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Par

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Thème

Groupes dont les sous-groupes propres de rang infini sont minimax-par-hypercentraux ou hypercentral-par-minimax

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Notation

Throughout this thesis, we shall adopt the following standardized notation that can be found in [48], unless we specify otherwise.

(i) Classes and functions

- $\mathfrak{Y}$, $\mathfrak{Z}$,... classes of groups.

- $\mathfrak{Y}\mathfrak{Z}$ is the class of all groups G containing a normal $\mathfrak{Y}$-subgroup N such that G/N belongs to $\mathfrak{Z}$.

Special classes of groups

- $\mathfrak{F}$ the class of finite groups.

- $\mathfrak{A}$ the class of abelian groups

- $\mathfrak{P}\mathfrak{F}$ the class of polycyclic-by-finite groups.

- $\mathfrak{C}$ the class of Chernikov groups.

- $\mathfrak{M}$ the class of soluble-by-finite minimax groups.

- $\mathfrak{R}$ the class of finite rank groups

- $\mathfrak{N}$ ($\mathfrak{N}_k$) the class of nilpotent groups (of class at most k).

- ZA the class of hypercentral groups.

- $\mathbf{L}\mathfrak{N}$ the class of locally nilpotent groups.

Closure operators on group classes

- $\mathbf{S}\mathfrak{Y}$ is the class of all groups whose subgroups belong to $\mathfrak{Y}$.

- $\mathbf{H}\mathfrak{Y}$ is the class of all groups whose homomorphic images belong to $\mathfrak{Y}$.

- $\mathbf{N}\mathfrak{Y}$ is the class of all groups generated by normal $\mathfrak{Y}$-subgroups.

- $\mathbf{L}\mathfrak{Y}$ is the class of all groups such that every finite subset is contained in a $\mathfrak{Y}$-subgroup;

in particular, if $\mathbf{S}\mathfrak{Y} = \mathfrak{Y}$, then $\mathbf{L}\mathfrak{Y}$ is the class of all groups whose finitely generated subgroups belong to $\mathfrak{Y}$.

- $\mathbf{P}\mathfrak{Y}$ is the class of all groups admitting a finite series with $\mathfrak{Y}$-factors.
- $\acute{\mathbf{P}}\mathfrak{Y}$ is the class of all groups admitting an ascending series with $\mathfrak{Y}$-factors.
- $\grave{\mathbf{P}}\mathfrak{Y}$ is the class of all groups admitting a descending series with $\mathfrak{Y}$-factors.
- $\mathbf{R}\mathfrak{Y}$ is the class of all groups which can be embedded in a cartesian product of $\mathfrak{Y}$-subgroups.

(ii) Elements and groups

- x, y, z,... group elements.
- The commutator of x and y, denoted $[x, y]$, is $x^{-1}y^{-1}xy$.
- For $n \in \mathbb{N}$, the iterated commutator is defined by $[x_1, x_2, ..., x_n] = [[x_1, x_2, ..., x_{n-1}], x_n]$.
- G, H,... groups, subgroups, etc.
- G^n subgroup generated by all nth powers g^n where $g \in G$.
- $C_G(H)$ and $N_G(H)$ denote, respectively, the centralizer and the normalizer of H in G.
- H^G and H_G denote, respectively, the normal closure and the normal interior of H in G.

(iii) Special subgroups

- G', $[G, G]$ the derived subgroup of G.
- $G^{(\alpha)}$ the α term of the derived series of G.
- $\gamma_\alpha(G)$ the α term of the lower central series of G.
- $\zeta(G)$ the centre of G.
- $\zeta_\alpha(G)$ the α term of the upper central series of G.

(iv) Miscellaneous

- $GF(q)$ Galois field with q elements.
- $GL(n, R)$ general linear group of all invertible $n \times n$ matrices over the ring R.
- $PGL(n, R)$ projective general linear group.
- $S_z(R)$ Suzuki group.

Introduction

One of the central topics of research in group theory (finite or infinite) is the study of the influence of given systems of subgroups of a group on the structure of the group itself. The structure of a group depends, to a significant extent, on the presence of a system of subgroups with the property of belonging to some class $\mathfrak{Y}$ of groups, the size of this system, and the interaction of this system with other subgroups. There is a wide variety of cases that can be studied. Sometimes the presence of a single subgroup with given properties can be very influential on the structure of a group whereas, in other cases, a group can have many subgroups with some given property, but the influence of this system of subgroups needs not to be significant.

A group G is said to be a *minimal non-$\mathfrak{Y}$ group*, MN$\mathfrak{Y}$-group in short, if all its proper subgroups belong to $\mathfrak{Y}$, while G itself is not in $\mathfrak{Y}$. The study of MN$\mathfrak{Y}$ groups, or more generally of groups whose proper subgroups are $\mathfrak{Y}$-groups has been one of the important themes in group theory. The first step of the study may be traced back to 1903 when G.A. Miller and H.C. Moreno [39] characterized finite MN$\mathfrak{A}$-groups, where $\mathfrak{A}$ is the class of abelian groups. They proved, among many things, that such groups are metabelian and have order divisible by at most two prime numbers. In 1924, O.J. Schmidt classified finite MN$\mathfrak{N}$-groups [50], where $\mathfrak{N}$ is the class of nilpotent groups. He proved, in particular, that such groups, which are now called *Schmidt groups*, are soluble of derived length at most 3 and have order divisible by exactly two primes. The existence of *Tarski groups*, those infinite simple 2-generator groups in which every proper subgroup is cyclic of prime order (see [42]), shows that the above ideas are much more complicated in the infinite case. To avoid such situation it is standard to restrict attention to the class of locally graded groups, where a group G is *locally graded* if every non-trivial finitely generated subgroup has a non-trivial finite image. It is easy to prove that any locally graded group whose proper subgroups are abelian is either finite or abelian. In other words, locally graded MN$\mathfrak{A}$-groups are finite. On the other hand, H. Heineken and I.J. Mohamed [32] constructed an extension of a countably infinite, elementary abelian p-group by a quasicyclic p-group (where p is a prime) with the following properties: all their proper subgroups are subnormal and nilpotent, but the centre is trivial. Heineken and Mohamed construction was studied and extended by many authors (see e.g. [12, 31]) and it became customary to call a group G of *Heineken-Mohamed type* if G is not nilpotent and all of its proper subgroups are nilpotent and subnormal. The structure of infinite locally graded MN$\mathfrak{N}$-groups was later investigated by M. Newman, H. Smith and J Weigold ([41, 51]), who proved in particular that such groups are either Chernikov groups or groups of Heineken-Mohamed type. Many authors have continued these investigations in both finite and infinite groups. The situation is especially promising, but challenging, in infinite group theory.

In the last two decades, research activity has moved to study the influence of the structure of the proper subgroups of infinite rank of a group on the entire group, and this theme is of course related to the previous one. Recall that a group G is said to have *finite (or Prüfer or special) rank* r if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with such property. If no such integer r exists then we say that the group has *infinite rank*. Thus groups of rank 1 are just the locally cyclic groups, while any free non-abelian group has infinite rank. A classical theorem of A.I. Mal'cev [37] states that a locally nilpotent group of infinite rank must contain an abelian subgroup of infinite rank. On the other hand, M.R. Dixon, M.J. Evans and H. Smith [27] proved that a (generalized) soluble group of infinite rank in which all proper subgroups of infinite rank are abelian is itself abelian. The investigation of groups whose proper subgroups of infinite rank belong to a given class $\mathfrak{Y}$ has been continued in a series of papers (see for instance [20], where a full reference list on this subject can be found). The results obtained show that the subgroups of infinite rank of a group of infinite rank have the power to influence the structure of the whole group and to force also the behavior of the subgroups of finite rank.

The main goal of this thesis is to give a further contribution to the above-mentioned themes. More specifically, for a chosen class of group $\mathfrak{Y}$, we consider groups in which the system of $\mathfrak{Y}$-subgroups coincide with the family of all proper subgroups and with the family of all proper subgroups of infinite rank.

We now describe the organization of the thesis. The thesis consists of four chapters. Chapter 1 is preliminary in nature and intended to present most of the results used in this thesis providing appropriate references for them. Then while using these results we refer to the first Chapter instead of referring to the original papers. Chapter 2, 3 and 4 report the work done by the author.

In Chapter 1, we give a brief summary of some results that have been obtained on groups in which the system of $\mathfrak{Y}$-subgroups coincide with the family of all proper subgroups and with the family of all proper subgroups of infinite rank. Our choice of what to include and that to omit has been guided by our own interests and undoubtedly there are many interesting results that will not be mentioned. For this reason, we have decided to frame our discussion in the context of properties such as nilpotency and its generalizations. We cite, for example, the results obtained in [1], [2], [6], [17], [18], [21], [23], [24], [27], [28], [29], [40], [41], [55].

In Chapter 2, we give a complete characterization of (generalized) soluble groups of infinite rank in which every proper subgroup of infinite rank is a $\mathfrak{Y}\mathfrak{N}$-group; where $\mathfrak{Y}$ is a subclass of the class of soluble-by-finite minimax groups with certain conditions of closure operations. As we will see, $\mathfrak{Y}$ can be chosen to be the class $\mathfrak{C}$, $\mathfrak{P}\mathfrak{F}$ or $\mathfrak{M}$ of Chernikov, polycyclic-by-finite or soluble-by-finite minimax groups, respectively. At the end of this chapter, we consider the dual situation–groups in which every proper subgroup of infinite rank is a $\mathfrak{N}\mathfrak{Y}$-group. Our results should be seen in relation with the description of groups whose proper subgroups are $\mathfrak{Y}\mathfrak{N}$-groups and $\mathfrak{N}\mathfrak{Y}$-groups, respectively.

In Chapter 3, we consider a (generalized) soluble group of infinite rank whose proper subgroups of infinite rank are $\mathfrak{C}ZA$, then we prove that such group have all their proper subgroups in $\mathfrak{C}ZA$. A corresponding result is holding for the dual property–groups in which every proper subgroup of infinite rank is in $ZA\mathfrak{C}$; where ZA denotes the class of hypercentral groups.

In Chapter 4, we show that a locally graded group whose proper subgroups are Engel (respectively, k-Engel) is either Engel (respectively, k-Engel) or finite. At the end of this chapter, we take the opportunity to study (generalized) soluble groups of infinite rank in which all proper subgroups of infinite rank are Engel (respectively, k-Engel).

Chapter 1

Groups with many subgroups in a given class of groups

1.1 Introduction

In a number of papers, authors studied groups many of whose subgroups belong to a given group class $\mathfrak{Y}$. Two interpretations of the requirement that "many $\mathfrak{Y}$-subgroups" have been that "all proper subgroups of infinite rank belong to $\mathfrak{Y}$", and, more generally, that "the family of all proper subgroups belong to $\mathfrak{Y}$". The results have included structure theorems, and sometimes assertions that such groups either belong to $\mathfrak{Y}$ or must be small in some sense, at least for several natural choices of $\mathfrak{Y}$. In this chapter, we will give an overview of the main results that have been obtained in this context. In particular the results that are used in this thesis.

1.2 Groups in which every proper subgroup is in $\mathfrak{Y}$

A group G is said to be *minimal non-*$\mathfrak{Y}$ if it is not a $\mathfrak{Y}$-group but all its proper subgroups belong to $\mathfrak{Y}$. We denote the minimal non-$\mathfrak{Y}$-groups by MN$\mathfrak{Y}$. Many results have been obtained on MN$\mathfrak{Y}$-groups, or more generally on groups whose proper subgroups are $\mathfrak{Y}$-

groups, for several choices of $\mathfrak{Y}$. Already in 1903, G.A. Miller and H.C. Moreno [39] investigated properties of finite MN$\mathfrak{A}$-groups, proving in particular that such groups are metabelian and that the orders of their elements are divisible by at most two primes; where $\mathfrak{A}$ is the class of abelian groups. Then Schmidt [50] classified finite MN$\mathfrak{N}$-groups, where $\mathfrak{N}$ is the class of nilpotent groups. Later, this kind of investigation has been dealt with also in the infinite case by several authors. In this section, we will give a brief summary of the main results obtained in this context.

1.2.1 Groups in which every proper subgroup is in $\mathfrak{N}$

Throughout this thesis, $\mathfrak{N}_k$ denotes the class of nilpotent groups of class at most k, where k is a positive integer. In 1964, M.F. Newman and J. Wiegold [41] described the structure of infinite MN$\mathfrak{N}$-groups. It is clear that such groups are either finitely generated or locally nilpotent groups.

Theorem 1.2.1.1. *Let G be an infinite finitely generated* MN$\mathfrak{N}$*-group or* MN$\mathfrak{N}_k$*-group, then $G/Frat(G)$ is non-abelian and simple.*

This theorem means that if infinite finitely generated MN$\mathfrak{N}$-groups and MN$\mathfrak{N}_k$-groups exist so do simple ones. This led M.F. Newman and J. Wiegold [41] to list some of the properties that such a simple group must-have.

Theorem 1.2.1.2. *Let G be a finitely generated simple* MN$\mathfrak{N}$*-group or* MN$\mathfrak{N}_k$*-group. Then:*

(i) every pair of maximal subgroups of G has trivial intersection;

(ii) to every non-identity element x of G, there is an element g of G such that x, x^g generate *G;*

(iii) there are no elements of order 2*.*

Since the publication of the paper of M.F. Newman and J. Wiegold there have appeared the examples due to A.Yu. Ol'shanskii (see [42]) and E. Rips of infinite simple 2-generator groups with all proper subgroup cyclic of prime order (the so-called *Tarski monsters*). Note that the Tarski monsters are, obviously, examples of finitely generated simple MN$\mathfrak{N}$-groups.

Theorem 1.2.1.3. *Let G be a locally nilpotent* MN$\mathfrak{N}$*-group. Then:*
(i) G is countably infinite;
(ii) the commutator factor-group of G is a locally cyclic p-group for some prime p; if $G/\gamma_2(G)$ is non-trivial, then G is a p-group.

