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Predictability of fuzzy discrete event systems

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Abstract This paper studies the problem of predictability in fuzzy discrete event systems (FDESs). FDESs combine fuzzy set theory with discrete events systems (DESs). They are proposed in Lin and Ying (2001) to cope with vagueness, impreciseness, and subjectivity in real-world problems. In this work: (1) We propose a fuzzy approach for predictability by introducing fuzzy predictability functions to characterize the predictability degree of a prefix, a faulty trace as well as a faulty event in a fuzzy DES. These functions take values in the interval [0, 1]. The degree of predictability gives a valuable measure to refine the decision about fault predictability. (2) We show that the degree of predictability of a faulty event is always at most equal to its degree of diagnosability. This captures the idea that predictability is stronger than diagnosability. (3) For checking predictability in FDESs, we propose an approach based on the so-called verifier. This approach results in a polynomial (in the number of states) complexity test for the verification of predictability. Our results generalize those of predictability for crisp DESs and allow one to deal with the problem of predictability for both crisp DESs and FDESs.

Keywords Fuzzy predictability · Fuzzy discrete event systems (FDESs) · Fuzzy finite automata · Prediction · Prognosis

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1 Introduction

Nowadays, a great number of technological systems such as queuing systems, traffic systems, manufacturing systems, communication protocols and computer networks can be modeled as discrete event systems (DES) at some abstraction level. A DES is a dynamical process with discrete states. Its dynamic evolves from a state to another state in response to some occurrences of discrete events. The behavior of a DES consists of all traces of events it can execute starting from the initial state. Some sets of traces are desirable but some others are undesirable. The execution of an undesirable trace constitutes a failure.

Besides, current systems are more and more demanding in terms of autonomy. One key notion in increasing system's autonomy is its diagnosability (Jiang et al. 2001; Sampath et al. 1995; Yoo and Lafortune 2002). Indeed, diagnosability is an important property that ensures the possibility to detect the presence of a fault after a finite time of its occurrence. However, diagnosability concerns the ability to determine whether a fault has effectively occurred based on the observations. Sometimes it is very expensive to recover the system after fault occurrence, which motivates the work on the analysis of a stronger property which is predictability, that is the ability to predict with certainty future occurrences of faults based on the observations from the system whose current state is normal.

Likewise, predictability is an important and crucial system's property. If the fault is predicted, the system operator can be warned and may decide to halt the system or otherwise take preventive measures. Predictability has received considerable attention in recent years. The first work that dealt with the problem of predicting the occurrence of a significant event such as a failure event was considered for partially observed discrete event systems in Genc and Lafortune (2009). After that, many works have been proposed including: predictability of pattern occurrences (Jeron et al. 2008), predictability in distributed systems (Kumar and Takai 2010; Takai and Kumar 2012; Ye et al. 2013, 2015), predictability in probabilistic DESs (Jun and Kumar 2014; Nouioua et al. 2016), predictability of events occurrences in partially observable timed systems modeled by timed automata (Cassez and Grastien 2013), bounded predictability (Brandan Briones and Madalinski 2011, 2013; Grastien 2015) and predictability in abstracted models with few states and transitions (Yokotani and Takai 2014). However, the problem of predictability in the framework of fuzzy discrete event systems (FDESs) has not been addressed and most of the recent research on predictability dealt with crisp and probabilistic DESs. In this paper, we try to fill this gap and address the problem of predictability in FDESs.

1.1 Motivation

Imperfection of data is ubiquitous in real applications since it is rarely possible to have a complete, consistent, certain and precise description of a studied system in applicative contexts (Parsons 1996). Imperfection may have different forms including incompleteness, inconsistency, uncertainty and imprecision of available data. Each of these forms requires specific appropriate tools to dealt with.

In the context of discrete event systems we find two main variants in the literature that cope with two different forms of possible imperfection in real applications. The first variant corresponds to probabilistic discrete event systems where from a given state, the uncertainty about the event which will occur and the target state which will be reached after the occurrence of this event are modeled by a probability value associated to each transition of the system. The second variant corresponds to fuzzy discrete event systems used when one has not a precise picture of the system to model. This imprecision concerns the membership in states where a real state is fuzzy and corresponds to being in several crisp states with different membership degrees. Likewise, imprecision may also concern transitions between states: a fuzzy transition may relate simultaneously different couples of states with different possibility degrees; observability of events where a degree of observability is associated with each event instead of a binary classification of events as completely observable or completely unobservable; and finally the faulty nature of events: an event may be faulty to some extent.

It is worth noticing that probabilistic DESs and fuzzy DESs are suitable for two completely different situations that may be encountered in real applications. Probabilistic DES are used when one is uncertain about the transition to be fired from a given state. The probability values may be induced from frequencies extracted for instance from a study of the history of the system's running for a sufficiently long time. Notice that in an effective running of the system, from a given state, one particular transition with one particular event and one particular target occurs. However fuzzy DES are used when one is confronted to a lack of granularity that prevents one to classify the states, the transitions and the events of the system in completely crisp categories. For instance, a fuzzy transition does not correspond in a real running to one particular transition that we have not enough certainty to determine at the modeling stage but to a simultaneous membership, with different degrees, to several crisp transitions.

Of course, these completely different natures of imperfection expressed by probability theory and fuzzy sets theory make them in general incomparable and do not represent two possible alternatives to model the same real situation. According to the nature of imperfection present in a real situation to model, one of the two approaches is more suitable: probabilistic approach is suitable in cases of uncertainty and fuzzy approach is suitable in cases of imprecision.

Much attention has been devoted to probability theory in general and on different applications around probabilistic DES. In the context of diagnosis, there are several works about diagnosability (see. eg. Thorsley and Teneketzis (2005) and Nouioua and Dague (2008)) and predictability (see e.g. Chang et al. (2013), Jun and Kumar (2014), Chen and Kumar (2015), and Nouioua et al. (2016)) in probabilistic DES. A main probabilistic tool used in that works is Markov chains. Indeed, this tool is quite suitable to capture the dynamics of probabilistic systems and the well established results on Markov chains, namely those concerning their asymptotic behavior, allow one to obtain valuable appreciation of the probability of non-predictability and/or non-diagnosability in probabilistic DES.

In contrast, actually the use of FDES follows the idea of computing with words as suggested by Zadeh (1996) as a methodology in which the objects of computation are words. For example, "young", "small", "old", "good", and "large", etc. can be regarded as a possibility distribution which may conform more to human's perception when describing the real world problems. Exactly, there exist countless DESs which can be treated better using computing with words methodology, especially those in biomedicine (Lin and Ying 2002). A convincing example given in Lin and Ying (2001) and Lin and Ying (2002): suppose that the patient's condition is represented with "Good", "Fair", and "Poor", where the patient's condition state can simultaneously belong to "good", "fair" and "poor" with some possibilities. Thus the condition can be written with a word "Patient's Condition" (PC): $PC = x/G + y/F + z/P$ where $x, y, z \in [0, 1]$ denote the possibilities distributions of "good", "fair" and "poor", respectively. In a same manner, the Patient's Condition can be represented with a vector $[xyz]$. In fact, this representation has lead Lin and Ying (2001) and Lin and Ying (2002) to the formulation and the proposition of fuzzy finite automata and fuzzy DESs. Lin and Ying (2001, 2002) initiated the study of FDESs by combining fuzzy logic (Zadeh 1996) to the conventional or crisp discrete event systems (DESs) (Cassandras and

Lafortune 1999), with the aim to solve those problems that cannot be treated by crisp DESs. From then on, increasing attention has been received by this domain, and many important concepts, methods and results have been proposed in the literature (see e.g. Qiu (2005), Cao and Ying (2005, 2006), Cao et al. (2007) among others). The works in (Qiu and Liu 2009; Liu and Dziong 2013) develop an FDES model with fuzzy observability and consider the issues of supervisory control (Qiu and Liu 2009) and reliable control (Liu and Dziong 2013). Especially, FDESs have been successfully applied to biomedical control for HIV/AIDS treatment (Ying et al. 2006, 2007; Lin et al. 2004), to Crisis Management (Traore et al. 2014, 2015) and to robotic control (Jayasiri et al. 2011; Huq et al. 2006a, b). To deal with the problems of failure diagnosis, fuzzy approaches are proposed in Liu and Qiu (2009), Luoa et al. (2012), and Liu (2014).

Clearly, the above mentioned works show the relevance of fuzzy DES for modeling imprecision in several real applications including medicine, robotics and industrial control. Hence, there is obviously a real need to generalize the study of problems related to fault diagnosis, namely those of faults predictability, to the FDES setting. To the best of our knowledge, the present paper is the first one which tackles the problem of defining and checking fault predictability in FDES.

1.2 Contributions of the paper

The main contributions of this paper are threefold:

- We propose fuzzy predictability functions to characterize the predictability degree of (i) a prefix (ii) a faulty trace and (iii) a faulty event in the fuzzy DES. These functions take values from the interval $[0, 1]$ rather than the set $\{0, 1\}$. The fuzzy predictability functions indicate that a significant faulty fuzzy event in a fuzzy system is predictable with some degree. In particular, if the fuzzy predictability function of a faulty event equals one, then we say that the fuzzy event is 1-predictable or completely predictable; if the fuzzy predictability function equals zero, then we say that the fuzzy event is 0-predictable or completely non-predictable; finally if the fuzzy predictability function equals to $\lambda \in]0, 1[$, then we say that the fuzzy event is λ-predictable or partially predictable with degree λ. The interval $]0, 1[$ of the values of λ characterizes a continuous spectrum of possible degrees of partial predictability. In fact, the predictability degree is a valuable measure which can be used to refine the decision about fault predictability. This kind of measure is beneficial in practice: It may be better in contexts where the consequences of a fault are not very critical, to tolerate a system with a sufficiently large (even not maximal) degree of predictability than to add the missing sensors which can be very expensive.
- We show how to capture, in the fuzzy setting, the idea that predictability is stronger than diagnosability by proving that the predictability degree of a faulty event in any FDES is at most equal to its diagnosability degree. We also show that binary predictability in crisp DES may be taken into account as a particular case of our predictability approach in FDES.
- We use the FDES model with fuzzy observability similar to that proposed in Qiu and Liu (2009) and Liu and Qiu (2009) and we design a verifier to check the predictability of occurrences of a faulty fuzzy event in a FDES. The advantage of the verifier approach is that the number of its states is polynomial. This makes more efficient the offline verification of fuzzy predictability. A necessary and sufficient condition for the predictability of FDESs is presented on the verifier.

1.3 Organization of the paper

The remainder of the paper is organized as follows. In Section 2, some preliminaries concerning diagnosability and predictability in crisp DESs are reviewed. In Section 2.3, the fuzzy discrete event systems (FDESs) model is presented. In Section 3, we formalize the notion of fuzzy predictability for FDESs by introducing predictability functions. Section 4 presents a comparison between fuzzy diagnosability and fuzzy predictability. Section 5 is devoted to the discussion of the proposed approach for the verification of predictability in FDESs. In Section 6, we provide an additional example, commonly used in the literature, which is about the treatment process of an animal modeled by a FDES. Finally, Section 7 concludes the paper and gives some perspectives of future work.

2 Preliminaries

In this section, we give some definitions (stated briefly since they are standard) concerning the conventionel (crisp) discrete event systems (DES) theory. After that we recall the notions of diagnosability and predictability in this context. For more details, we can refer to Sampath et al. (1995), Yoo and Lafortune (2002), and Genc and Lafortune (2009).

2.1 Crisp discrete event systems

Finite automata or finite state machines are one of the most used modeling formalisms to model discrete event systems. A finite automaton is denoted by $G = (Q, E, \delta, q_0)$, where Q is a finite set of states, E is a finite set of events, q_0 is the initial state and δ is a partial function from $Q \times E \times Q$ to $\{0, 1\}$. For $q, q' \in Q$ and $a \in E$, we write $(q, a, q') \in \delta$ iff $\delta(q, a, q') = 1$. A string is a finite-length sequence of events in E. Given a string s, the length of s is denoted by $\|s\|$. The set of all strings formed by events in E is denoted by E^* (the Kleene closure of E). δ extends to words $s \in E^*$ with $s = a_1 \dots a_k$ by: $(q, s, q') \in \delta$ iff there is sequence of states $q_{j_0}, \dots, q_{j_k}$ such that $q_{j_0} = q$, $q_{j_k} = q'$ and $(q_{j_{i-1}}, a_i, x_{j_i}) \in \delta$ for $1 \leq i \leq k$. Any subset of E^* is called a language over E. $L(G)$ (or just L if there is no ambiguity in the context) denotes the language over E of strings generated by G. Given a string $s \in L$, L/s is called the post-language of L after s and defined as $L/s = \{t \in E^* : st \in L\}$. Some of the events in E are observable, i.e., their occurrence can be observed (e.g. detected by sensors), while the other events are unobservable. Thus E is partitioned as $E = E_o \cup E_{uo}$ and $E_o \cap E_{uo} = \emptyset$, where E_o and E_{uo} represent the sets of observable and unobservable events, respectively. A language L is live if for any state $q \in Q$, there exists an outgoing transition: $\forall q \in Q, \exists e \in E, \exists q' \in Q$ such that $(q, e, q') \in \delta$.

The projection function $\Pi : E^* \to E_o^*$ is used to remove the unobservable events (elements of E_{uo}) from any trace. Given a string $s \in L$, $\Pi(s)$ is obtained by removing unobservable events in s. The projection Π is defined as: $\Pi(\epsilon) = \epsilon$, and $\Pi(s\sigma) = \Pi(s)\Pi(\sigma)$ for $\sigma \in E$ and $s \in E^*$, where:

$$\Pi(\sigma) = \begin{cases} \sigma, & if \;\; \sigma \in E_o \\ \epsilon, & if \;\; \sigma \in E_{uo} \end{cases} \tag{1}$$

The inverse projection operation of a string $t \in E_o^*$, denoted by $\Pi^{-1}(t)$, is the set of strings in L whose projection is equal to t, and it is given by:

$$\Pi^{-1}(t) = \{s \in L : \Pi(s) = t\} \tag{2}$$

Let $E_f \subseteq E_{uo}$ denote the set of (unobservable) failure events to be predicted.
The set of all sequences that end with a failure event is denoted Ψ, formally:

$$\Psi = \{se \in L : s \in E^*, e \in E_f\} \tag{3}$$

A prefix (resp. proper prefix) of a string $s = a_0a_1...a_n$ is a string $s' = a_0a_1...a_m$ where $m \leq n$ (resp. $m < n$). We denote by $Pref(s)$, the set of all proper prefixes of a string s.

Example 1 Figure 1 shows an example taken from Genc and Lafortune (2009) of a crisp DES $G = (Q, E, \delta, q_0)$, where $Q = \{q_0, q_1, q_2, q_3, q_4, q_5\}$. $E = E_o \cup E_{uo}$ with $E_o = \{\alpha, \beta, \gamma\}$ and $E_{uo} = \{\tau, \theta\}$. The set of fault events is $E_f = \{\theta\}$. The initial state is q_0 and the transition function is shown in Fig. 1.

2.2 Predictability and diagnosability in crisp DES

We recall in Definitions 1, 2 the notions of diagnosability and predictability as introduced in Sampath et al. (1995) and Genc and Lafortune (2009) respectively.

First, a language L is diagnosable with respect to a fault σ if it is possible to detect the occurrences of σ within a finite delay.

Definition 1 Sampath et al. (1995) Let L be a prefix-closed and live language. The occurrence of event σ is diagnosable in L with respect to the projection Π iff

$$(\exists n \in N)(\forall s \in \Psi(\sigma))(\forall t \in L/s)[\|t\| \geq n] \Rightarrow \mathbf{D} \qquad where$$
$$\mathbf{D} = \begin{cases} 1, & if\ \ \forall \omega \in \Pi^{-1}(\Pi(st)) \Rightarrow \sigma \in \omega \\ 0, & otherwise \end{cases}$$

Second, the occurrence of a faulty event in a language is predictable if it is possible to infer its future occurrences based on the observable record of strings that do not contain the event to be predicted.

Definition 2 Genc and Lafortune (2009) Let L be a prefix-closed and live language. The occurrence of event σ is predictable in L with respect to the projection Π iff

$$(\exists n \in N)(\forall s \in \Psi(\sigma))(\exists t \in Pref(s))(\sigma \notin t) \wedge \mathbf{P} \qquad where$$
$$\mathbf{P} = \begin{cases} 1, & if\ (\forall u \in L)(\forall v \in L/u)[\Pi(u) = \Pi(t) \\ \wedge(\sigma \notin u) \wedge \|v\| \geq n] \Rightarrow \sigma \in v \\ 0, & otherwise \end{cases}$$

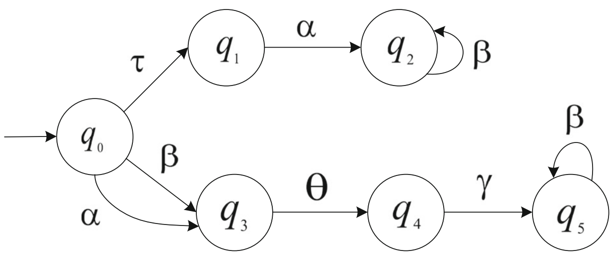

Fig. 1 A crisp Discrete event systems G (Genc and Lafortune 2009)

It has been shown in Genc and Lafortune (2009) that predictability is stronger than diagnosability: Every $\sigma \in E$ which is predictable in L with respect to the projection Π, is also diagnosable with respect to Π but the inverse is not true.

Example 1 (**Cont**) In Example 1 illustrated in Fig. 1, the occurrence of the event θ is not predictable in L, but it is diagnosable. Indeed, as stated in Genc and Lafortune (2009), the set of strings that end with θ is $\Psi(\theta) = \{\alpha\theta, \beta\theta\}$. Let $s = \alpha\theta$ and $t \in Pref(s)$. Let $t = \epsilon$ and $\Pi^{-1}\Pi(t) \cap (E\backslash\theta)^* = \{\tau\}$. When $u = \tau$, $L/u = \alpha\beta^*$. None of the strings in L/u contains θ. Let $t = \alpha$ and $\Pi^{-1}\Pi(t) \cap (E\backslash\theta)^* = \{\tau\alpha, \alpha\}$. If $u = \tau\alpha$, then $L/u = \beta^*$. None of the strings in L/u contains θ. If $u = \alpha$, then $L/u = \theta\gamma\beta^*$. The strings in L/u that have length 1 and greater contain θ. Thus, the condition **P** is not satisfied for all u such that $\Pi(u) = \Pi(t)$ and $\theta \notin u$. This means that there is a string s such that there is no $t \in Pref(s)$ which satisfies the condition in Definition 2. Thus, θ is not predictable in L.

However, θ is diagnosable in L. The set of strings that end with θ is $\Psi(\theta) = \{\alpha\theta, \beta\theta\}$. First let $s = \alpha\theta$ and $t \in L/s$. Let $t = \gamma$, so $\Pi^{-1}(\Pi(st)) = \{\alpha\theta\gamma\}$. All strings in $\Pi^{-1}(\Pi(st))$ contain θ. Now Let $s = \beta\theta$ and $t \in L/s$. Let $t = \gamma$, so $\Pi^{-1}(\Pi(st)) = \{\beta\theta\gamma\}$. All strings in $\Pi^{-1}(\Pi(st))$ contain θ. For all strings in $\Psi(\theta)$ there exists a continuation $t = \gamma$ which satisfies the condition **D** in Definition 1. Thus, θ is diagnosable in L.

