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YAZID FARES

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Global existence and stability for some hyperbolic systems

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Examination Committee Members:

Pr. Merouani Boubakeur	(Univ. Setif 1)	Chairman
Dr. Ouchenane Djamel	(Univ. Laghouat)	First Supervisor
Dr. Chougui Nadhir	(Univ. Setif 1)	Second Supervisor
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Pr. Hamici Nacerdine	(Univ. Setif 1)	Examinator
Dr. Zitouni Salah	(Univ. Souk Ahras)	Examinator
Pr. Drabla Salah	(Univ. Setif 1)	Invitee

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FARES YAZID

Under the Supervision of

First Supervisor: Dr. Ouchenane Djamel
Second Supervisor: Dr. Chougui Nadhir

This thesis is submitted in order to obtain the degree of **Doctor of Sciences**

Abstract

In this thesis we considered some elastic, thermoelastic and viscoelastic systems with the presence of different mechanisms of dissipation. The first part of the thesis is composed of two chapters. In chapter 1, we consider a Bresse-Timoshenko system with distributed delay term. Under suitable assumptions, we prove the global well-posedness and exponential stability results by Faedo-Galerkin approximations and some energy estimates. Chapter 2 is related to the one-dimensional linear thermoelastic system of Timoshenko type of second sound with distributed delay term. We prove the exponential stability result will be shown without the usual assumption on the wave speeds. The second part of the thesis is devoted to the study of some viscoelastic systems with voids. In Chapter 3, we consider a system of nonlinear viscoelastic wave equations with degenerate damping and strong source terms. We prove, with positive initial energy, the global nonexistence of solution by concavity method. Similar results were shown in Chapter 4 where we coupled nonlinear Klein-Gordon equations with degenerate damping and source terms. We prove the global nonexistence of solution.

Keywords: Timoshenko system, Bresse-Timoshenko system, Exponential decay, Distributed delay term, Source term, Positive initial energy, Concavity method, Energy method, Global nonexistence, Degenerate damping thermoelasticity, Nonlinear viscoelastic wave equations.

ملخص

في هذه الرسالة ندرس استقرار، اضمحلال و انفجار الحل لبعض الجمل الزائدية، وقد قسمنا الرسالة إلى قسمين كل قسم منهما يحوي فصلين.

ندرس في القسم الأول وجود و وحدانية الحل لجملتين زائديتين حيث نقوم في الفصل الأول بإثبات الوجود والوحدانية غير المحلي مع الاستقرار الأسي لحل جملة من نوع بريس-تيموشينكو Bresse-Timoshenko مع وجود دالة التوزيع للتأخير باستخدام تقريبات فايدو غالركين Faedo-Galerkin و تقديرات الطاقة، أما في الفصل الثاني نثبت الاستقرار الأسي لحل جملة زائدية أحادية البعد من نوع تيموشينكو Timoshenko مع الصوت الثاني بتأخير و ذلك باستخدام طريقة الطاقة.

في القسم الثاني ندرس انفجار الحل لجملتين زائديتين كل منهما جملة معادلات تفاضلية جزئية غير خطية بإضافة حد يمثل اللزوجة و ذلك باستخدام طريقة الطاقة.

Résumé

Dans cette thèse, nous avons considéré certains systèmes élastiques, thermoélastiques et viscoélastiques avec la présence de différents mécanismes de dissipation. La première partie de la thèse est composée de deux chapitres. Dans le chapitre 1, nous considérons un système de Bresse-Timoshenko avec un retard distribué. Sous des hypothèses appropriées, nous prouvons le bien-posé global et la stabilité exponentielle des résultats par approximations de Faedo-Galerkin et quelques estimations énergétiques. Le chapitre 2 est lié au système thermoélastique linéaire unidimensionnel de type Timoshenko du deuxième son avec un terme de retard distribué. Nous prouvons que le résultat de stabilité exponentielle sera montré sans l'hypothèse habituelle sur les vitesses des ondes. La deuxième partie de la thèse est consacrée à l'étude de certains systèmes viscoélastiques à vides. Dans le chapitre 3, nous considérons un système d'équations d'onde viscoélastiques non linéaires avec amortissement dégénéré et termes source. Nous prouvons, avec une énergie initiale positive, la non-existence globale de solution par méthode de concavité. Similaire les résultats ont été présentés au chapitre 4 où nous avons couplé des équations non linéaires de Klein-Gordon avec d'amortissement dégénéré et termes sources. Nous prouvons la non-existence globale de solution

Mots clés : Système de Timoshenko, Système de Bresse-Timoshenko, Décroissance exponentielle, Terme de retard déstibué, Terme source, Eenergie initiale positive, Méthode de concavité, Méthode d'énergie, Non-existence globale, Thermoélasticité d'amortissement d'énergie, Equations d'ondes visco-élastiques non linéaires.

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Yazid Fares

Contents

Introduction

The aim of this thesis is to investigate the stability of some elastic, thermoelastic and viscoelastic evolution problems, such as the Timoshenko systems, Bresse-Timoshenko systems and the system of viscoelastic wave equations.

The systems that we treated here are the following:
The Timoshenko systems
The Timoshenko system is usually considered as describing the transverse vibration of a beam and ignoring damping effects of any nature. Precisely, we have the following model, which was developed by Timoshenko in 1921(see in [126]), this is given by a system of two coupled hyperbolic equations of the form

$$\begin{cases} \rho\varphi_{tt} = (K(\varphi_x - \psi))_x, & \text{in } (0, L) \times \mathbb{R}_+, \\ I_\rho\psi_{tt} = (EI\psi_x)_x + K(\varphi_x - \psi), & \text{in } (0, L) \times \mathbb{R}_+, \end{cases} \tag{1}$$

where φ is the transverse displacement of the beam, and ψ is the rotation angle of the filament of the beam. The coefficients ρ, I_ρ, E, I and K are respectively the density (the mass per unit length), the polar moment of inertia of a cross section, Young‘s modulus of elasticity, the moment of inertia of a cross section and the shear modulus, respectively.
System (1), together with boundary conditions of the form

$$EI\varphi_x \mid_{x=0}^{x=L} = 0,$$
$$K(\varphi_x - \varphi) \mid_{x=0}^{x=L} = 0,$$

is conservative, and thus the total energy is preserved, as time goes to infinity. Several authors introduced different types of dissipative mechanisms to stabilize system (1), and several results concerning uniform and asymptotic decay of energy have been established. Let us now mention some known results related to the stabilization of the Timoshenko beam. There are a number of publications concerning the stabilization of the Timoshenko system with different kinds of damping. Kim and Renardy considered Timoshenko (1) in [61] with two boundary controls of the form

$$\begin{cases} K\varphi(L,t) - Ku_x(L,t) = \alpha\varphi_t(L,t), & \text{on } \mathbb{R}_+, \\ EI\psi_x(L,t) = -\beta\psi_t(L,t), & \text{on } \mathbb{R}_+, \end{cases} \tag{2}$$

they establish an exponential decay result for the natural energy of system (1). They also provided some numerical estimates to the eigenvalues of the operator associated with the system (1). An analogous result was also established by Feng et al. in [44], where a stabilization of vibrations in a Timoshenko system was studied. Raposo et al. studied in [108] Timoshenko (1) with homogeneous Dirichlet boundary conditions and two linear frictional dampings; i.e, they considered the following system

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x - \psi)_x + \varphi_t = 0, & \text{in } (0,L)\times\mathbb{R}_+ \\ \rho\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x - \psi) + \psi_t = 0, & \text{in } (0,L)\times\mathbb{R}_+ \\ \varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0, & \text{on } \mathbb{R}_+, \end{cases} \tag{3}$$

they showed that the Timoshenko system (3) is exponentially stable. This result is similar to that one by Taylor [125], but as they mentioned, the originality of their work lies in the method based on the semigroup theory developed by Liu and Zheng [73].
Soufyane and Wehbe in [123] considered Timoshenko (1) with one internal distributed dissipation law; i.e, they cosidered the following system:

$$\begin{cases} \rho\psi_{tt} = (K(\psi_x - \varphi))_x, & \text{in } (0,L)\times\mathbb{R}_+ \\ I_\rho\varphi_{tt} = (EI\varphi x)_x + K(\psi_x - \varphi) - b(x)\varphi t, & \text{in } (0,L)\times\mathbb{R}_+ \\ \varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0, & \text{on } \mathbb{R}_+, \end{cases} \tag{4}$$

where b is a positive and continuous function, which satisfies

$$b(x) \geq b_0 > 0, \quad \forall\, x \,\in\, [a_0, a_1] \subset [0, L].$$

They showed that the Timoshenko system (4) is exponentially stable if and only if the wave propagation speeds are equal (i.e., $\frac{K_1}{\rho} = \frac{EI}{I_\rho}$), otherwise, only the asymptotic stability has been proved. This result improves previous ones by Soufyane [122] and Shi and Feng [120] who proved an exponential decay of

the solution of (1) together with two locally distributed feedbacks.
Rivera and Racke in [109] improved the previous results and showed an exponential decay of the solution of the system (4) when the coefficient of the feedback admits an indefinite sign. Also, Rivera and Racke [94] treated a nonlinear Timoshenkotype system of the form

$$\begin{aligned} &\rho_1\varphi_{tt} - \sigma_1(\varphi_x, \psi)_x = 0 \\ &\rho_2\psi_{tt} - \chi(\psi_x)_x + \sigma_2(\varphi_x, \psi) + d\psi_t = 0 \end{aligned} \tag{5}$$

in a one-dimensional bounded domain. The dissipation is produced here through a frictional damping which is only present in the equation for the rotation angle. The authors gave an alternative proof for a necessary and sufficient condition for exponential stability in the linear case and then proved a polynomial stability in general. Moreover, they investigated the global existence of small smooth solutions and exponential stability in the nonlinear case.
Xu and Yung [132] studied a system of Timoshenko beams with pointwise feedback controls, looked for the information about the eigenvalues and eigenfunctions of the system, and used this information to examine the stability of the system.
A weaker type of dissipation was considered by Ammar-Khodja and *al.* [9]by introducing the memory term $\int_0^t g(t-s)\psi_{xx}(s)ds$ in the rotation angle equation of (11). They used the multiplier techniques and showed that the system is uniformly stable if and only if (9) holds and the kernel g decays uniformly. Precisely, they proved that the rate of decay is exponential (polynomial) if g decays exponentially (polynomially).

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0, & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_2\psi_{tt} - b\psi_{xx} + \displaystyle\int_0^t g(t-s)\psi_{xx}(s)ds + K(\varphi_x + \psi) = 0, & \text{in } (0, L) \times \mathbb{R}_+, \end{cases} \tag{6}$$

with homogeneous boundary conditions. They proved that the system (6) is uniformly stable if and only if the wave speeds are equal and g decays uniformly. Also, they proved an exponential decay if g decays at an exponential rate and polynomially if g decays at a polynomial rate. They also required some technical conditions on both g' and g'' to obtain their result, Ammar-Khodja and al. in [10] studied the decay rate of the energy of the nonuniform Timoshenko beam with two boundary controls acting in the rotation-angle equation. The feedback of memory type has also been studied by Santos [115]. He considered a Timoshenko system and showed that the presence of two feedbacks of memory type at a subset of the boundary stabilizes the system uniformly. He also obtained the energy decay rate which is exactly the decay rate of the relaxation functions. In fact, under the equal speed wave propagation condition, they established exponential decay results up to an unknown finite dimensional space of initial data. In addition, they showed that the equal speed wave propagation condition is necessary for the exponential stability. However, in the case of non equal speeds, no decay rate has been discussed. The result in [10] has been recently improved by Wehbe and al. in [15] where are established nonuniform stability and an optimal polynomial energy decay rate of the Timoshenko system with only one dissipation law on the boundary.

Shi and Feng [120] investigated a nonuniform Timoshenko beam and showed that the vibration of the beam decays exponentially under some locally distributed controls. To achieve their goal, the authors used the frequency multiplier method.

For Timoshenko systems of classical thermoelasticity, Muñoz Rivera and Racke in [95] studied nonlinear system of the form

$$\begin{cases} \rho_1\varphi_{tt} - \sigma(\varphi_x, \psi)_x = 0, & \text{in } (0, L) \times \mathbb{R}_+ \\ \rho_2\psi_{tt} - b(\psi_{xx})_x + k(\varphi_x + \psi) + \gamma\theta_x = 0, & \text{in } (0, L) \times \mathbb{R}_+ \\ \rho_3\theta_t - k\theta_{xx} + \gamma\psi_{xt} = 0, & \text{in } (0, L) \times \mathbb{R}_+ \end{cases} \tag{7}$$

where θ is the difference temperature, φ is the displacement and ψ is the rotation angle of filament of the beam and $\sigma, \rho_1, \rho_2, b, K$ and γ are constituve constants. They showed that, for the bouandary conditions

$$\varphi(x, t) = \psi_x(x, t) = \theta(x,) = 0, \quad for \quad x = 0, L \quad and \quad t \geqslant 0, \tag{8}$$

the energy of system (7) decays exponentially if and only if

$$\frac{\rho_1}{k} = \frac{\rho_2}{b}, \tag{9}$$

and that condition (9) suffices to stabilise system (7) exponentially for the bouandary conditions

$$\varphi(x, t) = \psi(x, t) = \theta_x(x,) = 0, \quad for \quad x = 0, L \quad and \quad t \geqslant 0,$$

and non-exponential stability result for the case of different wave speeds of propagation. Muñoz Rivera and Racke in [94] studied nonlinear Timoshenko system of the form

$$\begin{cases} \rho_1\varphi_{tt} - \sigma_1(\varphi_x, \psi)_x = 0, & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_2\psi_{tt} - \chi(\psi_x)_x + \sigma_2(\varphi_x, \psi) + d\psi_t = 0, & \text{in } (0, L) \times \mathbb{R}_+, \end{cases} \tag{10}$$

with homogeneous boundary conditions, they showed that the Timoshenko system (10) is exponentially stable if and only if the wave propagation speeds are equal, otherwise, only the polynomial stability holds.

In the above system, the heat flux is given by the Fourier's law. As a result, we obtain a physical discrepancy of infnite heat propagation speed. That is, any thermal disturbance at a single point has an instantaneous efect everywhere in the medium. Experiments showed that heat conduction in some dielectric crystals at low temperatures is free of this paradox. Moreover, the disturbances being almost entirely thermal, propagate at a finite speed. This phenomenon in dielectric crystals is called second sound.

To overcome this physical paradox, many theories have been developed. One of which suggests that we should replace the Fourier's law

$$q + \kappa\theta_x = 0$$

by so called Cattaneo's law

$$\tau q_t + q + \kappa\theta_x = 0$$

Alabau- Boussouira [3] extended the results of [94] to the case of nonlinear feedback $\alpha(\psi t)$, instead of $d\psi t$, where α is a globally Lipchitz function satisfying some growth conditions at the origin. Indeed, she considered the following system

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_2\psi_{tt} - b(\psi_{xx}) + k(\varphi_x + \psi) + \alpha(\psi_x) = 0, & \text{in } (0, L) \times \mathbb{R}_+, \end{cases} \tag{11}$$

with homogeneous boundary conditions. In fact, if the wave propagation speeds are equal she established a general semi-explicit formula for the decay rate of the energy at infinity. Otherwise, she proved polynomial decay in the case of different speed of propagation for both linear and nonlinear globally Lipschitz feedbacks.

Concerning the Timoshenko system with delay, the investigation started by Houari and Laskri with the paper [111] where the authors studied the following problem:

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - \alpha\psi_{xx}(x,t) + K(\varphi_x + \psi)(x,t) + \mu_1\psi_t(x,t) + \mu_2\psi_t(x,t-\tau) = 0. \end{cases} \tag{12}$$

Under the assumption $\mu_1 \geq \mu_2$ on the weights of the two feedbacks, they proved the well-posedness of the system. They also established for $\mu_1 > \mu_2$ an exponential decay result for the case of equal-speed wave propagation.

Subsequently, the work in [111] has been extended to the case of time-varying delay of the form $\psi_t(x,\ t - \tau(t))$ by Kirane, Said-Houari and Anwar [62]. The case where the damping $\mu_1\psi_t$ is replaced by history type $\int_0^\infty g(s)\psi_{xx}(x, t-s)ds$ (with either discrete delay $\mu_2\psi_t(t-\tau)$ or distributed one $\int_0^\infty f(s)\psi_t(t-s)ds$) has been treated in [49] (in case (9) and the opposite one), where several general decay estimates have been proved.

The Bresse systems

The problem with the arc is also known as the Bressse system, the elastic structures of this type are widely used for study in engineering, architecture, marine engineering, aeronautics and others. In particular, vibration on elastic structures is an important research topic in engineering and also in mathematics. In the Mathematical Analysis area, is interesting to know the properties that relate to energy behavior associated with the respective dynamic model. For feedback laws, by example, we can ask what conditions about the dynamic model must be taken to obtain the decrease in energy from solutions in time t. In this sense, the property of stabilization has been studied for dynamic problems in elastic structures are translated according to partial differential equations.

Originally the Bresse system consists of three wave equations where the main variables describing the longitudinal, vertical and shear angle displacements, which can be represented as (see [25]). The original Bresse system is given by the following equations

$$\begin{cases} \rho_1\varphi_{tt} = Q_x + lN + F_1, \\ \rho_2\psi_{tt} = M_x - Q + F_2, \\ \rho_1 w_{tt} = N_x - lQ + F_3, \end{cases} \tag{13}$$

These strengths are the relationships deformation (stress-strain) for elastic behavior and such are given by:

$$\begin{aligned} N &= k_0\,(w_x - l\varphi) \\ Q &= k\,(\varphi_x + lw + \psi) \\ M &= b\psi_x \end{aligned} \tag{14}$$

We use N, Q and M to denote the axial force, the shear force and the bending moment. By w, φ and ψ we are denoting the longitudinal, vertical and shear angle displacements. Here $\rho_1 = \rho A = \rho I, k_0 = EA, k = k'GA$ and $l = R^{-1}$. To material properties, we use ρ for density, E for the modulus of elasticity, G for the shear modulus, K for the shear factor, A for the cross-sectional area, I for the second moment of area of the cross-section and R for the radius of curvature and we assume that all this quantities are positives. Also by Fi we are denoting external forces.

Considering the coupling of equations (13) and (14), we obtain

$$\begin{cases} \rho_1\varphi_{tt} - \kappa\,(\varphi_x + l\omega + \psi)_x - \kappa_0 l\,(\omega_x - l\varphi) = F_1 \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa\,(\varphi_x + l\omega + \psi) = F_2 \\ \rho_3\omega_{tt} - \kappa_0\,(\omega_x - l\varphi)_x + \kappa l\,(\varphi_x + l\omega + \psi) = F_3 \end{cases} \tag{15}$$

System (15) is an undamped system and its associated energy remains constant when the time t evolves. To stabilize system (15), many damping terms have been considered by several authors. (see [21], [17], [23], [22],[50], [72], [116]). We start by recalling some results related to the stabilization of the elastic

Bresse system. Wehbe and Youssef [131], considered the elastic Bresse system subject to two locally internal dissipation laws. They proved that the system is exponentially stable if and only if the wave propagation speeds are equal. Otherwise, only a polynomial stability holds. Alabau-Boussouira et al. [21], considered the same system with one globally distributed dissipation law. The authors proved that, in general, the system is not exponentially stable but there exists polynomial decay with rates that depend on some particular relation between the coefficients. Using boundary conditions of Dirichlet-Dirichlet-Dirichlet type, they proved that the energy of the system decays at a rate $t^{-\frac{1}{3}}$ and at the rate $t^{-\frac{2}{3}}$ if $k \equiv k_0$. These results are completed by Fatori and Montiero [38]. Using boundary conditions of Dirichlet-Neumann- Neumann type, the authors showed that the energy of the elastic Bresse system decays polynomially at the rate $t^{-\frac{1}{2}}$ and at the rate t^{-1}if $k \equiv k_0$. Noun and Wehbe [99] extended the results of [21] and [38]. The authors considered the elastic Bresse system subject to one locally distributed feedback with Dirichlet-Neumann- Neumann or Dirichlet-Dirichlet-Dirichlet boundary conditions type. They proved By considering damping terms as infinite memories acting in the three equations.
For the thermoelastic Bresse system, subject of this paper, there exist two important results. The first result is due to Liu and Rao [72], when they considered the Bresse system with two thermal dissipation laws. The authors showed that the energy decays exponentially when the wave speed of the vertical displacement coincides with the wave speed of longitudinal displacement or of the shear angle displacement. Otherwise, they found polynomial decay rates depending on the boundary conditions. When the system is subject to Dirichlet-Neumann-Neumann boundary conditions, they showed that the energy decays at the rate $t^{-\frac{1}{2}}$ and for fully Dirichlet boundary conditions, they proved that the energy of the system decays as $t^{-\frac{1}{4}}$. This result has been recently improved by Fatori and Rivera [39] in the sense that the authors considered only one globally dissipative mechanism given by one temperature, and they established the rate of decay $t^{-\frac{1}{3}}$for Dirichlet- Neumann- Neumann and Dirichlet-Dirichlet-Dirichlet boundary conditions type. The main result of this paper is to extend the results from [39], by taking into consideration the important case when the thermal dissipation law is locally distributed on the angle displacement equation i.e the damping coefficient is not constant but it is a positive function in $W^{2,\infty}(0, L)$ and strictly positive in an open subinterval $]a, b[\subset]0, L[$ (the cases $a = 0$ or $b = L$ are not excluded) and to improve the polynomial energy decay rate.

Klein-Gordon equation

The Klein-Gordon equation is distinguished among other nonlinear hyperbolic equations b its theoretical and practical significance. The nonlinear Klein-Gordon equation appears in the study of some problems of mathematical physics. For example, this equation arises in general relativity, nonlinear optics (e.g., in the study of instability phenomena such as self-focusing), plasma physics, uid mechanics, radiation theory or in the theory of spin waves [119, 75, 77]. The Cauchy problem for nonlinear Klein-Gordon equation

$$\varphi_{tt} - \Delta\varphi + m\varphi + \varphi_t = f(\varphi), \quad t > 0, \quad x \in R^n \tag{16}$$

$$\varphi(0,x) = \varphi_0(x), \varphi_t(0,x) = \varphi_1(x), x \in R^n \tag{17}$$

has been studied by many authors (see e.g. [70, 138]). Existence and nonexistence of global solutions are the main points of study for the problem (16), (17) in the case $m = 0$, $f(\varphi) \sim |\varphi|^p$ (see e.g. [69]). In [128, 57], the problem (16), (17) has been investigated in the case $m = 0$, $f(\varphi) \sim |\varphi|^p$, where $1 < p \leq p_c = 1 + \frac{2}{n}$, and the existence of sufficiently small initial data $(\varphi_0; \varphi_1)$ was proved for which the corresponding Cauchy problem has no global solution. In [7, 8] the Klein-Gordon equation has been investigated in the case $m = 0$, $f(\varphi) \sim |\varphi|^p$ when $p > p_c = 1 + \frac{2}{n}$, and the existence of a global solution for the problem (16), (17) has been proved for suffciently small $(\varphi_0; \varphi_1)$. In the case $m > 0$, i.e for the Klein-Gordon equation with mass, the above efects do not occur. In this case, the main objects of study are the corresponding potential well and stability or instability of standing wave. There is a series of works devoted to that problem. The nonexistence of global solutions was studied in [67] for nonlinear wave equations with negative energy and in [68] for a class of abstract equations that, in particular, contains nonlinear wave equations. The nonexistence of global solutions of nonlinear wave equations with positive initial energy was considered in [104]. It was shown in the study of nonlinear wave equations in [65] that there exist initial data with fixed initial energy such that the corresponding Cauchy problem does not have a global solution. This result was improved in [127]. A mixed problem for systems of two semilinear wave equations with viscosity and with memory was studied in [74], where the nonexistence of global solutions with positive initial energy was proved. The nonexistence of global solutions of problem (16), (17) with negative initial energy was studied in [4] for $m = 2$ and in [71] for $m = 2$ and $p_1 = p_2$. The nonexistence of global solutions of a generalized fourth-order Klein–Gordon equation with positive initial energy was analyzed in [64]. A fairly comprehensive picture of the studies in this direction can be gained from the monograph [92]. Our main results in this thesis can be summarized as follows:

Chapter 1. In this chapter, we consider a Bresse-Timoshenko type system with distributed delay term. Under suitable assumptions, we establish the global well-posedness of the initial and boundary value problem by using the Faedo-Galerkin approximations and some energy estimates. By using the energy method, we show two exponential stability results for the system with delay in vertical displacement and in angular rotation, respectively. This extends earlier results in the literature [40] to the system with distributed delay. This work has been recently published in [30].

Chapter 2. In this chapter we studied a one-dimensional linear thermoelastic system of Timoshenko type with distributed delay term. The heat conduction is given by Cattaneo's law. Under an appropriate assumption between the weight of the delay and the weight of the damping, we prove the exponential stability result will be shown without the usual assumption on the wave speeds. To achieve our goals, we make use of the energy method. This work has been recently published in [133].

Chapter 3. In this chapter we will substantiation that the positive intial-energy solution for coupled nonlinear Klein-Gordon equations with degenerate damping and source terms. We prove, with positive initial energy, the global nonexistence of solution by concavity method. This work has been recently accepted in [134].

Chapter 4. In this chapter we will substantiation that the positive intial-energy solution for coupled nonlinear Klein-Gordon equations with nonlinear damping terms. We prove, with positive initial energy, the global nonexistence of solution by concavity method. This work has been recently accepted in [102].

Part I

Bresse-Timoshenko and Timoshenko systems

Chapter 1

Global Well-Posedness and Exponential Stability results of class of Bresse-Timoshenko type systems with Distributed delay term

1.1 Introduction

In this chapter, we deal with a nolinear Bresse-Timoshenko type system with distributed delay cases (1.9) and (1.72), under appropriate assumptions and we study the exponential decay of two cases of dissipative system based on the recent works by Almeida Júnior et al. in [5]-[8] and Feng in [40], we are working to establish the global well-posedness of the initial and boundary value problem by using the Faedo-Galerkin approximations and some energy estimates, we relied on the energy method to demonstrated these results with the help of convex functions. In the next, let's c positive constant.
Firstly, after an explanation about the meaning of second spectrum based on existing literature, they showed that the viscous damping acting on angle rotation of the original Timoshenko system in [126] given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + \beta(\varphi_x + \psi) + \mu\psi_t = 0 \end{cases} \tag{1.1}$$

Regarding relationship between dissipative effects and second spectrum, Nesterenko in [96] suggested that "dissipative processes are able of eliminating the damaging effect given by second spectrum. However, Nesterenko did not prove anything about his affirmation. According to our information, the first contribution in that direction was obtained by Manevich and Kolakowski [76]. They analyzed the dynamic of a Timoshenko model where the damping mechanism is viscoelastic. More precisely, they considered the dissipative system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x - \mu_1(\varphi_x + \psi)_{tx} = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \beta(\varphi_x + \psi) - \mu_2\psi_{ttx} + \mu_1(\varphi_x + \psi)_t = 0, \end{cases} \tag{1.2}$$

where $\mu_i, i = 1, 2$ are positive constants and they conclude, from dispersion analysis, that "internal friction damping eliminates the second branch for sufficiently short wavelengths, and there remains only the first branch.

