grave{\xi}} dr \right)^{\frac{1}{2}} \\ &\leq C_{p,G,\grave{\xi},\alpha} \left(\int_s^t (1+\||u(r)|^2_H\|_{L^{\frac{p}{2}}(\Omega,\mathbb{R})})(t-r)^{-\frac{4}{\alpha}\grave{\xi}} dr \right)^{\frac{1}{2}} \\ &= C_{p,G,\grave{\xi},\alpha} \left(\int_s^t (1+\|u(r)\|^2_{L^p(\Omega,H)})(t-r)^{-\frac{4}{\alpha}\grave{\xi}} dr \right)^{\frac{1}{2}} \\ &\leq C_{p,G,\grave{\xi},\alpha}(1+\sup_{r\in[0,1]}\|u(r)\|^2_{L^p(\Omega,H)})^{\frac{1}{2}} \left(\int_s^t (t-r)^{-\frac{4}{\alpha}\grave{\xi}} dr \right)^{\frac{1}{2}} \end{aligned}$$

From the condition $\grave{\xi} < \frac{\alpha}{4}$, it holds $\int_s^t (t-r)^{-\frac{4}{\alpha}\grave{\xi}} dr = \frac{1}{1-\frac{4}{\alpha}\grave{\xi}}(t-s)^{1-\frac{4}{\alpha}\grave{\xi}}$, and so

$$S4 \leq C_{(p,G,\grave{\xi},\alpha,\sup_{r\in[0,1]}\|u(r)\|^2_{L^p(\Omega,\mathbb{H})})}(t-s)^{\frac{1}{2}-\frac{2}{\alpha}\grave{\xi}}. \tag{6.32}$$

Comming back to Est.(6.26), then from Est.(6.29) and Est.(6.32), we arrive at

$$\begin{aligned} \| \int_0^t S_\alpha(t-r)G(u(r))dW(r) &- \int_0^s S_\alpha(s-r)G(u(r))dW(r)\|_{L^p(\Omega,H)} \\ &\leq C_{(p,\alpha,\grave{\xi},\grave{\gamma})}\left((t-s)^{\grave{\gamma}-\frac{2}{\alpha}\grave{\xi}} + (t-s)^{\frac{1}{2}-\frac{2}{\alpha}\grave{\xi}}\right). \end{aligned}$$

Since $\grave{\gamma} < \frac{1}{2}$, then $\min\{\grave{\gamma} - \frac{2}{\alpha}\grave{\xi}, 1 - \frac{2}{\alpha}\grave{\xi}\} = \grave{\gamma} - \frac{2}{\alpha}\grave{\xi}$, and so

$$\| \int_0^t S_\alpha(t-r)G(u(r))dW(r) - \int_0^s S_\alpha(s-r)G(u(r))dW(r)\|_{L^p(\Omega,H)} \leq C_{(p,\alpha,\grave{\xi},\grave{\gamma})}(t-s)^{\grave{\gamma}-\frac{2}{\alpha}\grave{\xi}}, \tag{6.33}$$

where $\grave{\xi} \in (\frac{1}{4}, \frac{\alpha}{4})$ and $\grave{\gamma} \in (\frac{2}{\alpha}\grave{\xi}, \frac{1}{2})$.

Now, replacing Est.(6.18), Est.(6.25) and Est.(6.33) in Est.(6.16), to get

$$\begin{aligned} \|u(t) - u(s)\|_{L^p(\Omega,H)} &\leq C_{(\beta,\|A^{\frac{\eta}{2}}u_0\|_{L^p(\Omega,H)})}(t-s)^\beta + C_{(p,\alpha,\rho,\gamma)}(t-s)^{\gamma-\frac{2}{\alpha}\rho} \\ &+ C_{(p,\alpha,\grave{\xi},\grave{\gamma})}(t-s)^{\grave{\gamma}-\frac{2}{\alpha}\grave{\xi}}, \end{aligned}$$

where $\eta > 0$, $\beta \in (0, \frac{\eta}{\alpha})$, $\rho \in [0, \frac{\alpha}{2})$, $\gamma \in (\frac{2}{\alpha}\rho, 1)$, $\grave{\xi} \in (\frac{1}{4}, \frac{\alpha}{4})$ and $\grave{\gamma} \in (\frac{2}{\alpha}\grave{\xi}, \frac{1}{2})$.

Hence,

$$\|u(t) - u(s)\|_{L^p(\Omega,H)} \leq C_{(\beta,\|A^{\frac{\eta}{2}}u_0\|_{L^p(\Omega,H)},p,\alpha,\rho,\gamma,\grave{\xi},\grave{\gamma})}(t-s)^{\min\{\beta,\gamma-\frac{2}{\alpha}\rho,\grave{\gamma}-\frac{2}{\alpha}\grave{\xi}\}}.$$

To improve the exponent of the temporal regularity, we take $\eta > \frac{\alpha-1}{2}$, $\rho \leq \frac{1}{2}$, and let $\beta \to \frac{\alpha-1}{2\alpha}$, $\gamma \to 1$, $\grave{\xi} \to \frac{1}{4}$ and $\grave{\gamma} \to \frac{1}{2}$. Thus, $\min\{\beta, \gamma - \frac{2}{\alpha}\rho, \grave{\gamma} - \frac{2}{\alpha}\grave{\xi}\} = \frac{\alpha-1}{2\alpha} - \kappa$, for any $\kappa > 0$. And so,

$$\|u(t) - u(s)\|_{L^p(\Omega,H)} \leq C_{(\|A^{\frac{\eta}{2}}u_0\|_{L^p(\Omega,H)},p,\alpha)}(t-s)^{\frac{\alpha-1}{2\alpha}-\kappa}.$$

6.4 Temporal approximation

This section is concerned with the time discretization. Let $M \in \mathbb{N}_0$ be large, in particular $M > \pi^\alpha$ and $\tau = \frac{1}{M}$ be a uniform time step. Then, for $t_m = m\tau$, $m \in \{1, ..., M\}$, we seek to approximate the values $(u(t_m))_{m=1}^M$, where u is the mild solution of Prb.(6.1) on the time interval $[0,1]$. We construct the sequence of random variables $(u^m)_{m=1}^M$ via implicit Euler scheme as follows:

$$\begin{cases} \dfrac{u^m - u^{m-1}}{\tau} & = -A_\alpha u^m + F(u^{m-1}) + G(u^{m-1})\dfrac{(W(t_m) - W(t_{m-1}))}{\tau}, \\ u^0 & := u_0. \end{cases} \tag{6.34}$$

The scheme described by (6.34) is informal and has to be understood in the following sense

$$\begin{cases} u^m & = (I+\tau A_\alpha)^{-1}u^{m-1} + (I+\tau A_\alpha)^{-1}\tau F(u^{m-1}) + (I+\tau A_\alpha)^{-1}G(u^{m-1})\Delta W_m, \\ u^0 & := u_0, \end{cases} \tag{6.35}$$

where $\Delta W_m := W(t_m) - W(t_{m-1})$, $\forall 1 \leq m \leq M$.

Theorem 6.4 *Let $\alpha \in (1,2]$ and $p \geq 2$. Assume that Assumptions $\mathcal{A}_1^r$ with $\eta \in (\frac{\alpha-1}{2}, \alpha]$, $\mathcal{B}_1^r$ with $\rho \in [0, \frac{1}{2}]$ and $\mathcal{C}_1$ are satisfied. Then, Scheme (6.35) is of L^p-order $\frac{\alpha-1}{2\alpha} - \kappa$,for any $\kappa > 0$ in H . More precisely, there exists a positive constant $C_{p,\alpha,\eta,\rho}$ s.t.*

$$\|u^m - u(t_m)\|_{L^p(\Omega,H)} \leq C_{p,\alpha,\eta,\rho}\tau^{\frac{\alpha-1}{2\alpha}-\kappa}, \text{ for every } m \in \{1, ..., M\}. \tag{6.36}$$

Moreover, if $p > \frac{2\alpha}{\alpha-1}$, then Scheme (6.35) converges in probability with the same order, and converges almost surely with order $\xi < (\frac{\alpha-1}{2\alpha} - \kappa) - \frac{1}{p}$, for any $\kappa > 0$.

Remark 6.5 1. *The order of convergence $\frac{\alpha-1}{2\alpha} - \kappa$ is optimal since it coincides with the exponent of the temporal regularity.*

2. *For $\alpha = 2$, the order is less than $\frac{1}{4}$ and this result has been already obtained in the classical case, see e.g. [58, 87].*

To prove Theorem 6.4, we need the following lemma.

Lemma 6.6 1. *For all $\xi \in [0, \frac{\alpha}{2})$, there exists $C_\xi > 0$, s.t.*

$$\|A^\xi (I+\tau A_\alpha)^{-m}\|_{\mathcal{L}(H)} \leq C_\xi t_m^{-\frac{2\xi}{\alpha}}. \tag{6.37}$$

2. *For all $\zeta \in [0, \alpha]$, there exists $C_\zeta > 0$ s.t.*

$$\|A^{-\frac{\zeta}{2}}\left((I+\tau\, A_\alpha)^{-m} - S_\alpha(t_m)\right)\|_{\mathcal{L}(H)} \leq C_\zeta\, \tau^{\frac{\zeta}{\alpha}}\, m^{-(1-\frac{\zeta}{\alpha})}. \tag{6.38}$$

3. For all $\xi \in [0, \frac{\alpha}{2})$ *and all* $\beta \in [0, \frac{\alpha}{2} - \xi)$ *there exists* $C_{\xi,\beta} > 0$ *s.t.*

$$\|A^{\xi}\left((I + \tau\, A_{\alpha})^{-(m-k)} - S_{\alpha}(t_m - s)\right)\|_{\mathcal{L}(H)} \leq C_{\xi,\beta}\, \tau^{\frac{2\beta}{\alpha}}\,(t_m - s)^{-\frac{2}{\alpha}(\xi+\beta)}, \qquad (6.39)$$

where $k \in \{0, ..., m-1\}$ *and* $s \in [t_k, t_{k+1}]$.

Proof. To get Est.(6.37), we consider separately the case $m = 1$, using Identity.(1.12) in Lemma 1.36 to get

$$\begin{aligned}
\|A^{\xi}(I \;+\; \tau\, A_{\alpha})^{-1}\|_{\mathcal{L}(H)} &= \sup_{1\leq j\leq\infty}\left(\lambda_j^{\xi}(1 + \tau\, \lambda_j^{\frac{\alpha}{2}})^{-1}\right) \\
&= \tau^{-\frac{2\xi}{\alpha}} \sup_{1\leq j\leq\infty}\left(\frac{\tau^{\frac{2\xi}{\alpha}}\, \lambda_j^{\xi}}{(1 + \tau\, \lambda_j^{\frac{\alpha}{2}})}\right) \leq \tau^{-\frac{2\xi}{\alpha}} \sup_{1\leq j\leq\infty}\left(\frac{\tau\, \lambda_j^{\frac{\alpha}{2}}}{(1 + \tau\, \lambda_j^{\frac{\alpha}{2}})}\right)^{\frac{2\xi}{\alpha}} \leq\; t_1^{-\frac{2\xi}{\alpha}}.
\end{aligned} \qquad (6.40)$$

Now, for $m \geq 2$, the use of Identity.(1.12) in Lemma 1.36 yields

$$\begin{aligned}
\|A^{\xi}(I + \tau\, A_{\alpha})^{-m}\|_{\mathcal{L}(H)} &= \sup_{1\leq j\leq\infty}\left(\lambda_j^{\xi}(1 + \tau\, \lambda_j^{\frac{\alpha}{2}})^{-m}\right) \\
&\leq \sup_{1\leq j<\tau^{-\frac{1}{\alpha}}\pi^{-1}}\left(\lambda_j^{\xi}(1 + \tau\, \lambda_j^{\frac{\alpha}{2}})^{-m}\right) \\
&+ \sup_{\tau^{-\frac{1}{\alpha}}\pi^{-1}\leq j\leq\infty}\left(\lambda_j^{\xi}(1 + \tau\, \lambda_j^{\frac{\alpha}{2}})^{-m}\right).
\end{aligned} \qquad (6.41)$$

Recall that $\lambda_j = (j\pi)^2$, then for $1 \leq j < \tau^{-\frac{1}{\alpha}}\pi^{-1}$, we have $\tau\, \lambda_j^{\frac{\alpha}{2}} < 1$. Owing to the fact that, for any $x \in [0,1]$, it holds $log(1 + x) \geq \frac{x}{2}$, we get

$$\begin{aligned}
\sup_{1\leq j<\tau^{-\frac{1}{\alpha}}\pi^{-1}}\left(\lambda_j^{\xi}(1 + \tau\, \lambda_j^{\frac{\alpha}{2}})^{-m}\right) &= \sup_{1\leq j<\tau^{-\frac{1}{\alpha}}\pi^{-1}}\left(\lambda_j^{\xi} e^{-m\, log(1+\tau\, \lambda_j^{\frac{\alpha}{2}})}\right) \\
&\leq \sup_{1\leq j<\tau^{-\frac{1}{\alpha}}\pi^{-1}}\left(\lambda_j^{\xi} e^{-\frac{m\,\tau\,\lambda_j^{\frac{\alpha}{2}}}{2}}\right).
\end{aligned}$$

Using LemmaA.8 with $x = \frac{m\tau\lambda_j^{\frac{\alpha}{2}}}{2}$ and $\gamma = \frac{2\xi}{\alpha}$ we conclude

$$\sup_{1\leq j<\tau^{-\frac{1}{\alpha}}\pi^{-1}}\left(\lambda_j^{\xi}(1 + \tau\, \lambda_j^{\frac{\alpha}{2}})^{-m}\right) \leq \lambda_j^{\xi}\left((m\tau)^{\frac{-2\xi}{\alpha}}\lambda_j^{-\xi}\right) \leq C_{\xi} t_m^{\frac{-2\xi}{\alpha}} \qquad (6.42)$$

For $\tau^{-\frac{1}{\alpha}}\pi^{-1} \leq j \leq \infty$, we have $\tau\, \lambda_j^{\frac{\alpha}{2}} \geq 1$, then by the same calculus as above it is true that

$$\begin{aligned}
\sup_{\tau^{-\frac{1}{\alpha}}\pi^{-1}\leq j\leq\infty}(\lambda_j^{\xi}(1 \;+\; \tau\,\lambda_j^{\frac{\alpha}{2}})^{-m}) &= \sup_{\tau^{-\frac{1}{\alpha}}\pi^{-1}\leq j\leq\infty}\left(\lambda_j^{\xi}\,(\tau\,\lambda_j^{\frac{\alpha}{2}})^{-m}(1+(\tau\,\lambda_j^{\frac{\alpha}{2}})^{-1})^{-m}\right)\\
&= \sup_{\tau^{-\frac{1}{\alpha}}\pi^{-1}\leq j\leq\infty}\left(\lambda_j^{\xi}\,(\tau\,\lambda_j^{\frac{\alpha}{2}})^{-m}e^{-m\,log(1+(\tau\,\lambda_j^{\frac{\alpha}{2}})^{-1})}\right)\\
&\leq \sup_{\tau^{-\frac{1}{\alpha}}\pi^{-1}\leq j\leq\infty}\left(\lambda_j^{\xi}\,(\tau\,\lambda_j^{\frac{\alpha}{2}})^{-m}e^{-\frac{m\,(\tau\,\lambda_j^{\frac{\alpha}{2}})^{-1}}{2}}\right)\\
&\leq C_{\xi}\sup_{\tau^{-\frac{1}{\alpha}}\pi^{-1}\leq j\leq\infty}\left(\lambda_j^{\xi}\,(\tau\,\lambda_j^{\frac{\alpha}{2}})^{-m}m^{-\frac{2\xi}{\alpha}}(\tau\,\lambda_j^{\frac{\alpha}{2}})^{\frac{2\xi}{\alpha}}\right)\\
&\leq C_{\xi}(\tau\,m)^{-\frac{2\xi}{\alpha}}\sup_{\tau^{-\frac{1}{\alpha}}\pi^{-1}\leq j\leq\infty}\left((\tau\,\lambda_j^{\frac{\alpha}{2}})^{-(m-\frac{4\xi}{\alpha})}\right). \qquad (6.43)
\end{aligned}$$

Thanks to the condition $0\leq\xi<\frac{\alpha}{2}$, the exponent $m-\frac{4\xi}{\alpha}$ is positive, moreover, $\tau\,\lambda_j^{\frac{\alpha}{2}}\geq 1$ by assumption, hence $(\tau\,\lambda_j^{\frac{\alpha}{2}})^{-(m-\frac{4\xi}{\alpha})}=(\frac{1}{\tau\,\lambda_j^{\frac{\alpha}{2}}})^{(m-\frac{4\xi}{\alpha})}\leq 1$ and

$$\sup_{\tau^{-\frac{1}{\alpha}}\pi^{-1}\leq j\leq\infty}(\lambda_j^{\rho}(1+\tau\,\lambda_j^{\frac{\alpha}{2}})^{-m})\leq\; C_{\rho}t_m^{-\frac{2\rho}{\alpha}}. \qquad (6.44)$$

Replace Est.(6.42) and Est.(6.44) in Est.(6.41) to arrive at the result.

