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THÈME

Interior-point methods of primal-dual central-path type for solving some classes of linear complementarity problems over symmetric cones.

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Dedication

There are a number of people without whom this thesis might not have been written, and to whom I am greatly indebted.

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Abstract

Since the 1950's, the theory of mathematical programming and game theory have been developed rapidly and extensively. The literature shows the rich theory of linear programming, convex quadratic programming, and bimatrix game, which are fundamental subjects in these areas. As a unifying framework of such problems, the linear complementarity problem was introduced in mathematical programming in the mid 1960's.
The linear complementarity problem is to find, for a square matrix and a vector, a vector satisfying linear constraints and complementarity conditions.
This thesis is concerned with the analysis, implementation of interior-point methods . In particular, we focus on two type of problems: horizontal linear complementarity problems (HLCPs) and Semidefinite linear complementarity problems (SDLCPs).
In chapter 1: we present the definitions and terms that will be used throughout the thesis.
In chapter 2: we present a full-Newton feasible step interior-point algorithm for solving monotone horizontal linear complementarity problems. The idea of this algorithm is to follow the centers of the perturbed HLCP by using only full-Newton steps with the advantage that no line search is required and restricts iterates in a small neighborhood of the central-path by introducing a suitable proximity measure during the solution process. Then we prove across a new appropriate choice of the defaults of the threshold of the parameter τ which defines the size of the neighborhood of the central-path and of the update barrier parameter θ that our algorithm is well-defined and the full-Newton step to the central-path is locally quadratically convergent. Moreover, we derive its complexity bound. which coincides with the best known iteration bound for such feasible IPMs. Finally, we report some numerical results to show the ability of this approach.
In chapter 3: we deal with the complexity analysis and the numerical implementation of primal-dual interior-point methods for monotone semidefinite linear complementarity

problems based on a new parametric kernel function. The proposed kernel function is neither a self-regular and nor the usual logarithmic barrier function. By means of the feature of the kernel function, we study the complexity analysis of primal-dual IPMs and derive the currently best known iteration bound for the large-update algorithm. Finally, we report some numerical results to show the practical performance of the proposed algorithm with different parameters.

keywords: Semidefinite linear complementarity problems; Horizontal linear complementarity problems; Interior-point methods; Kernel function; Full-Newton step; Polynomial complexity.

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Glossary of Notation

Problem Classes

(MP)	:	Mathematical Programming;
(SDO)	:	Semidefinite Optimization;
(LP)	:	Linear Programming or linear optimization;
(QP)	:	Quadratic Programming;
(P)	:	Primal of a mathematical programming;
(D)	:	Dual of **(P)**;
IPC	:	Interior-Point-Condition;
IPMs	:	Interior-Point Methods;
IP	:	Interior-Point;
K.K.T	:	Karush-Kuhn-Tucker;

Spaces

$\mathbb{R}^n$	:	the real n-dimensional space;
$\mathbb{R}^n_+$	:	the nonnegative orthant of $\mathbb{R}^n$;
$\mathbb{R}^n_{++}$	:	the positive orthant of $\mathbb{R}^n$;
$\mathbb{R}^{n\times n}$	:	the set of all $n \times n$ squared matrices;
$\mathbb{R}^{m\times n}$	:	the space of$m \times n$ real matrices;
$\mathbb{S}^n$	:	denotes the set of all symmetric $n \times n$ matrices;
$\mathbb{S}^n_+$	:	the cone of positive semidefinite symmetric matrices of order n;

Matrices

$X \succeq 0$	:	means $X \in \mathbb{S}^n$ is positive semidefinite;
$X \succ 0$	:	means $X \in \mathbb{S}^n$is positive definite.;
$\lambda_i(X), i = 1, \ldots, n$	:	the eigenvalues of X;
$\lambda_{\max}(X), i = 1, \ldots, n$	:	the largest eigenvalues of X;
$\lambda_{\min}(X), i = 1, \ldots, n$	:	the smallest eigenvalues of X;
$\mathbf{Tr}(X)$	=	$\sum_{i=1}^n X_{ii} = \sum_{i=1}^n \lambda_i(X)$;
$A \bullet X$	=	$\mathbf{Tr}(A^T X)$ the inner-product between$A \in \mathbb{S}^n$ and $X \in \mathbb{S}^n$;
I	:	identity matrix;
$\|X\|_F$	:	$\|X\|_F = \sqrt{X \bullet X}$. Frobenius matrix norm (Schur);
$\sigma(X)$	:	denote the spectrum, or the set of eigenvalues, of a matrix $X \in \mathbb{R}^{n \times n}$
A^T	:	the transpose of the matrix A;
$A \sim B$	:	P non-singular with $A = PBP^{-1}$;

Vectors

e	=	$(1, \ldots, 1)^T$;
x^T	=	$(x_1, \ldots, x_n)$ the transpose of a vector$x$ with components x_i;
xy	=	$(x_1 y_1, \ldots, x_n y_n)^T$ Hadamard product;
$x^T y$	=	$\sum_{i=1}^n x_i y_i$ the standard inner product in $\mathbb{R}^n$;
$\frac{x}{y}$	=	$\left(\frac{x_1}{y_1}, \frac{x_2}{y_2}, \ldots, \frac{x_n}{y_n}\right)^T (y \neq 0)$;
$\sqrt{x}$	=	$\left(\sqrt{x_1}, \ldots, \sqrt{x_n}\right)^T (x \geq 0)$;
x^{-1}	=	$\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)^T (x \neq 0)$;
$X = Diag(x)$	:	the diagonal matrix of X with $X_{ii} = x_i$;

Functions

$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x)\right)^T$	:	the gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$;
$\nabla^2 f(x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right)_{1 \leq i,j \leq n}$	:	the Hessian matrix of $f : \mathbb{R}^n \to \mathbb{R}$;
$\psi(t)$	:	a kernel function;

List of publications

Paper 1

M. Achache, N. Tabchouche. A full-Newton step feasible interior-point algorithm for monotone horizontal linear complementarity problems. **Optimization Letters**. 1-19(2018).

Paper 2

M. Achache, N. Tabchouche. Complexity analysis and numerical implementation of large-update interior-point methods for SDLCP based on a new parametric barrier kernel function. **Optimization (Taylor & Francis)**. **67**: 1-20(2018).

Introduction

This doctoral thesis deals with the complexity analysis and numerical implementation of interior point methods for symmetric cones linear complementarity problems (SCLCP for short). Mathematically, consider a Euclidean finite dimensional vector space $\mathbf{E}$ over the field $\mathbb{R}$ equipped with inner product $\langle ., . \rangle$. Let $\mathcal{L} : \mathbf{E} \to \mathbf{E}$ be a given linear transformation, and $q \in \mathbf{E}$. The SCLCP consists of finding a pair of elements (x, y) such that

$$x \in \mathbb{K}, y = \mathcal{L}(x) + q \in \mathbb{K}, \text{ and } \langle x, y \rangle = 0,$$

where $\mathbb{K}$ is a symmetric cone i.e., $\mathbb{K}$ is homogenous and self-dual.(see e.g. [25, 46])
The SCLCP is not an optimization problem but it is closely related to the optimality conditions for conic optimization i.e., linear, convex quadratic and semidefinite optimization over symmetric cones. Although if $\mathbb{K} = \mathbb{R}^n_+$, the orthant positive cone, then the SCLCP reduces to linear complementarity problems (LCP) meanwhile if $\mathbb{K} = \mathbb{S}^n_+$, the cone of symmetric semidefinite positive matrices, the SCLCP becomes the semidefinite linear complementarity problem (SDLCP)(see e.g. [21, 25, 46]).

Since the publication of the famous paper of Karamarkar [32] in 1984, for solving the linear programming problem (LP), the field of interior-point methods (IPMs) becomes one of the most active areas of research in mathematical programming. Interior point methods are known for their polynomial convergence, speed and efficiency and have proven to be true competitors of classical methods (simplex and Lemeke, etc.). These methods have been widely disseminated in the literature and have been enriched with several variants in order to improve their complexity and their numerical efficiency. This latter can be classified into four main categories:

- Projective methods,
- Affine-scaling methods,

- Potential-reduction methods,
- Path-following methods or central path methods.

In this thesis we focus on the primal-dual IPMs to solving two classes of monotone SCLCP, namely,

- Horizontal linear complementarity problems (HLCP).
- Semidefinite linear complementarity problems (SDLCP).

The HLCPs has been very popular in the literature due to its numerous applications [39]. In the literature of IPMs, this problem has also been an active subject since it includes the standard LCP, LP and convex quadratic optimization (CQO), and finds many applications in economic equilibrium problems, traffic assignment problems, and optimization problems [21]. There are a variety of solutions approaches for HLCP which have been studied intensively. Among them, path-following IPMs gained much more attention than other methods [43, 51]. These methods are powerful tools to solve a wide large of mathematical problems such as LP [43, 54, 58], CQO [2, 54], LCP [4, 21, 54, 59], linearly convex constrained optimization (LCOO) [7], the semidefinite programming (SDP) [3, 39], and the semidefnite linear complementarity problem (SDLCP) [8].

In the last years, the SDLCPs has received considerable attention from researchers because of its wide applications such as control theory linear and bilinear matrix inequalities, semidefinite linear complementarity problem (SDLCP) is a special class of mathematical programming, which became popular during the 1960's. The SDLCPs are extensions of LP and LCP, respectively, with the cone of nonnegative real vectors replaced by the cone of symmetric positive semidefinite real matrices.

This problem was introduced in a slightly different form by Kojima, Shindoh, and Hara [34] as a model unifying various problems arising from system and control theory and combinatorial optimization. The SDLCP can be regarded as a generalization of the LCP [8, 9]. There are many approaches for solving this class of problems. Among them, the path following IPMs are the most efficient and fundamental methods which obtain the best complexity bounds (see e.g. [2, 6, 22, 24, 27, 42, 49]).

In 2001, Peng et al. [43] designed a new paradigm of primal-dual algorithms based on the so-called self-regular proximity functions for LP. They improved iteration bound with large-update from $\mathcal{O}(n \log \frac{n}{\epsilon})$ with logarithmic kernel function to $\mathcal{O}(\sqrt{n} \log n \log \frac{n}{\epsilon})$. Subsequently, in 2004, Bai et al. [15], introduced new primal-dual IPMs based on new proximity functions which were called non logarithmic eligible barrier kernel functions. These functions enjoy useful properties and determine new search directions for primal-dual interior point algorithms. Based on these functions, they obtained the best known complexity results for large-update methods, namely, $\mathcal{O}(\sqrt{n} \log n \log \frac{n}{\epsilon})$ and good numerical results. Some examples of kernel functions which have been analyzed in earlier papers can be seen in references (see e.g.,[1, 3, 14, 16, 33, 36, 37, 38, 42, 50, 52]).

Kernel functions play an important role in the design of new primal-dual interior point algorithms for solving many instances such as LP, CQO, SDO and LCP. Their extensions to SDLCPs is with great importance since, on one hand the feasible set of this problem is not a polyhedral set and therefore the classical simplicial methods does not applicable. On the other hand, as for our knowledge there is no interior-point methods based on kernel functions are developed for solving this problem. To do so, we suggest the following parametric kernel function

$$\psi(t) = \frac{t^2 - 1}{2} + \frac{q^{\frac{1}{t}-1} - 1}{q \log q} - \frac{(q-1)}{q}(t-1),\ t > 0, \tag{1}$$

where $q > 1$ is a barrier parameter. The proposed kernel function is neither a self-regular and nor a logarithmic barrier function. Based on this kernel function, we show that the proposed large-update algorithm enjoy the favorable best iteration complexity for the specific choice of the barrier parameter's function.

The table 1 shows the iteration bound based on some studied kernel functions in the literature.

kernel function $\psi_i(t)$	Iteration bound	ref
$\psi_1(t) = \frac{t^2-1}{2} + \frac{q^{\frac{1}{t}-1}-1}{q\log q},\ q>1$	$\mathcal{O}(\sqrt{n}\log n\log\frac{n}{\epsilon})$	[1]
$\psi_2(t) = \frac{t^2-1}{2} - \log(t)$	$\mathcal{O}(n\log\frac{n}{\epsilon})$	[13]
$\psi_3(t) = \frac{1}{2}(t-\frac{1}{t})^2$	$\mathcal{O}(n^{\frac{2}{3}}\log\frac{n}{\epsilon})$	[13]
$\psi_4(t) = \frac{t^2-1}{2} + \frac{t^{1-q}-1}{q-1}, q>1$	$\mathcal{O}(qn^{\frac{q+1}{2q}}\log\frac{n}{\epsilon})$	[13]
$\psi_5(t) = \frac{t^2-1}{2} + \frac{t^{1-q}-1}{q(q-1)} - \frac{q-1}{q}(t-1), q>1$	$\mathcal{O}(qn^{\frac{q+1}{2q}}\log\frac{n}{\epsilon})$	[39]
$\psi_6(t) = \frac{t^2-1}{2} + \frac{e^{\frac{1}{t}}-e}{e}$	$\mathcal{O}(\sqrt{n}\log^2 n\log\frac{n}{\epsilon})$	[13]
$\psi_7(t) = t-1+\frac{t^{1-q}-1}{q-1}, q>1$	$\mathcal{O}(qn\log\frac{n}{\epsilon})$	[14]

Table 1: Examples of some kernel functions and its iteration bound for large-update methods

Short Outline of the Thesis

The thesis contains three chapters, followed by a bibliography. This thesis is organized as follows

- **Chapter 1**: In this chapter, we present the definitions and terms that will be used throughout the thesis.

- **Chapter 2**: This chapter deals with the study of the polynomial complexity and the numerical implementation for a full-Newton short-step primal-dual central-path interior-point method for solving the class of monotone HLCPs. We show that the corresponding algorithm enjoys the best known theoretical polynomial complexity, namely, $\mathcal{O}\left(\sqrt{n}\log\frac{n}{\epsilon}\right)$. Some numerical results are provided to show the efficiency of the proposed algorithm and to compare with an available method.

- **Chapter 3**: Based on a new parametric kernel function, this chapter presents a primal-dual large-update interior-point algorithm for SDLCPs. By the mean of this function we show that the proposed algorithm has the favorable complexity bound, namely, $\mathcal{O}\left(\sqrt{n}\log n\log\frac{n}{\epsilon}\right)$ with a suitable choice of the parameter barrier related to the kernel function. Finally, we report some numerical results on some practical

problems to show the practical performance of the proposed algorithm with different values of the parameter barrier. Finally, we end the thesis by a conclusion and future work.

Chapter 1

Convex Analysis and matrix theory

Contents

In this chapter, we present fundamental definitions and notation that will be used throughout the thesis.

1.1 Preliminaries

In this thesis we will be working in the space of vectors and in the space of matrices, first we will define a Euclidean space **E** to be a finite-dimensional vector space over the reals with which there is an inner product $\langle ., . \rangle : \mathbf{E} \times \mathbf{E} \to \mathbb{R}$ that satisfies the following conditions $x, y, z \in \mathbf{E},\ \alpha, \beta \in \mathbb{R}$

- $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$,
- $\langle x, y \rangle = \langle y, x \rangle$,
- $\langle x, x \rangle \geq 0$, where $\langle x, x \rangle = 0$ if and only if $x = 0$.

Examples of Euclidean spaces are $\mathbb{R}^n$ and $\mathbb{R}^{m \times n}$, which are the spaces of real column vectors $x = (x_1, x_2, \ldots, x_n)^T$ and real $m \times n$ matrices $A = (a_{ij})$, respectively. The standard inner product on $\mathbb{R}^n$ is defined as

$$x^T y = \sum_{i=1}^{n} x_i y_i,$$

and the inner product we will use on $\mathbb{R}^{m \times n}$ is defined as follows

$$\langle X, Y \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} y_{ij} = \mathbf{Tr}(X^T Y),$$

where $\mathbf{Tr}(A) = \sum_{i=1}^{n} a_{ii}$ is the trace of a matrix $A = (a_{ij}) \in \mathbb{R}^{m \times n}$.
Now, we briefly recall some known facts from linear algebra that will be used in the sequel of this thesis. For the details, we refer to the book [27].

Definition 1.1.1. *For $A, B \in \mathbb{R}^{n \times n}$, $c \in \mathbb{R}^n$ we recall a function $\|.\| : \mathbb{R}^{n \times n} \to \mathbb{R}$, a matrix norm if it satisfies the following axioms*

1. $\|A\| \geq 0$, $\|A\| = 0 \Leftrightarrow A = 0$,

2. $\|cA\| = |c|\|A\|$,

3. $\|A + B\| \leq \|A\| + \|B\|$,

4. $\|AB\| \leq \|A\|\|B\|$.

The Frobenius and the spectral norms are defined, respectively, as follows:

$$\|A\|_F = \sqrt{\mathbf{Tr}(A^T A)} \text{ and } \|A\|_F = \sqrt{\mathbf{Tr}(A^2)} \text{ if } A \in \mathbb{S}^n,$$

and

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)} \text{ and } \|A\|_2 = \lambda_{\max}(A) \text{ if } A \in \mathbb{S}^n.$$

We have the following properties.

Property 1.1.1. *Let $A, B \in \mathbb{R}^{n\times n}$, then*

- $\mathbf{Tr}(A) = \mathbf{Tr}(A^T)$.
- $\mathbf{Tr}(A) = \mathbf{Tr}(B)$ *if A is similar to B i.e., $A = PBP^{-1}$ for some non-singular matrix P.*
- $\mathbf{Tr}(AB) = \mathbf{Tr}(BA)$.
- $\mathbf{Tr}(A + B) = \mathbf{Tr}(A) + \mathbf{Tr}(B)$.
- $\|AB\|_2 \leq \|A\|_2\|B\|_F$.
- $\|A\|_2 \leq \|A\|_F \leq \sqrt{n}\|A\|_2$, $n \geq 1$.

We list below some well known matrix theoretic properties that are needed in this thesis

1. A is positive semidefinite (positive definite) if the usual inner product $\langle Ax, x\rangle \geq 0 (> 0)$ for all nonzero $x \in \mathbb{R}^n$.
2. $A \in \mathbb{S}^n$ is positive definite if and only if $\lambda_{min}(A) > 0$.
3. $A \in \mathbb{R}^{n\times n}$ is positive definite if and only if its symmetric part $\frac{A+A^T}{2}$ is positive definite.

