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Pseudo $\mathcal{PT}$-symmetry in time periodic non-Hermitian Hamiltonians systems

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A B S T R A C T

We investigate the concept of the pseudo-parity–time (pseudo-$\mathcal{PT}$) symmetry in periodic quantum systems. This pseudo parity–time symmetry manifests itself dynamically in the framework of the non-unitary evolution (Floquet) operator $U(\tau) = e^{-iL\tau}$, over a period τ, which shows that the stability of the dynamics occurs when the $\mathcal{PT}$-symmetry (or pseudo-$\mathcal{PT}$) of the time-independent non-Hermitian Hamiltonian L is unbroken i.e. its quasienergies $\mathcal{E}_n$ are real. Nevertheless, when the $\mathcal{PT}$-symmetry of the non-Hermitian Hamiltonian L is broken, which corresponds to the complex conjugate quasienergies $\mathcal{E}_n$, an instable dynamics arises. We investigate in greater detail a harmonic oscillator with imaginary time-dependent periodic driving term linear in x. The Floquet operator for the modulated system is pseudo-$\mathcal{PT}$ symmetric if the relative phase ϕ of the applied mode is not 0 or π.

1. Introduction

Parity–time ($\mathcal{PT}$) symmetry, the invariance under simultaneous parity and time reversal transformation, play an important role in non-Hermitian (NH) quantum mechanics and optics [1–3]. The physical condition of $\mathcal{PT}$ symmetry can be placed in a more general mathematical context known

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as pseudo-Hermiticity [4–9]. A Hamiltonian is said to be pseudo-Hermitian with respect to an invertible Hermitian operator η if it satisfies $H^{\dagger} = \eta H \eta^{-1}$. There is an intimate relation between the $\mathcal{PT}$-symmetric and the pseudo-Hermitian Hamiltonians, i.e., an exact antilinear symmetric system can be transformed into its corresponding Hermitian system through a similarity transformation; this similarity transformation can be realized by means of a positive definite metric operator η [8]. Specifically, a $\mathcal{PT}$-symmetric Hamiltonian can correspond to a Hermitian one. In the case of $\mathcal{PT}$-symmetric Hamiltonians, the role of η is played by $\mathcal{PC}$ where $\mathcal{C}$ is the charge conjugation operator.

While research on $\mathcal{PT}$-symmetry has focused on time-independent Hamiltonians [1–3,10–12], several recent works have studied time-dependent Hamiltonians [13–19], e.g., those with a time-dependent $\mathcal{PT}$-symmetric Hamiltonian and with a time-dependent metric. Motivated by this progress, the periodical systems with non-Hermitian Hamiltonians have been explored [20–27]. It is well known that the dynamics of a periodically driven system is dictated by its Floquet spectrum. This led some authors [25,26,28] to study the Floquet operator in a time evolution of a system with a time-dependent non-Hermitian Hamiltonian.

Most often, in nonrelativistic quantum mechanics and in relativistic quantum field theory, one finds the $\mathcal{T}$-operator employed as complex conjugation because the time coordinate t is a parameter and thus the time-reversal operator $\mathcal{T}$ does not actually reverse the sign of t [29]. There are however cases where T is not used as complex conjugation.

More recently, in a discussion of how to employ biharmonic modulations to manipulate $\mathcal{PT}$ symmetry, a new concept of pseudo-parity–time (pseudo-$\mathcal{PT}$) symmetry has been introduced in [22] for periodically modulated optical systems with balanced gain and loss. The time-reversal operator $\mathcal{T}$ has been defined by its action on the position operator, the momentum operator, the sign of i and the time t as

$$\mathcal{T} : p \to -p, \ x \to x, \ i \to -i \text{ and } t \to -t. \tag{1}$$

This definition of time-reversal operator $\mathcal{T}$ has been used in many papers related to time-dependent systems [20–23].

However, with a more careful examination of the quasi-energies obtained in [22], an important remark can be made. The reality of the spectrum and consequently the $\mathcal{PT}$ symmetry or "pseudo-$\mathcal{PT}$ symmetry" appears in the effective system corresponding to an averaged non-Hermitian Hamiltonian and not on the original periodic non-Hermitian Hamiltonian which does not predict the information on its spectrum. Moreover with the high-frequency approximation they used, they obtained a time-independent system. However in this case, one cannot go back to the original system, contrary to our work. Generally, the dynamics of a periodic time dependent non-Hermitian system is dictated by its Floquet spectrum which may become real and the $\mathcal{PT}$ symmetry can be associated to the Floquet Hamiltonian.

