People's Democratic Republic of Algeria

***Ministry of Higher Education and* Scientific Research.**

University Ferhat Abbas Setif-1

Faculty of sciences

Department of physics

Doctorat Thesis

Option : Theoretical Physics

By

Boumaza Roumaissa

Submitted for the Degree of Doctorat LMD

Dynamical properties of quantum gases following a sudden change of confining potential.

Doctoral committee :

Pr : Mamache Mustapha U. Ferhat Abbas Sétif-1 : President

Pr : Mohamed Benarous U. Hassiba Benbouali (Chlef) : Examinator

***Pr : Abdelhamid Bouldjedri U. Hadj Lakhdar (Batna) :* Examinator**

Pr : Lamari Saadi U. Ferhat Abbas Sétif-1 : Examinator

Pr : Bencheikh kamel U. Ferhat Abbas Sétif-1 : Supervisor

Table des matières

Acknowledgments

I would first like to thank my supervisor, Pr *Kamel Bencheikh* for his constant guidance over these years. I admire him for his skills as a researcher and also as teacher. His qualities as a human being greatly helped me to accomplish this thesis in a good working conditions.

I am very grateful to the membres of thesis jury for accepting to attend my thesis defense. Namely, The jury president, Pr *M.Mamache* from our university and the examiners : Pr *M. Benarous* from University of Chlef, Pr *A. bouldjedri* from University of Batna and Pr *S. lamari* from our university.

I am also grateful to my teachers who contributed to my education at university, in particular, *Dr Berkane*, *Pr Mansouri*, *Pr Houamer*, *Dr Guenoune* and *Pr Hachemi.*

Finally, I express my very profound gratitude to my amazing families *Boumaza* and *Boureghda* for the love, support and constant encouragement. In particular, I would like to thank my dear parents and my husband for providing me with unfailing support during process of researching and writing this thesis. This accomplishment would not have been possible without them. To all of them I say thank you so much.

0.1 Introduction

The physics of ultra-cold quantum gases has been one of the most exciting and rapidly growing fields of physics of the past two decades. The advances in cooling, where temperatures of the order of a fraction of the Fermi temperatures were reached, and trapping techniques have led to experimental realization of confined ultra-cold gases in various spatial geometries,[1, 2, 3] . Moreover advances in high resolution imaging cold cloud gases allow to probe these systems. Nowadays experimentalists can emulate properties of real materials [4] through the use of utra-cold gases by engineering a desired Hamiltonian.

These developments led to the emergence of traps like the square well [3] and the hard-wall optical box with a strong confinement in two directions [4]. This progress in experimental activity has allowed a remarkable control of the external parameters of the Hamiltonian governing the dynamics of the quantum system. The frequency of the trapping harmonic potential or the strength of the effective two-body interaction are among the main physical parameters which can be controlled.

The response of a quantum system to a sudden change of its parameters entering in the hamiltonian, such as the confining potential or two-body particle-particle interactions, is an interesting issue in physics. This issue is experimentally realized in the field of ultra-cold quantum gases, where the advances in such field allow a full control of the parameters of the confined gas. The subsequent time evolution of the system following a sudden change of a specific parameter, called quench, is interesting and allows the study of dynamical non-equilibrium properties of the system.

Understanding and describing the non-equilibrium dynamics of quantum many-body system constitutes a very interesting problem in modern theoretical physics. Fundamental questions about the time evolution of quantum systems have become the subject of intense study, such as whether thermalization or some more general equilibration occurs when starting from a non-equilibrium initial state, hence making contact with statistical mechanics. In this respect this has led to the introduction of a new ensemble called the generalized Gibbs ensemble [5], which is a more general one than the usual grand-canonical ensemble. When applied to specific system of hard-core bosons on a one-dimensional lattice, the study confirmed the relaxation hypothesis or the existence of an equilibrium state [5].

In this thesis we were interested in dynamical properties of quantum gas subjected to a sudden change of its initially trapping potential, and our study was mainly restriceted to one-dimensional confining systems. The dynamical quantities we were interested are the one-body density matrix and the resulting mass current density distribution. The manuscript presents the work we realized during three years and it is mainly organized in four chapters

In Chapter 1, we provide some basics physics of ultra-cold gases. Starting from how cold gases can be cooled and trapped in different geometries. We also discuss how to probe of these gases and how a quench generates a non-equilirium dynamics.

In Chapter 2 :

In this chapter we consider a system of N noninteracting particle initially confined in a harmonic potential and we calculate analytically one-body density matrix after a sudden change of the frequency of this potential. We generalized the scaling law known for local density to the non-local density. using this expression we derive an exact expression of the resulting collective current density distribution generated by this quench. We believe that this collective quantity is an interesting tool.

In Chapter 3 :

A one-dimensional boson system of atoms interacting through a very strong zero-range repulsive interactions is known as a hard-core system or Tonks-Girardeau Bose gas. For the case of two hard-core bosons trapped in a harmonic potential, we derive an alternative expression of one-body density matrix in terms of centre of mass and relative coordinates of the atoms. This form is suitable since it allows to easily obtain the short range expansion in terms of the relative coordinate and therefore get through an appropriate fourier transform the high tail behaviour of the momentum density distribution $n(p)$ .We recovered in a transparent way the $1/p^4$ dependence originating from short-range pair correlation with the corresponding Tan's contact coefficient.

It should be noted that the methods used in previous two chapters are original contributions. In chapter 4 we shall present and discuss the main results obtained in this thesis. Here we focus on the non-equilibrium dynamics of quantum gas released from an initially trapping potential and we discuss two different physical situations.

1) The first situation concerns on an extension of the work published in 2015 by Campbell et al in Physical Review Letters 114 125302. We derived a simple analytical expression for the long-time asymptotic one-body reduced density matrix during free expansion for a one-dimensional system of bosons with large atom number interacting through a repulsive zero-range interactions (Lieb-liniger gas) initially confined by a potential well. This density matrix allows direct access to the momentum distribution and we suggest to use the collective mass current density as an interesting physical quantity during the expansion. Some resuls are given for the class of confining power-law potentials.

2) The second work concerns the system of Tonks–Girardeau Bose gas with a given atom number. For this system, We derive an explicit exact analytical expression for the mass current distribution (mass transport) after change from one harmonic trap to another harmonic trap. It is shown that, the current distribution is a suitable collective observable and under the weak quench regime, it exhibits oscillations at the same frequencies as those recently predicted for the peak momentum distribution in the breathing mode [ref full]. The analysis is extended to other possible higher spatial dimensions quenched systems.

We end this thesis with a conculsion, where we summarize our results and suggest some possible extensions.

Chapitre 1

Some general aspects on ultracold quantum gases

1.1 Introduction

Ultra-cold quantum gases is a new field of condensed matter physics research which generates during the last two decades an intense activity both experimental and theoretical. Ultra-cold atomic physics plays an important role to study quantum many-body systems and allows us also vision the density and the momentum distribution through the new tools that it added, this kind of atoms led us to observation the Bose-Einstein condensate (BEC) in 1995 **[6, 7]**. There are many ways to obtain ultra-cold gas, the standard method used is Laser cooling (Doppler cooling)[fig] and the aim of this technique is to immobilize the atoms by the use the pressure force exerted through lasers, reducing the atoms velocity to extremely low values down to few centimetres per seconde (cm/s).

The atom velocity changes from the interaction between the atom and the light, when the atom absorbes a photon with momentum, $p = \hbar k$, it stores the energy and keeps the momentum during the retreating, the corresponding velocity change is : $v = \hbar k/m = h/m\lambda$. If the atom velocity is large so its wavelength is very small, we can say that we are in classical regime and in the weak velocity case (low temperature), the wavelength becomes appreciable and the quantum effect will can manifest more clearly. An important quantity that governs the behavior of gas, is phase space density $\rho = n\lambda^3$, where $n = \frac{N}{V}$ is the particle density and $\lambda = \sqrt{\frac{2\pi\hbar^2}{mk_BT}}$ thermal De Broglie wavelength. We define the volume occupied by single atom $l^3 = \frac{V}{N}$, whose l is the length, by simple words; when $\rho = \left(\frac{\lambda}{l}\right)^3 \leq 1$ we say that we are in classical regime. For a gas to ambient temperature, we have $\rho \backsimeq 10^{-17}$ and for more diluted sample we approach $\rho \backsimeq 10^{-12}$, while if the phase space density $\rho = \left(\frac{\lambda}{l}\right)^3 \geq 1$ we would be in quantum regime. This occurs for $\rho \backsimeq 2.618$ as condition for Bose-Einstein condensate(BEC) [the values of the phase space density are displayed in table(1)]. An others techniques of cooling and trapping are possible to overcome the difficulties, we quote two mechanisms of trapping, known as the magnetic trap and dipolar trap and both give a harmonic confinement, then we move to a technique of cooling, called evaporation

T	λ	n	ρ
$300\ K$	$0.02\ nm$	$10^{10}/cm^3$	10^{-17}
$30\ mK$	$2\ nm$	$10^8/cm^3$	10^{-12}
$1\ mK$	$10\ nm$	$10^9/cm^3$	10^{-9}
$70\ nK$	$1\ \mu m$	$10^{12}/cm^3$	2.618

Table 1 : The values of the phase space density

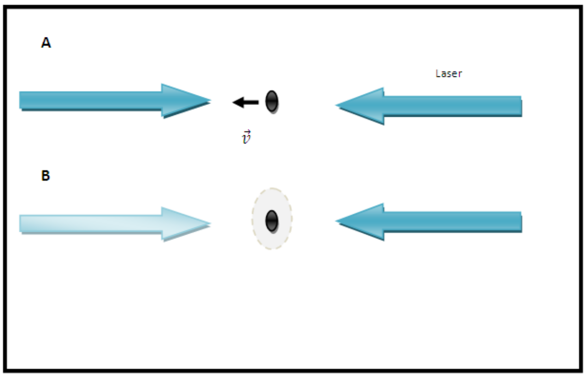

Fig. 1 : Laser cooling

1.2 magnetic confinement

The magnetic trapping considers the tank of cold atoms, it blockades the atoms by quadrupole magnetic field that assures the trapping them, through used Zeeman effect. This confinement depends on magnetic moment of atoms $\overrightarrow{\mu}$ and magnetic field (inhomogeneous and stationnary field) $\overrightarrow{B}(\overrightarrow{r})$. The magnetic confinement consequence of the interaction between the magnetic field and the atoms (Zeeman interaction) with energy $U(\overrightarrow{r}) = -\overrightarrow{\mu}.\overrightarrow{B}(\overrightarrow{r})$.

The energy U plays the role of potential energy for the centre of mass mouvement of atom. The resulting magnetic strength is

$\overrightarrow{F} = \| \overrightarrow{\mu} \| \overrightarrow{\nabla} \left\{ \| \overrightarrow{B}(\overrightarrow{r}) \| . \cos(\overrightarrow{\mu}, \overrightarrow{B}(\overrightarrow{r})) \right\}$ that allows the atoms confinement. If the magnetic momentum is aligned along the axis of the magnetic field, we will have $U(\overrightarrow{r}) = \pm\mu.B(\overrightarrow{r})$ according to $\overrightarrow{\mu}$ whether parallel or antiparallel to $\overrightarrow{B}(\overrightarrow{r})$.

To trap atoms, it is necessary that the potential magnetic energy of the atoms be on maximum or minimum values. When $\overrightarrow{\mu}$ have the same direction of $\overrightarrow{B}(\overrightarrow{r})$, the atoms will be attracted to the regions that have a strong magnetic field. While for the opposite direction of $\overrightarrow{\mu}$ and $\overrightarrow{B}(\overrightarrow{r})$, the atoms are pushed to minimum.

The simple configuration is Ioffe-Pritchard trap, this trap has been designed as a combination of a radial quadrupole field creates by four parallele wires are called Ioffe bars by circulating an electric current two to two opposite, these wires are responsible to radial confinement of atoms, and pairs of coils of "pinch" of parallele axis with same current, that assure the longitudinal trapping [fig.1]

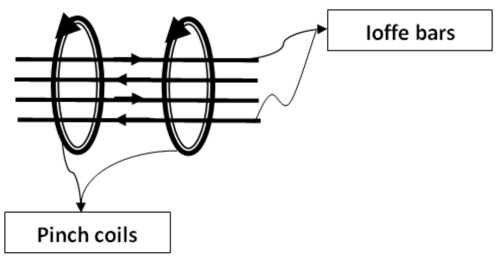

Fig. 2 : Ioffe-Pritchard trap

1.3 harmonic magnetic trap

If the axis of confinement in Ioffe Pritchard configuration is the axis (oz), the expression of total magnetic field created through trap is [8] :

$$\overrightarrow{B}(\overrightarrow{r}) = B_0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{B''}{2} \begin{pmatrix} xy \\ -yz \\ z^2 - (x^2+y^2)/2 \end{pmatrix} + B' \begin{pmatrix} x \\ -y \\ 0 \end{pmatrix} \tag{1.1}$$

where B_0,$B'$and $B''$are the angle, the gradient and curvature of field respectively. the two first terms represent the magnetic field creates by the two coils, and the last terms is created through Ioffe bars.

The module of this field in the lowest order of x, y et z are :

$$\|\overrightarrow{B}(\overrightarrow{r})\| = B_0 + \left(\frac{B'^2}{2B_0} - \frac{B''}{4}\right)(x^2+y^2) + \frac{B''}{2}z^2 \tag{2.1}$$

This expression of the module of magnetic field present a local minimum at point 0 on condition $2B'^2 \succ B_0B''$, in this case, the magnetic interaction energy is :

$$U(\overrightarrow{r}) = \mu B_0 + \frac{1}{2}\mu\left(\frac{B'^2}{B_0} - \frac{B''}{2}\right)(x^2+y^2) + \frac{\mu B''}{2}z^2 \tag{3.1}$$

we see that the potential of confinement is harmonic in the region around the centre of confinement. Indeed, it can be rewritten as

$$U(\overrightarrow{r}) = U_0 + \frac{1}{2}m\omega_\perp^2(x^2+y^2) + \frac{1}{2}m\omega_z^2 z^2 \tag{4.1}$$

with $\omega_\perp = \sqrt{\frac{\mu}{m}\left(\frac{B'^2}{B_0} - \frac{B''}{2}\right)}$ and $\omega_z = \sqrt{\frac{\mu B''}{m}}$ are respectively the frequencies of the trapping radial and axial We note that the frequency $\omega_\perp$ depends of the value of the angle of magnetic field B_0 whose can be modified using the two coils.

1.4 Dipolar confinement

The dipolar confinement of atoms results of the interaction between the electric field of light beam and the electric dipole of the particle.When a neutral atom is subjected to the electric field $\overrightarrow{E}(\overrightarrow{r})$ of light beam, this field changes polarize the particle so that the induced dipole $\overrightarrow{d}$ possesses the same direction as $\overrightarrow{E}(\overrightarrow{r})$.

The resulting interaction energy is given by[see for instance : [14]]

$$V(\overrightarrow{r}) = -\frac{1}{2}\overrightarrow{d}.\overrightarrow{E}(\overrightarrow{r}) \tag{5.1}$$

where the factor 1/2 accounts that the induced dipole is not permanent. The electric dipole $\overrightarrow{d}$ is given by $\overrightarrow{d} = \alpha.\overrightarrow{E}(\overrightarrow{r})$ where α is the complex polarisability of the atom. Since the light intensity is $I(\overrightarrow{r}) = \frac{\varepsilon_0 c}{2}\left|\overrightarrow{E}(\overrightarrow{r})\right|^2$, so the interaction energy becomes

$$V(\overrightarrow{r}) = -\frac{1}{2\varepsilon_0 c}\,\mathrm{Re}(\alpha).I(\overrightarrow{r}) \tag{6.1}$$

Here ε_0 and c are respectively the permittivity in the vacuum and the light velocity. The dipolar potential created is proportionnal to complex polarisability, we can say that α will determine the most attractive areas of atoms.