4. The square real matrix A is symmetric if and only if there exists an orthogonal matrix O such that $OO^T = I$ and $A = ODO^T$ where D is a diagonal matrix, i.e., $D := \text{Diag}(\lambda_i(A))$, $\lambda_i(A) \in \sigma(A)$.

Lemma 1.1.1. *Let $(X, Z) \in \mathbb{S}^n \times \mathbb{S}^n$, we have:*

1. $\mathbf{Tr}(XY) \geq 0$, *for every* $X, Y \succeq 0$.
2. $X, Y \succeq 0$, $\mathbf{Tr}(XY) = 0$ *if and only if* $XY = YX = 0$.
3. $X \in \mathbb{S}^n$ *and* $\mathbf{Tr}(XY) \geq 0,\ Y \succeq 0 \Rightarrow X \succeq 0$.

We will use also the following transformations:
The Lyapunov linear transformation,

$$\mathcal{L}(X) = \frac{1}{2}(AX + XA^T),\ \mathcal{L} : \mathbb{S}^n \to \mathbb{S}^n,\ \text{where } A \in \mathbb{R}^{n\times n}.$$

The two sided multiplicative transformation,

$$\mathcal{M}(X) = AXA^T,\ \mathcal{M} : \mathbb{S}^n \to \mathbb{S}^n.$$

1.2 Matrix theory

Every matrix $X \in \mathbb{S}^n$ is orthogonally diagonalizable, i.e., $X = UDU^T$ with an orthogonal matrix U and a diagonal matrix $D = \text{diag}(d_1, \ldots, d_n)$, where each d_i is the eigenvalue of X.
If $X \succeq 0$, we have

$$X^{1/2} = U \begin{bmatrix} \sqrt{d_1} & 0 & \ldots & 0 \\ 0 & \sqrt{d_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sqrt{d_n} \end{bmatrix} U^T.$$

The $X^{1/2}$ is called the square root matrix of the matrix X.

Definition 1.2.1. *We write $T : \mathcal{H} \to \mathcal{K}$ is a transformation from a set $\mathcal{H}$ to $\mathcal{K}$, the set $\mathcal{H}$ is called the domain of T, then $T^* : \mathcal{K} \to \mathcal{H}$ is called the adjoint of T and he is determined by the formula*

$$\langle Tx, y\rangle_{\mathcal{K}} = \langle x, T^*y\rangle_{\mathcal{H}},\ \ \forall x \in \mathcal{H}, y \in \mathcal{K},$$

the transformation T is called self-adjoint if $T = T^*$.

Theorem 1.2.1. *The properties of the adjoint:*

- $(A + B)^* = A^* + B^*, (\lambda A)^* = \overline{\lambda} A^*$ $(\lambda \in \mathbb{C})$,
- $(A^*)^* = A, (AB)^* = B^* A^*$,
- $(A^{-1})^* = (A^*)^{-1}$ *if A is non-singular,*
- $\|A\| = \|A^*\|$.

Proposition 1.2.1. *If all multiplications and additions make sense, the following hold for matrices, A, B, C and a, b scalars*

- $A(aB + bC) = a(AB) + b(AC)$,
- $(B + C)A = BA + CA$,
- $A(BC) = (AB)C$.

Lemma 1.2.1. *Let A be an $m \times n$ matrix and let B be a $n \times p$ matrix. Then*

$$(AB)^T = B^T A^T,$$

and if α and β are scalars

$$(\alpha A + \beta B)^T = \alpha A^T + \beta B^T,$$

if A and B are non-singular matrices, we have

$$(AB)^{-1} = B^{-1} A^{-1}.$$

Definition 1.2.2. *An $n \times n$ matrix A is said to be symmetric if $A = A^T$. It is said to be skew symmetric if $A^T = -A$.*

1.3 Tensor product

Definition 1.3.1. *Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$. Then the Kronecker product (or tensor product) of A and B is defined as the matrix*

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{mp\times nq}.$$

Let $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, then

$$\begin{aligned} x \otimes y &= [x_1 y^T, \dots, x_m y^T]^T \\ &= [x_1 y_1, \dots, x_1 y_n, x_2 y_1, \dots, x_m y_n]^T. \end{aligned}$$

Theorem 1.3.1. *The following rules hold:*

- $(A \otimes B)(C \otimes D) = AC \otimes BD$,
- $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$,
- $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$,
- $\lambda(A \otimes B) = \lambda A \otimes B = A \otimes \lambda B$, $(\lambda \in \mathbb{C})$,
- $(A \otimes B)^T = (A^T \otimes B^T)$,
- $(A \otimes B)^{-1} = (A^{-1} \otimes B^{-1})$, *if A and B are non-singular,*
- $\|A \otimes B\| = \|A\|\|B\|$.

Notice that the Kronecker product is non-commutative since in general we have $A \otimes B \neq B \otimes A$.

1.3.1 Application to Sylvester and Lyapunov Equations

In this section we study the linear matrix equation

$$AX + XB = C,$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{m\times m}$ and $C \in \mathbb{R}^{n\times m}$. This equation is now often called a Sylvester equation who studied general linear matrix equations of the form

$$\sum_{i=1}^{k} A_i X B_i = C,$$

a special case is the symmetric equation

$$AX + XA^T = C, \tag{1.1}$$

obtained by taking $B = A^T$. When C is symmetric, the equation (1.1) is known as the Lyapunov equation.

Definition 1.3.2. *Let $c_i \in \mathbb{R}^n$ denote the columns of $C \in \mathbb{R}^{n\times m}$, means that $C = [c_1, \ldots, c_m]$, then vec(C) is defined as follows*

$$vec(C) = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}.$$

Theorem 1.3.2. *$A, C \in \mathbb{R}^{n\times n}$. Then the Lyapunov equation*

$$AX + XA^T = C, \tag{1.2}$$

has a unique solution if and only if A and $-A^T$ have no eigenvalues in common. If C is symmetric and (1.1) has a unique solution, then that solution is symmetric.

Remark 1.3.1. *The Lyapunov equation $AX + XA^T = C$ can also be written using the vec notation in the equivalent form*

$$((I \otimes A) + (A \otimes I))vec(X) = vec\,(C).$$

Definition 1.3.3. *For any three matrices A, B, and X for which the matrix product AXB is defined as follows*

$$vec(AXB) = (B^T \otimes A)vecX,$$

in the other hand let $C \in \mathbb{S}^n$, the matrix equation

$$AXB = C,$$

is equivalent to following

$$(B^T \otimes A)vecX = vecC.$$

1.4 Convex sets and functions

We will begin by defining the concepts of convex sets and functions.

Definition 1.4.1. *A set Ω in a Euclidean space **E** is called convex if*

$$tx + (1-t)y \in \Omega, \ \text{ for all } \ x, y \in \Omega \ \text{ in } \ t \in [0,1]\,.$$

Given a convex set Ω, a function $f : \Omega \to \mathcal{R}$ is called convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

for all $x, y \in \Omega$ and $t \in [0,1]$, the function f is called strictly convex if the above inequality is strict whenever $x \neq y$ and $t \in (0,1)$.

1.4.1 Convex optimization

Theorem 1.4.1. *Let $f : C \to \mathbb{R}$ be a convex function defined on the convex set C. Let $x^* \in C$ be a local minimum of f over C. Then x^* is a global minimum of f over C.*

Theorem 1.4.2. *Let $f : C \to \mathbb{R}$ be a strictly convex function defined on the convex set C. Let $x^* \in C$ be a local minimum of f over C. Then x^* is a strict global minimum of f over C.*

Theorem 1.4.3. *(convexity of the optimal set in convex optimization)) Let $f : C \to \mathbb{R}$ be a convex function defined over the convex set $C \subseteq \mathbb{R}^n$. Then the set of optimal solutions of the problem*

$$\min\{f(x), \ x \in C\}, \tag{1.3}$$

which we denote by x^*, is convex. If, in addition, f is strictly convex over C, then there exists at most one optimal solution of the problem (1.3).

1.4.2 Convex Cones

Definition 1.4.2. *A non-empty subset $\mathbb{K} \subseteq \mathbb{R}^n$ is called a closed convex cone if it satisfies:*

- $\alpha x \in \mathbb{K}$ *whenever* $x \in \mathbb{K}$ *and* $\alpha > 0$.
- $x + y \in \mathbb{K}$ *whenever* $x \in \mathbb{K}$ *and* $y \in \mathbb{K}$.

Moreover, $\mathbb{K}$ is pointed if

$$x \text{ and } -x \in \mathbb{K} \Rightarrow x = 0.$$

A pointed convex cone in $\mathbb{R}^n$ defines a partial order on $\mathbb{R}^n$ by

$$x \geq y \Leftrightarrow x - y \in \mathbb{K},$$

or equivalently

$$(-\mathbb{K}) \cap \mathbb{K} = \{0\},$$

with $-\mathbb{K} = \{-x \mid x \in \mathbb{K}\}$.

This partial order satisfies the following conditions:

- $\forall x \in \mathbb{R}^n \ : \ x \geq x,$
- $\forall x, y \in \mathbb{R}^n \ : \ x \geq y, y \geq x \Rightarrow x = y,$
- $\forall x, y, z \in \mathbb{R}^n \ : \ x \geq y, y \geq z \Rightarrow x \geq z,$
- $\forall x, y \in \mathbb{R}^n, \ \forall \alpha \in \mathbb{R}_+ \ : \ x \geq y \Rightarrow \alpha x \geq \alpha y,$
- $\forall x, y, x', y' \in \mathbb{R}^n \ : \ x \geq y, x' \geq y' \Rightarrow x + x' \geq y + y',$

for $x, y \in \mathbb{R}^n$.

Lemma 1.4.1. *The set $\{X \in \mathbb{S}^n : X \succeq 0\}$ of positive semidefinite matrices is a closed convex cone.*

Definition 1.4.3. *Let $\mathbb{K} \subseteq \mathbb{R}^n$ be a closed convex cone. The set*

$$\mathbb{K}^* = \{y \in \mathbb{R}^n : \langle y, x \rangle \geq 0 \ \ \forall x \in \mathbb{K}\}$$

is called the dual cone of $\mathbb{K}$, with $\langle ., . \rangle$ the standard scalar product in $\mathbb{R}^n$.

Definition 1.4.4. *A cone $\mathbb{K}$ is called self-dual, if $\mathbb{K} = \mathbb{K}^*$.*

Definition 1.4.5. *$\mathbb{K} \subset \mathbb{R}^n$ is a proper cone if it is a cone which is closed, convex, pointed and solid, which means it has nonempty interior.*

Definition 1.4.6. *A regular convex cone $\mathbb{K} \subseteq \mathbb{R}^n$ is called homogeneous if the automorphism group $Aut(\mathbb{K})$ acts transitively on $\mathbb{K}^\circ$, i.e., if for every $x, y \in \mathbb{K}^\circ$ there exists an automorphism $A \in Aut(\mathbb{K})$ such that $Ax = y$.*

Definition 1.4.7. *A self-dual, homogeneous regular convex cone is called symmetric cone.*

1.4.3 Examples of symmetric and nonsymmetric cones

- The nonnegative orthant $\mathbb{R}^n_+$ of $\mathbb{R}^n$:

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n | \, x \geq 0\}.$$

- Lorentz cone $\mathcal{L}^n_+$: partition every element of $\mathbb{R}^n$ as $x = (x_0, x_1) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and define the Lorentz cone (or the ice-cream cone) as

$$\mathcal{L}^n_+ = \{(x_0, x_1) \in \mathbb{R} \times \mathbb{R}^{n-1} | \;\; \|x_1\|_2 \leq x_0\} \subseteq \mathbb{R}^n$$

- The cone of symmetric positive semidefinite matrices $\mathbb{S}^n_+$:

$$\mathbb{S}^n_+ = \{A \in \mathbb{S}^n | \;\; x^T A x \geq 0, \; \forall x \in \mathbb{R}^n\}.$$

- The cone of copositive matrices

$$\mathbb{C}_n = \{A \in \mathbb{S}^n | \;\; x^T A x \geq 0, \; \forall x \in \mathbb{R}^n_+\},$$

  is not self-dual and therefore is not a symmetric cone (see e.g. [24]).

1.5 Newton-Raphson's method for nonlinear systems

Now we are interested to the Newton-Raphson method for solving nonlinear system of equations. This latter plays an important role in the development of interior-point methods. We have the following definition.

Definition 1.5.1. *A system of nonlinear equations is a set of equations as the following:*

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0, \\ f_2(x_1, x_2, \ldots, x_n) &= 0, \\ &\vdots \\ f_n(x_1, x_2, \ldots, x_n) &= 0, \end{aligned}$$

where $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ and f_i is a nonlinear real function, $i = 1, \ldots, n$. To simplify the notation even further, define the vector valued function

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) \end{bmatrix}.$$

We now define the $n \times n$ Jacobian matrix $J(x)$ as

$$J(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) & \dots & \frac{\partial f_1}{\partial x_n}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) & \dots & \frac{\partial f_2}{\partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(x) & \frac{\partial f_n}{\partial x_2}(x) & \dots & \frac{\partial f_n}{\partial x_n}(x) \end{bmatrix},$$

where $\frac{\partial f_i}{\partial x_j}(x)$ is called the partial derivative of f_i at x_j.

The Hessian matrix is a matrix of second order partial derivatives, define as follow:

$$H(x) = \begin{bmatrix} \frac{\partial^2 f_1}{\partial x_1^2}(x) & \frac{\partial^2 f_1}{\partial x_1 \partial x_2}(x) & \dots & \frac{\partial^2 f_1}{\partial x_1 \partial x_n}(x) \\ \frac{\partial^2 f_2}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f_2}{\partial x_2^2}(x) & \dots & \frac{\partial^2 f_2}{\partial x_2 \partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f_n}{\partial x_n \partial x_1}(x) & \frac{\partial^2 f_n}{\partial x_n \partial x_2}(x) & \dots & \frac{\partial^2 f_n}{\partial x_n^2}(x) \end{bmatrix}.$$

1.5.1 Newton-Raphson's Method

If $x = x_0$ (a vector) represents the first guess for the solution, successive approximations to the solution are obtained from

$$x_{n+1} = x_n - J^{-1} f(x_n) = x_n - \Delta x_n, \quad \text{with} \quad x_{n+1} = x_n + \Delta x_n,$$

where Δx_n (Newton's direction) is the solution of the following system

$$J(x_n)\Delta x_n = -f(x_n).$$

1.6 Problem Description

In the mid 1960's, Lemke, Cottle, and Cottle and Dantzig [20] introduced the linear complementarity problem (LCP), as a unifying problem of the linear and convex quadratic programming, and the problem of finding Nash equilibrium of bimatrix games.

In this section, we start from a mathematical definition of LCP, we begin with some mathematical definitions, before introducing a standard form of semidefinite linear complementarity problem (SDLCP), in section 1.6.1 we introduce the standard linear complementarity problem, then in section 1.6.3 the Horizontal linear complementarity problem, and in section 1.6.5 the semidefinite linear complementarity problem (SDLCP).

1.6.1 Linear Complementarity Problem

LCP is not an optimization problem, but it has robust relationship with both linear programming (LP) and convex quadratic programming (CQP) problems. This strong relationship is based on the fact that Karush-Kuhn-Tucker (KKT) optimality conditions for LP and QP can be converted into LCP. Many optimization problems from engineering, finance, transportation, etc. can be directly written as LCP. Therefore, solving LCPs has been very important topic for many years.

LCPs is a problem of finding a particular vector in a finite real vector space that satisfies a certain system of inequalities. Mathematically, given a vector $q \in \mathbb{R}^n$ and a matrix $M \in \mathbb{R}^{n\times n}$, we want to find a vector $x \in \mathbb{R}^n$ (or to show such a vector does not exist) such that

$$x \geq 0,\ y \geq 0,\ y - Mx = q,\ x^T y = 0. \tag{1.4}$$

We will denote

$$\mathcal{F} = \left\{(x, y) \in \mathbb{R}^{2n} : y = Mx + q,\ x \geq 0, y \geq 0\right\}.$$

The feasible set of points of the LCP as defined in (1.4).

The solution set of the LCP is given by:

$$Sol(M, q) = \left\{(x, y) \in \mathcal{F} :\ x^T y = 0\right\}.$$

1.6.2 Classes of LCP

In the literature of LCPs there are more than 50 matrix classes. In this section, we recall some important used classes in an IPM framework.

Let $M \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$ and $\kappa \geq 0$.

- M is called a skew-symmetric matrix if

$$x \in \mathbb{R}^n\ :\ x^T M x = 0.$$

- M is called a positive semi-definite matrix if

$$x \in \mathbb{R}^n\ :\ x^T M x \geq 0.$$

- M is called a P_0-matrix if

$$(0 \neq x \in \mathbb{R}^n)(\exists i \in I)(x_i(Mx)_i \geq 0).$$

- M is called a P-matrix: Matrix with all principal minors positive or equivalently

$$(0 \neq x \in \mathbb{R}^n)(\exists i \in I)(x_i(Mx)_i > 0).$$

- M is called a $P_*(\kappa)$-matrix if

$$(1+4\kappa)\sum_{i\in I^+(x)}(x_i(Mx)_i) + \sum_{i\in I^-(x)}(x_i(Mx)_i) \geq 0, \forall x \in \mathbb{R}^n,$$

where

$$I^+(x) = \{i : x_i(Mx)_i \geq 0\}, I^-(x) = \{i : x_i(Mx)_i < 0\},$$

or

$$x^T Mx \geq -4\kappa \sum_{i\in I^+(x)}(x_i(Mx)_i), \forall x \in \mathbb{R}^n.$$

Example 1.6.1.