In order to estimate Est.(6.38), we introduce the rational function $r(x)=\frac{1}{1+x}$ for $x\in\mathbb{R}-\{-1\}$,. It is well known, see [104, Ch. 7], that for x close to 0

$$r(x)=e^{-x}+\mathcal{O}(x^2). \qquad (6.45)$$

and

$$\exists\, C>0,\; s.t.\;\; \mid r(x)-e^{-x}\mid\leq Cx^2,\;\forall\, x\in[0,1]. \qquad (6.46)$$

besides

$$\exists\, c\in(0,1),\; s.t.\;\; \mid r(x)\mid\leq e^{-cx},\;\forall\, x\in[0,1]. \qquad (6.47)$$

As we have seen before, $\forall\,\zeta\in[0,\alpha]$, we have,

$$\|A^{-\frac{\zeta}{2}}\left((I+\tau\,A_{\alpha})^{-m}-S_{\alpha}(t_m)\right)\|_{\mathcal{L}(H)}=\sup_{1\leq j\leq\infty}|\lambda_j^{-\frac{\zeta}{2}}\left((1+\tau\lambda_j^{\frac{\alpha}{2}})^{-m}-e^{-t_m\lambda_j^{\frac{\alpha}{2}}}\right)|.$$

Due to the fact that $t_m=m\tau$ and using the polynomial expansion: $x^m-y^m=(x-y)\sum_{i=0}^{m-1}x^{m-1-i}y^i$, we get

$$\begin{aligned}
(\lambda_j^{-\frac{\zeta}{2}}\left((1+\tau\lambda_j^{\frac{\alpha}{2}})^{-m}\right. \;-\; \left.e^{-t_m\lambda_j^{\frac{\alpha}{2}}}\right)) &= \left(\lambda_j^{-\frac{\zeta}{2}}(1+\tau\lambda_j^{\frac{\alpha}{2}})^{-m}-(e^{-\tau\lambda_j^{\frac{\alpha}{2}}})^m\right)\\
&= \lambda_j^{-\frac{\zeta}{2}}\left(\left(\left(1+\tau\lambda_j^{\frac{\alpha}{2}}\right)^{-1}-e^{-\tau\lambda_j^{\frac{\alpha}{2}}}\right)\right)\sum_{i=0}^{m-1}\left(1+\tau\lambda_j^{\frac{\alpha}{2}}\right)^{-(m-1-i)}e^{-\tau\lambda_j^{\frac{\alpha}{2}}i}.
\end{aligned}$$

(6.48)

For $\tau\lambda_j^{\frac{\alpha}{2}} < 1$, we use Inequality (6.46) with the definition of the rational function r to get

$$|\lambda_j^{-\frac{\zeta}{2}}\Big((1+\tau\lambda_j^{\frac{\alpha}{2}})^{-m} - e^{-t_m\lambda_j^{\frac{\alpha}{2}}}\Big)| \;\leq\; C\lambda_j^{-\frac{\zeta}{2}}(\tau\lambda_j^{\frac{\alpha}{2}})^2\sum_{i=0}^{m-1}\Big(1+\tau\lambda_j^{\frac{\alpha}{2}}\Big)^{-(m-1-i)} e^{-\tau\lambda_j^{\frac{\alpha}{2}}i}.$$

Now, we use Inequality (6.47) to arrive at

$$\begin{aligned}
|\lambda_j^{-\frac{\zeta}{2}}\Big((1+\tau\lambda_j^{\frac{\alpha}{2}})^{-m} &- e^{-t_m\lambda_j^{\frac{\alpha}{2}}}\Big)| \leq C\lambda_j^{-\frac{\zeta}{2}}(\tau\lambda_j^{\frac{\alpha}{2}})^2\sum_{i=0}^{m-1} e^{-c(m-1-i)\tau\lambda_j^{\frac{\alpha}{2}}}\; e^{-\tau\lambda_j^{\frac{\alpha}{2}}i} \\
&\leq C\lambda_j^{-\frac{\zeta}{2}}(\tau\lambda_j^{\frac{\alpha}{2}})^2\sum_{i=0}^{m-1} e^{-c(m-1)\tau\lambda_j^{\frac{\alpha}{2}}} \\
&\leq C\lambda_j^{-\frac{\zeta}{2}}e^{c\tau\lambda_j^{\frac{\alpha}{2}}}(\tau\lambda_j^{\frac{\alpha}{2}})^2\, m\, e^{-c\,m\,\tau\lambda_j^{\frac{\alpha}{2}}}.
\end{aligned}$$

The fact that, $c\tau\lambda_j^{\frac{\alpha}{2}} < 1$ yields to $e^{c\tau\lambda_j^{\frac{\alpha}{2}}} < e$, besides the use of Lemma A.8 gives us

$$|\lambda_j^{-\frac{\zeta}{2}}\Big((1+\tau\lambda_j^{\frac{\alpha}{2}})^{-m} - e^{-t_m\lambda_j^{\frac{\alpha}{2}}}\Big)| \leq C_{\alpha,\zeta}\lambda_j^{-\frac{\zeta}{2}}(\tau\lambda_j^{\frac{\alpha}{2}})^2 m\;\Big(cm\tau\lambda_j^{\frac{\alpha}{2}}\Big)^{-(2-\frac{\zeta}{\alpha})} \leq C_{c,\alpha,\zeta}\tau^{\frac{\zeta}{\alpha}}m^{-(1-\frac{\zeta}{\alpha})}. \tag{6.49}$$

For $\tau\lambda_j^{\frac{\alpha}{2}} \geq 1$ and arguing as before, we have

$$\begin{aligned}
|\lambda_j^{-\frac{\eta}{2}}\left(\Big(1+\tau\lambda_j^{\frac{\alpha}{2}}\Big)^{-m} - e^{-t_m\lambda_j^{\frac{\alpha}{2}}}\right)| &\leq \lambda_j^{-\frac{\eta}{2}}\left(\Big(1+\tau\lambda_j^{\frac{\alpha}{2}}\Big)^{-m} + e^{-t_m\lambda_j^{\frac{\alpha}{2}}}\right) \\
&= \lambda_j^{-\frac{\eta}{2}}\Big(e^{-m\,log(1+\tau\lambda_j^{\frac{\alpha}{2}})} + e^{-m\,\tau\,\lambda_j^{\frac{\alpha}{2}}}\Big) \\
&\leq \lambda_j^{-\frac{\eta}{2}}(\tau\lambda_j^{\frac{\alpha}{2}})^{\frac{\eta}{\alpha}}e^{-m(log2+1)}.
\end{aligned} \tag{6.50}$$

Again, the use of Lemma A.8 yields

$$|\lambda_j^{-\frac{\eta}{2}}\left(\Big(1+\tau\lambda_j^{\frac{\alpha}{2}}\Big)^{-m} - e^{-t_m\lambda_j^{\frac{\alpha}{2}}}\right)| \leq C_{\alpha,\eta}\tau^{\frac{\eta}{\alpha}}m^{-(1-\frac{\eta}{\alpha})}. \tag{6.51}$$

Thus, from Est.(6.49) and Est.(6.51) we obtain the result.

About Est.(6.39), the facts that $S_\alpha(t_m - s)) = e^{-(t_m-s)A_\alpha}$ and $t_{m-k} = (m-k)\tau = t_m - t_k$ help us to write

$$\begin{aligned}
(I+\tau\,A_\alpha)^{-(m-k)} &- S_\alpha(t_m - s) = (I+\tau\,A_\alpha)^{-(m-k)} - e^{-(t_m-s)A_\alpha} \\
&= (I+\tau\,A_\alpha)^{-(m-k)} - (I+\tau\,A_\alpha)^{-(m-k+1)} \\
&+ (I+\tau\,A_\alpha)^{-(m-k+1)} - (I+\tau\,A_\alpha)^{-1}e^{-(t_{m-k})A_\alpha} \\
&+ (I+\tau\,A_\alpha)^{-1}e^{-(t_{m-k})A_\alpha} - (I+\tau\,A_\alpha)^{-1}e^{-(t_m-s)A_\alpha} \\
&+ (I+\tau\,A_\alpha)^{-1}e^{-(t_m-s)A_\alpha} - e^{-(t_m-s)A_\alpha}.
\end{aligned} \tag{6.52}$$

For $0 \leq \xi < \frac{\alpha}{2}$, we have

$$\begin{aligned}
\|A^{\xi}((I+\tau A_\alpha)^{-(m-k)} &- S_\alpha(t_m-s))\|_{\mathcal{L}(H)} = \|A^{\xi}((I+\tau A_\alpha)^{-(m-k)} - e^{-(t_m-s)A_\alpha}\|_{\mathcal{L}(H)} \\
&\leq \|A^{\xi}(I+\tau A_\alpha)^{-(m-k)}\left(I-(I+\tau A_\alpha)^{-1}\right)\|_{\mathcal{L}(H)} \\
&+ \| A^{\xi}(I+\tau A_\alpha)^{-1}\left((I+\tau A_\alpha)^{-(m-k)} - e^{-(t_{m-k})A_\alpha}\right)\|_{\mathcal{L}(H)} \\
&+ \| A^{\xi}(I+\tau A_\alpha)^{-1} e^{-(t_m-s)A_\alpha}\left(I-e^{-(s-t_k)A_\alpha}\right)\|_{\mathcal{L}(H)} \\
&+ \| A^{\xi}e^{-(t_m-s)A_\alpha}\left(I-(I+\tau A_\alpha)^{-1}\right)\|_{\mathcal{L}(H)} \\
&:= a+b+c+d. \qquad (6.53)
\end{aligned}$$

In order to estimate the different terms appear in Est.(6.53), let us fix a parameter $\beta \in [0, \frac{\alpha}{2}-\xi)$. So, the first term a can be rewritten by using Identity.(1.12) in Lemma 1.36 as

$$\begin{aligned}
a &:= \|A^{\xi}(I+\tau A_\alpha)^{-(m-k)}\left(I-(I+\tau A_\alpha)^{-1}\right)\|_{\mathcal{L}(H)} \\
&= \sup_{1\leq j\leq\infty}\left(\lambda_j^{\xi}(1+\tau\,\lambda_j^{\frac{\alpha}{2}})^{-(m-k)}\frac{\tau\lambda_j^{\frac{\alpha}{2}}}{(1+\tau\lambda_j^{\frac{\alpha}{2}})}\right) \\
&= \tau^{\frac{2\beta}{\alpha}}\sup_{1\leq j\leq\infty}\left(\lambda_j^{(\xi+\beta)}(1+\tau\,\lambda_j^{\frac{\alpha}{2}})^{-(m-k)}\frac{(\tau\lambda_j^{\frac{\alpha}{2}})^{1-\frac{2\beta}{\alpha}}}{(1+\tau\lambda_j^{\frac{\alpha}{2}})}\right).
\end{aligned}$$

The fact that $\beta \in [0, \frac{\alpha}{2}-\xi)$ with $0 \leq \xi < \frac{\alpha}{2}$ gives $0 \leq \frac{2\xi}{\alpha} < 1-\frac{2\beta}{\alpha} < 1$, besides $(1+\tau\,\lambda_j^{\frac{\alpha}{2}}) > 1$, and so $(1+\tau\,\lambda_j^{\frac{\alpha}{2}}) > (1+\tau\,\lambda_j^{\frac{\alpha}{2}})^{1-\frac{2\beta}{\alpha}}$. Then,

$$\begin{aligned}
a &\leq \tau^{\frac{2\beta}{\alpha}}\sup_{1\leq j\leq\infty}\left(\lambda_j^{(\xi+\beta)}(1+\tau\,\lambda_j^{\frac{\alpha}{2}})^{-(m-k)}\left(\frac{\tau\lambda_j^{\frac{\alpha}{2}}}{(1+\tau\lambda_j^{\frac{\alpha}{2}})}\right)^{1-\frac{2\beta}{\alpha}}\right) \\
&\leq \tau^{\frac{2\beta}{\alpha}}\sup_{1\leq j\leq\infty}\left(\lambda_j^{(\xi+\beta)}(1+\tau\,\lambda_j^{\frac{\alpha}{2}})^{-(m-k)}\right) = \tau^{\frac{2\beta}{\alpha}} = \|A^{(\xi+\beta)}(I+\tau A_\alpha)^{-(m-k)}\|_{\mathcal{L}(H)}.
\end{aligned}$$

As $s \in [t_k, t_{k+1}]$ we get $t_m - s \leq t_m - t_k = t_{m-k}$, then an application of Est.(6.37) yields

$$a \leq C_{\xi,\beta}\tau^{\frac{2\beta}{\alpha}}\, t_{m-k}^{-\frac{2}{\alpha}(\xi+\beta)} \leq C_{\xi,\beta}\tau^{\frac{2\beta}{\alpha}}\,(t_m-s)^{-\frac{2}{\alpha}(\xi+\beta)}. \qquad (6.54)$$

To estimate b, for $\xi \in [0, \frac{\alpha}{2})$ and $\beta \in [0, \frac{\alpha}{2} - \xi)$ we argue as above to get

$$\begin{aligned}
b &:= \| A^{\xi}(I+\tau A_\alpha)^{-1}\left((I+\tau A_\alpha)^{-(m-k)} - e^{-(t_{m-k})A_\alpha}\right)\|_{\mathcal{L}(H)} \\
&= \sup_{1\le j\le\infty}\left(\lambda_j^{\xi}(1+\tau\lambda_j^{\frac{\alpha}{2}})^{-1}|(1+\tau\lambda_j^{\frac{\alpha}{2}})^{-(m-k)} - e^{-(t_{m-k})\lambda_j^{\frac{\alpha}{2}}}|\right) \\
&= \tau^{\frac{2\beta}{\alpha}}\sup_{1\le j\le\infty}\left(\frac{(\tau\lambda_j^{\frac{\alpha}{2}})^{\frac{2\xi}{\alpha}}}{(1+\tau\lambda_j^{\frac{\alpha}{2}})}\tau^{-\frac{2}{\alpha}(\xi+\beta)}|(1+\tau\lambda_j^{\frac{\alpha}{2}})^{-(m-k)} - e^{-(t_{m-k})\lambda_j^{\frac{\alpha}{2}}}|\right) \\
&\le \tau^{\frac{2\beta}{\alpha}}\tau^{-\frac{2}{\alpha}(\xi+\beta)}\sup_{1\le j\le\infty}\left(\left(\frac{\tau\lambda_j^{\frac{\alpha}{2}}}{(1+\tau\lambda_j^{\frac{\alpha}{2}})}\right)^{\frac{2\xi}{\alpha}}|(1+\tau\lambda_j^{\frac{\alpha}{2}})^{-(m-k)} - e^{-(t_{m-k})\lambda_j^{\frac{\alpha}{2}}}|\right). \\
&\le \tau^{\frac{2\beta}{\alpha}}\tau^{-\frac{2}{\alpha}(\xi+\beta)}\sup_{1\le j\le\infty}\left(|(1+\tau\lambda_j^{\frac{\alpha}{2}})^{-(m-k)} - e^{-(t_{m-k})\lambda_j^{\frac{\alpha}{2}}}|\right)
\end{aligned}$$

By using Est.(6.38) (with $\zeta = 0$) we arrive at

$$b \le \tau^{\frac{2\beta}{\alpha}}\tau^{-\frac{2}{\alpha}(\xi+\beta)}\|(I+\tau A_\alpha)^{-(m-k)} - e^{-t_{m-k}A_\alpha}\|_{\mathcal{L}(H)} \le C\,\tau^{\frac{2\beta}{\alpha}}\tau^{-\frac{2}{\alpha}(\xi+\beta)}(m-k)^{-1}. \quad (6.55)$$

Since $m \in \{1, ..., M\}$ and $k \in \{0, ..., m-1\}$ we deduce $(m-k) \ge 1$, with the fact that $\frac{2}{\alpha}(\xi+\beta) < 1$ thanks to $\xi \in [0, \frac{\alpha}{2})$ and $\beta \in [0, \frac{\alpha}{2} - \xi)$, then $(m-k) \ge (m-k)^{\frac{2}{\alpha}(\xi+\beta)}$. Now, we majorize the term b as

$$\begin{aligned}
b &\le C\,\tau^{\frac{2\beta}{\alpha}}\tau^{-\frac{2}{\alpha}(\xi+\beta)}(m-k)^{-\frac{2}{\alpha}(\xi+\beta)} = C\,\tau^{\frac{2\beta}{\alpha}}(\tau\,(m-k))^{-\frac{2}{\alpha}(\xi+\beta)} \\
&\le C\,\tau^{\frac{2\beta}{\alpha}}(t_m - s)^{-\frac{2}{\alpha}(\xi+\beta)}. \quad (6.56)
\end{aligned}$$

To estimate c, first we have

$$\begin{aligned}
c &:= \| A^{\xi}(I+\tau A_\alpha)^{-1}e^{-(t_m-s)A_\alpha}\left(I - e^{-(s-t_k)A_\alpha}\right)\|_{\mathcal{L}(H)} \\
&= \sup_{1\le j\le\infty}\left(\lambda_j^{\xi}(1+\tau\lambda_j^{\frac{\alpha}{2}})^{-1}e^{-(t_m-s)\lambda_j^{\frac{\alpha}{2}}}\left(1 - e^{-(s-t_k)\lambda_j^{\frac{\alpha}{2}}}\right)\right) \\
&= \sup_{1\le j\le\infty}\left(\tau^{-\frac{2\xi}{\alpha}}\frac{(\tau\lambda_j^{\frac{\alpha}{2}})^{\frac{2\xi}{\alpha}}}{(1+\tau\lambda_j^{\frac{\alpha}{2}})}e^{-(t_m-s)\lambda_j^{\frac{\alpha}{2}}}\left(1 - e^{-(s-t_k)\lambda_j^{\frac{\alpha}{2}}}\right)\right) \\
&\le \tau^{-\frac{2\xi}{\alpha}}\sup_{1\le j\le\infty}\left(\left(\frac{\tau\lambda_j^{\frac{\alpha}{2}}}{(1+\tau\lambda_j^{\frac{\alpha}{2}})}\right)^{\frac{2\xi}{\alpha}}\left(\lambda_j^{(\xi+\beta)}e^{-(t_m-s)\lambda_j^{\frac{\alpha}{2}}}\right)\lambda_j^{-(\xi+\beta)}\left(1 - e^{-(s-t_k)\lambda_j^{\frac{\alpha}{2}}}\right)\right)
\end{aligned}$$

The use of Lemma A.8 and Lemma A.9, besides the fact that $s - t_k \le t_{k+1} - t_k = \tau$, helps us to get

$$\begin{aligned}
c &\le C_{\xi,\beta}\tau^{-\frac{2\xi}{\alpha}}(t_m - s)^{-\frac{2}{\alpha}(\xi+\beta)}(s-t_k)^{\frac{2}{\alpha}(\xi+\beta)}\sup_{1\le j\le\infty}\left(\left(\frac{\tau\lambda_j^{\frac{\alpha}{2}}}{(1+\tau\lambda_j^{\frac{\alpha}{2}})}\right)^{\frac{2\xi}{\alpha}}\right) \\
&\le C_{\xi,\beta}\tau^{-\frac{2\rho}{\alpha}}(t_m - s)^{-\frac{2}{\alpha}(\xi+\beta)}\tau^{\frac{2}{\alpha}(\xi+\beta)} = C_{\xi,\beta}\,\tau^{\frac{2\beta}{\alpha}}(t_m - s)^{-\frac{2}{\alpha}(\xi+\beta)}. \quad (6.57)
\end{aligned}$$

It remains to estimate the last term d. By the same calculus as above with the use of Lemma A.8, we arrive at

$$\begin{aligned}
d &:= \| A^{\xi} e^{-(t_m-s)A_\alpha} \left(I-(I+\tau\, A_\alpha)^{-1}\right) \|_{\mathcal{L}(H)} = \sup_{1\leq j\leq\infty} \left(\lambda_j^{\xi}\, e^{-(t_m-s)\lambda_j^{\frac{\alpha}{2}}} \left(\frac{\tau\, \lambda_j^{\frac{\alpha}{2}}}{(1+\tau\, \lambda_j^{\frac{\alpha}{2}})} \right) \right) \\
&\leq \tau^{\frac{2\beta}{\alpha}} \sup_{1\leq j\leq\infty} \left(\left(\lambda_j^{(\xi+\beta)}\, e^{-(t_m-s)\lambda_j^{\frac{\alpha}{2}}} \right) \left(\frac{\tau\, \lambda_j^{\frac{\alpha}{2}}}{(1+\tau\, \lambda_j^{\frac{\alpha}{2}})} \right)^{1-\frac{2\beta}{\alpha}} \right) \\
&\leq C_{\xi,\beta}\, \tau^{\frac{2\beta}{\alpha}}\, (t_m-s)^{-\frac{2}{\alpha}(\xi+\beta)} \sup_{1\leq j\leq\infty} \left(\left(\frac{\tau\, \lambda_j^{\frac{\alpha}{2}}}{(1+\tau\, \lambda_j^{\frac{\alpha}{2}})} \right)^{1-\frac{2\beta}{\alpha}} \right) \leq C_{\xi,\beta}\, \tau^{\frac{2\beta}{\alpha}}\, (t_m-s)^{-\frac{2}{\alpha}(\xi+\beta)}.
\end{aligned} \tag{6.58}$$

Coming back to Est.(6.53), so by Est.(6.54), Est.(6.56), Est.(6.57) and Est.(6.58) we get the desired result.