We consider two levels system $|f\rangle$ (ground state) and $|\zeta\rangle$ (excited state), of atomic frequency ω_0. In this example, the atomic polarisability α is a complex function of light pulsation ω_L [9, 10] :

$$\alpha = 6\pi\varepsilon_0 m_e c^3 \frac{\Gamma/\omega_0^2}{\omega_0^2 - \omega_L^2 - i(\omega_L^3/\omega_0^2)\Gamma} \tag{7.1}$$

where $\Gamma = \frac{\omega_0^3}{3\pi\varepsilon_0 hc^3}\langle f|\,\widehat{D}\,|\zeta\rangle^2$ is the natural width of the transition. For disagreement of the laser by contribution to atomic transition $\delta = \omega_L - \omega_0$, the interaction energy takes the following form :

$$V(\overrightarrow{r}) = \frac{3\pi c^2\Gamma}{2\omega_0^3}\frac{I(\overrightarrow{r})}{\delta} \tag{8.1}$$

it the follows from this expression, we will have two regimes according to the frequency of external field is lower or greater than the atomic frequency. For $\delta\langle 0$, the interaction potential is attractive, and the atoms are attracted to the areas that have maxima light intensity. If $\delta\rangle 0$, the potential is repulsive so, the atoms are attracted to minima light intensity.

1.5 Harmonic dipole trap

In previous paragraph we have seen that the dipolar trapped creates through focalization of simple laser beam. considering a laser beam of gaussien type, the intensity distribution is given[11]

$$I(z,r) = \frac{I_0}{1+z^2/Z_r^2} \exp\left[-\frac{2r^2}{w^2(z)}\right] \tag{9.1}$$

with :

I_0 is the maximal intensity of the beam.

$w(z) = w_0\sqrt{1+z^2/Z_r^2}$ is the profil rayon of the intensity, at $1/e^2$, at

w_0 is the profil rayon of the intensity, at $1/e^2$, at

$Z_r = \pi w_0/\lambda$ is the Rayleigh length, the λ is wave length.

The expression (8.1) becomes

$$V(\overrightarrow{r}) = \frac{V_0}{1+z^2/Z_r^2} \exp\left[-\frac{2r^2}{w^2(z)}\right] \tag{10.1}$$

the $V_0 = \frac{3\pi c^2 \Gamma}{2\omega_0^3}\frac{I_0}{\delta}$ the depth of the trap. We can devlop the interaction energy to second order, so we obtain

$$V(x,y,z) = V_0 - \frac{2V_0}{w_0^2}(x^2+y^2) - \frac{V_0}{Z_r^2}z^2 \tag{11.1}$$

1.6 Evaporation cooling

Trapping technics described previously don't allow of cooling the atoms, but only to confine. So to increase its density. But we are then far to the threshold of quantum degeneracy. To reach to this regime, we have increase the phase space density ρ, because the magnetic and the dipolar confinement used to load the ultracold atoms in traps. doesn't provide a high atomic density. The phase space density stays 10^{-5} [12]. For obtain the higher phase space density, and among the cooling technics that allow to access to lower temperature, we quote **evaporation cooling**.

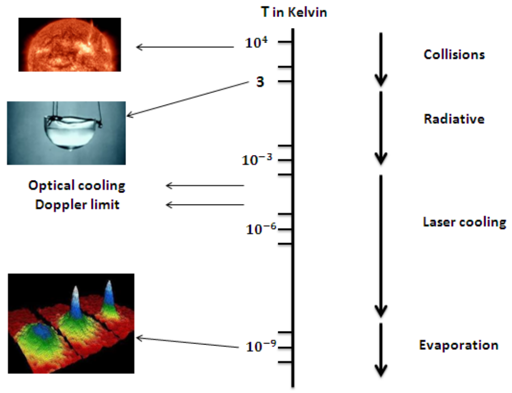

Fig.4 : Temperature scale "Laser cooling and trapping"

The evaporation cooling is a cooling technic of a gas of atoms initially trapped at a temperatures of the order mK to a temperature of order ηK. We can't reach to this step after a prior phase of cooling et of trapping (Laser cooling)[fig].

This kind of cooling is based on elimination of the most energetic atoms of a trapped gas, in parallel with a rethermalization of the atoms remaining at a lower temperature.

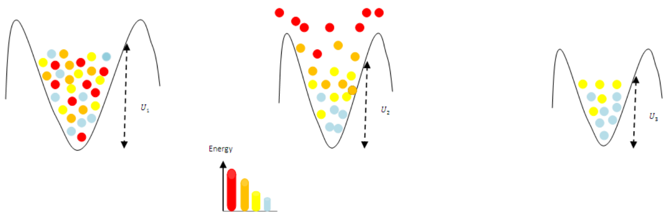

Fig. 5 : Evaporation cooling

After the overview on the cooling and trapping technics, a question comes to our mind, **why we cool the atoms ?**

The answer of this question is very clear, because the new physics hides behind this type of atoms. The new state of matter, the quantum phenomenon that manifest at lower temperature to macroscopic level, the possibility of study and understand the quantum phenomenons and their behavior in a accessible and controllable. The possibility to change the interaction force between atom-atom through a simple ajustement of magnetic field, this modification called **Feshbach resonance**[13].

1.7 Feshbach resonance

In ultracold-atoms, the interaction between atoms can be fully controlled through the so-colled Feshbach resonance. The latter is a subtle and beautiful phenomenon allowing to modify the atom-atom interaction, by a simple change of a magnetic field. The change of interactions can be performed over an enormous range, passing from strongly attractive to strongly repulsive configurations. This modification of the interaction strength allows to obtain atoms from unbound state (open channel) to bound state with different spin configuration (closed channel). Due to the difference in magnetic moments between open and closed channel, the zeeman energies change differently in a magnetic field, an interesting parameter also changes, the so-called **scattering length** a.

The scattering length depends only on the potential of interaction between the atoms as follows [14]

$$V(x-x^{'}) = \frac{4\pi a\hbar^2}{m}\delta(x-x^{'}) \tag{12.1}$$

where δ is the Dirac delta function , $g = \frac{4\pi a\hbar^2}{m}$ is the coupling constant and $V(x-x')$ is the pseudo-potential [14].

In reality, the exact form of the potential V is very difficult to calculate, because a small error in the development of the potential V can give us a very large error for the scattering length as a result we obtain a large number of bound states [14]. This makes it difficult to describe the ultracold quantum gases.

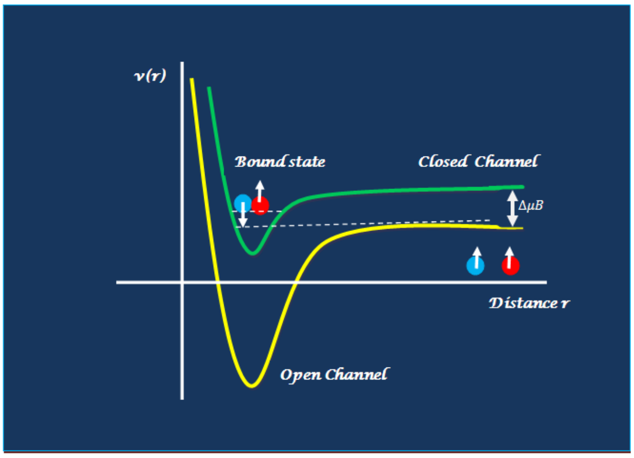

Fig. 6 : Basic two-channel model for Feshbach resonance

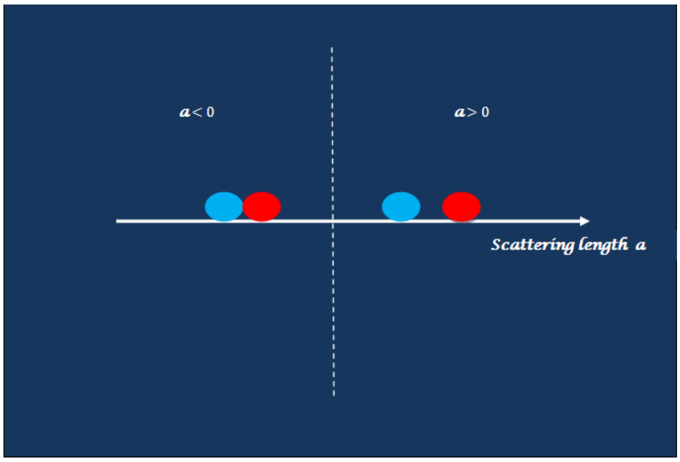

Fig. 7 : Scattering length

1.8 Non-equilibrium dynamics in ultra-cold gases through quantum quench.

1.8.1 Introduction

An interesting issue in the field of ultra-cold gases is the study of non-equilibrium dynamics. This subfield was driven by the spectacular experimental advances in controling and high resolution imaging cold cloud gases and allow to probe these latter systems. The most popular technique used to probe a cold gas is to use the time-of-flight technique (TOF), which consists of a sudden switch-off of trapping potential and simply observing the atoms expand as they fall under gravity. After a certain time, atoms are illuminated by laser light. The atoms of the gas absorb light and from the intensity of the outgoing light one can deduce the spatial density distribution $\rho^{TOF}(\overrightarrow{r})$ of the expanded gas [14].

1.8.2 Probe of momentum distribution of cold atoms.

It consists of :

- Suddenly switch-off the initial confining trap potential at time $t = 0$.

- Let the quantum gas expand freely for a time t, called time-of-flight.

Assuming a ballistic expansion, i.e ; neglect the interactions during the free expansion, one can show that for long times, the spatial distribution is proportional to the initial momentum distribution of the initially confined gas, that is

$$\rho(\overrightarrow{r}, t \gg 1) \rightarrow \left(\frac{m}{t}\right)^3 n_0\left(\overrightarrow{p} = \frac{m\overrightarrow{r}}{t}\right) \tag{13.1}$$

where $n_0\left(p = \frac{m\overrightarrow{r}}{t}\right)$ is the initial momentum density evaluated at momentum $p = m\overrightarrow{r}/t$. Experimentaly informations on the expanded cloud are obtained, as pointed out before, through the imaging absorption technique[14] which allows to extract, the number of atoms, temperature, cloud velocity, and cloud size and the spatial density. The measured latter density allows through Eq. (13.1) to deduce the momentum density of the initially confined cloud.

1.8.3 Probe of dynamical correlations and time dependent many-body problem.

The progress in the resolution imaging techniques are allowing nowdays cold atom nonequilibrium experiments to probe spatial dynamical correlation functions following a sudden change in experimental parameters such as confining trap or interaction strength. The sudden change of the potential or the interaction strength is called respectively quench of the potential and the quench of the interaction.

The quench generates a nonequilibrium state and allows the study of the broad domain of time-dependent problems of quantum many-particle system where one can study a number of fundamental issues. Among these issues a fundamental question related to statistical mechanics addressed by [5] on the possibility that a many-body quantum system with a full set of conserved quantities could relax to a stationary state (equilibrium state) and more generaly understanding how quantum systems approach equilibrium and the timescales involved. In this respect these authors introduced a new ensemble called the generalized Gibbs ensemble, which is a more general one than the usual grand-canonical ensemble, and examined the specific system of hard-core bosons on a one-dimensional lattice, the study confirmed the relaxation hypothesis or the existence of an equilibrium state [5].

1.8.4 Collective dynamics in quantum many-body systems

Collective dynamics in many-particle systems manifest as a result of interactions between particles. The resulting collective mode or state is characterised by a coherent or correlated behaviour of the particles. Collective dynamics can therefore serve as a probe of the such interactions. In the field of ultra-cold quantum gases, the simplest collective mode relate to the frequencies of monopole (breathing mode) and multipole oscillations in harmonic trapping potentials [see for instance :[15] and references therein]. To close this section we note the realization of trap to trap quench which consists, as will be discussed in the next chapters of this thesis, of a sudden change of the frequency of the confining trap potential passing from frequency ω_0 to ω so that $\omega > \omega_0$, resulting of an excitation of collective mode oscillations of the system.

Chapitre 2

Dynamical response of noninteracting Fermions to a sudden change of harmonic trap (quantum quench potential)

2.1 Introduction :

The ideal Fermi gas represents a natural starting point for explaining the physic of Fermi gases. In many cases the role of interactions can in fact be neglected, as in the case of spin polarized gases where interactions are suppressed at low temperature by the Pauli principle. To explore the properties of many-body physics, ultra-cold trapped gases have been used. In most of current experimental setups, cold atoms are confined in harmonic traps. In this chapter we consider a system of N noninteracting Fermions, the main aim of this chapter is to obtain analytical expression for one-body density matrix after a sudden release of the confining potential $V(x)$. Another interesting problem we are going to examine is the resulting dynamics of a gas through its one body density matrix after a sudden change of the frequency of the confining potential (trap to trap quench). It is well known that the time dependent density exhibites scaling properties and is related to the initial, at $t = 0$, density by a scaling law[18].

In the following we shall provide a different derivation based on the so-called Bloch propagator. As a result we use the time dependent density matrix to obtain the current density distribution generated by this quench and show which oscillates with time at frequency twice of the one of the confining trap. Although our derivation is given for one dimensional case it can easily be extended to deal with higher spatial dimensions.

2.2 Scaling law property of one-body density matrix for noninteracting Fermi gas initially confined in harmonic trap.

2.2.1 Free expension

Consider system of N noninteracting one dimensional harmonically trapped Fermions. For $t \leq 0$ the gas is in equilibrium and we denote by $\widehat{\rho}$ the first order reduced density matrix and called from now on the one-ody density matrix operator. At $t = 0$, the trap potential is suddenly released and the time evolution of the resulting density operator $\widehat{\rho}(t)$ after a finite time $t > 0$ is given according to the transformation law as

$$\widehat{\rho}(t) = e^{-i\frac{H}{\hbar}t}\widehat{\rho}(0)e^{+i\frac{H}{\hbar}t} \tag{1.2}$$

Here $H = p^2/2m$ stands for the hamiltonian of the particle after the release of the trap and reduces to the kinetic energy operator of a particle with mass m. We start by considering a simple situation where the system is formed by N noninteracting Fermions and initially confined in one dimension by a harmonic potential of the form $V(x) = m\omega_0^2 x^2/2$.

In spatial coordinates representation, Eq(1.2) reads

$$\begin{aligned} \rho(x,y;t) &= \langle x| e^{-i\frac{H}{\hbar}t}\widehat{\rho}(0)e^{+i\frac{H}{\hbar}t}|y\rangle \\ &= \int\int U(x,x_1;t)\rho_0(x_1,x_2)U^*(y,x_2;t)dx_1dx_2 \end{aligned} \tag{2.2}$$

Here $\rho_0(x_1,x_2) = \langle x_1|\widehat{\rho}(0)|x_2\rangle$ is the density matrix of the system before the removal of the trap potential and $U(x,x_1,t) = \langle x| e^{-i\frac{H}{\hbar}t}|x_1\rangle$ is nothing but the free particle time-evolution propagator given in one dimension by [16]

$$U(x,x_1,t)=\sqrt{\frac{m}{2\pi i\hbar t}}e^{i\frac{m}{2\hbar t}(x-x_1)^2} \tag{3.2}$$

Upon insertion into Eq.(2.2), we can write

$$\rho(x,y,t)=\left(\frac{m}{2\pi\hbar t}\right)\int\int e^{i\frac{m}{2\hbar t}\left[(x-x_1)^2-(y-x_2)^2\right]}\rho_0(x_1,x_2)dx_1dx_2 \tag{4.2}$$

or

$$\begin{aligned}\rho(x,y,t) &= \left(\frac{m}{2\pi\hbar t}\right)e^{i\frac{m}{2\hbar t}\left(x^2-y^2\right)}\int\int e^{i\frac{m}{2\hbar t}\left(x_1^2-x_2^2-2xx_1+2yx_2\right)}\rho_0(x_1,x_2)dx_1dx_2\\ &= \left(\frac{m}{2\pi\hbar t}\right)e^{i\frac{m}{2\hbar t}\left(x^2-y^2\right)}\int\int e^{i\frac{m}{2\hbar t}[(x_1-x_2)(x_1+x_2-2xx_1+2yx_2)]}\rho_0(x_1,x_2)dx_1dx_2\end{aligned} \tag{5.2}$$

Now, using the center of mass and relative coordinates respectively defined by $u=(x_1+x_2)/2$ and $v=x_1-x_2$, Eq.(5.2) becomes

$$\begin{aligned}\rho(x,y,t) &= \left(\frac{m}{2\pi\hbar t}\right)e^{i\frac{m}{2\hbar t}\left(x^2-y^2\right)}\int\int e^{i\frac{m}{2\hbar t}\left[2uv-2x\left(u+\frac{v}{2}\right)+2y\left(u-\frac{v}{2}\right)\right]}\rho_0(u+v/2,u-v/2)dudv\\ &= \left(\frac{m}{2\pi\hbar t}\right)e^{i\frac{m}{2\hbar t}\left(x^2-y^2\right)}\int\int e^{i\frac{m}{2\hbar t}[-2(x-y)u+(2u-x-y)v]}\rho_0(u+v/2,u-v/2)dudv\end{aligned} \tag{6.2}$$

As can be seen in Eq. (6.2) the main input to obtain $\rho(x,y,t)$ is the density matrix $\rho_0(x_1,x_2)\equiv\rho_0(u+v/2,u-v/2)$ of the initial confined system.