Recall that a pair of primal-dual LP programs is defined as

$$\begin{cases} \min\limits_x c^T x \\ \text{s.t. } Ax = b \\ x \geq 0, \end{cases} \text{ , and its dual } \begin{cases} \max\limits_{(y,z)} b^T y \\ \text{s.t. } A^T y + z = c, \\ z \geq 0, \end{cases}$$

with $x, z \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$. LP can be converted into LCP thanks to KKT conditions

$$\begin{aligned} Ax &= b, \ \ x \geq 0 \\ A^T y + z &= c, \ \ z \geq 0 \\ x^T z &= 0. \end{aligned}$$

The LCP is given as follows:

$$\text{LCP}(M, q) : \ \overline{z} = M\overline{x} + q$$

where $\overline{z} = \begin{bmatrix} z \\ 0 \end{bmatrix}, \overline{x} = \begin{bmatrix} x \\ y \end{bmatrix}, M = \begin{bmatrix} 0 & -A^T \\ A & 0 \end{bmatrix}$ and $q = \begin{bmatrix} c \\ -b \end{bmatrix}$,

1.6.3 Horizontal Linear Complementarity Problems

Given two square matrices $M, N \in \mathbb{R}^{n\times n}$ and a vector $q \in \mathbb{R}^n$, the horizontal linear complementarity problem (HLCP) consists in finding a pair $x, y \in \mathbb{R}^n$ such that

$$x \geq 0,\ y \geq 0,\ Ny - Mx = q,\ x^T y = 0. \tag{1.5}$$

We will denote

$$\mathcal{F}^* = \left\{(x, y) \in \mathbb{R}^{2n} : Ny = Mx + q,\ x \geq 0, y \geq 0\right\}.$$

The feasible set of points of the HLCP as defined in (1.5). The solution set of the HLCP is given by:

$$\text{Sol}(M, N, q) = \left\{(x, y) \in \mathcal{F}^* : x^T y = 0\right\}.$$

Remark 1.6.1. *If $N = I$ or N^{-1} exists, the HLCP reduced to the standard LCP.*

1.6.4 Classes of HLCP

In this section, we recall some important classes of HLCP. Let $M, N \in \mathbb{R}^{n\times n}$, $x, y \in \mathbb{R}^n$ and $\kappa \geq 0$.

- $\{M, N\}$ is called a monotone pair if

  $$-Mx + Ny = 0 \Rightarrow x^T y \geq 0.$$

  Or $[M, N]$ is full row rank (i.e., rank $[M, N] = n$).

- $\{M, N\}$ is called a P_0-pair if $-Mx + Ny = 0$ and $(x, y) \neq 0$

  $$(0 \neq x \in \mathbb{R}^n \text{ or } 0 \neq y \in \mathbb{R}^n)(\exists i \in I)(x_i y_i \geq 0).$$

- $\{M, N\}$ is called a $P_*(\kappa)$-pair if $-Mx + Ny = 0$ implies that

  $$(1 + 4\kappa) \sum_{i\in I^+(x)} x_i y_i + \sum_{i\in I^-(x)} x_i y_i \geq 0, \forall x, y \in \mathbb{R}^n,$$

  where

  $$I^+(x) = \{i : x_i y_i \geq 0\}, I^-(x) = \{i : x_i y_i < 0\},$$

  or

  $$x^T y \geq -4\kappa \sum_{i\in I^+(x)} x_i y_i, \forall x, y \in \mathbb{R}^n.$$

Example 1.6.2.

We consider the absolute value equation, which consists in finding $x \in \mathbb{R}^n$ such that

$$Ax - B|x| = b, \tag{1.6}$$

where $A,\ B \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ and $|x|$ denotes the vector in $\mathbb{R}^n$ with absolute values of components of x. Indeed, for $x \in \mathbb{R}^n$, we define $x^+ = \max(x, 0)$ and $x^- = \max(0, -x)$. Then it is easy to conclude that

$$x^+ \geq 0,\ x^- \geq 0,\ x = x^+ - x^-,\ |x| = x^+ + x^- \text{ and } x^{+T}x^- = 0.$$

Therefore, the AVE is equivalent to the following HLCP: find $x^+ \geq 0$ and $x^- \geq 0$ such that

$$\begin{cases} Nx^+ - Mx^- = q \\ x^{+T}x^- = 0, \end{cases}$$

with $N = A - B$, $M = A + B$ and $q = b$.

Example 1.6.3.

Let us consider the following convex quadratic program:

$$(\mathcal{P}) \quad \min_x \left[c^T x + \frac{1}{2}x^T Q x\right] :\ Ax \leq b,\ x \geq 0,$$

where Q is a symmetric semidefinite matrix in $\mathbb{R}^{n\times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m\times n}$ with rank $(A) = m$. Cottle [20] showed by invoking the K.K.T optimality conditions that x is an optimal solution of $\mathcal{P}$ if and only if there exists $y \in \mathbb{R}^m_+$, and $\lambda \in \mathbb{R}^n_+$ such that

$$\begin{cases} Qx + c + A^T y - \lambda = 0 \\ y^T(b - Ax) = 0, \\ \lambda^T x = 0 \end{cases} \Leftrightarrow \begin{cases} \lambda = Qx + c + A^T y \\ \mu = -Ax + 0y + b, \\ \lambda^T x + \mu^T y = 0,\ x, y, \lambda, \mu \geq 0. \end{cases}$$

Therefore the K.K.T optimality conditions are equivalent to the following monotone HLCP

$$Nw - Mz = q, w^T z = 0, w, z \geq 0$$

where

$$N = I,\ M = \begin{pmatrix} Q & A^T \\ -A & 0 \end{pmatrix}, q = \begin{pmatrix} c \\ b \end{pmatrix}, w = \begin{pmatrix} \lambda \\ \mu \end{pmatrix},\ z = \begin{pmatrix} x \\ y \end{pmatrix}.$$

1.6.5 Semidefinite Linear Complementarity Problem

Let $\mathbb{S}^n$ denotes the linear space of all $n \times n$ real symmetric matrices and $\mathbb{S}^n_+ \subset \mathbb{S}^n$ be the closed convex cone of symmetric positive semidefinite matrices. Given a linear transformation $\mathcal{L} : \mathbb{S}^n \to \mathbb{S}^n$ and a symmetric matrix $Q \in \mathbb{S}^n$, the semidefinite linear complementarity problem (SDLCP), is the problem of finding a pair of matrices $(X, Y) \in \mathbb{S}^n \times \mathbb{S}^n$ such that

$$X, Y \in \mathbb{S}^n_+, \;\; Y = \mathcal{L}(X) + Q, \;\; X \bullet Y = 0, \tag{1.7}$$

where $X \bullet Y$ denotes the trace of the matrix product XY.

1.6.6 Classes of SDLCP

For a linear transformation $\mathcal{L} : \mathbb{S}^n \to \mathbb{S}^n$, we say that

- $\mathcal{L}$ has the R_0-property if the zero matrix is the only solution of SDLCP($\mathcal{L}$,0).
- $\mathcal{L}$ has the Q-property if for every $Q \in \mathbb{S}^n$, SDLCP($\mathcal{L}, Q$) has a solution.
- $\mathcal{L}$ has the P-property if

$$X \;\text{ and }\; \mathcal{L}(X) \;\text{ commute, i.e., } X\mathcal{L}(X) = \mathcal{L}(X)X, \;\; X\mathcal{L}(X) \preceq 0 \Rightarrow X = 0.$$

- $\mathcal{L}$ has the the strict semimonotone property if

$$X \succeq 0, \; X \;\text{ and }\; \mathcal{L}(X) \;\text{ commute, }\; X\mathcal{L}(X) \preceq 0 \Rightarrow X = 0.$$

- $\mathcal{L}$ is strictly monotone (monotone) if

$$\langle \mathcal{L}(X), X \rangle > 0 (\geq 0), \;\; \text{for all } \; X \neq 0.$$

Definition 1.6.1. *A linear transformation $\mathcal{L} : \mathbb{S}^n \to \mathbb{S}^n$ is said to be nondegenerate if*

$$X\mathcal{L}(X) = 0 \Rightarrow X = 0.$$

1.6.7 Some results of the existence and uniqueness of solution of monotone SDLCP

Lemma 1.6.1 ([9, 29])**.** *The SDLCP has solutions if the transformation $\mathcal{L}$ is monotone and the set of feasibility is non empty. In addition, if $\mathcal{L}$ is strictly monotone, then the SDLCP has a unique solution for each $Q \in \mathbb{S}^n$.*

Remark 1.6.2. *These results are also valid for HLCPs.*

Example 1.6.4.

The standard linear complementarity problem is a special case of the semidefinite linear complementarity problems, let $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$, as in (1.4), and define the following transformation $\mathcal{L} : \mathbb{S}^n \to \mathbb{S}^n$ by

$$\mathcal{L}(X) = \text{Diag}(M\text{diag}(X)),$$

where diag(X) is a vector whose entries are the diagonal entries of the matrix X, and Diag(x) is a diagonal matrix whose diagonal is the vector x.
The semidefinite linear complementarity problem SDLCP($\mathcal{L}$,Diag(q)) corresponding to the standard linear complementarity problem LCP(M,q) is define as follows: Find $X \in \mathbb{S}^n$ such that

$$X \in \mathbb{S}^n_+, \;\; Y = \mathcal{L}(X) + \text{Diag}(q) \in \mathbb{S}^n_+, \;\; \text{and } X \bullet Y = 0,$$

if X is the solution of SDLCP($\mathcal{L}$,Diag(q)), then diag(X) is the solution of LCP(M, q). Conversely, if x is the solution of LCP(M, q), then Diag(x) is the solution of SDLCP($\mathcal{L}$,Diag(q)).

Example 1.6.5.

A pair of primal-dual semidefinite optimization SDO programs is defined as:

$$\begin{aligned} &\min_X C \bullet X \\ &\text{s.t. } \langle A_i, X\rangle = b_i, \; i = 1, 2, \ldots, m \qquad \text{(P)} \\ &X \succeq 0, \end{aligned}$$

and its dual

$$\begin{aligned} &\max_{(y,Z)} b^T y \\ &\text{s.t. } \sum_{i=1}^m y_i A_i + Z = C, \qquad \text{(D)} \\ &Z \succeq 0, \end{aligned}$$

with $A_i, C \in \mathbb{S}^n$, $b, y \in \mathbb{R}^m$ and $X, Z \in \mathbb{S}^n_+$. It is well-known, that finding an optimal solution X of P and an optimal solution (y, Z) of D is equivalent to solving the following implicit SDLCP problem characterized by the following optimality conditions

$$\begin{aligned} & A_i \bullet X = b_i,\ i = 1, 2, \ldots, m, \quad && X \succeq 0, \\ & \sum_{i=1}^{m} y_i A_i + Z = C, && Z \succeq 0, \\ & XZ = 0. \end{aligned}$$

Applications

Example 1.6.6.

Consider the symmetric semidefinite constrained least squares problem (SDLS) which is defined as the following convex optimization problem:

$$\begin{aligned} \min_X \ & \frac{1}{2}\|AX - B\|_F^2, \\ \text{s.t.} \quad & X \succeq 0, \end{aligned}$$

where $A, B \in \mathbb{R}^{m \times n}$ with $m \geq n$. It is shown [35], that if rank $(A) = n$, the SDLS has a unique solution $\mathbf{X}^* \succeq 0$. The optimality conditions for SDLS are equivalent to the following monotone SDLCP with $\mathcal{L} : \mathbb{S}^n \to \mathbb{S}^n$ and $Q \in \mathbb{S}^n$ are given by:

$$\mathcal{L}(X) = \frac{1}{2}(A^T AX + XA^T A) \text{ and } Q = -\frac{1}{2}(A^T B + B^T A).$$

Here $\mathcal{L}(X)$ is a Lyapunov linear transformation, with $A^T A$ is symmetric positive definite matrix if rank $(A) = n$. Next Lemma shows the strict monotony of it.

Lemma 1.6.2. *[35] The transformation $\mathcal{L}(X) = \frac{1}{2}(A^T AX + XA^T A)$ linked to the SDLS is a strictly monotone transformation.*

Proof. Let $X \in \mathbb{S}^n$, we have

$$\begin{aligned} \langle \mathcal{L}(X), X \rangle &= \frac{1}{2}\langle A^T AX, X \rangle + \frac{1}{2}\langle XA^T A, X \rangle \\ &= \frac{1}{2}\mathbf{Tr}((A^T AX)^T X) + \frac{1}{2}\mathbf{Tr}((XA^T A)^T X) \\ &= \mathbf{Tr}((AX)^T AX) = \|AX\|_2^2 > 0, \text{ since } rank(A) = n.\ \forall X \neq 0. \quad \square \end{aligned}$$

Lemma 1.6.3 (Theorem 3.5 in [17])**.** *If A is symmetric positive definite matrix then the two-sided linear transformation $\mathcal{M}(X) = AXA^T$ is strictly monotone and consequently the corresponding SDLCP has a unique solution for every $Q \in \mathbb{S}^n$.*

Chapter 2

A full-Newton step IP algorithm for HLCP

Contents

In this chapter, we present an Interior-Point Method (IPM) for monotone HLCP, we study first the central-path of HLCP and the search directions, then we state the generic full-Newton step feasible interior-point algorithm for HLCP in section 2.1. The details of the analysis of the proposed algorithm will be discussed in section 2.2. Finally, some numerical results are provided to show the efficiency of the proposed algorithm and to compare with an available method.

We recall that the horizontal linear complementarity problem (HLCP) consists in finding

a pair $x, y \in \mathbb{R}^n$ such that

$$x \geq 0,\ y \geq 0,\ Ny - Mx = q,\ x^T y = 0. \tag{2.1}$$

Such that $M, N \in \mathbb{R}^{n\times n}$ are a given two square matrices, and $q \in \mathbb{R}^n$ is a vector.

2.1 Central-path for HLCP

Throughout this chapter, we assume that (2.1) satisfies the following assumptions [18, 59].

- (Interior-Point-Condition (IPC)). There exists a pair of vectors (x^0, y^0) such that

$$Nx^0 - My^0 = q,\ y^0 > 0,\ x^0 > 0.$$

- (Monotonicity of (2.1)). The pair of matrices $[N,\ M]$ satisfies

$$Nu - Mv = 0 \Rightarrow u^T v \geq 0,\ \text{for any } u, v \in \mathbb{R}^n.$$

These two assumptions imply the existence of a solution for HLCP (see [60]). Finding an approximate solution of HLCP is equivalent to solving the following system:

$$\begin{cases} Ny - Mx = q, \\ xy = 0,\ x \geq 0,\ y \geq 0. \end{cases} \tag{2.2}$$

The basic idea of the path-following interior-point algorithm is to replace the second equation in (2.2), the so-called the complementarity condition for (2.1), by the parameterized equation $xy = \mu e$ with $\mu > 0$. Thus one may consider the following system:

$$\begin{cases} Ny - Mx = q, \\ xy = \mu e. \end{cases} \tag{2.3}$$

The system (2.3) has a unique solution denoted by $(x(\mu), y(\mu))$ for each $\mu > 0$ [53]. Then we call $(x(\mu), y(\mu))$ the $\mu-$center of HLCP. The set of $\mu-$center (with μ running through all positive real numbers) is called the central-path of HLCP. If $\mu \to 0$, then the limit of the central-path exists and since the limit point satisfy the complementarity condition $xy = 0$, the limit yields a solution for HLCP [18].

2.1.1 Search directions for HLCP

Now, we want to define search directions $(\Delta x, \Delta y)$ that move in the direction of the μ-centers $(y(\mu), x(\mu))$. Applying Newton's method for (2.3) for a given strictly feasible point (x, y), i.e., the IPC holds, we get the following linear system:

$$\begin{cases} N\Delta y - M\Delta x = 0, \\ X\Delta y + Y\Delta x = \mu e - xy, \end{cases} \tag{2.4}$$

where X= Diag (x) and Y= Diag (y). The unique solution $(\Delta x,\ \Delta y)$ of (2.4) is guaranteed by our assumptions since the bloc matrix

$$\begin{pmatrix} -M & N \\ Y & X \end{pmatrix}$$

is nonsingular (cf Proposition 3.1 in [58]). Therefore, the new iterate is obtained by taking a full-Newton step according to:

$$x_+ := x + \Delta x,\ y_+ := y + \Delta y.$$

Denote

$$v = \sqrt{\frac{xy}{\mu}}, \tag{2.5}$$

and

$$d_x = \frac{v\Delta x}{x},\ d_y = \frac{v\Delta y}{y}. \tag{2.6}$$

One can easily check that

$$\mu d_x d_y = \Delta x \Delta y,\ \text{and}\ x\Delta y + y\Delta x = \mu v(d_x + d_y). \tag{2.7}$$

Now due to (2.6) and (2.7), system (2.4) becomes

$$\begin{cases} \bar{N}d_y - \bar{M}d_x = 0, \\ d_x + d_y = p_v, \end{cases} \tag{2.8}$$

where $\bar{M} = MXV^{-1}$, $\bar{N} = NYV^{-1}$ with V:= Diag (v) and,

$$p_v = v^{-1} - v.$$

For the analysis of the algorithm, we use a norm-based proximity measure $\delta(v)$ defined by:

$$\delta := \delta(xy; \mu) = \frac{1}{2}\|p_v\|. \tag{2.9}$$

Clearly,

$$\delta(v) = 0 \Leftrightarrow v = e \Leftrightarrow xy = \mu e.$$

Hence, the value of $\delta(v)$ can be considered as a measure for the distance between the given pair (x, y) and the corresponding $\mu-$center $(x(\mu), y(\mu))$.

2.1.2 Algorithm for HLCP

The interior-point primal-dual algorithm for HLCP works as follows. First, we use a suitable threshold (default) value $\tau > 0$, with $0 < \tau < 1$ and we suppose that a strictly feasible initial point (x^0, y^0) exists such that $\delta(x^0 y^0; \mu_0) \leq \tau$, for certain μ_0 is known. Using the obtained search directions $(\Delta x, \Delta y)$ from (2.4) and taking a full-Newton step, the algorithm produces a new iterate $(x_+, y_+) = (x + \Delta x, y + \Delta y)$. Then, it updates the barrier parameter μ to $(1 - \theta)\mu$ with $0 < \theta < 1$ and solves the Newton system (2.4), and target a new μ-center and so on. This procedure is repeated until the stopping criterion $n\mu \leq \epsilon$ is satisfied for a given accuracy parameter ϵ. The generic feasible full-Newton step interior-point algorithm for HLCP is now presented in Fig 2.1, as follows.

The generic interior-point algorithm for HLCP

Input:

A threshold parameter $0 < \tau < 1$ (default $\tau = \frac{2}{\sqrt{10}}$);

an accuracy parameter $\epsilon > 0$;

a fixed barrier update parameter $0 < \theta < 1$ (default $\theta = (\frac{6}{23n})^{1/2}$);

a strictly feasible point (x^0, y^0) and $\mu_0 = \frac{1}{2}$ s.t. $\delta(x^0 y^0; \mu^0) \leq \tau$.

begin

$x := x^0$; $y := y^0$; $\mu := \mu_0$;

While $n\mu \geq \epsilon$ **do**

Solve system (2.4) to obtain $(\Delta x, \Delta y)$;

Update $x := x + \Delta x$; $y := y + \Delta y$;

$\mu := (1 - \theta)\mu$;

end while

end.

