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A Dynamic Frictionless Elastic-Viscoplastic Problem with Normal Damped Response and Damage

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Abstract. We consider a mathematical model for the process of a frictionless contact between an elastic-viscoplastic body and a reactive foundation. The material is elastic-viscoplastic with internal state variable which may describe the damage of the system caused by plastic deformations. We establish a variational formulation for the model and prove the existence and uniqueness result of the weak solution. The proof is based on arguments of nonlinear equations with monotone operators, on parabolic type inequalities and fixed point.

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1. Introduction

In this paper we study a frictionless contact problem with normal damped response for elastic-viscoplastic materials with a general constitutive law of the form

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \beta(s))\, ds, \quad (1.1)$$

where $\mathbf{u}$ denotes the displacement field and $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}(\mathbf{u})$ represent the stress and the linearized strain tensor, respectively. Here $\mathcal{A}$ and $\mathcal{E}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, and $\mathcal{G}$ is a nonlinear constitutive function which describes the viscoplastic behaviour of the material. We also consider that the function $\mathcal{G}$ depends on the internal state variable β describing the damage of the material caused by plastic deformations. In (1.1) and everywhere in this paper the

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dot above a variable represents derivative with respect to the time variable t. It follows from (1.1) that, at each time moment t, the stress tensor $\boldsymbol{\sigma}(t)$ is split into two parts: $\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}^V(t) + \boldsymbol{\sigma}^R(t)$, where $\boldsymbol{\sigma}^V(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t))$ represents the purely viscous part of the stress whereas $\boldsymbol{\sigma}^R(t)$ satisfies a rate-type elastic-viscoplastic relation with damage,

$$\boldsymbol{\sigma}^R(t) = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}^R(s), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \beta(s))ds. \tag{1.2}$$

When $\mathcal{G} = 0$ the constitutive law (1.1) reduces to the Kelvin-Voigt viscoelastic constitutive relation given by

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)). \tag{1.3}$$

Examples and mechanical interpretation of elastic-viscoplastic materials of the form (1.2) in which the function $\mathcal{G}$ does not depend on the damage parameter β were considered by many authors, see for instance [3, 14] and the references therein. Contact problems for materials of the form (1.1), (1.2) without damage parameter and (1.3) are the topic of numerous papers, e.g. [6, 11, 12, 15, 19, 20] and the recent references [1, 13]. Contact problems for elastic-viscoplastic materials of the form (1.2) are studied in [2, 5, 18, 20]. The damage subject is extremely important in design engineering, since it directly affects the useful life of the designed structure or component. There exists a very large engineering literature on it. Models taking into account the influence of the internal damage of the material on the contact process have been investigated mathematically. General models for damage were derived in [7, 8] from the virtual power principle. Mathematical analysis of one-dimensional problems can be found in [9]. The damage function β is restricted to have values between zero and one. When $\beta = 1$ there is no damage in the material, when $\beta = 0$ the material is completely damaged, when $0 < \beta < 1$ there is partial damage and the system has a reduced load carrying capacity. Contact problems with damage have been investigated in [10, 16, 17, 18, 20]. In this paper the inclusion used for the evolution of the damage field is

$$\dot{\beta} - k \triangle \beta + \partial\varphi_K(\beta) \ni \phi(\boldsymbol{\sigma} - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}), \boldsymbol{\varepsilon}(\mathbf{u}), \beta),$$

where K denotes the set of admissible damage functions defined by

$$K = \{\xi \in H^1(\Omega) \ / \ 0 \le \xi \le 1 \ \text{ a.e. in } \Omega\},$$

k is a positive coefficient, $\partial\varphi_K$ represents the subdifferential of the indicator function of the set K and ϕ is a given constitutive function which describes the sources of the damage in the system.

In the present paper we consider a mathematical model for the process of a frictionless contact between an elastic-viscoplastic body and a reactive foundation. The contact is modelled with the normal damped response condition, see, e.g., [15] . We derive the variational formulation and prove existence and uniqueness of the weak solution of the model.

The paper is organized as follows. In section 2 we present the notation and some preliminaries. In section 3 we present the mechanical problem, we list the assumptions on the data and give the variational formulation of the problem. In section 4 we state our main existence and uniqueness result. It is based on arguments of nonlinear equations with monotone operators, parabolic type inequalities and fixed point.

2. Notation and preliminaries

In this section we present the notation we shall use and some preliminary material. For further details, we refer the reader to [4]. We denote by S_d the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$, while "." and $|\,.\,|$ will represent the inner product and the Euclidean norm on S_d and $\mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ and let $\boldsymbol{\nu}$ denote the unit outer normal on Γ. Everywhere in the sequel the index i and j run from 1 to d, summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the independent spatial variable. We use the standard notation for Lebesgue and Sobolev spaces associated to Ω and Γ and introduce the spaces

$$
\begin{gathered}
H = L^2(\Omega)^d = \left\{ \mathbf{u} = (u_i) \ / \ u_i \in L^2(\Omega) \ \right\}, \\
\mathcal{H} = \left\{ \boldsymbol{\sigma} = (\sigma_{ij}) \ / \ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \ \right\}, \\
H_1 = \{ \mathbf{u} = (u_i) \ / \ \boldsymbol{\varepsilon}(\mathbf{u}) \in \mathcal{H} \ \}, \\
\mathcal{H}_1 = \{ \boldsymbol{\sigma} \in \mathcal{H} \ / \ Div \ \boldsymbol{\sigma} \in H \}.
\end{gathered}
$$