For a given Fermi energy μ determined so that the local density is normalized to the total particle number N, it is possible to write the density matrix $\rho_0(x_1,x_2)$ as an appropriate inverse Laplace transform with respect to a complex parameter denoted r [17], that is

$$\rho_0(x_1,x_2)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{z\mu}\frac{C_0(x_1,x_2;z)}{z}dz=L^{-1}_{z\to\mu}\left[\frac{C_0(x_1,x_2;z)}{z}\right],\quad c>0 \tag{7.2}$$

with $C_0(x_1,x_2;z)=C_0(u+v/2,u-v/2;z)$ is the so-called Bloch propagator [17]. Analytical expression of the latter propagator is known for the harmonic oscillator potential. In terms of centre of mass and relative coordinates, it is given as

$$C_0(u+v/2,u-v/2;z)=\sqrt{\frac{m\omega_0}{2\pi\hbar\sinh(z\hbar\omega_0)}}e^{-\frac{m\omega_0}{\hbar}u^2\tanh(\frac{z\hbar\omega_0}{2})}e^{-\frac{m\omega_0}{4\hbar}v^2\coth(\frac{z\hbar\omega_0}{2})} \tag{8.2}$$

Upon inserting Eq. (7.2) with (8.2) into (6.2), we can write

$$\begin{aligned}\rho(x,y,t) &= \frac{m}{2\pi\hbar t}e^{i\frac{m}{2\hbar t}\left(x^2-y^2\right)}L^{-1}_{z\to\mu}\left\{\int\int e^{\frac{1}{z}\sqrt{\frac{m\omega_0}{2\pi\hbar\sinh(z\hbar\omega_0)}}}e^{i\frac{m}{2\hbar t}[-2(x-y)u+(2u-x-y)v]}\times\right.\\ &\left. e^{-\frac{m\omega_0}{\hbar}u^2\tanh(\frac{z\hbar\omega_0}{2})}e^{-\frac{m\omega_0}{4\hbar}v^2\coth(\frac{z\hbar\omega_0}{2})}dudv\right\}\\ &= \frac{m}{2\pi\hbar t}e^{i\frac{m}{2\hbar t}\left(x^2-y^2\right)}L^{-1}_{z\to\mu}\left\{e^{\frac{1}{z}\sqrt{\frac{m\omega_0}{2\pi\hbar\sinh(z\hbar\omega_0)}}}\int_{-\infty}^{+\infty}e^{-\frac{m\omega_0}{\hbar}u^2\tanh(\frac{z\hbar\omega_0}{2})}e^{-i\frac{m}{\hbar t}(x-y)u}du\times\right.\\ &\left.\int_{-\infty}^{+\infty}e^{-\frac{m\omega_0}{4\hbar}v^2\coth(\frac{z\hbar\omega_0}{2})}e^{i\frac{m}{2\hbar t}(2u-x-y)v}dv\right\} \qquad (9.2)\end{aligned}$$

The above spatial integrals can be carried out by using the identity

$$\int e^{-\alpha x^2}e^{(b+ic)x}dx = \sqrt{\frac{\pi}{\alpha}}e^{\frac{(b^2-c^2)}{4\alpha}}e^{i\frac{bc}{2\alpha}} \qquad (10.2)$$

performing the v integration allows us to write

$$\begin{aligned}\rho(x,y,t) &= \frac{m}{2\pi\hbar t}e^{i\frac{m}{2\hbar t}\left(x^2-y^2\right)}L^{-1}_{z\to\mu}\left\{\frac{1}{z\cosh(\frac{z\hbar\omega_0}{2})}\int_{-\infty}^{+\infty}e^{-i\frac{m}{\hbar t}(x-y)u}e^{-\frac{m\omega_0}{\hbar}u^2\tanh(\frac{z\hbar\omega_0}{2})}\times\right.\\ &\left. e^{-\frac{m}{4t^2\hbar\omega_0}(2u-x-y)^2\tanh(\frac{z\hbar\omega_0}{2})}du\right\}\\ &= \frac{m}{2\pi\hbar t}e^{i\frac{m}{2\hbar t}\left(x^2-y^2\right)}L^{-1}_{z\to\mu}\left\{\frac{e^{-\frac{m\tanh(\frac{z\hbar\omega_0}{2})}{\hbar\omega_0}\frac{(x+y)^2}{4t^2}}}{z\cosh(\frac{z\hbar\omega_0}{2})}\int_{-\infty}^{+\infty}e^{-i\frac{m}{\hbar t}(x-y)u}e^{-\frac{m\omega_0}{\hbar}u^2\tanh(\frac{z\hbar\omega_0}{2})}\times\right.\\ &\left. e^{-\frac{m\tanh(\frac{z\hbar\omega_0}{2})}{t^2\hbar\omega_0}u^2}e^{+\frac{m\tanh(\frac{z\hbar\omega_0}{2})}{t^2\hbar\omega_0}(x+y)u}du\right\}\\ &= \frac{m}{2\pi\hbar t}e^{i\frac{m}{2\hbar t}\left(x^2-y^2\right)}L^{-1}_{z\to\mu}\left\{\frac{e^{-\frac{m\tanh(\frac{z\hbar\omega_0}{2})}{\hbar\omega_0}\frac{(x+y)^2}{4t^2}}}{z\cosh(\frac{z\hbar\omega_0}{2})}\int_{-\infty}^{+\infty}e^{-\left(\frac{m\omega_0}{\hbar}\frac{1+\omega_0^2t^2}{\omega_0^2t^2}\tanh(\frac{z\hbar\omega_0}{2})\right)u^2}\times\right.\\ &\left. e^{\left(\frac{m}{\hbar\omega_0}\frac{(x+y)}{t^2}\tanh(\frac{z\hbar\omega_0}{2})-i\frac{m}{\hbar t}(x-y)\right)u}du\right\} \qquad (11.2)\end{aligned}$$

using the idenity in Eq. (11.2), we have

$$\int_{-\infty}^{+\infty}e^{-\left(\frac{m\omega_0}{\hbar}\frac{1+\omega_0^2t^2}{\omega_0^2t^2}\tanh(\frac{z\hbar\omega_0}{2})\right)u^2}e^{\left(\frac{m}{\hbar\omega_0}\frac{(x+y)}{t^2}\tanh(\frac{z\hbar\omega_0}{2})-i\frac{m}{\hbar t}(x-y)\right)u}du = \sqrt{\frac{\pi\hbar\omega_0t^2}{m(1+\omega_0^2t^2)\tanh(z\hbar\omega_0/2)}}\times$$
$$e^{+\frac{m}{4\hbar\omega_0}\frac{(x+y)^2}{t^2(1+\omega_0^2t^2)}\tanh(\frac{z\hbar\omega_0}{2})}e^{-\frac{m\omega_0}{4\hbar}\frac{(x-y)^2}{(1+\omega_0^2t^2)}\coth(\frac{z\hbar\omega_0}{2})}e^{-i\frac{m}{2\hbar t}\frac{(x^2-y^2)}{(1+\omega_0^2t^2)}} \qquad (12.2)$$

Substituting this result into the last form of Eq. (12.2) we get

$$\rho(x,y,t)=\frac{e^{i\frac{m\omega_0^2t^2}{2\hbar t}\frac{(x^2-y^2)}{(1+\omega_0^2t^2)}}}{\sqrt{1+\omega_0^2t^2}}L^{-1}_{z\to\mu}\left[\frac{1}{z}\sqrt{\frac{m\omega_0}{2\pi\hbar\sinh(z\hbar\omega_0)}}e^{-\frac{m\omega_0}{\hbar}\frac{\left(\frac{x+y}{2}\right)^2}{(1+\omega_0^2t^2)}\tanh(\frac{z\hbar\omega_0}{2})}e^{-\frac{m}{4\hbar}\frac{(x-y)^2}{(1+\omega_0^2t^2)}\coth(\frac{z\hbar\omega_0}{2})}\right] \tag{13.2}$$

According to Eq. (5.2) with (6.2), it is easy to note that the above inverse Laplace transform is nothing but the expression of the density matrix before the release of the trap calculated at the rescaled position $x/\sqrt{1+\omega_0^2t^2}$, then Eq. (13.2) may be rewritten as

$$\begin{aligned}\rho(x,y,t) &= \frac{e^{i\frac{m}{2\hbar t}\omega_0^2t^2\frac{(x^2-y^2)}{(1+\omega_0^2t^2)}}}{\sqrt{1+\omega_0^2t^2}}\rho_0\left(\frac{x}{\sqrt{1+\omega_0^2t^2}},\frac{y}{\sqrt{1+\omega_0^2t^2}}\right)\\ &= \frac{e^{i\frac{m\omega_0}{2\hbar}\frac{\omega_0t}{(1+\omega_0^2t^2)}(x^2-y^2)}}{\sqrt{1+\omega_0^2t^2}}\rho_0\left(\frac{x}{\sqrt{1+\omega_0^2t^2}},\frac{y}{\sqrt{1+\omega_0^2t^2}}\right)\end{aligned} \tag{14.2}$$

2.2.2 Expansion from trap to trap ($\omega_0 \to \omega$)

In the following we shall study the response of the ultracold gas to to a sudden modification of the trapping potential (is known as the expansion from trap to trap). Initially the gas is confined in harmonic oscillator potential of the form $V(x) = m\omega_0^2x^2/2$ and at $t = 0$ a sudden change of the frequency is realized passing from frequency ω_0 to ω so that $\omega > \omega_0$. After this quench of the potential the evolution is still governed by Eq. (1.2) or (2.2) but now with, $H = p^2/2m + m\omega^2x^2/2$ is the Hamiltonien at times $t > 0$. We then have in coordinate representation

$$\rho_{\omega_0\to\omega}(x,y;t)=\int\int U^{osc}(x,x_1;t)\rho_0(x_1,x_2)U^{osc*}(y,x;t)dx_1dx_2 \tag{15.2}$$

Here $U^{osc}(x,x_1,t)$ is the well known harmonic oscillator propagator [16]

$$U^{osc}(x,x_1;t)=\sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega t}}e^{\frac{im\omega}{2\hbar\sin\omega t}[(x^2+x_1^2)\cos\omega t-2xx_1]} \tag{16.2}$$

using the centre of mass $u = (x_1 + x_2)/2$ and relative $v = x_1 - x_2$ coordinates, we can write

$$\rho_{\omega_0\to\omega}(x,x^{'};t) = \sqrt{\frac{m^2\omega^2}{4\pi^2\hbar^2\sin^2\omega t}}e^{\frac{im\omega\cos\omega t}{2\hbar\sin\omega t}(x^2-y^2)}\int\int\rho_0(u+v/2,u-v/2)\times$$
$$\times e^{\frac{im\omega}{2\hbar\sin\omega t}[(2u\cos\omega t-(x+y))v-2u(x-y)]}dudv \quad (17.2)$$

To calculate the above integral, we will go through the use of the Wigner transform[17].The Wigner transfom for $\rho_0(X,s)$ is defined as

$$\rho_0(X,s) = \frac{1}{2\pi\hbar}\int dp\rho_w(X,p)e^{-\frac{ips}{\hbar}} \quad (18.2)$$

and the Wigner transform version of Eq. (7.2) reads

$$\rho_w(X,p) = L^{-1}_{z\to\mu}\{\frac{1}{z}C_w(X,p)\} \quad (19.2)$$

and where the phase space function $C_w(X,p)$ is given by[22]

$$C_w(X,p) = \frac{1}{\cosh(\frac{z\hbar\omega_0}{2})}e^{\frac{-2}{\hbar\omega_0}\tanh(\frac{z\hbar\omega_0}{2})(\frac{p^2}{2m}+\frac{1}{2}m\omega_0^2X^2)} \quad (20.2)$$

we substitute Eq. (18.2) with (19.2), (20.2) into (17.2), to rewrite $\rho(x,x^{'},t)$ as

$$\rho(x,y,t) = \sqrt{\frac{m^2\omega^2}{4\pi^2\hbar^2\sin^2\omega t}}L^{-1}_{z\to\mu}\left\{\frac{1}{z}\frac{e^{\frac{im\omega\cos\omega t}{2\hbar\sin\omega t}(x^2-y^2)}}{\cosh(\frac{z\hbar\omega_0}{2})}e^{\frac{im\omega\cos\omega t}{4\omega_0\hbar\sin\omega t}\tanh(\frac{z\hbar\omega_0}{2})(x+y)^2}\times\right.$$
$$\left.\int e^{-X^2\tanh(\frac{z\hbar\omega_0}{2})(\frac{m\omega_0}{\hbar}+\frac{m\omega^2\cos^2\omega t}{\hbar\omega_0\sin^2\omega t})}e^{X(\frac{m\omega^2\cos\omega t}{\hbar\omega_0\sin^2\omega t}\tanh(\frac{z\hbar\omega_0}{2})(x+y)-\frac{im\omega}{\hbar\sin\omega t}(x-y))}dX\right\} \quad (21.2)$$

After a bit of manipulations, we arrived to final result

$$\rho(x,y,t) = \frac{e^{\frac{im\omega\sin 2\omega t(\omega_0^2/\omega^2-1)}{2\hbar(1+(\omega_0^2-\omega^2)\sin^2\omega t/\omega^2)}(x^2-y^2)}}{\sqrt{(1+(\omega_0^2-\omega^2)\sin^2\omega t/\omega^2)}}\times$$
$$\rho_0\left(\frac{x}{\sqrt{(1+(\omega_0^2-\omega^2)\sin^2\omega t/\omega^2)}},\frac{y}{\sqrt{(1+(\omega_0^2-\omega^2)\sin^2\omega t/\omega^2)}}\right) \quad (22.2)$$

2.3 Current density distribution generated by a sudden quench of the trap

These expressions of non local density for harmonic traps show how the initial density matrix is easily determined through the measure of density matrix in time t after the trap is switched off. The imaginary phase which appeared respectively in Eqs. (14.2), (22.2) indicates the existence of current density distribution defined in quantum mechanics as

$$j(x,t) = \frac{\hbar}{2mi}\left[\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\rho(x,y,t)\right]_{x=y} \tag{23.2}$$

Using the Eqs.(14.2and 22.2), we immediately obtain these current distribution

$$j(x,t) = \frac{\omega^2 tx}{(1+\omega^2t^2)^{3/2}}\rho_0\left(\frac{x}{\sqrt{1+\omega^2t^2}}\right) \tag{24.2}$$

$$j(x,t) = \frac{\omega\,(\sin 2\omega t)\,(\omega_0^2/\omega^2 - 1)x}{2(1+(\omega_0^2-\omega^2)\sin^2\omega t/\omega^2)^{3/2}}\rho_0\left(\frac{x}{\sqrt{1+(\omega_0^2-\omega^2)\sin^2\omega t/\omega^2}}\right) \tag{25.2}$$

The equation (25.2) represents the collective current generated by the abrupt change in the frequency of the confining potential. we can illustrate the evolution of this current at the time $t > 0$ after the suddenly change of the frequency from ω_0 to ω by the following curves.

