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DOI 10.1007/s12666-016-0949-x

TECHNICAL PAPER

A Theoretical Study of Indentation with an Oblate Spheroid Shape

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Received: 9 February 2016 / Accepted: 17 August 2016

Abstract In this study, firstly we introduced general mathematical concepts related to the geometrical form of ellipse and ellipsoid. Secondly, the hardness was calculated when we used an indenter having a geometrical shape of an oblate spheroid; the mathematical expressions of static hardness were determined as functions of the applied load, the radius or the depth of the imprint area. The general static hardness formula was expressed considering the projected and the real area imprint of the oblate spheroid indenter. Thirdly mathematical assumptions were introduced in order to simplify the mirohardness and nanohardness formulas when we used the real imprint area. Finally we did a comparative study between the oblate spheroid and the spherical indenters.

Keywords Hardness measurement · Mechanical characterization · Indenter · Geometrical modeling · Mathematical formulas

List of symbols

A, B, C	Semi-axes of an ellipsoid
R, C	Semi-axes of the oblate spheroid
D	Diameter of the spherical indenter
d	Diameter of the projected imprint area
H_{ac}	Static hardness of oblate spheroid indenter (case of projected imprint area)
H_{ar}	Static hardness of oblate spheroid indenter (case of real imprint area)
H_m	Microhardness
H_n	Nanohardness
F_a	Applied load on the oblate spheroid indenter
F_S	Applied load on the sphere indenter
S	The projected imprint area
r	Radius of the imprint projected area
h_a	Imprint depth of the oblate spheroid indenter
h_S	Imprint depth of the sphere indenter
S_a	Real imprint area of the oblate spheroid indenter
S_S	Real imprint area of the sphere indenter

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1 Introduction

The different dynamic and static hardness tests have been used to define the hardness of massive and covered materials. In all such cases, a harder indenter penetrates into a relatively softer body [1]. The material static hardness is defined as the mechanical resistance of material under test which opposes the indentation of another harder material [2–10]. It's defined by the ratio of the applied load to the imprint surface.

The indenter geometry form (pyramid, sphere cone… etc.) is a very significant factor in the hardness measured because the resulting imprint of the geometrical form and phenomena occurring during and after the tests (cracking, deformation… etc.) are dependent on it.

We have chosen the geometric form of an indenter as oblate spheroid, for the reason that, it has a rounded tip. However, when its big semi-axis (R) is equivalent to the radius of a spherical indenter, its projected imprint cap have

Published online: 27 August 2016

the same radius as that of the spherical indenter, but with indentation depth smaller than that of the spherical indenter ($h_S > h_a$), thus resulting in a less deformed material indented by the oblate spheroid. Accordingly, the applied load and the phenomena occurring during and after the tests (cracking, deformation…etc.) are reduced. Also, as a blunt indenter, the advantages of an oblate spheroid tip are:

- Possibility of measurement under low stresses
- Obtain elastic and viscoelastic parameters of testing materials without the influence of irreversible processes
- Extended elastic–plastic deformation
- Gradual increase of stresses and strains with increasing indenter depth, enabling the construction of stress–strain diagrams
- Minor pile-ups over little depths of imprint penetration.

However, the oblate spheroid indenter also have disadvantages as;

- Tip geometry is not very pointed
- The oblate spheroid surface is not always perfect.
- The different constitutive and large deformation of imprint can disturb the measurements of the material hardness.

Furthermore, in the present study, the mathematical expression of contact surface (the imprint) resulting from a hard body having an oblate spheroid geometrical shape (the indenter) on a plane surface (testing material) can be developed.

In the present study, an indenter of an oblate spheroid form has been used for measuring the static hardness. Its formulation is based on the semi- axes (R, C) of the indenter, the semi-axis (r) of the projected imprint, the applied load F_a and the imprint depth h_a. Furthermore, the hardness is expressed using a mathematical approach for the real imprint cap.

The paper is organized as follows: Primarily, general theoretical concepts related to the indenter theory are highlighted. Secondly, the hardness is calculated as functions of the imprint projected axes and the applied load. Thirdly, mathematical and geometrical assumptions are made in order to obtain a useful simplified microhardness and nanohardness expressions for the real imprint surface area of the oblate spheroid indenter. Finally, comparative study has been carried out between the imprint depths of the oblate spheroid and the spherical shapes.