In this paper we adopt the following properties of the space reflection (parity) operator $\mathcal{P}$ and the time-reflection operator $\mathcal{T}$. The parity operator $\mathcal{P}$ is linear and has the effect

$$\mathcal{P} : p \to -p \text{ and } x \to -x. \tag{2}$$

The time-reversal operator $\mathcal{T}$ is antilinear and has the effect

$$\mathcal{T} : p \to -p, \ x \to x \text{ and } i \to -i. \tag{3}$$

Therefore, it is shown that if an energy eigenstate of a non-Hermitian Hamiltonian H is simultaneously an eigenstate of $\mathcal{PT}$ then the energy eigenvalues are real. The regime in which this requirement holds is referred to as 'unbroken' $\mathcal{PT}$-symmetry; within this regime we see that the first requirement for the $\mathcal{PT}$ formulation of quantum mechanics is satisfied. However, the remaining required properties of the states, namely that they are orthonormal and evolve in time in such a way as to preserve the inner product, are not satisfied if we consider these properties with respect to the standard inner product [1]. The purpose of Bender et al. [2] is to identify a new symmetry, denoted $\mathcal{C}$, having properties very similar to the charge conjugation operator, inherent in all $\mathcal{PT}$-symmetric Hamiltonians that possess an unbroken $\mathcal{PT}$ symmetry. This has allowed the introduction of an inner-product structure associated with $\mathcal{CPT}$ conjugation for which the norms of quantum states are positive definite and unitary-invariant. In particular, $\mathcal{CPT}$ symmetry is shown to generalize the conventional Hermiticity requirement by replacing it with a dynamically determined inner product (one that is defined by the Hamiltonian itself). When an eigenstate of the Hamiltonian is not simultaneously an eigenstate of $\mathcal{PT}$, the symmetry is broken and the eigenvalues come in complex conjugate pairs.

However, $\mathcal{PT}$ symmetry and the $\mathcal{PT}$-symmetric operator have not yet been directly identified from the time-evolution operator itself, since the definition of $\mathcal{PT}$ symmetry on the time-evolution operator has not been established so far. In the present work, we study $\mathcal{PT}$ symmetry of the time-evolution operator after a period (or Floquet operator) of non-unitary quantum periodic evolution [26,28].

In light of the above discussion, one important question motivates our work here: How could one treat a periodic time-dependent quantum non-Hermitian problem and investigate the possibility of finding a $\mathcal{PT}$-symmetric or "pseudo-$\mathcal{PT}$ symmetric " non-Hermitian Hamiltonian Floquet operator and its associated real eigenvalues, without making use of reversing the sign of t under the time-reversal operator $\mathcal{T}$ which is a complex conjugation operator? In this paper we answer this question from a new perspective by studying the time-dependent periodic non-Hermitian Hamiltonian systems using a time-dependent non-unitary transformation. In Section 2, we introduce the Floquet theory of periodic time-dependent non-Hermitian quantum Hamiltonian $H(t) = H(t + \tau)$ and we analyze the dynamics of time-periodic $\mathcal{PT}$-symmetric Hamiltonians. To this end, we perform a non-unitary transformation on the state that corresponds to a change of the reference frame. It works such that the dynamics, as seen from the new reference frame, are governed by a time-independent non-Hermitian Hamiltonian (non-Hermitian Floquet Hamiltonian). In fact, this is the guiding principle behind the construction of the Floquet solutions to some integrable periodically time-dependent problems. Therefore, the quantity of main importance in the study of the periodic non-Hermitian Hamiltonian and cyclic states and the accompanying phases is the one-cycle evolution operator $U(\tau) = e^{-iL\tau}$, which is also known as the Floquet operator. The time-independent operator L plays here the role of the Hamiltonian for the transformed states. We show the connection between the existence of the stable dynamics and the *unbroken* $\mathcal{PT}$-symmetry of the time-independent operator L. For the purpose of illustration, we study in detail in Section 3, the periodic driven non-Hermitian harmonic oscillator. This allows us to introduce and discuss, at an elementary level several facets of the Floquet picture. More importantly, the $\mathcal{PT}$ symmetry can appear in the effective system corresponding to a non-$\mathcal{PT}$-symmetric and non-Hermitian Hamiltonian Floquet operator. Therefore, we call the induced symmetry associated with Floquet systems as the "pseudo-$\mathcal{PT}$ symmetry". Note that for this concrete example, we can obtain explicit form of the non-unitary transformation which converts a time-periodic non-Hermitian system to time-independent non-Hermitian problem; however for more general case (time-dependent problem), it is hard to find the explicit form for such transformation. The paper ends with concluding remarks.

2. $\mathcal{PT}$-symmetry and stable dynamics

We now put some elements which are required for the description of quantum systems possessing a discrete translational invariance in time. We start with the time-dependent Schrödinger equation governed by a periodic time-dependent non-Hermitian Hamiltonian $H(t) = H(t + \tau)$:

$$i\frac{\partial}{\partial t}\left|\Phi(t)\right\rangle = H(t)\left|\Phi(t)\right\rangle, \tag{4}$$

where $\tau = 2\pi/\omega$ is a period and we assume ($\hbar = 1$). The time evolution operator $U(t)$ for the period of $[0, t]$ satisfies

$$i\frac{\partial}{\partial t}U(t) = H(t)U(t), \tag{5}$$

with the initial condition $U(0) = 1$. The dynamics yielded by Eq. (5) with a time-dependent and non-Hermitian $H(t)$ is non unitary in general [17].

