Rend. Sem. Mat. Univ. Padova, Vol. 117 (2007)

A note on Torsion-by-Nilpotent Groups.

Tarek Rouabhi (*) - Nadir Trabelsi (**)

Abstract - In this note we prove that a finitely generated soluble-by-finite group G is torsion-by-nilpotent if and only if every infinite subset contains two distinct elements x, y such that $\langle x, x^y\rangle$ is torsion-by-nilpotent.

1. Introduction and results.

Let $\mathcal{X}$ be a class of groups. Denote by $(\mathcal{X}, \infty)$ (respectively, $(\mathcal{X}, \infty)^*$) the class of groups in which every infinite subset contains two distinct elements x, y such that $\langle x, y\rangle$ (respectively, $\langle x, x^y\rangle$) belongs to $\mathcal{X}$. The second author proves in [7] that a finitely generated soluble group in the class $(\mathcal{TN}, \infty)$ (respectively, $(\mathcal{CN}, \infty)^*$) is torsion-by-nilpotent (respectively, finite-by-nilpotent), where $\mathcal{N}$ (respectively, $\mathcal{C}$, $\mathcal{T}$) denotes the class of nilpotent (respectively, Chernikov, torsion) groups. In this note we extend the result obtained on $(\mathcal{TN}, \infty)$ to the class $(\mathcal{TN}, \infty)^*$. Many results have been obtained on $(\mathcal{X}, \infty)$ for various classes of groups $\mathcal{X}$, see for example the papers referred to in [7]. Recall that the origin of this type of problems is a result of B. H. Neumann [5] which asserts that a group is centre-by-finite if and only if every infinite subset contains a commuting pair of distinct elements. Our main result is as follows:

Theorem 1.1. A finitely generated soluble-by-finite group in the class $(\mathcal{TN}, \infty)^*$ is torsion-by-nilpotent.

(*) Indirizzo dell'A.: Department of Mathematics, Faculty of Sciences, University of Setif, Algeria.
E-mail: rtarek5@yahoo.fr
(**) Department of Mathematics, Faculty of Sciences, University of Setif, Algeria.
E-mail: nadir_trabelsi@yahoo.fr

2000 *Mathematics Subject Classification.* 20F99

Note that Theorem 1.1 is not true for arbitrary groups. Indeed, in [3, Corollary 4.2] it is proved that for every integer $d \geq 2$, there exists a non nilpotent residually nilpotent torsion-free group generated by d elements such that any $d-1$ elements generate a nilpotent group.

Let k be a positive integer and let $\mathcal{E}_k(\infty)$ be the class of groups in which every infinite subset contains two distinct elements x, y such that $[x,_k y] = 1$. In [1] Abdollahi proved that a finitely generated metabelian group G is in the class $\mathcal{E}_k(\infty)$ if, and only if, $G/Z_k(G)$ is finite, and if G is a finitely generated soluble group in the class $\mathcal{E}_k(\infty)$, then there exists an integer $c = c(k)$, depending only on k, such that $G/Z_c(G)$ is finite. Note that $(\mathcal{N}_k, \infty)^*$ is contained in $\mathcal{E}_{k+1}(\infty)$, where $\mathcal{N}_k$ stands for the class of nilpotent groups of class at most k. Combining the results of [1] and Theorem 1.1, we shall obtain the following consequences.

COROLLARY 1.2. Let k be a positive integer.

(i) If G is a finitely generated soluble-by-finite group in the class $(\mathcal{TN}_k, \infty)^*$, then there exists an integer $c = c(k)$, depending only on k, such that G belongs to $\mathcal{TN}_c$.

(ii) A finitely generated metabelian-by-finite group is in the class $(\mathcal{TN}_k, \infty)^*$ if, and only if, it belongs to $\mathcal{TN}_{k+1}$.

2. Proofs of the results.

In [4, Theorem 1] Longobardi and Maj proved that a finitely generated soluble group in the class $\mathcal{E}(\infty)$ is finite-by-nilpotent, where $\mathcal{E}(\infty)$ denotes the class of groups in which every infinite subset contains two distinct elements x, y such that $[x,_n y] = 1$ for some integer $n = n(x, y)$. To prove Theorem 1.1, we extend this result to finitely generated soluble-by-finite groups.