2.3 Fuzzy discrete event systems

Fuzzy discrete event systems (FDESs) combine fuzzy set theory (Zadeh 1996) with crisp DESs (Cassandras and Lafortune 1999). They have been successfully applied to many real-world complex systems such as biomedical systems in which vagueness, impreciseness, and subjectivity are typical features.

In this work, we use a FDES model with fuzzy observability which has been proposed in Qiu and Liu (2009) and Liu and Qiu (2009). FDESs are modeled as fuzzy automata with fuzzy states and fuzzy events denoted by vectors and matrices, respectively. In the FDESs framework, we suppose $Q = \{q_1, q_2, \cdots q_n\}$, be the set of the crisp states, then every fuzzy state $\tilde{q}$ is represented with a vector over the set of crisp states, denoted as $[a_1, a_2, ..., a_n]$, where $a_i \in [0, 1]$ represents the possibility that the system being in state q_i. In addition, each event $\tilde{\sigma}$ is a represented by a state transition matrix $matrix[a_{ij}]_{n\times n}$, where $a_{ij} \in [0, 1]$ represents the likelihood of the system changing from state q_i to state q_j. Formally, fuzzy discrete event systems are defined as follows.

Definition 3 Qiu (2005)

A fuzzy automaton is a system $\tilde{G} = (\tilde{Q}, \tilde{E}, \tilde{\delta}, \tilde{q}_0)$ where $\tilde{Q}$ is a set of fuzzy states, $\tilde{q}_0$ is the initial fuzzy state and $\tilde{E}$ is the set of fuzzy events. $\tilde{\delta}$ is partial function from $\tilde{Q} \times \tilde{E}$ to $\tilde{Q}$ and defined as $\tilde{\delta}(\tilde{q}, \tilde{\sigma}) = \tilde{q} \odot \tilde{\sigma}$. Note that $\odot$ is a max-min operation: for a matrix $A[a_{ij}]_{n\times m}$ and a matrix $B[b_{ij}]_{m\times k}$, $A \odot B = [c_{ij}]_{n\times k}$ is defined by:

$$c_{ij} = max_{l=1}^{m} min\{a_{il} b_{lj}\} \tag{4}$$

Example 2 Let $\tilde{G} = (\tilde{Q}, \tilde{E}, \tilde{\delta}, \tilde{q}_0)$ be an FDES. Suppose that we have a fuzzy state $\tilde{q} \in \tilde{Q}$ and a fuzzy event $\tilde{\sigma} \in \tilde{E}$ where $\tilde{q} = [0.3, 0.5]$ and $\tilde{\sigma} = \begin{pmatrix} 0.4 & 0.8 \\ 0.2 & 0.3 \end{pmatrix}$ then:

There is two crisp states in the system, say q_1 and q_2. Being in the fuzzy state $\tilde{q}$ means the fuzzy membership in q_1 with degree 0.3 and the fuzzy membership in q_2 with degree 0.5. The fuzzy event σ expresses the fuzzy membership in four transitions: from q_1 to q_1 with degree 0.4, from q_1 to q_2 with degree 0.8, from q_2 to q_1 with degree 0.2 and from q_2 to q_2 with degree 0.3.

Now, being in fuzzy state $\tilde{q}$ the occurrence of the fuzzy event $\tilde{\sigma}$ leads to the fuzzy state $\tilde{q}'$ given by:

$$\tilde{q}' = \tilde{\delta}(\tilde{q}, \tilde{\sigma}) = \tilde{q} \odot \tilde{\sigma} = [0.3, 0.5] \odot \begin{pmatrix} 0.4 & 0.8 \\ 0.2 & 0.3 \end{pmatrix} = [0.3, 0.3]$$

3 Predictability in fuzzy DES

In this section, we formalize the notion of predictability for FDESs. To do so, fuzzy predictability functions are introduced to characterize the predictability degree of (i) a non-faulty prefix of a trace ending by a faulty event, (ii) a trace ending by a faulty event and (iii) a faulty event in the FDES. Let us first start by giving some basic concepts.

3.1 Basic concepts

In the framework of FDESs, each fuzzy event has simultaneous membership in the observable event set, in the unobservable event set, and in the failure event set; with different degrees. We use three fuzzy subsets: the unobservable events fuzzy subset: $\tilde{\Sigma}_{uo} : \tilde{E} \to [0, 1]$, the observable events fuzzy subset $\tilde{\Sigma}_o : \tilde{E} \to [0, 1]$ and the failure events fuzzy subset $\tilde{\Sigma}_f : \tilde{E} \to [0, 1]$. $\tilde{\Sigma}_f(\tilde{\sigma})$ denotes the possibility of failure occurring on $\tilde{\sigma} \in \tilde{E}$ and $\tilde{\Sigma}_o(\tilde{\sigma})$ (resp. $\tilde{\Sigma}_{uo}(\tilde{\sigma})$) represents the observability (resp. unobservability) degree of $\tilde{\sigma}$. We have: $\tilde{\Sigma}_{uo}(\tilde{\sigma}) + \tilde{\Sigma}_o(\tilde{\sigma}) = 1$.

The language generated by $\tilde{G}$, denoted as $L_{\tilde{G}}$ (or simply L when it is clear from the context), is defined as follows:

$$L = \{\tilde{s} \in \tilde{E}^* : (\exists \tilde{q} \in \tilde{Q})\ \tilde{\delta}(\tilde{q}_0, \tilde{s}) = \tilde{q}\} \tag{5}$$

$\tilde{\Sigma}_f(\tilde{s}) = max\{\tilde{\Sigma}_f(\tilde{\sigma}) : \tilde{\sigma} \in \tilde{s}\}$ and $\tilde{\Sigma}_o(\tilde{s}) = min\{\tilde{\Sigma}_o(\tilde{\sigma}) : \tilde{\sigma} \in \tilde{s}\}$ represent the failure degree and observability degree of the string $\tilde{s} \in L_{\tilde{G}}$, respectively.

We use the maximal observable event set $\tilde{E}_{mo}$ composed of the events having the largest observability degree:

$$\tilde{E}_{mo} = \{\tilde{\sigma} \in \tilde{E} : (\forall \tilde{\sigma'} \in \tilde{E})\ \tilde{\Sigma}_o(\tilde{\sigma}) \geq \tilde{\Sigma}_o(\tilde{\sigma'})\} \tag{6}$$

Remark 1 Let $\tilde{G}$ be an FDES. As usual, we suppose that the language of $\tilde{G}$ is live (for every fuzzy state $\tilde{q}$, there is a fuzzy event $\tilde{\sigma}$ such that $\tilde{\delta}(\tilde{q}, \tilde{\sigma})$ is defined) and $\tilde{G}$ does not contain a cycle in where states are connected only with events whose observability is less than the observability of the event to be predicted.

The post language of L after $\tilde{s}$ is the set of continuations of $\tilde{s}$ in $\tilde{G}$, i.e.

$$L/\tilde{s} = \{\tilde{t} \in \tilde{E}^*\ (\exists \tilde{q} \in \tilde{Q})\ \tilde{\delta}(\tilde{q}_0, \tilde{s}\tilde{t}) = \tilde{q}\} \tag{7}$$

Definition 4 Qiu and Liu (2009) Let $\tilde{\sigma} \in \tilde{E}$, the $\tilde{\sigma}$-projection operation $\tilde{\Pi}_{\tilde{\sigma}} : \tilde{E}^* \to \tilde{E}^*$ is defined as: $\tilde{\Pi}_{\tilde{\sigma}}(\epsilon) = \epsilon$, and $\tilde{\Pi}_{\tilde{\sigma}}(\tilde{s}\tilde{a}) = \tilde{\Pi}_{\tilde{\sigma}}(\tilde{s})\tilde{\Pi}_{\tilde{\sigma}}(\tilde{a})$ for $\tilde{a} \in \tilde{E}$ and $\tilde{s} \in \tilde{E}^*$, where

$$\tilde{\Pi}_{\tilde{\sigma}}(\tilde{a}) = \begin{cases} \tilde{a}, & if\ \ \tilde{a} \in \tilde{E}_{mo}\ or\ \tilde{\Sigma}_o(\tilde{a}) > \tilde{\Sigma}_o(\tilde{\sigma}) \\ \epsilon, & otherwise \end{cases} \tag{8}$$

The inverse projection operation is given by:

$$\tilde{\Pi}_{\tilde{\sigma}}^{-1}(\tilde{t}) = \{\tilde{s} \in L\ : (\exists \tilde{q} \in \tilde{Q})\ \tilde{\delta}(\tilde{q}_0, \tilde{s}) = \tilde{q} \wedge \tilde{\Pi}_{\tilde{\sigma}}(\tilde{s}) = \tilde{t}\ \} \tag{9}$$

Remark 2 The aim of $\tilde{\sigma}$-projection is to erase the fuzzy events in which the observability is lower than $\tilde{\Sigma}_o(\tilde{\sigma})$.

In order to make a correct predictability decision, we specify an upper bound $\tilde{\Sigma}_f(\tilde{\sigma})$ for each $\tilde{\sigma} \in \tilde{E}$. If the possibility of the failure in a string $\tilde{s}$ exceeds a specified bound (i.e., $\tilde{\Sigma}_f(\tilde{s}) > \tilde{\Sigma}_f(\tilde{\sigma})$) then we consider that $\tilde{s}$ is a failure string. In the following, we formalize an approach of "fuzzy predictability" to predict the occurrence of the failure strings that exceed the specified upper bound.

We denote the set of faulty events by $\tilde{E}_f = \{\tilde{\sigma} \in \tilde{E} : \tilde{\Sigma}_f(\tilde{\sigma}) > 0\}$. For $\tilde{\sigma} \in \tilde{E}_f$, the set of all traces that end with an event where the possibility of failure occurring is greater than $\tilde{\Sigma}_f(\tilde{\sigma})$ is defined as follow:

$$\tilde{\Psi}_{\tilde{\sigma}} = \{\tilde{s}\tilde{e} \in L :\ \tilde{e} \in \tilde{E}_f, \tilde{\Sigma}_f(\tilde{e}) > \tilde{\Sigma}_f(\tilde{\sigma})\}.$$

3.2 Fuzzy predictability functions

Before presenting the fuzzy predictability functions, let us first introduce two new concepts that will be used in the definition of these functions.

The first concept, denoted by $//$, is a variant of the notion of a post-language of a trace after another one.

Definition 5 For $\tilde{\sigma} \in \tilde{E}_f$, let $\tilde{t} \in L_{\tilde{G}}$ such that $\tilde{\Sigma}_f(\tilde{t}) < \tilde{\Sigma}_f(\tilde{\sigma})$, We define $L//\tilde{t}$ as follow:

$$\begin{aligned} L//\tilde{t} = \{(\tilde{u}, \tilde{v}) : \tilde{u} \in L_{\tilde{G}} \wedge \tilde{v} \in L/\tilde{u} \wedge \tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(\tilde{t}) \\ \wedge \tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma})\} \end{aligned} \tag{10}$$

Intuitively, $L//\tilde{t}$ is the set of couples $(\tilde{u}, \tilde{v})$ where $\tilde{u}$ is a non-faulty word of L which shares the same observables as $\tilde{t}$ and $\tilde{v}$ is a continuation of $\tilde{u}$ in L. Notice that since one possible value of $\tilde{u}$ is $\tilde{t}$ itself, it holds that: $(\forall \tilde{v} \in L/\tilde{t})\ (\tilde{t}, \tilde{v}) \in L//\tilde{t}$.

The second concept is that of a maximal non-faulty prefix of a faulty trace:

Definition 6 Let $\tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}}$ be a trace of $L_{\tilde{G}}$ ending by a faulty event. The maximal prefix of $\tilde{s}$ without a fault denoted $MaxPref(\tilde{s})$ is defined by:

$\tilde{t} = MaxPref(\tilde{s})$ if and only if:

- $\tilde{t} \in Pref(\tilde{s})$ and $\tilde{\Sigma}_f(\tilde{t}) < \tilde{\Sigma}_f(\tilde{\sigma})$.
- for all $\tilde{t}' \in Pref(\tilde{s})$, if $\tilde{\Sigma}_f(\tilde{t}') < \tilde{\Sigma}_f(\tilde{\sigma})$ and $\tilde{t}' \neq \tilde{t}$ then $\tilde{t}' \in Pref(\tilde{t})$.

Intuitively, $MaxPref(\tilde{s})$ is the prefix of $\tilde{s}$ followed by the first occurrence of an event $\tilde{e}$ in $\tilde{s}$ such that $\tilde{\Sigma}_f(\tilde{e}) \geq \tilde{\Sigma}_f(\tilde{\sigma})$.

Now, we are ready to define fuzzy predictability functions that allow one to quantify predictability degree of a non-faulty trace, of a trace ending by a faulty event and more generally of a FDES with respect of a given faulty event.

We start by defining the predictability degree associated to a non-faulty prefix of a trace ending by a fault. We limit ourselves only to non-faulty prefixes of traces ending by faults because the study of the predictability degree associated to such traces is not an objective per se but it is used later on in determining the predictability degree of a trace ending by a given fault (the maximal degree over all its non-faulty prefixes) which in turn is used to compute the predictability degree of the system with respect to a given fault (the minimal degree over all traces ending by the fault). In addition to this technical argument, intuitively, it does not make sense to associate a fault predictability degree to a prefix which can never be extended to a trace containing a fault.

Definition 7 For $\tilde{\sigma} \in \tilde{E}_f$, let $\tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}}$ and $\tilde{t} \in Pref(\tilde{s})$ such that $\tilde{\Sigma}_f(\tilde{t}) < \tilde{\Sigma}_f(\tilde{\sigma})$ and let $n > 0$. We define the partial predictability function $FP^{(1)}_{\tilde{\sigma}} : \tilde{E}^* \times \mathbb{N} \rightarrow [0, 1]$ which mesures the predictability degree of the prefix $\tilde{t}$ to rank n as follows:

$$FP^{(1)}_{\tilde{\sigma}}(\tilde{t}, n) = \frac{min\{\tilde{\Sigma}_f(\tilde{\sigma}), \tilde{\Sigma}_f(\tilde{\omega}) : \tilde{\omega} \in L(\tilde{G}) \wedge \tilde{\omega} = \tilde{u}\tilde{v} \wedge (\tilde{u}, \tilde{v}) \in L//\tilde{t} \wedge \|\tilde{v}\| = n\}}{\tilde{\Sigma}_f(\tilde{\sigma})} \tag{11}$$

Then, the predictability degree of the prefix $\tilde{t}$ is defined as follows: $\tilde{t}$ is predictable with respect to the fuzzy event $\tilde{\sigma}$ with degree λ iff:

$$(\exists n_0 \in \mathbb{N})(\forall n > n_0)\ FP^{(1)}_{\tilde{\sigma}}(\tilde{t}, n) = \lambda \tag{12}$$

The function $FP^{(1)}_{\tilde{\sigma}}(\tilde{t}, n)$ is computed by taking the minimal degree of fault of all traces of the form $\tilde{u}\tilde{v}$ where $\tilde{u}$ is a non-faulty trace that shares the same observables as $\tilde{t}$ and $\tilde{v}$ is a continuation of $\tilde{u}$ of length n. If this degree (say d) is at least equal to $\tilde{\Sigma}_f(\tilde{\sigma})$ (resp. null) then $FP^{(1)}_{\tilde{\sigma}}(\tilde{t}, n)$ takes the maximal (resp. the minimal) value 1 (resp. 0). Otherwise, $FP^{(1)}_{\tilde{\sigma}}(\tilde{t}, n)$ takes a value in the interval $]0, 1[$ such that, the more the value of d is close to $\tilde{\Sigma}_f(\tilde{\sigma})$ (resp. close to 0), the more the value of $FP^{(1)}_{\tilde{\sigma}}(\tilde{t}, n)$ is close to 1 (resp. close to 0).

The following proposition says that given a prefix $\tilde{t}$, the function $FP^{(1)}_{\tilde{\sigma}}(\tilde{t}, n)$ is an increasing function (with respect to n).

Proposition 1 *Let $\tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}}$ and $\tilde{t} \in Pref(\tilde{s})$ such that $\tilde{\Sigma}_f(\tilde{t}) < \tilde{\Sigma}_f(\tilde{\sigma})$.*

$$(\forall n, n' > 0)\ \ if(n > n')\ \ then\ \ FP^{(1)}_{\tilde{\sigma}}(\tilde{t}, n) \geq FP^{(1)}_{\tilde{\sigma}}(\tilde{t}, n') \tag{13}$$

Notice that the predictability degree λ of a prefix $\tilde{t}$ corresponds to $FP^{(1)}_{\tilde{\sigma}}(\tilde{t}, n)$ for n sufficiently large. Thus it holds that: $\lambda = FP^{(1)}_{\tilde{\sigma}}(\tilde{t}, \infty)$.[1] However, λ is reached for a finite rank $n > 0$ after which it is kept for all ranks greater than n.

Now, using the previous definition, the predictability degree of a trace ending by a fault is defined as the maximal predictability degree over all its non-faulty prefixes.

Definition 8 For $\tilde{\sigma} \in \tilde{E}_f$, let $\tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}}$ and $n > 0$.

We define the partial function $FP^{(2)}_{\tilde{\sigma}} : \tilde{E}^* \times \mathbb{N} \rightarrow [0, 1]$ which measures the predictability degree of $\tilde{s}$ to rank n as follows:

$$FP^{(2)}_{\tilde{\sigma}}(\tilde{s}, n) = max\{FP^{(1)}_{\tilde{\sigma}}(\tilde{t}, n) : \tilde{t} \in Pref(\tilde{s})\ and\ \tilde{\Sigma}_f(\tilde{t}) < \tilde{\Sigma}_f(\tilde{\sigma})\} \tag{14}$$

[1] $FP^{(1)}_{\tilde{\sigma}}(\tilde{t}, \infty)$ is an abbreviation of : $\lim_{n\to\infty} FP^{(1)}_{\tilde{\sigma}}(\tilde{t}, n)$. It is clear that the sequence $FP^{(1)}_{\tilde{\sigma}}(\tilde{t}, n)$ is bounded and from Proposition 1, it is increasing. Hence it is convergent and its limit is its supremum which is λ (from Eq. 12).

Then, the predictability degree of the trace $\tilde{s}$ is defined as follows: $\tilde{s}$ is predictable with respect to the fuzzy event $\tilde{\sigma}$ with degree λ iff:

$$(\exists n_0 \in \mathbb{N})(\forall n > n_0)\ FP^{(2)}_{\tilde{\sigma}}(\tilde{s}, n) = \lambda \tag{15}$$

The following proposition shows that the function $FP^{(2)}_{\tilde{\sigma}}$ is an increasing function (with respect to n).

Proposition 2 *Let* $\tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}}$.

$$(\forall n, n' > 0)\ \ if(n > n')\ \ then\ \ FP^{(2)}_{\tilde{\sigma}}(\tilde{s}, n) \geq FP^{(2)}_{\tilde{\sigma}}(\tilde{s}, n') \tag{16}$$

Notice that the predictability degree λ of a trace $\tilde{s}$ ending by a fault corresponds to $FP^{(2)}_{\tilde{\sigma}}(\tilde{s}, n)$ for n sufficiently large. Thus it holds that: $\lambda = FP^{(2)}_{\tilde{\sigma}}(\tilde{s}, \infty)$.[2] However, λ is reached for a finite rank $n > 0$ after which it is kept for all ranks greater than n.