By using Floquet's theorem [30–34], the time-evolution operator $U(t)$ of a τ-periodically time dependent quantum system has the form [26,28,35]

$$U(t) = Z(t)e^{-iLt}, \tag{6}$$

where the non-unitary operator $Z(t) = Z(t+\tau)$ is τ-periodic, and the time-independent operator L is non-Hermitian. The initial condition $U(0) = 1$ implies $Z(0) = Z(\tau) = 1$. With the help of this exponential we now define a further non unitary operator

$$Z(t) = U(t)e^{iLt} \tag{7}$$

and perform the non unitary transformation

$$|\Phi(t)\rangle = Z(t)\,|\psi(t)\rangle\,. \tag{8}$$

Now the definition of $Z(t)$ (Eq. (7)) implies

$$i\dot{Z} = HZ - ZL, \tag{9}$$

this gives

$$i\frac{\partial}{\partial t}\,|\Phi(t)\rangle = (HZ - ZL)\,|\psi(t)\rangle + Zi\frac{\partial}{\partial t}\,|\psi(t)\rangle\,. \tag{10}$$

Next, Eq. (4) readily yields

$$i\frac{\partial}{\partial t}\,|\Phi(t)\rangle = HZ\,|\psi(t)\rangle\,, \tag{11}$$

and, leaving us with

$$i\frac{\partial}{\partial t}\,|\psi(t)\rangle = L\,|\psi(t)\rangle\,. \tag{12}$$

The time-independent non-Hermitian operator L plays here the role of the Hamiltonian for the transformed states $|\psi(t)\rangle$. In fact, this is the strategy for the construction of the Floquet solutions to some integrable periodically time-dependent problems. Thus, when the wavefunctions are simultaneous eigenstates of the Hamiltonian L and the $\mathcal{PT}$-operator one can easily argue that the spectrum has to be real. However, despite the fact that $[\mathcal{PT}, H] = 0$, one may also encounter conjugate pairs of eigenvalues for broken $\mathcal{PT}$-symmetry [1–3]. It should be stressed that $\mathcal{PT}$ symmetry cannot be regarded as the fundamental property, which always explains the reality of the spectrum for the non-Hermitian Hamiltonian; there are also examples with real spectra for which not even the Hamiltonian is $\mathcal{PT}$ symmetric. This induced symmetry is called the "pseudo-$\mathcal{PT}$ symmetry" [22].

Since, the operator L plays a very important role in studying the notion of $\mathcal{PT}$ (or pseudo $\mathcal{PT}$)-symmetry, the decomposition (6) is strongly linked to the stability of the dynamics of the system and depends on the nature of the $\mathcal{PT}$-symmetry of the operator L (*broken* or *unbroken*). Writing the set of eigenvalues of the Floquet propagator $U(\tau) = e^{-iL\tau}$ as $\{e^{-i\tau\mathcal{E}_n}\}$, and its eigenstates as $\{|\phi_n(0)\rangle\}$

$$U(\tau)\,|\phi_n(0)\rangle = e^{-i\epsilon_n\tau}\,|\phi_n(0)\rangle\,, \tag{13}$$

we have

$$e^{-iLt}\,|\phi_n(0)\rangle = e^{-i\mathcal{E}_n t}\,|\phi_n(0)\rangle\,. \tag{14}$$

Now we are in a position to monitor the time-evolution of an arbitrary initial state $|\Phi(0)\rangle$. Expanding with respect to the eigenstates $|\phi_n(0)\rangle$ of $U(\tau)$,

$$|\Phi(0)\rangle = \sum_n a_n\,|\phi_n(0)\rangle \tag{15}$$

and applying $U(t)$ (6), we then find

$$|\Phi(t)\rangle = U(t)\,|\Phi(0)\rangle = \sum_n a_n Z(t) e^{-iLt}\,|\phi_n(0)\rangle$$
$$= \sum_n a_n e^{-i\mathcal{E}_n t}\,|\phi_n(t)\rangle\,. \tag{16}$$

In the last step made here we use the special decomposition of Moore and Stedman's formalism to define the Floquet functions

$$|\phi_n(t)\rangle = Z(t)\,|\phi_n(0)\rangle\,. \tag{17}$$

which are τ-periodic i.e. $|\phi_n(t+\tau)\rangle = |\phi_n(t)\rangle$. We will refer to the states

$$|\Phi_n(t)\rangle = e^{-i\mathcal{E}_n t}\,|\phi_n(t)\rangle\,, \tag{18}$$

as Floquet states. Note that these states, in contrast to the τ-periodic Floquet functions $|\phi_n(t)\rangle$, are solutions of the time-dependent Schrödinger equation (4). Substituting (18) into the time-dependent Schrödinger equation (4) we get

$$K(t)\,|\phi_n(t)\rangle = \mathcal{E}_n\,|\phi_n(t)\rangle\,, \tag{19}$$

where the operator

$$K(t) = H(t) - i\frac{\partial}{\partial t}, \tag{20}$$

is the non-Hermitian Floquet Hamiltonian. In the physics literature $\mathcal{E}_n$ is often called a quasi-energy and $|\phi_n(t)\rangle$ a quasienergy state [36]. Thus, the analysis of systems with periodic non-Hermitian Hamiltonians $H(t)$ can be reduced to an equivalent "time-independent" form by using the non-Hermitian Floquet Hamiltonian $K(t) = H(t) - i\frac{\partial}{\partial t}$.