■

Proof of Theorem 6.4

From the one hand, we rewrite the mild solution u given by Eq.(6.9) as follows

$$u(t_m) = S_\alpha(t_m)u_0 + \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} S_\alpha(t_m-s)F(u(s))ds + \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} S_\alpha(t_m-s)G(u(s))dW(s). \tag{6.59}$$

From the other hand, in Identity.(6.35) we replace u^{m-1} by its value which can be deduced from the same Identity.(6.35) and we iterate this sequence up to m, besides the fact that $\int_{t_k}^{t_{k+1}} ds = \tau$, we get

$$\begin{aligned}
u^m = (I+\tau A_\alpha)^{-m}u_0 \; &+ \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} (I+\tau A_\alpha)^{-(m-k)} F(u^k)ds \\
&+ \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} (I+\tau A_\alpha)^{-(m-k)} G(u^k)dW(s).
\end{aligned} \tag{6.60}$$

Then

$$\begin{aligned}
\|u^m &- u(t_m)\|_{L^p(\Omega,H)} \leq \| \Big((I+\tau A_\alpha)^{-m} - S_\alpha(t_m)\Big) u_0\|_{L^p(\Omega,H)} \\
&+ \|\sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \Big((I+\tau A_\alpha)^{-(m-k)} F(u^k) - S_\alpha(t_m-s)F(u(s))\Big)\, ds\|_{L^p(\Omega,H)} \\
&+ \|\sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \Big((I+\tau A_\alpha)^{-(m-k)} G(u^k) - S_\alpha(t_m-s)G(u(s))\Big)\, dW(s)\|_{L^p(\Omega,H)}.
\end{aligned} \tag{6.61}$$

We estimate the terms of the RHS of Est.6.61 as follows:
The fact that $A^{\frac{\eta}{2}}$ commutes with the semigroup $S_\alpha(t)$ and the resolvant $(I+\tau A_\alpha)^{-1}$, with the use of Est.(6.38) in Lemma 6.6 (with $\zeta = \eta$) besides the Assumption $\mathcal{A}_1^r$ help us to get

$$\begin{aligned}
\|((I+\tau A_\alpha)^{-m} &- S_\alpha(t_m))u_0\|_{L^p(\Omega,H)} = \|A^{-\frac{\eta}{2}} \Big((I+\tau A_\alpha)^{-m} - S_\alpha(t_m)\Big) A^{\frac{\eta}{2}} u_0\|_{L^p(\Omega,H)} \\
&\leq \|A^{-\frac{\eta}{2}} \Big((I+\tau A_\alpha)^{-m} - S_\alpha(t_m)\Big) \|_{\mathcal{L}(H)} \|u_0\|_{L^p(\Omega,H_2^\eta)} \\
&\leq C_\eta\, \tau^{\frac{\eta}{\alpha}}\, m^{-(1-\frac{\eta}{\alpha})} \|u_0\|_{L^p(\Omega,H_2^\eta)} \leq C_{(\eta,\|u_0\|_{L^p(\Omega,H_2^\eta)})}\, \tau^{\frac{\eta}{\alpha}}\, m^{-(1-\frac{\eta}{\alpha})}.
\end{aligned}$$

Thanks to condition $\eta \leq \alpha$; the power $1-\frac{\eta}{\alpha}$ is positive, thus

$$\| \Big((I+\tau A_\alpha)^{-m} - S_\alpha(t_m)\Big) u_0\|_{L^p(\Omega,H)} \leq C_{(\eta,\|u_0\|_{L^p(\Omega,H_2^\eta)})}\, \tau^{\frac{\eta}{\alpha}}. \tag{6.62}$$

The second term in the RHS of Est.(6.61) can be splitted by the following way

$$\begin{aligned}
&\|\sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \Big((I+\tau A_\alpha)^{-(m-k)} F(u^k) - S_\alpha(t_m-s)F(u(s))\Big)\, ds\|_{L^p(\Omega,H)} \\
\leq\; & \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \| \Big((I+\tau A_\alpha)^{-(m-k)} F(u^k) - S_\alpha(t_m-s)F(u(s))\Big) \|_{L^p(\Omega,H)} ds \\
\leq\; & \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \|(I+\tau A_\alpha)^{-(m-k)} \Big(F(u^k) - F(u(t_k))\Big) \|_{L^p(\Omega,H)} ds \\
+\; & \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \|(I+\tau A_\alpha)^{-(m-k)} \left(F(u(t_k)) - F(u(s))\right) \|_{L^p(\Omega,H)} ds \\
+\; & \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \| \Big((I+\tau A_\alpha)^{-(m-k)} - S_\alpha(t_m-s)\Big) F(u(s))\|_{L^p(\Omega,H)} ds \\
:=\; & R_1 + R_2 + R_3.
\end{aligned} \tag{6.63}$$

By the commutativity of A^ρ and $(I+\tau A_\alpha)^{-(m-k)}$, Est.(6.37) in Lemma 6.6 (with $\xi=\rho$) and Assumption $\mathcal{B}_1^r$, we get

$$\begin{aligned}
R_1 &:= \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \|A^\rho (I+\tau A_\alpha)^{-(m-k)} A^{-\rho}\left(F(u^k)-F(u(t_k))\right)\|_{L^p(\Omega,H)} ds \\
&\leq \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \|A^\rho (I+\tau A_\alpha)^{-(m-k)}\|_{\mathcal{L}(H)} \|A^{-\rho}\left(F(u^k)-F(u(t_k))\right)\|_{L^p(\Omega,H)} ds \\
&\leq C_\rho \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} t_{m-k}^{-\frac{2\rho}{\alpha}} \|u^k-u(t_k)\|_{L^p(\Omega,H)} ds = C_\rho \tau \sum_{k=0}^{m-1} t_{m-k}^{-\frac{2\rho}{\alpha}} \|u^k-u(t_k)\|_{L^p(\Omega,H)}.
\end{aligned} \tag{6.64}$$

To estimate R_2, by arguing as above we get

$$\begin{aligned}
R_2 &:= \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \|(I+\tau A_\alpha)^{-(m-k)}\left(F(u(t_k))-F(u(s))\right)\|_{L^p(\Omega,H)} ds \\
&\leq \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \|A^\rho (I+\tau A_\alpha)^{-(m-k)}\|_{\mathcal{L}(H)} \|A^{-\rho}\left(F(u(t_k))-F(u(s))\right)\|_{L^p(\Omega,H)} ds \\
&\leq C_\rho \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} t_{m-k}^{-\frac{2\rho}{\alpha}} \|u(t_k)-u(t(s))\|_{L^p(\Omega,H)} ds.
\end{aligned} \tag{6.65}$$

According to the temporal regularity Est.(6.10) with the fact that $t_m - s \leq t_m - t_k = t_{m-k}$ we majorized R_2 as follows

$$\begin{aligned}
R_2 &\leq C_\rho \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} t_{m-k}^{-\frac{2\rho}{\alpha}} (s-t_k)^{(\frac{\alpha-1}{2\alpha})-\kappa} ds \leq C_{\rho,\alpha} \sum_{k=0}^{m-1} t_{m-k}^{-\frac{2\rho}{\alpha}} (t_{k+1}-t_k)^{1+(\frac{\alpha-1}{2\alpha}-\kappa)} \\
&= C_{\rho,\alpha} \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} t_{m-k}^{-\frac{2\rho}{\alpha}} \tau^{\frac{\alpha-1}{2\alpha}-\kappa} ds \leq C_{\rho,\alpha}\, \tau^{\frac{\alpha-1}{2\alpha}-\kappa} \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} (t_m-s)^{-\frac{2\rho}{\alpha}} ds.
\end{aligned} \tag{6.66}$$

Since $0\leq \rho < \frac{\alpha}{2}$ we have $\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}(t_m-s)^{-\frac{2\rho}{\alpha}}ds = \int_0^{t_m}(t_m-s)^{-\frac{2\rho}{\alpha}}ds \leq \frac{1}{1-\frac{2\rho}{\alpha}} < \infty$ and so

$$R_2 \leq C_{\alpha,\rho}\, \tau^{\frac{\alpha-1}{2\alpha}-\kappa}. \tag{6.67}$$

To estimate R_3 we use Est.(6.39) in Lemma 6.6 (with $\xi=\rho$), the Assumption $\mathcal{B}_1^r$ and choose $\beta\in(0,\frac{\alpha}{2}-\rho)$. Then

$$\begin{aligned}
R_3 &:= \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \|A^\rho \left((I+\tau A_\alpha)^{-(m-k)} - S_\alpha(t_m-s)\right) A^{-\rho}F(u(s))\|_{L^p(\Omega,H)} ds \\
&\leq \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \|A^\rho \left((I+\tau A_\alpha)^{-(m-k)} - S_\alpha(t_m-s)\right)\|_{\mathcal{L}(H)} \|A^{-\rho}F(u(s))\|_{L^p(\Omega,H)} ds \\
&\leq C_{\alpha,\rho}\, \tau^{\frac{2\beta}{\alpha}} \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \left((t_m-s)^{-\frac{2}{\alpha}(\rho+\beta)}\right) \left(1+\sup_{s\in[0,1]}\|u(s)\|_{L^p(\Omega,H)}\right) ds.
\end{aligned}$$

The term $\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}(t_m-s)^{-\frac{2}{\alpha}(\rho+\beta)}ds$ is finite because $\beta<\frac{\alpha}{2}-\rho$, and so

$$R_3 \leq C_{\alpha,\rho}\tau^{\frac{2\beta}{\alpha}}. \tag{6.68}$$

We replace Est.(6.64), Est.(6.67) and Est.(6.68) in Est.(6.63), to get

$$\begin{aligned}
&\|\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}((I+\tau A_\alpha)^{-(m-k)}F(u^k)-S_\alpha(t_m-s)F(u(s)))ds\|_{L^p(\Omega,H)}\\
\leq\ & C_{\alpha,\rho}\left(\tau^{\frac{\alpha-1}{2\alpha}-\kappa}+\tau^{\frac{2\beta}{\alpha}}\right)+C_\rho\tau\sum_{k=0}^{m-1}t_{m-k}^{-\frac{2\rho}{\alpha}}\|u^k-u(t_k)\|_{L^p(\Omega,H)}.
\end{aligned} \tag{6.69}$$

Now, we deal with the stochastic term in the RHS of Est.(6.61). First, by Burkholder-Davis-Gundy inequality we have the following sequence of inequalities

$$\begin{aligned}
&\|\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}\Big((I+\tau A_\alpha)^{-(m-k)}G(u^k)-S_\alpha(t_m-s)G(u(s))\Big)\,dW(s)\|_{L^p(\Omega,H)}\\
\leq\ & C_p\left(\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}\Big(\mathbb{E}\|(I+\tau A_\alpha)^{-(m-k)}G(u^k)-S_\alpha(t_m-s)G(u(s))\|_{HS}^p\Big)^{\frac{2}{p}}ds\right)^{\frac{1}{2}}\\
\leq\ & C_p\left(\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}\Big(\mathbb{E}\|(I+\tau A_\alpha)^{-(m-k)}\Big(G(u^k)-G(u(t_k))\Big)\|_{HS}^p\Big)^{\frac{2}{p}}ds\right)^{\frac{1}{2}}\\
+\ & C_p\left(\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}\Big(\mathbb{E}\|(I+\tau A_\alpha)^{-(m-k)}\left(G(u(t_k))-G(u(s))\right)\|_{HS}^p\Big)^{\frac{2}{p}}ds\right)^{\frac{1}{2}}\\
+\ & C_p\left(\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}\Big(\mathbb{E}\|\Big((I+\tau A_\alpha)^{-(m-k)}-S_\alpha(t_m-s)\Big)G(u(s))\|_{HS}^p\Big)^{\frac{2}{p}}ds\right)^{\frac{1}{2}}\\
:=\ & C_p(R_4+R_5+R_6).
\end{aligned} \tag{6.70}$$

In order to estimate R_4, we use Assumption $\mathcal{C}_1$ to arrive at

$$\begin{aligned}
(\mathbb{E}\|(I+\tau A_\alpha)^{-(m-k)}(G(u^k)\ &-\ G(u(t_k)))\|_{HS}^p)^{\frac{2}{p}}\\
&\leq\ \|(I+\tau A_\alpha)^{-(m-k)}\|_{HS}^2(\mathbb{E}\|G(u^k)-G(u(t_k))\|_{\mathcal{L}(H)}^p)^{\frac{2}{p}}\\
&\leq\ C_G^2\|A^{-\grave{\rho}}\|_{HS}^2\ \|A^{\grave{\rho}}(I+\tau A_\alpha)^{-(m-k)}\|_{\mathcal{L}(H)}^2\ \|u^k-u(t_k)\|_{L^p(\Omega,H)}^2.
\end{aligned}$$

The operator $A^{-\grave{\rho}}$ is Hilbert-Schmidt due to the choice of $\grave{\rho}>\frac{1}{4}$. By using Est.(6.37) (with $\frac{1}{4}<\xi=\grave{\rho}<\frac{\alpha}{4}$) we have

$$C_G^2\|A^{-\grave{\rho}}\|_{HS}^2\ \|A^{\grave{\rho}}(I+\tau A_\alpha)^{-(m-k)}\|_{\mathcal{L}(H)}^2\ \|u^k-u(t_k)\|_{L^p(\Omega,H)}^2\leq C_{\grave{\rho}}^2\ t_{m-k}^{-\frac{4\grave{\rho}}{\alpha}}\ \|u^k-u(t_k)\|_{L^p(\Omega,H)}^2.$$

Thus R_4 is majorized as

$$
\begin{aligned}
R_4 &:= \left(\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}\left(\mathbb{E}\|(I+\tau A_\alpha)^{-(m-k)}\left(G(u^k)-G(u(t_k))\right)\|_{HS}^p\right)^{\frac{2}{p}}ds\right)^{\frac{1}{2}}\\
&\leq C_{\grave{\rho}}\left(\tau\sum_{k=0}^{m-1}t_{m-k}^{-\frac{4\grave{\rho}}{\alpha}}\|u^k-u(t_k)\|_{L^p(\Omega,H)}^2\right)^{\frac{1}{2}}. \qquad (6.71)
\end{aligned}
$$

We estimate R_5, by the same manner used above (i.e. the use of Assumption $\mathcal{C}_1$ and Est.(6.37) (with $\frac{1}{4}<\xi=\grave{\rho}<\frac{\alpha}{4}$)) with an application of the temporal regularity Est.(6.10) to arrive at

$$
\begin{aligned}
R_5 &:= \left(\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}\left(\mathbb{E}\|(I+\tau A_\alpha)^{-(m-k)}\left(G(u(t_k))-G(u(s))\right)\|_{HS}^p\right)^{\frac{2}{p}}ds\right)^{\frac{1}{2}}\\
&\leq C_{\grave{\rho}}\left(\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}t_{m-k}^{-\frac{4\grave{\rho}}{\alpha}}\|u(t_k)-u(s)\|_{L^p(\Omega,H)}^2ds\right)^{\frac{1}{2}}\\
&\leq C_{\grave{\rho}}\left(\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}t_{m-k}^{-\frac{4\grave{\rho}}{\alpha}}(s-t_k)^{2(\frac{\alpha-1}{2\alpha}-\kappa)}ds\right)^{\frac{1}{2}}.
\end{aligned}
$$

By following the same steps used in the Est.(6.66) we get

$$
\begin{aligned}
R_5 &\leq C_{\grave{\rho}}\left(\sum_{k=0}^{m-1}t_{m-k}^{-\frac{4\grave{\rho}}{\alpha}}(t_{k+1}-t_k)^{1+2(\frac{\alpha-1}{2\alpha}-\kappa)}\right)^{\frac{1}{2}}=C_{\grave{\rho}}\,\tau^{\frac{\alpha-1}{2\alpha}-\kappa}\left(\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}t_{m-k}^{-\frac{4\grave{\rho}}{\alpha}}ds\right)^{\frac{1}{2}}\\
&\leq C_{\grave{\rho}}\,\tau^{\frac{\alpha-1}{2\alpha}-\kappa}\left(\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}(t_m-s)^{-\frac{4\grave{\rho}}{\alpha}}ds\right)^{\frac{1}{2}}.
\end{aligned}
$$

The fact that $\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}(t_m-s)^{-\frac{4\grave{\rho}}{\alpha}}ds=\int_0^{t_m}(t_m-s)^{-\frac{4\grave{\rho}}{\alpha}}ds\leq\frac{1}{1-\frac{4\grave{\rho}}{\alpha}}<\infty$ since $\frac{4\grave{\rho}}{\alpha}<1$ yields

$$
R_5\leq C_{\grave{\rho},\alpha}\,\tau^{\frac{\alpha-1}{2\alpha}-\kappa}. \qquad (6.72)
$$

To estimate R_6, we use Assumption $\mathcal{C}_1$. Then

$$
\begin{aligned}
R_6 &:= \left(\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}\left(\mathbb{E}\|\left((I+\tau A_\alpha)^{-(m-k)}-S_\alpha(t_m-s)\right)G(u(s))\|_{HS}^p\right)^{\frac{2}{p}}ds\right)^{\frac{1}{2}}\\
&\leq \left(\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}\|(I+\tau A_\alpha)^{-(m-k)}-S_\alpha(t_m-s)\|_{HS}^2\left(1+\|u(s)\|_{L^p(\Omega,H)}^2\right)ds\right)^{\frac{1}{2}}\\
&\leq \left(1+\sup_{s\in[0,1]}\|u(s)\|_{L^p(\Omega,H)}^2\right)\\
&\quad\left(\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}\|A^{-\grave{\rho}}\|_{HS}^2\|A^{\grave{\rho}}\left((I+\tau A_\alpha)^{-(m-k)}-S_\alpha(t_m-s)\right)\|_{\mathcal{L}(H)}^2ds\right)^{\frac{1}{2}}.
\end{aligned}
$$

The use of Est.(6.39) in lemma 6.6 (with $\frac{1}{4} < \xi = \grave{\rho} < \frac{\alpha}{4}$ and $0 < \beta = \grave{\beta} < \frac{\alpha}{4} - \grave{\rho}$) and the fact that $\|A^{-\grave{\rho}}\|_{HS} < \infty$ leads to

$$R_6 \leq \tau^{\frac{2\grave{\beta}}{\alpha}} \left(\sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} (t_m - s)^{-\frac{4}{\alpha}(\grave{\rho}+\grave{\beta})} ds \right)^{\frac{1}{2}} \leq C_{p,\grave{\rho},\grave{\beta}}\, \tau^{\frac{2\grave{\beta}}{\alpha}}. \tag{6.73}$$

From Est.(6.71), Est.(6.72) and Est.(6.73) we estimate the stochastic term in the RHS of Est.(6.61) as follows

$$\begin{aligned} &\| \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \left((I + \tau A_\alpha)^{-(m-k)} G(u^k) - S_\alpha(t_m - s) G(u(s)) \right) dW(s) \|_{L^p(\Omega,H)} \\ \leq\ & C_{p,\alpha,\grave{\rho},\grave{\beta}} (\tau^{\frac{\alpha-1}{2\alpha}-\kappa} + \tau^{\frac{2\grave{\beta}}{\alpha}}) + C_{p,\grave{\rho}} \left(\tau \sum_{k=0}^{m-1} t_{m-k}^{-\frac{4\grave{\rho}}{\alpha}} \|u^k - u(t_k)\|^2_{L^p(\Omega,H)} \right)^{\frac{1}{2}}. \end{aligned} \tag{6.74}$$