These are the only results that M.F. Newman and J. Wiegold have obtained on locally nilpotent MN$\mathfrak{N}$-groups. This led them to assume further that the groups in question have maximal subgroups. Thus, they obtained a complete description of locally nilpotent MN$\mathfrak{N}$-groups with maximal subgroups.
Let p be a prime number, n a positive integer and $r = 0$ or 1. Let $\mathfrak{B}$ be the set of groups $B(p, n, r)$, where $B(p, n, r)$ is the group generated by the set $\{b, h_1, h_2, ...\}$ with defining relations:

$$\begin{cases} [h_i, h_j] = [h_i, b^p] = 1, & [h_{i+1}, b] = h_i, \quad i, j : 1, 2, ..., \\ [h_1, b] = 1, & b^{p^n} = h_1^r. \end{cases}$$

Theorem 1.2.1.4. *(i) Every locally nilpotent* MN$\mathfrak{N}$*-group with maximal subgroups is isomorphic to a group in* $\mathfrak{B}$;
(ii) Every group in $\mathfrak{B}$ *is a locally nilpotent* MN$\mathfrak{N}$*-group with maximal subgroups;*
(ii) No two groups in $\mathfrak{B}$ *are isomorphic.*

The previous theorem admits the following consequence.

Corollary 1.2.1.5. *If G is a locally nilpotent* MN$\mathfrak{N}$*-group with maximal subgroups, then G is a metabelian Chernikov p-group, for some prime p, and hence hypercentral.*

The only information provided by M.F. Newman and J. Wiegold on MN$\mathfrak{N}$-groups without maximal subgroups is that they are countable and, if soluble, p-groups for some prime p (Theorem 1.2.1.3). Four years or so after the publication of [41], there appeared the examples of H. Heineken and I.J. Mohamed [32]: for every prime p there exists a metabelian, non-nilpotent p-group G having all proper subgroups nilpotent and subnormal; further, G has no maximal subgroups. These examples led H. Smith to begin the investigation of infinite locally nilpotent $\mathrm{MN}\mathfrak{N}$-groups without maximal subgroups, distinguishing the soluble case from the non-soluble case. However, a recent amazing theorem of A.O. Asar [2] shows that every locally graded group, all of whose proper subgroups are nilpotent, is soluble. The following result supplements this theorem of A.O. Asar very nicely and was obtained by H. Smith in [51].

Theorem 1.2.1.6. *Let G be a locally nilpotent $\mathrm{MN}\mathfrak{N}$-group, and suppose that G has no maximal subgroups. Then:*
(i) G is a countable p-group for some prime p and $G/G' \simeq C_{p^\infty}$;
(ii) Every subgroup of G is subnormal;
(iii) $(G')^p \neq G'$, and every hypercentral image of G is abelian. In particular, $G' = \gamma_n(G)$ for all $n \geq 2$;
(iv) Every radicable subgroup of G is central;
(v) The centraliser of G' is abelian, and G' is omissible (that is, $HG' = G$ implies $H = G$); in particular, G has no proper subgroups of finite index;
(vi) G' is not the normal closure in G of a finite subgroup;
(vii) The hypercentre of G coincides with its center.

When a bound is placed on the nilpotency classes things are more straightforward. In fact, B. Bruno and R.E. Phillips [10] studied locally graded $\mathrm{MN}\mathfrak{N}_k$-groups and they proved the following result.

Theorem 1.2.1.7. *Let k be a positive integer and let G be a locally graded $\mathrm{MN}\mathfrak{N}_k$-group. Then G is finite.*

It follows that locally graded $\mathrm{MN}\mathfrak{A}$-groups are finite, more generally, each locally graded group in which proper subgroups are $\mathfrak{N}_k$-groups is finite or in $\mathfrak{N}_k$.

1.2.2 Groups in which every proper subgroup is in $\mathfrak{FA}$ or $\mathfrak{AF}$

Let $\mathfrak{FA}$ and $\mathfrak{AF}$ denote the class of finite-by-abelian and abelian-by-finite groups, respectively. In 1983, B. Bruno and R.E. Phillips [11] investigated the structure of locally graded MN$\mathfrak{FA}$-groups. In particular they have shown.

Theorem 1.2.2.1. *Let G be a locally graded* MN$\mathfrak{FA}$*-group. Then:*
(i) for some prime p and positive integer $n \geq 0$ there is a normal subgroup A of G with $A \simeq C_{p^\infty}^n$ and an element y in G of prime power of order q^s with $G = A\langle y\rangle$, and
(ii) $V = \langle y\rangle / C_{\langle y\rangle}(A)$ has order q and A is a divisibly irreducible V-module.

B. Bruno in [6] began the study of periodic locally graded MN$\mathfrak{AF}$-groups and obtained important properties distinguishing the cases of p-groups and non p-groups. Later, combined efforts of a number of people; see B. Bruno and R.E. Phillips [13], F. Napolitani and E. Pegoraro [40] and A.O. Asar [2], showed that locally graded MN$\mathfrak{AF}$-groups are periodic and non-perfect. More precisely, we have the following result.

Theorem 1.2.2.2. *Let G be a locally graded.*
(i) If G is not a p-group, for any prime p. Then G is a MN$\mathfrak{AF}$*-group, if and only if, $G = G' \rtimes A$, where $A \simeq C_{r^\infty}$, for some prime r, and G' is an elementary abelian minimal normal q-subgroup of G, for some prime $q \neq r$.*
(ii) If G is a p-group. Then G is a MN$\mathfrak{AF}$*-group, if and only if, G' is abelian, the quotient $G/G' \simeq C_{p^\infty}$, and for all proper subgroups H of G, $HG' < G$.*

Note that the Heineken-Mohamed groups are examples of p-groups MN$\mathfrak{AF}$. This fact follows from [32, Lemma 1] since if $H < G$ then $HG' < G$. Hence HG'/G' is finite and H is abelian-by-finite.

1.2.3 Groups in which every proper subgroup is in $\mathfrak{FN}$ or $\mathfrak{NF}$

Let $\mathfrak{FN}$ and $\mathfrak{NF}$ denote the class of finite-by-nilpotent and nilpotent-by-finite groups, respectively. In 1996, M. Xu [55] investigated the structure of MN$\mathfrak{FN}$-groups, distinguishing the finitely generated case from the non-finitely generated.
In the finitely generated case, M. Xu has extended Theorem 1.2.1.1 of M.F. Newman and J. Wiegold for MN$\mathfrak{FN}$-groups. More precisely, he proved the following result.

Theorem 1.2.3.1. *If G is a finitely generated* MN$\mathfrak{FN}$*-group, then $G' = G$, G has no non-trivial finite factor groups, and $G/Frat(G)$ is a simple group.*

Note that the Tarski monsters assert the existence of finitely generated MN$\mathfrak{FN}$-groups.

Theorem 1.2.3.2. *A group G is an infinitely generated* MN$\mathfrak{FN}$*-group if and only if G is of one of the following types:*
(i) a non-perfect MN$\mathfrak{FA}$*-group;*
(ii) a MN$\mathfrak{N}$ *without maximal subgroup.*

B. Bruno in [7, 8] extended the results obtained on MN$\mathfrak{AF}$-groups by replacing the term "$\mathfrak{A}$-group" by "$\mathfrak{N}$-group", and gave the important properties of periodic locally graded MN$\mathfrak{NF}$-groups by distinguishing the cases of locally nilpotent and the non-locally nilpotent. Later, combined efforts of a number of people; see B. Bruno and R.E. Phillips [13], A.O. Asar [2] and F. Napolitani and E. Pegoraro [40], show that locally graded MN$\mathfrak{NF}$-groups are periodic and non-perfect. Thus, we have the following result.

Theorem 1.2.3.3. *(i) Let G be a locally graded non-locally nilpotent* MN$\mathfrak{NF}$*-group. Then $G = V \rtimes H$ is a semidirect product of a normal subgroup V and a quasicyclic p-subgroup H, where V is a special q-group, V is centralized by H and V/V' is a minimal normal subgroup of G/V (p and q are distinct primes).*
(ii) Let G be a locally nilpotent MN$\mathfrak{NF}$*-group. Then G is a p-group, for some prime p.*

1.2.4 Groups in which every proper subgroup is in $(\mathfrak{PF})\mathfrak{N}$ or $\mathfrak{N}(\mathfrak{PF})$

Denote by $(\mathfrak{PF})\mathfrak{N}$ and $\mathfrak{N}(\mathfrak{PF})$ the class of (polycyclic-by-finite)-by-nilpotent and nilpotent-by-(polycyclic-by-finite) groups, respectively. In 1999, S. Franciosi, F. De Giovanni and Y.P. Sysak [18] established results on locally graded MN$(\mathfrak{PF})\mathfrak{N}$-groups. They proved, in particular, that such groups are periodic so that they are also MN$\mathfrak{FN}$-groups.

Theorem 1.2.4.1. *A group G is a locally graded* MN$(\mathfrak{PF})\mathfrak{N}$ *if and only if it is a countable locally finite* MN$\mathfrak{FN}$*-group.*

In the same paper [18], they considered the dual situation studying locally graded MN$\mathfrak{N}(\mathfrak{PF})$-groups. They have shown in particular that such groups are precisely the locally finite MN$\mathfrak{NF}$-groups.

Theorem 1.2.4.2. *A group G is a locally graded* MN$\mathfrak{N}(\mathfrak{PF})$ *if and only if it is a locally finite* MN$\mathfrak{NF}$*-group.*

1.2.5 Groups in which every proper subgroup is in $\mathfrak{CN}$ or $\mathfrak{NC}$

Let $\mathfrak{CN}$ and $\mathfrak{NC}$ denote the class of Chernikov-by-nilpotent and nilpotent-by-Chernikov groups, respectively. In the paper [1], A. Arikan and N. Trabelsi classified locally graded MN$\mathfrak{CN}$-groups. In particular, they showed that locally graded MN$\mathfrak{CN}$-groups are locally nilpotent MN$\mathfrak{N}$-groups without maximal subgroups that have been described by H. Smith (Theorem 1.2.1.6).

Theorem 1.2.5.1. *A group G is a locally graded* MN$\mathfrak{CN}$*-group if and only if it is a* MN$\mathfrak{N}$*-group without maximal subgroup.*

Note that the Heineken-Mohamed groups are examples of MN$\mathfrak{CN}$-groups.

In 1997, F. Napolitani and E. Pegoraro [40] studied locally graded groups in which every proper subgroup belongs to $\mathfrak{NC}$. They proved, in particular, that such groups are themselves $\mathfrak{NC}$-groups, if they are not locally finite p-groups. But in 2000, A.O. Asar [2], studied this case where he proved, among other things, that the locally finite p-groups do not constitute an exception to the former statement. Combining these results, we have the following.

Theorem 1.2.5.2. *Let G be a locally graded group in which every proper subgroup is $\mathfrak{NC}$, then G is a $\mathfrak{NC}$-group.*

1.2.6 Groups in which every proper subgroup is in $\mathfrak{RN}$ or $\mathfrak{NR}$

A group G is said to have *finite (or Prüfer or special) rank* r if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with such property. In particular, a group has rank 1 if and only if it is locally cyclic. It is clear that the class of finite rank groups is **S, H** and **P**-closed.

The structure of locally graded groups with all proper subgroups of finite rank, and even the structure of locally graded groups of finite rank, is unknown, although well-researched in other classes of groups. In 1990, N.S. Černikov [16] considered *the class* $\mathfrak{X}$ obtained by taking the closure of the class of periodic locally graded groups by the closure operations $\acute{\mathbf{P}}$, $\grave{\mathbf{P}}$, $\mathbf{R}$, $\mathbf{L}$. Clearly $\mathfrak{X}$ is a subclass of the class of locally graded groups, it is unknown whether $\mathfrak{X}$ exhausts the class of all locally graded groups. In his paper [16], Černikov proved that an $\mathfrak{X}$-group of finite rank is (locally soluble)-by-finite. The $\mathfrak{X}$-groups form a large **S**-closed class of generalized soluble groups containing, in particular, the classes of locally (soluble-by-finite) groups, residually finite groups, and generalized radical groups. We remark that the class $\mathfrak{X}$ is not **H**-closed—as the consideration of the free non-abelian group shows.

The result obtained by J. Otal and J.M. Peña [45] has inspired many generalizations. For example, in [29], M.R. Dixon, M.J. Evans and H. Smith considered groups with all proper subgroups belongs to $\mathfrak{RN}$. The following result was obtained.

Theorem 1.2.6.1. *Let G be an $\mathfrak{X}$-group and suppose that every proper subgroup of G*

belongs to $\mathfrak{R}\mathfrak{N}$.

(i) If G *is a* p*-group for some prime* p *then either* $G \in \mathfrak{R}\mathfrak{N}$ *or* $G/G' \simeq C_{p^\infty}$ *and every proper subgroup of* G *is nilpotent.*

(ii) If G *is not a* p*-group then* $G \in \mathfrak{R}\mathfrak{N}$.

Combining Theorem 1.2.6.1 and Theorem 1.2.5.1, we obtain the following consequence.

Corollary 1.2.6.2. *Let* G *be an* $\mathfrak{X}$*-group. Then the following properties are equivalent.*

(i) G *is a* MN$\mathfrak{C}\mathfrak{N}$*;*

(ii) G *is a* MN$\mathfrak{R}\mathfrak{N}$*;*

(iii) G *is a* MN$\mathfrak{N}$ *without maximal subgroup.*

In [28], M.R. Dixon, M.J. Evans and H. Smith investigated the groups all of whose proper subgroups are $\mathfrak{N}\mathfrak{R}$-groups. More precisely, they established the following results.

Theorem 1.2.6.3. *Let* G *be a locally (soluble-by-finite) with all proper subgroups belongs to* $\mathfrak{N}\mathfrak{R}$*. Suppose that* G *is not a* p*-group. If* G *has no infinite simple images then* $G \in \mathfrak{N}\mathfrak{R}$.