Now, we engage in the study of the stability of the dynamics. For (arbitrary) N periods, the time evolution operator $U(N\tau) = e^{-iNL\tau}$ has the eigenvalues $e^{-iN\epsilon_n\tau}$. This means that the knowledge of the time evolution operator $U(\tau) = e^{-iL\tau}$ over a full period τ yields a stroboscopic view of the dynamics over long multiples of the fundamental period,

$$|\phi_n(N\tau)\rangle = U(N\tau)\,|\phi_n(0)\rangle = e^{-iN\epsilon_n\tau}\,|\phi_n(0)\rangle\,. \tag{21}$$

From this important feature, one can deduce the nature of the evolution (stable or unstable). As we have said above, if the operator L is $\mathcal{PT}$-symmetric, then the situation of particular interest is when the $\mathcal{PT}$-symmetry is in an unbroken phase. This means that the quasienergies ϵ_n of L are real. Thus the phase factor $e^{-iN\epsilon_n\tau}$ of $U(N\tau)$ contains only real factors $N\epsilon_n\tau$. The dynamics is therefore stable over an arbitrary number of driving periods N because there are no complex factors in $e^{-iN\epsilon_n\tau}$ which generate exponential growth or decay with N. For the *broken* $\mathcal{PT}$-symmetry phase, the quasienergies ϵ_n become complex conjugates and the dynamics is hence unstable. Thereby, the stability of the dynamics depends on the *unbroken* $\mathcal{PT}$-symmetry phase of the operator L, and does not depend on the *unbroken* $\mathcal{PT}$-symmetry phase of the Hamiltonian $H(t)$. Also an interesting situation is when the quasienergies are real, the operator L is not $\mathcal{PT}$-symmetric (but $\mathcal{PT}$ pseudo-symmetric), in this case the dynamics is stable. We discuss all these situations in a concrete example in the next section.

3. Driven non Hermitian harmonic oscillator: pseudo $\mathcal{PT}$-symmetric Floquet operator

This section considers one of the simplest Floquet problems: harmonic oscillator with a non-Hermitian time-dependent periodic driving term linear in x. Although the Floquet eigenenergies and eigenfunctions for the Hermitian system have been previously given in the literature [37,38], we will show how to derive, for the non-Hermitian system, the pseudo PT-symmetric Floquet operator and the associated real eigenenergies. We first consider a harmonic oscillator interacting with an imaginary time-dependent periodic driving term linear in x. In a semiclassical approach where the electric field is written $f(t) = \lambda\cos(\omega t + \phi)$ the non-Hermitian Hamiltonian is a periodic function of time

$$H(t) = \frac{1}{2m}p^2 + \frac{m\omega_0^2}{2}x^2 - if(t)x. \tag{22}$$

Now we begin to use an algebraic transformation method to solve the time-dependent Schrödinger equation (5). Usually, a solution of the time-dependent system is resorted to a time-dependent transformation as

$$|\Phi(t)\rangle = D(t)\,|\chi(t)\rangle \tag{23}$$

then an initial Schrödinger equation will be changed into

$$\begin{aligned} &i\frac{\partial}{\partial t}\,|\chi(t)\rangle = h(t)\,|\chi(t)\rangle \\ &\text{with } h(t) = D^{-1}(t)H(t)D(t) - iD^{-1}(t)\frac{\partial}{\partial t}D(t). \end{aligned} \tag{24}$$

In the above transformation, the time-dependent operators $D(t)$ should have its inverse operator and it can be properly chosen in order to obtain the new Hamiltonian $h(t)$ into a solvable form, we can choose

$$D(t) = e^{i(x\mathcal{P}_c(t) - p\mathcal{X}_c(t))} = e^{-\frac{i}{2}\mathcal{X}_c(t)\mathcal{P}_c(t)}e^{ix\mathcal{P}_c(t)}e^{-ip\mathcal{X}_c(t)} \tag{25}$$

and its reverse operator is

$$D^{-1}(t) = e^{+\frac{i}{2}\mathcal{X}_c(t)\mathcal{P}_c(t)}e^{ip\mathcal{X}_c(t)}e^{-ix\mathcal{P}_c(t)} \tag{26}$$

where

$$\begin{aligned} \mathcal{X}_c(t) &= x_c(t) - x_c(0) \\ \mathcal{P}_c(t) &= p_c(t) - p_c(0) \end{aligned} \tag{27}$$

$x_c(t)$ et $p_c(t)$ are classical complex solutions associated with the system ruled by the classical non-Hermitian Hamiltonian $H(t)$

$$\begin{aligned} \dot{x}_c &= \frac{1}{m}p_c \\ \dot{p}_c &= -m\omega_0^2 x_c + if(t) \\ \ddot{x}_c + m\omega_0^2 x_c &= if(t) \end{aligned} \tag{28}$$

whose classical solutions will be

$$x_c(t) = i\frac{\lambda\cos(\omega t + \phi)}{m\left(\omega_0^2 - \omega^2\right)}, \qquad \dot{x}_c = -i\frac{\lambda\omega\sin(\omega t + \phi)}{m\left(\omega_0^2 - \omega^2\right)} \tag{29}$$

and $x_c(0)$, $p_c(0)$ are the classical initial conditions. It can easily be shown that under the transformation $D(t)$ the coordinate and momentum operators change according to

$$\begin{aligned} D^{-1}(t)xD(t) &= x + \mathcal{X}_c(t) \\ D^{-1}(t)pD(t) &= p + \mathcal{P}_c(t). \end{aligned} \tag{30}$$