Figure 2.1: **Algorithm 2.1**

2.2 Complexity analysis of the algorithm

In this section, we will show across the new defaults θ and τ, described in Fig 2.1, that Algorithm 2.1, is well-defined and solves the HLCP in polynomial complexity.

2.2.1 Feasibility and locally quadratically convergence of the feasible full-Newton step

We first investigate the feasibility of a full-Newton step. Then we mainly prove that the iterate is locally quadratically convergent. We first quote the following technical lemma which will be used later.

Lemma 2.2.1. *Let $\delta > 0$ and $(d_x,\ d_y)$ be a solution of system (2.8). Then, we have*

$$0 \leq d_x^T d_y \leq 2\delta^2, \tag{2.10}$$

and

$$\|d_x d_y\|_\infty \leq \delta^2,\ \|d_x d_y\| \leq \sqrt{2}\delta^2. \tag{2.11}$$

Proof. For the first part of the first statement, let d_x and d_y be the unique solution of system (2.8), hence from the first equation of it, we have $\bar{N}d_y - \bar{M}d_x = 0$. Substitution $\bar{N} = NYV^{-1}$ and $\bar{M} = MXV^{-1}$, then $MXV^{-1}d_y - NYV^{-1}d_x = 0$. But since the pair $[N, M]$ is in the monotone HLCP, so by assumption 2, we conclude that

$$(XV^{-1}d_y)^T(YV^{-1}d_x) = d_x^T d_y \geq 0.$$

For the second part of it, it follows trivially from the following equality

$$4\delta^2 = \|p_v\|^2 = \|d_x + d_y\|^2 = \|d_x\|^2 + \|d_y\|^2 + 2d_x^T d_y \geq 2d_x^T d_y.$$

For the second statement, we have

$$d_x d_y = \frac{1}{4}((d_x + d_y)^2 - (d_x - d_y)^2),$$

and

$$\|d_x + d_y\|^2 = \|d_x - d_y\|^2 + 4d_x^T d_y.$$

But since $d_x^T d_y \geq 0$, it follows that

$$\|d_x - d_y\| \leq \|d_x + d_y\|.$$

On the other hand,

$$\begin{aligned}
\|d_x d_y\|_\infty &= \frac{1}{4}\|(d_x + d_y)^2 - (d_x - d_y)^2\|_\infty \\
&\leq \frac{1}{4}\max(\|d_x + d_y\|_\infty^2, \|d_x - d_y\|_\infty^2) \\
&\leq \frac{1}{4}\max(\|d_x + d_y\|^2, \|d_x - d_y\|^2) \\
&\leq \frac{1}{4}\|d_x + d_y\|^2 = \frac{1}{4}\|p_v\|^2 = \delta^2.
\end{aligned}$$

Therefore, $\|d_x d_y\|_\infty \leq \delta^2$. To prove the last part of it, we have,

$$\begin{aligned}
\|d_x d_y\|^2 &= e^T(d_x d_y)^2 = \frac{1}{16}e^T((d_x+d_y)^2-(d_x-d_y)^2)^2 \\
&= \frac{1}{16}\|(d_x+d_y)^2-(d_x-d_y)^2\|^2 \\
&\leq \frac{1}{16}(\|(d_x+d_y)^2\|^2+\|(d_x-d_y)^2\|^2) \\
&\leq \frac{1}{16}(\|d_x+d_y\|^4+\|d_x-d_y\|^4) \\
&\leq \frac{1}{8}\|d_x+d_y\|^4 = \frac{1}{8}\|p_v\|^4 = 2\delta^4.
\end{aligned}$$

Hence, $\|d_x d_y\| \leq \sqrt{2}\delta^2$. This completes the proof. □

Lemma 2.2.2. *The full-Newton step is positive if and only if* $e + d_x d_y > 0$.

Proof. We have,

$$x_+ y_+ = (x+\Delta x)(y+\Delta y) = xy + x\Delta y + y\Delta x + \Delta x \Delta y.$$

Using (2.6) and (2.7), we get

$$x_+ y_+ = \mu(e + d_x d_y).$$

If $x_+ > 0$ and $y_+ > 0$ then $x_+ y_+ > 0$ and so $e + d_x d_y > 0$. Conversely, let $0 \leq \alpha \leq 1$ and define $x(\alpha) := x + \alpha\Delta(x)$, $y(\alpha) := y + \alpha\Delta(y)$, we have

$$x(\alpha)y(\alpha) = xy + \alpha(\mu e - xy) + \alpha^2 \Delta x \Delta y.$$

If $e + d_x d_y > 0$ then $\mu e + \Delta x \Delta y > 0$ and $\Delta x \Delta y > -\mu e$. Hence

$$x(\alpha)y(\alpha) > (1-\alpha)xy + (\alpha - \alpha^2)\mu e \geq 0.$$

So $x(\alpha)y(\alpha) > 0$ for each $0 \leq \alpha \leq 1$. Since $x(\alpha)$ and $y(\alpha)$ are linear functions of α and $x(0) = x > 0$ and $y(0) = y > 0$, then $x(1) = x_+ > 0$ and $y(1) = y_+ > 0$. This completes the proof. □

For convenience, we may write

$$v_+^2 = \frac{x_+ y_+}{\mu}.$$

It is easy to deduce that

$$v_+^2 = e + d_x d_y \Leftrightarrow x_+ y_+ = \mu(e + d_x d_y). \tag{2.12}$$

Lemma 2.2.3. *If $\delta < 1$, then $x_+ > 0$ and $y_+ > 0$. Thus concludes that $x_+ > 0$ and $y_+ > 0$ are strictly feasible for (2.1).*

Proof. From Lemma 2.2.2, we have seen that $x_+ > 0, y_+ > 0$ if and only if $e + d_x d_y > 0$. Because,

$$1 + (d_x d_y)_i \geq 1 - |(d_x d_y)_i| \geq 1 - \|d_x d_y\|_\infty \text{ for all } i,$$

so, by (2.11), it follows that $1 + (d_x d_y)_i \geq 1 - \delta^2$. Thus $e + d_x d_y > 0$ if $\delta < 1$. This completes the proof. □

The next lemma shows the influence of the full-Newton step on the proximity measure.

Lemma 2.2.4. *If $\delta < 1$. Then*

$$\delta_+ := \delta(v_+; \mu) \leq \frac{\delta^2}{\sqrt{2(1-\delta^2}}.$$

Proof. We have

$$2\delta_+ = \|v_+ - v_+^{-1}\|.$$

Due to (2.12), we have $v_+ = \sqrt{e + d_x d_y}$ and $v_+^{-1} = \frac{e}{\sqrt{e+d_x d_y}}$. Thus implies that

$$2\delta_+ = \left\| \frac{d_x d_y}{\sqrt{e + d_x d_y}} \right\| \leq \frac{\|d_x d_y\|}{\sqrt{1 - \|d_x d_y\|_\infty}}.$$

It follows from (2.11) that $\delta_+ \leq \frac{\delta^2}{\sqrt{2(1-\delta^2)}}$. This completes the proof. □

Corollary 2.2.1. *Let $\delta \leq \frac{2}{\sqrt{10}}$, thus $\delta_+ \leq \delta^2$ which means the locally quadratically convergence of the full-Newton step to the central-path.*

2.2.2 Updating the barrier parameter

The following lemma gives an upper bound for the duality gap after a full-Newton step.

Lemma 2.2.5. *Let $\delta \leq \frac{2}{\sqrt{10}}$. Then after a full-Newton step the duality gap for the pair (x_+, y_+) satisfies*

$$x_+^T y_+ \leq 2\mu n. \tag{2.13}$$

Proof. Using (2.12), we have

$$x_+^T y_+ = \mu e^T (e + d_x d_y) = \mu(n + d_x^T d_y).$$

Using (2.10), it follows that

$$x_+^T y_+ \leq \mu(n + 2\delta^2).$$

Now, let $\delta \leq \frac{2}{\sqrt{10}}$, then

$$x_+^T y_+ \leq \mu\left(n + \frac{4}{5}\right).$$

But since for all $n \geq 1$, $n + \frac{4}{5} \leq 2n$, this gives the required result. □

In the next lemma, we investigate the effect on the proximity of a full-Newton step followed by an update of the barrier parameter μ.

Lemma 2.2.6. *Let $\delta \leq \frac{2}{\sqrt{10}}$ and $\mu_+ = (1-\theta)\mu$, where $0 \leq \theta \leq 1$. Then*

$$\delta^2(x_+y_+;\mu_+) \leq \frac{2}{15} + \frac{\theta^2(n+\frac{4}{5})}{4(1-\theta)} + \frac{4\theta}{15}.$$

In addition, if $\delta \leq \frac{2}{\sqrt{10}}$, $\theta = (\frac{6}{23n})^{1/2}$ and $n \geq 2$, then $\delta(x_+y_+;\mu_+) \leq \frac{2}{\sqrt{10}}$.

Proof. We have,

$$\begin{aligned}
4\delta^2(x_+y_+;\mu_+) &= \left\| \sqrt{1-\theta} v_+^{-1} - \frac{1}{\sqrt{1-\theta}} v_+ \right\|^2 \\
&= \left\| \sqrt{1-\theta}(v_+^{-1} - v_+) - \frac{\theta}{\sqrt{1-\theta}} v_+ \right\|^2 \\
&= (1-\theta)\|v_+^{-1} - v_+\|^2 + \frac{\theta^2}{1-\theta}\|v_+\|^2 - 2\theta(v_+^{-1} - v_+)^T v_+ \\
&= (1-\theta)\|v_+^{-1} - v_+\|^2 + \frac{\theta^2}{1-\theta}\|v_+\|^2 - 2\theta(v_+^{-1})^T v_+ + 2\theta(v_+)^T v_+.
\end{aligned}$$

As $(v_+^{-1})^T v_+ = n$ and (Lemma 2.2.5)

$$(v_+)^T v_+ = \|v_+\|^2 = \frac{1}{\mu} x_+^T y_+ \leq \left(n + \frac{4}{5}\right),$$

it follows that

$$\delta^2(x_+y_+;\mu_+) \leq (1-\theta)\delta_+^2 + \frac{\theta^2(n+\frac{4}{5})}{4(1-\theta)} + \frac{2\theta}{5}.$$

Let $\delta \leq \frac{2}{\sqrt{10}}$, then $\delta_+^2 \leq \frac{2}{15}$, and we deduce that

$$\delta^2(x_+y_+;\mu_+) \leq \frac{2(1-\theta)}{15} + \frac{\theta^2(n+\frac{4}{5})}{4(1-\theta)} + \frac{2\theta}{5}.$$

Now, taking $\theta = (\frac{6}{23n})^{1/2}$ then $\theta^2 = \frac{6}{23n}$, hence, we get

$$\delta^2(x_+y_+;\mu_+) \leq \frac{2(1-\theta)}{15} + \frac{\frac{6}{23n}(n+\frac{4}{5})}{4(1-\theta)} + \frac{2\theta}{5}.$$

Now, as $\frac{6(n+\frac{4}{5})}{23n} \leq \frac{42}{115}$ for all $n \geq 2$, then

$$\delta^2(x_+y_+;\mu_+) \leq \frac{2}{15} + \frac{21}{230(1-\theta)} + \frac{4\theta}{15}.$$

Again, for $n \geq 2$, we have $\theta \in [0, (\frac{3}{23})^{1/2}]$. Letting

$$f(\theta) = \frac{2}{15} + \frac{21}{230(1-\theta)} + \frac{4\theta}{15}.$$

The function $f(\theta)$ is continuous and monotone increasing on $[0, (\frac{3}{23})^{1/2}]$. Therefore,

$$f(\theta) \leq f\left(\left(\frac{3}{23}\right)^{1/2}\right) = 0.37256 < \frac{4}{10}, \text{ for all } \theta \in \left[0, \left(\frac{3}{23}\right)^{1/2}\right].$$

Then, after the barrier parameter is update to $\mu_+ = (1-\theta)\mu$ with $\theta = (\frac{6}{23n})^{1/2}$ and if $\delta \leq \frac{2}{\sqrt{10}}$, then

$$\delta(x_+y_+;\mu_+) \leq \frac{2}{\sqrt{10}}.$$

This completes the proof. □

From Lemma 2.2.6, we deduce that for the defaults $\tau = \frac{2}{\sqrt{10}}$ and $\theta = (\frac{6}{23n})^{1/2}$, Algorithm 2.1 is well-defined since the conditions $x > 0$, $y > 0$ and $\delta(x_+y_+;\mu_+) \leq \frac{2}{\sqrt{10}}$ are maintained during the solution process.

2.2.3 Iteration bound

We conclude this section with a theorem that gives us the iteration bound of Algorithm 2.1. Before doing this we apply the results obtained in the previous subsections and get the following lemma.

Lemma 2.2.7. *Assume that x^0 and y^0 are strictly feasible starting point such that $\delta(x^0y^0;\mu_0) \leq \frac{2}{\sqrt{10}}$ for certain $\mu_0 > 0$. Moreover, let x^k and y^k be the iterate produced by Algorithm 2.1, after k iterations. Then the inequality $(x^k)^T y^k \leq \epsilon$ is satisfied for*

$$k \geq \frac{1}{\theta} \log\left(\frac{2n\mu_0}{\epsilon}\right).$$

Proof. From (2.13), it follows that

$$(x^k)^T y^k \leq 2n\mu_k = 2n(1-\theta)^k\mu_0.$$

Then the inequality $(x^k)^T y^k \leq \epsilon$ holds if

$$2n(1-\theta)^k \mu_0 \leq \epsilon.$$

Taking logarithms, we obtain

$$k \log(1-\theta) \leq \log \epsilon - \log(2n\mu_0)$$

and using $-\log(1-\theta) \geq \theta$ for $0 \leq \theta \leq 1$, then we observe that the above inequality holds if

$$k\theta \geq \log \epsilon - \log(2n\mu_0) = \log\left(\frac{2n\mu_0}{\epsilon}\right).$$

This implies the lemma. □

Theorem 2.2.1. *If $\theta = (\frac{6}{23n})^{1/2}$ and $\mu_0 = \frac{1}{2}$, then Algorithm 2.1 requires at most*

$$\mathcal{O}\left(\sqrt{n} \log \frac{n}{\epsilon}\right)$$

iterations for getting an ϵ-approximate solution of (2.1).

Proof. Let $\theta = (\frac{6}{23n})^{1/2}$ and $\mu_0 = \frac{1}{2}$, by using Lemma 2.2.7, the proof is straightforward. □

2.3 Numerical results

In this section, we test Algorithm 2.1, on some monotone HLCPs which are reformulated from three interesting problems, namely the absolute value equation (AVE) (see, e.g.,[30, 39, 44]), convex quadratic optimization programs and the monotone standard LCP, respectively.

Recall that the AVE (Example 1.6.2) is defined as:

$$Ax - B|x| = b, \tag{2.14}$$

Therefore, the AVE is equivalent to the following HLCP: find $x^+ \geq 0$ and $x^- \geq 0$ such that

$$\begin{cases} Nx^+ - Mx^- = q \\ x^{+T} x^- = 0, \end{cases} \tag{2.15}$$

with $N = A - B$, $M = A + B$ and $q = b$. It shown under the condition that $\sigma_{\min}(A) > \sigma_{\max}(B)$ (cf. Proposition 2.3 in [30]), that the HCLP in (2.15), is reduced to a P-matrix standard LCP and therefore the HLCP has a unique solution (x^+, x^-) for every b (cf. Theorem 3.3.7 in [21]). Hence, the AVE has $x = x^+ - x^-$ as the unique solution. Now, for the implementation of Algorithm 2.1, our accuracy is set to $\epsilon = 10^{-6}$ and we use different values of the barrier parameters μ_0 and θ in order to improve its performances. The Algorithm 2.1 is implemented on software MATLAB 7.9 and run on a PC with CPU 2.67 GHz and 4G RAM memory and double precision format. Now, four problems of monotone HLCP are constructed from two randomly AVE problems and from a convex quadratic program and a monotone standard LCP. To this end we compare our Algorithm 2.1 with an iterative fixed point method.

Example1. Consider the AVE problem of type (2.14), where A, B and b are given by

$$A = \begin{pmatrix} 8 & 0 & -1 & 1 & -20 \\ 1 & 1 & 1 & 4 & 25 \\ 1 & -5 & 0 & 8 & -10 \\ 0 & 8 & 1 & -6 & 1 \\ 3 & 5 & -3 & 0 & 10 \end{pmatrix}, B = \begin{pmatrix} -1.5 & 0 & 1.5 & 0.5 & 0.1 \\ 0 & 0.25 & 1 & 0 & 0.5 \\ 1 & 0.6 & 1 & 0.4 & 0.5 \\ 0 & 0.3 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}, b = e \in \mathbb{R}^5.$$

One can easily check that the AVE in problem 1, has a unique solution since $\sigma_{\min}(A) = 2.8215 > \sigma_{\max}(B) = 2.7434$. Therefore the corresponding HLCP in (2.15) is strictly monotone and its data is given by

$$N = \begin{pmatrix} 9.5 & 0 & -2.5 & 0.5 & -20.1 \\ 1 & 0.75 & 0 & 4 & 24.5 \\ 0 & -5.6 & -1 & 7.6 & -10.5 \\ 0 & 7.7 & 0 & -7 & 1 \\ 2 & 5 & -4 & 0 & 10 \end{pmatrix}, \; M = \begin{pmatrix} 6.5 & 0 & 0.5 & 1.5 & -19.9 \\ 1 & 1.25 & 2 & 4 & 25.5 \\ 2 & -4.4 & 1 & 8.4 & -9.5 \\ 0 & 8.3 & 2 & -5 & 1 \\ 4 & 5 & -2 & 0 & 10 \end{pmatrix},$$

and $q = b$. The strictly feasible starting point for Algorithm 2.1, is

$$x_0^- = (2.6677,\ 0.4111,\ 1.3168,\ 0.3506,\ 1.6744)^T,$$

and

$$x_0^+ = (1.3825,\ 4.9548,\ 2.7173,\ 4.6145,\ 1.1166)^T.$$

The unique solution of the corresponding HLCP is

$$x^- = (0,\ 0,\ 0,\ 0,\ 0.0735)^T,$$

and

$$x^+ = (0.0286,\ 0.6808,\ 0.4270,\ 0.5953,\ 0)^T.$$

Therefore, the unique solution of the AVE is given by

$$x = (0.0285,\ 0.6808,\ 0.4270,\ 0.5953,\ -0.0753)^T.$$

The numerical results with different theoretical and relaxed barrier values of μ_0 and θ are shown in table 2.1.