It turns out that the predictability degree of a trace $\tilde{s}$ ending by a faulty event depends only on the predictability degree of its maximal non-faulty prefix.

Proposition 3 *Let* $\tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}}$ *and* $\tilde{t} = MaxPref(\tilde{s})$

$$(\forall n > 0)\ \ FP^{(2)}_{\tilde{\sigma}}(\tilde{s}, n) = FP^{(1)}_{\tilde{\sigma}}(\tilde{t}, n) \tag{17}$$

Finally, based on the previous definition, we define the predictability degree of a FDES with respect to a faulty event. This degree corresponds to the minimal predictability degree over all the traces ending by the faulty event at hand.

Definition 9 Let $\tilde{H}$ the set of FDES and $\tilde{\sigma} \in \tilde{E}_f$.

We define the function $FP_{\tilde{\sigma}} : \tilde{H} \times \mathbb{N} \rightarrow [0, 1]$ which measures the predictability degree of $\tilde{G}$ to rank n with respect to $\tilde{\sigma}$ as follows:

$$FP_{\tilde{\sigma}}(\tilde{G}, n) = min\{FP^{(2)}_{\tilde{\sigma}}(\tilde{s}, n) : \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}}\} \tag{18}$$

The predictability degree of the FDES $\tilde{G}$ is then defined as follows: $\tilde{G}$ is predictable with respect to the fuzzy event $\tilde{\sigma}$ with degree λ ($\tilde{G}$ is λ-predictable) iff:

$$(\exists n_0 \in \mathbb{N})(\forall n > n_0)\ FP_{\tilde{\sigma}}(\tilde{G}, n) = \lambda \tag{19}$$

Similarly to the previous analyses, the predictability degree λ of a fault $\tilde{\sigma}$ in a FDES $\tilde{G}$ corresponds to $FP_{\tilde{\sigma}}(\tilde{G}, n)$ for n sufficiently large. Thus it holds that: $\lambda = FP_{\tilde{\sigma}}(\tilde{G}, \infty)$.[3] However, λ is reached for a finite rank $n > 0$ after which it is kept for all ranks greater than n.

Using Proposition 3 and then Definition 7, the predictability degree of a FDES with respect to a faulty event may be equivalently characterized as follows:

[2] $FP^{(2)}_{\tilde{\sigma}}(\tilde{t}, \infty)$ is an abbreviation of : $\lim_{n\rightarrow\infty} FP^{(2)}_{\tilde{\sigma}}(\tilde{t}, n)$. It is clear that the sequence $FP^{(2)}_{\tilde{\sigma}}(\tilde{t}, n)$ is bounded and from Proposition 2, it is increasing. Hence it is convergent and its limit is its supremum which is λ (from Eq. 15).

[3] A same reasoning is valid for showing that $FP_{\tilde{\sigma}}(\tilde{G}, \infty) = \lambda$.

Corollary 1 *We have:*

$$FP_{\tilde{\sigma}}(\tilde{G}, n) = min\{FP^{(1)}_{\tilde{\sigma}}(\tilde{t}, n) : \tilde{t} = MaxPref(\tilde{s}) \text{ and } \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}}\} \tag{20}$$

hence:

$$FP_{\tilde{\sigma}}(\tilde{G}, n) = \frac{\begin{array}{r} min\{\tilde{\Sigma}_f(\tilde{\sigma}), \tilde{\Sigma}_f(\tilde{\omega}) : \tilde{\omega} \in L(\tilde{G}), \tilde{\omega} = \tilde{u}\tilde{v}, (\tilde{u}, \tilde{v}) \in L//\tilde{t}, \\ \tilde{t} = MaxPref(\tilde{s}), \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}}, \|\tilde{v}\| = n\} \end{array}}{\tilde{\Sigma}_f(\tilde{\sigma})} \tag{21}$$

Let us now analyze the cases where the fuzzy event $\tilde{\sigma}$ is: (1) Completely predictable $((\exists n_0 \in \mathbb{N})(\forall n > n_0)\ FP_{\tilde{\sigma}}(\tilde{G}, n) = 1)$ in the FDES $\tilde{G}$, (2) partially predicable $((\exists n_0 \in \mathbb{N})(\forall n > n_0)\ FP_{\tilde{\sigma}}(\tilde{G}, n) = \lambda$, with $0 < \lambda < 1)$ and (3) completely non-predictable $((\exists n_0 \in \mathbb{N})(\forall n > n_0)\ FP_{\tilde{\sigma}}(\tilde{G}, n) = 0)$.

Proposition 4 *Let $\tilde{G}$ be an FDES. $\tilde{G}$ is* 1*-predictable iff*

$$\begin{array}{l}(\exists n_0 \in \mathbb{N})(\forall n > n_0)(\forall \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})(\forall \tilde{u} \in L_{\tilde{G}})(\forall \tilde{v} \in L/\tilde{u}) : \\ [\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(MaxPref(\tilde{s})) \text{ and } \tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma}) \text{ and } \|\tilde{v}\| = n] \Rightarrow \\ \tilde{\Sigma}_f(\tilde{u}\tilde{v}) \geq \tilde{\Sigma}_f(\tilde{\sigma})\end{array}$$

Unsurprisingly, the characterization of 1-$predictability$ is very close to that of classical predictability. The main difference consists in replacing the fact that the continuation v must contain the fault by the fact that the degree of fault of $\tilde{u}\tilde{v}$ is not less than the degree of fault of the faulty event $\tilde{\sigma}$ at hand.[4] Now, the λ-predictability with $\lambda \in [0, 1[$ is characterized as follows:

Proposition 5 *Let $\tilde{G}$ be an FDES and $\lambda \in [0, 1[$. $\tilde{G}$ is $\lambda - predictable$ iff:*

$$(\exists n_0 \in \mathbb{N})(\forall n > n_0)\ (D_1 \wedge D_2) where :$$

1. D_1: $(\exists \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})(\exists \tilde{u} \in L_{\tilde{G}})(\exists \tilde{v} \in L/\tilde{u})$:
   $[\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(MaxPref(\tilde{s})) \text{ and } \tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma}) \text{ and } \|\tilde{v}\| = n] \text{ and } \tilde{\Sigma}_f(\tilde{u}\tilde{v}) = \lambda.\tilde{\Sigma}_f(\tilde{\sigma})$
2. D_2: $(\forall \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})(\forall \tilde{u} \in L_{\tilde{G}})(\forall \tilde{v} \in L/\tilde{u})$:
   $[\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(MaxPref(\tilde{s})) \text{ and } \tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma}) \text{ and } \|\tilde{v}\| = n] \Rightarrow \tilde{\Sigma}_f(\tilde{u}\tilde{v}) \geq \lambda.\tilde{\Sigma}_f(\tilde{\sigma})$

Condition D_1 means that we are not in the case of a complete predictability ($\lambda \neq 0$) while condition D_2 determines the exact value of λ for which $\tilde{G}$ is λ-predictable with respect to $\tilde{\sigma}$.

For $\lambda = 0$, condition D_2 in the previous proposition becomes trivial and 0-predictability (complete non-predictability) is given by the following corollary:

[4] Note that since $\tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma})$ and from the definition of the degree of fault of a trace, it follows that $\tilde{\Sigma}_f(\tilde{u}\tilde{v}) \geq \tilde{\Sigma}_f(\tilde{\sigma})$ if and only if $\tilde{\Sigma}_f(\tilde{v}) \geq \tilde{\Sigma}_f(\tilde{\sigma})$.

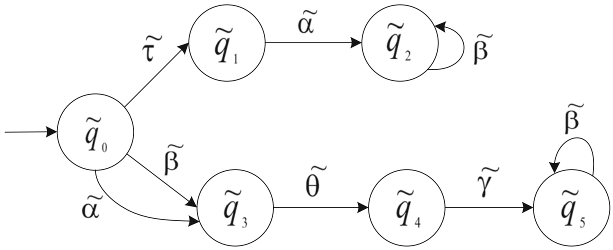

Fig. 2 Fuzzy Discrete event systems $\tilde{G}$

Corollary 2 *Let $\tilde{G}$ be an FDES. $\tilde{G}$ is* 0-*predictable iff:*

$$(\exists n_0 \in N)(\forall n > n_0)(\exists \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})(\exists \tilde{u} \in L_{\tilde{G}})(\exists \tilde{v} \in L/\tilde{u}) :$$
$$[\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(MaxPref(\tilde{s}))\ and\ \tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma})\ and\ \|\tilde{v}\| = n]\ and\ \tilde{\Sigma}_f(\tilde{u}\tilde{v}) = 0 \tag{22}$$

Example 3 Let us take the fuzzy FDES $\tilde{G}$ depicted in Fig. 2 which is a fuzzy version of the DES of Example 1 presented in Fig. 1.

In $\tilde{G} = (\tilde{Q}, \tilde{E}, \tilde{\delta}, \tilde{q}_0)$ the set of fuzzy states is $\tilde{Q} = \{\tilde{q}_0, \tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_4, \tilde{q}_5\}$ and the set of fuzzy events is $\tilde{E} = \{\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\theta}, \tilde{\tau}\}$. Each state in $\tilde{G}$ is denoted by a vector $\tilde{q} = [a_0, a_1, a_2]$ meaning that the system can simultaneously belong to three crisp states with membership degrees a_0 , a_1 and a_2, respectively. Furthermore, a fuzzy event is represented by a 3×3 matrix.

$$\tilde{\alpha} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$$

For instance, the fuzzy event $\alpha_{32} \in [0, 1]$ represents the fact that the system transits from the third crisp state to the second crisp state with membership degree α_{32}. $\tilde{q}_0 = [a_0, a_1, a_2]$ is the initial state. Then, we can calculate the other states by the max-min operation. The observable and fault membership degrees of the fuzzy events are given as follows:

$$\tilde{\Sigma}_o(\tilde{\alpha}) = 1, \tilde{\Sigma}_o(\tilde{\beta}) = 0.8, \tilde{\Sigma}_o(\tilde{\gamma}) = 0.6, \tilde{\Sigma}_o(\tilde{\theta}) = 0.1, \tilde{\Sigma}_o(\tilde{\tau}) = 0$$

$$\tilde{\Sigma}_f(\tilde{\alpha}) = 0, \tilde{\Sigma}_f(\tilde{\beta}) = 0.2, \tilde{\Sigma}_f(\tilde{\gamma}) = 0.3, \tilde{\Sigma}_f(\tilde{\theta}) = 0.7, \tilde{\Sigma}_f(\tilde{\tau}) = 0$$

Let us compute the predictability degree of the fuzzy event $\tilde{\theta}$ in the FDES $\tilde{G}$. We have: $\tilde{\Psi}_{\tilde{\theta}}(\tilde{\Sigma}_f) = \{\tilde{\alpha}\tilde{\theta}, \tilde{\beta}\tilde{\theta}\}$.

– For $\tilde{s} = \tilde{\alpha}\tilde{\theta}$, $\tilde{t} = MaxPref(\tilde{s}) = \tilde{\alpha}$. We have:[5]

$$\begin{cases} \tilde{\Sigma}_f(\tilde{t}) < \tilde{\Sigma}_f(\tilde{\theta}) \\ L//\tilde{t} = \{(\tilde{\tau}\tilde{\alpha}, \tilde{\beta}^*), (\tilde{\alpha}, \tilde{\theta}\tilde{\gamma}\tilde{\beta}^*)\} \end{cases}$$

[5] For the sake of simplification, we abuse notation and we write $L//\tilde{t} = \{(\tilde{\tau}\tilde{\alpha}, \tilde{\beta}^*), (\tilde{\alpha}, \tilde{\theta}\tilde{\gamma}\tilde{\beta}^*)\}$ to indicate the set of all couples of the form $(\tilde{\tau}\tilde{\alpha}, w)$ or of the form $(\tilde{\alpha}, \tilde{\theta}\tilde{\gamma}w)$ where w is any word of the language $\tilde{\beta}^*$.

By Definition 7, we calculate the predictability degree $FP^{(1)}_{\tilde{\theta}}(\tilde{t}, n)$ of $\tilde{t} = \tilde{\alpha}$ by:

$$FP^{(1)}_{\tilde{\theta}}(\tilde{t}, n) = \frac{\begin{array}{r} min\{\tilde{\Sigma}_f(\tilde{\theta}), \tilde{\Sigma}_f(\tilde{\omega}) : \tilde{\omega} \in L(\tilde{G}) \wedge \tilde{\omega} = \tilde{u}\tilde{v} \\ \wedge(\tilde{u}, \tilde{v}) \in L//\tilde{t} \wedge \|\tilde{v}\| = n\} \end{array}}{\tilde{\Sigma}_f(\tilde{\theta})}$$

For $n = 1$ we have:

$$FP^{(1)}_{\tilde{\theta}}(\tilde{t}, 1) = \frac{\begin{array}{r} min\{\tilde{\Sigma}_f(\tilde{\theta}), \tilde{\Sigma}_f(\tilde{\omega}) : \tilde{\omega} \in L(\tilde{G}) \wedge \tilde{\omega} = \tilde{u}\tilde{v} \\ \wedge(\tilde{u}, \tilde{v}) \in \{(\tilde{\tau}\tilde{\alpha}, \tilde{\beta}), (\tilde{\alpha}, \tilde{\theta})\}\} \end{array}}{\tilde{\Sigma}_f(\tilde{\theta})}$$

$$FP^{(1)}_{\tilde{\theta}}(\tilde{t}, 1) = \frac{min\{0.7, 0.2, 0.7\}}{0.7} \simeq 0.28$$

We can check that for all $n \geq 1$, $FP^{(1)}_{\tilde{\theta}}(\tilde{t}, n) = FP^{(1)}_{\tilde{\theta}}(\tilde{t}, 1) \simeq 0.28$. It follows from Proposition 3, for $\tilde{s} = \tilde{\alpha}\tilde{\theta}$ that: $\forall n \geq 1$, $FP^{(2)}_{\tilde{\theta}}(\tilde{s}, n) = FP^{(1)}_{\tilde{\theta}}(\tilde{t}, 1) \simeq 0.28$.

Then, the predictability degree of the faulty string $\tilde{s} = \tilde{\alpha}\tilde{\theta}$ is 0.28.

– For $\tilde{s} = \tilde{\beta}\tilde{\theta}$, $\tilde{t} = MaxPref(\tilde{s}) = \tilde{\beta}$. In a similar way, we use Definition 7 to calculate the predictability degree of $\tilde{t} = \tilde{\beta}$. For $n = 1$ we have:

$$FP^{(1)}_{\tilde{\theta}}(\tilde{t}, 1) = \frac{\begin{array}{r} min\{\tilde{\Sigma}_f(\tilde{\theta}), \tilde{\Sigma}_f(\tilde{\omega}) : \tilde{\omega} \in L(\tilde{G}) \wedge \tilde{\omega} = \tilde{u}\tilde{v} \\ \wedge(\tilde{u}, \tilde{v}) \in \{(\tilde{\beta}, \tilde{\theta})\} \end{array}}{\tilde{\Sigma}_f(\tilde{\theta})} = \frac{min\{0.7, 0.7\}}{0.7} = 1$$

We can check that for all $n \geq 1$, $FP^{(1)}_{\tilde{\theta}}(\tilde{t}, n) = FP^{(1)}_{\tilde{\theta}}(\tilde{t}, 1) = 1$. It follows from Proposition 3, for $\tilde{s} = \tilde{\beta}\tilde{\theta}$ that: $\forall n \geq 1$, $FP^{(2)}_{\tilde{\theta}}(\tilde{s}, n) = FP^{(1)}_{\tilde{\theta}}(\tilde{t}, 1) = 1$.

Then, the predictability degree of the faulty string $\tilde{s} = \tilde{\beta}\tilde{\theta}$ is 1.

Now, from Definition 9, the predictability of the fuzzy event $\tilde{\theta}$ in the FDES $\tilde{G}$ is calculated by: $FP_{\tilde{\theta}}(\tilde{G}, n) = min\{FP^{(2)}_{\tilde{\theta}}(\tilde{s}, n) : \tilde{s} \in \tilde{\Psi}_{\tilde{\theta}}(\tilde{\Sigma}_f)\}$. It follows that: $\forall n \geq 1$, $FP_{\tilde{\theta}}(\tilde{G}, n) = min\{FP^{(2)}_{\tilde{\theta}}(\tilde{s}, n) : \tilde{s} \in \tilde{\Psi}_{\tilde{\theta}}(\tilde{\Sigma}_f)\} = min\{0.28, 1\} = 0.28$

In conclusion, we can say that the fuzzy event $\tilde{\theta}$ is partially predictable in FDES $\tilde{G}$ with degree 0.28. In fact , the fuzzy predictability degree 0.28 means that we cannot exactly predict the occurring failure, but we may say that the possibility of predicting the failure event θ in all sequences of the system is 0.28. This means also that θ belongs only with degree 0.28 to the fuzzy set of predictable events. Note that in Example 1 we have used exactly the same DES but in a non-fuzzy setting. We have shown that event θ is (completely) non predictable. In the present example where the used DES is fuzzy, the decision is no longer binary but gradual. It says that θ has a relatively small degree of predictability which means that it is neither completely predictable nor completely non-predictable, but it is closer to non-predictable events.

3.3 Predictability of crisp DES as a special case of predictability of fuzzy DES

The aim of this section is to show that predictability of fuzzy DES as defined in this paper is a proper generalization of classical predictability defined for crisp DES (see Genc and Lafortune (2009)). In a crisp DES, the observable event set and the failure event set are crisp instead of fuzzy, i.e for each $\tilde{a} \in \tilde{E}$, $\tilde{\Sigma}_o(\tilde{a})$,$\tilde{\Sigma}_f(\tilde{a}) \in \{0, 1\}$. More precisely, for each $\tilde{a} \in \tilde{E}$, $\tilde{\Sigma}_f(\tilde{a}) = 1$ if $\tilde{a} \in \tilde{E}_f$ and $\tilde{\Sigma}_f(\tilde{a}) = 0$ otherwise. Similarly, $\tilde{\Sigma}_o(\tilde{a}) = 1$ if $\tilde{a} \in \tilde{E}_o$ and $\tilde{\Sigma}_o(\tilde{a}) = 0$ otherwise. Accordingly, For a given crisp DES $\tilde{G}$ and a fault σ, the predictability degree of $\tilde{G}$ with respect to σ is either 1 if σ is predictable in $\tilde{G}$ or 0 otherwise.