An important property of the transformation $D(t)$ which, when it acts on a wavefunction in the x-representation, gives

$$D(t)G(x) = \exp\left[-\tfrac{i}{2}\mathcal{X}_c(t)\mathcal{P}_c(t)\right]\exp\left[ix\mathcal{P}_c(t)\right]G(x - \mathcal{X}_c(t)). \tag{31}$$

Substituting Eq. (30) into Eq. (24) one can show that

$$h(t) = h_d^{OQL} + h_f^{OC}(t) + \frac{1}{2}\left(\dot{p}_c x_c - p_c\dot{x}_c\right) + \frac{1}{2}\left(\dot{p}_c x_c(0) - p_c(0)\dot{x}_c\right) \tag{32}$$

where

$$h_d^{OQL} = \frac{1}{2m}(p - p_c(0))^2 + \frac{m\omega_0^2}{2}(x - x_c(0))^2 \tag{33}$$

represents the time-independent quantum displaced harmonic oscillator and

$$h_f^{OC}(t) = \frac{1}{2m}(p_c)^2 + \frac{m\omega_0^2}{2}(x_c)^2 - if(t)x_c. \tag{34}$$

Then, the transformed Hamiltonian determined by Eq. (32) can be written as

$$h(t) = h_d^{OQL} + \mathcal{L}(t), \tag{35}$$

where

$$\begin{aligned}\mathcal{L}(t) &= h_f^{OC}(t) + \frac{1}{2}(\dot{p}_c x_c - p_c \dot{x}_c) + \frac{1}{2}(\dot{p}_c x_c(0) - p_c(0)\dot{x}_c) \\ &= \frac{1}{4}\frac{\lambda^2}{m\left(\omega_0^2 - \omega^2\right)} + \frac{1}{4}\frac{\lambda^2}{m\left(\omega_0^2 - \omega^2\right)}\cos 2(\omega t + \phi) + \frac{1}{2}\frac{\lambda^2\omega^2}{m\left(\omega_0^2 - \omega^2\right)^2}\cos\omega t\end{aligned} \tag{36}$$

can be removed from the Hamiltonian of Eq. (35) by performing the transformation

$$|\chi(t)\rangle = \exp\left[-i\int_0^t d\acute{t}\left(\frac{1}{4}\frac{\lambda^2}{m\left(\omega_0^2 - \omega^2\right)} + \mathcal{L}(\acute{t})\right)\right]|\psi(t)\rangle. \tag{37}$$

Thus, the Schrödinger equation associated with $h(t)$ reduces to a time-independent Schrödinger equation of a displaced harmonic oscillator

$$\begin{aligned} i\frac{\partial}{\partial t}|\psi(t)\rangle &= \left(h_d^{OQL} + \frac{1}{4}\frac{\lambda^2}{m\left(\omega_0^2 - \omega^2\right)}\right)|\psi(t)\rangle \\ &= L\,|\psi(t)\rangle. \end{aligned} \tag{38}$$

The solutions are well-known, with the quantum real quasienergy,

$$\mathcal{E}_n = \omega_0(n + 1/2) + \frac{1}{4}\frac{\lambda^2}{m\left(\omega_0^2 - \omega^2\right)}. \tag{39}$$

With definition of the parity operator as $\mathcal{P} : p \to -p$ and $x \to -x$ and the time operator as $\mathcal{T} : p \to -p, x \to x, i \to -i$, it may be checked that

$$\begin{aligned}\mathcal{PT}h_d^{OQL}\mathcal{TP} &= \mathcal{PT}\left\{\frac{1}{2m}(p - p_c(0))^2 + \frac{m\omega_0^2}{2}(x - x_c(0))^2\right\}\mathcal{TP} \\ &= \mathcal{PT}\left\{\frac{1}{2m}\left(p + i\frac{\lambda\omega\sin\phi}{\left(\omega_0^2 - \omega^2\right)}\right)^2 + \frac{m\omega_0^2}{2}\left(x - i\frac{\lambda\cos\phi}{m\left(\omega_0^2 - \omega^2\right)}\right)^2\right\}\mathcal{TP} \\ &= \left\{\frac{1}{2m}(p + p_c(0))^2 + \frac{m\omega_0^2}{2}(x - x_c(0))^2\right\} \neq h_d^{OQL}\end{aligned} \tag{40}$$

therefore the non-Hermitian Hamiltonian h_d^{OQL} (33) is not invariant under the $\mathcal{PT}$-symmetric transformation but pseudo-Hermitian with respect to an invertible Hermitian operator $\eta = D(0) = e^{-i(xp_c(0)-px_c(0))}$. The non-Hermitian Hamiltonian h_d^{OQL} (33) becomes $\mathcal{PT}$-symmetric if $\phi = 0$ or π. Consequently, $L = \left(h_d^{OQL} + \frac{1}{4}\frac{\lambda^2}{m\left(\omega_0^2-\omega^2\right)}\right)$ is $\mathcal{PT}$-symmetric for $\phi = 0$ or $\phi = \pi$. Otherwise, L becomes pseudo-$\mathcal{PT}$-symmetric.