Secondly, the papers due to Elishakoff and collaborators and their studies on truncated versions for classical Timoshenko equations [1] (see also recent contributions of Elishakoff et *al.* [36]-[35])have an important influence on our incursions in the paper of Almeida Junior and Ramos [5], where they showed that the total energy for viscous damping acting on angle rotation of the simplified Timoshenko system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x = 0 \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x + \psi) + \mu_1\psi_t = 0 \end{cases} \tag{1.3}$$

is exponentially stable regardless of any values of coefficients o_1, ρ_2, b and β It is noteworthy that both set of systems (1.1) and (1.3) without damping ($\mu = 0$) have all main features considered by Timoshenko in [126] from modelling point of view: curvature, vertical displacement, angle rotation, shear deformation and rotary inertia. We believe that the truncated version (1.3) is simpler than the classical ones for dealing with the exponential stability because the hypothesis of equal wave propagation velocities is not necessary anymore. The reason behind this is the absence of the second spectrum or non-physical spectrum [1, 35] and its damage consequences for wave propagation speeds [5]. We can find the historical and mathematical observations in [1, 35]. The same results are achieved for a dissipative truncated version where the viscous damping acts on vertical displacement

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x + \mu_1\varphi_t = 0 \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x + \psi) = 0 \end{cases} \tag{1.4}$$

Then, for more consistency and in light of the absence of the second spectrum resulting from exponential decay, Almeida Junior et *al.* [8] considered two cases of dissipative systems for Bresse-Timoshenko type systems with constant delay cases. For the first one, the authors proved the exponential decay for the system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x + \mu_1\varphi_t + \mu_2\varphi_t(x, t - \tau) = 0 \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x + \psi) = 0 \end{cases} \tag{1.5}$$

For the second one, the authors also proved the exponential decay result for the system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x = 0 \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x + \psi) + \mu_1\psi_t + \mu_2\psi_t(x, t - \tau) = 0 \end{cases} \tag{1.6}$$

Feng and *al.* [40] considered two cases of dissipative systems for Bresse-Timoshenko type systems with time-varying delay cases. For the first one, the authors proved the exponential decay for the system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x + \mu_1\varphi_t + \mu_2\varphi_t(x, t - \tau(t)) = 0 \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x + \psi) = 0 \end{cases} \tag{1.7}$$

For the second one, the authors also proved the exponential decay result for the system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x = 0 \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x + \psi) + \mu_1\psi_t + |\mu_2|\psi_t(x, t - \tau(t)) = 0 \end{cases} \tag{1.8}$$

Guesmia and Soufyane [52] studied a vibrating system of Timoshenko-type with one or two discrete time delays and one or two internal frictional dampings, and established exponential decay and polynomial decay of corresponding system. Said-Houari and Soufyane [112] studied exponential stability of a Timoshenko beam system together with two boundary controls and with delay terms. For Timoshenko system of thermo-viscoelasticity with delay term, one can refer [13], [29] and [59]. On the other hand, in [37], [53], [55], [98] and [140] the authors considered the case of a Timoshenko type systems with distributed delay terms, and established some stability results.

1.2 Distributed delay and viscous damping in vertical displacement

Here, we are concerned with the following system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x + \mu_1\varphi_t + \int_{\tau_1}^{\tau_2} |\mu_2(p)|\varphi_t(x, t-p)\,dp = 0 \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x + \psi) = 0 \end{cases} \tag{1.9}$$

where

$$(x, p, t) \in (0, 1) \times (\tau_1, \tau_2) \times (0, \infty)$$

Additionally, we consider initial conditions given by

$$\begin{cases} \varphi(x, 0) = \varphi_0(x), \varphi_t(x, 0) = \varphi_1(x), \varphi_{tt}(x, 0) = \varphi_2(x) \\ \varphi_{ttt}(x, 0) = \varphi_3(x),\ \psi(x, 0) = \psi_0(x), x \in (0, 1) \end{cases} \tag{1.10}$$

where $\varphi_0, \varphi_1, \varphi_2, \psi_0$, are given functions, and boundary conditions of Dirichlet-Dirichlet given by

$$\varphi(0, t) = \varphi(1, t) = \psi(0, t) = \psi(1, t) = 0,\ t > 0 \tag{1.11}$$

Wherever, φ is the transverse displacement of the beam, ψ is the angle of rotation, and $\rho_1, \rho_2, b, \beta > 0$ and the integral represents the distributed delay term with $\tau_1, \tau_2 > 0$ are a time delay, μ_1 is positive constant, μ_2 is an L^∞function.

(A1) $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2} |\mu_2(p)| dp \leq \mu_1 \tag{1.12}$$

In order to deal with the disributed delay feedback term, motivated by [98], let use introduce a new dependent variable

$$y(x, \tau, p, t) = \varphi_t(x, t - p\tau), \tag{1.13}$$

Using (1.13), we have

$$\begin{cases} p y_t(x, \tau, p, t) = -y_\tau(x, \tau, p, t) \\ y(x, 0, p, t) = \varphi_t(x, t). \end{cases} \tag{1.14}$$

Thus, the problem is equivalent to

$$\begin{cases} \rho_1 \varphi_{tt} - \beta(\varphi_x + \psi)_x + \mu_1 \varphi_t + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(p)| y\,(x, 1, p, t)\, dp = 0 \\ -\rho_2 \varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x + \psi) = 0 \\ p y_t(x, \tau, p, t) + y_\tau(x, \tau, p, t) = 0 \end{cases} \tag{1.15}$$

where

$$(x, \tau, p, t) \in (0, 1) \times (0, 1) \times (\tau_1, \tau_2) \times (0, \infty).$$

Additionally, we consider initial conditions given by

$$\begin{cases} \varphi\,(x, 0) = \varphi_0\,(x)\,, \varphi_t\,(x, 0) = \varphi_1\,(x)\,, \varphi_{tt}\,(x, 0) = \varphi_2\,(x) \\ \varphi_{ttt}(x, 0) = \varphi_3(x),\ \psi\,(x, 0) = \psi_0\,(x)\,, x \in (0, 1) \\ y\,(x, \tau, p, 0) = f_0(x, -p\tau), y_t\,(x, \tau, p, 0) = f_1\,(x, -p\tau)\,,\ in\ \ (0, 1) \times (0, 1) \times (0, \tau_2), \\ y_{tt}\,(x, \tau, p, 0) = f_2\,(x, -p\tau)\,,\ in\ \ (0, 1) \times (0, 1) \times (0, \tau_2) \end{cases} \tag{1.16}$$

where $\varphi_0, \varphi_1, \varphi_2, \psi_0, f_0, f_1$, are given functions, and boundary conditions of Dirichlet-Dirichlet given by

$$\varphi\,(0, t) = \varphi\,(1, t) = \psi\,(0, t) = \psi(1, t) = 0,\ \ t > 0 \tag{1.17}$$

Next we say that the global well-posedness of problem (1.15)-(1.17) given in the following theorem.

Theorem 1.1. *Assume the assumption (1.12) holds. Then, we have the following:*
(i) If the initial data $(\varphi_0, \varphi_1, \varphi_2, \varphi_3, \psi_0)$ *is in* $(H^2(0, 1) \cap H_0^1(0, 1)) \times (H^2(0, 1) \cap H_0^1(0, 1)) \times H^1(0, 1) \times L^2(0, 1) \times (H^2(0, 1) \cap H_0^1(0, 1))$, $f_0, f_1, f_2 \in H^1((0, 1) \times (0, 1) \times (\tau_1, \tau_2))$, *then problem (1.15)-1.17) has a*

stronger weak solution

$$\begin{array}{ll}\varphi \in L^{\infty}_{loc}(\mathbb{R}^+, H^2(0,1)\cap H^1_0(0,1)), & \psi \in L^{\infty}_{loc}(\mathbb{R}^+, H^2(0,1)\cap H^1_0(0,1)) \\ \varphi_t \in L^{\infty}_{loc}(\mathbb{R}^+, H^2(0,1)\cap H^1_0(0,1)), & \varphi_{tt} \in L^{\infty}_{loc}(\mathbb{R}^+, H^1_0(0,1)) \end{array}.$$

(ii)If the initial data $(\varphi_0,\varphi_1,\varphi_2,\varphi_3,\psi_0)$ *is in* $(H^1_0(0,1)\times H^1_0(0,1)\times L^2(0,1)\times L^2(0,1)\times H^1_0(0,1))$, $f_0, f_1, f_2 \in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2))$, *then problem (1.15)-(1.17) has a weak solution such that*

$$\begin{array}{ll}\varphi \in C([0,T], H^1_0(0,1))\cap C^1([0,T], L^2(0,1)), & \psi \in C([0,T], H^1_0(0,1)) \\ \varphi_t \in C([0,T], H^1_0(0,1)), & \varphi_{tt} \in C([0,T], L^2(0,1)) \end{array}.$$

(iii)In both cases, we have that the solution $(\varphi,\varphi_t,\varphi_{tt},\psi)$ *depends continuously on the initial data in* $H^1_0(0,1)\times H^1_0(0,1)\times L^2(0,1)\times H^1_0(0,1)$. *In particular, problem (1.15)-1.17) has a unique weak solution.*

1.2.1 The Global Well-Posedness

In this subsection, we will prove the global existence and the uniqueness of the solution of problem (1.9)-(1.17) by using the classical Faedo-Galerkin approximations along with some priori estimates. We only prove the existence of solution in (i). For the existence of stronger solution in (ii), we can use the same method as in (i) and one can refer to Andrade e al. [11] and Jorge Silva and Ma [121] and Feng [42].

1.2.1.1 Approximate Problem

Let u_j, θ_j be the Galerkin basis, For $m \geq 1$, let

$$W_m = span\{u_1, u_2, ..., u_n\}$$
$$K_m = span\{\theta_1, \theta_2, ..., \theta_n\}$$

We define for $1 \leq j \leq m$ the sequence $\phi_j(x,\tau,p)$ by

$$\phi_j(x,0,p) = u_j(x)$$

Then we can extend $\phi_j(x,0,p)$ by over $L^2((0,1)\times(0,1)\times(\tau_1,\tau_2))$ and denote $V_m = span\{\phi_1,\phi_2,....,\phi_n\}$. Given initial data $\varphi_0,\psi_0 \in H^1_0(0,1)$, $\varphi_1,\varphi_2 \in L^2(0,1)$, and $f_0, f_1 \in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2))$, we define the following approximations

$$\begin{aligned} \varphi_m &= \sum_{j=1}^{n} g_{jm}(t)u_j(x) \\ \psi_m &= \sum_{j=1}^{n} h_{jm}(t)\theta_j(x) \\ y_m &= \sum_{j=1}^{n} f_{jm}(t)\phi_j(x,\tau,p) \end{aligned} \tag{1.18}$$

which satisfy the following approximate problem:

$$\begin{aligned} &\rho_1(\varphi_{mtt}, u_j) + \beta((\varphi_{mx} + \psi_j), u_{jx}) + \mu_1(\varphi_{mt}, u_j) \\ &+(\int_{\tau_1}^{\tau_2} |\mu_2(p)| y_m\,(x,1,p,t)\,dp, u_j) = 0, \\ &b(\psi_{mx}, \theta_{jx}) + \rho_2(\varphi_{mtt}, \theta_{jx}) + \beta((\varphi_{mx} + \psi_j), \theta_{mj}) = 0 \\ &(py_{mt}(x,\tau,p,t), \phi_j) + (y_{m\tau}(x,\tau,p,t), \phi_j) = 0 \\ &(py_{mtt}(x,\tau,p,t), \phi_j) + (y_{m\tau t}(x,\tau,p,t), \phi_j) = 0 \end{aligned} \tag{1.19}$$

with initial conditions

$$\begin{aligned} &\varphi_m(0) = \varphi_0^m, \varphi_{mt}(0) = \varphi_1^m, \varphi_{mtt}(0) = \varphi_2^m \\ &\varphi_{mttt}(0) = \varphi_3^m, \psi_m(0) = \psi_0^m, \psi_{mt}(0) = \psi_1^m, \\ &y_m(0) = y_0^m, y_{mt}(0) = y_1^m, y_{mtt}(0) = y_2^m \end{aligned} \tag{1.20}$$

which satisfies

$$\begin{aligned} &\varphi_0^m \to \varphi_0, \ \ \textit{strongly in} \ \ H^2(0,1) \cap H_0^1(0,1) \\ &\varphi_1^m \to \varphi_1, \ \ \textit{strongly in} \ \ H_0^1(0,1) \\ &\varphi_2^m \to \varphi_2, \ \ \textit{strongly in} \ \ H_0^1(0,1) \\ &\varphi_3^m \to \varphi_3, \ \ \textit{strongly in} \ \ L^2(0,1) \\ &\psi_0^m \to \psi_0, \ \ \textit{strongly in} \ \ H^2(0,1) \cap H_0^1(0,1) \\ &\psi_1^m \to \psi_1, \ \ \textit{strongly in} \ \ H_0^1(0,1) \\ &y_0^m \to y_0, \ \ \textit{strongly in} \ \ L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)) \\ &y_1^m \to y_1, \ \ \textit{strongly in} \ \ L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)) \qquad (1.21) \\ &y_2^m \to y_2, \ \ \textit{strongly in} \ \ L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)) \qquad (1.22) \end{aligned}$$

By using standard ordinary differential equations theory, the problem (1.19)-(1.20) has a solution $(g_{jm}, h_{jm}, f_{jm})_{j=1,m}$ defined on $[0, t_m)$. The following estimate will give the local solution being extended to $[0, T]$, for any given $T > 0$.

1.2.1.2 A Priori Estimate I

Now multiplying The first four terms of the first approximate equation of (1.19) by g'_{jm}, and Now multiplying The last three terms of the first approximate equation of (1.19) by h'_{jm}, and using the fact that

$$\begin{aligned} -\beta\int_0^1 \varphi_{tt}\psi_{tx}dx &= \rho_1\int_0^1 \varphi_{ttt}\varphi_{tt}dx + \beta\int_0^1 \varphi_{txx}\varphi_{tt}dx + \mu_1\int_0^1 \varphi_{tt}^2 dx \\ &+ \int_0^1 \varphi_{tt}\int_{\tau_1}^{\tau_2} |\mu_2(p)|\varphi_{tt}(x, t-p)\,dpdx, \end{aligned}$$

we get

$$\begin{aligned} &\frac{d}{dt}\frac{1}{2}\Big[\rho_1\int_0^1 \varphi_{mt}^2 dx + \frac{\rho_1\rho_2}{\beta}\int_0^1 \varphi_{mtt}^2 dx + \rho_2\int_0^1 \varphi_{mtx}^2 dx + \beta\int_0^1 (\varphi_{mx}+\psi_m)^2 dx \\ &+b\int_0^1 \psi_{mx}^2 dx\Big] + \mu_1\int_0^1 \varphi_{mt}^2 dx + \int_0^1 \varphi_{mt}\int_{\tau_1}^{\tau_2} |\mu_2(p)|y_m(x, 1, p, t)\,dpdx \\ &+\mu_1\frac{\rho_2}{\beta}\int_0^1 \varphi_{mtt}^2 dx + \frac{\rho_2}{\beta}\int_0^1 \varphi_{mtt}\int_{\tau_1}^{\tau_2} |\mu_2(p)|y_{tm}(x, 1, p, t)\,dpdx. \end{aligned} \tag{1.23}$$

Multiplying the second approximate equation of (1.19) by $|\mu_2(p)|f_{jm}$, and then integrating over $(0, t)\times(0, 1)\times(0, 1)\times(\tau_1, \tau_2)$, we obtain

$$\begin{aligned} &\frac{1}{2}\int_0^t\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} p|\mu_2(p)|y_m^2(x, \tau, p, s)dpd\tau dxds \\ &= -\int_0^t\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(p)|y_m y_{m\tau}(x, \tau, p, s)\,dpd\tau dxds \\ &= -\frac{1}{2}\int_0^t\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(p)|\frac{d}{d\tau}y_m^2(x, \tau, p, s)\,dpd\tau dxds \\ &= \frac{1}{2}\int_0^t\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(p)|(y_m^2(x, 0, p, s) - y_m^2(x, 1, p, s))dpdxds \\ &= \frac{1}{2}\Big(\int_{\tau_1}^{\tau_2} |\mu_2(p)|dp\Big)\int_0^t\int_0^1 y_m^2(x, 0, p, s)dxds \\ &-\frac{1}{2}\int_0^t\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(p)|y_m^2(x, 1, p, s)\,dpdxds \end{aligned} \tag{1.24}$$

similarly, we have

$$
\begin{aligned}
&\frac{1}{2}\int_0^t\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} p|\mu_2(p)|y_{mt}^2(x,\tau,p,s)dpd\tau dxds\\
&=-\int_0^t\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(p)|y_{mt}y_{m\tau t}\,(x,\tau,p,s)\,dpd\tau dxds\\
&=-\frac{1}{2}\int_0^t\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(p)|\frac{d}{d\tau}y_{mt}^2\,(x,\tau,p,s)\,dpd\tau dxds\\
&=\frac{1}{2}\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(p)|(y_{mt}^2\,(x,0,p,s)-y_{mt}^2(x,1,p,s))dpdxds\\
&=\frac{1}{2}(\int_{\tau_1}^{\tau_2}|\mu_2(p)|dp)\int_0^t\int_0^1 y_{mt}^2(x,0,p,s)dxds \qquad (1.25)\\
&-\frac{1}{2}\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(p)|y_{mt}^2\,(x,1,p,s)\,dpdxds
\end{aligned}
$$

Now integrating (1.23) and using (1.24) and (1.25), we obtain

$$
\begin{aligned}
\mathcal{E}_m(t)\;&+\;(\mu_1-\frac{1}{2}\int_{\tau_1}^{\tau_2}p|\mu_2(p)|dp)\int_0^t\int_0^1\varphi_{mt}^2(s)dxds\\
&+\frac{\rho_2}{\beta}(\mu_1-\frac{1}{2}\int_{\tau_1}^{\tau_2}p|\mu_2(p)|dp)\int_0^t\int_0^1\varphi_{mtt}^2(s)dxds\\
&+\int_0^t\int_0^1\varphi_{mt}(s)\int_{\tau_1}^{\tau_2}|\mu_2(p)|y_m\,(x,1,p,s)\,dpdxds\\
&+\frac{\rho_2}{\beta}\int_0^t\int_0^1\varphi_{mtt}(s)\int_{\tau_1}^{\tau_2}|\mu_2(p)|y_{tm}\,(x,1,p,s)\,dpdxds\\
&+\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}p|\mu_2(p)|y_m^2(x,1,p,s)dpd\tau dxds\\
&+\frac{\rho_2}{\beta}\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}p|\mu_2(p)|y_{tm}^2(x,1,p,s)dpd\tau dxds\\
&=\;\mathcal{E}_m(0) \qquad (1.26)
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{E}_m(t)\;=\;&\frac{1}{2}\Big[\rho_1\int_0^1\varphi_{mt}^2dx+\frac{\rho_1\rho_2}{\beta}\int_0^1\varphi_{mtt}^2dx+\rho_2\int_0^1\varphi_{mtx}^2dx\\
&+\beta\int_0^1(\varphi_{mx}+\psi_m)^2dx+b\int_0^1\psi_{mx}^2dx\Big]\\
&+\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}p|\mu_2(p)|y_m^2(x,\tau,p,t)dpd\tau dx\\
&+\frac{\rho_2}{\beta}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}p|\mu_2(p)|y_{tm}^2(x,\tau,p,t)dpd\tau dx \qquad (1.27)
\end{aligned}
$$

Then we have the following cases.
(i) Consider ($\mu_1 > \int_{\tau_1}^{\tau_2} p|\mu_2(p)|dp$). Using Young's inequality, we have

$$\begin{aligned}\int_0^t \int_0^1 \varphi_{mt}(s) \int_{\tau_1}^{\tau_2} |\mu_2(p)| y_m\,(x,1,p,s)\,dpdxds \\ \geq -\frac{1}{2}(\int_{\tau_1}^{\tau_2} p|\mu_2(p)|dp) \int_0^t \int_0^1 \varphi_{mt}^2(s)dxds \\ -\frac{1}{2}\int_0^t \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(p)| y_m^2\,(x,1,p,s)\,dpdxds\end{aligned} \tag{1.28}$$

and

$$\begin{aligned}\int_0^t \int_0^1 \varphi_{mtt}(s) \int_{\tau_1}^{\tau_2} |\mu_2(p)| y_{mt}\,(x,1,p,s)\,dpdxds \\ \geq -\frac{1}{2}(\int_{\tau_1}^{\tau_2} p|\mu_2(p)|dp) \int_0^t \int_0^1 \varphi_{mtt}^2(s)dxds \\ -\frac{1}{2}\int_0^t \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(p)| y_{mt}^2\,(x,1,p,s)\,dpdxds\end{aligned} \tag{1.29}$$

which, together with (1.26), yields

$$\begin{aligned}\mathcal{E}_m(t) \quad + \quad & (\mu_1 - \int_{\tau_1}^{\tau_2} p|\mu_2(p)|dp) \int_0^t \int_0^1 \varphi_{mt}^2(s)dxds \\ & + \frac{\rho_2}{\beta}(\mu_1 - \int_{\tau_1}^{\tau_2} p|\mu_2(p)|dp) \int_0^t \int_0^1 \varphi_{mtt}^2(s)dxds \\ \leq \quad & \mathcal{E}_m(0)\end{aligned} \tag{1.30}$$

It follows from (1.12) that there exist a constants $\eta_0 = \mu_1 - \int_{\tau_1}^{\tau_2} p|\mu_2(p)|dp$ such that

$$\mathcal{E}_m(t) + \eta_0 \int_0^t \int_0^1 \varphi_{mt}^2(s)dxds + \frac{\rho_2}{\beta}\eta_0 \int_0^t \int_0^1 \varphi_{mtt}^2(s)dxds \leq \mathcal{E}_m(0) \tag{1.31}$$

(ii) Consider ($\mu_1 = \int_{\tau_1}^{\tau_2} p|\mu_2(p)|dp$) and using (1.19), we know that

$$\mathcal{E}_m(t) \leq \mathcal{E}_m(0) \tag{1.32}$$

(iii)Then, in both cases, we infer that there exists a positive constant C independent on m such that

$$\mathcal{E}_m(t) \leq C, \quad t \geq 0 \tag{1.33}$$

It follows from (1.12), and (1.33) that

$$\int_0^1 \varphi_{mt}^2 dx + \int_0^1 \varphi_{mtt}^2 dx + \rho_2 \int_0^1 \varphi_{mtx}^2 dx + \int_0^1 (\varphi_{mx} + \psi_m)^2 dx \\ + \int_0^1 \psi_{mx}^2 dx + \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)| y_m^2(x,\tau,p,t) dp d\tau dx \\ + \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)| y_{mt}^2(x,\tau,p,t) dp d\tau dx \leq C \tag{1.34}$$

Thus we can obtain $t_m = T$, for all $T > 0$.

1.2.1.3 A Priori Estimate II

Differentiating the first equation of (1.19) and multiplying by φ_{mtt}, and then integrating the result over $(0, 1)$, we have

$$\frac{\rho_1}{2}\frac{d}{dt}\int_0^1 \varphi_{mtt}^2 dx + \beta \int_0^1 (\varphi_{mx} + \psi_m)_t \varphi_{mttx} dx + \mu_1 \int_0^1 \varphi_{mtt}^2 dx \\ + \int_0^1 \varphi_{mtt} \int_{\tau_0}^{\tau_1} |\mu_2(p)| y_{mt}(x,1,p,t) dp dx = 0. \tag{1.35}$$

Differentiating the second equation of (1.19), multiplying by ψ_{mtt}, noting

$$\psi_{mttx} = \frac{1}{\beta}\left(\rho_1 \varphi_{mtttt} - \beta \varphi_{mttxx} + \mu_1 \varphi_{mttt} + \int_{\tau_0}^{\tau_1} |\mu_2(p)| y_{mtt}(x,1,p,t) dp\right),$$

and then integrating the result over $(0, 1)$, we get

$$\frac{\rho_1\rho_2}{2\beta}\frac{d}{dt}\int_0^1 \varphi_{mttt}^2 dx + \frac{\rho_2}{2}\frac{d}{dt}\int_0^1 \varphi_{mttx}^2 dx + \frac{\rho_2\mu_1}{\beta}\int_0^1 \varphi_{mtt}^2 dx \\ + \frac{b}{2}\frac{d}{dt}\int_0^1 \psi_{mxt}^2 dx + \frac{\rho_2}{\beta}\int_0^1 \varphi_{mttt} \int_{\tau_1}^{\tau_2} |\mu_2(p)| y_{mtt}(x,1,p,t) dp \\ + \beta \int_0^1 (\varphi_{mxt} + \psi_{mt}) \psi_{mtt} dx = 0. \tag{1.36}$$

The same arguments as in (1.24)-(1.25), and combining (1.35) and (1.36), we get

$$
\begin{aligned}
\mathcal{G}_m(t) &+ (\mu_1 - \frac{1}{2}\int_{\tau_1}^{\tau_2} p|\mu_2(p)|dp)\int_0^t\int_0^1 \varphi_{mtt}^2(s)dxds \\
&+\frac{\rho_2}{\beta}(\mu_1 - \frac{1}{2}\int_{\tau_1}^{\tau_2} p|\mu_2(p)|dp)\int_0^t\int_0^1 \varphi_{mttt}^2(s)dxds \\
&+ \int_0^t\int_0^1 \varphi_{mt}(s)\int_{\tau_1}^{\tau_2}|\mu_2(p)|y_m(x,1,p,s)\,dpdxds \\
&+ \frac{\rho_2}{\beta}\int_0^t\int_0^1 \varphi_{mttt}(s)\int_{\tau_1}^{\tau_2}|\mu_2(p)|y_{mtt}(x,1,p,s)\,dpdxds \\
&+ \int_0^t\int_0^1\int_{\tau_1}^{\tau_2} p|\mu_2(p)|y_{mt}^2(x,1,p,s)dpd\tau dxds \\
&+ \frac{\rho_2}{\beta}\int_0^t\int_0^1\int_{\tau_1}^{\tau_2} p|\mu_2(p)|y_{mtt}^2(x,1,p,s)dpd\tau dxds \\
&= \mathcal{G}_m(0)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{G}_m(t) &= \frac{1}{2}\Big[\rho_1\int_0^1 \varphi_{mtt}^2dx + \frac{\rho_1\rho_2}{\beta}\int_0^1\varphi_{mttt}^2dx + \rho_2\int_0^1\varphi_{mttx}^2dx \\
&\quad +\beta\int_0^1(\varphi_{mxt}+\psi_{mt})^2dx + b\int_0^1\psi_{mxt}^2dx\Big] \\
&\quad +\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} p|\mu_2(p)|y_{mt}^2(x,\tau,p,t)dpd\tau dx \\
&\quad +\frac{\rho_2}{2\beta}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} p|\mu_2(p)|y_{mtt}^2(x,\tau,p,t)dpd\tau dx
\end{aligned}
$$

Similarly to **A Priori Estimate I**, we can get there exists a positive constant C independent on m such that

$$
\mathcal{G}_m(t) \le C, \quad t \ge 0. \tag{1.37}
$$

1.2.1.4 A Priori Estimate III

Let $u_j = -\varphi_{mtxx}$ and $\theta_j = -\psi_{mtxx}$ in (1.19). We integrate the result over (0, 1) to get for any $t \geq 0$.