$\mu_0 \rightarrow$	0.5		0.05		0.005		0.0005		0.00005	
$\theta \downarrow$	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
$(\frac{6}{23n})^{1/2}$	51	0.011	42	0.0149	33	0.0194	24	0.0079	16	0.0087
$\frac{1}{2\sqrt{n}}$	52	0.0091	43	0.0112	34	0.0095	25	0.0110	16	0.0136

Table 2.1: The numerical results, after the algorithm reaches $n\mu \leq 10^{-6}$

Example 2. The matrices $A,\ B \in \mathbb{R}^{n\times n}$, and the vector $b \in \mathbb{R}^n$ of the AVE problem are given by

$$A = \begin{pmatrix} 6 & 0.5 & 0.5 & \dots & 0.5 & 0 \\ 0.5 & 6 & 0.5 & \dots & 0.5 & 0 \\ 0.5 & 0.5 & 6 & \cdots & 0.5 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0.5 & 0 \\ 0.5 & 0.5 & 0.5 & \dots & 6 & 0 \\ 0 & 0 & \dots & 0 & 0 & 6 \end{pmatrix}$$

and

$$B = \begin{pmatrix} -1 & 0.5 & 0.5 & \dots & 0.5 & 0 \\ 0.5 & -1 & 0.5 & \dots & 0.5 & 0 \\ 0.5 & 0.5 & -1 & \cdots & 0.5 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0.5 & 0 \\ 0.5 & 0.5 & 0.5 & \dots & -1 & 0 \\ 0 & 0 & \dots & 0 & 0 & -1 \end{pmatrix}$$

and

$$b = (21,\ 28, \ldots, 28,\ 21)^T.$$

The numerical results with different size of n are summarized in tables Table 2.2 and Table 2.3. For the initialization of the corresponding HLCP, we take

$$x_0^- = \frac{1}{2}e \text{ and } x_0^+ = (3.4286,\ 4.5, \ldots, 4.5,\ 3.4286)^T.$$

The unique solution of HLCP is given by

$$x^+ = (3,\ 4, \ldots,\ 4,\ 3)^T \text{ and } x^- = (0, \ldots, 0)^T.$$

Therefore the unique solution of the AVE is

$$x = (3,\ 4, \ldots, 4,\ 3)^T.$$

$\mu_0 \rightarrow$	0.5		0.05		0.005		0.0005		0.00005	
Size $n \downarrow$	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
6	57	0.0284	47	0.0265	37	0.0271	27	0.0281	17	0.0269
12	83	0.0346	68	0.0283	54	0.0259	39	0.0265	25	0.0206
18	103	0.0452	85	0.0407	67	0.0312	49	0.0372	31	0.0265
24	120	0.0381	99	0.0340	78	0.0349	57	0.0368	36	0.0318
50	176	0.1385	145	0.1169	114	0.0734	83	0.0794	53	0.0510
100	251	0.7036	207	0.5885	163	0.4844	119	0.3945	75	0.3019
200	357	10.2619	295	9.4354	232	7.7834	169	5.7430	107	3.5279
1100	846	3549.9	698	2148	549	1511.3	401	1635	253	771.8659

Table 2.2: Numerical results with $\theta = (\frac{6}{23n})^{1/2}$, after the algorithm reaches $n\mu \leq 10^{-6}$.

$\mu_0 \rightarrow$	0.5		0.05		0.005		0.0005		0.00005	
Size $n \downarrow$	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
6	58	0.0292	48	0.0193	38	0.0256	28	0.0216	18	0.0316
12	85	0.0379	70	0.033	55	0.0307	40	0.0277	26	0.0199
18	105	0.0505	87	0.0321	68	0.0468	50	0.033	32	0.0338
24	122	0.0349	101	0.0354	80	0.0552	58	0.0436	37	0.0212
50	179	0.1258	148	0.1109	117	0.0964	85	0.0825	54	0.0604
100	256	0.7376	211	0.6271	167	0.5164	122	0.4129	77	0.3067
200	365	10.5878	301	9.5922	237	7.9824	173	5.8425	109	3.6946
1100	864	4170	713	2149.9	561	1552.9	410	1684.3	258	828.3743

Table 2.3: Numerical results with $\theta = \frac{1}{2\sqrt{n}}$, after the algorithm reaches $n\mu \leq 10^{-6}$

Next example is constructed from a convex quadratic program.

Example 3. Let us consider the convex quadratic program $\mathcal{P}$ define in Example 1.6.3, with the following data:

$$A = \begin{pmatrix} 3 & 4 & -2 \\ -3 & 2 & 1 \end{pmatrix},\ Q = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix},\ b = (10,\ 2)^T,\ c = (1,\ -2,\ 4)^T.$$

Therefore the data of the corresponding monotone HLCP is given by

$$M = \begin{pmatrix} 2 & 1 & 0 & 3 & -3 \\ 1 & 4 & 0 & 4 & 2 \\ 0 & 0 & 6 & -2 & 1 \\ -3 & -4 & 2 & 0 & 0 \\ 3 & -2 & -1 & 0 & 0 \end{pmatrix},\ N = I, q = (1,\ -2,\ 4,\ 10,\ 2)^T.$$

The starting point for Algorithm 2.1, is given by

$$z^0 = e \in \mathbb{R}^5 \text{ and } w^0 = (4,\ 9,\ 9,\ 5,\ 2)^T.$$

The unique solution of the HLCP is given by

$$z^* = (0,\ 0.5,\ 0,\ 0,\ 0)^T$$

and

$$w^* = (1.5,\ 0,\ 4,\ 8,\ 1)^T.$$

So the unique minimum x^* of $\mathcal{P}$ is attained at the point

$$x^* = (0,\ 0.5,\ 0)^T.$$

The numerical results with different theoretical and relaxed barrier values of μ_0 and θ are shown in table 2.4.

$\mu_0 \rightarrow$	0.5		0.05		0.005		0.0005		0.00005	
$\theta \downarrow$	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
$(\frac{6}{23n})^{1/2}$	51	0.0046	42	0.0052	33	0.0042	24	0.0049	16	0.0036
$\frac{1}{2\sqrt{n}}$	52	0.0051	43	0.0057	34	0.0044	25	0.0050	16	0.0037

Table 2.4: The numerical results, after the algorithm reaches $n\mu \leq 10^{-6}$

In the next example, we compare the performance of Algorithm 2.1 with an available non-interior-point method, namely, an iterative fixed point algorithm developed earlier by Yong [56]. In fact, if we take $y = |z| - z \geq 0$ and $x = |z| + z \geq 0$ in (2.1), then the HLCP can be equivalently transformed into a system of fixed point equation

$$(N + M)z = (N - M)\,|z| - q. \tag{2.16}$$

Based on (2.16), and assume that the matrix $(N + M)$ is nonsingular, then the iterative fixed point algorithm can be described below as follows.

Fixed Point Algorithm.

- Given an initial point $z^0 \in \mathbb{R}^n$;
- compute $z^{k+1} = (N + M)^{-1}(N - M)\left|z^k\right| - (N + M)^{-1}q$;
- set $x^{k+1} = z^{k+1} + \left|z^{k+1}\right|$; until the iteration sequence $\{x^k\}_{k\geq 0}$ is convergent.

Yong specialized in the HLCP with $N = I$ i.e., in monotone standard LCP, he proved under suitable conditions that the iterative fixed point method is globally convergent to a solution of HLCP. For more details the interested reader can consult the reference [56].

Example 4. [56] Let us consider the following monotone standard LCP with the following data:

$$M = \begin{pmatrix} 4 & -1 & 0 & \dots & 0 \\ -1 & 4 & -1 & \dots & \vdots \\ 0 & -1 & 4 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & -1 \\ 0 & \dots & 0 & -1 & 4 \end{pmatrix}, \; N = I, \; q = -e.$$

The starting point for Algorithm 2.1, is

$$x^0 = e \text{ and } y^0 = (2, 1, \dots, 1, 2)^T.$$

The unique solution of the HLCP is given by

$$x^* = (0.3660,\ 0.4641,\ 0.4904,\ 0.4974,\ 0.5,\ \dots,\ 0.5,\ 0.4974,\ 0.4904,\ 0.4641,\ 0.3660)^T$$

and

$$y^* = (0,\ \dots,\ 0)^T.$$

So the unique solution z^* obtained by the iterative fixed point method is

$$z^* = (0.1830,\ 0.2321,\ 0.2452,\ 0.2487,\ 0.25,\ \dots,\ 0.25,\ 0.2487,\ 0.2452,\ 0.2321,\ 0.1830)^T.$$

The numerical results obtained by these two algorithms are shown in table 2.5.

Algorithms $\rightarrow$	Fixed Point Algorithm		Algorithm 2.1	
Size $n \downarrow$	Iter	CPU	Iter	CPU
5	37	0.0401	16	0.0903
10	55	0.0215	23	0.0133
50	122	0.2030	53	0.1650
100	255	0.6410	75	0.2021
500	810	991.2144	170	348.6089

Table 2.5: The numerical results, after the algorithm reaches $n\mu \leq 10^{-6}$

Comments. Across the obtained numerical results stated in the above tables, we see that Algorithm 2.1offers with these new defaults a solution for monotone HLCPs. Moreover, Algorithm 2.1 obtains better numerical results with the relaxed parameter $\mu_0 = 0.00005$

and the update barrier $\theta = (\frac{6}{23n})^{1/2}$, since the number of iterations is significantly reduced. Also our obtained numerical results compete well with those obtained with the classical strategy of the update barrier parameter, namely, $\theta = \frac{1}{2\sqrt{n}}$ (see [7]).

Contributions and some remarks

We end Chapter I with some contributions and remarks.

- We contribute on suggesting two new suitable parameters, namely, the barrier θ and the threshold τ which lead to the convergence of the algorithm.
- The best polynomial complexity of the short-step algorithm is achieved with these two new parameters.
- If an initial point is available i.e., strictly feasible and close to the central-path, then the advantage of this algorithm is to generate a sequence that still strictly feasible and close to the central-path during the solution process.

Some practical difficulties of the proposed algorithm.

- It is mentioned that these methods suffer to obtain a strictly feasible starting point and close to central-path to initiate their associate algorithm. This because the used neighborhood of the central-path is narrow since the threshold $\tau < 1$).
- The update barrier parameter θ depends on the size n of the problem. If n is large enough this leads to a slow convergence of the algorithm.

Chapter 3

IP Methods for SDLCP based on a new kernel function

Contents

Kernel functions play an important role in defining new search directions for interior-point algorithms for solving monotone linear complementarity problems, for this reason we have studied primal-dual interior point algorithms for solving SDLCP.

Based on a new parametric kernel function, this chapter presents a primal-dual large-update interior-point algorithm (IPM) for semi-definite linear complementarity problems (SDLCPs).

The proposed kernel function is defined as follows:

$$\psi(t) = \frac{t^2-1}{2} + \frac{q^{\frac{1}{t}-1}-1}{q\log q} - \frac{(q-1)}{q}(t-1),\ t>0, \tag{3.1}$$

where $q > 1$ is the barrier parameter.

In this chapter, we will discuss the IPM method for solving a SDLCP. First, we will explain the concept of the IPM (see e.g. [8, 34, 40]), then we will analyze its convergence and finally we will give an estimate on the number of iterations needed to find an ϵ-approximate solution of SDLCP.

Recall that the semidefinite linear complementarity problem (SDLCP), is the problem of finding a pair matrices $(X, Y) \in \mathbb{S}^n \times \mathbb{S}^n$ such that

$$X, Y \in \mathbb{S}^n_+,\ \ Y = \mathcal{L}(X) + Q,\ \ X \bullet Y = 0, \tag{3.2}$$

such that $\mathcal{L} : \mathbb{S}^n \to \mathbb{S}^n$ is a given linear transformation, $Q \in \mathbb{S}^n$ is a symmetric matrix and $X \bullet Y$ denotes the trace of the matrix product XY.

3.1 Some results on matrices and matrix functions

Definition 3.1.1. *Let $V \in \mathbb{S}^n_+$ and*

$$V = Q^T diag(\lambda_1(V), \lambda_2(V), \ldots, \lambda_n(V))Q,$$

where Q is any orthonormal matrix ($Q^T = Q^{-1}$) that diagonalizes V. The matrix valued function $\psi(V)$ is defined by

$$\psi(V) = Q^T diag(\psi(\lambda_1(V)), \psi(\lambda_2(V)), \ldots, \psi(\lambda_n(V)))Q. \tag{3.3}$$

It should be noted that the matrix Q is not unique, but $\psi(V)$ is well defined whenever $\psi(t)$ is well defined on the eigenvalues of V.

Furthermore, replacing $\psi(\lambda_l(V))$ in (3.3) by $\psi'(\lambda_l(V))$, we can conclude that the matrix functions $\psi(V)$

$$\psi'(V) = Q^T diag(\psi'(\lambda_1(V)), \psi'(\lambda_2(V)), \ldots, \psi'(\lambda_n(V)))Q, \tag{3.4}$$

is defined as well.

Definition 3.1.2. *A matrix $X(t)$ is said to be a matrix of functions if each entry of $X(t)$ is a function of t, that is, $X(t) = [X_{ij}(t)]$. The concepts of continuity, differentiability and integrability naturally extended to matrix-valued functions of a scalar by interpreting them component wise. Thus we can say that*

$$\frac{d}{dt}X(t) = \frac{d}{dt}\left[X_{ij}(t)\right] = X'(t).$$

Suppose that the matrix-valued functions $H(t)$ and $G(t)$ are differentiable with respect to t. Then we have

$$\begin{aligned}\frac{d}{dt}(\mathbf{Tr}(G(t))) &= \mathbf{Tr}\left(\frac{d}{dt}G(t)\right) = \mathbf{Tr}(G'(t))\\ \frac{d}{dt}(G(t)H(t)) &= G'(t)H(t) + G(t)H'(t).\end{aligned}$$

For any function $\psi(t)$, let us denote by $\triangle\psi(t)$ the divided difference of $\psi(t)$:

$$\triangle\psi(t_1, t_2) = \frac{\psi(t_1) - \psi(t_2)}{t_1 - t_2},\ \forall t_1 \neq t_2,$$

if $t_1 = t_2$, we simply write $\triangle\psi(t_1, t_2) = \psi'(t)$.

Lemma 3.1.1 (Lemma 16 [43])**.** *Suppose that $H(t)$ is a matrix of functions such that the matrix $H(t)$ is positive definite with eigenvalues $\lambda_1(t) \geq \lambda_2(t) \geq \cdots \geq \lambda_n(t) > 0$. If $H(t)$ is twice differentiable with respect to $t \in (l_t, u_t)$ and $\psi(t)$ is twice continuously differentiable function in a domain that contains all the eigenvalues of $H(t)$, then*

$$\begin{aligned}\frac{d}{dt}\boldsymbol{Tr}(\psi(H(t))) &= \boldsymbol{Tr}(\psi'(H(t)H'(t)),\\ \frac{d^2}{dt^2}\boldsymbol{Tr}(\psi(H(t))) &\leq \omega\left\|H'(t)\right\|^2 + \boldsymbol{Tr}(\psi'(H(t)H''(t)),\end{aligned}$$

where

$$\omega = \max\left\{\left|\Delta\psi'(\lambda_j(t), \lambda_k(t))\right| : t \in (l_t, u_t), j, k = 1, 2, \ldots, n\right\}$$

is a number depending on $H'(t)$ and $\psi(t)$ with

$$\Delta\psi'(t_1, t_2) = \frac{\psi'(t_1) - \psi'(t_2)}{t_1 - t_2},\ \forall t_1, t_2 \in [l_t, u_t].$$

3.2 The central-path for SDLCP

Throughout this chapter, we assume that:

- Interior-Point-Condition (IPC). There exists a pair of matrices (X^0, Y^0) such that:

$$Y^0 - \mathcal{L}(X^0) = Q,\ X^0 \succ 0,\ Y^0 \succ 0.$$

- Self-adjoint and monotonicity.

$$\forall\, X, Y \in \mathbb{S}^n, \mathcal{L}(X) \bullet Y = X \bullet \mathcal{L}(Y),\ \mathcal{L}(X) \bullet X \geq 0.$$

Under these assumptions, it is known that (3.2) has at least one solution [29]. Since for $X, Y \in \mathbb{S}^n_+$, we have $\langle X, Y\rangle = 0$ if and only if $XY = 0$. Then, finding a solution of (3.2) is equivalent to solving the following system:

$$\begin{cases} Y - \mathcal{L}(X) = Q, \\ XY = 0, \quad X, Y \succeq 0. \end{cases} \tag{3.5}$$

The basic idea of primal-dual IPMs is to replace the second equation in (3.5) by the parameterized equation $XY = \mu I$, $\mu > 0$. Hence, we consider the following system:

$$\begin{cases} Y - \mathcal{L}(X) = Q, \\ XY = \mu I, \quad X, Y \succeq 0. \end{cases} \tag{3.6}$$

If IPC holds, then for each $\mu > 0$ system (3.6) has a unique solution. This solution, denoted by $(X(\mu), Y(\mu))$, is called the μ-center of SDLCP. The set of μ-centers gives the central-path of SDLCP. It has been shown by Kojima et al.[34] that the limit of the central-path exists as μ tends to zero and it is a solution of (3.2).