We now derive the time-evolution operator associated to Hamiltonian $H(t)$ (22). The solution $|\Phi(t)\rangle$ of the Schrödinger equation associated with the non-Hermitian Hamiltonian $H(t)$ is written as

$$|\Phi(t)\rangle = \exp\left[-i\int_0^t d\acute{t}\left(\mathcal{L}(\acute{t}) - \frac{1}{4}\frac{\lambda^2}{m\left(\omega_0^2-\omega^2\right)}\right)\right] D(t)\,|\psi(t)\rangle$$

where

$$|\psi(t)\rangle = \exp\left[-i\left(h_d^{OQL} + \frac{1}{4}\frac{\lambda^2}{m\left(\omega_0^2-\omega^2\right)}\right)t\right]|\psi(0)\rangle\,. \tag{41}$$

Using (36) and evaluating the following integral

$$\int_0^t d\acute{t}\mathcal{L}(\acute{t}) = \frac{1}{4}\frac{\lambda^2 t}{m\left(\omega_0^2-\omega^2\right)} + \frac{\lambda^2}{8m\omega\left(\omega_0^2-\omega^2\right)}\left(\sin 2(\omega t+\phi) - \sin 2\phi\right) + \frac{\lambda^2\omega}{2m\left(\omega_0^2-\omega^2\right)^2}\sin\omega t$$

allows one to identify the desired non unitary time-evolution operator for the periodic non-Hermitian Hamiltonian (22) as

$$U(t) = \exp\left[-i\left(\frac{\lambda^2}{8m\omega\left(\omega_0^2-\omega^2\right)}\left(\sin 2(\omega t+\phi) - \sin 2\phi\right) + \frac{\lambda^2\omega}{2m\left(\omega_0^2-\omega^2\right)^2}\sin\omega t\right)\right] \times D(t)\exp\left[-i\left(h_d^{OQL} + \frac{1}{4}\frac{\lambda^2}{m\left(\omega_0^2-\omega^2\right)}\right)t\right]. \tag{42}$$

Note that the time evolution operator $U(t)$ (42) is written as the product

$$U(t) = Z(t)e^{-iLt} \tag{43}$$

$Z(t)$ is constrained to be periodic

$$Z(t) = \exp\left[-i\left(\frac{\lambda^2}{8m\omega\left(\omega_0^2-\omega^2\right)}\left(\sin 2(\omega t+\phi) - \sin 2\phi\right) + \frac{\lambda^2\omega}{2m\left(\omega_0^2-\omega^2\right)^2}\sin\omega t\right)\right] D(t) \tag{44}$$

while L is constant and not self-adjoint and eventually reads

$$L = h_d^{OQL} + \frac{1}{4}\frac{\lambda^2}{m\left(\omega_0^2-\omega^2\right)}. \tag{45}$$

Its eigenvalues $\mathcal{E}_n$ (39), called quasienergies, are illustrated in Fig. 1.

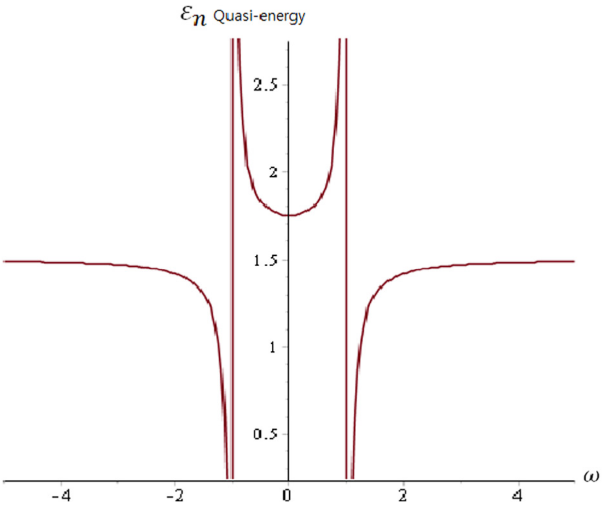

Fig. 1. Quasienergies with respect to ω. We have used $\omega_0 = n = m = \lambda = 1$. At resonance $\omega = \omega_0$, the quasienergies are no longer correct. Instead, the spectrum assumes an absolutely continuous form.