Here $\boldsymbol{\varepsilon}$ and Div are the deformation and divergence operators, respectively, defined by

$$
\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \ Div\, \boldsymbol{\sigma} = (\sigma_{ij,\,j}).
$$

The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$
(\mathbf{u}, \mathbf{v})_H = \int_\Omega u_i v_i \ dx \qquad \forall\, \mathbf{u}, \mathbf{v} \in H,
$$

$$
(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij} \tau_{ij} \ dx \qquad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H},
$$

$$
(\mathbf{u}, \mathbf{v})_{H_1} = (\mathbf{u}, \mathbf{v})_H + (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} \qquad \forall\, \mathbf{u}, \mathbf{v} \in H_1,
$$

$$
(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (Div \ \boldsymbol{\sigma}, Div \ \boldsymbol{\tau})_H \ \ \forall \ \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H}_1.
$$

The associated norms on the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $|\,.\,|_H$, $|\,.\,|_{\mathcal{H}}$, $|\,.\,|_{H_1}$ and $|\,.\,|_{\mathcal{H}_1}$, respectively. For every element $\mathbf{v} \in H_1$ we also use the notation $\mathbf{v}$ for the trace of $\mathbf{v}$ on Γ and we denote by v_ν and $\mathbf{v}_\tau$ the normal and the tangential components of $\mathbf{v}$ on Γ given by

$$
v_\nu = \mathbf{v}.\boldsymbol{\nu}, \ \mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}. \tag{2.1}
$$

We also denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the normal and the tangential traces of a function $\boldsymbol{\sigma} \in \mathcal{H}_1$, and we recall that when $\boldsymbol{\sigma}$ is a regular function then

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}).\boldsymbol{\nu}, \ \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}, \tag{2.2}$$

and the following Green's formula holds:

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + (Div\ \boldsymbol{\sigma}, \mathbf{v})_H = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}.\mathbf{v}\ da \qquad \forall \mathbf{v} \in H_1. \tag{2.3}$$

Let $T > 0$. For every real Banach space X we use the notation $C(0,T;X)$ and $C^1(0,T;X)$ for the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively; $C(0,T;X)$ is a real Banach space with the norm

$$\mid \mathbf{f} \mid_{C(0,T;X)} = \max_{t\in[0,T]} \mid \mathbf{f}(t) \mid_X$$

while $C^1(0,T;X)$ is a real Banach space with the norm

$$\mid \mathbf{f} \mid_{C^1(0,T;X)} = \max_{t\in[0,T]} \mid \mathbf{f}(t) \mid_X + \max_{t\in[0,T]} \mid \dot{\mathbf{f}}(t) \mid_X .$$

Finally, for $k \in \mathbb{N}$ and $p \in [1, \infty]$, we use the standard notation for the Lebesgue spaces $L^p(0,T;X)$ and for the Sobolev spaces $W^{k,p}(0,T;X)$. Moreover, if X_1 and X_2 are real Hilbert spaces then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(.,.)_{X_1\times X_2}$.

3. Mechanical and variational formulations

The physical setting is the following. An elastic-viscoplastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ with a Lipschitz surface Γ that is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that *meas* $(\Gamma_1) > 0$. Let $T > 0$ and let $[0,T]$ be the time interval of interest. The body is clamped on $\Gamma_1 \times (0,T)$, and, therefore, the displacement field vanishes there. Surface tractions of density $\mathbf{f}_2$ act on $\Gamma_2 \times (0,T)$ and a body force of density $\mathbf{f}_0$ is applied in $\Omega \times (0,T)$. The body is in frictionless contact with a reactive foundation over the potential contact surface $\Gamma_3 \times (0,T)$. Moreover, the process is dynamic, and thus the inertial terms are included in the equation of motion. Then, the classical formulation of the mechanical contact problem of an elastic-viscoplastic material with damage is as follows.

Problem P. Find a displacement field $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$*, a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to S_d$ *and a damage field* $\beta : \Omega \times [0,T] \to \mathbb{R}$ *such that*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \beta(s))ds \quad \text{in } \Omega \times (0,T), \tag{3.1}$$

$$\dot{\beta} - k \triangle \beta + \partial\varphi_K(\beta) \ni \phi(\boldsymbol{\sigma} - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}), \boldsymbol{\varepsilon}(\mathbf{u}), \beta) \quad \text{in } \Omega \times (0,T), \tag{3.2}$$

$$\rho\ddot{\mathbf{u}} = Div\ \boldsymbol{\sigma} + \mathbf{f}_0 \quad \text{in } \Omega \times (0,T), \tag{3.3}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1 \times (0,T), \tag{3.4}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \mathbf{f}_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{3.5}$$

$$\sigma_\nu = -\alpha \mid \dot{u}_\nu \mid, \ \boldsymbol{\sigma}_\tau = \mathbf{0} \quad \text{on } \Gamma_3 \times (0,T)\,, \tag{3.6}$$

$$\frac{\partial \beta}{\partial \boldsymbol{\nu}} = 0 \ \text{ on } \Gamma \times (0,T)\,, \tag{3.7}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \ \dot{\mathbf{u}}(0) = \mathbf{v}_0, \ \beta(0) = \beta_0 \text{ in } \Omega. \tag{3.8}$$

Here (3.1) is the elastic-viscoplastic constitutive law with damage introduced in section I. (3.2) represents the inclusion used for the evolution of the damage field. (3.3) represents the equation of motion where ρ is the mass density. (3.4)-(3.5) are the displacement-traction conditions. The first equality in (3.6) represents the normal damped response condition (see [15]) while the second relation in (3.6) indicates that the contact is frictionless, i.e., the tangential stress vanishes on the contact surface during the process. (3.7) represents a homogeneous Neumann boundary condition where $\frac{\partial \beta}{\partial \boldsymbol{\nu}}$ is the normal derivative of β. In (3.8) $\mathbf{u}_0$ is the initial displacement, $\mathbf{v}_0$ is the initial velocity and β_0 is the initial material damage.