Theorem 1.2.6.4. *Let* G *be a group with all proper subgroups belongs to* $\mathfrak{N}\mathfrak{R}$ *and suppose that* G *is locally (soluble-by-finite) and locally of finite rank (and therefore locally minimax). Suppose that* G *is not locally finite, then* $G \in \mathfrak{N}\mathfrak{R}$.

1.2.7 Groups in which every proper subgroup is in $\mathbf{L}\mathfrak{N}$ or ZA

Let $\mathbf{L}\mathfrak{N}$ denotes the class of locally nilpotent groups. In a paper published in 1988, J. Otal and J.M. Peña [44] characterized locally graded MN$\mathbf{L}\mathfrak{N}$-groups. More precisely, they have shown the following result.

Theorem 1.2.7.1. *Let* G *be a locally graded* MN$\mathbf{L}\mathfrak{N}$*-group. Then* G *is finite.*

Recall that a group is called *hypercentral* (or a ZA*-group*) if it admits an ascending central series. The hypercentral groups are known to be locally nilpotent (see [49, P.

365]). In 2015, F. De Giovanni and M. Trombetti [24] described the main properties of infinite MNZA-groups, extending the result of H. Smith on infinite MN$\mathfrak{N}$-groups (Theorem 1.2.1.6).

Theorem 1.2.7.2. *Let G be an infinite locally graded* MNZA*-group. Then:*
(i) G is a locally finite p-group for some prime number p;
(ii) G is hyperabelian;
(iii) If $G' \neq G$, then G/G' is a group of type p^∞;
(iv) $Z(G)$ is the last term of the upper central series of G, and G' is the last term of the lower central series of G;
(v) The centraliser of G' is abelian;
(vi) All subgroups of G are ascendant;
(vii) If N is any proper normal subgroup of G, then $HN \neq G$ for every proper subgroup H of G;
(viii) G' cannot be the normal closure of a finite subgroup of G.

1.3 Groups in which every proper subgroup of infinite rank is in $\mathfrak{Y}$

A group G is said to have *finite rank* if there exists a positive integer r such that all finitely generated subgroups of G can be generated by at most r elements and r is the least positive integer with such property; otherwise, if such an r does not exist, the group is said to have *infinite rank*. It is not difficult to see that in some universes of generalized soluble groups, if a group G has infinite rank, then it must be rich in subgroups of infinite rank. Moreover, in recent years, a series of papers have been published by many authors (among which M. De Falco, F. De Giovanni, M.R. Dixon, J. Evans, C. Musella, H. Smith, N. Trabelsi) which show that the subgroups of infinite rank of a group of infinite rank have the power to influence the structure of the whole group and to force also the behavior of the subgroups of finite rank. In fact, it has been proved that, for some choices of group classes $\mathfrak{Y}$, if G is a group in which all subgroups of infinite rank belong to $\mathfrak{Y}$, then the samehappens also to the subgroups of finite rank. So, now it will give a brief summary of results on this subject. The results described in this section will be usually stated for strongly locally graded groups that can be defined as follows. Let $\mathfrak{X}$ be the Černikov's class of groups defined in Chapter 1. Following [17], we shall say that a group G is *strongly locally graded* if every section of G is an $\mathfrak{X}$-group. In particular every strongly locally graded group is an $\mathfrak{X}$-group. Thus strongly locally graded groups form a large class of generalized soluble groups, which is **S** and **H**-closed, and contains all locally (soluble-by-finite) groups.

It is not our intention here to dwell too much on the history of groups satisfying rank conditions, but it is helpful to put some of the results in context by briefly discussing some of the main results. The structure of abelian groups A of finite rank r is well-known (For example see L. Fuchs [19]). Thus for each prime p, the p-component of A is a direct product of at most r groups which are either cyclic groups or quasicyclic groups. The factor group of A modulo its torsion subgroup embeds into a direct product of at most r copies of the additive group of the rationals. As expected, classes of groups containing many abelian sections can be described particularly well if they have finite rank. Appropriate classes of groups considered in the literature are generalized nilpotent and generalized soluble groups. The first result that we state here as a lemma is due to V.S. Čarin [14], and shows in particular that locally soluble groups of finite rank are hyperabelian (i.e. they have an ascending normal series with abelian factors).

Lemma 1.3.1. *Let G be a locally soluble group of finite rank r. Then there exists a positive integer $k = k(r)$ such that the subgroup $G^{(k)}$ is a periodic hypercentral group with Chernikov p-components.*

A natural question to ask here, of course, is whether locally soluble groups of finite rank are soluble. Kegel's example (see Baer [4]) shows that a periodic locally soluble or even hypercentral group with finite rank need not be soluble.

For locally nilpotent groups of finite rank, very precise structure results are the following.

Lemma 1.3.2. *Let G be a locally nilpotent group of finite rank r.*

(i) G is hypercentral (see [48, Corollary 1 of Theorem 6.36]).

(ii) If G is torsion-free, then it is nilpotent of nilpotency class at most r (V.S. Čarin [15, Theorem 9]).

(iii) If G is periodic, then it has Chernikov p-components (see [48, Corollary 2 of Theorem 6.36]).

The structural results on hyperabelian groups of finite rank are listed in the next lemma.

Lemma 1.3.3. *Let G be a hyperabelian group of finite rank r.*

(i) If G is torsion-free, then G is soluble of r-bounded derived length (see [48, Corollary 1 of Theorem 10.38]).

(ii) If G is periodic, then for any prime p, the Sylow p-subgroups of G are conjugate (R. Baer and H. Heineken [5]).

(iii) Each chief factor of G is finite and every maximal subgroup of G has finite index in G (D.J.S. Robinson [47, Theorems C and D]).

(iv) G is $\mathfrak{F}$-perfect if and only if G is radicable nilpotent with central torsion group (see [48, Theorem 9.31]).

Here, a group is $\mathfrak{F}$-*perfect* if it has no non-trivial finite homomorphic images. The group G is called *radicable* if for every $n \in \mathbb{N}$ and every $g \in G$, there exists an element $h \in G$ such that $h^n = g$.

A relevant result by A.I. Mal'cev [37] states that any locally nilpotent group of infinite rank must contain an abelian subgroup of infinite rank, and a corresponding result for locally finite groups was later proved by V.P. Šunkov [52]. Mal'cev's theorem was improved by R. Baer and H. Heineken [5], to the case of radical groups of infinite rank. For locally soluble groups things are not quite so nice since Y.I. Merzljakov [38] has shown that there exist locally soluble groups of infinite rank in which every abelian subgroup has finite rank. Consequently there is no analogue of the Baer-Heineken theorem for locally soluble groups. More recently, M.R. Dixon, M.J. Evans and H. Smith [27] has shown that a locally soluble group of infinite rank contains a proper subgroup of infinite rank. As in many problems concerning groups of infinite rank, also in our case, the existence of proper subgroups of infinite rank plays a crucial role.

Lemma 1.3.4. *Let G be an $\mathfrak{X}$-group of infinite rank. Then G contains a proper subgroup of infinite rank.*

Proof. Suppose, for a contradiction, that every proper subgroup of G has finite rank. If G is finitely generated then, since G is locally graded, there exists $N \triangleleft G$ such that G/N is a non-trivial finite group and clearly N has infinite rank, a contradiction. Thus G is not finitely generated and it follows that every finitely generated subgroup of G has finite rank. A result of Černikov [16] implies that G is locally (soluble-by-finite). Since G has infinite rank, it must contain a locally soluble subgroup H of infinite rank [26], and hence $H = G$ is locally soluble. Therefore G has finite rank [27, Lemma 1], and this contradiction proves the lemma. ■

The first relevant result for our purposes was obtained 20 years ago by M.R. Dixon, M.J. Evans and H. Smith [27], and deals respectively with the class $\mathfrak{N}_c$ and $\mathbf{L}\mathfrak{N}$ of nilpotent groups of given class at most c and locally nilpotent groups.

Theorem 1.3.5. *Let c be a positive integer and let G be an $\mathfrak{X}$-group of infinite rank.*
(i) If all proper subgroups of infinite rank of G are in $\mathfrak{N}_c$, then G itself belongs to $\mathfrak{N}_c$.
(ii) If all proper subgroups of infinite rank of G are in $\mathbf{L}\mathfrak{N}$, then G itself belongs to $\mathbf{L}\mathfrak{N}$.

Of course there is no corresponding result if we replace $\mathfrak{N}_c$ by $\mathfrak{N}$ the class of all nilpotent groups, since the Heineken-Mohamed type groups are locally nilpotent, non-nilpotent and of infinite rank but have all proper subgroups nilpotent. However, the existence of such groups is essentially the only reason why we cannot replace $\mathfrak{N}_c$ by $\mathfrak{N}$ in the results mentioned above.

The structure of groups of infinite rank in which all proper subgroups of infinite rank are in $\mathfrak{A}\mathfrak{F}$ (respectively, in $\mathfrak{N}\mathfrak{F}$) was investigated by F. De Giovanni and F. Saccomanno [21]. They proved in particular that such groups either belong to $\mathfrak{A}\mathfrak{F}$ or are MN$\mathfrak{A}\mathfrak{F}$ (respectively, belong to $\mathfrak{N}\mathfrak{F}$ or are MN$\mathfrak{N}\mathfrak{F}$). More specifically, the following theorem is valid.

Theorem 1.3.6. *Let G be a strongly locally graded group of infinite rank whose proper subgroups of infinite rank are $\mathfrak{A}\mathfrak{F}$-groups (and $\mathfrak{N}\mathfrak{F}$-groups). Then so are all proper subgroups of G.*

M. De Falco, F. De Giovanni, C. Musella and N. Trabelsi have proved that, in [17], if G is a group of infinite rank in which all proper subgroups of infinite rank are in $(\mathbf{L}\mathfrak{F})\mathfrak{A}$, then some informations can be obtained on the structure of such group.

Theorem 1.3.7. *Let G be a strongly locally graded group of infinite rank whose proper subgroups of infinite rank are in $(\mathbf{L}\mathfrak{F})\mathfrak{A}$. Then G itself belongs to $(\mathbf{L}\mathfrak{F})\mathfrak{A}$.*

Some interesting results in this context are due to F. De Giovanni and M. Trombetti [23].

Theorem 1.3.8. *Let G be a strongly locally graded group of infinite rank.*
(i) If all proper subgroups of infinite rank of G are in $(\mathbf{L}\mathfrak{F})\mathfrak{N}$, then G itself belongs to $(\mathbf{L}\mathfrak{F})\mathfrak{N}$.
(ii) If all proper subgroups of infinite rank of G are in $(\mathbf{L}\mathfrak{F})(\mathbf{L}\mathfrak{N})$, then G itself belongs to $(\mathbf{L}\mathfrak{F})(\mathbf{L}\mathfrak{N})$.

Chapter 2

Groups with many subgroups in $\mathfrak{NM}$ or $\mathfrak{MN}$

2.1 Introduction

As we have already mentioned, many results have been obtained on groups with many subgroups in a given group class $\mathfrak{Y}$. A result of A. Arikan and N. Trabelsi states that locally graded MN$\mathfrak{CN}$-groups are precisely the locally nilpotent MN$\mathfrak{N}$-groups without maximal subgroups (Theorem 1.2.5.1). Whereas F. Napolitani, E. Pegoraro and A.O. Asar have proved that locally graded groups in which every proper subgroup is in $\mathfrak{NC}$ are themselves $\mathfrak{NC}$-groups (Theorem 1.2.5.2). In other words, there are no locally graded MN$\mathfrak{NC}$-groups. In this chapter we give the soluble-by-finite minimax $\mathfrak{M}$ version of the above results.

It follows from a result of B. Bruno and R.E. Phillips (Theorem 1.2.2.1) that $\mathfrak{X}$-groups of infinite rank all of whose subgroups of infinite rank belong to $\mathfrak{FA}$ are themselves $\mathfrak{FA}$-groups. Moreover, F. De Giovanni and F. Saccomanno considered strongly locally graded groups of infinite rank in which all proper subgroups of infinite rank belong to $\mathfrak{AF}$, they proved in particular that these groups have all their proper subgroups are $\mathfrak{AF}$-groups (Theorem 1.3.6). Here we provide a further contribution by replacing the terms "$\mathfrak{A}$-

group" by "$\mathfrak{N}$-group" and "$\mathfrak{F}$-group" by "$\mathfrak{Y}$-group", where $\mathfrak{Y}$ is a subclass of the class of soluble-by-finite minimax groups with certain conditions of closure operations. As we will see, $\mathfrak{Y}$ can be chosen to be the class $\mathfrak{C}$, $\mathfrak{PF}$ or $\mathfrak{M}$ of Chernikov, polycyclic-by-finite or soluble-by-finite minimax groups, respectively.

Recall that a group G is called *minimax* (cf. Baer [3]) if there is in G a series of finite length in which every factor satisfies *either* Max (the maximal condition on subgroups) *or* Min (the minimal condition on subgroups). Such a series is called a *minimax series*. The structure of *soluble minimax groups* has been described by D.J.S. Robinson (see [48, Part 2, Chapter 10]). Note that soluble minimax groups have finite rank, because polycyclic and Chernikov groups do, while a finitely generated soluble group of finite rank is minimax, as follows from Theorem 10.38 of [48]. Thus the class of locally soluble minimax groups is precisely the class of locally soluble groups that are locally of finite rank. It is also the case to observe that a periodic soluble minimax group is a Chernikov group, since a periodic polycyclic group is finite. For our convenience we quote a well-known fact about nilpotent groups, which will be freely used in what follows: Let G be a nilpotent group. Then G is minimax if and only if G/G' is minimax.

2.2 The class of $\mathfrak{MN}$-groups

In this section we are interested in the classes $\mathfrak{FN}$, $\mathfrak{CN}$, $(\mathfrak{PF})\mathfrak{N}$, $\mathfrak{MN}$. We shall give the complete characterization of $\mathfrak{X}$-groups of infinite rank in which all proper subgroups of infinite rank are in these classes. The knowledge of the structure of MN$\mathfrak{FN}$-groups (and MN$\mathfrak{CN}$-groups, MN$(\mathfrak{PF})\mathfrak{N}$-groups, MN$\mathfrak{MN}$-groups) will be relevant in our considerations.