Thus our original wave function assumes

$$\begin{aligned}\Phi_n(x,t) = {} & \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega_0}{\pi}\right)^{1/4} \exp\left[-i\frac{\omega}{m}\left(\frac{\lambda\sin(\omega t+\phi)}{2(\omega_0^2-\omega^2)}\right)^2\right] \\ & \times \exp\left[i\left((n+1/2)\omega_0 + \frac{1}{4}\frac{\lambda^2}{m\left(\omega_0^2-\omega^2\right)}\right)t\right] \\ & \times \exp\left[-i\left(\frac{\lambda^2}{8m\omega\left(\omega_0^2-\omega^2\right)}\left(\sin 2(\omega t+\phi)-\sin 2\phi\right)\right.\right. \\ & \left.\left. + \frac{\lambda^2\omega}{2m\left(\omega_0^2-\omega^2\right)^2}\sin\omega t\right)\right] \\ & \times \exp\left[ip_c(x-x_c)\right]\exp\left[-\frac{m\omega_0}{2}(x-x_c)^2\right]\mathcal{H}_n(\sqrt{\omega_0 m}(x-x_c)) \end{aligned} \tag{46}$$

corresponding to time-periodic Floquet modes proportional to the Hermite functions $\mathcal{H}_n$.

When $n = 0$, the corresponding quasienergy is $\mathcal{E}_0 = \omega_0/2 + \lambda^2/4m\left(\omega_0^2-\omega^2\right)$, the ground state wave function is

$$\begin{aligned}\Phi_0(x,t) = {} & \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega_0}{\pi}\right)^{1/4} \exp\left[-i\frac{\omega}{m}\left(\frac{\lambda\sin(\omega t+\phi)}{2(\omega_0^2-\omega^2)}\right)^2\right] \\ & \times \exp\left[i\left(\frac{1}{2}\omega_0 + \frac{1}{4}\frac{\lambda^2}{m\left(\omega_0^2-\omega^2\right)}\right)t\right] \\ & \times \exp\left[-i\left(\frac{\lambda^2}{8m\omega\left(\omega_0^2-\omega^2\right)}\left(\sin 2(\omega t+\phi)-\sin 2\phi\right)\right.\right. \\ & \left.\left. + \frac{\lambda^2\omega}{2m\left(\omega_0^2-\omega^2\right)^2}\sin\omega t\right)\right]\exp\left[ip_c(x-x_c)\right]\exp\left[-\frac{m\omega_0}{2}(x-x_c)^2\right]. \end{aligned} \tag{47}$$

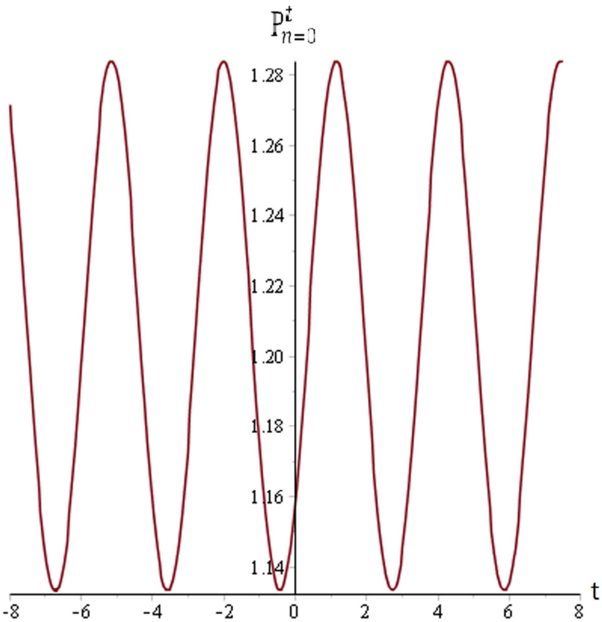

Fig. 2a. Occupation probability $P^t_{n=0}$ with respect to time t. We have used $\omega = 3$, $\phi = \pi$, $\omega_0 = m = \lambda = 1$.

It is straightforward to obtain the occupation probability $P^t_{n=0}$ for the ground state wave function with respect to time:

$$P^t_{n=0} = \int_{-\infty}^{+\infty} dx\Phi_0^*(x,t)\Phi_0(x,t) = \exp\left[\frac{\lambda^2\omega^2}{m\omega_0\left(\omega_0^2-\omega^2\right)^2}\right]\exp\left[\frac{\lambda^2\cos^2(\omega t+\phi)}{m\omega_0\left(\omega_0^2-\omega^2\right)}\right]. \quad (48)$$

The above expression shows that the occupation probability $P^t_{n=0}$ becomes nonconservative. The variation of $P^t_{n=0}$ with t and ϕ is shown in Figs. 2 and 3.