To simplify the notation, we do not indicate explicitely the dependence of various functions on the variables $\mathbf{x} \in \Omega \cup \Gamma$ and $t \in [0,T]$. To obtain a variational formulation of the problem (3.1)-(3.8) we need additional notation.

Let V be the closed subspace of H_1 given by

$$V = \{\mathbf{v} \in H_1 \ / \ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1\}\,.$$

Then, the following Korn's inequality holds:

$$\mid \boldsymbol{\varepsilon}(\mathbf{v}) \mid_{\mathcal{H}} \geq C_k \mid \mathbf{v} \mid_{H_1} \ \forall \mathbf{v} \in V,$$

where $C_k > 0$ is a constant depending only on Ω and Γ_1. On the space V we consider the inner product given by

$$(\mathbf{u}, \mathbf{v})_V = (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} \tag{3.9}$$

and let $\mid . \mid_V$ be the associated norm. It follows from Korn's inequality that $\mid . \mid_{H_1}$ and $\mid . \mid_V$ are equivalent norms on V. Therefore $(V, \mid . \mid_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem there exists a positive constant C_0 which depends only on Ω, Γ_1 and Γ_3 such that

$$\mid \mathbf{v} \mid_{L^2(\Gamma_3)^d} \leq C_0 \mid \mathbf{v} \mid_V \qquad \forall \mathbf{v} \in V. \tag{3.10}$$

In the study of the mechanical problem (3.1)-(3.8) we assume that

The viscosity operator $\mathcal{A} : \Omega \times S_d \to S_d$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_{\mathcal{A}} > 0 \text{ such that} \\ \mid \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_2) \mid \leq L_{\mathcal{A}} \mid \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 \mid \ \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_d, \text{ a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ (\mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_2)).(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{A}} \mid \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 \mid^2 \ \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_d, \\ \ \text{a.e. } \mathbf{x} \in \Omega. \\ (c) \ \mathbf{x} \to \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega. \\ (d) \text{ The mapping } \mathbf{x} \to \mathcal{A}(\mathbf{x}, \mathbf{0}) \in \mathcal{H}. \end{cases} \tag{3.11}$$

The elasticity operator $\mathcal{E} : \Omega \times S_d \to S_d$ satisfies

$$\left\{\begin{array}{l} (a) \text{ There exists a constant } L_{\mathcal{E}} > 0 \text{ such that} \\ \mid \mathcal{E}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{E}(\mathbf{x}, \boldsymbol{\varepsilon}_2) \mid \leq L_{\mathcal{E}} \mid \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 \mid \ \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_d, \text{ a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ For any } \boldsymbol{\varepsilon} \in S_d,\ \mathbf{x} \to \mathcal{E}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega. \\ (c) \text{ The mapping } \mathbf{x} \to \mathcal{E}(\mathbf{x}, \mathbf{0}) \in \mathcal{H}. \end{array}\right. \tag{3.12}$$

The visco-plasticity operator $\mathcal{G} : \Omega \times S_d \times S_d \times \mathbb{R} \to S_d$ satisfies

$$\left\{\begin{array}{l} (a) \text{ There exists a constant } L_{\mathcal{G}} > 0 \text{ such that} \\ \mid \mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1, \beta_1) - \mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \beta_2) \mid \\ \leq L_{\mathcal{G}}(\mid \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2 \mid + \mid \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 \mid + \mid \beta_1 - \beta_2 \mid) \\ \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_d \text{ and } \beta_1, \beta_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ For any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in S_d \text{ and } \beta \in \mathbb{R},\ \mathbf{x} \to \mathcal{G}(\mathbf{x}, \sigma, \boldsymbol{\varepsilon}, \beta) \text{ is Lebesgue} \\ \text{measurable on } \Omega. \\ (c) \text{ The mapping } \mathbf{x} \to \mathcal{G}(\mathbf{x}, \mathbf{0}, \mathbf{0}, 0) \in \mathcal{H}. \end{array}\right. \tag{3.13}$$

The damage source function $\phi : \Omega \times S_d \times S_d \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\left\{\begin{array}{l} (a) \text{ There exists a constant } L_{\phi} > 0 \text{ such that} \\ \mid \phi(\mathbf{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1, \beta_1) - \phi(\mathbf{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \beta_2) \mid \\ \leq L_{\phi}(\mid \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2 \mid + \mid \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 \mid + \mid \beta_1 - \beta_2 \mid) \\ \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_d \text{ and } \beta_1, \beta_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ For any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in S_d \text{ and } \beta \in \mathbb{R},\ \mathbf{x} \to \phi(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \beta) \text{ is Lebesgue} \\ \text{measurable on } \Omega. \\ (c) \text{ The mapping } \mathbf{x} \to \phi(\mathbf{x}, \mathbf{0}, \mathbf{0}, 0) \in \mathcal{H}. \end{array}\right. \tag{3.14}$$

The mass density satisfies

$$\rho \in L^\infty(\Omega), \text{ there exists } \rho^* > 0 \text{ such that } \rho(\mathbf{x}) \geq \rho^* \text{ a.e. } \mathbf{x} \in \Omega. \tag{3.15}$$

The function α satisfies

$$\alpha \in L^\infty(\Gamma_3),\ \alpha(x) \geq \alpha_* > 0 \text{ a.e. on } \Gamma_3. \tag{3.16}$$