LEMMA 2.1. A finitely generated soluble-by-finite group in the class $\mathcal{E}(\infty)$ is finite-by-nilpotent.

PROOF. Let G be an infinite finitely generated soluble-by-finite group in the class $\mathcal{E}(\infty)$ and let H be a normal soluble subgroup of finite index. First of all note that by [4, Theorem 1] G is a finitely generated nilpotent-by-finite group, so it satisfies the maximal condition on subgroups and therefore all its subgroups are finitely generated. To prove our lemma we proceed by induction on the derived length d of H. If $d = 1$, then G is a finitely generated abelian-by-finite group, so it has a normal torsion-free abelian subgroup A of finite index. Let $1 \neq a \in A$ and let $g \in G$, then the subset $\{a^i g : i > 0\}$ is infinite. So there are two distinct positive integers m, n such that $[a^m g,_c a^n g] = 1$ for some positive integer c. Since A is normal and abelian we obtain that $[a,_c g]^{m-n} = 1$, and this gives $[a,_c g] = 1$ as A is torsion-free. It follows that a is a right Engel element of G. Since G satisfies the maximal condition on subgroups, the set of its right Engel elements coincides with a term of the upper central series [6, Theorem 7.21]. Hence $A \leq Z_i(G)$, for some integer $i > 0$. So $G/Z_i(G)$ is finite, and this gives that G is finite-by-nilpotent by a result of Baer [6, Corollary 2 of Theorem 4.21]. Now assume that $d > 1$. Then by the inductive hypothesis $G/H^{(d-1)}$ is finite-by-nilpotent. So G belongs to the class $(\mathcal{AF})\mathcal{N}$, where $\mathcal{A}$ (respectively, $\mathcal{F}$) stands for the class of abelian (respectively, finite) groups. Hence G is in the class $(\mathcal{FN})\mathcal{N}$ by the first part of the proof. It follows that G is a finitely generated finite-by-soluble group. Consequently by [4, Theorem 1] G is finite-by-nilpotent, as required. □

LEMMA 2.2. Let G be a soluble-by-finite group in the class $(\mathcal{TN}, \infty)^*$. If G is abelian-by-torsion then it is torsion-by-abelian.

PROOF. Let G be a soluble-by-finite group in the class $(\mathcal{TN}, \infty)^*$ and suppose that G is abelian-by-torsion. Firstly, we show that G has a torsion subgroup. Let $x, y \in G$ be two elements of finite order. Then $H = \langle x, y \rangle$ is a finitely generated soluble-by-finite group which belongs to $\mathcal{AT}$, so it is abelian-by-finite. Clearly we may assume that H is infinite. Therefore H has a torsion-free normal abelian subgroup A of finite index. Let $1 \neq a \in A$ and let $h \in H$, then the subset $\{a^i h : i > 0\}$ is infinite. Therefore there are two distinct positive integers m, n such that $\langle a^n h, (a^n h)^{a^m h} \rangle$ is torsion-by-nilpotent, so $\langle [a^m h, a^n h], a^n h \rangle$ is also torsion-by-nilpotent. Hence $[[a^m h, a^n h],_c a^n h]^d = 1$ for some positive integers c, d. Since A is abelian and normal in H, we obtain that $[a,_{c+1} h]^{(m-n)d} = 1$ and this gives that $[a,_{c+1} h] = 1$ as A is torsion-free. It follows that A is contained in $R(H)$ the set of right Engel elements of H. Since H satisfies the maximal condition on subgroups, $R(H)$ coincides with $Z_i(H)$ for some integer $i > 0$ [6, Theorem 7.21] and hence $H/Z_i(H)$ is finite. Thus by a result of Baer [6, Corollary 2 of Theorem 4.21] H is a finite-by-nilpotent group generated by two elements of finite order. So H is finite, which is a contradiction. Consequently H is a finite group and this means that G

has a torsion subgroup T, as claimed. Now G/T is a torsion-free group in the class $(\mathcal{TN},\infty)^*$. Therefore G/T belongs to $(\mathcal{N},\infty)^*$ which is contained in $\mathcal{E}(\infty)$. It follows that G/T is a soluble-by-finite group in the class $\mathcal{E}(\infty)$ and this gives, by Lemma 2.1, that G/T is locally finite-by-nilpotent. Therefore G/T is a locally nilpotent torsion-free group in the class $\mathcal{AT}$; so by [6, Lemma 6.33] G/T is abelian. It follows that G is torsion-by-abelian, as claimed. □