In the first part we study MN$\mathfrak{MN}$-groups as a continuation of characterizations of locally graded MN$\mathfrak{N}$-groups (Theorem 1.2.1.4, 1.2.1.6), MN$\mathfrak{FN}$-groups (Theorem 1.2.3.2), MN$(\mathfrak{PF})\mathfrak{N}$-groups (Theorem 1.2.4.1) and MN$\mathfrak{CN}$-groups (Theorem 1.2.5.1).

The following result will be often used in our proofs.

Let $\mathfrak{Y}$ and $\mathfrak{Z}$ be two group classes, we say that $\mathfrak{Y}$ is $\mathfrak{Z}$-*characteristic* (a generalization of the idea of [25]) if any $\mathfrak{Y}\mathfrak{Z}$-group G contains a characteristic $\mathfrak{Y}$-subgroup A such that G/A is a $\mathfrak{Z}$-group. Note that if $\mathfrak{Y}$ is any subgroup closed class, then $\mathfrak{Y}$ is $\mathfrak{N}$-characteristic.

Lemma 2.2.1. *Let $\mathfrak{Y}$ and $\mathfrak{Z}$ be two group classes such that $\mathfrak{Y}$ is $\mathfrak{Z}$-characteristic and $\mathbf{N}_0$-closed and $\mathfrak{Z}$ is $\{\mathbf{H}, \mathbf{N}_0\}$-closed. Then $\mathfrak{Y}\mathfrak{Z}$ is also $\mathbf{N}_0$-closed. Therefore if G is a* MN$\mathfrak{Y}\mathfrak{Z}$*-group, then every nilpotent image of G is a (possibly trivial) locally cyclic p-group for some prime p.*

Proof. Let H and K be normal $\mathfrak{Y}\mathfrak{Z}$-subgroups of a group G. Then there exists two subgroups A and B such that A (resp. B) characteristic in H (resp. K), A, B are $\mathfrak{Y}$-groups and H/A, K/B are $\mathfrak{Z}$-groups. Clearly AB is a normal $\mathfrak{Y}$-subgroup of G by the hypothesis. We also have that HK/AB is a $\mathfrak{Z}$-group, as it is the product of HB/AB and KA/AB. Hence HK is a $\mathfrak{Y}\mathfrak{Z}$-group. The rest of the claim follows by [41, Theorem 2.12].

Our next result is well-known, but we give a proof.

Lemma 2.2.2. *Let G be an hypercentral group. If G/G' is periodic, then G is periodic.*

Proof. Assume that the result is false and let T denote the torsion subgroup of G. Obviously, G/T is likewise a counterexample, so without lose of generality it can be assumed that G is torsion-free. Since G/G' is periodic, we deduce that G is non-abelian, and let h be an element in the set $Z_2(G) \setminus Z(G)$. Then the map

$$\theta : g \in G \to [h, g] \in Z(G)$$

is a non-zero homomorphism of G onto a torsion-free abelian group, so that $G/ker(\theta)$ is an abelian non-periodic group which is a contradiction. ■

Theorem 2.2.3. *A group G is a locally graded* MN$\mathfrak{M}\mathfrak{N}$ *if and only if it is a* MN$\mathfrak{N}$*-group without maximal subgroup.*

Proof. Let G be a locally graded MN$\mathfrak{M}\mathfrak{N}$-group. In particular, G cannot be finitely generated. As finitely generated $\mathfrak{MN}$-groups are obviously in $\mathfrak{M}$, it follows from the hypothesis that every finitely generated subgroup of G is a $\mathfrak{M}$-group. So G is locally in $\mathfrak{M}$ and, by Theorem 1.2.6.1, it is either a $\mathfrak{RN}$-group or G/G' is p-quasicyclic for some prime p and every proper subgroup of G is nilpotent. Assume for a contradiction that G is a $\mathfrak{RN}$-group; then G is of finite rank By Lemma 2.2.1.

Let us suppose first that G has a proper normal subgroup N of a finite index. Then N belongs to $\mathfrak{MN}$ and $\gamma_c(N)$ is a $\mathfrak{M}$-group for some positive integer c; there is no loss of generality if we assume that $\gamma_c(N) = 1$ and so N is nilpotent. If G is periodic, then all its proper subgroups are in $\mathfrak{CN}$, and hence so is G by Theorem 1.2.5.1. Therefore G is non-periodic, and N is non-periodic. We deduce by [49, 5.2.6], that also the factor group N/N' is non-periodic. Let M/N' be a maximal free abelian subgroup of N/N'; then $(N/N')/(M/N')$ is periodic. Since G/N' has finite rank and M/N' has a finite number of conjugates in G/N', the abelian group $(M/N')^G$ is finitely generated. Furthermore, Theorem 1.2.5.1 shows that the periodic group $(G/N')/(M/N')^G$ belongs to $\mathfrak{CN}$ hence, since every nilpotent image of G is Chernikov, $(G/N')/(M/N')^G$ is a Chernikov group. It follows that G/N' belongs to $\mathfrak{M}$, and therefore N/N' is an $\mathfrak{M}$-group. Since N is nilpotent we deduce that N also belongs to $\mathfrak{M}$. But this contradict our assumption.

Thus G has no proper subgroup of finite index. By [16], G is locally soluble, and hence it has an ascending normal abelian series whose factors are of rank at most r and either torsion or torsion-free. Let F be a factor of the series. If F is periodic, then $A = \langle x \rangle^G$ $(x \in F)$ has finite exponent and is therefore finite and hence centralized by G. If F is torsion-free, then $G/C_G(F)$ embeds in a rationally irreducible linear group of degree r. But locally soluble linear groups are soluble by [48, Corollary to Theorem 3.23], so applying [48, Corollary 1 to Lemma 5.29.1] gives that $G/C_G(F)$ is free abelian-by-finite. We conclude again that F is central. Consequently, G is hypercentral. Since G/G' is periodic, we deduce that G is also periodic by Lemma 2.2.2. This contradiction gives the result. ■

We turn now to give information about groups of infinite rank in which all proper subgroups of infinite rank belong to $\mathfrak{Y}\mathfrak{N}$, where $\mathfrak{Y}$ is an $\mathbf{S}$-closed subclass of the class of soluble-by-finite minimax groups.

Theorem 2.2.4. *Let $\mathfrak{Y}$ be a subclass of the class of soluble-by-finite minimax groups which is $\mathbf{S}$-closed. If G is an $\mathfrak{X}$-group of infinite rank whose proper subgroups of infinite rank are $\mathfrak{Y}\mathfrak{N}$-groups, then so are all proper subgroups of G.*

Proof. It is an immediate consequence of Theorem 1.2.6.1. Clearly we may assume that G is a $\mathfrak{R}\mathfrak{N}$-group. Then G contains a proper normal subgroup N such that G/N is nilpotent. Hence G/G' is of infinite rank, and so there exists a subgroup M of G of infinite rank such that $G' \leq M$ and G/M has infinite rank. The product HM is a proper subgroup of infinite rank of G, and hence it is a $\mathfrak{Y}\mathfrak{N}$-group. Therefore all proper subgroups of G are $\mathfrak{Y}\mathfrak{N}$-groups, as desired. ■

Next, we derive some results on $\mathfrak{X}$-groups of infinite rank in which all proper subgroups of infinite rank are $\mathfrak{F}\mathfrak{N}$ (resp. $\mathfrak{C}\mathfrak{N}$, $(\mathfrak{P}\mathfrak{F})\mathfrak{N}$, $\mathfrak{M}\mathfrak{N}$).
The class $\mathfrak{F}$ of finite groups satisfies the hypotheses of Theorem 2.2.4; moreover, MN$\mathfrak{F}\mathfrak{N}$ of infinite rank are MN$\mathfrak{N}$-groups without maximal subgroups by the results quoted in Theorem 1.2.3.2, 1.2.2.1. The following result is valid.

Corollary 2.2.5. *Let G be an $\mathfrak{X}$-group of infinite rank. If all proper subgroups of infinite rank of G are in $\mathfrak{F}\mathfrak{N}$, then G is either in $\mathfrak{F}\mathfrak{N}$ or it is a* MN$\mathfrak{N}$*-group without maximal subgroup.*

Since the class $\mathfrak{P}\mathfrak{F}$ of polycyclic-by-finite groups satisfies the hypotheses of Theorem 2.2.4. Then by Theorem 1.2.4.1 and Corollary 2.3.5, we have the following consequence.

Corollary 2.2.6. *Let G be an $\mathfrak{X}$-group of infinite rank. If all proper subgroups of infinite rank of G are in $(\mathfrak{P}\mathfrak{F})\mathfrak{N}$, then G is either in $(\mathfrak{P}\mathfrak{F})\mathfrak{N}$ or it is a* MN$\mathfrak{N}$*-group without maximal subgroup.*

The class $\mathfrak{C}$ of Chernikov groups also satisfies the hypotheses of Theorem 2.2.4, together with the result quoted in Theorem 1.2.5.1, we deduce the following result.

Corollary 2.2.7. *Let G be $\mathfrak{X}$-group of infinite rank. If all proper subgroups of infinite rank of G are in $\mathfrak{C}\mathfrak{N}$, then G is either in $\mathfrak{C}\mathfrak{N}$ or it is a* MN$\mathfrak{N}$*-group without maximal subgroup.*

As the class $\mathfrak{M}$ of soluble-by-finite minimax groups satisfies the hypotheses of Theorem 2.2.4, 2.2.3, we have the following consequence.

Corollary 2.1.8. *Let G be $\mathfrak{X}$-group of infinite rank. If all proper subgroups of infinite rank of G are in $\mathfrak{M}\mathfrak{N}$, then G is either in $\mathfrak{M}\mathfrak{N}$ or it is a* MN$\mathfrak{N}$*-group without maximal subgroup.*

2.3 The class of $\mathfrak{N}\mathfrak{M}$-groups

In this section, we consider the dual property, that is the case in which proper subgroups of infinite rank of an $\mathfrak{X}$-group of infinite rank are $\mathfrak{N}\mathfrak{C}$ (resp. $\mathfrak{N}(\mathfrak{P}\mathfrak{F})$, $\mathfrak{N}\mathfrak{M}$). Our results should be seen in relation with the structure of MN$\mathfrak{N}\mathfrak{C}$-groups (resp. MN$\mathfrak{N}(\mathfrak{P}\mathfrak{F})$-groups, MN$\mathfrak{N}\mathfrak{M}$-groups).

In view of Theorem 1.2.5.2, locally graded groups whose proper subgroups are in $\mathfrak{N}\mathfrak{C}$ are themselves $\mathfrak{N}\mathfrak{C}$-groups. A corresponding result is proved for groups whose proper subgroups are $\mathfrak{N}\mathfrak{M}$, where $\mathfrak{M}$ is the class of soluble-by-finite minimax groups. The proof requires a few lemmas.

The following result is a key ingredient in the proof of Lemma 2.3.2.

Lemma 2.3.1. *Let $\mathfrak{Y}$ be a class of groups contained in the class of finite rank groups which is* **H***-closed. If G is a $\mathfrak{N}\mathfrak{Y}$-group, then G has a characteristic nilpotent subgroup N such that G/N is a $\mathfrak{Y}$-group.*

Proof. We follow almost the same way of the proof of [28, Lemma 1]. Let A be a normal nilpotent subgroup of G of nilpotency class c such that G/A is a $\mathfrak{Y}$-group and has rank

r. Consider the characteristic closure of A in G, and we write $\overline{A} = \langle \sigma(A), \sigma \in Aut(G) \rangle$. Then $\overline{A}$ is characteristic in G and $G/\overline{A}$ is a $\mathfrak{Y}$-group. Since $\sigma(A) \lhd G$ and $\sigma(A) \simeq A$ for all $\sigma \in Aut(G)$, $\overline{A}$ is generated by normal nilpotent subgroups and hence it is locally nilpotent. Let H be a finitely generated subgroup of $\overline{A}$–it suffices to prove that the nilpotency class of H is bounded by a function of r and c. The subgroup H is generated by some products of elements in the $\sigma(A)$. Therefore, H is contained in a subgroup H_1 generated by finitely many elements in the $\sigma(A)$. Now let K be a normal subgroup of H_1 with H_1/K a finite p-group for some prime p. Applying the Burnside Basis Theorem [49, 5.3.2] on $H_1/(H_1 \cap A)K$, gives that H_1/K is nilpotent of class at most $(r+1)c$. Since H is finitely generated nilpotent, the intersection of all such subgroups K (as p varies) is trivial and the lemma is proved.

Lemma 2.3.2. *Let $\mathfrak{Y}$ be a class of groups contained in the class of finite rank groups and $\{\mathbf{H}, \mathbf{P}\}$-closed. If G is a group containing a normal $\mathfrak{N}\mathfrak{Y}$-subgroup N such that G/N is in $\mathfrak{Y}$, then G is in $\mathfrak{N}\mathfrak{Y}$.*

Proof. By Lemma 2.3.1, there exists a characteristic nilpotent subgroup B of N such that N/B is a $\mathfrak{Y}$-group. Since G/N and N/B are $\mathfrak{Y}$-groups, and the class $\mathfrak{Y}$ is $\mathbf{P}$-closed, we deduce that G/B is likewise a $\mathfrak{Y}$-group. It follows that G is a $\mathfrak{N}\mathfrak{Y}$-group. ■

Corollary 2.3.3. *Let G be a group whose proper normal subgroups belong to $\mathfrak{N}\mathfrak{M}$.* If G is imperfect, then G itself belongs to $\mathfrak{N}\mathfrak{M}$.