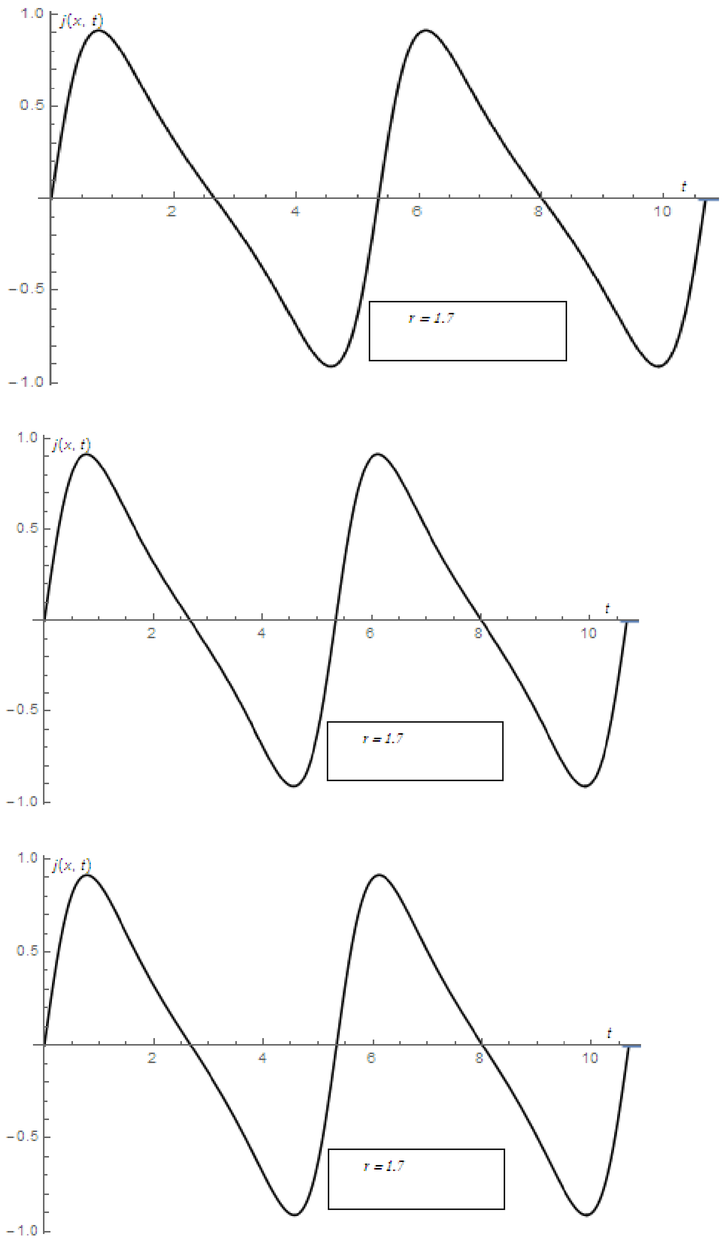

Fig. 8 : Evolution of the current as a function of time for fixed position after

the abrupt change of confining potential

Chapitre 3

Alternative expression for one body density matrix of harmonically trapped two hard-core bosons in one spatial dimension and Tan's contact of momentum distribution.

3.1 Introduction

A one dimensional bosons interacting through a repulsive pointlike interactions is known as the homogeneous Lieb-Liniger model [19]. This model is among a few quantum many body systems where the ground state and the excited states are exactly known [20] The associated many body Hamitonian of N identical bosons each with mass m, reads

$$H^{LL} = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + g_{1D}\sum_{1\leq i<j<N}\delta\left(x_i - x_j\right) \tag{1.3}$$

where $\{x_1, x_2, ...x_N\}$ are the positions of the bosons and g_{1D} is an effective $1d$ coupling strength. If the gas is confined by an external trapping potential $V_{ext}(x_i)$ acting on the particle located at x_i, the hamiltonian becomes

$$H = \sum_{i=1}^{N} \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_i^2} + V_{ext}(x_i) \right] + g_{1D} \sum_{1 \leq i < j = N} \delta \left(x_i - x_j \right) \tag{2.3}$$

The above system can be realized experimentally with ultracold bosonic atoms trapped in effectively $1d$ atomic waveguide [21]. By varying the interaction strength, the system can be tuned from the mean field regime up to the strongly interacting regime ($g_{1D} \mapsto \infty$) or hard core regime called Tonks–Girardeau (TG) Bose gas [23] with infinitely repulsive contact interaction. In the latter case of strongly interacting regim, the interaction term of the Hamiltonian can be properly taken into account by imposing boundary condition on the many-body wavefunction $\Psi_{TG}(x_1, x_2, ...x_N)$ so that, $\Psi_{TG}(x_1, ..x_i, ..x_j...x_N) = 0$ when $x_i = x_j$. In the limit of infinitely zero-range respulsive interaction corresponding to $g_{1D} \to +\infty$, the interaction term plays the same role as the Pauli principle for noninteracting Fermions and the TG gas system becomes similar to a spin polarized non-interacting Fermions. This analogy is exhibited by the so-called Bose-Fermi mapping theorem [23, 24, 25]. This mapping allows to obtain the TG many body wavefunction Ψ_{TG} in terms of the wavefunction Φ_F of noninteracting Fermions, that is $\Psi_{TG}(x_1, ..x_i, ..x_j...x_N) = |\Phi_F(x_1, ..x_i, ..x_j...x_N)|$, which maps a strongly interacting problem onto a simpler one of noninteracting Fermions with wavefunction Φ_F is a slater determinant . As a consequence of this analogy between the TG Bose gas system and the non-interacting Fermions, the Bose-Fermi mapping yields to identical local densities of the two systems. In fact, from the very definition of the local density

$$\rho(x) = N \int dx_2....dx_N \left| \Psi(x, x_2, ..., x_N) \right|^2 \tag{3.3}$$

and since $\Psi_{TG}(x, x_2, ..., x_N) = |\Phi_F(x_1, ..x_i, ..x_j...x_N)|$, we conclude that $\rho_{TG}(x) = \rho_F(x)$, where $\rho_F(x)$ is the local density of N Fermions and is given in terms of occupied single particle wavefunctions as $\rho_F(x) = \sum_{j=1}^{N} \int dx \left| \varphi_j(x) \right|^2$. Moreover the pair distribution function normalized to $N(N-1)$ defined as

$$D(x, x') = N(N-1) \int dx_3....dx_N \left| \Psi(x, x', x_3..., x_N) \right|^2 \tag{4.3}$$

which measures the joint probability of finding one atom at position x and another at x', are identical, i.e; $D_{TG}(x, x') = D_F(x, x')$. However, if we consider the off-diagonal correlations the analogy between the two systems is no longer true. This can be easily seen if we consider the one-body reduced density matrix defined as

$$\begin{aligned}\rho_{TG}(x,y) &= N\int dx_2dx_3....dx_N\Psi_{TG}(x,x_2,x_3...,x_N)\Psi_{TG}(y,x_2,x_3...,x_N)\\ &= N\int dx_2dx_3....dx_N\,|\Phi_F(x,x_2,x_3...,x_N)|\,|\Phi_F(y,x_2,x_3...,x_N)| \qquad (5.3)\end{aligned}$$

where we have used, $\Psi_{TG} = |\Phi_F|$ to write the second form.

Since $\rho_F(x,y) = N\int dx_2dx_3....dx_N\Phi_F(x,x_2,x_3...,x_N)\Phi^*_F(y,x_2,x_3...,x_N)$, we conclude that, $\rho_{TG}(x,y) \neq \rho_F(x,y)$. As a consequence the momentum density distributions, defined as the Fourier transform of the one body-density matrix, that is

$$n(p) = \frac{1}{2\pi\hbar}\int\int dxdye^{-i\frac{p(x-y)}{\hbar}}\rho(x,y) \qquad (6.3)$$

are completely different $(n_{TG}(p) = n_F(p))$ for the two systems.

After this brief introduction on Tonks-Girardeau gas, let us now come to our main concern of this chapter. For a harmonically trapped system consisting of two one-dimensional bosons with infinitely zero-range repulsive interaction (two-hard core bosons), we derive an alternative expression for the one-body density matrix in terms of centre of mass and relative coordinates of the particles. This form is suitable and allows us to derive an analytical expression of the momentum density distribution $n(p)$. For large-values of the momentum we recover the $1/p^4$ dependence originating from short-range pair correlation.

3.2 One-body density matrix of two harmonically trapped hard-core Bosons in one dimension.

Two-body models have the merits that they can be solved exactly and give direct access to the wave function and to the one-body density matrix. We consider two Bose atoms with mass m in one dimension with infinitely repulsive two-body zero-range interaction (two hard core bosons or the so-called Tonks-Girardeau regime) and the system is subjected to external harmonic well with angular trapping frequency ω_0. According to Girardeau [26], the ground-state energy of a Bose gas in the strongly-interacting limit is the same as that of an ideal spin polarized Fermi gas. It follows from the Bose-Fermi mapping theorem[23, 24, 25], that the ground state wavefunction of the system $\psi_B(x, y)$ can be written as $\psi_B(x, y) = |\psi_F(x, y)|$ with the fermionic ground state ψ_F is a Slater determinant built on the two lowest single particle eigenfunctions φ_0 and φ_1, so that

$$\psi_F(x, y) = \frac{1}{\sqrt{2}} \left(\varphi_0(x)\varphi_1(y) - \varphi_0(y)\varphi_1(x)\right) \tag{7.3}$$

with $\varphi_0(x) = (m\omega/\pi\hbar)^{1/4} e^{-m\omega x^2/2\hbar}$ and $\varphi_1(x) = (m\omega/\pi\hbar)^{1/4}(2m\omega/\hbar)^{1/2} x e^{-m\omega x^2/2\hbar}$, we get after insertion

$$\psi_B(x, y) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \sqrt{\frac{m\omega}{\hbar}} e^{-\frac{m\omega}{2\hbar}(x^2+y^2)} |x - y| \tag{8.3}$$

The one body reduced density matrix is given by

$$\rho(x, y) = 2 \int_{-\infty}^{+\infty} \psi_B(x; x')\psi_B^*(y; x')dx' \tag{9.3}$$

and using Eq. (8.3) we obtain

$$\rho(x, y) = 2 \left(\frac{m\omega}{\pi\hbar}\right) \left(\frac{m\omega}{\hbar}\right) e^{-\frac{m\omega}{2\hbar}(x^2+y^2)} \int_{-\infty}^{+\infty} e^{-\frac{m\omega}{\hbar}x'^2} |x - x'||y - x'|dx' \tag{10.3}$$

or

$$\rho(x, y) = \frac{2}{\pi} \left(\frac{m\omega}{\hbar}\right)^2 e^{-\frac{m\omega}{2\hbar}(x^2+y^2)} \int_{-\infty}^{+\infty} e^{-\frac{m\omega}{\hbar}x'^2} |(x' - \frac{x+y}{2})^2 - \frac{(x-y)^2}{4}|dx' \tag{11.3}$$

making the change of variable $u = (x' - X\ )$

$$\rho(x, y) = \frac{2}{\pi} \left(\frac{m\omega}{\hbar}\right)^2 e^{-\frac{m\omega}{\hbar}(X^2+s^2/4)} \int_{-\infty}^{+\infty} e^{-\frac{m\omega}{\hbar}(u+X)^2} |(u^2 - \frac{s^2}{4})|du \tag{12.3}$$

with $X = (x + y)/2$ and $s = x - y$ are respectively the centre of mass and relative coordinates of the two particles. To evaluate the above integral, We remove the absolute value symbol and write the density matrix as

$$\rho(x, y) = \frac{2}{\pi} \left(\frac{m\omega}{\hbar}\right)^2 e^{-\frac{m\omega}{\hbar}(X^2+s^2/4)} B \tag{13.3}$$

with B contains three contributions

$$B = I_1 + I_2 - I_3 \tag{14.3}$$

where

$$I_1 = \int_{-\infty}^{-|s|/2} e^{-\frac{m\omega}{\hbar}(u+X)^2}(u^2 - \frac{s^2}{4})du \tag{15.3}$$

$$I_2 = \int_{|s|/2}^{+\infty} e^{-\frac{m\omega}{\hbar}(u+X)^2}(u^2 - \frac{s^2}{4})du \tag{16.3}$$

$$I_3 = \int_{-|s|/2}^{|s|/2} e^{-\frac{m\omega}{\hbar}(u+X)^2}(u^2 - \frac{s^2}{4})du \tag{17.3}$$

The three terms I_1, I_2 and I_3 and their combination B are evaluated in the appendix A. Here we quote the final form obtained for B

$$B = \frac{\hbar}{m\omega}\left[\left(X + \frac{|s|}{2}\right)e^{-\frac{m\omega}{\hbar}(X-\frac{|S|}{2})^2} - \left(X - \frac{|s|}{2}\right)e^{-\frac{m\omega}{\hbar}(X+\frac{|S|}{2})^2}\right] + \sqrt{\frac{\pi\hbar}{m\omega}}\left(\frac{\hbar}{2m\omega} + X^2 - \frac{s^2}{4}\right)\left[1 + \mathrm{erf}\left(\sqrt{\frac{m\omega}{\hbar}}\left(X - \frac{|s|}{2}\right)\right) - \mathrm{erf}\left(\sqrt{\frac{m\omega}{\hbar}}\left(X + \frac{|s|}{2}\right)\right)\right] \tag{18.3}$$

where, erf denotes the error function defined by $\mathrm{erf}(x) = 2/\sqrt{\pi}\int_0^x e^{-t^2}dt$. Substituting the above result into Eq. (13), we obtain the one-body density matrix $\rho(x, y)$ as

$$\rho(x,y) = \frac{2}{\pi}\left(\frac{m\omega}{\hbar}\right)^2 e^{-\frac{m\omega}{\hbar}(X^2+\frac{S^2}{4})}\left\{\frac{\hbar}{m\omega}\left[\left(X + \frac{|s|}{2}\right)e^{-\frac{m\omega}{\hbar}(X-\frac{|S|}{2})^2} - \left(X - \frac{|s|}{2}\right)e^{-\frac{m\omega}{\hbar}(X+\frac{|S|}{2})^2}\right] + \sqrt{\frac{\pi\hbar}{m\omega}}\left(\frac{\hbar}{2m\omega} + X^2 - \frac{s^2}{4}\right)\left[1 + \mathrm{erf}\left(\sqrt{\frac{m\omega}{\hbar}}\left(X - \frac{|s|}{2}\right)\right) - \mathrm{erf}\left(\sqrt{\frac{m\omega}{\hbar}}\left(X + \frac{|s|}{2}\right)\right)\right]\right\} \tag{19.3}$$

Denoting $a_{ho} = \sqrt{\hbar/(m\omega)}$ the harmonic oscillator length, we can rewrite Eq. (19.3) as

$$\rho(x,y) = \frac{2}{\pi a_{ho}}e^{-\left(\frac{X^2}{{a_{ho}}^2}+\frac{S^2}{4a_{ho}^2}\right)}\left\{\left[\left(\frac{X}{a_{ho}} + \frac{|s|}{2a_{ho}}\right)e^{-\left(\frac{X}{a_{ho}}-\frac{|s|}{2a_{ho}}\right)^2} - \left(\frac{X}{a_{ho}} - \frac{|s|}{2a_{ho}}\right)e^{-\left(\frac{X}{a_{ho}}+\frac{|s|}{2a_{ho}}\right)^2}\right] + \sqrt{\pi}\left(\frac{1}{2} + \frac{X^2}{{a_{ho}}^2} - \frac{s^2}{4{a_{ho}}^2}\right)\left[1 + \left(erf\left(\frac{X}{a_{ho}} - \frac{|s|}{2a_{ho}}\right) - erf\left(\frac{X}{a_{ho}} + \frac{|s|}{2a_{ho}}\right)\right)\right]\right\} \tag{20.3}$$

At this level it is instructive to display the result obtained long time ago for the one-body density matrix [26]. For that, let us first re-express our Eq. (20.3) in terms of the original variables x, y and we use the dimensionless units so that $a_{ho} = 1$, and we rewrite it as follows

$$\rho(x,y) = \frac{2e^{-\left(x^2+y^2\right)/2}}{\pi}\left\{\sqrt{\pi}\left(\frac{1}{2}+xy\right)+ \right.$$
$$\sqrt{\pi}\left(\frac{1}{2}+xy\right)\left[erf\left(\frac{(x+y)-|x-y|}{2}\right)-erf\left(\frac{(x+y)+|x-y|}{2}\right)\right]+$$
$$\left.\left[\frac{((x+y)+|x-y|)}{2}e^{-\frac{1}{4}((x+y)-|x-y|)^2}-\frac{((x+y)-|x-y|)}{2}e^{-\frac{1}{4}((x+y)+|x-y|)^2}\right]\right\} \quad (21.3)$$

and the one-body density matrix denoted by $\rho_{LGW}(x,y)$ obtained for two hard-core bosons in ref. [26] is

$$\begin{aligned}\rho_{LGW}(x,y) &= \frac{2e^{-\left(x^2+y^2\right)/2}}{\pi}\left\{\sqrt{\pi}\left(\frac{1}{2}+xy\right)+\right.\\ &\sqrt{\pi}\left(\frac{1}{2}+xy\right)sgn(y-x)\left[(erf(x)-erf(y))\right]+\\ &\left.sgn(y-x)\left[\left(ye^{-x^2}-xe^{-y^2}\right)\right]\right\}\end{aligned} \quad (22.3)$$

where sgn stands for the sign function defined as $sgn(u) = +1$ for $u > 0$ and $sgn(u) = -1$ for $u < 0$.