4. Concluding remarks

In summary, the study of the dynamics of time-dependent periodic non-Hermitian Hamiltonians possesses certain delicacy because of their specific properties. This is due to the non-unitarity of the evolution operators, the dynamics of such systems is not stable in general. To solve the dynamics of such systems, we take advantage of its periodicity by applying Floquet theory [30,31,35]. This method is based on a factorization of the non-unitary evolution operator [26,28] and is in the spirit of the theory of systems of linear differential equations with periodic coefficients [30]. The use of this approach in practical situations is greatly facilitated by exploiting the Fourier decomposition of the Hamiltonian. This converts the problem into an equivalent time-independent form. The solution to the problem is then expressible in terms of the eigenvectors and eigenvalues of a certain operator called the Floquet Hamiltonian. The $\mathcal{PT}$-symmetry is well implemented for time-periodic quantum systems because the behavior of the time is like that of space for *spatially* periodic Hamiltonians. This is due to the fact that the Floquet modes are the *time* equivalents of the Bloch functions which arise for *spatially* periodic Hamiltonians [32–34]. We have established that the stability of the dynamics occurs when the $\mathcal{PT}$-symmetry of the time-independent operator L which defines the Floquet operator $U(\tau) = e^{-iL\tau}$ is unbroken and correspond to the real quasienergies $\mathcal{E}_n$. Nevertheless, when the $\mathcal{PT}$-symmetry of L is *broken*, which corresponds to the complex conjugates quasienergies $\mathcal{E}_n$, an instable dynamics arises. We have also shown that the stability of the dynamics depends on the *unbroken* $\mathcal{PT}$-symmetry phase of the operator L, and not on the *unbroken* $\mathcal{PT}$-symmetry phase

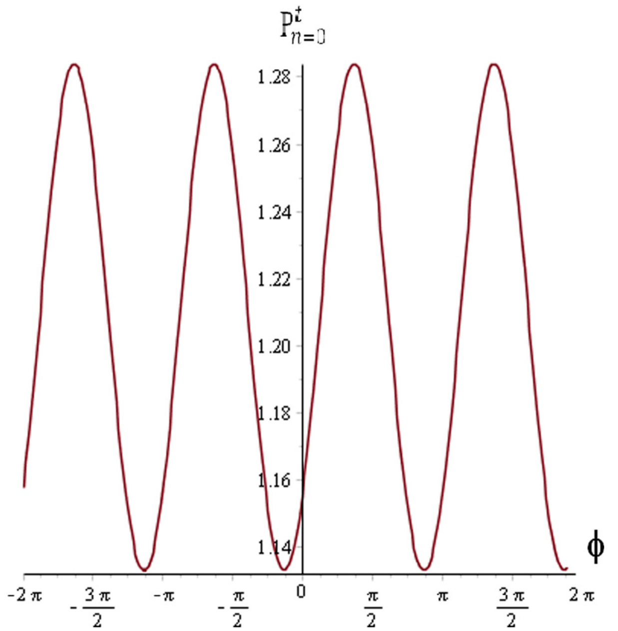

Fig. 2b. Occupation probability $P^t_{n=0}$ with respect to angle ϕ. We have used $\omega = 3$, $t = 2$, $\omega_0 = m = \lambda = 1$.

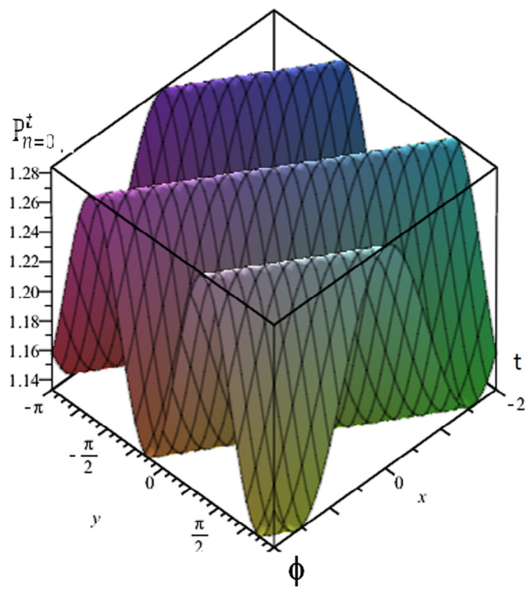

Fig. 3. Occupation probability $P^t_{n=0}$ with respect to angle ϕ and time. We have used $\omega = 3$, $\omega_0 = m = \lambda = 1$.

of the Hamiltonian $H(t)$. More importantly, the "pseudo-$\mathcal{PT}$ symmetry" can appear in the system corresponding to a non-Hermitian Floquet Hamiltonian.

Then, we convert the properties of $\mathcal{PT}$-symmetry for the Hamiltonian into those for the time-evolution Floquet operator in Eq. (6). Assuming the operator L to be invariant under simultaneous parity transform $\mathcal{P}$ and time-reversal $\mathcal{T}$ and using the relation between the time-evolution operator and $\mathcal{PT}$-symmetric non-Hermitian Hamiltonian L: $U(\tau) = e^{-iL\tau}$, we derive the relation between $U(\tau)$ and its inverse as

$$\mathcal{PT}U(\tau)\mathcal{PT} = \mathcal{PT}e^{-iL\tau}\mathcal{PT} = e^{i\mathcal{PT}L\mathcal{PT}\tau} = e^{iL\tau} = U^{-1}(\tau). \tag{49}$$

This shows that the $\mathcal{PT}$ transformed of Floquet operator $\mathcal{PT}U(\tau)\mathcal{PT}$ is just the inverse of $U(\tau)$. This relation, obtained also in the context of quantum walks [39], replaces the well known standard $\mathcal{PT}$-symmetry relation in the context of non-Hermitian quantum mechanics. Then, we considered a simple model, the driven periodic non-Hermitian oscillator, allowing us to introduce and discuss at an elementary level several facets of the Floquet picture which also govern the evolution operator obtained in closed form.

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