Let $\tilde{G}$ be a crisp DES and $\tilde{u} \in L$ be a trace of $\tilde{G}$. Then, $\tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma})$ (resp. $\tilde{\Sigma}_f(\tilde{u}) \geq \tilde{\Sigma}_f(\tilde{\sigma})$) is simply replaced by $\sigma \notin \tilde{u}$ (resp. $\sigma \in \tilde{u}$). Proposition 4 is then rewritten as follows:

$$\tilde{G} \text{ is } 1\text{-}predictable \text{ iff} : (\exists n_0 \in \mathbb{N})(\forall n > n_0)(\forall \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})(\forall \tilde{u} \in L_{\tilde{G}})(\forall \tilde{v} \in L/\tilde{u}) :$$
$$[\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(MaxPref(\tilde{s})) \text{ and } (\sigma \notin \tilde{u}) \text{ and } \|\tilde{v}\| = n] \Rightarrow (\sigma \in \tilde{u}\tilde{v})$$

Or equivalently:

$$\tilde{G} \text{ is } 1\text{-}predictable \text{ iff} : (\exists n \in \mathbb{N})(\forall \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})(\forall \tilde{u} \in L_{\tilde{G}})(\forall \tilde{v} \in L/\tilde{u}) :$$
$$[\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(MaxPref(\tilde{s})) \text{ and } (\sigma \notin \tilde{u}) \text{ and } \|\tilde{v}\| \geq n] \Rightarrow (\sigma \in \tilde{v})$$

This is equivalent to the[6] classical predictability characterization for crisp DESs given in Genc and Lafortune (2009).[7]

4 Predictability vs diagnosability in FDES

In this section we show that fuzzy predictability is stronger than fuzzy diagnosability. This idea is captured by the fact that the predictability degree of any FDES $\tilde{G}$ with respect to a fuzzy event $\tilde{\sigma}$ is at most equal to diagnosability degree with respect to the same fuzzy event. We paraphrase the definition of fuzzy diagnosability function used in Liu and Qiu (2009) to adapt it to a form similar to our predictability functions.

Definition 10 Let $\tilde{H}$ be the set of FDESs and $\tilde{\sigma} \in \tilde{E}_f$. The fuzzy diagnosability function is defined as a partial function $FD_{\tilde{\sigma}} : \tilde{H} \times \mathbb{N} \to [0, 1]$ where:

$$FD_{\tilde{\sigma}}(\tilde{G}, n) = \frac{min\{\tilde{\Sigma}_f(\tilde{\sigma}), \tilde{\Sigma}_f(\tilde{\omega}) : \tilde{\omega} \in \tilde{\Pi}^{-1}(\Pi(\tilde{s}.\tilde{s}')), \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}}, \tilde{s}' \in L/\tilde{s}, \|\tilde{s}'\| = n\}}{\tilde{\Sigma}_f(\tilde{\sigma})} \tag{23}$$

$\tilde{G}$ is diagnosable with respect to the fuzzy event $\tilde{\sigma}$ with degree γ iff:

$$(\exists n_0 \in \mathbb{N})(\forall n > n_0)\ FD_{\tilde{\sigma}}(\tilde{G}, n) = \gamma \tag{24}$$

The diagnosability degree γ of a fault $\tilde{\sigma}$ in a FDES $\tilde{G}$ corresponds to $FD_{\tilde{\sigma}}(\tilde{G}, n)$ for n sufficiently large. Thus it holds that: $\gamma = FD_{\tilde{\sigma}}(\tilde{G}, \infty)$. However, γ is reached for a finite rank $n > 0$ after which it is kept for all ranks greater than n. The predictability and the diagnosability degrees are related as follows:

Proposition 6 *Let $\tilde{G}$ be a FDES and $\tilde{\sigma}$ be a fuzzy event in $\tilde{G}$. The degree of predictability of $\tilde{G}$ with respect to $\tilde{\sigma}$ is at most equal to its degree of diagnosability with respect to $\tilde{\sigma}$, i.e., $FP_{\tilde{\sigma}}(\tilde{G}, \infty) \leq FD_{\tilde{\sigma}}(\tilde{G}, \infty)$.*

[6] We have replaced: "$(\exists n_0 \in \mathbb{N})(\forall n > n_0) \ldots \|\tilde{v}\| = n] \Rightarrow \ldots$" by the equivalent form: "$(\exists n \in \mathbb{N}) \ldots \|\tilde{v}\| \geq n] \Rightarrow \ldots$". We replaced also $(\sigma \in \tilde{u}\tilde{v})$ by $(\sigma \in \tilde{v})$. Indeed, since $\sigma \notin \tilde{u}$, $(\sigma \in \tilde{u}\tilde{v})$ if and only if $(\sigma \in \tilde{v})$.

[7] Notice that here, we use the maximal non-faulty prefix of $\tilde{s}$ while in Genc and Lafortune (2009) any non-faulty prefix of $\tilde{s}$ may be used. It is easy to show by a similar idea as that used in the proof of Proposition 3 that the two forms are equivalent.

This proposition is important since it states that the predictability degree of a FDES with respect to a faulty fuzzy event is always at most equal to its diagnosability degree with respect to the same event. This captures, in the fuzzy setting, the fact that predictability is stronger than diagnosability.

Example 3 (Cont) Let us take again the example of the FDES depicted in Fig. 2 and compare the degrees of diagnosability and predictability of the fuzzy event $\tilde{\theta}$. We have shown in Section 3, Example 3 that the fuzzy event $\tilde{\theta}$ in FDES $\tilde{G}$ is partially predictable with degree 0.28.
Now let us use Definition 10 to show that $\tilde{\theta}$ is completely diagnosable in $\tilde{G}$. For the fuzzy event $\tilde{\theta}$, we have: $\tilde{\Psi}_{\tilde{\theta}}(\tilde{\Sigma}_f) = \{\tilde{\alpha}\tilde{\theta}, \tilde{\beta}\tilde{\theta}\}$.

For $n = 1$:

$$FD_{\tilde{\theta}}(\tilde{G}, 1) = \frac{min\{\tilde{\Sigma}_f(\tilde{\theta}), \tilde{\Sigma}_f(w) : w \in \tilde{\Pi}^{-1}(\Pi(\tilde{s}\tilde{t})) \; where \; \tilde{s}\tilde{t} \in \{\tilde{\alpha}\tilde{\theta}\tilde{\gamma}, \tilde{\beta}\tilde{\theta}\tilde{\gamma}\}\}}{\tilde{\Sigma}_f(\tilde{\theta})}$$

$$FD_{\tilde{\theta}}(\tilde{G}, 1) = \frac{min\{\tilde{\Sigma}_f(\tilde{\theta}), \tilde{\Sigma}_f(w) : w \in \{\tilde{\alpha}\tilde{\theta}\tilde{\gamma}, \tilde{\beta}\tilde{\theta}\tilde{\gamma}\}\}}{\tilde{\Sigma}_f(\tilde{\theta})} = \frac{min\{0.7, 0.7, 0.7\}}{0.7} = 1$$

It follows that for $n \geq 1$, $FD_{\tilde{\theta}}(\tilde{G}, 1) = 1$. In summary, the fuzzy event $\tilde{\theta}$ is completely diagnosable but it is only partially predictable with degree 0.28.

5 Verification of predictability in FDES

Until now, we have given the definition of predictability in FDESs and we have shown the relationship between fuzzy diagnosability and fuzzy predictability. In this section, we tackle the problem of verifying predictability in Fuzzy DES. For that purpose, one can adapt two approaches used in verifying predictability in crisp DESs (see (Genc and Lafortune 2009)). The first approach is based on the so-called diagnoser whereas the second one is based on the so-called verifier. For efficiency reasons, we have chosen to develop here the adaptation of the second approach to the fuzzy setting. Indeed, unlike the diagnoser in which the number of states may be exponential on the number of states of the input system, the verifier contains only a polynomial number of states.

5.1 Constructing of the verifier

The verifier is a nondeterministic finite state machine first proposed in Yoo and Lafortune (2002).[8] It has initially been used to check diagnosability (Yoo and Lafortune 2002) and then predictability (Genc and Lafortune 2009) of crisp DES by polynomial time algorithms. We use here a fuzzy version of the verifier with a modified state transition function inspired from (Liu 2014).

Let $\tilde{G} = (\tilde{Q}, \tilde{E}, \tilde{\delta}, \tilde{q}_0)$ be a FDES. We denote by $\tilde{V}_{\tilde{G}}$ the verifier built for $\tilde{G}$ and the fuzzy event $\tilde{\sigma} \in \tilde{E}_f$: $\tilde{V}_{\tilde{G}} = (\tilde{Q}_v, \tilde{E}, \tilde{\delta}_v, \tilde{q}_{v0}, \tilde{\sigma})$ where:

[8] A similar structure called the *twin plant* has been proposed independently in (Jiang et al. 2001) for the same reason. In the twin plant, only observables are kept whereas the verifier used in this paper, keeps both observable and unobservable events.

$\tilde{Q}_v$ is the set of verifier states. It is a subset of $\tilde{Q} \times \{F, N^{\mu}\} \times \tilde{Q} \times \{F, N^{\mu}\}$.[9] Each verifier state $\tilde{q}_v \in \tilde{Q}_v$ is of the form: $\tilde{q}_v = [(\tilde{q}_1, l_1), (\tilde{q}_2, l_2)]$ where $\tilde{q}_1, \tilde{q}_2 \in \tilde{Q}$ and $l_1, l_2 \in \{F, N^{\mu}\}$. The label F stands for "faulty" and means that the possibility of the occurring failure exceeds the specified degree $\tilde{\Sigma}_f(\sigma)$. The label N^{μ} stands for "normal" with degree μ such that $\mu \in \{\tilde{\Sigma}_f(\tilde{a}) : \tilde{a} \in \tilde{E}\}$ where $\mu < \tilde{\Sigma}_f(\tilde{\sigma})$.

$\tilde{q}_{v0} = [(\tilde{q}_0, N^0), (\tilde{q}_0, N^0)]$ is the initial state of the verifier and N^0 means that the verifier starts from a normal state with failure degree 0.

The transition function of the verifier $\tilde{V}_{\tilde{G}}$ is a partial function proposed in Liu (2014) and defined as follows:

Let $\tilde{q}_v = [(\tilde{q}_1, l_1), (\tilde{q}_2, l_2)] \in \tilde{Q}_v$ and $\tilde{a} \in \tilde{E}$. The transition $\tilde{\delta}_v([(\tilde{q}_1, l_1), (\tilde{q}_2, l_2)], \tilde{a})$ is formalized by the following four cases:

1. case (1): if $\tilde{\Sigma}_o(\tilde{a}) \geq \tilde{\Sigma}_o(\tilde{\sigma})$ and $\tilde{\Sigma}_f(\tilde{a}) \geq \tilde{\Sigma}_f(\tilde{\sigma})$ then:
$$\tilde{\delta}_v([(\tilde{q}_1, l_1), (\tilde{q}_2, l_2)], \tilde{a}) = [(\tilde{\delta}(\tilde{q}_1, \tilde{a}), F), (\tilde{\delta}(\tilde{q}_2, \tilde{a}), F)]$$
2. case (2): if $\tilde{\Sigma}_o(\tilde{a}) \geq \tilde{\Sigma}_o(\tilde{\sigma})$ and $\tilde{\Sigma}_f(\tilde{a}) < \tilde{\Sigma}_f(\tilde{\sigma})$ then:
$$\begin{aligned}
\tilde{\delta}_v([(\tilde{q}_1, F), (\tilde{q}_2, F)], \tilde{a}) &= [(\tilde{\delta}(\tilde{q}_1, \tilde{a}), F), (\tilde{\delta}(\tilde{q}_2, \tilde{a}), F)]\\
\tilde{\delta}_v([(\tilde{q}_1, N^{\mu1}), (\tilde{q}_2, F)], \tilde{a}) &= [(\tilde{\delta}(\tilde{q}_1, \tilde{a}), N^{\mu1'}), (\tilde{\delta}(\tilde{q}_2, \tilde{a}), F)]\\
\tilde{\delta}_v([(\tilde{q}_1, F), (\tilde{q}_2, N^{\mu2})], \tilde{a}) &= [(\tilde{\delta}(\tilde{q}_1, \tilde{a}), F), (\tilde{\delta}(\tilde{q}_2, \tilde{a}), N^{\mu2'})]\\
\tilde{\delta}_v([(\tilde{q}_1, N^{\mu1}), (\tilde{q}_2, N^{\mu2})], \tilde{a}) &= [(\tilde{\delta}(\tilde{q}_1, \tilde{a}), N^{\mu1'}), (\tilde{\delta}(\tilde{q}_2, \tilde{a}), N^{\mu2'})]
\end{aligned}$$
where $\mu1' = max\{\mu1, \tilde{\Sigma}_f(\tilde{a})\}$, $\mu2' = max\{\mu2, \tilde{\Sigma}_f(\tilde{a})\}$.
3. case (3):if $\tilde{\Sigma}_o(\tilde{a}) < \tilde{\Sigma}_o(\tilde{\sigma})$ and $\tilde{\Sigma}_f(\tilde{a}) \geq \tilde{\Sigma}_f(\tilde{\sigma})$ then:
$$\tilde{\delta}_v([(\tilde{q}_1, l_1), (\tilde{q}_2, l_2)], \tilde{a}) = \begin{cases} [(\tilde{\delta}(\tilde{q}_1, \tilde{a}), F), (\tilde{q}_2, l_2)] \\ [(\tilde{q}_1, l_1), (\tilde{\delta}(\tilde{q}_2, \tilde{a}), F)] \\ [(\tilde{\delta}(\tilde{q}_1, \tilde{a}), F), (\tilde{\delta}(\tilde{q}_2, \tilde{a}), F)] \end{cases}$$
4. case (4):if $\tilde{\Sigma}_o(\tilde{a}) < \tilde{\Sigma}_o(\tilde{\sigma})$ and $\tilde{\Sigma}_f(\tilde{a}) < \tilde{\Sigma}_f(\tilde{\sigma})$ then:
$$\tilde{\delta}_v([(\tilde{q}_1, F), (\tilde{q}_2, F)], \tilde{a}) = \begin{cases} [(\tilde{\delta}(\tilde{q}_1, \tilde{a}), F), (\tilde{q}_2, F)] \\ [(\tilde{q}_1, F), (\tilde{\delta}(\tilde{q}_2, \tilde{a}), F)] \\ [(\tilde{\delta}(\tilde{q}_1, \tilde{a}), F), (\tilde{\delta}(\tilde{q}_2, \tilde{a}), F)] \end{cases}$$
$$\tilde{\delta}_v([(\tilde{q}_1, N^{\mu1}), (\tilde{q}_2, F)], \tilde{a}) = \begin{cases} [(\tilde{\delta}(\tilde{q}_1, \tilde{a}), N^{\mu1'}), (\tilde{q}_2, F)] \\ [(\tilde{q}_1, N^{\mu1}), (\tilde{\delta}(\tilde{q}_2, \tilde{a}), F)] \\ [(\tilde{\delta}(\tilde{q}_1, \tilde{a}), N^{\mu1'}), (\tilde{\delta}(\tilde{q}_2, \tilde{a}), F)] \end{cases}$$
$$\tilde{\delta}_v([(\tilde{q}_1, F), (\tilde{q}_2, N^{\mu2})], \tilde{a}) = \begin{cases} [(\tilde{\delta}(\tilde{q}_1, \tilde{a}), F), (\tilde{q}_2, N^{\mu2})] \\ [(\tilde{q}_1, F), (\tilde{\delta}(\tilde{q}_2, \tilde{a}), N^{\mu2'})] \\ [(\tilde{\delta}(\tilde{q}_1, \tilde{a}), F), (\tilde{\delta}(\tilde{q}_2, \tilde{a}), N^{\mu2'})] \end{cases}$$
$$\tilde{\delta}_v([(\tilde{q}_1, N^{\mu1}), (\tilde{q}_2, N^{\mu2})], \tilde{a}) = \begin{cases} [(\tilde{\delta}(\tilde{q}_1, \tilde{a}), N^{\mu1'}), (\tilde{q}_2, N^{\mu2})] \\ [(\tilde{q}_1, N^{\mu1}), (\tilde{\delta}(\tilde{q}_2, \tilde{a}), N^{\mu2'})] \\ [(\tilde{\delta}(\tilde{q}_1, \tilde{a}), N^{\mu1'}), (\tilde{\delta}(\tilde{q}_2, \tilde{a}), N^{\mu2'})] \end{cases}$$
where $\mu1' = max\{\mu1, \tilde{\Sigma}_f(\tilde{a})\}$, $\mu2' = max\{\mu2, \tilde{\Sigma}_f(\tilde{a})\}$.

[9] Here we abused notation for the sake of simplification. To take into account all possible values of μ, he exact expression is: $\tilde{Q} \times (\{F\} \cup \{N^{\mu} : \mu \in [0, 1]\}) \times \tilde{Q} \times (\{F\} \cup \{N^{\mu} : \mu \in [0, 1]\})$.

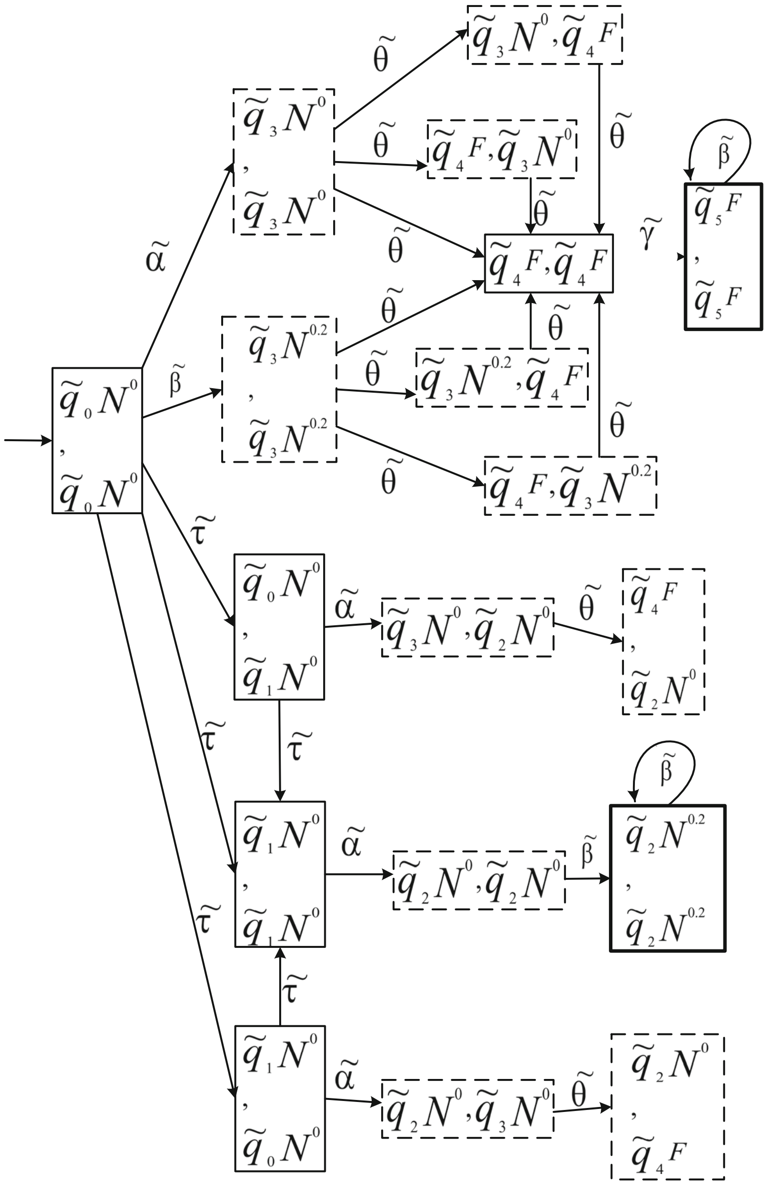

Fig. 3 Verifier $\tilde{V}_{\tilde{G}}$ with respect to $\tilde{\theta}$ of Example 3 FDES $\tilde{G}$

We say that a verifier state $\tilde{q}_v = [(\tilde{q}_1, l_1), (\tilde{q}_2, l_2)]$ is: *normal* if $l_1 = l_2 = N^{\mu}$ or $l_1 = N^{\mu 1}$ and $l_2 = N^{\mu 2}$; *certain* if $l_1 = l_2 = F$; and *uncertain* if $l_1 = F$ and $l_2 = N^{\mu}$ or *vice versa*. We denote by $\tilde{Q}_v^N$ the set of verifier states that are normal, by $\tilde{Q}_v^C$ the set of states that are certain, and by $\tilde{Q}_v^U$ the set of the verifier states that are uncertain.