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\Big[\rho_1\int_0^1 \varphi_{mtt}^2 dx + \frac{\rho_1\rho_2}{\beta}\int_0^1 \varphi_{mttt}^2 dx + \rho_2\int_0^1 \varphi_{mttx}^2 dx \\
&+\beta\int_0^1 (\varphi_{mxt}+\psi_{mt})^2 dx + b\int_0^1 \psi_{mxt}^2 dx\Big] + \mu_1\int_0^1 \varphi_{mtx} dx \\
&+\int_0^1 \varphi_{mtx}\int_{\tau_1}^{\tau_2} |\mu_2(p)| y_{mt}^2(x,\tau,p,t)\,dp\,dx + \mu_1\frac{\rho_2}{2\beta}\int_0^1 \varphi_{mtt} dx \\
&+\frac{\rho_2}{\beta}\int_0^1 \varphi_{mttx}\int_{\tau_1}^{\tau_2} |\mu_2(p)| y_{mtt}^2(x,\tau,p,t)\,dp\,dx
\end{aligned}$$

By using the same arguments as in (1.24) and (1.35), we can get from (1.38) that

$$\begin{aligned}
&\mathcal{F}(t) + (\mu_1 - \int_{\tau_1}^{\tau_2} p|\mu_2(p)\,dp)\int_0^t\int_0^1 \varphi_{mtx}(s)\,dx\,ds \\
&+\frac{\rho_2}{\beta}(\mu_1 - \int_{\tau_1}^{\tau_2} p|\mu_2(p)\,dp)\int_0^t\int_0^1 \varphi_{mttx}(s)\,dx\,ds \leq \mathcal{F}(0),
\end{aligned} \tag{1.38}$$

where

$$\begin{aligned}
&\frac{1}{2}\Big[\rho_1\int_0^1 \varphi_{mtx}^2 dx + \frac{\rho_1\rho_2}{\beta}\int_0^1 \varphi_{mttx}^2 dx + \rho_2\int_0^1 \varphi_{mtxx}^2 dx \\
&+\beta\int_0^1 (\varphi_{mxx}+\psi_{mx})^2 dx + b\int_0^1 \psi_{mxx}^2 dx\Big] \\
&+\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} p|\mu_2(p)| y_{mt}^2(x,\tau,p,t)\,dp\,d\tau\,dx \\
&+\frac{\rho_2}{\beta}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} p|\mu_2(p)| y_{mtt}^2(x,\tau,p,t)\,dp\,d\tau\,dx
\end{aligned}$$

Therefore, we obtain that there exists a positive constant C independent on m such that

$$\mathcal{F}_m(t) \leq C, t \geq 0.$$

1.2.1.5 Passage to Limit

From (1.34) and (1.37), we conclude that for any $m \in \mathbb{N}$,

$$
\begin{aligned}
\varphi_m &\quad \textit{is bounded in } L^\infty(\mathbb{R}_+, H^2 \cap H_0^1)\\
\varphi_{mt} &\quad \textit{is bounded in } L^\infty(\mathbb{R}_+, H^2 \cap H_0^1)\\
\varphi_{mtt} &\quad \textit{is bounded in } L^\infty(\mathbb{R}_+, H_0^1)\\
\psi_m &\quad \textit{is bounded in } L^\infty(\mathbb{R}_+, H^2 \cap H_0^1)\\
\psi_{mt} &\quad \textit{is bounded in } L^\infty(\mathbb{R}_+, L^2)\\
y_m &\quad \textit{is bounded in } L^\infty(\mathbb{R}_+, L^2((0,1)\times(0,1)\times(\tau_1,\tau_2))\\
y_{mt} &\quad \textit{is bounded in } L^\infty(\mathbb{R}_+, L^2((0,1)\times(0,1)\times(\tau_1,\tau_2))
\end{aligned} \tag{1.39}
$$

Thus we get

$$
\begin{aligned}
\varphi_m &\quad \textit{weakly star in } L^2(\mathbb{R}_+, H^2 \cap H_0^1)\\
\varphi_{mt} &\quad \textit{weakly star in } L^2(\mathbb{R}_+, H^2 \cap H_0^1)\\
\varphi_{mtt} &\quad \textit{weakly star in } L^2(\mathbb{R}_+, H^1)\\
\psi_m &\quad \textit{weakly star in } L^2(\mathbb{R}_+, H^2 \cap H_0^1)\\
\psi_{mt} &\quad \textit{weakly star in } L^2(\mathbb{R}_+, L^2)\\
y_m &\quad \textit{weakly star in } L^2(\mathbb{R}_+, L^2((0,1)\times(0,1)\times(\tau_1,\tau_2))\\
y_{mt} &\quad \textit{weakly star in } L^2(\mathbb{R}_+, L^2((0,1)\times(0,1)\times(\tau_1,\tau_2))
\end{aligned} \tag{1.40}
$$

By (1.40), we can also deduce that φ_m, ψ_m is bounded in $L^2(\mathbb{R}_+, H^2 \cap H_0^1)$ and $\varphi_{mt}, \varphi_{mtt}$ is bounded in $L^2(\mathbb{R}_+, L^2)$. Then from Aubin-Lions theorem [70], we infer that for and, $T > 0$,

$$
\begin{aligned}
\varphi_m &\quad \textit{strongly in } L^\infty(0, T, H_0^1(0,1))\\
\varphi_{mt} &\quad \textit{strongly in } L^\infty(0, T, H_0^1(0,1))\\
\psi_m &\quad \textit{strongly in } L^\infty(0, T, H_0^1(0,1))
\end{aligned} \tag{1.41}
$$

We also obtain by Lemma 1.4 in Kim [129] that

$$
\begin{aligned}
\varphi_m &\quad \textit{strongly in } C(0, T, H_0^1(0,1))\\
\varphi_{mt} &\quad \textit{strongly in } C(0, T, H_0^1(0,1))\\
\psi_m &\quad \textit{strongly in } C(0, T, H_0^1(0,1))
\end{aligned} \tag{1.42}
$$

Then we can pass to limit the approximate problem (1.19)-(1.20) in order to get a weak solution of problem (1.15)-(1.17).

1.2.1.6 Continuous Dependence and Uniqueness

Firstly we prove the continuous dependence and uniqueness for stronger solutions of problem (1.15)-(1.17).

Let $(\varphi, \varphi_t, \varphi_{tt}, \psi, \Upsilon, \Upsilon_t)$, and $(\Gamma, \Gamma_t, \Gamma_{tt}, \Xi, \Pi, \Pi_t)$ be two global solutions of problem (1.15)-(1.17) with respect to initial data $(\varphi_0, \varphi_1, \varphi_2, \psi_0, \Theta_0, \Theta_1)$, and $(\Gamma_0, \Gamma_1, \Gamma_0, \Xi_0, \Phi_0, \Phi_1)$ respectively. Let

$$\begin{aligned} \Lambda(t) &= \varphi - \Gamma \\ \Sigma(t) &= \psi - \Xi \\ \chi(t) &= \Pi - \Phi \end{aligned} \tag{1.43}$$

Then (Λ, Σ, χ) verifies (1.15)-(1.17), and we have

$$\begin{aligned} &\rho_1 \Lambda_{tt} - \beta(\Lambda_x + \Sigma)_x + \mu_1 \Lambda_t + \int_{\tau_1}^{\tau_2} |\mu_2(p)| \Lambda_t(x, t-p)\, dp \\ &-\rho_2 \Lambda_{ttx} - b\Sigma_{xx} + \beta(\Lambda_x + \Sigma) = 0 \\ &p\chi_t(x, \tau, p, t) + \chi_\tau(x, \tau, p, t) = 0 \end{aligned} \tag{1.44}$$

Multiplying $(1.44)_1$ by Λ_t, $(1.44)_2$ by Σ_t and integrating the result over (0,1), and using the fact that

$$\begin{aligned} \beta \int_0^1 \Lambda_{tt}\Sigma_{tx} dx &= \rho_1 \int_0^1 \Lambda_{ttt}\Lambda_{tt} dx + \beta \int_0^1 \Lambda_{txx}\Lambda_{tt} dx + \mu_1 \int_0^1 \Lambda_{tt}^2 dx \\ &\quad + \int_0^1 \Lambda_{tt} \int_{\tau_1}^{\tau_2} |\mu_2(p)| \Lambda_{tt}(x, t-p)\, dp dx \end{aligned}$$

we get

$$\begin{aligned} &\frac{d}{dt}\frac{1}{2}\Big[\rho_1 \int_0^1 \Lambda_t^2 dx + \frac{\rho_1\rho_2}{\beta} \int_0^1 \Lambda_{tt}^2 dx + \rho_2 \int_0^1 \Lambda_{tx}^2 dx + \beta \int_0^1 (\Lambda_x + \Sigma)^2 dx \\ &+b \int_0^1 \Sigma_x^2 dx\Big] + \mu_1 \int_0^1 \Lambda_t^2 dx + \int_0^1 \Lambda_{mt} \int_{\tau_1}^{\tau_2} |\mu_2(p)| \chi(x, 1, p, t)\, dp dx \\ &+\mu_1 \frac{\rho_2}{\beta} \int_0^1 \Lambda_{tt}^2 dx + \frac{\rho_2}{\beta} \int_0^1 \Lambda_{tt} \int_{\tau_1}^{\tau_2} |\mu_2(p)| \chi_t(x, 1, p, t)\, dp dx \end{aligned} \tag{1.45}$$

Using Young's inequality, we get

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) &\leq c(\int_0^1 \Lambda_t^2 dx + \int_0^1 \Lambda_{tt}^2 dx)\\ &\leq c(\int_0^1 \Lambda_t^2 dx + \int_0^1 \Lambda_{tt}^2 dx + \int_0^1 \Lambda_{tx}^2 dx + \int_0^1 \Sigma_x^2 dx + \int_0^1 (\Lambda_x + \Sigma)^2 dx\\ &+ \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)|\chi^2(x,1,p,t)\, dp d\tau dx)\end{aligned}$$

where

$$\begin{aligned}\mathcal{E}(t) &= \frac{1}{2}\Big[\rho_1 \int_0^1 \Lambda_t^2 dx + \frac{\rho_1\rho_2}{\beta}\int_0^1 \Lambda_{tt}^2 dx + \rho_2 \int_0^1 \Lambda_{tx}^2 dx + \beta \int_0^1 (\Lambda_x + \Sigma)^2 dx\\ &+ b\int_0^1 \Sigma_x^2 dx\Big] + \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)|\chi^2(x,1,p,t)\, dp d\tau dx \end{aligned} \tag{1.46}$$

by integrating (1.45), we get

$$\begin{aligned}\mathcal{E}(t) &\leq \mathcal{E}(0) + C_1 \int_0^t (||\Lambda_t||^2 + ||\Lambda_{tt}||^2 + ||\Lambda_{tx}||^2 + ||\Sigma_x||^2 + ||(\Lambda_x + \Sigma)||^2\\ &+ \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)|||\chi(x,1,p,t)||^2 dp d\tau)\end{aligned} \tag{1.47}$$

on the other hand, we have

$$\begin{aligned}\mathcal{E}(t) &\geq c_0(||\Lambda_t||^2 + ||\Lambda_{tt}||^2 + ||\Lambda_{tx}||^2 + ||\Sigma_x||^2 + ||(\Lambda_x + \Sigma)||^2\\ &+ \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)|||\chi(x,1,p,t)||^2 dp d\tau)\end{aligned} \tag{1.48}$$

Applying Gronwall's inequality to (1.47), we get

$$\begin{aligned}&(||\Lambda_t||^2 + ||\Lambda_{tt}||^2 + ||\Lambda_{tx}||^2 + ||\Sigma_x||^2 + ||(\Lambda_x + \Sigma)||^2\\ &+ \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)|||(x,1,p,t)||_\chi^2 dp d\tau) \leq e^{C_2 t}\mathcal{E}(0)\end{aligned} \tag{1.49}$$

This shows that solution of problem (1.15)-(1.17) depends continuously on the initial data. This ends the proof of Theorem.

1.2.1.7 Weak solution

If $(\varphi_0, \varphi_1, \varphi_2, \varphi_3, \psi_0)$ lies in $(H_0^1(0,1) \times L^2(0,1) \times L^2(0,1) \times L^2(0,1) \times H_0^1(0,1))$, we can use density arguments to get problems (1.15)-(1.17) admit a global weak solution. This ends the proof of theorem.

1.2.2 Exponential stability

In this subsection, we will prove an exponential stability estimate for problem (1.15) – (1.17), under the assumption (1.12), and by using a multiplier technique.
Define the energy of solution as

$$\begin{aligned}\mathcal{E}(t) &= \frac{1}{2}\int_0^1\left[\rho_1\varphi_t^2 + b\psi_x^2 + \beta(\varphi_x+\psi)^2 + \frac{\rho_1\rho_2}{\beta}\varphi_{tt}^2 + \rho_2\varphi_{tx}^2\right]dx \\ &+\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} p|\mu_2(p)|y^2(x,\tau,p,t)\,dpd\tau dx\end{aligned} \tag{1.50}$$

Then we have the following lemma.

Lemma 1.2. *The energy $\mathcal{E}(t)$ satisfies*

$$\begin{aligned}\mathcal{E}'(t) &\leq -\left(\mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(p)|dp\right)\int_0^1\varphi_t^2dx - \left(\mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(p)|dp\right)\int_0^1\varphi_{tt}^2dx \\ &\leq -\eta_0\int_0^1\varphi_t^2dx - \eta_0\frac{\rho_2}{\beta}\int_0^1\varphi_{tt}^2dx \leq 0\end{aligned} \tag{1.51}$$

where $\eta_0 = \mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(p)|dp \geq 0.$

Proof. Multiplying the first equation of (1.15) by φ_t, the second equation of (1.15) by ψ_t, using integration by parts, and (1.17), we get

$$\begin{cases}\frac{\rho_1}{2}\frac{d}{dt}\int_0^1\varphi_t^2dx + \beta\int_0^1(\varphi_x+\psi)\varphi_{tx}dx + \mu_1\int_0^1\varphi_t^2dx \\ +\int_0^1\varphi_t\int_{\tau_1}^{\tau_2}|\mu_2(p)|\varphi_t(x,t-p)\,dpdx = 0 \\ \rho_2\int_0^1\varphi_{tt}\psi_{tx}dx + \frac{b}{2}\frac{d}{dt}\int_0^1\psi_x^2dx + \beta\int_0^1(\varphi_x+\psi)\psi_tdx = 0\end{cases} \tag{1.52}$$

Now, substituting: $\psi_{tx} = \frac{\rho_1}{\beta}\varphi_{ttt} - \varphi_{xxt} + \frac{\mu_1}{\beta}\varphi_{tt} + \frac{1}{\beta}\int_{\tau_1}^{\tau_2} |\mu_2(p)|\varphi_{tt}(x, t-p)\, dp$, obtained from (1.15)$_1$, into first integral of (1.52)$_2$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho_1\varphi_t^2 + b\psi_x^2 + \beta(\varphi_x + \psi)^2 + \frac{\rho_1\rho_2}{\beta}\varphi_{tt}^2 + \rho_2\varphi_{tx}^2\right] dx \\
&+\mu_1\int_0^1 \varphi_t^2 dx + \int_0^1 \varphi_t \int_{\tau_1}^{\tau_2} |\mu_2(p)| y(x,1,p,t)\, dp dx \\
&+\mu_1\frac{\rho_2}{\beta}\int_0^1 \varphi_{tt}^2 dx + \frac{\rho_2}{\beta}\int_0^1 \varphi_{tt}\int_{\tau_1}^{\tau_2} |\mu_2(p)| y_t(x,1,p,t)\, dp dx = 0
\end{aligned} \tag{1.53}$$

Secondly, multiplying (1.15)$_3$ by $y|\mu_2(p)|$, Then integrate the result, we get

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} p|\mu_2(p)| y^2(x,\tau,p,t) dp d\tau dx \\
&= -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(p)| y y_\tau(x,\tau,p,t)\, dp d\tau dx \\
&= -\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(p)| \frac{d}{d\tau} y^2(x,\tau,p,t)\, dp d\tau dx \\
&= \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(p)| (y^2(x,0,p,t) - y^2(x,1,p,t)) dp dx \\
&= \frac{1}{2}(\int_{\tau_1}^{\tau_2} |\mu_2(p)| dp)\int_0^1 \varphi_t^2 dx \\
&\quad -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(p)| y^2(x,1,p,t)\, dp dx
\end{aligned} \tag{1.54}$$

and, using (1.14), we have

$$\begin{cases} p y_{tt}(x,\tau,p,t) = -y_{t\tau}(x,\tau,p,t) \\ y_t(x,0,p,t) = \varphi_{tt}(x,t) \end{cases} \tag{1.55}$$

Multiplying (1.55) by $y_t|\mu_2(p)|$, Then integrate the result, we get

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} p|\mu_2(p)|y_t^2(x,\tau,p,t)dpd\tau dx \\
&= -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(p)|y_t y_{t\tau}(x,\tau,p,t)\,dpd\tau dx \\
&= -\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(p)|\frac{d}{d\tau}y_t^2(x,\tau,p,t)\,dpd\tau dx \\
&= \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(p)|(y_t^2(x,0,p,t)-y_t^2(x,1,p,t))dpdx \\
&= \frac{1}{2}(\int_{\tau_1}^{\tau_2} |\mu_2(p)|dp)\int_0^1 \varphi_{tt}^2 dx \\
&\quad -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(p)|y_t^2(x,1,p,t)\,dpdx. \qquad (1.56)
\end{aligned}$$

From (1.50), (1.53), (1.54), and (1.56) we get (1.51).

$$\mathcal{E}'(t) \leq -(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(p)|dp)\int_0^1 \varphi_t^2 dx - \frac{\rho_2}{\beta}(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(p)|dp)\int_0^1 \varphi_{tt}^2 dx, \qquad (1.57)$$

then, by $(A1)$, $\exists\eta_0 > 0$ so that

$$\mathcal{E}'(t) \leq -\eta_0\int_0^1 \varphi_t^2 dx - \frac{\rho_2}{\beta}\eta_0\int_0^1 \varphi_{tt}^2 dx \qquad (1.58)$$

then we obtain $\mathcal{E}$ is decreasing. □

Lemma 1.3. *The functional*

$$F_1(t) := -\frac{\mu_1}{2}\int_0^1 \varphi_t^2 dx - \beta\int_0^1 \varphi_{tx}\varphi_x dx \qquad (1.59)$$

satisfies

$$\begin{aligned}
F_1'(t) \quad \leq \quad & -\beta\int_0^1 \varphi_{tx}^2 dx + \varepsilon_1\int_0^1 \psi_x^2 dx + c(1+\frac{1}{\varepsilon_1})\int_0^1 \varphi_{tt}^2 dx \\
& + c\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(p)|y^2(x,1,p,t)dpdx \qquad (1.60)
\end{aligned}$$

Proof. Direct computation using integration by parts, we get

$$\begin{aligned} F_1'(t) &= \rho_1 \int_0^1 \varphi_{tt}^2 dx - \beta \int_0^1 \psi_x \varphi_{tt} dx - \beta \int_0^1 \varphi_{tx}^2 dx \\ &+ \int_0^1 \varphi_{tt} \int_{\tau_1}^{\tau_2} |\mu_2(p)| y(x,1,p,t) dp dx \end{aligned} \quad (1.61)$$

estimate (1.60) easily follows by using Young's and Poincare's inequalities. □

Lemma 1.4. *The functional*

$$F_2(t) := \rho_1 \int_0^1 \varphi\varphi_t dx + \frac{\mu_1}{2} \int_0^1 \varphi^2 dx + \frac{\mu_1\rho_2}{2\beta} \int_0^1 \varphi_t^2 dx + \rho_2 \int_0^1 \varphi_{tx}\varphi_x dx$$

satisfies,

$$\begin{aligned} F_2(t) &\leq -\frac{\rho_1\rho_2}{2\beta} \int_0^1 \varphi_{tt}^2 dx - \frac{\beta}{2} \int_0^1 (\varphi_x + \psi)^2 dx - b \int_0^1 \psi_x^2 dx \\ &+\rho_2 \int_0^1 \varphi_{tx}^2 dx + \rho_1 \int_0^1 \varphi_t^2 dx \\ &+c \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(p)| y^2(x,1,p,t) dp dx. \end{aligned} \quad (1.62)$$

Proof. Differentiating F_2, using integrating by parts and (1.17), we get

$$\begin{aligned} F_2'(t) &= \rho_1 \int_0^1 \varphi_t^2 dx - \beta \int_0^1 (\varphi_x + \psi)^2 dx - b \int_0^1 \psi_x^2 dx \\ &-\frac{\rho_1\rho_2}{2\beta} \int_0^1 \varphi_{tt}^2 dx + \int_0^1 \varphi \int_{\tau_1}^{\tau_2} |\mu_2(p)| y(x,1,p,t) dp dx \\ &+\rho_2 \int_0^1 \varphi_{tx}^2 dx + \frac{\rho_2}{\beta} \int_0^1 \varphi_{tt} \int_{\tau_1}^{\tau_2} |\mu_2(p)| y(x,1,p,t) dp dx. \end{aligned} \quad (1.63)$$

Using Young's, Cauchy-Schwarz and Poincaré's inequalities, we obtain (1.62) □

At this point, let us introduce the functional used by:

Lemma 1.5. *The functional*

$$\begin{aligned} F_3(t) &:= \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p e^{-p\tau} |\mu_2(p)| y^2(x,\tau,p,t) \, dp d\tau dx \\ &+ \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p e^{-p\tau} |\mu_2(p)| y_t^2(x,\tau,p,t) \, dp d\tau dx. \end{aligned}$$

satisfies,

$$
\begin{aligned}
F_3'(t) \leq & -\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)|y^2(x,\tau,p,t)\,dpd\tau dx + \mu_1 \int_0^1 \varphi_t^2 dx \\
& -\eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(p)|y^2(x,1,p,t)\,dpdx \\
& -\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)|y_t^2(x,\tau,p,t)\,dpd\tau dx + \mu_1 \int_0^1 \varphi_{tt}^2 dx \\
& -\eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(p)|y_t^2(x,1,p,t)\,dpdx \qquad (1.64)
\end{aligned}
$$

where $\eta_1 > 0$.

Proof. By differentiating F_3, and use the equation $(1.15)_3$, we get

$$
\begin{aligned}
F_3'(t) = & -2 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-p\tau}|\mu_2(p)|yy_\tau(x,\tau,p,t)\,dpd\tau dx \\
& -2 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-p\tau}|\mu_2(p)|y_t y_{t\tau}(x,\tau,p,t)\,dpd\tau dx \\
= & -\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} pe^{-p\tau}|\mu_2(p)|y^2(x,\tau,p,t)\,dpd\tau dx \\
& -\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(p)|[e^{-p}y^2(x,1,p,t) - y^2(x,0,p,t)]dpdx \\
& -\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} pe^{-p\tau}|\mu_2(p)|y_t^2(x,\tau,p,t)\,dpd\tau dx \\
& -\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(p)|[e^{-p}y_t^2(x,1,p,t) - y_t^2(x,0,p,t)]dpdx
\end{aligned}
$$

Using the equality $y(x,0,p,t) = \varphi_t(x,t)$, $y_t(x,0,p,t) = \varphi_{tt}(x,t)$ and $e^{-p} \leq e^{-p\tau} \leq 1$, for any $0 < \tau < 1$, we get

$$
\begin{aligned}
F_3'(t) = & -\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} pe^{-p\rho}|\mu_2(p)|y^2(x,\rho,p,t)\,dpd\rho dx \\
& -\int_0^1 \int_{\tau_1}^{\tau_2} e^{-p}|\mu_2(p)|y^2(x,1,p,t)\,dpdx + \int_{\tau_1}^{\tau_2} |\mu_2(p)|dp \int_0^1 \varphi_t^2 dx \\
& -\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} pe^{-p\tau}|\mu_2(p)|y_t^2(x,\tau,p,t)\,dpd\tau dx \\
& -\int_0^1 \int_{\tau_1}^{\tau_2} e^{-p}|\mu_2(p)|y_t^2(x,1,p,t)\,dpdx + \int_{\tau_1}^{\tau_2} |\mu_2(p)|dp \int_0^1 \varphi_{tt}^2 dx
\end{aligned}
$$

□

As $-e^{-p}$ is a increasing function, we have $-e^{-p} \leq -e^{-\tau_2}$, for any $p \in [\tau_1, \tau_2]$.
Then, setting $\eta_1 = e^{-\tau_2}$ and using $(A1)$, we obtain (1.64).