Note that in our expression in Eq. (21.3) the sign function does not appear contrary to the expression in (22.3). Our expression seems to have a less simpler analytical form than the one in Eq. (22.3). However, as we will show later on, our expression which is expressed in terms of center of mass and relative coordinates, is suitable when examining the short range behaviour (expansion for small values of the relative coordinate s) of the one-body density matrix. Indeed, this yields to an easy acces to the high-momentum tails of momentum density.

To exhibit the deviation from the one-body density matrix of two noninteracting Fermions, it is better to split equation (20.3) as follows

$$\rho(x,y) = \rho_F(x,y) + \rho_D(x,y) \quad (23.3)$$

with $\rho_F(x,y)$ denotes the one body density matrix of two noninteracting Fermions,

$$\rho_F(x,y) = \frac{2}{\sqrt{\pi}a_{ho}}\left(\frac{1}{2}+\frac{X^2}{{a_{ho}}^2}-\frac{s^2}{4{a_{ho}}^2}\right)e^{-X^2/{a_{ho}}^2}e^{-s^2/4{a_{ho}}^2} \quad (24.3)$$

and $\rho_D(x,y) = \rho(x,y) - \rho_F(x,y)$ is the deviation from the noninteracting Fermions given by

$$\rho_D(x,y) = \frac{2}{\pi a_{ho}} e^{-X^2/a_{ho}{}^2} e^{-s^2/4a_{ho}{}^2} \left\{ \left[\left(\frac{X}{a_{ho}} + \frac{|s|}{2a_{ho}} \right) e^{-\left(\frac{X}{a_{ho}} - \frac{|s|}{2a_{ho}}\right)^2} - \left(\frac{X}{a_{ho}} - \frac{|s|}{2a_{ho}} \right) e^{-\left(\frac{X}{a_{ho}} + \frac{|s|}{2a_{ho}}\right)^2} \right] + \sqrt{\pi} \left(\frac{1}{2} + \frac{X^2}{a_{ho}{}^2} - \frac{s^2}{4a_{ho}{}^2} \right) \left[erf\left(\frac{X}{a_{ho}} - \frac{|s|}{2a_{ho}} \right) - erf\left(\frac{X}{a_{ho}} + \frac{|s|}{2a_{ho}} \right) \right] \right\} \quad (25.3)$$

The above part of the one-body density matrix containing short-range correlations (terms of the form $|s|$) induce, as we shall see further, to the existence of a large-momentum asymptotics in the momentum density. Notice that, for $y = x$ corresponding to $X = x$ and $s = 0$, we obtain $\rho_D(x,y) = 0$ yielding to $\rho(x,x) = \rho_F(x,x)$ as required by the Bose-Fermi mapping theorem [23, 24, 25]. For latter use we rewrite Eq. (25.3) as

$$\rho_D(x,y) = \frac{2}{\pi a_{ho}} e^{-2X^2/a_{ho}{}^2} e^{-s^2/2a_{ho}{}^2} \left[\frac{2X}{a_{ho}} \sinh\left(\frac{X|s|}{a_{ho}{}^2} \right) + \frac{|s|}{a_{ho}} \cosh\left(\frac{X|s|}{a_{ho}{}^2} \right) \right] + \frac{2}{\sqrt{\pi} a_{ho}} e^{-X^2/a_{ho}{}^2} e^{-s^2/4a_{ho}{}^2} \left(\frac{1}{2} + \frac{X^2}{a_{ho}{}^2} - \frac{s^2}{4a_{ho}{}^2} \right) \left[erf\left(\frac{X}{a_{ho}} - \frac{|s|}{2a_{ho}} \right) - erf\left(\frac{X}{a_{ho}} + \frac{|s|}{2a_{ho}} \right) \right] \quad (26.3)$$

Inserting Eq. (26.3) together with Eq. (24.3) into (23) we have the full density matrix as [ref : R. Boumaza and K. Bencheikh, conf Chlef nov 2018]

$$\rho(x,y) = \left\{ \frac{2}{\sqrt{\pi} a_{ho}} \left(\frac{1}{2} + \frac{X^2}{a_{ho}{}^2} - \frac{s^2}{4a_{ho}{}^2} \right) e^{-X^2/a_{ho}{}^2} e^{-s^2/4a_{ho}{}^2} \right\} + \left\{ \frac{2}{\pi a_{ho}} \left[\frac{2X}{a_{ho}} \sinh\left(\frac{X|s|}{a_{ho}{}^2} \right) + \frac{|s|}{a_{ho}} \cosh\left(\frac{X|s|}{a_{ho}{}^2} \right) \right] e^{-2X^2/a_{ho}{}^2} e^{-s^2/2a_{ho}{}^2} + \frac{2}{\sqrt{\pi} a_{ho}} \left(\frac{1}{2} + \frac{X^2}{a_{ho}{}^2} - \frac{s^2}{4a_{ho}{}^2} \right) \left[erf\left(\frac{X}{a_{ho}} - \frac{|s|}{2a_{ho}} \right) - erf\left(\frac{X}{a_{ho}} + \frac{|s|}{2a_{ho}} \right) \right] e^{-X^2/a_{ho}{}^2} e^{-s^2/4a_{ho}{}^2} \right\} \quad (27.3)$$

To obtain the short-range expansion, we Taylor expand $\widetilde{\rho}(X,s)$ for small inter-particle separations s, and thanks to Mathematica software, we immediately find, up to $o(s^4)$

$$\rho(x,y) \equiv \widetilde{\rho}(X,s) \approx \frac{1}{\sqrt{\pi} a_{ho}} \left[\left(1 + 2\frac{X^2}{a_{ho}{}^2} \right) e^{-X^2/a_{ho}{}^2} \right] - \frac{3}{4\sqrt{\pi} a_{ho}{}^3} \left[\left(1 + \frac{2X^2}{3a_{ho}{}^2} \right) e^{-X^2/a_{ho}{}^2} \right] |s|^2 + \frac{2}{3\pi a_{ho}{}^4} \left[e^{-2X^2/a_{ho}{}^2} \right] |s|^3 + o(s^4) \quad (28.3)$$

This expression is suitable to find asymptotic form of the momentum density, the latter is exactly defined by

$$\begin{aligned} n(p) &= \frac{1}{2\pi\hbar}\int\int dx dy e^{i\frac{p(x-y)}{\hbar}}\rho(x,y) \\ &= \frac{1}{\pi\hbar}\int_{-\infty}^{+\infty} dX \int_0^{+\infty} ds\widetilde{\rho}(X,s)\cos\left(\frac{ps}{\hbar}\right) \end{aligned} \tag{29.3}$$

where to get the second form, we have used the fact that the density matrix in Eq. (27.3), is an even function with respect of the relative variable s, so that $\widetilde{\rho}(X,-s) \equiv \widetilde{\rho}(X,s)$. Substituting Eq. (28.3) into (29.3), we obtain the momentum density $n(p)$, for large values of p

$$n(p)_{p\gg 1} = K_1 + K_2 + K_3 \tag{30.3}$$

with

$$K_1 = +\frac{1}{\sqrt{\pi}a_{ho}}\frac{1}{\pi\hbar}\int_{-\infty}^{+\infty} dX\left(1+2\frac{X^2}{{a_{ho}}^2}\right)e^{-X^2/{a_{ho}}^2}\int_0^{+\infty} ds\cos\left(\frac{ps}{\hbar}\right) \tag{31.3}$$

$$K_2 = -\frac{3}{4\sqrt{\pi}{a_{ho}}^3}\frac{1}{\pi\hbar}\int_{-\infty}^{+\infty} dX\left(1+\frac{2X^2}{3{a_{ho}}^2}\right)e^{-X^2/{a_{ho}}^2}\int_0^{+\infty} ds s^2\cos\left(\frac{ps}{\hbar}\right) \tag{32.3}$$

$$K_3 = +\frac{2}{3\pi{a_{ho}}^4}\frac{1}{\pi\hbar}\int_{-\infty}^{+\infty} dX e^{-2X^2/{a_{ho}}^2}\int_0^{+\infty} ds s^3\cos\left(\frac{ps}{\hbar}\right) \tag{33.3}$$

In Eqs. (31.3)-(32.3) the X integration can be easily carried out, yielding to

$$K_1 = +\frac{2}{\pi\hbar}\int_0^{+\infty} ds\cos\left(\frac{ps}{\hbar}\right) \tag{34.3}$$

$$K_2 = -\frac{1}{\pi\hbar{a_{ho}}^2}\int_0^{+\infty} ds s^2\cos\left(\frac{ps}{\hbar}\right) \tag{35.3}$$

$$K_3 = +\frac{2}{3\sqrt{2}\pi^{3/2}\hbar{a_{ho}}^3}\int_0^{+\infty} ds s^3\cos\left(\frac{ps}{\hbar}\right) \tag{36.3}$$

Integrals of the structure $\int_0^\infty t^n\cos(t)\,dt$ are defined by the limit : $\lim_{\mu\to 0}\int_0^\infty e^{-\mu t}t^n\cos(at)\,dt$ [28] and where the latter integral is nothing but the Laplace transformation. For a sake of clarity we display below results for $n = 0, 1, 2, 3$ values

$$\int_0^{+\infty} e^{-\mu t}\cos(at)\,dt = \mu/(\mu^2+a^2) \tag{37.3}$$

$$\int_0^{+\infty} e^{-\mu t}t\cos(at)\,dt = \frac{(-\mu^2+a^2)}{(\mu^2+a^2)^2} \tag{38.3}$$

$$\int_0^{+\infty} e^{-\mu t}t^2\cos(at)\,dt = \frac{2\mu(-3\mu^2+a^2)}{(\mu^2+a^2)^3} \tag{39.3}$$

$$\int_0^{+\infty} e^{-\mu t}t^3\cos(at)\,dt = \frac{6(\mu^4-6\mu^2a^2+a^4)}{(\mu^2+a^2)^4} \tag{40.3}$$

Using the above results with $a = p/\hbar$ and taking the limit $\mu \to 0$, equations (32.3)-(24.3) reduce to

$$K_1 = 0 \tag{41.3}$$

$$K_2 = 0 \tag{42.3}$$

$$\begin{aligned} K_3 &= \frac{4\hbar^3}{\sqrt{2}\pi^{3/2}{a_{ho}}^3 p^4} \\ &= \left(\frac{2}{\pi}\right)^{3/2} \frac{(m\omega\hbar)^{3/2}}{p^4} \end{aligned} \tag{43.3}$$

where we have used $a_{ho} = \sqrt{\hbar/(m\omega)}$, to obtain the last form of K_3. With these results, the asymptotic form of the momentum density in Eq. (30) reads [ref : R. Boumaza and K. Bencheikh, conf Chlef nov 2018]

$$n(p)_{p\gg 1} = \left(\frac{2}{\pi}\right)^{3/2} \frac{(m\omega\hbar)^{3/2}}{p^4} \tag{44.3}$$

As can be seen within our study we recover at equilibrium the well known $1/p^4$ dependence of $n(p)$ for large values of the momentum p with a contact parameter, $(2/\pi)^{3/2}(m\omega\hbar)^{3/2}$. It should be noted that our contact parameter differs from the one reported long time ago in the literature [29], which was found to be, $2(2/\pi)^{1/2}(m\omega\hbar)^{-3/2}$. From a basic dimensional analysis, equation (38.3) yields correct units for the momentum density. We believe that our result is correct.

Chapitre 4

Dynamical one-body reduced density matrix of an interacting zero-range repulsive Bose gas following a sudden change of trap potential and the resulting current density distribution.

4.1 Introduction

In this chapter we shall present and discuss the main results obtained in this thesis, and these results have been the subject of a publication in Journal of physics A [30]. We focus on the non-equilibrium dynamics of quantum gas released from an initially trapping potential and we discuss in the following two different physical situations.

1) The first situation concerns on an extension of the work published in 2015 by Campbell et al in Physical Review Letters **114** 125302 where they considered for the long-time asymptotic only the local density. Using the so-called operator product expansion to lowest order, we derive a simple analytical expression for the long-time asymptotic one-body reduced density matrix during free expansion for a one-dimensional system of bosons with large atom number

interacting through a repulsive zero-range repulsive interactions initially confined by a potential well. This density matrix allows direct access to the momentum distribution and we suggest to use the collective mass current density as an interesting physical quantity during the expansion. For initially confining power-law potentials we give explicit expressions, in the limits of very weak and very strong interaction, for the current density distributions during the free expansion.

2) The second work deals with a study on an expansion of ultracold gas from a confining harmonic trap to another harmonic trap with a different frequency (called a quench from trap to trap). For the particular case of a quantum impenetrable gas of bosons (a Tonks–Girardeau gas) with a given atom number, we present an exact analytical expression for the mass current distribution (mass transport) after release from one harmonic trap to another harmonic trap. It is shown that, for a harmonically quenched Tonks–Girardeau gas, the current distribution is a suitable collective observable and under the weak quench regime, it exhibits oscillations at the same frequencies as those recently predicted for the peak momentum distribution in the breathing mode [15, 37]. The analysis is extended to other possible quenched systems.

4.2 Asymptotic long-time one-body reduced density matrix of an interacting zero-range repulsive Bose gas system with large N.