Example 3 (Cont)

We continue with Example 3. Figure 3 shows the verifier $\tilde{V}_{\tilde{G}}$ with respect to the fuzzy event $\tilde{\theta}$ built from the FDES depicted in Fig. 2. The verifier state $[(\tilde{q}_5, F), (\tilde{q}_5, F)]$ is certain while the state $[(\tilde{q}_2, N^{0.2}), (\tilde{q}_2, N^{0.2})]$ is normal and the state $[(\tilde{q}_2, N^0), (\tilde{q}_4, F)]$ is uncertain.

5.2 Necessary and sufficient condition of predictability for FDESs

After having presented the method of constructing the verifier, let us now give some of its related definitions before giving the necessary and sufficient conditions of the fuzzy predictability for FDESs using the verifier approach. Similarly to the verifier-based predictability checking approach for crisp DES used in Genc and Lafortune (2009) we need to define the set F'_V of verifier states that are normal but immediately followed in the verifier by non-normal states as well as the set F_V of verifier states that share at least on of the two components of a state of F'^V.

Definition 11 Genc and Lafortune (2009) A subset of normal verifier states F'_V is defined as follows:

$$F'_V = \{x_v \in \tilde{Q}_v^N : y_v = \tilde{\delta}_v(x_v, \tilde{t}_{uo}\tilde{\sigma}_p)\ is\ defined\ for\ \tilde{t}_{uo} \in \tilde{E}^*\ and$$
$$\tilde{\Sigma}_f(\tilde{t}_{uo}) < \tilde{\Sigma}_f(\tilde{\sigma})\ and\ \tilde{\Sigma}_o(\tilde{t}_{uo}) < \tilde{\Sigma}_o(\tilde{\sigma})\ and\ \tilde{\Sigma}_f(\tilde{\sigma}_p) \geq \tilde{\Sigma}_f(\tilde{\sigma})\ and\ y_v \notin \tilde{Q}_v^N\}$$

Then, the set F_V is constructed using the states in F'_V as follows:

$$F_V = \{\tilde{q}_v = [(x, l_x), (y, l_y)] \in \tilde{Q}_v^N : [(x, l_x), (.,.)]\ or\ [(.,.), (y, l_y)] \in F'_V\} \tag{25}$$

where (., .) denotes any one of the state components in a verifier state.

The different kinds of cycles present in the verifier are defined as follows:

Definition 12 Let $\tilde{V}_{\tilde{G}} = (\tilde{Q}_v, \tilde{E}, \tilde{\delta}_v, \tilde{q}_{v0}, \tilde{\sigma})$ be the verifier of $\tilde{G}$ with respect to $\tilde{\sigma}$. a set $\{\tilde{q}_{v,1}, \tilde{a}_1, \tilde{q}_{v,2}, \tilde{a}_2, ..., \tilde{q}_{v,k}, \tilde{a}_k, \tilde{q}_{v,1}\}$ is said to form:

- a $\mu - normal$ cycle in $\tilde{V}_{\tilde{G}}$ if $\{\tilde{q}_{v,1}, \tilde{q}_{v,2}, ..., \tilde{q}_{v,k}\}$ are normal states, and the set $\{\tilde{q}_{v,1}, \tilde{a}_1, \tilde{q}_{v,2}, \tilde{a}_2, ..., \tilde{q}_{v,k}, \tilde{a}_k, \tilde{q}_{v,1}\}$ form a cycle in $\tilde{V}_{\tilde{G}}$ and $\mu < \tilde{\Sigma}_f(\tilde{\sigma})$ is the minimum value appearing in the normal labels of the states of the cycle;
- a $certain$ cycle in $\tilde{V}_{\tilde{G}}$ if $\{\tilde{q}_{v,1}, \tilde{q}_{v,2}, ..., \tilde{q}_{v,k}\}$ are certain states, and the set $\{\tilde{q}_{v,1}, \tilde{a}_1, \tilde{q}_{v,2}, \tilde{a}_2, ..., \tilde{q}_{v,k}, \tilde{a}_k, \tilde{q}_{v,1}\}$ form a cycle in $\tilde{V}_{\tilde{G}}$;
- a $\mu - uncertain$ cycle in $\tilde{V}_{\tilde{G}}$ if $\{\tilde{q}_{v,1}, \tilde{q}_{v,2}, ..., \tilde{q}_{v,k}\}$ are uncertain states, and the set $\{\tilde{q}_{v,1}, \tilde{a}_1, \tilde{q}_{v,2}, \tilde{a}_2, ..., \tilde{q}_{v,k}, \tilde{a}_k, \tilde{q}_{v,1}\}$ form a cycle in $\tilde{V}_{\tilde{G}}$ and $\mu < \tilde{\Sigma}_f(\tilde{\sigma})$ is the minimum value appearing in the normal labels of the states of the cycle.

The accessible part of the verifier from a given state is defined as follows:

Definition 13 Genc and Lafortune (2009) Let $\tilde{V}_{\tilde{G}} = (\tilde{Q}_v, \tilde{E}, \tilde{\delta}_v, \tilde{q}_{v0}, \tilde{\sigma})$ be the verifier of $\tilde{G}$ with respect to $\tilde{\sigma}$ and $\tilde{q}_v \in \tilde{Q}_v$. The accessible part of $\tilde{V}_{\tilde{G}}$ with respect to $\tilde{q}_v$ is denoted by $Ac(\tilde{V}_{\tilde{G}}, \tilde{q}_v)$ such that:

$$Ac(\tilde{V}_{\tilde{G}}, \tilde{q}_v) = (\tilde{Q}_{Ac}, \tilde{E}, \tilde{\delta}_{Ac}, \tilde{q}_v) \tag{26}$$

Where $\tilde{Q}_{Ac} = \{\tilde{q}'_v \in \tilde{Q}_v : (\exists \tilde{s} \in \tilde{E}^*)$ s.t. $\tilde{\delta}(\tilde{q}_v, \tilde{s}) = \tilde{q}'_v$ is defined and $\tilde{\delta}_{Ac} = \tilde{\delta}|_{\tilde{Q}_{Ac} \times \tilde{E}}$.[10]

[10]This refers to the restriction of the transition function to the subset $\tilde{Q}_{Ac}$ of states.

Before stating the necessary and sufficient condition of fuzzy predictability, let us first state some useful lemmas.

Lemma 1 Genc and Lafortune (2009) *Let $\tilde{G} = (\tilde{Q}, \tilde{E}, \tilde{\delta}, \tilde{q}_0)$ be a FDES that generates $L_{\tilde{G}}$. and let $\tilde{s}_1, \tilde{s}_2 \in L_{\tilde{G}}$ such that $\tilde{q}_{\tilde{s}_1} = \tilde{\delta}(\tilde{q}_0, \tilde{s}_1)$ and $\tilde{q}_{\tilde{s}_2} = \tilde{\delta}(\tilde{q}_0, \tilde{s}_2)$ are defined, and let $\tilde{V}_{\tilde{G}} = (\tilde{Q}_v, \tilde{E}, \tilde{\delta}_v, \tilde{q}_{v0}, \tilde{\sigma})$ the verifier of $\tilde{G}$ and $\tilde{\sigma}$ then:*

$$\tilde{\Pi}_{\tilde{\sigma}}(\tilde{s}_1) = \tilde{\Pi}_{\tilde{\sigma}}(\tilde{s}_2) \Leftrightarrow \tilde{q}_v = [(\tilde{q}_{\tilde{s}_1}, l_{\tilde{s}_1}), (\tilde{q}_{\tilde{s}_2}, l_{\tilde{s}_2})] \in \tilde{Q}_v \text{ where } l_{\tilde{s}_1}, l_{\tilde{s}_2} \in \{F, N^{\mu}\}$$

Lemma 2 Genc and Lafortune (2009) *Any uncertain or certain verifier state is reached from a verifier state in F_V.*

Lemma 3 *Let $\tilde{G} = (\tilde{Q}, \tilde{E}, \tilde{\delta}, \tilde{q}_0)$ be a FDES and $\tilde{V}_{\tilde{G}} = (\tilde{Q}_v, \tilde{E}, \tilde{\delta}_v, \tilde{q}_{v0}, \tilde{\sigma})$ its verifier. Let $\mu < \tilde{\Sigma}_f(\tilde{\sigma})$. There is a state $\tilde{q}_v \in F_V$ and a cycle C in $Ac(\tilde{V}_{\tilde{G}}, \tilde{q}_v)$ which is normal or uncertain such that each state in C contains a state of $\tilde{G}$ labelled by N^{μ} if and only if the condition H holds:*

$$H : (\exists n_0 \in \mathbb{N})(\forall n > n_0)(\exists \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})(\exists \tilde{u} \in L_{\tilde{G}})(\exists \tilde{v} \in L/\tilde{u}) :$$
$$\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(MaxPref(\tilde{s})) \text{ and } \tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma}) \text{ and } \|\tilde{v}\| = n \text{ and } \tilde{\Sigma}_f(\tilde{u}\tilde{v}) = \mu$$

Now, 1-predictability (complete predictability) is characterized in the verifier in exactly the same manner as in the crisp DES (see Genc and Lafortune (2009)):

Theorem 1 *Let $\tilde{G} = (\tilde{Q}, \tilde{E}, \tilde{\delta}, \tilde{q}_0)$ be a FDES and $\tilde{V}_{\tilde{G}} = (\tilde{Q}_v, \tilde{E}, \tilde{\delta}_v, \tilde{q}_{v0}, \tilde{\sigma})$ its verifier. The fuzzy event $\tilde{\sigma}$ is 1-predictable (completely predictable) in $\tilde{V}_{\tilde{G}}$ if and only if P_v holds where:*

$$P_v : for\ all\ \tilde{q}_v \in F_V, there\ is\ only\ certain\ cycles\ in\ Ac(\tilde{V}_{\tilde{G}}, \tilde{q}_v).$$

The case of partial predictability i.e., predictability of degree $\lambda < 1$ is characterized by the following theorem.

Theorem 2 *Let $\tilde{G} = (\tilde{Q}, \tilde{E}, \tilde{\delta}, \tilde{q}_0)$ be a FDES and $\tilde{V}_{\tilde{G}} = (\tilde{Q}_v, \tilde{E}, \tilde{\delta}_v, \tilde{q}_{v0}, \tilde{\sigma})$ its verifier. The fuzzy event $\tilde{\sigma}$ is λ-predictable (partially predictable with the degree λ) in $\tilde{G}$ if and only if conditions $R1$ and $R2$ hold where:*
$R1$: there is $\tilde{q}_v \in F_V$, there is cycle C in $Ac(\tilde{V}_{\tilde{G}}, \tilde{q}_v)$ which is normal or uncertain.
$R2$: the minimal μ-normal or μ-uncertain in $Ac(\tilde{V}_{\tilde{G}}, \tilde{q}_v)$ for all $\tilde{q}_v \in F_V$ satisfies $\mu = \lambda.\tilde{\Sigma}_f(\tilde{\sigma})$

Condition $R1$ states that we are not in the situation of completely predictability. Condition $R2$ allows one to quantify the predictability degree by using the minimal μ-normal cycle (having the minimal value of μ in its labels of the form N^{μ}) accessible from a state of F_V. The particular case of 0-predictability is then given as follows:

Corollary 3 *Let $\tilde{G} = (\tilde{Q}, \tilde{E}, \tilde{\delta}, \tilde{q}_0)$ be a FDES and $\tilde{V}_{\tilde{G}} = (\tilde{Q}_v, \tilde{E}, \tilde{\delta}_v, \tilde{q}_{v0}, \tilde{\sigma})$ its verifier. The fuzzy event $\tilde{\sigma}$ is 0-predictable (completely non-predictable) in $\tilde{G}$ if and only if there is $\tilde{q}_v \in F_V$ and there is a 0-normal or a 0-uncertain cycle in $Ac(\tilde{V}_{\tilde{G}}, \tilde{q}_v)$.*

Example 3 (Cont) We take again Example 3 given above. Let us use Theorem 2 to calculate the predictability degree of the fuzzy event $\tilde{\theta}$ in the FDES depicted in Fig. 2, where its verifier

$\tilde{V}_{\tilde{G}}$ is shown in the Fig. 3. (In Section 3.2, we have used Definitions 7 and 9 to prove that $\tilde{\theta}$ is partially predictable with a degree $\lambda = 0.28$.) First, we determine from the verifier that the set F'_V containing the normal states that have a certain or uncertain verifier as immediate successor: $F'_V = \{[(\tilde{q}_3, N^0), (\tilde{q}_3, N^0)], [(\tilde{q}_3, N^{0.2}), (\tilde{q}_3, N^{0.2})], [(\tilde{q}_3, N^0), (\tilde{q}_2, N^0)], [(\tilde{q}_2, N^0), (\tilde{q}_3, N^0)]\}$. And from F'_V we construct the set $F_V = F'_V \cup \{[(\tilde{q}_2, N^{0.2}), (\tilde{q}_2, N^{0.2})], [(\tilde{q}_3, N^0), (\tilde{q}_4, F)], [(\tilde{q}_4, F), (\tilde{q}_3, N^0)], [(\tilde{q}_3, N^{0.2}), (\tilde{q}_4, F)], [(\tilde{q}_4, F), (\tilde{q}_3, N^{0.2})], [(\tilde{q}_4, F), (\tilde{q}_2, N^0)], [(\tilde{q}_2, N^0), (\tilde{q}_4, F)]\}$. The states of the set F_V are presented with dashed rectangles.

Clearly, $Ac(\tilde{V}_{\tilde{G}}, [(\tilde{q}_2, N^{0.2}), (\tilde{q}_2, N^{0.2})])$ contains a single cycle that is a minimal 0.2-normal cycle and it is presented in the verifier with bold-border rectangle. Thus, based on Theorem 2 the fuzzy event $\tilde{\theta}$ is partially predictable with a degree $\lambda = \dfrac{\mu}{\tilde{\Sigma}_f(\tilde{\sigma})} = 0.2/0.7 = 0.28$.

6 Another illustrative example: Treatment process on an animal

Consider the treatment process of an animal, modeled by fuzzy DES $\tilde{G}_3$ depicted in Fig. 4 (this example is inspired from Liu and Qiu (2009)). This animal becoming sick with a new disease. The drugs Theophylline, Ipratropium Bromide, Erythromycin Ethylsuccinate, and Dopamine are denoted by fuzzy event $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ and $\tilde{\theta}$, respectively. The doctor believes that these drugs may be useful in treating the disease. A state in this fuzzy DES is denoted by a vector $\tilde{q} = (a_1, a_2, a_3)$ which means that the animal's condition can simultaneously belong to "good", "fair" and "poor" with respective membership degrees a_1, a_2 and a_3. The initial state is $\tilde{q}_0 = [0.9, 0.1, 0]$ and the other states calculated using max-min operation (see Definition 3) are: $\tilde{q}_1 = [0.4, 0.9, 0.4]$, $\tilde{q}_2 = [0.9, 0.4, 0.4]$, $\tilde{q}_3 = [0.4, 0.9, 0.4]$, $\tilde{q}_4 = [0.9, 0.9, 0.4]$, $\tilde{q}_5 = [0.5, 0.1, 0]$, $\tilde{q}_6 = [0.5, 0.4, 0.4]$ and $\tilde{q}_7 = [0.5, 0.4, 0.4]$.

Since it is imprecise to determine the exact point at which the animal has evolved from one state to another after a drug treatment, each fuzzy event is modeled by a 3×3 matrix.

$$\tilde{\alpha} = \begin{pmatrix} 0.4 & 0.9 & 0.4 \\ 0 & 0.4 & 0.4 \\ 0 & 0 & 0.4 \end{pmatrix} \tilde{\beta} = \begin{pmatrix} 0.4 & 0 & 0 \\ 0.9 & 0.4 & 0 \\ 0.4 & 0.4 & 0.4 \end{pmatrix} \tilde{\gamma} = \begin{pmatrix} 0.9 & 0.9 & 0.4 \\ 0 & 0.4 & 0.4 \\ 0 & 0 & 0.4 \end{pmatrix} \tilde{\theta} = \begin{pmatrix} 0.5 & 0 & 0 \\ 0.1 & 0.1 & 0 \\ 0.1 & 0.1 & 0.1 \end{pmatrix}$$

Suppose that the observability degrees and the failure possibilities of the events are defined as follows:

$$\tilde{\Sigma}_o(\tilde{\alpha}) = 0.5,\ \tilde{\Sigma}_o(\tilde{\beta}) = 0.4,\ \tilde{\Sigma}_o(\tilde{\gamma}) = 0.6,\ \tilde{\Sigma}_o(\tilde{\theta}) = 0.3$$
$$\tilde{\Sigma}_f(\tilde{\alpha}) = 0.1,\ \tilde{\Sigma}_f(\tilde{\beta}) = 0.2,\ \tilde{\Sigma}_f(\tilde{\gamma}) = 0.3,\ \tilde{\Sigma}_f(\tilde{\theta}) = 0.4.$$

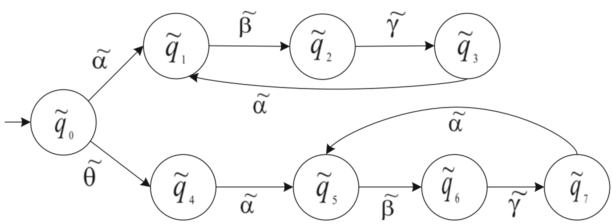

Fig. 4 Treatment process (Liu and Qiu 2009) modeled by FDES $\tilde{G}_3$

6.1 Using the fuzzy predictability definition

1. For the failure event $\tilde{\alpha}$, we have: $\tilde{\Psi}_{\tilde{\alpha}}(\tilde{\Sigma}_f) = \{\tilde{\alpha}, \tilde{\theta}\}$. Notice that $\tilde{\theta}$ is added to $\tilde{\Psi}_{\tilde{\alpha}}(\tilde{\Sigma}_f)$ because $\tilde{\Sigma}_f(\tilde{\theta}) > \tilde{\Sigma}_f(\tilde{\alpha})$.