The body forces and surface tractions have the regularity

$$\mathbf{f}_0 \in L^2(0, T; H),\ \mathbf{f}_2 \in L^2(0, T; L^2(\Gamma_2)^d). \tag{3.17}$$

Finally, we assume that the initial data satisfy

$$\mathbf{u}_0 \in V,\ \mathbf{v}_0 \in H,\ \beta_0 \in K. \tag{3.18}$$

We define the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$a(\xi, \varphi) = k \int_\Omega \nabla\xi . \nabla\varphi \, dx. \tag{3.19}$$

We will use a modified inner product on the Hilbert space $H = L^2(\Omega)^d$ given by

$$((\mathbf{u}, \mathbf{v}))_H = (\rho\mathbf{u}, \mathbf{v})_H \quad \forall \mathbf{u}, \mathbf{v} \in H,$$

that is, it is weighted with ρ, and we let $\| . \|_H$ be the associated norm, i.e.,

$$\| \mathbf{v} \|_H = (\rho\mathbf{v}, \mathbf{v})_H^{\frac{1}{2}} \quad \forall \mathbf{v} \in H.$$

It follows from assumptions (3.15) that $\| \,.\, \|_H$ and $| \,.\, |_H$ are equivalent norms on H, and also the inclusion mapping of $(V, | \,.\, |_V)$ into $(H, \| \,.\, \|_H)$ is continuous and dense. We denote by $V^{'}$ the dual space of V. Identifying H with its own dual, we can write the Gelfand triple

$$V \subset H \subset V^{'}.$$

We use the notation $(.,.)_{V' \times V}$ to represent the duality pairing between $V^{'}$ and V and recall that

$$(\mathbf{u}, \mathbf{v})_{V' \times V} = ((\mathbf{u}, \mathbf{v}))_H \quad \forall \mathbf{u} \in H, \forall \mathbf{v} \in V.$$

Assumptions (3.17) allow us, for a.e. $t \in (0, T)$, to define $\mathbf{f}(t) \in V^{'}$ by

$$(\mathbf{f}(t), \mathbf{v})_{V' \times V} = \int_\Omega \mathbf{f}_0(t).\mathbf{v}\; dx + \int_{\Gamma_2} \mathbf{f}_2(t).\mathbf{v}\; da \qquad \forall \mathbf{v} \in V, \tag{3.20}$$

and note that

$$\mathbf{f} \in L^2(0, T; V^{'}). \tag{3.21}$$

Finally, we consider the functional $j : V \times V \to \mathbb{R}$ defined by

$$j(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} \alpha \mid u_\nu \mid v_\nu \; ds. \tag{3.22}$$

Using standard arguments we obtain the following variational formulation of the mechanical problem (3.1)–(3.8).

Problem PV. Find a displacement field $\mathbf{u} : [0, T] \to V$, a stress field $\boldsymbol{\sigma} : [0, T] \to \mathcal{H}$ and a damage field $\beta : [0, T] \to H^1(\Omega)$ such that

$$\begin{aligned}\boldsymbol{\sigma}(t) &= \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)) \\ &\quad + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \beta(s))ds \qquad \text{a.e. } t \in (0, T)\,,\end{aligned} \tag{3.23}$$

$$\begin{aligned}&(\ddot{\mathbf{u}}(t), \mathbf{v})_{V' \times V} + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + j(\dot{\mathbf{u}}(t), \mathbf{v}) \\ &= (\mathbf{f}(t), \mathbf{v})_{V' \times V} \qquad \forall \mathbf{v} \in V \qquad \text{a.e. } t \in (0, T)\,,\end{aligned} \tag{3.24}$$

$$\begin{aligned}&\beta(t) \in K,\; (\dot{\beta}(t), \xi - \beta(t))_{L^2(\Omega)} + a(\beta(t), \xi - \beta(t)) \\ &\qquad \geq (\phi(\boldsymbol{\sigma}(t) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)), \boldsymbol{\varepsilon}(\mathbf{u}(t)), \beta(t)), \xi - \beta(t))_{L^2(\Omega)} \\ &\qquad \forall \xi \in K, \qquad \text{a.e. } t \in (0, T)\,,\end{aligned} \tag{3.25}$$

$$\mathbf{u}(0) = \mathbf{u}_0,\;\; \dot{\mathbf{u}}(0) = \mathbf{v}_0,\;\; \beta(0) = \beta_0. \tag{3.26}$$

4. An existence and uniqueness result

Our main existence and uniqueness result is the following.

Theorem 4.1 (Main Theorem). *Assume* (3.11)–(3.18)*. Then there exists a constant* $\alpha_0 > 0$ *which depends only on* Ω, Γ_1, Γ_3 *and* $\mathcal{A}$ *such that if* $|\alpha|_{L^\infty(\Gamma_3)} < \alpha_0$, *then problem PV has a unique solution* $\{\mathbf{u}, \boldsymbol{\sigma}, \beta\}$ *with the regularity*

$$\mathbf{u} \in H^1(0,T;V)) \cap C^1(0,T;H),\ \ddot{\mathbf{u}} \in L^2(0,T;V'), \tag{4.1}$$

$$\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H}),\ Div\, \boldsymbol{\sigma} \in L^2(0,T;V'), \tag{4.2}$$

$$\beta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)). \tag{4.3}$$

We conclude that under the assumptions (3.11)-(3.18) the mechanical problem (3.1)-(3.8) has a unique weak solution with the regularity (4.1)-(4.3), provided α is sufficiently small. The proof of this theorem will be carried out in several steps. It is based on arguments of nonlinear equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities ([20]) and a fixed point.