First we consider a large N system of zero range interacting bosons in one dimension (called Lieb-Liniger model and discussed in Chapter 3.), for which we use the time-dependent evolution in free space of the many-body wave function $\Psi(x_1, x_2, ..., x_N; t)$ after the release of the trap to obtain analytical expression for the long time asymptotic one-body density matrix after a sudden release of arbitrary confining potential $V(x)$. The time dependent evolution in free space after the release of the trap potential $V(x)$ of the many-body wave function $\Psi(x_1, x_2, ..., x_N; t)$ has been calculated in the work [31] and after a long time, the wave function denoted by $\Psi_\infty(x_1, x_2, ..., x_N; t)$, was shown to be

$$\Psi_\infty(x_1, x_2, ..., x_N; t) = \left(\frac{m}{\hbar t}\right)^{N/2} e^{-i\pi\frac{N}{4}} b\left(\frac{mx_1}{\hbar t}, ..., \frac{mx_N}{\hbar t}\right) \times e^{i\theta\left(\frac{mx_1}{\hbar t},...,\frac{mx_N}{\hbar t}\right)+i\frac{m}{2\hbar t}\left(x_1^2+...x_N^2\right)} \tag{1.4}$$

In above, $b\,(k_1, k_2..., k_N)$ and $\theta\,(k_1, k_2..., k_N)$ are functions of N different quasimomenta (also referred to as rapidities) evaluated at stationary phase points $k_i = mx_i/(\hbar t)$ [31]. Let us consider the corresponding asymptotic one-body reduced density matrix

$$\rho_\infty(x, y, t) = N \int dx_2....dx_N \Psi_\infty(x, x_2, x_{3,...}x_N; t)\Psi^*_\infty(y, x_2, x_{3,...}x_N; t) \tag{2.4}$$

Upon inserting Eq. (1.4) into (2.4), we obtain

$$\rho_\infty(x, y; t) = N e^{i\frac{m}{2\hbar t}\left(x^2-y^2\right)} \left(\frac{m}{\hbar t}\right)^N \times \int dx_2....dx_N b\left(\frac{mx}{\hbar t}, ..., \frac{mx_N}{\hbar t}\right) b^*\left(\frac{my}{\hbar t}, ..., \frac{mx_N}{\hbar t}\right) e^{i\left[\theta\left(\frac{mx}{\hbar t},...,\frac{mx_N}{\hbar t}\right)-\theta\left(\frac{my}{\hbar t},...,\frac{mx_N}{\hbar t}\right)\right]} \tag{3.4}$$

or

$$\rho_\infty(x, y; t) = N e^{i\frac{m}{2\hbar t}\left(x^2-y^2\right)} \left(\frac{m}{\hbar t}\right) \times \int du_2....du_N b\left(\frac{mx}{\hbar t}, u_2..., u_N\right) b^*\left(\frac{my}{\hbar t}, u_2..., u_N\right) e^{i\left[\theta\left(\frac{mx}{\hbar t},u_2...,u_N\right)9-\theta\left(\frac{my}{\hbar t},u_2...,u_N\right)\right]} \tag{4.4}$$

Using the centre of mass $X = (x+y)/2$ and the relative $s = x-y$ coordinates, Eq.(4.4) becomes

$$\begin{aligned} \rho_\infty(x, y; t) &\equiv \rho_\infty(X + s/2, X - s/2; t) \\ &= N e^{i\frac{mX}{\hbar t}s} \left(\frac{m}{\hbar t}\right) \times \\ &\int du_2....du_N b\left(\frac{m}{\hbar t}(X + \frac{s}{2}), u_2..., u_N\right) b^*\left(\frac{m}{\hbar t}(X - \frac{s}{2}), u_2..., u_N\right) \times \\ &e^{i\left[\theta\left(\frac{m}{\hbar t}(X+\frac{s}{2}),u_2...,u_N\right)-\theta\left(\frac{m}{\hbar t}(X-\frac{s}{2}),u_2...,u_N\right)\right]} \end{aligned} \tag{5.4}$$

where we have used $\left(x^2 - y^2\right)/2 = X.s$. Since we consider a system of large particle number, we shall obtain an expression for $\rho_\infty(x, y; t)$ which is consistent with the Thomas-Fermi approximation. For fixed values of $u_2, ..., u_N$, we use a Taylor series of the above functions b, b^* and θ

around the non-locality s, to first order we can write

$$b\left(\frac{m}{\hbar t}(X+\frac{s}{2}),u_2...,u_N\right) \approx b\left(\frac{m}{\hbar t}X,u_2...,u_N\right)+\frac{s}{2}\left(\frac{\partial b}{\partial X}\right)_{s=0} \tag{6.4}$$

$$b^*\left(\frac{m}{\hbar t}(X-\frac{s}{2}),u_2...,u_N\right) \approx b^*\left(\frac{m}{\hbar t}X,u_2...,u_N\right)-\frac{s}{2}\left(\frac{\partial b^*}{\partial X}\right)_{s=0} \tag{7.4}$$

$$\theta\left(\frac{m}{\hbar t}(X+\frac{s}{2}),u_2...,u_N\right) \approx \theta\left(\frac{m}{\hbar t}X,u_2...,u_N\right)+\frac{s}{2}\left(\frac{\partial \theta}{\partial X}\right)_{s=0} \tag{8.4}$$

$$\theta\left(\frac{m}{\hbar t}(X-\frac{s}{2}),u_2...,u_N\right) \approx \theta\left(\frac{m}{\hbar t}X,u_2...,u_N\right)-\frac{s}{2}\left(\frac{\partial \theta}{\partial X}\right)_{s=0} \tag{9.4}$$

To be consistent with the Thomas-Fermi approximation, we keep in the above expansions only the zero order term since higher contributions will involve gradients of the density. We shall now substitute in Eq. (5.4) the product $b\left(\frac{m}{\hbar t}(X+\frac{s}{2}),u_2...,u_N\right)b^*\left(\frac{m}{\hbar t}(X-\frac{s}{2}),u_2...,u_N\right)$ by its zero order expansion $b\left(\frac{m}{\hbar t}X,u_2...,u_N\right)b^*\left(\frac{m}{\hbar t}X,u_2...,u_N\right)$. In the literature this is known as the operator product expansion (OPE) of local operators [32]. Then Eq. (5) reduces to

$$\begin{aligned}\rho_\infty(x,y;t) &\equiv \rho_\infty(X+s/2,X-s/2;t)\\ &\approx Ne^{i\frac{mX}{\hbar t}s}\left(\frac{m}{\hbar t}\right)\int du_2....du_N b^*\left(\frac{m}{\hbar t}X,u_2...,u_N\right)b\left(\frac{m}{\hbar t}X,u_2...,u_N\right)\times\\ &\quad e^{i\left[\theta\left(\frac{m}{\hbar t}(X+\frac{s}{2}),u_2...,u_N\right)-\theta\left(\frac{m}{\hbar t}(X-\frac{s}{2}),u_2...,u_N\right)\right]}\end{aligned} \tag{10.4}$$

where we have also used the OPE for the product; $e^{i\theta\left(\frac{mx}{\hbar t},u_2...,u_N\right)}e^{-i\theta\left(\frac{my}{\hbar t},u_2...,u_N\right)}$. Using the definition of the quasi-momenta distribution, $g(k)=\int dk_2....dk_N\left|b\left(k,k_2...,k_N\right)\right|^2$[31], Eq. (10.4) becomes after returning to the original variables x,y, we find

$$\rho_\infty(x,y;t)=e^{i\frac{m}{2\hbar t}\left(x^2-y^2\right)}\frac{m}{\hbar t}g\left(\frac{m}{2\hbar t}(x+y)\right) \tag{11.4}$$

For $y=x$, Eq. (11.4) reduces to $\rho_\infty(x;t)=\frac{m}{\hbar t}g\left(\frac{m}{\hbar t}x\right)$ which is the result obtained previously or the local density in [31]. Notice that at this level the hermiticity property of Eq. (11) is still ensured ,i.e; $\rho^*_\infty(x,y;t)=\rho_\infty(y,x;t)$.

It is well known that the momentum density $n(k)$ is the Fourier transform of one-body density matrix $\rho(x,x')$. We immediately get the asymptotic long-time momentum density $n_\infty(k,t)$ as

$$n_\infty(k,t)=\frac{1}{2\pi}\int\int dXdse^{-iks}\rho_\infty(X+s/2,X-s/2;t) \tag{12.4}$$

To check the correctness of our result in Eq. (11.4), we substitute it into Eq. (12.4) to find

$$n_\infty(k,t) = \frac{1}{2\pi}\frac{m}{\hbar t}\int dX g\left(\frac{m}{\hbar t}X\right)\int ds e^{i\left(\frac{mX}{\hbar t}-k\right)s} \tag{13.4}$$

Upon using, $\int ds e^{i\left(\frac{mX}{\hbar t}-k\right)s} = 2\pi\frac{\hbar t}{m}\delta\left(X-\frac{\hbar t}{m}k\right)$, we obtain

$$n_\infty(k,t) = g(k) \tag{14.4}$$

Hence we recover the result of work [31], that the initial quasimomentum distribution $g(k)$ of the trapped gas is mapped to real momenta distribution of the expanded cloud.

It should be noted that the role of the phase factor $e^{i\frac{m}{2\hbar t}(x^2-y^2)}$ in Eq. (11.4) is to impart a collective motion with a nonvanishing velocity field, to the system initially confined after the release of the trap. This collective motion or mass transport is an important observable and is well described by the current density distribution defined as

$$j(x,t) = \frac{\hbar}{2mi}\left[\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)\rho(x,y;t)\right]_{y=x} \tag{15.4}$$

While the use of the current density distribution as a tool to describe physical situation where suddenly the trap is removed, goes back to the earlier work of Moshinsky [33] on the shutter problem where it is shown in that work that the transient current has a close mathematical resemblance with the intensity of light in the Fresnel diffraction by a straight edge [see also [34]], the current growing interest in the out-of-equilibrium protocol realised by a quantum quench, motivated us to examine the structure of current distribution $j(x,t)$ as a dynamical variable describing the post-quench mass transport.

Substituting Eq. (11.4) into (15.4) gives the asymptotic long-time current distribution in the form

$$j_\infty(x,t) = \frac{x}{t}\left[\frac{m}{\hbar t}g(\frac{m}{\hbar t}x)\right] \tag{16.4}$$

According to Eq. (11.4) we may write, $j_\infty(x,t) = \frac{x}{t}\ \rho_\infty(x,t)$, showing that the associated velocity field $\upsilon(x,t) = j(x,t)/\rho(x,t)$ exhibits a linear spatial dependence, given by $\upsilon(x,t) = x/t$.

In the following we shall present the results for the density matrix, current density distribution in the case of class of initially trapping potentials of the form $V(x) = \alpha_\nu\left|x\right|^\nu/2$, where

α_ν is the confinement strength These power law potentials are suitable for studying the effects of adiabatically changing the shape of the trap.

We start with the weakly-interacting regime, where the local density is known to be $\rho(x) = (\mu_0 - V(x))/g$ where g is the strength of the interaction, the quasimomentum distribution was obtained as $g(k) = \frac{RI_\nu}{\pi}\left(\frac{4\hbar^2\rho_0}{mg}\right)^{1/2}\left(1-\frac{\hbar^2k^2}{4mg\rho_0}\right)^{1/\nu+1/2}$[31].

Here $I_\nu = \sqrt{\pi}\Gamma(1/\nu)/\left[(2+\nu)\Gamma\left(1/2+1/\nu\right)\right]$, $\rho_0 = \rho(x=0)$ is the density in the centre of the trap and μ_0 is the chemical potential of the initially trapped gas. Moreover $R = (2\mu_0/\alpha_\nu)^{1/\nu}$ is the Thomas-Fermi radius obtained through $V(R) = \mu_0$. On substituting respectively into Eq. (11.4) and (16.4) we find

$$\rho_\infty(x,y;t) = e^{i\frac{m}{2\hbar t}(x^2-y^2)}\frac{4I_\nu}{\pi}\frac{\rho_0}{\lambda(t)}\left(1-\frac{(x+y)^2}{4\lambda(t)^2R^2}\right)^{\frac{1}{\nu}+\frac{1}{2}} \quad (17.4)$$

$$j_\infty(x,t) = \frac{x}{t}\frac{4I_\nu}{\pi}\frac{\rho_0}{\lambda(t)}\left(1-\frac{x^2}{\lambda(t)^2R^2}\right)^{\frac{1}{\nu}+\frac{1}{2}} \quad (18.4)$$

with $\lambda(t) = 2\sqrt{\frac{\mu_0}{m}}\frac{t}{R}$ is a dimensionless parameter.

For strong interacting regime, the spatial density is $\rho(x) = \frac{1}{\pi}\sqrt{2m\left(\mu_0 - V(x)\right)/\hbar^2}$, the initial quasimomentum distribution $g(k)$ was shown to be $g(k) = \frac{R}{\pi}\left(1-\frac{k^2}{\pi^2\rho_0^2}\right)^{1/\upsilon}$ [31].

Here $\rho_0 = \rho(x=0)$ and $R = (2\mu_0/\alpha_\nu)^{1/\nu}$. Using the dimensionless parameter $\lambda(t) = \sqrt{\frac{2\mu_0}{m}}t/R$, we get after combining with Eqs. (11.4) and (16.4)

$$\rho_\infty(x,y;t) = e^{i\frac{m}{2\hbar t}(x^2-y^2)}\frac{\rho_0}{\lambda(t)}\left(1-\frac{(x+y)^2}{4\lambda(t)^2R^2}\right)^{\frac{1}{\nu}} \quad (19.4)$$

$$j_\infty(x,t) = \frac{x}{t}\frac{\rho_0}{\lambda(t)}\left(1-\frac{x^2}{\lambda(t)^2R^2}\right)^{\frac{1}{\nu}} \quad (20.4)$$

In the following we shall show that Eqs. (20.4) obtained for strong coupling case can be recovered for the specific harmonic oscillator potential $V(x) = m\omega_0^2x^2/2$, through an exact treatment. This corresponds to $\nu = 2$ and setting $\alpha_2 = m\omega_0^2$ for the confinement strength of the potential. The Thomas-Fermi radius R is then given by $R = (2\mu_0/m\omega_0^2)^{1/2}$ and the paramertr $\lambda(t) = \omega_0 t$, so Eq. (20.4) reduces

$$j_\infty^{osc}(x,t) = \frac{x}{t}\frac{\rho_0}{\omega_0 t}\left(1-\frac{x^2}{\omega_0^2t^2R^2}\right)^{\frac{1}{2}} \quad (21.4)$$

Using the Bose-Fermi mapping theorem [35] and see also [in chapter 3], which links a gas of bosons in strong interacting regime (Tonks-Girardeau gas) to a system of noninteracting spinless

Fermions, the authors of [36] have derived an exact analytic expression for the time dependent one-body density matrix $\rho(x, y; t)$ in an external harmonic oscillator trap $V(x) = m\omega^2(t)x^2/2$ with time dependent frequency $\omega(t)$. It is given by the scaling and gauge transformation

$$\rho(x, y; t) = e^{i\frac{\dot{b}}{b}\frac{m}{2\hbar}(x^2-y^2)}\frac{1}{b}\rho(\frac{x}{b}, \frac{y}{b}; 0) \tag{22.4}$$

where the scaling factor $b(t)$ is a solution of the differential equation, $\ddot{b} + \omega^2 b = \omega_0^2/b^3$ with initial conditions $b(0) = 1$, $\dot{b}(0) = 0$ and $\omega_0 = \omega(t = 0)$.

Now if we assume that the one-body reduced density matrix at $\rho(x, y; 0)$ at $t = 0$ is a real function, it follows from the hermiticity property that this density matrix is symmetric, i.e $\rho(\frac{x}{b}, \frac{y}{b}; 0) = \rho(\frac{y}{b}, \frac{x}{b}; 0)$. Upon using Eq. (15.4) we deduce the quantum mechanical current density as

$$j(x; t) = x\frac{\dot{b}}{b^2}\rho(\frac{x}{b}; 0) \tag{23.4}$$

Here $\rho(\frac{x}{b}; 0)$ is the scaled density profile at $t = 0$. Notice that the above current obeys the equation of continuity, $\partial j(x; t)/\partial x + \partial\rho(x; t)/\partial t = 0$ with $\rho(x; t) = \rho(\frac{x}{b}; 0)/b$, as it should.

Let us now consider the case of a free expansion after turning off the confining harmonic trap. This corresponds to the situation where $\omega(t) = 0$ for $t > 0$, yielding to a scaling factor $b(t) = \sqrt{1 + \omega_0^2 t^2}$[36]. It is easy to check that in the long-time ($\omega_0 t >> 1$), we have $b(t) \approx \omega_0 t$, the current distribution in Eq. (23.4) reduces to the one obtained in (21.4) providing one considers a sufficiently high number of particles where the density profile is given by the well-known Thomas-fermi expression, $\rho(x; 0) = \rho_0\sqrt{1 - x^2/R^2}$.

The next section is devoted to study the breathing-mode oscillations of a harmonically trapped TonksGirardeau gas [37, 15], so we will apply the exact relation in Eq.(23.4) to the case of a rapid change of the external trapping potential (quench from trap to another trap).