2 Mathematical concept

2.1 Standard equation Ellipse

For an ellipse having a radius that changes in between the major radius A, along the x axis and the minor radius B, along the y axis (Fig. 1) [11], the standard equation with its center at the origin of a Cartesian coordinates system can be written as the following:

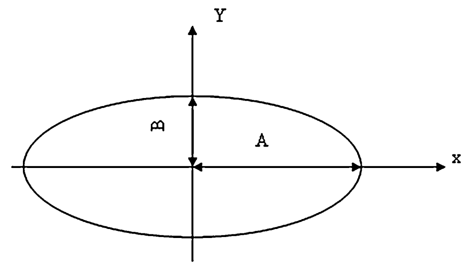

Fig. 1 Ellipse dimensions

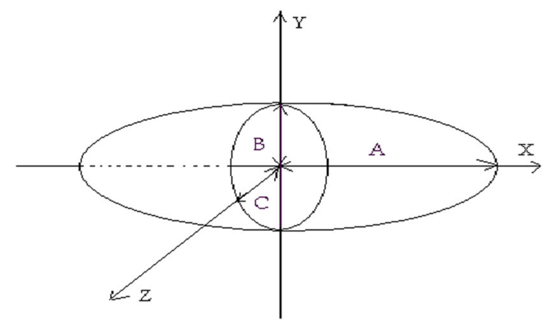

Fig. 2 Ellipsoid dimensions

$$\frac{x^2}{A^2} + \frac{y^2}{B_2} = 1 \tag{1}$$

The area (S) of ellipse surface is:

$$S = \pi AB \tag{2}$$

2.2 General equation of Ellipsoid

The general equation of a tri-axial ellipsoid, centered at the origin of a Cartesian coordinate system and with its semi-principal diameters [11] aligned with the x, y and z axes (Fig. 2) is:

$$\left(\frac{x}{A}\right)^2 + \left(\frac{y}{B}\right)^2 + \left(\frac{z}{C}\right)^2 = 1 \tag{3}$$

For an oblate spheroid indenter of semi-axes (R, R, and C) along OX, OY and OZ axes respectively (Fig. 3), its equation is as follows:

$$\left(\frac{x}{R}\right)^2 + \left(\frac{y}{R}\right)^2 + \left(\frac{z}{C}\right)^2 = 1 \tag{4}$$

By using

$$x^2 + y^2 = r^2 \tag{5}$$

Then the Eq. (6) becomes:

$$\left(\frac{r}{R}\right)^2 + \left(\frac{z}{C}\right)^2 = 1 \tag{6}$$

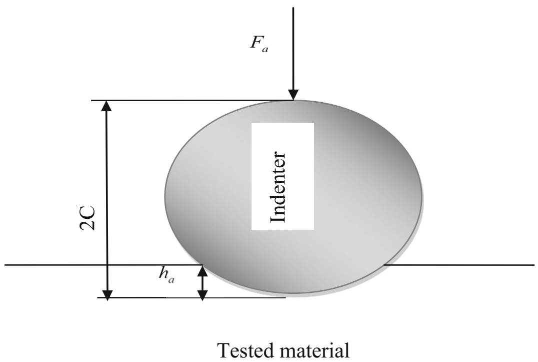

Fig. 3 Penetration of the oblate spheroid indenter

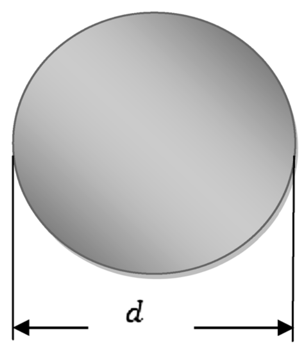

Fig. 4 Area of projected imprint of an oblate spheroid indenter

3 The hardness when the indenter is an oblate spheroid

3.1 Projected imprint area

For an oblate spheroid indenter, the projected imprint is a discus of radius $\frac{d}{2} = r$ (Fig. 4). The projected imprint surface is expressed by the following equation:

$$S = \pi r^2 \tag{7}$$

Then; the hardness can be written as following:

$$H_{ac} = \frac{F_a}{\pi r^2} = \frac{4F_a}{\pi d^2} \tag{8}$$

3.2 Real imprint area

The imprint surface of an oblate spheroid indenter may be computed as a surface of revolution of an oblate spheroid about the z-axis. Its geometrical form is a discus of radius, r and height, h_a (Fig. 3). Accordingly, the surface area of the imprint is expressed by the following formula:

$$S_a = \int_{C-h}^{C} 2\pi r \sqrt{1 + \left(\frac{dr}{dz}\right)^2} dz \tag{9}$$