In the first step we let $\boldsymbol{\eta} \in L^2(0,T;V')$ be given, then there exists a constant α_0 which depends only on Ω, Γ_1, Γ_3 and $\mathcal{A}$ such that if $|\alpha|_{L^\infty(\Gamma_3)} < \alpha_0$, there exists a unique solution $\mathbf{u}_\eta$ of the following intermediate problem.

Problem PV_η. *Find a displacement field* $\mathbf{u}_\eta : [0,T] \to V$ *such that*

$$(\ddot{\mathbf{u}}_\eta(t), \mathbf{v})_{V'\times V} + (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + j(\dot{\mathbf{u}}_\eta(t), \mathbf{v}) + (\boldsymbol{\eta}(t), \mathbf{v})_{V'\times V}$$
$$= (\mathbf{f}(t), \mathbf{v})_{V'\times V} \quad \forall \mathbf{v} \in V \text{ a.e. } t \in (0,T), \tag{4.4}$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0,\ \dot{\mathbf{u}}_\eta(0) = \mathbf{v}_0. \tag{4.5}$$

In the study of the problem PV_η we have the following result.

Lemma 4.2. *There exists a unique solution to problem* PV_η *satisfying the regularity expressed in* (4.1)*.*

Proof. We use an abstract existence and uniqueness result which may be found in [20, p.48] and proceed like in [20, p.105] with another choice of the operator. □

In the second step we let $\theta \in L^2(0,T;L^2(\Omega))$ be given and consider the following variational problem for the damage field.

Problem PV_θ. *Find a damage field* $\beta_\theta : [0,T] \to H^1(\Omega)$ *such that*

$$\beta_\theta(t) \in K,\ (\dot{\beta}_\theta(t), \xi - \beta_\theta(t))_{L^2(\Omega)} + a(\beta_\theta(t), \xi - \beta_\theta(t))$$
$$\geq (\theta(t), \xi - \beta_\theta(t))_{L^2(\Omega)}\ \forall \xi \in K \text{ a.e. } t \in (0,T), \tag{4.6}$$

$$\beta_\theta(0) = \beta_0. \tag{4.7}$$

In the study of the problem PV_θ we have the following result.

Lemma 4.3. *Problem* PV_θ *has a unique solution* β_θ *satisfying*

$$\beta_\theta \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)). \tag{4.8}$$

Proof. We use a classical existence and uniqueness result on parabolic inequalities (see, e.g., [20, p.47]). □

In the third step we use the displacement field $\mathbf{u}_\eta$ obtained in Lemma 4.2 and β_θ obtained in lemma 4.3 to construct the following Cauchy problem for the stress field.

Problem $PV_{\eta\theta}$. Find a stress field $\boldsymbol{\sigma}_{\eta\theta} : [0,T] \to \mathcal{H}$ such that

$$\boldsymbol{\sigma}_{\eta\theta}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}_{\eta\theta}(s), \boldsymbol{\varepsilon}(\mathbf{u}_\eta(s)), \beta_\theta(s))ds \quad \forall t \in [0,T]. \quad (4.9)$$

In the study of problem $PV_{\eta\theta}$ we have the following result.

Lemma 4.4. *There exists a unique solution of problem $PV_{\eta\theta}$ and it satisfies $\boldsymbol{\sigma}_{\eta\theta} \in W^{1,2}(0,T,\mathcal{H})$. Moreover, if $\boldsymbol{\sigma}_i$, $\mathbf{u}_i$ and β_i represent the solutions of problem $PV_{\eta_i\theta_i}$, PV_{η_i} and PV_{θ_i}, respectively, for $(\eta_i, \theta_i) \in L^2(0,T;V' \times L^2(\Omega))$, $i = 1,2$, then there exists $C > 0$ such that*

$$| \boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t) |_{\mathcal{H}}^2 \leq C(| \mathbf{u}_1(t) - \mathbf{u}_2(t) |_V^2$$
$$+ \int_0^t | \mathbf{u}_1(s) - \mathbf{u}_2(s) |_V^2 \, ds + \int_0^t | \beta_1(s) - \beta_2(s) |_{L^2(\Omega)}^2 \, ds) \;\; \forall t \in [0,T]. \quad (4.10)$$

Proof. Let $\Lambda_{\eta\theta} : L^2(0,T,\mathcal{H}) \to L^2(0,T,\mathcal{H})$ be the operator given by

$$\Lambda_{\eta\theta}\boldsymbol{\sigma}\,(t) = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}\,(s), \boldsymbol{\varepsilon}(\mathbf{u}_\eta(s)), \beta_\theta(s))ds, \quad (4.11)$$

for all $\boldsymbol{\sigma} \in L^2(0,T,\mathcal{H})$ and $t \in [0,T]$. For $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in L^2(0,T,\mathcal{H})$ we use (4.11) and (3.13) to obtain for all $t \in [0,T]$

$$| \Lambda_{\eta\theta}\boldsymbol{\sigma}_1(t) - \Lambda_{\eta\theta}\boldsymbol{\sigma}_2(t) |_{\mathcal{H}} \leq L_{\mathcal{G}} \int_0^t | \boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s) |_{\mathcal{H}} \, ds.$$

It follows from this inequality that for p large enough, a power $\Lambda_{\eta\theta}^p$ of the operator $\Lambda_{\eta\theta}$ is a contraction on the Banach space $L^2(0,T;\mathcal{H})$ and, therefore, there exists a unique element $\boldsymbol{\sigma}_{\eta\theta} \in L^2(0,T;\mathcal{H})$ such that $\Lambda_{\eta\theta}\boldsymbol{\sigma}_{\eta\theta} = \boldsymbol{\sigma}_{\eta\theta}$. Moreover, $\boldsymbol{\sigma}_{\eta\theta}$ is the unique solution of problem $PV_{\eta\theta}$ and, using (4.9), the regularity of $\mathbf{u}_\eta$, the regularity of β_θ and the properties of the operators $\mathcal{E}$ and $\mathcal{G}$, it follows that $\boldsymbol{\sigma}_{\eta\theta} \in W^{1,2}(0,T,\mathcal{H})$.