   – For $\tilde{s} = \tilde{\alpha}, \tilde{t} = MaxPref(\tilde{s}) = \epsilon$. We have:

   $$\begin{cases} \tilde{\Sigma}_f(\tilde{t}) < \tilde{\Sigma}_f(\tilde{\alpha}) \\ L//\tilde{t} = \{(\epsilon, \tilde{\alpha}(\tilde{\beta}\tilde{\gamma}\tilde{\alpha})^*), (\epsilon, \tilde{\theta}\tilde{\alpha}(\tilde{\beta}\tilde{\gamma}\tilde{\alpha})^*)\} \end{cases}$$

   For $n = 1$ we have:

   $$FP^{(1)}_{\tilde{\alpha}}(\tilde{t}, 1) = \frac{\begin{array}{r} min\{\tilde{\Sigma}_f(\tilde{\alpha}), \tilde{\Sigma}_f(\tilde{\omega}) : \tilde{\omega} \in L(\tilde{G}) \wedge \tilde{\omega} = \tilde{u}\tilde{v} \\ \wedge(\tilde{u}, \tilde{v}) \in \{(\epsilon, \tilde{\alpha}), (\epsilon, \tilde{\theta})\}\} \end{array}}{\tilde{\Sigma}_f(\tilde{\alpha})} = 1$$

   We can check that for all $n \geq 1$ we have: $FP^{(1)}_{\tilde{\alpha}}(\tilde{t}, n) = FP^{(1)}_{\tilde{\alpha}}(\tilde{t}, 1) = 1$.
   It follows from Proposition 3, for $\tilde{s} = \tilde{\alpha}$ that:

   $$\forall n \geq 1, FP^{(2)}_{\tilde{\alpha}}(\tilde{s}, n) = FP^{(1)}_{\tilde{\alpha}}(\tilde{t}, 1) = 1$$

   Then, the predictability degree of the faulty string $\tilde{s} = \tilde{\alpha}$ is 1.

   – For $\tilde{s} = \tilde{\theta}, \tilde{t} = MaxPref(\tilde{s}) = \epsilon$.
   For $n = 1$ we have:

   $$FP^{(1)}_{\tilde{\alpha}}(\tilde{t}, 1) = \frac{\begin{array}{r} min\{\tilde{\Sigma}_f(\tilde{\alpha}), \tilde{\Sigma}_f(\tilde{\omega}) : \tilde{\omega} \in L(\tilde{G}) \wedge \tilde{\omega} = \tilde{u}\tilde{v} \\ \wedge(\tilde{u}, \tilde{v}) \in \{(\epsilon, \tilde{\alpha}), (\epsilon, \tilde{\theta})\}\} \end{array}}{\tilde{\Sigma}_f(\tilde{\alpha})} = 1$$

   We can check that for all $n \geq 1$ we have: $FP^{(1)}_{\tilde{\alpha}}(\tilde{t}, n) = FP^{(1)}_{\tilde{\alpha}}(\tilde{t}, 1) = 1$.
   It follows from Proposition 3, for $\tilde{s} = \tilde{\theta}$ that:

   $$\forall n \geq 1, FP^{(2)}_{\tilde{\alpha}}(\tilde{s}, n) = FP^{(1)}_{\tilde{\alpha}}(\tilde{t}, 1) = 1$$

   Then, the predictability degree of the faulty string $\tilde{s} = \tilde{\theta}$ is 1.

   Now, from Definition 9, the predictability of the fuzzy event $\tilde{\alpha}$ in the FDES $\tilde{G}_3$ is calculated by: $FP_{\tilde{\alpha}}(\tilde{G}_3, n) = min\{FP^{(2)}_{\tilde{\alpha}}(\tilde{s}, n) : \tilde{s} \in \tilde{\Psi}_{\tilde{\alpha}}(\tilde{\Sigma}_f)\}$.

   It follows that: $\forall n \geq 1, FP_{\tilde{\alpha}}(\tilde{G}_3, n) = min\{FP^{(2)}_{\tilde{\alpha}}(\tilde{s}, n) : \tilde{s} \in \{\tilde{\alpha}, \tilde{\theta}\}\} = 1$.

   So, the fuzzy event $\tilde{\alpha}$ is completelly predictable in $\tilde{G}_3$ which means that the possibility of failure occurring (i.e., high white blood cell count) is predictable on each treatment strategy.

2. For the failure event $\tilde{\beta}$, we have, $\tilde{\Psi}_{\tilde{\beta}}(\tilde{\Sigma}_f) = \{\tilde{\alpha}\tilde{\beta}, \tilde{\theta}\}$. We have added $\tilde{\theta}$ because $\tilde{\Sigma}_f(\tilde{\theta}) > \tilde{\Sigma}_f(\tilde{\beta})$. In a similar way as for the fuzzy event $\tilde{\alpha}$ we obtain: $\forall n \geq 1, FP_{\tilde{\beta}}(\tilde{G}_3, n) = min\{FP^{(2)}_{\tilde{\beta}}(\tilde{s}, n) : \tilde{s} \in \tilde{\Psi}_{\tilde{\beta}}(\tilde{\Sigma}_f)\} = min\{1, 1\} = 1$

   It follows that the fuzzy event $\tilde{\beta}$ is completely predictable in $\tilde{G}_3$.

3. For the failure event $\tilde{\gamma}$, we can use the same method as for $\tilde{\alpha}$ and $\tilde{\beta}$, we obtain:

   $$\forall n \geq 1, FP_{\tilde{\gamma}}(\tilde{G}_3, n) = 1$$

We conclude that the fuzzy event $\tilde{\gamma}$ is also completely predictable in $\tilde{G}_3$.

4. For the failure event $\tilde{\theta}$, we have: $\tilde{\Psi}_{\tilde{\theta}}(\tilde{\Sigma}_f) = \{\tilde{\theta}\}$.

   For $\tilde{s} = \tilde{\theta}, \tilde{t} = MaxPref(\tilde{s}) = \epsilon$. We have:

$$\begin{cases} \tilde{\Sigma}_f(\tilde{t}) < \tilde{\Sigma}_f(\tilde{\theta}) \\ L//\tilde{t} = \{(\epsilon, \tilde{\alpha}(\tilde{\beta}\tilde{\gamma}\tilde{\alpha})^*), (\epsilon, \tilde{\theta}\tilde{\alpha}(\tilde{\beta}\tilde{\gamma}\tilde{\alpha})^*)\} \end{cases}$$

   For $n = 1$ we have:

$$FP^{(1)}_{\tilde{\theta}}(\tilde{t}, 1) = \frac{\begin{array}{r} min\{\tilde{\Sigma}_f(\tilde{\theta}), \tilde{\Sigma}_f(\tilde{\omega}) : \tilde{\omega} \in L(\tilde{G}) \wedge \tilde{\omega} = \tilde{u}\tilde{v} \\ \wedge(\tilde{u}, \tilde{v}) \in \{(\epsilon, \tilde{\alpha}), (\epsilon, \tilde{\theta})\}\} \end{array}}{\tilde{\Sigma}_f(\tilde{\theta})} = 0.25$$

For $n = 2$ we have:

$$FP^{(1)}_{\tilde{\theta}}(\tilde{t}, 2) = \frac{\begin{array}{r} min\{\tilde{\Sigma}_f(\tilde{\theta}), \tilde{\Sigma}_f(\tilde{\omega}) : \tilde{\omega} \in L(\tilde{G}) \wedge \tilde{\omega} = \tilde{u}\tilde{v} \\ \wedge(\tilde{u}, \tilde{v}) \in \{(\epsilon, \tilde{\alpha}\tilde{\beta}), (\epsilon, \tilde{\theta}\tilde{\alpha})\}\} \end{array}}{\tilde{\Sigma}_f(\tilde{\theta})} = 0.5$$

For $n = 3$ we have:

$$FP^{(1)}_{\tilde{\theta}}(\tilde{t}, 3) = \frac{\begin{array}{r} min\{\tilde{\Sigma}_f(\tilde{\theta}), \tilde{\Sigma}_f(\tilde{\omega}) : \tilde{\omega} \in L(\tilde{G}) \wedge \tilde{\omega} = \tilde{u}\tilde{v} \\ \wedge(\tilde{u}, \tilde{v}) \in \{(\epsilon, \tilde{\alpha}\tilde{\beta}\tilde{\gamma}), (\epsilon, \tilde{\theta}\tilde{\alpha}\tilde{\beta})\}\} \end{array}}{\tilde{\Sigma}_f(\tilde{\theta})} = 0.75$$

We can check that for all $n \geq 3$ we have: $FP^{(1)}_{\tilde{\theta}}(\tilde{t}, n) = FP^{(1)}_{\tilde{\theta}}(\tilde{t}, 3) = 0.75$.

   It follows from Proposition 3, for $\tilde{s} = \tilde{\theta}$ that:

$$\forall n \geq 3, FP^{(2)}_{\tilde{\theta}}(\tilde{s}, n) = FP^{(1)}_{\tilde{\theta}}(\tilde{t}, 3) = 0.75$$

   Then, the predictability degree of the faulty string $\tilde{s} = \tilde{\theta}$ is 0.75.

Now, from Definition 9, the predictability of the fuzzy event $\tilde{\theta}$ in the FDES $\tilde{G}_3$ is calculated by: $FP_{\tilde{\theta}}(\tilde{G}_3, n) = min\{FP^{(2)}_{\tilde{\theta}}(\tilde{s}, n) : \tilde{s} \in \tilde{\Psi}_{\tilde{\theta}}(\tilde{\Sigma}_f)\}$.

   It follows that: $\forall n \geq 3, FP_{\tilde{\theta}}(\tilde{G}_3, n) = min\{FP^{(2)}_{\tilde{\theta}}(\tilde{s}, n) : \tilde{s} \in \tilde{\Psi}_{\tilde{\theta}}(\tilde{\Sigma}_f)\} = 0.75$.

   In conclusion, we can say that the fuzzy event $\tilde{\theta}$ is 0.75-predictable in $\tilde{G}_3$, i.e., partially predictable with degree $\lambda = 0.75$. The result shows that we cannot predict the failure occurring on the treatment strategy $\tilde{\omega}$ containing the drug Dopamine, but we may ensure that the possibilities of failure occurring on all sequences of the system are at least 0.75 times that $\tilde{\Sigma}_f(\tilde{\theta})$. So, we can say that the fuzzy FDES $\tilde{G}$ depicted in Fig. 4 is 75% predictable.

6.2 Using the verifier approach

Now we use the verifier approach to verify the fuzzy predictability of fuzzy events in the FDES $\tilde{G}_3$.

1. The verifier with respect to $\tilde{\beta}$ is shown in Fig. 5. The set F'_V of normal verifier states with an uncertain or a certain verifier state as an immediate successor, is $F'_V = \{[(\tilde{q}_0, N^0), (\tilde{q}_0, N^0)], [(\tilde{q}_1, N^{0.1}), (\tilde{q}_1, N^{0.1})]\}$. We can check that $F_V = F'_V \cup \{[(\tilde{q}_2, F), (\tilde{q}_1, N^{0.1})], [(\tilde{q}_1, N^{0.1}), (\tilde{q}_2, F)], [(\tilde{q}_4, F), (\tilde{q}_0, N^0)], [(\tilde{q}_5, F), (\tilde{q}_1, N^{0.1})], [(\tilde{q}_6, F), (\tilde{q}_1, N^{0.1})], [(\tilde{q}_0, N^0), (\tilde{q}_4, F)], [(\tilde{q}_1, N^{0.1}), (\tilde{q}_5, F)],$

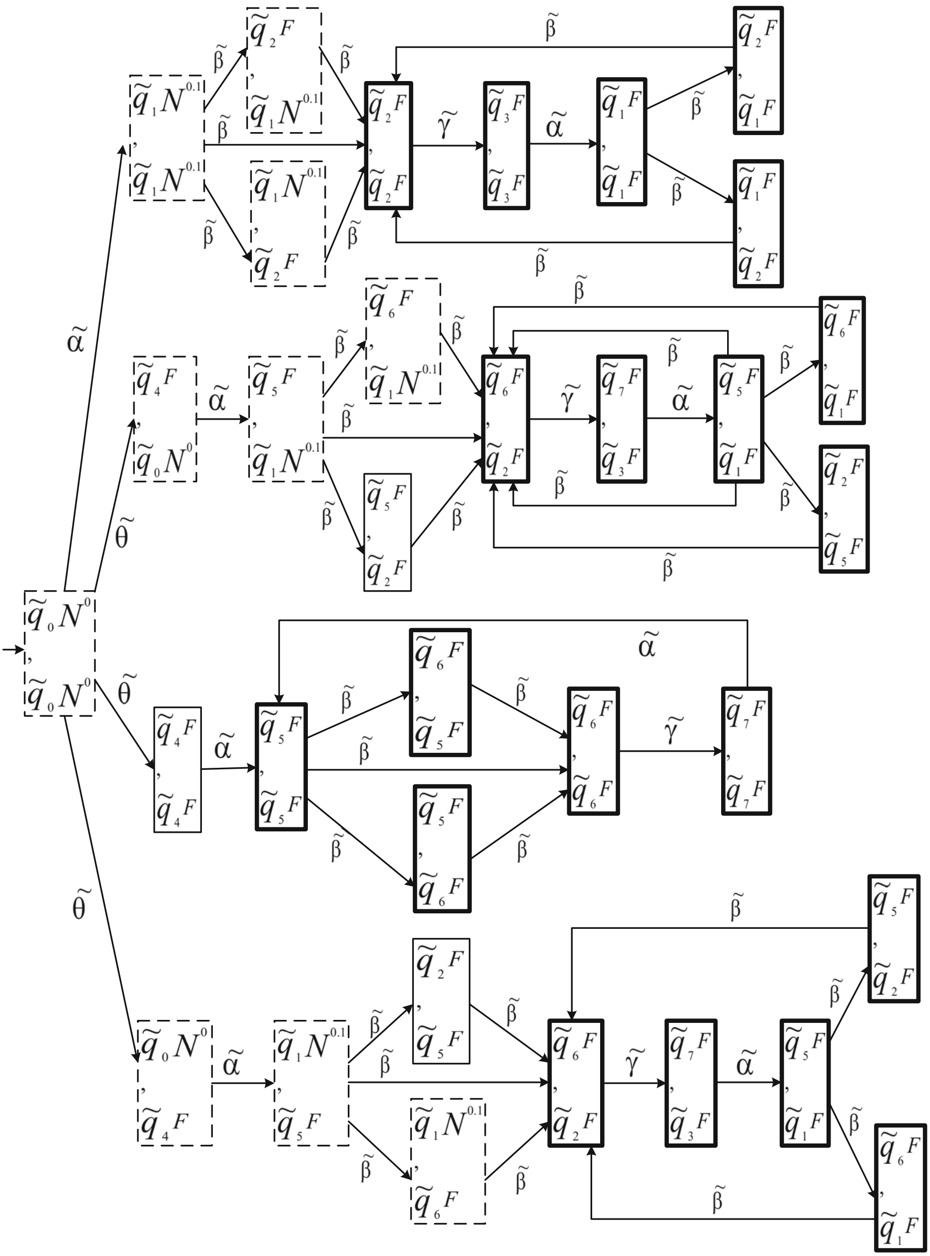

Fig. 5 Verifier $\tilde{V}_{\tilde{G}_3}$ of FDES $\tilde{G}_3$ with respect to $\tilde{\beta}$

$[(\tilde{q}_1, N^{0.1}), (\tilde{q}_6, F)]\}$. All states in cycles of the accessible part for each $\tilde{q}_v \in F_V$, $Ac(\tilde{V}_{\tilde{G}_3}, \tilde{q}_v)$ are certain states. Thus, from Theorem 1, the occurrences of the fuzzy event $\tilde{\beta}$ are completely predictable in $\tilde{G}_3$. In a similar way, we can check that $\tilde{\alpha}$ and $\tilde{\gamma}$ are also completely predictable in $\tilde{G}_3$.

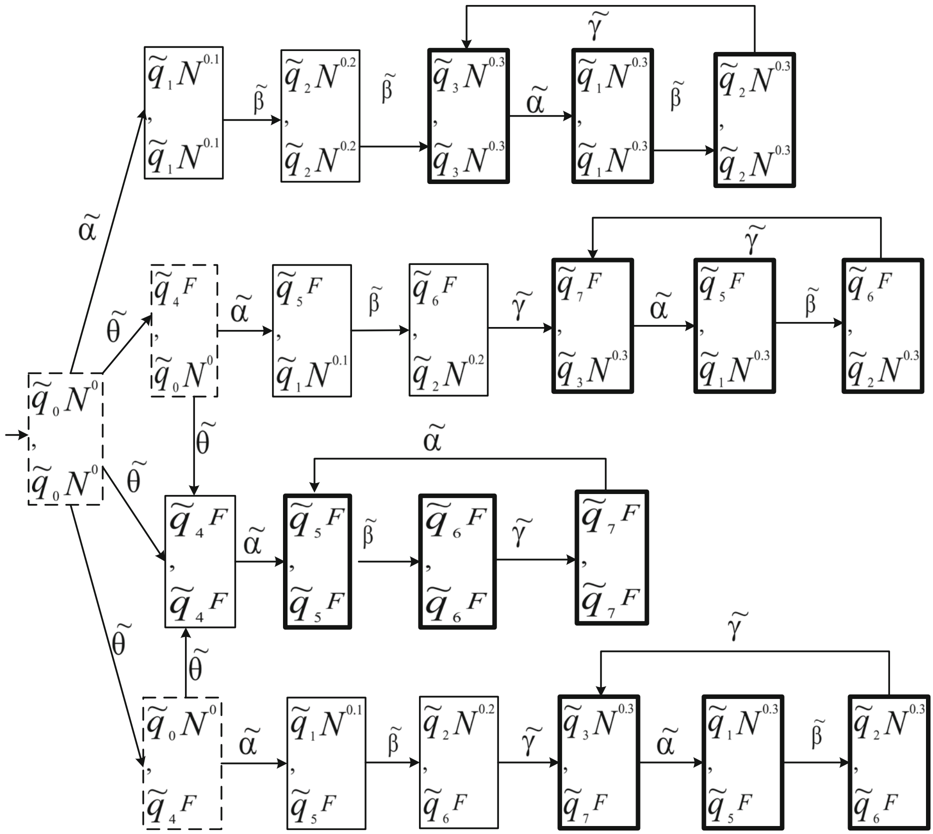

Fig. 6 Verifier $\tilde{V}_{\tilde{G}_3}$ of FDES $\tilde{G}_3$ with respect to $\tilde{\theta}$

2. The verifier with respect to $\tilde{\theta}$ is shown in Fig. 6. The set F'_V of normal states with an uncertain or a certain state as an immediate successor is $F'_V = \{[(\tilde{q}_0, N^0), (\tilde{q}_0, N^0)]\}$. Then, $F_V = F'_V \cup \{[(\tilde{q}_4, F),$ $(\tilde{q}_0, N^0)], [(\tilde{q}_0, N^0), (\tilde{q}_4, F)]\}$. There is a minimal μ-normal and two minimal μ-uncertain cycles in the accessible part $Ac(\tilde{V}_{\tilde{G}_3}, \tilde{q}_v)$ such that $\tilde{q}_v \in F_V$. It is easy to see in Fig. 6 that $\mu = 0.3 < \tilde{\Sigma}_f(\tilde{\theta}) = 0.4$. Since $\mu = \lambda \tilde{\Sigma}_f(\tilde{\theta})$, it follows that: $\lambda = 0.75$. Therefore, by Theorem 2, the fuzzy event $\tilde{\theta}$ is partially predictable with degree 0.75 in the FDES $\tilde{G}_3$. This is in accordance with the result found in Section 6.1 using the predictability definition.