Now, to unify the two terms $\left(\tau \sum_{k=0}^{m-1} t_{m-k}^{-\frac{2\rho}{\alpha}} \|u^k - u(t_k)\|_{L^p(\Omega,H)} \right)^2$ and $\left(\tau \sum_{k=0}^{m-1} t_{m-k}^{-\frac{4\grave{\rho}}{\alpha}} \|u^k - u(t_k)\|^2_{L^p(\Omega,H)} \right)$, we apply Hölder inequality; Est.A.11 and the fact that $t_m - s \leq t_{m-k}$ to obtain

$$\begin{aligned} \tau \sum_{k=0}^{m-1} t_{m-k}^{-\frac{2\rho}{\alpha}} \|u^k - u(t_k)\|_{L^p(\Omega,H)} &\leq \left(\tau \sum_{k=0}^{m-1} t_{m-k}^{-\frac{2\rho}{\alpha}} \right)^{\frac{1}{2}} \left(\tau \sum_{k=0}^{m-1} t_{m-k}^{-\frac{2\rho}{\alpha}} \|u^k - u(t_k)\|^2_{L^p(\Omega,H)} \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} (t_m - s)^{-\frac{2\rho}{\alpha}} ds \right)^{\frac{1}{2}} \left(\tau \sum_{k=0}^{m-1} t_{m-k}^{-\frac{2\rho}{\alpha}} \|u^k - u(t_k)\|^2_{L^p(\Omega,H)} \right)^{\frac{1}{2}} \\ &\leq C_{\alpha,\rho} \left(\tau \sum_{k=0}^{m-1} t_{m-k}^{-\frac{2\rho}{\alpha}} \|u^k - u(t_k)\|^2_{L^p(\Omega,H)} \right)^{\frac{1}{2}}. \end{aligned} \tag{6.75}$$

Now, coming back to Est.(6.61) and replace Est.(6.62), Est.(6.69) and Est.(6.74) to conclude that

$$\begin{aligned} \|u^m - u(t_m)\|^2_{L^p(\Omega,H)} &\leq C_{(p,\alpha,\eta,\rho,\grave{\rho},\grave{\beta},\|u_0\|_{L^p(\Omega,H^{\eta}_{2})})} \left(\tau^{\frac{\eta}{\alpha}} + \tau^{\frac{\alpha-1}{2\alpha}-\kappa} + \tau^{\frac{2\beta}{\alpha}} + \tau^{\frac{2\grave{\beta}}{\alpha}} \right)^2 \\ &+ C_{\alpha,\rho} \left(\tau \sum_{k=0}^{m-1} t_{m-k}^{-\frac{2\rho}{\alpha}} \|u^k - u(t_k)\|^2_{L^p(\Omega,H)} \right) \\ &+ C_{p,\grave{\rho}} \left(\tau \sum_{k=0}^{m-1} t_{m-k}^{-\frac{4\grave{\rho}}{\alpha}} \|u^k - u(t_k)\|^2_{L^p(\Omega,H)} \right). \end{aligned} \tag{6.76}$$

Therefore, for $0 < \eta \leq \alpha$, $0 < \beta < \frac{\alpha}{2} - \rho$, $0 \leq \rho < \frac{\alpha}{2}$, $0 < \grave{\beta} < \frac{\alpha}{4} - \grave{\rho}$ and $\frac{1}{4} < \grave{\rho} < \frac{\alpha}{4}$.

$$\begin{aligned} \|u^m - u(t_m)\|^2_{L^p(\Omega,H)} &\leq C_{(p,\alpha,\eta,\rho,\grave{\rho},\grave{\beta},\|u_0\|_{L^p(\Omega,H^{\eta}_{2})})} \left(\tau^{\frac{\eta}{\alpha}} + \tau^{\frac{\alpha-1}{2\alpha}-\kappa} + \tau^{\frac{2\beta}{\alpha}} + \tau^{\frac{2\grave{\beta}}{\alpha}} \right)^2 \\ &+ C_{p,\rho,\grave{\rho}} \tau \sum_{k=0}^{m-1} t_{m-k}^{-\frac{2}{\alpha} max(\rho, 2\grave{\rho})} \|u^k - u(t_k)\|^2_{L^p(\Omega,H)}, \end{aligned} \tag{6.77}$$

By denoting $\mu := \min\{\frac{\eta}{\alpha}, \frac{\alpha-1}{2\alpha} - \kappa, \frac{2\beta}{\alpha}, \frac{2\grave{\beta}}{\alpha}\}$ with the application of the descrite version of Gronwall lemma A.2 we arrive at

$$\|u^m - u(t_m)\|^2_{L^p(\Omega,H)} \leq C_{(p,\alpha,\eta,\rho,\grave{\rho},\grave{\beta},\beta,\|u_0\|_{L^p(\Omega,H^{\eta}_{2})})}\tau^{2\mu} \times e^{(C_{p,\rho,\grave{\rho}}\tau \sum_{k=0}^{m-1} t_{m-k}^{-\frac{2}{\alpha} max(\rho,2\grave{\rho})})}. \tag{6.78}$$

Thanks to the fact that $\frac{2}{\alpha}\, max(\rho, 2\grave{\rho}) < 1$, we obtain

$$e^{(C_{p,\rho,\grave{\rho}}\tau \sum_{k=0}^{m-1} t_{m-k}^{-\frac{2}{\alpha} max(\rho,2\grave{\rho})})} \leq e^{C_{p,\rho,\grave{\rho}} \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} (t_m-s)^{-\frac{2}{\alpha} max(\rho,2\grave{\rho})}} \leq e^{C_{p,\alpha,\rho,\grave{\rho}}} \tag{6.79}$$

In order to improve the order of convergence, we take $\eta > \frac{\alpha-1}{2}$, $\rho \leq \frac{1}{2}$, and let $\beta \to \frac{\alpha}{2} - \rho$ and $\grave{\beta} \to \frac{\alpha}{4} - \grave{\rho}$ with $\grave{\rho} \to \frac{1}{4}$. Then, we have

$$\|u^m - u(t_m)\|_{L^p(\Omega,H)} \leq C_{p,\alpha,\eta,\rho}\tau^{\frac{\alpha-1}{2\alpha}-\kappa}, \text{ for any } \kappa > 0. \tag{6.80}$$

Now, if $p > \frac{2\alpha}{\alpha-1}$, the sufficient condition of Lemma 4.17 is fulfilled and therefore our scheme converges in probability with the same order $\frac{\alpha-1}{2\alpha} - \kappa$. Moreover, it is almost surely convergent with the order $\xi < (\frac{\alpha-1}{2\alpha}-) - \frac{1}{p}$.

6.5 Spacial approximation

This section is devoted to fulfill the spacial approximation of the fractional stochastic nonlinear heat equation Prb.(6.1), with $\alpha \in (1, 2]$. First, we establish the spacial discrete version of Eq.(6.1). We fix $N \in \mathbb{N}_0$, $h = \frac{1}{N}$ and $(H_h)_{h\in(0,1]}$ being a sequence of finite dimensional subspaces of $H := L^2(0, 1)$ generated by the N first eigenfunctions $(e_k)_{k=1}^N$, i.e.

$$H_h := span\{e_k, k = 1, ..., N\},$$

where $(e_k(.) := \sqrt{2}sin(k\pi.))_{k=1}^{\infty}$ is the ONB of H which can be considred as the eigenvectors of minus Laplacian corresponding to the eigenvalues $(\lambda_k := (k\pi)^2)_{k\in\mathbb{N}_0}$, see Section 1.2.3.

Definition 6.7 ***(Discrete version of*** A_α***).*** *The discrete version of* A_α *denoted* $A_{\alpha,h}$ *is defined from* H_h *to* H_h *by*

$$A_{\alpha,h}v_h := \sum_{i=1}^{N} v_h^i A_\alpha e_i = \sum_{i=1}^{N} v_h^i \lambda_i^{\frac{\alpha}{2}} e_i, \ \forall\, v_h \in H_h, ,$$

where $v_h^i := \langle v_h, e_i\rangle$ *with* $\langle ., .\rangle$ *is the* H*-inner product.*

It is easy to see that, for every $k \in \{1, ..., N\}$,

$$A_{\alpha,h} e_k = \sum_{i=1}^{N} \langle e_k, e_i \rangle A_\alpha e_i = A_\alpha e_k = \lambda_k^{\frac{\alpha}{2}} e_k.$$

Then, $(\lambda_k^{\frac{\alpha}{2}})_{k=1}^N$ is the set of eigenvalues the operator $A_{\alpha,h}$ corresponding to the set of eigenvectors $(e_k)_{k=1}^N$.

Lemma 6.8

- *the operator $-A_{\alpha,h}$ is a generator of a semigroup of contraction $(e^{-tA_{\alpha,h}})_{t\in[0,1]}$ on H_h, let us denote it by $(S_{\alpha,h}(t))_{t\in[0,1]}$, acting on the spectrum as; for any $k \in \{1, ..., N\}$ it holds $e^{-tA_{\alpha,h}} e_k = e^{-t\lambda_k^{\frac{\alpha}{2}}} e_k,$.*
- *there exists a positive constant C_β s.t.*

$$\|A_{\alpha,h}^{\beta} e^{-tA_{\alpha,h}}\|_{\mathcal{L}(H)} \leq C_\beta t^{-\beta}, \text{ for all } t \in (0, 1]. \tag{6.81}$$

Proof. Let $u_h := \sum_{i=1}^{N} u_h^i A_\alpha e_i \in H_h$ and $v_h := \sum_{j=1}^{N} v_h^j A_\alpha e_j \in H_h$, where $u_h^i := \langle u_h, e_i \rangle$ and $v_h^j := \langle v_h, e_j \rangle$. We have

$$\begin{aligned} \langle A_{\alpha,h} u_h, v_h \rangle &= \langle \sum_{i=1}^{N} u_h^i A_\alpha e_i, \sum_{j=1}^{N} v_h^j e_j \rangle = \sum_{i,j=1}^{N} u_h^i v_h^j \langle A_\alpha e_i, e_j \rangle = \sum_{i,j=1}^{N} u_h^i v_h^j \langle e_i, A_\alpha e_j \rangle \\ &= \sum_{i,j=1}^{N} \langle u_h^i e_i, v_h^j A_\alpha e_j \rangle = \langle \sum_{i=1}^{N} u_h^i e_i, \sum_{j=1}^{N} v_h^j A_\alpha e_j \rangle = \langle u_h, A_{\alpha,h} v_h \rangle. \end{aligned}$$

Thus, $A_{\alpha,h}$ is symmetric with $D(A_{\alpha,h}) = H_h$, and so, it is self-adjoint via Corolarry 1.32. Further

$$\begin{aligned} \langle A_{\alpha,h} u_h, u_h \rangle &= \langle \sum_{i=1}^{N} u_h^i A_\alpha e_i, \sum_{j=1}^{N} u_h^j e_j \rangle = \sum_{i,j=1}^{N} u_h^i u_h^j \langle A_\alpha e_i, e_j \rangle \\ &= \sum_{i,j=1}^{N} u_h^i u_h^j \lambda_i^{\frac{\alpha}{2}} \langle e_i, e_j \rangle = \sum_{i=1}^{N} (u_h^i)^2 \lambda_i^{\frac{\alpha}{2}} \geq 0. \end{aligned}$$

Then, $A_{\alpha,h}$ is nonnegative. According to Proposition 1.58, the operator $-A_{\alpha,h}$ is a generator of a semigroup of contraction $(e^{-tA_{\alpha,h}})_{t\in[0,1]}$ on H_h.
About the smoothing property Est.(6.81), the use of Est.(1.12) in Lemma 1.36 and Lemma A.8 gives

$$\|A_{\alpha,h}^{\beta} e^{-tA_{\alpha,h}}\|_{\mathcal{L}(H)} = \sup_{1\leq i\leq N} \left(\lambda_i^{\frac{\alpha\beta}{2}} e^{-t\lambda_i^{\frac{\alpha}{2}}} \right) \leq C_\beta t^{-\beta}.$$

■

Let us recall the Galerkin projection studied in Subsection 5.1.4. In this chapter we denote it by P_h instead of P_N, with $h = \frac{1}{N}$. Then, for all $u \in H$ we have

$$(P_h u)(x) := \sum_{k=1}^{N} \langle u, e_k \rangle e_k(x), \text{ for all } x \in [0,1].$$

We add to the list of properties of P_h stated in Lemma 5.11 the following two properties.

Lemma 6.9 *The operator P_h satisfies,*

- *for all $\beta \in \mathbb{R}$, the operators A_β and P_h commute.*
- *For all $\alpha \in (1,2]$ and all $\beta > \frac{1}{2}$ there exists $C_{\alpha,\beta} > 0$ s.t.*

$$\|P_h S_\alpha(t)\|_{HS}^2 \leq C_{\alpha,\beta}\, t^{-\frac{2\beta}{\alpha}}, \tag{6.82}$$

where $\| \,.\, \|_{HS}$ denotes the Hilbert-Shmidt norm, see Remark 1.13.

Proof. As the operator A_β is defined on the spectrum of A and P_h is the projection on H_h construct via the elements of the ONB of the eigenvalues of A, the proof of the first property is easy, so we omit.
To proof Est.(6.82), we use the definition of the Hilbert-Schmidt norm (see Remark 1.13), Est.(5.31), Est.(1.23) to get

$$\begin{aligned} \|P_h S_\alpha(t)\|_{HS}^2 &= \sum_{k=1}^{\infty} |P_h S_\alpha(t) e_k|_H^2 \leq \sum_{k=1}^{\infty} \|P_h\|_{\mathcal{L}(H)}^2 |S_\alpha(t) e_k|_H^2 \leq C \sum_{k=1}^{\infty} |S_\alpha(t) e_k|_H^2 \\ &= C \sum_{k=1}^{\infty} |A^{\frac{\beta}{2}} S_\alpha(t) A^{-\frac{\beta}{2}} e_k|_H^2 = C \sum_{k=1}^{\infty} \lambda_k^{-\beta} |A^{\frac{\beta}{2}} S_\alpha(t) e_k|_H^2 \leq C \sum_{k=1}^{\infty} \lambda_k^{-\beta} \|A^{\frac{\beta}{2}} S_\alpha(t)\|_{\mathcal{L}(H)}^2 \, |e_k|_H^2. \end{aligned}$$

The facts that $|e_k|_H^2 = 1$ and $\lambda_k = (k\pi)^2$ leads to

$$\|P_h S_\alpha(t)\|_{HS}^2 \leq C_{\alpha,\beta} t^{-\frac{2\beta}{\alpha}} \sum_{k=1}^{\infty} \lambda_k^{-\beta} |e_k|_H^2 \leq C_{\alpha,\beta} t^{-\frac{2\beta}{\alpha}} \sum_{k=1}^{\infty} k^{-2\beta} \;\leq\; C_{\alpha,\beta} t^{-\frac{2\beta}{\alpha}}.$$

■

Now, we are able to introduce the following discrete version of Prb.(6.1) given in the evolution form by using the spectral Galerkin method as:

$$\begin{cases} du_h(t) &= (-A_{\alpha,h} u_h(t) + P_h F(u_h(t)))dt + P_h G(u_h(t)) dW(t), \; t \in (0,1], \\ u_h(0) &= P_h u_0, \end{cases} \tag{6.83}$$

We recall here the result of the wellposedness for Prb.(6.83) (see [48, Theorems 2.2 and 2.3]).

Theorem 6.10 *There exists a continous H_h-valued $\mathcal{F}_t$-adapted process $(u_h(t))_{t\in[0,1]}$ solution of Prb.(6.83) s.t. for all $t \in [0,1]$ the following equality holds in H_h, $\mathbb{P}$- a.s.*

$$u_h(t) = S_{\alpha,h}(t)P_h u_0 + \int_0^t S_{\alpha,h}(t-s)P_h F(u_h(s))\,ds + \int_0^t S_{\alpha,h}(t-s)P_h G(u_h(s))\,dW(s). \tag{6.84}$$

Moreover, there exists a constant $C_h > 0$ s.t.

$$\mathbb{E}\sup_{t\in[0,1]} |u_h(t)|_H^p \le C_h(1+\mathbb{E}|u_h(0)|_H^p), \text{ for all } \quad p\in[1,\infty). \tag{6.85}$$

Our main result of this section is the following.

Theorem 6.11 *Let $\alpha \in (1,2]$. Under Assumptions $\mathcal{A}_1^r$ with $\eta > 0$, $\mathcal{B}_1^r$ with $\rho \in [0, \frac{\alpha}{2})$ and $\mathcal{C}_1$, there exists a constant $C_{\alpha,\eta,p} > 0$ independent of $h \in (0,1]$ s.t.*

$$\|u_h(t) - u(t)\|_{L^p(\Omega,H)} \le C_{\alpha,\eta,p} h^{\nu}, \text{ for all } t\in(0,1], \tag{6.86}$$

with $\nu := \min\{\eta, \alpha - 2\rho, \frac{\alpha}{2} - \kappa\}$, for any $\kappa > 0$. In particular, for $\eta > \frac{\alpha}{2}$ and $\rho \in [0, \frac{\alpha}{4})$, we get $\nu = \frac{\alpha}{2} - \kappa$.

In order to prove Theorem 6.11, we introduce and study some basic estimates of the family of operators $(E_{\alpha,h}(t))_{t\in[0,1]}$ s.t.

$$\forall\, t\in[0,1],\ E_{\alpha,h}(t) := S_\alpha(t) - S_{\alpha,h}(t)P_h.$$

Remark 6.12 *It is easy to see that $S_{\alpha,h}(t)P_h = P_h S_\alpha(t)$, and so $E_{\alpha,h}(t) = (I - P_h)S_\alpha(t)$.*

Lemma 6.13 *Let $\alpha \in (1,2]$ and $t \in (0,1]$. We have*

- *For all $\gamma \ge 0$ and all $\beta \in \mathbb{R}$ there exists $C_{\gamma,\beta} > 0$ s.t.*

$$|E_{\alpha,h}(t)x|_H \le C_{\gamma,\beta} h^{\gamma}\, t^{-\frac{(\gamma-\beta)}{\alpha}}\, |A^{\frac{\beta}{2}}x|_H, \ \ \forall x\in H_2^{\beta}, \tag{6.87}$$

  where H_2^β is the fractional Sobolev space given by Definition 1.3 and Remark 1.4.

- *For all $0 \le \delta < \frac{\alpha}{2}$ and $\gamma > \frac{4\delta}{\alpha}$ there exists $C_{\alpha,\gamma,\delta} > 0$ s.t.*

$$|E_{\alpha,h}(t)x|_H \le C_{\alpha,\gamma,\delta} h^{(\frac{\alpha\gamma}{2} - 2\delta)}\, t^{-\frac{\gamma}{2}}\, |A^{-\delta}x|_H, \tag{6.88}$$

  for all $x \in D(A^{-\delta})$.