3.2.1 The search directions determined by kernel functions

The core idea of primal-dual IPMs is to follow the central-path approximately and to approach the solution of (3.2) as $\mu \to 0$. A direct application of Newton's method to (3.6), produces the following system for the search direction ΔX and ΔY:

$$\begin{cases} \Delta Y - \mathcal{L}(\Delta X) = 0 \\ \Delta XY + X\Delta Y = \mu I - XY, \end{cases} \tag{3.7}$$

or equivalently

$$\begin{cases} \Delta Y - \mathcal{L}(\Delta X) = 0 \\ \Delta X + X\Delta Y Y^{-1} = \mu Y^{-1} - X. \end{cases} \tag{3.8}$$

Clearly, ΔX is not symmetric due to the matrix $X\Delta YY^{-1}$. Therefore, a way to symmetrizing the second equation in (3.8) is to introduce an non-singular matrix P and to replace it by the equation $\Delta X + P\Delta Y P^T = \mu Y^{-1} - X$. Thus, we obtain

$$\begin{cases} \Delta Y - \mathcal{L}(\Delta X) = 0 \\ \Delta X + P\Delta Y P^T = \mu Y^{-1} - X. \end{cases} \tag{3.9}$$

Now, ΔX is symmetric and ΔY is automatically so. Among several symmetrization scheme, we use the Nesterov-Todd (NT) direction where the matrix P is defined as

$$P = X^{1/2}(X^{1/2}YX^{1/2})^{-1/2}X^{1/2} = Y^{-1/2}(Y^{1/2}XY^{1/2})^{1/2}Y^{-1/2}.$$

To simplify the analysis, we adopt the matrix $D = P^{1/2}$. Then D can be used to rescale X and Y to the same symmetric positive definite matrix V because

$$V := \frac{1}{\sqrt{\mu}}D^{-1}XD^{-1} = \frac{1}{\sqrt{\mu}}DYD. \tag{3.10}$$

Moreover, let us further define

$$D_X = \frac{1}{\sqrt{\mu}}D^{-1}\Delta XD^{-1},\ D_Y = \frac{1}{\sqrt{\mu}}D\Delta YD. \tag{3.11}$$

Now, due to (3.10) and using the scaled directions (3.11), system (3.9) can be rewritten as follows:

$$\begin{cases} \overline{\mathcal{L}}(D_X) = D_Y \\ D_X + D_Y = V^{-1} - V, \end{cases} \tag{3.12}$$

where

$$\overline{\mathcal{L}}(D_X) = \frac{1}{\sqrt{\mu}}D\mathcal{L}(\Delta X)D = D\mathcal{L}(DD_XD)D.$$

It is not difficult to verify that the right-hand side $V^{-1}-V$ in the second equation in (3.12) equals minus the derivative of the classical logarithmic barrier function

$$\begin{aligned} \Psi_c(V) &:= \mathbf{Tr}(\psi_c(V)) = \sum_{i=1}^{n}(\psi_c(\lambda_i(V))) \\ &= \sum_{i=1}^{n}\left(\frac{\lambda_i(V)^2-1}{2} - \log(\lambda_i(V))\right). \end{aligned} \tag{3.13}$$

That is to say

$$\nabla\Psi_c(V) = V^{-1} - V.$$

Moreover, we call $\psi_c(t)$ the kernel function of the logarithmic barrier function $\Psi_c(V)$. Thus, the system (3.12) becomes

$$\begin{cases} \overline{\mathcal{L}}(D_X) = D_Y \\ D_X + D_Y = -\nabla\Psi_c(V). \end{cases} \tag{3.14}$$

The approach in this chapter, is to replace the logarithmic barrier $\Psi_c(V)$ by a scaled barrier function $\Psi(V)$ defined $\Psi(V) = \sum_{i=1}^{n} \psi(\lambda_i(V))$, where ψ is defined in (3.1).
Now, we replace the right hand side of the second equation in (3.14) by $-\nabla\Psi(V)$. Thus the new search directions D_X and D_Y are obtained by solving the following system

$$\begin{cases} \overline{\mathcal{L}}(D_X) = D_Y \\ D_X + D_Y = -\nabla\Psi(V). \end{cases} \tag{3.15}$$

This system has a unique solution (D_X, D_Y) for each $\mu > 0$ which can be used to compute ΔX and ΔY via (3.11). If $(X, Y) \neq (X(\mu), Y(\mu))$, then $(\Delta X, \Delta Y)$ is non zero. By taking a step-size α along the search directions, we construct a new pair according to

$$X_+ := X + \alpha\Delta X,\ Y_+ := Y + \alpha\Delta Y.$$

Note that for the SDO case [3, 22], we have $\langle D_X, D_Y\rangle = 0$, i.e., D_X and D_Y are orthogonal. However, for the case of SDLCP this relation does not hold. Since $\mathcal{L}$ is a monotone transformation we have

$$\langle D_X, D_Y\rangle = \frac{1}{\mu}\langle \Delta X, \Delta Y\rangle = \frac{1}{\mu}\langle \Delta X, \mathcal{L}(\Delta X)\rangle \geq 0. \tag{3.16}$$

This is the only difference between the SDLCP problem and the SDO problem in their analysis.
In the algorithm, we use the barrier function $\Psi(V)$ as a measure function and also we introduce the norm-based proximity measure $\delta(V)$ to the central-path as follows:

$$\delta(X, Y; \mu) := \delta(V) = \frac{1}{2}\left\|\psi'(V)\right\| = \frac{1}{2}\sqrt{\sum_{i=1}^{n} \psi'(\lambda_i(V))^2}. \tag{3.17}$$

Clearly,

$$XY = \mu I \Leftrightarrow V = I \Leftrightarrow \psi'(V) = 0 \Leftrightarrow \Psi(V) = 0.$$

Hence, the value of $\delta(V)$ can be considered as a measure for the distance between the matrix pair (X, Y) and the central-path.

3.3 The generic primal-dual IPM for SDLCP

The general outline of the generic primal-dual kernel-function-based (IPMs) for SDLCP works as follows. We suppose that there exists a strictly feasible initial point (X^0, Y^0) in a τ-neighborhood i.e., $\Psi(V) \leq \tau$. Then the value of the barrier parameter μ is reduced by the factor $(1-\theta)\mu$ with some fixed $0 < \theta < 1$ which changes the value of V and obtains a new μ-center $(X(\mu), Y(\mu))$. Hence the value of $\Psi(V)$ exceeds the threshold τ i.e., $\Psi(V) \geq \tau$. Now, we start the inner iteration by solving the Newton system (3.15) and use (3.11), to obtain the unique search direction. In order to reduce the value of $\Psi(V)$. a step-size is introduced. If necessary, the procedure is repeated until we find the iterate that again belongs to the τ- neighborhood of the current μ-center and so on. At this point we start a new outer iteration by reducing the value of μ again. This procedure is repeated until the stopping criterion $n\mu \leq \epsilon$ is satisfied for a given accuracy parameter ϵ. The details of our generic IPM outlined above is summarized in Fig 3.1 as follows.

The generic interior-point algorithm for SDLCP

Input:

A threshold parameter default $\tau \geq 1$;

an accuracy parameter $\epsilon > 0$;

a fixed barrier update parameter θ, $0 < \theta < 1$;

a strictly feasible pair (X^0, Y^0) and $\mu_0 = 1$ s.t. $\Psi(X^0, Y^0; \mu_0) \leq \tau$.

begin

$X := X^0; Y := Y^0;\ \mu := \mu_0$;

while $n\mu \geq \epsilon$ **do**

$\mu := (1 - \theta)\mu$;

while $\Psi(V) > \tau$ **do**

solve system (3.15) and use (3.11) to obtain $\Delta X, \Delta Y$;

determine the default step-size α;

update $X := X + \alpha\Delta X, Y := Y + \alpha\Delta Y$;

end while

end while

end

Figure 3.1: **Algorithm 3.1**

We will prove that Algorithm 3.1 is well-defined and the currently best known iteration bound for large-update method, namely, $\mathcal{O}(\sqrt{n}\log n \log \frac{n}{\epsilon})$ is derived.

3.4 Properties of the Kernel (barrier) function

In this section, we study some basic properties of our kernel function $\psi(t)$ in (3.1) and the corresponding barrier function $\Psi(V)$. As we need the first three derivatives of $\psi(t)$ with

respect to t frequently, we provide them as follows:

$$\psi'(t) = t - \frac{1}{qt^2}q^{\frac{1}{t}-1} - \frac{(q-1)}{q}, \tag{3.18}$$

$$\psi''(t) = 1 + \frac{\log q + 2t}{qt^4}q^{\frac{1}{t}-1}, \tag{3.19}$$

$$\psi'''(t) = -\frac{\log^2 q + 6t^2 + 6t\log q}{qt^6}q^{\frac{1}{t}-1}, \text{ for all } q \geq 1. \tag{3.20}$$

It follows that

$$\psi(1) = \psi'(1) = 0, \ \psi''(t) > 1,$$

and

$$\lim_{t\to 0}\psi(t) = \lim_{t\to+\infty}\psi(t) = +\infty, \ \ \psi'''(t) < 0, \text{ for all } t > 0 \text{ and } q \geq 1.$$

Moreover, $\psi(t)$ is strictly convex and $\psi''(t)$ is monotonically decreasing in $t \in (0, \infty)$. As mentioned before we will restrict ourselves in this chapter to the case where $\psi(V)$ is separable with identical coordinate functions. Thus, letting ψ denote the function on the coordinates, we have

$$\Psi(V) = \sum_{i=1}^{n}\psi_i(V). \tag{3.21}$$

Let us denote by $\varrho : [0,\infty) \longrightarrow [1,\infty)]$ the inverse function of $\psi(t)$ for $t \geq 1$ and by $\rho : [0,\infty) \longrightarrow (0,1]$ the inverse function of the restriction of $-\frac{1}{2}\psi'(t)$ to the interval $(0,1]$. In what follows, we state the following lemma. For a detailed proofs, see [15].

Lemma 3.4.1. *Let $\psi_b(t)$ be the barrier term of $\psi(t)$ and let $\overline{\rho} : [0,\infty) \longrightarrow [0,1]$ be the inverse function of the restriction of $-\psi_b(t)$ to the interval $[0,1]$. Then one has.*

$$\rho(s) \geq \overline{\rho}(1+2s).$$

Lemma 3.4.2. *For $\psi(t)$ with $q > 1$, we have*

$$\begin{aligned}\sqrt{1+2s} \ &\leq \ \varrho(s) \leq 1 + \sqrt{2s}, \ s \geq 0,\\ \rho(s) \ &\geq \ \frac{1}{2 + \frac{\log(1+2s)}{\log q}}, \ s \geq 0.\end{aligned} \tag{3.22}$$

Proof. From the equation$-\psi_b(t) = s$ we have

$$\begin{aligned}-\psi_b(t) &= \frac{1}{qt^2}q^{\frac{1}{t}-1} = s,\\ q^{\frac{1}{t}-1} &= qst^2 \Leftrightarrow \left(\frac{1}{t}-1\right)\log q = \log s + 2\log t + \log q.\end{aligned}$$

For $t \in (0, 1]$, we find that $\log t \leq \log 1 = 0$ and as a result

$$\left(\frac{1}{t} - 1\right) \log q \leq \log s + \log q \Leftrightarrow t = \overline{\rho}(s) \geq \frac{1}{2 + \frac{\log s}{\log q}}.$$

By using Lemma 3.4.1, we derive a lower bound for $\rho(s)$ as follows:

$$\rho(s) \geq \frac{1}{2 + \frac{\log(1+2s)}{\log q}}. \tag{3.23}$$

This implies the lemma. □

The following property plays an important role in the analysis of kernel-function based (IPMs).

Lemma 3.4.3 (Exponential convexity. Lemma 2.1 [15]). *If $t_1 > 0$ and $t_2 > 0$, then one has*

$$\psi(\sqrt{t_1 t_2}) \leq \frac{1}{2}(\psi(t_1) + \psi(t_2)).$$

We have the following theorems.

Theorem 3.4.1 (Proposition 3(II) [43]). *Let $V_1, V_2 \in \mathbb{S}^n_{++}$, then*

$$\Psi\left(\left[V_1^{\frac{1}{2}} V_2 V_1^{\frac{1}{2}}\right]\right) \leq \frac{1}{2}(\Psi(V_1) + \Psi(V_2)).$$

Theorem 3.4.2 (Theorem 3.2 [15]). *Let $V \in \mathbb{S}^n_{++}$, and $\beta \geq 1$. Then*

$$\Psi(\beta V) \leq n\psi\left(\frac{\Psi(V)}{n}\right).$$

Corollary 3.4.1. *Let $0 \leq \theta < 1$ and $V_+ = \frac{V}{\sqrt{1-\theta}}$, if $\Psi \leq \tau$. Then*

$$\Psi(V_+) \leq n\psi\left(\frac{\varrho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}\right).$$

Theorem 3.4.3 (Theorem 4.8 [15]). *Let $V \in \mathbb{S}^n_{++}$. Then*

$$\delta(V) \geq \frac{1}{2}\psi'(\varrho(\Psi(V))).$$

Corollary 3.4.2. *Let $V \in \mathbb{S}^n_{++}$. Then*

$$\delta(V) \geq \frac{1}{\sqrt{2}}(\Psi(V))^{\frac{1}{2}}.$$

Proof. We have

$$\begin{aligned}\delta(V) &\geq \frac{1}{2}\psi'(\varrho(\Psi(V))) \geq \frac{1}{2}\left(\sqrt{1+2\Psi(V)} - \frac{q^{\frac{1}{\sqrt{1+2\Psi(V)}}-1}}{q(1+2\Psi(V))} - \frac{q-1}{q}\right)\\ &\geq \frac{1}{2}\left(\sqrt{1+2\Psi(V)} - \frac{q^{\frac{1}{\sqrt{1+2\Psi(V)}}-1}}{1+2\Psi(V)}\right) \geq \frac{1}{2}(\sqrt{1+2\Psi(V)}-1)\\ &= \frac{\Psi(V)}{\sqrt{1+2\Psi(V)}+1} \geq \frac{1}{\sqrt{2}}(\Psi(V))^{\frac{1}{2}}.\end{aligned}$$

This implies the lemma. □

3.5 Analysis of the interior-point algorithm

In the next subsection, we compute a default step-size α and the resulting decrease in the barrier function.

3.5.1 Decrease of the barrier function and choice of the default step-size

After a damped step, we have

$$X_+ = X + \alpha\Delta X, \;\; Y_+ = Y + \alpha\Delta Y. \tag{3.24}$$

From (3.10) and (3.11) we have

$$X_+ = X + \alpha\Delta X = X + \alpha\sqrt{\mu}DD_XD = \sqrt{\mu}D(V + \alpha D_X)D, \tag{3.25}$$

and

$$Y_+ = Y + \alpha\Delta Y = Y + \alpha\sqrt{\mu}DD_YD = \sqrt{\mu}D^{-1}(V + \alpha D_Y)D^{-1}. \tag{3.26}$$

Thus we obtain

$$V_+^2 = \frac{D^{-1}X_+Y_+D}{\mu} = (V + \alpha D_X)(V + \alpha D_Y), \tag{3.27}$$

we can verify that $V_+^2 \sim X_+^{\frac{1}{2}}Y_+X_+^{\frac{1}{2}}$ and consequently is similar to

$$(V + \alpha D_X)^{\frac{1}{2}}(V + \alpha D_Y)(V + \alpha D_X)^{\frac{1}{2}}.$$

We define the following matrix

$$\overline{V}_+ = \left((V+\alpha D_X)^{\frac{1}{2}}(V+\alpha D_Y)(V+\alpha D_X)^{\frac{1}{2}}\right)^{\frac{1}{2}}. \tag{3.28}$$

Hence, since Ψ is convex and from Theorem 3.4.1 we have

$$\Psi(V_+) = \Psi(\overline{V}_+) \leq \frac{1}{2}(\Psi(V+\alpha D_X) + \Psi(V+\alpha D_Y)). \tag{3.29}$$

Defining

$$f(\alpha) = \Psi(V_+) - \Psi(V) = \Psi(\overline{V}_+) - \Psi(V). \tag{3.30}$$

We thus have $f(\alpha) \leq f_1(\alpha)$, where

$$f_1(\alpha) = \frac{1}{2}(\Psi(V+\alpha D_X) + \Psi(V+\alpha D_Y)) - \Psi(V). \tag{3.31}$$

Obviously,

$$f(0) = f_1(0).$$

Taking the derivative with respect to α , we get

$$f'(\alpha) = \frac{1}{2}(\Psi'(V+\alpha D_X)D_X + \Psi'(V+\alpha D_Y)D_Y). \tag{3.32}$$

This gives, using the second equation in (3.15),

$$f'(0) = \frac{1}{2}(\Psi'(V)(D_X+D_Y)) = \frac{1}{2}\mathbf{Tr}(-\psi'(V)^2) = -2\delta(V)^2. \tag{3.33}$$

Differentiating once more, we obtain

$$\begin{aligned} f''(\alpha) &= \frac{1}{2}(\Psi''(V+\alpha D_X)D_X^2 + \Psi''(V+\alpha D_Y)D_Y^2) \\ &\leq \frac{1}{2}(\omega_1\|D_X\|^2 + \omega_2\|D_Y\|^2), \end{aligned} \tag{3.34}$$

with

$$\omega_1 = \max\{|\Delta\psi'(\lambda_j(V+\alpha D_X), \lambda_k(V+\alpha D_X))| : j,k = 1,2,\ldots,n\},$$

and

$$\omega_2 = \max\{|\Delta\psi'(\lambda_j(V+\alpha D_Y), \lambda_k(V+\alpha D_Y))| : j,k = 1,2,\ldots,n\}.$$

Below we use the following notation:

$$V_1 := \lambda_{min}(V),\ \delta := \delta(V),$$

with $\delta(V)$ as defined in (3.17).