Proof. If the quotient group G/G' is decomposable, then $G = MN$ is a product of two proper normal subgroups M and N. Since M and N are $\mathfrak{N}\mathfrak{M}$-groups, G is such as well by Lemma 2.3.1 and Lemma 2.2.1. Now let G/G' be an indecomposable group. Then by [41, Lemma 2.9] G/G' is a Chernikov group and so G lies in $\mathfrak{N}\mathfrak{M}$ by Lemma 2.3.2. ■

For the next result, we recall that if G be a group and H is a subgroup of G, *the isolator*

$I_G(H)$ of H in G is the set,

$$I_G(H) = \{g \in G : g^n \in H \text{ for some } n \geq 1\}.$$

For a good account of the properties of isolators the reader is referred to [36, Section 2.3] (see also [46]). A group in which the isolator of every subgroup is a subgroup is said to satisfy the isolator property. Thus a locally nilpotent group G satisfies the isolator property and if in addition G is torsion-free, then a subgroup H is hypercentral or soluble implies that $I_G(H)$ also is hypercentral or soluble respectively by [36, 2.3.9 (ii) and (iv)].

Lemma 2.3.4. *Let G be a locally nilpotent group whose proper subgroups are in $\mathfrak{NM}$. Then $G \in \mathfrak{NM}$.*

Proof. Assume for a contradiction that G is a MN$\mathfrak{NM}$-group. Then G is perfect and has no non-trivial finite images. Since the class of soluble groups is countably recognizable, G is countable. Let T be the torsion subgroup of G and assume that $T \neq G$. Then G/T is a countable locally nilpotent torsion-free group, so it admits a proper subgroup H/T with $I_{G/T}(H/T) = G/T$. Now H (and H/T) is soluble, whence G/T is soluble by [36, 2.3.9 (iv)], contradicting our assumption. Thus $T = G$, and so G lies in $\mathfrak{NC}$ by Theorem 1.2.5.2. ■

The following lemma is a part of Proposition 1 and Theorem 2 of [30].
We recall that the *Hirsch-Plotkin radical* of a group G is the unique maximal normal locally nilpotent subgroup of G containing all the ascendant locally nilpotent subgroups of G (see [49, 12.1.4]).

Lemma 2.3.5. *Let G be a countable infinite simple locally (soluble-by-finite) group. Then G contains a locally soluble and residually finite proper subgroup R such that G is periodic over R, that is, for every $g \in G$ there exists an integer $n > 0$ such that $g^n \in R$.*

Moreover, if R has non-trivial Hirsch-Plotkin radical, then G is locally finite.

We are now ready to prove the following result.

Theorem 2.3.6. *Let G be a locally graded group whose proper subgroups are in $\mathfrak{NM}$. Then $G \in \mathfrak{NM}$.*

Proof. Suppose for a contradiction that G is a MN$\mathfrak{NM}$-group. In particular, G has no non-trivial finite images, see Lemma 2.3.2. Moreover, from Corollary 2.3.3 and Lemma 2.3.4, we deduce that G is perfect non-locally nilpotent. Let V be the Hirsch-Plotkin radical of G. Of course, the factor group G/V is a MN$\mathfrak{NM}$-group and hence it is likewise a counterexample. Replacing G by G/V, it can be assumed without loss of generality that all ascendant subgroups of G are of finite rank. It is easy to see, using [28, Theorem 2], that if G has no simple images, then it is radicable nilpotent. By this contradiction G contains a normal subgroup M such that $\overline{G} = G/M$ is an infinite simple group.
Since the class of (locally soluble)-by-finite groups is countably recongnizable by [26, Lemma 3.5] and $\overline{G}$ has no non-trivial finite images, we deduce that $\overline{G}$ is countable. It follows that the hypotheses of Lemma 2.3.5 are satisfied, and let $\overline{R}$ be the proper subgroup of $\overline{G}$ as described in Lemma 2.3.5. So $\overline{R}$ belongs to $\mathfrak{NM}$ and hence, as $\overline{G}$ is non-locally finite by Theorem 1.2.5.2, the Hirsch-Plotkin radical of $\overline{R}$ is trivial. It follows that $\overline{R}$ is finite and therefore $\overline{G}$ is periodic, as $\overline{G}$ is periodic over $\overline{R}$. But $\overline{G}$ is non-locally finite, and we have a contradiction that establishes the result. ■

It is known that a strongly locally graded group of infinite rank whose proper subgroups of infinite rank are $\mathfrak{AF}$-groups have all their proper subgroups $\mathfrak{AF}$-groups (see Theorem 1.3.6). In the last part we prove that similar results are holding for $\mathfrak{X}$-groups of infinite rank in which the proper subgroups of infinite rank have certain slightly stronger properties.
Note that the main tool we used in the proof of our theorems is the description of infinite locally finite simple groups in which all proper subgroups are (locally soluble)-by-finite

given in [35, 43] and without this description our results would be difficult to prove.

Lemma 2.3.7. *If G is a locally (soluble-by-finite) group of infinite rank whose proper subgroups of infinite rank are hypercentral-by-$\mathfrak{M}$, then G is not simple.*

Proof. Assume for a contradiction that G is simple. Note that the proper subgroups of G are either locally (soluble-by-finite) of finite rank or hypercentral-by-$\mathfrak{M}$. Since G is locally (soluble-by-finite), we deduce that the hypercentral-by-$\mathfrak{M}$ subgroups are (locally soluble)-by-finite. Therefore all proper subgroups of G are (locally soluble)-by-finite. We deduce that G is countable as the class of (locally soluble)-by-finite groups is countably recognizable by [26, Lemma 3.5]. The hypotheses of Lemma 2.3.5 are satisfied and so G has a proper subgroup R as described in Lemma 2.3.5. If R has a non-trivial normal locally nilpotent subgroup then its Hirsch-Plotkin radical is non-trivial, while if R has finite rank then its Hirsch-Plotkin radical is also non-trivial by Lemma 1.3.1. We deduce, using Lemma 2.3.5, that G is locally finite. By [35], G is isomorphic to either $PSL(2, F)$ or $Sz(F)$ for some infinite locally finite field F containing no infinite proper subfield. But each of these groups has a proper non hypercentral-by-Chernikov subgroup of infinite rank [43], our final contradiction. ■

Theorem 2.3.8. *Let $\mathfrak{Y}$ be a subclass of the class of soluble-by-finite minimax groups which is $\{\mathbf{P}, \mathbf{S}, \mathbf{R}_0\}$-closed. If G is a perfect $\mathfrak{X}$-group of infinite rank whose proper subgroups of infinite rank are in $\mathfrak{N}\mathfrak{Y}$, then all proper subgroups of G are in $\mathfrak{N}\mathfrak{Y}$.*

Proof. Assume for a contradiction that the statement is false. Let N be a proper normal subgroup of finite index in G. So there is a normal nilpotent subgroup A of N such that N/A is a $\mathfrak{Y}$-group. Since A has only finitely many conjugates in G and $\mathfrak{Y}$ is $\mathbf{R}_0$-closed, N/A_G is a $\mathfrak{Y}$-group, where, of course, A_G is the core of A in G. But then, as A_G is nilpotent and G/N belongs to $\mathfrak{Y}$, G is a $\mathfrak{N}\mathfrak{Y}$-group. Hence G has no non-trivial finite images and so it cannot be finitely generated. Now using [16], all proper subgroups of G are (locally soluble)-by-finite; in particular G is locally (soluble-by-finite).

Let N be a proper normal subgroup of infinite rank of G. If G/N has finite rank, then, in particular, all its proper subgroups are $\mathfrak{NM}$-groups, and hence G/N itself a $\mathfrak{NM}$-group by Theorem 2.3.6. Then G/N is soluble, a contradiction, because G is perfect. It follows that G/N has infinite rank, so that if H is a proper subgroup of G of finite rank, then HN is a proper subgroup of infinite rank. We deduce that HN belongs to $\mathfrak{NY}$, and so H is a $\mathfrak{NY}$-group. Therefore all proper normal subgroups of G have finite rank, and hence G contains a normal subgroup M such that $\overline{G} = G/M$ is a simple group of infinite rank [28, Theorem 2]. But Lemma 2.3.7 shows that $\overline{G}$ cannot be simple, and we have a contradiction that completes the proof of the theorem. ■

The following result shows that the subgroups of infinite rank of a group G force the behaviour of its normal subgroups.

Lemma 2.3.9. *Let $\mathfrak{Y}$ be a subgroup closed class of groups. If G is a strongly locally graded group of infinite rank whose proper subgroups of infinite rank are $\mathfrak{Y}$-groups, then each proper normal subgroup of G is a $\mathfrak{Y}$-group.*

Proof. Let M be a proper normal subgroup of G, and it can be obviously assumed that M has finite rank. Then the factor group G/M has infinite rank, and Lemma 1.3.4 ensures that G/M contains a proper subgroup of infinite rank, say K/M. Thus K, and hence M, is a $\mathfrak{Y}$-group. ■

Now combining Theorem 2.3.6 and 2.3.8, we obtain the following consequence.

Corollary 2.3.10. *Let G be an $\mathfrak{X}$-group of infinite rank. If all proper subgroups of infinite rank of G are in $\mathfrak{NM}$, then G itself is in $\mathfrak{NM}$.*

Proof. Clearly we may assume that G is infinitely generated, see Lemma 2.3.2. So G is locally (soluble-by-finite) and by Lemma 2.3.9, we conclude that all proper normal subgroups are in $\mathfrak{NM}$. By Corollary 2.3.3 we may assume that G is perfect. Application of Theorem 2.3.8 yields that all proper sugroups of G belong to $\mathfrak{NM}$, so that G itself is in $\mathfrak{NM}$ by Theorem 2.3.6. ■

Following a similar argument as in Corollary 2.3.10 by using Theorem 2.3.8 and 1.2.5.2, we can deduce the following result.

Corollary 2.3.11. *Let G be an $\mathfrak{X}$-group of infinite rank. If all proper subgroups of infinite rank of G are in $\mathfrak{NC}$, then G itself is in $\mathfrak{NC}$.*

We turn now to $\mathfrak{X}$-groups of infinite rank whose proper subgroups of infinite rank are in $\mathfrak{N}(\mathfrak{PF})$, but we are immediately confronted with the non-perfect case of these groups. The objective, then, is to settle that situation.

Lemma 2.3.12. *If G is a non-perfect group of infinite rank whose proper subgroups of infinite rank belong to $\mathfrak{N}(\mathfrak{PF})$. Then either $G \in \mathfrak{N}(\mathfrak{PF})$-group or G/G' is quasicyclic and G' is nilpotent.*

Proof. Assume that G does not belong to $\mathfrak{N}(\mathfrak{PF})$. In particular, G has no non-trivial finite images and so it is locally (soluble-by-finite). By Lemma 2.3.9, all proper normal subgroups of G are in $\mathfrak{N}(\mathfrak{PF})$. If the factor group G/G' is decomposable, then $G = MN$ is a product of two proper normal subgroups M and N. Since M and N are $\mathfrak{N}(\mathfrak{PF})$-groups, G is such as well by Lemma 2.3.1 and Lemma 2.2.1. Now let G/G' be an indecomposable group. Then by [41, Lemma 2.9] G/G' is a quasicyclic group. Let L be the G-invariant nilpotent subgroup of G' such that G'/L is polycyclic-by-finite. Then $(G/L)/C_G(G'/L)$ is polycyclic-by-finite by [48, Theorem 3.27]. So that $C_G(G'/L) = G/L$, as G has no finite images, and hence the factor group G/L is centre-by-locally cyclic. It follows that G/L is even abelian, and so $G' = L$ is nilpotent. ■

The next lemma is useful for our purpose, was proved by B. Bruno and R.E. Phillips (see [9, Lemma 2.3]). Before stating it, we have to recall some notions we shall use. Let G be a group and A an additive abelian group. We say, following [54, Chapter 3], that A is a *right G-module* if there is a map of $A \times G$ into A, denoted by $(a, g) \to ag$, such that

$$(a_1 + a_2)g = a_1 g + a_2 g,\ \ (ag_1)g_2 = a(g_1 g_2) \text{ and } a1 = a.$$

for all a, a_1, a_2 in A and g, g_1, g_2 in G. The first identity here says that $g \in G$ defines an endomorphism φ_g of the abelian group A. The other two imply that $(ag)g^{-1} = a(gg^{-1}) = a$, so φ_g is actually an automorphism of A. The second identity then says that $\varphi : g \to \varphi_g$ is a (group) homomorphism of G into the automorphism group $Aut_{\mathbf{Z}}A$ of A. Observe that φ can be extended to a ring homomorphism $\tilde{\varphi}$ of $\mathbf{Z}G$ (the free $\mathbf{Z}$-module with basis G) into $End_{\mathbf{Z}}A$. Define an action of $\mathbf{Z}G$ on A by $ax = a(x\tilde{\varphi})$ for $a \in A$ and $x \in \mathbf{Z}G$. Then A becomes a *right* $\mathbf{Z}G$*-module*. Hence this amounts to gives a $\mathbf{Z}G$-module structure on A. To summarize, if A is a right G-module, then A becomes a right $\mathbf{Z}G$-module by linearly extending, that is by setting $a(\sum_g \alpha_g g) = \sum_g \alpha_g(ag)$, where the α_g lie in $\mathbf{Z}$ and almost all are zero. Conversely if A is a right $\mathbf{Z}G$-module, then A becomes a right G-module simply by restricting the acting set from $\mathbf{Z}G$ to G.

Recall also that, for any group G, the symbol $\pi(G)$ denotes the set of all prime numbers p such that G has elements of order p.

Lemma 2.3.13. *Let G be a periodic abelian group and $A \neq \{0\}$ be a $\mathbf{Z}G$-module which is torsion-free as a group. Then for any finite set π of prime numbers, there is a $\mathbf{Z}G$-submodule N of A such that the quotient module A/N is periodic as a group and, for all p in π, contains an element of order p.*

Lemma 2.3.14. *Let E be group, and let A be a torsion-free abelian non-trivial normal subgroup of E such that E/A is periodic. Then for any finite set π of prime numbers, there exists an E-invariant subgroup N of A such that A/N is periodic and π is contained in $\pi(A/N)$.*

Proof. Set $G = E/A$ and consider the action of G on A by $aeA = e^{-1}ae$ for $a \in A$ and $e \in E$. Then A becomes a right G-module and the linearly extending gives the $\mathbf{Z}G$-module structure on A. Since the submodules of A are exactly the normal subgroups of E contained in A then, using Lemma 2.3.13, we deduce that N contains an E-invariant subgroup A such that A/N is periodic and, for all p in π, contains an element of order p.