- *For all $\gamma > \frac{1}{\alpha}$ there exists $C_{\gamma,\alpha} > 0$ s.t.*

$$\|E_{\alpha,h}(t)\|_{HS}^2 \leq C_{\gamma,\alpha}\, h^{\alpha\gamma}\, t^{-\gamma}. \tag{6.89}$$

Proof. Fix $t \in (0,1]$ and let $\gamma \geq 0$ and $\beta \in [0,\gamma]$, then for all $x \in H_2^{\beta}$ and from Remark 6.12 we have

$$E_{\alpha,h}(t)x = \sum_{k=N+1}^{\infty} \langle x, e_k\rangle e^{-t\lambda_k^{\frac{\alpha}{2}}} e_k.$$

Now, about Est.(6.87) we use Lemma A.8 to end up with

$$\begin{aligned} |E_{\alpha,h}(t)x|_H^2 &= \sum_{k=N+1}^{\infty} \langle x, e_k\rangle^2 e^{-2\,t\lambda_k^{\frac{\alpha}{2}}} = \sum_{k=N+1}^{\infty} \lambda_k^{-\gamma}\left(\lambda_k^{(\gamma-\beta)} e^{-2\,t\lambda_k^{\frac{\alpha}{2}}}\right)\, \langle x, e_k\rangle^2\, \lambda_k^{\beta} \\ &\leq \lambda_{N+1}^{-\gamma} \sum_{k=N+1}^{\infty} t^{-\frac{2(\gamma-\beta)}{\alpha}} \left((t\,\lambda_k^{\frac{\alpha}{2}})^{\frac{2(\gamma-\beta)}{\alpha}}\, e^{-2\,t\lambda_k^{\frac{\alpha}{2}}}\right)\, \langle x, e_k\rangle^2\, \lambda_k^{\beta} \\ &\leq C_{\gamma,\beta}(N+1)^{-2\gamma}\, t^{-\frac{2(\gamma-\beta)}{\alpha}} \sum_{k=1}^{\infty} \langle x, e_k\rangle^2\, \lambda_k^{\beta} \;\leq\; C_{\gamma,\beta} h^{2\gamma}\, t^{-\frac{2(\gamma-\beta)}{\alpha}} |A^{\frac{\beta}{2}} x|_H^2, \end{aligned}$$

which achieve the Est.(6.87). The proof of Est.(6.88) follows the same lines as that of Est.(6.87) and the fact that $\lambda_k = (k\pi)^2$, for this let $0 \leq \delta < \frac{\alpha}{2}$, $\gamma > \frac{4\delta}{\alpha}$ and $x \in D(A^{-\delta})$ then

$$\begin{aligned} |E_{\alpha,h}(t)x|_H^2 &= \sum_{k=N+1}^{\infty} \langle x, e_k\rangle^2 e^{-2\,t\lambda_k^{\frac{\alpha}{2}}} \leq C_{\alpha,\gamma} t^{-\gamma} \sum_{k=N+1}^{\infty} \lambda_k^{-\frac{\alpha\gamma}{2}} \langle x, e_k\rangle^2 \\ &= C_{\alpha,\gamma} t^{-\gamma} \sum_{k=N+1}^{\infty} \lambda_k^{-(\frac{\alpha\gamma}{2}-2\delta)} \langle x, e_k\rangle^2 \lambda_k^{-2\delta} = C_{\alpha,\gamma} t^{-\gamma} \sum_{k=N+1}^{\infty} (k\pi)^{-2(\frac{\alpha\gamma}{2}-2\delta)} \langle x, e_k\rangle^2 \lambda_k^{-2\delta} \\ &\leq C_{\alpha,\gamma,\delta} t^{-\gamma} (N+1)^{-2(\frac{\alpha\gamma}{2}-2\delta)} \sum_{k=1}^{\infty} \langle x, e_k\rangle^2 \lambda_k^{-2\delta} \leq C_{\alpha,\gamma,\delta} t^{-\gamma} h^{-2(\frac{\alpha\gamma}{2}-2\delta)} |A^{-\delta} x|_H^2. \end{aligned}$$

About Est.(6.89), let $\gamma > \frac{1}{\alpha}$, the use of definitions of $E_{\alpha,h}$ and HS-norm besides the orthogonality of $(e_k)_k$ leads to

$$\|E_{\alpha,h}(t)\|_{HS}^2 = \sum_{k=1}^{\infty} |E_{\alpha,h}(t)e_k|_H^2 \;=\; \sum_{k=1}^{\infty}\sum_{i=N+1}^{\infty} \langle e_k, e_i\rangle^2 e^{-2t\,\lambda_i^{\frac{\alpha}{2}}} \;=\; \sum_{k=N+1}^{\infty} e^{-2t\,\lambda_k^{\frac{\alpha}{2}}}.$$

We apply Lemma A.8 with the definition of λ_k allows us to get

$$\begin{aligned} \|E_{\alpha,h}(t)\|_{HS}^2 &= (N+1)^{-\alpha\gamma} \sum_{k=N+1}^{\infty} (N+1)^{\alpha\gamma}\, e^{-2t\,\lambda_k^{\frac{\alpha}{2}}} \leq C_{\gamma}(N+1)^{-\alpha\gamma}\, t^{-\gamma} \sum_{k=N+1}^{\infty} (N+1)^{\alpha\gamma} \lambda_k^{-\frac{\alpha\gamma}{2}} \\ &\leq C_{\gamma,\alpha} h^{\alpha\gamma}\, t^{-\gamma} \sum_{k=N+1}^{\infty} \left(\frac{k}{N+1}\right)^{-\alpha\gamma} \leq C_{\gamma,\alpha} h^{\alpha\gamma}\, t^{-\gamma} \int_1^{\infty} s^{-\alpha\gamma}\, ds, \end{aligned}$$

The integral $\int_1^{\infty} s^{-\alpha\gamma}\, ds$ is finite, thanks to the fact that $\gamma > \frac{1}{\alpha}$, and so the desired estimate is achived. ■

Now we are ready to prove our main result.

Proof of Theorem 6.11

From equations (6.9) and (6.84) we have,

$$\begin{aligned}\|u_h(t) \;-\; u(t)\|_{L^p(\Omega,H)} &\leq \|E_{\alpha,h}(t)u_0\|_{L^p(\Omega,H)} \\ &+ \|\int_0^t S_{\alpha,h}(t-s)P_hF(u_h(s))\,ds - \int_0^t S_\alpha(t-s)F(u(s))\,ds\|_{L^p(\Omega,H)} \\ &+ \|\int_0^t S_{\alpha,h}(t-s)P_hG(u_h(s))\,dW(s) - \int_0^t S_\alpha(t-s)G(u(s))\,dW(s)\|_{L^p(\Omega,H)}.\end{aligned} \tag{6.90}$$

By using Est.(6.87) (with $\beta = \gamma = \eta$), with η given in Assumption A_1^r we get

$$\|E_{\alpha,h}(t)u_0\|_{L^p(\Omega,H)} \leq C_\eta\; h^\eta\; \|A^{\frac{\eta}{2}}u_0\|_{L^p(\Omega,H)}. \tag{6.91}$$

The second term in the RHS of Est.(6.90) is splitted to two additional terms as follows

$$\begin{aligned}\| &\int_0^t S_{\alpha,h}(t-s)P_hF(u_h(s))\,ds - \int_0^t S_\alpha(t-s)F(u(s))\,ds\|_{L^p(\Omega,H)} \\ \leq\;& \|\int_0^t S_{\alpha,h}(t-s)P_h\left(F(u_h(s)) - F(u(s))\right)ds\;\|_{L^p(\Omega,H)} \\ +\;& \|\int_0^t \left(S_{\alpha,h}(t-s)P_h - S_\alpha(t-s)\right)F(u(s))ds\;\|_{L^p(\Omega,H)} = S_1 + S_2.\end{aligned}$$

Thanks to commutativity of P_h and $S_{\alpha,h}$, we have

$$\begin{aligned}S_1 &\leq \int_0^t \|S_{\alpha,h}(t-s)P_h\left(F(u_h(s)) - F(u(s))\right)\|_{L^p(\Omega,H)}ds \\ &= \int_0^t \|P_hS_\alpha(t-s)\left(F(u_h(s)) - F(u(s))\right)\|_{L^p(\Omega,H)}ds \\ &= \int_0^t \|P_hA^\rho S_\alpha(t-s)A^{-\rho}\left(F(u_h(s)) - F(u(s))\right)\|_{L^p(\Omega,H)}ds.\end{aligned}$$

An application of Est.(5.31), Est.(1.23) and the Assumption $\mathcal{B}_1^r$ leads to

$$\begin{aligned}S_1 &\leq \int_0^t \|P_h\|_{\mathcal{L}(H)}\|A^\rho S_\alpha(t-s)\|_{\mathcal{L}(H)}\|\left(A^{-\rho}\left(F(u_h(s)) - F(u(s))\right)\right)\|_{L^p(\Omega,H)}ds \\ &\leq C\int_0^t \|A^\rho S_\alpha(t-s)\|_{\mathcal{L}(H)}\|A^{-\rho}\left(F(u_h(s)) - F(u(s))\right)\|_{L^p(\Omega,H)}ds \\ &\leq C_{\alpha,\rho}\int_0^t (t-s)^{-\frac{2\rho}{\alpha}}\|A^{-\rho}\left(F(u_h(s)) - F(u(s))\right)\;\|_{L^p(\Omega,H)}ds \\ &\leq C_{\alpha,\rho}\int_0^t (t-s)^{-\frac{2\rho}{\alpha}}\|u_h(s) - u(s)\|_{L^p(\Omega,H)}ds.\end{aligned} \tag{6.92}$$

To estimate S_2 we use the definition of $E_{\alpha,h}$, Remark 6.1 and Est.(6.88) (with $\delta=\rho<\frac{\alpha}{2}$ and $\frac{4\rho}{\alpha}<\gamma<2$) and Assumption $\mathcal{B}_1^r$ we get

$$\begin{aligned}
S_2 &= \| \int_0^t \left(S_{\alpha,h}(t-s)P_h - S_\alpha(t-s)\right) F(u(s))ds \|_{L^p(\Omega,H)} = \| \int_0^t E_{\alpha,h}(t-s)F(u(s))ds \|_{L^p(\Omega,H)} \\
&\leq \int_0^t \|E_{\alpha,h}(t-s)F(u(s))\|_{L^p(\Omega,H)}ds \leq C_{\alpha,\gamma,\rho,p}h^{(\frac{\alpha\gamma}{2}-2\rho)} \int_0^t (t-s)^{-\frac{\gamma}{2}}\|A^{-\rho}F(u(s))\|_{L^p(\Omega,H)}ds \\
&\leq C_{\alpha,\gamma,\rho,p}h^{(\frac{\alpha\gamma}{2}-2\rho)}\left(1+\sup_{t\in[0,1]}\|u(t)\|_{L^p(\Omega,H)}\right)\int_0^t (t-s)^{-\frac{\gamma}{2}}ds,
\end{aligned}$$

thanks to $\gamma<2$, the integral $\int_0^t (t-s)^{-\frac{\gamma}{2}}ds$ is finite and so

$$S_2 \leq C_{\alpha,\gamma,\rho,p}h^{(\frac{\alpha\gamma}{2}-2\rho)}. \tag{6.93}$$

By gathering Est.(6.92) and Est.(6.93) we arrive at

$$\begin{aligned}
\| &\int_0^t S_{\alpha,h}(t-s)P_hF(u_h(s))\,ds - \int_0^t S_\alpha(t-s)F(u(s))\,ds\|_{L^p(\Omega,H)} \\
&\leq C_{\alpha,\gamma,\rho,p}h^{(\frac{\alpha\gamma}{2}-2\rho)} + C_{\alpha,\rho}\int_0^t (t-s)^{-\frac{2\rho}{\alpha}}\|u_h(s)-u(s)\|_{L^p(\Omega,H)}ds. \qquad (6.94)
\end{aligned}$$

To estimate the stochastic term in the RHS of Est.(6.90), we use Burkholder-Davis-Gundy inequality with $p\geq 2$ and the definition of the norm in $L^p(\Omega,\mathbb{R})$ to get

$$\begin{aligned}
\| &\int_0^t S_{\alpha,h}(t-s)P_hG(u_h(s))\,dW(s) - \int_0^t S_\alpha(t-s)G(u(s))\,dW(s)\|_{L^p(\Omega,H)} \\
&\leq C_p\left(\mathbb{E}\left\{\left(\int_0^t \|S_{\alpha,h}(t-s)P_hG(u_h(s)) - S_\alpha(t-s)G(u(s))\|_{HS}^2ds\right)^{\frac{p}{2}}\right\}\right)^{\frac{1}{p}} \\
&= C_p\|\left(\int_0^t \|S_{\alpha,h}(t-s)P_hG(u_h(s)) - S_\alpha(t-s)G(u(s))\|_{HS}^2ds\right)^{\frac{1}{2}}\|_{L^p(\Omega,\mathbb{R})}. \qquad (6.95)
\end{aligned}$$

Owing to the basic inequality: $(a+b)^2\leq C(a^2+b^2)$, $\forall\, a,b\geq 0$, for some $C>0$, we get

$$\begin{aligned}
&\|S_{\alpha,h}(t-s)P_hG(u_h(s)) - S_\alpha(t-s)G(u(s))\|_{HS}^2 \\
&= \|S_{\alpha,h}(t-s)P_h\left(G(u_h(s))-G(u(s))\right) + \left(S_{\alpha,h}(t-s)P_h - S_\alpha(t-s)\right)G(u(s))\|_{HS}^2 \\
&\leq \|S_{\alpha,h}(t-s)P_h\left(G(u_h(s))-G(u(s))\right)\|_{HS}^2 + \int_0^t \|\left(S_{\alpha,h}(t-s)P_h - S_\alpha(t-s)\right)G(u(s))\|_{HS}^2
\end{aligned}$$

The fact that: $(a+b)^{\frac{1}{2}}\leq C(a^{\frac{1}{2}}+b^{\frac{1}{2}})$, $\forall\, a,b\geq 0$, for some $C>0$ yields

$$\begin{aligned}
(\int_0^t \|S_{\alpha,h}(t-s)P_hG(u_h(s)) &- S_\alpha(t-s)G(u(s))\|_{HS}^2ds)^{\frac{1}{2}} \\
&\leq \left(\int_0^t \|S_{\alpha,h}(t-s)P_h\left(G(u_h(s))-G(u(s))\right)\|_{HS}^2ds\right)^{\frac{1}{2}} \\
&+ \left(\int_0^t \|\left(S_{\alpha,h}(t-s)P_h - S_\alpha(t-s)\right)G(u(s))\|_{HS}^2ds\right)^{\frac{1}{2}},
\end{aligned}$$

and so

$$
\begin{aligned}
\| & \int_0^t S_{\alpha,h}(t-s)P_h G(u_h(s))\, dW(s) - \int_0^t S_\alpha(t-s)G(u(s))\, dW(s)\|_{L^p(\Omega,H)} \\
\leq & \; C_p \| \left(\int_0^t \|S_{\alpha,h}(t-s)P_h \left(G(u_h(s)) - G(u(s))\right)\|_{HS}^2 ds \right)^{\frac{1}{2}} \|_{L^p(\Omega,\mathbb{R})} \\
+ & \; C_p \| \left(\int_0^t \| \left(S_{\alpha,h}(t-s)P_h - S_\alpha(t-s)\right) G(u(s))\|_{HS}^2 ds \right)^{\frac{1}{2}} \|_{L^p(\Omega,\mathbb{R})} := C_p(S_3 + S_4).
\end{aligned}
\tag{6.96}
$$

To estimate S_3, we have $S_{\alpha,h}(t)P_h = P_h S_\alpha(t)$ then

$$
\begin{aligned}
S_3 &= \| \left(\int_0^t \|S_{\alpha,h}(t-s)P_h \left(G(u_h(s)) - G(u(s))\right)\|_{HS}^2 ds \right)^{\frac{1}{2}} \|_{L^p(\Omega,\mathbb{R})} \\
&= \| \left(\int_0^t \|P_h S_\alpha(t-s) \left(G(u_h(s)) - G(u(s))\right)\|_{HS}^2 ds \right)^{\frac{1}{2}} \|_{L^p(\Omega,\mathbb{R})},
\end{aligned}
$$

Now, we use the fact that $\|AB\|_{HS} \leq \|A\|_{\mathcal{L}(H)}\|B\|_{HS}$ for $A \in \mathcal{L}(H)$ and $B \in HS$ and Est.(6.82) (with $\frac{1}{2} < \beta < \frac{\alpha}{2}$) together with the Assumption $\mathcal{C}_1$, to get

$$
\begin{aligned}
S_3 &\leq \| \left(\int_0^t \|P_h S_\alpha(t-s)\|_{HS}^2 \|G(u_h(s)) - G(u(s))\|_{\mathcal{L}(H)}^2 ds \right)^{\frac{1}{2}} \|_{L^p(\Omega,\mathbb{R})} \\
&\leq C_{\alpha,\beta} \| \left(\int_0^t (t-s)^{-\frac{2\beta}{\alpha}} |u_h(s) - u(s)|_H^2 ds \right)^{\frac{1}{2}} \|_{L^p(\Omega,\mathbb{R})} \\
&= C_{\alpha,\beta} \left(\left(\mathbb{E} \left(\int_0^t (t-s)^{-\frac{2\beta}{\alpha}} |u_h(s) - u(s)|_H^2 ds \right)^{\frac{p}{2}} \right)^{\frac{2}{p}} \right)^{\frac{1}{2}} \\
&= C_{\alpha,\beta} \left(\| \int_0^t (t-s)^{-\frac{2\beta}{\alpha}} |u_h(s) - u(s)|_H^2 ds \|_{L^{\frac{p}{2}}(\Omega,\mathbb{R})} \right)^{\frac{1}{2}} \\
&\leq C_{\alpha,\beta} \left(\int_0^t (t-s)^{-\frac{2\beta}{\alpha}} \| |u_h(s) - u(s)|_H^2 \|_{L^{\frac{p}{2}}(\Omega,\mathbb{R})} ds \right)^{\frac{1}{2}} \\
&= C_{\alpha,\beta} \left(\int_0^t (t-s)^{-\frac{2\beta}{\alpha}} \|u_h(s) - u(s)\|_{L^p(\Omega,H)}^2 ds \right)^{\frac{1}{2}}.
\end{aligned}
\tag{6.97}
$$

Now, we move to estimate S_4, we follow the same steps as above with the use of

Est.(6.89) (with $\frac{1}{\alpha} < \gamma = \xi < 1$) and Assumption $\mathcal{C}_1$, thus

$$
\begin{aligned}
S_4 &= \| \left(\int_0^t \|E_{\alpha,h}(t-s)G(u(s))\|_{HS}^2 ds\right)^{\frac{1}{2}} \|_{L^p(\Omega,\mathbb{R})} \\
&\leq \| \left(\int_0^t \|E_{\alpha,h}(t-s)\|_{HS}^2 \|G(u(s))\|_{\mathcal{L}(H)}^2 ds\right)^{\frac{1}{2}} \|_{L^p(\Omega,\mathbb{R})} \\
&\leq C_{\xi,\alpha} h^{\frac{\alpha\xi}{2}} \| \left(\int_0^t (t-s)^{-\xi}(1+|u(s)|_H)^2 ds\right)^{\frac{1}{2}} \|_{L^p(\Omega,\mathbb{R})} \\
&\leq C_{\xi,\alpha} h^{\frac{\alpha\xi}{2}} \| \left(\int_0^t (t-s)^{-\xi}(1+|u(s)|_H^2) ds\right)^{\frac{1}{2}} \|_{L^p(\Omega,\mathbb{R})} \\
&\leq C_{\xi,\alpha} h^{\frac{\alpha\xi}{2}} \left(\| \int_0^t (t-s)^{-\xi}(1+|u(s)|_H^2) ds \|_{L^{\frac{p}{2}}(\Omega,\mathbb{R})}\right)^{\frac{1}{2}} \\
&\leq C_{\xi,\alpha} h^{\frac{\alpha\xi}{2}} \left(\int_0^t (t-s)^{-\xi} \|1+|u(s)|_H^2\|_{L^{\frac{p}{2}}(\Omega,\mathbb{R})} ds\right)^{\frac{1}{2}} \\
&\leq C_{\xi,\alpha} h^{\frac{\alpha\xi}{2}} \left(\int_0^t (t-s)^{-\xi}(1+\|u(s)\|_{L^p(\Omega,H)}^2) ds\right)^{\frac{1}{2}} \\
&\leq C_{\xi,\alpha} h^{\frac{\alpha\xi}{2}} \left(1+\sup_{t\in[0,1]} \|u(s)\|_{L^p(\Omega,H)}\right) \left(\int_0^t (t-s)^{-\xi} ds\right)^{\frac{1}{2}}.
\end{aligned}
$$