Proof of Theorem 1.1. Let G be a finitely generated soluble-by-finite group in the class $(\mathcal{TN},\infty)^*$ and let H be a normal soluble subgroup of finite index. We proceed by induction on the derived length d of H. From Lemma 2.2, this is true if $d = 1$, so we can assume that $d > 1$. By the inductive hypothesis we have that $G/H^{(d-1)}$ is torsion-by-nilpotent. Thus G is in the class $(\mathcal{AT})\mathcal{N}$, and by Lemma 2.2 it belongs to $(\mathcal{TA})\mathcal{N}$. Let N be a normal torsion-by-abelian subgroup of G such that G/N is nilpotent and let T be the torsion subgroup of N. Clearly, T is normal in G and G/T is (torsion-free abelian)-by-nilpotent. Since $(\mathcal{TN},\infty)^*$ is a quotient closed class of groups, we may, therefore, suppose G (torsion-free abelian)-by-nilpotent. Let A be a normal torsion-free abelian subgroup of G such that G/A is nilpotent. Let $1 \neq a \in A$ and let $g \in G$, then the subset $\{a^i g : i > 0\}$ is infinite. Hence there are two distinct positive integers m, n such that $\langle a^n g, (a^n g)^{a^m g}\rangle$ is torsion-by-nilpotent. So $\langle [a^m g, a^n g], a^n g\rangle$ is also torsion-by-nilpotent. Thus $[[a^m g, a^n g],_c a^n g]^d = 1$ for some positive integers c and d. Therefore $[a,_{c+1} g]^{(m-n)d} = 1$ as A is abelian and normal in G, and this gives that $[a,_{c+1} g] = 1$ as A is torsion-free. It follows that a is a right Engel element of G. Since G is a finitely generated soluble group, the set of its right Engel elements coincides with its hypercenter $\overline{Z}(G)$ [2, Theorem A]. Consequently $A \leq \overline{Z}(G)$, hence $G/\overline{Z}(G)$ is nilpotent from which we deduce that G is hypercentral. Therefore G is nilpotent since it is finitely generated. □

Proof of Corollary 1.2. Let k be a positive integer.

(*i*) It suffices to factor G by its torsion subgroup and to apply [1, Theorem 3].

(*ii*) Let T be the torsion subgroup of G. Then by [1, Theorem 2] G/T is a torsion-free nilpotent group in the class $\mathcal{N}_{k+1}\mathcal{F}$. So by [6, Lemma 6.33] G/T is in $\mathcal{N}_{k+1}$. Hence G belongs to $\mathcal{TN}_{k+1}$. The converse follows from the fact that $\langle x, x^y\rangle$ is contained in $\langle x, \langle x, y\rangle'\rangle$. Therefore if $\langle x, y\rangle$ belongs to $\mathcal{TN}_{k+1}$, then $\langle x, x^y\rangle$ is in $\mathcal{TN}_k$. □

REFERENCES

[1] A. Abdollahi, *Some Engel conditions on infinite subsets of certain groups*, Bull. Austral. Math. Soc. **62** (2000), pp. 141–148.
[2] C. J. B. Brookes, *Engel elements of soluble groups*, Bull. London Math. Soc., **18** (1986), pp. 7–10.
[3] L. Hammoudi, *Burnside and Kuroch Problems*, Internat. J. Algebra Comput., **14** (2004), pp. 197–211.
[4] P. Longobardi - M. Maj, *Finitely generated soluble groups with an Engel condition on infinite subsets*, Rend. Sem. Mat. Univ. Padova, **89** (1993), 97–102.
[5] B. H. Neumann, *A problem of Paul Erdös on groups*, J. Austral. Math. Soc. ser. A, **21** (1976), pp. 467–472.
[6] D. J. S. Robinson, *Finiteness conditions and generalized soluble groups*, (Springer-Verlag, Berlin, Heidelberg, New York 1972).
[7] N. Trabelsi, *Soluble groups with many 2-generated torsion-by-nilpotent subgroups*, Publ. Math. Debrecen, **67**/1-2 (2005), pp. 93–102.