Consider now $(\boldsymbol{\eta}_1, \theta_1), (\boldsymbol{\eta}_2, \theta_2) \in L^2(0,T;V' \times L^2(\Omega))$ and, for $i = 1,2$, denote $\mathbf{u}_{\eta_i} = \mathbf{u}_i, \boldsymbol{\sigma}_{\eta_i\theta_i} = \boldsymbol{\sigma}_i$ and $\beta_{\theta_i} = \beta_i$. We have

$$\boldsymbol{\sigma}_i(t) = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_i(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}_i(s), \boldsymbol{\varepsilon}(\mathbf{u}_i(s)), \beta_i(s)) \, ds \quad \forall t \in [0,T],$$

and, using the properties (3.12) and (3.13) of $\mathcal{E}$ and $\mathcal{G}$, we find

$$| \boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t) |_{\mathcal{H}}^2 \leq C(| \mathbf{u}_1(t) - \mathbf{u}_2(t) |_V^2 + \int_0^t | \mathbf{u}_1(s) - \mathbf{u}_2(s) |_V^2 \, ds$$
$$+ \int_0^t | \boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s) |_{\mathcal{H}}^2 \, ds + \int_0^t | \beta_1(s) - \beta_2(s) |_{L^2(\Omega)}^2 \, ds) \quad \forall t \in [0,T]. \quad (4.12)$$

Using now a Gronwall argument in the previous inequality we deduce (4.10), which concludes the proof of Lemma 4.4. □

Finally, as a consequence of these results and using the properties of the operator $\mathcal{G}$, the operator $\mathcal{E}$ and the function ϕ, for $t \in [0, T]$, we consider the element

$$\Lambda(\boldsymbol{\eta}, \theta)(t) = (\Lambda^1(\boldsymbol{\eta}, \theta)(t), \Lambda^2(\boldsymbol{\eta}, \theta)(t)) \in V' \times L^2(\Omega), \tag{4.13}$$

defined by the equalities

$$(\Lambda^1(\boldsymbol{\eta}, \theta)(t), \mathbf{v})_{V' \times V} = (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}$$
$$+ (\int_0^t \mathcal{G}(\boldsymbol{\sigma}_{\eta\theta}(s), \boldsymbol{\varepsilon}(\mathbf{u}_\eta(s)), \beta_\theta(s))ds, \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} \ \forall \mathbf{v} \in V, \tag{4.14}$$

$$\Lambda^2(\boldsymbol{\eta}, \theta)(t) = \phi(\boldsymbol{\sigma}_{\eta\theta}(t), \boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)), \beta_\theta(t)). \tag{4.15}$$

Here, for every $(\boldsymbol{\eta}, \theta) \in L^2(0, T; V' \times L^2(\Omega))$, $\mathbf{u}_\eta$, β_θ and $\boldsymbol{\sigma}_{\eta\theta}$ represent the displacement field, the damage field and the stress field obtained in lemmas 4.2, 4.3 and 4.4 respectively. We have the following result.

Lemma 4.5. *The operator Λ has a unique fixed point $(\boldsymbol{\eta}^*, \theta^*) \in L^2(0, T; V' \times L^2(\Omega))$ such that $\Lambda(\boldsymbol{\eta}^*, \theta^*) = (\boldsymbol{\eta}^*, \theta^*)$.*

Proof. Let now $(\boldsymbol{\eta}_1,\theta_1), (\boldsymbol{\eta}_2,\theta_2) \in L^2(0, T; V' \times L^2(\Omega))$. We use the notation $\mathbf{u}_{\eta_i} = \mathbf{u}_i, \dot{\mathbf{u}}_{\eta_i} = \mathbf{v}_{\eta_i} = \mathbf{v}_i, \boldsymbol{\sigma}_{\eta_i\theta_i} = \boldsymbol{\sigma}_i, \beta_{\theta_i} = \beta_i$ for $i = 1, 2$. Using (3.9), (3.12) and (3.13), we have

$$| \Lambda^1(\boldsymbol{\eta}_1,\theta_1)(t) - \Lambda^1(\boldsymbol{\eta}_2,\theta_2)(t) |_{V'}^2$$
$$\leq | \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_1(t)) - \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_2(t)) |_{\mathcal{H}}^2$$
$$+ \int_0^t | \mathcal{G}(\boldsymbol{\sigma}_1(s), \boldsymbol{\varepsilon}(\mathbf{u}_1(s)), \beta_1(s)) - \mathcal{G}(\boldsymbol{\sigma}_2(s), \boldsymbol{\varepsilon}(\mathbf{u}_2(s)), \beta_2(s)) |_{\mathcal{H}}^2 \, ds$$
$$\leq C(| \mathbf{u}_1(t) - \mathbf{u}_2(t) |_V^2 + \int_0^t | \boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s) |_{\mathcal{H}}^2 \, ds$$
$$+ \int_0^t | \mathbf{u}_1(s) - \mathbf{u}_2(s) |_V^2 \, ds + \int_0^t | \beta_1(s) - \beta_2(s) |_{L^2(\Omega)}^2 \, ds). \tag{4.16}$$