The choice of $\xi < 1$ ensures the convergence of $\int_0^t (t-s)^{-\xi} ds$ and so

$$
S_4 \leq C_{p,\xi,\alpha} h^{\frac{\alpha\xi}{2}}. \tag{6.98}
$$

Replacing Est.(6.97) and Est.(6.98) in (6.96) in order to estimate the third term in (6.90) as follows

$$
\begin{aligned}
& \| \int_0^t S_{\alpha,h}(t-s) P_h G(u_h(s)) dW(s) - \int_0^t S_\alpha(t-s)\, G(u(s)) dW(s) \|_{L^p(\Omega,H)} \\
\leq\; & C_{p,\xi,\alpha} h^{\frac{\alpha\xi}{2}} + C_{p,\alpha,\beta} \left(\int_0^t (t-s)^{-\frac{2\beta}{\alpha}} \|u_h(s)-u(s)\|_{L^p(\Omega,H)}^2 ds\right)^{\frac{1}{2}}.
\end{aligned} \tag{6.99}
$$

Finally, we replace Est.(6.91) with $\eta > 0$, Est.(6.94) with $0 \leq \rho < \frac{\alpha}{2}$, $\frac{4\rho}{\alpha} < \gamma < 2$ and Est.(6.99) with $\frac{1}{2} < \beta < \frac{\alpha}{2}$, $\frac{1}{\alpha} < \xi < 1$ in Est.(6.90) we conclude that

$$
\begin{aligned}
\|u_h(t)-u(t)\|_{L^p(\Omega,H)} &\leq \left(C_\eta h^\eta \|A^{\frac{\eta}{2}} u_0\|_{L^p(\Omega,H)} + C_{\alpha,\gamma,\rho,p} h^{(\frac{\alpha\gamma}{2}-2\rho)} + C_{p,\xi,\alpha} h^{\frac{\alpha\xi}{2}}\right) \\
&+ C_{\alpha,\rho} \int_0^t (t-s)^{-\frac{2\rho}{\alpha}} \|u_h(s)-u(s)\|_{L^p(\Omega,H)} ds \\
&+ C_{p,\alpha,\beta} \left(\int_0^t (t-s)^{-\frac{2\beta}{\alpha}} \|u_h(s)-u(s)\|_{L^p(\Omega,H)}^2 ds\right)^{\frac{1}{2}}.
\end{aligned} \tag{6.100}
$$

To unify the two terms $\int_0^t (t-s)^{-\frac{2\rho}{\alpha}} \|u_h(s)-u(s)\|_{L^p(\Omega,H)} ds$ and

$\left(\int_0^t (t-s)^{-\frac{2\beta}{\alpha}} \|u_h(s)-u(s)\|^2_{L^p(\Omega,H)} ds\right)^{\frac{1}{2}}$, we apply Hölder inequality A.10 to get

$$\begin{aligned}
&\int_0^t (t-s)^{-\frac{2\rho}{\alpha}} \|u_h(s)-u(s)\|_{L^p(\Omega,H)} ds \\
\leq\ & \left(\int_0^t (t-s)^{-\frac{2\rho}{\alpha}} ds\right)^{\frac{1}{2}} \left(\int_0^t (t-s)^{-\frac{2\rho}{\alpha}} \|u_h(s)-u(s)\|^2_{L^p(\Omega,H)} ds\right)^{\frac{1}{2}} \\
\leq\ & C_{\alpha,\rho} \left(\int_0^t (t-s)^{-\frac{2\rho}{\alpha}} \|u_h(s)-u(s)\|^2_{L^p(\Omega,H)} ds\right)^{\frac{1}{2}}.
\end{aligned}$$

Now, by taking $\nu := min\left\{\eta, (\frac{\alpha\gamma}{2}-2\rho), \frac{\alpha\xi}{2}\right\}$ with $\eta > 0$, $\gamma \in [\frac{4\rho}{\alpha}, 2)$, $\rho \in [0, \frac{\alpha}{2})$, $\xi \in (\frac{1}{\alpha}, 1)$ and $\beta \in (\frac{1}{2}, \frac{\alpha}{2})$ we have

$$\begin{aligned}
\|u_h(t)-u(t)\|_{L^p(\Omega,H)} \leq\ & C_{\alpha,\eta,\rho,p,\gamma,\xi} h^{\nu} \\
+\ & C_{\alpha,\rho,p,\beta} \left(\int_0^t (t-s)^{-\frac{2}{\alpha}\max(\rho,\beta)} \|u_h(s)-u(s)\|^2_{L^p(\Omega,H)} ds\right)^{\frac{1}{2}}
\end{aligned} \tag{6.101}$$

An application of the basic inequality: $(a+b)^2 \leq C(a^2+b^2)$, $\forall\, a, b \geq 0$, for some $C > 0$, leads to

$$\begin{aligned}
\|u_h(t)-u(t)\|^2_{L^p(\Omega,H)} \leq\ & C_{\alpha,\eta,\rho,p,\gamma,\xi} h^{2\nu} \\
+\ & C_{\alpha,\rho,p,\beta} \int_0^t (t-s)^{-1+(1-\frac{2}{\alpha}\ \max(\rho,\beta))} \|u_h(s)-u(s)\|^2_{L^p(\Omega,H)} ds.
\end{aligned} \tag{6.102}$$

Let $\gamma \to 2$ and $\xi \to 1$, we apply the Gronwall lemma A.1 which is possible since $1 - \frac{2}{\alpha}\max(\rho,\beta) > 0$ to infer that

$$\|u_h(t)-u(t)\|^2_{L^p(\Omega,H)} \leq C_{\alpha,\eta,\rho,p} h^{2\nu}, \tag{6.103}$$

with $\nu = min\left\{\eta, (\alpha-2\rho), \frac{\alpha}{2}-\kappa\right\}$, for any $\kappa > 0$. In particular, for $\eta > \frac{\alpha}{2}$ and $\rho \in [0, \frac{\alpha}{4})$ it holds

$$\|u_h(t)-u(t)\|_{L^p(\Omega,H)} \leq C_{\alpha,\eta,\rho,p} h^{\frac{\alpha}{2}-\kappa}, \text{ for any } \kappa > 0. \tag{6.104}$$

6.6 Full approximation

In this section we deal with the full discretization of Prb.(6.1). By making a combination of the spectral Galerkin method and the implicit Euler method, we elaborate the space-time scheme. We use the same techniques stated in Section 6.4 and we construct the sequence of random variables $(u_h^m)_{m=1}^M$, $m \in \{1, ..., M\}$, $M > \pi^\alpha$, $h = \frac{1}{N}$ and $N \in \mathbb{N}_0$ as follows

$$\begin{cases} \dfrac{u_h^m - u_h^{m-1}}{\tau} & = -A_{\alpha,h} u_h^m + P_h F(u_h^{m-1}) + P_h G(u_h^{m-1}) \dfrac{W(t_m) - W(t_{m-1})}{\tau}, \\ u_h^0 := P_h u_0. \end{cases} \tag{6.105}$$

Theorem 6.14 *Let $\alpha \in (1, 2]$. Under Assumptions $\mathcal{A}_1^r$ (with $\eta \in (\frac{\alpha}{2}, \alpha]$), $\mathcal{B}_1^r$ (with $\rho \in [0, \frac{\alpha}{4})$) and $\mathcal{C}_1$, there exists a positive constant $C_{p,\alpha}$ independent of τ and h s.t.*

$$\|u_h^m - u(t_m)\|_{L^p(\Omega,H)} \leq C_{p,\alpha,\eta,\rho} \left(h^{\frac{\alpha}{2}-\kappa} + \tau^{\frac{\alpha-1}{2\alpha}-\kappa} \right), \text{ for any } \kappa > 0,$$

for all $m \in \{1, ..., M\}$.

Proof. We have

$$\|u_h^m - u(t_m)\|_{L^p(\Omega,H)} \leq \|u_h^m - u_h(t_m)\|_{L^p(\Omega,H)} + \|u_h(t_m) - u(t_m)\|_{L^p(\Omega,H)}.$$

Using Theorem 6.11, in particular Est.(6.86), we get

$$\|u_h^m - u(t_m)\|_{L^p(\Omega,H)} \leq \|u_h^m - u_h(t_m)\|_{L^p(\Omega,H)} + C_{\alpha,\eta,p} h^{\frac{\alpha}{2}-\kappa}, \tag{6.106}$$

for any $\kappa > 0$. In order to estimate $\|u_h^m - u_h(t_m)\|_{L^p(\Omega,H)}$, we have on the one hand, Scheme (6.105) is equivalent to

$$\begin{aligned} u_h^m = (I + \tau A_{\alpha,h})^{-m} P_h u_0 \quad &+ \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} (I + \tau A_{\alpha,h})^{-(m-k)} P_h F(u_h^k) ds \\ &+ \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} (I + \tau A_{\alpha,h})^{-(m-k)} P_h G(u_h^k) dW(s). \end{aligned} \tag{6.107}$$

In the other hand, we rewrite Eq.(6.84) as the form of Eq.(6.59). Hence,

$$\begin{aligned} \|u_h^m - u_h(t_m)\|_{L^p(\Omega,H)} &\leq \| \left((I + \tau A_{\alpha,h})^{-m} P_h - S_{\alpha,h}(t_m) P_h \right) u_0 \|_{L^p(\Omega,H)} \\ + \quad & \| \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \left((I + \tau A_{\alpha,h})^{-(m-k)} P_h F(u_h^k) - S_{\alpha,h}(t_m - s) P_h F(u_h(s)) \right) ds \|_{L^p(\Omega,H)} \\ + \quad & \| \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \left((I + \tau A_\alpha)^{-(m-k)} G(u^k) - S_\alpha(t_m - s) G(u(s)) \right) dW(s) \|_{L^p(\Omega,H)}. \end{aligned} \tag{6.108}$$

The fact that $A_{\alpha,h}$ commutes with P_h since $A_{\alpha,h} = A_\alpha|_{H_h}$, the boundedness of P_h (i.e. Est.(5.31)) and the use of Est.(6.62) leads to

$$\begin{aligned} \|((I + \tau A_{\alpha,h})^{-m} P_h - S_{\alpha,h}(t_m) P_h) u_0\|_{L^p(\Omega,H)} &= \|P_h \left((I + \tau A_\alpha)^{-m} - S_\alpha(t_m) \right) u_0\|_{L^p(\Omega,H)} \\ \leq \quad \|P_h\|_{\mathcal{L}(H)} \| \left((I + \tau A_\alpha)^{-m} - S_\alpha(t_m) \right) u_0\|_{L^p(\Omega,H)} &\leq C_{(\eta, \|u_0\|_{L^p(\Omega, H_2^\eta)})} \tau^{\frac{\eta}{\alpha}}. \end{aligned} \tag{6.109}$$

To estimate the second term in the RHS of Est.(6.108), we argue as above to infer that

$$\begin{aligned}
&\| \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \left((I+\tau A_{\alpha,h})^{-(m-k)}P_hF(u_h^k)-S_{\alpha,h}(t_m-s)P_hF(u_h(s))\right)ds\|_{L^p(\Omega,H)}\\
\leq\ & \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \| \left((I+\tau A_{\alpha,h})^{-(m-k)}P_hF(u_h^k)-S_{\alpha,h}(t_m-s)P_hF(u_h(s))\right)\|_{L^p(\Omega,H)}ds\\
=\ & \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \|P_h\left((I+\tau A_{\alpha})^{-(m-k)}F(u_h^k)-S_{\alpha}(t_m-s)F(u_h(s))\right)\|_{L^p(\Omega,H)}ds\\
\leq\ & \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \|P_h\|_{\mathcal{L}(H)}\|\left((I+\tau A_{\alpha})^{-(m-k)}F(u_h^k)-S_{\alpha}(t_m-s)F(u_h(s))\right)\|_{L^p(\Omega,H)}ds\\
\leq\ & \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \| \left((I+\tau A_{\alpha})^{-(m-k)}F(u_h^k)-S_{\alpha}(t_m-s)F(u_h(s))\right)\|_{L^p(\Omega,H)}ds. \qquad (6.110)
\end{aligned}$$

By following the same steps used in the sequence of inequalities Est.(6.63) with u replaced by u_h and u^k replaced by u_h^k together with the techniques used to estimate $R1$, $R2$ and $R3$, we arrive at

$$\begin{aligned}
&\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \|((I+\tau A_{\alpha})^{-(m-k)}F(u_h^k)-S_{\alpha}(t_m-s)F(u_h(s)))\|_{L^p(\Omega,H)}ds\\
\leq\ & C_{\alpha,\rho}\left(\tau^{\frac{\alpha-1}{2\alpha}-}+\tau^{\frac{2\beta}{\alpha}}\right)+C_\rho\tau\sum_{k=0}^{m-1} t_{m-k}^{-\frac{2\rho}{\alpha}}\|u_h^k-u_h(t_k)\|_{L^p(\Omega,H)}, \qquad (6.111)
\end{aligned}$$

where $\rho\in[0,\frac{\alpha}{2})$ and $\beta\in(0,\frac{\alpha}{2}-\rho)$.

In order to estimate the third term in the RHS of Est.(6.108), we use Burkholder-Davis-Gundy inequality and the boundedness of P_h to obtain

$$\begin{aligned}
&\| \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} \left((I+\tau A_{\alpha,h})^{-(m-k)}P_hG(u_h^k)-S_{\alpha,h}(t_m-s)P_hG(u_h(s))\right)dW(s)\|_{L^p(\Omega,H)}\\
\leq\ & C_p\left(\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}\left(\mathbb{E}\|(I+\tau A_{\alpha,h})^{-(m-k)}P_hG(u_h^k)-S_{\alpha,h}(t_m-s)P_hG(u_h(s))\|_{HS}^p\right)^{\frac{2}{p}}ds\right)^{\frac{1}{2}}\\
\leq\ & C_p\left(\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}\left(\mathbb{E}\|P_h\left((I+\tau A_{\alpha})^{-(m-k)}G(u_h^k)-S_{\alpha}(t_m-s)G(u_h(s))\right)\|_{HS}^p\right)^{\frac{2}{p}}ds\right)^{\frac{1}{2}}\\
\leq\ & C_p\left(\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}\left(\mathbb{E}\left(\|P_h\|_{\mathcal{L}(H)}^p\|(I+\tau A_{\alpha})^{-(m-k)}G(u_h^k)-S_{\alpha}(t_m-s)G(u_h(s))\|_{HS}^p\right)\right)^{\frac{2}{p}}ds\right)^{\frac{1}{2}}\\
\leq\ & C_p\left(\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}\left(\mathbb{E}\|(I+\tau A_{\alpha})^{-(m-k)}G(u_h^k)-S_{\alpha}(t_m-s)G(u_h(s))\|_{HS}^p\right)^{\frac{2}{p}}ds\right)^{\frac{1}{2}}. \qquad (6.112)
\end{aligned}$$

We use the same steps followed in the sequence of inequalities Est.(6.70) besides the techniques used to estimate the terms $R4$, $R5$ and $R6$, to get

$$\begin{aligned}
&\Big\| \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \Big((I+\tau A_{\alpha,h})^{-(m-k)} P_h G(u_h^k) - S_{\alpha,h}(t_m - s) P_h G(u_h(s)) \Big)\, dW(s) \Big\|_{L^p(\Omega,H)} \\
\leq\ & C_{p,\alpha,\grave{\rho},\grave{\beta}} (\tau^{\frac{\alpha-1}{2\alpha}-\kappa} + \tau^{\frac{2\grave{\beta}}{\alpha}}) + C_{p,\grave{\rho}} \left(\tau \sum_{k=0}^{m-1} t_{m-k}^{-\frac{4\grave{\rho}}{\alpha}} \|u_h^k - u_h(t_k)\|^2_{L^p(\Omega,H)} \right)^{\frac{1}{2}}, \qquad (6.113)
\end{aligned}$$

for any $\kappa > 0$, where $\grave{\rho} \in (\frac{1}{4}, \frac{\alpha}{4})$ and $\grave{\beta} \in (0, \frac{\alpha}{4} - \grave{\rho})$.

Coming back to Est.(6.108), by gathering Est.(6.109), Est.(6.111) and Est.(6.113) we arrive at

$$\begin{aligned}
\|u_h^m - u_h(t_m)\|^2_{L^p(\Omega,H)} \ \leq\ & C_{(p,\alpha,\eta,\rho,\grave{\rho},\grave{\beta},\|u_0\|_{L^p(\Omega,H^{\eta}_2)})} \left(\tau^{\frac{\eta}{\alpha}} + \tau^{\frac{\alpha-1}{2\alpha}-\kappa} + \tau^{\frac{2\beta}{\alpha}} + \tau^{\frac{2\grave{\beta}}{\alpha}} \right)^2 \\
+\ & C_{\rho} \left(\tau \sum_{k=0}^{m-1} t_{m-k}^{-\frac{2\rho}{\alpha}} \|u_h^k - u_h(t_k)\|_{L^p(\Omega,H)} \right)^2 \\
+\ & C_{p,\grave{\rho}} \left(\tau \sum_{k=0}^{m-1} t_{m-k}^{-\frac{4\grave{\rho}}{\alpha}} \|u_h^k - u_h(t_k)\|^2_{L^p(\Omega,H)} \right). \qquad (6.114)
\end{aligned}$$

For $\eta > \frac{\alpha}{2}$, $\rho \in [0, \frac{\alpha}{4})$ and let $\beta \to \frac{\alpha}{2} - \rho$, $\grave{\beta} \to \frac{\alpha}{4} - \grave{\rho}$ with $\grave{\rho} \to \frac{1}{4}$, we apply the discrete Gronwall Lemma A.2 as stated before in Section 6.4 we infer that

$$\|u_h^m - u_h(t_m)\|_{L^p(\Omega,H)} \leq C_{p,\alpha,\eta,\rho} \tau^{\frac{\alpha-1}{2\alpha}-\kappa}, \qquad (6.115)$$

for any $\kappa > 0$.

Finally, from Est.(6.106) and Est.(6.115) we get the desired result. ■

Chapter 7

The temporal approximation for fractional stochastic Burgers equation in Hilbert space

In this third contribution chapter, we study a weak temporal convergence of Euler scheme for the multiplicative fractional stochastic Burgers equation in the Hilbert space $L^2(0,1)$. Let us mention that, in Chapter 5, we studied the additive fractional stochastic Burgers-type equation in the Hölder space $C^\delta(0,1)$. Regarding the relaxed condition imposed on the spacial functional space, we seek to improve not only the diffusion dissipation index α but the order of convergence as well, by proving a weaker convergence. Precisely, we prove the convergence in probability which is weaker than the L^p convergence and the pathwise convergence.