Lemma 3.5.1 (Lemma 2 [47]). *Let $A, A+B \in \mathbb{S}^n_+$, one has*

$$\lambda_i(A+B) \geq \lambda_n(A) - |\lambda_1(B)|\,,\ i = 1, 2, \ldots, n.$$

Lemma 3.5.2. *One has $f''(\alpha) \leq 2\delta^2\psi''(V_1 - 2\alpha\delta)$.*

Proof. Due to (3.16) and (3.17), we have

$$4\delta(V)^2 := 4\delta^2 = \|D_X + D_Y\|^2 = \|D_X\|^2 + \|D_Y\|^2 + 2\langle D_X, D_Y\rangle \geq \|D_X\|^2 + \|D_Y\|^2\,.$$

This implies that

$$|\lambda_1(D_X)| \leq 2\delta,\ \text{and}\ |\lambda_1(D_Y)| \leq 2\delta, \tag{3.35}$$

on the other hand, using Lemma 3.5.1, we have

$$\lambda_i(V + \alpha D_X) \geq \lambda_n(V) - \alpha\,|\lambda_1(D_X)| \geq \lambda_n(V) - 2\alpha\delta,\ i = 1, 2, \ldots n. \tag{3.36}$$

Similarly, we have

$$\lambda_i(V + \alpha D_Y) \geq \lambda_n(V) - 2\alpha\delta,\ i = 1, 2, \ldots n.$$

Thus from the choice of ω_1, we can conclude that

$$\omega_1 = \{|\Delta\psi'(\lambda_{j_*}(V + \alpha D_X), \lambda_{k_*}(V + \alpha D_X))| : j_*, k_* = 1, 2, \ldots, n\}.$$

Due to the definition of $\Delta\psi'(t_1, t_2)$ and the mean value theorem, there exists a constant

$$\xi_* \in [\min\{\lambda_{j_*}(V + \alpha D_X), \lambda_{k_*}(V + \alpha D_X)\}, \max\{\lambda_{j_*}(V + \alpha D_X), \lambda_{k_*}(V + \alpha D_X)\}]\,,$$

the function ψ satisfies

$$\psi''(\xi_*) = \Delta\psi'\left(\lambda_{j_*}(V + \alpha D_X), \lambda_{k_*}(V + \alpha D_X)\right).$$

Due to (3.18), $\psi''(t)$ is monotonically decreasing in $t \in (0, \infty)$. Therefore, from (3.36), we obtain

$$\begin{aligned} \omega_1 &= \psi''(\xi_*) \leq \psi''\left(\min\{\lambda_{j_*}(V + \alpha D_X), \lambda_{k_*}(V + \alpha D_X)\}\right) \\ &\leq \psi''\left(\lambda_n(V) - 2\alpha\delta\right) \leq \psi''\left(\lambda_{min}(V) - 2\alpha\delta\right). \end{aligned}$$

Similarly, we have

$$\omega_2 \leq \psi''\left(\lambda_{min}(V) - 2\alpha\delta\right).$$

Then it follows from (3.34) that

$$f_1''(\alpha) \leq \frac{1}{2}\psi''\left(\lambda_{min}(V) - 2\alpha\delta\right)\left(\|D_X\|^2 + \|D_Y\|^2\right) \leq 2\delta^2\psi''\left(V_1 - 2\alpha\delta\right).$$

This proves the lemma. □

Lemma 3.5.3 (Lemma 4.2 [15]). *$f_1'(\alpha) \leq 0$ certainly holds if α satisfies the inequality*

$$-\psi'(V_1 - 2\alpha\delta) + \psi'(V_1) \leq 2\delta. \tag{3.37}$$

Lemma 3.5.4 (Lemma 4.3 [15]). *Let $\rho : [0, \infty) \longrightarrow (0, 1]$ denote the inverse function of the restriction of $-\frac{1}{2}\psi'(t)$ to the interval $(0, 1]$. Then the largest step-size α is given by*

$$\overline{\alpha} = \frac{1}{2\delta}(\rho(\delta) - \rho(2\delta)). \tag{3.38}$$

Lemma 3.5.5 (Lemma 4.4 [15]). *Let ρ and $\overline{\alpha}$ be as defined in Lemma 3.5.4. Then*

$$\frac{1}{\psi''(\rho(2\delta))} \leq \overline{\alpha} \leq \frac{1}{\psi''(\rho(\delta))}. \tag{3.39}$$

In what follows we use the notation

$$\tilde{\alpha} = \frac{1}{\psi''(\rho(2\delta))}. \tag{3.40}$$

Lemma 3.5.6 (Lemma 4.5 [15]). *If the step-size α is such that $\alpha \leq \tilde{\alpha}$, then*

$$f(\alpha) \leq -\alpha\delta^2. \tag{3.41}$$

Combining the results of Lemmas 3.5.5 and 3.5.6, we obtain the following theorem.

Theorem 3.5.1. *We have*

$$\alpha \geq \frac{1}{1 + (\log q + 2)(1 + 4\delta)\left(2 + \frac{\log(1+4\delta)}{\log q}\right)^2}. \tag{3.42}$$

Proof. The function $\psi''(t)$ is monotonically decreasing, from Lemma 3.4.1, we obtain

$$\overline{\alpha} \geq \frac{1}{\psi''(\rho(2\delta))} \geq \frac{1}{\psi''(\overline{\rho}(1 + 4s))}.$$

Putting $t = \overline{\rho}(1 + 4s)$, we have $t \leq 1$. Then

$$\overline{\alpha} \geq \frac{1}{\psi''(t)} \geq \frac{1}{1 + \frac{\log q + 2t}{qt^4} q^{\frac{1}{t} - 1}}. \tag{3.43}$$

In other hand we have

$$t = \overline{\rho}(1+4s) \Leftrightarrow 1+4s = -\psi'(t) = \frac{1}{qt^2} q^{\frac{1}{t}-1},$$

the equation (3.43) becomes

$$\begin{aligned} \overline{\alpha} &\geq \frac{1}{1+\frac{\log q+2t}{t^2}(1+4s)} \\ &\geq \frac{1}{1+\frac{\log q+2}{t^2}(1+4s)}. \end{aligned} \tag{3.44}$$

We have

$$\frac{1}{t^2} = \frac{1}{\overline{\rho}(1+2s)^2} \leq \frac{1}{\rho(2\delta)^2} \leq \left(2+\frac{\log(1+4\delta)}{\log q}\right)^2.$$

Substituting this equation in (3.44), we obtain

$$\overline{\alpha} \geq \frac{1}{1+(\log q+2)(1+4\delta)\left(2+\frac{\log(1+4\delta)}{\log q}\right)^2}. \tag{3.45}$$

This completes the proof of the theorem. □

Theorem 3.5.2. *With $\tilde{\alpha}$ being the default step-size, as given by (3.40), one has*

$$f(\tilde{\alpha}) \leq -\frac{\Psi(V)^{\frac{1}{2}}}{20(\log q+2)\left(2+\frac{\log(1+\sqrt{\Psi_0})}{\log q}\right)^2}. \tag{3.46}$$

Proof. We have

$$\delta \geq \sqrt{\frac{\Psi}{2}}. \tag{3.47}$$

According to Lemma 3.5.6, with (3.42) and (3.47), we obtain

$$\begin{aligned} f(\tilde{\alpha}) \leq -\tilde{\alpha}\delta^2 &\leq -\frac{\delta^2}{1+(\log q+2)(1+4\delta)\left(2+\frac{\log(1+4\delta)}{\log q}\right)^2} \\ &\leq -\frac{\delta^2}{10\delta(\log q+2)\left(2+\frac{\log(1+4\delta)}{\log q}\right)^2} \\ &\leq -\frac{\delta}{10(\log q+2)\left(2+\frac{\log(1+4\delta)}{\log q}\right)^2} \\ &\leq -\frac{\Psi(V)^{\frac{1}{2}}}{10\sqrt{2}(\log q+2)\left(2+\frac{\log(1+\sqrt{\Psi_0})}{\log q}\right)^2}. \end{aligned}$$

This completes the proof. □

Lemma 3.5.7 (Lemma 12 [43])**.** *Let $h(t)$ be twice differentiable convex function with $h(0) = 0$, $h'(0) \leq 0$, and let $h(t)$ attain its (global) minimum at $t^* > 0$, if $h''(t)$) is monotonically increasing for $t \in \{0, t^*\}$ then $h(t) \leq \frac{th'(0)}{2}$, $0 \leq t \leq t^*$.*

3.5.2 Iteration bound

We need to count how many inner iterations are required to return to the situation where $\Psi(V) \leq \tau$ after a μ-update. We denote the value of $\Psi(V)$ after μ-update as Ψ_0 the subsequent values in the same outer iteration are denoted as Ψ_k, $k = 1, \ldots, K$, K denotes the total number of inner iterations in the outer iteration.

Lemma 3.5.8 (Lemma 14 [43]). *Let $t_0, t_1, \ldots, t_k$ be a sequence of positive numbers such that*

$$t_{k+1} \leq t_k - \beta_k^{1-\gamma}, \ k = 0, 1, \ldots, K-1.$$

Where $\beta > 0$ and $0 < \gamma \leq 1$, then $K \leq \left\lceil \frac{t_0^\gamma}{\beta\gamma} \right\rceil$.

Thus, it follows that

$$\Psi_{k+1} \leq \Psi_k - \kappa(\Psi)^{1-\gamma}, \quad k = 0, 1, \ldots, K-1. \tag{3.48}$$

With

$$\kappa = \frac{1}{10\sqrt{2}(2+\log q)\left(2+\frac{\log(1+\sqrt{\Psi_0})}{\log q}\right)^2}, \ \gamma = \frac{1}{2}.$$

Hence, by Lemma 3.5.8 the number K of inner iterations is bounded above by

$$K \leq \Psi_0^{\frac{1}{2}}\left(20\sqrt{2}(2+\log q)\left(2+\frac{\log(1+\sqrt{\Psi_0})}{\log q}\right)^2\right). \tag{3.49}$$

As a result we have that if $\Psi(V) \leq \tau$ and $\beta = \frac{1}{\sqrt{1-\theta}}$, then

$$L_\Psi(n, \theta, \tau) = n\Psi\left(\frac{\varrho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}\right), \tag{3.50}$$

is an upper bound for $\Psi\left(\frac{V}{\sqrt{1-\theta}}\right)$, the value of $\psi(V)$ after the μ-update.
By lemma 3.4.1 we have $\varrho(s) \leq 1 + \sqrt{2s}$, and also using (3.50) we have

$$\Psi_0 \leq L_\Psi(n, \theta, \tau) = n\Psi\left(\frac{\varrho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}\right) \leq n\Psi\left(\frac{1+\sqrt{\frac{2\tau}{n}}}{\sqrt{1-\theta}}\right).$$

Since $\psi(t) \leq \frac{t^2-1}{2}$ for $t \geq 1$, we have

$$\Psi_0 \leq \frac{n}{2}\left[\left(\frac{1+\sqrt{\frac{2\tau}{n}}}{\sqrt{1-\theta}}\right)^2 - 1\right] \leq \left(\frac{\tau+\sqrt{n\tau}}{1-\theta}\right) = \mathcal{O}(n). \tag{3.51}$$

So (3.49) becomes

$$K \leq 20\sqrt{2}(2+\log q)\left(\frac{\tau+\sqrt{2n\tau}}{1-\theta}\right)^{\frac{1}{2}}\left(2+\frac{\log\left(1+\sqrt{\frac{\tau+\sqrt{2n\tau}}{1-\theta}}\right)}{\log q}\right)^{2}\frac{1}{\theta}\log\frac{n}{\epsilon}. \quad (3.52)$$

We have

$$\Psi_0 \leq \mathcal{O}(n),$$

by choosing

$$q = 1+\mathcal{O}(n).$$

the best iteration bound is obtained. We have the following theorem.

Theorem 3.5.3. *For large-update methods with $\tau = \mathcal{O}(n)$, and $\theta = \Theta(1)$, Algorithm 3.1 requires at most*

$$\mathcal{O}\left(\sqrt{n}\log n\log\frac{n}{\epsilon}\right),$$

iterations. This is the best well-known complexity results for large-update methods.

3.6 Numerical results

In this section, we test Algorithm 3.1 on some monotone SDLCPs where its implementation is done under the software MATLAB 7.9 and run on a pc. The initial strictly feasible point (X^0, Y^0) with $\mu_0 > 0$ is chosen such that $\Psi(X^0, Y^0; \mu_0) \leq \tau$. However, the implementation with the selected theoretical step-size α during each inner iteration guarantees the convergence of the method but it yields unfavorable numerical results i.e., a slow convergence of the algorithm. Therefore, we compute at each inner iteration a practical step-size $\alpha_{\max}$ such that $X+\alpha_{\max}\Delta X \succ 0$ and $Y+\alpha_{\max}\Delta Y \succ 0$ with $\alpha_{\max} = \rho\min(\alpha_X, \alpha_Y)$ and $\rho \in (0,1)$, where

$$\alpha_X = \begin{cases} -\frac{1}{\lambda_{\min}(X^{-1}\Delta X)} & \text{if } \lambda_{\min}(X^{-1}\Delta X) < 0 \\ 1 & \text{if } \lambda_{\min}(X^{-1}\Delta X) \geq 0, \end{cases}$$

and

$$\alpha_Y = \begin{cases} -\frac{1}{\lambda_{\min}(Y^{-1}\Delta Y)} & \text{if } \lambda_{\min}(Y^{-1}\Delta Y) < 0 \\ 1 & \text{if } \lambda_{\min}(Y^{-1}\Delta Y) \geq 0. \end{cases}$$

The kernel functions used here are:

$$\psi(t) = \frac{t^2-1}{2} + \frac{q^{\frac{1}{t}-1}-1}{q\log q} - \frac{q-1}{q}(t-1), \quad q > 1.$$

In the implementation, we take different values of the parameters θ, q and α as follows. $\theta \in \{0.15,\, 0.35,\, 0.55,\, 0.65,\, 0.75,\, 0.9\}$, $q \in \{1.1,\, 1{+}\mathcal{O}(n),\, 5\}$ and $\alpha \in \{0.2,\, 0.5,\, \alpha_{\max}\}$. The favored threshold is $\tau = \sqrt{n}$ with $\rho \in [0.95,\, 0.99]$ and $\epsilon = 10^{-6}$. The number of iterations and the time produced by Algorithm 3.1, are denoted by "It " and "CPU", respectively.

Now, three test problems of monotone semidefinite linear complementarity problems are considered where the first is reformulated from the symmetric semidefinite constrained least squares problem (SDLS), the second one is linked to the two-sided linear transformation, and the last one is given by a pair of primal-dual semidefinite optimization problem.

Example 1.

For example for $m = 6$ and $n = 5$, the data of the SDLS (Example 1.6.6) is given by:

$$A = \begin{pmatrix} 6 & 1 & 1 & 1 & 1 \\ 0 & 6 & 1 & 1 & 1 \\ 0 & 0 & 6 & 1 & 1 \\ 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -0.4 & 1 & 0 & 0 & 0 \\ -0.4 & -0.4 & 1 & 0 & 0 \\ -0.4 & 0 & -0.4 & 1 & 0 \\ -0.4 & 0 & 0 & -0.4 & 1 \\ -0.4 & 0 & 0 & 0 & -0.4 \end{pmatrix}.$$

The unique optimal solution $X^* \in \mathbb{S}^5_+$ is given by:

$$X^* = \begin{pmatrix} 0.1929 & -0.0333 & -0.0346 & -0.0391 & -0.0506 \\ -0.0333 & 0.1778 & -0.0409 & -0.0066 & -0.0064 \\ -0.0346 & -0.0409 & 0.1808 & -0.0403 & -0.0069 \\ -0.0391 & -0.0066 & -0.0403 & 0.1791 & -0.0426 \\ -0.0506 & -0.0064 & -0.0069 & -0.0426 & 0.1558 \end{pmatrix}.$$

For the initialization, we take

$$X^0 = \text{Diag}\,[0.2920,\, \ldots,\, 0.2920],$$

as the strictly feasible starting point.

$\theta \rightarrow$	0.15	0.35	0.55	0.65	0.75	0.9
$\alpha \downarrow$	It CPU	It CPU	It CPU	It CPU	It CPU	It CPU
$\alpha = 0.2$	98 0.4134	93 0.3152	88 0.2733	87 0.2673	78 0.2365	73 0.2303
$\alpha = 0.5$	86 0.2843	33 0.1661	31 0.1564	29 0.1383	28 0.1324	25 0.1193
$\alpha = \alpha_{\max}$	86 0.2252	33 0.1342	18 0.1051	14 0.0989	10 0.0871	10 0.0832

Table 3.1: Numerical results with $q = 1.1$, after the algorithm reaches $n\mu \leq 10^{-6}$

$\theta \rightarrow$	0.15	0.35	0.55	0.65	0.75	0.9
$\alpha \downarrow$	It CPU	It CPU	It CPU	It CPU	It CPU	It CPU
$\alpha = 0.2$	88 0.4090	85 0.2971	81 0.2618	81 0.2499	73 0.2334	70 0.2396
$\alpha = 0.5$	86 0.2719	33 0.1786	28 0.1539	28 0.1365	25 0.1249	24 0.1201
$\alpha = \alpha_{\max}$	86 1.2907	33 0.1338	18 0.1133	14 0.1017	11 0.0891	10 0.0753

Table 3.2: Numerical results with $q = 5$, after the algorithm reaches $n\mu \leq 10^{-6}$

$\theta \rightarrow$	0.15	0.35	0.55	0.65	0.75	0.9
$\alpha \downarrow$	It CPU	It CPU	It CPU	It CPU	It CPU	It CPU
$\alpha = 0.2$	86 0.4533	83 0.3390	78 0.3577	79 0.2934	72 0.2885	68 0.2472
$\alpha = 0.5$	86 0.3090	33 0.2185	27 0.2150	27 0.1922	25 0.2545	24 0.1378
$\alpha = \alpha_{\max}$	86 0.2879	33 0.1213	18 0.1534	14 0.1721	10 0.1295	10 0.1364

Table 3.3: Numerical results with $q = 1 + \mathcal{O}(n)$, after the algorithm reaches $n\mu \leq 10^{-6}$

Example 2.