Theorem 1.6. *Assume (A1), there exist positive constants λ_1 and λ_2 such that the energy functional (1.50) satisfies*

$$\mathcal{E}(t) \leq \lambda_2 e^{-\lambda_1 t}, \forall t \geq 0 \tag{1.65}$$

Proof. We define a Lyapunov functional

$$\mathcal{L}(t) := N\mathcal{E}(t) + N_1 F_1(t) + N_2 F_2(t) + N_3 F_3(t). \tag{1.66}$$

where $N, N_1, N_2, N_3 > 0$. By differentiating (1.66) and using (1.51), (1.60), (1.62), and (1.64) we have

$$\begin{aligned}
\mathcal{L}'(t) \leq & -[N\eta_0 - \rho_2 N_2 - \mu_1 N_3] \int_0^1 \varphi_t^2 dx \\
& -\left[\eta_0 \frac{\rho_2}{\beta} N + \frac{\rho_1 \rho_2}{\beta} N_2 - cN_1(1 + \frac{1}{\varepsilon_1}) - \mu_1 N_3\right] \int_0^1 \varphi_{tt}^2 dx \\
& -[bN_2 - \varepsilon_1 N_1] \int_0^1 \psi_x^2 dx \\
& -\frac{\beta}{2} \int_0^1 (\varphi_x + \psi)^2 dx - [\beta N_1 - cN_2] \int_0^1 \varphi_{tx}^2 dx \\
& -[N_3 \eta_1 - cN_1 - cN_2] \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(p)| y^2(x, 1, p, t)\, dp dx \\
& -N_3 \eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)| y^2(x, \tau, p, t)\, dp d\tau dx \\
& -N_3 \eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(p)| y_t^2(x, 1, p, t)\, dp dx \\
& -N_3 \eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)| y_t^2(x, \tau, p, t)\, dp d\tau dx.
\end{aligned}$$

By setting $\varepsilon_1 = \frac{bN_2}{2N_1}$, we fixed N_2 and we choose N_1 large enough so that

$$\alpha_1 = \beta N_1 - cN_2 > 0,$$

After that, we can choose N_3 large enough such that

$$\alpha_2 = \eta_1 N_3 - cN_1 - cN_2 > 0,$$

thus, we arrive at

$$\begin{aligned}\mathcal{L}'(t) \leq & -[N\eta_0 - c]\int_0^1 \varphi_t^2 dx - \left[\eta_0 \frac{\rho_2}{\beta} N + \alpha_3 - c\right]\int_0^1 \varphi_{tt}^2 dx - \alpha_4 \int_0^1 \psi_x^2 dx \\ & -\alpha_5 \int_0^1 (\varphi_x + \psi)^2 dx - \alpha_1 \int_0^1 \varphi_{tx}^2 dx \\ & -\alpha_2 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(p)| y^2(x,1,p,t)\, dp dx \\ & -\alpha_6 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)| y^2(x,\tau,p,t)\, dp d\tau dx \\ & -\alpha_6 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(p)| y_t^2(x,1,p,t)\, dp dx \\ & -\alpha_6 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)| y_t^2(x,\tau,p,t)\, dp d\tau dx. \end{aligned} \tag{1.67}$$

where $\alpha_3 = \frac{\rho_1\rho_2}{2\beta} N_2, \alpha_4 = \frac{b}{2} N_2, \alpha_5 = \frac{\beta}{2} N_2, \alpha_6 = N_3 \eta_1$.
On the other hand, if we let

$$\mathcal{H}(t) = N_1 F_1(t) + N_2 F_2(t) + N_3 F_3(t)$$

then

$$\begin{aligned}|\mathcal{H}(t)| \leq & \frac{\mu_1}{2} N_1 \int_0^1 \varphi_t^2 dx + \beta N_1 \int_0^1 |\varphi_{tx}\varphi_x| dx + \rho_1 N_2 \int_0^1 |\varphi\varphi_t| dx \\ & + \frac{\mu_1}{2} N_2 \int_0^1 \varphi^2 dx + \frac{\mu_1\rho_2}{2\beta} N_2 \int_0^1 \varphi_t^2 dx + \rho_2 N_2 \int_0^1 |\varphi_{tx}\varphi_x| dx \\ & + N_3 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p e^{-p\tau} |\mu_2(p)| y^2(x,\tau,p,t)\, dp d\tau dx. \\ & + N_3 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p e^{-p\tau} |\mu_2(p)| y_t^2(x,\tau,p,t)\, dp d\tau dx. \end{aligned}$$

By using the inequalities of Young, Cauchy-Schwarz, and Poincaré, we get

$$\begin{aligned}|\mathcal{H}(t)| \leq & c \int_0^1 \left(\varphi_t^2 + \varphi_{tx}^2 + \psi_x^2 + \varphi_{tt}^2 + (\varphi_x + \psi)^2\right) dx \\ & + c \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)| y^2(x,\tau,p,t)\, dp d\tau \\ & + c \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)| y_t^2(x,\tau,p,t)\, dp d\tau \\ \leq & c\mathcal{E}(t). \end{aligned}$$

Consequently,

$$|\mathcal{H}(t)| = |\mathcal{L}(t) - N\mathcal{E}(t)| \leq c\mathcal{E}(t)$$

which yield

$$(N-c)\,\mathcal{E}(t) \leq \mathcal{L}(t) \leq (N+c)\,\mathcal{E}(t) \tag{1.68}$$

Now, we choose N large enough so that

$$\eta_0 \frac{\rho_2}{\beta} N + \alpha_3 - c > 0, N\eta_0 - c > 0, N - c > 0,$$

we get

$$c_1\mathcal{E}(t) \leq \mathcal{L}(t) \leq c_2\mathcal{E}(t), \forall t \geq 0 \tag{1.69}$$

and used (1.50), estimates (1.67), (1.68), respectively, we get

$$\mathcal{L}'(t) \leq -h_1\mathcal{E}(t), \forall t \geq t_0 \tag{1.70}$$

for some $h_1, c_1, c_2 > 0$.
A combination (1.70) with (1.69), gives

$$\mathcal{L}'(t) \leq -\lambda_1 \mathcal{L}(t), \tag{1.71}$$

where $\lambda_1 = \frac{h_1}{c_2}$. Finally, a simple integration of (1.71)we obtain (1.65). This completes the proof. □

1.3 Distributed delay and viscous damping in angular rotation

In this Section, we are concerned with the following system

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x + \psi)_x = 0 \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x + \psi) + \mu_1\psi_t + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(p)|\psi_t(x, t-p)\,dp = 0 \end{cases} \tag{1.72}$$

where

$$(x, p, t) \in (0,1) \times (\tau_1, \tau_2) \times (0, \infty)$$

with the initial conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), \varphi_{tt}(x,0) = \varphi_2(x) \\ \psi(x,0) = \psi_0(x), x \in (0,1) \\ \psi_t(x,-t) = f_0(x,t), x \in (0,1), t \in (0,\tau_2) \end{cases} \tag{1.73}$$

where $\varphi_0, \varphi_1, \varphi_2, \psi_0, f_0$, are given functions.
and the Dirichlet conditions

$$\varphi_x(0,t) = \varphi_x(1,t) = \psi(0,t) = \psi(1,t) = 0, \ t > 0 \tag{1.74}$$

As in [98], let use introduce a new dependent variable

$$y(x, \tau, p, t) = \psi_t(x, t - p\tau), \tag{1.75}$$

Then, by (1.75), we obtain

$$\begin{cases} py_t(x, \tau, p, t) = -y_\tau(x, \tau, p, t) \\ y(x, 0, p, t) = \psi_t(x, t) \end{cases} \tag{1.76}$$

consequently, the problem is equivalent to

$$\begin{cases} \rho_1 \varphi_{tt} - \beta(\varphi_x + \psi)_x = 0 \\ -\rho_2 \varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x + \psi) + \mu_1 \psi_t + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(p)| y(x, 1, p, t)\, dp = 0 \\ py_t(x, \tau, p, t) + y_\tau(x, \tau, p, t) = 0 \end{cases} \tag{1.77}$$

where

$$(x, \tau, p, t) \in (0, 1) \times (0, 1) \times (\tau_1, \tau_2) \times (0, \infty).$$

with the initial conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), \varphi_{tt}(x,0) = \varphi_2(x) \\ \varphi_{ttt}(x,0) = \varphi_3(x), \psi(x,0) = \psi_0(x), x \in (0,1) \\ y(x, \tau, p, 0)) = f_0(x, -\tau p), y_t(x, \tau, p, 0)) = f_1(x, -\tau p) \ \ in \ \ (0,1) \times (0,1) \times (0, \tau_2), \\ y_{tt}(x, \tau, p, 0)) = f_2(x, -\tau p) \ \ in \ \ (0,1) \times (0,1) \times (0, \tau_2), \end{cases} \tag{1.78}$$

and the Dirichlet conditions

$$\varphi_x(0,t) = \varphi_x(1,t) = \psi(0,t) = \psi(1,t) = 0, \ t > 0 \tag{1.79}$$

Similarly to Theorem 2.1, we can get the global well-posedness of problem (1.77)-1.79) given in the following theorem.

Theorem 1.7. *Assume the assumption (1.12) holds. Then, we have the following:*

(i) If the initial data $(\varphi_0, \varphi_1, \varphi_2, \varphi_3, \psi_0)$ *is in* $(H^2(0,1) \cap H^1_*(0,1)) \times (H^2(0,1) \cap H^1_*(0,1)) \times H^1_*(0,1) \times L^2_*(0,1) \times (H^2(0,1) \cap H^1_0(0,1))$, $f_0, f_1, f_2 \in H^1((0,1) \times (0,1) \times (\tau_1, \tau_2))$, *then problem (1.77)-1.79) has a*

unique stronger weak solution satisfuing

$$\begin{array}{ll} \varphi \in L^{\infty}_{loc}(\mathbb{R}^+, H^2(0,1) \cap H^1_*(0,1)), & \psi \in L^{\infty}_{loc}(\mathbb{R}^+, H^2(0,1) \cap H^1_*(0,1)) \\ \varphi_t \in L^{\infty}_{loc}(\mathbb{R}^+, H^2(0,1) \cap H^1_*(0,1)), & \varphi_{tt} \in L^{\infty}_{loc}(\mathbb{R}^+, H^1_0(0,1)) \end{array}.$$

(ii)If the initial data $(\varphi_0, \varphi_1, \varphi_2, \varphi_3, \psi_0)$ *is in* $(H^1_*(0,1) \times H^1_*(0,1) \times L^2_*(0,1) \times L^2_*(0,1) \times H^1_0(0,1))$, $f_0, f_1, f_2 \in L^2((0,1) \times (0,1) \times (\tau_1, \tau_2))$, *then problems (1.77)-(1.79) have a unique weak solution such that for any* $T > 0$,

$$\begin{array}{ll} \varphi \in C([0,T], H^1_*(0,1)) \cap C^1([0,T], L^2_*(0,1)), & \psi \in C([0,T], H^1_0(0,1)) \\ \varphi_t \in C([0,T], H^1_*(0,1)), & \varphi_{tt} \in C([0,T], L^2_*(0,1)) \end{array}.$$

where

$$H^1_*(0,1) = \varphi \in H^1(0,1) : \varphi_x = \varphi_x 1 = 0,$$

and

$$L^2_*(0,1) = \varphi \in L^2(0,1) : \varphi_x = \varphi_x 1 = 0.$$

1.3.1 Exponential stability

In this subsection, we will prove an exponential stability estimate for problem (1.77) – (1.79), under the assumption (1.12), and by using a multiplier technique.
To work, we use a several Lemmas.

Lemma 1.8. *Define the energy of solution as*

$$\begin{aligned} \mathcal{E}(t) &= \frac{1}{2}\int_0^1 \left[\rho_1 \varphi_t^2 + b\psi_x^2 + \beta(\varphi_x + \psi)^2 + \frac{\rho_1 \rho_2}{\beta}\varphi_{tt}^2 + \rho_2 \varphi_{tx}^2\right] dx \\ &+ \frac{1}{2}\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)|y^2(x, \tau, p, t)\, dp d\tau dx \end{aligned} \tag{1.80}$$

satisfies

$$\begin{aligned} \mathcal{E}'(t) &\le -\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(p)| dp\right) \int_0^1 \psi_t^2 dx \\ &\le -\eta_0 \int_0^1 \psi_t^2 dx \le 0 \end{aligned} \tag{1.81}$$

where $\eta_0 = \mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(p)|dp \geq 0.$

Proof. Multiplying the first equation of (1.77) by φ_t, the second equation of (1.77) by ψ_t, using integration by parts, and (1.79), we get

$$\begin{cases} \frac{\rho_1}{2}\frac{d}{dt}\int_0^1 \varphi_t^2 dx + \beta\int_0^1 (\varphi_x+\psi)\varphi_{tx}dx = 0 \\ +\rho_2\int_0^1 \varphi_{tt}\psi_{tx}dx + \frac{b}{2}\frac{d}{dt}\int_0^1 \psi_x^2 dx + \beta\int_0^1 (\varphi_x+\psi)\psi_t dx + \mu_1\int_0^1 \psi_t^2 dx \\ +\int_0^1 \psi_t \int_{\tau_1}^{\tau_2} |\mu_2(p)|\psi_t(x,t-p)\,dpdx = 0 \end{cases} \tag{1.82}$$

Now, substituting: $\psi_{tx} = \frac{\rho_1}{\beta}\varphi_{ttt} - \varphi_{xxt}$, obtained from $(1.77)_1$, into first integral of $(1.82)_2$ we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho_1\varphi_t^2 + b\psi_x^2 + \beta(\varphi_x+\psi)^2 + \frac{\rho_1\rho_2}{\beta}\varphi_{tt}^2 + \rho_2\varphi_{tx}^2\right]dx \\ &+\mu_1\int_0^1 \psi_t^2 dx + \int_0^1 \psi_t \int_{\tau_1}^{\tau_2} |\mu_2(p)|y(x,1,p,t)\,dpdx = 0 \end{aligned} \tag{1.83}$$

Secondly, multiplying $(1.77)_3$ by $|\mu_2(p)|$, Then integrate the result, we get

$$\begin{aligned} &\frac{d}{dt}\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} p|\mu_2(p)|y^2(x,\tau,p,t)dpd\tau dx \\ =\ & -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(p)|yy_\tau(x,\tau,p,t)\,dpd\tau dx \\ =\ & -\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(p)|\frac{d}{d\tau}y^2(x,\tau,p,t)\,dpd\tau dx \\ =\ & \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(p)|(y^2(x,0,p,t) - y^2(x,1,p,t))dpdx \\ =\ & \frac{1}{2}(\int_{\tau_1}^{\tau_2} |\mu_2(p)|dp)\int_0^1 \psi_t^2 dx \\ & -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(p)|y^2(x,1,p,t)\,dpdx \end{aligned} \tag{1.84}$$

From (1.80), (1.83), and (1.84), we get (1.81).

$$\mathcal{E}'(t) \leq -(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(p)|dp)\int_0^1 \psi_t^2 dx \tag{1.85}$$

then, by $(A1)$, there exists a $\eta_0 > 0$ so that

$$\mathcal{E}'(t) \leq -\eta_0 \int_0^1 \psi_t^2 dx \leq 0 \tag{1.86}$$

then we obtain E is decreasing. □

Lemma 1.9. *The functional*

$$F_1(t) := -\rho_1 \int_0^1 \varphi_t \varphi dx \tag{1.87}$$

satisfies

$$F_1'(t) \leq -\rho_1 \int_0^1 \varphi_t^2 dx + \varepsilon_1 \int_0^1 \psi_x^2 dx + c(1 + \frac{1}{\varepsilon_1}) \int_0^1 (\varphi_x + \psi)^2 dx \tag{1.88}$$

Proof. Direct computation using integration by parts, we get

$$\begin{aligned} F_1'(t) &= -\rho_1 \int_0^1 \varphi_t^2 dx + \beta \int_0^1 (\varphi_x + \psi)\varphi_x dx \\ &= -\rho_1 \int_0^1 \varphi_t^2 dx + \beta \int_0^1 (\varphi_x + \psi)^2 dx - \beta \int_0^1 (\varphi_x + \psi)\psi dx \end{aligned} \tag{1.89}$$

estimate (1.88) easily follows by using Young's and Poincare's inequalities. □

Lemma 1.10. *The functional*

$$F_2(t) := -\rho_2 \int_0^1 \varphi_{tx} \psi dx + \frac{\mu_1}{2} \int_0^1 \psi^2 dx + \rho_1 \int_0^1 \varphi_t \varphi dx$$

satisfies,

$$\begin{aligned} F_2(t) \leq & -\frac{b}{2} \int_0^1 \psi_x^2 dx - \beta \int_0^1 (\varphi_x + \psi)^2 dx + \varepsilon_2 \int_0^1 \varphi_{tx}^2 dx \\ & + \rho_1 \int_0^1 \varphi_t^2 dx + \frac{c}{\varepsilon_2} \int_0^1 \psi_t^2 dx \\ & + c \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(p)| y^2(x, 1, p, t) dp dx. \end{aligned} \tag{1.90}$$

Proof. Differentiating F_2, using integrating by parts and (1.79), we get

$$\begin{aligned} F_2'(t) &= -\rho_2 \int_0^1 \varphi_{ttx}\psi dx - \rho_2 \int_0^1 \varphi_{tx}\psi_t dx + \mu_1 \int_0^1 \psi dx - b \int_0^1 \psi\psi_t dx \\ &\quad + \rho_1 \int_0^1 \varphi_t^2 dx + \rho_1 \int_0^1 \varphi\varphi_{tt} dx \\ &= -b \int_0^1 \psi_x^2 dx - \beta \int_0^1 (\varphi_x + \psi)\psi dx - \int_0^1 \psi \int_{\tau_1}^{\tau_2} |\mu_2(p)| y(x,1,p,t) dp dx \\ &\quad -\rho_2 \int_0^1 \varphi_{tx}\psi_x dx + \beta \int_0^1 (\varphi_x + \psi)_x \psi dx. \end{aligned} \tag{1.91}$$

Using Young's, Cauchy-Schwarz and Poincaré's inequalities, we arrive to (1.90) □

Lemma 1.11. *The functional*

$$F_3(t) := \beta \int_0^1 \varphi_{tx}\varphi_x dx \tag{1.92}$$

satisfies

$$F_3'(t) \leq -\frac{\rho_1}{2} \int_0^1 \varphi_{tt}^2 dx + c \int_0^1 \psi_x^2 dx + \beta \int_0^1 \varphi_{tx}^2 dx \tag{1.93}$$

Proof. Direct computation using integration by parts, we obtain

$$\begin{aligned} F_3'(t) &= \beta \int_0^1 \varphi_{ttx}\varphi_x dx + \beta \int_0^1 \varphi_{tx}^2 dx \\ &= -\rho_1 \int_0^1 \varphi_{tt}^2 dx + \beta \int_0^1 \varphi_{tt}\psi_x dx + \beta \int_0^1 \varphi_{tx}^2 dx. \end{aligned} \tag{1.94}$$

Then (1.93) follows by using Young's inequality. □

Lemma 1.12. *The functional*

$$F_4(t) := -\rho_2 \int_0^1 \varphi_{tx}(\varphi_x + \psi) dx + \frac{b\rho_1}{\beta} \int_0^1 \varphi_t \psi_x dx.$$

satisfies,

$$\begin{aligned} F_4(t) &\leq -\frac{\rho_2}{2} \int_0^1 \varphi_{tx}^2 dx - \frac{\beta}{2} \int_0^1 (\varphi_x + \psi)^2 dx + c \int_0^1 \psi_t^2 dx \\ &\quad + c \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(p)| y^2(x,1,p,t) dp dx. \end{aligned} \tag{1.95}$$

Proof. Differentiating F_4, using integrating by parts and (1.79), we get

$$\begin{aligned} F_4'(t) &= -\rho_2 \int_0^1 \varphi_{ttx}(\varphi_x + \psi)dx - \rho_2 \int_0^1 \varphi_{tx}^2 dx - \rho_2 \int_0^1 \varphi_{tx}\psi_t dx \\ &\quad + \frac{b\rho_1}{\beta} \int_0^1 \varphi_{tt}\psi_x dx + \frac{b\rho_1}{\beta} \int_0^1 \varphi_t \psi_{tx} dx \\ &= -\beta \int_0^1 (\varphi_x + \psi)^2 dx - \mu_1 \int_0^1 (\varphi_x + \psi)\psi_t dx \\ &\quad - \int_0^1 (\varphi_x + \psi) \int_{\tau_1}^{\tau_2} |\mu_2(p)| y(x, 1, p, t) dp dx \\ &\quad - \rho_2 \int_0^1 \varphi_{tx}^2 dx - (\rho_2 + \frac{b\rho_1}{\beta}) \int_0^1 \varphi_{tx}\psi_t dx. \end{aligned} \tag{1.96}$$

Using Young's, Cauchy-Schwarz and Poincaré's inequalities, we obtain (1.95) □

At this point, let us introduce the functional used by:

Lemma 1.13. *The functional*

$$F_5(t) := \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p e^{-p\tau} |\mu_2(p)| y^2(x, \tau, p, t)\, dp d\tau dx.$$

satisfies,

$$\begin{aligned} F_5'(t) &\leq -\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p |\mu_2(p)| y^2(x, \tau, p, t)\, dp d\tau dx + \mu_1 \int_0^1 \psi_t^2 dx \\ &\quad - \eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(p)| y^2(x, 1, p, t)\, dp dx. \end{aligned} \tag{1.97}$$

where $\eta_1 > 0$.

Proof. By differentiating F_5, and use the (1.77), we obtain

$$\begin{aligned} F_5'(t) &= -2 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-p\tau} |\mu_2(p)| y y_\tau(x, \tau, p, t)\, dp d\tau dx \\ &= -\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p e^{-p\rho} |\mu_2(p)| y^2(x, \rho, p, t)\, dp d\rho dx \\ &\quad - \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(p)| [e^{-p} y^2(x, 1, p, t) - y^2(x, 0, p, t)] dp dx \end{aligned}$$

Using the equality $y(x,0,p,t) = \psi_t(x,t)$, and $e^{-p} \leq e^{-p\tau} \leq 1$, for any $0 < \tau < 1$, we arrive at

$$F_5'(t) = -\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} pe^{-p\rho}|\mu_2(p)|y^2(x,\rho,p,t)\,dpd\rho dx$$
$$-\int_0^1 \int_{\tau_1}^{\tau_2} e^{-p}|\mu_2(p)|y^2(x,1,p,t)\,dpdx + \int_{\tau_1}^{\tau_2} |\mu_2(p)|dp \int_0^1 \psi_t^2 dx$$

□

where $-e^{-p}$ is a increasing function, we have $-e^{-p} \leq -e^{-\tau_2}$, for any $p \in [\tau_1, \tau_2]$.
Now, choosing $\eta_1 = e^{-\tau_2}$ and using $(A1)$, we arrive at (1.97).

Theorem 1.14. *Under the assumption (A1), there exist positive constants k_1 and k_2 such that the energy functional (1.80) satisfies*

$$\mathcal{E}(t) \leq k_2 e^{-k_1 t}, \forall t \geq 0 \tag{1.98}$$

Proof. We define a Lyapunov functional

$$\mathcal{L}(t) := N\mathcal{E}(t) + N_1F_1(t) + N_2F_2(t) + F_3(t) + N_4F_4(t) + N_5F_5(t). \tag{1.99}$$

where $N, N_1, N_2, N_4, N_5 > 0$. By differentiating (1.99) and using (1.81), (1.88), (1.90), (1.93), (1.95) and (1.97) we have

$$\begin{aligned}
\mathcal{L}'(t) \leq & -[\rho_1 N_1 - cN_2] \int_0^1 \varphi_t^2 dx - \rho_1 \int_0^1 \varphi_{tt}^2 dx \\
& - \left[\frac{b}{2}N_2 - \varepsilon_1 N_1 - c\right] \int_0^1 \psi_x^2 dx \\
& - \left[\eta_0 N - \frac{c}{\varepsilon_2}N_2 - cN_4 - \mu_1 N_5\right] \int_0^1 \psi_t^2 dx \\
& - \left[\frac{\beta}{2}N_4 + \beta N_2 - cN(1 + \frac{1}{\varepsilon_1})\right] \int_0^1 (\varphi_x + \psi)^2 dx \\
& - \left[\frac{\rho_2}{2}N_4 - \beta - \varepsilon_2 N_2\right] \int_0^1 \varphi_{tx}^2 dx \\
& - [N_5\eta_1 - cN_2 - cN_4] \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(p)|y^2(x,1,p,t)\,dpdx \\
& - N_5\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)|y^2(x,\tau,p,t)\,dpd\tau dx.
\end{aligned}$$

By setting

$$\varepsilon_1 = \frac{bN_2}{4N_1}, \varepsilon_2 = \frac{\rho_2 N_4}{4N_2}.$$

At this point taking N_2 large enough so that

$$\alpha_1 = \frac{b}{4}N_2 - c > 0,$$

After that, taking N_1 large enough such that

$$\alpha_2 = \rho_1 N_1 - cN_2 > 0,$$

Next, choosing N_4 large enough such that

$$\begin{aligned} \alpha_3 &= \frac{\rho_2}{4}N_4 - \beta > 0 \\ \alpha_4 &= \frac{\beta}{2}N_4 + \beta N_2 - cN_1(1 + \frac{N_1}{N_2}) > 0 \end{aligned}$$

Finally, taking N_5 large enough such that

$$\alpha_5 = \eta_1 N_5 - cN_2 - cN_4 > 0,$$

thus, we arrive at

$$\begin{aligned} \mathcal{L}'(t) \leq\ & -\alpha_2 \int_0^1 \varphi_t^2 dx - \rho_1 \int_0^1 \varphi_{tt}^2 dx - \alpha_1 \int_0^1 \psi_x^2 dx \\ & -\alpha_3 \int_0^1 (\varphi_x + \psi)^2 dx - (\eta_0 N - c) \int_0^1 \psi_t^2 dx - \alpha_4 \int_0^1 \varphi_{tx}^2 dx \\ & -\alpha_5 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(p)| y^2 (x, 1, p, t)\, dp dx \\ & -\alpha_6 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} p|\mu_2(p)| y^2 (x, \tau, p, t)\, dp d\tau dx. \end{aligned} \tag{1.100}$$

where $\alpha_6 = N_5 \eta_1$.
On the other hand, if we let

$$\mathcal{H}(t) = N_1 F_1(t) + N_2 F_2(t) + F_3(t) + N_4 F_4(t) + N_5 F_5(t)$$

then

$$\begin{aligned}
|\mathcal{H}(t)| \leq\ & N_1\rho_1\int_0^1 \varphi_t\varphi dx + \rho_2 N_2\int_0^1 |\varphi_{tx}\psi| dx + \rho_1 N_2\int_0^1 |\varphi\varphi_t| dx \\
&+\frac{\mu_1}{2}N_2\int_0^1 \psi^2 dx + \beta\int_0^1 |\varphi_{tx}\varphi_x| dx \\
&+\rho_2 N_4\int_0^1 |\varphi_{tx}(\varphi_x+\psi)| dx + \frac{b\rho_1}{\beta}N_4\int_0^1 |\varphi_t\psi_x| dx \\
&+N_5\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} pe^{-p\tau}|\mu_2(p)| y^2(x,\tau,p,t)\, dpd\tau dx.
\end{aligned}$$

By using the inequalities of Young, Cauchy-Schwarz, and Poincaré, we obtain

$$\begin{aligned}
|\mathcal{H}(t)| \leq\ & c\int_0^1 \left(\varphi_t^2+\varphi_{tx}^2+\psi_x^2+\varphi_{tt}^2+(\varphi_x+\psi)^2\right) dx \\
&+c\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} p|\mu_2(p)| y^2(x,\tau,p,t)\, dpd\tau \\
\leq\ & c\mathcal{E}(t).
\end{aligned}$$

Consequently,

$$|\mathcal{H}(t)| = |\mathcal{L}(t) - N\mathcal{E}(t)| \leq c\mathcal{E}(t)$$

which yield

$$(N-c)\,\mathcal{E}(t) \leq \mathcal{L}(t) \leq (N+c)\,\mathcal{E}(t) \tag{1.101}$$

Now, we choose N large enough so that

$$N\eta_0 - c > 0, N - c > 0,$$

we get

$$c_3\mathcal{E}(t) \leq \mathcal{L}(t) \leq c_4\mathcal{E}(t), \forall t \geq 0 \tag{1.102}$$

and used (1.80), estimates (1.100), (1.101), respectively, we get

$$\mathcal{L}'(t) \leq -h_1\mathcal{E}(t), \forall t \geq t_0 \tag{1.103}$$

for some $h_1, c_3, c_4 > 0$.
A combination (1.103) with (1.102), gives

$$\mathcal{L}'(t) \leq -k_1\mathcal{L}(t), \tag{1.104}$$

where $k_1 = \frac{h_1}{c_4}$. Finally, a simple integration of (1.104)we obtain (1.98). This completes the proof. □

Chapter 2

A stability result for Thermoelastic Timoshenko system of second sound with distributed delay term

2.1 Introduction

In this chapter, we study the following thermoelastic Timoshenko system of second sound with distirubted delay term,

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) + \mu_1\varphi_t(x,t) + \int_{\tau_1}^{\tau_2}\mu_2(s)\varphi_t(x,t-s)\,ds = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) + \gamma\theta_x(x,t) = 0, \\ \rho_3\theta_t(x,t) + \kappa q_x(x,t) + \gamma\psi_{tx}(x,t) = 0, \\ \tau_0 q_t(x,t) + \delta q(x,t) + \kappa\theta_x(x,t) = 0, \end{cases} \tag{2.1}$$

where $t \in (0,\infty)$ denotes the time variable and $x \in (0,1)$ is the space variable, the functions φ and ψ are respectively, the transverse displacement of the solid elastic material and the rotation angle, the function θ is the temperature difference, $q = q(t,x) \in \mathbb{R}$ is the heat flux, and ρ_1, ρ_2, ρ_3, γ, τ_0, δ, κ, μ_1, and K are positive constants, $\mu_2 : [\tau_1,\tau_2] \to \mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2} |\mu_2(s)|\,ds < \mu_1, \tag{2.2}$$

where τ_1 and τ_2 two real numbers satisfying $0 \le \tau_1 \le \tau_2$. We propose the following initial and boundary conditions:

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \\ \theta(x,0) = \theta_0(x), \quad q(x,0) = q_0(x), \quad \varphi_t(x,-t) = f_0(x,t) \text{ in } (0,1)\times(0,\tau_2), \end{cases} \tag{2.3}$$

$$\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = q(0,t) = q(1,t) = 0, \qquad \forall t \geq 0. \tag{2.4}$$

Where $x \in (0,1)$ and f_0 is the history function.

Our main interest in this work is to study the existence and prove the exponential stability of the solution to problem (2.1)-(2.4). Before going on, let us first review some related results which seem to us interesting.

Questions related to stability/instability of wave equations with delay have attracted considerable attention in recent years and many authors have shown that delays can destabilize a system that is asymptotically stable in the absence of delays (see [33] for more details).

As it has been proved by Datko [32, Example 3.5], systems of the form

$$\begin{cases} w_{tt} - w_{xx} - aw_{xxt} = 0, & x \in (0,1),\ t > 0, \\ w(0,t) = 0, \qquad w_x(1,t) = -kw_t(1,t-\tau), & t > 0, \end{cases}$$

where a, k and τ are positive constants become unstable for any arbitrarily small values of τ and any values of a and k.

D. Ouchenane in [101] considered the following Timoshenko system with a delay term in the internal feedback

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) + \mu_1\varphi_t(x,t) + \mu_2\varphi_t(x,t-s)\,ds = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x + \psi)(x,t) + \gamma\theta_x(x,t) = 0, \\ \rho_3\theta_t(x,t) + \kappa q_x(x,t) + \gamma\psi_{tx}(x,t) = 0, \\ \tau_0 q_t(x,t) + \delta q(x,t) + \kappa\theta_x(x,t) = 0, \end{cases} \tag{2.5}$$

he proved the well-posedness of the system, he also established for $\mu_1 > \mu_2$ an exponential decay result.