7 Conclusion and future work

In this paper, we have dealt with the problem of failures predictability in the framework of fuzzy DESs. First, we have formalized the concept of fuzzy predictability in fuzzy DESs by means of fuzzy predictability functions that take their values in the interval [0, 1] rather than in the binary set {0, 1}. These functions quantifies the predictability degree of a failure event at different levels including (i) the predictability degree of a non-faulty prefix of a word ending by a failure event (ii) the predictability degree of a word ending by a failure event, and (iii) the predictability degree of a failure event in the fuzzy DES. We have also shown that the new setting generalizes the classical setting of predictability in crisp DESs. Indeed, by using an adapted presentation of any crisp DES as a special fuzzy DES, our proposed fuzzy approach leads to the correct decision about the (classical) predictability of any failure event in the crisp DES.

Then, we have shown that the well-known result stating that the predictability property is stronger than the diagnosability property continue to hold in the fuzzy framework. This result is expressed in the fuzzy framework by the fact that for any fuzzy DES, the predictability degree of any failure event is at most equal to its degree of diagnosability.

After that we have proposed an approach for the verification of failures predictability in fuzzy DESs based on discrete event structure commonly used in the literature. From a computation complexity point of view, the number of states in the verifier is polynomial with respect to the number of states in the input system.

Our proposal may be used at the design stage of the general context of failure diagnosis of DESs tainted with imprecision in their states, events, observability as well as failure status. The importance of our proposal comes from the fact that these kinds of imperfections are common in real-life application contexts where it is rare to have a precise and complete amount of information about the functioning of the studied system. The evaluation of the predictability degrees of the different failure events occurring in the system provides the designer of the model with valuable hints about the possible changes to perform on the model, according to the criticality of the application and the gravity of failures. The objective being to make the predictability degree of a critical failure as high as possible in the limits of the available resources. Indeed, increasing the predictability degree of a failure often comes down to use additional sensors (which may be costly) in order to increase its observability degree or the observability degree of other events. Thus, the indications given by the predictability degrees helps the designer in finding a reasonable tradeoff between the global predictability of the system and the cost necessary to ensure it.

As future work, a natural further issue worthy of consideration is to generalize our predictability approach to the framework of distributed fuzzy DESs. Another future work that we plan to do is to study predictability of patterns in fuzzy DESs where a failure in no longer a single event but rather a pattern, i.e., a set of sequences of events that are considered as "abnormal". Finally, another interesting future issue to consider is the study of safe predictability in fuzzy DESs.

Appendix: Proofs

Proof of Proposition 1 The proof follows immediately from Definition 7 and the definition of the degree of fault of a trace in the language. □

Proof of Proposition 2 Follows immediately from Definition 8 and Propositions 1, 3. □

Proof of Proposition 3 Let $\tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}}$ and $\tilde{t} = MaxPref(\tilde{s})$. Let us first show that: $(\forall \tilde{t}_1 \in Pref(\tilde{t}))(\forall n > 0)(\exists n' > n)\ E(\tilde{t}, n) \subseteq E(\tilde{t}_1, n')$ where $E(\tilde{r}, n) = \{\tilde{\omega} \in L_{\tilde{G}} : \tilde{\omega} = \tilde{u}\tilde{v}, (\tilde{u}, \tilde{v}) \in L//\tilde{r}, \|\tilde{v}\| = n\}$.

Let $\tilde{t}_1 \in Pref(\tilde{t})$ and suppose that $\tilde{t} = \tilde{t}_1\tilde{s}_1$. If $\tilde{\omega} \in E(\tilde{t}, n)$, then:

$\tilde{\omega} = \tilde{u}\tilde{v}$ where $\tilde{u} \in L_{\tilde{G}}$, $\tilde{v} \in L/\tilde{u}$, $\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(\tilde{t})$, $\tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma})$ and $\|\tilde{v}\| = n$, i.e.,

$\tilde{\omega} = \tilde{u}\tilde{v}$ where $\tilde{u} \in L_{\tilde{G}}$, $\tilde{v} \in L/(\tilde{u})$, $\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(\tilde{t}_1)\tilde{\Pi}_{\tilde{\sigma}}(\tilde{s}_1)$, $\tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma})$ and $\|\tilde{v}\| = n$

Let us put $\tilde{u} = \tilde{u_1}\tilde{u_2}$ such that $\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}_1) = \tilde{\Pi}_{\tilde{\sigma}}(\tilde{t}_1)$ and $\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}_2) = \tilde{\Pi}_{\tilde{\sigma}}(\tilde{s}_1)$. Let $\tilde{v}' = \tilde{u_2}\tilde{v}$ and $n' = \|\tilde{v}\| + \|\tilde{u_2}\|$, then we have:

$\tilde{\omega} = \tilde{u_1}\tilde{v}'$where $\tilde{u_1} \in L_{\tilde{G}}$, $\tilde{v}' \in L/\tilde{u_1}$, $\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u_1}) = \tilde{\Pi}_{\tilde{\sigma}}(\tilde{t}_1)$, $\tilde{\Sigma}_f(\tilde{u_1}) < \tilde{\Sigma}_f(\tilde{\sigma})$ and $\|\tilde{v}'\| = n'$

This means that $\tilde{\omega} \in E(\tilde{t}_1, n')$ and hence, $E(\tilde{t}, n) \subseteq E(\tilde{t}_1, n')$. It follows that:

$$\begin{aligned}
&(\forall \tilde{t}_1 \in Pref(\tilde{t}))(\forall n > 0)(\exists n' > n):\\
&min\,\{\tilde{\Sigma}_f(\tilde{\omega}) : \tilde{\omega} \in L_{\tilde{G}} \ and\ \tilde{\omega} = \tilde{u}\tilde{v}, (\tilde{u}, \tilde{v}) \in L//\tilde{t}_1, \|\tilde{v}\| = n'\} \leq\\
&min\,\{\tilde{\Sigma}_f(\tilde{\omega}) : \tilde{\omega} \in L_{\tilde{G}} \ and\ \tilde{\omega} = \tilde{u}\tilde{v}, (\tilde{u}, \tilde{v}) \in L//\tilde{t}, \|\tilde{v}\| = n\}, i.e.,\\
&(\forall \tilde{t}_1 \in Pref(\tilde{t}))(\forall n > 0)(\exists n' > n),\ FP^{(1)}_{\tilde{\sigma}}(\tilde{t}_1, n') \leq FP^{(1)}_{\tilde{\sigma}}(\tilde{t}, n)
\end{aligned}$$

But from Proposition 1 and the fact that $n < n'$, it holds that: $FP^{(1)}_{\tilde{\sigma}}(\tilde{t_1}, n) \leq FP^{(1)}_{\tilde{\sigma}}(\tilde{t_1}, n')$. We deduce finally that: $(\forall \tilde{t_1} \in Pref(\tilde{t}))(\forall n > 0),\ FP^{(1)}_{\tilde{\sigma}}(\tilde{t_1}, n) \leq FP^{(1)}_{\tilde{\sigma}}(\tilde{t}, n)$.

The proposition follows then directly from Definition 8. □

Proof of Corollary 1 The proof follows immediately from Definition 9, Definition 7 and Proposition 3. □

Proof of Proposition 4 $\tilde{G}$ is 1-$predictable \Leftrightarrow (\exists n_0 \in \mathbb{N})(\forall n > n_0) FP_{\tilde{\sigma}}(\tilde{G}, n) = 1$ .... (1)

$(1) \Leftrightarrow (\exists n_0 \in \mathbb{N})(\forall n > n_0)\ min\{FP^{(1)}_{\tilde{\sigma}}(\tilde{t}, n) : \tilde{t} = MaxPref(\tilde{s}),\ \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}}\} = 1$

$(1) \Leftrightarrow (\exists n_0 \in \mathbb{N})(\forall n > n_0)(\forall \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})\ FP^{(1)}_{\tilde{\sigma}}(MaxPref(\tilde{s}), n) = 1$

$$(1) \Leftrightarrow (\exists n_0 \in \mathbb{N})(\forall n > n_0)(\forall \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})\ \frac{\begin{array}{c} min\{\tilde{\Sigma}_f(\tilde{\sigma}), \tilde{\Sigma}_f(\tilde{\omega}) : \tilde{\omega} \in L(\tilde{G}) \wedge \tilde{\omega} = \tilde{u}\tilde{v} \\ \wedge(\tilde{u}, \tilde{v}) \in L//MaxPref(\tilde{s}) \wedge \|\tilde{v}\| = n\} \end{array}}{\tilde{\Sigma}_f(\tilde{\sigma})} = 1$$

$(1) \Leftrightarrow (\exists n_0 \in \mathbb{N})(\forall n > n_0)(\forall \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})$
$min\{\tilde{\Sigma}_f(\tilde{\sigma}), \tilde{\Sigma}_f(\tilde{\omega}) : \tilde{\omega} \in L(\tilde{G}) \wedge \tilde{\omega} = \tilde{u}\tilde{v} \wedge (\tilde{u}, \tilde{v}) \in L//MaxPref(\tilde{s}) \wedge \|\tilde{v}\| = n\} = \tilde{\Sigma}_f(\tilde{\sigma})$

$(1)\ (\exists n_0 \in \mathbb{N})(\forall n > n_0)(\forall \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})(\forall \tilde{u} \in L_{\tilde{G}})(\forall \tilde{v} \in L/\tilde{u}):$
$[\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(MaxPref(\tilde{s}))\ and\ \tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma})\ and\ \|\tilde{v}\| = n] \Rightarrow \tilde{\Sigma}_f(\tilde{u}\tilde{v}) \geq \tilde{\Sigma}_f(\tilde{\sigma})$ □

Proof of Proposition 5 Let $\tilde{G}$ be an FDES. $\tilde{G}$ and $\lambda \in [0, 1[$.
$\tilde{G}$ is λ-$predictable \Leftrightarrow (\exists n_0 \in \mathbb{N})(\forall n > n_0) FP_{\tilde{\sigma}}(\tilde{G}, n) = \lambda$ .... (1)

$(1) \Leftrightarrow (\exists n_0 \in \mathbb{N})(\forall n > n_0)\ min\{FP^{(1)}_{\tilde{\sigma}}(\tilde{t}, n) : \tilde{t} = MaxPref(\tilde{s}),\ \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}}\} = \lambda$

$(1) \Leftrightarrow (\exists n_0 \in \mathbb{N})(\forall n > n_0)(H_1 \wedge H_2)$ where: H_1: $(\exists \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})\ FP^{(1)}_{\tilde{\sigma}}(MaxPref(\tilde{s}), n) = \lambda$ and H_2: $(\forall \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})\ FP^{(1)}_{\tilde{\sigma}}(MaxPref(\tilde{s}), n) \geq \lambda$

$H_1 \Leftrightarrow (\exists \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})$
$min\{\tilde{\Sigma}_f(\tilde{\sigma}), \tilde{\Sigma}_f(\tilde{\omega}) : \tilde{\omega} \in L(\tilde{G}) \wedge \tilde{\omega} = \tilde{u}\tilde{v} \wedge (\tilde{u}, \tilde{v}) \in L//MaxPref(\tilde{s}) \wedge \|\tilde{v}\| = n\} = \lambda.\tilde{\Sigma}_f(\tilde{\sigma})$

$H_1 \Leftrightarrow (\exists\tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})(H_1' \wedge H_2')$ where:

$H_1' : (\exists\tilde{u} \in L_{\tilde{G}})(\exists\tilde{v} \in L/\tilde{u})$

$[\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(MaxPref(\tilde{s}))\ and\ \tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma})\ and\ \|\tilde{v}\| = n]\ and \tilde{\Sigma}_f(\tilde{u}\tilde{v}) = \lambda.\tilde{\Sigma}_f(\tilde{\sigma})$

$H_2' : (\forall\tilde{u} \in L_{\tilde{G}})(\forall\tilde{v} \in L/\tilde{u})$

$[\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(MaxPref(\tilde{s}))\ and\ \tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma})\ and\ \|\tilde{v}\| = n] \Rightarrow \tilde{\Sigma}_f(\tilde{u}\tilde{v}) \geq \lambda.\tilde{\Sigma}_f(\tilde{\sigma})$

$H_2 \Leftrightarrow (\forall\tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})$

$min\{\tilde{\Sigma}_f(\tilde{\sigma}), \tilde{\Sigma}_f(\tilde{\omega}) : \tilde{\omega} \in L(\tilde{G}) \wedge \tilde{\omega} = \tilde{u}\tilde{v} \wedge (\tilde{u}, \tilde{v}) \in L//MaxPref(\tilde{s}) \wedge \|\tilde{v}\| = n\} \geq \lambda.\tilde{\Sigma}_f(\tilde{\sigma})$

$H_2 \Leftrightarrow (\forall\tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})(\forall\tilde{u} \in L_{\tilde{G}})(\forall\tilde{v} \in L/\tilde{u})$

$[\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(MaxPref(\tilde{s}))\ and\ \tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma})\ and\ \|\tilde{v}\| = n] \Rightarrow \tilde{\Sigma}_f(\tilde{u}\tilde{v}) \geq \lambda.\tilde{\Sigma}_f(\tilde{\sigma}) \Leftrightarrow D_2$

It is easy to check that:

$$\begin{aligned}(H_1 \wedge H_2) &\Leftrightarrow (((\exists\tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})(H_1' \wedge H_2')) \wedge D_2)\\ &\Leftrightarrow (((\exists\tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})\ H_1') \wedge D_2)\\ &\Leftrightarrow (D_1 \wedge D_2).\end{aligned}$$ □

Proof of Corollary 2 Follows immediately from Proposition 4 and Proposition 5. □

Proof of Proposition 6 Suppose that the degree of predictability (resp. of diagnosability) of $\tilde{\sigma}$ in $\tilde{G}$ is $\lambda = FP_{\tilde{\sigma}}(\tilde{G}, \infty)$ (resp. $\gamma = FG_{\tilde{\sigma}}(\tilde{G}, \infty)$), then:

$$\gamma = \frac{min\{\tilde{\Sigma}_f(\tilde{\sigma}), \tilde{\Sigma}_f(\tilde{\omega}) : \tilde{\omega} \in E\}}{\tilde{\Sigma}_f(\tilde{\sigma})}$$

where: $E = \{\tilde{\omega} : \tilde{\omega} \in \tilde{\Pi}^{-1}(\Pi(\tilde{s}.\tilde{s}')), \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}}, \tilde{s}' \in L/\tilde{s}, \|\tilde{s}'\| = \infty\}$ and

$$\lambda = \frac{min\{\tilde{\Sigma}_f(\tilde{\sigma}), \tilde{\Sigma}_f(\tilde{\omega}) : \tilde{\omega} \in E'\}}{\tilde{\Sigma}_f(\tilde{\sigma})}$$

where: $E' = \{\tilde{\omega} : \tilde{\omega} = \tilde{u}\tilde{v}, (\tilde{u}, \tilde{v}) \in L//\tilde{t}, \tilde{t} = MaxPref(\tilde{s}), \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}}, \|\tilde{v}\| = \infty\}$
We have: $E = \{\tilde{\omega} : \Pi(\tilde{\omega}) = \Pi(\tilde{s}\tilde{s}'), \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}}, \tilde{s}' \in L/\tilde{s}, \|\tilde{s}'\| = \infty\}$
For all $\tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}}$, we put $\tilde{s} = \tilde{t}\tilde{t}'$ where $\tilde{t} = MaxPref(\tilde{s})$. Then:
$E = \{\tilde{\omega} : \tilde{\omega} = \tilde{u}\tilde{v}, \Pi(\tilde{u}) = \Pi(\tilde{t}), \Pi(\tilde{v}) = \Pi(\tilde{t}'\tilde{s}'), \tilde{t} = MaxPref(\tilde{s}), \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}}, \tilde{s}' \in L/\tilde{s}, \|\tilde{s}'\| = \infty\}$
Since $\tilde{u}$ may or may not be faulty, E may be written: $E = E_1 \cup E_2$ where:
$E_1 = \{\tilde{\omega} : \tilde{\omega} = \tilde{u}\tilde{v}, \Pi(\tilde{u}) = \Pi(\tilde{t}), \tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma}), \Pi(\tilde{v}) = \Pi(\tilde{t}'\tilde{s}'), \tilde{t} = MaxPref(\tilde{s}), \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}}, \tilde{s}' \in L/\tilde{s}, \|\tilde{s}'\| = \infty\}$. i.e., $E_1 = \{\tilde{\omega} : \tilde{\omega} = \tilde{u}\tilde{v}, (\tilde{u}, \tilde{v}) \in L//\tilde{t}, \Pi(\tilde{v}) = \Pi(\tilde{t}'\tilde{s}'), \tilde{t} = MaxPref(\tilde{s}), \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}}, \tilde{s} = \tilde{t}\tilde{t}', \tilde{s}' \in L/\tilde{s}, \|\tilde{v}'\| = \infty\}$ and $E_2 = \{\tilde{\omega} : \tilde{\omega} = \tilde{u}\tilde{v}, \Pi(\tilde{u}) = \Pi(\tilde{t}), \tilde{\Sigma}_f(\tilde{u}) \geq \tilde{\Sigma}_f(\tilde{\sigma}), \Pi(\tilde{v}) = \Pi(\tilde{t}'\tilde{s}'), \tilde{t} = MaxPref(\tilde{s}), \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}}, \tilde{s}' \in L/\tilde{s}, \|\tilde{s}'\| = \infty\}$
Note that, for every $\tilde{\omega} \in E_2$, it holds that $\tilde{\Sigma}_f(\tilde{\omega}) \geq \tilde{\Sigma}_f(\tilde{\sigma})$ since every $\tilde{\omega} \in E_2$ is of the form $\tilde{u}\tilde{v}$ where $\tilde{\Sigma}_f(\tilde{u}) \geq \tilde{\Sigma}_f(\tilde{\sigma})$. It follows that: $min\{\tilde{\Sigma}_f(\tilde{\omega}) : \tilde{\omega} \in E_2\} \geq \tilde{\Sigma}_f(\tilde{\sigma})$ and hence:

$$\gamma = \frac{min\{\tilde{\Sigma}_f(\tilde{\sigma}), \tilde{\Sigma}_f(\tilde{\omega}) : \tilde{\omega} \in E_1\}}{\tilde{\Sigma}_f(\tilde{\sigma})}$$

It is clear that $E_1 \subseteq E'$. Indeed, among all the traces $\tilde{u}\tilde{v} \in E'$, E_1 contains only those traces where v_1 shares the same observables as $\tilde{t}'\tilde{s}'$. It follows that:

$min\{\tilde{\Sigma}_f(\tilde{\omega}) : \tilde{\omega} \in E'\} < min\{\tilde{\Sigma}_f(\tilde{\omega}) : \tilde{\omega} \in E_1\}$, i.e., $min\{\tilde{\Sigma}_f(\tilde{\sigma}), \tilde{\Sigma}_f(\tilde{\omega}) : \tilde{\omega} \in E'\} < min\{\tilde{\Sigma}_f(\tilde{\sigma}), \tilde{\Sigma}_f(\tilde{\omega}) : \tilde{\omega} \in E_1\}$ and hence: $\lambda \leq \gamma$ □

Proof of Lemma 1 The proof is done by induction on the sequence of observable events. □

Proof of Lemma 2 The proof is done by induction on the sequence of observable events. □

Proof of Lemma 3 ($\Rightarrow$) Suppose $\tilde{q}_v \in F_V$ and C is a cycle in $Ac(\tilde{V}_{\tilde{G}}, \tilde{q}_v)$ such that C is normal or uncertain and each state in C contains a state of $\tilde{G}$ labelled by N^{μ}. Suppose C formed by the states $\tilde{x}_{v,1}, \tilde{x}_{v,2}, ..., \tilde{x}_{v,m}$ and events $\tilde{\sigma}_1, \tilde{\sigma}_2, ..., \tilde{\sigma}_m \in \tilde{E}^*$ i.e. $\tilde{\delta}_d(\tilde{x}_{v,i}, \tilde{\sigma}_i) = \tilde{x}_{v,i+1}$ for $1 \leq i \leq m$ and $\tilde{\delta}_v(\tilde{x}_{v,m}, \tilde{\sigma}_m) = \tilde{x}_{v,1}$. Each state $\tilde{x}_{v,i}$ contains a state of $\tilde{G}$ (say $\tilde{x}_i$) labelled by N^{μ} in $\tilde{V}_{\tilde{G}}$ i.e. $(\tilde{x}_i, N^{\mu}) \in \tilde{x}_{v,i}$.