Theorem 2.3.15. *Let G be an $\mathfrak{X}$-group of infinite rank. If all proper subgroups of infinite rank of G are $\mathfrak{N}(\mathfrak{PF})$, then all proper subgroups of G are $\mathfrak{N}(\mathfrak{PF})$.*

Proof. Assume for a contradiction that the statement is false. In particular, G has no non-trivial finite images and an application of Theorem 2.3.8 and Lemma 2.3.12 yields that G is imperfect and G/G' is quasicyclic with G' is nilpotent. Suppose first that G' is periodic, then G is periodic and hence all it's proper subgroups of infinite rank are nilpotent-by-finite, the result follows by [21]. Thus G' is non-periodic and by [49, 5.2.6], $G'/\gamma_2(G')$ is non-periodic. Let $T/\gamma_2(G')$ be the torsion subgroup of $G'/\gamma_2(G')$. So G'/T is non-trivial, abelian and torsion-free. Now G/G' is a periodic abelian (in fact quasicyclic) group, and G'/T is abelian and torsion-free group. Hence we can apply Lemma 2.3.14 and get that for each pair of primes p_1 and p_2 with $p_1 \neq p_2$, there exists a G-invariant subgroup M of G', such that $T < M$ and G'/M is an abelian group contaning elements of orders p_1 and p_2. If the periodic group G/M has finite rank then all its proper subgroups are $\mathfrak{NF}$-groups, while if G/M has infinite rank then all its proper subgroups are also $\mathfrak{NF}$-groups by Theorem 1.3.6. We deduce that G/M is either in $\mathfrak{NF}$ or it is a MN$\mathfrak{NF}$-group. Assume that G/M is in $\mathfrak{NF}$, then it is nilpotent. It follows, as G has no proper subgroup of finite index, that G/M is even abelian, and so $G' = M$. Hence G/M is a MN$\mathfrak{NF}$-group. But the commutator subgroup of G/M cannot be cannot be a p-group for any prime p, contradicting Theorem 1.2.3.3. The statement is proved. ■

We may, therefore, draw the following consequence.

Corollary 2.3.16. *Let G be an $\mathfrak{X}$-group of infinite rank. If all proper subgroups of infinite rank of G are in $\mathfrak{N}(\mathfrak{PF})$, then G is either in $\mathfrak{N}(\mathfrak{PF})$ or it is a* MN$\mathfrak{NF}$*-group.*

Chapter 3

Groups in which every proper subgroups of infinite rank is in $\mathfrak{C}$ZA or ZA$\mathfrak{C}$

3.1 Introduction

In this chapter, we focus on the influence of the proper subgroups of infinite rank on the structure of the groups themselves. The description of groups of infinite rank in which the proper subgroups of infinite rank satisfy the condition of being $(\mathbf{L}\mathfrak{F})(\mathbf{L}\mathfrak{N})$ was given by F. de Giovanni and M. Trombetti. In particular, they proved that strongly locally graded groups of infinite rank with this property are themselves belong to $(\mathbf{L}\mathfrak{F})(\mathbf{L}\mathfrak{N})$, and hence so does all their proper subgroups (see Theorem 1.3.8). Here we will extend this result by proving that $\mathfrak{X}$-groups of infinite rank whose proper subgroups of infinite rank are in the class $\mathfrak{C}$ZA have all their proper subgroups $\mathfrak{C}$ZA-groups. Moreover, a similar result is obtained for groups whose proper subgroups of infinite rank are in the class ZA$\mathfrak{C}$.

3.2 The class of $\mathfrak{C}$ZA-groups

In this section we give the characterization of $\mathfrak{X}$-groups of infinite rank in which the proper subgroups of infinite rank are $\mathfrak{C}$ZA-groups.
First, we need some preliminary results.

Lemma 3.2.1. *If G is a $\mathfrak{C}$ZA-group, then G has a characteristic Chernikov subgroup N such that G/N is hypercentral.*

Proof. Let C be a normal Chernikov subgroup of G such that G/C is hypercentral. Consider the non-empty set,

$$\mathcal{H} := \{E \mid E \leq C,\, E \lhd G \text{ and } G/E \text{ is hypercentral}\}.$$

Since C satisfies the minimal condition on subgroups, the set $\mathcal{H}$ admits a minimal element, noted M. Let $\alpha \in Aut(G)$. Then $\alpha(M)$ is normal in G and $G/\alpha(M) \simeq G/M$. We next consider the map $\Psi : G \to G/M \times G/\alpha(M)$ with $g \mapsto (gM, g\alpha(M))$, so $Ker\Psi = M \cap \alpha(M)$, and therefore $G/M \cap \alpha(M) \simeq \operatorname{Im}\Psi$. Since the direct product of two hypercentral groups is hypercentral, we deduce that $\operatorname{Im}\Psi$ is hypercentral and hence so is $G/M \cap \alpha(M)$. Then $M = M \cap \alpha(M)$ by minimality of M, and it follows that $M \leq \alpha(M)$. But $\alpha^{-1} \in Aut(G)$, then $M \leq \alpha^{-1}(M)$. Therefore G contains a characteristic Chernikov subgroup M such that G/N is hypercentral. ■

By Lemma 3.2.1, one can prove the following lemma.

Lemma 3.2.2. *Let G be a group without proper subgroups of finite index. If N is a normal subgroup of G in the class $\mathfrak{C}$ZA, then N is hypercentral.*

Proof. Since N is in $\mathfrak{C}ZA$, then it has a characteristic Chernikov subgroup K such that N/K is hypercentral. Let D be the divisible part of K. Put $\overline{G} = G/D$ so that $\overline{K}$ is a finite normal subgroup of $\overline{G}$, and hence $\overline{G}/C_{\overline{G}}(\overline{K})$ is finite. Since $\overline{G}$ is $\mathfrak{F}$-perfect we have that $\overline{K}$ is central in $\overline{G}$, and we deduce that $\overline{N}$ is an hypercentral group. Now let D_q

denote the q-component of D, for some prime q. By [53, Lemma 2.4], the automorphism group of a divisible abelian q-group of finite rank is residually finite, so $G/C_G(D_q)$ is trivial. Hence D_q is central in G and so does D. It follows that N is hypercentral. ■

Now we give a general result on $\mathfrak{X}$-groups.

Lemma 3.2.3. *Let G be a group containing a proper normal subgroup N such that G/N is an $\mathfrak{X}$-group of finite rank. Then G has non-trivial homomorphic image which is either finite or abelian.*

Proof. Assume that G has no non-trivial finite images. By [16], G/N is (locally soluble)-by-finite, and hence it is locally soluble. We deduce by Lemma 1.3.1 that there exists a positive integer n such that $(G/N)^{(n)}$ is hypercentral, and it follows that the commutator subgroup G' is properly contained in G. ■

Our next result is useful, in order to reduce our arguments to certain special situation.

Lemma 3.2.4. *Let G be a locally (soluble-by-finite) group of infinite rank. If all proper subgroups of infinite rank of G are in* $\mathfrak{C}$ZA*, then G is either in* $\mathfrak{C}$ZA *or it is locally finite.*

Proof. Assume for a contradiction that G is neither a $\mathfrak{C}$ZA-group nor locally finite. As all proper subgroups of infinite rank of G belong to (L$\mathfrak{F}$)(L$\mathfrak{N}$), it follows from Theorem 1.3.8 that G itself belongs to (L$\mathfrak{F}$)(L$\mathfrak{N}$). Let T be the torsion part of G. Then T is locally finite and G/T is locally nilpotent and torsion-free. It is easy to see, using Lemma 1.3.2 (i), that all proper subgroups of G/T are hypercentral. Hence G/T is hypercentral by Theorem 1.2.7.2 (i). Since G/G' is periodic by Lemma 3.2.1 and 2.2.1, we deduce that G/T is also periodic, a contradiction. Therefore G is as claimed. ■

Lemma 3.2.5. *Let G be a locally finite group of finite rank having a non-trivial finite homomorphic image. If all proper subgroups of G are $\mathfrak{C}$ZA-group, then so is G.*

Proof. Let $G = FN$, where N is a proper normal subgroup of G and F is a finite subgroup of G. Then N has a characteristic subgroup K, $K \in \mathfrak{C}$, such that N/K is hypercentral;

there is no loss of generality in assuming that $K = 1$ and so N is an hypercentral group. Since locally finite p-groups of finite rank are Chernikov groups, we may assume that N is the direct product of infinitely many maximal p-subgroups for different primes p, and hence we can write $N = Dr_{p\in\pi(N)}N_p$; where $\pi(N)$ is the set of primes dividing the orders of the elements of N. Now we choose $q \in \pi(N)$ such that $q \nmid |F|$. Write $N = N_q \times A$, where $A := Dr_{q\neq p\in\pi(N)}N_p$. Then we have that $B := FA$ is a proper subgroup of G and hence it is a $\mathfrak{C}ZA$-group. Therefore G itself is a $\mathfrak{C}ZA$-group and the result follows. ■

There is no doubt that a major difficulty one encounters in proving theorems concerned groups of infinite rank in which proper subgroups of infinite rank are $\mathfrak{Y}$-groups, is the possible presence of simple groups. However, our next lemma shows that this situation cannot occur in our case.

Lemma 3.2.6. *If G is a locally finite group of infinite rank whose proper subgroups of infinite rank are $\mathfrak{C}$ZA-groups, then G is not simple.*

Proof. Assume for a contradiction that G is simple. First we claim that periodic Chernikov-by-hypercentral subgroups are hypercentral-by-Chernikov. Let H be a Chernikov-by-hypercentral subgroup of G. Then H has a normal Chernikov subgroup A such that H/A is hypercentral. Since G is periodic, we deduce that $H/C_H(A)$ is a Chernikov group by [48, Theorem 3.29]. Now the group $C_H(A)/A \cap C_H(A)$ is hypercentral, as H/A is hypercentral. On the other hand, $H \cap C_H(A)$ is contained in the centre of $C_H(A)$, and hence $C_H(A)$ is hypercentral. Consequently, H is hypercentral-by-Chernikov. Moreover, as locally finite groups of finite rank are (locally soluble)-by-finite [16], we deduce that every proper subgroup of G is (locally soluble)-by-finite. Hence G is isomorphic either to $PSL(2, F)$ or to $Sz(F)$ for some some infinite locally finite field (see [35]). But each of these groups has a proper non-(hypercentral-by-Chernikov) of infinite rank [43], and therefore a proper non-(Chernikov-by-hypercentral) of infinite rank. This leads to a contradiction. ■

Our final requirement for the proof of Theorem 3.2.8 is the following.

Lemma 3.2.7. *Let G be a locally finite group of infinite rank whose proper subgroups of infinite rank are $\mathfrak{C}$ZA-groups. If G has a non-trivial finite image, then all proper subgroups of G are $\mathfrak{C}$ZA-groups.*

Proof. Assume for a contradiction that the lemma is false. Let N be a proper normal subgroup with G/N is finite. Then N is a $\mathfrak{C}$ZA-group and, by Lemma 3.2.2, there is no loss in assuming that N is hypercentral. Hence N is the direct product of its maximal p-subgroups, and we can write $N = Dr_{p\in\pi(N)}N_p$, where $\pi(N)$ is the set of primes dividing the orders of the elements of N. Let p a prime that belongs to $\pi(N)$ such that $r(N_p) = \infty$. If the factor group G/N_p has infinite rank, then XN_p is a proper subgroup of infinite rank of G, where X is a proper subgroup of finite rank of G. We deduce that XN_p and fortiori X is a $\mathfrak{C}$ZA-group. This contradiction shows that G/N_p has finite rank, so that its all proper subgroups are in the class $\mathfrak{C}$ZA, and by Lemma 3.2.5 G/N_p is itself in $\mathfrak{C}$ZA. Since X is not contained in any proper subgroup of infinite rank, $G = XN_p$. Then $X/X \cap N_p$ is a $\mathfrak{C}$ZA-group. Moreover, $X \cap N_p$ is a locally finite p-subgroup of finite rank of G, so it is Chernikov. Thus X is a $\mathfrak{C}$ZA-group, and this contradiction shows that for all primes p belong to $\pi(N)$, N_p have finite but unbounded ranks. We can write $N = B \times C$, where B has bounded rank and C is a direct product of countably many maximal p-subgroups, the bound is strictly increasing i.e, $C = F_1 \times F_2 \times F_3 \times \cdots$, where F_i is a p_i-group, p_i's are distinct primes, $r(F_i)$ of each F_i is finite, $r(F_i) < r(F_j)$ if $i < j$ and $r(F_i)$'s are unbounded. Let $C = T \times R$, where $T = F_1 \times F_3 \times F_5 \times \cdots$ and $R = F_2 \times F_4 \times F_6 \times \cdots$. Then, T and R are proper subgroups of infinite rank of G. It follows that XT is proper subgroup of infinite rank of N. This last contradiction completes the proof. ■

It is now easy to prove the main result of this section. It shows in particular that groups of infinite rank in which all proper subgroups of infinite rank are in the class $\mathfrak{C}$ZA have all their proper subgroups in the class $\mathfrak{C}$ZA.

Theorem 3.2.8. *Let G be an $\mathfrak{X}$-group of infinite rank. If all proper subgroups of infinite rank of G are in the class $\mathfrak{C}$ZA, then G is either in $\mathfrak{C}$ZA or it is a* MN$\mathfrak{C}$ZA-group.