4.3 Exact analytical expression of current density for Tonks-Girardeau gas in trap to trap quench.

We will focus on the persistent collective mass current distribution generated by the quantum quench. The fast change of the trap frequency induce a collective motion described by current density and we shall derive analytical expression of this current density for a TG gas with a fixed number of particles N.

We consider the situation where the gas is initially harmonically trapped in a well with frequency ω_0 and at $t > 0$ the frequency suddenly is changed and becomes ω such that $\omega < \omega_0$. As shown in [36], Eq. (22.4) can be used with

$$b(t) = \sqrt{1 + \epsilon \sin^2 \omega t} \tag{24.4}$$

where $\epsilon = (\omega_0^2/\omega^2 - 1)$ is the quench strength. The current density distribution is still given by Eq. (23.4) which combined with (24.4) yields

$$j(x;t) = \frac{\epsilon\omega \sin 2\omega t}{2(1 + \epsilon \sin^2 \omega t)^{3/2}} x\rho(\frac{x}{b};0) \tag{25.4}$$

Here $\rho\,(x/b(t);0)$ is the local density of the TG gas before the release of the trap calculated at rescaled position $x/\sqrt{1 + \epsilon \sin^2 \omega t}$. This local density is identical to the density profile of an other system, namely the fermionic system [35]. For this latter system of noninteracting Fermions an exact analytical expression of the density profile exists for some time and has been obtained by Shea and Van Zyl [38]. Considering a system of N noninteracting Fermions filling $(M + 1)$ oscillator shells, in a $d - dimensional$ isotropic harmonic oscillator with ω_0 is the frequency and the Fermi energy $\mu = (M + d/2)\hbar\omega$, these authors obtained an exact expression for the one-body density matrix, which in one spatial dimension ($d = 1$) and for spin polarized gas becomes

$$\rho_F(x,y) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega_0}{2\hbar}(x^2+y^2)} \sum_{n=0}^{M} (-1)^n L_n^{-\frac{1}{2}} \left(\frac{m\omega_0}{2\hbar}(x+y)^2\right) L_{M-n}^{\frac{1}{2}} \left(\frac{m\omega_0}{2\hbar}(x-y)^2\right) \tag{26.4}$$

Here $L_n^{\alpha}(x)$ is the associated Laguerre polynomial [?]. The local density $\rho_F = [\rho_F(x,y)]_{y=x}$ is then given by

$$\rho_F(x) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega_0}{\hbar}x^2} \sum_{n=0}^{M} (-1)^n L_{M-n}^{\frac{1}{2}}(0)\, L_n^{-\frac{1}{2}} \left(\frac{2m\omega_0}{\hbar}x^2\right) \tag{27.4}$$

As stated before and due to the Bose-Fermi mapping we can write, $\rho\,(x/b(t);0) = \rho_F(x/b(t))$. Eq. (27.4) may be substituted into Eq. (25.4), we then find the exact result

$$j(x;t) = \frac{\epsilon\omega \sin 2\omega t}{2(1 + \epsilon \sin^2 \omega t)^{3/2}} x \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2} e^{-\left(\frac{m\omega_0}{\hbar}\right)\frac{x^2}{(1+\epsilon \sin^2 \omega t)}} \sum_{n=0}^{M} (-1)^n L_{M-n}^{\frac{1}{2}}(0)\, L_n^{-\frac{1}{2}} \left(\frac{2m\omega_0}{\hbar}\frac{x^2}{(1 + \epsilon \sin^2 \omega t)}\right) \tag{28.4}$$

which shows that the system responds through a periodic function of time for the current density distribution having a period $T = \pi/\omega$. Mathematically speaking the Fourier series of this periodic odd function of time contains only sine terms (harmonics) having frequencies, $2\omega, 4\omega, 6\omega..., 2n\omega, ...$ Notice that this current density is the same as for the ideal Fermi gas (discussed in chapter 2.).

It is interesting to examine the weak quench regime corresponding to $\epsilon \ll 1$, since this regime has been the subject of a recent study [15]. Rather than using Fourier expansion, here we simply use the approximation, $(1+\epsilon\sin^2\omega t)^{-3/2} \approx 1-(3/4)\epsilon(1-\cos(2\omega t))$, in Eq. (28.4) yielding to

$$\begin{aligned} j(x;t) &\approx \frac{\epsilon}{2}\omega\sin 2\omega t\,[1-(3/4)\epsilon(1-\cos(2\omega t))]\times \\ & x\left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2} e^{-\left(\frac{m\omega_0}{\hbar}\right)\frac{x^2}{(1+\epsilon\sin^2\omega t)}} \sum_{n=0}^{M}(-1)^n L_{M-n}^{\frac{1}{2}}(0)\, L_n^{-\frac{1}{2}}\left(\frac{2m\omega_0}{\hbar}\frac{x^2}{(1+\epsilon\sin^2\omega t)}\right) \end{aligned} \tag{29.4}$$

After arrangement, we get up to order ϵ^2

$$\begin{aligned} j(x;t) &\approx \frac{\omega}{2}\left[\epsilon\left(1-\frac{3}{4}\epsilon\right)\sin 2\omega t+\frac{3}{8}\epsilon^2\sin(4\omega t))\right]\times \\ & x\left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2} e^{-\left(\frac{m\omega_0}{\hbar}\right)x^2} \sum_{n=0}^{M}(-1)^n L_{M-n}^{\frac{1}{2}}(0)\, L_n^{-\frac{1}{2}}\left(\frac{2m\omega_0}{\hbar}x^2\right) \end{aligned} \tag{30.4}$$

which shows that the current density is a superposition of two harmonics at frequencies 2ω and 4ω. This collective breathing mode oscillations in a harmonic trap has been recently predicted [15] in the time evolution of the peak momentum distribution $n(k=0;t)$ in the weak quench limit of the harmonic trap. We believe that the current distribution offers good insight and complement the study of these oscillations in the breathing mode.

4.4 Generalization to higher spatial dimensions

In the following we shall show that the behavior discussed in the previous section concerning the collective current density can be generalized to other many-body interacting systems. First , consider a system of interacting Fermions at unitarity [see : [39] and references therein] with a three-dimensional isotropic harmonic trap with time dependent frequency $\omega(t)$. This problem has been exactly solved [39] and a relationship between the time dependent many-body wave function $\Psi(\overrightarrow{r}_1, \overrightarrow{r}_2, ..., \overrightarrow{r}_N; t)$ and the wave function $\Psi_0(\overrightarrow{r}_1, \overrightarrow{r}_2, ..., \overrightarrow{r}_N)$ at $t = 0$ through a scaling and gauge transformation law was then derived

$$\Psi(\overrightarrow{r}_1, \overrightarrow{r}_2, ..., \overrightarrow{r}_N; t) = e^{i\frac{m}{2\hbar}\frac{\dot{b}(t)}{b(t)}\sum_{j=1}^{N}\overrightarrow{r}_j^2} \frac{1}{[b(t)]^{3N/2}} \Psi_0\left(\frac{\overrightarrow{r}_1}{b(t)}, \frac{\overrightarrow{r}_2}{b(t)}, ..., \frac{\overrightarrow{r}_N}{b(t)}\right) \tag{31.4}$$

Here the scaling factor $b(t)$ is a solution of the differential equation, $\ddot{b}+\omega^2 b = \omega_0^2/b^3$ and we assume the initial conditions $b(0) = 1$, $\dot{b}(0) = 0$ and $\omega_0 = \omega(t = 0)$. Using this transformation, we immediately obtain the time dependence of the one-body reduced density matrix $\rho(\overrightarrow{r}_1, \overrightarrow{r}_2; t)$ in the form

$$\rho(\overrightarrow{r}_1, \overrightarrow{r}_2; t) = e^{i\frac{m}{2\hbar}\frac{\dot{b}(t)}{b(t)}(\overrightarrow{r}_1^2 - \overrightarrow{r}_2^2)} \frac{1}{[b(t)]^3} \rho\left(\frac{\overrightarrow{r}_1}{b(t)}, \frac{\overrightarrow{r}_2}{b(t)}; 0\right) \tag{32.4}$$

yielding to exact scaling solution similar to Eq. (22.4). in a similar way as was done previously for a Tonks Girardeau Bose gas, one can calculate the collective current density defined by, $\overrightarrow{j}(\overrightarrow{r}; t) = \frac{\hbar}{2mi}\left[\left(\overrightarrow{\nabla}_{\overrightarrow{r}_1} - \overrightarrow{\nabla}_{\overrightarrow{r}_2}\right)\rho(\overrightarrow{r}_1, \overrightarrow{r}_2; t)\right]_{\overrightarrow{r}_2 = \overrightarrow{r}_1 = \overrightarrow{r}}$.

Starting from a physical situation where the gas is initially harmonically confined in isotropic harmonic trap with frequency ω_0 and at $t > 0$, the frequency suddenly is changed and becomes ω such that $\omega < \omega_0$. We assume that the static solution $\rho_0(\overrightarrow{r}_1, \overrightarrow{r}_2)$ does not support mass current at $t \le 0$ and using Eq. (32.4) the current density at $t > 0$ is then given by

$$\overrightarrow{j}(\overrightarrow{r}; t) = \overrightarrow{r}\frac{\dot{b}}{b^4}\rho(\frac{\overrightarrow{r}}{b}; 0) \tag{33.4}$$

Here $b(t) = \sqrt{1 + \epsilon \sin^2 \omega t}$ and if we since the initial density profile has spherical symmetry, we obtain periodic current density distribution along the radial direction $\overrightarrow{e}_r = \overrightarrow{r}/r$

$$\begin{aligned}\overrightarrow{j}(\overrightarrow{r}; t) &= \frac{\dot{b}}{b^4} r\rho(\frac{r}{b}; 0)\overrightarrow{e}_r \\ &= \frac{\epsilon\omega \sin 2\omega t}{2(1 + \epsilon \sin^2 \omega t)^{5/2}} r\rho\left(\frac{r}{\sqrt{1 + \epsilon \sin^2 \omega t}}; 0\right)\overrightarrow{e}_r \end{aligned} \tag{34.4}$$

For the case of weak quench, we have $\epsilon \ll 1$, the time dependent parameter $b(t)$ reduces to, $b(t) \approx (1 - 5\epsilon/4) - (5\epsilon/4)\cos(2\omega t)$, yielding after substitution into Eq. (34.4), to order ϵ^2

$$\overrightarrow{j}(\overrightarrow{r}; t) \approx \frac{\omega}{2}\left[\epsilon\left(1 - \frac{5}{4}\epsilon\right)\sin 2\omega t + \frac{5}{8}\epsilon^2 \sin(4\omega t)\right] r\rho(r; 0)\,\overrightarrow{e}_r \tag{35.4}$$

which clearly shows that in this weak quench regime the mass current density is a sum of two harmonics with frequencies 2ω and 4ω respectively.

It is worth mentioning that the above analysis and the use of current density as a tool may be extended to the class of time-dependent many-body interacting systems for which the corresponding Schrödinger equation is solved through a scaling transformation. In this context the authors of ref. [40] showed that, a time dependent many-body wave function Ψ, solution of time dependent Schrödinger equation with time dependent parameters, such as mass, interaction constant, external potential, can be mapped, under certain conditions satisfied by these parameters, to an equilibrium many-body wave function Φ with time independent parameters. As a consequence of this mapping similar relation to Eq. (32.4) was obtained between the one-body reduced density matrices of the Ψ system and the time independent Φ system. This relation allows a direct calculation of the mass current density.

Chapitre 5

Conclusion

In the last two decades, the study of ultra-cold quantum gases has been a central topic in the experimental and theoretical physics. The extraordinary degree of control for such systems of external parameters governing their Hamiltonians allows to investigate the fundamental behavior of quantum matter in various regimes and geometries. This thesis has been devoted to the theoretical study of one-dimensional quantum interacting systems subjected to a sudden change of their trapping potentials, called quench of the potential, and study the resulting behaviour.These systems can be realized using ultracold atoms.

In chapter I, we have presented general considerations on the physics of ultra-cold quantum gases. We have briefly discussed some experimental aspects related to this field such as cooling and trapping gases. We have also discussed on the control of the external parameters acting on the confined gas which has led to the interesting issue of the study of non-equilibrium dynamics of many-body system.

In chapter II, we have examined a simple quench problem which we have solved through the use of time evolution propagator. For a system of N noninteracting Fermions in one-spatial dimension initially confined in harmonic trap potential, we have calculated the time dependent one-body density matrix, this quantity is of extreme theoretical importance, after a sudden release of the confining potential (free expansion). After that, we calculated the one-body density matrix a sudden change of the frequency of the confining potential from an initial value to a different value (trap to trap quench). We derived a scaling law for the time dependent density matrix, extending the result known in the litterature for the local density.

The knowledge of density matrix allows to calculate the current density distribution. We have proposed to use the current density distribution as an interesting tool to describe the collective motion of the particles following the quench.

In chapter III, we tackled a somewhat different problem. Considering the system of two one-dimensional zero-range strongly repulsive bosons confined in harmonic trap, we present an alternative expression of the corresponding reduced one-body density matrix in terms of centre of mass and relative coordinates. Using a short range expression of our expression, we calculated the momentum density of the system. We have recovered the well-known behaviour of momentum density at large values of momentum (high momentum tail) and we compared our Tan's coefficient with the one obtained in the litterature.

Finally we discussed in the last chapter of this thesis (chapter 4) the dynamics of system of one-dimensional zero- range repulsive interacting bosons. We notice that, while the methods used in the previous chapters two and three are new, the results discussed in chapter 4 constitute the main part of this thesis and these results have been the subject of a publication in Journal of physics A [**31**]. We focus on the non-equilibrium dynamics of quantum gas released from an initially trapping potential and we discuss in the following two different physical situations. In the first part of this chapter, we used the operator product expansion to lowest order to derive a simple analytical expression for the long-time asymptotic one-body reduced density matrix during free expansion for a one-dimensional system of bosons with large atom number interacting through a repulsive zero-range repulsive interactions initially confined by a potential well. The second part of this chapter 4 deals with a study on an expansion of ultracold gas following a quench from trap to trap of a system of impenetrable gas of bosons (a Tonks–Girardeau gas) with a given atom number. We pointed out the use of current mass distribution as an important tool describing the collective motion following the quench. We have shown that this current exhibites the same oscillations as those of momentum density. The study has been extended to the so-called interacting three-dimensional Fermi gas at unitarity.

The previous results were derived at zero temperature and a natural perspective would to extent non-zero temperatures. Another interesting issue would be to study the quench of spin-orbit coupled systems.

5.1 References

Bibliographie

[1] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).

[2] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008).

[3] A. L. Fetter, Rev. Mod. Phys. **81**, 647 (2009).

[4] M. Lewenstein et al., Adv. Phys. **56**, 243–379 (2007).

[5] M. Rigol, V. Dunjko, V. Yurovsky and M. Olshanii, Phys. Rev. Lett. **98** 050405 (2007).

[6] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Science 269, 198 (1995).

[7] K. B. Davis, M. O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Phys. Rev. Lett. **75**, 3969 (1995).

[8] Jean Dalibard , Cours Atomes ultra-froids, année (2006). see website

[9] R. Grimm, M. Weidemüller and Y. B. Ovchinnikov, Adv. At. Mol. Opt. Phys., **42** 95–170 (2000).

[10] Benoît Darquié, Manipulation d'atomes dans des pièges dipolaires microscopiques et émission contrôlée de photons par un atome unique. Thèse de doctorat, Université Paris XI (2005).

[11] Nicolas. Schlosser, Étude et réalisation de micro-pièges dipolaires optiques pour atomes neutres, Thèse de doctorat, Université Paris XI (2001).

[12] C. G. Townsend, et al., Phys. Rev. A **52** (2), 1423 (1995).

[13] C. Chin et al, Rev. Mod. Phys. **82**, 1225 (2010) .

[14] David Guéry-Odelin and Claude Cohen-Tannoudji, Advances in atomic physics, World. Scientific, Singapore, 2011.

[15] Y. Y.Atas, I. Bouchoule, D. M. Gangardt and K. V. Kheruntsyan, Phys. Rev. A **96** 041605 (2017).