Nicaise and Pignotti [97] considered the following problem

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + +\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \geq 0, \\ u(x,-t) = u_0(x,t), \qquad u_t(x,0) = u_1(x), & x \in \Omega,\ t \geq 0 \\ u_t(x,t-\tau) = f_0(x,t-\tau), & x \in \Omega,\ t \in (0,\tau). \end{cases} \tag{2.6}$$

Using an observability inequality obtained with a Carleman estimate, they proved that, under the assumption

$$\mu_2 < \mu_1, \tag{2.7}$$

the energy is exponentially stable. On the contrary, if (2.7) does not hold, they found a sequence of delays for which the corresponding solution of (2.6) is unstable. The same results were shown if both the damping and the delay act in the boundary of the domain.

The Timoshenko system goes back to Timoshenko [126] in 1921 who proposed a coupled hyperbolic system which is similar to (12) (with $\mu_1 = \mu_2 = 0$), describing the transverse vibration of a beam, but without the presence of any damping. For a physical derivation of Timoshenko systems, we refer the reader to [46].

In the absence of the delay in system (12), that is for $\mu_2 = 0$, the question of the stability of the Timoshenko-type systems has received a lot of attention in the last years, and quite a number of results concerning uniform and asymptotic decay of energy have been established.

An important issue of research is to look for a minimum dissipation by which solutions of the Timoshenko system decay uniformly to zero as time goes to infinity. In this regard, several types of dissipative mechanisms have been introduced, such as: frictional damping, viscoelastic damping and thermal dissipation. We recall here only some results related to the thermal dissipation in the Timoshenko systems. The interested reader is referred to [3, 83, 82, 94, 109, 123] for the Timoshenko systems with frictional damping and to [9, 51, 86, 110] for Timoshenko systems with viscoelastic damping.

To the best of our knowledge, the paper [95] is the first paper in which the authors dealt with the Timoshenko system with thermal dissipation. More precisely, they treated the problem

$$\begin{cases} \rho_1\varphi_{tt} - \sigma(\varphi_x, \psi)_x = 0, & \text{in } (0,L)\times(0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0, & \text{in } (0,L)\times(0,+\infty), \\ \rho_3\theta_t - k\theta_{xx} + \gamma\psi_{tx} = 0, & \text{in } (0,L)\times(0,+\infty), \end{cases} \tag{2.8}$$

where φ, ψ and θ are functions of (x,t) which model the transverse displacement of the beam, the rotation angle of the filament, and the difference temperature respectively. Under appropriate conditions on $\sigma, \rho_i, b, k, \gamma$, they proved several exponential decay results for the linearized system and a non exponential stability result for the case of different wave speeds.

Modeling heat conduction with the so-called Fourier law (as in (2.8)), which assumes the flux q to be proportional to the gradient of the temperature θ at the same time t,

$$q + \kappa\nabla\theta = 0, \qquad (\kappa > 0),$$

leads to the phenomenon of infinite heat propagation speed. To overcome this physical paradox in the Fourier, a number of modifications of the basic assumption on the relation between the heat flux and the temperature have been made. The common feature of these theories is that all lead to hyperbolic differential equation and the speed of propagation becomes finite. See [28] for more details. Among them Cattaneo's law,

$$\tau q_t + q + \kappa \nabla \theta = 0,$$

leading to the system with *second sound*, ([124], [105], [84]) and a suggestion by Green and Naghdi [48, 47], for thermoelastic systems introducing what is called *thermoelasticity of type III*, where the constitutive equations for the heat flux is characterized by

$$q + \kappa^* p_x + \tilde{\kappa} \nabla \theta = 0, \qquad (\tilde{\kappa} > \kappa^* > 0, \quad p_t = \theta).$$

Messaoudi and *al.* [89] studied the following problem

$$\begin{cases} \rho_1 \varphi_{tt} - \sigma(\varphi_x, \psi)_x + \mu \varphi_t = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \beta \theta_x = 0, \\ \rho_3 \theta_t + \gamma q_x + \delta \psi_{tx} = 0, \\ \tau_0 q_t + q + \kappa \theta_x = 0, \end{cases} \tag{2.9}$$

where $(x, t) \in (0, L) \times (0, \infty)$, $\varphi = \varphi(t, x)$ is the displacement vector, $\psi = \psi(t, x)$ is the rotation angle of the filament, $\theta = \theta(t, x)$ is the temperature difference, $q = q(t, x)$ is the heat flux vector, ρ_1, ρ_2, ρ_3, b, k, γ, δ, κ, μ, τ_0 are positive constants. The nonlinear function σ is assumed to be sufficiently smooth and satisfy

$$\sigma_{\varphi_x}(0, 0) = \sigma_\psi(0, 0) = k$$

and

$$\sigma_{\varphi_x \varphi_x}(0, 0) = \sigma_{\varphi_x \psi}(0, 0) = \sigma_{\psi\psi} = 0.$$

Several exponential decay results for both linear and nonlinear cases have been established.

2.2 Preliminaries

The proposed is discussed in this section, we present some materials needed in the proof of our result. We also state, without proof, a local existence result for problem (2.1). The proof can be established by using Faedo-Galerkin method. As in [97] (see also [111]). Let us introduce the following new dependent variable

$$z(x,\rho,\tau,t) = \varphi_t(x, t-\tau\rho), \qquad \text{in } (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty).$$

Then, we get the following equation

$$\begin{cases} \tau z_t(x,\rho,\tau,t) + z_\rho(x,\rho,\tau,t) = 0, \\ z(x,0,\tau,t) = \varphi_t(x,t). \end{cases}$$

Hence, we can rewrite the problem (2.1) as

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) + \mu_1 z(x,0,\tau,t) + \int_{\tau_1}^{\tau_2}\mu_2(s)\, z(x,1,s,t)ds = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x+\psi) + \gamma\theta_x = 0, \\ \rho_3\theta_t + \kappa q_x + \gamma\psi_{tx} = 0, \\ \tau_0 q_t + \delta q + \kappa\theta_x = 0, \\ \tau z_t(x,\rho,\tau,t) + z_\rho(x,\rho,\tau,t) = 0, \end{cases} \tag{2.10}$$

where $x \in (0,1)$, $\rho \in (0,1)$, and $t > 0$. The above system subjected to the following initial conditions

$$\begin{cases} \left.\begin{array}{ll} \varphi(x,0) = \varphi_0(x), & \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,0) = \psi_0(x), & \psi_t(x,0) = \psi_1(x), \\ \theta(x,0) = \theta_0(x), & q(x,0) = q_0(x), \end{array}\right\} \quad x \in (0,1) \\ z(x,\rho,\tau,0) = f_0(x,\rho,\tau), \qquad \text{in } (0,1)\times(0,1)\times(0,\tau_2). \end{cases} \tag{2.11}$$

In addition to the above initial conditions, we consider the following boundary conditions

$$\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = q(0,t) = q(1,t) = 0, \qquad \forall t \geq 0. \tag{2.12}$$

Our existence and uniqueness result reads as follows. We now consider the energy space

$$\boldsymbol{H} := \left[H_0^1(0,1)\times L^2(0,1)\right]^2 \times \left[L^2(0,1)\right]^2 \times L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)). \tag{2.13}$$

Theorem 2.1. *Let $U_0 = (\varphi_0,\varphi_1,\psi_0,\psi_1,\theta_0,q_0,f_0) \in \boldsymbol{H}$, be given, such that (2.2) satisfied, then there exists a unique global (weak) solution of problem (2.10)-(2.12) satisfying*

$$U = (\varphi, u, \psi, v, \theta, q, z) \in C([0,+\infty), \boldsymbol{H})$$

Proof. To prove this theorem, we can either the Faedo-Galerkin method, like the method used in the first chapter of this thesis or a method of semigroup like the method gollowed in the paper of Djamel

Ouchenane [101].

2.3 Exponential stability

In this section, we show that, under the assumption (2.2), the solution of problem (2.10)-(2.12) decays exponentially, independently of the wave speed assumption[1]. To achieve our goal we use the energy method to produce a suitable Lyapunov functional which leads to an exponential decay result.

In order to use the Poincaré's inequality for θ, we introduce, as in [105],

$$\overline{\theta}(x,t) = \theta(x,t) - \int_0^1 \theta_0(x)dx.$$

Then by the third equation in (2.1) we easily verify that

$$\int_0^1 \overline{\theta}\,(x,t)\,dx = 0,$$

for all $t \geq 0$. In this case the Poincaré's inequality is applicable for $\overline{\theta}$. On the other hand $(\varphi, \psi, \overline{\theta}, q, z)$ satisfies the same system (2.10) and the boundary conditions (2.12). We define the functional energy of the solution of problem (2.10)-(2.12) as

$$\begin{aligned} E\,(t) &= E\,(t, z, \varphi, \psi, \theta, q) \\ &= \frac{1}{2}\int_0^1 \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2\right)dx + \frac{1}{2}\int_0^1 \left\{K\,(\varphi_x + \psi)^2 + b\psi_x^2 + \rho_3\theta^2\right\}dx \\ &\quad + \frac{1}{2}\int_0^1 \tau_0 q^2 dx + \frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2\,(s)|\,z^2\,(x,\rho,s,t)\,dsd\rho dx. \end{aligned} \tag{2.14}$$

We multiply the first equation in (2.10) by φ_t, the second equation by ψ_t, the third equation in (2.10) by θ, and the fourth equation in (2.10) by q, we integrate by parts, using Young's inéquality we get

$$\frac{dE\,(t)}{dt} \leq -\delta\int_0^1 q^2 dx - C\int_0^1 \varphi_t^2\,(x,t)\,dx, \tag{2.15}$$

such that $C > 0$, the last inequality implies that the energy E is a non-increasing function with respect to t.

At this point we establish sevral lemmas needed for the proof of our main result.

[1]The wave speed assumption is significant only from the mathematical point of view since in practice the velocities of waves propagations may be different, see [72]. So, it is very interesting to obtain some stability results for the Timoshenko systems without the wave speed condition.

First, let us consider the functional I_1 given by

$$I_1(t) := \rho_1 \int_0^1 \varphi_t \left(-\varphi + \int_0^x \psi(y)\, dy \right) dx. \tag{2.16}$$

Then we have the following estimate.

Lemma 2.2. *Let $(\varphi, \psi, \theta, q, z)$ be the solution of (2.10)-(2.12). Then we have for any $\varepsilon_1 > 0$,*

$$\begin{aligned} \frac{dI_1(t)}{dt} \leq & -\frac{K}{2} \int_0^1 (\varphi_x + \psi)^2 dx + \varepsilon_1 \int_0^1 \psi_t^2 dx + c\left(1 + \frac{1}{\varepsilon_1}\right) \int_0^1 \varphi_t^2 dx \\ & + \frac{\mu_1}{K} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x, 1, s, t)\, ds dx, \end{aligned} \tag{2.17}$$

where $c = 1/\pi^2$ is the Poincaré constant.

Proof. Taking the derivative of (2.16), integrating by parts, we obtain

$$\begin{aligned} \frac{dI_1(t)}{dt} = & \rho_1 \int_0^1 \varphi_t \left(\int_0^x \psi_t(y)\, dy \right) dx \\ & - \int_0^1 \left(\varphi + \int_0^x \psi(y)\, dy \right) \int_{\tau_1}^{\tau_2} \mu_2(s)\, z(x, 1, s, t)\, ds dx \\ & - K \int_0^1 (\varphi_x + \psi)^2 dx + \rho_1 \int_0^1 \varphi_t^2 dx - \mu_1 \int_0^1 \varphi_t \left(\varphi + \int_0^x \psi(y)\, dy \right) dx. \end{aligned}$$

Using Young's, Poincaré's, and Cauchy-Schwarz inequalities and the fact that

$$-\mu_1 \int_0^1 \varphi_t \left(\varphi + \int_0^x \psi(y)\, dy \right) dx \leq \frac{K}{4} \int_0^1 (\varphi_x + \psi)^2 dx + c \int_0^1 \varphi_t^2 dx,$$

then lead to estimate (2.17).

Now, let w be the solution of

$$-w_{xx} = \psi_x, \qquad w(0) = w(1) = 0, \tag{2.18}$$

then we get

$$w(x, t) = -\int_0^x \psi(y, t)\, dy + x \left(\int_0^1 \psi(y, t)\, dy \right).$$

We have the following inequalities.

Lemma 2.3. *The solution of (2.18) satisfies*

$$\int_0^1 w_x^2 dx \leq \int_0^1 \psi^2 dx$$

and

$$\int_0^1 w_t^2 dx \leq \int_0^1 \psi_t^2 dx.$$

Let w be the solution of (2.18). We introduce the following functional

$$I_2(t) := \int_0^1 \left(\rho_2 \psi_t \psi + \rho_1 \varphi_t w - \frac{\gamma \tau_0}{\kappa} \psi q \right) dx. \tag{2.19}$$

Then we have the following estimate.

Lemma 2.4. *Let* $(\varphi, \psi, \theta, q, z)$ *be the solution of (2.10)-(2.12). Then we have for any* $\varepsilon_2 > 0$,

$$\begin{aligned} \frac{dI_2(t)}{dt} &\leq \left(-b + \frac{c\mu_1 \varepsilon_2}{2} + \frac{\delta \gamma \varepsilon_2 c}{2\kappa} \right) \int_0^1 \psi_x^2 dx + \left(\frac{\mu_1}{2\varepsilon_2} + \frac{\rho_1}{2\varepsilon_2} \right) \int_0^1 \varphi_t^2 dx \\ &+ \left(\rho_2 + \frac{\gamma \tau_0 \varepsilon_2}{2\kappa} + \frac{\rho_1 \varepsilon_2}{2} \right) \int_0^1 \psi_t^2 dx + \left(\frac{\gamma \tau_0}{2\kappa \varepsilon_2} + \frac{\delta \gamma}{2\kappa \varepsilon_2} \right) \int_0^1 q^2 dx \\ &+ \frac{1}{2\varepsilon_2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) \, ds dx, \end{aligned} \tag{2.20}$$

Proof. Taking the derivative of (2.16), we conclude

$$\begin{aligned} \frac{dI_2(t)}{dt} &= -b \int_0^1 \psi_x^2 dx + K \int_0^1 \varphi \psi_x dx - K \int_0^1 \psi^2 dx + \rho_2 \int_0^1 \psi_t^2 dx \\ &\quad -K \int_0^1 \varphi_x w_x dx - K \int_0^1 \psi w_x dx - \mu_1 \int_0^1 \varphi_t w dx + \rho_1 \int_0^1 \varphi_t w_t dx \\ &\quad -\frac{\gamma \tau_0}{\kappa} \int_0^1 \psi_t q dx + \frac{\delta \gamma}{\kappa} \int_0^1 \psi q dx - \int_0^1 w \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t) \, ds dx. \end{aligned}$$

We apply Young's inequality and Poincaré's inequality, we find (2.20).

Now, following [111], we define the functional

$$I_3(t) := \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)| z^2(x,1,s,t) \, ds d\rho dx. \tag{2.21}$$

Then the following result holds.

Lemma 2.5. *Let* $(\varphi, \psi, \theta, q, z)$ *be the solution of (2.10)-(2.12), then for* $C_1 > 0$, *we have*

$$\begin{aligned} \frac{dI_3(t)}{dt} &\leq -C_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)| z^2(x,1,s,t) \, ds d\rho dx \\ &\quad -C_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) \, ds dx + \mu_1 \int_0^1 \varphi_t^2 dx, \end{aligned} \tag{2.22}$$

where C_1 is a positive constant.

Proof. Differentiating (2.21), and using $z(x,0,s,t)=\varphi_t$, $e^{-s}\le e^{-s\rho}$, we get for all $\rho\in[0,1]$

$$\begin{aligned}\frac{dI_3(t)}{dt} \le & \int_0^1\int_{\tau_1}^{\tau_2} e^{-s}|\mu_2(s)|z^2(x,1,s,t)\,dsdx+\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\int_0^1\varphi_t^2dx\\ & -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s}|\mu_2(s)|z^2(x,1,s,t)\,dsd\rho dx.\end{aligned}$$

Since $s\to -e^{-s}$ is an increasing function, we have $-e^{-s}\le -e^{-\tau_2}$ for all $s\in[\tau_1,\tau_2]$. Finally, setting, $C_1=-e^{-\tau_2}$ and recalling (2.2), we obtain (2.22)

In order to obtain a negative term of $\int_0^1\psi_t^2dx$, we introduce, the following functional (see [89])

$$I_4(t):=\rho_2\rho_3\int_0^1\left(\int_0^x\theta(t,y)dy\right)\psi_t(t,x)\,dx. \tag{2.23}$$

Then we have the following estimate.

Lemma 2.6. *Let $(\varphi,\psi,\theta,q,z)$ be the solution of (2.10)-(2.12). Then for any ε_4, $\varepsilon_4'>0$, we have*

$$\begin{aligned}\frac{d}{dt}I_4(t) \le & \left(-\gamma\rho_2+\frac{\varepsilon_4\rho_2\kappa}{2}\right)\int_0^1\psi_t^2dx+\left(\frac{\varepsilon_4'\rho_3}{2}(b+\kappa c)\right)\int_0^1\psi_x^2dx\\ & +\frac{\varepsilon_4'\kappa\rho_3 c}{2}\int_0^1\varphi_x^2dx+\left(\gamma\rho_3+\frac{\rho_3}{2\varepsilon_4'}(b+2\kappa)\right)\int_0^1\theta^2dx\\ & +\frac{\rho_2\kappa}{2\varepsilon_4}\int_0^1 q^2dx.\end{aligned} \tag{2.24}$$

Proof Differentiating (2.23) and using the third equation in (2.10), we have

$$\begin{aligned}\frac{d}{dt}I_4(t) = & \int_0^1\left(\int_0^x\rho_3\theta_t dy\right)\rho_2\psi_t dx+\int_0^1\left(\int_0^x\rho_3\theta dy\right)\rho_2\psi_{tt}dx\\ = & -\int_0^1\left(\int_0^x(\kappa q_x+\gamma\psi_{tx})\,dy\right)\rho_2\psi_t dx\\ & +\int_0^1\left(\int_0^x\rho_3\theta dy\right)(b\psi_{xx}-\kappa(\varphi_x+\psi)-\gamma\theta_x)\,dx,\\ = & -\gamma\rho_2\int_0^1\psi_t^2dx-\rho_2\kappa\int_0^1 q\psi_t dx-b\rho_3\int_0^1\theta\psi_x dx\\ & +\kappa\rho_3\int_0^1\theta\varphi dx-\kappa\rho_3\int_0^1\left(\int_0^x\theta dy\right)\psi dx+\gamma\rho_3\int_0^1\theta^2dx,\end{aligned}$$

by using Young's inequality and Poincaré's inequality, we obtain (2.24).

Differentiating (2.23) and using the third equation in (2.10)

$$\begin{aligned}\frac{d}{dt}I_4(t) &= \int_0^1\left(\int_0^x \rho_3\theta_t dy\right)\rho_3\psi_t dx + \int_0^1\left(\int_0^x \rho_3\theta dy\right)\rho_2\psi_{tt}dx,\\ &= -\int_0^1\left(\int_0^x (\kappa q_x+\gamma\psi_{tx})\,dy\right)\rho_2\psi_t dx\\ &\quad+\int_0^1\left(\int_0^x \rho_3\theta dy\right)(b\psi_{xx}-K(\varphi_x+\psi)-\gamma\theta_x)\,dx,\\ &= -\gamma\rho_2\int_0^1 \psi_t^2dx-\rho_2K\int_0^1 q\psi_t dx-b\rho_3\int_0^x\theta\psi_x dx\\ &\quad+\kappa\rho_3\int_0^1\theta\varphi dx-\kappa\rho_3\int_0^1\left(\int_0^x\theta dy\right)\psi dx+\gamma\rho_3\int_0^1\theta^2dx,\end{aligned}$$

by using Young's inequality, we obtain (2.24)

Now, in order to obtain a negative term of $\int_0^1\theta^2dx$ we introduce the following functional

$$I_5(t) := -\tau_0\rho_3\int_0^L q(t,x)\left(\int_0^x\theta(t,y)dy\right)dx. \tag{2.25}$$

Then we have the following estimate.

Lemma 2.7. *Let $(\varphi,\psi,\theta,q,z)$ be the solution of (2.10)-(2.12). Then for any $\varepsilon_5,\varepsilon_5'>0$, we have*

$$\begin{aligned}\frac{dI_5(t)}{dt} &\le \left(-\rho_3\kappa+\frac{\varepsilon_5\rho_3\delta c}{2}\right)\int_0^1\theta^2dx+\frac{\varepsilon_5'\tau_0\gamma}{2}\int_0^1\psi_t^2dx\\ &\quad+\left(\tau_0\kappa+\frac{\rho_3\delta}{2\varepsilon_5}+\frac{\tau_0\gamma}{2\varepsilon_5'}\right)\int_0^1 q^2dx.\end{aligned}\tag{2.26}$$

The above Lemma was proved in [89, Inequality (33)].

We define for N, N_1, N_2 the Lyapunov functional L

$$L(t) := NE(t)+N_1I_1(t)+I_2(t)+I_3(t)+I_4(t)+N_2I_5(t). \tag{2.27}$$

Moreover, we have the following:

Lemma 2.8. *For N large enough, we have*

$$L(t)\sim E(t), \qquad \forall t\ge 0. \tag{2.28}$$

Proof. We consider the functional

$$H(t) = N_1 I_1(t) + I_2(t) + I_3(t) + I_4(t) + N_2 I_5(t),$$

we have

$$\begin{aligned} H(t) \leq\ & N_1 \int_0^1 \left| \rho_1 \varphi_t \left(\varphi + \int_0^x \psi(y)\, dy \right) \right| dx \\ & + \int_0^1 \left| \left(\rho_2 \psi_t \psi + \rho_1 \varphi_t w - \frac{\gamma \tau_0}{\kappa} \psi q \right) \right| dx \\ & + \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s \left| e^{-s\rho} \mu_2(s) \right| z^2(x, 1, s, t)\, ds d\rho dx \\ & + \int_0^1 \left| \left(\int_0^x \rho_2 \rho_3 \theta(t, y) dy \right) \psi_t(t, x) \right| dx \\ & + N_2 \int_0^L \left| \tau_0 \rho_3 q(t, x) \left(\int_0^x \theta(t, y) dy \right) \right| dx. \end{aligned} \tag{2.29}$$

By the same techniques used in the proof of Lemma (2.2)-(2.7), and exploiting Young's; Poincaré's, Cauchy-Schwarz inequalities, (2.14) and the fact that $e^{-s\rho} \leq 1$ for all $\rho \in [0, 1]$, we obtain

$$\begin{aligned} |L(t)| \leq\ & c \int_0^1 \left[\varphi_t^2 + \psi_t^2 + \psi_x^2 + (\varphi_x + \psi)^2 + \theta^2 + q^2 \right] dx \\ & + c \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)| z^2(x, 1, s, t)\, ds d\rho dx \\ \leq\ & cE(t). \end{aligned}$$

Thus, $|L(t) - NE(t)| \leq cE(t)$, which yields

$$(N - c) E(t) \leq L(t) \leq (N + c) E(t).$$

Choosing N large enough, then there exist two positive constants β_1 and β_2 such that

$$\beta_1 E(t) \leq L(t) \leq \beta_2 E(t). \tag{2.30}$$

This completes the proof.

Theorem 2.9. *Assume that (2.2). Then there exist two positive constants C and γ independent of t such that for any solution of problem (2.10)-(2.12), we have*

$$E(t) \leq C e^{-\gamma t}, \qquad \forall t \geq 0. \tag{2.31}$$

To derive the exponential decay of the solution, it is enough to construct a functional $L(t)$, equivalent to the energy $E(t)$, and satisfying

$$\frac{dL(t)}{dt} \leq -\Lambda L(t), \qquad \forall t \geq 0,$$

for some constant $\Lambda > 0$.

Proof of Theorem 2.9

To prove Theorem 2.9, Combining (2.15), (2.17), (2.20), (2.22), (2.24) and (2.26), we get

$$\begin{aligned} \frac{d}{dt}L(t) \leq & \left\{\left(-b + \frac{c\mu_1\varepsilon_2}{2} + \frac{\delta\gamma\varepsilon_2 c}{2\kappa}\right) + \left(\frac{\varepsilon_4'\rho_3}{2}(b+\kappa c)\right)\right\} \int_0^1 \psi_x^2 dx \\ & + \left\{-N_1\frac{K}{2} + \frac{\varepsilon_4'\kappa\rho_3 c}{2}\right\} \int_0^1 (\varphi_x + \psi)^2 dx \\ & + \left\{\frac{1}{2\varepsilon_2} + \frac{\mu_1}{K} - N_2 C_1\right\} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)\, dx \\ & + \left\{-CN + \left(\frac{\mu_1}{2\varepsilon_2} + \frac{\rho_1}{2\varepsilon_2}\right) + N_2\mu_1 + N_1 c\left(1 + \frac{1}{\varepsilon_1}\right)\right\} \int_0^1 \varphi_t^2 dx \\ & + \left\{\left(\rho_2 + \frac{\gamma\tau_0\varepsilon_2}{2\kappa} + \frac{\rho_1\varepsilon_2}{2}\right) + \frac{1}{2\tau} + \varepsilon_1 + \left(-\gamma\rho_2 + \frac{\varepsilon_4\rho_2\kappa}{2}\right) + \frac{\varepsilon_5'\tau_0\gamma}{2}\right\} \int_0^1 \psi_t^2 dx \\ & + \left\{-N\delta + \left(\frac{\gamma\tau_0}{2\kappa\varepsilon_2} + \frac{\delta\gamma}{2\kappa\varepsilon_2}\right) + \frac{\rho_2\kappa}{2\varepsilon_4} + \left(\tau_0\kappa + \frac{\rho_3\delta}{2\varepsilon_5} + \frac{\tau_0\gamma}{2\varepsilon_5'}\right)\right\} \int_0^1 q^2 dx \\ & + \left\{\left(\gamma\rho_3 + \frac{\rho_3}{2\varepsilon_4'}(b+2\kappa)\right) + \left(-\rho_3\kappa + \frac{\varepsilon_5\rho_3\delta c}{2}\right)\right\} \int_0^1 \theta^2 dx. \end{aligned} \tag{2.32}$$

At this point, we have to choose our constants very carefully. First, choosing $\varepsilon_1, \varepsilon_2, \varepsilon_4$ and ε_5 small enough, such that

$$\varepsilon_2\left(\frac{c\mu_1}{2} + \frac{\delta\gamma c}{2\kappa}\right) \leq \frac{b}{2}, \quad \varepsilon_1\left(\frac{K}{2} + \frac{\mu_2 c}{2}\right) \leq \frac{K}{2}, \quad \varepsilon_4 \leq \frac{\gamma}{\kappa}, \quad \varepsilon_5 \leq \frac{\kappa}{\delta c}.$$

After that, we can choose N_1, N_2 large enough such that

$$\max\left(-N_1\frac{K}{2} + \frac{\varepsilon_4'\kappa\rho_3 c}{2}, \frac{1}{2\varepsilon_2} + \frac{\mu_1}{K} - N_2 C_1\right) < 0.$$

We take ε_4' small enough such that

$$\varepsilon_4' \leq \min\left\{\frac{b}{4\rho_3(b+\kappa c)}, \frac{K}{2\kappa\rho_3 c}\right\}.$$

After that, we fix ε_5' small enough such that

$$\varepsilon_5' \leq \frac{\gamma\rho_2}{4\tau_0\gamma}.$$

Finally, once all the above constants are fixed, we choose N large enough such that

$$\begin{cases} \dfrac{CN}{2} \geq \left(\dfrac{\mu_1}{2\varepsilon_2} + \dfrac{\rho_1}{2\varepsilon_2}\right) + N_2\mu_1 + N_1 c\left(1 + \dfrac{1}{\varepsilon_1}\right), \\ \dfrac{N\delta}{2} \geq \left(\dfrac{\gamma\tau_0}{2\kappa\varepsilon_2} + \dfrac{\delta\gamma}{2\kappa\varepsilon_2}\right) + \dfrac{\rho_2\kappa}{2\varepsilon_4} + \left(\tau_0\kappa + \dfrac{\rho_3\delta}{2\varepsilon_5} + \dfrac{\tau_0\gamma}{2\varepsilon_5'}\right). \end{cases}$$

Consequently, there exists a positive constant η_1, such that (2.32) becomes

$$\begin{aligned} \frac{d}{dt}L(t) \leq & -\eta_1 \int_0^1 \left(\psi_t^2 + \psi_x^2 + \varphi_t^2 + (\varphi_x + \psi)^2 + \theta^2 + q^2\right) dx \\ & -\eta_1 \int_0^1 \int_0^1 z^2(x, \rho, t)\, d\rho dx, \end{aligned} \tag{2.33}$$

which implies by (2.14), that there exists also $\Lambda > 0$, such that

$$\frac{d}{dt}L(t) \leq -\Lambda E(t), \quad \forall t \geq 0. \tag{2.34}$$

A simple integration of (2.34) leads to.

$$L(t) \leq L(0)\, e^{-\Lambda t}, \quad \forall t \geq 0. \tag{2.35}$$

Again, the us (2.30) and (2.35) yeilds the desired result (2.31). This completes the proof of theorem 2.9.