Let $\tilde{q}_v = [(\tilde{q}_1, N^{\mu 1}), (\tilde{q}_2, N^{\mu 2})] \in \tilde{\delta}(\tilde{q}_{v0}, \tilde{w}_1)$ where $\tilde{q}_1, \tilde{q}_2 \in \tilde{Q}$ and $\tilde{w}_1 \in \tilde{E}^*$. If $\tilde{q}_v \in F_V$ then there exist $\tilde{y}_v$ and $\tilde{z}_v$, such that $\tilde{y}_v = \tilde{\delta}(\tilde{q}'_v, \tilde{t}_{uo})$ and $\tilde{z}_v = \tilde{\delta}(\tilde{y}_v, \tilde{\sigma}_p)$ where: $\tilde{t}_{uo} \in \tilde{E}^*$, $\tilde{\Sigma}_f(\tilde{t}_{uo}) < \tilde{\Sigma}_f(\tilde{\sigma})$, $\tilde{\Sigma}_o(\tilde{t}_{uo}) < \tilde{\Sigma}_o(\tilde{\sigma})$, $\tilde{\Sigma}_f(\tilde{\sigma}_p) \geq \tilde{\Sigma}_f(\tilde{\sigma})$ and $\tilde{q}'_v = [(\tilde{q}_1, N^{\mu 1}), (.,.)]$ or $\tilde{q}'_v = [(.,.), (\tilde{q}_2, N^{\mu 2})]$. The existence of $\tilde{y}_v$ and $\tilde{z}_v$ follows from Lemma 1 and the existence of $\tilde{q}'_v$ follows from the definition of F_V.

There is $\tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}}$ such that $\tilde{\Pi}_{\tilde{\sigma}}(\tilde{s}) = \tilde{\Pi}_{\tilde{\sigma}}(\tilde{w}_1\tilde{t}_{uo}\tilde{\sigma}_p)$. let $\tilde{s} = \tilde{s}_1\tilde{\sigma}_p$ where $\tilde{s}_1 \in \tilde{E}^*$ and $\tilde{\Sigma}_f(\tilde{\sigma}_p) \geq \tilde{\Sigma}_f(\tilde{\sigma})$ .take a state from the cycle C . without lost of generality take the state $\tilde{x}_{v,1}$. let $\tilde{t} = \tilde{s}_1 = MaxPref(\tilde{s})$. let $\tilde{x}_{v,1} = [(\tilde{x}_1, N^{\mu}), (\tilde{x}_2, l_{x2})]$ where $\tilde{x}_1, \tilde{x}_2 \in \tilde{Q}$ and $l_{x2} \in \{N^{\mu}, N^{\mu'}, F\}$ and suppose that $\tilde{x}_{v,1}$ is reached from $\tilde{q}_v$ by $\tilde{w}_2 \in \tilde{E}^*$: $\tilde{x}_{v,1} = \tilde{\delta}(\tilde{q}_v, \tilde{w}_2)$.

There is $\tilde{u} \in L_{\tilde{G}}$ and $\tilde{u}' \in L_{\tilde{G}}/\tilde{u}$ such that $\tilde{\Pi}_{\tilde{\sigma}}(\tilde{w}_1) = \tilde{\Pi}_{\tilde{\sigma}}(\tilde{u})$ and $\tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma})$ and $\tilde{\delta}(\tilde{q}_0, \tilde{u}\tilde{u}') = \tilde{x}_1$, since
$(\tilde{x}_1, N^{\mu}) \in \tilde{x}_{v,1}$ and since $\tilde{\Pi}_{\tilde{\sigma}}(\tilde{w}_1) = \tilde{\Pi}_{\tilde{\sigma}}(\tilde{w}_1\tilde{t}_{uo}) = \tilde{\Pi}_{\tilde{\sigma}}(\tilde{s}_1) = \tilde{\Pi}_{\tilde{\sigma}}(\tilde{t})$ then $\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(\tilde{t})$. because there is a cycle C in the verifier $\tilde{V}_{\tilde{G}}$ fromed by $\tilde{x}_{v,1}, \tilde{x}_{v,2}, ..., \tilde{x}_{v,m}$ and events $\tilde{\sigma}_1, \tilde{\sigma}_2, ..., \tilde{\sigma}_m \in \tilde{E}^*$, so there exist a corresponding cycle in $\tilde{G}$ formed by normal states in $\tilde{x}_{v,i}$ for $i = 1, \cdots, m$ and a subsequence $\tilde{w}'_1, ..., \tilde{w}'_{m'} \in \tilde{E}^*$ where $m' \leq m$ is a positive integer. Moreover we have: $\forall k \geq 1$, $\tilde{x}_1 = \tilde{\delta}(\tilde{q}_0, \tilde{u}\tilde{u}'(\tilde{w}'_1, ..., \tilde{w}'_{m'})^k)$ and $\tilde{\Sigma}_f(\tilde{u}\tilde{u}'(\tilde{w}'_1, ..., \tilde{w}'_{m'})^k) = \mu$.
Take $n_0 = \|\tilde{u}'\| + k$ and $\tilde{v} = \tilde{u}'(\tilde{w}'_1, ..., \tilde{w}'_{m'})^k)$ then:

$$(\exists n_0 \in \mathbb{N})(\forall n > n_0)(\exists \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})(\exists \tilde{u} \in L_{\tilde{G}})(\exists \tilde{v} \in L/\tilde{u}) :$$
$$\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(MaxPref(\tilde{s}))\ and\ \tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma})\ and\ \|\tilde{v}\| = n\ and\ \tilde{\Sigma}_f(\tilde{u}\tilde{v}) = \mu$$

($\Leftarrow$) Suppose that:

$$(\exists n_0 \in \mathbb{N})(\forall n > n_0)(\exists \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})(\exists \tilde{u} \in L_{\tilde{G}})(\exists \tilde{v} \in L/\tilde{u}) :$$
$$\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(MaxPref(\tilde{s}))\ and\ \tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma})\ and\ \|\tilde{v}\| = n\ and\ \tilde{\Sigma}_f(\tilde{u}\tilde{v}) = \mu$$

Let us take $\tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}}$ and let $\tilde{s} = \tilde{s}_1\tilde{\sigma}_p$ where $\tilde{\Sigma}_f(\tilde{s}_1) < \tilde{\Sigma}_f(\tilde{\sigma})$ and $\tilde{\Sigma}_f(\tilde{\sigma}_p) \geq \tilde{\Sigma}_f(\tilde{\sigma})$. $\tilde{t} = \tilde{s}_1 = MaxPref(\tilde{s})$. We have $\tilde{\Sigma}_f(\tilde{t}) < \tilde{\Sigma}_f(\tilde{\sigma})$. Take $\tilde{u} \in L_{\tilde{G}}$ such that $\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(\tilde{t}) = \tilde{\Pi}_{\tilde{\sigma}}(\tilde{s}_1)$ and $\tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma})$.

Let $\tilde{x} = \tilde{\delta}(\tilde{q}_0, \tilde{s}_1)$ and $\tilde{y} = \tilde{\delta}(\tilde{x}, \tilde{\sigma}_p)$. There exist $\tilde{x}_v = [(\tilde{x}, N^{\mu 1}), (\tilde{x}', l'_x)]$ and $\tilde{y}_v = [(\tilde{x}, F), (\tilde{y}', l'_y)] \in \tilde{Q}_v$ such that $\tilde{y}_v \in \tilde{\delta}_v(\tilde{x}_v, \tilde{\sigma}_p)$ where $\tilde{x}', \tilde{y}' \in \tilde{Q}$ and $l'_x, l'_y \in \{N^{\mu 3}, F\}$ then $\tilde{x}_v \in \tilde{Q}^N_v \cup \tilde{Q}^U_v$ and $\tilde{y}_v \in \tilde{Q}^u_v \cup \tilde{Q}^c_v$. We distinguish two cases for $\tilde{x}_v$:

case (i): $\tilde{x}_v \in \tilde{Q}_v^N$ and $\tilde{y}_v \notin \tilde{Q}_v^N$ then $\tilde{x}_v \in F_V$.
case (ii): $\tilde{x}_v \notin \tilde{Q}_v^N$ but in this case , by Lemma 2, there is a state $\tilde{w}_v \in F_V$ reachable from $\tilde{q}_{v0}$ such that $\tilde{x}_v$ is reachable from $\tilde{w}_v$.
So, in both cases, there is a state $\tilde{q}_v \in F_V$ ($\tilde{q}_v = \tilde{x}_v$ or $\tilde{q}_v = \tilde{w}_v$) from which $\tilde{y}_v$ is reached from $\tilde{q}_v$ by σ_p: $\tilde{y}_v = \tilde{\delta}_v(\tilde{q}_v, \sigma_p)$. On the other hand $\tilde{v}$ may be chosen arbitrarily long and then has the from: $\tilde{v} = \tilde{u}'(\tilde{w}_1...\tilde{w}_m)^k)$ for $k \geq 1$ with $\tilde{\Sigma}_f(\tilde{u}\tilde{v}) = \mu$.

Let $\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}') = \tilde{s}_1'$ and $\tilde{\Pi}_{\tilde{\sigma}}(\tilde{w}_1...\tilde{w}_m) = \tilde{s}_1'' = \tilde{\sigma}_1...\tilde{\sigma}_l, (l \leq m)$. We have two cases:
case A: $\tilde{s}_1' \neq \epsilon$ then let $\tilde{x}_v' = \tilde{\delta}_v(\tilde{x}_v, \tilde{s}_1')$ we have: $\tilde{x}_v' = \tilde{\delta}_v(\tilde{x}_v', \tilde{\sigma}_1...\tilde{\sigma}_l)$.
case B: $\tilde{s}_1' = \epsilon$ then let $\tilde{x}_v' = \tilde{\delta}_v(\tilde{x}_v, \tilde{\sigma}_1)$ we have: $\tilde{x}_v' = \tilde{\delta}_v(\tilde{x}_v', \tilde{\sigma}_2...\tilde{\sigma}_l\tilde{\sigma}_1)$
In both cases: $\tilde{x}_v' = \tilde{\delta}_v(\tilde{q}_{v0}, \tilde{s}_1\tilde{s}_1'(\tilde{\sigma}_1...\tilde{\sigma}_l)^k)$, $k \geq 1$. Since $\tilde{\Sigma}_f(\tilde{u}\tilde{v}) = \mu$, it is clear that there is $(\tilde{x}_i, N^\mu) \in \tilde{x}_v'$ and $\tilde{x}_v'$ is involved in cycle C reached from $\tilde{q}_v$. From the fault propagation in the verifier, each state of C contains a state labelled by N^μ. □

Proof of Theorem 1 (⇒) Suppose that $\tilde{\sigma}$ is 1-predictable. Then we have:

$$(\exists n_0 \in \mathbb{N})(\forall n > n_0)(\forall \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})(\forall \tilde{u} \in L_{\tilde{G}})(\forall \tilde{v} \in L/\tilde{u}) :$$
$$[\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(MaxPref(\tilde{s}))\ and\ \tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma})\ and\ \|\tilde{v}\| = n] \Rightarrow \tilde{\Sigma}_f(\tilde{u}\tilde{v}) \geq \tilde{\Sigma}_f(\tilde{\sigma}) (\star)$$

For the sake of contradiction, suppose that P_v does not holds. So, there is $\tilde{q}_v \in F_V$ and there is a cycle $C \in Ac(\tilde{V}_{\tilde{G}}, \tilde{q}_v)$ such that C is uncertain or normal. This means that there is a state $\tilde{x}_{v,i} \in C$ containing $(\tilde{x}_i, N^\mu)$ for some $\tilde{x}_i \in \tilde{Q}$ and $\mu < 1$. Then from Lemma 3 we obtain:

$$(\exists n_0 \in \mathbb{N})(\forall n > n_0)(\exists \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})(\exists \tilde{u} \in L_{\tilde{G}})(\forall \tilde{v} \in L/\tilde{u}) :$$
$$\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(MaxPref(\tilde{s}))\ and\ \tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma})\ and\ \|\tilde{v}\| = n\ and\ \tilde{\Sigma}_f(\tilde{u}\tilde{v}) = \mu < \tilde{\Sigma}_f(\tilde{\sigma})$$

This contradicts $(\star)$
(⇐) Suppose that P_v holds: for all $\tilde{q}_v \in F_V$, there is only certain cycles in $Ac(\tilde{V}_{\tilde{G}}, \tilde{q}_v)$. For the sake of contradiction, suppose that $\tilde{G}$ is not 1-predictable i.e. $\tilde{G}$ is λ-predictable for some $0 \leq \lambda < 1$. From Proposition 4 it follows that:

$$(\exists n_0 \in \mathbb{N})(\forall n > n_0)(\exists \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})(\exists \tilde{u} \in L_{\tilde{G}})(\exists \tilde{v} \in L/\tilde{u}) :$$
$$\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(MaxPref(\tilde{s}))\ and\ \tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma})\ and\ \|\tilde{v}\| = n\ and\ \tilde{\Sigma}_f(\tilde{u}\tilde{v}) = \lambda.\tilde{\Sigma}_f(\tilde{\sigma})$$

Let us put $\mu = \lambda.\tilde{\Sigma}_f(\tilde{\sigma})$. We have $\mu < \tilde{\Sigma}_f(\tilde{\sigma})$. From Lemma 3 it follows that there is $\tilde{q}_v \in F_V$ and there is a cycle C in $Ac(\tilde{V}_{\tilde{G}}, \tilde{q}_v)$ which is normal or uncertain (hence, not certain). This contradicts P_v. □

Proof of Theorem 2 (⇒) $\tilde{\sigma}$ is λ-predictable in $\tilde{G}$. from Proposition 5 we have:

(i) $(\exists n_0 \in \mathbb{N})(\forall n > n_0)(\exists \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})(\exists \tilde{u} \in L_{\tilde{G}})(\exists \tilde{v} \in L/\tilde{u}) :$
$$\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(MaxPref(\tilde{s}))\ and\ \tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma})\ and\ \|\tilde{v}\| = n\ and\ \tilde{\Sigma}_f(\tilde{u}\tilde{v}) = \lambda.\tilde{\Sigma}_f(\tilde{\sigma})$$

(ii) $(\exists n_0 \in \mathbb{N})(\forall n > n_0)(\forall \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})(\forall \tilde{u} \in L_{\tilde{G}})(\forall \tilde{v} \in L/\tilde{u}) :$
$$[\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(MaxPref(\tilde{s}))\ and\ \tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma})\ and\ \|\tilde{v}\| = n] \Rightarrow \tilde{\Sigma}_f(\tilde{u}\tilde{v}) \geq \lambda.\tilde{\Sigma}_f(\tilde{\sigma})$$

Let us put $\lambda.\tilde{\Sigma}_f(\tilde{\sigma}) = \mu$. Then $R1$ follows from (i) and Lemma 3.
To show $R2$, suppose that μ is not minimal, i.e. there is $\mu' < \mu$, there is $\tilde{q}_v \in F_V$ and a cycle

C in $Ac(\tilde{V}_{\tilde{G}}, \tilde{q}_v)$ such that C is μ'-normal or μ'-uncertain. From Lemma 3 it follows that:

$$(\exists n_0 \in \mathbb{N})(\forall n > n_0)(\exists \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})(\exists \tilde{u} \in L_{\tilde{G}})(\exists \tilde{v} \in L/\tilde{u}):$$
$$\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(MaxPref(\tilde{s}))\ and\ \tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma})\ and\ \|\tilde{v}\| = n\ and\ \tilde{\Sigma}_f(\tilde{u}\tilde{v}) = \mu' < \lambda.\tilde{\Sigma}_f(\tilde{\sigma})$$

This contradicts the condition (ii) of λ-predictability.
($\Leftarrow$) Suppose that the two conditions $R1$ and $R2$ hold where:
$R1$: there is $\tilde{q}_v \in F_V$, there is cycle C in $Ac(\tilde{V}_{\tilde{G}}, \tilde{q}_v)$ which is normal or uncertain.
$R2$: the minimal μ-normal or μ-uncertain in $Ac(\tilde{V}_{\tilde{G}}, \tilde{q}_v)$ for all $\tilde{q}_v \in F_V$ satisfies $\mu = \lambda.\tilde{\Sigma}_f(\tilde{\sigma})$.
From $R1$ and Lemma 3we have:

$$(i)\ (\exists n_0 \in \mathbb{N})(\forall n > n_0)(\exists \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})(\exists \tilde{u} \in L_{\tilde{G}})(\exists \tilde{v} \in L/\tilde{u}):$$
$$\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(MaxPref(\tilde{s}))\ and\ \tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma})\ and\ \|\tilde{v}\| = n\ and\ \tilde{\Sigma}_f(\tilde{u}\tilde{v}) = \lambda.\tilde{\Sigma}_f(\tilde{\sigma})$$

From $R2$ we have:

$$(ii)\ (\exists n_0 \in \mathbb{N})(\forall n > n_0)(\forall \tilde{s} \in \tilde{\Psi}_{\tilde{\sigma}})(\forall \tilde{u} \in L_{\tilde{G}})(\forall \tilde{v} \in L/\tilde{u}):$$
$$[\tilde{\Pi}_{\tilde{\sigma}}(\tilde{u}) = \tilde{\Pi}_{\tilde{\sigma}}(MaxPref(\tilde{s}))\ and\ \tilde{\Sigma}_f(\tilde{u}) < \tilde{\Sigma}_f(\tilde{\sigma})\ and\ \|\tilde{v}\| = n] \Rightarrow \tilde{\Sigma}_f(\tilde{u}\tilde{v}) \geq \lambda.\tilde{\Sigma}_f(\tilde{\sigma})$$

From (i), (ii) and from Proposition 5 we deduce that $\tilde{\sigma}$ is λ-predictable in $\tilde{G}$. □

Proof of Corollary 3 Follows immediately from Theorem 2. □

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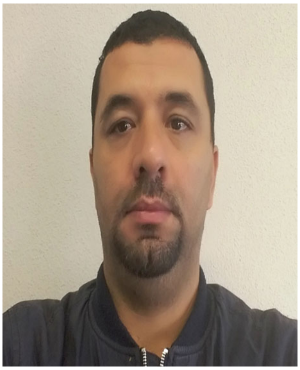

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