The monotone SDLCP considered in this example is linked to the two-sided multiplicative transformation $\mathcal{L} : \mathbb{S}^n \rightarrow \mathbb{S}^n$ given by:

$$\mathcal{L}(X) = AXA^T.$$

Balaji et al. [17] showed that if A is a positive definite matrix, then $\mathcal{L}(X)$ is strictly monotone and therefore the associated SDLCPs, is uniquely globally solvable for every

$Q \in \mathbb{S}^n$. For $n = 5$, the data of our example is given by:

$$A = \begin{pmatrix} 36.04 & 7.2 & 0.2 & 0 & 0 \\ 7.2 & 37.04 & 7.2 & 0.2 & 0 \\ 0.2 & 7.2 & 37.04 & 7.2 & 0.2 \\ 0 & 0.2 & 7.2 & 37.04 & 7.2 \\ 0 & 0 & 0.2 & 7.2 & 38 \end{pmatrix},$$

and

$$Q = \begin{pmatrix} -5.92 & 0.64 & 1.44 & 1.44 & 1.6 \\ 0.64 & -5.92 & 0.6 & 0.2 & 0 \\ 1.44 & 0.6 & -5.92 & 0.6 & 0.2 \\ 1.44 & 0.2 & 0.6 & -5.92 & 0.6 \\ 1.6 & 0 & 0.2 & 0.6000 & -5.6 \end{pmatrix}.$$

The optimal solution $X^* \in \mathbb{S}^5_+$ is given by:

$$X^* = \begin{pmatrix} 0.0053 & -0.0024 & -0.0002 & -0.0009 & -0.0010 \\ -0.0024 & 0.0058 & -0.0025 & 0.0008 & 0.0001 \\ -0.0002 & -0.0025 & 0.0058 & -0.0026 & 0.0005 \\ -0.0009 & 0.0008 & -0.0026 & 0.0058 & -0.0023 \\ -0.0010 & 0.0001 & 0.0005 & -0.0023 & 0.0045 \end{pmatrix}.$$

The numerical results for this example with the feasible starting point:

$$X^0 = \text{Diag}\,[0.0357, \ldots, 0.0357],$$

are summarized in tables 3.4, 3.5 and 3.6.

$\theta \rightarrow$	0.15	0.35	0.55	0.65	0.75	0.9
$\alpha \downarrow$	It CPU	It CPU	It CPU	It CPU	It CPU	It CPU
$\alpha = 0.2$	99 0.4078	94 0.3030	88 0.2598	87 0.2494	79 0.2224	74 0.2078
$\alpha = 0.5$	86 0.2797	33 0.1648	32 0.1408	30 0.1279	29 0.1138	26 0.1054
$\alpha = \alpha_{\max}$	86 0.2163	33 0.2793	19 0.0916	14 0.0807	10 0.0665	10 0.0604

Table 3.4: Numerical results with $q = 1.1$, after the algorithm reaches $n\mu \leq 10^{-6}$

$\theta \rightarrow$	0.15	0.35	0.55	0.65	0.75	0.9
$\alpha \downarrow$	It CPU	It CPU	It CPU	It CPU	It CPU	It CPU
$\alpha = 0.2$	89 0.3854	85 0.2775	81 0.2443	81 0.2292	75 0.2159	70 0.1964
$\alpha = 0.5$	86 0.2642	33 0.1644	29 0.1320	29 0.1234	26 0.1089	25 0.0995
$\alpha = \alpha_{\max}$	86 0.2678	33 0.1173	18 0.0899	14 0.0815	11 0.0723	10 0.0672

Table 3.5: Numerical results with $q = 5$, after the algorithm reaches $n\mu \leq 10^{-6}$

$\theta \rightarrow$	0.15	0.35	0.55	0.65	0.75	0.9
$\alpha \downarrow$	It CPU	It CPU	It CPU	It CPU	It CPU	It CPU
$\alpha = 0.2$	86 0.3938	84 0.2812	79 0.2408	80 0.2335	73 0.2184	69 0.1903
$\alpha = 0.5$	86 0.2635	33 0.1561	28 0.1296	28 0.1165	26 0.1069	24 0.0966
$\alpha = \alpha_{\max}$	86 0.2704	33 0.1158	18 0.0884	14 0.2144	10 0.0685	9 0.0628

Table 3.6: Numerical results with $q = 1 + \mathcal{O}(n)$, after the algorithm reaches $n\mu \leq 10^{-6}$

Problem 1.

Recall that a pair of primal-dual SDO programs is defined as

$$\begin{aligned} &\min_X C \bullet X \\ &\text{s.t. } \langle A_i, X\rangle = b_i,\ i = 1, 2, \ldots, m \qquad \text{(P)} \\ &X \succeq 0, \end{aligned}$$

and its dual

$$\begin{aligned} &\max_{(y,Z)} b^T y \\ &\text{s.t. } \sum_{i=1}^{m} y_i A_i + Z = C, \qquad \text{(D)} \\ &Z \succeq 0, \end{aligned}$$

with $A_i, C \in \mathbb{S}^n$, $b, y \in \mathbb{R}^m$ and $X, Z \in \mathbb{S}^n_+$.

Example 1.

Let us consider the following pair of primal-dual problems P and D with the following data:

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & -1 \\ 0 & -1 & 1 & -1 & -2 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & 0 & -2 & 2 & 0 \\ 0 & 2 & 1 & 0 & 2 \\ -2 & 1 & -2 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 2 & 2 & -1 & -1 & 1 \\ 2 & 0 & 2 & 1 & 1 \\ -1 & 2 & 0 & 1 & 0 \\ -1 & 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & 0 & -2 \end{pmatrix}, \; C = \begin{pmatrix} 3 & 3 & -3 & 1 & 1 \\ 3 & 5 & 3 & 1 & 2 \\ -3 & 3 & -1 & 1 & 2 \\ 1 & 1 & 1 & -3 & -1 \\ 1 & 2 & 2 & -1 & -1 \end{pmatrix},$$

$$b = (-2, 2, -2)^T.$$

An optimal-dual solution of P and D is given by

$$X = \begin{pmatrix} 0.0714 & -0.0718 & 0.0169 & 0.0649 & -0.1583 \\ -0.0718 & 0.0724 & -0.0183 & -0.0602 & 0.1676 \\ 0.0169 & -0.0183 & 0.0103 & -0.0084 & -0.0772 \\ 0.0649 & -0.0602 & -0.0084 & 0.1481 & 0.0056 \\ -0.1583 & 0.1676 & -0.0772 & 0.0056 & 0.6022 \end{pmatrix},$$

$$y = \begin{pmatrix} 0.8585 \\ 1.0937 \\ 0.7831 \end{pmatrix}, \; Z = \begin{pmatrix} 1.4338 & 0.5754 & -0.0295 & -0.4043 & 0.2169 \\ 0.5754 & 1.0956 & 0.3401 & 0.2169 & -0.1120 \\ -0.0295 & 0.3401 & 1.1874 & 0.2169 & 0.0478 \\ -0.4043 & 0.2169 & 0.2169 & 0.2831 & -0.1415 \\ 0.2169 & -0.1120 & 0.0478 & -0.1415 & 0.0957 \end{pmatrix},$$

The numerical results with the strictly feasible triplet starting point $(X^0, y^0, Z^0) = (I, e, I)$ and with different values of θ, q and α are stated in tables 3.7, 3.8 and 3.9.

$\theta \rightarrow$	0.15	0.35	0.55	0.65	0.75	0.9
$\alpha \downarrow$	It CPU	It CPU	It CPU	It CPU	It CPU	It CPU
$\alpha = 0.3$	81 4.9977	56 0.2295	52 0.1957	51 0.1879	49 0.1805	47 0.1678
$\alpha = 0.5$	81 0.3236	31 0.1688	29 0.1452	26 0.1402	27 0.1250	24 0.1121
$\alpha = \alpha_{\max}$	81 0.2108	31 0.1305	17 0.0984	14 0.0963	12 0.0927	13 0.0885

Table 3.7: Numerical results with $q = 1.1$, after the algorithm reaches $n\mu \leq 10^{-6}$

$\theta \rightarrow$	0.15	0.35	0.55	0.65	0.75	0.9
$\alpha \downarrow$	It CPU	It CPU	It CPU	It CPU	It CPU	It CPU
$\alpha = 0.3$	81 1.2777	51 0.2144	50 0.1935	47 0.1718	47 0.1633	44 0.1592
$\alpha = 0.5$	81 0.2535	31 0.1696	26 0.1377	26 0.1258	25 0.1191	24 0.1105
$\alpha = \alpha_{\max}$	81 0.2157	31 0.1307	17 0.0969	14 0.0962	12 0.0935	13 0.0926

Table 3.8: Numerical results with $q = 5$, after the algorithm reaches $n\mu \leq 10^{-6}$

$\theta \rightarrow$	0.15	0.35	0.55	0.65	0.75	0.9
$\alpha \downarrow$	It CPU	It CPU	It CPU	It CPU	It CPU	It CPU
$\alpha = 0.3$	81 0.3264	51 0.2390	48 0.2044	46 0.1955	46 0.1863	43 0.1790
$\alpha = 0.5$	81 0.2983	31 0.2021	25 0.1362	25 0.1640	24 0.1655	23 0.1648
$\alpha = \alpha_{\max}$	81 0.2085	31 0.1307	16 0.1077	13 0.0923	11 0.0852	11 0.0842

Table 3.9: Numerical results of $\psi(t)$ with $q = 1 + \mathcal{O}(n)$, after the algorithm reaches $n\mu \leq 10^{-6}$

Contributions and some remarks

Some contributions and remarks for Chapter II:

- Extension of a new primal-dual IP paradigm based on the barrier kernel functions for SDLCP.
- Introduction of a new parametric kernel function and elaboration of a primal-dual with large-step algorithm.

- With the choice of the parameter $q = 1 + \mathcal{O}(n)$, the best polynomial complexity of this algorithm is reached.
- Improvement of the complexity obtained by EL Ghami et al. for LP and SDP.
- Presentation of important numerical experiences for some problems in practice.

Some difficulties and advantages of this paradigm.

- Contrary to the short-step algorithm used for HLCP, a linear search is made to ensure the positive definite of the iterates during the solution procedure.
- The theoretical step given in this algorithm ensures the convergence of the method but numerically leads to a slow convergence.
- To remedy this handicap, a practical procedure was used to accelerate the convergence.
- The advantage of this method is that the size of the neighborhood of the central-path is wide because the threshold $\tau \geq 1$, which gives a flexibility to have a strictly feasible starting point.

General conclusion and future work

This manuscript presents a theoretical analysis and numerical study of primal-dual path-following interior point methods (IPMs) applied to the symmetric cones linear complementarity problems (SCLCP). Recall that given a symmetric cone $\mathbb{K} \subset \mathbf{E}$ where $\mathbf{E}$ is a Euclidean space equipped with an inner product $\langle ., . \rangle_{\mathbf{E}\times\mathbf{E}}$ on the corps $\mathbb{R}$,a linear transformation $\mathcal{L} : \mathbf{E} \to \mathbf{E}$ and $q \in \mathbb{E}$, the SCLCP consists in finding a pair $(x, y) \in \mathbb{K} \times \mathbb{K}$ such that:

$$y = \mathcal{L}(x) + q \in \mathbb{K} \quad \langle x, y \rangle = 0_{\mathbf{E}}.$$

In particular, we have studied two primal-dual IPMs for solving the monotone horizontal LCP (HLCP) and the SDLCP that act on the cone of nonnegative orthant $\mathbb{K} = \mathbb{R}^n_+$ and of symmetric semidefinite matrices $\mathbb{K} = \mathbb{S}^n_+$, respectively.

In the first part, we have deal with the short-step primal-dual IPMs for solving the HLCPs. The advantage of this algorithm is to use only full-Newton steps to obtain a solution, i.e., the step-size $\alpha = 1$, and avoid to use the classical line search methods for its determination because it costs highly in practice. On the other hand, the main fundamental ingredient in the analysis of this algorithm is to maintain during the solution process the strictly feasibility of the full-Newton steps. This task is very hard and is based in choosing a suitable neighborhood of size τ and an updating barrier parameter θ to stay near the central-path. This difficulty is overcome where, through a suitable analysis, the best values have been given where $\tau = \frac{2}{\sqrt{10}}$. and $\theta = (\frac{6}{23n})^{1/2}$. Again and across these two parameters, we have shown that this algorithm has the best polynomial complexity, namely, $\mathcal{O}\left(\sqrt{n}\log\frac{n}{\epsilon}\right)$ to get an $\epsilon-$solution of HLCP. Meanwhile, for its numerical performances, we have used different values of the barrier parameter $\mu > 0$, and θ. Thus lead to reduce significantly the number of iterations and the time produced by the algorithm. Our testing examples are selected from some practical instances such as the absolute value

equations (partial differential equations), the convex quadratic optimization, and the standard monotone linear complementarity problems. We have end this first part of the thesis, with comparison of our obtained numerical results with those obtained by an iterative (non interior point methods) fixed point method. Finally, we deduce from this study the following facts.

1. These methods have the best known polynomial complexity.

2. These algorithms are simple and easy to implement and give the solution of the considered problems.

3. The main difficulty is the strictly feasible initial point to start their algorithm. This handicap is a very hard task to release in practice.

4. The theoretical and numerical treatment of the HLCPs have a great applications for some other instances of mathematical problems.

For the second part of the thesis, we have developed a large-update primal-dual path-following IPMs for solving the monotone SDLCPs based on a new eligible parametric kernel function. In brief, we have analyzed across this latter the polynomial complexity of large-update IP algorithms. The iteration bound is calculated by choosing a suitable value of the barrier parameter $q > 1$, namely $q = 1 + \mathcal{O}(n)$ and therefore the best well-known polynomial complexity is $\mathcal{O}(\sqrt{n}\log n \log \frac{n}{\epsilon})$. We have also deduce from this study the following general facts.

1. We mention that for our knowledge, is the first primal-dual IPMs for SDLCPs based on the barrier kernel functions.

2. The feasible set of the SDLCPs is not a polyhedral therefore the necessity of interior-point methods for solving it.

3. The study of SDLCPs needs a great background of Matrix Theory and Convex Analysis.

4. The kernel function used in this work is a kind of generalization of the one used by Bai for LP and SDO.

5. Their polynomial complexity is improved by a factor of $\log n$.

6. The Nesterov-Todd direction are computed via the Kronecker operator.

7. The theoretical step-size determined by the analysis of these methods along this direction leads to a slow convergence of the algorithm. To remedy this drawback an economical and practical step-size is proposed and this leads to a good numerical results.

8. The testing examples are selected from three problems such as the symmetric semidefinite constrained least squares problem (SDLS), the second one is linked to the two-sided linear transformation, and the last one is given by a pair of primal-dual semidefinite optimization problem.

Future Work

In this section, we will give some interesting topics for further research.

- The generalization of primal-dual (IPMs) for HLCPs and SDLCPs to an arbitrary symmetric cone linear complementarity problems (SCLCPs) is interesting topic.

- The development of infeasible interior-point algorithm based on the analysis (feasible) given in the first part seems to be an interesting.

- The generalization of the results obtained in the first part to other class of matrices such us $P_*(\kappa)$.

- It would be interesting to use other directions rather to Nesterov-Todd, such us Alizadeh-HaeberlyOverton (AHO), Monteiro-Zhang (MZ) family and Helmberg et al, Kojima et al and Monteiro (HKM) family, directions for solving the monotone SDLCPs.

- The implementation of Algorithm 3.1 on problems of SDLCPs with large size (high-dimension) is an important topic of research.

- The weighted path method is also an interesting study in order to give a relaxation of getting a good initial starting point for these methods.

- The comparison of our obtained numerical results to other kernel functions be an interesting topic in the future.

- The extension of this analysis for the second-order cone linear complementarity problems (SOCLCP), such that the SOCLCP is to find $x, y \in \mathbb{R}^n$ satisfying

$$y = M(x) + q,\ x^T y = 0,\ x \in \mathcal{L}^n,\ y \in \mathcal{L}^n,$$

  with

$$\mathcal{L}^n = \{(x_0, x_1) \in \mathbb{R} \times \mathbb{R}^{n-1} |\ \ \|x_1\|_2 \leq x_0\} \subseteq \mathbb{R}^n$$

- The extension of these methods for more general semidefinite complementarity problems such as the semidefinite nonlinear complementarity problems (SNLCP).

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ملخص

نقدم في هذه الأطروحة دراسة نظرية و عددية لمسألة التتام الخطية الأفقية ونصف المعرفة الرتيبة .هذه المسألة هي الأكثر دراسة في السنوات الأخيرة. الجزء الأول يهتم بتطوير خوارزمية ذات النقطة الداخلية لحل مسألة التتام الخطي الأفقي باستعمال طريقة المسار المركزي ذات الخطوة الصغيرة . هذه الدراسة تثبت أن الخوارزمية ذات تكلفة حدودية. أخيرا، قمنا بإعطاء بعض النتائج العددية لإظهار فعالية الخوارزمية المقترحة. في الجزء الثاني نقدم خوارزمية النقطة الداخلية من صنف أولى-مرادف ذو خطوة كبيرة اعتمادا على دالة حاجز ذات نواة جديدة و ذلك لحل مسألة التتام الخطية نصف المعرفة. التكلفة الحدودية لهذه الخوارزمية عينت و قمنا بتقديم نتائج عددية و ذلك لإظهار فعالية هذه الأخيرة.

الكلمات المفتاحية : مسألة التتام، طرق النقطة الداخلية، خوارزمية الأولى-المرادف، التكلفة الحدودية

Résumé

Dans cette thèse, nous présentons une étude théorique et algorithmique concernant les problèmes de complémentarité linéaire horizontale (PCLH) et semi-défini (PCLSD) monotone. Ces problèmes sont les plus étudiés ces dernières années. La première partie est concerné par le développement d'un algorithme de trajectoire centrale de type primal-dual à petit pas pour résoudre le problème de complémentarité linéaire horizontale. On montre que l'algorithme admet une complexité polynomiale. Finalement, quelques résultats numériques sont donnés pour montrer l'efficacité de cet algorithme. Dans la deuxième partie un nouvel algorithme de point intérieur de type primal-dual à grand-pas basé sur une nouvelle fonction barrière de noyau pour résoudre le problème de complémentarité linéaire semi-défini est présenté. Son complexité polynomiale est calculée de plus des résultats numériques sont donnés pour montrer l'efficacité de ce dernier.

Mots clés : Problème de complémentarité, méthodes de point-intérieur, algorithme primal-dual, complexité polynomiale

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Abstract

In this thesis, we present a theoretical and algorithmic study for the monotone horizontal linear complementarity problems (HLCPs) and monotone semi-definite problems (SDLCPs). These problems are the most studied in recent years. The first part is concerned with the development of a short-step primal-dual interior-point algorithm for monotone horizontal linear complementarity problems. We prove that the corresponding algorithm has polynomial complexity. Finally, some numerical results are given to show the efficiency of the proposed algorithm. In the second part, a new long-step primal-dual interior-point algorithm for solving SDLCPs based on a new barrier kernel function is presented. Moreover, its complexity polynomial is computed and some numerical results are given to show the effectiveness of this latter.

Keywords: Complementarity problem, interior-point methods, primal-dual path algorithm, polynomial complexity

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