Part II

Damped wave equations of Klein-Gordon

Chapter 3

Global nonexistence of solutions to system of Klein-Gordon equations with degenerate damping and strong source terms in viscoelasticity

3.1 Introduction

In this chapter, we consider a system of viscoelastic wave equations with degenerate damping and strong nonlinear source terms

$$\begin{cases} u_{tt} - \Delta u + m_1 u^2 + \int_0^t g(t-s)\,\Delta u(x,s)\,ds + \left(a\,|u|^k + b\,|v|^l\right)|u_t|^{m-1}u_t &= f_1(u,v), \\ v_{tt} - \Delta v + m_2 v^2 + \int_0^t h(t-s)\,\Delta v(x,s)\,ds + \left(c\,|v|^\theta + d\,|u|^\varrho\right)|v_t|^{r-1}v_t &= f_2(u,v), \end{cases} \tag{3.1}$$

where $m, r > 0, k, l, \theta, \varrho \geq 1$ and the functions $f_1(u,v), f_2(u,v)$ are defined by

$$\begin{aligned} f_1(u,v) &= a_1|u+v|^{2(\rho+1)}(u+v) + b_1|u|^\rho u|v|^{(\rho+2)} \\ f_2(u,v) &= a_1|u+v|^{2(\rho+1)}(u+v) + b_1|u|^{(\rho+2)}|v|^\rho v, a_1, b_1 > 0, \end{aligned} \tag{3.2}$$

where $\rho > -1$. In (3.1), $u = u(x,t), v = v(x,t)$, where $x \in \Omega$ is a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary $\partial\Omega$ and $t > 0, a, b, c, d, m_1, m_2 > 0$.
To above system (3.1), we add the initial conditions given by

$$(u(0), v(0)) = (u_0, v_0), (u_t(0), v_t(0)) = (u_1, v_1), x \in \Omega \tag{3.3}$$

and boundary conditions given by

$$u(x) = v(x) = 0, x \in \partial\Omega. \tag{3.4}$$

This kind of problems arise in viscoe-lasticity. Dafermos was the first who study this type in [31], where the general decay was treated. In the last decades, problems related to system (3.1) had a lot of attention and many results appeared on the existence and long time behavior of solutions. See in this directions ([24, 16, 18, 19, 20, 26, 27, 56, 79, 88, 103, 100, 107, 136, 135]) and references therein.

In the absence of viscoelastic term, some special cases of the single wave equations with nonlinear damping and nonlinear source terms in the form

$$u_{tt} - \Delta u + a|u_t|^{m-1}u_t = b|u|^{p-1}u. \tag{3.5}$$

With nonlinear damping and source terms, it arises in the quantum-field and used to describe the movement of charged electromagnetic fields. Equation (3.5) equipped with initial and bounded conditions of Dirichlet type has been extensively studied and many results regarding existence, blow up and asymptotic behavior of solutions have been obtained. Many authors have studied the single wave equations in the presence of various mechanisms of dissipation, damping and non-linear sources. See ([12, 80, 93, 45, 58, 68, 119, 130, 137]) and references therein.

In [87], authors considered the nonlinear viscoelastic system

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds + |u_t|^{m-1} u_t = f_1(u,v), \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(x,s)ds + |v_t|^{r-1} v_t = f_2(u,v), \end{cases} \tag{3.6}$$

where

$$\begin{aligned} f_1(u,v) &= a|u+v|^{2(\rho+1)}(u+v) + b|u|^{\rho}u|v|^{(\rho+2)} \\ f_2(u,v) &= a|u+v|^{2(\rho+1)}(u+v) + b|u|^{(\rho+2)}|v|^{\rho}v, \end{aligned} \tag{3.7}$$

The global nonexistence theorem for some solutions with positive energy was proved using a method applied in [114].

In [113], the authors studied the nonlinear viscoelastic system in (3.6), where they obtained the decay of solutions for system. Under some restrictions on the nonlinearities of damping and source terms, they proved that, for some class of relaxation functions and some restrictions on the initial data, the rate of decay of relaxation functions affects the rate of decay of solution for system.

In this paper, we consider system (3.1)-(3.4) and proved a global nonexistence result of solutions. We extended to result in [87] and [136] to more general cases.

3.2 Preliminaries

In this section, we present some notations and Lemmas.

We assume that the relaxation functions $g, h \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfying

$$\begin{cases} 1 - \int_0^\infty g(s)ds = l' > 0, \quad g(t) \geq 0, \quad g'(t) \leq 0, \\ 1 - \int_0^\infty h(s)ds = k' > 0, \quad h(t) \geq 0, \quad h'(t) \leq 0, \end{cases} \quad t \geq 0. \tag{3.8}$$

We introduce the "*modified*" energy functional E associated to our system

$$2E(t) = \|u_t\|_2^2 + \|v_t\|_2^2 + 2\left(m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right) + J(u,v) - 2\int_\Omega F(u,v)\,dx, \tag{3.9}$$

where $F(u, v)$ is defined for all $(u, v) \in \mathbb{R}^2$,

$$\begin{aligned} F(u,v) &= \frac{1}{2(\rho+2)}\left[uf_1(u,v) + vf_2(u,v)\right], \\ &= \frac{1}{2(\rho+2)}\left[|u+v|^{2(\rho+2)} + 2|uv|^{\rho+2}\right] \geq 0, \end{aligned}$$

where

$$\frac{\partial F}{\partial u} = f_1(u,v), \frac{\partial F}{\partial v} = f_2(u,v),$$

and

$$\begin{aligned} J(u,v) &= \left(1 - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + \left(1 - \int_0^t h(s)\,ds\right)\|\nabla v\|_2^2 \\ &+ (g \circ \nabla u) + (h \circ \nabla v). \end{aligned} \tag{3.10}$$

Noting by

$$\begin{cases} (g \circ u)(t) = \int_0^t g(t-\tau)\|u(t) - u(\tau)\|_2^2\,d\tau, \\ (h \circ v)(t) = \int_0^t h(t-\tau)\|v(t) - v(\tau)\|_2^2\,d\tau. \end{cases} \tag{3.11}$$

We suppose that ρ satisfies

$$\begin{cases} -1 < \rho, & \text{if} \quad N = 1, 2, \\ -1 < \rho \leq \frac{4-N}{N-2} & \text{if} \quad N \geq 3. \end{cases} \tag{3.12}$$

Lemma 3.1. *[114] There exist two positive constants c_0 and c_1 with the end goal that*

$$\frac{c_0}{2(\rho+2)}\left(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}\right) \leq F(u,v) \leq \frac{c_1}{2(\rho+2)}\left(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}\right).$$

Lemma 3.2. *Assume that (3.12) holds. There exists $\eta > 0$, such that for any $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$, the inequality*

$$2(\rho + 2) \int_\Omega F(u, v)\, dx \leq \eta \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right)^{\rho+2} \tag{3.13}$$

holds.

Lemma 3.3. *Let $\nu > 0$, be a real positive number and let $L(t)$ be a solution of the ordinary differential inequality*

$$\frac{dL(t)}{dt} \geq \xi L^{1+\nu}(t) \tag{3.14}$$

defined in $[0, \infty)$.
If $L(0) > 0$, then the solution does not exist for $t \geq L(0)^{-\nu} \xi^{-\nu} \nu^{-1}$.

Proof. By simple integration of (3.14), we have

$$L^{-\nu}(0) - L^{-\nu}(t) \geq \xi \nu t.$$

Then, we obtain the following estimate

$$L^{\nu}(t) \geq [L^{-\nu}(0) - \xi \nu t]^{-1}. \tag{3.15}$$

Then, the RHS of (3.15) is unbounded for

$$\xi \nu t = L^{-\nu}(0)\,.$$

□

The proof is completed.

3.3 Blow up result

Lemma 3.4. *Assume that (3.12) holds. Let (u, v) be the solution of the system (3.1)–(3.4) then the energy functional is a non-increasing function, that is, for all $t \geq 0$,*

$$\begin{aligned} E'(t) &= -\int_{\Omega}\left(|u(t)|^{k}+|v(t)|^{l}\right)|u_t(t)|^{m+1}\,dx \\ &- \int_{\Omega}\left(|v(t)|^{\theta}+|u(t)|^{\varrho}\right)|v_t(t)|^{r+1}\,dx \\ &+ \frac{1}{2}(g'\circ\nabla u)+\frac{1}{2}(h'\circ\nabla v)-\frac{1}{2}g(s)\|\nabla u\|_2^2-\frac{1}{2}h(s)\|\nabla v\|_2^2. \end{aligned} \tag{3.16}$$

Lemma 3.5. *Suppose that (3.12) holds. Let (u, v) be the solution of the system (3.1)-(3.4), then the energy functional is a non-increasing function, that is, for all $t > 0$,*

$$\begin{aligned} \frac{dE(t)}{dt} &= -\int_{\Omega}\left(|u(t)|^{k}+|v(t)|^{l}\right)|u_t(t)|^{m+1}\,dx \\ &- \int_{\Omega}\left(|v(t)|^{\theta}+|u(t)|^{\vartheta}\right)|v_t(t)|^{r+1}\,dx. \end{aligned} \tag{3.17}$$

The proof of Lemma 3.4 can be done by using a classical calculations.

Our main result reads as follows

Theorem 3.6. *Suppose that (3.12) holds. Assume further that*

$$\rho > \max\left(\frac{k+m-3}{2}, \frac{l+m-3}{2}, \frac{\theta+r-3}{2}, \frac{\varrho+r-3}{2}\right), \tag{3.18}$$

and that there exists p such that $2 < p < 2(\rho+2)$, for which

$$\max\left(\int_0^\infty g(s)\,ds, \int_0^\infty h(s)\,ds\right) < \frac{(p/2)-1}{(p/2)-1+1/(2p)}, \tag{3.19}$$

holds. Then any solution of problem (3.1)–(3.4), with initial data satisfying

$$\|\nabla u_0\|_2^2+\|\nabla v_0\|_2^2+m_1^2\|u_0\|_2^2+m_2^2\|v_0\|_2^2 > \alpha_1^2,\ \textit{and } E(0) < E_2 \tag{3.20}$$

blows up in finite time, where the constants α_1 and E_2 are defined in (3.21).

We take $a = b = c = d = 1, a_1 = b_1 = 1$ for convenience. We introduce the following constants

$$\begin{array}{rclcrcl} & & B = \eta^{\frac{1}{2(\rho+2)}}, & & \alpha_1 & = & B^{-\frac{\rho+2}{\rho+1}}, \\ E_1 & = & \left(\frac{1}{2} - \frac{1}{2(\rho+2)}\right)\alpha_1^2, & & E_2 & = & \left(\frac{1}{p} - \frac{1}{2(\rho+2)}\right)\alpha_1^2, \end{array} \tag{3.21}$$

where η is the optimal constant in (3.13).

Lemma 3.7. *[114] Suppose that (3.12), (3.18) and (3.19) hold. Let (u, v) be a solutions of (3.1)–(3.4). Assume further that $E(0) < E_2$ and*

$$\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2 + m_1^2 \|u_0\|_2^2 + m_2^2 \|v_0\|_2^2 > \alpha_1^2. \tag{3.22}$$

Then, there exists a constant $\alpha_2 > \alpha_1$ such that

$$J(t) > \alpha_2^2, \tag{3.23}$$

and

$$2(\rho+2)\int_\Omega F(u, v)\,dx \geq (B\alpha_2)^{2(\rho+2)}, \forall t \geq 0. \tag{3.24}$$

Proof of Theorem 3.6. The proof is similar to one given in [88] with the necessary modification imposed by the nature of our problem. We assume that the solutions exists for all t and we get a contradiction. We set

$$H(t) = E_2 - E(t). \tag{3.25}$$

By using the definition of $H(t)$, we obtain

$$\begin{array}{rcl} H'(t) & = & -E'(t) \\ & = & \int_\Omega \left(|u(t)|^k + |v(t)|^l\right)|u_t(t)|^{m+1}\,dx \\ & + & \int_\Omega \left(|v(t)|^\theta + |u(t)|^\varrho\right)|v_t(t)|^{r+1}\,dx \\ & - & \frac{1}{2}\left(g' \circ \nabla u\right) - \frac{1}{2}\left(h' \circ \nabla v\right) + \frac{1}{2}g(s)\|\nabla u\|_2^2 + \frac{1}{2}h(s)\|\nabla v\|_2^2 \\ & \geq & 0, \forall t \geq 0. \end{array} \tag{3.26}$$

Therefore,

$$H(0) = E_2 - E(0) > 0. \tag{3.27}$$

Then,

$$\begin{aligned} 0 &< H(0) \leq H(t) \\ &= E_2 - \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right) - \frac{J(t)}{2} \\ &+ \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right]. \end{aligned} \tag{3.28}$$

Note that from (3.8) and (3.23), we get

$$\begin{aligned} E_2 - \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) - \frac{J(t)}{2} &< E_2 - \frac{1}{2}\alpha_2^2 \\ &< E_2 - \frac{1}{2}\alpha_1^2 \\ &< E_1 - \frac{1}{2}\alpha_1^2 \\ &= -\frac{1}{2(\rho+2)}\alpha_1^2 < 0, \forall t \geq 0. \end{aligned} \tag{3.29}$$

Thus, by using (3.29) and Lemma 3.1, we get

$$\begin{aligned} 0 &< H(0) \leq H(t) \leq \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right] \\ &\leq \frac{c_1}{2(\rho+2)}\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right), \forall t \geq 0. \end{aligned} \tag{3.30}$$

We define the function M as

$$M(t) = \frac{1}{2}\int_\Omega \left(u^2 + v^2\right)(x,t)\,dx, \tag{3.31}$$

and let

$$L(t) = H^{1-\sigma}(t) + \varepsilon M'(t), \tag{3.32}$$

for ε small to be chosen later and

$$\begin{aligned} 0 &< \sigma \leq \min\left\{\frac{1}{2}, \frac{2\rho+3-(k+m)}{2(m+1)(\rho+2)}, \frac{2\rho+3-(l+m)}{2(m+1)(\rho+2)}, \right. \\ &\left. \frac{2\rho+3-(\varrho+r)}{2(r+1)(\rho+2)}, \frac{2\rho+3-(\theta+r)}{2(r+1)(\rho+2)}, \frac{2\rho+2}{4(\rho+2)}\right\}. \end{aligned} \tag{3.33}$$

By differentiation of (3.32) with respect to time and using (3.1), we get

$$\begin{aligned} L'(t) &= (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ &- \varepsilon\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \\ &- \varepsilon \int_\Omega u\left(|u(t)|^k + |v(t)|^l\right)|u_t|^{m-1} u_t dx \\ &- \varepsilon \int_\Omega v\left(|v(t)|^\theta + |u(t)|^\varrho\right)|v_t|^{r-1} v_t dx \\ &+ \varepsilon \int_\Omega \left(u f_1(u,v) + v f_2(u,v)\right) dx \\ &+ \varepsilon \int_\Omega \nabla u(t) \int_0^t g(t-s) \nabla u(\tau)\, dx ds \\ &\quad +\varepsilon \int_\Omega \nabla v(t) \int_0^t h(t-s) \nabla v(\tau)\, dx ds. \end{aligned} \tag{3.34}$$

Then,

$$\begin{aligned} L'(t) &= (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right) \\ &- \varepsilon\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \\ &- \varepsilon \int_\Omega u\left(|u(t)|^k + |v(t)|^l\right)|u_t|^{m-1} u_t dx \\ &- \varepsilon \int_\Omega v\left(|v(t)|^\theta + |u(t)|^\varrho\right)|v_t|^{r-1} v_t dx \\ &+ \varepsilon\left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right) \\ &+ \varepsilon\left(\int_0^t g(s)\, ds\right)\|\nabla u\|_2^2 + \left(\int_0^t h(s)\, ds\right)\|\nabla v\|_2^2 \\ &+ \varepsilon \int_0^t g(t-s) \int_\Omega \nabla u(t) . \left[\nabla u(\tau) - \nabla u(t)\right] dx ds \\ &\quad +\varepsilon \int_0^t h(t-s) \int_\Omega \nabla v(t) . \left[\nabla v(\tau) - \nabla v(t)\right] dx ds. \end{aligned} \tag{3.35}$$

By using Cauchy-Schwartz and Young's inequalities, we obtain the following estimate

$$\begin{aligned} &\int_0^t g(t-s) \int_\Omega \nabla u(t) . \left[\nabla u(\tau) - \nabla u(t)\right] dx ds \\ \leq\ & \int_0^t g(t-s) \|\nabla u\|_2 \|\nabla u(\tau) - \nabla u(t)\|_2\, d\tau \\ \leq\ & \lambda (g \circ \nabla u) + \frac{1}{4\lambda}\left(\int_0^t g(s)\, ds\right)\|\nabla u\|_2^2, \qquad \lambda > 0 \end{aligned} \tag{3.36}$$

and

$$\begin{aligned} & \int_0^t h(t-s) \int_\Omega \nabla v(t) \,.\, [\nabla v(\tau) - \nabla v(t)] \, dx ds \\ \leq \; & \lambda (h \circ \nabla v) + \frac{1}{4\lambda} \left(\int_0^t h(s) \, ds \right) \|\nabla v\|_2^2, \qquad \lambda > 0. \end{aligned} \tag{3.37}$$

Adding $pE(t)$ and using the definition of $H(t), E_2$ leads to

$$\begin{aligned} L'(t) \;\geq\; & (1-\sigma) H^{-\sigma}(t) H'(t) \\ & + \varepsilon \left(1 + \frac{p}{2}\right) \left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2 \right) \\ & + \varepsilon \left(\frac{p}{2} - \lambda\right) [(g \circ \nabla u) + (h \circ \nabla v)] + p\varepsilon H(t) - p\varepsilon E_2 \\ & - \varepsilon \int_\Omega u \left(|u(t)|^k + |v(t)|^l \right) |u_t|^{m-1} u_t dx \\ & - \varepsilon \int_\Omega v \left(|v(t)|^\theta + |u(t)|^\varrho \right) |v_t|^{r-1} v_t dx \\ & + \varepsilon \left(1 - \frac{p}{2(\rho+2)}\right) \left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2} \right) \\ & + \varepsilon \left[\left(\frac{p}{2} - 1\right) - \left(\frac{p}{2} - 1 + \frac{1}{4\lambda}\right) \int_0^\infty g(s) \, ds \right] \|\nabla u\|_2^2 \\ & + \varepsilon \left[\left(\frac{p}{2} - 1\right) - \left(\frac{p}{2} - 1 + \frac{1}{4\lambda}\right) \int_0^\infty h(s) \, ds \right] \|\nabla v\|_2^2, \end{aligned} \tag{3.38}$$

for some λ such that

$$a_1 = \frac{p}{2} - \lambda > 0,$$

and

$$a_2 = \left[\left(\frac{p}{2} - 1\right) - \left(\frac{p}{2} - 1 + \frac{1}{4\lambda}\right) \max \left(\int_0^\infty g(s) \, ds, \; \int_0^\infty h(s) \, ds \right) \right] > 0.$$

Then, (3.38) can be estimated as follows

$$\begin{aligned} L'(t) \;\geq\; & (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon \left(1 + \frac{p}{2}\right) \left(\|u_t\|_2^2 + \|v_t\|_2^2 \right) \\ & + \varepsilon a_1 [(g \circ \nabla u) + (h \circ \nabla v)] + p\varepsilon H(t) - p\varepsilon E_2 \\ & - \varepsilon \int_\Omega u \left(|u(t)|^k + |v(t)|^l \right) |u_t|^{m-1} u_t dx \\ & - \varepsilon \int_\Omega v \left(|v(t)|^\theta + |u(t)|^\varrho \right) |v_t|^{r-1} v_t dx \\ & + \varepsilon \left(1 - \frac{p}{2(\rho+2)}\right) \left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2} \right) \\ & + \varepsilon a_2 \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right). \end{aligned} \tag{3.39}$$

By taking $c_3 = 1 - \frac{p}{\rho+2} - 2E_2 (B\alpha_2)^{-2(\rho+2)} > 0$, since $\alpha_2 > B^{-\frac{2(\rho+2)}{\rho+1}}$. Consequently, (3.39) takes the form

$$\begin{aligned} L'(t) &\geq (1-\sigma) H^{-\sigma}(t) H'(t) \\ &+ \varepsilon\left(1+\frac{p}{2}\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2\right) \\ &+ \varepsilon a_1 \left[(g \circ \nabla u) + (h \circ \nabla v)\right] \\ &+ \varepsilon a_2 \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + p\varepsilon H(t) \\ &+ \varepsilon c_3 \left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right) \\ &- \varepsilon \int_\Omega u \left(|u(t)|^k + |v(t)|^l\right) |u_t|^{m-1} u_t dx \\ &- \varepsilon \int_\Omega v \left(|v(t)|^\theta + |u(t)|^\varrho\right) |v_t|^{r-1} v_t dx. \end{aligned} \tag{3.40}$$

By using Young's inequality, we have

$$XY \leq \frac{\delta^\alpha X^\alpha}{\alpha} + \frac{\delta^{-\beta} Y^\beta}{\beta}, \tag{3.41}$$

where $X, Y \geq 0$, $\delta > 0$ and $\alpha, \beta > 0$ such that $1/\alpha + 1/\beta = 1$, we obtain

$$\left|u\, |u_t|^{m-1} u_t\right| \leq \frac{\delta_1^{m+1}}{m+1} |u|^{m+1} + \frac{m}{m+1} \delta_1^{-(m+1)/m} |u_t|^{m+1}, \forall \delta_1 \geq 0 \tag{3.42}$$

and

$$\begin{aligned} &\int_\Omega \left(|u(t)|^k + |v(t)|^l\right) \left|u\, |u_t|^{m-1} u_t\right| dx \\ \leq\ & \frac{\delta_1^{m+1}}{m+1} \int_\Omega \left(|u(t)|^k + |v(t)|^l\right) |u|^{m+1} dx \\ &+ \frac{m}{m+1} \delta_1^{-(m+1)/m} \int_\Omega \left(|u(t)|^k + |v(t)|^l\right) |u_t|^{m+1} dx. \end{aligned} \tag{3.43}$$

Similarly, for any $\delta_2 > 0$,

$$\left|v\, |v_t|^{r-1} v_t\right| \leq \frac{\delta_2^{r+1}}{r+1} |v|^{r+1} + \frac{r}{r+1} \delta_2^{-(r+1)/r} |v_t|^{r+1}, \tag{3.44}$$

which gives

$$\begin{aligned}
&\int_{\Omega}\left(|v(t)|^{\theta}+|u(t)|^{\varrho}\right)\left|v\,|v_t|^{r-1}\,v_t\right|dx \\
\leq\;& \frac{\delta_2^{r+1}}{r+1}\int_{\Omega}\left(|v(t)|^{\theta}+|u(t)|^{\varrho}\right)|v|^{r+1}\,dx \\
+\;& \frac{r}{r+1}\delta_2^{-(r+1)/r}\int_{\Omega}\left(|v(t)|^{\theta}+|u(t)|^{\varrho}\right)|v_t|^{r+1}\,dx.
\end{aligned} \tag{3.45}$$

Then, we obtain

$$\begin{aligned}
L'(t) \;\geq\;& (1-\sigma)\,H^{-\sigma}(t)\,H'(t) \\
+\;& \varepsilon\left(1+\frac{p}{2}\right)\left(\|u_t\|_2^2+\|v_t\|_2^2+m_1^2\,\|u\|_2^2+m_2^2\,\|v\|_2^2\right) \\
+\;& \varepsilon a_1\left[(g\circ\nabla u)+(h\circ\nabla v)\right] \\
+\;& \varepsilon a_2\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)+p\varepsilon H(t) \\
+\;& \varepsilon c_3\left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2}\right) \\
-\;& \varepsilon\frac{\delta_1^{m+1}}{m+1}\int_{\Omega}\left(|u(t)|^{k}+|v(t)|^{l}\right)|u|^{m+1}\,dx \\
-\;& \varepsilon\frac{m}{m+1}\delta_1^{-\frac{(m+1)}{m}}\int_{\Omega}\left(|u(t)|^{k}+|v(t)|^{l}\right)|u_t|^{m+1}\,dx \\
-\;& \varepsilon\frac{\delta_2^{r+1}}{r+1}\int_{\Omega}\left(|v(t)|^{\theta}+|u(t)|^{\varrho}\right)|v|^{r+1}\,dx \\
-\;& \varepsilon\frac{r}{r+1}\delta_2^{-\frac{(r+1)}{r}}\int_{\Omega}\left(|v(t)|^{\theta}+|u(t)|^{\varrho}\right)|v_t|^{r+1}\,dx.
\end{aligned} \tag{3.46}$$

Choosing δ_1 and δ_2 such that

$$\delta_1^{-\frac{(m+1)}{m}} = M_1 H(t)^{-\sigma}\,,\delta_2^{-\frac{(r+1)}{r}} = M_2 H(t)^{-\sigma}\,, \tag{3.47}$$

for M_1 and M_2 large constants to be fixed later. Thus, by using (3.47), we obtain

$$\begin{aligned}
L'(t) &\geq ((1-\sigma) - M\varepsilon) H^{-\sigma}(t) H'(t) \\
&+ \varepsilon\left(1+\frac{p}{2}\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right) \\
&+ \varepsilon a_1\left[(g\circ\nabla u) + (h\circ\nabla v)\right] \\
&+ \varepsilon a_2\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + p\varepsilon H(t) \\
&+ \varepsilon c_3\left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right) \\
&- \varepsilon M_1^{-m} H^{\sigma m}(t)\int_\Omega\left(|u(t)|^k + |v(t)|^l\right)|u|^{m+1}\,dx \\
&- \varepsilon\frac{m}{m+1}\delta_1^{-\frac{(m+1)}{m}}\int_\Omega\left(|u(t)|^k + |v(t)|^l\right)|u_t|^{m+1}\,dx \\
&- \varepsilon M_2^{-r} H^{\sigma r}(t)\int_\Omega\left(|v(t)|^\theta + |u(t)|^\varrho\right)|v|^{r+1}\,dx \\
&- \varepsilon\frac{r}{r+1}\delta_2^{-\frac{(r+1)}{r}}\int_\Omega\left(|v(t)|^\theta + |u(t)|^\varrho\right)|v_t|^{r+1}\,dx,
\end{aligned} \tag{3.48}$$

where $M = m/(m+1)M_1 + r/(r+1)M_2$. Therefore, we have

$$\int_\Omega\left(|u(t)|^k + |v(t)|^l\right)|u|^{m+1}\,dx = \|u\|_{k+m+1}^{k+m+1} + \int_\Omega |v|^l\,|u|^{m+1}\,dx, \tag{3.49}$$

and

$$\int_\Omega\left(|v(t)|^\theta + |u(t)|^\varrho\right)|v|^{r+1}\,dx = \|v\|_{\theta+r+1}^{\theta+r+1} + \int_\Omega |u|^\varrho\,|v|^{r+1}\,dx. \tag{3.50}$$