We use (4.10) to obtain

$$| \Lambda^1(\boldsymbol{\eta}_1,\theta_1)(t) - \Lambda^1(\boldsymbol{\eta}_2,\theta_2)(t) |_{V'}^2$$
$$\leq C(| \mathbf{u}_1(t) - \mathbf{u}_2(t) |_V^2 + \int_0^t | \mathbf{u}_1(s) - \mathbf{u}_2(s) |_V^2 \, ds$$
$$+ \int_0^t | \beta_1(s) - \beta_2(s) |_{L^2(\Omega)}^2 \, ds). \tag{4.17}$$

By a similar argument, from (4.15), (4.10) and (3.14) it follows that

$$| \Lambda^2(\boldsymbol{\eta}_1,\theta_1)(t) - \Lambda^2(\boldsymbol{\eta}_2,\theta_2)(t) |_{L^2(\Omega)}^2$$
$$\leq C(| \boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t) |_{\mathcal{H}}^2 + | \mathbf{u}_1(t) - \mathbf{u}_2(t) |_V^2 + | \beta_1(t) - \beta_2(t) |_{L^2(\Omega)}^2)$$
$$\leq C(| \mathbf{u}_1(t) - \mathbf{u}_2(t) |_V^2 + \int_0^t | \mathbf{u}_1(s) - \mathbf{u}_2(s) |_V^2 \, ds$$
$$+ | \beta_1(t) - \beta_2(t) |_{L^2(\Omega)}^2 + \int_0^t | \beta_1(s) - \beta_2(s) |_{L^2(\Omega)}^2 \, ds). \tag{4.18}$$

Therefore,

$$\begin{aligned}
&| \Lambda(\boldsymbol{\eta}_1,\theta_1)(t) - \Lambda(\boldsymbol{\eta}_2,\theta_2)(t) |^2_{V'\times L^2(\Omega)} \leq C(| \mathbf{u}_1(t) - \mathbf{u}_2(t) |^2_V \\
&+ \int_0^t | \mathbf{u}_1(s) - \mathbf{u}_2(s) |^2_V \, ds + | \beta_1(t) - \beta_2(t)) |^2_{L^2(\Omega)} \\
&+ \int_0^t | \beta_1(s) - \beta_2(s) |^2_{L^2(\Omega)} \, ds). \qquad (4.19)
\end{aligned}$$

Moreover, from (4.4) we obtain that

$$(\dot{\mathbf{v}}_1 - \dot{\mathbf{v}}_2, \mathbf{v}_1 - \mathbf{v}_2)_{V'\times V} + (\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{v}_1) - \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{v}_2), \boldsymbol{\varepsilon}(\mathbf{v}_1 - \mathbf{v}_2))_{\mathcal{H}}$$

$$+j(\mathbf{v}_1, \mathbf{v}_1 - \mathbf{v}_2) - j(\mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2) + (\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2, \mathbf{v}_1 - \mathbf{v}_2)_{V'\times V} = 0 \quad \text{a.e. } t \in \ (0,T)\,.$$

We integrate this equality with respect to time, we use the initial conditions $\mathbf{v}_1(0) = \mathbf{v}_2(0) = \mathbf{v}_0$, condition (3.10), (3.11) and (3.22) to find

$$\begin{aligned}
&(m_{\mathcal{A}} - C_0^2 \mid \alpha \mid_{L^\infty(\Gamma_3)}) \int_0^t | \mathbf{v}_1(s) - \mathbf{v}_2(s) |^2_V \, ds \\
&\leq - \int_0^t (\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s), \mathbf{v}_1(s) - \mathbf{v}_2(s))_{V'\times V} ds,
\end{aligned}$$

for all $t \in [0,T]$. Then, using the inequality $2ab \leq \frac{a^2}{\gamma} + \gamma b^2$ we obtain

$$\int_0^t | \mathbf{v}_1(s) - \mathbf{v}_2(s) |^2_V \, ds \leq C \int_0^t | \boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s) |^2_{V'} \, ds \quad \forall t \in [0,T]\,. \quad (4.20)$$

On the other hand, from (4.6) we have

$$\begin{aligned}
&(\dot{\beta}_1 - \dot{\beta}_2, \beta_1 - \beta_2)_{L^2(\Omega)} + a(\beta_1 - \beta_2, \beta_1 - \beta_2) \\
&\leq (\theta_1 - \theta_2, \beta_1 - \beta_2)_{L^2(\Omega)} \text{ a.e. } t \in (0,T)\,.
\end{aligned}$$

Integrating the previous inequality on $[0,t]$, after some manipulations we obtain

$$\frac{1}{2} | \beta_1(t) - \beta_2(t) |^2_{L^2(\Omega)} \leq C \int_0^t < \theta_1(s) - \theta_2(s), \beta_1(s) - \beta_2(s) >_{L^2(\Omega)} ds,$$

which implies that

$$| \beta_1(t) - \beta_2(t) |^2_{L^2(\Omega)} \leq \int_0^t | \theta_1(s) - \theta_2(s) |^2_{L^2(\Omega)} \, ds + \int_0^t | \beta_1(s) - \beta_2(s) |^2_{L^2(\Omega)} \, ds. \quad (4.21)$$

Applying Gronwall's inequality to the previous inequality, we find

$$| \beta_1(t) - \beta_2(t) |^2_{L^2(\Omega)} \leq C \int_0^t | \theta_1(s) - \theta_2(s) |^2_{L^2(\Omega)} \, ds. \qquad (4.22)$$