From the characteristic Eq. (6), the imprint radius, r as a function of z, can be written as follows:

$$r^2 = R^2\left(1 - \frac{z^2}{C^2}\right) \tag{10}$$

The derivative of Eq. (10) leads to the following expression:

$$rr' = -\frac{R^2 z}{C^2} \tag{11}$$

By substituting Eqs. (10) and (11) in (9), the expression of the surface area of the imprint is obtained as:

$$S_a = 2\pi R \int_{C-h}^{C} \sqrt{1 + \frac{z^2}{c^2}\left(\frac{R^2}{C^2} - 1\right)} dz \tag{12}$$

$$\text{Let}: t = \mu\frac{Z}{C}, \quad \text{with} \quad \mu^2 = \frac{R^2 - C^2}{R^2} \tag{13}$$

Then the imprint area becomes:

$$S_a = \frac{2\pi C}{\mu} \int_{\frac{\mu(C-h)}{C}}^{\mu} \sqrt{1 + t^2} dt \tag{14}$$

After integration, the expression of S is:

$$S_a = \frac{\pi C}{\mu}\left[t\sqrt{1+t^2} + \ln\left(t + \sqrt{1+t^2}\right)\right]_0^{\frac{\mu h}{C}} \tag{15}$$

Then, the developed form of the Eq. (15) is:

$$S_a = \frac{\pi C}{\mu}\left[\frac{\mu h}{C}\sqrt{1 + \left(\frac{\mu h_a}{C}\right)^2} + \ln\left(\frac{\mu h_a}{C} + \sqrt{1 + \left(\frac{\mu h_a}{C}\right)^2}\right)\right] \tag{16}$$

From the Eq. (6), the penetration expression h_a becomes a function of both the dimensions of the oblate spheroid indenter (C, R) and the radius r of the projected imprint.

$$h_a = \frac{CR - C\sqrt{R^2 - r^2}}{R} \quad \text{(Accepted)} \tag{17}$$

and

$$h_a = \frac{CR + C\sqrt{R^2 - r^2}}{R} > C \quad \text{(rejected)} \tag{18}$$

Thus, the indentation depth of the oblate spheroid takes the following expression;

$$h_a = C\left(1 - \sqrt{1 - \left(\frac{r}{R}\right)^2}\right) \tag{19}$$

Then, the static hardness becomes:

$$H_{ar} = \frac{\mu F}{\pi C\left(\left(\frac{\mu h_a}{C}\sqrt{\left(\frac{\mu h_a}{C}\right)^2 + 1}\right) + \ln\left(\frac{\mu h_a}{C} + \sqrt{1 + \left(\frac{\mu h_a}{C}\right)^2}\right)\right)} \tag{20}$$

with; $h_a = C\left(1-\sqrt{1-\left(\frac{r}{R}\right)^2}\right)$

For microhardness, considering that ($C \gg h_a$) for a very small imprint, so $\left(\frac{h_a}{C}\right)^2 \approx 0$, the real surface area becomes:

$$S_a = \frac{\pi C}{\mu}\left[\frac{\mu h_a}{C} + \ln\left(\frac{\mu h_a}{C}+1\right)\right] \tag{21}$$

Then, the microhrdness can be written as follows;

$$H_m = \frac{\mu F}{\pi C\left(\left(\frac{\mu h_a}{C}\right) + \ln\left(\frac{\mu h_a}{C} + 1\right)\right)} \tag{22}$$

For a very small imprint, the ratio, $\left(\frac{r}{R}\right)$ is negligible [12]:

$$\left(\sqrt{1-\left(\frac{r}{R}\right)^2}\right) = 1 - \frac{1}{2}\left(\frac{r}{R}\right)^2 \tag{23}$$

Thus, the penetration, h_a takes the following formula:

$$h_a = \frac{C}{2}\left(\frac{r}{R}\right)^2 \tag{24}$$

Then, the expression of the imprint surface becomes:

$$S_a = \frac{\pi C}{\mu}\left[\frac{\mu r^2}{2R^2} + \ln\left(\frac{\mu r^2}{2R^2}+1\right)\right] \tag{25}$$

Thus the microhrdness is;

$$H_m = \frac{\mu F}{\pi C\left(\left(\frac{\mu r^2}{2R^2}\right) + \ln\left(\frac{\mu r^2}{2R^2} + 1\right)\right)} \tag{26}$$