Also by using Young's inequality, we obtain

$$\begin{aligned}
\int_\Omega |v|^l\,|u|^{m+1} &\leq \frac{l}{l+m+1}\delta_1^{(l+m+1)/l}\|v\|_{l+m+1}^{l+m+1} \\
&+ \frac{m+1}{l+m+1}\delta_1^{-(l+m+1)/(m+1)}\|u\|_{l+m+1}^{l+m+1}, \\
\int_\Omega |u|^\varrho\,|v|^{r+1} &\leq \frac{\varrho}{\varrho+r+1}\delta_2^{(\varrho+r+1)/\varrho}\|u\|_{\varrho+r+1}^{\varrho+r+1} \\
&+ +\frac{r+1}{\varrho+r+1}\delta_2^{-(\varrho+r+1)/(r+1)}\|v\|_{\varrho+r+1}^{\varrho+r+1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& H^{\sigma m}(t)\int_\Omega\left(|u(t)|^k + |v(t)|^l\right)|u|^{m+1}\,dx \\
= \; & H^{\sigma m}(t)\|u\|_{k+m+1}^{k+m+1} + \frac{l}{l+m+1}\delta_1^{(l+m+1)/l}H^{\sigma m}(t)\|v\|_{l+m+1}^{l+m+1} \\
+ \; & \frac{m+1}{l+m+1}\delta_1^{-(l+m+1)/(m+1)}H^{\sigma m}(t)\|u\|_{l+m+1}^{l+m+1},
\end{aligned} \tag{3.51}$$

and

$$
\begin{aligned}
& H^{\sigma r}(t) \int_{\Omega} \left(|v(t)|^{\theta} + |u(t)|^{\varrho} \right) |v|^{r+1} \, dx \\
= \; & H^{\sigma r}(t) \|v\|_{\theta+r+1}^{\theta+r+1} + \frac{\varrho}{\varrho+r+1} \delta_2^{\frac{\varrho+r+1}{\varrho}} H^{\sigma r}(t) \|u\|_{\varrho+r+1}^{\varrho+r+1} \\
+ \; & \frac{r+1}{\varrho+r+1} \delta_2^{-\frac{(\varrho+r+1)}{r+1}} H^{\sigma r}(t) \|v\|_{\varrho+r+1}^{\varrho+r+1}.
\end{aligned} \tag{3.52}
$$

Since (3.18) holds, we get by using (3.33)

$$
\begin{cases}
H^{\sigma m}(t) \|u\|_{k+m+1}^{k+m+1} \le c_5 \left(\|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)+k+m+1} + \|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)} \|u\|_{k+m+1}^{k+m+1} \right), \\
H^{\sigma r}(t) \|v\|_{\theta+r+1}^{\theta+r+1} \le c_6 \left(\|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)+\theta+r+1} + \|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)} \|v\|_{\theta+r+1}^{\theta+r+1} \right).
\end{cases} \tag{3.53}
$$

This implies

$$
\begin{aligned}
& \tfrac{l}{l+m+1} \delta_1^{\frac{l+m+1}{l}} H^{\sigma m}(t) \|v\|_{l+m+1}^{l+m+1} \\
& \le c_7 \tfrac{l}{l+m+1} \delta_1^{\frac{l+m+1}{l}} \left(\|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)+l+m+1} + \|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)} \|v\|_{l+m+1}^{l+m+1} \right),
\end{aligned} \tag{3.54}
$$

and

$$
\begin{aligned}
& \tfrac{\varrho}{\varrho+r+1} \delta_2^{\frac{\varrho+r+1}{\varrho}} H^{\sigma r}(t) \|u\|_{\varrho+r+1}^{\varrho+r+1} \\
& \le c_8 \tfrac{\varrho}{\varrho+r+1} \delta_2^{\frac{\varrho+r+1}{\varrho}} \left(\|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)+\varrho+r+1} + \|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)} \|u\|_{\varrho+r+1}^{\varrho+r+1} \right).
\end{aligned} \tag{3.55}
$$

Using (3.33) and the algebraic inequality

$$
z^{\nu} \le (z+1) \le \left(1 + \frac{1}{a}\right)(z+a), \forall z \ge 0, 0 < \nu \le 1, a > 0, \tag{3.56}
$$

we get, for all $t \ge 0$,

$$
\begin{cases}
\|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)+k+m+1} \le d \left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + H(0) \right) \le d \left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + H(t) \right), \\
\|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)+\theta+r+1} \le d \left(\|v\|_{2(\rho+2)}^{2(\rho+2)} + H(t) \right), \forall t \ge 0,
\end{cases} \tag{3.57}
$$

where $d = 1 + 1/H(0)$. Similarly

$$
\begin{cases}
\|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)+l(m+1)} \le d \left(\|v\|_{2(\rho+2)}^{2(\rho+2)} + H(0) \right) \le d \left(\|v\|_{2(\rho+2)}^{2(\rho+2)} + H(t) \right), \\
\|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)+\varrho(r+1)} \le d \left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + H(t) \right), \forall t \ge 0.
\end{cases} \tag{3.58}
$$

Also, since

$$
(X+Y)^s \le C(X^s + Y^s), \qquad X, Y \ge 0, s > 0, \tag{3.59}
$$

by using (3.33) and (3.56) we have

$$\begin{aligned} \|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)} \|u\|_{k+m+1}^{k+m+1} &\leq c_9 \left(\|v\|_{2(\rho+2)}^{2(\rho+2)} + \|u\|_{k+m+1}^{2(\rho+2)} \right) \\ &\leq c_{10} \left(\|v\|_{2(\rho+2)}^{2(\rho+2)} + \|u\|_{2(\rho+2)}^{2(\rho+2)} \right), \end{aligned} \tag{3.60}$$

similarly

$$\|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)} \|v\|_{\theta+r+1}^{\theta+r+1} \leq c_{11} \left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} \right), \tag{3.61}$$

$$\|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)} \|v\|_{l+m+1}^{l+m+1} \leq c_{12} \left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} \right) \tag{3.62}$$

and

$$\|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)} \|u\|_{\varrho+r+1}^{\varrho+r+1} \leq c_{13} \left(\|v\|_{2(\rho+2)}^{2(\rho+2)} + \|u\|_{2(\rho+2)}^{2(\rho+2)} \right). \tag{3.63}$$

Taking into account (3.51)-(3.63), then (3.48) written as

$$\begin{aligned} L'(t) &\geq ((1-\sigma) - M\varepsilon) H^{-\sigma}(t) H'(t) \\ &+ 2\varepsilon \left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2 \right) \\ &+ \varepsilon \left[2 - C M_1^{-m} \left(1 + \frac{l}{l+m+1} \delta_1^{\frac{l+m+1}{l}} + \frac{m+1}{l+m+1} \delta_1^{-\frac{(l+m+1)}{m+1}} \right) \right. \\ &- \left. C M_2^{-r} \left(1 + \frac{\varrho}{\varrho+r+1} \delta_2^{\frac{\varrho+r+1}{\varrho}} + \frac{r+1}{\varrho+r+1} \delta_2^{-\frac{(\varrho+r+1)}{r+1}} \right) \right] H(t) \\ &+ \varepsilon \left[c_4 - C M_1^{-m} \left(1 + \frac{l}{l+m+1} \delta_1^{\frac{l+m+1}{l}} + \frac{m+1}{l+m+1} \delta_1^{-\frac{(l+m+1)}{m+1}} \right) \right. \\ &- \left. C M_2^{-r} \left(1 + \frac{\varrho}{\varrho+r+1} \delta_2^{\frac{\varrho+r+1}{\varrho}} + \frac{r+1}{\varrho+r+1} \delta_2^{-\frac{(\varrho+r+1)}{r+1}} \right) \right] \\ &\times \left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} \right). \end{aligned} \tag{3.64}$$

At this point and for large values of M_1 and M_2, we can find positive constants Λ_1 and Λ_2 such that (3.64) becomes

$$\begin{aligned} L'(t) &\geq ((1-\sigma) - M\varepsilon) H^{-\sigma}(t) H'(t) \\ &+ 2\varepsilon \left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2 \right) \\ &+ \varepsilon \Lambda_1 \left(\|u(t)\|_{2(\rho+2)}^{2(\rho+2)} + \|v(t)\|_{2(\rho+2)}^{2(\rho+2)} \right) + \varepsilon \Lambda_2 H(t). \end{aligned} \tag{3.65}$$

Once M_1 and M_2 are fixed (hence Λ_1 and Λ_2), we choose ε small enough so that $((1-\sigma) - M\varepsilon) \geq 0$ and

$$L(0) = H^{1-\sigma}(0) + \varepsilon \int_\Omega [u_0.u_1 + v_0.v_1]\, dx > 0. \tag{3.66}$$

Therefore, there exists $\Gamma > 0$ such that (3.65) can be written as

$$L^{'}(t) \geq \varepsilon\Gamma\left(H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right). \tag{3.67}$$

Then, we have $L(t) \geq L(0) > 0$, for all $t \geq 0$. Next, by using Holder's and Young's inequalities, we have the estimate

$$\begin{aligned} &\left(\int_\Omega u.u_t(x,t)\,dx + \int_\Omega v.v_t(x,t)\,dx\right)^{\frac{1}{1-\sigma}} \\ \leq\ & C\left(\|u\|_{2(\rho+2)}^{\frac{\tau}{1-\sigma}} + \|u_t\|_2^{\frac{s}{1-\sigma}} + \|v\|_{2(\rho+2)}^{\frac{\tau}{1-\sigma}} + \|v_t\|_2^{\frac{s}{1-\sigma}}\right), \end{aligned} \tag{3.68}$$

for $1/\tau + 1/s = 1$. We takes $s = 2(1-\sigma)$, to get $\dfrac{\tau}{1-\sigma} = \dfrac{2}{1-2\sigma}$. From (3.25) and (3.56), we have

$$\|u\|_{2(\rho+2)}^{\frac{2}{1-2\sigma}} \leq d\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + H(t)\right), \tag{3.69}$$

and

$$\|v\|_{2(\rho+2)}^{\frac{2}{1-2\sigma}} \leq d\left(\|v\|_{2(\rho+2)}^{2(\rho+2)} + H(t)\right), \forall t \geq 0. \tag{3.70}$$

Consequently, (3.68) can be written as

$$\begin{aligned} &\left(\int_\Omega uu_t(x,t)\,dx + \int_\Omega vv_t(x,t)\,dx\right)^{\frac{1}{1-\sigma}} \\ \leq\ & c_{14}\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|u_t\|_2^2 + \|v_t\|_2^2\right) \\ +\ & c_{14}\left(m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2 + H(t)\right), \forall t \geq 0. \end{aligned}$$

Also, we have

$$\begin{aligned} L^{\frac{1}{1-\sigma}}(t) &= \left(H^{1-\sigma}(t) + \varepsilon\int_\Omega (u.u_t + v.v_t)(x,t)\,dx\right)^{\frac{1}{(1-\sigma)}} \\ &\leq c_{15}\left(H(t) + \left|\int_\Omega (u.u_t(x,t) + v.v_t(x,t))\,dx\right|^{\frac{1}{(1-\sigma)}}\right) \\ &\leq c_{16}\left[H(t) + \|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|u_t\|_2^2\right] \\ &+ c_{16}\left[\|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right], \forall t \geq 0, \end{aligned} \tag{3.71}$$

from (3.71) and (3.67), we get

$$L^{'}(t) \geq a_0 L^{\frac{1}{1-\sigma}}(t), \forall t \geq 0. \tag{3.72}$$

Finally, a simple integration of (3.72) gives the desired result. □

Chapter 4

Global nonexistence of solution for coupled nonlinear Klein-Gordon with degenerate damping and source terms

4.1 Introduction

In this chapter, we consider the following system:

$$\begin{cases} u_{tt} - \Delta u_t - div\left(|\nabla u|^{\alpha-2}\nabla u\right) - div\left(|\nabla u_t|^{\beta_1-2}\nabla u_t\right) + a_1|u_t|^{m-2}u_t + m_1u^2 & = f_1(u,v), \\ v_{tt} - \Delta v_t - div\left(|\nabla v|^{\alpha-2}\nabla v\right) - div\left(|\nabla v_t|^{\beta_2-2}\nabla v_t\right) + a_2|v_t|^{r-2}v_t + m_2v^2 & = f_2(u,v), \end{cases} \tag{4.1}$$

where $u = u(t,x)$, $v = v(t,x)$, $x \in \Omega$, a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary $\partial\Omega$, $t > 0$ and $a_1, a_2, b_1, b_2, m_1, m_2 > 0$ and $\beta_1, \beta_2, m, r \geq 2$, $\alpha > 2$, and the two functions $f_1(u,v)$ and $f_2(u,v)$ given by

$$\begin{aligned} f_1(u,v) &= b_1|u+v|^{2(\rho+1)}(u+v) + b_2|u|^{\rho}u|v|^{(\rho+2)} \\ f_2(u,v) &= b_1|u+v|^{2(\rho+1)}(u+v) + b_2|u|^{(\rho+2)}|v|^{\rho}v, \end{aligned} \tag{4.2}$$

The System (4.1) is supplemented by the following initial and boundary conditions

$$\begin{cases} (u(0),v(0)) = (u_0,v_0), (u_t(0),v_t(0)) = (u_1,v_1), & x \in \Omega \\ u(x) = v(x) = 0 & x \in \partial\Omega, \end{cases} \tag{4.3}$$

Originally the interaction between the source term and the damping term in the wave equation is given by :

$$u_{tt} - \Delta u + a|u_t|^{m-2}u_t = b|u|^{p-2}u, \text{ in } \Omega \times (0,T), \tag{4.4}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$ with a smooth boundary $\partial\Omega$, has an exciting history.
It has been shown that the existence and the asymptotic behavior of solutions depend on a crucial way on the parameters m, p and on the nature of the initial data. More precisely, it is well known that in the absence of the source term $|u|^{p-2}\,u$ then a uniform estimate of the form

$$\|u_t(t)\|_2 + \|\nabla u(t)\|_2 \leq C, \tag{4.5}$$

holds for any initial data $(u_0, u_1) = (u(0), u_t(0))$ in the energy space $H_0^1(\Omega)\times L^2(\Omega)$, where C is a positive constant independent of t. The estimate (4.5) shows that any local solution u of problem (4.4) can be continued in time as long as (4.5) is verified. This result has been proved by several authors. See for example [54, 63]. On the other hand in the absence of the damping term $|u_t|^{m-2}\,u_t$, the solution of (4.4) ceases to exist and there exists a finite value T^* such that

$$\lim_{t\to T^*} \|u(t)\|_p = +\infty, \tag{4.6}$$

the reader is refereed to Ball [14] and Kalantarov & Ladyzhenskaya [60] for more details.

When both terms are present in equation (4.4), the situation is more delicate. This case has been considered by Levine in [66, 67], where he investigated problem (4.4) in the linear damping case ($m = 2$) and showed that any local solution u of (4.4) cannot be continued in $(0, \infty) \times \Omega$ whenever the initial data are large enough (negative initial energy). The main tool used in [66] and [67] is the "concavity method". This method has been a widely applicable tool to prove the blow up of solutions in finite time of some evolution equations. The basic idea of this method is to construct a positive functional $\theta(t)$ depending on certain norms of the solution and show that for some $\gamma > 0$, the function $\theta^{-\gamma}(t)$ is a positive concave function of t. Thus there exists T^* such that $\lim_{t\to T^*}\theta^{-\gamma}(t) = 0$. Since then, the concavity method became a powerful and simple tool to prove blow up in finite time for other related problems. Unfortunately, this method is limited to the case of a linear damping. Georgiev and Todorova [45] extended Levine's result to the nonlinear damping case ($m > 2$). In their work, the authors considered the problem (4.4) and introduced a method different from the one known as the concavity method. They showed that solutions with negative energy continue to exist globally 'in time' if the damping term dominates the source term (i.e.$m \geq p$) and blow up in finite time in the other case (i.e.$p > m$) if the initial energy is sufficiently negative. Their method is based on the construction of an auxiliary function L which is a perturbation of the total energy of the system and satisfies the differential inequality

$$\frac{dL(t)}{dt} \geq \xi L^{1+\nu}(t) \tag{4.7}$$

In $[0, \infty)$, where $\nu > 0$. Inequality (4.7) leads to a blow up of the solutions in finite tim $t \geq L(0)^{-\nu}\,\xi^{-1}\nu^{-1}$, provided that $L(0) > 0$. However the blow up result in [45] was not optimal in terms of the initial data

causing the finite time blow up of solutions. Thus several improvement have been made to the result in [45] (see for example [68, 78, 130]. In particular, Vitillaro in [130] combined the arguments in [45] and [68] to extend the result in [45] to situations where the damping is nonlinear and the solution has positive initial energy.
In [139], Yang.Z, studied the problem

$$u_{tt} - \Delta u_t - div\left(|\nabla u|^{\alpha-2} \nabla u\right) - div\left(|\nabla u_t|^{\beta-2} \nabla u_t\right) + a\,|u_t|^{m-2} u_t = b|u|^{p-2}u, \tag{4.8}$$

in $(0,T)\times\Omega$ with initial conditions and boundary condition of Dirichlet type. He showed that solutions blow up in finite time T^* under the condition $p > \max\{\alpha, m\}$, $\alpha > \beta$, and the initial energy is sufficiently negative (see condition (ii) in [139, Theorem 2.1]). In fact this condition made it clear that there exists a certain relation between the blow-up time and $|\Omega|$[139, Remark 2].

Messaoudi and Said-Houari [81] improved the result in [139] and showed that the blow up of solutions of problem (4.8) takes place for negative initial data only regardless of the size of Ω.

The absence of the terms m_1u^2 and m_2v^2 , equations (4.1) take the form :

$$\begin{cases} u_{tt} - \Delta u_t - div\left(|\nabla u|^{\alpha-2} \nabla u\right) - div\left(|\nabla u_t|^{\beta_1-2} \nabla u_t\right) + a_1\,|u_t|^{m-2} u_t & = f_1(u,v), \\ v_{tt} - \Delta v_t - div\left(|\nabla v|^{\alpha-2} \nabla v\right) - div\left(|\nabla v_t|^{\beta_2-2} \nabla v_t\right) + a_2\,|v_t|^{r-2} v_t & = f_2(u,v), \end{cases}$$

In [106] Rahmoune. A and Ouchenane. D proved the global nonexistence result, Under an appropriate assumptions on the initial data and under some restrictions on the parameter ; β_1;β_2; m; r and on the nonlinear functions f_1 and f_2.

4.2 Preliminaries

In this section, we introduce some notations and some technical lemmas to be used throughout this paper. By $\|.\|_q$, we denote the usual $L^q(\Omega)$-norm. The constants $C, c, c_1, c_2, \ldots$, used throughout this paper are positive generic constants, which may be different in various occurrences. We define

$$F(u,v) = \frac{1}{2(\rho+2)}\left[b_1\,|u+v|^{2(\rho+2)} + 2b_2\,|uv|^{\rho+2}\right].$$

Then , it is clear that, from (4.2),we have

$$uf_1(u,v) + vf_2(u,v) = 2(\rho+2)F(u,v). \tag{4.9}$$

The following lemma was introduced and proved in [87]

Lemma 4.1. *There exist two positive constants c_0 and c_1 such that*

$$\frac{c_0}{2(\rho+2)}\left(|u|^{2(\rho+2)}+|v|^{2(\rho+2)}\right) \leq F(u,v) \leq \frac{c_1}{2(\rho+2)}\left(|u|^{2(\rho+2)}+|v|^{2(\rho+2)}\right). \tag{4.10}$$

And the energy functional

$$E(t)=\frac{1}{2}\left(\|u_t\|_2^2+\|v_t\|_2^2\right)+\frac{1}{\alpha}\left(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha\right)+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2-\int_\Omega F(u,v)\,dx. \tag{4.11}$$

Let us know define a constant r_α as follows :

$$r_\alpha=\frac{N\alpha}{N-\alpha}, \quad if\ N>\alpha,\ r_\alpha>\alpha\ if\ N=\alpha,\ and\ r_\alpha=\infty\ if\ N<\alpha. \tag{4.12}$$

The inequality below is a key element in proving the global existence of solution. A similar version of this lemma was first introduced in [114]

Lemma 4.2. *Suppose that $\alpha>2$, and $2<2(\rho+2)<r_\alpha$. Then there exists $\eta>0$ such that the inequality*

$$\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2} \leq \eta\left(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha\right)^{\frac{2(\rho+2)}{\alpha}} \tag{4.13}$$

holds.

Proof. It is clear that by using the Minkowski inequality, we get

$$\|u+v\|_{2(\rho+2)}^2 \leq 2(\|u\|_{2(\rho+2)}^2+\|v\|_{2(\rho+2)}^2),$$

the embedding $W_0^{1,\alpha} \hookrightarrow L^{2(\rho+2)}(\Omega)$, gives

$$\|u\|_{2(\rho+2)}^2 \leq C\|\nabla u\|_\alpha^2 \leq C(\|\nabla u\|_\alpha^\alpha)^{\frac{2}{\alpha}} \leq C(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha)^{\frac{2}{\alpha}},$$

and similary , we have

$$\|v\|_{2(\rho+2)}^2 \leq C\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha)^{\frac{2}{\alpha}}$$

Thus, we deduce from the above estimates that

$$\|u+v\|_{2(\rho+2)}^2 \leq C(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha)^{\frac{2}{\alpha}} \tag{4.14}$$

also, Hölder's and Young's inequalities give us

$$\|uv\|_{(\rho+2)} \leq \|u\|_{2(\rho+2)}\|v\|_{2(\rho+2)} \leq C(\|\nabla u\|_{2(\rho+2)}^2+\|\nabla v\|_{2(\rho+2)}^2) \leq C(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha)^{\frac{2}{\alpha}}. \tag{4.15}$$

Collecting the estimates (4.14) and (4.15), then (4.13) holds. This completes the proof of lemma (4.2) □

In the following lemma, we show that the total energy of our system is a nonincreasing function of t.

Lemma 4.3. *Let* (u, v) *be the solution of system* (4.1)-(4.3)*, then the energy functional is a non-increasing function for all* $t \geq 0$

$$\begin{aligned} \frac{dE(t)}{dt} &= -\|\nabla u_t\|_2^2 - \|\nabla v_t\|_2^2 - \|\nabla u_t\|_{\beta_1}^{\beta_1} - \|\nabla v_t\|_{\beta_2}^{\beta_2} \\ &\quad -a_1\|u_t\|_m^m - a_2\|v_t\|_r^r - m_1^2\|u\|_2^2 - m_2^2\|v\|_2^2 . \end{aligned} \tag{4.16}$$

Proof. We multiply the first equation in (4.1) by u_t and second equation by v_t and integrate over Ω, using integration by parts, we obtain (4.16). □

4.3 Global nonexistence result

In this section, we prove that, under some restrictions on the initial data and under som restrictions on the parameter α,β_1,β_2,m, r then the lifespan of solution of problem (4.1)- (4.3) is finite

Theorem 4.4. *Suppose that* $\beta_1, \beta_2, m, r \geq 2, \alpha > 2, \rho > -1$ *such that* $\beta_1, \beta_2 < \alpha$*, and* $\max\{m, r\} < 2(\rho+2) < r_\alpha$*, where* r_α *is the Sobolev critical exponent of* $W_0^{1,\alpha}(\Omega)$ *. defined in* (4.12)*.Assume further that*

$$E(0) < E_1, \qquad \left(\|\nabla u_0\|_\alpha^\alpha + \|\nabla v_0\|_\alpha^\alpha\right)^{\frac{1}{\alpha}} + m_1^2\|u_0\|_2^2 + m_2^2\|v_0\|_2^2 > \zeta_1$$

Then, any weak solution of (4.1)-(4.3) *cannot exist for all time .Here the constants* E_1 *and* ζ_1 *are defined in* (4.4).

In ordre to prove our result and for the sake of simplicity , we take $b_1 = b_2 = 1$ and introduce the following :

$$B = \eta^{\frac{1}{2(\rho+2)}}, \qquad \zeta_1 = B^{\frac{-2(\rho+2)}{2(\rho+2)-\alpha}}, \qquad E_1 = \left(\frac{1}{\alpha} - \frac{1}{2(\rho+2)}\right)\zeta_1^\alpha, \tag{4.17}$$

where η is the optimal constant in (4.13).

The following lemma allows us to prove a blow up result for a large class of initial data. This lemma is similar to the one in [114] and has its origin in [130]

Lemma 4.5. *Let* (u, v) *be a solution of* (4.1)-(4.3)*. Assume that* $\alpha > 2$, $\rho > -1$*. Assume further that* $E(0) < E_1$ *and*

$$(\|\nabla u_0\|_\alpha^\alpha + \|\nabla v_0\|_\alpha^\alpha)^{\frac{1}{\alpha}} + m_1^2 \|u_0\|_2^2 + m_2^2 \|v_0\|_2^2 > \zeta_1. \tag{4.18}$$

Then there exists a constant $\zeta_2 > \zeta_1$ *such that*

$$(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha)^{\frac{1}{\alpha}} + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2 > \zeta_2, \tag{4.19}$$

and

$$\left[\|u + v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right]^{\frac{1}{2(\rho+2)}} \geq B\zeta_2, \ \forall t \geq 0. \tag{4.20}$$

Proof. We first note that, by (4.11) and the definition of B, we have

$$\begin{aligned} E(t) &\geq \frac{1}{\alpha}(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha) + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2 - \frac{1}{2(\rho+2)}\left[|u + v|^{2(\rho+2)} + 2|uv|^{\rho+2}\right] \\ &\geq \frac{1}{\alpha}(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha) + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2 - \frac{\eta}{2(\rho+2)}(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha)^{\frac{2(\rho+2)}{\alpha}} \\ &\geq \frac{1}{\alpha}\zeta^\alpha - \frac{\eta}{2(\rho+2)}\zeta^{2(\rho+2)}, \end{aligned} \tag{4.21}$$

where $\zeta = \left[\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha + m_1^2 \|u\|_\alpha^\alpha + m_2^2 \|v\|_\alpha^\alpha\right]^{\frac{1}{\alpha}}$. It is not hard to verify that g is increasing for $0 < \zeta < \zeta_1$, decreasing for $\zeta > \zeta_1$, $g(\zeta) \to -\infty$ as $\zeta \to +\infty$, and

$$g(\zeta_1) = \frac{1}{\alpha}\zeta_1^\alpha - \frac{B^{2(\rho+2)}}{2(\rho+2)}\zeta_1^{2(\rho+2)} = E_1,$$

where ζ_1 is given in (4.17). Therefore, since $E(0) < E_1$, there exists $\zeta_2 > \zeta_1$ such that $g(\zeta_2) = E(0)$.