Recall that, the fractional stochastic Burgers equation is given by Prb.(3.33) in Subsection 3.5.2, with $A_\alpha := (-\Delta)^{\frac{\alpha}{2}}$, and the Laplacian being endowed with the homogeneous Dirichlet boundary conditions and $F(u(.)) := \partial_x u^2$.

7.1 Main result

By keeping the same notations used in Section 6.4, we study the following time-discretization of Eq.(3.33) given as follows

$$\begin{cases} u^m & = (I+\tau A_\alpha)^{-1}u^{m-1} + (I+\tau A_\alpha)^{-1}\tau F(u^{m-1}) + (I+\tau A_\alpha)^{-1}G(u^{m-1})\Delta W_m, \\ u^0 & := u_0, \end{cases} \tag{7.1}$$

where $\Delta W_m := W(t_m) - W(t_{m-1})$, $\tau = \frac{T}{M}$, $M \in \mathbb{N}_0$ and $t_m = m\tau$ with $m = 1, ..., M$.

The main result of this chapter is the following

Theorem 7.1 *Let $\alpha \in (\frac{3}{2}, 2]$ and $p > \frac{2\alpha}{\alpha-1}$. Then, under Assumptions $\mathcal{A}_1^r$ with $\eta \in (\frac{\alpha-1}{2}, \alpha]$, $\mathcal{B}_1^r$ with $\rho \in [0, \frac{1}{2}]$ and $\mathcal{C}_1$, the approximate solution $(u^m)_{m=0}^M$ defined by the numerical scheme* (7.1) *converges in probability with order $\frac{\alpha-1}{2\alpha} - \kappa$, for any $\kappa > 0$, in $L^2(0,1)$, to the mild solution u. More precisely, we have*

$$\lim_{c\to+\infty} \limsup_{M\to+\infty} \mathbb{P}\left\{ \max_{0\le m\le M} |u(m\tau) - u^m|_{L^2} \ge c\tau^{\frac{\alpha-1}{2\alpha}-\kappa} \right\} = 0. \tag{7.2}$$

7.2 Some auxiliary results

In order to prove our main theorem, we introduce the following truncated Cauchy problem:

$$\begin{cases} du_n(t) & = (-A_\alpha u_n(t) + F_n(u_n(t)))dt + G(u_n(t))dW(t), \quad t \in (0,T], \\ u_n(0) & := u_0, \end{cases} \tag{7.3}$$

where for fixed $n > 0$,

$$F_n(u) := F(\Pi_n(u)), \ \forall u \in H. \tag{7.4}$$

and Π_n is the projection from $L^2(0,1)$ onto the ball $B(0,n) \subset L^2(0,1)$ defined by

$$\Pi_n(u) := \begin{cases} u, \text{ if } |u|_{L^2} \le n, \\ \frac{n}{|u|_{L^2}}u, \text{ if } |u|_{L^2} > n \end{cases} \tag{7.5}$$

Let $\frac{3}{2} < \alpha \le 2$ and $p > \frac{2\alpha}{\alpha-1}$. According to [16, Theorem 1.3] (see also Theorem 3.15 in Subsection 3.5.2) respectively [16, Theorem 2.14], the problems (3.33) and (7.3) admit unique global solutions $u = \{u(t),\ t \in [0,T]\}$ respectively $u_n = \{u_n(t),\ t \in [0,T]\}$ with u satisfying

$$\mathbb{E}[\sup_{t\in[0,T]} |u(t)|^p_{L^2}] < \infty. \tag{7.6}$$

Using Markov inequality (A.13), we have

$$\lim_{C\to+\infty} \mathbb{P}\left\{\sup_{t\in[0,T]} |u(t)|_{L^2} \geq C\right\} = 0. \tag{7.7}$$

By construction, see [16], the processes u and u_n are $\mathbb{P}$-a.s. equal up to the stopping time:

$$\tau_n = inf\ \{t \geq 0,\ |u_n(t)|_{L^2} \geq n\}. \tag{7.8}$$

Consequently, we have from Markov inequality (A.13) that

$$\lim_{C\to+\infty} \limsup_{n\to+\infty} \mathbb{P}\left\{\sup_{t\in[0,T]} |u_n(t)|_{L^2} \geq C\right\} = 0. \tag{7.9}$$

Therefore, there exists $C > 0$, such that for all $n \geq 1$,

$$\sup_n \mathbb{E} \sup_{t\in[0,T]} |u_n(t)|^p_{L^2} \leq C. \tag{7.10}$$

The proof of the main result is related to the following lemma.

Lemma 7.2 *For any $\epsilon \in (0,1)$ and any $C \geq 1$, we have*

$$\lim_{M\to+\infty} \mathbb{P}\left\{\max_{0\leq m\leq M} |u(m\tau) - u^m|_{L^2} \geq \epsilon\right\} = 0, \tag{7.11}$$

and

$$\lim_{C\to+\infty} \limsup_{M\to+\infty} \mathbb{P}\left\{\max_{0\leq m\leq M} |u^m|_{L^2} \geq C\right\} = 0. \tag{7.12}$$

Proof. To prove the convergence in the probability of the temporal approximation $(u^m)_{m=0}^M$, first we need the convergence in probability of the temporal approximation of the truncated Cauchy problem (7.3). For this, we construct as above the following sequence of random variables $(u_n^m)_{m=0}^M$,

$$\begin{cases} u_n^m &= (I+\tau A_\alpha)^{-1}u_n^{m-1} + (I+\tau A_\alpha)^{-1}\tau F_n(u_n^{m-1}) + (I+\tau A_\alpha)^{-1}G(u_n^{m-1})\Delta W_m, \\ u_n^0 &:= u_0. \end{cases} \tag{7.13}$$

Second, for $\epsilon \in (0,1)$, let the event

$$A := \{w \in \Omega,\ \max_{0\leq m\leq M} |u(m\tau, w) - u^m(w)|_{L^2} \geq \epsilon\}. \tag{7.14}$$

We define as in [87], the random variables:

$$m_\epsilon = min\ \{0 \leq m \leq M,\ |u(m\tau) - u^m|_{L^2} \geq \epsilon\},\ \ a.s.$$

and, for $n > 1$

$$\Theta_n = inf\ \{t \leq T,\ |u(t)|_{L^2} \geq n - 1\}.\ a.s. \tag{7.15}$$

We can rewrite the event A by the following form

$$\begin{aligned} A &= A \cap (\{\Theta_n < T\} \cup \{\Theta_n \geq T\}) = (A \cap \{\Theta_n < T\}) \cup (A \cap \{\Theta_n \geq T\}) \\ &= (A \cap \{\Theta_n < T\}) \cup (A \cap \{\Theta_n \geq T\} \cap \{m_\epsilon \tau < T\}) \cup (A \cap \{\Theta_n \geq T\} \cap \{m_\epsilon \tau \geq T\}) \end{aligned} \tag{7.16}$$

Since $A \cap \{m_\epsilon \tau \geq T\} = \emptyset$, we deduce that

$$\begin{aligned} A &= (A \cap \{\Theta_n < T\}) \cup (A \cap \{\Theta_n \geq T\} \cap \{m_\epsilon \tau < T\}) \\ &\subset \{\Theta_n < T\} \cup (\{\Theta_n \geq T\} \cap \{m_\epsilon \tau < T\})\,. \end{aligned} \tag{7.17}$$

Now, we need to prove the following main inclusion

$$(\{\Theta_n \geq T\} \cap \{m_\epsilon \tau < T\}) \subset \{\max_{0 \leq m \leq M} |u_n(m\tau) - u_n^m|_{L^2} \geq \epsilon\}. \tag{7.18}$$

To do this, we have from the event $(\{\Theta_n \geq T\} \cap \{m_\epsilon \tau < T\})$ two facts:

- First, $\forall\ m \in \{0, ..., M\}$, $m\tau \leq T \leq \Theta_n$. Hence, for any $m \in \{0, ..., M\}$, we have $|u(m\tau)|_{L^2} < n - 1 < n$, and so

$$u(m\tau) = u_n(m\tau),\ \forall\ m \in \{0, ..., M\}. \tag{7.19}$$

- Second, if $m_\epsilon < M$ then, $\forall\ m \in \{0, ..., m_\epsilon - 1\}$

$$|u(m\tau) - u^m|_{L^2} < \epsilon < 1, \tag{7.20}$$

  Thus, from (7.20), we have $\forall\ m \in \{0, ..., m_\epsilon - 1\}$,

$$|u^m|_{L^2} \leq |u(m\tau) - u^m|_{L^2} + |u(m\tau)|_{L^2} \leq 1 + |u(m\tau)|_{L^2},$$

  and since $|u(m\tau)|_{L^2} < n - 1$ from the first fact, it holds

$$|u^m|_{L^2} < n,\ \forall\ m \in \{0, ..., m_\epsilon - 1\},$$

and so

$$\Pi_n(u^m) = u^m, \ \forall\, m \in \{0, ..., m_\epsilon - 1\}. \tag{7.21}$$

We use the definitions of u^m and u_n^m introduced by (7.1) and (7.13) respectively, besides (7.21) to get

$$u^m = u_n^m, \ \forall\, m \in \{0, ..., m_\epsilon - 1\}. \tag{7.22}$$

Similarly, the definitions of u^{m_ϵ} and $u_n^{m_\epsilon}$, by using (7.1) and (7.13) respectively, yields

$$u^{m_\epsilon} = u_n^{m_\epsilon}. \tag{7.23}$$

As a result, from the identities (7.19) and (7.23), we obtain

$$u(m_\epsilon \tau) - u^{m_\epsilon} = u_n(m_\epsilon \tau) - u_n^{m_\epsilon}.$$

Hence, by using the definition of the random variable m_ϵ we deduce that

$$|u_n(m_\epsilon \tau) - u_n^{m_\epsilon}|_{L^2} \geq \epsilon,$$

and so

$$\max_{0 \leq m \leq M} \ |u_n(m\tau) - u_n^m|_{L^2} \geq \epsilon.$$

Thus, the main inclusion (7.18) is proved.
Consequently, from (7.17) and (7.18) we have

$$\mathbb{P}(A) \leq \mathbb{P}(\Theta_n < T) + \mathbb{P}(\max_{0 \leq m \leq M} \ |u_n(m\tau) - u_n^m|_{L^2} \geq \epsilon).$$

Then

$$\lim_{M \to +\infty} \ \mathbb{P}(A) \leq \mathbb{P}(\Theta_n < T) + \lim_{M \to +\infty} \ \mathbb{P}(\max_{0 \leq m \leq M} \ |u_n(m\tau) - u_n^m|_{L^2} \geq \epsilon). \tag{7.24}$$

To get, $\lim_{M \to +\infty} \ \mathbb{P}(\max_{0 \leq m \leq M} \ |u_n(m\tau) - u_n^m|_{L^2} \geq \epsilon)$, we use Theorem 6.4. This theorem ensures the strong temporal convergence of the heat equation in the space $L^p(\Omega, L^2(0,1))$, for the time interval $[0,1]$. The proof is still valid for the time interval $[0,T]$, for any $T > 0$. Then, for any $n \geq 1$ we have

$$\mathbb{E}[\max_{0 \leq m \leq M} \ |u_n(m\tau) - u_n^m|_{L^2}^p] \leq C_{p,\alpha,\eta,\rho,n} \tau^{\frac{\alpha-1}{2\alpha} - \kappa}, \ \text{for any } \kappa > 0.$$

Let us mention that, the dependence of the constant $C_{p,\alpha,\eta,\rho,n}$ on n, is due to the dependence on $\sup_{t\in[0,T]}\ \mathbb{E}[|u_n(t)|^p_{L^2}]$. However, from Est.(7.10) (see also [16]), there exists $C>0$ for any $n\geq 1$ s.t.

$$\sup_{n\in\mathbb{N}_0}\left(\sup_{t\in[0,T]}\ \mathbb{E}[|u_n(t)|^p_{L^2}]\right)\leq \sup_{n\in\mathbb{N}_0}\left(\mathbb{E}[\sup_{t\in[0,T]}\ |u_n(t)|^p_{L^2}]\right)\leq\left(\mathbb{E}[\sup_{t\in[0,T]}\ |u(t)|^p_{L^2}]\right)<C.$$

Therefore, there exists $C_{p,\alpha,\eta,\rho}>0$ s.t. for any $n\geq 1$,

$$\mathbb{E}[\max_{0\leq m\leq M}\ |u_n(m\tau)-u_n^m|^p_{L^2}]\leq C_{p,\alpha,\eta,\rho}\tau^{\frac{\alpha-1}{2\alpha}-\kappa}. \tag{7.25}$$

Now, by taking the limit when M goes to $+\infty$ (recall that $\tau=\frac{T}{M}$, and so $\tau\to 0$), after applying Markov inequality (A.13) we get

$$\lim_{M\to+\infty}\mathbb{P}(\max_{0\leq m\leq M}\ |u_n(m\tau)-u_n^m|_{L^2}\geq\epsilon)=0. \tag{7.26}$$

To estimate $\mathbb{P}(\Theta_n<T)$, let us recall from [16, page 148] that, $\lim_{n\to+\infty}\ \tau_n=+\infty$, a.s. Then

$$\lim_{n\to+\infty}\ \mathbb{P}(\Theta_n<T)=\lim_{n\to+\infty}\ \mathbb{P}(\tau_n<T)=0, \tag{7.27}$$

From Est.(7.24), Est.(7.26) and Est.(7.27), the convergence in probability of $(u^m)_{m=0}^M$ has been ensured.

Now, it remains to prove the boundedness in probability of $(u^m)_{m=0}^M$. For this end, we fix $C\geq 1$ and $\epsilon\in(0,1)$. First, we have the following inclusion

$$(\max_{0\leq m\leq M}\ |u^m|_{L^2}\geq C)\subset(\max_{0\leq m\leq M}\ |u(m\tau)-u^m|_{L^2}\geq\epsilon)\cup(\sup_{t\in[0,T]}\ |u(t)|_{L^2}\geq C-\epsilon).$$

Indeed, if we suppose the inverse, i.e. there exists $w\in\Omega$ s.t. $\max_{0\leq m\leq M}\ |u^m(w)|_{L^2}\geq C$, besides $\max_{0\leq m\leq M}\ |u(m\tau,w)-u^m(w)|_{L^2}<\epsilon$ and $\sup_{t\in[0,T]}\ |u(t,w)|_{L^2}<C-\epsilon$, we get for all $m\in\{0,...,M\}$ and all $t\in[0,T]$

$$\begin{aligned}|u^m(w)|_{L^2} &\leq |u^m(w)-u(m\tau,w)|_{L^2}+|u(m\tau,w)|_{L^2}\\ &\leq \max_{0\leq m\leq M}\ |u^m(w)-u(m\tau,w)|_{L^2}+\sup_{t\in[0,T]}\ |u(t,w)|_{L^2}<C,\end{aligned} \tag{7.28}$$

and this is a contradiction. As a result,

$$\mathbb{P}(\max_{0\leq m\leq M}\ |u^m|_{L^2}\geq C)\leq\mathbb{P}(\max_{0\leq m\leq M}\ |u(m\tau)-u^m|_{L^2}\geq\epsilon)+\mathbb{P}(\sup_{t\in[0,T]}\ |u(t)|_{L^2}\geq C-\epsilon).$$

Est.(7.11) allows us to take the supermum limit when M goes to $+\infty$, for a fixed $C \geq 1$, then

$$\begin{aligned}\limsup_{M\to+\infty} \mathbb{P}(\max_{0\leq m\leq M} |u^m|_{L^2} \geq C) &\leq \limsup_{M\to+\infty} \mathbb{P}(\max_{0\leq m\leq M} |u(m\tau)-u^m|_{L^2} \geq \epsilon) \\ &+ \mathbb{P}(\sup_{t\in[0,T]} |u(t)|_{L^2} \geq C-\epsilon),\end{aligned}$$

which implies

$$\limsup_{M\to+\infty} \mathbb{P}(\max_{0\leq m\leq M} |u^m|_{L^2} \geq C) \leq \mathbb{P}(\sup_{t\in[0,T]} |u(t)|_{L^2} \geq C-\epsilon).$$

To obtain the desired result, we use Est.(7.7) after taking the limit to $+\infty$ with respect to C. ■

7.3 Proof of the main result

To prove the main result, let $\alpha \in (\frac{3}{2}, 2]$, $p > \frac{2\alpha}{\alpha-1}$, $M \geq 1$, $n \geq 1$ and $c > 0$. First, for any $\kappa > 0$, we have

$$\begin{aligned}\mathbb{P}\left(\max_{0\leq m\leq M} |u(m\tau)-u^m|_{L^2} \geq c\tau^{\frac{\alpha-1}{2\alpha}-\kappa}\right) &\leq \mathbb{P}\left(\max_{0\leq m\leq M} |u_n(m\tau)-u_n^m|_{L^2} \geq c\tau^{\frac{\alpha-1}{2\alpha}-\kappa}\right) \\ &+ \mathbb{P}\left(\max_{0\leq m\leq M} |u^m|_{L^2} \geq n\right) \\ &+ \mathbb{P}\left(\sup_{t\in[0,T]} |u(t)|_{L^2} \geq n\right). \qquad (7.29)\end{aligned}$$

By using Est.(7.25) with Markov inequality (A.13), we estimate the first term in the RHS of (7.29) as follows

$$\mathbb{P}\left(\max_{0\leq m\leq M} |u_n(m\tau)-u_n^m|_{L^2} \geq c\tau^{\frac{\alpha-1}{2\alpha}-\kappa}\right) \leq \frac{(C_{p,\alpha,\eta,\rho})^p}{c^p}. \qquad (7.30)$$

Again, the use of Markov inequality allows us to estimate the third term in the RHS of (7.29) as follows

$$\mathbb{P}\left(\sup_{t\in[0,T]} |u(t)|_{L^2} \geq n\right) \leq \frac{\mathbb{E}[\sup_{t\in[0,T]} |u(t)|^p_{L^2}]}{n^p}. \qquad (7.31)$$

We replace Est.(7.30) and Est.(7.31) in Est.(7.29), to get

$$\begin{aligned}\mathbb{P}\left(\max_{0\leq m\leq M} |u(m\tau)-u^m|_{L^2} \geq c\tau^{\frac{\alpha-1}{2\alpha}-\kappa}\right) &\leq \frac{(C_{p,\alpha,\eta,\rho})^p}{c^p} \\ &+ \mathbb{P}\left(\max_{0\leq m\leq M} |u^m|_{L^2} \geq n\right) \\ &+ \frac{\mathbb{E}[\sup_{t\in[0,T]} |u(t)|^p_{L^2}]}{n^p}. \qquad (7.32)\end{aligned}$$

We take first the limisup when M goes to $+\infty$, then the limit when n goes to $+\infty$, to get

$$\begin{aligned}\limsup_{M\to+\infty} \mathbb{P}\left(\max_{0\leq m\leq M} |u(m\tau)-u^m|_{L^2} \geq c\tau^{\frac{\alpha-1}{2\alpha}-\kappa}\right) &\leq \frac{(C_{p,\alpha,\eta,\rho})^p}{c^p} \\ &+ \limsup_{n\to+\infty}\limsup_{M\to+\infty} \mathbb{P}\left(\max_{0\leq m\leq M}|u^m|_{L^2}\geq n\right) \\ &+ \limsup_{n\to+\infty} \frac{\mathbb{E}[\sup_{t\in[0,T]} \ |u(t)|^p_{L^2}]}{n^p}. \end{aligned} \tag{7.33}$$

Thus, Est.(7.6) and Est.(7.12), yield

$$\limsup_{M\to+\infty} \mathbb{P}\left(\max_{0\leq m\leq M} |u(m\tau)-u^m|_{L^2} \geq c\tau^{\frac{\alpha-1}{2\alpha}-\kappa}\right) \leq \frac{(C_{p,\alpha,\eta,\rho})^p}{c^p}. \tag{7.34}$$

Finally, after taking the limit when c goes to $+\infty$, we get the desired result.