Since $\mathbf{u}_1(0) = \mathbf{u}_2(0) = \mathbf{u}_0$, we have

$$|\mathbf{u}_1(t) - \mathbf{u}_2(t)|^2_V \leq C \int_0^t | \mathbf{v}_1(s) - \mathbf{v}_2(s) |^2_V \, ds,$$

we consider the previous inequality and (4.19) to obtain

$$| \Lambda(\boldsymbol{\eta}_1,\theta_1)(t) - \Lambda(\boldsymbol{\eta}_2,\theta_2)(t) |^2_{V' \times L^2(\Omega)} \leq C(\int_0^t | \mathbf{v}_1(s) - \mathbf{v}_2(s) |^2_V \, ds$$

$$+ | \beta_1(t) - \beta_2(t) |^2_{L^2(\Omega)} + \int_0^t | \beta_1(s) - \beta_2(s) |^2_{L^2(\Omega)} \, ds).$$

It follows now from the previous inequality, the estimates (4.20) and (4.22) that

$$| \Lambda(\boldsymbol{\eta}_1,\theta_1)(t) - \Lambda(\boldsymbol{\eta}_2,\theta_2)(t) |^2_{V' \times L^2(\Omega)}$$
$$\leq C \int_0^t | (\boldsymbol{\eta}_1, \theta_1)(s) - (\boldsymbol{\eta}_2, \theta_2)(s) |^2_{V' \times L^2(\Omega)} \, ds.$$

Reiterating this inequality m times leads to

$$| \Lambda^m(\boldsymbol{\eta}_1,\theta_1) - \Lambda^m(\boldsymbol{\eta}_2,\theta_2) |^2_{L^2(0,T;V' \times L^2(\Omega))}$$
$$\leq \frac{C^m T^m}{m!} | (\boldsymbol{\eta}_1, \theta_1) - (\boldsymbol{\eta}_2, \theta_2) |^2_{L^2(0,T;V' \times L^2(\Omega))} .$$

Thus, for m sufficiently large, Λ^m is a contraction on the Banach space $L^2(0, T; V' \times L^2(\Omega))$, and so Λ has a unique fixed point. □

Now, we have all the ingredients to prove Theorem 4.1.

Proof. Let $(\boldsymbol{\eta}^*, \theta^*) \in L^2(0, T; V' \times L^2(\Omega))$ be the fixed point of Λ defined by (4.13)-(4.15) and denote

$$\mathbf{u} = \mathbf{u}_{\eta^*}, \beta = \beta_{\theta^*},$$

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) + \boldsymbol{\sigma}_{\eta^*\theta^*}. \tag{4.24}$$

We prove that the triplet $(\mathbf{u}, \boldsymbol{\sigma},\beta)$ satisfies (3.23)-(3.26) and (4.1)-(4.3). Indeed, we write (4.9) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$, $\theta = \theta^*$ and use (4.23)-(4.24) to obtain that (3.23) is satisfied. Then we use (4.4) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ and use the first equality in (4.23) to find

$$(\ddot{\mathbf{u}}(t), \mathbf{v})_{V' \times V} + (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + (\boldsymbol{\eta}^*(t), \mathbf{v})_{V' \times V}$$
$$+ j(\dot{\mathbf{u}}(t), \mathbf{v}) = (\mathbf{f}(t), \mathbf{v})_{V' \times V} \quad \forall \mathbf{v} \in V \quad \text{a.e. } t \in (0, T) . \tag{4.25}$$

Equalities $\Lambda^1(\boldsymbol{\eta}^*, \theta^*) = \boldsymbol{\eta}^*$ and $\Lambda^2(\boldsymbol{\eta}^*, \theta^*) = \theta^*$ combined with (4.14), (4.15), (4.23) and (4.24) show that

$$(\boldsymbol{\eta}^*(t), \mathbf{v})_{V' \times V} = (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}$$
$$+ (\int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \beta(s)) ds, \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} \ \forall \mathbf{v} \in V, \tag{4.26}$$

$$\theta^*(t) = \phi(\boldsymbol{\sigma}(t) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)), \boldsymbol{\varepsilon}(\mathbf{u}(t)), \beta(t)). \tag{4.27}$$

We now substitute (4.26) in (4.25) and use (3.23) to see that $(\mathbf{u}, \boldsymbol{\sigma},\beta)$ satisfies (3.24). We write (4.6) for $\theta = \theta^*$ and use (4.27) to find that $(\mathbf{u}, \boldsymbol{\sigma},\beta)$ satisfies (3.25). Next, (3.26) and the regularity (4.1) and (4.3) follow from Lemmas

4.2 and 4.3. The regularity $\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H})$ follows from Lemmas 4.2, 4.4, assumption (3.11) and (4.24). Finally (3.24) implies that

$$\rho\ddot{\mathbf{u}}(t) = Div\ \boldsymbol{\sigma}(t) + \mathbf{f}_0(t) \quad \text{in } V^{'} \text{ a.e. } t \in (0,T)\,,$$

and therefore by (3.15) and (3.17) we obtain that $Div\ \boldsymbol{\sigma} \in L^2(0,T;V^{'})$. We deduce that the regularity (4.2) holds which concludes the existence part of Theorem 4.1. The uniqueness part of Theorem 4.1 is a consequence of the uniqueness of the fixed point of the operator Λ defined by (4.13)-(4.15) and the unique solvability of the problems PV_η, PV_θ and $PV_{\eta\theta}$. □

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