If we set $\zeta_0 = [\|\nabla u(0)\|_\alpha^\alpha + \|\nabla v(0)\|_\alpha^\alpha]^{\frac{1}{\alpha}} + m_1^2 \|u(0)\|_2^2 + m_2^2 \|v(0)\|_2^2$, then by (4.21) we have $g(\zeta_0) \leq E(0) = g(\zeta_2)$, which implies that $\zeta_0 \geq \zeta_2$.

Now, establish (4.19), we suppose by contradiction that

$$(\|\nabla u_0\|_\alpha^\alpha + \|\nabla v_0\|_\alpha^\alpha)^{\frac{1}{\alpha}} + m_1^2 \|u_0\|_2^2 + m_2^2 \|v_0\|_2^2 < \zeta_2,$$

for some $t_0 > 0$; by the continuity of $\|\nabla u(.)\|_\alpha^\alpha + \|\nabla v(.)\|_\alpha^\alpha + m_1^2 \|u(.)\|_2^2 + m_2^2 \|v(.)\|_2^2$ we can choose t_0 such that

$$(\|\nabla u(t_0)\|_\alpha^\alpha + \|\nabla v(t_0)\|_\alpha^\alpha)^{\frac{1}{\alpha}} + m_1^2 \|u(t_0)\|_2^2 + m_2^2 \|v(t_0)\|_2^2 > \zeta_1.$$

Again, the use of (4.21) leads to

$$E(t_0) \geq g\left(\|\nabla u(t_0)\|_\alpha^\alpha + \|\nabla v(t_0)\|_\alpha^\alpha\right) + m_1^2\|u(t_0)\|_2^2 + m_2^2\|v(t_0)\|_2^2 > g(\zeta_2) = E(0).$$

This is impossible since $E(t) \leq E(0)$, for all $t \in [0, T)$. Hence, (4.19) is established.

To prove (4.20), we make use of (4.11) to get

$$\frac{1}{\alpha}\left(\|\nabla u_0\|_\alpha^\alpha + \|\nabla v_0\|_\alpha^\alpha\right) + m_1^2\|u_0\|_2^2 + m_2^2\|v_0\|_2^2 \leq E(0) + \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right].$$

Consequently, (4.19) yields

$$\begin{aligned}
\frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right] &\geq \frac{1}{\alpha}\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right) - E(0) \\
&\geq \frac{1}{\alpha}\zeta_2^\alpha - E(0) \\
&\geq \frac{1}{\alpha}\zeta_2^\alpha - g(\zeta_2) \\
&= \frac{B^{2(\rho+2)}}{2(\rho+2)}\zeta_2^{2(\rho+2)}.
\end{aligned} \tag{4.22}$$

Therefore, (4.22) and (4.17) yield the desired result. □

Proof. Proof of Theorem 4.4

We suppose that the solution exists for all time and set

$$H(t) = E_1 - E(t). \tag{4.23}$$

By using (4.11) and (4.23) we get

$$H'(t) = \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 + \|\nabla u_t\|_{\beta_1}^{\beta_1} + \|\nabla v_t\|_{\beta_2}^{\beta_2} + a_1\|u_t\|_m^m + a_2\|v_t\|_r^r + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2.$$

From (4.16) , It is clear that for all $t \geq 0$, $H'(t) > 0$. Therefore , we have

$$\begin{aligned}
0 &< H(0) \leq H(t) \\
&= E_1 - \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right) - \frac{1}{\alpha}\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right) \\
&\quad + \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right].
\end{aligned} \tag{4.24}$$

From (4.11) and (4.19), we obtain, for all $t \geq 0$,

$$E_1 - \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right) - \frac{1}{\alpha}\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right) < E_1 - \frac{1}{\alpha}\zeta_1^\alpha = -\frac{1}{2(\rho+2)}\zeta_1^\alpha < 0.$$

Hence,

$$0 < H(0) \le H(t) \le \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right], \ \forall t \ge 0.$$

Then by (4.10), we have

$$0 < H(0) \le H(t) \le \frac{c_1}{2(\rho+2)}\left[\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right], \ \forall t \ge 0. \tag{4.25}$$

We then define

$$L(t) = H^{1-\sigma}(t) + \varepsilon\int_\Omega (uu_t + vv_t))dx, \tag{4.26}$$

for ε small to be chosen later and

$$\begin{aligned} 0 \quad < \quad & \sigma \le \min\left\{\frac{1}{2}, \frac{\alpha - m}{2(\rho+2)(m-1)}, \frac{\alpha - r}{2(\rho+2)(r-1)},\right. \\ & \left.\frac{(\alpha-2)}{2(\rho+2)}, \frac{\alpha-\beta_1}{2(\rho+2)(\beta_1-1)}, \frac{\alpha-\beta_2}{2(\rho+2)(\beta_2-1)}\right\} \end{aligned} \tag{4.27}$$

Our goal is to show that $L(t)$ satisfies the differential inequality (4.7). Indeed, taking the derivative of (4.26), using (4.1)and adding subtracting $\varepsilon kH(t)$, we obtain

$$\begin{aligned} L'(t) \quad = \quad & (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon kH(t) + \varepsilon\left(1 + \frac{k}{2}\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right) \\ & +\varepsilon(1-k)\int_\Omega F(u,v) - \varepsilon kE_1 \\ & -\varepsilon\int_\Omega \nabla u\nabla u_t dx - \varepsilon\int_\Omega \nabla v\nabla v_t dx \\ & +\varepsilon\left(\frac{k}{\alpha} - 1\right)\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right) \\ & -\varepsilon\int_\Omega |\nabla u_t|^{\beta_1-2}\nabla u_t\nabla u dx - \varepsilon\int_\Omega |\nabla v_t|^{\beta_2-2}\nabla v_t\nabla v dx \\ & -\varepsilon a_1\int_\Omega |u_t|^{m-2}u_t u dx - \varepsilon a_2\int_\Omega |v_t|^{r-2}v_t v dx. \end{aligned} \tag{4.28}$$

We then exploit Young's inequality to get for $\mu_i, \lambda_i, \delta_i > 0$ $i = 1, 2$

$$\int_\Omega \nabla u\nabla u_t dx \le \frac{1}{4\mu_1}\|\nabla u\|_2^2 + \mu_1\|\nabla u_t\|_2^2$$

$$\int_{\Omega} \nabla v \nabla v_t dx \leq \frac{1}{4\mu_2} \|\nabla v\|_2^2 + \mu_2 \|\nabla v_t\|_2^2 \tag{4.29}$$

and

$$\int_{\Omega} |\nabla u_t|^{\beta_1 - 1} \nabla u dx \leq \frac{\lambda_1^{\beta_1}}{\beta_1} \|\nabla u\|_{\beta_1}^{\beta_1} + \frac{\beta_1 - 1}{\beta_1} \lambda_1^{-\beta_1/(\beta_1 - 1)} \|\nabla u_t\|_{\beta_1}^{\beta_1}$$

$$\int_{\Omega} |\nabla v_t|^{\beta_2 - 1} \nabla v dx \leq \frac{\lambda_2^{\beta_2}}{\beta_2} \|\nabla v\|_{\beta_2}^{\beta_2} + \frac{\beta_2 - 1}{\beta_2} \lambda_2^{-\beta_2/(\beta_2 - 1)} \|\nabla v_t\|_{\beta_1}^{\beta_1} \tag{4.30}$$

and also

$$\int_{\Omega} |u_t|^{m-2} u_t u dx \leq \frac{\delta_1^m}{m} \|u\|_m^m + \frac{m-1}{m} \delta_1^{-m/(m-1)} \|u_t\|_m^m$$

$$\int_{\Omega} |v_t|^{r-2} v_t v dx \leq \frac{\delta_2^r}{r} \|v\|_r^r + \frac{r-1}{r} \delta_2^{-r/(r-1)} \|v_t\|_r^r \tag{4.31}$$

A substitution of (4.29)-(4.31)) in (4.28) and using (4.10) yields

$$\begin{aligned}
L^{'}(t) \geq\; & (1-\sigma) H^{-\sigma}(t) H^{'}(t) + \varepsilon k H(t) + \varepsilon \left(1 + \frac{k}{2}\right) \left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2\right) \\
& + \varepsilon \left(\frac{c_0}{2(\rho+2)} - \frac{k c_1}{2(\rho+2)}\right) \left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right) - \varepsilon k E_1 \\
& - \frac{\varepsilon}{4\mu_1} \|\nabla u\|_2^2 - \mu_1 \varepsilon \|\nabla u_t\|_2^2 - \frac{\varepsilon}{4\mu_2} \|\nabla v\|_2^2 - \varepsilon \mu_2 \|\nabla v_t\|_2^2 \\
& + \varepsilon \left(\frac{k}{\alpha} - 1\right) (\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha) - \varepsilon \frac{\lambda_1^{\beta_1}}{\beta_1} \|\nabla u\|_{\beta_1}^{\beta_1} - \varepsilon \frac{\beta_1 - 1}{\beta_1} \lambda_1^{-\beta_1/(\beta_1 - 1)} \|\nabla u_t\|_{\beta_1}^{\beta_1} \\
& - \varepsilon \frac{\lambda_2^{\beta_2}}{\beta_2} \|\nabla v\|_{\beta_2}^{\beta_2} - \varepsilon \frac{\beta_2 - 1}{\beta_2} \lambda_2^{-\beta_2/(\beta_2 - 1)} \|\nabla v_t\|_{\beta_1}^{\beta_1} - a_1 \varepsilon \frac{\delta_1^m}{m} \|u\|_m^m \\
& - a_1 \varepsilon \frac{m-1}{m} \delta_1^{-m/(m-1)} \|u_t\|_m^m - a_2 \varepsilon \frac{\delta_2^r}{r} \|v\|_r^r - a_2 \varepsilon \frac{r-1}{r} \delta_2^{-r/(r-1)} \|v_t\|_m^m .
\end{aligned} \tag{4.32}$$

Let us choose $\delta_1, \delta_2, \mu_1, \mu_2, \lambda_1$, and λ_2 such that

$$\begin{cases} \delta_1^{-m/(m-1)} = M_1 H^{-\sigma}(t) \\ \delta_2^{-r/(r-1)} = M_2 H^{-\sigma}(t) \\ \mu_1 = M_3 H^{-\sigma}(t) \\ \mu_2 = M_4 H^{-\sigma}(t) \\ \lambda_1^{-\beta_1/(\beta_1-1)} = M_5 H^{-\sigma}(t) \\ \lambda_2^{-\beta_2/(\beta_2-1)} = M_6 H^{-\sigma}(t) \end{cases} \tag{4.33}$$

for M_1, M_2, M_3, M_4, M_5 and M_6 large constants to be fixed later. Thus, by using (4.33),and for $M = M_3 + M_4 + (\beta_1 - 1)M_5/\beta_1 + (\beta_2 - 1)M_6/\beta_2 + (m-1)M_1/m + (r-1)M_2/r$

then, inequality (4.32) takes the form

$$\begin{aligned} L^{'}(t) \geq\; & ((1-\sigma) - \varepsilon M) H^{-\sigma}(t) H^{'}(t) + \varepsilon k H(t) \\ & + \varepsilon \left(1 + \frac{k}{2}\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2\right) \\ & + \varepsilon \left(\frac{c_0}{2(\rho+2)} - \frac{kc_1}{2(\rho+2)}\right)\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right) - \varepsilon k E_1 \\ & + \varepsilon \left(\frac{k}{\alpha} - 1\right)\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right) \\ & - \frac{\varepsilon}{4M_3} H^{\sigma}(t) \|\nabla u\|_2^2 - \frac{\varepsilon}{4M_4} H^{\sigma}(t) \|\nabla v\|_2^2 \\ & - \frac{a_1 \varepsilon}{m} M_1^{-(m-1)} H^{\sigma(m-1)}(t) \|u\|_m^m - \frac{a_2 \varepsilon}{r} M_2^{-(r-1)} H^{\sigma(r-1)}(t) \|v\|_r^r \\ & - \varepsilon \frac{M_5^{-(\beta_1-1)}}{\beta_1} H^{\sigma(\beta_1-1)}(t) \|\nabla u\|_{\beta_1}^{\beta_1} - \varepsilon \frac{M_6^{-(\beta_2-1)}}{\beta_2} H^{\sigma(\beta_2-1)}(t) \|\nabla u\|_{\beta_2}^{\beta_2}, \end{aligned} \tag{4.34}$$

We then use the two embedding $L^{2(\rho+2)}(\Omega) \hookrightarrow L^m(\Omega)$, $W_0^{1,\alpha} \hookrightarrow L^{2(\rho+2)}(\Omega)$ and (4.25) to get

$$\begin{aligned} H^{\sigma(m-1)}(t) \|u\|_m^m &\leq c_2 \left(\|u\|_{2(\rho+2)}^{2\sigma(m-1)(\rho+2)+m} + \|v\|_{2(\rho+2)}^{2\sigma(m-1)(\rho+2)} \|u\|_{2(\rho+2)}^m\right) \\ &\leq c_2 \left(\|\nabla u\|_\alpha^{2\sigma(m-1)(\rho+2)+m} + \|\nabla v\|_\alpha^{2\sigma(m-1)(\rho+2)} \|\nabla u\|_\alpha^m\right). \end{aligned} \tag{4.35}$$

Similarly, the embedding $L^{2(\rho+2)}(\Omega) \hookrightarrow L^r(\Omega)$, $W_0^{1,\alpha} \hookrightarrow L^{2(\rho+2)}(\Omega)$ and (4.25) give

$$
\begin{aligned}
H^{\sigma(r-1)}(t)\,\|v\|_r^r &\leq c_3\left(\|v\|_{2(\rho+2)}^{2\sigma(r-1)(\rho+2)+r} + \|u\|_{2(\rho+2)}^{2\sigma(r-1)(\rho+2)}\,\|v\|_{2(\rho+2)}^r\right) \\
&\leq c_3\left(\|\nabla v\|_\alpha^{2\sigma(r-1)(\rho+2)+r} + \|\nabla u\|_\alpha^{2\sigma(r-1)(\rho+2)}\,\|\nabla v\|_\alpha^r\right). \qquad (4.36)
\end{aligned}
$$

Furthermore, the two embedding $W_0^{1,\alpha} \hookrightarrow L^{2(\rho+2)}(\Omega)$, $L^\alpha(\Omega) \hookrightarrow L^2(\Omega)$, yields

$$
\begin{aligned}
H^{\sigma}(t)\,\|\nabla u\|_2^2 &\leq c_4\left(\|u\|_{2(\rho+2)}^{2\sigma(\rho+2)}\,\|\nabla u\|_2^2 + \|v\|_{2(\rho+2)}^{2\sigma(\rho+2)}\,\|\nabla u\|_2^2\right) \\
&\leq c_4\left(\|\nabla u\|_\alpha^{2\sigma(\rho+2)+2} + \|\nabla v\|_\alpha^{2\sigma(\rho+2)}\,\|\nabla u\|_\alpha^2\right) \qquad (4.37)
\end{aligned}
$$

and

$$
\begin{aligned}
H^{\sigma}(t)\,\|\nabla v\|_2^2 &\leq c_5\left(\|\nabla u\|_\alpha^{2\sigma(\rho+2)}\,\|\nabla v\|_\alpha^2 + \|\nabla v\|_\alpha^{2\sigma(\rho+2)}\,\|\nabla v\|_\alpha^2\right) \qquad (4.38) \\
&= c_5\left(\|\nabla u\|_\alpha^{2\sigma(\rho+2)}\,\|\nabla v\|_\alpha^2 + \|\nabla v\|_\alpha^{2\sigma(\rho+2)+2}\right).
\end{aligned}
$$

Since $max(\beta_1, \beta_2) < \alpha$ then we have

$$
\begin{aligned}
H^{\sigma(\beta_1-1)}(t)\,\|\nabla u\|_{\beta_1}^{\beta_1} &\leq c_6\left(\|\nabla u\|_\alpha^{2\sigma(\beta_1-1)(\rho+2)}\,\|\nabla u\|_\alpha^{\beta_1} + \|\nabla v\|_\alpha^{2\sigma(\beta_1-1)(\rho+2)}\,\|\nabla u\|_\alpha^{\beta_1}\right) \\
&= c_6\left(\|\nabla u\|_\alpha^{2\sigma(\beta_1-1)(\rho+2)+\beta_1} + \|\nabla v\|_\alpha^{2\sigma(\beta_1-1)(\rho+2)}\,\|\nabla u\|_\alpha^{\beta_1}\right). \qquad (4.39)
\end{aligned}
$$

and

$$
\begin{aligned}
H^{\sigma(\beta_2-1)}(t)\,\|\nabla v\|_{\beta_2}^{\beta_2} &\leq c_7\left(\|\nabla u\|_\alpha^{2\sigma(\beta_2-1)(\rho+2)}\,\|\nabla v\|_\alpha^{\beta_2} + \|\nabla v\|_\alpha^{2\sigma(\beta_2-1)(\rho+2)}\,\|\nabla v\|_\alpha^{\beta_2}\right) \\
&= c_7\left(\|\nabla u\|_\alpha^{2\sigma(\beta_2-1)(\rho+2)}\,\|\nabla v\|_\alpha^{\beta_2} + \|\nabla v\|_\alpha^{2\sigma(\beta_2-1)(\rho+2)+\beta_2}\right). \qquad (4.40)
\end{aligned}
$$

for some positive constants c_2, c_3, c_4, c_5, c_6 and c_7. By using (4.27) and the algebraic inequality

$$
z^\nu \leq (z+1) \leq \left(1 + \frac{1}{a}\right)(z+a), \quad \forall z \geq 0,\ 0 < \nu \leq 1,\ a \geq 0, \qquad (4.41)
$$

we have, for all $t \geq 0$,

$$\begin{cases} \|\nabla u\|_{\alpha}^{2\sigma(m-1)(\rho+2)+m} \leq d\left(\|\nabla u\|_{\alpha}^{\alpha} + H(0)\right) \leq d\left(\|\nabla u\|_{\alpha}^{\alpha} + H(t)\right), \\ \|\nabla v\|_{\alpha}^{2\sigma(r-1)(\rho+2)+r} \leq d\left(\|\nabla v\|_{\alpha}^{\alpha} + H(t)\right), \\ \|\nabla u\|_{\alpha}^{2\sigma(\rho+2)+2} \leq d\left(\|\nabla u\|_{\alpha}^{\alpha} + H(t)\right), \\ \|\nabla v\|_{\alpha}^{2\sigma(\rho+2)+2} \leq d\left(\|\nabla v\|_{\alpha}^{\alpha} + H(t)\right), \\ \|\nabla u\|_{\alpha}^{2\sigma(\beta_1-1)(\rho+2)+\beta_1} \leq d\left(\|\nabla u\|_{\alpha}^{\alpha} + H(t)\right), \\ \|\nabla v\|_{\alpha}^{2\sigma(\beta_2-1)(\rho+2)+\beta_2} \leq d\left(\|\nabla v\|_{\alpha}^{\alpha} + H(t)\right), \end{cases} \tag{4.42}$$

where $d = 1 + 1/H(0)$. Also keeping in mind the fact that $max(m, r) < \alpha$, using Yong's inequality, the inequality (4.41) togrther withe (4.27), we conclude

$$\begin{cases} \|\nabla v\|_{\alpha}^{2\sigma(m-1)(\rho+2)} \|\nabla u\|_{\alpha}^{m} \leq C\left(\|\nabla v\|_{\alpha}^{\alpha} + \|\nabla u\|_{\alpha}^{\alpha}\right), \\ \|\nabla u\|_{\alpha}^{2\sigma(r-1)(\rho+2)} \|\nabla v\|_{\alpha}^{r} \leq C\left(\|\nabla u\|_{\alpha}^{\alpha} + \|\nabla v\|_{\alpha}^{\alpha}\right), \\ \|\nabla v\|_{\alpha}^{2\sigma(\rho+2)} \|\nabla u\|_{\alpha}^{2} \leq C\left(\|\nabla v\|_{\alpha}^{\alpha} + \|\nabla u\|_{\alpha}^{\alpha}\right), \\ \|\nabla u\|_{\alpha}^{2\sigma(\rho+2)} \|\nabla v\|_{\alpha}^{2} \leq C\left(\|\nabla u\|_{\alpha}^{\alpha} + \|\nabla v\|_{\alpha}^{\alpha}\right), \\ \|\nabla v\|_{\alpha}^{2\sigma(\beta_1-1)(\rho+2)} \|\nabla u\|_{\alpha}^{\beta_1} \leq C\left(\|\nabla v\|_{\alpha}^{\alpha} + \|\nabla u\|_{\alpha}^{\alpha}\right), \\ \|\nabla u\|_{\alpha}^{2\sigma(\beta_2-1)(\rho+2)} \|\nabla v\|_{\alpha}^{\beta_2} \leq C\left(\|\nabla u\|_{\alpha}^{\alpha} + \|\nabla v\|_{\alpha}^{\alpha}\right), \end{cases} \tag{4.43}$$

where C is a generic positive constant. Taking into account (4.35)- (4.43) , then, (4.34) takes the form

$$\begin{aligned} L^{'}(t) \;\geq\;& ((1-\sigma)-\varepsilon M)\,H^{-\sigma}(t)\,H^{'}(t)+\varepsilon\left(1+\frac{k}{2}\right)\left(\|u_t\|_2^2+\|v_t\|_2^2+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2\right) \\ &+\varepsilon\Big(\left[k/\alpha-1-kE_1\zeta_2^{-a}\right]-CM_1^{-(m-1)}-CM_2^{-(r-1)} \\ &-\frac{C}{4}M_3^{-1}-\frac{C}{4}M_4^{-1}-CM_5^{-(\beta_1-1)}-CM_6^{-(\beta_2-1)}-1\Big)\left(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha\right) \\ &+\varepsilon\Big(k-CM_1^{-(m-1)}-CM_2^{-(r-1)}-\frac{C}{4}M_3^{-1}-\frac{C}{4}M_4^{-1} \\ &-CM_5^{-(\beta_1-1)}-CM_6^{-(\beta_2-1)}\Big)H(t) \\ &+\varepsilon\left(\frac{c_0}{2(\rho+2)}-\frac{kc_1}{2(\rho+2)}\right)\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}\right), \end{aligned} \tag{4.44}$$

for some constant k. Using $k=c_0/c_1$, we arrive at

$$\begin{aligned} L^{'}(t) \;\geq\;& ((1-\sigma)-\varepsilon M)\,H^{-\sigma}(t)\,H^{'}(t)+\varepsilon\left(1+\frac{c_0}{2c_1}\right)\left(\|u_t\|_2^2+\|v_t\|_2^2+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2\right) \\ &+\varepsilon\Big(\overline{c}-CM_1^{-(m-1)}-CM_2^{-(r-1)}-\frac{C}{4}M_3^{-1}-\frac{C}{4}M_4^{-1} \\ &-CM_5^{-(\beta_1-1)}-CM_6^{-(\beta_2-1)}-1\Big)\left(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha\right) \\ &+\varepsilon\Big(c_0/c_1-CM_1^{-(m-1)}-CM_2^{-(r-1)}-\frac{C}{4}M_3^{-1}-\frac{C}{4}M_4^{-1} \\ &-CM_5^{-(\beta_1-1)}-CM_6^{-(\beta_2-1)}\Big)H(t), \end{aligned} \tag{4.45}$$

where $\overline{c}=k/\alpha-1-kE_1\zeta_2^{-2}=c_0/(c_1\alpha)-1-(c_0/c_1)\,E_1\zeta_2^{-2}>0$ since $\zeta_2>\zeta_1$.

At this point, and for large values of M_1, M_2, M_3, M_4, M_5 and M_6, we can find positive constants Λ_1 and Λ_2 such that (4.45) becomes

$$\begin{aligned} L^{'}(t) \;\geq\;& ((1-\sigma)-M\varepsilon)\,H^{-\sigma}(t)\,H^{'}(t)+\varepsilon\left(1+\frac{c_0}{2c_1}\right)\left(\|u_t\|_2^2+\|v_t\|_2^2+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2\right) \\ &+\varepsilon\Lambda_1\left(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha\right)+\varepsilon\Lambda_2 H(t). \end{aligned} \tag{4.46}$$

Once M_1, M_2, M_3, M_4, M_5 and M_6 are fixed (hence, Λ_1 and Λ_2), we pick ε small enough so that $((1-\sigma)-M\varepsilon)\geq 0$ and

$$L(0)=H^{1-\sigma}(0)+\int_\Omega [u_0.u_t+v_0.v_t]\,dx>0.$$

From these and (4.46) becomes

$$L^{'}(t)\geq\varepsilon\Gamma\left(H(t)+\|u_t\|_2^2+\|v_t\|_2^2+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2+\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha\right). \tag{4.47}$$

Thus, we have $L(t) \geq L(0) > 0$, for all $t \geq 0$. Next, by Holder's and Young's inequalities, we estimate

$$\begin{aligned}
&\left(\int_{\Omega} u.u_t(x,t)\,dx + \int_{\Omega} v.v_t(x,t)\,dx\right)^{\frac{1}{1-\sigma}} \\
\leq\; & C\left(\|u\|_{2(\rho+2)}^{\frac{\tau}{1-\sigma}} + \|u_t\|_2^{\frac{s}{1-\sigma}} + \|v\|_{2(\rho+2)}^{\frac{\tau}{1-\sigma}} + \|v_t\|_2^{\frac{s}{1-\sigma}}\right) \\
\leq\; & C\left(\|\nabla u\|_{\alpha}^{\frac{\tau}{1-\sigma}} + \|u_t\|_2^{\frac{s}{1-\sigma}} + \|\nabla v\|_{\alpha}^{\frac{\tau}{1-\sigma}} + \|v_t\|_2^{\frac{s}{1-\sigma}}\right) \qquad (4.48)
\end{aligned}$$

for $\frac{1}{\tau} + \frac{1}{s} = 1$. We take $s = 2(1-\sigma)$, to get $\frac{\tau}{1-\sigma} = \frac{2}{1-2\sigma}$. By using (4.27) and (4.41) we get

$$\|\nabla u\|_{\alpha}^{\frac{2}{(1-2\sigma)}} \leq d\left(\|\nabla u\|_{\alpha}^{\alpha} + H(t)\right),$$

and

$$\|\nabla v\|_{\alpha}^{\frac{2}{(1-2\sigma)}} \leq d\left(\|\nabla v\|_{\alpha}^{\alpha} + H(t)\right), \;\forall t \geq 0.$$

Therefore, (4.48) becomes

$$\begin{aligned}
&\left(\int_{\Omega} u.u_t(x,t)\,dx + \int_{\Omega} v.v_t(x,t)\,dx\right)^{\frac{1}{1-\sigma}} \\
\leq\; & C\left(\|\nabla u\|_{\alpha}^{\alpha} + \|\nabla v\|_{\alpha}^{\alpha} + \|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2 + H(t)\right), \;\forall t \geq 0. \qquad (4.49)
\end{aligned}$$

Also, since

$$\begin{aligned}
L^{\frac{1}{1-\sigma}}(t) \;&=\; \left(H^{1-\sigma}(t) + \varepsilon\int_{\Omega}(u.u_t + v.v_t)(x,t)\,dx\right)^{\frac{1}{(1-\sigma)}} \\
&\leq\; C\left(H(t) + \left|\int_{\Omega}(u.u_t(x,t) + v.v_t(x,t))\,dx\right|^{\frac{1}{(1-\sigma)}}\right) \qquad (4.50) \\
&\leq\; C\left[H(t) + \|\nabla u\|_{\alpha}^{\alpha} + \|\nabla v\|_{\alpha}^{\alpha} + \|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right], \;\forall t \geq 0,
\end{aligned}$$

combining withe (4.50) and (4.47), we arrive at

$$L'(t) \geq a_0 L^{\frac{1}{1-\sigma}}(t), \;\forall t \geq 0. \qquad (4.51)$$

Finally, a simple integration of (4.51) gives the desired result.This completes the proof of Theorem (4.4) □

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