Proof. If G is finitely generated then since it is locally graded it contains a proper normal subgroup of finite index. It follows that G is (Chernikov-by-polycyclic)-by-finite, which gives the contradiction that G has finite rank. Therefore finitely generated subgroups of G are proper, so that they are either Chernikov-by-hypercentral or $\mathfrak{X}$-groups of finite rank and hence soluble-by-finite by [16]. It follows that G is locally (soluble-by-finite). Assume for a contradiction that the statement is false, so that in particular G has a proper subgroup H of finite rank which is not a $\mathfrak{C}$ZA-group. Lemma 3.2.4 implies that G is locally finite. Let us suppose first that the commutator subgroup G' is properly contained in G. By Lemma 3.2.1 and Lemma 3.2.7, we deduce that G/G' is quasicyclic. An application of Lemma 3.2.2 yields that all proper normal subgroups of G are hypercentral. In particular, $\langle G', E\rangle$, is hypercentral for all finitely generated subgroups E of G, and we see that G is locally nilpotent. Now a locally nilpotent group of finite rank is hypercentral and so all proper subgroups of G are in the class $\mathfrak{C}$ZA. This contradiction shows that G is a perfect group. Let N be any normal subgroup of infinite rank of G. Then the rank of the quotient G/N is finite. This leads to the contradiction that G has a finite non-trival homomorphic image. Thus all proper normal subgroups of G have finite rank. By Lemma 3.2.6, G has no simple images. Hence we may apply [28, Theorem 2] to deduce that G is nilpotent, and we have a contradiction that establishes the result. ■

3.3 The class of ZA$\mathfrak{C}$-groups

The aim of this section is to prove that $\mathfrak{X}$-groups of infinite rank in which all proper subgroups of infinite rank are ZA$\mathfrak{C}$-groups have all their proper subgroups ZA$\mathfrak{C}$-groups. It turns out that, the behavior of subgroups of finite rank in these groups can be neglected.

The proof is accomplished through some lemmas that have been already mentioned above.

Theorem 3.3.1. *Let G be an $\mathfrak{X}$-group of infinite rank. If all proper subgroups of infinite rank of G are* ZA$\mathfrak{C}$*, then all proper subgroups of G are* ZA$\mathfrak{C}$*.*

Proof. Since finitely generated subgroups of G of finite rank are soluble-by-finite [16] and since finitely generated hypercentral-by-Chernikov are nilpotent-by-finite, G is locally (soluble-by-finite). Assume for a contradiction that the statement is false. In particular, G has no proper subgroups of finite index. For, if there is a proper normal subgroup N of finite index in G, then N has a normal hypercentral subgroup A such that N/A is Chernikov. Since the normalizer $N_G(A)$ of A in G contains N, the normal closure A^G of A in G is the product of finitely many normal hypercentral subgroups of N and hence A^G is hypercentral. Therefore G is hypercentral-by-Chernikov. Suppose first that all proper normal subgroups of G are of finite rank. Clearly G has no simple images by Lemma 2.3.7 and so Theorem 2 of [28] gives that G is nilpotent, which is a contradiction. Thus G has a proper normal subgroup M of infinite rank. If G/M has infinite rank, then XM is a proper subgroup of infinite rank, where X is a proper subgroup of G of finite rank. So XM and hence X is in the class ZA$\mathfrak{C}$. By this contradiction G/M is a group of finite rank. Then, by Lemma 3.2.3, G has a non-trivial abelian image and so G is not perfect. If G/G' is decomposable, then $G = MN$ is a product of two proper normal subgroups M and N. Since M and N are locally nilpotent-by-Chernikov, G is such as well. We deduce that, by Lemma 1.3.2 (i), X is an ZA$\mathfrak{C}$-group. Therefore G/G' is indecomposable; in particular, G/G' is a quasicyclic group. Now, as G' is an ZA$\mathfrak{C}$-group there exists a normal hypercentral subgroup A of G' such that G'/A is a Chernikov group. Consider $A^G = \langle A^g \mid g \in G\rangle$ the normal closure of A in G. Clearly A^g is an hypercentral normal subgroup of G' for all $g \in G$, since we have $(A^g)^{g'_1} = A^{gg'_1} = A^{g'_2 g} = A^g$, for some $g'_1, g'_2 \in G'$. It follows that A^G is locally nilpotent. But G'/A^G and G/G' are Chernikov and therefore G is locally nilpotent-by-Chernikov. We deduce that, using Lemma 1.3.2

(i), all proper subgroups of G are ZA$\mathfrak{C}$-groups. ■

By this observation it is natural to ask the following question:

Question. Let G be a group with every proper subgroup ZA$\mathfrak{C}$-group. Is G a ZA$\mathfrak{C}$-group? In other words, does there exist a MNZA$\mathfrak{C}$-group?

Chapter 4

Groups with many Engel subgroups

4.1 Introduction

In the previous chapters we attempted to answer the question "What happens if a group G has many $\mathfrak{Y}$-subgroups", for various classes $\mathfrak{Y}$ concerning nilpotency and related ideas. In this chapter we pursue our search for further structure theorems to that question. The structure of locally graded MN$\mathbf{L}\mathfrak{N}$-groups has been described by J. Otal and J.M. Peña; in paticular they showed that such groups are finite and hence they are Schmidt groups (Theorem 1.2.7.1). B. Bruno and R.E. Phillips considered the locally graded MN$\mathfrak{N}_k$-groups, where $\mathfrak{N}_k$ denotes the class of nilpotent groups of class at most the integer $k \geq 0$, which turn out to be just the Schmidt groups (Theorem 1.2.1.7). Our first object is to improve these results by proving that a locally graded minimal non-Engel group is finite and hence is a Schmidt group, and that a locally graded minimal non-k-Engel group is finite where k is a positive integer.

Investigating groups in terms of their subgroups of infinite rank is one of the most flourishing and active areas of research. Thus, M.R. Dixon, M.J. Evans and H. Smith proved that an $\mathfrak{X}$-group of infinite rank in which all proper subgroups of infinite rank are in $\mathbf{L}\mathfrak{N}$ (respectively, in $\mathfrak{N}_k$) is itself belongs to $\mathbf{L}\mathfrak{N}$ (respectively, to $\mathfrak{N}_k$), see Theorem 1.3.5. Here we obtain similar results by showing that an $\mathfrak{X}$-group of infinite rank whose proper subgroups of infinite rank are Engel (respectively, k-Engel) is itself Engel (respectively, k-Engel), where k is a positive integer.

In the case of this chapter we considered independently the above-mentioned topics. This work started when the author visited the Department of Mathematics of the University of Isfahan. Professor Alireza Abdollahi suggested to work with him on minimal non-Engel groups.

We recall that a group G is said to be *Engel* if for every $x, y \in G$ there exists a positive integer n (possibly depending on x, y) such that $[x,_n y] = 1$. Clearly, every locally nilpotent group is an Engel group. However we have the famous examples of E.S. Golod [22] that give for each prime p an infinite 3-generator residually finite p-group with all 2-generator subgroups finite. In particular these groups are non-locally nilpotent Engel. Thus Engel groups represent a rather wide generalization of nilpotent groups. By a *k-Engel group* is meant a group G such that $[x,_k y] = 1$ for all $x, y \in G$. Thus the class of k-Engel groups is the variety determined by the law $[x,_k y] = 1$. For example, a nilpotent group of class k is a k-Engel group. On the other hand , k-Engel groups need not be nilpotent; indeed, for any given prime p, the standard wreath product $C_p wr C_p^{\infty}$ is a $(p+1)$-Engel p-group that is non-nilpotent. A group is a *bounded Engel group* if it is k-Engel for some k.

4.2 The class of Engel groups

In this section, we consider groups which are rich in Engel subgroups. In particular, it turns out that such groups either Engel groups or must be small in some sense.

Our first theorem shows that locally graded minimal non-Engel groups must be finite. The proof will be accomplished in two steps.

Lemma 4.2.1. *Let G be a locally graded group containing a proper normal subgroup N such that G/N is polycyclic and N is a finitely generated Engel group. If all soluble* *factors of G have bounded derived length, then G is polycyclic.*

Proof. Let G be a group as stated. Since N is a finitely generated restrained group, $N^{(i)}$ is finitely generated for each positive integer i [34, Corollary 4]. We claim that N is soluble. We argue by contradiction and suppose that N is non-soluble. Then for each positive integer i, $G/N^{(i)}$ is a proper soluble factor group of G. Therefore there exists a postive integer d such that $N^{(d)} \leq N^{(i)}$ and hence $N^{(d)}$ is perfect. It follows that $N^{(d)} = R$, where R is the soluble residual subgroup of N. Since N/R is soluble, R has no non-trivial soluble images. But R is a non-trivial finitely generated locally graded group, so it must contains a non-trivial finite image. Now [49, Theorem 12.3.4] ensures that such image is nilpotent, a contradiction. Thus N is soluble. Now using [49, Theorem 12.3.3], we get that N is nilpotent and so it is polycyclic.This implies that G is polycyclic, as claimed. ■

Lemma 4.2.2. Let $\mathfrak{Y}$ be a subgroup closed class of groups and let G be a group whose proper normal subgroups are $\mathfrak{Y}$-groups. If G/G' is non-cyclic, then $\langle x^y, x\rangle$ is a $\mathfrak{Y}$-group for all x, y in G.

Proof. Let G be a group whose proper normal subgroups are $\mathfrak{Y}$-groups and let $x, y \in G$. Since the factor G/G' is non-cyclic, then there is a proper normal subgroup K of G such that $\langle x^y, x\rangle \subseteq K$, and hence $\langle x^y, x\rangle$ is a $\mathfrak{Y}$-group. ■

We are now in a position to prove the first main result.

Theorem 4.2.3. *If G is a locally graded minimal non-Engel group, then G is finite minimal non-$\mathfrak{N}_2$ group, where $\mathfrak{N}_2$ denotes the class of nilpotent groups of class at most 2.*

Proof. Let G be a locally graded minimal non-Engel group. Then $G = \langle x, y\rangle$ for some x, y in G. Let N be a proper finitely generated normal subgroup of G such that G/N is finite. Since every finite homomorphic image of G is either a Schmidt group or nilpotent group, we conclude that G/N is polycyclic. Further, Lemma 4.2.2 shows that G/G' is cyclic.

Thus every nilpotent image of G is cyclic. Since locally graded minimal non-(locally nilpotent) groups are finite, we have that each soluble image of G is either nilpotent or a Schmidt group by [49, Theorem 12.3.3]. So such images are soluble of length at most 3. An application of Lemma 4.2.1 yields that G is polycyclic, and hence G is soluble satisfies max. Again by [49, Theorem 12.3.3], we conclude that every proper subgroup of G is nilpotent. But in Theorem 1.2.1.1, it is it is proved that a finitely generated minimal non-$\mathfrak{N}$ group is either finite or its Frattini factor-group is a non-abelian simple group, hence G is finite. Therefore G is a Schmidt group and hence for some primes p and q, there is a unique Sylow p-subgroup P and a Sylow q-subgroup Q such that $G = QP$, Q is cyclic and P is nilpotent of class at most 2. Now let H be any proper subgroup of G. Since H is nilpotent, $H \simeq P_1 \times Q_1$, where Q_1 is cyclic and P_1 is nilpotent of class at most 2. It follows that H is nilpotent of class 2 and hence G is a minimal non-$\mathfrak{N}_2$ group, as required. ■

We turn now to determine the structure of groups of infinite rank in which all proper subgroups of infinite rank are Engel.

Theorem 4.2.4. *Let G be an $\mathfrak{X}$-group of infinite rank. If all proper subgroups of infinite rank of G are Engel, then so is G.*

Proof. Assume that G is not an Engel group. If G is finitely generated, then G contains a proper finitely generated normal subgroup N such that G/N is finite. Since every finite homomorphic image of G is either nilpotent or a Schmidt group, we deduce that G/N is polycyclic and so G is certainly imperfect. Moreover, since G is not an Engel group, there exist $x, y \in G$ such that $\langle x, [x, y]\rangle$ is not an Engel group, and hence $\langle x, G'\rangle$ is not an Engel group. Then the factor group $G/\langle x, G'\rangle$ cannot contain proper subgroups of infinite rank. Now as abelian groups of infinite rank contain proper subgroups of infinite rank, we deduce that $G/\langle x, G'\rangle$ has finite rank, and in particular $\langle x, G'\rangle$ has infinite rank. It follows that $G = \langle x, G'\rangle$, and hence G/G' is cyclic. Suppose that G/M is a soluble image of G of infinite rank. Then, by [49, Theorem 12.3.3], proper subgroups of infinite rank of G/M are locally nilpotent and hence so is G/M by Theorem 1.3.5 (ii). This implies the contradiction that G/M has finite rank. Therefore each soluble image, G/M, of G is of finite rank. It follows that M has infinite rank and so all proper subgroups of G/M are Engel and hence locally nilpotent. We deduce that G/M is either nilpotent or a Schmidt group. Therefore each soluble image of G is of length at most 3. An application of Lemma 4.2.1 yields that G is polycyclic, and in particular G has finite rank, a contradiction. Thus G is infinitely generated. Since $\langle x, y \rangle$ is a non-Engel subgroup of G, we deduce that all finitely generated subgroups of G are of finite rank, since $\langle \langle x, y \rangle, F \rangle$ is also non-Engel, for each finitely generated subgroup F of G. Now we conclude, using [16], that any finitely generated subgroup of G is soluble-by-finite. Hence G is locally (soluble-by-finite). Now let K be any proper subgroup of infinite rank of G and let E be a finitely generated subgroup of K. By [49, Theorem 12.3.4] we have that E is soluble and hence it is nilpotent. Thus, K is locally nilpotent. Again by by Theorem 1.3.5 (ii), we get that G is locally nilpotent. So that G is an Engel group, which is a contradiction. ■

4.3 The class of bounded Engel groups

The aim of this section is to prove that for a positive integer k and a locally graded group, the property of being minimal non-k-Engel is enough to force the finiteness of that group. In the last section, we study groups of infinite rank in which all proper subgroups of infinite rank are k-Engel.

In the first part we give the description of locally graded minimal non-k-Engel groups. Let begin with a very easy result.