Appendix A

Some definitions and useful results

A.1 Some useful versions of Grönwall's Lemma

Grönwall's Lemma plays an important role in the analytical or numerical theory of differential equations. The original version is due Grönwall T. H. [51], but there exists a huge number of its versions. In this section we collect some useful versions needed in this thesis.

Lemma A.1 *Let $T > 0$ and C_1, C_2 be two positive constants and let $f : [0,T] \to \mathbb{R}$ be a positive and continous function. If for $\beta > 0$ we have*

$$f(t) \leq C_1 + C_2 \int_0^t (t-s)^{-(1-\beta)} f(s)ds, \ \forall\, t \in (0,T],$$

then there exists $C_{C_2,T,\beta}$ s.t.

$$f(t) \leq C_1 C_{C_2,T,\beta}.$$

There also exists discrete analogues, see for instance [64] for the the next lemma.

Lemma A.2 *Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be positive sequences and C a positive constant. If for any $n \geq 1$*

$$x_n \leq C + \sum_{k=0}^{n-1} x_k y_k,$$

then

$$x_n \leq Ce^{(\sum_{k=0}^{n-1} y_k)}.$$

A.2 Some basic results

Fix $T > 0$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Lemma A.3 *[72, Lemma 2.1]. Let $\tau > 0$ and $C_p \in [0, \infty)$ for $p \geq 1$. In addition, Let Z_n, $n \in \mathbb{N}$, be a sequence of random variables such that*

$$(\mathbb{E}|Z_n|^p)^{\frac{1}{p}} \leq C_p\, n^{-\tau}, \tag{A.1}$$

for all $p \geq 1$ and all $n \in \mathbb{N}$. Then

$$\mathbb{P}\left(\sup_{n\in\mathbb{N}}(n^{\tau-\epsilon}|Z_n|) < \infty\right) = 1, \tag{A.2}$$

for all $\epsilon \in (0, \tau)$.

Lemma A.4 *[67, Lemma 10, P. 49] Let $Y : \Omega \to \mathbb{R}$ be a $\mathcal{F}/\mathcal{B}(\mathbb{R})-$measurable mapping, that is centered and normal distributed. Then for every $p \in \mathbb{N}$,*

$$\mathbb{E}|Y|^p \leq p!(\mathbb{E}|Y|^2)^{\frac{p}{2}}. \tag{A.3}$$

Lemma A.5 *[67, Lemma 12, P. 49] Let $B : [0, T] \times \Omega \to \mathbb{R}$ be a standard Brownian motion. Then*

$$\mathbb{E}\left(|\int_0^{t_2} e^{-\lambda(t_2-s)}dB(s) - \int_0^{t_1} e^{-\lambda(t_1-s)}dB(s)|^2\right) \leq \lambda^{r-1}|t_2 - t_1|^r, \tag{A.4}$$

for every $t_1, t_2 \in [0, T]$, $r \in [0, 1]$ and every $\lambda \in (0, \infty)$.

Lemma A.6 *[67, Lemma 9, P. 49] Let $\eta \in (0, 1)$. Then, we have*

$$\int_0^1 \int_0^1 |x - y|^{-\eta} dxdy \leq \frac{3}{1-\eta}.$$

Lemma A.7 *Let $\delta \in (0, 1]$. There exists a constant $c_\delta > 0$ (independent of k) such that*

$$|e_k|_{C^\delta} \leq c_\delta k^\delta. \tag{A.5}$$

Proof. Since the function sin $\in C^1$(in reality it is well known that sin $\in C^\infty$, but for our proof, it is enough to be in C^1), then it is Lipschitz ($\delta = 1$) and $\delta-$Hölder continuous for any $\delta \in (0,1)$. Therefore, there exists $C_\delta > 0$, such that $\forall x, y$, we have;

$$|\sin(x) - \sin(y)| \leq C_\delta |x-y|^\delta.$$

Hence, $\forall k \in \mathbb{N}_0, \ \ \forall x, y \in [0,1]$, we infer that

$$|e_k(x) - e_k(y)| \ = \ \sqrt{2}|\sin(k\pi x) - \sin(k\pi y)| \leq \sqrt{2} C_\delta (\pi k)^\delta |x-y|^\delta. \tag{A.6}$$

Moreover, using the properties of the function sin, it is well known that

$$|e_k|_{C^0} := \sup_{x\in[0,1]} |e_k(x)| := \sqrt{2} \sup_{x\in[0,1]} |\sin(k\pi x)| \leq \sqrt{2}. \tag{A.7}$$

$$\begin{aligned} |e_k|_{C^\delta} \ &:= \ \sup_{x\in[0,1]} |e_k(x)| + \sup_{x\neq y\in[0,1]} \frac{|e_k(x) - e_k(y)|}{|x-y|^\delta} \\ &\leq \ \sqrt{2}(1 + C_\delta \pi^\delta k^\delta) \leq \sqrt{2}(k^\delta + C_\delta \pi^\delta k^\delta) \leq c_\delta k^\delta. \end{aligned} \tag{A.8}$$

■

Lemma A.8 $\forall \gamma > 0, \exists \ C_\gamma > 0, s.t \, \forall x \geq 0, \ \ x^\gamma e^{-x} \leq C_\gamma.$

Proof. Let $f_\gamma(x) : x \in [0, +\infty) \to [0, +\infty) \ni x^\gamma e^{-x}$. The real function f_γ is differentiable and $\forall x \in (0, +\infty), \ \ f'_\gamma(x) = x^{\gamma-1} e^{-x}(\gamma - x)$, therefore f has a maximum on $(0, +\infty)$ at γ, i.e. $\forall x \geq 0, \ x^\gamma e^{-x} \leq C_\gamma := f_\gamma(\gamma) = \gamma^\gamma e^{-\gamma}$. ■

Lemma A.9 $\forall \beta \in [0,1], \exists \ C_\beta > 0, s.t \, \forall x > 0, \ \ x^{-\beta}(1 - e^{-x}) \leq C_\beta.$

Proof. First, we assume $x \geq 1$. As $1 - e^{-x} \leq 1$, then $x^{-\eta}(1 - e^{-x}) \leq x^{-\eta} \leq 1$. Now, for $x < 1$, we use Taylor expansion of order 1 of the function e^{-x}, we infer the existence of $c_x \in (0, x]$, s.t. $1 - e^{-x} = x e^{-c_x}$. Hence $x^{-\eta}(1 - e^{-x}) = x^{1-\eta} e^{-c_x} \leq x^{1-\eta} \leq 1$. ■

A.3 Some elementary inequalities

Following is a collection of elementary, but fundamental inequalities.

- **Young inequality with ϵ.** Let $1 < p, q < \infty$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$, $a, b > 0$ and $\epsilon > 0$. Then,

$$ab \leq \epsilon a^p + C_\epsilon b^q, \tag{A.9}$$

where $C_\epsilon := (\epsilon p)^{-\frac{q}{p}} q^{-1}$.

- **Hölder inequality.** Let D be a domain in $\mathbb{R}$ and $1 \leq p, q \leq \infty$ s.t., $\frac{1}{p} + \frac{1}{q} = 1$. Then

  1. **Continuous version.** For all $u \in L^p(D)$ and all $v \in L^q(D)$ it holds

$$\|uv\|_{L^1(D)} \leq \|u\|_{L^p(D)} \|v\|_{L^q(D)}. \tag{A.10}$$

  2. **Discrete version.** For all $a = (a_i)_{i=1}^n \in \mathbb{R}^n$ and all $b = (b_i)_{i=1}^n \in \mathbb{R}^n$ it holds

$$|\sum_{i=1}^n a_i b_i| \leq (|a_i|^p)^{\frac{1}{p}} (|b_i|^q)^{\frac{1}{q}}. \tag{A.11}$$

- **Interpolation inequality for L^p-norms.** Let $1 \leq p \leq r \leq q \leq \infty$ and $\theta \in (0, 1)$ s.t. $\frac{1}{r} = \frac{\theta}{p} + \frac{(1-\theta)}{q}$. For all $u \in L^p(D) \cap L^q(D)$. Then, $u \in L^r(D)$ and

$$\|u\|_{L^r(D)} \leq \|u\|^{\theta}_{L^p(D)} \|u\|^{1-\theta}_{L^q(D)}. \tag{A.12}$$

- **Markov inequality**. Let ξ be an $\mathbb{R}_+$-valued random variable. Then, for any $p \geq 1$ and any $a > 0$, we have

$$\mathbb{P}(\xi \geq a) \leq \frac{\mathbb{E}(\xi^p)}{a^p}, \tag{A.13}$$

provided that ξ is p-th integrable.

- **Polarization identity**. Let $(H, \langle ., . \rangle_H, |.|_H)$ be a real Hilbert space. Then, for all $u, v \in H$ it holds

$$\langle u, v \rangle_H = \frac{1}{4} \left(|u + v|_H^2 - |u - v|_H^2 \right). \tag{A.14}$$

A.4 Basic definitions on functional spaces

Linear spaces are the standard setting for studying and solving a large number of the problems in differential and integral equations, and other topics in applied mathematics. Let us recall them here.

Definition A.10 ***(Linear space).*** *Let X be a set of objects and $\mathbb{K}$ be a set of scalars, i.e. $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. If there exsit an addition operation $(x, y) \to x+y \in X$, for any $x, y \in X$ and a scalar multiplication operation $(\lambda, x) \to \lambda \times x \in X$, for any $\lambda \in \mathbb{K}$ and any $x \in X$ such that the following rules hold.*

1. *for any $x, y \in X$, $x + y = y + x$,*
2. *for any $x, y, z \in X$, $(x + y) + z = x + (y + z)$,*
3. *there is an element $0 \in X$ s.t. $0 + x = x$ for any $x \in X$,*
4. *for any $x \in X$, there is an element $-x \in X$ s.t. $x + (-x) = 0$,*
5. *for any $x \in X$, $1 \times x = x$,*
6. *for any $x \in X$ and any $\lambda, \gamma \in \mathbb{K}$, $\lambda \times (\gamma \times x) = (\lambda\gamma) \times x$,*
7. *for any $x, y \in X$ and any $\lambda, \gamma \in \mathbb{K}$, $\lambda \times (x + y) = \lambda \times x + \lambda \times y$ and $(\lambda + \gamma) \times x = \lambda \times x + \gamma \times x$.*

Then X is called ***real linear space****, for $\mathbb{K} = \mathbb{R}$ and called* ***complex linear space*** *for $\mathbb{K} = \mathbb{C}$.*

Definition A.11 ***(Banach space).*** *Let X be a real Linear space.*

- *A mapping $\|.\| : X \to [0, \infty)$ is called a* ***norm*** *if*
  1. *$\|x\| = 0 \Leftrightarrow x = 0$,*
  2. *$\|\lambda x\| = |\lambda| \, \|x\|$, $\forall x \in X$, $\forall \lambda \in \mathbb{R}$,*
  3. *$\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in X$.*
- *A* ***normed space*** *X is a real vector space endowed with a norm $\|.\|_X$.*
- *A* ***Banach space*** *$(X, \|.\|_X)$ is a complete normed space, in the sense that each Cauchy sequence in X converges in X with respect to the norm $\|.\|_X$.*

Definition A.12 ***(Separable space).*** *Let $(X, \|.\|_X)$ be a Banach space. We say that X is a* ***separable*** *if it contains a countable dense subset.*

Definition A.13 ***(Hilbert space).*** *Let H be a real vector space.*

- *A mapping $\langle ., . \rangle : H \times H \to \mathbb{R}$ is called an inner product (or **scalar product**) if*

  1. $\langle x, x \rangle \; = 0 \Leftrightarrow x = 0$,
  2. $\langle x, x \rangle \; \geq 0, \; \forall x \in H$,
  3. $\langle x, y \rangle = \langle y, x \rangle, \; \forall x, y \in H$,
  4. *the mapping $x \mapsto \langle x, y \rangle$ is linear for each $y \in H$.*

- *each inner product $\langle ., . \rangle$ generates a norm defined by $\|x\| := \langle x, x \rangle^{\frac{1}{2}}$, for $x \in H$.*

- *A **pre-Hilbertian space** H is a real vector space endowed with an inner product $\langle ., . \rangle_H$.*

- *A **Hilbert space** H is a complete pre-Hilbertian space with respect to the norm $\|.\|_H$ which is generated by $\langle ., . \rangle_H$.*

Definition A.14 (Schwartz space). *The space of all functions $f \in C^\infty$ of rapid decrease in the sense that, for all $\alpha, \beta \geq 0$*

$$\sup_{x \in \mathbb{R}} |x^\beta d^\alpha u(x)| < \infty,$$

*is called **Schwartz space** and is denoted by $\mathcal{S}$.*

Definition A.15 (Space of tempered distributions). *The space of all continuous functionals $\varphi : \mathcal{S} \to \mathbb{R}$ is called **space of tempered distributions** and denoted by $\mathcal{S}'$.*

Conclusion

In this thesis, we dealt with the numerical approximations of some fractional stochastic partial differential equations, driven by the fractional Laplacian and perturbed by a Gaussian noise. By applying Blömker method used in [12], we have generalized their results in the fractional case. Precisely, we have used the spacial discretization scheme in order to prove the wellposedness of the fractional stochastic Burgers-type equation in the $C_t^{\gamma}C_x^{\delta}$-topology, where $\gamma < \frac{\alpha-1-2\delta}{2\alpha}$ and $\delta \in (1-\frac{\alpha}{2}, \frac{2\alpha-3}{2})$, for $\alpha \in (\frac{7}{4}, 2)$. In addition, we have fulfilled the pathwise convergence of the obtained spacial and full approximations, with the orders of convergence $(\frac{\alpha-1-2\delta}{2}) - \kappa$ in space and $(\frac{\alpha-1-2\delta}{2\alpha}) - \kappa$ in time, for any $\kappa > 0$. These results have been proved here for the first time in the Hölder space $C^{\delta}(0,1)$, not only for the fractional stochastic Burgers-type equation, but for the fractional and classical stochastic Burgers equations as well.

In order to improve either the orders of convergence and the constraint on the diffusion dissipation index α, we have relaxed the condition imposed on the nonlinear term and have studied such equation in the Hilbert space $L^2(0,1)$ instead of the Hölder space $C^{\delta}(0,1)$. Precisely, we have dealt with the fractional stochastic nonlinear heat equation with multiplicative noise. We have fulfilled the strong convergence in the space $L^p(\Omega, L^2(0,1))$ of the temporal, the spacial and the full approximations of the mild solution with orders of convergence $\frac{\alpha}{2} - \kappa$ in space and $(\frac{\alpha-1}{2\alpha}) - \kappa$ in time, for any $\kappa > 0$. We have filled the gap $\alpha \in (1, \frac{7}{4}]$, for the heat equation. For Burgers equation, we have proved a weaker conver-

gence, namely the convergence in probability of the temporal approximation obtained via the implicit Euler scheme with the same order $(\frac{\alpha-1}{2\alpha}) - \kappa$, provided that $\alpha \in (\frac{3}{2}, 2]$.
The perspectives are:

- Extend the obtained results by considering alternative definitions of the fractional Laplacian, for instance in terms of Riesz operator. Moreover, also in case of spectral Laplacian, it would be interesting to consider Neumann or Robin boundary conditions.

- Study the numerical approximations of the fractional stochastic partial differential equations driven by other kind of fractional operators, like Riemann-Liouville.

- Study the numerical approximations of the fractional stochastic partial differential equations perturbed by other kind of stochastic noises, like Lévy noise.

- Extend the results obtained in this thesis for the d-dimensional case, for $d > 1$.

In the last let us mention that, by our study here we are in the neighborhood of the millennium problem, i.e. Navier-Stokes equation.

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ملخص:

قدمت هذه الأطروحة تحليلا دقيقا لمختلف التقريبات العددية لفئة من المعادلات التفاضلية الجزئية العشوائية الكسرية، الليبتشيزية محليا وغير محليا، في فضاء احادي البعد، مدفوعة بمؤثر لابلاس الكسري ومشوشة بضوضاء غوصية. وتتضمن الدراسة تحقيق تقريبات زمانية، مكانية وزمانية-مكانية وتقارباتها المختلفة. على وجه الخصوص، حددنا معدلات التقارب وبينا ارتباطها بالاس الكسري لمؤثر لابلاس.

الكلمات والعبارات المفتاحية: معادلة تفاضلية جزئية عشوائية كسرية من نوع بورغرس، معادلة الحرارة غير الخطية العشوائية الكسرية، مؤثر لابلاس الكسري، ضوضاء غوصية، ضوضاء بيضاء زمانية-مكانية مضاعفة ، ضوضاء بيضاء زمانية-مكانية غير مضاعفة ، فضاءات هولدر ، فضاءات صوبولاف الكسرية، مؤثر هيلبرت-شميدت، حلول معتدلة، تقريب غالاركين الطيفي، طريقة اولر الضمنية، طريقة اولر الاسية، التقريب الزماني-المكاني، رتبة التقارب.

Résumé :

Cette thèse a fourni une analyse rigoureuse de divers schémas numériques pour une classe d’équations aux dérivées partielles stochastiques fractionnaires non linéaires Lipschitzienne localement et globalement, entraînées par le Laplacien fractionnaire dans l’espace unidimensionnel et perturbées par un bruit gaussien. L’étude contient l’élaboration de schémas temps, espace et espace-temps et leurs différentes convergences. Spécialement, nous exprimons pour chaque schéma le taux de convergence en termes de la puissance fractionnaire du Laplacien.

Mots clés et phrases : Équation stochastique fractionnaire de type Burgers, équation de chaleur non linéaire stochastique fractionnaire, Laplacien fractionnaire, Bruit gaussien, bruit blanc en espace et en temps multiplicatif, bruit blanc en espace et en temps additif, espaces de Hölder, espaces de Sobolev fractionnaires, opérateur de Hilbert-Schmidt, solution mild, approximation spectrale de Galerkin, schéma implicite d’Euler, schéma d’Euler exponentiel, approximation complète, ordre de convergence.

Abstract:

This dissertation has provided a rigorous analysis of various numerical schemes for a class of local and global Lipschitz nonlinear fractional stochastic partial differential equations, driven by the fractional Laplacian in the one-dimensional space and perturbed by a Gaussian noise. The study contains the elaboration of time, space and space-time schemes and their different convergences. Specially, we express for every scheme the rate of convergence in terms of the fractional power of the Laplacian.

Key words and phraces: Fractional stochastic Burgers-type equation, fractional stochastic nonlinear heat equation, fractional Laplacian, Gaussian noise, multiplicative space-time white noise, additive space-time white noise, Hölder spaces, fractional Sobolev spaces, Hilbert-Schmidt operator, mild solutions, spectral Galerkin approximation, implicit Euler scheme, exponential Euler scheme, full approximation, order of convergence.