[16] R.P. Feynman and A.R. Hibbs, Quantum Mechanics and Path Integrals (McGraw-Hill, New York, 1964).

[17] M. Brack, R.K. Bhaduri, Semiclassical Physics, Frontiers in Physics, vol. 96, Westview, Boulder, 2003.

[18] D. S. Dean et al, arXiv :1902.02594v2 [cond-mat.stat-mech] 18 Mar 2019

[19] E. H. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963).

[20] E. H. Lieb and W. Liniger, Phys. Rev. **130** 1616 (1963).

[21] T. Kinoshita, T. Wenger and D. S. Weiss, Science **305** 1125 (2004).

[22] M. Hillery9, R.F. O'Connel, M.O. Scully, E.P. Wigner, Phys. Rep. **106**, 121 (1984).

[23] M. D. Girardeau, J. Math. Phys. **1** (1960) 516.

[24] M. D. Girardeau, Phys. Rev. **139**, B 500 (1965).

[25] M. D. Girardeau and E.M. Wright, Phys. Rev. Lett. **84**, 5691 (2000).

[26] M. D. Girardeau, E. M. Wright and J. M. Triscari, Phys. Rev. A **63** 033601 (2001).

[27] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products (Academic Press, New York 1994, 5th edition).

[28] E. K. U. Gross and R. M. Dreizler, Z. Phys. A **302** 103 (1981).

[29] A. Minguzzi, P. Vignolo and M. P. Tosi, Phys. Lett. A **294** (2002) 222.

[30] M. A. Rajabpour and S. Sotiriadis, Phys. Rev. A **89** 033620 (2014).

[31] A. S. Campbell , D. M. Gangardt and K.V. Kheruntsyan ,Phys. Rev. Lett. **114** 125302 (2015).

[32] K.G. Wilson, Phys. Rev. **179** 1499 (1969).

[33] M. Moshinsky, Phys. Rev. **88** 625 (1952).

[34] S. Godoy, Phys. Rev. A **65** 042111 (2002).

[35] M. D. Girardeau and E. M. Wright, Phys. Rev. Lett. **84** 5691 (2000).

[36] A. Minguzzi and D.M. Gangardt, Phys. Rev. Lett. **94** 240404 (2005).

[37] Y. Y. Atas , D. M. Gangardt, I. Bouchoule and K.V. Kheruntsyan, Phys. Rev. A **95** 043622 (2017).

[38] P. Shea and B. Van Zyl J. Phys A : Math. Theor. **40** 10589 (2007).

[39] Y. Castin, C. R. Phys. **5** 407 (2004).

[40] V. Gritsev, P. Barmettler and E. Demler, New J. Phys. **12** 11300517 (2010).

5.2 Appendix A

In this appendix we will evaluate the terms given in Eqs. (8)-(11). Returning to Eq. (9), and making the change of variable $t = u + X$, this equation becomes

$$\begin{aligned} I_1 &= \int_{-\infty}^{X-|s|/2} e^{\frac{-m\omega}{\hbar}t^2}\left[(t-X)^2 - \frac{s^2}{4}\right] dt \\ &= \int_{-\infty}^{X-|s|/2} t^2 e^{\frac{-m\omega}{\hbar}t^2} dt - 2X\int_{-\infty}^{X-|s|/2} t e^{\frac{-m\omega}{\hbar}t^2} dt + \left(X^2 - \frac{s^2}{4}\right)\int_{-\infty}^{X-|s|/2} e^{\frac{-m\omega}{\hbar}t^2} dt \quad \text{(A1)} \end{aligned}$$

performing an integration by parts for the first integral, and since the second integral can be exactly computed, we get after arrangements the result

$$I_1 = \frac{\hbar}{2m\omega}\left(X + \frac{|s|}{2}\right) e^{\frac{-m\omega}{\hbar}(X-\frac{|s|}{2})^2} + \left(\frac{\hbar}{2m\omega} + X^2 - \frac{s^2}{4}\right)\int_{-\infty}^{X-|s|/2} e^{\frac{-m\omega}{\hbar}t^2} dt \quad \text{(A2)}$$

To compute I_2 in Eq. (10) we rewrite it as

$$I_2 = \int_{-\infty}^{-|s|/2} e^{\frac{-m\omega}{\hbar}(u-X)^2}(u^2 - \frac{s^2}{4})du \quad \text{(A3)}$$

and putting $t = u - X$, and following the same steps as was done for I_1, we arrive to

$$I_2 = -\frac{\hbar}{2m\omega}\left(X + \frac{|s|}{2}\right) e^{\frac{-m\omega}{\hbar}(X+\frac{|s|}{2})^2} + \left(\frac{\hbar}{2m\omega} + X^2 - \frac{s^2}{4}\right)\int_{-\infty}^{-X-|s|/2} e^{\frac{-m\omega}{\hbar}t^2} dt \quad \text{(A4)}$$

Let us now come to the third term given in Eq. (11), and using the change of variable $t = u+X$, to write

$$I_3 = \int_{X-|s|/2}^{X+|s|/2} t^2 e^{\frac{-m\omega}{\hbar}t^2} dt - 2X\int_{X-|s|/2}^{X+|s|/2} t e^{\frac{-m\omega}{\hbar}t^2} dt + \left(X^2 - \frac{s^2}{4}\right)\int_{X-|s|/2}^{X+|s|/2} e^{\frac{-m\omega}{\hbar}t^2} dt \quad \text{(A5)}$$

carrying out a double integration by parts for the first integral, the equation (A5) becomes

$$\begin{aligned} I_3 &= \frac{\hbar}{2m\omega}\left(X - \frac{|s|}{2}\right) e^{\frac{-m\omega}{\hbar}(X+\frac{|s|}{2})^2} - \frac{\hbar}{2m\omega}\left(X + \frac{|s|}{2}\right) e^{\frac{-m\omega}{\hbar}(X-\frac{|s|}{2})^2} + \\ &\left(\frac{\hbar}{2m\omega} + X^2 - \frac{s^2}{4}\right)\int_{X-|s|/2}^{X+|s|/2} e^{\frac{-m\omega}{\hbar}t^2} dt \quad \text{(A6)} \end{aligned}$$

Collecting the results in Eq. (A2), (A4) and (A6), the expression of B in Eq. (8) reduces to

$$\begin{aligned} B &= \Bigg\{ \frac{\hbar}{m\omega} \left[(X + \frac{|s|}{2}) e^{\frac{-m\omega}{\hbar}(X - \frac{|S|}{2})^2} - (X - \frac{|s|}{2}) e^{\frac{-m\omega}{\hbar}(X + \frac{|S|}{2})^2} \right] \\ & \left(\frac{\hbar}{2m\omega} + X^2 - \frac{s^2}{4} \right) \left[\int_{-\infty}^{X-|s|/2} e^{\frac{-m\omega}{\hbar}t^2} dt + \int_{-\infty}^{-X-|s|/2} e^{\frac{-m\omega}{\hbar}t^2} dt - \int_{X-|s|/2}^{X+|s|/2} e^{\frac{-m\omega}{\hbar}t^2} dt \right] \Bigg\} \end{aligned} \tag{A7}$$

To further simplicity Eq. (A8), we denote by Z the following combination

$$Z = \int_{-\infty}^{X-|s|/2} e^{\frac{-m\omega}{\hbar}t^2} dt + \int_{-\infty}^{-X-|s|/2} e^{\frac{-m\omega}{\hbar}t^2} dt - \int_{X-|s|/2}^{X+|s|/2} e^{\frac{-m\omega}{\hbar}t^2} dt \tag{A8}$$

using the identities,

$$\int_{X-|s|/2}^{X+|s|/2} e^{\frac{-m\omega}{\hbar}t^2} dt = \int_{-\infty}^{X+|s|/2} e^{\frac{-m\omega}{\hbar}t^2} dt - \int_{-\infty}^{X-|s|/2} e^{\frac{-m\omega}{\hbar}t^2} dt \tag{A9}$$

$$\int_{-\infty}^{-X-|s|/2} e^{\frac{-m\omega}{\hbar}t^2} dt = \int_{X+|s|/2}^{+\infty} e^{\frac{-m\omega}{\hbar}t^2} dt \tag{A10}$$

equation (A9) can be rewritten as

$$\begin{aligned} Z &= 2 \left[\int_{-\infty}^{X-|s|/2} e^{\frac{-m\omega}{\hbar}t^2} dt - \int_{-\infty}^{X+|s|/2} e^{\frac{-m\omega}{\hbar}t^2} dt \right] + \int_{-\infty}^{+\infty} e^{\frac{-m\omega}{\hbar}t^2} dt \\ &= 2 \left[\int_{-\infty}^{X-|s|/2} e^{\frac{-m\omega}{\hbar}t^2} dt - \int_{-\infty}^{X+|s|/2} e^{\frac{-m\omega}{\hbar}t^2} dt \right] + \sqrt{\frac{\pi\hbar}{m\omega}} \end{aligned} \tag{A11}$$

The above term between brackets can be expressed in terms of error function, $erf(x) = 2/\sqrt{\pi} \int_0^x e^{-t^2} dt$ [4] through the use of the identity

$$\int_{-\infty}^{X-|s|/2} e^{\frac{-m\omega}{\hbar}t^2} dt - \int_{-\infty}^{X+|s|/2} e^{\frac{-m\omega}{\hbar}t^2} dt = \int_{0}^{X-|s|/2} e^{\frac{-m\omega}{\hbar}t^2} dt - \int_{0}^{X+|s|/2} e^{\frac{-m\omega}{\hbar}t^2} dt \tag{A12}$$

yielding to

$$\int_{-\infty}^{X-|s|/2} e^{\frac{-m\omega}{\hbar}t^2} dt - \int_{-\infty}^{X+|s|/2} e^{\frac{-m\omega}{\hbar}t^2} dt = \sqrt{\frac{\pi\hbar}{m\omega}} \left\{ \mathrm{erf} \left[\sqrt{\frac{m\omega}{\hbar}} \left(X - \frac{|s|}{2} \right) \right] - \mathrm{erf} \left[\sqrt{\frac{m\omega}{\hbar}} \left(X + \frac{|s|}{2} \right) \right] \right\} \tag{A13}$$

which upon insertion into Eq. (A11) gives

$$Z = \sqrt{\frac{\pi\hbar}{m\omega}} \left[1 + \mathrm{erf} \left(\sqrt{\frac{m\omega}{\hbar}} \left(X - \frac{|s|}{2} \right) \right) - \mathrm{erf} \sqrt{\frac{m\omega}{\hbar}} \left(X + \frac{|s|}{2} \right) \right] \tag{A14}$$

Substituting this result into Eq. (A7) we arrive to Eq. (12) of the main text.

Abstract.

The response of a quantum system to a sudden change of its parameters entering in the hamiltonian, such as the confining potential or two-body particle-particle interactions, is an interesting issue in physics. This issue is experimentally realized in the field of ultra-cold quantum gases, where the advances in such field allow a full control of the parameters of the confined gas. The subsequent time evolution of the system following a sudden change of a specific parameter, called quench, is interesting and allows the study of dynamical non-equilibrium properties of the system. In this thesis we considered the case of quench of trapping potential for different one-dimensional systems. The dynamical quantities we were interested are the one-body density matrix and the resulting mass current density distribution.

Using the time evolution propagator, we derived the time dependent one-body density matrix and the current density of an initially harmonically trapped noninteracting system following a quench of the potential. For a one-dimensional system of bosons with large atom number interacting through a repulsive delta potential initially confined by a potential well, we derived the so-called long-time asymptotic one-body density matrix during free expansion. The third system we examined is a quantum impenetrable gas of bosons (a Tonks–Girardeau gas) with a given atom number. We derived an explicit exact analytical expression for the mass current distribution (mass transport) after quench from one harmonic trap to another harmonic trap. We showed that, the current distribution is a suitable collective observable and under the weak quench regime, it exhibits oscillations at the same frequencies as those predicted for the peak momentum distribution in the breathing mode. The analysis is extended to other possible quenched systems..

Key-words: Ultra-cold gas, confining potential, quench potential, one-body density matrix, current density.

Résumé.

La réponse d'un système quantique à un soudain changement des paramètres de son hamiltonian, tel que le potentiel confinant ou les interactions à 2 corps atome-atome, constitue un sujet intéressant en physique. Ceci est réalisable expérimentalement dans le domaine des gaz quantiques ultra-froids, où les avancées technologiques permettent un contrôle total des paramètres du gaz ainsi confiné. L'évolution temporelle résultante du soudain changement d'un paramètre spécifique, appelé quench, est intéressant pour l'étude des propriétés dynamiques de non-équilibre. Dans cette thèse on a considéré le cas de quench du potentiel confinant pour différent systèmes unidimensionnels. Les grandeurs dynamiques auxquelles on s'est intéressé sont, la matrice densité à un corps et le courant de densité résultant. En utilisant le propagateur d'évolution, nous avons dérivé les expressions de la matrice densité et du courant de densité en fonction du temps, pour un système de particules indépendantes initialement confiné par un potentiel d'oscillateur harmonique, suite à un quench du potentiel. Pour un système de bosons à une dimension avec un nombre d'atomes grand et interagissant avec une interaction de type-delta répulsive et initialement confine par un potentiel, nous avons obtenu l'expression de la matrice densité asymptotique après des temps longs durant l'expansion libre. The troisième système que nous avons examine est celui d'un de bosons dit de sphères dures (Tonks–Girardeau gas) avec un nombre fixé d'atomes. Nous avons dérivé une expression analytique exacte de la densité de courant distribution de masse suite à un quench d'un potentiel harmonique vers un autre potentiel harmonique. Nous avons montré que le courant de distribution est une grandeur collective convenable et dans le cas d'un quench faible, ce courant présente des oscillations aux mêmes fréquences que celles prédites pour le pic de la densité des moments dans le breathing mode (mode respiratoire). Notre analyse à été étendue à d'autres systèmes subissant le quench.

Mots-clés : Gaz ultra-froid, potentiel confinant, quench du potentiel, matrice densité à un corps, densité de courant.

ملخص:

استجابة نظام الكم الى حدوث تغيير مفاجئ في معايير هاميلتون, مثل امكانية الحصر او التفاعلات الجسم ذرة–ذرة, هو موضوع مثير للاهتمام في الفيزياء. و هذا الاخير يمكن تحقيقه تجريبيا في مجال الغازات الكمية فائقة البرودة, حيث تسمح التطورات التكنولوجية بالتحكم الكامل في معايير الغاز المشمولة. ان التطور الزمني الناتج عن التغيير المفاجئ لمعايير معينة تدعى quench الاخماد, وهذا هام جدا لدراسة الخصائص الديناميكية للاختلال. في هذه الاطروحة درسنا حالة التبريد للعديد من الانظمة احادية البعد, الكميات الديناميكية ذات الاهمية هي مصفوفة الجسم الواحد و تدفقات الكثافة الناتجة. باستخدام المتصل التطوري, استنتجنا شروط وظيفة كثافة المصفوفة و كثافة الوقت الحالية في نظام الجسيمات. بالنسبة للبزونات احادية البعد ذات عدد كبير من الذرات المتفاعلة و المحدودة مبدئيا بالامكانات ,وجدنا عبارة مصفوفة الكثافة الطبيعية بغد فترة طويلة من التوسع الحر. النظام الثالث و الذي قمنا بفحصه هو البوزون, والذي يطلق عليه اسم الكرات الصلبة مع عدد محدد من الذرات. نحصل على عبارة تحليلية دقيقة لكثافة التيار بعد حذف الامكانات و لقد وضحنا ان توزيع كثافة التيار يتوافق مع الحجم المناسب للمجموعة, وانه في حالة الاخماد الاقل فانه يعرض الاهتزازات الحالية على نفس الترددات التي تنبأ بها. و هكذا فقد تم توسيع تحليلنا ليشمل انظمة اخرى يتم تبريدها.

الكلمات المفتاحية : غازات فائقة البرودة، امكانات الحصر، اخماد الامكانات، مصفوفة الكثافة للجسم الواحد، كثافة التيار.