Lemma 4.3.1. *Let G be a locally graded k-Engel group. Then G is locally nilpotent.*

Proof. By [34, 33], we have that G is locally (polycyclic-by-finite). But polycyclic-by-finite Engel groups are nilpotent. Hence the result. ■

Theorem 4.3.2. *Let k be a positive integer and let G be a locally graded minimal non-k-Engel group. Then G is finite.*

Proof. Let G be a locally graded minimal non-k-Engel group and assume that G is infinite. By Theorem 4.2.3, G is an Engel group and there exist x, y in G such that $G = \langle x, y \rangle$. Then G contains a normal proper subgroup N such that G/N is finite. Now locally graded k-Engel groups are locally nilpotent by Lemma 4.3.1 and so N is nilpotent. Since every finite homomorphic image of G is either nilpotent or a Schmidt group, G is a polycyclic group. Therefore G is nilpotent by [49, Theorem 12.3.3]. It follows that G has a normal torsion-free subgroup K such that G/K is finite. Let p be any prime number not dividing $|G/K|$. Since K is a residually finite p-group, there exists a family of normal subgroups $(L_i)_{i \in I}$ of K such that K/L_i is a p-group and $\cap_{i \in I} L_i = 1$. Consider the factor group $G/(L_i)_G$ which is a finite nilpotent group with at least two distinct Sylow subgroups. Thus $G/(L_i)_G$ is a k-Engel group, and it follows that $[a,_k b] \leq \cap_{i \in I} (L_i)_G$ for all $a, b \in G$ and so $[a,_k b] = 1$. This implies the contradiction that G is k-Engel. Therefore G is finite, as claimed. ■

Theorem 4.3.3. *Let k be a positive integer and let G be a finite minimal non-k-Engel group. Then G is either a minimal non-$\mathfrak{N}_2$ group or a $(k+1)$-Engel p-group that satisfies $[x^p,_k y] = 1$ for all x, y in G, where p is a prime.*

Proof. Obviously, proper subgroups of G are nilpotent. If G is not nilpotent, then G is a minimal non-$\mathfrak{N}_2$ group by Theorem 4.2.3. Now suppose that G is nilpotent. Since G cannot be the direct product of two proper subgroups, then only one primary component may exist, and so G is a p-group for some prime p. It follows that G/G' is not cyclic since $G' \leq Frat(G)$. Then, by Lemma 4.2.2, we deduce that $\langle x^y, x \rangle$ is k-Engel for all $x, y \in G$, and this implies that G is $(k+1)$-Engel. Now let M be any maximal subgroup of G, then M is normal and has index p. Since $G/Frat(G)$ cannot be cyclic, $\langle x^p, y \rangle \neq G$

and hence is k-Engel, so that $[x^p,_k y] = 1$ for all x, y in G. ■

The consideration of the symetric group of degree 3 and the Quaternion group of order 8 shows that both cases in Theorem 4.3.3 can occur.

Now we analyze groups of infinite rank with k-Engel proper subgroups of infinite rank.

Lemma 4.3.4. *Let G be a locally nilpotent group and let R be a perfect normal subgroup of G. Then $\langle x \rangle^G$ is a proper subgroup of R, for every element x in R.*

Proof. We argue by contradiction and suppose that $R := \langle x \rangle^G$ for some $1 \neq x \in R$. Since R is perfect, $R = R' = [x, G]$. Then $x = [x, g_1]^{\varepsilon_1} \cdot [x, g_2]^{\varepsilon_2} \cdots [x, g_r]^{\varepsilon_r}$ for some g_i in G and $\varepsilon_i = \pm 1$. Put $H := \langle x, g_1, ..., g_r \rangle$ and $A := \langle x \rangle^H$. Then $x \in [A, H]$ and hence $A = [A, H]$. Thus there exists a positive integer n such that $A \leq \gamma_n(H) = 1$ as H is nilpotent, a contradiction. ■

Our next two lemmas are useful in order to reduce our arguments to certain special situations.

Lemma 4.3.5. *Let k be a positive integer and let G be a locally nilpotent group of infinite rank whose proper subgroups of infinite rank are k-Engel. If G is not k-Engel, then G is imperfect and G/G' is 2-generated and hence it has finite rank.*

Proof. Assume that G is perfect. As G is not k-Engel, there exist x, y in G such that $\langle x, y \rangle$ is not k-Engel. It follows from Lemma 2.3.9 that $\langle x, y \rangle$ cannot be contained in any proper normal subgroup of G, and so $G = \langle x, y \rangle^G$. By Lemma 4.3.4 we also have that $H := \langle x \rangle^G < G$. By Zorn's Lemma we conclude that G contains a normal subgroup K such that $H \leq K$ and $y \notin K$ and K is maximal for these conditions. Thus G/K is a simple group and hence it is cyclic of prime order, a contradiction which gives that G is imperfect. Now, since G is not k-Engel, we have that $G = G' \langle x, y \rangle$. Therefore G/G' is a 2-generated group. ■

Lemma 4.3.6. *Let k be a positive integer and let G be a locally nilpotent group of infinite rank whose proper subgroups of infinite rank are k-Engel. If G/G' is finite, then G is k-Engel.*

Proof. Assume that G is not k-Engel. By Lemma 4.3.5, G is imperfect. Let N be a proper normal subgroup of finite index in G. Then N has infinite rank and $G = \langle x, y \rangle N$ for some $x, y \in G$. Thus the nilpotency class of G/N is at most the nilpotency class of $\langle x, y \rangle$. Therefore there exists a positive integer c such that $\gamma_c(G) \leq N$ and hence G/R is nilpotent, where R is the finite residual subgroup of G. Since G/G' is finite, then every nilpotent homomorphic image of G is finite. So R has no non-trivial finite images. Suppose that R is perfect and let $A := a^G$, where a is an arbitrary element of R. If A has infinite rank, then $G = \langle x, y \rangle A$ and hence G/A is finite, a contradiction. Thus A has finite rank. Let T be the torsion subgroup of A. By Lemma 1.3.2 (iii), each maximal p-subgroup T_p of T is Chernikov. Consider the subgroup $M := \langle R_p : p \in \pi(T) \rangle$, where R_p is the finite residual subgroup of T_p. Clearly M is abelian. So in order to prove that A is soluble, one can assume that $M = 1$, so that each T_p is finite and hence it is contained in $Z(R)$ as R has no proper subgroup of finite index. Therefore $T \leq Z(A)$. But A/T, as a torsion-free locally nilpotent group of finite rank, is nilpotent. Consequently, A is soluble. Therefore A has a G-invariant series of finite length whose factors are either elementary abelian p-groups of order $\leq p^r$ for certain primes p, or torsion-free abelian of rank $\leq r$ where r is the rank of A. Let X_i be a factor of the lattest type. Then $R/C_R(X_i)$ embeds as a subgroup in $GL(r, \mathbb{R})$, therefore is soluble [48, Corollary to Theorem 3.23] and hence X_i is contained in the center of R. Now if X_i is finite, then again $X_i \leq Z(R)$. We deduce that $a \in A \leq Z_\infty(R)$. Hence $Z_\infty(R) = R$ a contradiction, since R is perfect. It follows that R is imperfect. Let $N \lneqq R$ such that $R' \leq N$ and R/N is of finite rank. Since R/R' is of finite index in G/R', we have that $K/R' := (N/R')_{G/R'}$ is the intersection of a finite number of conjugates of N/R', so R/K is of finite rank, since the class of finite rank is $\mathbf{R}_0$-closed. Therefore G/K is of finite rank and hence K has infinite rank, so that $G = \langle x, y \rangle K$. We have that G/K is finite as it is nilpotent, this leads to the final contradiction that R has a proper subgroup of finite index. ■

The following remark is nearly obvious and we omit its proof.

Lemma 4.3.7. *Let G be a non-periodic finitely generated abelian group. Then for each prime p, G contains a proper subgroup M such that G/M is a p-group.*

Now we are able to prove our last result.

Theorem 4.3.8. *Let k be a positive integer and let G be an $\mathfrak{X}$-group of infinite rank. If all proper subgroups of infinite rank of G are k-Engel, then G itself is k-Engel.*

Proof. Assume that G is not k-Engel. As locally graded k-Engel groups are locally nilpotent, we deduce that G is also a locally nilpotent group by Theorem 1.3.5 (ii). It follows from Lemmas 4.3.5 and 4.3.6 that the abelian group G/G' is 2-generated non-periodic. Thus we may apply Lemma 4.3.7 and get that for each prime p, there exists a G-invariant subgroup M of G, such that G/M is an abelian p-group. Suppose that $\theta_k(x, y)$ is a k-word of G defined by $\theta_k(x, y) = [x,_k y]$ for some x, y in G, and denote by $\theta_k(G)$ the subgroup generated by all the k-Engel words of G. Notice that G is the p-isolator of M in G and so by [36, Thorem 2.3.5], we get that $\theta_k(G)$ is a p-group, for all primes p, since $\theta_k(M)$ is trivial. Hence we must have that $\theta_k(G) = 1$. Thus G is k-Engel as claimed. ■

Conclusion

One of the classical and broad topics in group theory is defined by the investigations of groups many of whose subgroups have a prescribed property. With researches on this subject, many important notions such as finiteness conditions, locally nilpotency, locally solubility, ranks, and many others have been introduced. Imposing of some natural restrictions on specifically chosen subgroups, we define concrete classes of groups having these properties. Here, we gave a further contribution to this topic by studying groups satisfying conditions somewhat larger than many of those hitherto studied in this context. We were primarily concerned with groups in which all proper subgroups belong to a group class $\mathfrak{Y}$, for certain values of $\mathfrak{Y}$ concerning to nilpotency and its generalizations. We also proved that in a group of infinite rank the behavior of subgroups of finite rank with respect to the main generalizations of nilpotency is neglectable.

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<u>ملخص:</u> تهدف هذه الرسالة إلى دراسة تأثير النظم المعينة للزمر الجزئية لزمرة ما على بنية الزمرة نفسها. إذا كان Y صف من زمر، عندئذٍ يقال أن الزمرة G ليست-Y أصغرية إذا لم تكن Y-زمرة ولكن تنتمي جميع زمرها الجزئية إلى Y. من المعروف أن أي زمرة ليست-A أصغرية متدرجة محليا هي زمرة منتهية، حيث A هو صف الزمر الآبلية. من ناحية أخرى، أعطى كل من هاينيكان، محمد، نيومان، سميث وفيقولد أوصاف زمر غير منتهية ليست-N أصغرية متدرجة محليا، حيث N هي صف الزمر عديمة القوة. هنا نقدم المسائل المذكورة أعلاه إلى الصفوف: MN(NM) و أنجل (k-أنجل)، حيث M هي صف الزمر (قابلة للحل-بواسطة-منتهية) مينيماكس. برهن ديكسون، إيفانز و سميث أن كل زمرة قابلة للحل (معممة) من رتبة غير منتهية حيث جميع زمرها الجزئية من الرتب الغير منتهية هي آبلية هي نفسها آبلية، وبالتالي جميع زمرها الجزئية. توصلنا إلى نتائج مماثلة للصفوف: MN(NM) ، CZA(ZAC) و أنجل (k- أنجل)، حيث يشير ZA و C إلى صف الزمر الفوق مركزية وتشارنيكوف ، على التوالي.

كلمات مفتاحية: فوق مركزية، أنجل، مينيماكس، رتبة بروفار، متدرجة محليا.

تصنيف الجمعية الأمريكية 2010 : 20E07، 20F18، 20F19، 20F45.

<u>Abstract</u>: This thesis aims to study the influence of given systems of subgroups of a group on the structure of the group itself. If Y is a class of groups, then a group G is said to be minimal non-Y if it is not a Y-group but all its proper subgroups belong to Y. It is known that any locally graded minimal non-A-group is finite, where A is the class of abelian groups. On the other hand, the descriptions of infinite locally graded minimal non-N were given by H. Heineken, I.J. Mohamed, M. Newman, H. Smith, and J. Wiegold, where N is the class of nilpotent groups. Here we extend the above problems to the classes: MN(NM) and Engel (k-Engel), where M denotes the class of soluble-by-finite minimax groups. M.R. Dixon, M.J. Evans and H. Smith proved that a (generalized) soluble group of infinite rank in which all proper subgroups of infinite rank are abelian is itself abelian, and hence so does all its proper subgroups. We have established similar results for the classes: MN(NM), CZA(ZAC) and Engel (k-Engel), where ZA and C denote the class of hypercentral and Chernikov groups, respectively.

Key words: Hypercentral, Engel, minimax, Prüfer rank, locally graded.

AMS Subject Classification 2010: 20E07, 20F18, 20F19, 20F45.

<u>Résumé</u>: Le but de cette thèse est d’étudier l’influence de systèmes donnés de sous-groupes d’un groupe sur la structure du groupe lui-même. Si Y une classe de groupes, alors un groupe G est dit non-Y minimal s'il n'est pas un Y-groupe mais tous ses sous-groupes propres le sont. On sait que tout groupe non-A minimal localement gradué est fini, où A est la classe des groupes abéliens. D'autre part, les descriptions des groupes non-N minimaux infini localement gradué ont été données par H. Heineken, I.J. Mohamed, M. Newman, H. Smith et J. Wiegold, où N est la classe des groupes nilpotents. Ici, nous étendons les problèmes ci-dessus aux classes: MN(NM) et Engel (k-Engel), où M désigne la classe des groupes résolubles-par-finis minimax. M.R. Dixon, M.J. Evans et H. Smith ont prouvé qu'un groupe résoluble (généralisé) de rang infini dans lequel tous les sous-groupes propres de rang infini sont abéliens est lui-même abélien, et donc tous ses sous-groupes le sont. Nous avons établi des résultats similaires pour les classes: MN(NM), CZA(ZAC) et Engel (k-Engel), où ZA et C désignent respectivement la classe des groupes hypercentraux et de Chernikov.

Mots clés: Hypercentral, Engel, minimax, rang de Prüfer, localement gradué.

Classification AMS 2010 : 20E07, 20F18, 20F19, 20F45.