Where, $\frac{r^2}{R^2} \ll 1$, so

$$\left[\ln\left(\frac{\mu r^2}{2R^2}+1\right)\right] \approx 0 \tag{27}$$

The imprint area becomes:

$$S_a = \frac{\pi C r^2}{2R^2} \tag{28}$$

By using this area, the nanohardness is a function of semi-axes (R, C), applied load F_a, the load and the radius (r) can be written as:

$$H_n = \frac{2R^2 F_a}{\pi C r^2} \tag{29}$$

3.3 Comparison between the oblate spheroid and the spherical shape

For an oblate spheroid having a big semi-axis, R equals the spherical indenter radius, $\frac{D}{2}$ (Fig. 5). However, corresponding to each of the oblate sphreroid's bases radius r, the imprint cap also has the same value as the radius of imprint cap base of a spherical indenter (Fig. 6), but with an indentation depth and an applied load (h_a, F_a) smaller than those of the spherical indenter (h_S, F_S).

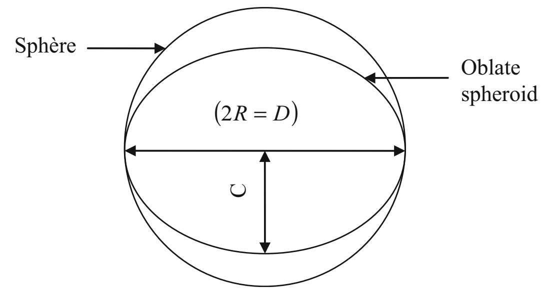

Fig. 5 Schematic to demonstrate the indenter's geometrical forms

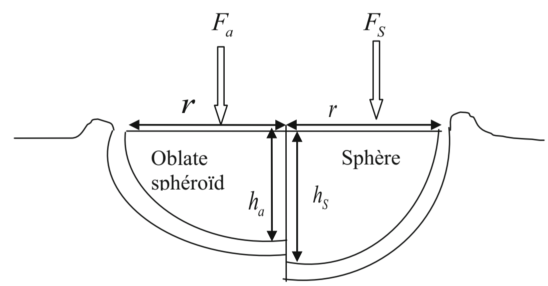

Fig. 6 Indentation data for the two indenter geometries

3.4 Proof of the comparison

The general equation of a sphere centered at the origin of a Cartesian coordinate system and with the diameter ($D = 2R$), is as follow:

$$x^2 + y^2 + z^2 = R^2 \tag{30}$$

From the Eq. (30), the depth expression, h_S becomes a function of both the sphere radius ($D/2 = R$) and the radius r of the projected imprint.

$$h_S = R - \sqrt{R^2 - r^2} \quad \text{(Accepted)} \tag{31}$$

and

$$h_S = R + \sqrt{R^2 - r^2} > R \quad \text{(rejected)} \tag{32}$$

Thus, the spherical indentation depth is;

$$h_S = R\left(1 - \sqrt{1-\left(\frac{r}{R}\right)^2}\right) \tag{33}$$

If we divide the Eq. (19) by (33), we get:

$$\frac{h_a}{h_s} = \frac{C}{R} \tag{34}$$

Since, $R > C$ therefore

$$\frac{h_a}{h_s} > 1 \tag{35}$$

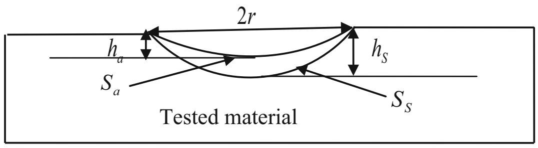

Fig. 7 Imprints of spherical and oblate spheroid indenters

So we can deduce

$$h_S > h_a \quad (36)$$

It's clear from Figs. 6 and 7, that the imprint area resulting from the oblate spheroid, S_a is smaller than that of the spherical indenter S_s, thus it needs a low applied load (F_a) than that in the case of the spherical indenter (F_S)

4 Conclusion

- New geometrical shape of an oblate spheroid indenter was proposed for measuring the static hardness of materials.
- The hardness expression of the oblate spheroid indenter was simple if the projected imprint surface area was considered. In the case of the real imprint (a cap), the mathematical expressions of the hardness became more complex. Simplified expressions for the case of micro-hardness and nanohardness were obtained using geometrical and mathematical assumptions.
- The proposed indenter presented theoretical and experimental interests which widened the field of different static hardness test methods.
- We could conclude, that the indentation by an oblate spheroid resulted a projected imprint area equal to that of the spherical indenter, but with a less depth ($h_S > h_a$) and a low applied load. Therefore the surface of testing material was less deformed and the phenomena occurring during and after the tests (cracking, deformation…etc.) were reduced.

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